ORTHO-POSITRONIUM ANNIHILATION:

STEPS TOWARD COMPUTING THE FIRST ORDER RADIATIVE CORRECTIONS

Thesis by

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A special thanks is due Professor A. C. Hearn, whose guest at the University of Utah the author was for five months. Professor Hearn provided the author with full access to the University of Utah time-sharing computer system, lessons in the use of Reduce language, complete software support, and extensive advice concerning the practical problems which were encountered in the course of the computation. For initiating the author into the world of computer algebra, thanks is due as well to Kevin Kay, Reudiger Loos, and

ABSTRACT

The 0.2% experimental accuracy of the 1968 Beers and Hughes measurement of the annihilation lifetime of ortho-positronium motivates the attempt to compute the first order quantum electrodynamic corrections to this lifetime. The theoretical problems arising in this computation are here studied in detail up to the point of preparing the necessary computer programs and using them to carry out some of the less demanding steps -- but the computation has not yet been completed. Analytic evaluation of the contributing Feynman diagrams is superior to numerical evaluation, and for this process can be carried out with the aid of the Reduce algebra manipulation computer program.

The relation of the positronium decay rate to the electron-positron annihilation-in-flight amplitude is derived in detail, and it is shown that at threshold annihilation-in-flight, Coulomb divergences appear while infrared divergences vanish. The threshold Coulomb divergences in the amplitude cancel against like divergences in the modulating continuum wave function.

Using the lowest order diagrams of electron-positron annihilation into three photons as a test case, various pitfalls of computer algebraic manipulation are discussed along with ways of avoiding them. The computer manipulation of artificial polynomial expressions is preferable to the direct treatment of rational expressions, even though redundant variables may have to be introduced.

Special properties of the contributing Feynman diagrams are discussed, including the need to restore gauge invariance to the sum of the virtual photon-photon scattering box diagrams by means of a finite subtraction.

A systematic approach to the Feynman-Brown method of decomposition of single loop diagram integrals with spin-related tensor numerators is developed in detail. This approach allows the Feynman-Brown method to be straightforwardly programmed in the Reduce algebra manipulation language.

The fundamental integrals needed in the wake of the application of the Feynman-Brown decomposition are exhibited and the methods which were used to evaluate them -- primarily dispersion techniques -- are briefly discussed.

Finally, it is pointed out that while the techniques discussed have permitted the computation of a fair number of the simpler integrals and diagrams contributing to the first order correction of the ortho-positronium annihilation rate, further progress with the more complicated diagrams and with the evaluation of traces is heavily contingent on obtaining access to adequate computer time and core capacity.

DEDICATION

This thesis is dedicated to Siao Tieh, my wife. Her
gentle patience and selfless sacrifice helped me to somehow overcome
what have sometimes seemed to be cruelly endless setbacks and
obstacles.

TABLE OF CONTENTS

INTRODUCTION

In 1968 Beers and Hughes[1] measured the annihilation rate of ortho-positronium to be $(7.262 \pm .015) \times 10^6$/sec, the percentage experimental error being about 0.2%.[2] As was expected, this accurate measurement was discrepant by significantly more than its experimental error with the annihilation rate of $(7.212 \pm .004) \times 10^6$/sec computed from lowest order quantum electrodynamics[2] by Ore and Powell[3].

The author set out to compute the first order corrections to the lowest order result found by Ore and Powell. Because of the great labor involved in this computation, it is as of this writing still incomplete, but by utilizing the Reduce[4] algebra manipulation program on some of the largest computers presently available, the author hopes to complete the calculation in the next eight to ten months. This thesis is a progress report on the calculation, the theoretical and computational details of which have been largely

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[1] R. H. Beers and V. W. Hughes, Amer. Phys. Soc. Bul. 13, 633 (1968).

[2] V. W. Hughes, "Muonium and Positronium," Physics of the One and Two Electron Atoms, ed. F. Bopp and H. Kleinpoppen (North Holland, Amsterdam, 1970), p. 407.

[3] A. Ore and J. L. Powell, Phys. Rev. 75, 1696 (1949).

[4] A. C. Hearn, "Reduce 2 User's Manual" (Stanford University Computer Science Department Report 181, 1970).

comprehended, but whose completion demands a further investment of time and, even more crucially, of computer resources.

The numerical integration of Feynman diagrams is an extraordinarily tricky business. Partly this is because multi-dimensional integrals over spiky rational functions are involved and partly it is because of the existence of ultraviolet, infrared, and even Coulomb divergences. Standard subtraction procedures are generally used to eliminate the ultraviolet divergences, but this, of course, tends to define the integrand as a small difference of large numbers in the ultraviolet region. Infrared divergences can sometimes be so intractable that curve fitting with increasingly smaller values of the photon mass must be resorted to.[5] Coulomb divergences have not, to the author's knowledge, occurred in Feynman diagrams on which numerical integration has been attempted, but they are endemic to higher order positronium calculations. Finally, most diagram integrals which have been treated numerically, such as those contributing to the anomalous electron magnetic moment, yield single numerical values, while the corresponding diagram integrals for the three photon annihilation of ortho-positronium yield functions of two variables.

Thus, it seems that the exact analytic integration of all Feynman diagrams contributing to the correction of the anni-

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[5]Private conversation with Jaques Calmet.

hilation rate of ortho-positronium is preferable to attempted numerical integration. Fortunately, through the use of dispersion and double dispersion methods as well as the computerized application of an extension of a method of Feynman and Brown for algebraically calculating numerator factors in a single loop diagram integrals[6], is now appears quite feasible, though tedious and demanding of considerable computer time, to carry out all the integrals analytically. Traces can be carried out quite straightforwardly, if again at a cost of considerable computer time, by also using the Reduce algebra program. Finally, the resulting differential annihilation rate, expected on physical groundsto be a fairly smooth function, can be satisfactorily integrated numerically over its two variables to give the total annihilation rate.

The author has recently learned that Pascual and de Rafael have published a numerical computation of the photon-photon scattering contribution -- given by six diagrams of the type shown in Figure 8 -- to the ortho-positronium annihilation rate.[7] Numerical integration was aided by the lack of Coulomb, infrared, and ultraviolet divergences in the sum of these six diagrams, but there is no known reason for these diagrams to be dominant.

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[6]L. M. Brown and R. P. Feynman, Phys. Rev. 85, 231 (1952), pp. 243-244.

[7]P. Pascual and E. de Rafael, Lett. al Nuovo Cim., 4, 1144 (1970).

CHAPTER I

THE THEORETICAL GROUNDWORK

The first theoretical problem that must be examined in
positronium annihilation is the relationship of the electron-positron
annihilation-in-flight amplitude to the positronium annihilation
amplitude. We will here generalize a treatment given by Jauch and
Rohrlich[1] such that its validity is extended at least up to cor-
rections of first order in the fine structure constant α. The
author can discern no reason in principle why this generalization
should not be valid in yet higher order corrections, but this
contention hasn't been examined in great detail. An alternative
and computationally more difficult approach is followed by Harris
and Brown in their treatment of the first order corrections to
para-positronium annihilation.[2] Note that throughout this thesis
the units and conventions used are those of Bjorken and Drell,
but with their optional spinor normalization of $\bar{u}(p,s)u(p,s)=2m$.[3]

The different interaction forces between electron and
positron may be considered in terms of the typical distances over

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[1] J. M. Jauch and F. Rohrlich, The Theory of Photons and
Electrons (Addison-Wesley, Reading, Mass., 1955), p. 283.

[2] I. Harris and L. M. Brown, Phys. Rev. 105, 1656 (1957).

[3] J. D. Bjorken and S. D. Drell, Relativistic Quantum
Mechanics (McGraw-Hill, New York, 1964).

which they produce variations in the electron-positron quantum state function. Thus, the static Coulomb potential produces variations typically over a Bohr radius (1/αm), while annihilation itself, being a crossed Compton scattering process, produces variations over about a Compton wavelength (1/m). All the other interactions, including among others, the spin-orbit, spin-spin, and virtual annihilation forces, produce variations over about a Compton wavelength or less. The static Coulomb potential, then, occupies a special position in virtue of the "long range" variations it produces.

The perturbation Feynman diagram approach breaks down in any finite order as a description of a bound state such as positronium. The failure is most severe at large electron-positron separations. But it is just at such large distances that the positronium wave function is shaped essentially entirely by the static Coulomb potential. Thus, the static Coulomb potential must be treated properly to all orders in positronium, which can be achieved by making use of the Schroedinger Coulomb bound state wave functions. To obtain the positronium annihilation amplitude, these Schroedinger Coulomb bound state wave functions must be folded into the electron-positron annihilation amplitude computed to a given order in perturbation theory, as set forth by Jauch and Rohrlich.[1] However, the perturbation theory electron-positron annihilation amplitude already contains the effect of the static Coulomb potential

up to a certain order. To avoid double counting the static Coulomb contribution, the electron-positron annihilation amplitude must be Coulomb "purified" to the appropriate order by being unfolded from the static Coulomb Schroedinger <u>continuum</u> wave function (which need only be computed to that order). This Coulomb "purified" electron-positron annihilation amplitude is dubbed $M^P(\vec{p}, \vec{q})$ and may be considered to be defined by equation (4), given further on, in which $M(\vec{p}\,', \vec{q}\,')$ is the ordinary electron-positron annihilation amplitude and $\emptyset_{\vec{p}}(\vec{p}\,')$ is the momentum space Schroedinger Coulomb continuum wave function. The assertion that the positronium annihilation amplitude $M(E_n)$ is obtained by folding the Schroedinger Coulomb bound state wave function into the Coulomb "purified" electron-positron annihilation amplitude $M^P(\vec{p}, \vec{q})$ is expressed formally by equation (3).

$M^P(\vec{p}, \vec{q})$, since it has been purged of the static Coulomb influence, possesses, as discussed above, a Compton wavelength variation property which can be expressed schematically in terms of its $\vec{p}$ and $\vec{q}$ gradients at zero momenta:

$$\frac{\partial^{k+j} M^P(\vec{p}, \vec{q})}{\partial p^k \, \partial q^j} \bigg|_{\vec{p}=\vec{q}=0} \sim (1/m)^{k+j} M^P(0,0) \tag{1}$$

To supply the Coulomb effects missing from $M^P(\vec{p}, \vec{q})$, there are the standard Schroedinger center of mass Coulomb wave functions, both those of the bound states $\Psi_{E_n}(\vec{r})$ and those

of the continuum states $\Psi_{\overrightarrow{p}}(\overrightarrow{r})$. The $\Psi_{\overrightarrow{p}}(\overrightarrow{r})$ satisfy:

$$\lim_{\alpha \to 0} \Psi_{\overrightarrow{p}}(\overrightarrow{r}) = \frac{e^{i\overrightarrow{p}\cdot\overrightarrow{r}}}{\sqrt{V}} \tag{2}$$

V is the very large quantization volume, and all the $\Psi(\overrightarrow{r})$ are normalized to one in this volume. Actually, the perturbative plane wave condition (2) cannot be satisfied for the Coulomb potential. We get around this by assigning the photon a very small mass λ in order to truncate the Coulomb potential:

$$-\alpha(1/r) \to -\alpha(e^{-\lambda r}/r)$$

Of course, in the field theoretic part of the calculation, the same small photon mass λ is again needed.

We Fourier transform the Coulomb wave functions to momentum space:

$$\phi_{E_n}\left(\frac{\overrightarrow{p}'-\overrightarrow{q}'}{2}\right) = \frac{\sqrt{V}}{(2\pi)^3} \int_V d^3\overrightarrow{r}\ \Psi_{E_n}(\overrightarrow{r})\ e^{-i\left(\frac{\overrightarrow{p}'-\overrightarrow{q}'}{2}\right)\cdot\overrightarrow{r}}$$

$$\phi_{\overrightarrow{p}}\left(\frac{\overrightarrow{p}'-\overrightarrow{q}'}{2}\right) = \frac{\sqrt{V}}{(2\pi)^3} \int_V d^3\overrightarrow{r}\ \Psi_{\overrightarrow{p}}(\overrightarrow{r})\ e^{-i\left(\frac{\overrightarrow{p}'-\overrightarrow{q}'}{2}\right)\cdot\overrightarrow{r}}$$

Now, in the manner of Jauch and Rohrlich,[1] we have that
the amplitude $M(E_n)$ for positronium at rest to annihilate from the n'th
Coulomb bound state   may be expressed as:

$$M(E_n) = \int d^3\vec{p}' \; d^3\vec{q}' \; M^P(\vec{p}',\vec{q}')\delta^{(3)}(\vec{p}'+\vec{q}') \; \phi_{E_n}(\frac{\vec{p}'-\vec{q}'}{2}) \tag{3}$$

The total amplitude for electron-positron annihilation
in flight $M(\vec{p}, \vec{q})$, as is computed by field theoretic methods, may
be expressed as:

$$M(\vec{p},\vec{q}) = \int d^3\vec{p}' \; d^3\vec{q}' \; M^P(\vec{p}',\vec{q}') \; \delta^{(3)}(\vec{p}'+\vec{q}'-\vec{p}-\vec{q})\phi_{(\frac{\vec{p}-\vec{q}}{2})}(\frac{\vec{p}'-\vec{q}'}{2})$$

$$\tag{4}$$

It is worthwhile to again note that (4) may be regarded
as the definition of $M^P(\vec{p}', \vec{q}')$ while (3) is a recipe for computing
$M(E_n)$, set forth in the spirit of Jauch and Rohrlich.[1]

Moving all expressions to the center of mass frame, we
write $M^P(\vec{p}') \equiv M^P(\vec{p}', -\vec{p}')$ and $M(\vec{p}) \equiv M(\vec{p}, -\vec{p})$.

Thus:

$$M(E_n) = \int d^3\vec{p}' \; M^P(\vec{p}') \; \phi_{E_n}(\vec{p}') \tag{5}$$

$$M(\vec{p}) = \int d^3\vec{p}' \ M^P(\vec{p}') \ \phi_{\vec{p}}(\vec{p}') \tag{6}$$

Because of the perturbative plane wave condition (2), we may write:

$$\phi_{\vec{p}}(\vec{p}') = \delta^{(3)}(\vec{p}-\vec{p}') + \alpha\Omega_{\vec{p}}(\vec{p}')$$

Thus, we may rewrite (6) in the form:

$$M^P(\vec{p}) = M(\vec{p}) - \alpha\int d^3\vec{p}' \ \Omega_{\vec{p}}(\vec{p}') \ M^P(\vec{p}') \tag{7}$$

If $M(\vec{p})$ is computed from field theory, (7) may be regarded as an integral equation for $M^P(\vec{p}')$. Its iteration solution yields $M^P(\vec{p})$ as a perturbation series in α. This $M^P(\vec{p})$ may then be inserted into (5) to yield the desired positronium annihilation amplitude. In line with the perturbative approach, and making use of (1), a further simplification of (5) and (7) can be made.

