THE
SUBVARIETIES OF A FIELD

Thesis

by

Carleton R. Worth

In partial fulfillment of the requirements
for the degree of Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

1933

Summary

Number systems which satisfy part but not all of the postulates for a field are called subvarieties of a field. The purpose of this paper is the determination of as great as possible a number of such varieties by suitable definitions of the class of elements and of the two operations involved.

Two postulate systems are considered. The first gives rise to 284 varieties, instances of all of which are given for infinite classes of elements, and of all except three for finite classes.

Of the 8192 combinations of postulates arising from the second system, not more than 1146 can be consistent. Instances are given of 1054 of these. As the postulates of this system are not independent, no conclusion has been reached regarding the remaining cases.

THE SUBVARIETIES OF A FIELD

1. Introduction

The properties of a number field may be regarded in a more general sense as definitions of the behavior of a set K of arbitrary elements under two binary operations denoted by $\oplus$ and $\circ$.

Consider a set of postulates for a field. These may be weakened either by the assumption of the falsity of any particular postulate or postulates, or by the omission of certain postulates. Any instance of a class K of elements together with definitions of $\oplus$ and $\circ$ such that the modified postulate system is satisfied will be termed a subvariety of a field. When a subvariety can be determined, the modified postulate system is evidently consistent. The case of systems from which certain postulates have been obtained is covered by the consideration of instances in which the omitted postulates are true, as well as examples in which they are false.

The problem proposed is that of determining the number of existent subvarieties of a field. To this end, the most extensive postulate system yet published, due to Huntington, has been selected. After modification of the original system by the introduction as postulates of two theorems which when so regarded offer instances not otherwise distinct, there is a total of thirteen postulates to be considered. Of the resulting 2^{13} varieties, not more than 1146 can be consistent, and instances of most of these have been obtained.

Another system, formulated by Dickson, and involving only nine postulates, is examined first. The fundamental distinction between the two systems lies in the treatment of zeros and units: in Huntington's system their existence, unicity, and distinctness are postulated, while in the other system none of these are required.

Instances are given of the 284 varieties resulting from the Dickson system, for both infinite and finite classes.

The Moore symbol ($\pm \pm \ldots \pm$) is used to denote the properties of a variety, where + indicates that a particular postulate holds for all elements of a class, - that it is false for at least one element of the class, and 0 that it is without significance. As the closure property of the sets under the defined operations is assumed throughout the paper, these postulates are not considered in writing the symbol of a variety. For convenience the symbols for $\oplus$, $\circ$, and the distributive law are written in separate parentheses.

The instances themselves are for the most part systems of number ennuples, for which it is most convenient to define the operation $\circ$ by means of multiplication tables such as are used for linear algebras. These tables for systems in which multiplication is associative have all been selected from those for known algebras.

I am greatly indebted to Professor E. T. Bell for his generous advice throughout the preparation of this paper, to Dr. R.S. Martin for a particularly useful definition of addition, and to Dr. Neal H. McCoy for suggestions derived from a yet unpublished paper.

2. Varieties for Which the Existence of Zero and Unit is not Postulated

2.1 The postulates The first set of postulates to be examined was formulated by Dickson[1,2]; to this has been added the postulate of commutativity of addition. We shall consider a set $\underline{K} = \{\underline{a}, \underline{b}, \underline{c}, \ldots\}$ of elements with the following properties:

A0. For every two equal or distinct elements $\underline{a}$ and $\underline{b}$ of the set $\underline{a} \oplus \underline{b}$ is a uniquely determined element of the set.

A1. $\underline{a} \oplus \underline{b} = \underline{b} \oplus \underline{a}$ for every $\underline{a}$ and $\underline{b}$ in the set.

A2. $(\underline{a} \oplus \underline{b}) \oplus \underline{c} = \underline{a} \oplus (\underline{b} \oplus \underline{c})$ for every $\underline{a}$, $\underline{b}$, $\underline{c}$, in the set.

A3. There exists an element $\underline{z}$ in $\underline{K}$ such that for every $\underline{a}$ in the set $\underline{a} \oplus \underline{z} = \underline{a}$

A4. If such elements $\underline{z}$ occur, then for a particular $\underline{z}$, and for every $\underline{a}$ in the set, there is an element $\underline{a}'$ in $\underline{K}$ such that $\underline{a} \oplus \underline{a}' = \underline{z}$.

M0. For every two equal or distinct elements $\underline{a}$ and $\underline{b}$ of the set $\underline{a} \circ \underline{b}$ is a uniquely determined element of the set.

M1. $\underline{a} \circ \underline{b} = \underline{b} \circ \underline{a}$ for every $\underline{a}$ and $\underline{b}$ in the set.

M2. $(a \circ b) \circ c = \underline{a} \circ (\underline{b} \circ \underline{c})$ for every $\underline{a}$, $\underline{b}$, $\underline{c}$, in the set.

M3. There exists an element $\underline{u}$ in $\underline{K}$ such that for every $\underline{a}$ in $\underline{K}$ $\underline{a} \circ \underline{u} = \underline{a}$

M4. If such elements $\underline{u}$ occur, then for a particular $\underline{u}$, and for every $\underline{a}$ different from each $\underline{z}$, there is an element $\underline{a}'$ in $\underline{K}$ such that $\underline{a} \circ \underline{a}' = \underline{u}$

D. $\underline{a} \circ (\underline{b} \oplus \underline{c}) = \underline{a} \circ \underline{b} \oplus \underline{a} \circ \underline{c}$ for every $\underline{a}$, $\underline{b}$, $\underline{c}$, in $\underline{K}$.

We shall assume that AO, MO hold throughout the discussion. It is to be noted that the falsity of postulate 3 for either operation implies the suppression of postulate 4. Hence the Moore symbols for postulates 1 - 4 for either operation are:

$$(\pm \pm + \pm), (\pm \pm - 0).$$

As the property of commutativity of addition can be provided from certain other postulates [2], the symbols

$$(- + + +)\ (+ + + +)\ (+),\ (- + + +)\ (+ + + -)\ (+),$$
$$(- + + +)\ (+ - + +)\ (+),\ (- + + +)\ (+ - + -)\ (+),$$

cannot occur. The total number of possible varieties is therefore 12 x 12 x 2 - 4 = 284.

2.2 Infinite Varieties. We shall first consider classes $\underline{K}$ which contain an infinite number of elements. These elements will be regarded as ennuples of real numbers $(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) = (\underline{a})$ and $\underline{a} \oplus \underline{b}$ will be interpreted as

$$(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) \oplus (\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n) = (\underline{a}_1 \oplus b_1, \underline{a}_2 \oplus b_2, \ldots, \underline{a}_n \oplus b_n)$$

except where the matrix notation is used. The product $\underline{a} \circ \underline{b}$ may be more compactly considered as

$$\sum_{i=1}^{n} \underline{a}_i e_i \circ \sum_{j=1}^{n} \underline{b}_j e_j = \sum_{i,j=1}^{n} \underline{a}_i \underline{b}_j e_i e_j$$

where $e_i e_j$ is defined by a multiplication table. For all except certain cases to be considered later, a variety having any desired Moore symbol can be constructed by combining suitable definitions of addition and multiplication.

The operations defining $\oplus$ are as follows:

1. (+ + + +) (+) Reals; $\underline{a} \oplus \underline{b} = \underline{a} + \underline{b}$.
2. (+ + + +) (-) Reals; $\underline{a} + \underline{b} + 1$ (for varieties in which the multiplication in verse is not required)*
3. (- + + +) (-) Rationals; $\underline{a}_1 / \underline{b}_1 \oplus \underline{a}_2 / \underline{b}_2$ is defined by writing, in the denominator, the digits of $\underline{b}_1$ in front of those in $\underline{b}_2$; and similarly in the numerator, with the restrictions that if the last m digits of $\underline{a}_1$ are the first m digits of $\underline{a}_2$ in reverse order, all these are to be suppressed, and further that a zero at the beginning or end of a number is to be suppressed, except that $0 \oplus 0 = 0$. Thus, $230/71 \oplus 25/17 = 2325/7117$; $135/11 \oplus 53/17 = 1/1117$.
4. (+ - + +) (+) Reals; $(\underline{a}) \oplus (\underline{b}) = (\underline{a} + \underline{b})$ if $(\underline{a}) \neq (\underline{b})$; $(\underline{a}) \oplus (\underline{a}) = (0)$.
5. (+ - + +) (-) Reals; Sgn $\underline{a}$ sgn $\underline{b}\, |\underline{a} + \underline{b}|$; sgn 0 = 1.
6. (+ + - 0) (+) Reals; 0.
7. (+ + - 0) (-) Reals; 1.*
8. (+ + + -) (+) Reals ≥ 0; $\underline{a} + \underline{b}$ *
9. (+ + + -) (-) Reals; $\underline{a}$ sgn $\underline{b}$ + $\underline{b}$ sgn $\underline{a}$; sgn 0 = 1.
10. (- - + +) (+) Reals; $\underline{a} + 2\underline{b}$.
11. (- - + +) (-) Reals; $\underline{a}\underline{b} + \underline{a}$.
12. (- + - 0) (+) Reals; $\underline{b}$.
13. (- + - 0) (-) Reals; $|\underline{a}|$.

