OPTIMAL GUIDANCE OF LOW-THRUST INTERPLANETARY SPACE VEHICLES

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology
Pasadena, California
1969

(Submitted October 14, 1968)

ACKNOWLEDGEMENT

The author wishes to express sincere appreciation to Dr. R. Sridhar for his continued help and challenging guidance throughout the course of this investigation. He is also indebted to Mr. Boris Dobrotin for many useful discussions during the initial phase of this work, and for his constant interest and encouragement; to Professor M. Aoki for suggesting pathways into the literature concerning the later phases of this study; to the National Aeronautics and Space Administration for fellowship support; and to Mrs. Ruth Stratton for an outstanding job of typing.

This work is dedicated to Lyn and Stephen, who truly have contributed the essence of its motivation.

ABSTRACT

The low-thrust guidance problem is defined as the minimum terminal variance (MTV) control of a space vehicle subjected to random perturbations of its trajectory. To accomplish this control task, only bounded thrust level and thrust angle deviations are allowed, and these must be calculated based solely on the information gained from noisy, partial observations of the state. In order to establish the validity of various approximations, the problem is first investigated under the idealized conditions of perfect state information and negligible dynamic errors. To check each approximate model, an algorithm is developed to facilitate the computation of the open loop trajectories for the nonlinear bang-bang system. Using the results of this phase in conjunction with the Ornstein-Uhlenbeck process as a model for the random inputs to the system, the MTV guidance problem is reformulated as a stochastic, bang-bang, optimal control problem. Since a complete analytic solution seems to be unattainable, asymptotic solutions are developed by numerical methods. However, it is shown analytically that a Kalman filter in cascade with an appropriate nonlinear MTV controller is an optimal configuration. The resulting system is simulated using the Monte Carlo technique and is compared to other guidance schemes of current interest.

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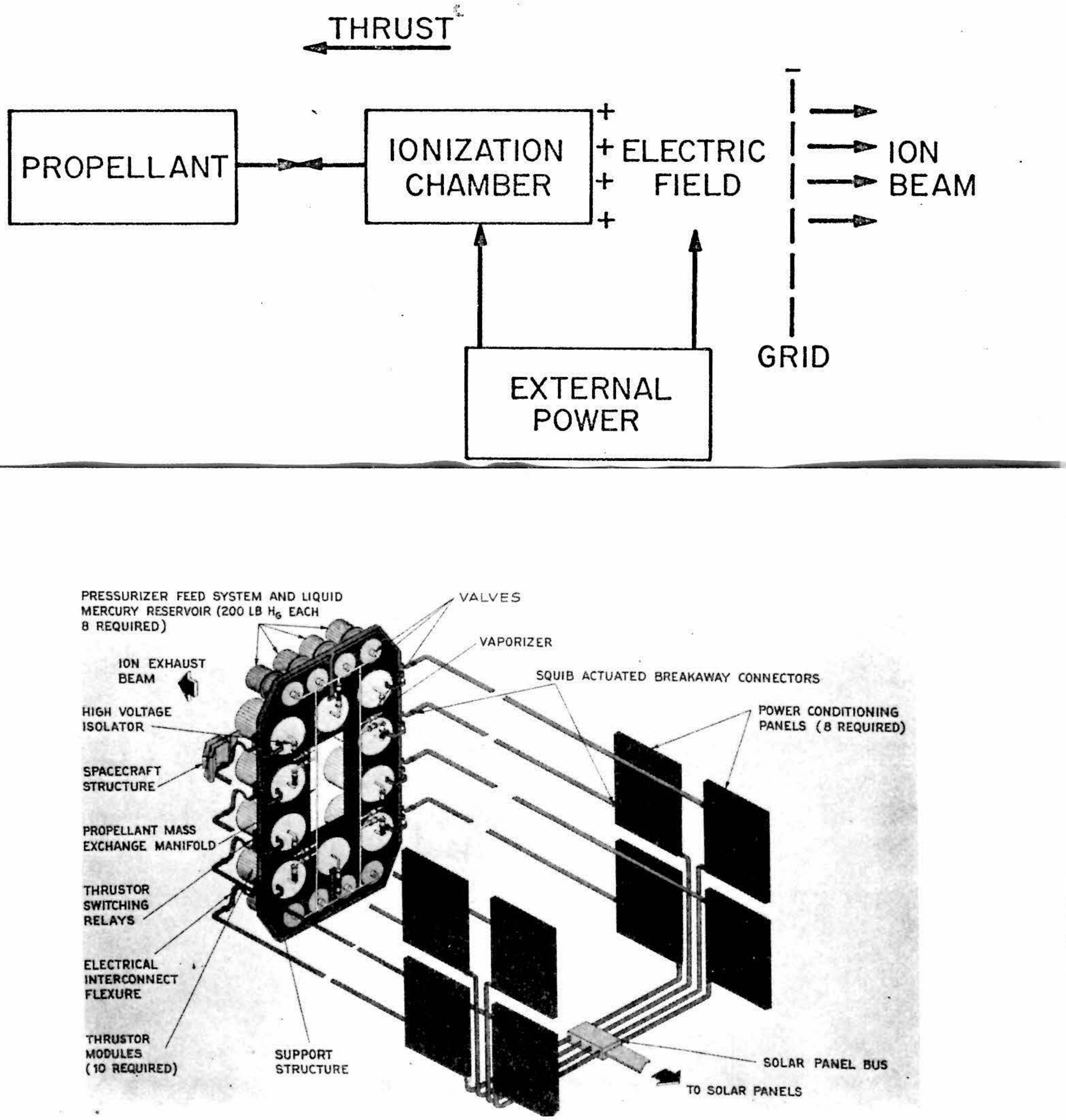
I. INTRODUCTION AND PROBLEM FORMULATION

1.1 General Discussion

In response to a relentless curiosity, man has accepted the challenge of space exploration. The space journeys already ventured seem almost fictional in character--yet the future holds endless possibilities. Many of the future challenges will become feasible conquests through the use of advanced propulsion systems. For this reason, the application of ion engines to space missions is currently under intense theoretical and experimental evaluation.

The low-thrust ion engine will probably find its most important application in missions to the outer planets where the retarding effect of the sun's gravity will require a large space vehicle energy. Up to the present, all the energy has been provided by the launch vehicle. For high energy missions, such as those to the outer planets, it seems desirable to use high impulse low-thrust engines to augment the energy supplied by the boost vehicle. These low-thrust devices would operate during the long flight times between launch and encounter, supplying a higher specific impulse than that available from present chemical boosters.

The principal components of an ion engine are illustrated in Figure 1. Basically, the generation of thrust involves two distinct phases. During the first phase the propellant is fed into the ionization chamber and converted into charged particles, called ions. Then, in the second phase, the particles are accelerated by the electric field and expelled through openings in the grid structure. The power



Courtesy of J.P.L.

Figure 1. A low-thrust ion engine

required to produce and accelerate the ions is supplied externally, and is derived perhaps from solar cells, or possibly from a nuclear reactor.

Ion engines have the ability to accelerate propellant to extremely high exhaust velocities. In contrast the exhaust velocities achieved by chemical rockets are limited by the inherent properties of the chemical reaction. The significance of this observation can best be illustrated by considering the equation relating the change in space vehicle velocity $v(t) - v(t_o)$ to the engine exhaust velocity c (in free space):

$$v(t) - v(t_o) = c \ \ln \frac{M(t_o)}{M(t)} \tag{1.1}$$

where $M(t)$ is the instantaneous space vehicle mass. Thus, for a fixed expenditure of fuel, the change in v is directly proportional to the exhaust velocity c, which explains the dramatic gain in the payload fraction* attained using ion-propulsion over that using chemical thrusting, (1),(38).

Low thrust level is the primary disadvantage of ion engines. This characteristic results from the power limitation of the thrustor systems. Assuming, however, that the propulsion is applied over the long flight time when a conventional vehicle would be coasting, then a considerable velocity increment can be obtained. Hence ion propulsion is well suited to long duration space voyages.

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* The ratio of useful payload weight to injected weight.

1.2 The Low-Thrust Guidance Problem

Prior to the initiation of an interplanetary flight, a nominal or desired trajectory is determined, and is specified in terms of the injection conditions and the nominal thrust program. Ideally, the nominal thrust program is optimized so that maximum payload is delivered to the vicinity of the target planet. The optimization is subject to constraints, and the final orbit selection will compromise many conflicting factors such as launch energy required, arrival date, telemetering and tracking considerations, etc. The design of nominal, or open loop trajectories has received considerable attention in the literature (2-10). For our purposes here, it is well to assume that this trajectory has been decided.

Because of launch energy dispersion and random effects in flight, the spacecraft will inevitably be perturbed from its standard path. Ideal behavior is further deteriorated when the state of the vehicle becomes uncertain. As nearly as possible, the guidance system should eradicate the effect of these disturbances and insure that the vehicle approaches the destination in the intended fashion. The guidance problem for a low-thrust vehicle has been approached in a variety of ways (11-24) and discussion of some relevant literature is the topic of the next section.

1.3 Discussion of the Literature on Low-Thrust Guidance

1.3.1. Second variation technique. One guidance technique that has attracted much attention recently is the method of neighboring optimal trajectories, otherwise known as the second variation technique (11-13). The application of this scheme yields a linear feedback equalizer whose control outputs guide the vehicle along alternative nominal trajectories. Within a certain neighborhood of the standard orbit, the method is akin to continuously recomputing the reference path. Kelley (14) and Breakwell (15) have applied the second variation technique to low-thrust guidance problems.

Several authors have investigated potential modifications of the second variation approach. Kelley (16) has suggested the following improvements:

(i) higher order terms in the control law approximation

(ii) the method of transverse state comparison in computing state deviations.

Evidently the second refinement provides a worthwhile increase in guidance accuracy. Kelley and Denham (17) have considered the simplification of making polynomial approximations to both the nominal trajectory and the feedback gain functions. Since the "best" polynomial coefficients will vary as the initial state changes, uniqueness is retained by optimizing over the statistics of the initial conditions.

1.3.2. λ-Matrix technique. Similar to second variation systems, the λ-matrix control scheme (18) yields a linear feedback control law. In the course of providing trajectory corrections, the λ-matrix controller minimizes the mean square deviations of the control variables from their nominal values. The method has been applied by Friedlander (19) who extends the method to include the statistical aspects of low-thrust guidance. In a lucid presentation, Friedlander reveals some aspects of this controller which will be of interest presently.

1.3.3. Minimum effort control. Another guidance scheme is the minimum effort control method (20), an approach which selects the optimal linear controller that minimizes the expected value of the average absolute control deviation. It is assumed that the variance of one terminal component is specified, and that control mechanization errors are negligible. When the method is used to control several terminal components (21), there is apparently some difficulty in computing the variable feedback gains. Minimum effort techniques have been applied by Breakwell, Tung, and Smith (22).

1.3.4. Other methods. Tung (23) has applied linear control theory to interplanetary guidance and compared his results with minimum effort control. Jordan (24) has investigated low-thrust interplanetary guidance using the stochastic calculus of variations, a technique formalized by Kushner (25-28). Using this method, he computes the stochastic open loop control function, and shows that this differs, in general, from the deterministic open loop function. This phenomenon is a result of the biasing effects of noise when it forces a nonlinear, dynamic system. Meditch (29) has considered the problem of nulling the state deviations while expending a minimum amount of absolute thrust control. The controller for this problem turns out to be bang-bang.

1.4 Critique of the Existing Methods

1.4.1. Second variation. When a space vehicle deviates from the nominal trajectory, any guidance scheme which corrects the path will produce a change in the nominal performance index. For the deterministic case, the first order change will be independent of the guidance law by virtue of the Euler equations. The objective of the second variation technique is to minimize the second order effect. It is an effective approach when the system is truly deterministic. Care must be exercised, though, when applying this method to a stochastic system. To be specific, Breakwell and Rauch (15) have realized that noisy system actuators cause random first order effects* on the performance index, and this presumably explains the exclusion of dynamic noise in their model. Indeed, the second variation controller would seem to offer little advantage in minimizing deterministic second order changes when there will be random first order changes. Because of process noise in the form of thrust level and attitude variations, a low-thrust vehicle would encounter perturbations resulting from dynamic noise.

A second aspect of the neighboring optimal guidance technique deserves consideration; namely, the fact that the control variables are the same in both the nominal trajectory and guidance sub-systems. To see the significance of this observation, consider a low-thrust vehicle on a minimum time, interplanetary trajectory. The properties of this trajectory dictate the use of maximum thrust throughout the entire orbit transfer maneuver. Hence the neighboring optimal guidance system effectively has no engine throttling capability, and the velocity errors which ensue are not immediately correctable. For this reason, deviations will continue to build up until the final portion of the mission. Then, in a summary effort to correct the accumulated errors, the engine cutoff

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* See Appendix D.

time will be used as a control input, and varied from its nominal value.

The disadvantage of this strategy is indicated by comparison with a ballistic space flight, and the familiar midcourse guidance maneuver. Generally speaking, the impulsive velocity corrections are applied as early in the mission as practical. Doing this gives the injection errors a minimum amount of time to propagate and also permits a larger influence to be exercised on the spacecraft trajectory. When applied to a low-thrust flight, the above philosophy suggests that both thrust level and thrust angle control be made available to a continuous time guidance system. This capability is necessitated not only by the injection errors, but also by the continuous in-flight anomalies associated with low-thrust missions. As explained above, this auxiliary thrust control will not be incorporated by the neighboring optimal guidance system and a logical rapport between guidance requirements and nominal trajectory design is not achieved using this method. The resolution of this discrepancy is one objective of this study.

1.4.2. λ-Matrix control. Friedlander (19) has pointed out that λ-matrix control will provide maximum terminal accuracy in spite of persistent disturbances and imperfectly known state. The truth of this statement is associated with the infinite terminal gain that a λ-matrix system will often require. This singularity produces a major control effort toward the end of a mission, and will almost always involve large control deviations. Such a characteristic is not peculiar to λ-matrix control, but crops up, in general, when the terminal state and terminal time are fixed, and the control force is left unconstrained.

This difficulty can be side-stepped by employing various artifices. For instance, Tung (23) proposes to specify the desired terminal variances and obtain the optimal linear regulator which satisfies these requirements. Proper choice of the terminal conditions will keep the gains finite, but there are still no restrictions on the control magnitude required at any particular time.

1.4.3. Minimum effort. The main drawbacks of this technique are as follows:

(i) the exclusion of mechanization errors

(ii) the restriction to linear controllers

(iii) the difficulty of controlling more than one terminal component

(iv) the absence of magnitude constraints on the control forces.

One further criticism could be made concerning the performance index; namely, the expected integral of the absolute control deviation does not seem to be simply related to fuel consumption, as claimed, because negative thrust deviations correspond to a fuel saving. Therefore, they should retain their negative value in a fuel consumption performance index.

1.4.4. Discussion. In general, many of the proposed schemes seem to tailor a performance index to fit the problem. Often there is no clear justification for what is defined to be the "performance." For example, the idea of associating a least squares performance index with a guidance system does not seem to have any specific physical or practical motivation, except that it yields a linear control law. Such a goal is of questionable merit, though, when the resulting implementation requirements are compared with those of a bang-bang type controller. To be sure, there are practical problems where a quadratic performance index is extremely meaningful; see, for instance, Reference (62).

Another consideration which has been somewhat disregarded in the conception of guidance schemes is the boundedness of the control variables. In practice the engine would certainly have a limited throttling capability, and this fact should be reflected in the problem formulation. If the control variables are bounded, then precise satisfaction of the terminal boundary conditions can no longer be guaranteed. In this situation the degree of terminal accuracy would become a meaningful criterion by which to judge the application of control efforts. Such a design viewpoint is consistent with the primary motivation for employing a space vehicle guidance system. Therefore attention will be focused on the synthesis of the minimum terminal variance (MTV) controller subject to control variable limitations.

1.5 Formulation of the Minimum Terminal Variance Control Problem

The formulation of the MTV control problem is made in three dimensions, where the coordinate frames of interest are illustrated in Figure 2. The $\underline{z}$ (meaning $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$) coordinate frame is assumed to have its origin fixed at the sun. An auxiliary coordinate frame $\hat{\underline{z}}$ is body fixed and used to define the thrust vector angles α and γ. Using Newton's law, the differential equations describing body motion in the $\underline{z}$ coordinate frame can be written as

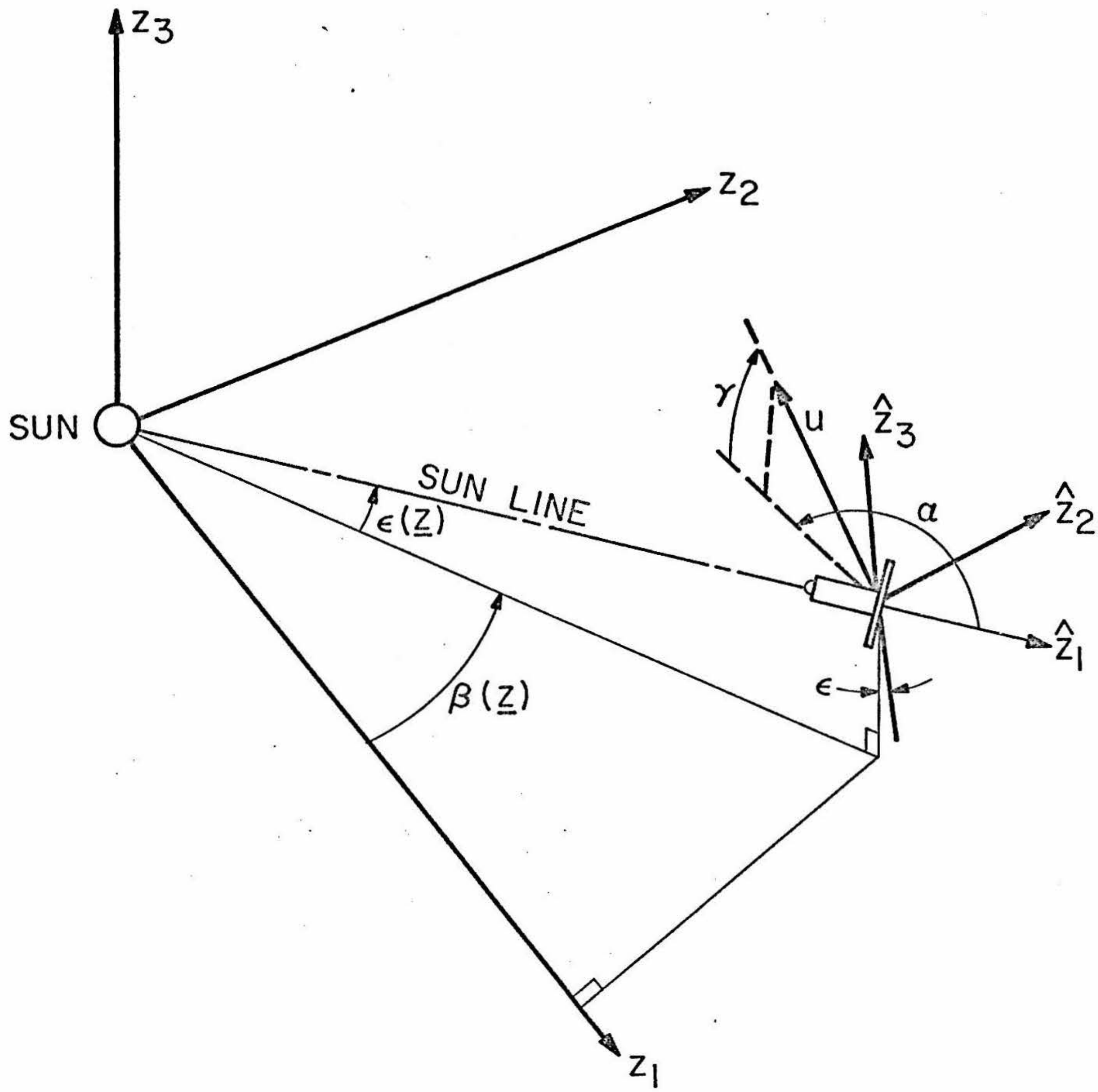

Figure 2. Coordinate frame definitions

$$\dot{\underline{z}} = \underline{v}$$

$$\dot{\underline{v}} = \underline{g}(\underline{z},t) + R_2^{-1}(\gamma(t)+\xi_3(t))\ R_1^{-1}(\alpha(t)+\xi_2(t))\ R_2^{-1}(\varepsilon(\underline{z}))$$

$$\times\ R_1^{-1}(\beta(\underline{z}))\ [u(t)+\xi_1(t)]\ \underline{1} + \underline{\phi}(t) \qquad (1.2)$$

where

(i) $\underline{v}$ is the three-dimensional velocity vector

(ii) $\underline{g}$ is the three-dimensional vector function representing gravitational accelerations and other deterministic accelerations (e.g., from solar pressure)

(iii) ξ_2 and ξ_3 are stochastic processes accounting for attitude control variations, and thrust vector pointing inaccuracies

(iv) ξ_1 is the stochastic process accounting for random thrust acceleration variations

(v) $\underline{\phi}$ is the stochastic process accounting for other random accelerations (e.g., from micrometeorites)

(vi) R_1 and R_2 are rotation matrices defined as follows:

$$R_1(\cdot) \triangleq \begin{bmatrix} \cos(\cdot) & \sin(\cdot) & 0 \\ -\sin(\cdot) & \cos(\cdot) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2(\cdot) \triangleq \begin{bmatrix} \cos(\cdot) & 0 & \sin(\cdot) \\ 0 & 1 & 0 \\ -\sin(\cdot) & 0 & \cos(\cdot) \end{bmatrix} \qquad (1.3)$$

(vii) $\underline{1} \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Note that the time variation of the thrust acceleration includes the known variations of both the vehicle mass and the power availability.

The state of the system $\begin{bmatrix} \underline{z} \\ \underline{v} \end{bmatrix}$ is observed via nonlinear, noisy measurements which could originate either from earth or from the spacecraft itself. Some candidates for these measurements might be

(i) earth-based doppler and range measurements

(ii) on-board optical sightings

(iii) on-board gyro and accelerometer measurements.

In general, it is assumed that m observations $\underline{y}$ are available and can be represented in the form

$$\underline{y} = \underline{h}(\underline{z},\underline{v},t) + \underline{n}(t) \tag{1.4}$$

where $\underline{h}$ is the m-dimensional observation vector function and $\underline{n}$ is the m-dimensional measurement error.

The MTV controller seeks to minimize the following performance functional

$$E\left[\sum_{i=1}^{3} k_i(v_i(t_e) - v_{in}(t_e))^2 + \ell_i(z_i(t_e) - z_{in}(t_e))^2\right] \tag{1.5}$$

where k_i and ℓ_i are weighting factors, E is the statistical expectation over the ξ_1, ξ_2, ξ_3, $\underline{\phi}$ and $\underline{n}$ stochastic processes, and t_e is the nominal time of planet encounter. If position deviations are the only concern, it is a simple matter to set the k_i equal to zero. The subscript n will always represent the nominal value of the indicated quantity. The control variables for this problem are taken to be the quantities α, γ and u , which are instantaneously constrained in accordance with their physical limitations.

In the course of this investigation, a bang-bang controller will often appear as the optimal solution. This type of controller requires only discrete levels of control action and does not seek a continuous range of values. Hence the bounded control problem is dealt with in a very natural way. Indeed, the optimality of bang-bang control systems for certain performance indices is one of the most interesting and valuable consequences of optimal control theory. The problem was originally studied by Bellman, Gamkrelidze, Krasovskii and LaSalle.

1.6 Summary of Principal Results

1.6.1. Results of Chapter II. The low-thrust guidance problem is defined as the minimum terminal variance control of a space vehicle subjected to random perturbations of its trajectory. In the general formulation, only noisy, nonlinear observations of the state vector are available. To accomplish this control task, only bounded thrust level and thrust angle deviations are allowed.

The first phase of the study is the consideration of the deterministic guidance problem (40),(41). It is assumed in this part that perfect knowledge of the state $\left(y = \begin{bmatrix} \underline{z} \\ \underline{v} \end{bmatrix}\right)$ is attainable, and also that no dynamic errors are incurred ($\xi_1 \equiv \xi_2 \equiv \xi_3 \equiv \underline{\phi} \equiv 0$). The analysis of the deterministic system is performed primarily to give insight into the accuracy of various approximations.

The deterministic problem admits a multitude of zero terminal error solutions. As a result, the requirement for a uniquely defined controller implies that additional performance indices are needed.

Using the minimum fuel and minimum time criteria for this purpose, a hierarchy of approximate control laws are derived, and substantial credence is given to their validity when they are checked against an exact open loop solution. An algorithm is developed to facilitate the computation of the open loop trajectories for the nonlinear, bang-bang system.

1.6.2. Results of Chapter III. The stochastic problem with perfect state information is the first topic considered in Chapter III. The Ornstein-Uhlenbeck (30) process is used to model the stochastic inputs to the system. Calling on the approximations which were successful in the deterministic part, it is possible to reformulate the minimum terminal variance (MTV) problem as a stochastic bang-bang problem. Asymptotic solutions are developed by numerical methods since a complete analytic solution seems to be unattainable. (However analytical methods are applied to the case when the stochastic disturbances are white noise, and partial results are obtained..) The MTV controller is then analyzed with regard to the following items:

(i) a comparison to the stochastic minimum time controller, which is obtained by using the approximation-in-policy-space algorithm; a comparison to the stochastic bang-bang controller obtained by Wonham's (55) approach; and a comparison to the analytic solution of the steady state Fokker-Planck equation

(ii) a discussion of the fuel consumption and implementation requirements

(iii) a comparison between the MTV controller and linear control designs reported in the literature--specifically, λ-matrix and second variation systems, including a Monte Carlo simulation of the MTV and λ-matrix systems.

The comparison in item (iii) indicates both the feasibility of the MTV controller and the validity of the approximations used in its derivation.

The stochastic problem with state estimation is the topic of the next section. It is shown that if the noisy, nonlinear observations can be approximated by noisy, linear observations, then the MTV guidance system consists of a Kalman (31) estimator in cascade with a bang-bang controller. In general, this bang-bang controller can be obtained by the same method used in deriving the known state solution. It is pointed out, however, that the "separation" property of linear, stochastic systems is not applicable to this controller since the switching law will depend on the type and accuracy of the observations. The separation property refers to the interesting consequence that if an optimal stochastic system is completely linear, then it consists of a Kalman estimator driving the <u>deterministic</u> optimal controller.

The final consideration of this chapter is the conceptual treatment of the full nonlinear problem. Several difficulties in obtaining the exact solution are disclosed, but it is made clear that the bang-bang property is still retained.

<u>1.6.3. Conclusion</u>. The study is concluded with summary remarks and an examination of the potential extensions of the methods into other areas of application.

II. THE DETERMINISTIC PROBLEM

2.1 Introduction

This chapter is concerned with the deterministic phase of the low-thrust guidance problem. Study of this aspect is motivated largely by questions concerning the validity of certain approximations and assumptions, and is earmarked for obtaining a solution which provides a reasonable compromise between mathematical tractability and solution accuracy. Later, in connection with the stochastic problem, the results of this chapter will be of fundamental importance.

The deterministic formulation is constructed upon certain idealizations. Namely, it is assumed that perfect knowledge of the state is attainable, and also that dynamic errors are negligible. Therefore, Equation 1.2 is modified by setting $\xi_1 \equiv \xi_2 \equiv \xi_3 \equiv \underline{\phi} \equiv 0$:

$$\dot{\underline{z}} = \underline{v}$$

$$\dot{\underline{v}} = \underline{g}(\underline{z},t) + R_2^{-1}(\gamma(t))\; R_1^{-1}(\alpha(t))\; R_2^{-1}(\varepsilon(\underline{z}))\; R_1^{-1}(\beta(\underline{z}))\; u(t)\; \underline{1} \qquad (2.1)$$

When these assumptions are valid, guidance is necessary only to correct injection errors which, in spite of control variable limitations, can be totally nullified by employing any one of a large number of different controllers. A unique controller is then obtained only by defining a performance index which is auxiliary to the MTV criterion. To meet this need, the minimum fuel and minimum time criteria are chosen as meaningful, although somewhat arbitrary, performance indices.

2.2 Linearization and Transformation of Plant Equations: The First Approximation

2.2.1. Mathematical development. If the actual spacecraft trajectory remains sufficiently close to the nominal path, then linear perturbation of the plant equations provides a good approximation to the dynamical behavior of the state deviations. However, this assumption does not imply that the uncontrolled state errors are tolerable from the standpoint of guidance requirements. Linear perturbation of the dynamic equations is equivalent to making a first order Taylor series expansion on both sides, and cancelling the zeroth order terms. The expansions are centered on the nominal trajectory and thus yield a system of time varying linear differential equations:

$$\underline{\dot{\delta z}} = \underline{\delta v}$$

$$\underline{\dot{\delta v}} = \{g_{\underline{z}_n} + [T^{-1} u_n \underline{1}]_{\underline{z}_n}\} \underline{\delta z} + R^{-1}_{2\gamma_n}(\gamma_n)\, R_1^{-1}(\alpha_n)\, R_2^{-1}(\varepsilon(\underline{z}_n))$$

$$R_1^{-1}(\beta(\underline{z}_n)) u_n \underline{1}\, \delta\gamma + R_2^{-1}(\gamma_n) R^{-1}_{1\alpha_n}(\alpha_n)\, R_2^{-1}(\varepsilon(\underline{z}_n))\, R_1^{-1}(\beta(\underline{z}_n))\, u_n \underline{1}\, \delta\alpha$$

$$+ T^{-1}\, \delta u\, \underline{1} \qquad (2.2)$$

where

(i) δ indicates a small deviation from the nominal value

(ii) $T = R_1(\beta(\underline{z}_n))\, R_2(\varepsilon(\underline{z}_n))\, R_1(\alpha_n)\, R_2(\gamma_n)$

(iii) the subscript $\underline{z}_n$ indicates the Jacobian matrix of the subscripted vector: e.g., $\left[g_{\underline{z}_n}\right]_{ij} = \frac{\partial g_i}{\partial z_j}$; $i,j \in \{1,2,3\}$

(iv)

$$R_{1\alpha_n}(\alpha_n) = \begin{bmatrix} -\sin\alpha_n & \cos\alpha_n & 0 \\ -\cos\alpha_n & -\sin\alpha_n & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{2\gamma_n}(\gamma_n) = \begin{bmatrix} -\sin\gamma_n & 0 & \cos\gamma_n \\ 0 & 0 & 0 \\ -\cos\gamma_n & 0 & -\sin\gamma_n \end{bmatrix}$$

At this point the following transformation is introduced:

$$\underline{\hat{X}} = T\,\underline{\delta z} \qquad (2.3)$$

Differentiating these equations twice with respect to time yields

$$\begin{bmatrix} \dot{\underline{\hat{X}}} \\ \ddot{\underline{\hat{X}}} \end{bmatrix} = \left[\begin{array}{c|c} \dot{T} & 0 \\ \hline \ddot{T} & 0 \end{array}\right] \begin{bmatrix} \underline{\delta z} \\ \underline{\delta v} \end{bmatrix} + \left[\begin{array}{c|c} T & 0 \\ \hline 2\dot{T} & T \end{array}\right] \begin{bmatrix} \underline{\delta \dot{z}} \\ \underline{\delta \dot{v}} \end{bmatrix} \qquad (2.4)$$

where

(i) 0 is the null matrix of order three

(ii) $\dot{T}_{ij} \triangleq \dfrac{dT_{ij}}{dt}$; $\ddot{T}_{ij} \triangleq \dfrac{d^2T_{ij}}{dt^2}$

Defining the six dimensional state vector

$$\underline{X} = \begin{bmatrix} \underline{\hat{X}} \\ \dot{\underline{\hat{X}}} \end{bmatrix}$$

and making use of Equation 2.2, Equation 2.4 then becomes

$$\underline{\dot{X}} = \left\{ \begin{bmatrix} \dot{T} & 0 \\ \ddot{T} & 0 \end{bmatrix} + \begin{bmatrix} T & 0 \\ 2\dot{T} & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ D & 0 \end{bmatrix} \right\} \begin{bmatrix} \underline{\delta z} \\ \underline{\delta v} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad (2.5)$$

where

(i) $D \equiv g_{\underline{z}_n} + [T^{-1} u_n \underline{1}]_{\underline{z}_n}$

(ii) I is the identity matrix of order three

(iii) control variables: $u_1 \triangleq \delta u$; $u_2 \triangleq u_n \delta\alpha$; $u_3 = u_n \delta\gamma$

Noting that

$$\underline{X} = \begin{bmatrix} T & 0 \\ \dot{T} & T \end{bmatrix} \begin{bmatrix} \underline{\delta z} \\ \underline{\delta v} \end{bmatrix}$$

which implies

$$\begin{bmatrix} \underline{\delta z} \\ \underline{\delta v} \end{bmatrix} = \begin{bmatrix} T^{-1} & 0 \\ -T^{-1}\dot{T}\,T^{-1} & T^{-1} \end{bmatrix} \underline{X} \qquad (2.6)$$

then Equation 2.5 becomes

$$\underline{\dot{X}} = \begin{bmatrix} 0 & I \\ \ddot{T}\,T^{-1} + D - 2\dot{T}\,T^{-1}\dot{T}\,T^{-1} & 2\dot{T}\,T^{-1} \end{bmatrix} \underline{X} + \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad (2.7)$$

Equation 2.7 displays the state dynamics in a form convenient for studying particular approximations. In connection with this study, a little foresight into the form of the control solution will be helpful. Since the plant is linear, a minimum time or, in this case, a minimum fuel criterion will yield a bang-bang control law. Unfortunately it is impossible to determine analytic, closed-loop, bang-bang controllers for systems of fourth order or higher (32). No hope can be held, therefore, for obtaining the closed form solution of the sixth order bang-bang problem under consideration here. A solution will only be possible if somehow the dimension of the state of the system can be reduced.