$M^P(\vec{p})$ may be expanded about $\vec{p} = 0$:

$$M^P(\vec{p}) = M^P(0) + \vec{p}\cdot\nabla \ M^P\Big|_{\vec{p}=0} + \tfrac{1}{2}p_i p_j \frac{\partial^2 M^P}{\partial p_i \ \partial p_j}\Big|_{\vec{p}=0} + \cdots$$

Also:

$$\int d^3\vec{p}\; p_i p_j \;\cdots\; \phi(\vec{p}) = \sqrt{V}\left(-i\frac{\partial}{\partial r_i}\right)\left(-i\frac{\partial}{\partial r_j}\right)\cdots \Psi(\vec{r})\Big|_{\vec{r}\,=\,0}$$

Denoting the Fourier transform of $\Omega_{\vec{p}}(\vec{p}\,')$ as $\omega_{\vec{p}}(\vec{r})$, we have:

$$M(E_n) = \sqrt{V}\left(M^P(0)\Psi_{E_n}(0) - i\left(\nabla_{\vec{p}} M^P(0)\right)\cdot\left(\nabla_{\vec{r}}\Psi_{E_n}(0)\right)\right.$$

$$\left. -\frac{1}{2}\frac{\partial^2 M^P}{\partial p_i \partial p_j}\Big|_{\vec{p}=0}\;\frac{\partial^2 \Psi_{E_n}}{\partial r_i \partial r_j}\Big|_{\vec{r}=0} + \cdots\right)\tag{8}$$

$$M^P(\vec{p}) = M(\vec{p}) - \alpha\sqrt{V}\left(M^P(0)\,\omega_{\vec{p}}(0) - i\left(\nabla_{\vec{p}} M^P(0)\right)\cdot\left(\nabla_{\vec{r}}\,\omega_{\vec{p}}(0)\right)\right.$$

$$\left. -\frac{1}{2}\frac{\partial^2 M^P}{\partial p_i \partial p_j}\Big|_{\vec{p}=0}\;\frac{\partial^2 \omega_{\vec{p}}}{\partial r_i \partial r_j}\Big|_{\vec{r}=0} + \cdots\right)\tag{9}$$

It is worthwhile to note that in some cases where integrals such as $\int d^3\vec{p}\;\vec{p}\,\phi(\vec{p})$ are well defined, the corresponding gradients $\nabla_{\vec{r}}\Psi(\vec{r})\Big|_{\vec{r}=0}$ can involve such ill-defined expressions as $\lim\limits_{\vec{r}\to 0}\frac{\vec{r}}{|\vec{r}|}$. A convenient recipe for properly resolving such ambiguities is to define gradients in terms of central differences, e.g.:

$$\hat{u} \cdot \nabla_{\vec{r}} \Psi(\vec{r}) = \lim_{h \to 0} \frac{\Psi(\vec{r} + h\hat{u}) - \Psi(\vec{r} - h\hat{u})}{2h} \tag{10}$$

For the gradients of Coulomb wave functions, we have the following "range" conditions to add to (1):

$$\left. \frac{\partial^k \Psi_{E_n}}{\partial r^k} \right|_{\vec{r}=0} \sim (\alpha m)^k \Psi_{E_n}(0)$$

$$\lim_{\vec{p} \to 0} \left. \frac{\partial^k \Psi_{\vec{p}}}{\partial r^k} \right|_{\vec{r}=0} \sim (\alpha m)^k \lim_{\vec{p} \to 0} \Psi_{\vec{p}}(0)$$

Thus, for small values of $|\vec{p}|$, both (8) and (9) are perturbation series in powers of α. Of course, (9) need only be solved (by iteration) for $M^P(\vec{p})$ at small values of $|\vec{p}|$ to provide all the gradients of $M^P(\vec{p})$ at $\vec{p} = 0$ needed in (8).

In the present calculation, we only need go to corrections of first order in α. Solving (9) to this order, we have for $|\vec{p}| \ll m$:

$$M^P(\vec{p}) = M(\vec{p}) - \alpha \sqrt{V} M(0) \omega_{\vec{p}}(0) + O(\alpha^2 M^P(0)) \tag{11}$$

We are considering the annihilation of the 1^3S_1 state with wave function $\Psi_{E_1}(\vec{r})$, which satisfies the Schroedinger equation:

$$\left(-\frac{1}{m}\nabla^2 - \frac{\alpha}{r}\right)\Psi_{E_1}(\vec{r}) = E_1\,\Psi_{E_1}(\vec{r})$$

For this ground state, we have:

$$\Psi_{E_1}(\vec{r}) = \frac{m^{3/2}\,\alpha^{3/2}}{\sqrt{8\pi}}\; e^{\frac{-m\alpha r}{2}} \tag{12}$$

and $E_1 = -\dfrac{m\alpha^2}{4}$

Since $\Psi_{E_1}(\vec{r})$ is spherically symmetric, we find in accord with our gradient definition (10), that $\nabla_{\vec{r}}\,\Psi_{E_1}(0) = 0$. Thus, to terms of relative order α, we may write (8) as:

$$M(E_1) = \sqrt{V}(M^P(0)\,\Psi_{E_1}(0))$$

$$= \sqrt{V}M(0)\,\Psi_{E_1}(0)\,(1-\alpha\sqrt{V}\omega_{\vec{p}=0}(0)) \tag{13}$$

In the second step, we have used (11).

$M(0)$ is to be computed from the Feynman diagrams for annihilation from rest up to relative order α, and $\Psi_{E_1}(0)$ follows from (12). We still need $\omega_{\vec{p}=0}(0)$. We know that:

$$\Psi_{\vec{p}=0}(\vec{r}) = \frac{1}{\sqrt{V}} + \alpha\,\omega_{\vec{p}=0}(\vec{r}) \tag{14}$$

We also know that $\Psi_{\vec{p}=0}(\vec{r})$ satisfies the Schroedinger equation with the small photon mass λ cutoff:

$$\left(-\frac{1}{m}\nabla^2 - \alpha\left(\frac{e^{-\lambda r}}{r}\right)\right)\Psi_{\vec{p}=0}(\vec{r}) = 0 \tag{15}$$

Substituting (14) into (15), we have to order α the equation:

$$\nabla^2 \omega_{\vec{p}=0}(\vec{r}) = -\frac{m}{\sqrt{V}}\left(\frac{e^{-\lambda r}}{r}\right) \tag{16}$$

For the spherically symmetric solution of (16), we have:

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r\,\omega_{\vec{p}=0}(r)\right) = -\frac{m}{\sqrt{V}}\left(\frac{e^{-\lambda r}}{r}\right)$$

$$\frac{d^2}{dr^2}\left(r\,\omega_{\vec{p}=0}(r)\right) = -\frac{m}{\sqrt{V}}\,e^{-\lambda r}$$

$$r\,\omega_{\vec{p}=0}(r) = -\frac{m}{\sqrt{V}\,\lambda^2}\,e^{-\lambda r} + C_1 r + C_2$$

$$\omega_{\vec{p}=0}(r) = \frac{\sqrt{V}\,\lambda^2 C_2 - m e^{-\lambda r}}{\sqrt{V}\,\lambda^2 r} + C_1$$

For large r, we expect $\Psi_{\vec{p}=0}(\vec{r}) \to 1/\sqrt{V}$, since the potential is localized. Thus, for large r, $\omega_{\vec{p}=0}(r) \to 0$, and so $C_1 = 0$. At $r = 0$, we expect $\Psi_{\vec{p}=0}(\vec{r})$ to be finite. This must also be true

of $\omega_{\vec{p} = 0}(r)$, so that $\sqrt{V} \lambda^2 C_2 = m$. Thus:

$$\omega_{\vec{p} = 0}(r) = \frac{1}{\sqrt{V}} \left(\frac{m}{\lambda}\right) \left(\frac{1 - e^{-\lambda r}}{\lambda r}\right)$$

$$\omega_{\vec{p} = 0}(0) = \frac{1}{\sqrt{V}} \left(\frac{m}{\lambda}\right) \tag{17}$$

Thus, from (12), (13), and (17) we have:

$$M(E_1) = \sqrt{V} M(0) \frac{m^{3/2} \alpha^{3/2}}{\sqrt{8\pi}} \left(1 - \frac{\alpha m}{\lambda}\right) \tag{18}$$

In the approximately $2m$ energy available to the three annihilation photons of momenta $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ and energies ω_1, ω_2, and ω_3, we may, to this first relative order in α, ignore the corrections due to the binding energy of positronium ($O(\alpha^2 m)$) and to the width of the 1^3S state ($O(\alpha^6 m)$). Thus, from the standard rate formula of perturbation theory,[3] we may write the rest frame 1^3S annihilation rate as:

$$R_1 = \left(\frac{1}{4m^2}\right) \int (2\pi)^4 \delta(2m - \omega_1 - \omega_2 - \omega_3) \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \left(\frac{4}{3}\right) \cdot$$

$$\frac{|M(E_1)|^2}{V} \frac{d^3 \vec{k}_1}{2\omega_1 (2\pi)^3} \frac{d^3 \vec{k}_2}{2\omega_2 (2\pi)^3} \frac{d^3 \vec{k}_3}{2\omega_3 (2\pi)^3} \left(\frac{1}{6}\right) \tag{19}$$

Here $\omega_i = |\vec{k}_i|$, $i = 1$, 2, 3, are the photon energies.

The 1/6 is the statistical factor for the three identical annihilation photons, while the 4/3 is inserted because it is assumed that $|M(E_1)|^2$ has been averaged over four initial fermion spin states, while the triplet level has only three such states. The kinematics of (19) is that of three photon electron-positron annihilation from rest. It is convenient to make use of the sets of kinematic variables for this process whose properties are given below:

In the rest frame $k_i = (\omega_i, \vec{k}_i)$, $i = 1$, 2, 3 and $p = (m, \vec{0})$.

$$2p = k_1 + k_2 + k_3, \quad p^2 = m^2, \quad k_1^2 = k_2^2 = k_3^2 = 0$$

$$\omega_i = \frac{p \cdot k_i}{m}, \quad i = 1, 2, 3$$

$$s \equiv (k_1 + k_2)^2 = (2p - k_3)^2 = 4m^2 - 4m\omega_3 \equiv m^2 s_3$$

$$u \equiv (k_2 + k_3)^2 = (2p - k_1)^2 = 4m^2 - 4m\omega_1 \equiv m^2 s_1 \tag{20}$$

$$t \equiv (k_3 + k_1)^2 = (2p - k_2)^2 = 4m^2 - 4m\omega_2 \equiv m^2 s_2$$

$$s + t + u = 4m^2, \quad s_1 + s_2 + s_3 = 4, \quad \omega_1 + \omega_2 + \omega_3 = 2m$$

$$0 \leq s \leq 4m^2, \quad 0 \leq t \leq 4m^2, \quad 0 \leq u \leq 4m^2, \quad 0 \leq s_1 \leq 4,$$

$$0 \leq \omega_i \leq m, \quad i = 1, 2, 3$$

In terms of the dimensionless variables s_1, s_2, and s_3, we can simplify (19) to:

$$R_1 = \left(\frac{1}{2}\right)^{13} \left(\frac{1}{3}\right)^2 \left(\frac{1}{\pi}\right)^4 \alpha^3 m^3 \int_0^4 ds \int_0^{4-s_1} ds_3 \left| M(0)\left(1 - \frac{\alpha m}{\lambda}\right)\right|^2$$

$$= \left(\frac{1}{2}\right)^{13} \left(\frac{1}{3}\right)^2 \left(\frac{1}{\pi}\right)^4 \alpha^3 m^3 \int_0^4 ds_1 \int_0^4 ds_2 \int_0^4 ds_3\, \delta(s_1 + s_2 + s_3 - 4) \left| M(0)\left(1 - \frac{\alpha m}{\lambda}\right)\right|^2$$

$$(21)$$

We turn now to the computation of $M(0)$, the electron-positron annihilation amplitude from rest into three photons.

CHAPTER II

LESSONS FROM THE LOWEST ORDER DIAGRAMS

The lowest order contribution to M(0) consists of six diagrams of the type shown in Figure 1 formed by permuting the order of the photons 1, 2, and 3 along the electron line.

The amplitude corresponding to Figure 1 can be written:

$$(-ie^3) \; \frac{\overline{v}(p)\not{e}_3(\not{k}_3 - \not{p} + m)\not{e}_2(\not{p} - \not{k}_1 + m)\not{e}_1 u(p)}{((k_3 - p)^2 - m^2)((p - k_1)^2 - m^2)} \;\; =$$

$$(\frac{4ie^3}{m^4}) \; \frac{\overline{v}(p)\not{e}_3(\not{p} - \not{k}_3 - m)\not{e}_2(\not{p} - \not{k}_1 + m)\not{e}_1 u(p)}{(4 - s_3)\,(4 - s_1)} \tag{1}$$

In spite of its simplicity, the lowest order amplitude reveals some of the subtleties and pitfalls of computer algebra. The Reduce program inexorably expands the numerator of (1) out to a sum of nine terms.[1] If instead of using the external particle momenta p, k_1, and k_3, we use the internal (virtual) momenta $q_1 = p - k_1$ and $q_3 = p - k_3$, the numerator of (1) becomes a sum of only four terms:

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[1]A. C. Hearn, "Reduce 2 User's Manual" (Stanford University Computer Science Department Report 181, 1970).

$$\left(\frac{4ie^3}{m^4}\right) \ \frac{\overline{v}(p) \not{\epsilon}_3 (\not{q}_3 - m) \not{\epsilon}_2 (\not{q}_1 + m) \not{\epsilon}_1 \ u(p)}{(4 - s_3)(4 - s_1)} \tag{2}$$

The use of virtual rather than real momenta is a useful economizing measure which is adopted in all computer portions of the calculation. The kinematic properties of the virtual momenta follow:

$$q_i = p - k_i, \quad i = 1, 2, 3 \qquad p = q_1 + q_2 + q_3$$

$$q_i^2 = m^2 \left(\frac{s_i - 2}{2}\right); \ p \cdot q_i = m^2 \frac{s_i}{4} \ ; \quad q_i \cdot q_j = m^2 \frac{s_k}{4} \ ,$$

$i, j, k = 1, 2, 3$ in cyclic order.

The symmetrization of the lowest order amplitude with respect to the three photons requires that the six permutations with respect to 1, 2, and 3 of (2) must be summed. The computer puts this sum over a common denominator and expands it out completely. This results in tens of thousands of terms!

Such a disaster is easily evaded by abbreviating:

$$s_{iw} = \frac{1}{4 - s_i} \qquad i = 1, 2, 3 \tag{3}$$

In the course of the calculation, all denominator factors which are not monomials are similarly abbreviated.

Thus, for computer manipulation, (2) is entered as:

$$\frac{4ie^3 s_{3w} s_{1w}}{m^4} \quad \not{s}_3 (\not{s}_3 - m) \not{s}_2 (\not{s}_1 + m) \not{s}_1 \tag{4}$$

The outer spinors $\overline{v}(p)$ and $u(p)$ are implicitly understood.

The use of abbreviations such as s_{1w} in addition to s_1 is a redundancy which can make complex computed expressions larger than they strictly need to be. Thus, $(4 - s_1) s_{1w}$ can be simply rewritten as 1. A large, automatically computed expression containing redundant variables can generally be written as a sum of much smaller subexpressions so chosen that each one will, in isolation, tend to contract rather than expand in size when the redundant variables are removed, a common denominator is invoked, and the greatest common divisor cancelled. Then the abbreviations can be reinstated by hand, and the shortened subexpressions resummed. As a simple example the expression $(s_1{}^2 s_{1w}^3 - 16 s_{1w}^3 + 8 s_{1w}^2) \not{s}_1 + (8 s_{3w}^2 - 2 s_3 s_{3w}^2 - s_{3w}) \not{s}_3$ is separated into the two subexpressions $(s_1{}^2 s_{1w}^3 - 16 s_{1w}^3 + 8 s_{1w}^2) \not{s}_1$ and $(8 s_{3w}^2 - 2 s_3 s_{3w}^2 - s_{3w}) \not{s}_3$. When the abbreviations are removed and a common denominator is invoked in each of the two subexpressions, there result

$$\left(\frac{s_1{}^2 - 8 s_1 + 16}{64 - 48 s_1 + 12 s_1{}^2 - s_1{}^3} \right) \not{s}_1 \quad \text{and} \quad \left(\frac{4 - s_3}{16 - 8 s_3 + s_3{}^2} \right) \not{s}_3 . \quad \text{If a common}$$

denominator had been invoked before the separation into subexpressions, the result would have clearly had many more terms.

When the greatest common divisor is cancelled in each subexpression, $(\frac{1}{4-s_1})\phi_1$ and $(\frac{1}{4-s_3})\phi_3$ result. (Reduce can perform greatest common divisor elimination automatically, at the user's option.) Reinstating the abbreviations and then resumming the subexpressions, we see that the original expression has simplified to $s_{1w}\phi_1 + s_{3w}\phi_3$. Such a "shrinkage" procedure is extremely tedious, and consumes a fair amount of computer time in the running of checks to make sure the shrunken expressions are equal to the original ones. Moreover, the re-introduction of the abbreviations must be carried out by hand, a necessity that makes the "shrinkage" of extremely large expressions more labor than it is worth. Typically, computed expressions containing redundant variables can be shrunk to about one third their "raw" size. This saves both computer processing time and core in subsequent manipulations of the expressions, but such savings must be balanced against the sheer labor involved in the shrinkage process. Most of the intermediate computed expressions containing a few hundreds of terms or less will be shrunken to as compact a form as possible. For expressions running over a thousand terms, it is most likely only feasible to shrink a small fraction of the contributing subexpressions. The automatic greatest common divisor elimination option in Reduce is a most useful aid to shrinkage.[2]

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[2] Hearn, "Reduce 2 User's Manual", p. 3-9. The author learned the techniques of abbreviation and shrinkage from A. C. Hearn and R. Loos.