* See Exceptional Cases

14. (- + + -) (+) Reals; $\underline{a}$.

15. (- + + -) (-) Reals; $\underline{a}$ sgn $\underline{b}$.

16. (+ - - 0) (+) Reals; $[\underline{a} + \underline{b}]/2$.

17. (+ - - 0) (-) Reals; $|\underline{a} + \underline{b}|$.

18. (+ - + -) (+) Reals; $(\underline{a} + \underline{b})$ if $(\underline{a}) \neq (-\underline{b})$;
$(\underline{a}) \oplus (-\underline{a}) = (-\underline{a}) \oplus (\underline{a}) = (2\underline{a})$;
to determine $(\underline{a})$, choose that one of $(\underline{a})$, $(-\underline{a})$ for which the first $\underline{a}_i$ not zero is positive.

19. (+ - + -) (-) Reals; $\frac{1}{2}$ sgn $\underline{a}$ $\underline{b}[|\underline{a}| + |\underline{b}| + |\underline{a} + \underline{b}|]$.

20. (- - + -) (+) Reals; $(\underline{a}) \oplus (0) = (\underline{a})$, $(\underline{a}) + (\underline{b}) = (2\underline{b})$ if $(\underline{b}) \neq (0)$.

21. (- - + -) (-) Reals; $\underline{a} + \underline{b}^2$.

22. (- - - 0) (+) Reals; $\underline{a}/2$.

23. (- - - 0) (-) Reals; $\underline{a} + 1$.

* See Exceptional Cases

The multiplication tables for $\underline{a} \circ \underline{b}$ are:

a) (+ + + +)

e_1

b) (- + + +)

e_1	e_2	e_3	e_4
e_2	$-e_1$	e_4	$-e_3$
e_3	$-e_4$	$-e_1$	e_2
e_4	e_3	$-e_2$	$-e_1$

c) (+ - + +)

e_1	e_2	e_3
e_2	$-e_1$	e_1
e_3	e_1	$-2e_1$

d) (+ + - 0)

e_2	e_3	0
e_3	0	0
0	0	0

e) (+ + + -)

e_1	e_2
e_2	e_1

f) (- - + +)

e_1	0
e_2	e_1

g) (- + - 0)

e_2	0	0
0	0	0
e_2	0	0

h) (- + + -)

e_1	e_2	e_3	e_4
e_2	e_3	0	0
e_3	0	0	0
e_4	e_3	0	0

i) (+ - - 0)

0	0	e_2
0	e_1	0
e_2	0	0

j) (+ - + -)

e_1	e_2	e_3
e_2	e_1	0
e_3	0	e_1

k) $(- - + -)$			l) $(- - - 0)$		
e_1	0	0	0	0	e_3
e_2	e_1	0	0	e_2	0
e_3	0	e_1	e_1	0	0

The following are the exceptional cases referred to above:

$(+ + + +)\ (+ + + +)\ (-)$ $\underline{K} = \{\text{reals}\}$; $\underline{a} \oplus \underline{b} = \underline{a} + \underline{b}$; $\underline{a} \circ \underline{b} = \underline{a} + \underline{b} + 1$.

$(+ + + +)\ (- + + +)\ (-)$ $\underline{K} = \{\text{rationals}\}$; $\underline{a} \oplus \underline{b} = \underline{a} + \underline{b}$; $\underline{a} \circ \underline{b}$ is defined as in case 3 for addition, except that 0 is to be replaced by 1 in this definition.

$(+ + + +)\ (+ - + +)\ (-)$ $\underline{K} = \{\text{reals}\}$; $\underline{a} \oplus \underline{b} = \underline{a} + \underline{b}$; $\underline{a} \circ \underline{b} = \operatorname{sgn} \underline{a}\ \underline{b}\ (\underline{a} + \underline{b})$, where $\operatorname{sgn} \underline{a} \circ 0 = \operatorname{sgn} \underline{a}$.

$(+ + + +)\ (- - + +)\ (-)$ $\underline{K} = \{\text{reals}\}$; $\underline{a} \oplus \underline{b} = \underline{a} + \underline{b}$; $\underline{a} \circ \underline{b} = \underline{a} - \underline{b}$.

$(+ + - 0)\ (+ + + +)\ (-)$ $\underline{K} = \{\text{reals}\}$; $\underline{a} \oplus \underline{b} = 1$; $\underline{a} \circ \underline{b} = \underline{a} + \underline{b}$.

$(+ + - 0)\ (- + + +)\ (-)$ $\underline{K} = \{\text{reals}\}$; $\underline{a} \oplus \underline{b} = 1$; $\underline{a} \circ \underline{b}$ is defined as is the numerator in definition 3 for addition.

$(+ + - 0)\ (+ - + +)\ (-)$ $\underline{K} = \{\text{reals}\}$; $\underline{a} \oplus \underline{b} = 1$; $\underline{a} \circ \underline{b} = \underline{a} + \underline{b}$ if $a \neq b$; $\underline{a} \circ \underline{a} = 0$

$(+ + - 0)\ (- - + +)\ (-)$ $\underline{K}$ = reals ; $\underline{a} \oplus \underline{b} = 1$; $\underline{a} \circ \underline{b} = \underline{a}\ (\underline{b} + 1)$

$(+ + + -)\ (- + + +)\ (+)$ $\underline{K} = \{\text{integers}\}$; $\underline{a} \oplus b$ = the numerically greater of $\underline{a}$, $\underline{b}$;

a ∘ b is defined as is the numerator in 3 for addition.

(+ + + -) (+ - + +) (+) $\underline{K}$ = reals ; $\underline{a} + \underline{b}$ = the numerically greater of $\underline{a}$, $\underline{b}$; $\underline{a} \circ \underline{b} = \underline{a}\,\underline{b}$ if $a \neq \underline{b}$; $\underline{a} \circ \underline{a} = 1$.

We next make use of number systems $\underline{G} \equiv \{\underline{a} \hat{+} \underline{g}\}$, where $\underline{a}$ is a real number, $\underline{g}$ an element of a finite non-abelian group, with the operations

$$(\underline{a}_1 \hat{+} \underline{g}_1) \oplus (\underline{a}_2 \hat{+} \underline{g}_2) = (\underline{a}_1 + \underline{a}_2) \hat{+} \underline{g}_1\underline{g}_2$$

$$(\underline{a}_1 \hat{+} \underline{g}_1) \circ (\underline{a}_2 \hat{+} \underline{g}_2) = \underline{a}_1\underline{a}_2 \hat{+} \underline{e},$$

where $\underline{e}$ is the identity element of the group. This system has the symbol (- +.+ +) (+ + - 0) (+). Using the elements of $G\_$ as coordinates of number ennuples with multiplication tables g, i, k, we have the respective symbols

(- + + +) (- + - 0) (+)

(- + + +) (+ - - 0) (+)

(- + + +) (- - - 0) (+)

if the operation of addition of group elements is always that in the definition of ⊕ . Using the same definition of addition, and

$$(\underline{a}_1 \hat{+} \underline{g}_1) \circ (\underline{a}_2 \hat{+} \underline{g}_2) = \underline{a}_1\underline{a}_2 + \underline{g}_1\underline{g}_2$$

with the restriction that if a factor $\underline{g}_i$ occurs more than once in a term, buthhas a $g_k \neq g_i$ between two g_i, such g_i are to be suppressed after the first occurrence, we have the symbol

(- + + +) (- + + -) (+);

this definition used with multiplication table gives a variety with symbol

$$(- + + +)\ (- - + -)\ (+).$$

Varieties with properties satisfying the remaining two symbols are:

(- + + +) (- + + +) (+) $\underline{K}$ = {integers}; $\underline{a} \oplus \underline{b}$ is defined as in case 3 for addition, with the additional restrictions that if a sequence of digits occurs twice in a number, the second sequence is to be suppressed, and that if a digit is repeated, its second occurence is likewise suppressed: i.e, 1231 ⊕ 1423 = 12314; $\underline{a} \circ \underline{b}$ is also defined as in 3, except that here 0 must be replaced by 1 in the definition.

(- + + +) (- - + +) (+) $\underline{K}$ = {integers}; $\underline{a} \oplus \underline{b}$ is defined as in the preceding example; $\underline{a} \circ \underline{b} = \underline{a}$ if $\underline{b} \neq 1$, $\underline{a} \circ 1 = 0$.

<u>2.3 Finite Varieties</u> Finite systems of elements may be defined in much the same way as infinite systems. Here the class $\underline{K}$ is a set of residues modulo a prime $\underline{p}$. The operations defining ⊕ are as follows:

1. (+ + + +) (+) $\underline{a} + \underline{b}$ (mod $\underline{p}$).
2. (+ + + +) (-) $\underline{a} + \underline{b} + 1$ (mod $\underline{p}$) (to be used only where the multiplication inverse is not required. For the other cases,

see the end of this section).