Fortunately a form of state reduction is possible if the following approximations are adopted: (i) The first approximation concerns the elements of the matrix D which are proportional to changes in the gravity vector $\underline{g}$ and the angles β and ε over a region in space near the nominal trajectory. Evaluating these elements for the case of heliocentric ($\underline{g}$ = sun's gravity only), planar ($z_3 = 0$) flight yields:

$$D_{11} = -\frac{GM_s + u_n z_{1n}}{(z_{1n}^2 + z_{2n}^2)^{3/2}} + \frac{3GM_s\, z_{1n}^2}{(z_{1n}^2 + z_{2n}^2)^{5/2}}$$

$$D_{12} = -\frac{u_n\, z_{1n}^2}{(z_{1n}^2 + z_{2n}^2)^{3/2}} + \frac{3GM_s\, z_{1n}\, z_{2n}}{(z_{1n}^2 + z_{2n}^2)^{5/2}}$$

$$D_{21} = \frac{3GM_s\, z_{2n}\, z_{1n}}{(z_{1n}^2 + z_{2n}^2)^{5/2}} + \frac{u_n(2z_{1n}^2 + z_{2n}^2)}{(z_{1n}^2 + z_{2n}^2)^{3/2}}$$

$$D_{22} = \frac{GM_s(2z_{2n}^2 - z_{1n}^2)}{(z_{1n}^2 + z_{2n}^2)^{5/2}} - \frac{u_n z_{2n}\, z_{1n}}{(z_{1n}^2 + z_{2n}^2)^{3/2}}$$

$$D_{13} = D_{23} = D_{31} = D_{32} = D_{33} = 0$$

where G is the constant of gravitation and M_s is the mass of the sun. For typical injection conditions, these quantities are of the order of 10^{-12} seconds^{-2}, and will therefore be neglected. This approximation becomes less accurate as the sun-vehicle distance decreases. (ii) Secondly, it is observed that the matrices $\dot{T}$ and $\ddot{T}$ are related to the rotation rates of both the vehicle about the sun and also the thrust vector about the sun-vehicle line. Generally these rates are very slow. Typical values for the elements of $\ddot{T}T^{-1}$ and $2\dot{T}T^{-1}\dot{T}T^{-1}$ are of the order of 10^{-14}seconds^{-2} and can therefore be neglected with small error. The validity of this approximation is compromised when the thrust vector goes through rapidly turning situations. (iii) Finally it will be assumed that the quantities $2\dot{T}T^{-1}\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$ are negligible with respect to the control variables $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$. Representative values for the components of these vectors would be 10^{-7} and 10^{-5} meters/second^2, respectively. This last assumption is expected to yield the largest over-all error.

Employing these approximations greatly simplifies the original problem and results in the following system of equations:

$$\dot{x}_1 = x_4 \qquad \dot{x}_2 = x_5 \qquad \dot{x}_3 = x_6$$

$$\dot{x}_4 = u_1 \qquad \dot{x}_5 = u_2 \qquad \dot{x}_6 = u_3 \tag{2.8}$$

These equations represent three, decoupled, purely inertial systems which can be discussed independently. Optimal control of an inertial plant has been studied for a number of performance indices (34),(35). Minimum fuel is the first criterion of interest here.

<u>2.2.2. Minimum fuel criterion.</u> Consider the integral of the nominal thrust + control thrust, $u_n + u_1$, over the time interval [0,t]:

$$\int_0^t (u_n + u_1)dt = \int_0^t u\, dt = \int_0^t \frac{\dot{M}c}{M}\, dt = c\, \ell n \frac{M(0)}{M(t)} \qquad (2.9)$$

where M is the space vehicle mass, and c is the constant exhaust velocity. Fuel consumption is now seen to be directly related to the integral of u . This fact suggests the following formulation for the minimum fuel guidance problem (note that only one inertial plant need be considered):

PLANT:

$$\dot{x}_1 = x_4$$

$$\dot{x}_4 = u_1$$

INITIAL CONDITIONS:

$$x_1(0) = x_{10}$$

$$x_4(0) = x_{40}$$

PERFORMANCE INDEX:

$$\int_0^{t_f} (u_n + u_1)\, dt; \quad x_1(t_f) = x_4(t_f) = 0;\ t_f \text{ free} \qquad (2.10)$$

where the assumption has been made that t_f will always occur before the planet encounter time t_e .

In order to illustrate some tools of optimal control theory, the problem will be worked out in some detail. First, the Hamiltonian is specified and is, by definition

$$H = u_n + u_1 + \lambda_1 x_4 + \lambda_2 u_1 \tag{2.11}$$

where λ_1 and λ_2 are Lagrange multipliers. A necessary condition for optimality is that the extremal control minimizes the Hamiltonian at each instant of time. This is the celebrated maximum principle of Pontryagin (35) and implies that the optimal control satisfies

$$u_1^* = -\, k \,\mathrm{sgn}(1 + \lambda_2) \tag{2.12}$$

where k is the maximum attainable value of the control variable u_1. This equation displays the bang-bang property of the optimal controller. Substitution of Equation 2.12 into Equation 2.11 yields the extremal Hamiltonian

$$H^* = u_n + \lambda_1 x_4 - \mathrm{sgn}(1 + \lambda_2)\, k \,\mathrm{sgn}(1 + \lambda_2) \tag{2.13}$$

Applying further necessary conditions yields the following canonic equations:

$$\begin{aligned}
\dot{x}_1 &= \frac{\partial H^*}{\partial \lambda_1} = x_4 \\
\dot{x}_4 &= \frac{\partial H^*}{\partial \lambda_2} = -k \,\mathrm{sgn}(1 + \lambda_2) \\
\dot{\lambda}_1 &= -\frac{\partial H^*}{\partial x_1} = 0 \\
\dot{\lambda}_2 &= -\frac{\partial H^*}{\partial x_4} = -\lambda_1
\end{aligned} \tag{2.14}$$

with boundary conditions

$$x_1(0) = x_{10};\; x_4(0) = x_{40};\;\; x_1(t_f) = x_4(t_f) = 0;\;\; H^*(t_f) = 0 \tag{2.15}$$

The last condition $H^*(t_f) = 0$ is termed the transversality condition. Solution of Equation 2.14 with conditions 2.15 yields the optimal control function $u_1^*(t)$. In general it is very difficult to solve two point boundary value problems of this type.

However, in the case of a low dimensional bang-bang problem, there is an alternative approach which considerably simplifies the analysis. From Equation 2.14, the Lagrange multipliers have the following solutions:

$$\lambda_1(t) = \lambda_1(0) \qquad \lambda_2(t) = -\lambda_1(0)t + \lambda_2(0) \tag{2.16}$$

Equations 2.16 imply that a maximum of one control switching is possible on any given trajectory. Solving the x_1 and x_4 equations for constant u_1 yields

$$x_1(t) = \frac{1}{2} kt^2 + x_{40}t + x_{10}$$

$$x_4(t) = kt + x_{40}$$

Eliminating t implies

$$2k(x_1 - x_{10}) = (x_4 - x_{40})^2 + 2x_{40}(x_4 - x_{40}) \tag{2.17}$$

Equation 2.17 shows that the vehicle will follow a parabolic trajectory in the (x_1, x_4) plane for constant u_1 . Coupling this fact with the fact that only one switching is optimal, the "switching boundary" is obtained and is illustrated in Figure 3.

It should be noted that the deviation of the fuel consumption is constant for any controller which nulls the state errors. This

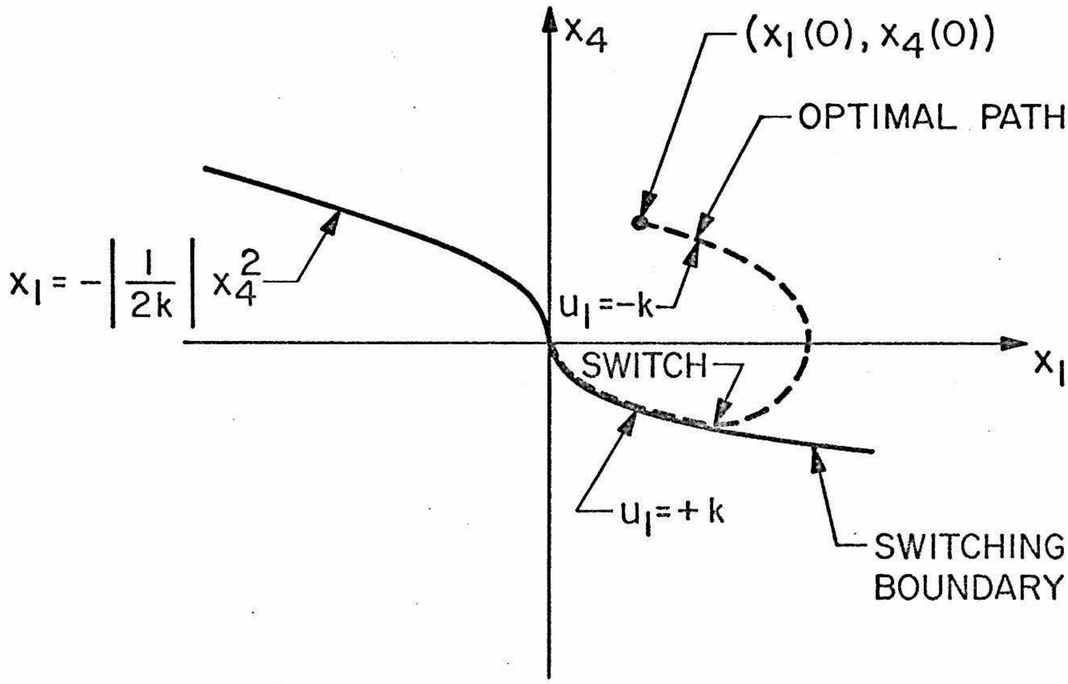

Figure 3. Definition of the "switching boundary"

statement is valid to the accuracy of the approximations used above. To see this, consider the integral of the $\dot{x}_4$ equation given in Equation 2.10

$$\int_0^{t_f} \dot{x}_4 \, dt = x_4(0) = \int_0^{t_f} u_1 dt$$

since $x_4(t_f) = 0$. Hence fuel deviation is not a meaningful performance index for this problem.

2.2.3. Minimum time criterion. Selection of a minimum time criterion reflects the desire to null the injection errors as quickly as possible, and therefore gives these errors a minimum amount of time to propagate. It will be shown in this section that this criterion yields exactly the same controller as the minimum fuel criterion. The performance index for the minimum time problem is

$$\int_0^{t_f} dt \tag{2.18}$$

which results in the following Hamiltonian:

$$H = 1 + \lambda_1 x_4 + \lambda_2 u_1 \tag{2.19}$$

Minimizing the Hamiltonian with respect to u_1 yields

$$u_1^* = -k \text{ sgn } \lambda_2$$

$$H^* = 1 + \lambda_1 x_4 - k\lambda_2 \text{ sgn } \lambda_2 \tag{2.20}$$

The canonic equations become

$$\dot{x}_1 = x_4$$

$$\dot{x}_4 = -k \operatorname{sgn} \lambda_2$$

$$\dot{\lambda}_1 = 0$$

$$\dot{\lambda}_2 = -\lambda_1 \tag{2.21}$$

Solving the Lagrange multiplier equations gives

$$\lambda_1(t) = \lambda_1(0) \qquad \lambda_2(t) = -\lambda_1(0)t + \lambda_2(0)$$

Once again only one control switching is optimal, and once again the same controller is obtained.

2.3 Control Variables: The Problems of Level Variation and Implementation

2.3.1. The problem of control level variations. In the foregoing discussion it has been implicitly assumed that the maximum control magnitude is constant throughout the mission. Realistically, such is not the case since these values would actually vary with the nominal acceleration level. A more reasonable assumption is that guidance maneuvers are completed rapidly with respect to the slowly varying control level magnitudes.

The usefulness of this assumption can be revealed by examining the Bellman-Hamilton-Jacobi equation for the optimization problem. For the minimum time criterion this equation can be written as follows:

$$\frac{\partial V}{\partial t} + \min_{u_1} \left[1 + x_4 \frac{\partial V}{\partial x_1} + u_1 \frac{\partial V}{\partial x_4}\right] = 0 \tag{2.22}$$

where V is the value function defined by

$$V(t,\underline{C}) \triangleq \min_{\substack{u_1(\tau) \\ t \leq \tau \leq t_f}} \int_t^{t_f} d\tau \qquad \text{with } \underline{X}(t) = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \underline{C}\,;\ \underline{X}(t_f) = \underline{0}$$

Performing the indicated minimization in Equation 2.22 yields

$$\frac{\partial V}{\partial t} + 1 + x_4 \frac{\partial V}{\partial x_1} - k(t) \left| \frac{\partial V}{\partial x_4} \right| = 0 \tag{2.23}$$

where $k(t)$ is the maximum value of the control u_1 at time t. In accordance with the discussion above, $k(t)$ is assumed to be slowly varying and is therefore written in the form

$$k(t) = k(t_o) + \varepsilon(t - t_o) \tag{2.24}$$

where t_o is an appropriately chosen initial time, and ε is the (small) slope of the control magnitude at time t_o. Using Equation 2.24 in Equation 2.23 yields

$$\frac{\partial V}{\partial t} + 1 + x_4 \frac{\partial V}{\partial x_1} - (k(t_o) + \varepsilon(t - t_o)) \left| \frac{\partial V}{\partial x_4} \right| = 0 \tag{2.25}$$

A solution of Equation 2.25 is sought in the form of the general perturbation expansion

$$V(t,\underline{C},\varepsilon) = \sum_{n=0}^{\infty} V_n \varepsilon^n = V_o + \varepsilon V_1 + o(\varepsilon) \tag{2.26}$$

where $\displaystyle \lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$.

Substitution of Equation 2.26 into Equation 2.25 yields

$$\frac{\partial V_o}{\partial t} + \varepsilon \frac{\partial V}{\partial t} + 1 + x_4 (\frac{\partial V_o}{\partial x_1} + \varepsilon \frac{\partial V_1}{\partial x_4}) - (k(t_o) + \varepsilon(t - t_o))$$

$$\times \left| \frac{\partial V_o}{\partial x_4} + \varepsilon \frac{\partial V_1}{\partial x_4} \right| = 0 \qquad (2.27)$$

Letting $\varepsilon \to 0$ in Equation 2.27 yields the Hamilton-Jacobi equation for the zeroth approximation:

$$\frac{\partial V_o}{\partial t} + 1 + x_4 \frac{\partial V_o}{\partial x_1} - k(t_o) \left| \frac{\partial V_o}{\partial x_4} \right| = 0 \qquad (2.28)$$

Equation 2.28 represents the Hamilton-Jacobi equation for a constant control level, and the solution is identical to that given in the previous section when t_o is identified as the current time.

If the second term of the expansion V_1 is desired, then the following expression must be noted

$$|A + \varepsilon B| = ((A + \varepsilon B)^2)^{1/2} = (A^2 + 2\varepsilon AB + \varepsilon^2 B)^{1/2}$$

$$= |A| \ (1 + 2\varepsilon B/A + o(\varepsilon))^{1/2} = |A| \ (1 + \frac{\varepsilon B}{A}) + o(\varepsilon) \qquad (2.29)$$

$$= |A| + \varepsilon B(\operatorname{sgn} A) + o(\varepsilon)$$

Using Equation 2.29 in Equation 2.27 and collecting terms of order ε yields the desired equation for V_1 . This function may be difficult to obtain, however, because of the behavior of the derivatives of V_o .

As was stated to begin with, the rate of guidance corrections is assumed greater than the magnitude of ε . Hence it is reasonable to approximate the true solution with the fundamental solution obtained from Equation 2.28, and this approximation will be adopted in this study.

2.3.2. Control variable implementation requirements. A few comments are warranted concerning the engineering aspects of controller implementation. The requirement of bang-bang thrust level control could be met by on-off throttle valve action. This control would be facilitated by employing the auxiliary thrustor modules required by reliability needs (36),(39). At constant specific impulse, variable thrust will sometimes require the use of reserve power. The constraint on guidance acceleration is then implicit in the engineering design, power availability, and reliability considerations.

Discrete thrust angle commands could be implemented in a variety of ways. Among these are the following:

(i) electronically deflecting the ion beam

(ii) engine rotation

(iii) vehicle rotation.

The first method is very attractive, since it effectively allows zero inertia switching. Success in so deflecting ion beams has been reported by Hughes Research Laboratories (37). The second method is achieved by conventional engine gimbal techniques, and the third is effected by sun sensor biasing. Since large variations of the engine angles are not desirable on a continuous basis, such control action will be limited to small deviations from the nominal angle program.

2.4 Experimental Results Using the First Approximation

One practical and meaningful way to check the validity of the results is by digital computer simulation. For this purpose, the following assumptions are made:

(i) the space vehicle is in heliocentric planar flight

(ii) $\alpha = 90^{\circ}$ during the initial phase of the mission

(iii) vehicle weight = 10,000 pounds (4535 kilograms)

(iv) constant initial acceleration = 10^{-3} meters/second^2

(v) maximum control accelerations taken at 10^{-4} meters/second^2

(vi) injection errors: velocity = 11.2 meters/second; position = 3350 kilometers (three days of velocity error propagation).

Referring to Figure 4 it can be seen that the accuracy of the controller is very good. The inaccuracies of ignoring the $2\dot{T}T^{-1}\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$ terms--when integrated over five days of initial error reduction--results in a small miss at the terminal time. This small error is easily eliminated by reapplication of the original control action. These results lend motivation for applying this approximate dynamic characterization to the MTV guidance system to be considered in Chapter III where, in a loose sense, the disregarded information can be considered to be included in the random forces which disturb the plant.

2.5 The Second Approximation

As could be expected, the accuracy achieved by the first approximation can be improved if digital computer capabilities are made available to the guidance system. In this section a set of four,

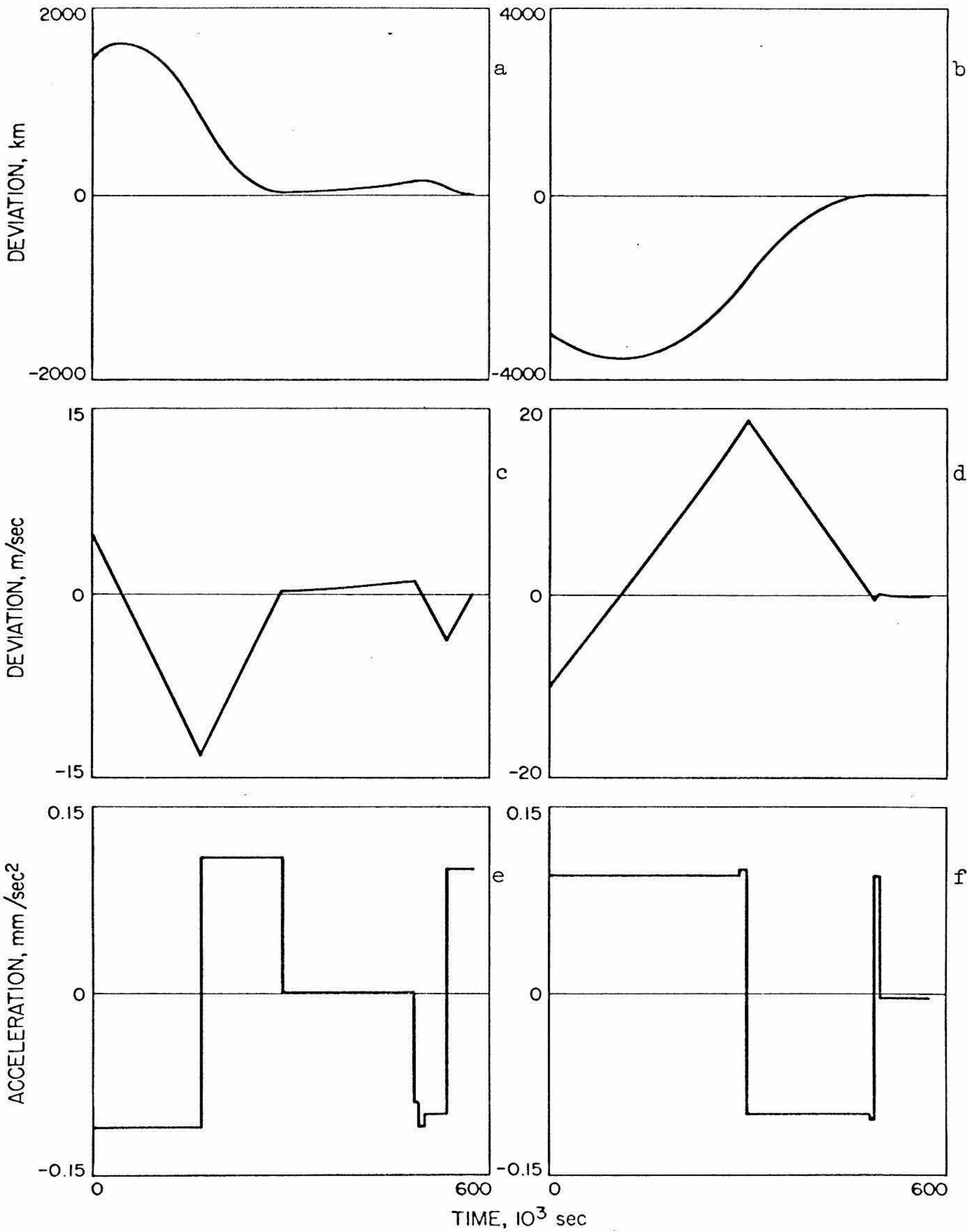

Figure 4. Results obtained using the first approximation: (a) x_1, (b) x_2, (c) x_4, (d) x_5, (e) u_2, (f) u_1 .

simultaneous, transcendental equations are derived, requiring on-line solution by the digital computer. For this derivation the following assumptions are made

(i) the vehicle is in heliocentric planar flight

(ii) the matrix D of Equation 2.5 is negligible

(iii) the maximum control level is either slowly varying or constant.

Hence, ignoring motion in the z_3 direction, Equation 2.2 becomes

$$\begin{bmatrix} \underline{\delta \dot{z}} \\ \underline{\delta \dot{v}} \end{bmatrix} = \left[\begin{array}{c|c} 0 & I \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} \underline{\delta z} \\ \underline{\delta v} \end{bmatrix} + \begin{bmatrix} 0 \\ \cos\mu \\ 0 \\ \sin\mu \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ -\sin\mu \\ 0 \\ \cos\mu \end{bmatrix} u_2 \tag{2.30}$$

where $\mu(t) = \alpha(t) + \beta(\underline{z}_n)$. Using the minimum time criterion, the multiplier equations are given by*

$$\underline{\dot{\lambda}} = - \left[\begin{array}{c|c} 0 & 0 \\ \hline I & 0 \end{array}\right] \underline{\lambda} \tag{2.31}$$

which have the solutions

$$\lambda_1(t) = \lambda_1(0) \qquad \lambda_2(t) = -\lambda_1(0)t + \lambda_2(0)$$

$$\lambda_3(t) = \lambda_3(0) \qquad \lambda_4(t) = -\lambda_3(0)t + \lambda_4(0) \tag{2.32}$$

The optimal controls are therefore given by

---

*A review of the linear minimum time problem is given in Appendix A.

$$u_1^*(t) = -k \ \mathrm{sgn}[(-\lambda_1(0)t + \lambda_2(0)) \cos \mu(t) + (-\lambda_3(0)t + \lambda_4(0)) \sin \mu(t)]$$

$$u_2^*(t) = -k \ \mathrm{sgn}[(-\lambda_1(0)t + \lambda_2(0))(-\sin \mu(t)) + (-\lambda_3(0)t + \lambda_4(0))\cos \mu(t)] \tag{2.33}$$

Some possible realizations of the arguments of the above sgn functions are illustrated in Figure 5. (Note that μ is not expected to exceed 90 degrees before nominal trajectory acquisition.) These realizations suggest that each control would have a maximum of two switchings.

Now, given the initial conditions on Equation 2.30, the explicit solution for $\begin{bmatrix} \delta z \\ \delta v \end{bmatrix}$ can be represented in the following form (where t_f is the nominal trajectory acquisition time):

$$\begin{bmatrix} \underline{\delta z}(t_f) \\ \underline{\delta v}(t_f) \end{bmatrix} = \Phi(t_f,0) \begin{bmatrix} \underline{\delta z}(0) \\ \underline{\delta v}(0) \end{bmatrix} + \int_0^{t_f} \Phi(t_f,t) \begin{bmatrix} 0 \\ \cos \mu(t) \\ 0 \\ \sin \mu(t) \end{bmatrix} dt$$

$$+ \int_0^{t_f} \Phi(t_f,t) \begin{bmatrix} 0 \\ -\sin \mu(t) \\ 0 \\ \cos \mu(t) \end{bmatrix} u_2(t)dt \tag{2.34}$$

Here $\Phi(t_2,t_1)$ is the fundamental matrix which satisfies the matrix differential equation

$$\dot{\Phi}(t_2,t_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Phi(t_2,t_1) \tag{2.35}$$

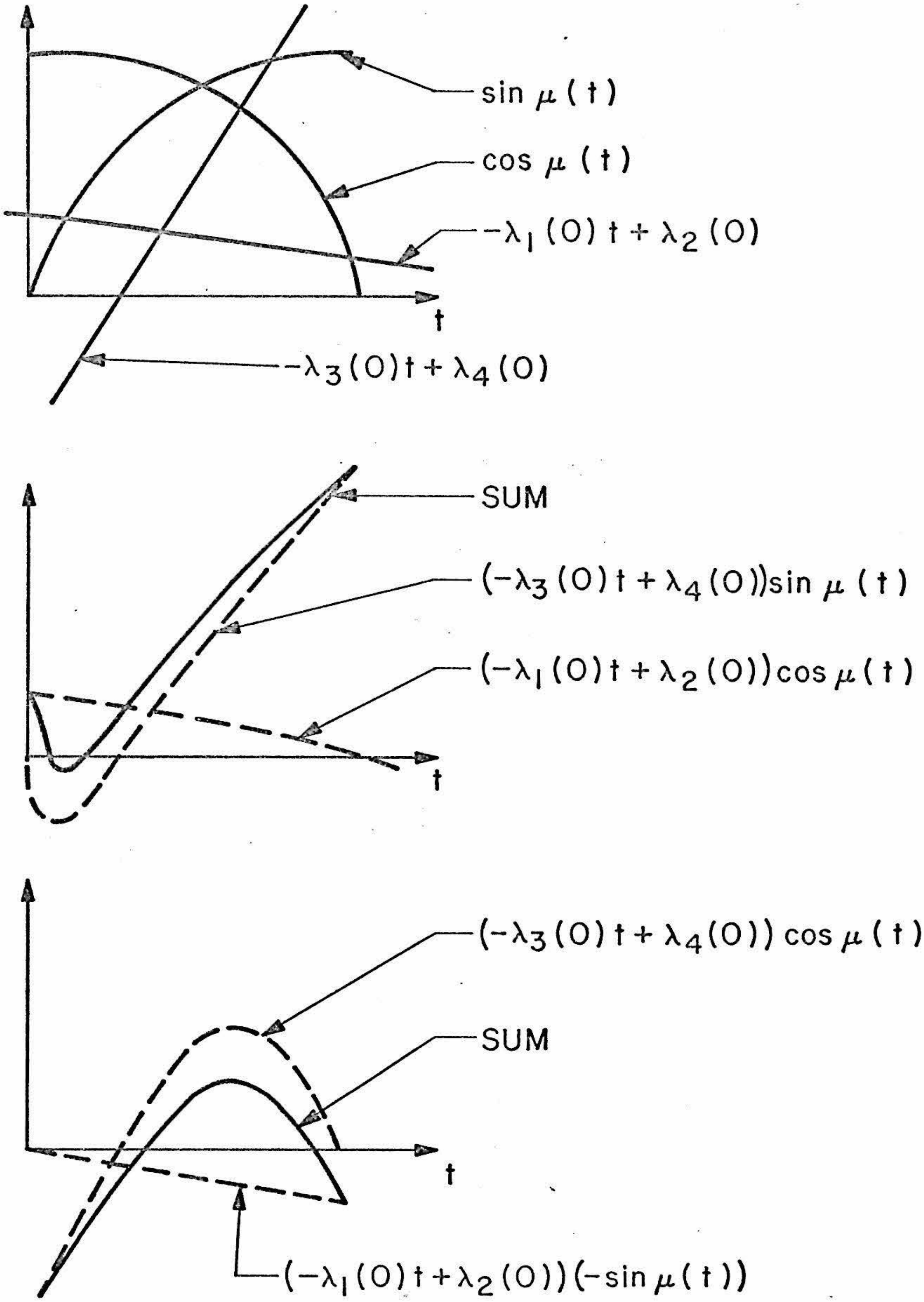

Figure 5. Arguments of the switching function

where $\Phi(t_1,t_1) = I$. The solution of Equation 2.35 is

$$\Phi(t_2,t_1) = \begin{bmatrix} 1 & (t_2-t_1) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (t_2-t_1) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.36}$$

Since the absolute values of u_1 and u_2 are constant, only the sign of these quantities is needed inside the integrals of Equation 2.34. If $u_1(0)$ and $u_2(0)$ are designated as the initial values of u_1 and u_2 , t_1 and t_2 as the switching times of u_1, and t_3 and t_4 as the switching times of u_2 , then Equation 2.34 becomes

$$\begin{bmatrix} \underline{\delta z}(t_f) \\ \underline{\delta v}(t_f) \end{bmatrix} = \Phi(t_f,0)\begin{bmatrix} \underline{\delta z}(0) \\ \underline{\delta v}(0) \end{bmatrix} + u_1(0)\left[\int_0^{t_1} - \int_{t_1}^{t_2} + \int_{t_2}^{t_f}\right] + u_2(0)\left[\int_0^{t_3} - \int_0^{t_4} + \int_{t_4}^{t_f}\right] \tag{2.37}$$

The integrals of Equation 2.34 can be explicitly evaluated if the assumption is made that μ varies at a constant rate. This is a good approximation for the trajectories of interest. Hence let

$$\mu(t) = \omega t \qquad \omega = \text{constant} \doteqdot \dot{\mu}$$

Now defining

$$I_1 = u_1(0)\left[\frac{2t_1}{\omega}\sin\omega t_1 + \frac{2}{\omega^2}\cos\omega t_1 - \frac{2t_2}{\omega}\sin\omega t_2 - \frac{2\cos\omega t_2}{\omega^2} + \frac{t_f \sin\omega t_f}{\omega}\right.$$

$$\left. + \frac{1}{\omega^2}\cos\omega t_f - \frac{1}{\omega^2}\right] - u_2(0)\left[-\frac{2t_3\cos\omega t_3}{\omega} + \frac{2\sin\omega t_3}{\omega^2} + \frac{2t_4\cos\omega t_4}{\omega}\right.$$

$$\left. - \frac{2\sin\omega t_4}{\omega^2} - \frac{t_f\cos\omega t_f}{\omega} + \frac{\sin\omega t_f}{\omega^2}\right]$$

$$I_2 = u_1(0)\left[\frac{2}{\omega}\sin\omega t_1 - \frac{2}{\omega}\sin\omega t_2 + \frac{1}{\omega}\sin\omega t_f\right] - u_2(0)\left[-\frac{2}{\omega}\cos\omega t_3\right.$$

$$\left. + \frac{2}{\omega}\cos\omega t_4 - \frac{1}{\omega}\cos\omega t_f + \frac{1}{\omega}\right]$$

$$I_3 = u_1(0)\left[\frac{-2t_1\cos\omega t_1}{\omega} + \frac{2\sin\omega t_1}{\omega^2} + \frac{2t_2\cos\omega t_2}{\omega} - \frac{2\sin\omega t_2}{\omega^2}\right.$$

$$\left. - \frac{t_f\cos\omega t_f}{\omega} + \frac{\sin\omega t_f}{\omega^2}\right] + u_2(0)\left[\frac{2t_3}{\omega}\sin\omega t_3\right.$$

$$\left. + \frac{2}{\omega^2}\sin\omega t_3 - \frac{2t_4}{\omega}\sin\omega t_4 - \frac{2\cos\omega t_4}{\omega^2} + \frac{t_f\sin\omega t_f}{\omega} + \frac{1}{\omega^2}\cos\omega t_f - \frac{1}{\omega^2}\right]$$

$$I_4 = u_1(0)\left[\frac{-2\cos\omega t_1}{\omega} + \frac{2}{\omega}\cos\omega t_2 - \frac{1}{\omega}\cos\omega t_f + \frac{1}{\omega}\right] + u_2(0)\left[\frac{2}{\omega}\sin\omega t_3\right.$$

$$\left. - \frac{2}{\omega}\sin\omega t_4 + \frac{1}{\omega}\sin\omega t_f\right]$$

then Equation 2.37 becomes

$$\delta z_1(t_f) = \delta z_1(0) + t_f\,\delta v_1(0) - I_1 + t_f I_2$$

$$\delta v_1(t_f) = \delta v_1(0) + I_2$$

$$\delta z_2(t_f) = \delta z_2(0) + t_f\, v_2(0) - I_3 + t_f I_4$$

$$\delta v_2(t_f) = \delta v_2(0) + I_4 \qquad (2.38)$$

Equations 2.38 are four equations in five unknowns. Since it is desired that

$$\begin{bmatrix} \underline{\delta z}(t_f) \\ \underline{\delta v}(t_f) \end{bmatrix} = \underline{0}$$

the problem is now to find the minimum value of t_f for which Equations 2.38 are satisfied. Fortunately these equations can be solved by the Newton-Raphson technique, and such analysis indicates the minimum value of t_f is achieved either when $t_f = t_2$ or $t_f = t_4$. Hence one control will have one switching and the other will have two switchings. It is fairly easy to determine the correct $u_1(0)$ and $u_2(0)$, and thereby Equations 2.8 can be solved for the minimum value of t_f and for the switching times of the control variables.