We denote by $\mathcal{M}_0$ the symmetrization of (4) with respect to photons 1, 2, and 3, and by $\mathcal{M}_1$ the order α correction to $\mathcal{M}_0$. $\mathcal{M}_1$, of course, includes a term $-\frac{\alpha m}{\lambda}\mathcal{M}_0$ as noted in I-(21), to cancel out the "double-counting" of the static Coulomb potential. The rest of $\mathcal{M}_1$ comes from the types of diagrams illustrated in Figures 2-9.

We have:

$$\left| M(0)(1 - \frac{\alpha m}{\lambda}) \right|^2 = \frac{1}{4} \sum_{\text{spins}} \left| \bar{v}(p)(\mathcal{M}_0 + \mathcal{M}_1)u(p) \right|^2$$

$$= \sum_{\text{photon spins}} \left[\frac{1}{4} \mathrm{Sp}((\not{p}-m)\mathcal{M}_0(\not{p}+m)\overline{\mathcal{M}}_0) \right.$$

$$\left. + 2\mathrm{Re}(\frac{1}{4}\mathrm{Sp}(\mathcal{M}_1(\not{p}+m)\overline{\mathcal{M}}_0(\not{p}-m))) \right] \tag{5}$$

The first term of (5) was computed by Ore and Powell,[3] while we propose to compute the second. It is to be noted that if $\mathcal{M}_0$ is symmetrized with respect to the six permutations of the three photons, then such symmetrization can be postponed for $\mathcal{M}_1$. Indeed, it is preferable to carry out the spin sums in the second term of (5) with the unsymmetrized $\mathcal{M}_1$, since it is only one sixth the length of the properly symmetrized $\mathcal{M}_1$. Then the symmetrization

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[3]A. Ore and J. L. Powell, Phys. Rev. 75, 1696 (1949).

of the whole of the relative order α part of $\left| M(0)(1 - \frac{\alpha m}{\lambda}) \right|^2$ can be economically carried out numerically just before the numerical integration of I-(21). Numerical symmetrization is vastly less demanding of computer resources than is explicit algebraic symmetrization. For future convenience in the evaluation of (5), the expression $2(\not{p} + m)\overline{\mathcal{M}}_0(\not{p} - m)$ has been explicitly computed. It runs to about seventy terms. Since some of the diagrams contributing to $\mathcal{M}_1$ are expected to run to many thousands of terms, it is clear that a major commitment of computer resources will be needed to compute the spin sums. Fortunately, it is expected that the spin sums themselves will be quite compact, running to no more than perhaps a hundred terms after shrinkage. Reduce tends to carry out a lengthy calculation with considerably augmented speed if the <u>result</u> of that calculation is fairly compact.

CHAPTER III

SPECIAL FEATURES OF THE FIRST ORDER CORRECTION DIAGRAMS

The most complicated of the diagrams contributing to $\mathcal{M}_1$
is that of Figure 9. We will see later that in a sense all the other
diagrams can be looked upon as arising from subparts of this pentagon.
The pentagon has no ultraviolet divergence, but it is undefined with-
out the small photon mass λ. Then it is found to give rise to a
term of the form $(\frac{\alpha m}{\lambda} + \frac{\alpha}{\pi} \log (\frac{\lambda}{m}))$ times the diagram in Figure 1.
The $\frac{\alpha m}{\lambda}$ term cancels against the like term in $-\frac{\alpha m}{\lambda} \mathcal{M}_0$ (Coulomb
correction). The $\frac{\alpha}{\pi} \log (\frac{\lambda}{m})$ term cancels against the infrared diver-
gence in the fermion wave function renormalization factor $\sqrt{Z_2}$,
which arises from the diagrams of Figure 2.

It is thus that all terms depending on the small photon
mass λ cancel. There can be no true infrared term in electron-
positron annihilation from rest, because no "acceleration" of charge
occurs.

The box diagram of Figure 8 has been noted as particularly
interesting by V. W. Hughes because of its possible relation to the
as yet experimentally inaccessible phenomenon of photon-photon
scattering.[1] However, the mass of the virtual photon (2m), far from

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[1] V. W. Hughes, "Muonium and Positronium", Physics of the
One and Two Electron Atoms, ed. F. Bopp and H. Kleinpoppen (North
Holland, Amsterdam, 1970), p. 407.

being negligible, perches the diagram directly atop a dynamical threshold. In consequence, the analytic form of this diagram seems to be rather simpler than that of true photon-photon scattering.

Although each box diagram is individually logarithmically divergent, it has often been noted that the symmetric sum of all six box diagrams is convergent.[2] However, an elementary check of this symmetrized sum reveals that it is not gauge invariant.[3] Technically, this is because the origin of integration in momentum space cannot be simply shifted in a linearly divergent integral. A correct solution is to cut off the individual box diagrams with Pauli-Villars regulators -- the origin shift can now be simply carried out since at most logarithmically divergent integrals are involved, and gauge invariance holds for the cutoff independent sum of the regularized box diagrams. The effect of the regularization procedure on the finite sum of the six diagrams is simply to subtract out a finite, gauge non-invariant polarization contact term.

Thus, for Figure 8, we must consider the following regularized integral:

[2]J. M. Jauch and F. Rohrlich, The Theory of Photons and Electrons (Addison-Wesley, Reading, Mass., 1955), p. 290.

[3]J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields (McGraw-Hill, New York, 1965), p. 200.

$$\int \frac{d^4\ell \ (-\Lambda^2) \ \mathrm{Sp} \ [\gamma^\mu(\ell\!\!\!/ + p\!\!\!/ + m)\epsilon\!\!\!/_1(\ell\!\!\!/ + q\!\!\!/_1 + m)\epsilon\!\!\!/_2(\ell\!\!\!/ - q\!\!\!/_3 + m)\epsilon\!\!\!/_3}{(\ell^2 - \Lambda^2)((\ell + p)^2 - m^2) \ ((\ell + q_1)^2 - m^2) \ ((\ell - q_3)^2 - m^2)}$$

$$\frac{(\ell\!\!\!/ - p\!\!\!/ + m)]}{((\ell - p)^2 - m^2)}$$

$$-\int \frac{d^4\ell \ (-\Lambda^2) \mathrm{Sp} \ [\gamma^\mu(\ell\!\!\!/ + M)\epsilon\!\!\!/_1(\ell\!\!\!/ + M)\epsilon\!\!\!/_2(\ell\!\!\!/ + M)\epsilon\!\!\!/_3(\ell\!\!\!/ + M)]}{(\ell^2 - \Lambda^2) \qquad (\ell^2 - M^2)^4} \tag{1}$$

where $\Lambda^2 \to \infty$.

The second integral in (1) evaluates to:

$$(\tfrac{4}{3})(i\pi^2)(\log(\frac{\Lambda^2}{M^2}) - \tfrac{5}{6})(\epsilon_1^\mu(\epsilon_3\cdot\epsilon_2) + \epsilon_3^\mu(\epsilon_1\cdot\epsilon_2) - 2\epsilon_2^\mu(\epsilon_1\cdot\epsilon_3))$$

$$+ (\tfrac{2}{3})(i\pi^2)(\epsilon_1^\mu(\epsilon_3\cdot\epsilon_2) + \epsilon_3^\mu(\epsilon_1\cdot\epsilon_2) - \epsilon_2^\mu(\epsilon_1\cdot\epsilon_3)) \tag{2}$$

Upon symmetrization, (2) becomes:

$$(\tfrac{4}{3})(i\pi^2)(\epsilon_1^\mu(\epsilon_2\cdot\epsilon_3) + \epsilon_2^\mu(\epsilon_3\cdot\epsilon_1) + \epsilon_3^\mu(\epsilon_1\cdot\epsilon_2)) \tag{3}$$

This is the finite, gauge non-invariant polarization
contact term which must be removed from the symmetric sum of the
six box diagrams in the interests of gauge invariance. It is inter-

esting to note that Dyson in 1949 apparently did not believe that this gauge violating term existed.[4]   Karplus and Neuman in their classic paper on photon-photon scattering properly disposed of it through regularization.[5] For their trouble, they earned a tongue lashing from Jauch and Rohrlich, who were mistakenly convinced that since the sum of the six box diagrams is finite, it "must" be gauge invariant.[2]

The author wishes to express the pious hope that future generations will have a better grasp of this subtle matter. It is perhaps the only instance in quantum electrodynamics where a <u>convergent</u> sum of diagrams requires a subtraction.

The diagrams in Figure 7 have neither ultraviolet nor infrared divergences. Speculations have been made that these diagrams might be dominant because of a big spike as the odd external photon goes to zero energy.[6] But the analytic form of many of the terms in these diagrams seems to be repeated in some terms of the pentagon, Figure 9.

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[4] F. J. Dyson, Phys. Rev. <u>75</u>, 1739 (1949), p. 1747.

[5] R. Karplus and M. Neuman, Phys. Rev. <u>80</u>, 380 (1950).

[6] Private conversation with J. Espinosa concerning the interpretation of an idea of C. Fronsdal.

The self energy diagrams, Figure 3, require mass renormalization counter terms, Figure 4. This being done, Ward's identity guarantees the cancellation of all further ultraviolet divergences among the vertex diagrams of Figure 5, the mass renormalized self energy diagrams of Figure 3 and 4, and the fermion wave function renormalization diagrams of Figure 2.

The photon wave function renormalization diagrams of Figure 6 are entirely absorbed into charge renormalization and require no explicit consideration.

CHAPTER IV

DECOMPOSITION OF DIAGRAMS INTO BASIC INTEGRALS

The diagrams which must be evaluated, Figures 1-9, involve at most single loop integrals. The evaluation of these integrals is considerably complicated by the presence of spin related tensor factors in the numerators of the integrands. In an all scalar theory with non-derivative coupling, the integrands would only have the number one in the numerator and products of simple scalar propagators in the denominator. Feynman and Brown pointed out that the tensors in the numerator could be expanded as linear combinations of the propagator factors occurring in the denominator.[1] Successive applications of this procedure, interspersed with preparatory shifts of the origin of the integration where necessary, allow the decomposition of an integral with a tensorial numerator into a linear combination of integrals with one in the numerator, such as occur in scalar theory. The basic single loop integrals, those with one in the numerator, form a reasonably small set, often possess fairly compact analytic forms, and tend to be especially amenable to dispersion theoretic computation. The decomposition procedure, however, is usually extremely bulky and tedious, and thus cries out for the use of computer algebra.

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[1] L. M. Brown and R. P. Feynman, Phys. Rev. 85, 231 (1952), pp. 243-244.

For programming purposes, it was found expeditious to sharpen the Feynman-Brown decomposition method into a small collection of formulas. From these it became pleasantly clear that explicit matrix inversion was only necessary for matrices of dimension at most four by four, however high the rank of the tensorial integrand. The remainder of the Feynman-Brown coefficients follow from an automatic orthogonality property. This is important, because the ability of computers to invert algebraic matrices decays rapidly with the growth of those matrices' dimensions. It was also found necessary to clarify the application of the Feynman-Brown method to divergent integrals, as well as to derive some formulas for the origin shift in such integrals. Finally, it is pointed out that difficulties arise in trying to extend the Feynman-Brown decomposition to problems involving multi-loop integrals.

Every integral occurring in the diagrams of Figure 2-9 can be written as a linear combination of Dirac gamma expressions con- tracted into integrals of the following form:

$$\int d^4\ell \; \frac{\ell^{\mu_1} \ell^{\mu_2} \dots \ell^{\mu_k}}{\prod\limits_{j=0}^{r} ((\ell-p_j)^2 - m_j^2)} \tag{1}$$

The propagator factors $[(\ell-p_j)^2 - m_j^2]$ in the denominator of (1) must all be distinct elements of the set of five propagator factors $\left\{ \ell^2, \; (\ell+p)^2-m^2, \; (\ell-p)^2-m^2, \; (\ell+p-k_1)^2-m^2, \; (\ell-p+k_3)^2-m^2 \right\}$, as we see upon examination of the diagrams. The total number of

subsets of this set of five propagator factors is $2^5 = 32$. Of course, not all 32 possible denominator types occur directly in the quantum electrodynamics, but as we decompose the numerators of (1), we will see the gaps being indirectly filled. We glimpse, then, a sense in which the pentagon of Figure 9, in which all five propagators occur is the "grand daddy" diagram. A further look at the diagrams shows that in (1) the rank k of the numerator tensor is never greater than the number $r + 1$ of propagator factors. This is, of course, a general feature of Q. E. D. single loop diagrams. In all our diagrams $k \leq 4$.

Performing a shift or origin, if necessary, we can rewrite integrals of the type given by (1) as linear combinations of integrals of the following form:

$$I^{\mu_1 \cdots \mu_k}(q_1, \ldots, q_n) \equiv \int d^4 \ell \; \frac{\ell^{\mu_1} \ell^{\mu_2} \ldots \ell^{\mu_k}}{(\ell^2 - m_0^2) \prod_{i=1}^{r} [(\ell - q_i)^2 - m_i^2]} \tag{2}$$

Of the q_i, $i = 1, \ldots, r$, we select the first n, i.e., $q_1, \ldots, q_n$, to be a maximal linearly independent set. Of course $n \leq 4$. It is crucial to note that we can now write:

$$q_i \cdot \ell = \frac{1}{2} \left\{ [\ell^2 - m_o^2] - [(\ell - q_i)^2 - m_i^2] + q_i^2 + m_o^2 - m_i^2 \right\}$$

for $i = 1, \ldots, n$

$$\ell^2 = [\ell^2 - m_o^2] + m_o^2 \tag{3}$$

The formulas (3) will, after expansion, allow the cancellation of ℓ-dependent numerator factors against the denominator.

Let us first expand (2) in the simplest case, namely when the tensor rank $k = 1$. As pointed out by Feynman and Brown:[1]

$$I^\mu(q_1, \ldots, q_n) = \sum_{i=1}^{n} a_i(q_1, \ldots, q_n) q_i^\mu$$

The $a_i(q_1, \ldots, q_n)$ are scalars and may be explicitly computed. We need to consider the Gram matrix $G_{ij} = q_i \cdot q_j$ for $i, j = 1, \ldots, n$. Since the q_i for $i = 1, \ldots, n$ are linearly independent, the Gram matrix is nonsingular[2] and may be inverted. It may then be readily verified that:

$$a_i(q_1, \ldots, q_n) = \sum_{j=1}^{n} (G^{-1})_{ij} (q_j \cdot I(q_1, \ldots, q_n))$$

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[2]An exception to this proposition will be discussed later in this chapter. See also Appendix B.

Thus:

$$I^\mu = \sum_{i=1,j=1}^{n} q_i^\mu (G^{-1})_{ij} (q_{j\alpha} I^\alpha) \tag{4}$$

The higher rank tensor integrals will involve the explicit appearance of the metric tensor $g^{\mu\nu}$ unless $n = 4$. To this end it is convenient to introduce the projective metric tensor $\mathscr{g}^{\mu\nu}$:

$$\mathscr{g}^{\mu\nu} \equiv g^{\mu\nu} - \sum_{i=1,j=1}^{n} (G^{-1})_{ij} q_i^\mu q_j^\nu \tag{5}$$

We note the "orthogonality" property:

$$q_{i\mu} \mathscr{g}^{\mu\nu} = 0 \qquad i = 1, \ldots, n \tag{6}$$

Of course, if $n = 4$, $\mathscr{g}^{\mu\nu} = 0$.