3. (+ − + +) (+) $(\underline{a} + \underline{b})$ (mod $\underline{p}$) if $(\underline{a}) \not\equiv (\underline{b})$ (mod $\underline{p}$);
$(\underline{a}) \oplus (\underline{a}) = (0)$.

4. (+ − + +) (−) (1) if $(\underline{a}) \not\equiv 0$, $(\underline{b}) \not\equiv 0$, $(\underline{a} + \underline{b}) \not\equiv (0)$;
$(\underline{a})$ if $(\underline{b}) \equiv (0)$, $(\underline{b})$ if $(\underline{a}) \equiv (0)$;
(0) if $(\underline{a} + \underline{b}) \equiv (0)$ (mod $\underline{p}$).

5. (+ + − 0) (+) 0.

6. (+ + − 0) (−) 1.

7. (+ + + −) (+) The coordinates consist of the numbers 0, 1, with the operations

$\oplus$	0	1
0	0	1
1	1	1

0	0	1
0	0	0
1	0	1

8. (+ + + −) (−) $\underline{a} + \underline{b} + \underline{a}\,\underline{b}$ (mod $\underline{p}$).

9. (− − + +) (+) $\underline{a} + 2\underline{b}$ (mod $\underline{p}$) ($\underline{p} > 2$)

10. (− − + +) (+) $\underline{a} - \underline{a}\,\underline{b}$ (mod $\underline{p}$).

11. (− + − 0) (+) $\underline{b}$ (mod $\underline{p}$).

12. (− + − 0) (−) $|\underline{a}|$ (mod $\underline{p}$), if $\underline{K}$ is the system of least residues.

13. (− + + −) (+) $\underline{a}$ (mod $\underline{p}$).

14. (− + + −) (−) $\underline{a}$ sgn $\underline{b}$ (mod $\underline{p}$) (system of least residues).

15. (+ − − 0) (+) $2\underline{a} + 2\underline{b}$ (mod $\underline{p}$). ($\underline{p} > 2$).

16. (+ − − 0) (−) $\underline{a}^2 + \underline{b}^2$ (mod $\underline{p}$).

17. (+ − + −) (+) $(\underline{a} + \underline{b})$ (mod $\underline{p}$) if $(\underline{a}) \not\equiv (-\underline{b})$ (mod $\underline{p}$);
$(\underline{a}) \oplus (-\underline{a}) = (-\underline{a}) \oplus (\underline{a}) = (2\underline{a})$ (mod $\underline{p}$)
($p > 2$)

18. (+ − + −) (−) The greatest of $\underline{a}$, $\underline{b}$, $\underline{ab}$, (mod $\underline{p}$).

19. (- - + -) (+) $(\underline{a}) \oplus (0) = (\underline{a})$;

$(\underline{a}) \oplus (\underline{b}) = (2\underline{b}) \pmod{p}, \ (p > 2)$ if $(\underline{b}) \not\equiv (0)$.

20. (- - + -) (-) $\underline{a} + \underline{b}^2 \pmod{p}$.

21. (- - - 0) (+) $2\underline{a} + \underline{b} \pmod{p}$. $(p > 2)$

22. (- - - 0) (-) $\underline{a}^2 + \underline{b} \pmod{p}$.

The multiplication tables used are the same as for the infinite case except that for b) and c) must be substituted:

b′) $(\underline{a}\, e_1 + \underline{b}\, e_2) \circ (\underline{c}\, e_1 + \underline{d}\, e_2) = \{\underline{a}\,\underline{c} - (-1)^{ab}\, \underline{b}\,\underline{d}\} e_1 + \{\underline{b}\,\underline{c} + (-1)^{ab}\, \underline{a}\,\underline{d}\} e_2$

where the coefficients and $\underline{a}\ \underline{b}$ are restricted to the values 1, 0, -1[1]. This definition of addition may be any except 7.

c′)

e_1	e_2	e_3
e_2	e_3	$2e_1 + e_2$
e_3	$2e_1 + e_2$	$4e_1 + 4e_2 + 3e_3$

taken modulo 5[2].

For varieties which have the addition symbol (- + + +) we consider first the numbers $\underline{G} = \{\underline{a} \hat{+} \underline{g}\}$, where the quantities $\underline{a}$ are elements of a finite field. Treating these numbers exactly as in § 2.2 we have systems with the symbols

(- + + +) (+ + - 0) (+) (- + + +) (- + + -) (+)

(- + + +) (- + - 0) (+) (- + + +) (- - + -) (+)

(- + + +) (+ - - 0) (+) (- + + +) (- - - 0) (+)

Let $\oplus$ be defined as above, and let

$$(\underline{a} \hat{+} \underline{g}_1) \circ (\underline{b} + \underline{g}_2) = \underline{ab} \hat{+} \underline{g}$$

where $\underline{g}$ is a fixed element, other than the identity, of the group. This system has the symbol

$$(- + + +)\ (+ + - 0)\ (-);$$

and when used as coefficients of a system with table g, the system is

$$(- + + +)\ (- + - 0)\ (-).$$

Next let $\underline{K} = \{\underline{s}_i\}$ be a finite non-abelian group. In the next ten systems, $\underline{a} \oplus \underline{b} = \underline{s}_i \oplus \underline{s}_j = \underline{s}_i \underline{s}_j$

(- + + +) (+ + + +) (-) $\underline{s}_i \circ \underline{s}_j = \underline{s}_{i+j}$, where i + j is taken modulo the order of the group.

(- + + +) (- + + +) (-) $\underline{s}_i \circ \underline{s}_j = \underline{s}_i \underline{s}_j$.

(- + + +) (+ - + +) (-) $\underline{s}_i \circ \underline{s}_j = \underline{e}$ if $\underline{s}_i \neq \underline{e}$, $\underline{s}_j \neq e$, $i \neq j$; $= \underline{s}_i$ if $\underline{s}_j = \underline{e}$; $= \underline{s}_j$ if $s_i = \underline{e}$; $= \underline{s}_i$ if i = j.

(- + + +) (+ + + -) (-) $\underline{s}_i \circ \underline{s}_j$ = whichever of $\underline{s}_i$, $\underline{s}_j$, involves the greater number of transpositions; if both involve the same number, then whichever introduces the greater subscript first. (Here $\underline{K}$ is a substitution group on n letters.)

(- + + +) (- - + +) (-) $\underline{s}_i \circ \underline{s}_j = \underline{s}_j^{-1} \underline{s}_i$.

(- + + +) (- + + -) (-) $\underline{s}_i \circ \underline{s}_j = \underline{s}_i$.

$(-+++)\ (+--0)\ (-)$ $\underline{s}_i \circ \underline{s}_j = \underline{s}_i^2$ or $\underline{s}_j^2$ determined as in the variety $(-+++)\ (+++-)\ (-)$.

$(-+++)\ (+-+-)\ (-)$ $\underline{s}_i \circ \underline{s}_j = \underline{s}_{i \oplus j}$ where $\underline{i} \oplus \underline{j} = i + j$ if $i \neq -j$, $(-i) \oplus i = i \oplus (-i) = 2i$, all taken modulo the order of the group.

$(-+++)\ (---+-)\ (-)$ $\underline{s}_i \circ \underline{s}_j = \underline{s}_j^{-1} s_i s_j$.

$(-+++)\ (---0)\ (-)$ $\underline{s}_i \circ \underline{s}_j = (\underline{s}_i \underline{s}_j)^2$.

Taking $\underline{s}_i \oplus \underline{s}_j = \underline{s}_{i+j}$, where $i+j$ is taken modulo the order of the group, and $\underline{s}_i \circ \underline{s}_j = \underline{s}_i \underline{s}_j$, we have

$(++++)\ (-++)\ (-)$.

To complete the cases not covered by definition 2 for addition, we may use

$(++++)\ (++++)\ (-)$ $\underline{a} \oplus \underline{b} = \underline{a} + \underline{b} \pmod{\underline{p}}$; $\underline{a} \circ \underline{b} = \underline{a} + \underline{b} + 1 \pmod{p}$.

$(++++)\ (+-++)\ (-)$ $\underline{a} \oplus \underline{b} = \underline{a} + \underline{b} \pmod{p}$; $\underline{a} \circ \underline{b} = \underline{a} + \underline{b} + 1 \pmod{p}$ if $\underline{a} \neq \underline{b}$, $\underline{a} \circ \underline{a} = -1$.

$(++++)\ (--++)\ (-)$ $\underline{a} \oplus \underline{b} = \underline{a} + \underline{b} \pmod{p}$; $\underline{a} \circ \underline{b} = \underline{a} - \underline{b} + 1 \pmod{p}$.

No examples have been found of finite varieties with the properties

$(-+++)\ (+++-)\ (+)$

$(-+++)\ (+-+-)\ (+)$

$(+++-)\ (+-++)\ (+)$.