2.6 Experimental Results Using the Second Approximation

The computer simulation of the second approximate solution indicates that there is a need to account for second-order effects in the control variables. Hence, for the case when $t_f = t_4$, Equation 2.37 can be modified as follows:

$$\begin{bmatrix} \underline{\delta z}(t_f) \\ \underline{\delta v}(t_f) \end{bmatrix} = \Phi(t_f,0)\begin{bmatrix} \underline{\delta z}(0) \\ \underline{\delta v}(0) \end{bmatrix} + u_1(0)\left(\text{FAC 1}\int_0^{t_1} - \text{FAC 1}\int_{t_1}^{t_3}\right.$$

$$\left. - \text{FAC 2}\int_{t_3}^{t_2} + \text{FAC 2}\int_{t_2}^{t_f}\right) + u_2(0)\left(\text{FAC 3}\int_0^{t_3} - \text{FAC 4}\int_{t_3}^{t_f}\right)$$

where FAC 1, FAC 2, FAC 3, and FAC 4 are the factors that account for the second-order effects. Introducing these refinements adds little analytical difficulty, and the solution can again be found by using the Newton-Raphson technique.

The performance achieved by using this modified solution is illustrated in Figure 6. A comparison with Figure 4 indicates that the second solution (1) requires about 24 hours less time to acquire the nominal trajectory and (2) requires three fewer commands, or switchings, to be sent to the vehicle for the initial deviations considered. It is also found that smaller values (i.e., $.25 \times 10^{-4}$ meters/second^2) of u_1 and u_2 could be used with no loss of accuracy (see Reference (40)).

It is probable that the second solution represents a close approximation to the exact optimal solution. This conjecture can be partially verified by actual comparison with a computed open loop trajectory. In the next section the problem of finding such a trajectory is undertaken.

2.7 The Open Loop Problem--An Algorithm for Determining Minimum Fuel and Minimum Time Trajectories

Numerical solutions for optimal bang-bang control systems have long been a topic of interest to control engineers. Much of the literature (32), (42), (43) deals with time optimal control of time invariant linear plants. At best, finding the optimal trajectories is a very difficult task, and unfortunately the powerful quasilinearization method is inapplicable to bang-bang systems since the associated differential equations possess discontinuities.

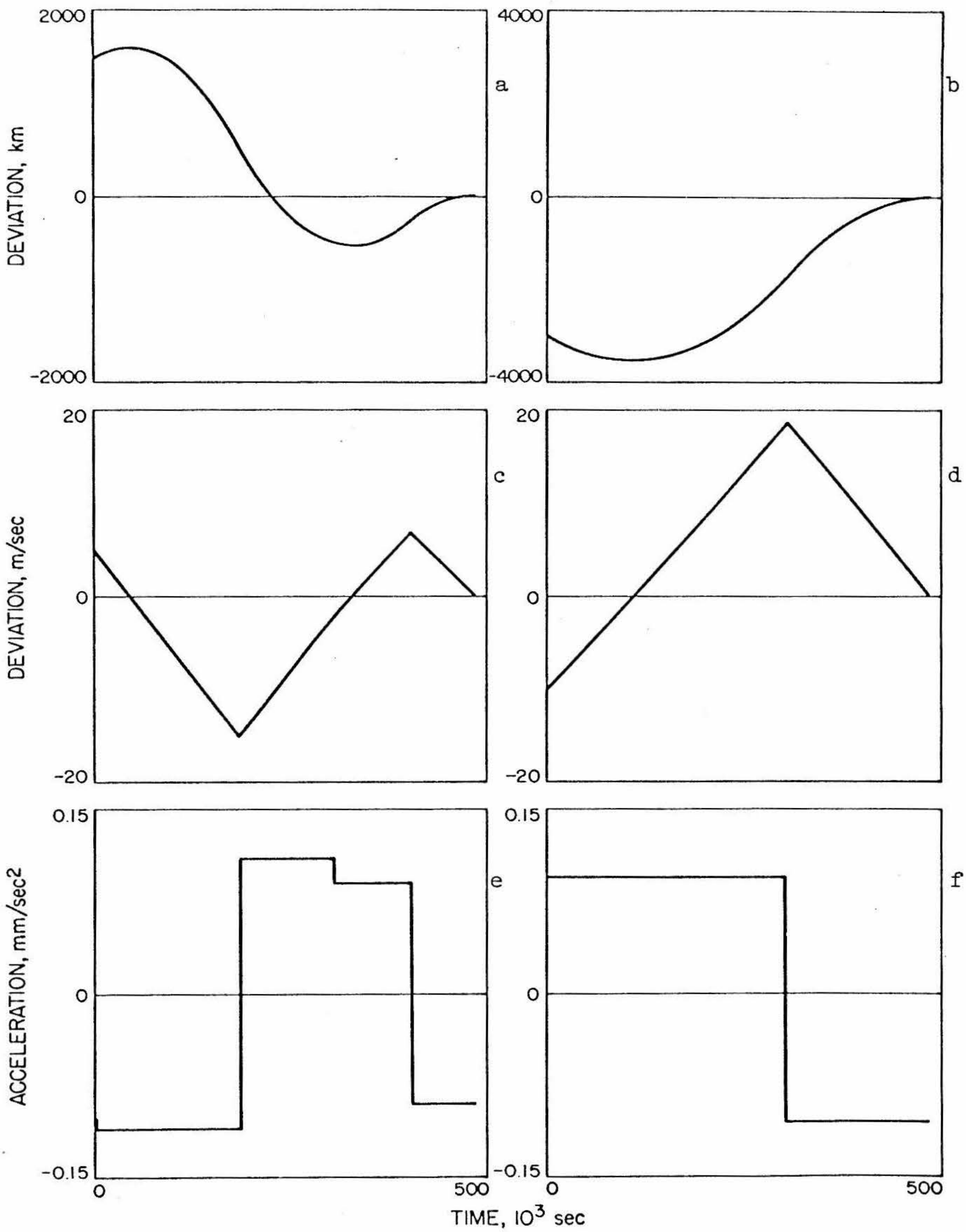

Figure 6. Results obtained using the second approximation: (a) x_1, (b) x_2, (c) x_4, (d) x_5, (e) u_2, (f) u_1 .

For the present guidance problem the difficulties become even greater. In addition to the nonlinear time varying differential equations, there is also the problem that the minimization of the Hamiltonian with respect to control cannot be obtained in closed form. These circumstances preclude the direct application of the methods cited above. For this reason the algorithm described in this section is proposed as a means of overcoming these difficulties.

In compact notation the differential equations for the deviations of the state vector from nominal values are defined as follows:

$$\begin{bmatrix} \dot{\underline{\delta z}} \\ \dot{\underline{\delta v}} \end{bmatrix} \triangleq \underline{G}(t, \underline{\delta z}, \underline{\delta v}, u_1, \delta\alpha) \tag{2.39}$$

For convenience the case of heliocentric planar flight has been assumed. The performance indices being considered are the following:

$$\text{(a)} \quad \int_0^{t_f} u \, dt \qquad \text{(b)} \quad \int_0^{t_f} dt \quad t_f \text{ free}, \begin{bmatrix} \underline{\delta z}(t_f) \\ \underline{\delta v}(t_f) \end{bmatrix} = \underline{0}$$

where the initial conditions $\begin{bmatrix} \underline{\delta z}(0) \\ \underline{\delta v}(0) \end{bmatrix} = \underline{C}$ are given.

The Hamiltonians for the two problems become

$$\text{(a)} \quad H = u + \langle \underline{\lambda}, \underline{G} \rangle \qquad \text{(b)} \quad H = 1 + \langle \underline{\lambda}, \underline{G} \rangle \tag{2.40}$$

The optimal control minimizes the Hamiltonian at each instant of time. In particular, it minimizes $(\mu_n \triangleq \alpha_n + \beta(\underline{z}_n))$:

(a) $M(u_1, \delta\alpha, z_1, z_2, \lambda_2, \lambda_4)$

$$= (u_n + u_1)\left[\frac{1 - \lambda_2(z_2\cos(\mu_n + \delta\alpha) + z_1\sin(\mu_n + \delta\alpha))}{(z_1^2 + z_2^2)^{1/2}}\right.$$

$$\left. + \frac{\lambda_4(z_1\cos(\mu_n + \delta\alpha) + z_2\sin(\mu_n + \delta\alpha)}{(z_1^2 + z_2^2)^{1/2}}\right] \quad (2.41a)$$

(b) $$M(u_1, \delta\alpha, z_1, z_2, \lambda_2, \lambda_4) = 1 - \frac{(u_n + u_1)\lambda_2(z_2\cos(\mu_n + \delta\alpha) + z_1\sin(\mu_n + \delta\alpha))}{(z_1^2 + z_2^2)^{1/2}}$$

$$+ \frac{(u_n + u_1)\,\lambda_4(z_1\cos(\mu_n + \delta\alpha) + z_2\sin(\mu_n + \delta\alpha))}{(z_1^2 + z_2^2)^{1/2}} \quad (2.41b)$$

Since the minimization functions cannot be written in compact form, the following symbolic functions are defined to meet this need:

$$\Phi_1(z_1, z_2, \lambda_2, \lambda_4) = u_1^* \; \varepsilon \; U = \{-k_1, 0, k_1\}: u_1^* \text{ minimizes } M$$

$$\Phi_2(z_1, z_2, \lambda_2, \lambda_4) = \delta\alpha^* \; \varepsilon \; \Lambda = \{-k_2, 0, k_2\}: \delta\alpha^* \text{ minimizes } M \quad (2.42)$$

Note that Φ_1 and Φ_2 are discontinuous functions whose partial derivatives are zero with respect to all arguments (except at discontinuities). Also Φ_1 and Φ_2 are not explicitly known functions, but can easily be calculated on the computer since only nine combinations of u_1 and $\delta\alpha$ need to be checked. Substituting Equation 2.42 into Equation 2.40 yields

$$\text{(a)} \qquad H^* = \Phi_1 + \langle \underline{\lambda}, \underline{G}(t, \underline{\delta z}, \underline{\delta v}, \Phi_1, \Phi_2) \rangle$$

$$\text{(b)} \qquad H^* = 1 + \langle \underline{\lambda}, \underline{G}(t, \underline{\delta z}, \underline{\delta v}, \Phi_1, \Phi_2) \rangle \qquad (2.43)$$

The canonic equations are

$$\begin{bmatrix} \underline{\delta \dot{z}} \\ \underline{\delta \dot{v}} \end{bmatrix} = H^*_{\underline{\lambda}}$$

$$\dot{\underline{\lambda}} = - H^*_{\begin{bmatrix} \underline{\delta z} \\ \underline{\delta v} \end{bmatrix}} \qquad (2.44)$$

and the transversality condition gives

$$H^*(t_f) = 0$$

The motivation for the algorithm which follows is the Newton-Raphson technique for solving nonlinear equations. The basic idea is to determine how the end conditions on $\begin{bmatrix} \underline{\delta z} \\ \underline{\delta v} \end{bmatrix}$ and H^* vary as functions of t_f and the initial conditions on $\underline{\lambda}$. Such behavior could normally be approximated by first linearizing the nonlinear equations, and then using linear differential equation techniques. This approach is applied in the quasilinearization method, but fails here owing to the discontinuity of Φ_1 and Φ_2.

Proceeding directly, the following quantities are defined

$$\begin{bmatrix} \underline{\lambda}(0) \\ t_f \end{bmatrix} \triangleq \underline{C}$$

$$\begin{bmatrix} \underline{\delta z}(t_f) \\ \underline{\delta v}(t_f) \\ H^*(t_f) \end{bmatrix} \triangleq \underline{E}(\underline{\lambda}(0),t_f) = \underline{E}(\underline{C}) \tag{2.46}$$

Let $\underline{C}^{(n)}$ be the nth estimate of $\underline{C}$. Then in general $\underline{E}(\underline{C}^{(n)}) \neq \underline{0}$ or

$$\underline{E}(\underline{C}^{(n)}) = \underline{\varepsilon}^{(n)} \neq \underline{0}$$

The problem is to find $\underline{\Delta C}^{(n)}$ such that

$$\underline{E}(\underline{C}^{(n)} + \underline{\Delta C}^{(n)}) = \underline{0}$$

Expanding this equation to first order about $\underline{C}^{(n)}$ yields

$$\underline{E}(\underline{C}^{(n)}) + \underline{E}_{\underline{C}} \, \underline{\Delta C}^{(n)} = \underline{0}$$

This implies

$$\underline{\Delta C}^{(n)} = - \underline{E}_{\underline{C}}^{-1} \, \underline{\varepsilon}^{(n)} \tag{2.47}$$

Equation 2.47 would yield $\underline{\Delta C}^{(n)}$ except for the fact that an explicit expression for $\underline{E}_{\underline{C}}$ is not available. This matrix is approximated in the algorithm by a perturbation technique. In summary the algorithm suggested is the following:

(i) guess $\underline{C}^{(1)}$

(ii) integrate Equations 2.44 to obtain $\underline{\varepsilon}^{(1)} = \underline{E}(\underline{C}^{(1)})$

(iii) perturb $\underline{C}^{(1)}$ by an amount $f\underline{C}^{(1)}$ where the scalar $f \ll 1$

(iv) compute $\underline{E}_C$ according to the following approximate formula:

$$\underline{E}_C = \begin{bmatrix} \dfrac{E_1(\underline{C}^{(1)} + f\underline{C}^{(1)}) - E_1(\underline{C}^{(1)})}{fC_1^{(1)}} & \cdots & \dfrac{E_1(\underline{C}^{(1)} + f\underline{C}^{(1)}) - E_1(\underline{C}^{(1)})}{fC_5^{(1)}} \\ \vdots & & \\ \dfrac{E_5(\underline{C}^{(1)} + f\underline{C}^{(1)} - E_5(\underline{C}^{(1)})}{fC_1^{(1)}} & \cdots & \dfrac{E_5(\underline{C}^{(1)} + f\underline{C}^{(1)}) - E_5(\underline{C}^{(1)})}{fC_5^{(1)}} \end{bmatrix}$$

(v) calculate $\underline{E}_C^{-1}$ and obtain $\underline{\Delta C}^{(1)}$ from Equation 2.47

(vi) repeat this process until the solution converges.

The computer results for both the minimum fuel and minimum time problems are shown in Figure 7. For purposes of comparison the trajectories obtained by using the closed loop controller derived in the second approximation are included. It is noted that the differences between the trajectories for the nonlinear minimum fuel and minimum time problems are small,and this supports the analysis given in Section 2.2. Also it is seen that the linearized controller gives a very good approximation to the exact optimal solution. One aspect of the extremal trajectories that was lost by linearization, however, is the time interval during which $\delta\alpha = 0$. But it is interesting to note that this "coast period" has little effect on the performance index, which suggests that the rather negligible degradation in system performance resulting from linearization is more than compensated for by the comparative simplicity

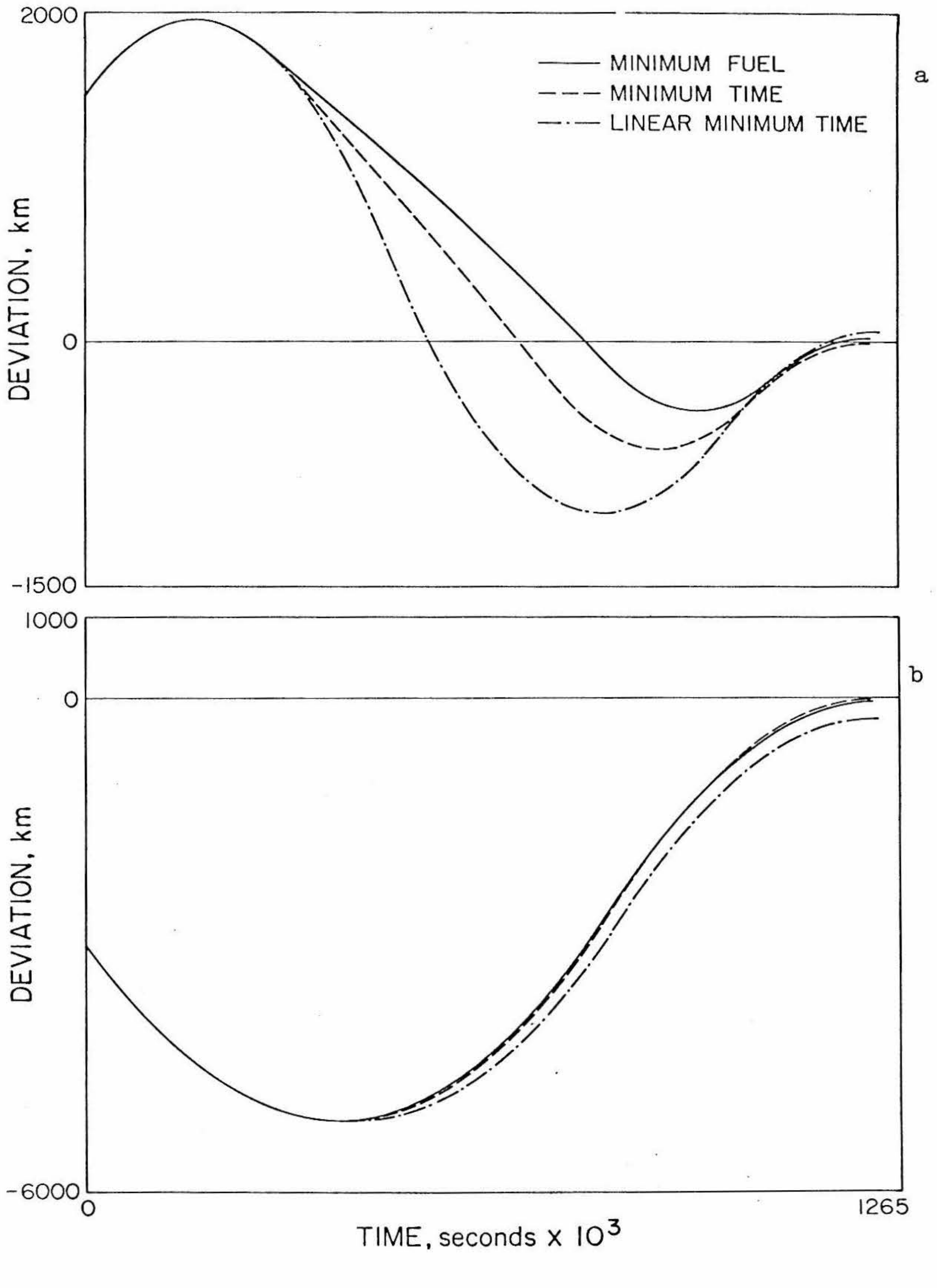

Figure 7. Results obtained for the nonlinear minimum time, nonlinear minimum fuel, and linear control systems: (a) x_1, (b) x_2 .

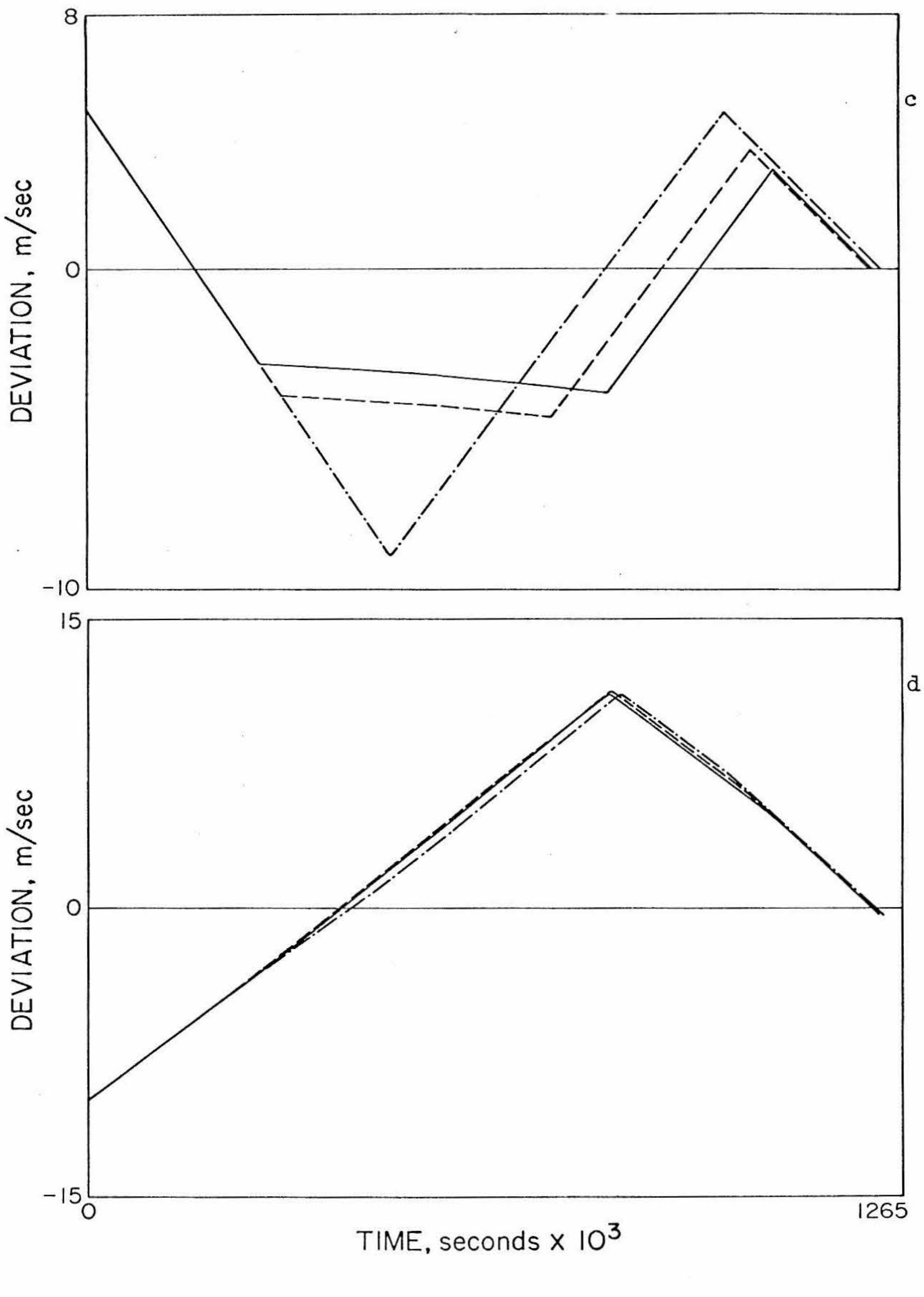

Figure 7 (continued). (c) x_4, (d) x_5 .

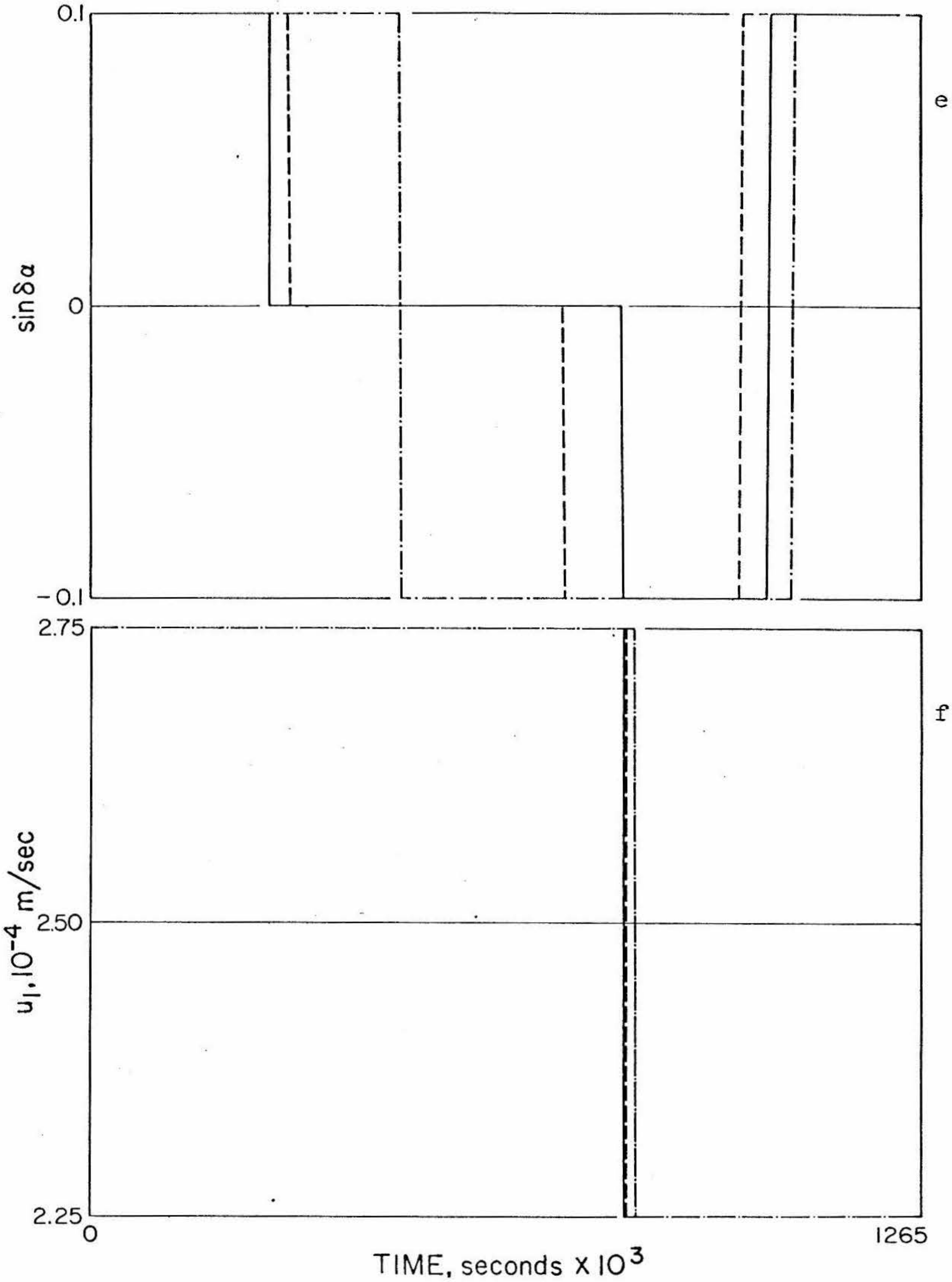

Figure 7 (continued). (e) sin $\delta\alpha$, (f) u_1 .

of controller implementation.

2.8 Conclusion of the Deterministic Problem

There have been several interesting consequences resulting directly from the study of the deterministic low-thrust guidance problem. For instance it was found that a surprisingly close approximation to the optimal controller could be obtained by merely solving a set of nonlinear equations, and that the difficult and time-consuming solution of the two point boundary value problem was, for all practical purposes, unnecessary.

In addition the first approximation showed that the system dynamics could be approximated quite well by three, decoupled, purely inertial plants. This important result provides a large step toward the solution of the stochastic problem, which is the next topic of interest.

III. THE STOCHASTIC PROBLEM

3.1 Introduction

One of the truly unique features of the low-thrust guidance problem is the manner in which random disturbances act to produce in-flight errors which are comparable in severity to injection errors. Undoubtedly the number of independent disturbances is great, but the ones of significant magnitude can be categorized as follows:

(i) attitude control variations and thrust vector pointing inaccuracies

(ii) thrust acceleration variations.

The less influential noise effects are considered to be lumped into the above processes. In this chapter the guidance system is sought which most accurately corrects the effect of these stochastic errors when constrained by control variable limitations. This minimum terminal variance (MTV) control problem is a stochastic optimization problem whose solution is more difficult to obtain, even approximately, than the deterministic cases treated in the last chapter.

3.2 The Noise Model

In making a statistical analysis it is necessary to model the noises acting on the system. Since a complete characterization of a random process is virtually impossible, there will invariably be a certain amount of arbitrariness in its representation. What is usually done is to select a model which contains features of physical significance, but which also retains properties conducive to mathematical analysis.

To obtain an approximation which is accurate to second order, the following essential features should be included in the stochastic model:

(i) the mean value (u)

(ii) the range of variation, or standard deviation (σ)

(iii) the rate at which the process varies, or the correlation time ($1/\beta$)

A stochastic model which retains all of these features as parameters is the Ornstein-Uhlenbeck (30) (OU) process*. If these parameters are not available from preflight test data, then an adaptive procedure (61),(63) of estimating them in flight is conceivable.

Mathematically, the OU process z can be represented as follows

$$z = u + n(t) \tag{3.1}$$

where n is a Gaussian, exponentially correlated process having zero mean, variance σ^2 and correlation time $1/\beta$. If a suitable interpretation is given to the solution of a stochastic differential equation, then it can be shown that the n(t) process satisfies the following Langevin equation

$$\dot{n} = -\beta n + \xi(t) \tag{3.2}$$

where ξ is a zero mean, Gaussian, white noise process with variance $2\beta\sigma^2$. A typical sample function for the z process is illustrated in Figure 8.

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*Jordan (24) has already applied this model to low-thrust guidance problems.