It is convenient to define the complimentary projector $\mathscr{H}^{\mu\nu} \equiv g^{\mu\nu} - \mathscr{g}^{\mu\nu}$ as well:

$$\mathscr{H}^{\mu\nu} = \sum_{i=1,j=1}^{n} (G^{-1})_{ij} q_i^\mu q_j^\nu$$

We may now write the expansions for tensor integrals up to fourth rank:

$$I^{\mu} = \mathcal{H}^{\mu}{}_{\alpha} I^{\alpha} \tag{7}$$

$$I^{\mu\nu} = \mathcal{H}^{\mu}{}_{\alpha} I^{\alpha\nu} + \frac{\mathcal{Y}^{\mu\nu}(\mathcal{Y}_{\alpha\beta} I^{\alpha\beta})}{(4-n)} \tag{8}$$

$$I^{\mu\nu\lambda} = \mathcal{H}^{\mu}{}_{\alpha} I^{\alpha\nu\lambda} + \frac{\mathcal{Y}^{\mu\nu}(\mathcal{Y}_{\alpha\beta} I^{\alpha\beta\lambda})}{(4-n)} + \frac{\mathcal{Y}^{\mu\lambda}(\mathcal{Y}_{\alpha\gamma} I^{\alpha\nu\gamma})}{(4-n)} \tag{9}$$

$$I^{\mu\nu\lambda\rho} = \mathcal{H}^{\mu}{}_{\alpha} I^{\alpha\nu\lambda\rho} + \frac{\mathcal{Y}^{\mu\nu}(\mathcal{H}^{\lambda}{}_{\gamma} \mathcal{Y}_{\alpha\beta} I^{\alpha\beta\gamma\rho})}{(4-n)}$$

$$+ \frac{\mathcal{Y}^{\mu\lambda}(\mathcal{H}^{\rho}{}_{\delta} \mathcal{Y}_{\alpha\gamma} I^{\alpha\nu\gamma\delta})}{(4-n)} + \frac{\mathcal{Y}^{\mu\rho}(\mathcal{H}^{\nu}{}_{\beta} \mathcal{Y}_{\alpha\delta} I^{\alpha\beta\lambda\delta})}{(4-n)}$$

$$\tag{10}$$

$$+ \frac{(\mathcal{Y}^{\mu\nu} \mathcal{Y}^{\lambda\rho} + \mathcal{Y}^{\mu\lambda} \mathcal{Y}^{\rho\nu} + \mathcal{Y}^{\mu\rho} \mathcal{Y}^{\nu\lambda})}{3(4-n)\ (6-n)} \cdot$$

$$(\mathcal{Y}_{\alpha\beta} \mathcal{Y}_{\gamma\delta} + \mathcal{Y}_{\alpha\gamma} \mathcal{Y}_{\delta\beta} + \mathcal{Y}_{\alpha\delta} \mathcal{Y}_{\beta\gamma})(I^{\alpha\beta\gamma\delta})$$

Note that when n = 4, all terms in (7) - (10) containing $\mathcal{J}^{\mu\nu}$ vanish.

The tensor contractions in (7) - (10) may be moved under the integral sign, and the substitutions (3) may then be linearly instituted. We define:

$$\mathcal{l}^{\mu} = \mathcal{H}^{\mu}{}_{\alpha}\, \mathcal{l}^{\alpha} = \sum_{i=1,j=1}^{n} q_i^{\mu}\, (G^{-1})_{ij}\, (q_j \cdot \ell)$$

$$= \sum_{i=1,j=1}^{n} q_i^{\mu}\, (G^{-1})_{ij}\, (\tfrac{1}{2}) \left\{ [\,\ell^2 - m_o^2\,] - [\,(\ell - q_j)^2 - m_j^2\,] + q_j^2 + m_o^2 - m_j^2 \right\}$$

$$\ell^2 = [\,\ell^2 - m_o^2\,] + m_o^2 \tag{11}$$

Using (7) - (11), we may expand the integrand numerator tensors through fourth rank as follows:

$$\ell^{\mu} \rightarrow \mathcal{l}^{\mu} \tag{12}$$

$$\ell^{\mu}\, \ell^{\nu} \rightarrow \mathcal{l}^{\mu}\, \ell^{\nu} + \frac{\mathcal{J}^{\mu\nu}(\ell^2 - \mathcal{h}_{\alpha}\, \ell^{\alpha})}{(4-n)} \tag{13}$$

$$\ell^{\mu}\, \ell^{\nu}\, \ell^{\lambda} \rightarrow \mathcal{l}^{\mu}\ell^{\nu}\ell^{\lambda} + \frac{(\mathcal{J}^{\mu\nu}\ell^{\lambda} + \mathcal{J}^{\mu\lambda}\, \ell^{\nu})(\ell^2 - \mathcal{h}_{\alpha}\, \ell^{\alpha})}{(4-n)} \tag{14}$$

$$\ell^\mu \ell^\nu \ell^\lambda \ell^\rho \rightarrow \mathcal{L}^\mu \ell^\nu \ell^\lambda \ell^\rho + (\tfrac{1}{4-n})(\mathcal{G}^{\mu\nu}\mathcal{L}^\lambda \ell^\rho + \mathcal{G}^{\mu\lambda}\mathcal{L}^\rho \ell^\nu + \mathcal{G}^{\mu\rho}\mathcal{L}^\nu \ell^\lambda)(\mathcal{G}_{\alpha\beta}\ell^\alpha \ell^\beta)$$

$$+ \frac{(\mathcal{G}^{\mu\nu}\mathcal{G}^{\lambda\rho} + \mathcal{G}^{\mu\lambda}\mathcal{G}^{\rho\nu} + \mathcal{G}^{\mu\rho}\mathcal{G}^{\nu\lambda})(\mathcal{G}_{\alpha\beta}\ell^\alpha \ell^\beta)(\ell^2 - \mathcal{L}_\gamma \ell^\gamma)}{(4-n)(6-n)}$$

$$(15)$$

Of course, when n = 4, all terms in (12) - (15) containing $\mathcal{G}^{\mu\nu}$ vanish.

The important point is that the whole ℓ-dependence of the $\mathcal{L}^\mu$ and ℓ^2 terms appearing in (12) - (15) may be cancelled against propagator factors in the denominator because of the representation (11). Since $\mathcal{L}^\mu$ and ℓ^2 appear to linear order throughout (12) - (15), the upshot is the lowering of the tensor rank of numerator terms by at least one order in all integrals in the resulting linear combination. The number of propagator factors in the denominator may decrease by up to one as well. The repeated application of (12) - (15), with interspersed origin shifts where necessary, can finally reduce an integral of type (1) to a linear combination of integrals of tensor rank zero, i.e., our 32 basic integrals.

After programming the definitions (5) and (11) and performing the substitutions (12) - (15) under the integral sign in (2), the LINEAR OPERATOR feature[3] of the Reduce language proves to be of

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[3] A new feature of Reduce which will be described in an upcoming version of A. C. Hearn's "Reduce 2 User's Manual".

particular value. It automatically performs the separation of the integral into a linear combination of integrals of the form (1) but of lower tensor rank. Then, alternating with possibly needed origin shifts, the process may be conveniently repeated until zero tensor rank is everywhere obtained. After an interspersed origin shift, the LINEAR OPERATOR feature is equally useful in restoring the form (2).

The Feynman-Brown decomposition tends to express a given integral as a linear combination of less convergent integrals, since a numerator factor ℓ^μ and a denominator factor $[(\ell-q_i)^2 - m_i^2]$ are often simultaneously eliminated. When divergent integrals finally come to be involved, it is most convenient to cut them off in a cascading manner, so that the cutoffs automatically drop out singly where appropriate.

Thus, for quadratically or cubically divergent integrals there is inserted under the integral sign of (1) the cascading double cutoff:

$$\frac{(-\Lambda_2^{\ 2})\ (-\Lambda_1^{\ 2})}{(\ell^2 - \Lambda_2^{\ 2})\ (\ell^2 - \Lambda_1^{\ 2})}$$

$$\Lambda_2^{\ 2} \gg \Lambda_1^{\ 2} \rightarrow \infty \tag{16}$$

Thus, if a linear combination of quadratically divergent integrals equals a linearly divergent integral, all terms dependent

on Λ_2 will automatically cancel. The use of cascading cutoffs permits the Feynman-Brown method to be in all cases very straight-forwardly pursued down to the 32 basic integrals.

In some cases where (1) is divergent, it is necessary to perform a shift of origin in order to obtain integrals of the form (2). Rules for performing the shift of origin of all the types of divergent integrals which occur and need shifting in this computation are listed in Appendix A. The rules are quite simple and were easily incorporated into the Reduce program.

It is interesting to point out that in one class of integrals which occur in this computation the Gram matrix is singular. This occurs when the number of linearly independent vectors n is equal to one, and that one is lightlike, e.g., a photon momentum. The formulas (12) - (15), which are the keystone of the Reduce program, then fail. In such a case the lightlike vector can be made slightly timelike, e.g., the photon given a small mass, and the Feynman-Brown program followed through as usual. Then, in the final answer, the lightlike limit is taken. In the present computation, the lightlike single vector case only occurs in integrals of up to tensor rank k = 2. These integrals were worked out by hand by the method described and included as special cases in the Reduce program. They may be found in Appendix B.

More generally, the Gram matrix fails to be non-singular for linearly independent q_i if and only if no linear combination of the q_i is timelike and there exists a linear combination of the q_i

which is lightlike. If this lightlike linear combination of the q_i is made "slightly" timelike, then the Gram matrix becomes non-singular, and the Feynman-Brown decomposition may be carried through. At the end, the lightlike limit of this linear combination is taken.

It is notable that the Feynman-Brown method treats the indices of a symmetric tensor in a highly asymmetric manner, as can be seen from an inspection of (13) - (15). Thus, making sure the final result is indeed symmetric provides an excellent overall check of the computation of integrals of type (1). Also, for tensor rank $k \geq 1$ in (1), there should be no Coulomb or infrared divergences present, providing a further check in certain cases. On the ultra-violet side, the lack of proper cancellation of the cascading cut-offs would provide a warning signal for some kinds of mistakes. In sum, it seems there are enough error warnings implicit in the computerized Feynman-Brown method to buoy one's confidence in its results.

Finally, a few words are in order about the possibility of extending the very useful Feynman-Brown method to multi-loop integrals. The main difference between the single loop and multi-loop cases appears to be that the set of basic integrals would be considerably larger for multi-loop diagrams. If we let ℓ_1, ℓ_2, $\ldots$, ℓ_n be the loop four vector variables of integration, then the numerators of the basic integrals would not just be one, as in the single loop case, but products of the following form:

$$\prod_{\substack{i=1, j=1 \\ i < j}}^{n} (\ell_i \circ \ell_j)^{\alpha_{ij}} \tag{17}$$

The α_{ij} are non-negative integers. The maximum size of the α_{ij} would be related to the maximum tensor rank of the integrals to be decomposed into linear combinations of such basic integrals. Given the considerable increase in size of the basic integral set due to the presence of numerators of the type (17), it is questionable whether this extension to multi-loop diagrams of the Feynman-Brown decomposition method could be of practical value.

CHAPTER V

COMPUTATION OF THE BASIC INTEGRALS

The basic integrals with one and two scalar propagator factors in their denominators can be computed quite generally, and the results will be displayed here. General results for integrals having three or more propagator factors are inordinately complicated and usually fairly useless for quantum electrodynamics since very difficult limits must generally be taken. Thus, the basic integrals with three or more propagator factors are computed one at a time, rather than making any attempt to obtain them from general results. These integrals are given in Appendix C, with this chapter touching only on some of the highlights of the integration methods used.

The actual carrying out of the integrations which will be so lightly touched in this chapter cost the author a very considerable amount of time and labor. Tremendous care had to be exercised whenever limits were involved, as, for example, in the Coulomb and infrared divergent basic integrals. Subtle changes of variable had to be discovered. A fairly arcane body of knowledge concerning dilogarithmic functions had to be mastered.[1] In the course of all the drudgery of integration, nothing truly new or interesting came to light. Very similar messy and, from a reader's point of view,

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[1]L. Lewin, Dilogarithms and Associated Functions (MacDonald, London, 1958).

indigestable calculations in quantum electrodynamics have appeared many times previously in the literature. However, for the reader who may be interested in reproducing or checking the evaluation of of the integrals given in Appendix C, the key steps, transformations, and formulas which the author made use of are set forth in Appendix D.

The general one propagator basic integral evaluated with cascading cutoffs is:

$$\int d^4\ell \ \frac{(-\Lambda_2^2)(-\Lambda_1^2)}{(\ell^2 - \Lambda_2^2)(\ell^2 - \Lambda_2^2)((\ell-a)^2 - \mu^2)} \tag{1}$$

$$\underbrace{\quad\quad}_{\Lambda_2^2 \gg \Lambda_1^2 \to \infty} (-i\pi^2) \ [\ (\Lambda_1^2)\log\left(\frac{\Lambda_2^2}{\Lambda_1^2}\right) - \mu^2 \log\left(\frac{\Lambda_1^2}{\mu^2}\right) + \frac{a^2}{2}\ ]$$

It is most convenient to evaluate the first rank tensor integral with a single propagator directly rather than by means of a Feynman-Brown decomposition. That is because cubically divergent integrals do not obey such simple origin shift relations as the less divergent integrals listed in Appendix A. We have:

$$\int d^4\ell \ \frac{(-\Lambda_2^2)(-\Lambda_1^2) \ \ell^\mu}{(\ell^2 - \Lambda_2^2)(\ell^2 - \Lambda_1^2)((\ell-a)^2 - \mu^2)} \qquad \underbrace{\quad\quad}_{\Lambda_2^2 \gg \Lambda_1^2 \to \infty}$$

$$(\frac{-i\pi^2}{2}) \ [\ (\Lambda_1^2)\log\left(\frac{\Lambda_2^2}{\Lambda_1^2}\right) - 2\mu^2\log\left(\frac{\Lambda_1^2}{\mu^2}\right) + \mu^2 + \frac{2}{3}a^2]\ (a^\mu) \tag{2}$$

The two propagator basic integral is not needed in this computation in full generality. Only the equal mass and the one mass equal to zero cases are necessary, and those only in the appropriate physical region. The results are given below:

$$\int d^4\ell \;\; \frac{(-\Lambda^2)}{(\ell^2-\Lambda^2)((\ell-q_1)^2-m^2)((\ell-q_2)^2-m^2)} \quad \overbrace{\Lambda^2 \to \infty}$$

$$(-i\pi^2)\left\{ -(\log(\frac{\Lambda^2}{m^2}) + 1) + 2\left(\sqrt{\frac{4m^2-Q^2}{Q^2}}\right)\arctan\left(\sqrt{\frac{Q^2}{4m^2-Q^2}}\right)\right\}$$

where $\;\; Q^2 \equiv (q_1 - q_2)^2 \; , \; 0 \le Q^2 \le 4m^2 \; .$
<div align="right">(3)</div>

$$\int d^4\ell \;\; \frac{(-\Lambda^2)}{(\ell^2 - \Lambda^2)((\ell-q_1)^2)((\ell-q_2)^2-m^2)} \quad \overbrace{\Lambda^2 \to \infty}$$

$$(-i\pi^2)\left\{ -(\log(\frac{\Lambda^2}{m^2}) + 1) + \frac{(Q^2-m^2)}{Q^2}\;\log(\frac{m^2-Q^2}{m^2})\right\}$$

where $\;\; Q^2 \equiv (q_1 - q_2)^2 \; , \; Q^2 \le m^2 \; .$
<div align="right">(4)</div>

Most of the 16 three, four, and five propagator basic
integrals listed in Appendix C were evaluated by dispersion methods
and the use of Cutkosky's rules. For every member of this set of
16 basic integrals there exists a corresponding (possibly identical)
member of the same set in which the roles of photon momenta k_1 and k_3
and thus the kinematic variables s and u have simply been inter-
changed. In Appendix C these "partners" have been omitted as under-
stood, leaving 10 rather than 16 integrals.

