PROJECTIONS IN A NORMED LINEAR SPACE
AND A GENERALIZATION OF THE PSEUDO-INVERSE

Thesis by

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ABSTRACT

The concept of a "projection function" in a finite-dimensional real or complex normed linear space H (the function P_M which carries every element into the closest element of a given subspace M) is set forth and examined.

If $\dim M = \dim H - 1$, then P_M is linear. If P_N is linear for all k-dimensional subspaces N, where $1 \leq k < \dim M$, then P_M is linear.

The projective bound Q, defined to be the supremum of the operator norm of P_M for all subspaces, is in the range $1 \leq Q < 2$, and these limits are the best possible. For norms with $Q = 1$, P_M is always linear, and a characterization of these norms is given.

If H also has an inner product (defined independently of the norm), so that a dual norm can be defined, then when P_M is linear its adjoint P_M^H is the projection on $(\text{kernel } P_M)^\perp$ by the dual norm. The projective bounds of a norm and its dual are equal.

The notion of a pseudo-inverse F^+ of a linear transformation F is extended to non-Euclidean norms. The distance from F to the set of linear transformations G of lower rank (in the sense of the operator norm $\|F - G\|$) is $c / \|F^+\|$, where $c = 1$ if the range of F fills its space, and $1 \leq c \leq Q$ otherwise. The norms on both domain and range spaces have $Q = 1$ if and only if $(F^+)^+ = F$ for every F. This condition is also sufficient to prove that we have $(F^+)^H = (F^H)^+$, where the latter pseudo-inverse is taken using dual norms.

In all results, the real and complex cases are handled in a completely parallel fashion.

CONTENTS

CHAPTER I

PRELIMINARY DEFINITIONS AND RESULTS

Let H_n be an n-dimensional real or complex Hilbert space, where n is a positive integer. The elements x, y of H_n may be thought of as column vectors, and the inner product (x, y) may be thought of as $\sum_{i=1}^{n} \bar{x}_i y_i$. For the fundamental properties of H_n, see any standard text (e.g., Halmos).

The real and complex cases will be handled in a completely parallel fashion, although most treatments of this subject handle the real case only, or handle the two cases separately. The term "scalar" will therefore be used to denote either a real or a complex number.

A function $F : H_m \to H_n$, where H_m and H_n have the same scalar field, is called homogeneous if it is continuous and $F(cx) = cF(x)$ for every $x \in H_m$ and every scalar c. The homogeneous function F is called linear if $F(x+y) = F(x) + F(y)$ for all $x, y \in H_m$.

A real-valued function α on H_n is called a norm if $x \neq 0 \Rightarrow \alpha(x) > 0$, $\alpha(cx) = |c|\alpha(x)$ and $\alpha(x+y) \leq \alpha(x) + \alpha(y)$ for all $x, y \in H_m$ and every scalar c.

We shall need the following concepts from convexity theory (see also Householder pp. 38-45).

A set of the form $B = \{x \in H_n \mid \alpha(x) \leq c\}$, where α is a norm and c is a positive constant, is called an equilibrated convex body. Since

$$\alpha(x) = \inf \{ b \mid b > 0, \ (c/b)x \in B \},$$

B and c determine α uniquely. It is sometimes convenient, especially in illustrations, to use an equilibrated convex body to represent a norm (see Fig. 1).

A k-dimensional <u>flat</u> (sometimes called a linear manifold) in H_n is a subset of the form $u+M$ (that is, the set $\{u+x \mid x \in M\}$), where M is a k-dimensional subspace of H_n. If $u' \in u+M$, then clearly $u+M = u'+M$. An (n-1)-dimensional flat is called a <u>hyperplane</u>.

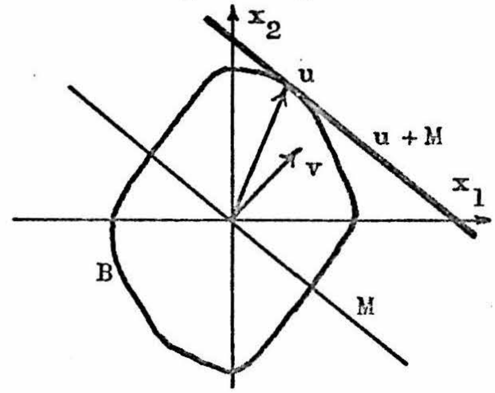

Fig. 1. An equilibrated convex body B and a support hyperplane at u in real H_2.