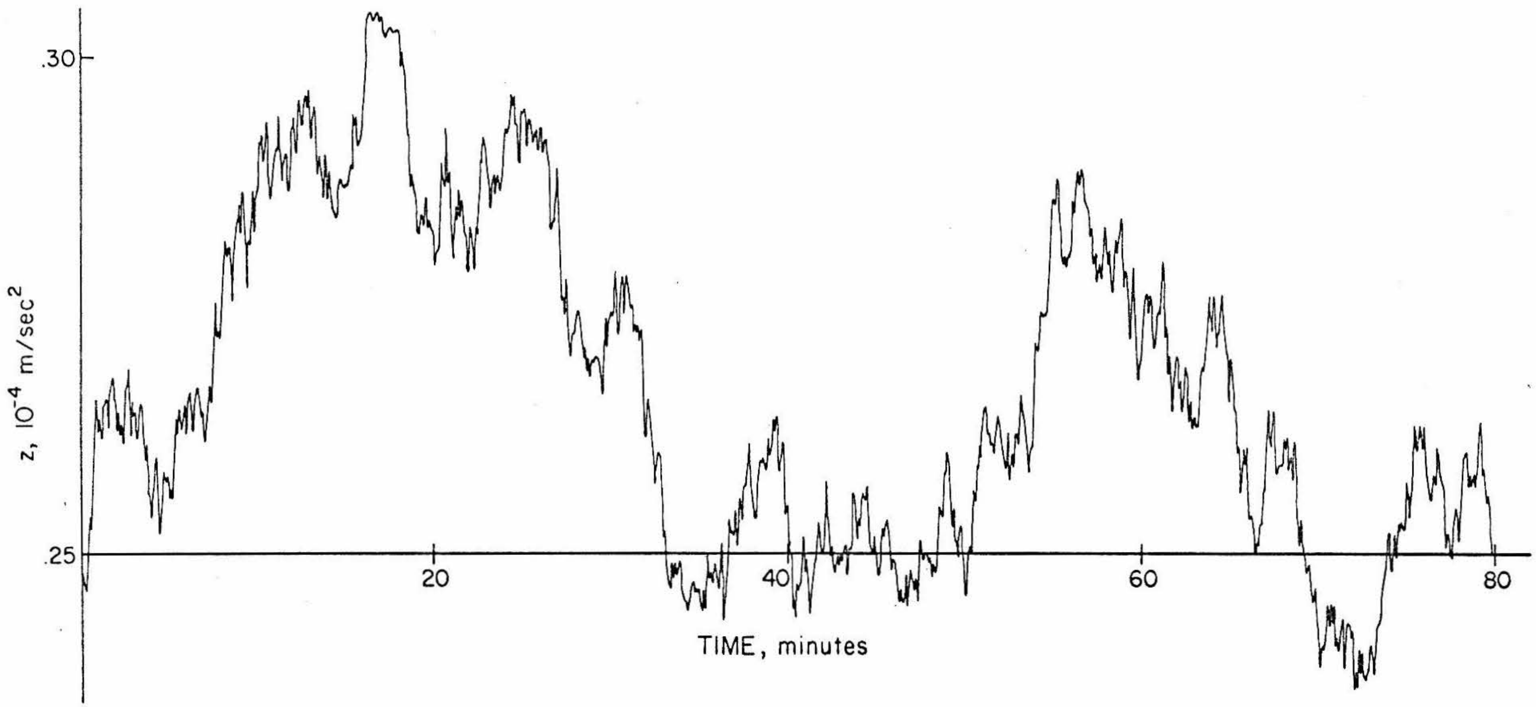

Figure 8. A sample function of the Ornstein-Uhlenbeck process: $1/\beta = 20$ min., $u = .25 \times 10^{-4} m/sec.^2$, $\sigma = .2u = .05 \times 10^{-4} m/sec.^2$

3.3 The Minimum Terminal Variance (MTV) Guidance

Using the motivation gathered from the previous deterministic cases, the procedure detailed in Section 2.2 is used to linearize, transform, and simplify the plant equations 1.2, only this time the ξ_1, ξ_2, and ξ_3 stochastic processes are retained (as explained in Section 3.1, the $\underline{\phi}$ process is lumped with these terms.) The result of carrying out these operations is the following set of plant equations:

$$\begin{aligned} \dot{x}_1 &= x_4 & \dot{x}_2 &= x_5 & \dot{x}_3 &= x_6 \\ \dot{x}_4 &= u_1 + \xi_1 & \dot{x}_5 &= u_2 + u_n\xi_2 & \dot{x}_6 &= u_3 + u_n\xi_3 \end{aligned} \tag{3.3}$$

where the coordinates x_i are identical to those expressed in Equation 2.8. Once again three, independent, purely inertial systems are obtained which can be studied individually. If now the noise terms ξ_1, $u_n\xi_2$, and $u_n\xi_3$ are identified with the OU process as defined in Equation 3.2, then the following formulation can be made for the MTV control problem:

PLANT:

$$\dot{x}_1 = x_4$$

$$\dot{x}_4 = u_1 + \xi_1$$

$$\dot{\xi}_1 = -\beta\xi_1 + \eta$$

INITIAL CONDITIONS:

$$\begin{aligned} x_1(0) &\sim N(0,\sigma_{x_{10}}) \\ x_4(0) &\sim N(0,\sigma_{x_{40}}) \\ \xi_1(0) &\sim N(0,\sqrt{2\beta}\,\sigma) \end{aligned} \tag{3.4}$$

PERFORMANCE INDEX:

$$E[k_1 x_1^2(t_e) + \ell_1 x_4^2(t_e)]$$

where

(i) $N(m,\hat{\sigma})$ indicates the normal distribution with mean m and variance $\hat{\sigma}^2$

(ii) k_1 and ℓ_1 are weighting factors as used in Equation 1.5

(iii) E is the statistical expectation over $\eta(t)$, $x_1(0)$, and $x_4(0)$.

Note that a similar formulation would apply to the other two of the three inertial systems. At first glance the above stochastic problem may seem quite innocuous, but to date no analytic solution has been found.

To illustrate the difficulties, consider a dynamic programming approach to the problem and let the following value function be defined

$$V(t,\underline{C}) \triangleq \min_{u_1} E[k_1 x_1^2(t_e) + \ell_1 x_4^2(t_e)]$$

$$\text{with} \quad \underline{X}(t) = \begin{bmatrix} x_1(t) \\ x_4(t) \\ \xi_1(t) \end{bmatrix} = \underline{C} \tag{3.5}$$

Associated with this function is the stochastic Bellman-Hamilton-Jacobi equation (44) which it satisfies:

$$0 = \min_{u_1} \left[\frac{\partial V}{\partial t} + x_4 \frac{\partial V}{\partial x_1} + u_1 \frac{\partial V}{\partial x_4} + \xi_1 \frac{\partial V}{\partial x_4} - \beta\xi_1 \frac{\partial V}{\partial \xi_1} + \beta\sigma^2 \frac{\partial^2 V}{\partial \xi_1^2} \right] \tag{3.6}$$

Performing the indicated minimization yields

$$u_1^*(x_1, x_4, \xi_1, t) = -k(t)\ \mathrm{sgn}\left(\frac{\partial V}{\partial x_4}\right) \tag{3.7}$$

where $k(t)$ is the maximum value of the control u_1 . Equations 3.6 and 3.7 represent a stochastic bang-bang control problem as well as the stopping point for elegant analysis. However a few important observations can be made regarding qualitative aspects of the optimal system. Specifically the bang-bang control policy implies that full control effort should be applied whenever it becomes available, and should not be delayed until the final part of a mission. In this manner the auxiliary control reserves referred to in Section 1.2 are incorporated in a very natural way. Several authors have already studied stochastic bang-bang systems, and a brief review is warranted here.

3.4 Literature Review of Stochastic Bang-Bang Control (44)-(55)

Much of the effort in this area has been directed toward singular perturbation expansions of the stochastic Bellman-Hamilton-Jacobi equation. Such an approach was first taken by Stratonovich (45), then extended by Lim (46), and further applied by Dorato (47), Hsieh (48), and Robinson (49). In making this expansion it is assumed that the noise covariance ε is sufficiently small so that the expansion can be made in powers of ε . This procedure leads to utilizing the deterministic solution as the zeroth order term.

There are several aspects of this approach which might affect the accuracy of the results. (i) First, the deterministic value

function for a bang-bang problem is usually badly behaved in that discontinuous and even unbounded partial derivatives are frequently encountered. Unfortunately these derivatives are required for the second and higher order terms of the perturbation expansion, and their behavior might overshadow the smallness of the parameter ε .
(ii) Second, a singular perturbation expansion normally requires the addition of boundary layer terms in order to insure that all boundary conditions are satisfied. As pointed out by the above authors, however, these terms are ignored in their expansions and the omission evidently causes difficulty in some of the examples given (46).

Even if these items cause small errors, the method cannot be applied here since the MTV guidance problem has no deterministic analog. In fact it is easily shown that the deterministic minimum terminal error problem is singular*. This property makes it necessary to define the auxiliary performance indices in the deterministic case.

Aoki (50) has applied dynamic programming and approximation-in-policy-space to solve stochastic minimum time problems. The numerical method is straightforward and effective for systems up to order three and possibly four. Two authors, Robinson (49) and Novosel'tsev (51), have found fault with Aoki's approach. Both criticisms seem unjustified. Robinson, on the one hand, has erroneously interpreted Aoki's dynamic programming equation (Equation 3) as a statement of the stochastic Hamilton-Jacobi equation. Novosel'tsev, on the other hand, has tried

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*i.e., the Hamiltonian is independent of the control. See Appendix B for this derivation.

in his paper to derive the stochastic Hamilton-Jacobi equation, but has failed to include the diffusion terms in his analysis (Equation 14). The result is the deterministic Hamilton-Jacobi equation for the mean state of the system and, of course, this error leads to results quite different from Aoki's. A stochastic minimum time problem is considered later in this study (Section 3.6) and the qualitative aspects of the solution agree with those noticed by Aoki.

There has been work (52),(53) on the direct numerical solution of the stochastic Hamilton-Jacobi equation. This approach is appealing but is computationally burdensome and needs perfecting for systems of order higher than one. One difficulty is finding enough boundary conditions to allow the application of known numerical algorithms for solving partial differential equations. In Reference (53) the approach of arbitrarily specifying the solution on a selected perimeter seems very artificial.

Van Mellaert (54) has done work related to the inclusion probability of a stochastic system (i.e., the probability of remaining in a given region of the state space over a specified time interval). The MTV control program would be similar to maximizing the inclusion probability in a neighborhood of the origin over the interplanetary flight time.

Wonham (55) suggests an interesting approach to the minimum mean square error problem, and bases his analysis on Booton's (56) statistical linearization. More will be said about this idea in Section 3.6.2.

In summary, there have been a host of approaches to stochastic optimization problems but none offers a cook-book solution to each and every example. It can safely be said that the approach must be selected which best suits the problem of interest. In the following section an algorithm is designed to solve the MTV guidance problem.

3.5 An Algorithm for Determining the MTV Guidance System

In this section a method is proposed for obtaining the control law associated with Equation 3.6. Fundamentally the algorithm involves three stages:

(i) the switching curve is parameterized using a finite set of parameters $\underline{a}$

(ii) for each control law $u_1(\underline{X},\underline{a}^{(i)},k)$ the steady state probability density function of the state deviations is computed

(iii) the parameters are optimized so that the performance index, Equation 3.4, is minimized.

Special attention is required to successfully execute the most difficult phase of the algorithm; i.e., step (ii). In order to find the stationary probability density function it is assumed that the control is discretized. In other words, over small intervals of time of length Δ , let the control input u_1 be equal to the constant value determined by the state of the system at the beginning of each time interval, and the value of the control function $u_1(\underline{X},\underline{a},k)$ at that state. Actually discretization is advantageous here since it limits the maximum switching rate of the control variables and thus prohibits the chattering sometimes encountered in stochastic bang-bang systems. In computing the stationary density it is implicitly assumed that the system is either time invariant or else slowly varying; in the latter case, the same perturbation argument used in Section 2.3 is again invoked here. An asymptotic solution is used primarily to eliminate the time dependence of the control law which results from a finite (although large) orbit transfer time t_e .

When the control u_1 is constant, then the transition probability density function associated with Equations 3.4 is obtained in a straightforward fashion from the Fokker-Planck equation*. This function is a Gaussian density which is totally characterized by its mean $\underline{\mu}$ and covariance M , and is used to compute successive probability distributions of the state at intervals of Δ seconds, assuming an arbitrary initial distribution. The sequence of distributions approaches the steady state in a manner suggesting geometric convergence. Using this technique to obtain the steady state density, the parameter set $\underline{a}$ is then optimized to yield the minimum value of the performance index.

In the present problem, two simple parameterizations are given for the switching curve:

(i) parabolic with multiplicative parameter a (see Figure 9); i.e., $u_1 = -k \operatorname{sgn} \{x_4 + a \sqrt{2k|x_1|} \operatorname{sgn}(x_1)\}$

(ii) same as (i) with a zero, or resting, control region (see Figure 10).

For easier implementation the switching law is made independent of

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*See Appendix C.

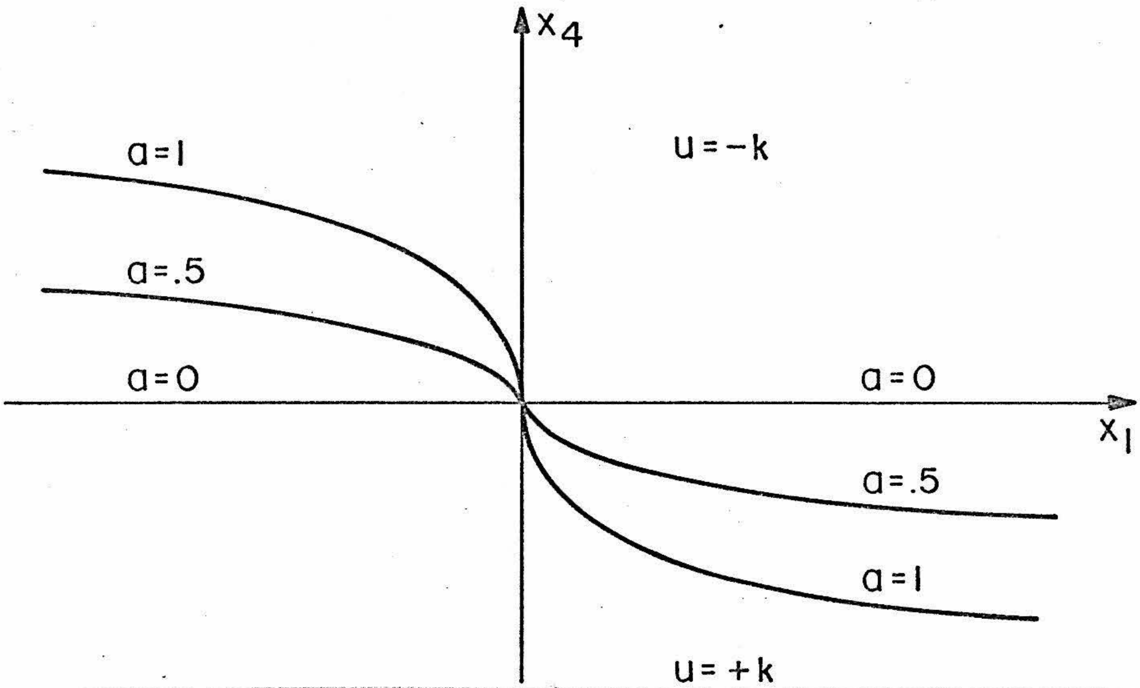


Figure 9. First parameterization of the switching curve

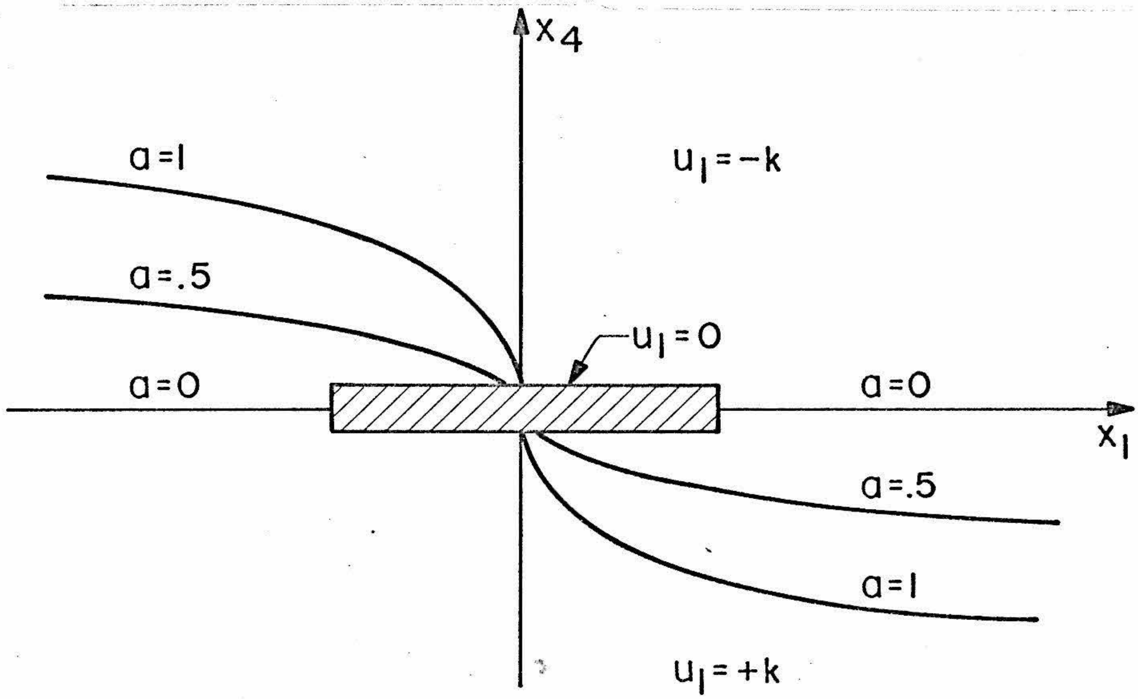


Figure 10. Second Parameterization of the switching curve

$\xi_1(t)$ and is chosen parabolic based on the results of the stochastic minimum time investigation (see Section 3.6). Clearly the motivation for these choices is heuristic and there is no conceptual difficulty in choosing more complicated parameterizations if desired.

The results for case (i) are plotted in Figures 11-12 for two different transition probability density functions M_1 and M_2 which correspond to small and large noise processes, respectively. In addition to the performance index, the steady state probability density function also yields the following information:

(1) the probability of being in a particular control state

(2) the probability of a transition from one control state to another, at each Δ-interval

(3) the total probability of a control switching every Δ seconds.

Quantities (1) and (3) are plotted in Figure 13. In order to gain insight into the nature of the steady state density function, the results for covariance M_1 with $a = 1$ and $a = .5$ are shown in Figures 17 and 18, respectively.

It is seen from Figure 11 that for small noise the performance index (with $\ell_1 = 0$) is minimized for $a = .6$. For the larger noise example, the minimum point occurs at $a = .25$. This counterclockwise swing of the switching curve (for increasing noise) agrees with the results of Aoki (50) and Wonham (55), and is consistent with the results of Section 3.6. A loose interpretation of rate-gain increase can be associated with this phenomenon. It is interesting to observe, though, that the optimum value of the parameter a is relatively

Fig.11

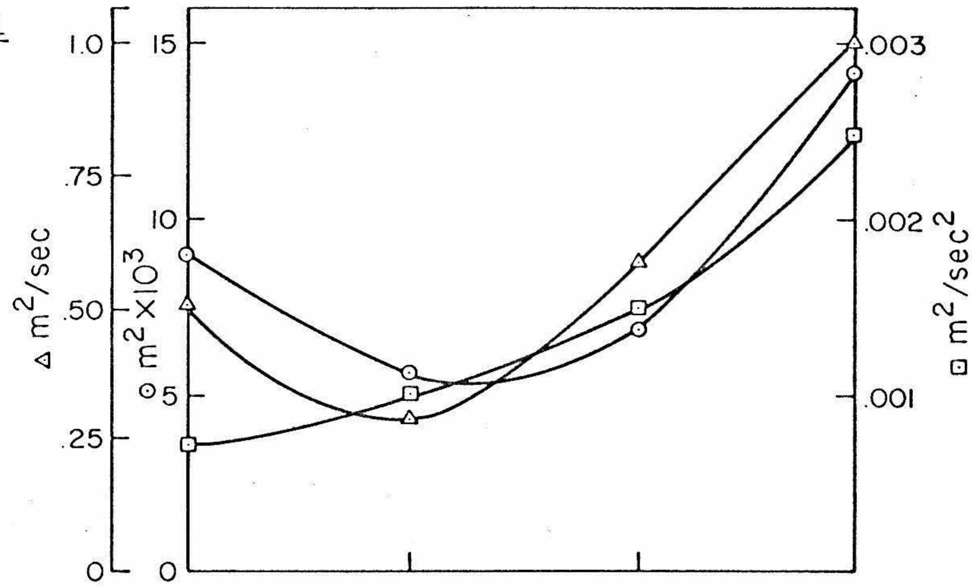
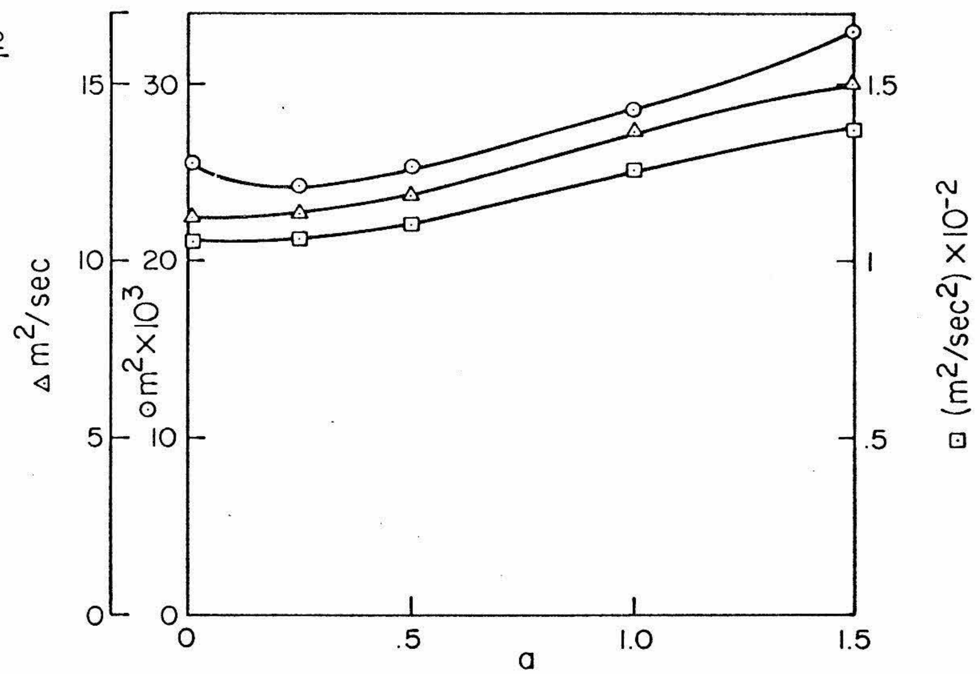


Fig.12



Figures 11 and 12. Steady state covariances vs. parameter 'a' for the first parameterization ($k = .25 \times 10^{-4}$ m/sec^2): ⊙ = $E[x_1^2]$, △ = $E[x_1 x_4]$, ⊡ = $E[x_4^2]$; Figure 11, small noise, $M_1 = \begin{bmatrix} 180.0 & .251 \\ .251 & .000416 \end{bmatrix}$; Figure 12, large noise, $M_2 = \begin{bmatrix} 935.0 & 1.98 \\ 1.98 & .00477 \end{bmatrix}$

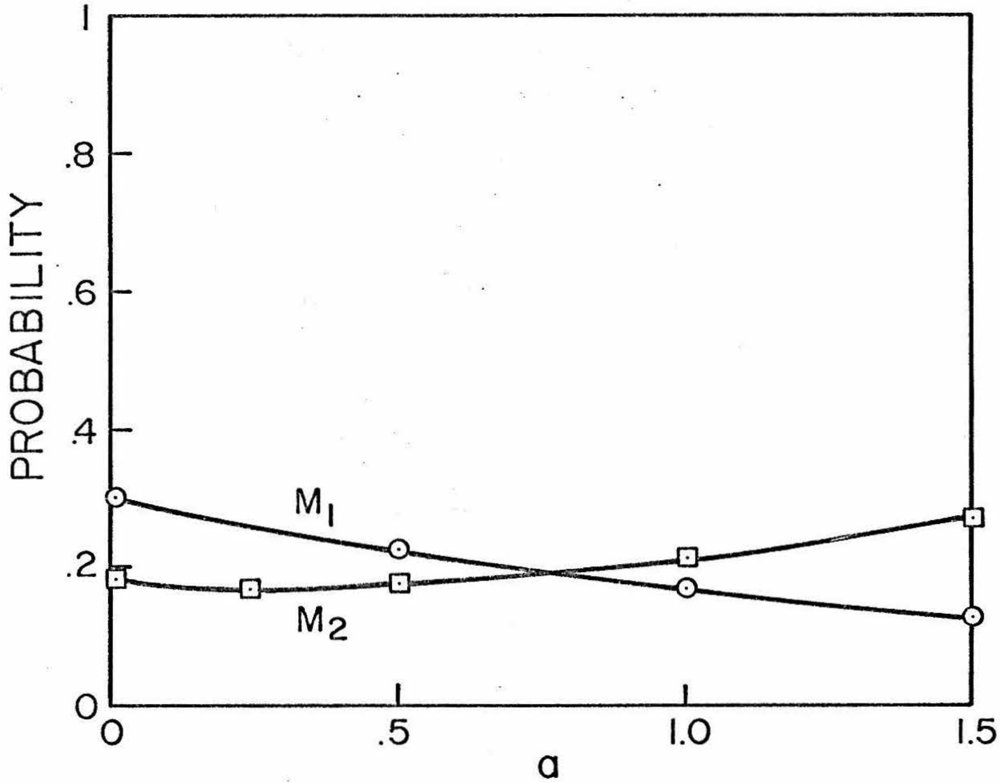


Figure 13. Total switching probabilities vs. parameter 'a' for the first parameterization ($k = .25 \times 10^{-4} m/sec^2$): Small noise ($M_1$) and large noise ($M_2$) are as defined in Figures 11 and 12.

insensitive to large variations of process noise levels.

The results for case (ii) are shown in Figures 14-16, and indicate the responses to both small and large noise. Introducing the zero control region violates the bang-bang requirement, but is appealing from the standpoint of reducing the total number of control switchings. It is not difficult to imagine other control configurations which could offer specific advantages, and which could be easily tested using the above algorithm.

The magnitude of Δ represents a trade-off in control system design. As Δ is decreased, better guidance accuracy is obtained, but mechanization simplicity is sacrificed because the expected number of control switchings increases. Hence there will be a best value of Δ according to design specifications. Also there is no reason why Δ must remain a constant throughout the entire mission, and it is likely that there would be advantages in allowing it to vary.

It would be of at least theoretical interest to find the result of letting Δ approach zero. At best, the procedure described above will provide a rough answer by extrapolation. Figures 18-20 show the stationary density function as Δ is decreased and, as expected, the probability distribution becomes concentrated around the origin as Δ becomes small. In Figure 21 the standard deviations of the states corresponding to $\Delta = 0$ are found by extrapolating the curve through the vertical axis. As Δ vanishes the expected number of control switchings is seen to increase without bound.

Fig.14

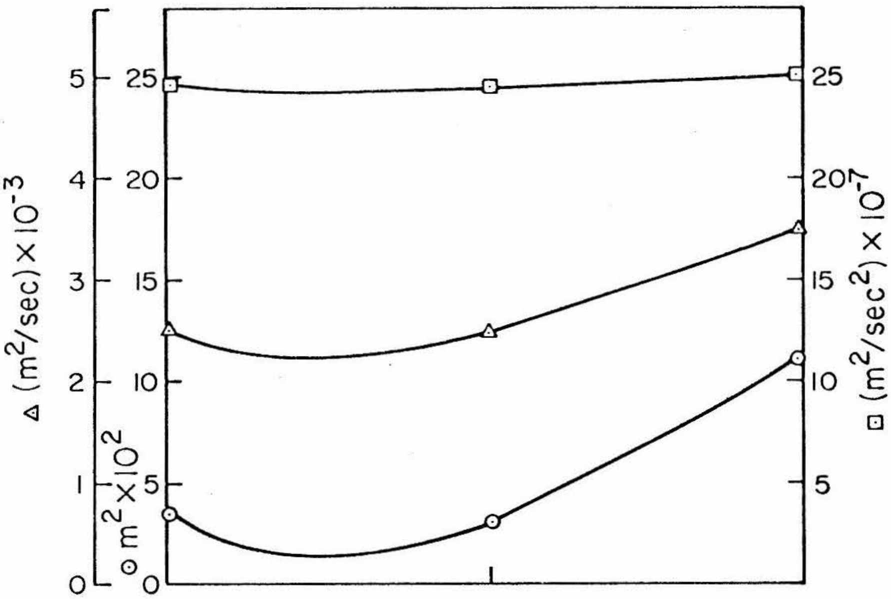
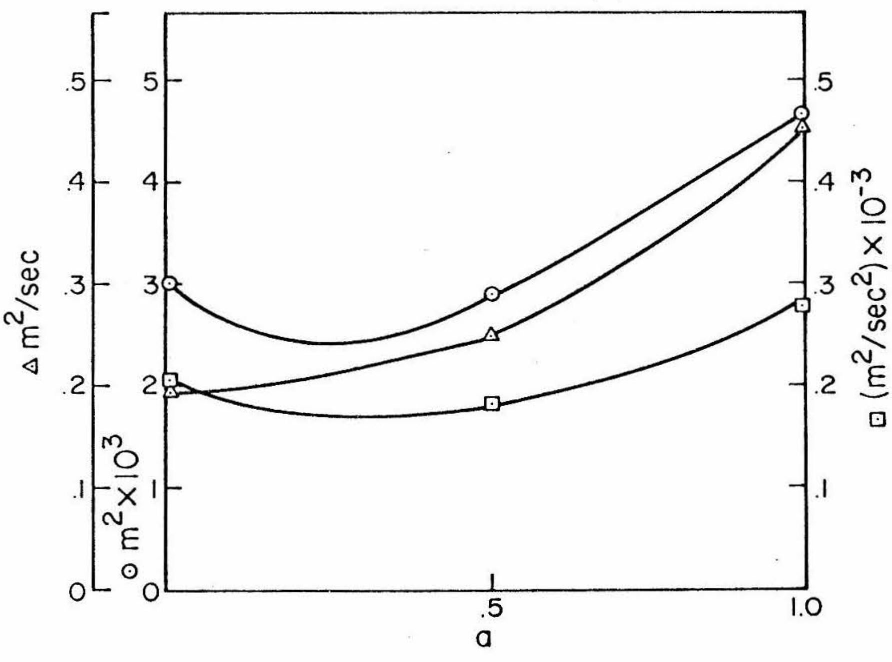


Fig.15



Figures 14 and 15. Steady state covariances vs. parameter 'a' for the second parameterization ($k = .25 \times 10^{-4}$ m/sec^2): ⊙ = $E[x_1^2]$, △ = $E[x_1 x_4]$, ⊡ = $E[x_4^2]$; Figure 14, small noise,

$M_1 = \begin{bmatrix} 31.2 & .05 \\ .05 & .00005 \end{bmatrix}$; Figure 15, large noise, $M_2 = \begin{bmatrix} 250.0 & .2 \\ .2 & .0002 \end{bmatrix}$

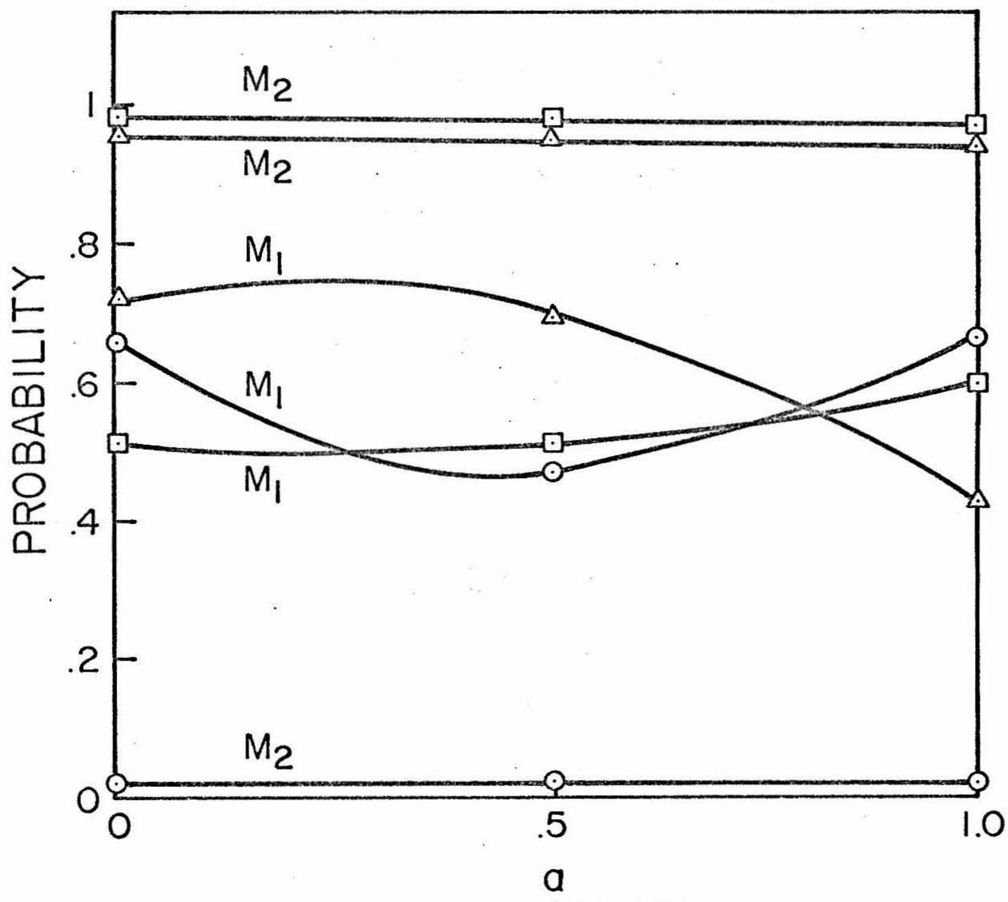


Figure 16. Switching probabilities vs. parameter 'a' for the second parameterization ($k = .25 \times 10^{-4} m/sec^2$): ⊙ = total switching probability every Δ seconds, △ = probability of remaining in zero control region every Δ seconds, ⊡ = probability of being in zero control region; small noise (M_1) and large noise (M_2) are as defined in Figures 14 and 15.