In the application of Cutkosky's rules to these integrals,
the threshold condition, i.e., the equality of the electron and
positron four momenta, meant that care had to be taken not to over-
look resulting multiple singularities in s or u, each of whose
discontinuities had to be separately computed and the results summed.[2]
In Appendix C these multiple singularity cases are written as a sum
of dispersion integrals.

For those four and five propagator basic integrals
dependent on both the variables s and u, it was found computationally
simplest to utilize the Mandelstam double dispersion representation.
Cutkosky briefly sets forth a simple method for computing the
Mandelstam double discontinuity of four propagator integrals as the
normalized inverse Jacobian evaluated at the point where the four

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[2]A private conversation with James S. Ball confirming
this point is gratefully acknowledged.

internal lines are taken to the mass shell.[3]   This is a highly
unphysical point, requiring the deformation of the integration
contour such that one of the components of the four-momentum
variable of integration is pure imaginary. The Jacobian also ana-
lytically continues to a pure imaginary value at this point. It
is found as well that the usual Cutkosky recipe for the number
of $2\pi i$'s needed to normalize a discontinuity properly fails by an
extra factor of two in this case, apparently due to the necessary
drastic deformation of the integration contour. Although Cutkosky's
paper does not give adequate warning of these minor points,[3] it is
easy to straighten out the matters of phase and normalization by
comparing in a few cases the independently computed ordinary dis-
continuity in a single variable with the double discontinuity com-
puted by Cutkosky's simple method. Also, Mandelstam develops a
closed form for the double disontinuity of a general integral with
four propagators in which the phase and normalization are given
correctly.[4]

Two interesting features occur among the Mandelstam double
dispersion integrals found in Appendix C. First, in the basic
integrals (C.9) represented by the spinless particle version of the
diagrams in Figure 7, the Mandelstam region has as one boundary the
line $s = u$. That there are singularities along this line may be

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[3]R. E. Cutkosky, Journ. Math. Phys. 1, 429 (1960),
pp. 432-433.

[4]S. Mandelstam, Phys. Rev. 115, 1741 (1959), pp. 1746-1750.

verified from the Landau equations for these basic integrals. Further, the independent (and tedious) calculation of the ordinary single variable discontinuities confirms this boundary feature of the double discontinuities. Second, the basic integral with all five propagators (C.10), represented by the spinless version of Figure 9, has single variable discontinuities which can each be decomposed into simple linear combinations of single variable discontinuities of integrals having only four propagators. Of course, these resulting single variable discontinuities of four propagator integrals can be represented by dispersion integrals over double discontinuities in the standard manner set forth by Mandelstam[4] for four propagator integrals. Thus is the rather complicated double dispersion representation of the five propagator basic integral in Appendix C derived from ordinary four propagator double dispersion representations.

On all computational matters, more detail is to be found in Appendix D.

Finally, it is to be noted that for many of the three propagator basic integrals and for all of the four and five propagator basic integrals, dispersion methods are computationally far simpler than the parametric integration method. Also, where applicable, double dispersion methods are computationally simpler than the ordinary single variable dispersion approach. This latter fact is due mostly to the simplicity of Cutkosky's double dis-

continuity recipe. The author, at an early stage having nothing but the parametric integration method in his arsenal, was about to give up the four and five propagator integrals as intractable, when others[5] made him aware of the potential relative simplicity of the dispersion approach. Later, after reading with delight the beautiful article of Mandelstam[4] and struggling with the terse but profound work of Cutkosky,[3] the author realized the further computational simplifications inherent in utilizing double dispersion theory. It now seems to the author that, as a rule of thumb, the larger are the number of propagators per loop in a basic integral, the greater is the computational advantage of dispersion over parametric methods.

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[5]Private conversations with M. Kislinger and F. Zachariasen.

CONCLUSION

The methods of the forgoing chapters have been used in conjunction with the Reduce algebra program to compute the self energy and vertex correction diagrams found in Figures 2 - 5. In addition, most of the necessary tensor integrals of first and second rank have been calculated, although many of these have not yet been subjected to "shrinkage." Typically, the largest expressions calculated so far have run to hundreds of terms. It is estimated that some of the third and fourth rank tensor integrals yet to be computed, as well as the more complicated diagrams, will each run to several thousand terms. After traces are taken, hundreds of terms or even less should again be the rule. To make it over the "hump," computer resources must be available in a very generous measure.

It is the author's heartfelt hope that even in this era of contracting support for physics research, enough funds will somehow be forthcoming to permit this calculation to be carried to its goal of comparison with experiment.

APPENDIX A

ORIGIN SHIFT FORMULAS FOR SOME DIVERGENT INTEGRALS

$$\int d^4\ell \ \frac{\ell^\mu \ell^\nu}{((\ell-q_1)^2-m_1^2)((\ell-q_2)^2-m_2^2)}$$

$$- \int d^4\ell \ \frac{(\ell-c)^\mu (\ell-c)^\nu}{((\ell-c-q_1)^2-m_1^2)((\ell-c-q_2)^2-m_2^2)} \tag{A.1}$$

$$= \left(\frac{i\pi^2}{6}\right)\left\{g^{\mu\nu}(c^2+c\cdot(q_1+q_2))-c^\mu c^\nu+ (q_1+q_2)^\mu c^\nu + (q_1+q_2)^\nu c^\mu\right\}$$

$$\int d^4\ell \ \frac{\ell^\mu \ell^\nu \ell^\rho}{((\ell-q_1)^2-m_1^2)((\ell-q_2)^2-m_2^2)((\ell-q_3)^2-m_3^2)}$$

$$- \int d^4\ell \ \frac{(\ell-c)^\mu (\ell-c)^\nu (\ell-c)^\rho}{((\ell-c-q_1)^2-m_1^2)((\ell-c-q_2)^2-m_2^2)((\ell-c-q_3)^2-m_3^2)} \tag{A.2}$$

$$= \left(\frac{i\pi^2}{12}\right) \left(c^\mu g^{\nu\rho} + c^\nu g^{\rho\mu} + c^\rho g^{\mu\nu}\right)$$

$$\int d^4\ell \; \frac{\ell^\mu}{((\ell-q_1)^2-m_1^2)((\ell-q_2)^2-m_2^2)} \;-\; \int d^4\ell \; \frac{(\ell-c)^\mu}{((\ell-c-q_1)^2-m_1^2)((\ell-c-q_2)^2-m_2^2)}$$

$$= \left(\frac{i\pi^2}{2}\right)\left(c^\mu\right) \tag{A.3}$$

Any logarithmically divergent single loop Feynman integral may be origin shifted at will.

APPENDIX B

TWO TENSOR INTEGRALS DEPENDENT ON A LIGHTLIKE VECTOR

$$\int d^4\ell \; \frac{(-\Lambda^2) \; \ell^\mu}{(\ell^2 - \Lambda^2)(\ell^2 - m^2)((\ell + k)^2 - m^2)} \qquad \xrightarrow[\substack{\Lambda^2 \to \infty \\ k^2 = 0}]{}$$

$$\left(\frac{i\pi^2}{2}\right) \; \left(\frac{3}{2} - \log\left(\frac{\Lambda^2}{m^2}\right)\right) \left(k^\mu\right) \tag{B.1}$$

$$\int d^4\ell \; \frac{(-\Lambda_2^{\,2}) \; (-\Lambda_1^{\,2}) \; \ell^\mu \ell^\nu}{(\ell^2 - \Lambda_2^{\,2})(\ell^2 - \Lambda_1^{\,2})(\ell^2 - m^2)((\ell+k)^2 - m^2)} \qquad \xrightarrow[\substack{\Lambda_2^{\,2} \gg \Lambda_1^{\,2} \to \infty \\ k^2 = 0}]{}$$

$$\left(-\frac{i\pi^2}{4}\right) \left[\left(\frac{22}{9} - \frac{4}{3}\log\left(\frac{\Lambda_1^{\,2}}{m^2}\right)\right) \; k^\mu k^\nu + \left(\Lambda_1^{\,2}\log\left(\frac{\Lambda_2^{\,2}}{\Lambda_1^{\,2}}\right)\right.\right.$$

$$\left.\left. -2m^2\log\left(\frac{\Lambda_1^{\,2}}{m^2}\right) + m^2\right) g^{\mu\nu}\right] \tag{B.2}$$

APPENDIX C

THE THREE TO FIVE PROPAGATOR BASIC INTEGRALS

$$\int d^4\ell \quad \frac{1}{(\ell^2 - \lambda^2)((\ell+p_1)^2 - m^2)((\ell-p_2)^2 - m^2)} \quad =$$

$$(i\pi^2) \int_{4m^2}^{\infty} dr' \quad \frac{\log\left(\frac{\lambda^2}{r' - 4m^2 + \lambda^2}\right)}{\sqrt{r'(r' - 4m^2)} \, (r' - r)} \qquad \text{where } r \equiv (p_1 + p_2)^2$$

$$\text{and } p_1^2 = p_2^2 = m^2$$

Thus:

$$\int d^4\ell \quad \frac{1}{(\ell^2 - \lambda^2)((\ell+p)^2 - m^2)((\ell-p)^2 - m^2} \qquad \left(\frac{\lambda}{m}\right) \to 0$$

$$\frac{(-i\pi^2)}{m^2} \left[\pi \left(\frac{m}{\lambda}\right) + \log\left(\frac{\lambda}{m}\right) - 1 \right] \tag{C.1}$$

$$\int d^4\ell \; \frac{1}{(\ell^2)((\ell+p)^2-m^2)((\ell-p+k_3)^2-m^2)} \quad =$$

$$(4i\pi^2)\left[\int_{4m^2}^{\infty} ds' \; \frac{\text{arctanh}\left(\frac{s'-4m^2}{s'}\right)}{(s'-s)(s'-4m^2)} \; - \int_{4m^2}^{\infty} ds' \; \frac{\text{arctanh}\left(\sqrt{\frac{s'-4m^2}{s'}}\right)}{(s'-s)(s'-4m^2)}\right] \quad =$$

$$\frac{(2i\pi^2)}{(4m^2-s)}\left[2\left(\arctan\left(\sqrt{\frac{s}{4m^2-s}}\right)\right)^2 \; - \; \mathsf{L}i_2\left(\frac{s-2m^2}{2m^2}\right) \; - \; \frac{\pi^2}{3}\right] \qquad (C.2)$$

where $\mathsf{L}i_2(x) = \int_0^x dt \; \frac{(-1)\log(1-t)}{t}$, the standard dilogarithm.

$$\int d^4\ell \; \frac{1}{(\ell^2)((\ell-q_1)^2-m^2)((\ell-q_2)^2-m^2)} \quad =$$

$$(i\pi^2)\int_0^1 dx \; \frac{1}{(x(v-w)+w)} \; \log\left(\frac{x(v-w)+w-m^2}{-m^2}\right) \quad =$$

$$\frac{(-i\pi^2)}{(v-w)}\left[\mathsf{L}i_2\left(\frac{v}{m^2}\right) \; - \; \mathsf{L}i_2\left(\frac{w}{m^2}\right)\right]$$

where $q_1^2 = v$, $q_2^2 = w$, and $(q_1-q_2)^2 = 0$.

The above integral was done by the parametric method, in which propagators are combined by successive applications of Feynman's integral:

$$\frac{1}{ab} = \int_0^1 dx \; \frac{1}{[ax+b(1-x)]^2} \qquad .$$

Thus:

$$\int d^4\ell \; \frac{1}{(\ell^2)((\ell-(k_1-p))^2-m^2)((\ell-(p-k_3))^2-m^2)} \quad =$$

$$\frac{(-2i\pi^2)}{(s-u)} \left[Li_2\left(\frac{s-2m^2}{2m^2}\right) - Li_2\left(\frac{u-2m^2}{2m^2}\right) \right] \qquad (C.3)$$

$$\int d^4\ell \; \frac{1}{(\ell^2)((\ell-(p-k_3))^2-m^2)((\ell-p)^2-m^2)} \quad =$$

$$\frac{(-2i\pi^2)}{(4m^2-s)} \left[\frac{\pi^2}{6} - Li_2\left(\frac{s-2m^2}{2m^2}\right) \right] \qquad (C.4)$$

since $Li_2(1) = \frac{\pi^2}{6}$

$$\int d^4\ell \ \frac{1}{\left((\ell+p)^2-m^2\right)\left((\ell+p-k_1)^2-m^2\right)\left((\ell-p+k_3)^2-m^2\right)} \quad =$$

$$(-2i\pi^2)\int_{4m^2}^{\infty} ds' \ \frac{\operatorname{arctanh}\left(\sqrt{\frac{s'-4m^2}{s'}}\right)}{s'(s'-s)} \quad =$$

$$\frac{(-2i\pi^2)}{s} \quad \left(\arctan\left(\sqrt{\frac{s}{4m^2-s}}\right)\right)^2 \tag{C.5}$$

$$\int d^4\ell \ \frac{1}{\left((\ell+p)^2-m^2\right)\left((\ell-p+k_3)^2-m^2\right)\left((\ell-p)^2-m^2\right)} \quad =$$

$$(-2i\pi^2)\int_{4m^2}^{\infty} ds' \ \frac{\operatorname{arctanh}\left(\sqrt{\frac{s'-4m^2}{s'}}\right)}{(s'-4m^2)(s'-s)} \quad =$$

$$\frac{(-2i\pi^2)}{(4m^2-s)} \quad \left(\frac{\pi^2}{4} - \left(\arctan\left(\sqrt{\frac{s}{4m^2-s}}\right)\right)^2\right) \tag{C.6}$$

$$\int d^4\ell \; \frac{1}{(\ell^2-\lambda^2)((\ell+p)^2-m^2)((\ell-p+k_3)^2-m^2)((\ell-p)^2-m^2)} \qquad =$$

$$\left(\frac{2i\pi^2}{\lambda^2}\right)\Bigg[\int_{4m^2}^{\infty} ds' \; \frac{\operatorname{arctanh}\left(\dfrac{\left(\dfrac{\lambda}{m}\right)2 \; \sqrt{\dfrac{s'-4m^2}{s'}}}{\left(\dfrac{s'-4m^2}{s'}\right)+\left(\dfrac{\lambda}{m}\right)^2} \right)}{(s'-4m^2)(s'-s)}$$

$$- \int_{(4m^2+4m\lambda+2\lambda^2)}^{\infty} ds' \; \frac{\operatorname{arctanh}\left[\dfrac{\lambda^2\sqrt{(s'-4m^2)^2-4\lambda^2 s'+4\lambda^4}}{m^2(s'-4m^2)+\lambda^2 s'-2\lambda^4} \right]}{(s'-4m^2)(s'-s)} \Bigg]$$

$$\underset{\lambda^2\to 0}{\sim} \; \frac{(2\pi^2 i)}{(m^2)(4m^2-s)} \Bigg[\frac{\pi m}{\lambda} + \log\left(\frac{\lambda}{m}\right)-1-\log\left(\frac{4m^2-s}{2m^2}\right) -$$

$$(2)\;\sqrt{\frac{s}{4m^2-s}} \; \arctan\left(\sqrt{\frac{s}{4m^2-s}}\right)\Bigg] \qquad (C.7)$$

$$\int d^4\ell \; \frac{1}{\left((\ell+p)^2-m^2\right)\left((\ell+p-k_1)^2-m^2\right)\left((\ell-p+k_3)^2-m^2\right)\left((\ell-p)^2-m^2\right)} \quad =$$

$$(2i\pi^2) \int_{4m^2}^{\infty} ds' \int_{4m^2}^{\infty} du' \; \frac{1}{\sqrt{s'(s'-4m^2)} \; \sqrt{u'(u'-4m^2)} \; (s'-s)(u'-u)} \quad =$$

$$(8i\pi^2) \left(\frac{1}{\sqrt{s(4m^2-s)}} \right) \left(\arctan\left(\sqrt{\frac{s}{4m^2-s}} \right) \right) \left(\frac{1}{\sqrt{u(4m^2-u)}} \right) \; \cdot$$

$$\left(\arctan \left(\sqrt{\frac{u}{4m^2-u}} \right) \right) \tag{C.8}$$

$$\int d^4\ell \; \frac{1}{\left((\ell+p)^2-m^2\right)\left((\ell+p-k_1)^2-m^2\right)\left((\ell-p+k_3)^2-m^2\right)(\ell^2)} \quad =$$

$$(4i\pi^2) \int_{4m^2}^{\infty} ds' \int_{4m^2}^{\infty} du' \; \frac{\Theta(u'-s')}{\sqrt{s'(s'-4m^2)} \; \sqrt{u'(u'-4m^2)} \; (s'-s)(u'-u)} \quad =$$

$$\frac{(8i\pi^2)}{\sqrt{s(4m^2-s)} \; \sqrt{u(4m^2-u)}} \quad (f(\alpha,\beta)) \tag{C.9}$$

where $\alpha \equiv \sqrt{\dfrac{s}{4m^2-s}}$, $\beta \equiv \sqrt{\dfrac{u}{4m^2-u}}$, and $f(\alpha,\beta) = \displaystyle\int_0^1 dx \; \frac{(2\beta)\arctan(\alpha x)}{(1+\beta^2 x^2)}$

Note that $\oint (\alpha,\beta) + \oint(\beta,\alpha) = (2)(\arctan(\alpha))(\arctan(\beta))$.