3. Varieties for which the Existence of Zero and Unit is Postulated

3.1 The postulates The set of postulates which will now be considered is based on one by Huntington[3], to which have been added the postulates of commutativity of addition (A 1), and of an identity element for multiplication (M5), both of which can be proved from the original set. While the independence of the postulates is thus destroyed, the loss is more than compensated by the determination of distinct varieties which would by abstractly equivalent under the original system of postulates.

We shall consider a class $\underline{K}$ of elements and the operations $\oplus$ and $\circ$ on those elements with regard to the following properties:

A0. If $\underline{a}$ and $\underline{b}$ are any elements of $\underline{K}$, then $\underline{a} \oplus \underline{b}$ is a uniquely determined element of $\underline{K}$.

M0. If $\underline{a}$ and $\underline{b}$ are any elements of $\underline{K}$, then $\underline{a} \circ \underline{b}$ is a uniquely determined element of $\underline{K}$.

These two postulates are assumed to hold throughout the discussion.

A1. If $\underline{a}$ and $\underline{b}$ are any elements of $\underline{K}$, then $\underline{a} \oplus \underline{b} = \underline{b} \oplus \underline{a}$.

A2. If $\underline{a}$, $\underline{b}$, and $\underline{c}$ are any elements of $\underline{K}$, then $(\underline{a} \oplus \underline{b}) \oplus \underline{c} = \underline{a} \oplus (\underline{b} \oplus \underline{c})$

A3. There is at least one element $\underline{z}$ in K such that $\underline{z} \oplus \underline{z} = \underline{z}$.

A4. There is not more than one element $\underline{z}$ in $\underline{K}$ such that $\underline{z} \oplus \underline{z} = \underline{z}$

A5. If there is a unique $\underline{z}$ such that $\underline{z} \oplus \underline{z} = \underline{z}$, then either $\underline{a} \oplus \underline{z} = \underline{a}$ for every $\underline{a}$ in $\underline{K}$, or $\underline{z} \oplus \underline{a} = \underline{a}$ for every $\underline{a}$ in $\underline{K}$.

A6. If there is a unique $\underline{z}$ such that $\underline{z} \oplus \underline{z} = z$, then either for every $\underline{a}$ in $\underline{K}$ there is an element $\underline{a}'$ in $\underline{K}$ such that $\underline{a} \oplus \underline{a}' = \underline{z}$, or for every $\underline{a}$ in $\underline{K}$ there is an $\underline{a}''$ in $\underline{K}$ such that $\underline{a}'' \oplus \underline{a} = \underline{z}$.

M1. If $\underline{a}$ and $\underline{b}$ are any elements of $\underline{K}$, then $\underline{a} \circ \underline{b} = \underline{b} \circ \underline{a}$.

M2. If $\underline{a}$, $\underline{b}$, and $\underline{c}$ are any elements of $\underline{K}$, then $(\underline{a} \circ \underline{b}) \circ \underline{c} = \underline{a} \circ (\underline{b} \circ \underline{c})$.

M3. There is at least one element $\underline{u}$ in $\underline{K}$ such that $\underline{u} \circ \underline{u} = \underline{u}$ and $\underline{u} \oplus \underline{u} \neq \underline{u}$.

M4. There is not more than one element $\underline{u}$ in $\underline{K}$ such that $\underline{u} \circ \underline{u} = \underline{u}$ and $\underline{u} \oplus \underline{u} \neq \underline{u}$.

M5. If there is a unique $\underline{u}$ such that $\underline{u} \circ \underline{u} = \underline{u}$, $\underline{u} \oplus \underline{u} \neq \underline{u}$, then either $\underline{a} \circ \underline{u} = \underline{a}$ for every $\underline{a}$ in $\underline{K}$, or $\underline{u} \circ \underline{a} = \underline{a}$ for every $\underline{a}$ in $\underline{K}$.

M6. If there is a unique $\underline{u}$ such that $\underline{u} \circ \underline{u} = \underline{u}$, $\underline{u} \oplus \underline{u} \neq \underline{u}$, then either for every $\underline{a}$ in $\underline{K}$ there is an element $\underline{a}'$ in $\underline{K}$ such that $\underline{a} \circ \underline{a}' = \underline{u}$, or for every $\underline{a}$ in $\underline{K}$ there is an $\underline{a}''$ such that $\underline{a}'' \circ \underline{a} = \underline{u}$.

D. Either $\underline{a} \circ (\underline{b} \oplus \underline{c}) = \underline{a} \circ \underline{b} \oplus \underline{a} \circ \underline{c}$ for every $\underline{a}$, $\underline{b}$, $\underline{c}$, in $\underline{K}$, or $(\underline{b} \oplus \underline{c}) \circ \underline{a} = \underline{b} \circ \underline{a} \oplus \underline{c} \circ \underline{a}$ for every $\underline{a}$, $\underline{b}$, $\underline{c}$, in $\underline{K}$.

Postulate M5 may be proved from A0, A2 -- A6, M0 -- M4, M6, and D; A1 may be proved by the use of all the other postulates. Hence the following symbols are inconsistent:

(- + + + + +) (+ + + + + +) (+)

(- + + + + +) (+ - + + + +) (+)

(- + + + + +) (+ + + + - +) (+)

(- + + + + +) (+ + + + + -) (+)

(- + + + + +) (+ - + + + -) (+)

(+ + + + + +) (+ + + + - +) (+).

Upon examination of the postulates it will be noted that numbers 5 and 6 for either operation have significance only when postulates 3 and 4 for the same operation are both true; also that if 3 is false, 4 is redundant. Hence of the 2^{13} varieties arising from the thirteen postulates under consideration, there remain only those with the symbols (for either operation): $(\underline{\pm}\ \underline{\pm} + + \underline{\pm}\ \underline{\pm})$, $(\underline{\pm}\ \underline{\pm} + - 0\ 0)$, $(\underline{\pm}\ \underline{\pm} - 0\ 0\ 0)$. These are in number $2^4 + 2^2 + 2^2$, so that on combining both operations and considering the distributive law, as well as the inconsistent cases listed above, there is a total of 24 x 24 x 2 - 6 = 1146 varieties to be considered.

3.2 Definitions of K and of the operations. As in 2, the varieties chosen as examples of the Moore symbols are for the most part systems of ennuples of numbers. Here, however, the properties of any particular definition of $\circ$ vary with the definition of $\oplus$, so that it is necessary to state precisely the nature of K, $\oplus$, and $\circ$ for each of the Moore symbols. To do this, we first catalogue all classes and operations to be used, and then combine these into a table according to the properties exhibited.

The elements of K are the coordinates of the ennuples. The nature of K is indicated as follows:

R, real numbers; I, integers;

r rational numbers; C, complex numbers.

A prime (') after one of these letters restricts K to non-negative values, a double prime (") to properly positive values, and the subscript 1 to values $\geqq 1$. Further, for ennuples $(a_1, a_2, \ldots, a_n)$, we use the classes

R^*, reals $\geqq 0$ such that $\Sigma a_i > 0$;

R^*_1, reals $\geqq 0$ such that $\Sigma a_i \geqq 1$;

R^{**}, reals $\geqq 0$ such that $a_1 > 0$.

These notations will occasionally be used for other classes than that of all real numbers.

In case the coordinates of an ennuple are themselves coordinates of an ennuple, these will be denoted by, for example, "R - d", where d denotes a multiplication table for ennuples. In this example the coordinates of the ennuple under consideration are real ennuples of the system defined by d. While such a system should be written in detail if the binary relations are interpreted strictly, for brevity of expression this indication of its composition will be used.

The operations defining $\oplus$ are always to be regarded as operating on coordinates of like subscript, $(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) \oplus (\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n) = (\underline{a}_1 \oplus \underline{b}_1, \underline{a}_2 \oplus \underline{b}_2, \ldots, \underline{a}_n \oplus \underline{b}_n)$, except when the matrix notation indicates that the operation is to be applied to the entire ennuple. In defining the operations + denotes ordinary addition, $\underline{a}\,\underline{b}$ ordinary multiplication of $\underline{a}$ and $\underline{b}$.