Fig.17

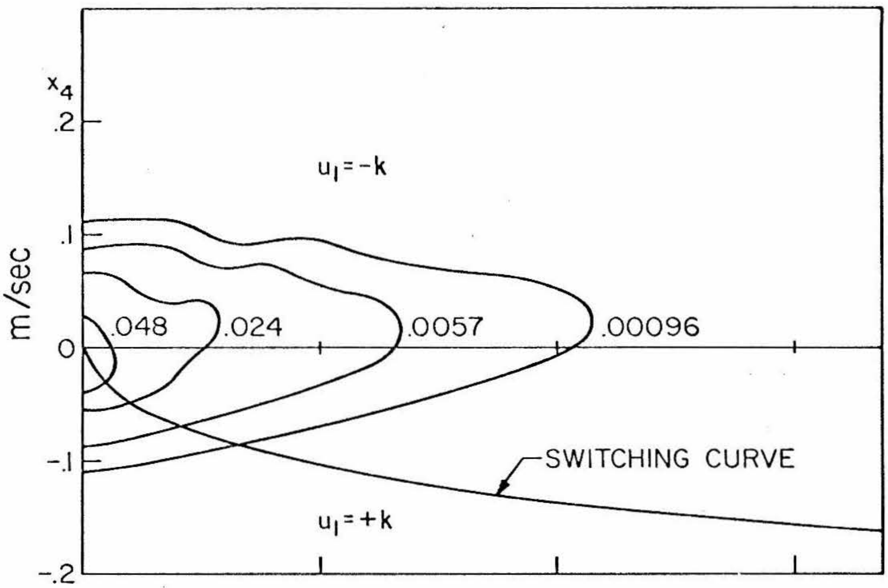
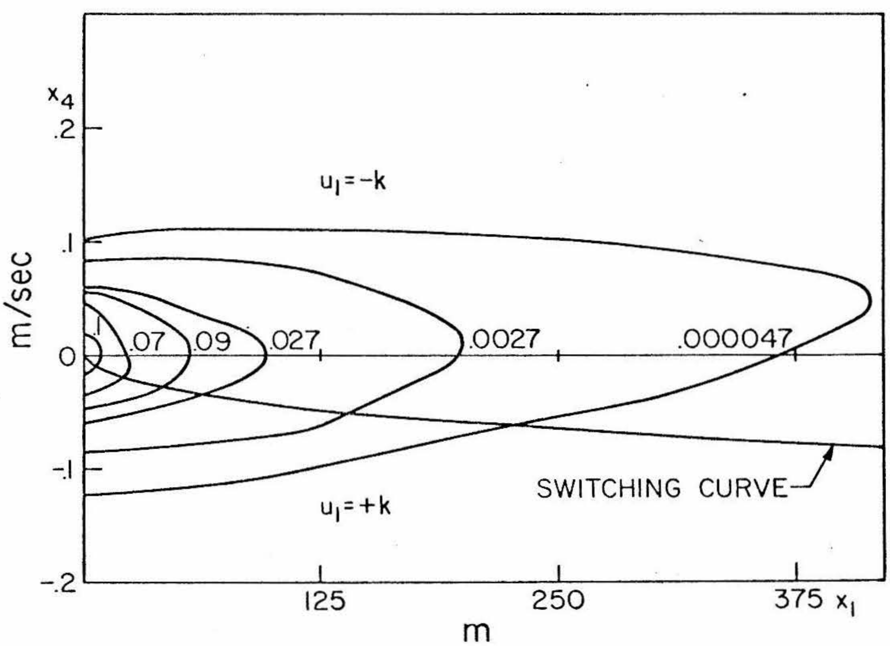


Fig.18



Figures 17 and 18. Steady state probability density functions for small noise, $M_1 = \begin{bmatrix} 180.0 & .251 \\ .251 & .000416 \end{bmatrix}$ and $\Delta = 1000$ sec.:

Figure 17, a = 1; Figure 18, a = .5 .

Fig.19

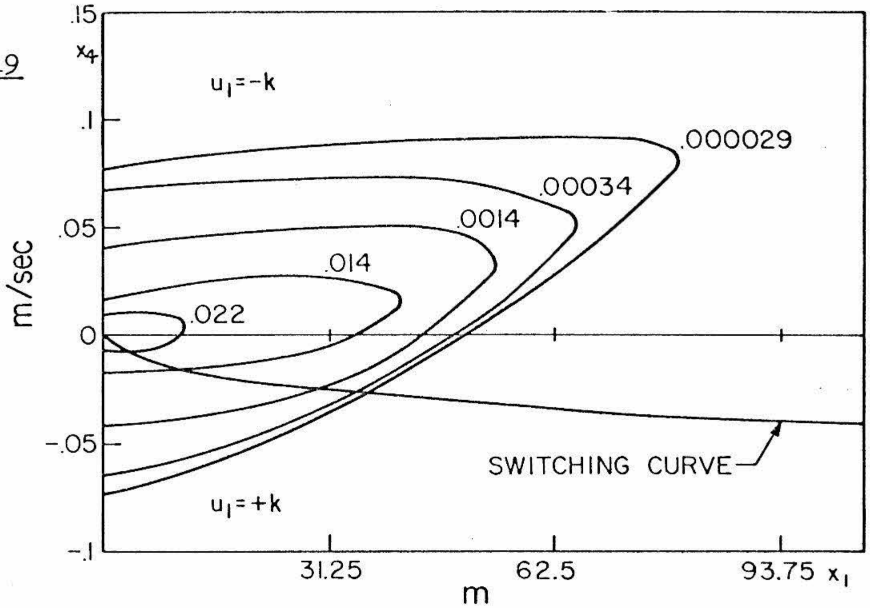
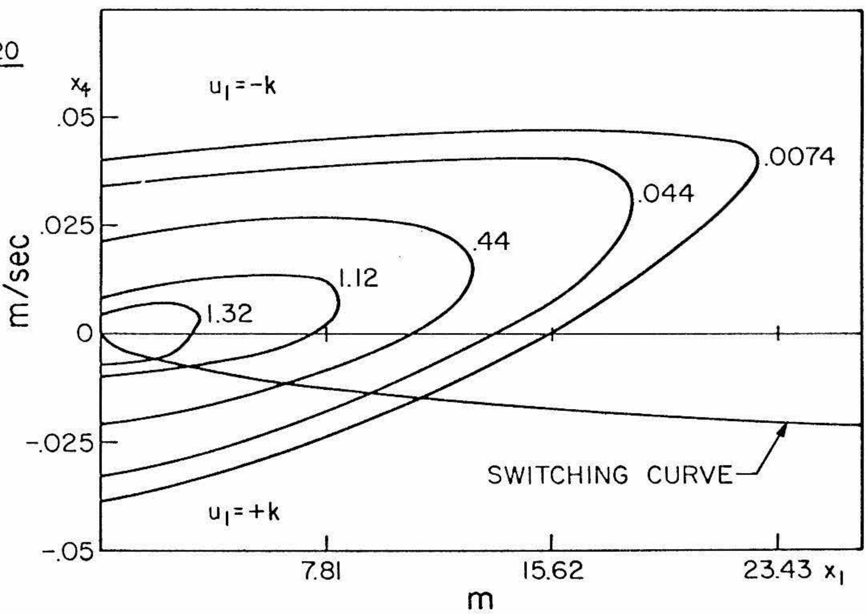


Fig.20



Figures 19 and 20. Steady state probability density functions for small noise and a = .5: Figure 19, Δ = 500 seconds; Figure 20, Δ = 250 seconds.

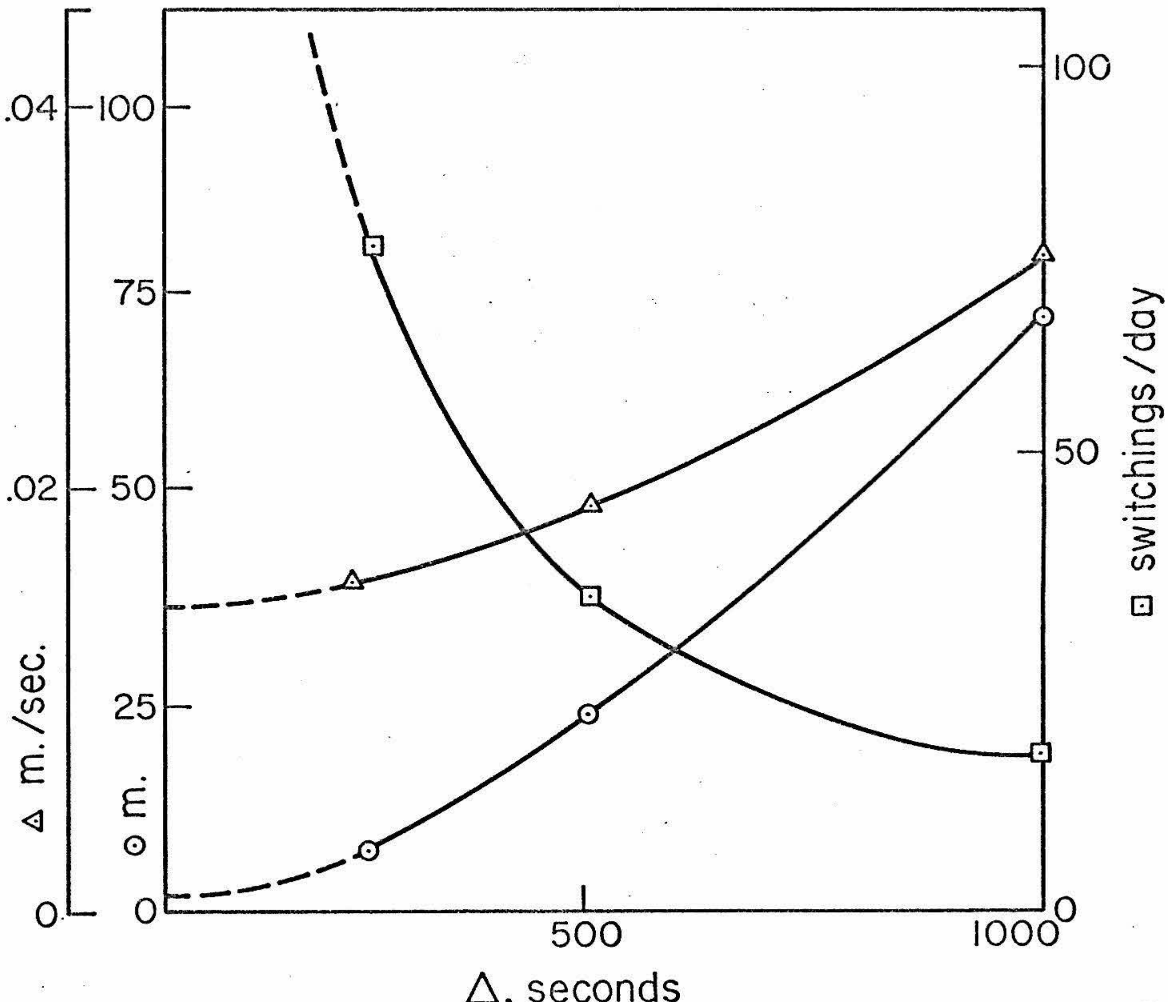


Figure 21. Steady state covariances and switching rate versus Δ (extrapolated to Δ = 0):

△ $= \sqrt{E[x_4^2]}$, ⊙ $= \sqrt{E[x_1^2]}$, ⊡ = average switching rate.

The deviation in fuel consumption from nominal is found by integrating the differential equation for $\dot{x}_4$ given in Equation 3.3:

$$\int_0^{t_e} \dot{x}_4 \, dt = \int_0^{t_e} (u_1 + \xi_1) \, dt \tag{3.8}$$

If the final velocity deviations are zero (or approximately zero as in the MTV controller) then

$$\int_0^{t_e} u_1 dt = - \int_0^{t_e} \xi_1 \, dt - v(0) \tag{3.9}$$

Hence, to the accuracy of the approximations employed, any controller which damps out the velocity errors will have the same total fuel consumption. At most, differences will be of second order.

In summary then, a controller has been found which provides maximum guidance accuracy subject to bounds on the control variables when noisy disturbances enter the system. To support these results it would be profitable to compare them to other stochastic bang-bang controllers. In the next section three such controllers are investigated.

3.6 Investigation of Other Stochastic Bang-Bang Controllers

3.6.1. The stochastic minimum time controller. The performance index to be minimized is the average time required to bring the state of the system into a specified neighborhood of the origin N_o . Corresponding to this criterion, the following value function is defined:

$$V(\tau,\underline{C}) = \min_{\substack{u_1(t) \\ \tau \leq t \leq t_f}} E\left[\int_{\tau}^{t_f} dt \,\middle|\, \underline{X}(\tau) = \underline{C} \right] ; \quad \underline{X}(t_f) \in N_o \qquad (3.10)$$

A set of target points is necessary since there is zero probability of hitting a single point in a stochastic system with bounded control. Following Aoki (50) the solution is sought by the method of dynamic programming (58) and the right side of Equation 3.10 is expanded as follows:

$$V(\tau,\underline{C}) = \min_{\substack{u_1(t) \\ \tau \leq t \leq \tau+\Delta}} \left[\Delta + E[V(\tau + \Delta, \underline{X}(\tau+\Delta) \mid \underline{X}(\tau) = \underline{C}] + O(\Delta^2) \right]$$

$$= \Delta + \min_{u_1} \int_{\underline{X}} p(\underline{X}(\tau+\Delta) | \underline{X}(\tau) = \underline{C}) \cdot V(\tau+\Delta, \underline{X}(\tau+\Delta))\underline{dX} + O(\Delta^2) \qquad (3.11)$$

where $p(\underline{X}(\tau+\Delta)|\underline{X}(\tau) = \underline{C})$ is the probability density of the state vector at time $\tau+\Delta$ given that $\underline{X}(\tau) = \underline{C}$; i.e., the transition probability density function.

Special attention is required to find p . In principle the Fokker-Planck equation associated with the system described in Equation 3.4 could be solved for this function, while in reality this solution is not available (57) since $u_1(\underline{X},t)$ is bang-bang. However, if at any given time the state of the system is in one control region (for instance, the state is above the switching curve in Figure 3), then over Δ seconds it looks to the system as though $u_1(\underline{X},t)$ is actually a constant function. There is a natural temptation, then, to use the solution of the Fokker-Planck equation for a constant control input*. Of course, the approximation will become less accurate as the state of the system nears the switching curve.

With p now in hand the solution of Equation 3.11 is sought by means of the approximation-in-policy-space algorithm (59). This method can be broken down into the following steps:

(i) guess an initial value function $V_o(\tau,\underline{C})$

(ii) compute $V_n(\tau,\underline{C}) = \Delta + \min_{u_1} E[V_{n-1}(\tau+\Delta,\underline{X}(\tau+\Delta)) \mid \underline{X}(\tau)]$ and store u_1 at each grid point; $n=2,3,\cdots$

(iii) iterate until the solution converges.

For completeness a few details of the numerical solution will be discussed. A convenient initial guess for the value function is given by the deterministic solution of the minimum time problem. And again, for simplicity, the switching curve is assumed to be independent of ξ_1. The target neighborhood N_o is defined as a rectangular area with boundaries at $\pm$ 40 meters in the x_1 direction, and $\pm$ .05 meters/second in the x_4 direction, while grid points are spaced 10 meters apart in the x_1 direction and .025 meters/second apart in the x_4 direction. In deference to considerations of computing time, the area of interest is limited to $\pm$ 1 kilometer by $\pm$ .25 meters/second, and even though solutions in larger regions could be obtained, the general solution characteristics are evidenced by the results in the area that was considered. In all cases Δ was taken as 1000 seconds.

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*See Appendix C.

The results appear in Figure 22 for various noise cases. As the noise gets large, it is interesting to note that the switching curve is found to optimize nearer to the x_1 axis. This effect is the same as that noticed by Aoki (50) and Wonham (55) for similar problems, and was also encountered in Section 3.5 of this chapter.

3.6.2. Wonham's approach to stochastic bang-bang control. Wonham's (55) technique is geared to finding stationary, or steady state, solutions associated with integral quadratic performance indices; but here an attempt is made to extend it to the nonstationary MTV guidance problem.

The Bellman-Hamilton-Jacobi equation which must be solved is the following:

$$\frac{\partial V}{\partial t} + \min_{\underline{u}} \left\langle (A\underline{X} + b\underline{u}), V_{\underline{X}} \right\rangle + \frac{1}{2} \operatorname{tr}(QV_{\underline{XX}}) = 0$$

$$V(t_e) = \underline{X}^T(t_e)\, W\underline{X}(t_e) \quad ; \quad V_{\underline{X}}(t_e) = 2W\underline{X}(t_e) \tag{3.12}$$

where $\underline{X}$ is the n-dimensional state deviation vector, $\underline{u}$ is the m-dimensional control deviation vector, and A and b are the Jacobian matrices of the plant dynamics with respect to nominal state and nominal control, respectively. In addition

$$V(\underline{C},\tau) \triangleq \min_{\underline{u}} E\left[\, \underline{X}^T(t_e)\, W\underline{X}(t_e) \mid \underline{X}(\tau) = \underline{C} \,\right] \tag{3.13}$$

Performing the indicated minimization in Equation 3.12 yields

$$\underline{u} = -\, K \operatorname{sgn}(b^T V_{\underline{X}}) \tag{3.14}$$

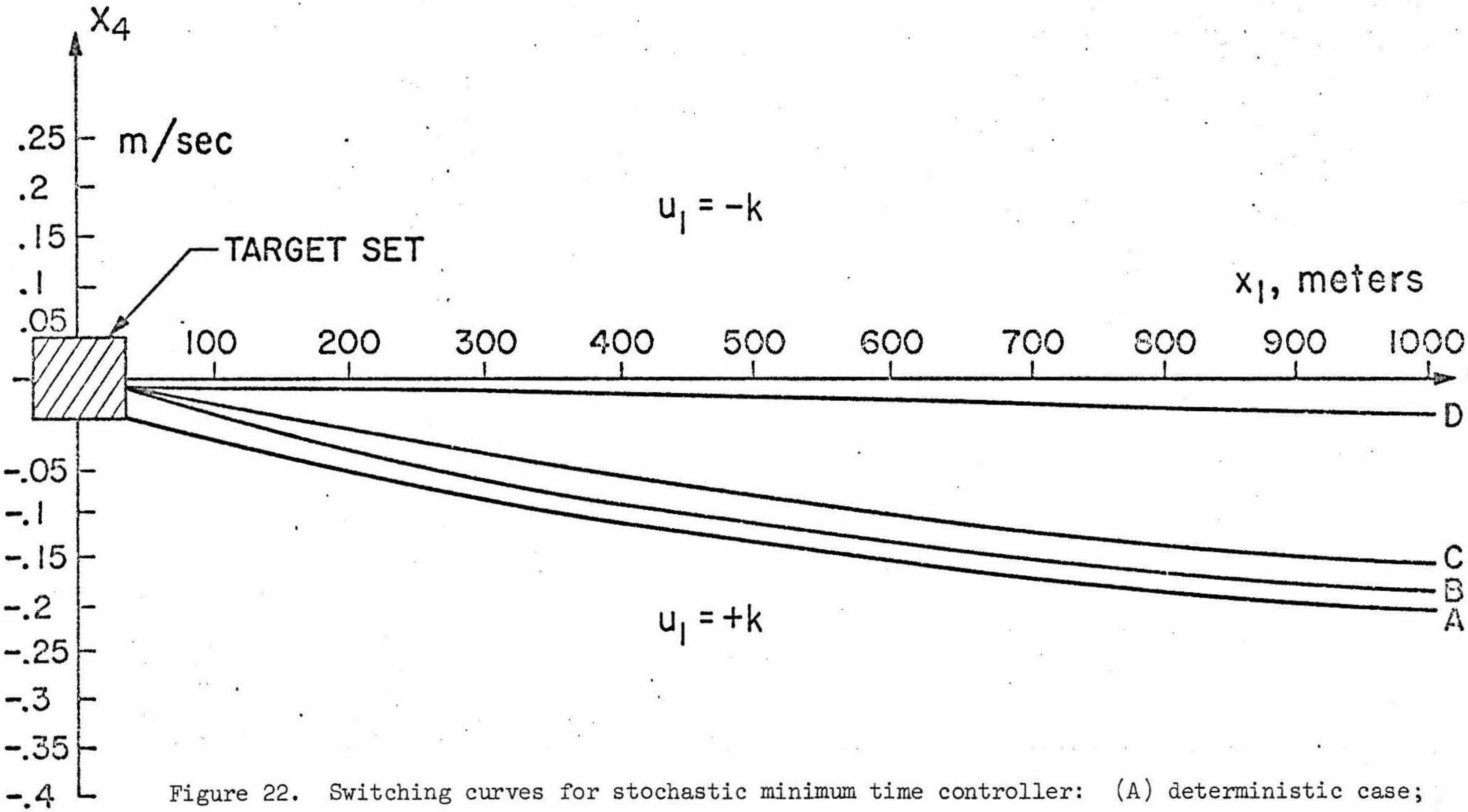


Figure 22. Switching curves for stochastic minimum time controller: (A) deterministic case; (B) transition covariance

$$M = \begin{bmatrix} 878 & 2.02 \\ 2.02 & .0056 \end{bmatrix}; \quad (C)\ M = \begin{bmatrix} 657 & .61 \\ .61 & .0021 \end{bmatrix}; \quad (D)\ M = \begin{bmatrix} 20220 & 46.46 \\ 46.46 & .116 \end{bmatrix}$$

where

$$K = \begin{bmatrix} k_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & k_m \end{bmatrix}$$

where k_i represents the bound on the control u_i. Wonham's approach relies heavily on the statistical linearization technique of Booton (56) to convert the nonlinear problem into a tractable linear problem. The object of Booton's method is to find an equivalent gain matrix C which has columns C_i that minimize

$$\int_{R^n} [u_i(\underline{X}) - C_i^T \underline{X}]^2 \, p(\underline{X},t) \, d\underline{X} \tag{3.15}$$

Here p is the probability density of the state deviations. This density is assumed to be approximately Gaussian with zero mean and covariance M . Wonham's final assumption is that the value function V is approximately quadratic, i.e.,

$$V = \underline{X}^T P \underline{X} + r \tag{3.16}$$

Under these assumptions the minimization indicated in Equation 3.15 can be determined for the control law in Equation 3.14

$$C_i = - k_i (2/\pi)^{1/2} \{ (b_i^T P M Pb_i) \}^{-1/2} Pb_i \tag{3.17}$$

where b_i is the i^{th} column of b . By substituting Equation 3.16 into Equation 3.12 and using 3.17 we get the following differential equations for P and r :

$$\dot{P} + (A + bC^T)^T\, 2P + 2P(A + bC^T) = 0 \; ; \; P(t_f) = W$$

$$\dot{r} + tr(QP) = 0 \; ; \quad r(t_f) = 0 \tag{3.18}$$

Finally, the covariance matrix M must satisfy

$$\dot{M} = (A + bC^T)M + M(A + bC^T)^T + Q \; ; \; M(0) = M_o \tag{3.19}$$

Equations 3.18 and 3.19 represent a two-point boundary value problem whose solution approximates the optimal, stochastic, bang-bang controller.

The solution of these equations was first attempted using the simplified dynamics (i.e., three decoupled, inertial plants) derived in Section 3.3. It was found that the divisor in Equation 3.17 almost invariably went to zero on any given iteration. The reason for this can be explained by examining the steady state solution of Equation 3.19. First, C can be obtained from Equation 3.17 as

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{-k(\frac{2}{\pi})^{1/2}}{(m_{11} + 2m_{12}p_{22} + m_{22}p_{22}^2)^{1/2}} \begin{bmatrix} 1 \\ p_{22} \end{bmatrix} \tag{3.20}$$

while the steady state solution of Equation 3.19 is given by (assuming white noise disturbances):

$$\begin{bmatrix} 0 & 1 \\ C_1 & C_2 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ 1 & C_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma^2 \end{bmatrix} = 0$$

Hence

$$2\dot{m}_{12} = 0$$

$$m_{22} + C_1 m_{11} + C_2 m_{12} = 0$$

$$2(C_1 m_{12} + C_2 m_{22}) + \sigma^2 = 0$$

which implies

$$m_{12} = 0$$

$$m_{22} = -\sigma^2/2\,C_2$$

$$m_{11} = \sigma^2/2\,C_2 C_1 = -m_{22}/C_1$$

Therefore

$$m_{22} = \frac{\sigma^2 (m_{11} + m_{22} p_{22}^2)^{1/2}}{k\, p_{22}^2} \left(\frac{\pi}{2}\right)^{1/2}$$

$$m_{11} = \frac{\sigma^2 (m_{11} + m_{22} p_{22}^2)}{2k^2\, p_{22}} \tag{3.21}$$

Solving these equations for m_{11} yields ($p_{22} \triangleq p$)

$$m_{11} = \frac{\sigma^4 \theta^2 (4\sigma^4 \theta^4 p^2 + 2(p - 2\sigma^2\theta^2)\, \sigma^2 \theta^2 p)}{(p - 2\sigma^2\theta^2)^2} \tag{3.22}$$

where $\theta = (\pi/2)^{1/2} / 2k$. Now, defining

$$p - r \triangleq x \quad ; \quad r \triangleq 2\sigma^2\theta^2$$

Equation 3.22 becomes

$$m_{11} = \sigma^6\theta^4\left(1 + r + \frac{2r^2 + r}{x} + \frac{r^3}{x^2}\right) \qquad (3.23)$$

If now it is desired to minimize m_{11} with respect to p --which is the same as minimization with respect to x --then standard analysis yields

$$x = \frac{2r^3}{2r^2 + r} \qquad (3.24)$$

Substituting Equation 3.24 into Equation 3.23 yields

$$m_{11} = \frac{-\sigma^6\theta^4}{4r} = \frac{-\sigma^4\pi}{64\ k^2} \qquad (3.25)$$

Equation 3.25 shows that the optimal m_{11} should be negative; this is, of course, an impossibility. Evidently this result is linked to the difficulty of solving the nonstationary problem referred to earlier, and unfortunately renders Wonham's method inapplicable to the MTV guidance problem.

3.6.3. The MTV controller and the steady state Fokker-Planck equation. It would be of interest to consider an analytical approach to the MTV control problem. If ξ_1 is interpreted as a white noise process with variance σ^2, then the following steady state Fokker-Planck equation is associated with Equation 3.4 (setting $b = \sigma^2/2$; $x = x_1$; $y = x_4$; $u = u_1$):

$$b\frac{\partial^2 p}{\partial y^2} - y\frac{\partial p}{\partial x} - \frac{\partial}{\partial y}(u(x,y)p) = 0 \qquad (3.26)$$

The MTV control problem can now be cast as follows: (i) find the solution of Equation 3.26 for arbitrary switching curves associated with $u(x,y)$, and (ii) choose the curve which minimizes the performance index given in Equation 3.4.

The first problem is approached by expanding the solution about the x axis (on which the solution is parameterized by unknown constants), and extending the expansion throughout the region where $u(x,y)$ is constant. Then various switching-boundary conditions are used to solve for the unknown constants.

To be specific, assume that the following quantities are known

$$p(x,0) = \phi(x) \quad , \qquad x \geqq 0$$

$$p_y(x,0) = \psi(x) \quad , \qquad x \geqq 0$$

and also assume that the switching boundary is below the x axis for positive values of x. In order to find the partial derivatives along the x axis, Equation 3.26 is used to obtain

$$\frac{\partial^2 p}{\partial y^2} = \frac{y}{b}\frac{\partial p}{\partial x} + \frac{u}{b}\frac{\partial p}{\partial y}$$

$$= \frac{y}{b}\phi_x + \frac{u}{b}\psi \quad , \qquad x \geqq 0 \ , \ y = 0$$

In addition, the third partial derivative is obtained from this equation

$$\frac{\partial^3 p}{\partial y^3} = \frac{y}{b}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{b}\frac{\partial p}{\partial x} + \frac{u}{b}\frac{\partial^2 p}{\partial y^2}$$

$$= \frac{y}{b}\psi_x + \left(\frac{1}{b} + \frac{u}{b}\left(\frac{y}{b}\right)\right)\phi_x + \frac{u}{b}\left(\frac{u}{b}\right)\psi$$

Similarly it is found that

$$\frac{\partial^4 p}{\partial y^4} = \left(2\,\frac{y}{b}\left(\frac{u}{b}\right) + \frac{2}{b}\right)\psi_x + \frac{u}{b}\left(\frac{1}{b} + \frac{u}{b}\left(\frac{y}{b}\right)\right)\phi_x + \frac{y}{b}\left(\frac{y}{b}\right)\phi_{xx} + \frac{u}{b}\left(\frac{u}{b}\right)\psi, \text{ and so on.}$$

The solution is then given by

$$p(x,y) = \sum_{n=0}^{\infty} \frac{\partial^n p(x,0)}{\partial y^n}\,\frac{y^n}{n!} \quad ; \qquad \frac{\partial^o p}{\partial y^o} \triangleq p(x,0) = \phi(x) \tag{3.27}$$

which can be written in the form

$$p(x,y) = \phi(x) + \psi(x)\,g_o(y) + \phi_x(x)\,f_1(y) + \psi_x(x)\,g_1(y)$$
$$+ \phi_{xx}(x)\,f_2(y) + \psi_{xx}(x)\,g_2(y) + \cdots \tag{3.28}$$

where

$$g_o(y) = y + \frac{1}{2!}\,\frac{u}{b} + \frac{1}{3!}\left(\frac{u}{b}\right)^2 + \frac{1}{4!}\left(\frac{u}{b}\right)^4 + \cdots$$

$$f_1(y) = \frac{y}{2!b}\,y^2 + \frac{1}{3!}\left(\frac{1}{b} + \frac{u}{b}\left(\frac{y}{b}\right)\right)y^3 + \frac{1}{4!}\,\frac{u}{b}\left(\frac{1}{b} + \frac{u}{b}\left(\frac{y}{b}\right)\right)y^4 + \cdots$$

$$g_1(y) = \frac{1}{3!}\,\frac{y}{b}\,y^3 + \frac{1}{4!}\left(2\,\frac{y}{b}\left(\frac{u}{b}\right) + \frac{2}{b}\right)y^4 + \cdots$$

$$f_2(y) = \frac{1}{4!}\,\frac{y}{b}\left(\frac{y}{b}\right)y^4 + \cdots$$

To obtain the conditions which must be satisfied at the switching boundary, let the boundary be defined by $y = h(x)$ for positive x. Symmetry and continuity imply that

$$p(x,h(x)) = p(-x,-h(x)) \tag{3.29}$$

while, in addition, the differential equation must be satisfied on the boundary, i.e.,

$$b \left.\frac{\partial^2 p}{\partial y^2}\right|_{(x,h(x))} = \left[u \frac{\partial p}{\partial y} + y \frac{\partial p}{\partial x} + 2u \ \delta(x,h(x))p \right] \Bigg|_{(x,h(x))} \tag{3.30}$$

where δ represents the Dirac delta function. This equation implies that

$$\frac{\partial^2 p}{\partial y^2} = \frac{2u}{b} \delta(x,h(x)) \ p$$

or

$$\frac{\partial p}{\partial y} (-x,-h(x)) = \frac{2u}{b} p(x,h(x)) - \frac{\partial p}{\partial y}(x,h(x)) \tag{3.31}$$

Since $u = 0$ on the boundary (the result when the step function is interpreted as the limit of symmetric, continuous functions) Equation 3.30 also implies

$$\frac{\partial p}{\partial x} (x,h(x)) = 0 \tag{3.32}$$

Hence Equations 3.29, 3.31 and 3.32 are three conditions which must be satisfied at every boundary point.