$\oint (\alpha,\beta)$ may be expressed in terms of dilogrithms of complex argument, but such a form is not enlightening.

$$\int d^4\ell \ \frac{1}{(\ell^2-\lambda^2)((\ell+p)^2-m^2)((\ell+p-k_1)^2-m^2)((\ell-p+k_3)^2-m^2)((\ell-p)^2-m^2)} \ \widetilde{\lambda^2\to 0}$$

$$\frac{(2i\pi^2)}{\lambda^2} \int_{4m^2}^{\infty} \frac{ds'}{(s'-s)\sqrt{s'}} \left[\left\{ \int^{\infty} du' \frac{1}{(u'-u)\sqrt{u'}\sqrt{(s'-4m^2)(u'-4m^2-4\lambda^2)}-16m^2\lambda^2} \right. \right.$$

$$\left. \left(\frac{16m^2\lambda^2}{s'-4m^2} + 4m^2 + 4\lambda^2 \right) \right\}$$

$$-\left\{ \int_{4m^2}^{\infty} du' \frac{1}{(u'-u)\sqrt{u'}\sqrt{(s'-4m^2)(u'-4m^2)}} \right\} \left. \right] \ \div$$

(See next page for continuation)

$$\frac{(2i\pi^2)}{(\lambda^2)} \int_{4m^2+4m\lambda}^{\infty} \frac{ds'}{(s'-s)\sqrt{s'}} \qquad .$$

$$\left[\left\{ \int_{\left(4m^2+\frac{16m^2\lambda^2}{s'-4m^2-4\lambda^2}\right)}^{\infty} du' \frac{\Theta(s'-u')}{(u'-u)\sqrt{u'}\sqrt{s'-4m^2-4\lambda^2}\sqrt{u'-4m^2-\frac{16m^2\lambda^2}{s'-4m^2-4\lambda^2}}} \right\} \right.$$

$$\left. - \left\{ \int_{\left(4m^2+\frac{4\lambda^2 s'}{s'-4m^2}\right)}^{\infty} du' \frac{\Theta(s'-u')}{(u'-u)\sqrt{u'}\sqrt{s'-4m^2}\sqrt{u'-4m^2-\frac{4\lambda^2 s'}{s'-4m^2}}} \right\} \right] \widetilde{\lambda^2 \to 0}$$

$$
\frac{(2i\pi^2)}{(4m^2-s)(4m^2-u)(m^2)} \left[\log\left(\frac{4m^2-s}{2m^2}\right) + \log\left(\frac{4m^2-u}{2m^2}\right) \right.
$$

$$
+ 2\sqrt{\frac{s}{4m^2-s}} \ \arctan\left(\sqrt{\frac{s}{4m^2-s}}\right) + 2\sqrt{\frac{u}{4m^2-u}} \ \arctan\left(\sqrt{\frac{u}{4m^2-u}}\right)
$$

$$
+ \ \frac{(4m^2)(s\,f(\alpha,\beta) + u\,f(\beta,\alpha))}{\sqrt{s(4m^2-s)\,u\,(4m^2-u)}} + 2 - 2\log\left(\frac{\lambda}{m}\right) - 2\left(\frac{\pi m}{\lambda}\right) \left. \right] \qquad (C.10)
$$

f , α, and β have been defined after (C.9).

APPENDIX D

COMPUTATION METHODS USED FOR THE BASIC INTEGRALS

A basic single loop Feynman integral may be regarded as an analytic function of any one of the kinematic variables on which it depends. Of course this requires the analytic continuation of the integral beyond the physical domain of the kinematic variable in question. Landau and others explored the nature of the singularities which occur in analytically continued single loop Feynman integrals and found them to be branch points which occur for those values of the kinematic variables which allow two or more virtual particles in a loop to become real. The mass shell and physical propagation require-ments for the existence of a branch point are known as the Landau equations, and should be familiar to the reader. Since it is known that the three or more denominator basic single loop integrals have only branch point singularities (which are readily located with the aid of Landau's equations) and that they vanish at infinite values of their kinematic variables, knowledge of the discontinuities across their branch lines is sufficient to express them as Cauchy integrals restricted to those branch lines - known to physicists as dispersion integrals. As shown by Cutkosky and others, the dis-continuities across the branch lines are far easier to compute than the full integrals themselves. The recipes for computing dis-continuities are known as Cutkosky's rules. If a basic integral

depends on more than one kinematic variable, one can go further, and, in the manner of Mandelstam, consider the discontinuities in a second variable of a discontinuity of the integral in one of the other variables. Cutkosky has given a very simple recipe for computing such a double discontinuity in certain cases. The original integral, of course, is expressed as a double dispersion integral over the double discontinuity.

The detailed application of Cutkosky's rules to the computation of single variable discontinuities will be illustrated with the general three propagator basic integral. From Cutkosky's work we know that:

$$\text{disc}_{a^2} \int d^4\ell \; \frac{1}{(\ell^2-\Lambda^2)((\ell-a)^2-m_1^2)((\ell-b)^2-m_2^2)}$$

$$=(-2\pi i)^2 \int d^4\ell \; \frac{\Theta(\ell^0)\,\delta(\ell^2-\Lambda^2)\Theta(a^0-\ell^0)\,\delta((\ell-a)^2-m_1^2)}{((\ell-b)^2-m_2^2)}$$

Θ is the Heaviside function which satisfies $\Theta(x)=1$ for $x > 0$ and $\Theta(x)=0$ for $x < 0$. disc_z means the discontinuity in the variable z. In the discontinuity region $a^2 > 0$, so we may choose a coordinate system such that:

$$a = \left(\sqrt{a^2}, \ 0, \ 0, \ 0 \right)$$

$$b = \left(\frac{a \cdot b}{\sqrt{a^2}}, \ 0, \ 0, \ \frac{\sqrt{(a \cdot b)^2 - a^2 b^2}}{\sqrt{a^2}} \right)$$

We are, of course, using the four vector dot product convention of Bjorken and Drell in which, for the four vectors $c = (c^0, \ \vec{c})$ and $d = (d^0, \ \vec{d})$, we have $c \cdot d = c^0 d^0 - \vec{c} \cdot \vec{d}$.

For the four vector variable of integration ℓ, we have:

$$\ell = (\ell^0, \ |\vec{\ell}| \ \sin\theta \ \cos\phi, \ |\vec{\ell}| \ \sin\theta \ \sin\phi, \ |\vec{\ell}| \ \cos\theta)$$

$$d^4\ell = d\ell^0 \ |\vec{\ell}|^2 \ d(|\vec{\ell}|) d\Omega$$

$$= d\ell^0 \ \frac{|\vec{\ell}|}{2} \ d(|\vec{\ell}|^2) \ d\phi \ d(\cos\theta)$$

Now:

$$\int d^4\ell \ \frac{\Theta(\ell^0)\Theta(a^0-\ell^0)\ \delta(\ell^2-\Lambda^2)\ \delta((\ell-a)^2-m_1^2)}{((\ell-b)^2-m_2^2)} \quad =$$

$$\int d\Omega \int d\ell^0 \Theta(\ell^0)\Theta(\sqrt{a^2}-\ell^0) \quad .$$

$$\int \frac{|\vec{\ell}|}{2}\, d(|\vec{\ell}|^2)\ \frac{\delta((\ell^0)^2-|\vec{\ell}|^2-\Lambda^2)\ \delta((\ell^0)^2-|\vec{\ell}|^2-2\ell^0\sqrt{a^2}+a^2-m^2)}{((\ell^0)^2-|\vec{\ell}|^2-2\ell^0\,\frac{(a\cdot b)}{\sqrt{a^2}}+2|\vec{\ell}|\,\frac{\sqrt{(a\cdot b)^2-a^2b^2}}{\sqrt{a^2}}\cos\theta+b^2-m_2^2)}$$

Now we integrate away the δ-functions one by one to get:

$$\int d\Omega \int d\ell^0 \Theta(\ell^0)\Theta(\sqrt{a^2}-\ell^0)\Theta((\ell^0)^2-\Lambda^2)\ \frac{\sqrt{(\ell^0)^2-\Lambda^2}}{2}$$

$$\frac{\delta(\Lambda^2-2\ell^0\sqrt{a^2}+a^2-m_1^2)}{(\Lambda^2-2\ell^0\frac{(a\cdot b)}{\sqrt{a^2}}+2\sqrt{(\ell^0)^2-\Lambda^2}\frac{\sqrt{(a\cdot b)^2-a^2 b^2}}{\sqrt{a^2}}\cos\Theta+b^2-m_2^2)} \quad =$$

$$\Theta(a^2-(\Lambda+m_1)^2)\frac{\sqrt{(a^2-(\Lambda+m_1)^2)(a^2-(\Lambda-m_1)^2)}}{8\ a^2}\ \int d\Omega\ \frac{1}{D_1(\cos\Theta)}$$

where $D_1(\cos\Theta)=$

$$(\Lambda^2-\frac{(a\cdot b)(\Lambda^2+a^2-m_1^2)}{a^2}\ +$$

$$\frac{\sqrt{(a^2-(\Lambda+m_1)^2)(a^2-(\Lambda-m_1)^2)((a\cdot b)^2-a^2 b^2)}}{a^2}\cos\Theta+b^2-m_2^2)\ .$$

After performing the trivial azimuthal angle ϕ integration and changing variables to $x = \cos\theta$ we have:

$$\theta(a^2-(\Lambda+m_1)^2)\;\frac{\sqrt{(a^2-(\Lambda+m_1)^2)(a^2-(\Lambda-m_1)^2)}}{8}\;(2\pi)\int_{-1}^{1}dx\;\frac{1}{D_2(x)}$$

where $D_2(x) = (a^2(\Lambda^2+b^2-m_2^2)-(a\cdot b)(\Lambda^2+a^2-m_1^2)+$

$$\sqrt{(a^2-(\Lambda+m_1)^2)(a^2-(\Lambda-m_1)^2)((a\cdot b)^2-a^2b^2)}\;x)$$

Integrating over x we get:

$$(\frac{\pi}{2})\theta(a^2-(\Lambda+m_1)^2)\;\frac{1}{\sqrt{(a\cdot b)^2-a^2b^2}}\;\cdot$$

$$\text{arctanh}\left(\frac{\sqrt{(a^2-(\Lambda+m_1)^2)(a^2-(\Lambda-m_1)^2)((a\cdot b)^2-a^2b^2)}}{a^2(\Lambda^2+b^2-m_2^2)-(a\cdot b)(\Lambda^2+a^2-m_1^2)}\right)\tag{D.1}$$

When (D.1) is inserted into the dispersion integral over a^2, $(a\cdot b)$ must be re-expressed as $\frac{1}{2}[a^2+b^2-(a-b)^2]$, because the variables in which the Landau threshold singularities occur are a^2, b^2, and $(a-b)^2$.

If we wish to analytically continue (D.1) in the variable b^2 or the variable $(a-b)^2$, it is to be noted that m_2^2 has a small negative imaginary part. Depending on the kinematic region, the log

or arctan functions may be more appropriate to (D.1) than the arctanh:

$$\text{arctanh } x = \frac{1}{2} \log \left(\frac{1 + x}{1 - x} \right) = i \arctan (-ix).$$

(D.1) can be used in the computation of any basic integral containing three propagators, such as (C.5), for example.

When the Cutkosky single variable discontinuity procedure is applied to basic integrals containing four or more propagators, a non-trivial integration over azimuthal as well as polar angle may be required. This can be performed with the aid of the formula:

$$\int_0^{2\pi} d\phi \; \frac{1}{A + C \sin\phi} = \frac{2\pi}{\sqrt{A^2 - C^2}}$$

The subsequent polar angle integration will involve the square root of a quadratic, and for such integrals the variable change given below is useful:

$$\int_{-1}^{1} \frac{dx}{\sqrt{ax^2 + bx + c} \;\; (dx + e)} \;\; =$$

$$2 \int_{\left(\sqrt{a-b+c} + \sqrt{a}\right)}^{\left(\sqrt{a+b+c} - \sqrt{a}\right)} dy \; \frac{1}{(dy^2 - 2e\sqrt{a}\, y + (eb-cd))} \quad \text{where } y = \sqrt{ax^2 + bx + c} - \sqrt{a}\, x$$

Actually, all the integrals which would have required non-trivial azimuthal integration in the direct computation of their single variable discontinuities are amenable to the much simpler double discontinuity approach. Nevertheless, the very tiresome azimuthal and polar integration sequence was pursued in detail for the integrals given by (C.8) and (C.9). Doing (C.8) the hard way confirmed that the easy way (double discontinuity) really works and settled the question of normalization and phase to be used with the easy method. Doing (C.9) the hard (single discontinuity) way confirmed the presence of the s = u boundary for the double discontinuity -- a feature which follows as well from the Landau equations. It is always comforting to see the single and double discontinuity approaches yielding the same answer.

Having examined in detail the computation of single variable discontinuities, let us now examine methods of carrying out

the dispersion integration over these discontinuities in individual cases.

In the dispersion integral preceding (C.1) the following change of variables is decisive:

$$\int_{4m^2}^{\infty} dr' \frac{\log\left(\frac{\lambda^2}{r'-4m^2+\lambda^2}\right)}{\sqrt{r'(r'-4m^2)}\ (r'-r)} \quad =$$

$$2\int_{0}^{1} dx \frac{\log\left(\frac{\lambda^2}{4m^2-\lambda^2}\right)\left(\frac{1-x^2}{x^2+\frac{\lambda^2}{4m^2-\lambda^2}}\right)}{(rx^2-r+4m^2)} \quad \text{where } x = \sqrt{\frac{r'-4m^2}{r'}}$$

$$(D.2)$$

We are interested in (D.2) only at $r = 4m^2$. At this value of r, (D.2) can be evaluated by an elementary , if slightly tedious, integration by parts.

The integral in (C.2) has two singularities, the discontinuities of which must be summed:

$$\text{disc}_s \int d^4\ell \; \frac{1}{(\ell^2)((\ell+p)^2-m^2)((\ell-p+k_3)^2-m^2)} \quad =$$

$$(-2\pi i)^2 \left\{ \int d^4\ell \; \frac{\delta_+((\ell+p)^2-m^2)\,\delta_+((\ell-p+k_3)^2-m^2)}{(\ell^2)} \right.$$

$$\left. + \int d^4\ell \; \frac{\delta_+(\ell^2)\,\delta_+((\ell-p+k_3)^2-m^2)}{((\ell+p)^2-m^2)} \right\}$$

Note $\delta_+(a^2-m^2)$ means $\theta(a^o)\,\delta(a^2-m^2)$.