Definitions of $\underline{a} \oplus \underline{b}$ are as follows:

1. $\underline{a} + \underline{b}$
2. $\underline{a} + \underline{b} + 1$
3. $(\underline{a} + \underline{b})$ if $(\underline{a}) \neq (\underline{b})$, $(\underline{a}) \oplus (\underline{a}) = (0)$
4. 0
5. 1
6. $2\underline{a} + \underline{b}$
7. $\underline{a}^2 + \underline{b}$
8. $(\underline{a} + \underline{b}) / 3$
9. $|\underline{a} + \underline{b}|$
10. $|\underline{a}| + |\underline{b}|$
11. $\underline{a} / 2$
12. $\underline{a}^2 - \underline{b}^2$
13. $(\underline{a}) + (0) = (2\underline{a})$, $(\underline{a}) + (\underline{b}) = (2\underline{b})$, if $(\underline{b}) \neq (0)$
14. $(|\underline{a}| + |\underline{b}|) / 3$
15. $(\underline{a}) \oplus (0) = (0) \oplus (\underline{a}) = (\underline{a})$, $(\underline{a}) \oplus (\underline{b}) = (2\underline{a} + 2\underline{b})$ if $(\underline{a}) \neq (\underline{b})$, $(\underline{a}) \oplus (-\underline{a}) = (-\underline{a}) \oplus (\underline{a}) = (\underline{a})$*
16. $2|\underline{a}| + 3|\underline{b}|$
17. $(\underline{a}) \oplus (\underline{b}) = (0)$ if $(\underline{a}) \neq (\underline{b})$; $(\underline{a}) \oplus (\underline{a}) = (\underline{a})$
18. $\underline{a} + \underline{b} + \underline{ab}$
19. $(\underline{a} + \underline{b}) / 2$
20. $\underline{a}^2 + \underline{b}^2$
21. $\underline{a}$

22. $-|\underline{a}|$

23. $2\underline{a} - \underline{b}$

24. $\underline{ab} - \underline{a}$

25. $|\underline{a}| + |\underline{b}| + 1$

26. $1 + \underline{ab}$

27. $\underline{a} + 1$

28. $(\underline{a}) \oplus (0) = (0) + (\underline{a}) = (2\underline{a})$;

$(\underline{a}) \oplus (\underline{b}) = (\underline{a} + \underline{b})$ if $(\underline{a}) \neq (\underline{b})$, $(\underline{a}) + (-\underline{a}) = (\underline{a}) = (-\underline{a}) + (\underline{a})$.*

29. $(\underline{a}) + (0) = (\underline{a})$,

$(\underline{a}) + (\underline{b}) = (2\underline{b})$ if $(\underline{b}) \neq (0)$.

In the following, sgn $(0.\ \underline{a})$ = sgn $\underline{a}$.

30. sgn $\underline{a}\ \underline{b}\ |\underline{a} + \underline{b}|$

31. $\underline{a}$ sgn b + $\underline{b}$ sgn $\underline{a}$

32. $\frac{1}{2}$ sgn $\underline{a}\ \underline{b}\ \left[|\underline{a}| + |\underline{b}| + |\underline{a} + \underline{b}|\right]$.

In the next set of definitions,** $\underline{a}$ and $\underline{b}$ are integers, and if B is the number of digits in $\underline{b}$, then $10^{B}\ \underline{a} + \underline{b}$ is equivalent to writing the digits of $\underline{a}$ in front of those in $\underline{b}$: thus if $\underline{a} = 51$, $\underline{b} = 37$, $\underline{a} + \underline{b} = 5137$. These definitions are designed for application to rational fractions; if $\underline{K} = I$, the law for the denominator of the sum is to be ignored. If $\underline{a}$ (or $\underline{b}$) = 0, take A (or B) = 0. Considering fractions a_i / b_i we define $a_1/b_1 \oplus a_2/b_2$ to be

---

* For $(\underline{a})$, choose that one of $(\underline{a})$, $(-\underline{a})$ for which the first a_i not zero is positive.

** I am indebted to Dr. R.S. Martin for a definition of addition which led to these.

33. $\operatorname{sgn} \underline{a}_1 \; 10^{A_2} |\underline{a}_1| / [|10^{B_2} \underline{b}_1| + |\underline{b}_2|]$

34. $10^{A_2} |\underline{a}_1| / [|10^{B_2} \underline{b}_1| + |\underline{b}_2|]$

35. $\operatorname{sgn} \underline{a}_1 \; [|10^{A_2} \underline{a}_1| + |\underline{a}_2|] / [|10^{B_2} \underline{b}_1| + |\underline{b}_2|]$

36. $[|10^{A_2} \underline{a}_1| + |\underline{a}_2|] / [|10^{B_2} \underline{b}_1| + |\underline{b}_2|]$

37. $\underline{a} + \underline{b} i \quad (\underline{K} = C)$

38. $\Re(\underline{a} + \underline{b}) \quad (\underline{K} = C)$

In the following, $a = \Sigma \alpha_i e_i$, where the multiplication table for the e_i is indicated by the letter following that for $\underline{K}$.

39. $\underline{a}\, e_1 + \underline{b}\, e_2$

40. $\underline{a}\, e_1$

41. $\underline{a}\, e_1 - \underline{b}\, e_2$

42. $\underline{a}\, e_2$

43. $\dfrac{\underline{a} + \underline{b}}{2}\, e_2$

We shall now consider two systems of elements and binary operations $\underline{R} = \{\underline{r} \;\; +_1, \circ_1\}$, $S = \{\underline{s} \;\; +_2, \circ_2\}$ and from these form a new system $\underline{K} = \{\underline{k} = \underline{r}e + \underline{s}\,\epsilon \;\; \oplus, \circ\}$, with the properties

$$\underline{k}_1 \oplus \underline{k}_2 \equiv (\underline{r}_1 +_1 \underline{r}_2)e + (\underline{s}_1 +_2 \underline{s}_2)\epsilon$$

$$\underline{k}_1 \circ \underline{k}_2 \equiv \underline{r}_1 \circ_1 \underline{r}_2 e + \underline{s}_1 \circ_2 \underline{s}_2\, \epsilon.$$

As the first instances of this sort, let the elements of both $\underline{R}$ and $\underline{S}'$ be real numbers, both $\circ_1$ and $\circ_2$ ordinary multiplications, and:

44. $\underline{r}_1 +_1 \underline{r}_2 = \underline{r}_1 + r_2, \quad \underline{s}_1 +_2 \underline{s}_2 = 0$

45. $\underline{r}_1 +_1 \underline{r}_2 = \underline{r}_1 + \underline{r}_2, \; \underline{s}_1 +_2 \underline{s}_2 = 0$ if $\underline{s}_1 \neq \underline{s}_2$, $\underline{s}_1 +_2 \underline{s}_1 = \underline{s}_1$

46. $\underline{r}_1 +_1 \underline{r}_2 = \underline{r}_1 + \underline{r}_2 \quad \underline{s}_1 +_2 \underline{s}_2 = 2\underline{s}_1 - \underline{s}_2.$

A more extensive set of definitions is obtained by taking for the elements $\underline{r}$ real numbers, for $\underline{s}$ the elements $\underline{g}$ of a finite non-abelian group of which $\underline{i}$ represents the identity element, while $\bullet_1$ and $\oplus_2$ are defined to be ordinary multiplication and group multiplication respectively, and $\oplus_1$, $\circ_2$ are defined as follows:

47. $\underline{r}_1 +_1 \underline{r}_2 = \underline{r}_1 + \underline{r}_2$, $\underline{g}_1 \circ_2 \underline{g}_2 = \underline{i}$

48. $\underline{r}_1 +_1 \underline{r}_2 = 0$ $\qquad \underline{g}_1 \circ_2 \underline{g}_2 = \underline{i}$

In definitions 49 and 50, $\underline{g}_1 \circ_2 \underline{g}_2 = \underline{g}_1\underline{g}_2$, except that in a product $\underline{g}_i\underline{g}_i \cdots g_l$ if an element $\underline{g}_k$ occurs more than once, and elements $\underline{g}_h$ $(h \neq k)$. separate the $\underline{g}_k$, then every $\underline{g}_k$ after the first is to be suppressed; and

49. $\underline{r}_1 +_1 \underline{r}_2 = \underline{r}_1 + \underline{r}_2$

50. $\underline{r}_1 +_1 \underline{r}_2 = 0$

51. $\underline{r}_1 +_1 \underline{r}_2 = \underline{r}_1 + \underline{r}_2$, $\underline{g}_1 +_2 \underline{g}_2 = \underline{g}_1\underline{g}_2$.