In order to find the optimal switching curve, the following numerical method is suggested. First write $\phi(x)$ and $\psi(x)$ in their Taylor series expansions about the origin

$$\phi(x) = \phi_o + \phi_1 x + \frac{\phi_2 x^2}{2} + \frac{\phi_3 x^3}{3!} + \cdots$$

$$\psi(x) = \psi_o + \psi_1 x + \frac{\psi_2 x^2}{2} + \frac{\psi_3 x^3}{3!} + \cdots$$

These series are truncated after 3M (M integer) terms which means that there are a total of 6M unknown constants. If 2M switching boundary points are selected, then Equations 3.29, 3.31 and 3.32 will yield 6M boundary conditions--the number required to solve for the constants. Note that the f_i's and g_i's in Equation 3.28 are known functions and can be computed to any degree of accuracy. Finally, the switching curve is parameterized, and the optimum parameter set is found by a search technique such as steepest descent.

To put the MTV controller in the proper perspective it is necessary to relate it to other control schemes of current interest. In the next section, certain aspects of the second variation and λ-matrix controllers are considered.

3.7 Characteristics of Linear Least Squares Controllers

3.7.1. General characteristics. It is significant that both the second variation and the λ-matrix controllers fall into the general category of linear regulators. In both of these cases, as in the MTV controller, the plant equations are obtained by linearization about a predetermined nominal trajectory:

$$\dot{\underline{X}} = H^*_{\underline{\lambda}_n \underline{X}_n} \underline{X} + H^*_{\underline{\lambda}_n \underline{u}_n} \underline{u} \qquad (3.33)$$

where

(i) H represents the prehamiltonian of the original problem; $\underline{X}_n$ and $\underline{u}_n$ are the nominal state and controls; and $\underline{X}$ and $\underline{u}$ are the deviations of state and control

(ii) $\underline{\lambda}_n$ is the Lagrange multiplier vector

(iii) the * indicates that a quantity is to be evaluated along the nominal trajectory.

In the usual notation, $A(t) = H^*_{\underline{\lambda}_n \underline{X}_n}$ and $b(t) = H^*_{\underline{\lambda}_n \underline{X}_n}$. The performance indices for both control schemes are of the least squares type:

$$\int_0^{t_e} \| \underline{u} \|_R^2 + \| \underline{X} \|_Q^2 + \| \underline{X}\ \underline{u} \|_W \, dt \tag{3.34}$$

where in the second variation case

$$R = H^*_{\underline{u}_n \underline{u}_n} ; \quad Q = H^*_{\underline{X}_n \underline{X}_n} ; \quad W = H^*_{\underline{u}_n \underline{X}_n}$$

and in the λ-matrix case

$$Q \equiv W \equiv 0 \quad ; \quad R = \text{arbitrary positive definite matrix}$$

With the specification of the appropriate terminal boundary conditions, the solution to problems of this type is well known and is completely specified in terms of a gain matrix which satisfies a Riccati differential equation.

There are several characteristics of controllers of this class which would be appropriate to discuss at this point. It will be attempted to structure the problem with a general framework, and for that reason it is necessary to make some rather gross approximations.

First it is assumed that the weighting matrices in Equation 3.34 act in such a way that $\underline{u}$ and $\underline{X}$ are penalized with approximately equal value for a given percentage change in nominal values. That is, if the optimal path lies in a field of neighboring optimal trajectories, then small changes in the state $\underline{X}$ are caused by proportionately small changes in the control $\underline{u}$. It will therefore be assumed that the weighting matrices can be approximated by constants whose values tend to produce the effect described above. Of course an exact analysis would be totally dependent on the individual problem, and justification depends on comparison with particular cases.

Since Equation 3.33 simply represents the linearized plant equations, it is reasonably accurate to use the rotated equations, 3.4. Also, in keeping with the discussion above, the following performance index is defined

$$\int_0^{t_e} \left\{ \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} + \frac{u_1^2}{C_2^2} \right\} dt \qquad (3.35)$$

where the constants C_1 and C_2 are chosen such that $z_1^2(0)$, $C_1 v_1^2(0)$, and $u^2(0)/C_2^2$ are all of equal value. From the theory of the linear regulator problem, the optimal control u_1^* is given by

$$u_1^* = -\frac{1}{2} R^{-1} b^T P \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \qquad (3.36)$$

where the matrix P satisfies the differential equation

$$\dot{P} + PA + A^TP - \frac{1}{2} PbR^{-1} b^TP + 2Q = 0 \qquad (3.37)$$

Here

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad R = \frac{1}{C_2^2} \quad ; \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix}$$

and the boundary conditions on Equation 3.37 depend on the transversality condition of the particular optimization problem. Since for all space missions the value of the terminal time t_e is very large, the matrix P would assume its stationary or asymptotic value through most of the flight. Therefore Equation 3.37 is solved as an algebraic equation by setting $\dot{P} = 0$, and doing this yields the following control law

$$u_1^* = -C_2 x_1 - \sqrt{2C_2 + C_1 C_2^2} \; x_4$$

Therefore the spacecraft state deviations will obey the following differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -C_2 & -\sqrt{2C_2 + C_1 C_2^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \qquad (3.38)$$

It is now possible to analyze the performance of this system in the presence of noise. Using the same noise model described in Section 3.2, the following Langevin equations serve as the appropriate dynamic model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_4 \\ \dot{\xi}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -C_2 & -\sqrt{2C_2 + C_1 C_2^2} & 1 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \\ \xi_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \eta \end{bmatrix} \qquad (3.39)$$

where the new state ξ_1 is the Ornstein-Uhlenbeck process, and η is a zero mean, Gaussian white noise process with variance $2\beta\sigma^2$. According to Fokker-Planck theory, the state of the system described by Equation 3.39 is completely represented by a Gaussian density function whose covariance matrix M satisfies the following differential equation:

$$\dot{M} = DM + MD^T + GG^T \qquad (3.40)$$

where

$$D = \begin{bmatrix} 0 & 1 & 0 \\ -C_2 & -\sqrt{2C_2 + C_1 C_2^2} & 1 \\ 0 & 0 & -\beta \end{bmatrix} ; \quad G = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2\beta}\,\sigma \end{bmatrix}$$

and the boundary condition is the prespecified initial covariance matrix $M(0)$. Since the time necessary for the space flight is very large, the matrix M will necessarily converge to its asymptotic value. Therefore Equation 3.40 is solved by setting $\dot{M} = 0$ and this yields

$$M = \begin{bmatrix} \left(\frac{1}{C_2} + \frac{\sqrt{2C_2 + C_1C_2^2}}{\beta C_2}\right) M_{22} & 0 & \frac{\sqrt{2C_2 + C_1C_2^2}}{\beta} M_{22} \\ 0 & M_{22} & \sqrt{2C_2 + C_1C_2^2}\; M_{22} \\ \frac{\sqrt{2C_2 + C_1C_2^2}}{\beta} M_{22} & \sqrt{2C_2 + C_1C_2^2}\; M_{22} & \sigma^2 \end{bmatrix}$$

where

$$M_{22} = \frac{\sigma^2}{\frac{C_2\sqrt{2C_2 + C_1C_2^2}}{\beta} + 2C_2 + C_1C_2^2 + \beta\sqrt{2C_2 + C_1C_2^2}}$$

The values of M_{11} and M_{22} which represent the variances of the spacecraft position deviations and velocity deviations, respectively, are plotted in Figure 23 as a function of the correlation time of the ξ_1 process. Examination of the results reveals that significant steady state deviations build up when using these guidance strategies. An intuitive picture of the mechanism which causes this effect is fairly easy to construct. Namely, when the spacecraft deviates from the nominal trajectory, the new optimal thrust program assumes a form such that the course of the vehicle is corrected in a relatively gradual fashion. This is a logical strategy in a deterministic system where there is no noise present to produce any further deviations. In the stochastic system of interest here, though, the policy of slowly correcting the course of the space vehicle only gives the noise more time to build up

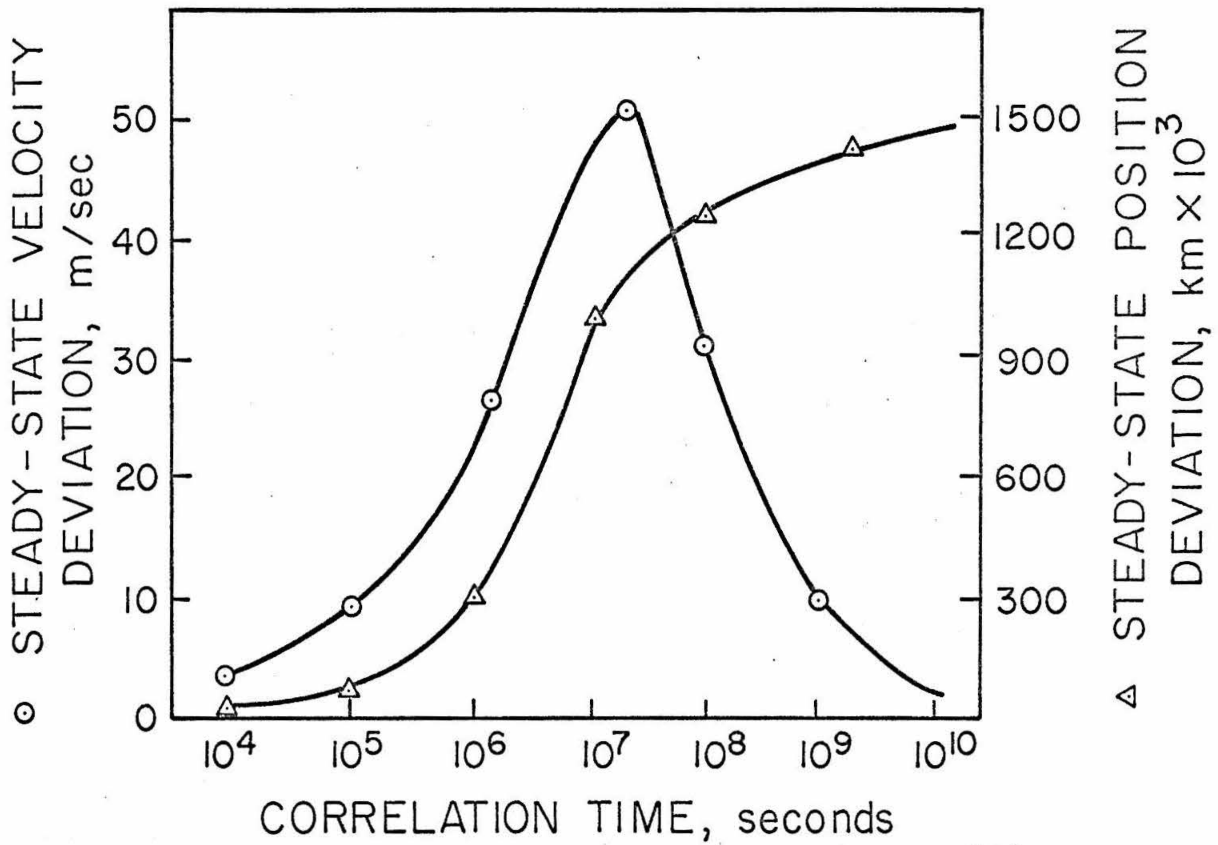


Figure 23. Steady state position and velocity deviations vs. OU process correlation time (least squares guidance system): $z_1(0) = 1.5 \times 10^{11}$m; $v_1(0) = 3 \times 10^4$m/sec; $u_n(0) = 10^{-3}$m/sec^2; $C_1 = .25 \times 10^{14}$; $C_2 = .666 \times 10^{-14}$.

larger deviations. This effect builds upon itself until the steady state deviations are attained.

The value of finding these steady state deviations is related to an important property of a linear regulator with a fixed terminal boundary condition. It is the characteristic of such systems to delay the major control effort until near the terminal time, so as to insure that the terminal conditions are satisfied. The magnitude of this effort will be directly related to the errors near the terminal time, which in turn are indicated by the steady state deviations found above. Hence, the larger the steady state errors, then the larger will be the average terminal control variable deviations which are necessary to correct these inaccuracies. Indeed, there are cases when the control variables will almost always be unbounded at the terminal time; see, for instance, Reference (65).

In contrast the MTV controller must guard against the eventuality of building up large errors which, because of the bounded control levels, it has no chance to eliminate. In that system, intuition agrees with mathematics in calling on all the available control all of the time in order to combat the noise inputs to the system. Specific examples will now be given which illustrate the phenomena discussed above.

<u>3.7.2. Analysis of the neighboring optimal guidance system accuracy for a constant acceleration, minimum time Mars rendezvous mission.</u> In this section the covariance of the state deviations is determined numerically for a guidance system obtained by means of the

second variation optimization technique. The first step is to obtain the nominal, or open loop trajectory, and for that purpose a constant acceleration level of $.78 \times 10^{-3}$ meters/second^2 is assumed, which corresponds to a 3 ounce thrust applied to a 2500 pound space vehicle. Since the minimum time Mars rendezvous is a free terminal time problem, it is convenient to use the analytical artifice of normalized time to convert the free terminal time problem into the more usual fixed time problem. This is done by defining

$$t = t_e \tau \qquad \tau \in [0,1] \tag{3.41}$$

where t is the true time and τ is the normalized time. Here t_e represents the unknown terminal time which is treated as a state variable by adjoining its dynamical equation

$$\dot{t}_e = 0$$

It is easily seen that

$$\frac{dx}{d\tau} = t_e \frac{dx}{dt}$$

and thus it is possible to consider the following equivalent dynamical system (dots indicate derivatives with respect to τ):

$$\dot{u} = \left(\frac{v^2}{r} - \frac{\mu}{r^2} - a \sin \hat{\alpha}\right) t_e$$

$$\dot{v} = \left(- \frac{uv}{r} + a \cos \hat{\alpha}\right) t_e$$

$$\dot{r} = u t_e$$

$$\dot{\theta} = \frac{v t_e}{r}$$

$$\dot{t}_e = 0$$

where, referring to Figure 2,

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$r = \sqrt{z_1^2 + z_2^2}$$

$$\theta = \beta$$

$$\hat{\alpha} = \alpha - \pi/2$$

$$\mu = GM_s$$

Forming the Hamiltonian of the optimization problem, and carrying out its minimization with respect to $\hat{\alpha}$ results in the canonic differential equations for the system state variables and Lagrange multipliers:

$$\dot{u} = \left(\frac{v^2}{r} - \frac{\mu}{r^2} - \frac{a\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} \right) t_e$$

$$\dot{v} = \left(- \frac{uv}{r} - \frac{a\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \right) t_e$$

$$\dot{r} = ut_e$$

$$\dot{\theta} = \frac{vt_e}{r}$$

$$\dot{t}_e = 0$$

$$\dot{\lambda}_1 = \frac{\lambda_2 vt_e}{r} - \lambda_3 t_e$$

$$\dot{\lambda}_2 = -\frac{2\lambda_1 t_e v}{r} + \frac{\lambda_2 t_e u}{r} - \frac{\lambda_4 t_e}{r}$$

$$\dot{\lambda}_3 = \frac{\lambda_1 v^2 t_e}{r^2} - \frac{2\mu\lambda_1 t_e}{r^3} - \frac{\lambda_2 vu t_e}{r^2} + \frac{\lambda_4 v t_e}{r^2}$$

$$\dot{\lambda}_4 = 0$$

$$\dot{\lambda}_5 = -1 - \lambda_1\left(\frac{v^2}{r} - \frac{\mu}{r^2} - \frac{a\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\right)$$

$$- \lambda_2\left(-\frac{uv}{r} - \frac{a\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}\right) - \lambda_3 u - \frac{\lambda_4 v}{r} \qquad (3.42)$$

The given boundary conditions and the transversality conditions associated with the optimization problem yield the boundary values necessary for the solution of Equation 3.42:

$$u(0) = 0$$

$$u(1) = 0$$

$$v(0) = v_{earth}$$

$$v(1) = v_{mars}$$

$$r(0) = r_{earth}$$

$$r(1) = r_{mars}$$

$$\theta(0) = \theta_{earth}(0)$$

$$\theta(1) = \theta_{mars}(1)$$

$$\lambda_5(0) = 0$$

$$\lambda_5(1) = -\lambda_4(1)\, \dot{\theta}_{mars} \qquad (3.43)$$

Using the quasilinearization method, a solution of Equation 3.42 is determined which satisfies the required boundary conditions.

The second variation (or neighboring optimal) guidance system is now considered. This method is equivalent to linearizing Equations 3.42 and using these to approximate the behavior of the system. Therefore suppose that at time τ there are known state variable deviations equal to $\text{col}[\delta u(\tau), \delta v(\tau), \delta r(\tau), \delta\theta(\tau)]$, and that it is desired to determine the manner in which these errors are nulled in the optimum system. From the theory of linear differential equations, the following relationship must hold

$$\Phi(1,\tau)\begin{bmatrix} \delta u(\tau) \\ \delta v(\tau) \\ \delta r(\tau) \\ \delta\theta(\tau) \\ \delta t_e(\tau) \\ \delta\lambda_1(\tau) \\ \delta\lambda_2(\tau) \\ \delta\lambda_3(\tau) \\ \delta\lambda_4(\tau) \\ \delta\lambda_5(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{\theta}_{mars}\delta t_e(1) \\ \delta t_e(1) \\ \delta\lambda_1(1) \\ \delta\lambda_2(1) \\ \delta\lambda_3(1) \\ \delta\lambda_4(1) \\ -\delta\lambda_4(1)\,\dot{\theta}_{mars} \end{bmatrix} \tag{3.44}$$

where Φ is the (10×10) fundamental matrix of the linearized equations which satisfies

$$\dot{\Phi} = A(\tau)\,\Phi \qquad\qquad \Phi(0) = I$$

Here A is the Jacobian matrix of Equations 3.42, evaluated along the nominal state and Lagrange multiplier vectors. Since Equations 3.44 are 10 linear equations in 10 unknowns, it is straightforward to determine the unknown initial values in the following form

$$\begin{bmatrix} \delta t_e(\tau) \\ \delta\lambda_1(\tau) \\ \delta\lambda_2(\tau) \\ \delta\lambda_3(\tau) \\ \delta\lambda_4(\tau) \end{bmatrix} = B(\tau) \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \end{bmatrix}$$

where $B(\tau)$ is a 5×4 matrix (note that $\delta\lambda_5(\tau) = 0$). Hence

$$\begin{bmatrix} \delta\dot{u} \\ \delta\dot{v} \\ \delta\dot{r} \\ \delta\dot{\theta} \end{bmatrix} = [A^{(1)}(1,\tau) + A^{(2)}(1,\tau)\, B(\tau)] \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \end{bmatrix} \qquad (3.45)$$

where $A^{(1)}$ is defined to be the first four terms of the first four rows of the A matrix, and $A^{(2)}$ represents the fifth through the ninth terms of the first four rows of the A matrix. Equation 3.45 therefore represents the differential equation satisfied by the state variable deviations.

In order to complete the analysis, the noise terms representing the attitude and thrust level variations must be adjoined to the system. The result is analogous to Equation 3.4:

$$\dot{\underline{X}} = \hat{A}\underline{X} + G \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{3.46}$$

where $\underline{X} \triangleq \text{col}[\delta u, \delta v, \delta r, \delta\theta, \xi_1, \xi_2]$

$$\hat{A} = \begin{bmatrix} \begin{bmatrix} \\ A^{(1)} + A^{(2)} B \\ \\ \end{bmatrix} & \begin{matrix} \cos\hat{\alpha} & -\sin\hat{\alpha} \\ \sin\hat{\alpha} & \cos\hat{\alpha} \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{matrix} \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sqrt{2\beta_1\sigma_1} & 0 \\ 0 & \sqrt{2\beta_2\sigma_2} \end{bmatrix}$$

and η_1 and η_2 are independent, zero mean, Gaussian, white noise processes with unit variances. The covariance matrix M of the state deviations at time τ can be shown to satisfy the following differential equation:

$$\dot{M} = \hat{A}M + M\hat{A}^T + GG^T \qquad M(0) = M_o \tag{3.47}$$

The state deviation time histories can be obtained, in principle, using Equation 3.47. However, it has been found that the matrix

B of Equation 3.46 has badly behaved elements at two points along the trajectory under study, and for this reason the complete history of the covariance matrix cannot be obtained using this method. But since it is of primary interest to see how the errors build up, it would be of value to compute the covariances up to the first singularity (about 25% into the mission). For that purpose four cases were considered:

Case I. $\sigma_1 = \sigma_2 = .0013$ meters/second^2 = 1% u ; all cases

$\beta_1 = \beta_2 = (.5 \text{ second})^{-1}$

velocity error = .027 meters/second

position error = 165 kilometers

Case II. $\beta_1 = \beta_2 = (30 \text{ seconds})^{-1}$

velocity error = .21 meters/second

position error = 1350 kilometers

Case III. $\beta_1 = \beta_2 = (30 \text{ minutes})^{-1}$

velocity error = 1.5 meters/second

position error = 10000 kilometers

Case IV. $\beta_1 = \beta_2 = (100 \text{ hours})^{-1}$

velocity error = 21 meters/second

position error = 135000 kilometers

These results verify the build-up of errors which was predicted by the rough analysis given in the last section. In the following discussion the MTV controller is actually compared to a realization of the λ-matrix control scheme.

3.7.3 Comparison of λ-matrix control with minimum terminal variance control. As discussed in previous sections, the λ-matrix control scheme yields a linear feedback control law. In the process of making trajectory corrections, the λ-matrix controller minimizes the mean square deviations of the control variations from their nominal values.

In order to derive the control gains it is necessary that the plant equations be linearized to yield a differential system in the form (i.e., Equation 3.19)

$$\dot{\underline{X}} = A(t)\,\underline{X} + b(t)\,\underline{u} \tag{3.48}$$

where $\underline{X}$ is an n-dimensional state deviation vector, and $\underline{u}$ is an m-dimensional control deviation vector. For the deterministic low-thrust guidance problem, $n = 6$ and $m = 3$. The performance index to be minimized is of the form

$$\int_0^{t_e} \underline{u}^T R\,\underline{u}\,dt \tag{3.49}$$

with given initial conditions $\underline{X}(0) = \underline{C}$ and given terminal conditions $\underline{X}(t_e) = \underline{0}$. The control function for this problem can be shown to satisfy

$$\underline{u} = -\frac{1}{2} R^{-1} b^T G^{-1} \underline{X} \tag{3.50}$$

where G is an $n \times n$ matrix which is the solution of

$$\dot{G} = GA^T + AG - \frac{1}{2} b R^{-1} b^T \; ; \quad G(t_e) = 0 \tag{3.51}$$

Note that the gain matrix G^{-1} is unbounded at the terminal time t_e .

3.7.4. Monte Carlo simulation of the MTV and λ-matrix control systems. In order to compare the λ-matrix controller to the MTV controller, the Mars minimum time rendezvous trajectory described in Section 3.7.2 is used as a trial mission. Since the MTV controller is decidedly nonlinear, it is very difficult to obtain the covariance equations for the state deviations, which are easily found in the case of linear systems such as second variation or λ-matrix systems. For this reason it is necessary to use the Monte Carlo simulation technique. The noise model for the thrust acceleration and thrust angle disturbance processes is taken as a sequence of 10000 independent, Gaussian random variables with zero mean and variances set at 3% of acceleration and 30 milliradians of control angle. This process approximates an OU noise process with the same variance and a correlation time of Δ seconds, where Δ is chosen so that each control variable has 10,000 switching opportunities. Errors at injection are assumed to be 8.4 meters/second in velocity, and 2700 kilometers in position.

Based on the optimization studies in Section 3.5, the MTV controller is chosen to have no zero control region and a switching curve constant a equal to .25 . Three control configurations are to be investigated, namely ($u = .78 \times 10^{-3}$ meters/second^2):

(i) $u_i(t) \; \varepsilon \left[- .05 \, u\left(\frac{t_e - .8t}{t_e}\right) \; , \; .05 \, u\left(\frac{t_e - .8t}{t_e}\right) \right]; \quad i = 1,2$

(ii) $u_i(t) \; \varepsilon \; [- .02u, .02u] \; ; \quad i = 1,2$

(iii) $u_i(t) \; \varepsilon \left[- .02u \left(\frac{t_e - .8t}{t_e}\right) \; , \; .02u \left(\frac{t_e - .8t}{t_e}\right) \right]; \quad i = 1,2$

Hence, in the first and third cases, the control levels are time varying. Also note that in the third case the magnitude of the process noise is more than seven times larger than the control force at the encounter time t_e . The simulation results are presented in Figures 24-31. In case (i), Figures 24-25, it is seen that the injection errors are nulled rapidly and state deviations are kept small over the entire trajectory, thus showing that the varying control level has little effect in this instance. For case (ii), Figures 26-27, there is slower damping of injection errors because of the reduced control force, and the approximation errors resulting from the rapid turning of the thrust vector at mid-trajectory are evident. However, the mission accuracy is still very high. Under the extreme conditions hypothesized in case (iii), Figures 28-29, it is found that remarkable accuracy is achieved in spite of the overpowering magnitude of the noise in comparison with the available control.

For each case the number of control variable switchings is given in the appropriate figures. The decrease in this number as control level decreases is indicative of the loss of control "tightness," and

Fig.24

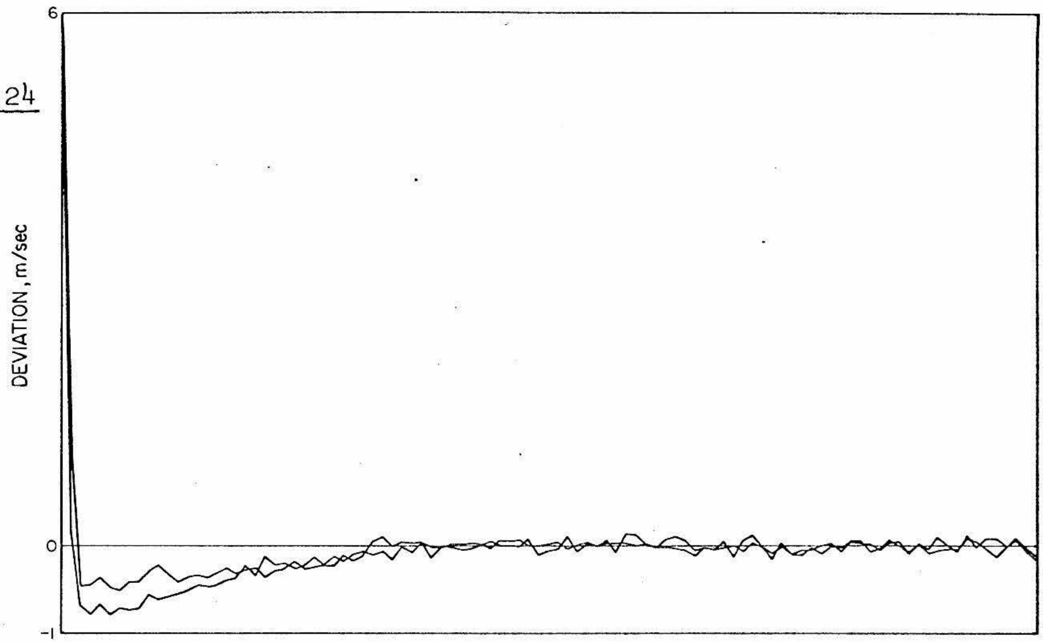
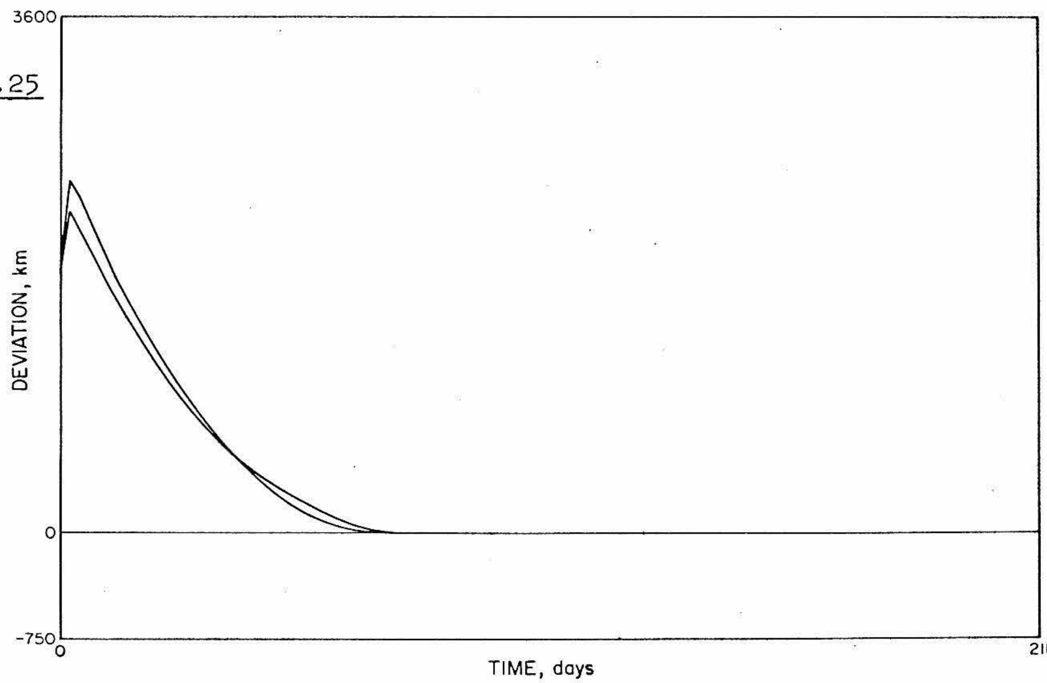


Fig.25



Figures 24 and 25. Velocity (24) and position (25) deviations for MTV guidance system with control configuration (i): u_1 switchings = 3187; $\delta\alpha$ switchings = 3320 .