In the second resulting dispersion integral in (C.2) we make a change of variable like that of (D.2):

$$\int^\infty ds' \; \frac{\text{arctanh} \sqrt{\dfrac{s'-4m^2}{s}}}{(s'-s)(s'-4m^2)} \quad =$$

$$\left(\frac{2}{s}\right) \int_0^1 dx \; \frac{\text{arctanh}(x)}{x\left(x^2+\dfrac{4m^2-s}{s}\right)} \quad = \quad (D.3)$$

$$\left(\frac{2}{4m^2-s}\right) \int_0^1 dx \left[\left(\frac{1}{x} - \frac{x}{x^2+\left(\dfrac{4m^2-s}{s}\right)}\right)\text{arctanh}(x)\right] \quad \text{where } x = \sqrt{\frac{s'-4m^2}{s'}}$$

From the table of integrals and the properties of dilogarithms of real and complex argument to be found in Chapter IX of L. Lewin's <u>Dilogrithms and Associated Functions</u> (Macdonald, London, 1958), the following integral may be readily deduced:

$$\int_0^1 dx \left(\frac{x}{x^2+b^2}\right) \operatorname{arctanh}(x) = \frac{1}{2} \left(\frac{\pi}{2} - \arctan(b)\right)^2 \qquad (D.4)$$

(D.4) allows (D.3) to be readily evaluated.

For the first dispersion integral in (C.2) we have the change of variables:

$$\int_{4m^2}^\infty ds' \frac{\operatorname{arctanh}\left(\frac{s'-4m^2}{s'}\right)}{(s'-s)(s'-4m^2)} = \frac{1}{(4m^2-s)} \qquad .$$

$$\int_0^1 dy \left(\frac{\left(\frac{4m^2-s}{s}\right)}{y\left(y+\left(\frac{4m^2-s}{s}\right)\right)}\right) \operatorname{arctanh}(y) \qquad (D.5)$$

$$\text{where} \quad y = \left(\frac{s'-4m^2}{s'}\right)$$

We can again readily deduce from Lewin's integral tables that:

$$\int_0^1 dy \left(\frac{c}{y(y+c)}\right) \operatorname{arctanh}(y) = \left(\tfrac{1}{2}\right)\left[\frac{\pi^2}{6} - \operatorname{Li}_2\left(\frac{1-c}{1+c}\right)\right] \qquad (D.6)$$

With (D.6) we evaluate (D.5).

Moving now to the parametric integral which precedes and permits the evaluation of (C.3) and (C.4), we see that the following change of variable lucidly reveals the dilogarithmic nature of the answer:

$$\int_0^1 dx \ \frac{1}{(x(v-w)+w)} \log\left(\frac{x(v-w)+w-m^2}{-m^2}\right) \quad =$$

$$\left(\frac{1}{v-w}\right)\int_{\frac{w}{m^2}}^{\frac{v}{m^2}} \frac{dz}{z} \log(1-z) \qquad \text{where } z = \frac{x(v-w)+w}{m^2}$$

The dispersion integrals in both (C.5) and (C.6) are readily evaluated by making the change of variable $x = \sqrt{\frac{s'-4m^2}{s'}}$ and then using (D.4).

The integral in (C.7) has four propagators, but since it is a function of s alone, the azimuthal angle integrations to be done in deriving its discontinuity structure are trivial. However, it possesses two singularities whose discontinuities must be summed:

$$\text{disc}_s \int d^4\ell \; \frac{1}{(\ell^2-\lambda^2)((\ell+p)^2-m^2)((\ell-p+k_3)^2-m^2)((\ell-p)^2-m^2)} \quad =$$

$$(-2\pi i)^2 \left\{ \int d^4\ell \; \frac{\delta_+((\ell+p)^2-m^2)\delta_+((\ell-p+k_3)^2-m^2)}{(\ell^2-\lambda^2)((\ell-p)^2-m^2)} \right.$$

$$\left. + \int d^4\ell \; \frac{\delta_+(\ell^2-\lambda^2)\delta_+((\ell-p+k_3)^2-m^2)}{((\ell+p)^2-m^2)((\ell-p)^2-m^2)} \right\}$$

The resulting dispersion integrals in (C.7) require a careful limiting procedure as $\lambda^2 \to 0$. Beginning with the familiar change of variables $x = \sqrt{\frac{s'-4m^2}{s'}}$, we give below the steps in evaluating the first dispersion integral in (C.7):

$$\left(\frac{1}{\lambda^2}\right) \int_{4m^2}^{\infty} \frac{\text{arctanh}\left(\frac{\left(\frac{\lambda}{m}\right)^2 \sqrt{\frac{s'-4m^2}{s'}}}{\left(\frac{s'-4m^2}{s'}\right)+\left(\frac{\lambda}{m}\right)^2}\right)}{(s'-4m^2)(s'-s)} =$$

$$\left(-\frac{2}{s\lambda^2}\right) \int_{0}^{1} dx \, \frac{\text{arctanh}\left(\frac{\left(\frac{\lambda}{m}\right)^2 x}{x^2 + \left(\frac{\lambda}{m}\right)^2}\right)}{(x)\left(x^2 + \left(\frac{4m^2-s}{s}\right)\right)} \quad \underset{\left(\frac{\lambda}{m}\right)^2 \to 0}{\overbrace{}}$$

$$\left(\frac{2}{sm^2}\right) \int_{0}^{1} dx \, \frac{1}{\left(x^2+\left(\frac{\lambda}{m}\right)^2\right)\left(x^2+\left(\frac{4m^2-s}{2}\right)\right)} =$$

$$\frac{(2)}{(sm^2)\left(\left(\frac{4m^2-s}{s}\right)-\left(\frac{\lambda}{m}\right)^2\right)} \left[\left(\frac{m}{\lambda}\right) \arctan\left(\frac{m}{\lambda}\right) - \sqrt{\frac{s}{4m^2-s}} \, \arctan\left(\sqrt{\frac{s}{4m^2-s}}\right)\right] \underset{\left(\frac{\lambda}{m}\right)^2 \to 0}{\overbrace{}}$$

$$\frac{(2)}{(m^2)(4m^2-s)} \left[\frac{m\pi}{2\lambda} - 1 - \sqrt{\frac{s}{4m^2-s}} \, \arctan\left(\sqrt{\frac{s}{4m^2-s}}\right)\right]$$

For the second dispersion integral in (C.7) we have:

$$\left(\frac{1}{\lambda^2}\right) \int\limits_{(4m^2+4m\lambda+2\lambda^2)}^{\infty} ds' \frac{1}{(s'-s)(s'-4m^2)} \text{ arctanh} \left[\frac{\lambda^2 \sqrt{(s'-4m^2)^2-4\lambda^2 s'+4\lambda^4}}{m^2(s'-4m^2)+\lambda^2 s'-2\lambda^4}\right]_{\widetilde{(\frac{\lambda}{m})^2 \to 0}}$$

$$\left(\frac{1}{m^2}\right) \int\limits_{(4m^2+4m\lambda)}^{\infty} ds' \frac{1}{(s'-s)(s'-4m^2)} \left(\frac{\sqrt{(s'-4m^2)^2-4\lambda^2 s'}}{(s'-4m^2)}\right) \qquad\qquad (D.7)$$

The kinematic region of interest is $4m^2 > s > 0$. In order to make further progress in evaluating (D.7) in this region we must split the integration interval into two parts. We introduce the parameter α which has dimensions of mass squared and satisfies:

$$(4m^2-s) \gg \alpha \gg 4m\lambda \to 0 \qquad .$$

We split the integration interval into the two parts $s' > 4m^2 + \alpha$ and $4m^2 + \alpha > s' > 4m^2 + 4m\lambda$. In each part we can approximate the integrand differently as follows:

$$\left(\frac{1}{m^2}\right)\int_{(4m^2+4m\lambda)}^{\infty} ds' \frac{1}{(s'-s)(s'-4m^2)} \left(\frac{\sqrt{(s'-4m^2)^2-4\lambda^2 s'}}{(s'-4m^2)}\right) \quad \overbrace{(\frac{\lambda}{m})^2 \to 0}$$

$$(D.8)$$

$$\left(\frac{1}{m^2}\right)\int_{(4m^2+\alpha)}^{\infty} ds' \frac{1}{(s'-s)(s'-4m^2)} + \left(\frac{1}{m^2}\right)\int_{(4m^2+4m\lambda)}^{(4m^2+\alpha)} \frac{1}{(4m^2-s)(s'-4m^2)} \frac{\sqrt{(s'-4m^2)^2-16m^2\lambda^2}}{(s'-4m^2)}$$

The first integral in (D.8) can be evaluated immediately, while in the second we make a variable change to:

$$z = \frac{\sqrt{(s'-4m^2)^2-16m^2\lambda^2}}{(s'-4m^2)}$$

Thus, (D.8) is equal to:

$$\left(\frac{1}{m^2}\right)\left(\frac{1}{4m^2-s}\right)\left\{\log\left(\frac{4m^2-s+\alpha}{\alpha}\right)+\int_0^{\sqrt{1-\left(\frac{4m\lambda}{\alpha}\right)^2}}dz\,\frac{z^2}{1-z^2}\right\}\quad=$$

$$\left(\frac{1}{m^2}\right)\left(\frac{1}{4m^2-s}\right)\left\{\log\left(\frac{4m^2-s+\alpha}{\alpha}\right)\right.$$

$$\left.+\ \text{arctanh}\left(\sqrt{1-\left(\frac{4m\lambda}{\alpha}\right)^2}\right)-\sqrt{1-\left(\frac{4m\lambda}{\alpha}\right)^2}\right\}_{(4m^2-s)\ \gg\ \alpha\ \gg\ 4m\lambda\ \to\ 0}$$

$$\left(\frac{1}{m^2}\right)\left(\frac{1}{4m^2-s}\right)\left\{\log\left(\frac{4m^2-s}{2m^2}\right)-\log\left(\frac{\lambda}{m}\right)-1\right\}$$

The technique of separating the integration region into two parts in order to compute small photon mass limits is set forth in detail by Feynman and Brown (Phys.Rev. 85, 231 (1952)). It is very useful once again for evaluating the dispersion integrals in (C.10).

Going on now to (C.8) we will indicate the method of computing double discontinuities. A simple alternative would be to just utilize the closed formula given by Mandelstam (Phys. Rev. 115, 1741 (1959), p. 1750). In line with Cutkosky's work, however:

$$\text{disc}_s \ \text{disc}_u$$

$$\int d^4\ell \ \frac{1}{((\ell+p)^2-m^2)((\ell+p-k_1)^2-m^2)((\ell-p+k_3)^2-m^2)((\ell-p)^2-m^2)} \ =$$

$$(2)(-2\pi i)^4 \ . \tag{D.9}$$

$$\int d^4\ell \ \delta((\ell+p)^2-m^2)\delta((\ell+p-k_1)^2-m^2)\delta((\ell-p+k_3)^2-m^2)\delta((\ell-p)^2-m^2)$$

Note that (D.9) has an extra factor of two, as mentioned in Chapter V. After shifting the origin in (D.9) by $-p$ and recalling that $2p = k_1 + k_2 + k_3$, we can rewrite the integral in (D.9) as:

$$\int d^4\ell \ \delta(\ell^2-m^2)\delta((\ell-k_1)^2-m^2)\delta((\ell-k_1-k_2)^2-m^2)\delta((\ell-k_1-k_2-k_3)^2-m^2)$$

$$\tag{D.10}$$

From (D.10) we see that if we could change the variables of integration from the four components of ℓ^μ to ℓ^2, $(\ell \cdot k_1)$, $(\ell \cdot k_2)$,

and $(\ell \cdot k_3)$, then the integral would drop out immediately.

We note:

$$d(\ell^2)d(2\ell \cdot k_1)d(2\ell \cdot k_2)d(2\ell \cdot k_3) = 16 \left| \epsilon^{\mu\alpha\beta\gamma} \ell_\mu k_{1\alpha} k_{2\beta} k_{3\gamma} \right| d^4\ell \qquad \text{(D.11)}$$

Now we can conveniently choose a reference frame in which:

$$k_1 = \left(\frac{\sqrt{s}}{2}, \ 0, \ 0, \ \frac{\sqrt{s}}{2} \right), \quad k_2 = \left(\frac{\sqrt{s}}{2}, \ 0, \ 0, \ -\frac{\sqrt{s}}{2} \right), \quad \text{and}$$

$$\qquad \text{(D.12)}$$

$$k_3 = \left(\frac{4m^2-s}{2\sqrt{s}}, \ 0, \ -\frac{\sqrt{ut}}{\sqrt{s}}, \ \frac{u-t}{2\sqrt{s}} \right)$$

In this frame the four components of ℓ^μ satisfy:

$$\ell^{(0)} = \frac{(k_1 \cdot \ell) + (k_2 \cdot \ell)}{\sqrt{s}}, \quad \ell^{(3)} = \frac{(k_2 \cdot \ell) - (k_1 \cdot \ell)}{\sqrt{s}}$$

$$\ell^{(2)} = \frac{1}{\sqrt{stu}} \left(s(k_3 \cdot \ell) - u(k_1 \cdot \ell) - t(k_2 \cdot \ell) \right) \qquad \text{(D.13)}$$

$$\ell^{(1)} = \frac{2 \, \epsilon^{\mu\alpha\beta\gamma} \ell_\mu k_{1\alpha} k_{2\beta} k_{3\gamma}}{\sqrt{stu}}$$

Thus we see that $\ell^{(1)}$ is simply related to the Jacobian we need in (D.11). Now $\ell^{(1)}$ satisfies a simple property which permits the Jacobian to be computed in terms of our new variables ℓ^2, $(\ell \cdot k_1)$, $(\ell \cdot k_2)$, and $(\ell \cdot k_3)$:

$$(\ell^{(1)})^2 = (\ell^{(0)})^2 - (\ell^{(2)})^2 - (\ell^{(3)})^2 - \ell^2 \qquad (D.14)$$

Combining (D.14) with (D.13) and inserting the result into (D.11), we have the required change of variables:

$$d^4\ell = \frac{d(\ell^2)d(2\ell \cdot k_1)d(2\ell \cdot k_2)d(2\ell \cdot k_3)}{8\sqrt{4ut(k_1 \cdot \ell)(k_2 \cdot \ell) - stu\ \ell^2 - (s(k_3 \cdot \ell) - u(k_1 \cdot \ell) - t(k_2 \cdot \ell))^2}}$$

$$(D.15)$$

Noting that the δ-functions in (D.10) enforce the conditions $\ell^2 = m^2$, $(\ell \cdot k_1) = 0$, $(\ell \cdot k_2) = \frac{s}{2}$, and $(\ell \cdot k_3) = \frac{4m^2 - s}{2}$, we get for the double discontinuity:

$$(2)(-2\pi i)^4 \ \frac{1}{(4)\sqrt{(-su)(s - 4m^2)(u - 4m^2)}} \qquad (D.16)$$

In the Mandelstam region we have $s > 4m^2$ and $u > 4m^2$, so the argument of the square root in the denominator of (D.16) is negative. The correct recipe is to factor a $(+i)$ out of the square

root, yielding the final answer for the double discontinuity (D.9)
as:

$$\frac{(-2\pi i)^4}{(2i)} \frac{1}{\sqrt{(su)(s-4m^2)(u-4m^2)}} \tag{D.17}$$

We note as well that in the Mandelstam region the δ-functions
have required the $\ell^{(2)}$ component of ℓ^{μ} given in (D.13) to be pure
imaginary. This forced distortion of the $d^4\ell$ integration region
presumably accounts for the anomalous factor of two required in
(D.9) as well as the strange $+i$ phase of the Jacobian. Lest such
normalization and phase anomalies make the reader uneasy, he may
check them against Mandelstam's closed formula or the single variable
discontinuity approach.