$\underline{r}_1 \circ_1 \underline{r}_2 = \underline{r}_1\underline{r}_2$. $\qquad \underline{g}_1 \circ_2 \underline{g}_2 = \underline{g}_1\underline{g}_2$

52. $\underline{r}_1 +_1 \underline{r}_2 = \underline{r}_1 + \underline{r}_2$, $\underline{g}_1 +_2 \underline{g}_2 = \underline{g}_1$

$\underline{r}_1 \circ_1 \underline{r}_2 = \underline{r}_1\underline{r}_2$, $\qquad \underline{g}_1 \circ_2 \underline{g}_2 = \underline{g}_1\underline{g}_2$

53. $\underline{r}_1 +_1 \underline{r}_2 = \underline{r}_1 + \underline{r}_2 + 1$, $\underline{g}_1 +_2 \underline{g}_2 = \underline{g}_1$

$\underline{r}_1 \circ_1 \underline{r}_2 = \underline{r}_1\underline{r}_2$, $\qquad \underline{g}_1 \circ_2 \underline{g}_2 = \underline{g}_1\underline{g}_2$

For systems $\underline{K} = \{k = \underline{r}e_1 \hat{+} \underline{s}e_2 \hat{+} \underline{g}e_3 \oplus , \circ\}$, where $\underline{r}$ and $\underline{s}$ are real numbers, $\underline{g}$ is an element of a finite non-abelian group, we use the definitions

54. $\underline{k} \oplus \underline{k}_2 = (\underline{r}_1 + \underline{r}_2)e_1 + 0e_2 + \underline{g}_1\underline{g}_2 e_3$

$\underline{k}_1 \circ \underline{k}_2 = \underline{r}_1\underline{r}_2 e_1 + \underline{s}_1\underline{s}_2 e_2 + \underline{i}\ e_3$

55. $\underline{k}_1 \oplus \underline{k}_2 = (\underline{r}_1 + \underline{r}_2)\, e_1 + 0\, e_2 + \underline{g}_1\underline{g}_2 e_3$

$\underline{k}_1 \circ \underline{k}_2 = \underline{r}_1\underline{r}_2 e_1 + \underline{s}_1\underline{s}_2 e_2 + (\underline{g}_1\underline{g}_2)\, e_3$

where $(\underline{g}_1\underline{g}_2)$ denotes that the restriction of definition 49 is to hold here

The tables for **multiplication** of ennuples follow. (For convenience $(\underline{a}_1, \ldots, \underline{a}_n)$ will be regarded as $\sum \underline{a}_i \underline{e}_i$.)

a)

e_1

b)

e_1	e_2	e_3	e_4
e_2	$-e_1$	e_4	$-e_3$
e_3	$-e_4$	$-e_1$	e_2
e_4	e_3	$-e_2$	$-e_1$

c) [2]

e_1	e_2	e_3
e_2	$-e_1$	e_1
e_3	e_1	$-2e_1$

d) [4]

e_1	e_1
e_1	e_1

e)

e_1	e_2
e_2	0

f)

e_1	e_2	e_3
0	e_1	0
0	0	e_1

g)

e_1	e_2	e_3	e_4
e_2	e_3	0	0
e_3	0	0	0
e_4	e_3	0	0

h)

0	0	e_2
0	e_2	0
e_2	0	0

i) $$\begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ e_2 & e_3 & e_4 & -e_3 & e_6 & -e_5 \\ e_3 & e_4 & e_5 & -e_6 & 0 & 0 \\ e_4 & -e_3 & -e_6 & -e_5 & 0 & 0 \\ e_5 & e_6 & 0 & 0 & 0 & 0 \\ e_6 & -e_5 & 0 & 0 & 0 & 0 \end{matrix}$$

j) $$\begin{matrix} e_1 & 0 & -e_3 \\ 0 & -e_1 & e_2 \\ e_3 & e_2 & e_1 \end{matrix}$$

k) $$\begin{matrix} e_1 & e_2 & e_1 \\ e_2 & 0 & 0 \\ e_3 & 0 & 0 \end{matrix}$$

l) $$\begin{matrix} e_2 & 0 & e_3 \\ 0 & e_2 & 0 \\ e_3 & 0 & 0 \end{matrix}$$

m) $$\begin{matrix} e_1 & e_1 & 0 \\ 0 & 0 & 0 \\ e_1 & 0 & 0 \end{matrix}$$

n) $$\begin{matrix} e_1 & e_2 \\ e_2 & e_1 \end{matrix}$$

o) $$\begin{matrix} e_1 & 0 & e_3 \\ 0 & 0 & e_2 \\ e_3 & e_2 & e_1 \end{matrix}$$

p) $$\begin{matrix} e_1 & e_2 & e_3 & e_4 \\ e_2 & 0 & e_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

q) $$\begin{matrix} e_1 & e_2 & e_3 \\ 0 & e_1 & 0 \\ e_3 & 0 & e_1 \end{matrix}$$

s) $$\begin{matrix} e_2 & e_3 & 0 \\ e_3 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

t) $$\begin{matrix} 0 & 0 & e_2 \\ 0 & e_1 & 0 \\ e_2 & 0 & 0 \end{matrix}$$

u) $$\begin{matrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ e_2 & 0 & 0 \end{matrix}$$

v) $\begin{matrix} e_2 & 0 & 0 \\ e_2 & 0 & 0 \\ e_2 & 0 & 0 \end{matrix}$

w) $\begin{matrix} e_1 & e_2 \\ e_2 & -e_1 \end{matrix}$

x) $\begin{matrix} 0 & 0 & e_3 \\ 0 & e_2 & 0 \\ e_1 & 0 & 0 \end{matrix}$

y) $\begin{matrix} 0 & 0 & e_2 \\ 0 & e_2 & e_1 \\ e_2 & 0 & e_1 \end{matrix}$

z) $\begin{matrix} e_1 & e_2 & e_3 \\ e_2 & 0 & e_2 \\ e_3 & e_2 & 0 \end{matrix}$

a') $\begin{matrix} e_1 & e_2 \\ e_2 & e_2 \end{matrix}$

b') $\begin{matrix} e_1 & e_2 & 0 \\ e_2 & e_2 & 0 \\ 0 & 0 & e_3 \end{matrix}$

c') $\begin{matrix} e_1 & 0 \\ 0 & e_2 \end{matrix}$

d') $\underline{a} \circ \underline{b} = 1$

e') $\underline{a} \circ \underline{b} = 10^A + 1$, where A is the number of digits in $\underline{a}$.

f') $\begin{matrix} e_1 & e_1 \\ e_1 & e_2 \end{matrix}$

The following examples are listed by number in the table:

	$\underline{K}$	$\underline{a} \oplus \underline{b}$	$\underline{a} \circ \underline{b}$
M1.	R	$\underline{a} + \underline{b}$	$\underline{a} + \underline{b} + 1$
M2.	R	$\underline{a} + \underline{b}$	$\underline{a}\ \underline{b}$ if $\underline{a} \neq \underline{b}$; $\underline{a} \circ \underline{a} = 1$

M3	R	$\underline{a} + \lvert\underline{b} + 1\rvert$	$\underline{a} + \underline{b}$ if $\underline{a} \neq \underline{b}$; $\underline{a} \circ \underline{a} = 0$
M4	R-d	$(\underline{a}) + (0) = \underline{a}$, $(\underline{a}) + (\underline{b}) = (0)$ if $(\underline{b}) \neq (0)$	Table d.
M5	R-d	$(\underline{a}) + (0) = (2\underline{a})$ $(\underline{a}) + (\underline{b}) = (0)$ if $(\underline{b}) \neq (0)$	Table d
M6	I	$\underline{a}\ \underline{b}$ sgn $\underline{a}\ \underline{b}$	$\underline{a}\ \underline{b}$
M7	R	$\underline{a} + \underline{b}$	$\underline{a} + 2\underline{b} + 2$
M8	I	$\underline{a} + \underline{b}$	Sgn $\underline{a}\ \underline{b}$ $[\lvert a\rvert + \lvert b\rvert - \underline{1}]$
M9	R"	$\underline{a} + \underline{b}$	$2\underline{a} + 2\underline{b}$ if $\underline{a}$, $\underline{b}$, $\neq 1$; $\underline{a}\ 1 = 1\ \underline{a} = \underline{a}$
M10	R"	$(\underline{a} + \underline{b})/3$	Same as M9
M11	R"	$2\underline{a} + \underline{b}$	Same as M9
M12	I	$\underline{a} + \underline{b}$	$\underline{a}\ \underline{b}$
M13	C	$\underline{a} + \underline{b}$	$2\underline{a} + \underline{b} + 1/12$
M14	$\{\pm 2^m,\ m \gtreqless 0$ (integers)$\}$	$2^m \oplus 2^n = 2^{m+n}$	$2^m \circ 2^n = 2^{m^2+mn-1}$
M15	I	$\underline{a} + \underline{b}$	$\underline{a}^2\ \underline{b}^2$
M16	I	$\underline{a} + \underline{b}$	No. 36 for a addition, 0 to be replaced by 1 whenever it occurs in a product

M17	R	$\underline{a} + \underline{b}$	$2\underline{a}^2 + \underline{b}^2 + 1/12$
M18	C	$-\underline{a}\,\underline{b}$	-1
M19	R	$\underline{a} + \underline{b}$	$\lvert\underline{a}\rvert + \lvert\underline{b}\rvert + 1$
M20	I	$\underline{a} + \underline{b}$	$10^B\underline{a}$, where B is the number of digits in $\underline{b}$
M21	I	$\underline{a} + \underline{b}$	$\underline{a}^2 + \underline{b}^2$
M22	R	$\underline{a} + \underline{b}$	$\underline{a} - \underline{b}$
M23	r"	Def. 35	$\underline{ab}$ if $\underline{a} \neq \underline{b}$, $\underline{a}\ \underline{a} = 1$
M24	$I \geq 2$	$10^B\underline{a} + \underline{b}$	$\underline{a}^2\underline{b}^2$
M25	I	$\underline{a} + \underline{b}$	$\lvert\underline{a}\,\underline{b}\rvert$
M26	R	$\lvert\underline{a}\rvert + \lvert\underline{b}\rvert$	$\lvert\underline{a}\,\underline{b}\rvert$
M27	I'	Digits of $\underline{a}$ followed by those of $\underline{b}$; if a sequence is repeated, suppress after first occurrence; if repeated in reverse order, suppress both; write 0 for a number which is suppressed.	Digits of $\underline{a}$ followed by those of $\underline{b}$; if a sequence is repeated in reverse order, suppress both; write 1 for a suppressed number.