Fig.26

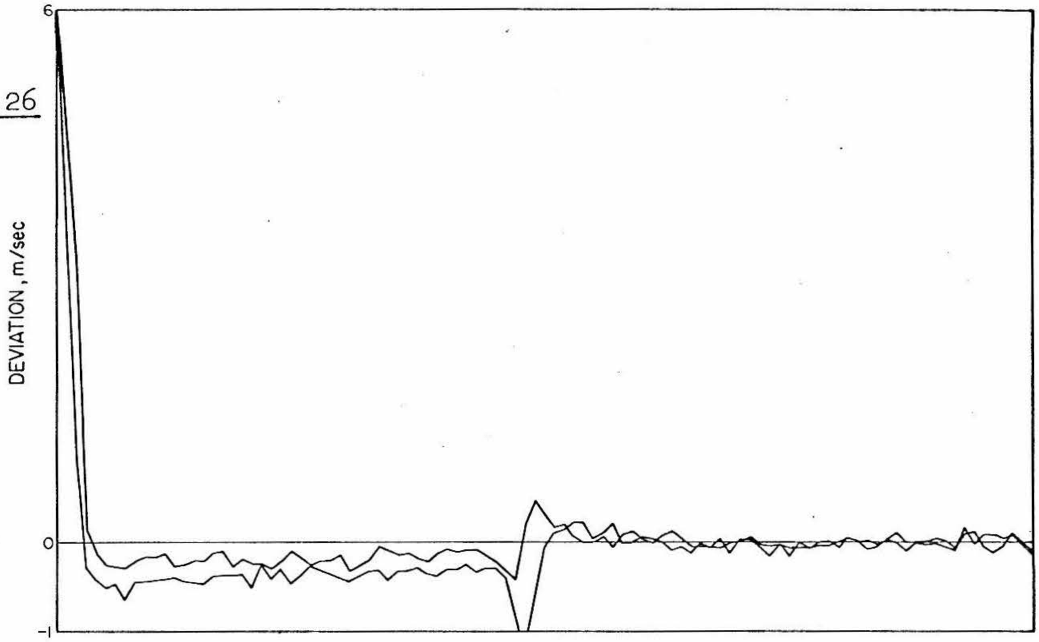
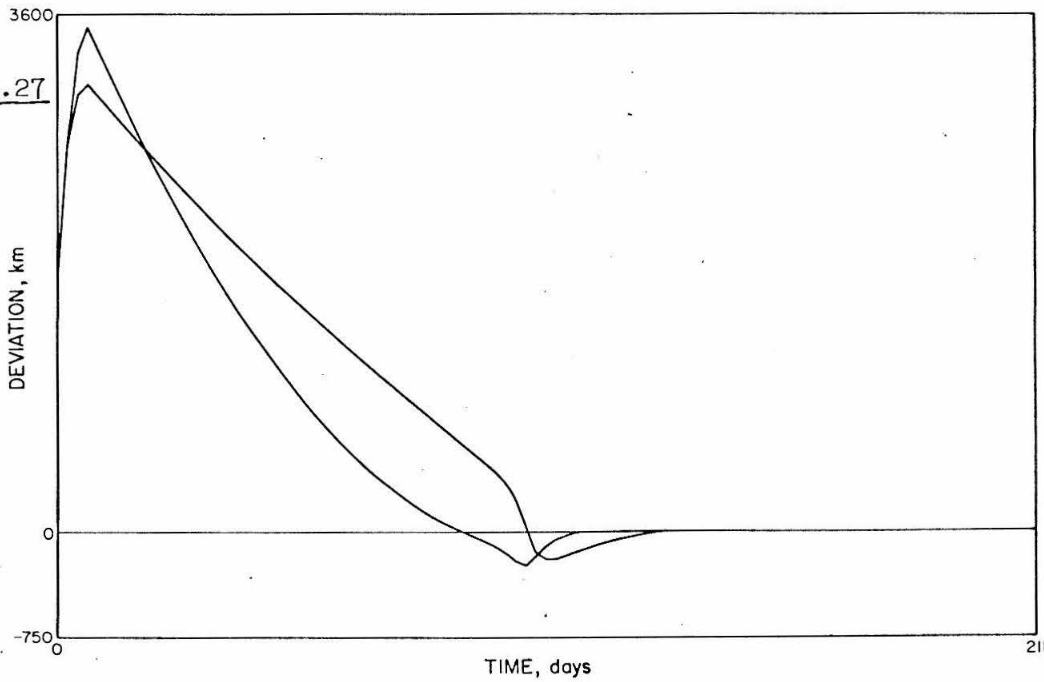


Fig.27



Figures 26 and 27. Velocity (26) and position (27) deviations for MTV guidance system with control configuration (ii): u_1 switchings = 2721; $\delta\alpha$ switchings = 2850 .

Fig.28

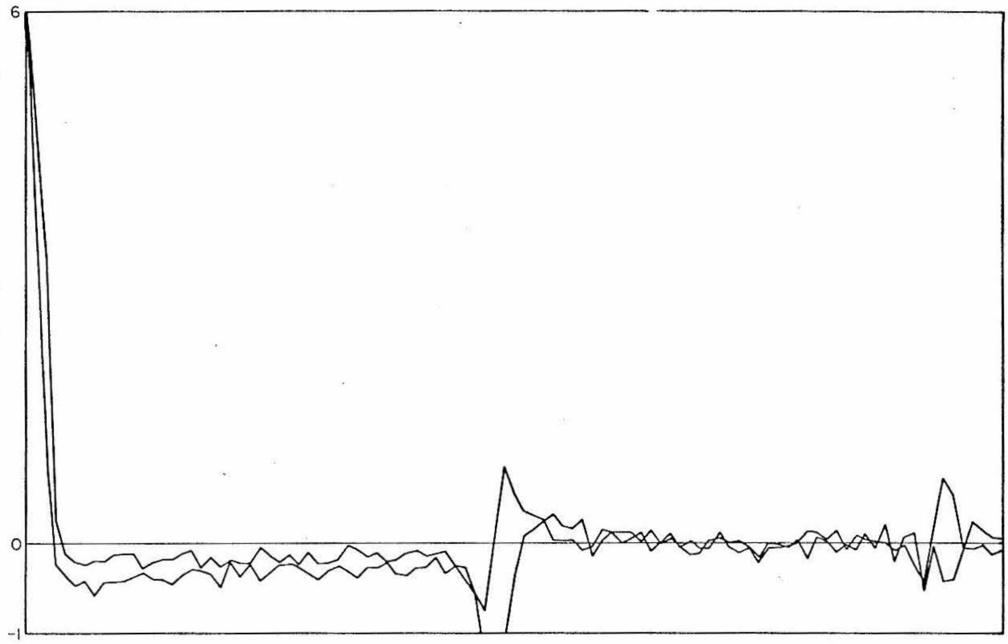
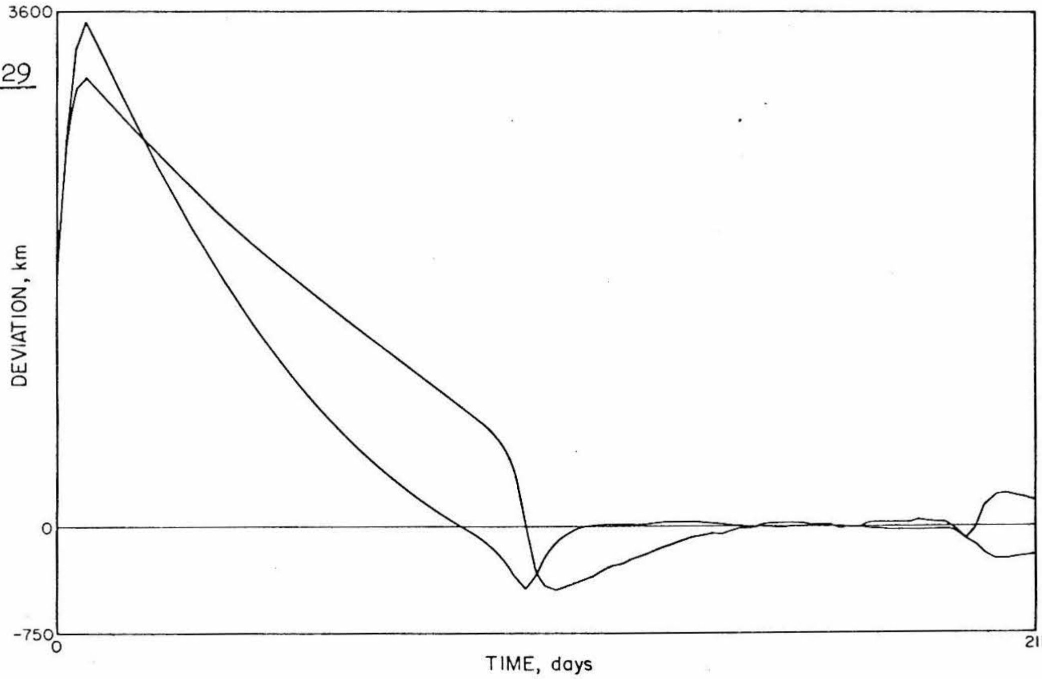


Fig.29



Figures 28 and 29. Velocity (28) and position (29) deviations for MTV guidance system with control configuration (iii): u_1 switchings = 1832; $\delta\alpha$ switchings = 1967.

Fig.30

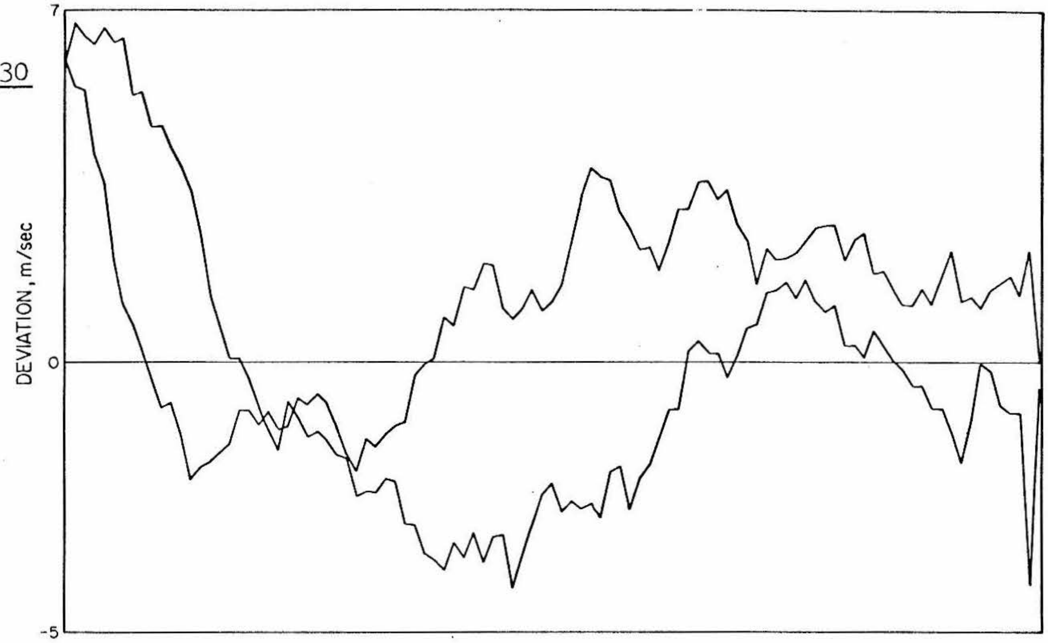
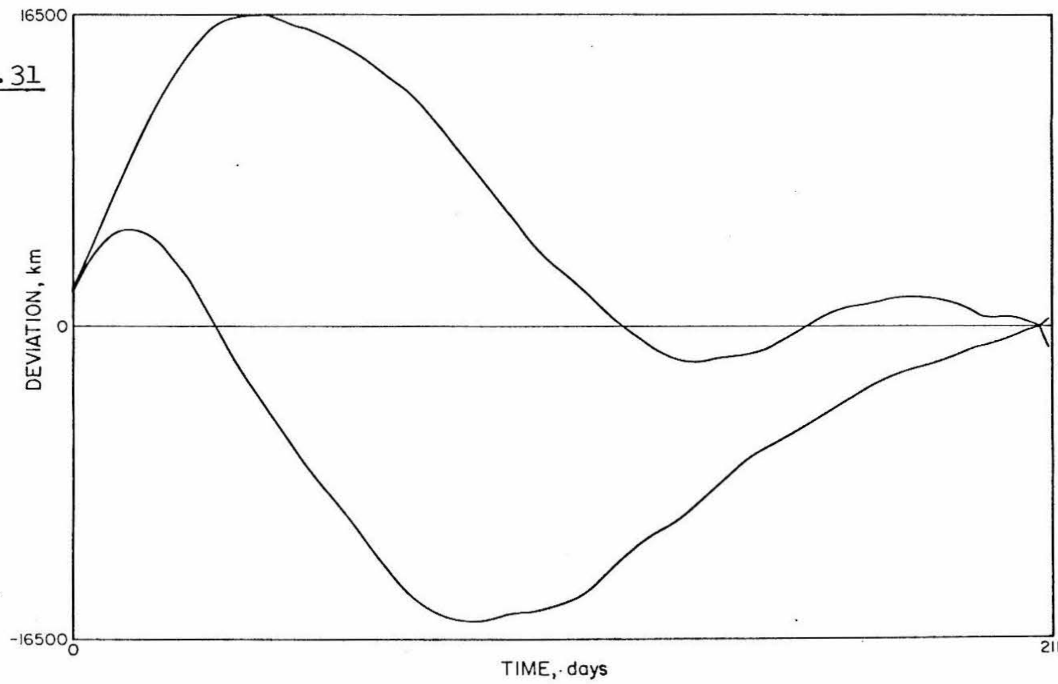


Fig.31



Figures 30 and 31. Velocity (30) and position (31) deviations for λ-matrix guidance system.

is in good agreement with the computed switching probabilities shown in Figure 13.

Figures 30-31 indicate that the λ-matrix controller allows the state deviations to wander much more freely than does the MTV controller. In addition the λ-matrix controller demands a 30% maximum thrust deviation and a 14° maximum angle deviation near the final time, which amounts to a relatively large control effort. These effects occur just as predicted by the analysis in Section 3.7.1.

Since the MTV controller has bounded control variables, it has the tendency to keep the state deviations as low as possible at all times. In effect the controller anticipates that there will not be a large control capability near the final time and takes appropriate action to accomplish guidance maneuvers whenever control becomes available.

In the following section the technique of MTV control will be extended to the case when the state of the system is not known exactly, but must be estimated.

3.8 Minimum Terminal Variance Control with State Estimation

In this section the MTV controller will be discussed for the general case of linear plant and linear observation equations. The plant equations are therefore represented by

$$\dot{\underline{X}} = A(t)\,\underline{X} + b(t)\,\underline{u} + \underline{\xi} \qquad (3.52)$$

where $\underline{X}$ is the n-dimensional state deviation vector, $\underline{u}$ is the m-dimensional control deviation vector, and $\underline{\xi}$ is the Gaussian white noise disturbance vector. Owing to the disturbance processes, the state $\underline{X}$ may contain variables which augment the deterministic state vector. This occurs, for example, when the Ornstein-Uhlenbeck process is used as a stochastic model.

Linearization about the nominal trajectory is assumed to yield an adequate approximation to the observation equations, which then become

$$\underline{y} = H(t)\,\underline{X} + \underline{\eta} \tag{3.53}$$

where $\underline{y}$ is the p-dimensional observation vector, and $\underline{\eta}$ is the Gaussian white noise error vector. The following covariance matrices are assumed to be known:

$$\left.\begin{aligned} E[\,\underline{\eta}(\tau)\,\underline{\eta}(t)^T] &= R(t)\,\delta(\tau-t) \\ E[\,\underline{\xi}(\tau)\,\underline{\eta}^T(t)] &= S(t)\,\delta(\tau-t) \\ E[\,\underline{\xi}(\tau)\,\underline{\xi}^T(t)] &= Q(t)\,\delta(\tau-t) \end{aligned}\right\} \tag{3.54}$$

The problem is to choose a control law $\underline{k}$ which is a function of past observations and past control inputs; i.e.,

$$\underline{u}(t) = \underline{k}[y(s),\ 0 \le s \le t\ ;\quad u(s),\ 0 \le s < t];\ 0 \le t \le t_e \tag{3.55}$$

and this control law must minimize

$$E[\,\sum_i w_i x_i^2(t_e)] \tag{3.56}$$

where E is the expectation over both the $\underline{\xi}$ and $\underline{\eta}$ processes.

Since $\underline{u}(s)$, $0 \leq s < t$, is known, the information in brackets in Equation 3.55 can be summarized by the Kalman estimate $\underline{\mu}$ and error covariance Γ, which satisfy the following differential equations

$$\dot{\underline{\mu}} = (A - SR^{-1}H - \Gamma H^TR^{-1}H)\,\underline{\mu} + b\underline{u}(t) + (SR^{-1} + \Gamma H^TR^{-1})\,(H\underline{X} + \underline{n})$$

$$\dot{\Gamma} = -\Gamma H^TR^{-1}H\Gamma + (A - SR^{-1}H)\Gamma + \Gamma(A^T - H^TR^{-1}S^T) + (Q - SR^{-1}S^T) \tag{3.57}$$

Thus $\underline{\mu}$ is a sufficient statistic for the posterior density of $\underline{X}$ which means that Equation 3.55 becomes

$$\underline{u}(t) = \underline{k}(t, \underline{\mu}(t)) \tag{3.58}$$

Hence

$$\dot{\underline{X}} = A\underline{X} + b\underline{k}\,(t, \underline{\mu}) + \underline{\xi} \tag{3.59}$$

$$\dot{\underline{\mu}} = (A - SR^{-1}H - \Gamma H^TR^{-1}H)\underline{\mu} + b\underline{k}(t, \underline{\mu})$$

$$+ (SR^{-1} + \Gamma H^TR^{-1})(H\underline{X} + \underline{n})$$

Following Wonham (60), the error process $\underline{z}$ is defined as

$$\underline{z} = \underline{X} - \underline{\mu} \tag{3.60}$$

Using Equation 3.59, the $\underline{z}$ process is found to satisfy

$$\dot{\underline{z}} = (A - KH)\,\underline{z} + \underline{\rho} \tag{3.61}$$

where

$$K = SR^{-1} + \Gamma H^TR^{-1}$$

$$\underline{\rho} = \underline{\xi} - K\,\underline{n} \tag{3.62}$$

Also the following equation is readily established

$$\dot{\underline{\mu}} = A\underline{\mu} + KH\underline{z} + b\underline{k}(t,\underline{\mu}) + K\underline{\eta} \tag{3.63}$$

By direct calculation

$$E[\underline{\rho}\,\underline{\rho}^T] = L\delta(\tau - t)$$

$$E[\underline{\rho}(K\underline{\eta})^T] = J\delta(\tau - t) \tag{3.64}$$

where

$$L = Q - SR^{-1}S^T + \Gamma H^T R^{-1} H\Gamma$$

$$J = -\Gamma H^T K^T \tag{3.65}$$

If $\overline{p}(t,\underline{z},\underline{\mu};\ s,\underline{v},\underline{r};\ \underline{k})$ is defined to be the transition probability density function of the $\begin{bmatrix} \underline{z} \\ \underline{\mu} \end{bmatrix}$ process with control law $\underline{k}(\underline{\mu},t)$, it can be shown (60) that

$$\overline{p}(t,\underline{z},\underline{\mu};\ s,\underline{v},\underline{r};\ \underline{k}) = q(t,\underline{z};\ s,\underline{v})\ \overline{q}(t,\underline{\mu};\ s,\underline{r};\ \underline{k}) \tag{3.66}$$

where q satisfies

$$\frac{\partial q}{\partial s} = \frac{1}{2} L \circ \frac{\partial^2 p}{\partial \underline{v}^2} - \left\langle (A-KH)\ \underline{z},\ \frac{\partial p}{\partial \underline{v}} \right\rangle - [tr(A - KH)]p \tag{3.67}$$

and $\overline{q}$ satisfies

$$\frac{\partial \overline{q}}{\partial s} = \frac{1}{2}(KRK^T) \circ \frac{\partial^2 \overline{q}}{\partial \underline{r}^2} - \left\langle A\underline{r} + b\underline{k}(s,\underline{r}), \frac{\partial \overline{q}}{\partial \underline{r}} \right\rangle - \left(tr\ A + b \circ \frac{\partial \underline{k}(s,\underline{r})}{\partial \underline{r}}\right) \overline{q} \tag{3.68}$$

Equation 3.56 can now be expanded as follows

$$(W \triangleq \begin{bmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{bmatrix}) :$$

$$E[\sum_i w_i x_i^2(t_e)] \triangleq E[\underline{X}^T W \underline{X}_{t_e}] = E[(\underline{r}+\underline{v})^T W(\underline{r}+\underline{v})]$$

$$= \int_{\underline{r}} \int_{\underline{v}} (\underline{r}^T W \underline{r} + \underline{v}^T W \underline{r} + \underline{r}^T W \underline{v} + \underline{v}^T W \underline{v})\, q(0,\underline{z};\, t_e,\underline{v})\, \overline{q}(0,\underline{\mu};t_e,\underline{r};\underline{k}) d\underline{r} d\underline{v}$$

$$= \int \underline{r}^T W \underline{r}\, d\underline{r} + 2\left[\int \underline{r}\, \overline{q}\, d\underline{r}\right]^T W \left[\int \underline{v}\, q\, d\underline{v}\right] + \int \underline{v}^T W \underline{v}\, q\, d\underline{v} \qquad (3.69)$$

The second term in Equation 3.69 is zero because the mean of the error process is always zero. Also, the third term is fixed and independent of the control law $\underline{k}$. Hence the minimization depends only on the first integral which in turn depends solely on $\overline{q}$. The stochastic differential equation described by Equation 3.68 can be deduced to be the following

$$\dot{\underline{\mu}} = A\, \underline{\mu} + b\underline{k}(t,\underline{\mu}) + \hat{\underline{\xi}} \qquad (3.70)$$

where

$$E[\hat{\underline{\xi}}(\tau)\, \hat{\underline{\xi}}^T(t)] = KRK^T\, \delta(\tau - t)$$

This equation is equivalent to the Kalman estimation equation if the forcing term associated with the observations is viewed as an equivalent white noise process with covariance KRK^T. Hence the problem is now in the same form as the known state case, except that the noise term has a different covariance; and therefore the same techniques can be applied to solve this problem. However it must be realized that "separation" does not hold for the MTV system, even though a Kalman filter may be used in the control loop. This is because the control law will depend on and will change with the type of observations used.

To illustrate the effect of observations, consider the plot in Figure 32 of the distribution of the uncontrolled terminal variance between $E[\underline{\mu}^T(t_e)W\underline{\mu}(t_e)]$ and $tr[W\Gamma(t_e)]$ as a function of the "goodness" of the observations. The measure of the "goodness" of the observations is rather arbitrary since it will be inherently dependent on the number, type, and accuracy of the observations made. However, in this case "goodness" is chosen to be a linear function of $tr[W\Gamma(t_e)]$ where perfect observations nullify this error, and no observations maximize it. From Equation 3.69 it is found that

$$E[\underline{X}(t_e)^T W \underline{X}(t_e)] = E[\underline{\mu}^T(t_e) W \underline{\mu}(t_e)] + tr[W\Gamma(t_e)] \tag{3.71}$$

Hence the sum of the uncontrolled estimate and the error covariances must be constant.

Now referring to Figure 32, it is seen that perfect observations result in the largest value of $E[\underline{\mu}(t_e) W\underline{\mu}(t_e)]$, and hence the largest equivalent noise on Equation 3.70. In a sense, this means that the case of known state represents the hardest eventuality for the MTV controller. The dotted curves in Figure 32 are used to indicate the potential reductions in the terminal variances using MTV control action.

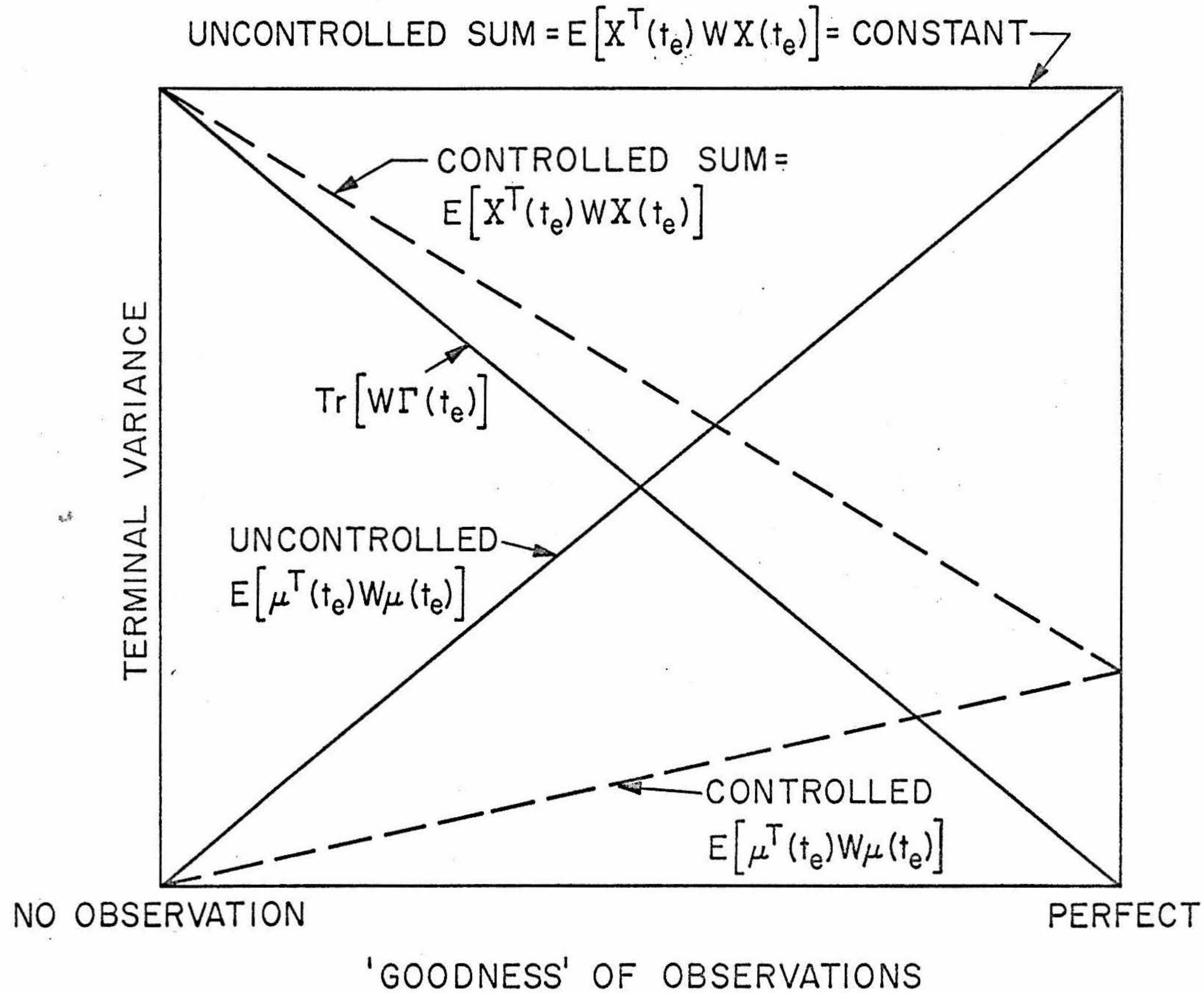


Figure 32. Terminal variance vs. "goodness" of observations

Since the control does not affect $tr[W\Gamma(t_e)]$, this quantity represents the upper bound on terminal accuracy. Note that in general a bounded controller will not be able to completely nullify $E[\underline{\mu}(t_e)\, W\underline{\mu}(t_e)]$ in a noisy (observed) system. It will be recalled that this fact was used to justify the MTV criterion from the outset.

The design and optimization of the observations could constitute a study in itself, and thus only the limiting case of perfect observations is considered. However this case represents the greatest challenge from the MTV controller point of view. The composite MTV control system is illustrated in Figure 33.

In the final part of this chapter, the full nonlinear MTV problem is investigated, and the difficulties still existing in stochastic optimization problems are illustrated.

3.9 Combined Navigation and Guidance of the Interplanetary Vehicle

In this section, the state vector $\underline{X}$ represents both the deviation of the position and velocity components from their nominal values, and also the OU process models for the dynamic noise. The differential equations for the state deviations can be written in the general form

$$\dot{\underline{X}} = \underline{f}(\underline{X},\underline{u},t) + \sum_k F_{ik}\, \xi_k \qquad (3.72)$$

where the state $\underline{X}$ is an n-dimensional state deviation vector, $\underline{u}$ is an m-dimensional control deviation vector, and $\underline{\xi}$ is a vector-valued Gaussian white noise disturbance process with independent components. The p observations $\underline{y}$ are given by

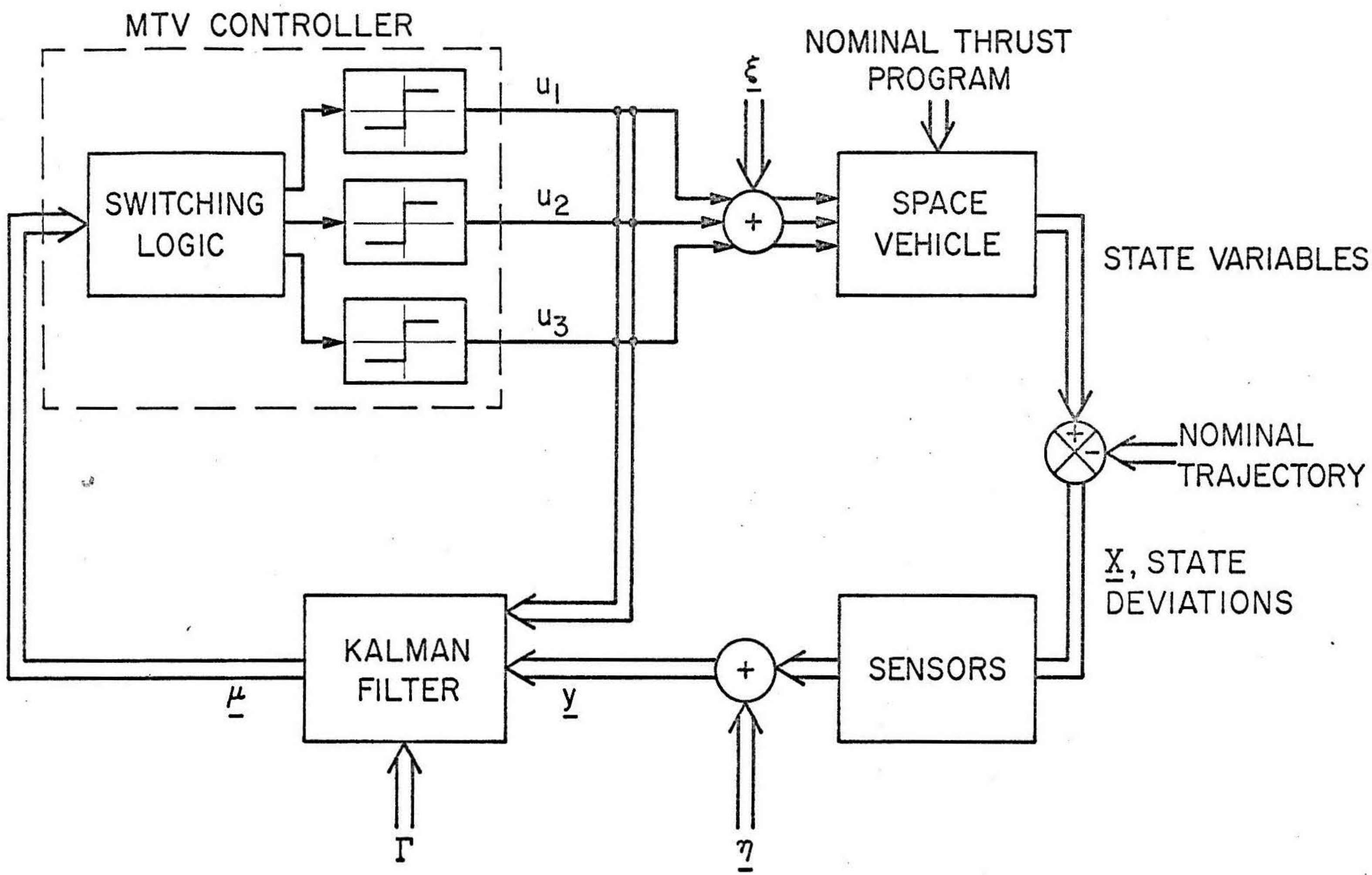


Figure 33. The composite MTV guidance system

$$\underline{y} = \underline{h}(\underline{X},t) + \underline{\eta}(t) \tag{3.73}$$

where $\underline{h}$ is the p-dimensional observation function, and $\underline{\eta}$ is a vector-valued Gaussian white noise process. If the symbol $P(a,t|t)$ is used to represent the conditional probability density function of the state $\underline{a}$ at time t based on the observations up to time t, then this function can be shown (66) to satisfy a stochastic, partial differential, integral, functional equation of the form:

$$P(\underline{a},t+dt|t+dt) - P(a,t|t) = dV(\underline{a},t) - \sum_{i=1}^{n} (\overline{df}_i(\underline{a},\underline{u},t) \;\; P(\underline{a},t|t))_{a_i}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} (\overline{Q}_{ij}(\underline{a}) \cdot P(\underline{a},t|t))_{a_i a_j} \; dt \tag{3.74}$$

where

$$\overline{d\underline{f}}(\underline{a},\underline{u},t) = \underline{f}(\underline{a},\underline{u},t)\; dt + FSR^{-1}(\underline{y}-\underline{h}(\underline{a},t))\; dt$$

$$\overline{Q} = Q - FSR^{-1}(FS)^T$$

$$dV(\underline{a},t) = P(\underline{a},t|t) \;\; (\underline{y}dt - E_t\underline{h}(\underline{a},t)dt)^T \; R^{-1}(\underline{h}(\underline{a},t) - E_t(\underline{h}(\underline{a},t))$$

$$Q = FF^T \; ; \quad E_t[\underline{\eta}\,\underline{\eta}^T] = R \; ; \quad E_t[\underline{\xi}\underline{\eta}^T] = S$$

and E_t represents the conditional expectation using $P(a,t|t)$. Indeed, Equation 3.74 is equivalent to Equations 3.57 when the plant and observations are linear.