The reader's attention is directed to the unusual and
pleasant s and u factorization property of the double discontinuity
(D.17). Photon-photon scattering integrals with all four photons
on mass shell don't have this pleasant feature, as was noted in
Chapter III. The factorization makes the double dispersion integral
in (C.8) factor into a product of elementary integrals -- we give
below a formula for the general type of elementary integral:

$$\int_{a}^{u} \frac{ds'}{\sqrt{s'(s'-a)}\,(s'-s)} \;=\; \frac{2}{\sqrt{s(a-s)}} \arctan\left(\sqrt{\frac{u-a}{u}}\,\sqrt{\frac{s}{a-s}}\right) \qquad (D.18)$$

for $0 < s < a < u$

Now we turn our attention to the integral in (C.9), which
may be regarded as coming from a spinless version of Figure 7.
This diagram is not of the normal Mandelstam form, since one of the
lines leading into the four propagator loop has a virtual mass which
varies along with s, one of the two Mandelstam variables. Indeed,
examination of the Landau equations shows the existence of singu-
larities along the line $s = u$ as well as along the usual Mandelstam
boundary of the region $s > 4m^2$, $u > 4m^2$. Thus, the double dis-
continuity is computed in the usual fashion, but the additional
boundary constraint must be incorporated into the double dispersion
integral, appearing in (C.9) as $\Theta(u'- s')$. The single variable
dispersion approach confirms the correctness of this procedure.

The integral over s' in (C.9) is readily carried out by
using (D.18). The subsequent u' integral can be simplified by usual
change of variable $x = \sqrt{\dfrac{u'-4m^2}{u'}}$ as below:

$$\int_{4m^2}^{\infty} \frac{du'}{\sqrt{u'(u'-4m^2)}} \qquad \frac{\arctan\left(\sqrt{\frac{s}{4m^2-s}}\ \sqrt{\frac{u'-4m^2}{u'}}\right)}{(u'-u)} =$$

$$\left(\frac{1}{\sqrt{u(4m^2-u)}}\right) \int_{0}^{1} dx \ \frac{2\left(\sqrt{\frac{u}{4m^2-u}}\right)\ \arctan\left(\left(\sqrt{\frac{s}{4m^2-s}}\right)x\right)}{1+\left(\frac{u}{4m^2-u}\right)x^2} =$$

$$\left(\frac{1}{\sqrt{u(4m^2-u)}}\right) f(\alpha,\beta) \qquad \text{as in (C.9).}$$

Using Lewin, $f(\alpha,\beta)$ can be expressed as the difference of the real parts of two dilogarithms of complex argument, but as stated below (C.9), this is not an enlightening form.

Turning now to the pentagon in (C.10), we see that we cannot apply directly to this five propagator integral the double discontinuity procedures which are tailored to four propagator integrals, such as Mandelstam's closed formula or Cutkosky's four δ-function recipe. Thus we temporarily fall back on the well-defined Cutkosky procedure for the single variable discontinuity in s. There are two singularities in s, and their discontinuities must be summed:

$$\text{disc}_s \int d^4\ell \; \frac{1}{(\ell^2-\lambda^2)((\ell+p)^2-m^2)((\ell+p-k_1)^2-m^2)((\ell-p+k_3)^2-m^2)((\ell-p)^2-m^2)} =$$

$$(-2\pi i)^2 \left\{ \int d^4\ell \; \frac{\delta_+((\ell+p)^2-m^2)\delta_+((\ell-p+k_3)^2-m^2)}{(\ell^2-\lambda^2)((\ell+p-k_1)^2-m^2)((\ell-p)^2-m^2)} \right.$$

$$\left. + \int d^4\ell \; \frac{\delta_+(\ell^2-\lambda^2)\delta_+((\ell-p+k_3)^2-m^2)}{((\ell+p)^2-m^2)((\ell+p-k_1)^2-m^2)((\ell-p)^2-m^2)} \right\} \qquad (D.19)$$

Looking at the first integral in (D.19) we note the identity:

$$\frac{\delta((\ell+p)^2-m^2)}{(\ell^2-\lambda^2)((\ell-p)^2-m^2)} = \frac{\delta(\ell^2+2\ell\cdot p)}{(\ell^2-\lambda^2)(\ell^2-2\ell\cdot p)} = \frac{\delta(\ell^2+2\ell\cdot p)}{(\ell^2-\lambda^2)(2\ell^2)} =$$

$$\frac{\delta((\ell+p)^2-m^2)}{(2\lambda^2)} \left(\frac{1}{\ell^2-\lambda^2} - \frac{1}{\ell^2} \right) \qquad (D.20)$$

Similarly, looking at the second integral in (D.19) we note:

$$\frac{\delta(\ell^2-\lambda^2)}{((\ell+p)^2-m^2)((\ell-p)^2-m^2)} = \frac{\delta(\ell^2-\lambda^2)}{(\ell^2+2\ell\cdot p)(\ell^2-2\ell\cdot p)} = \tag{D.21}$$

$$\frac{\delta(\ell^2-\lambda^2)}{(2\ell^2)}\left(\frac{1}{\ell^2-2\ell\cdot p} + \frac{1}{\ell^2+2\ell\cdot p}\right) = \frac{\delta(\ell^2-\lambda^2)}{2\lambda^2}\left(\frac{1}{(\ell-p)^2-m^2} + \frac{1}{(\ell+p)^2-m^2}\right)$$

Using (D.20) and (D.21) we can rewrite the discontinuity (D.19) as a linear combination of four integrals, each of which involves only four propagators:

$$\frac{(-2\pi i)^2}{(2\lambda^2)} \left\{ \int d^4\ell \ \frac{\delta_+((\ell+p)^2-m^2)\delta_+((\ell-p+k_3)^2-m^2)}{(\ell^2-\lambda^2)((\ell+p-k_1)^2-m^2)} \ - \right.$$

$$\int d^4\ell \ \frac{\delta_+((\ell+p)^2-m^2)\delta_+((\ell-p+k_3)^2-m^2)}{(\ell^2)((\ell+p-k_1)^2-m^2)} + \int d^4\ell \ \frac{\delta_+(\ell^2-\lambda^2)\delta_+((\ell-p+k_3)^2-m^2)}{((\ell+p-k_1)^2-m^2)((\ell-p)^2-m^2)}$$

$$\left. + \int d^4\ell \ \frac{\delta_+(\ell^2-\lambda^2)\delta_+((\ell-p+k_3)^2-m^2)}{((\ell+p)^2-m^2)((\ell+p-k_1)^2-m^2)} \right\} \tag{D.22}$$

The four s discontinuity integrals entering into (D.22) can each be expressed as u dispersion integrals over double discontinuities which are calculable by the standard methods used for (C.8) and (C.9), for these s discontinuity integrals only have four propagators apiece. The five propagator problem has been reduced to a linear combination of four propagator problems, which can each be tackled by a standard double discontinuity approach. (D.22) leads directly to the four double dispersion integrals given in (C.10).

Alternately, we could have chosen to write the original five propagator integral as a linear combination of four propagator integrals before taking any discontinuities at all. The possibility of doing this springs from the identity:

$$\frac{1}{(\ell^2-\lambda^2)((\ell+p)^2-m^2)((\ell-p)^2-m^2)} \quad =$$

$$\left(\frac{1}{2\lambda^2}\right)\left(\frac{1}{(\ell^2-\lambda^2)((\ell+p)^2-m^2)} - \frac{1}{(\ell^2)((\ell+p)^2-m^2)} + \frac{1}{(\ell^2-\lambda^2)((\ell-p)^2-m^2)}\right.$$

$$\left. - \frac{1}{(\ell^2)((\ell-p)^2-m^2)}\right) \tag{D.23}$$

Again, the resulting four propagator integrals would be treated by a standard double discontinuity approach.

The result of performing the four double dispersion integrals in (C.10) involves only functions which have already been encountered in (C.7), (C.8), and (C.9). Just so, no really new techniques are needed to carry out the dispersion integrations in (C.10), just a wonderfully tedious melange of the methods learned from carrying out the integrations in (C.7), (C.8), and (C.9). The gruesome details of the bookkeeping task that is required occupy eleven pages of scratch paper in the author's desk. Even in an appendix so pedantic as this one it is too embarassing to present such a mass of complicated trivia. Anyway, it would be shabby of the author to try to deny the truly masochistic reader the exquisite pleasures of intimate rather than vicarious experience with prolonged tedium.
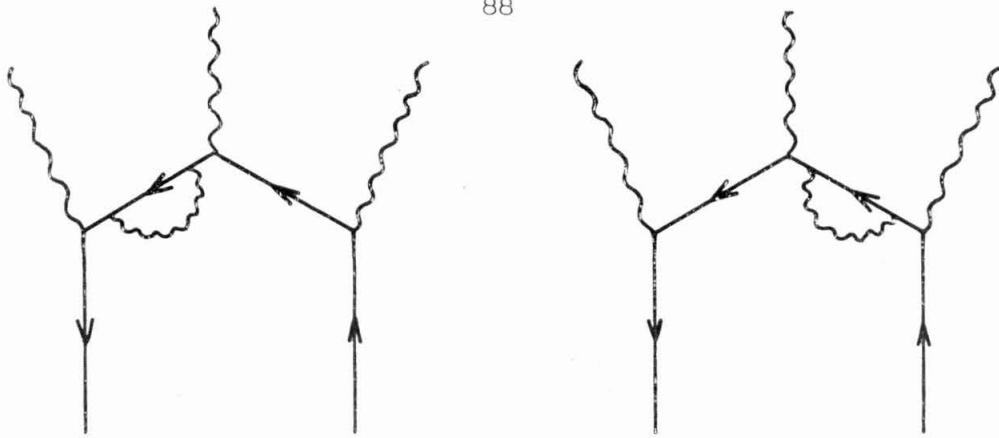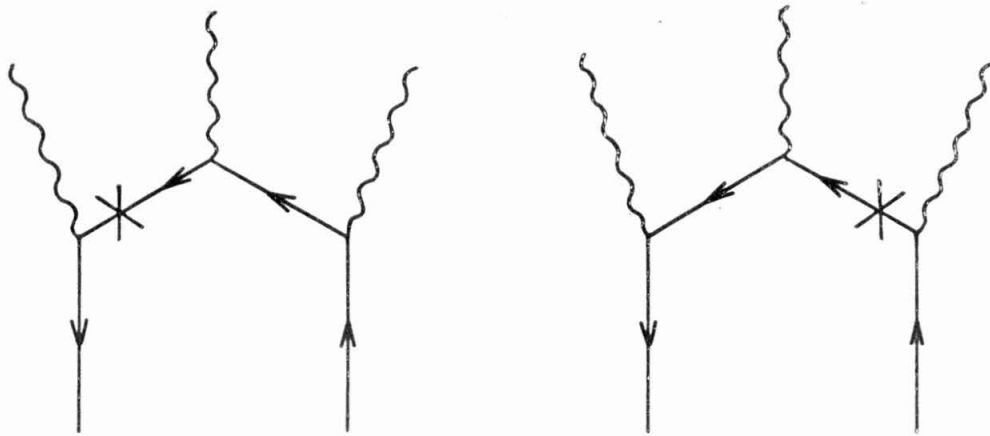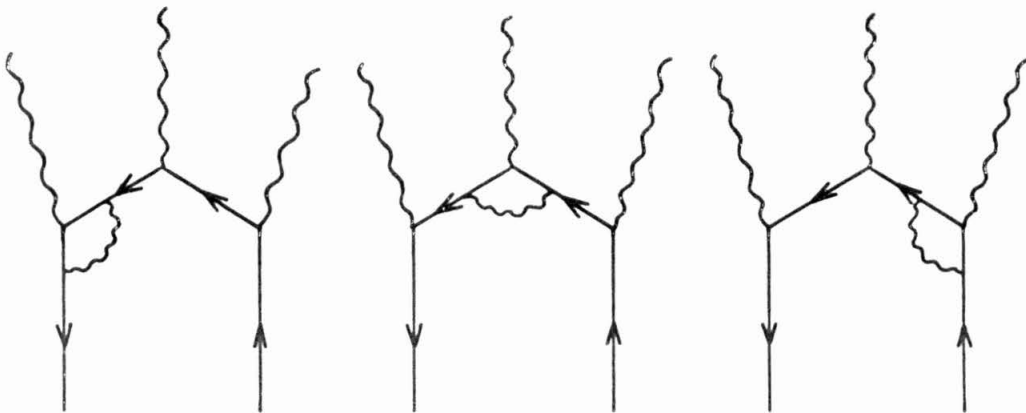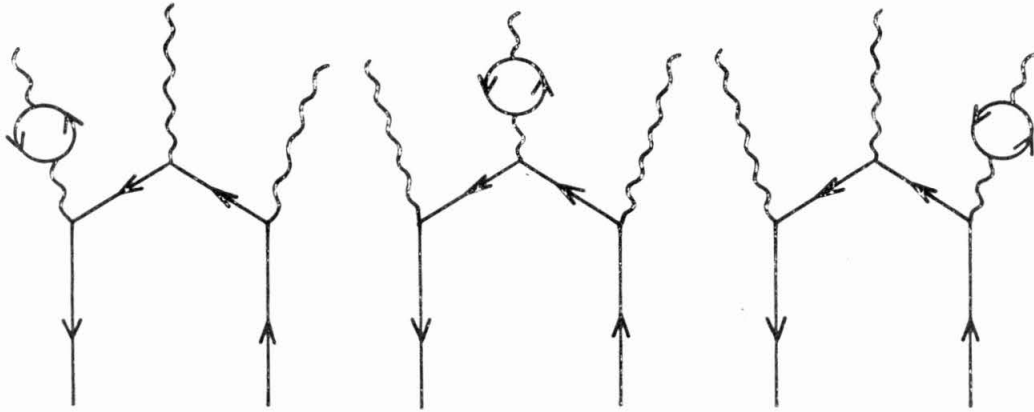
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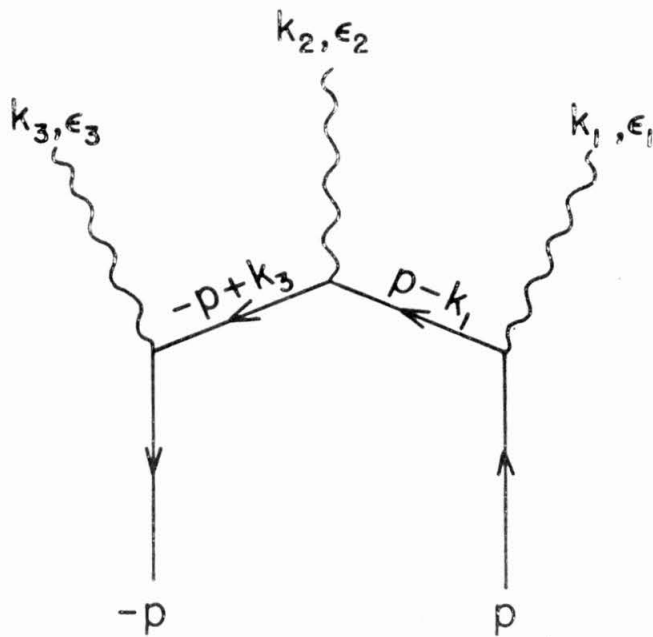
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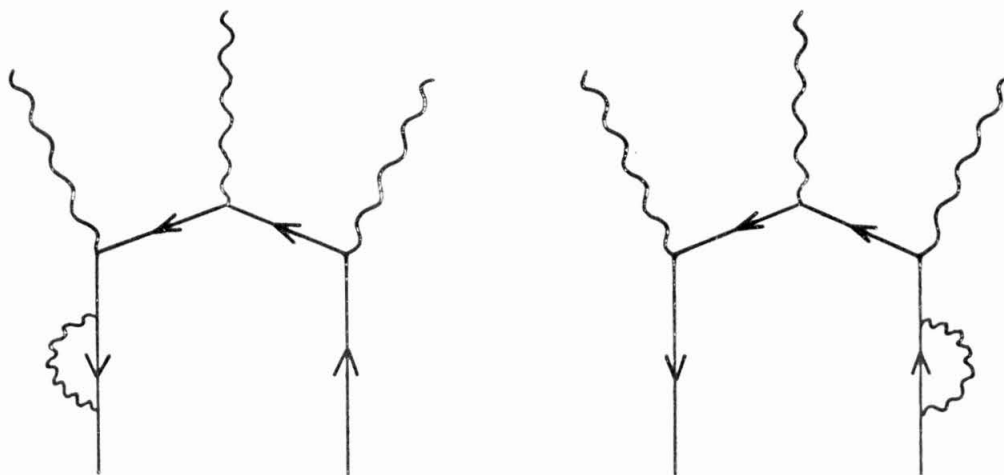
Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9