M28	I'	Same as M27, except that 0 is written for a suppressed sequence, and between $\underline{a}$ and $\underline{b}$.	Same as M27
M29	I'	Digits of $\underline{a}$ followed by those of $\underline{b}$; suppress repeated sequences after first occurrence; suppress zero.	Same as M27
M30	I"	Same as M29	Same as M27
M31	I"	$10^{p}a + b$	Same as M27
M32	I'	Same as M29 with modifications of M28	Same as M27

<u>3.3</u> <u>Table</u> <u>of</u> <u>the</u> <u>varieties.</u> The Moore symbols for addition and for multiplication are indicated separately by number as follows:

1. (+ + + + + +)
2. (- + + + + +)
3. (+ - + + + +)
4. (+ + + + - +)
5. (+ + + + + -)
6. (- - + + + +)
7. (- + + + - +)
8. (- + + + + -)
9. (+ - + + - +)
10. (+ - + + + -)
11. (+ + + + - -)
12. (- - + + - +)
13. (- - + + + -)
14. (+ - + + - -)
15. (- + + + - -)
16. (- - + + - -)
17. (+ + + - 0 0)
18. (+ - + - 0 0)
19. (- + + - 0 0)
20. (- - + - 0 0)
21. (+ + - 0 0 0)
22. (- + - 0 0 0)
23. (+ - - 0 0 0)
24. (- - - 0 0 0)

Addition symbols are indicated in the right hand column, multiplication symbols in the top row of each page of the tables. The letter D indicates that the distributive law holds, N that it does not hold.

The first letter of each formula defines $\underline{K}$, the second (number) the rule for $\oplus$, the final letter that for $\circ$. Thus 6,14D, with formula R61 represents the symbol

$$(- - + + + +)\ (+ - + + - -)\ (+)$$

for which the instance offered is the system of triples of real numbers $(\underline{a}_1, \underline{a}_2, \underline{a}_3)$ defined by the operations $(\underline{a}_1, \underline{a}_2, \underline{a}_3) \oplus (\underline{b}_1, \underline{b}_2, \underline{b}_3) = (2\underline{a}_1 + \underline{b}_1,\ 2\underline{a}_2 + \underline{b}_2,\ 2\underline{a}_3 + \underline{b}_3)$; $(a_1, a_2, a_3) \bullet (\underline{b}_1, \underline{b}_2, \underline{b}_3) = (0,\ \underline{a}_1\underline{b}_1 + \underline{a}_2\underline{b}_2,\ \underline{a}_1\underline{b}_3 + \underline{a}_3\underline{b}_1)$.

When the definition of addition is any one of 47 -- 55, that of K refers to the real number components of the system. In the cases where multiplication is indicated by a number, the addition rule listed under this number is to be used as that for multiplication.

⊕	o: 1D	1N	2D	2N
1	R1a	M1	R1b	I1,33
2	**	r33a	M27	r33b
3	R3a	R30a	R3b	R30b
4	R4a	C38w	R4b	C38w-b
5	R'1a	R31a		R31b
6	R6a	R7a	R6b	R7b
7		r34a	M28	r34b
8		r35a	M29	r35b
9	R8a	R9a	R8b	R9b
10	R15a	R32a	R15b	R32b
11		R10a		R10b
12	R11a	R12a	R11b	R12b
13	R29a	C37w	R29b	C37w-b
14	R28a	R14a	R28b	R14b
15		r36a	M32	r36b
16	R13a	R16a	R13b	R16b
17		R18a		R18b
18		R20a		R20b
19		R22a		R22b
20		R24a		R24b
21	R"1a	R"2a		I25,33
22	R"52a	R"53a	M30	M31
23	R"8a	R"26a		I 26,33
24	R"6a	R"27a		I 27,33

	3D	3N	4D	4N
1	R1c	M2	* *	R2s
2	* *	r33c	* *	I33d'
3	R3c	R30c	R'3d	R30d'
4	R4c	R5,4	R'4d	R5s
5		R31c	R'1d	R31d'
6	R6c	R7c	M4	R7d'
7		r34c		I34d'
8		r35c		I35d'
9	R8c	R9c		R9d'
10	R15c	R32c	R'15d	R32d'
11		R10c	M26	R'45d'
12	R11c	R12c	M5	R12d'
13	R29c	M3	R'29d	R'7d
14	R28c	R14c	R'28d	R14d'
15		r36c		I36d'
16	R36c	R16c	R'13d	R16d'
17		R18c		R18d'
18		R20c		R20d'
19		R22c		R22d'
20		R24c		R24d'
21		R*25,3	R*1d	R*25d
22		M23	R*52d	R*53d
23		R*26,3	R*8d	R*26d
24		R*27,3	R*6d	R*27d

	5D	5N	6D	6N
1	R1e	M6	R1f	M7
2	* *	I33e		r33f
3	R3e	R30e	R3f	R30f
4	R4e	C38w-e	R4f	C38w-f
5	R'1e	R31e	R'1f	R31f
6	R6e	R7e	R6f	R7f
7		I34e		r34f
8		I35e		r35f
9	R8e	R9e	R8f	R9f
10	R15e	R32e	R15f	R32f
11	R'44e	R10e		R10f
12	R11e	R12e	R11f	R12f
13	R29e	C37w-e	R29f	C37w-f
14	R28e	R14e	R28f	R14f
15		I36e		r36f
16	R13e	R16e	R13f	R16f
17	R45n	R18e		R18f
18	R*-c'43n	R20e		R20f
19	R'-a'42n	R41n		R22f
20	R* 40n	R24e		R24f
21	R"1n	R"25n	R*1f	R*25f
22	R*52e	R*53e	R*52f	R*53f
23	R"8n	R"26n	R*8f	R*26f
24	R"6n	R"27n	R*6f	R*27f

	7N	8D	8N
1	R2u	R1g	I2,35
2	I33e'	R49a	r33g
3	I3e'	R3g	R30g
4	R5u	R4g	I5,35
5	I'1e'	R'1g	R30g
6	I'6e'	R6g	R7g
7	I34e'	R50a	r34g
8	I35e'	R'49a	r35g
9	I9e'	R8g	R9g
10	I32e'	R15g	R32g
11	I10e'	R'44g	R10g
12	I12e'	R11g	R12g
13	I13e'	R29g	C37w-g
14	I28e'	R28g	R14g
15	I36e'	R'55a	r36g
16	I15e'	R13g	R16g
17	I18e'	R45g	R18g
18	I20e'	R*-c'43g	R20g
19	I22e'	R'-a'42g	R22g
20	I24e'	R*-n40g	R24g
21	R25u	R*1g	I25,35
22	I'36e'	R"49a	R'59a
23	R26u	R*8g	I26,35
24	R27u	R*6g	I27,35

	9D	9N	10D	10N
1	R1h	R2t	R1i	M8
2		I33h	* *	r33i
3	R3h	R30h	R3i	R30i
4	R4h	R5h	R4i	C38w-i
5	R'1h	R31h	R'1z	R31i
6	R6h	R7h	R6i	R7i
7		I34h		r34i
8		I35h		r35i
9	R8h	R9h	R8i	R9i
10	R15h	R32h	R15z	R32i
11		R10h	R'44z	R10i
12	R11h	R12h	R11i	R12i
13	R29h	C37w-h	R29i	C37w-i
14	R28h	R14h	R28z	R14i
15		I36h		r36i
16	R13h	I16h	R13i	R16i
17		R18h	R45i	R18i
18		R20h	R*-c'43z	R20i
19		R22h	R'-a'42z	R22i
20		R24h	R*-n40z	R24i
21	R*1h	R*25h	R*1z	M9
22	R*52h	R*53h	R*52z	R*53z
23	R*8h	R*26h	R*8z	M10
24	R*6h	R*27h	R*6z	M11

	11D	11N	12D	12N
1	R1d	M25	R1j	M13
2	R47a	I33d		r33j
3	R3d	R30d	R3j	R30j
4	R4d	C38w-d	R4j	C38w-j
5	R'-d1e	R31d	R*1y	R31j
6	R6d	R7d	R6j	R7j
7	R48a	I34d		r34j
8	R'47a	I35d		r35j
9	R8d	R9d	R8j	R9j
10	R-d15e	R32d	R*3y	R32j
11	M12	R10d	R*37y	R10j
12	R11d	R12d	R11j	R12j
13	R29d	C37w-d	R29j	C37w-j
14	R-e28d	R14d	R*28y	R14j
15	R'55a	I36d		r36j
16	R13d	R16d	R13y	R16j
17	R45d	R18d		R18j
18	R*-c'43d	R20d		R20j
19	R'-a'42d	R22d		R22j
20	R*-n40d	R24d		R24j
21	R*-e1d	R*-e25d	R*1y	R*25y
22	R*47d	R*53d	R*-52y	R*-53y
23	R*-e8d	R*-e26d	R*8y	R*26y
24	R*-e6d	R*-e27d	R*6y	R*27y