The control problem is to find the optimal feedback controller, $\underline{u}(t,P(a,t|t))$, such that

$$\hat{E}E_{t_e}\left[\sum_k x_k^2 \, \ell_k \right] \tag{3.75}$$

is minimized subject to control variable limitations. Here $\hat{E}$ represents the expectation over the observation process and the ℓ_k are suitable weighting factors.

Following Detchmendy (67), the value function is defined as

$$V[m_1(t), m_2(t), \cdots; t] = \min_{\underline{u}} \hat{E}E_{t_e}\left[\sum_k \ell_k x_k^2 \middle| m_1(t), m_2(t) \cdots \right] \tag{3.76}$$

where the m_i are the moments of $P(\underline{a}, t|t)$. Note that for dimensions greater than one, the m_i will be multivariate. Using the principle optimality, Equation 3.76 is expanded as follows:

$$V[m_1(t), m_2(t) \cdots; t] = \min_{\underline{u}} \hat{E}[V(m_1(t+\Delta), m_2(t+\Delta), \cdots; t+\Delta]$$

$$= \min_{\underline{u}} \hat{E}[V(m_1(t), m_2(t) \cdots; t) + \frac{\partial V}{\partial t}\Delta + \sum_i \frac{\partial V}{\partial m_i}(m_k(t+\Delta) - m_i(t))$$

$$+ \frac{1}{2}\sum_i \sum_j \frac{\partial^2 V}{\partial m_i \partial m_j}(m_i(t+\Delta) - m_i(t))(m_j(t+\Delta) - m_j(t)) + 0(\Delta^2)] \tag{3.77}$$

Letting $\Delta \to dt$ and cancelling the common V yields

$$0 = \min_{\underline{u}} \left[\frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial m_i} \frac{\hat{E}(dm_i)}{dt} + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 V}{\partial m_i \partial m_j} \frac{\hat{E}(dm_i\, dm_j)}{dt} \right] \tag{3.78}$$

If now the expectation operator $\hat{E}$ is applied to a few terms, certain qualitative information about the solution can be obtained. For example, keeping terms up to order dt, the following expressions are obtained:

$$\hat{E}[dm_1]_i = E_t \bar{f}_i(\underline{a},\underline{u},t)\ dt$$

$$\hat{E}[dm_2]_{ij} = E_t[\bar{f}_i(a_j - m_{1j})^T]\ dt + E_t[(a_i - m_{1i})^T\ \bar{f}_j]\ dt$$

$$+ \bar{Q}_{ij} dt - [E_t(a_i - m_{1i})h^T]\ R^{-1}\ E_t[\underline{h}(a_j - m_{1j})]\ dt$$

$$\hat{E}[dm_{1i}\,dm_{1j}] = E_t[a_i - m_{1i})\underline{h}^T]\ R^{-1}\ E_t[\underline{h}(a_j - m_{1j})]\ dt$$

$$\hat{E}[dm_{2ij}\,dm_{2k\ell}] = [E_t a_i a_j \underline{h}^T - E_t \underline{h}^T E_t a_i a_j - 2m_{1i}\{E_t \underline{h}^T(a_j - m_{1j})\}]\ R^{-1}$$

$$\times\ [E_t a_k a_\ell \underline{h} - E_t \underline{h} E_t a_k a_\ell - 2m_{1k}\{E_t \underline{h}(a_\ell - m_{1\ell})\}]\ dt$$

$$\hat{E}[dm_{2ij}\,dm_{1k}] = [E_t a_i a_j \underline{h}^T - E_t \underline{h}^T E_t a_i a_j$$

$$- 2m_{1i}\{E_t \underline{h}^T(a_j - m_j)\}]\ R^{-1} E_t[\underline{h}(a_k - m_{1k})]\ dt \tag{3.79}$$

Thus the cross product terms do not involve the control $\underline{u}$ if R is independent of these variables. Since $\bar{f}$ is linear in the thrust control, the minimization indicated in Equation 3.78 will yield a bang-bang solution for this control variable, implying that this situation has not changed significantly from the previous cases considered. However, minimization over the thrust angle control variables $\delta\alpha$ and $\delta\Upsilon$ will not yield bang-bang variables since these quantities enter nonlinearly into $\bar{f}$. It is not until the differential equations are linearized that these control variables turn out to be bang-bang.

There is considerable difficulty in finding solutions to equations such as Equation 3.78. Even in the case when the plant and observations are linearized, the bang-bang nature of the problem seems to make the exact solution unattainable. Indeed, the Hamilton-Jacobi equation for deterministic bang-bang problems has rarely been solved for third order problems, and never for fourth order. As pointed out by Wonham (55), it is doubtful that an exact solution of the stochastic Hamilton-Jacobi equation is worth attempting. At best only modest improvements in system performance could be expected over suboptimal design techniques. With present day computers, this modest improvement would materialize at the expense of at least an order of magnitude jump in computational effort.

3.10 Conclusion of the Stochastic Problems

In this chapter the realization of the minimum terminal variance guidance system is consummated through the development of a number of straightforward design techniques. These methods are not only easy to apply, but also provide a design vehicle by which a simple, accurate and practical control system can be synthesized. In particular, there are two aspects of the results which are very appealing with regard to state-of-the-art engineering practice:

(i) the simplicity of relay control systems

(ii) the applicability of Kalman filtering.

Both the power and versatility of the results are dramatically emphasized by the Monte Carlo simulation of the MTV system.

IV. CONCLUSIONS

4.1 The Gap Between Theory and Practice: The Digital Computer

The underlying purpose of this investigation has been to map existing theoretical concepts into synthesis techniques, computational algorithms, and slight theoretical extensions which apply to the solution of a specific engineering problem. The author feels that the greatest advantage of this approach has been that even though the results are supported by a theoretical foundation, they are at the same time grounded to practical engineering reality. Consequently the difficulties encountered have been the traditional ones which separate theory from practice.

One of the primary difficulties in applying optimization theory to engineering problems is the specification of the system performance criterion. Often the final engineering design should represent the best compromise among a myriad of conflicting goals, but it is usually difficult to interpret and properly weight all the factors in terms of a mathematical expression of performance. For example, it is often argued that a guidance system must consume a minimum amount of fuel. But considering the fact that the Mariner midcourse guidance systems have typically carried on-board five times the fuel required to correct the maximum expected initial velocity deviation, it would appear that efforts to minimize only the guidance fuel consumption would be somewhat wasted. Other considerations such as system accuracy, simplicity, and implementation should certainly be stressed in the specification of the final design.

The first attempt in this study has been to make the problem formulation as realistic as possible, and only then to seek a solution. There has been virtually no attempt to force the problem into a form in which there already exists an analytic solution, and then turn about and try to justify the desired formulation. Indeed, the solutions and synthesis techniques presented in this study have time and again relied heavily on the use of one of the most powerful design tools in modern technology, the digital computer. There can be little doubt that the computer allows the solution of problems which would otherwise be deemed impossible. Yet while it is very shortsighted to ignore its capabilities in favor of gross simplifications and approximations, it is equally undesirable to allow the computer to inspire laziness and poorly conceived solution algorithms. In the MTV guidance problem the computer has been employed to the maximum possible advantage, but only after the mathematical development has been carried out as far as possible.

4.2 Stochastic Optimal Control

It is popular nowadays to write off stochastic optimization theory as a somewhat futile endeavor. The mainstream of discontent apparently comes from the immense difficulties involved in solving even what seem to be the easiest examples. There are also those who claim that the differences between a deterministic design and a stochastic design will not usually be very great, and therefore the extra design effort will hardly be worth it. These arguments may be well taken in many instances but their general veracity cannot be asserted without becoming rather arbitrary about the scope of problems being considered. For instance, it has been made clear that the MTV guidance problem has no deterministic analog; therefore a stochastic design is essential. And even though the exact solution has not been attained, it has been seen that the approximate stochastic system has a great deal to offer. In this way, the results establish at least one concrete example of the practical benefits of stochastic optimal control.

4.3 Extensions and Future Efforts

It is usually quite tempting to try to extend a once successful idea beyond its original point of application. While this is essentially a very good idea, it is also tempting to claim far more generality than is warranted. For the present problem there are two areas where the results would appear to have clear application. The first is the more general class of powered flight guidance problems, e.g., booster guidance. These other targeting guidance problems would possess essential similarities to low-thrust, interplanetary guidance in the aspects of bounded control levels, stochastic disturbance inputs, and similar dynamic behavior. The second area is the space vehicle attitude-stabilization problem in which it is desired to minimize the total number of stepper-motor actions, or reaction-jet firings, on any given mission. The solution of this problem would seem to be straightforward in view of the identical dynamic response (i.e., purely inertial), and also the built-in capability of the MTV guidance method to yield the switching probabilities for each control configuration.

Future effort in this area could certainly branch into many different directions. In each instance there is really no way of telling initially to what extent any given idea will be significant. It is the contention of the author that rather than document a long list of alternative pathways, it is perhaps more effective to leave the reader unbiased in choosing new and interesting branches to explore.

Appendix A

SOLUTION OF THE MINIMUM TIME PROBLEM

Consider the dynamical equations

$$\dot{\underline{X}} = A\underline{X} + b(t)u \qquad |\underline{u}| \leq \underline{k} \tag{A.1}$$

where $\underline{X}$ is an n-vector, b is an $n \times m$ matrix, and $\underline{k}$ and $\underline{u}$ are m-vectors. Also

$$\underline{X}(t_o) = \underline{X}_o$$

$$\underline{X}(t_f) = \underline{0} \qquad (t_f \text{ is minimum}) \tag{A.2}$$

The Hamiltonian for this problem is

$$H(t,\underline{X},\underline{u},\underline{\lambda}) = \langle \underline{\lambda}, A\underline{X} \rangle + \langle \underline{\lambda}, b(t)\underline{u} \rangle$$

The optimal $\underline{u}$ minimizes the Hamiltonian. Hence

$$\underline{u}^* = K(-\text{sgn}(b^T(t)\underline{\lambda})) \,, \qquad K = \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & & \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot\,\cdot \\ 0 & \cdot & \cdot & \cdot\; k_m \end{bmatrix}$$

where the sgn function is defined as

$$\text{sgn}(y) \triangleq \begin{cases} +1 & y > 0 \\ -1 & y < 0 \end{cases}$$

Applied to a vector, the sgn function acts on each component. Thus

$$H(t,\underline{X},\underline{\lambda}) = \langle \underline{\lambda}, A\underline{X} \rangle + \langle \underline{\lambda}, b(t)(-K \operatorname{sgn}(b^T(t)\underline{\lambda})) \rangle$$

The equations of motion are

$$\dot{\underline{X}} = H^*_{\underline{\lambda}} = A\underline{X} - b(t)\, K \operatorname{sgn}(b^T(t)\underline{\lambda})$$

$$\dot{\underline{\lambda}} = -H^*_{\underline{X}} = -A^T\underline{\lambda} \qquad (A.3)$$

The transversality condition yields

$$|\langle \underline{\lambda}(t_f),\, b(t_f)\underline{k} \rangle| = 1 \qquad (A.4)$$

Equations A.3, with conditions A.2 and A.4, yield a two-point boundary value problem that must be solved in order to obtain the optimal control.

APPENDIX B

THE DETERMINISTIC MINIMUM TERMINAL ERROR PROBLEM

Given (refer to Appendix A)

$$\dot{\underline{X}} = A\underline{X} + b\underline{u} \qquad |\underline{u}| \leq \underline{k} \tag{B.1}$$

the problem is to minimize

$$\underline{X}^T(t_f)\,\underline{X} \quad , \qquad t_f \text{ free} \tag{B.2}$$

Forming the Hamiltonian yields

$$H = \langle \underline{\lambda}, A\underline{X} \rangle + \langle \underline{\lambda}, b\underline{u} \rangle \tag{B.3}$$

and the maximum principle yields

$$\underline{u}^* = K(-\text{sgn}(b^T\underline{\lambda})) \tag{B.4}$$

where K is defined in Appendix A. The transversality condition yields

$$\underline{\lambda}(t_f) = 2\underline{X}(t_f) \tag{B.5}$$

Since t_f is free and the plant deterministic, it will normally be possible to drive the state variables to zero even though the control levels are bounded. Hence

$$\underline{\lambda}(t_f) = \underline{0} \tag{B.6}$$

and since

$$\dot{\underline{\lambda}} = -A^T\underline{\lambda} \tag{B.7}$$

it is found that

$$\underline{\lambda}(t) \equiv \underline{0} \qquad 0 \leq t \leq t_f$$

Therefore the argument of the sgn function in Equation B.3 is zero which makes the control indeterminate. This is the same as saying that the optimization problem is singular.

APPENDIX C

THE FOKKER-PLANCK EQUATION

Doob (64) has shown that the solution of Equation 3.4 will be a Markov process which can be defined by its transition probability density function

$$p(0,\underline{X}_o;\ t,\underline{X}) \tag{C.1}$$

which is the probability density that $\underline{X}(t) = \underline{X}$ given $\underline{X}(0) = \underline{X}_o$. In addition, it can be shown that this probability density function satisfies the Fokker-Planck equation associated with Equation 3.4 ($b \triangleq \beta\sigma^2;\ x_2 \triangleq x_4;\ x_3 \triangleq \xi_1;\ u \triangleq u_1$):

$$\frac{\partial p}{\partial t} = b\frac{\partial^2 p}{\partial x_3^2} - x_2\frac{\partial p}{\partial x_1} - u\,\frac{\partial p}{\partial x_2} - x_3\,\frac{\partial p}{\partial x_2} + \beta p + \beta x_3\,\frac{\partial p}{\partial x_3} \tag{C.2}$$

In order to solve Equation C.2, the assumption has been made in Sections 3.5 and 3.6 that u should be regarded as a constant.

The boundary condition on Equation C.2 is

$$\lim_{t \to 0}\ p(0,X_o;\ t,X) = \delta(x_1-x_{10})\ \delta(x_2-x_{20})\ \delta(x_3-x_{30}) \tag{C.3}$$

Equation C.2 is now solved using the Fourier transform technique. Transforming first in x_1 yields

$$\int \frac{\partial p}{\partial t}\, e^{-ik_1x_1}dx_1 - \int b\,\frac{\partial^2 p}{\partial x_3^2}\, e^{-ik_1x_1}dx_1 + \int x_2\,\frac{\partial p}{\partial x_1}\, e^{-ik_1x_1}dx_1 \quad +$$

$$+ \int u \frac{\partial p}{\partial x_2} e^{-ik_1x_1} dx_1 + \int x_3 \frac{\partial p}{\partial x_2} e^{-ik_1x_1} dx_1 - \beta \int pe^{-ik_1x_1} dx_1$$

$$- \beta \int x_3 \frac{\partial p}{\partial x_3} e^{-ik_1x_1} dx_1 = 0 \qquad (C.4)$$

Defining

$$\int pe^{-ik_1x_1} dx_1 \triangleq p'(t,k_1,x_2,x_3)$$

implies

$$\frac{\partial p'}{\partial t} - b \frac{\partial^2 p'}{\partial x_3^2} + ik_1x_2p' + u \frac{\partial p'}{\partial x_2} + x_3 \frac{\partial p'}{\partial x_2} - \beta p' - \beta x_3 \frac{\partial p'}{\partial x_3} = 0 \qquad (C.5)$$

Transforming Equation C.5 in x_2 and defining

$$\int p'e^{-ik_2x_2} dx_2 \triangleq p''(t,k_1,k_2,x_3)$$

yields

$$\frac{\partial p''}{\partial t} - b \frac{\partial^2 p''}{\partial x_3^2} + ik_1 \frac{\partial p''}{\partial k_2} (i) + u\, ik_2p'' + x_3ik_2p'' - \beta p'' - \beta x_3 \frac{\partial p''}{\partial x_3} = 0 \qquad (C.6)$$

Finally, defining

$$\int p'' e^{-ik_3x_3} dx_3 \triangleq \hat{p}(t,k_1,k_2,k_3)$$

implies

$$\frac{\partial \hat{p}}{\partial t} - b(ik_3)^2 \hat{p} - k_1 \frac{\partial \hat{p}}{\partial k_2} + iuk_2\hat{p} + ik_2(i) \frac{\partial \hat{p}}{\partial k_3} - \beta\hat{p} + \beta\hat{p}$$

$$+ \beta k_3 \frac{\partial \hat{p}}{\partial k_3} = 0$$

which becomes

$$\frac{\partial \hat{p}}{\partial t} - k_1 \frac{\partial \hat{p}}{\partial k_2} + (\beta k_3 - k_2) \frac{\partial \hat{p}}{\partial k_3} + (bk_3^2 + iuk_2)\, \hat{p} = 0 \qquad \text{(C.7)}$$

Equation C.7 is a linear first order partial differential equation which has the characteristic equations

$$\frac{dt}{ds} = 1 \; ; \qquad \frac{dk_1}{ds} = 0 \; ; \qquad \frac{dk_2}{ds} = -k_1$$

$$\frac{dk_3}{ds} = \beta k_3 - k_2 \; ; \qquad \frac{d\hat{p}}{ds} = -\,(bk_3^2 + iuk_2)\, \hat{p}$$

These equations have the solutions (subscript o indicates initial values)

$$t = s \; ; \quad k_1 = k_{10} \; ; \quad k_2 = -k_{10}s + k_{20} \; ;$$

$$k_3 = e^{\beta s}k_{30} - \frac{k_{10}s}{\beta} - \frac{k_{10}}{\beta^2} + \frac{k_{10}e^{\beta s}}{\beta^2} + \frac{k_{20}}{\beta}(1 - e^{\beta s}) \; ; \qquad \text{(C.8)}$$

$$\hat{p} = \hat{p}(0) \exp -\left\{ \int_0^s b\left(e^{\beta s'}k_{30} - \frac{k_{10}s'}{\beta} - \frac{k_{10}}{\beta^2} + \frac{k_{10}e^{\beta s'}}{\beta^2} + \frac{k_{20}}{\beta}(1-e^{\beta s'})\right)^2 + iu(-k_{10}s' + k_{20})\, ds' \right\} \qquad \text{(C.9)}$$

Performing the integration in Equation C.9 and collecting terms yields

$$\hat{p} = \hat{p}(0) \exp - \left[b \left\{ k_{10}^2 \left(\frac{s^3}{3\beta^2} + \frac{s}{\beta^4} - \frac{2se^{\beta s}}{\beta^4} + \frac{s^2}{\beta^3} + \frac{e^{2\beta s}}{2\beta^5} - \frac{1}{2\beta^5} \right) \right. \right.$$

$$+ k_{20}^2 \left(\frac{s}{\beta^2} + \frac{e^{2\beta s}}{2\beta^3} - \frac{2e^{\beta s}}{\beta^3} + \frac{3}{2\beta^3} \right) + k_{30}^2 \left(\frac{e^{2\beta s}}{2\beta} - \frac{1}{2\beta} \right) + k_{10}k_{20}$$

$$\left(- \frac{s^2}{\beta^2} + \frac{2se^{\beta s}}{\beta^3} + \frac{2e^{\beta s}}{\beta^4} - \frac{1}{\beta^4} - \frac{2s}{\beta^3} - \frac{e^{2\beta s}}{\beta^4} \right) + k_{10}k_{30} \left(- \frac{2se^{\beta s}}{\beta^2} + \frac{e^{2\beta s}}{\beta^3} - \frac{1}{\beta^3} \right)$$

$$\left. + k_{20}k_{30} \left(\frac{2e^{\beta s}}{\beta^2} - \frac{1}{\beta^2} - \frac{e^{2\beta s}}{\beta^2} \right) \right\} - \frac{ius^2}{2} k_{10} + iu\, k_{20} s \Bigg] \qquad (C.10)$$

Now imposing the boundary conditions implies

$$p(0,\underline{X}_o;\, s,x_1,x_2,x_3) =$$

$$\left(\frac{1}{2\pi}\right)^3 \iiint \hat{p}(s,k_{10},k_{20},k_{30})\, e^{ik_{10}x_1}\, e^{ik_{20}x_2}\, e^{ik_{30}x_3} dk_{10}dk_{20}dk_{30}$$

But

$$p(0,x_{10},x_{20},x_{30};\, 0,x_{10},x_{20},x_{30}) = \delta(x_1-x_{10})\, \delta(x_2-x_{20})\, \delta(x_3-x_{30})$$

$$= \left(\frac{1}{2\pi}\right)^3 \iiint \hat{p}(0,k_{10},k_{20},k_{30})\, e^{ik_{10}x_{10}}\, e^{ik_{20}x_{20}}\, e^{ik_{30}x_{30}} dk_{10}dk_{20}dk_{30}$$

Therefore,

$$\iiint \delta(x_1-x_{10})\ \delta(x_2-x_{20})\ \delta(x_3-x_{30})\ e^{-ik_{10}x_1}\ e^{-ik_{20}x_2}\ e^{-ik_{30}x_3} dx_1 dx_2 dx_3 =$$

$$= \hat{p}(0,k_{10},k_{20},k_{30}) = e^{-ik_{10}x_{10}}\ e^{-ik_{20}x_{20}}\ e^{-ik_{30}x_{30}} \qquad \text{(C.11)}$$

Using Equation C.11 and solving Equations C.8 for s, k_{10}, k_{20}, and k_{30}; Equation C.10 becomes

$$\begin{aligned}
\hat{p} = \exp - \Bigg[& ik_1x_{10} + i(k_2 + k_1t)\, x_{20} + i\left(\frac{k_1e^{-\beta t}}{\beta^2} - \frac{k_1}{\beta^2} - \frac{e^{-\beta t}k_2}{\beta} + \frac{k_2}{\beta} + \frac{k_1t}{\beta}\right. \\
& \left. + k_3e^{-\beta t}\right) x_{30} + b\left\{k_1^2\left(\frac{t^3}{3\beta^2} + \frac{t}{\beta^4} - \frac{2te^{\beta t}}{\beta^4} + \frac{t^2}{\beta^3} + \frac{e^{2\beta t}}{2\beta^5} - \frac{1}{2\beta^5}\right)\right. \\
& + (k_2 + k_1t)^2\left(\frac{t}{\beta^2} + \frac{e^{2\beta t}}{2\beta^3} - \frac{2e^{\beta t}}{\beta^3} + \frac{3}{2\beta^3}\right) + \left(\frac{k_1e^{-\beta t}}{\beta^2} - \frac{k_1}{\beta^2} - \frac{e^{-\beta t}k_2}{\beta} + \frac{k_2}{\beta}\right. \\
& \left. + \frac{k_1t}{\beta} + k_3e^{-\beta t}\right)^2\left(\frac{e^{2\beta t}}{2\beta} - \frac{1}{2\beta}\right) + k_1(k_2 + k_1t)\left(-\frac{t^2}{\beta^2} + \frac{2te^{\beta t}}{\beta^3} + \frac{2e^{\beta t}}{\beta^4}\right. \\
& \left. - \frac{1}{\beta^4} - \frac{2t}{\beta^3} - \frac{e^{2\beta t}}{\beta^4}\right) + k_1\left(\frac{k_1e^{-\beta t}}{\beta^2} - \frac{k_1}{\beta^2} - \frac{e^{-\beta t}k_2}{\beta} + \frac{k_2}{\beta} + \frac{k_1t}{\beta} + k_3e^{-\beta t}\right) \\
& \times\left(\frac{-2te^{\beta t}}{\beta^2} + \frac{e^{2\beta t}}{\beta^3} - \frac{1}{\beta^3}\right) + (k_2 + k_1t)\left(\frac{k_1e^{-\beta t}}{\beta^2} - \frac{k_1}{\beta^2} - \frac{e^{-\beta t}k_2}{\beta} + \frac{k_2}{\beta} + \frac{k_1t}{\beta}\right. \\
& \left.\left. + k_3e^{-\beta t}\right)\left(\frac{2e^{\beta t}}{\beta^2} - \frac{1}{\beta^2} - \frac{e^{2\beta t}}{\beta^2}\right)\right\} - \frac{iut^2}{2}k_1 + iu(k_2 + k_1t)\, t \Bigg] \qquad \text{(C.12)}
\end{aligned}$$

Simplifying Equation C.12 and collecting terms yields

$$\hat{p} = \exp - \Bigg[k_1^2 \left(\frac{bt^3}{3\beta^2} - \frac{bt^2}{\beta^3} + \frac{bt}{\beta^4}\,(1 - 2e^{-\ t}) + \frac{b}{\beta^5}\left(\frac{1}{2} - \frac{e^{-2\beta t}}{2} \right) \right)$$

$$+ k_2^2 \left(\frac{bt}{\beta^2} + \frac{b}{\beta^3}\left(-3/2 + 2e^{-\beta t} - \frac{e^{-2\beta t}}{2} \right) \right) + k_3^2 \left(\frac{b}{2\beta} - \frac{be^{-2\beta t}}{2} \right)$$

$$+ k_1 k_2 \left(\frac{bt^2}{\beta^2} + \frac{bt}{\beta^3}\left(-2 + 2e^{-\beta t} \right) + \frac{b}{\beta^4}\left(1 + e^{-2\beta t} - 2e^{-\beta t} \right) \right)$$

$$+ k_2 k_3 \left(\frac{b}{\beta^3}\,(1 + e^{-2\beta t} - 2e^{-\beta t}) \right) + k_1 k_3 \left(\frac{bt}{\beta^2} - 2e^{-\beta t} + \frac{b}{\beta^3}(1 - e^{-2\beta t}) \right)$$

$$+ ik_1 \left(x_{10} + x_{20}t + x_{30}\left(\frac{e^{-\beta t}}{\beta^2} - \frac{1}{\beta^2} + \frac{t}{\beta} \right) + \frac{ut^2}{2} \right)$$

$$+ ik_2 \left(x_{20} + x_{30}\left(\frac{1}{\beta} - \frac{e^{-\beta t}}{\beta} \right) + ut \right) + ik_3 (e^{-\beta t}\ x_{30}) \Bigg] \qquad \text{(C.13)}$$

This expression can be recognized as the characteristic function of a Gaussian density with mean values

$$\bar{x}_1 = x_{10} + x_{20}t + x_{30}\left(\frac{e^{-\beta t}}{\beta^2} - \frac{1}{\beta^2} + \frac{t}{\beta} \right) + \frac{ut^2}{2}$$

$$\bar{x}_2 = x_{20} + x_{30}\left(\frac{1}{\beta} - \frac{e^{-\beta t}}{\beta} \right) + ut$$

$$\bar{x}_3 = e^{-\beta t} x_{30}$$

and covariance matrix, M

$$m_{11} = E[(x_1 - \bar{x}_1)^2] = 2\left(\frac{bt^3}{3\beta^2} - \frac{bt^2}{\beta^3} + \frac{bt}{\beta^4}\,(1 - 2e^{-\beta t}) + \frac{b}{\beta^5}\left(\frac{1}{2} - \frac{e^{-2\beta t}}{2} \right) \right)$$

$$m_{22} = E[(x_2 - \bar{x}_2)^2] = 2\left(\frac{bt}{\beta^2} + \frac{b}{\beta^3}\left(-3/2 + 2e^{-\beta t} - \frac{e^{-2\beta t}}{2}\right)\right)$$

$$m_{33} = E[(x_3-\bar{x}_3)^2] = 2\left(\frac{b}{2\beta} - \frac{be^{-2\beta t}}{2\beta}\right)$$

$$m_{12} = E[(x_1-\bar{x}_1)(x_2-\bar{x}_2)] = \frac{bt^2}{\beta^2} + \frac{bt}{\beta^3}(-2 + 2e^{-\beta t}) + \frac{b}{\beta^4}(1 + e^{-2\beta t} - 2e^{-\beta t})$$

$$m_{23} = E[(x_2-\bar{x}_2)(x_3-\bar{x}_3)] = \frac{b}{\beta^2}(1 + e^{-2\beta t} - 2e^{-\beta t})$$

$$m_{13} = E[(x_1-\bar{x}_1)(x_3-\bar{x}_3)] = \frac{bt}{\beta^2}(-2e^{-\beta t}) + \frac{b}{\beta^3}(1 - e^{-2\beta t})$$

Since the switching curves in Sections 3.5 and 3.6 are assumed to be independent of x_3 (and therefore x_{30}), these quantities are integrated out of the above equations. The result is

$$\bar{x}_1 = x_{10} + x_{20}t + \frac{ut^2}{2}$$

$$\bar{x}_2 = x_{20} + ut$$

$$m_{11} = 2\sigma^2\left[\frac{e^{-\beta t}}{\beta^4} + \frac{1}{6}\frac{t^3}{\beta} - \frac{1}{2}\frac{t^2}{\beta^2} + \frac{t}{\beta^3} - \frac{1}{\beta^4}\right]$$

$$m_{22} = \frac{2\sigma^2}{\beta^2}[e^{-\beta t} - 1 + \beta t]$$

$$m_{12} = \frac{\sigma^2}{\beta^3}[1 - \beta t + \frac{1}{2}(\beta t)^2 - e^{-\beta t}]$$

APPENDIX D

ON RANDOM FIRST ORDER COST CHANGES IN AN OPTIMAL CONTROL SYSTEM

Consider the dynamic system

$$\dot{\underline{X}} = f(t,\underline{X},\underline{u}) \tag{D.1}$$

and assume that an optimal trajectory and control, denoted $\underline{X}^*$ and $\underline{u}^*$ have been determined such that the following performance functional is minimized:

$$J(\underline{C},\tau) = \min_{\underline{u}} \int_{\tau}^{t_f} g(t,\underline{X},\underline{u})\, dt; \qquad \underline{X}(0) = \underline{C} \tag{D.2}$$

It is now desired to compute the first order loss when the state deviates from the optimal path under the assumption that random process noise $\underline{\xi}$ enters additively into the dynamic equations D.1. The first order loss is given by

$$\delta J = \int_{\tau}^{t_f} (g_{\underline{X}}^{*T}\, \delta\underline{X} + g_{\underline{u}}^{*T}\, \delta\underline{u})\, dt \tag{D.3}$$

where the asterisk indicates the quantity is evaluated on the nominal trajectory.

Using the optimality conditions

$$g_{\underline{u}}^{*} + \underline{f}_{\underline{u}}^{*T}\, \underline{\lambda}^* = \underline{0}$$

$$\dot{\underline{\lambda}}^* = -g_{\underline{X}}^{*} - \underline{f}_{\underline{X}}^{*T}\, \underline{\lambda}^* \tag{D.4}$$

Equation D.3 becomes

$$\delta J = \int_{\tau}^{t_f} \left[(- \dot{\underline{\lambda}}^{*T} - \underline{\lambda}^{*T} f^*_{\underline{X}}) \, \underline{\delta X} - (\underline{\lambda}^{*T} \underline{f}^*_{\underline{u}}) \, \underline{\delta u} \right] dt \tag{D.5}$$

Simplification yields

$$\delta J = \int_{\tau}^{t_f} \left[-\dot{\underline{\lambda}}^{*T} \, \underline{\delta X} - \underline{\lambda}^{*T} (\underline{f}^*_{\underline{X}} \, \underline{\delta X} + \underline{f}^*_{\underline{u}} \, \underline{\delta u}) \right] dt \tag{D.6}$$

But the first order dynamics are described by

$$\underline{\delta \dot{X}} = \underline{f}^*_{\underline{X}} \, \underline{\delta X} + \underline{f}^*_{\underline{u}} \, \underline{\delta u} + \underline{\xi} \tag{D.7}$$

Hence

$$\delta J = \int_{\tau}^{t_f} \left[-\dot{\underline{\lambda}}^{*T} \, \underline{\delta X} - \underline{\lambda}^{*T} \, \underline{\delta \dot{X}} + \underline{\lambda}^{*T} \, \underline{\xi} \right] dt$$

$$= \int_{\tau}^{t_f} \left[\frac{d}{dt} (-\underline{\lambda}^{*T} \, \underline{\delta X}) + \underline{\lambda}^{*T} \underline{\xi} \right] \tag{D.8}$$

Now assuming initial deviations equal to $\underline{\delta X}(\tau)$ and fixed terminal boundary conditions, Equation D.8 becomes

$$\delta J = \underline{\lambda}^{*T}(\tau) \, \underline{\delta X}(\tau) + \int_{\tau}^{t_f} \underline{\lambda}^{*T} \, \underline{\xi} \, dt \tag{D.9}$$

The first order contribution of the process noise is clearly evident in Equation D.9.

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