	13D	13N	14D	14N
1	R1k	M14	R1ł	M15
2	R49k	I33k	R47ł	I33ł
3	R3k	R30k	R3ł	R30ł
4	R4k	C38w-k	R4ł	C38w-ł
5	R'1k	R31k	R'1ł	R31ł
6	R6k	R7k	R6ł	R7ł
7	R50k	I34k	R48ł	r34ł
8	R'49k	I35k	R'47ł	r35ł
9	R8k	R9k	R8ł	R9ł
10	R15k	R32k	R15ł	R32ł
11	R'44k	R10k	R'44ł	R10ł
12	R11k	R12k	R11ł	R12ł
13	R29k	C37w-k	R29ł	C37w-ł
14	R28k	R14k	R28ł	R14ł
15	R*52k	I36k	R'55ł	I36ł
16	R13k	R16k	R13ł	R16ł
17	R45k	R18k	R45ł	R18ł
18	R*-c'43k	R20k	R*-d'43ł	R20ł
19	R'-a'42k	R22k	R'-a'42ł	R22ł
20	R*-n40k	R24k	R46ł	R24ł
21	R*1k	R*25k	R*1ł	R*25ł
22	R*49k	R*51k	R*47ł	R*53ł
23	R*8k	R*26k	R*8ł	R*26ł
24	R*6k	R*27k	R*6ł	R*27ł

	15D	15N	16D	16N
1	R-d1g	M16	R1m	M17
2	R-d49g	I-d33g	R49m	r33m
3	R-d3g	R-d30g	R3m	R30m
4	R-d4g	C-d38w-g	R4m	C38w-m
5	R'-d1g	R-d31g	R'1m	R31m
6	R-d 6g	R-d7g	R6m	R7m
7	R-d50g	I-d34g	R50m	r34m
8	R'-d49g	I-d35g	R'49m	r35m
9	R-d8g	R-d9g	R8m	R9m
10	R-d15g	R-d32g	R15m	R32m
11	R'-d44g	R-d10g	R'44m	R10m
12	R-d11g	R-d12g	R11m	R12m
13	R-d29g	R-d37w-g	R29m	C37w-m
14	R-d28g	R-d14g	R28m	R14m
15	R'-d55g	I-d36g	R'55m	r36m
16	R-d13g	R-d16g	R13m	R16m
17	R-d45g	R-d18g	R45m	R18m
18	R*-c'-d43g	R-d20g	R*-c'43m	R20m
19	R'-a'-d42g	R-d22g	R'-a'42m	R22m
20	R*-n-d40g	R-d24g	R*-n40m	R24m
21	R*-d1g	R*-d25g	R,1x	R,25x
22	R*-d49g	R*-d51g	R**49m	R**51m
23	R*-d8g	R*-d26g	R,8x	R,26x
24	R*-d6g	R*-d27g	R,6x	R,27x

	17D	17N	18D	18N
1	R1n	R2n	R1o	R2c
2	R47n	r33n	R47o	I33o
3	R3n	R30n	R3o	R30o
4	R4n	R5n	R4o	R5c
5	R'1n	R31n	R'1o	R30o
6	R6n	R7n	R6o	R7o
7	R48n	r34n	R48o	I34o
8	R'-a'39n	r35n	R'-a'39o	I35o
9	R8n	R9n	R8o	R9o
10	R15n	R32n	R15o	R32o
11	R'44n	R10n	R'44o	R10o
12	R11n	R12n	R11o	R12o
13	R29n	C37w-n	R29o	C37w-o
14	R28n	R14n	R28o	R14o
15	R'-b'44(b')	r36n	R'-b'44o	I36o
16	R13n	R16n	R13o	R16o
17	M18	R18n	R45o	R18o
18	R'-c'43(c')	R20n	R'-c'43o	R20o
19	C-f'39(f')	R22n	C-f'39c	R22o
20	R-a'41n	R24n	R-a'41o	R24o
21	R*1n	R*25n	R*1o	R25c
22	R*-a'39n	R*53n	R*-a'39o	R*53o
23	R*8n	R*26n	R*8o	R26c
24	R*6n	R*27n	R*6o	R27c

	19D	19N	20D	20N
1	R1b	R2b	R1q	R2f
2	R47p	I33p	R47q	I33q
3	R3p	R30p	R3q	R30q
4	R4p	R5b	R4q	R5f
5	R'1p	R31p	R'1q	R31q
6	R6p	R7p	R6q	R7q
7	R48p	I34p	R48q	I34q
8	R'-a'39p	I35p	R'-a'49q	I35q
9	R8p	R9p	R8q	R9q
10	R15p	R32p	R15q	R32q
11	R'44p	R10p	R'44q	R10q
12	R11p	R12p	R11q	R12q
13	R29p	C37w-p	R29q	C37w-q
14	R28p	R14p	R28q	R14q
15	R'-b'39p	I36p	R'-b'39q	I36q
16	R13p	R16p	R13q	R16q
17	R45p	R18p	R45q	R18q
18	R'-c'43p	R20p	R'-c'43q	R20q
19	C-f'39p	R22p	C-f'39q	R22q
20	R-a'41p	R24p	R-a'41q	R24q
21	R*1p	R25p	R*'1q*	R25q
22	R*-a'39p	R*51p	R*-a'39q	R"51q
23	R*8p	R26p	R*'8q*	R26q
24	R*6p	R27p	R*'6q*	R27q

R' = {reals $\geq$ 0 such that $a_1 \geq 1$}

	21D	21N	22D	22N
1	R1s	M19	R1u	M20
2	R47s	I33s	R47u	I33u
3	R3s	R30s	R3u	R30u
4	R4s	R5a	R4u	C38w-u
5	R'1s	R31s	R'1u	R31u
6	R6s	R7s	R6u	R7u
7	R48s	I34s	R48u	I34u
8	R'-a'39s	I35s	R'-a'39u	I35u
9	R8s	R9s	R8u	R9u
10	R15s	R32s	R15u	R32u
11	R'44s	R10s	R'44u	R10u
12	R11s	R12s	R11u	R12u
13	R29s	C37w-s	R29u	C37w-u
14	R28s	R14s	R28u	R14u
15	R'-b'39s	I36s	R'-b'39u	I36u
16	R13s	R16s	R13u	R16u
17	R17n	R18n	R17u	R18u
18	R19n	R20n	R19u	R20u
19	R21n	R41e	R21u	R22u
20	R23n	R24n	R23u	R24u
21	R**1e	R**25e	R"1g	R, 25g
22	R**39n	R"53n	R**39g	R"51g
23	R**8e	R**26e	R"8g	R, 26g
24	R**6e	R**27e	R"6g	R, 27g

	23D	23N	24D	24N
1	R1t	M21	R1v	M22
2	R47t	I33t	R47v	I33v
3	R3t	R30t	R3v	R30v
4	R4t	C38w-t	R4v	R5v
5	R'1t	R31t	R'1v	R31v
6	R6t	R7t	R6v	R7v
7	R48t	I34t	R48v	I34v
8	R'-a'39t	I35t	R'-a'39v	I35v
9	R8t	R9t	R8v	R9v
10	R15t	R32t	R15v	R32v
11	R'44t	R10t	R'44v	R10v
12	R11t	R12t	R11v	R12v
13	R29t	C37w-t	R29v	C37w-v
14	R28t	R14t	R28v	R14v
15	R'-b'39t	I36t	R'-b'39v	I36v
16	R13t	R16t	R13v	R16v
17	R17t	R18t	R17v	R18v
18	R19t	R20t	R19v	R20v
19	R21t	R22t	R21v	R22v
20	R23t	R24t	R23v	R24v
21	R,1o	R,25o	R"1f	R"25f
22	R*-a'39t	M24	R*-a'39v	R"51f
23	R,8o	R,26o	R"8f	R"26f
24	R,6o	R,27o	R"6f	R"27f

References

1. Dickson, Göttingen Nachrichten, 1905, p. 358

2. Dickson, Trans. Amer. Math. Soc., 6(1905), p. 198 - 204

3. Huntington, Trans. Amer. Math. Soc., 6(1905), p.181 - 197

4. Dickson, Algebren und Ihre Zahlentheorie, p. 94

5 Dickson, Linear Algebra, p. 23

The associative algebras are taken from

Pierce, Amer. Jour. of Math, 4(1881), p. 97 - 215

Encyclopädie der Mathematischen Wissenschaften, I. 1, p. 166