For $v \neq 0$, a set of the form $\{x \in H_n \mid (v,x) = 1\}$ is the hyperplane $[v/(v,v)] + \{v\}^{\perp}$, which does not contain 0. Conversely, if the hyperplane $u+M$ does not contain the point 0, then it is equal to the set $\{x \in H_n \mid (v,x) = 1\}$, where $u = u_1 + u_2$, $u_1 \in M$, $u_2 \in M$ and $v = u_2/(u_2,u_2)$.

A hyperplane $u+M$ such that $\alpha(u) = c$ and $\alpha(x) \geq c$ for all $x \in u+M$ is called a <u>support hyperplane</u> for B at u, where B is the equilibrated convex body described previously. (See Fig. 1.)

<u>Theorem 1.1</u> Let $B = \{x \in H_n \mid \alpha(x) \leq \alpha(u)\}$ be an equilibrated convex body. Let $u+N$ be a flat of dimension $\leq n-1$ such that $\alpha(x) \geq \alpha(u)$ for every $x \in u+N$. Then there exists a support hyperplane $u+M$ for B at u such that $u+N \subset u+M$.

<u>Proof.</u> If H_n is a real Hilbert space, convexity theory will supply the proof (for example, see Eggleston p. 19).

If H_n is complex, then H_n with the new inner product $Re(x,y)$ is a 2n-dimensional real Hilbert space, α is still a norm, and $u+N$ is a flat of dimension $\leq 2n-1$. Hence there is a support hyperplane $u+M'$ for B at u in this new space such that $u+N \subset u+M'$. Moreover, we can write

$$u + M' = \{x \in H_n \mid \mathrm{Re}(v,x) = 1\}$$

for some $v \in H_n$.

Since $\mathrm{Re}(v,u) = 1$, $(v,u) \neq 0$ and we can define $u' = u/(v,u)$. Then $(v,u') = 1$ and hence u' is on the support hyperplane $u + M'$. Therefore

$$\alpha(u) \leq \alpha(u') = \frac{\alpha(u)}{|(v,u)|},$$

which implies that $|(v,u)| \leq 1$. Since $\mathrm{Re}(v,u) = 1$, this implies that $(v,u) = 1$.

Therefore, consider the set $\{x \in H_n \mid (v,x) = 1\}$ in the original Hilbert space. Since $(v,u) = 1$, this set can be represented as $u + M$. It is a support hyperplane for B at u because $x \in u + M \Rightarrow x \in u + M' \Rightarrow \alpha(x) \geq \alpha(u)$.

Now let x be an arbitrary element of $u + N$. Then $x \in u + M'$ and hence $\mathrm{Re}(v,x) = 1$. We can express x as $x = u + x'$ where $x' \in N$. Then

$$1 = \mathrm{Re}(v,x) = \mathrm{Re}(v,u) + \mathrm{Re}(v,x') = 1 + \mathrm{Re}(v,x'),$$

and hence $\mathrm{Re}(v,x') = 0$ for all $x' \in N$. Therefore $\mathrm{Re}(v,-ix') = \mathrm{Im}(v,x') = 0$ also, and $(v,x') = 0$. Consequently $(v,x) = (v,u) + (v,x') = 1$ and $x \in u + M$. This shows that $u + N \subset u + M$, which completes the proof. ∎

For the special case $\dim N = 0$, we have the following important result.

<u>Corollary 1.2</u> An equilibrated convex body has at least one support hyperplane at each boundary point.

The norm of a homogeneous function $F : H_m \to H_n$ induced by the norm α on H_n and the norm β on H_m is defined by

$$\|F\|_{\alpha\beta} = \sup_{x\neq 0} \frac{\alpha(F(x))}{\beta(x)} = \sup_{\beta(x)=1} \alpha(F(x)).$$

The supremum is actually attained for some nonzero x, since the set $\{x \in H_m \mid \beta(x) = 1\}$ is compact.

If $F : H_m \to H_n$ is linear, F^H will represent the adjoint of F, that is, $(F(x),y) = (x,F^H(y))$ for all $x \in H_m$, $y \in H_n$.

CHAPTER II

DEFINITION OF THE PROJECTION FUNCTION

A norm α on H_n is called strictly convex if

$$\alpha(cx+(1-c)y) \geqslant \alpha(x) = \alpha(y)$$

for all scalars c implies that $x=y$. In terms of the equilibrated convex body $B = \{x \mid \alpha(x) \leqslant 1\}$, α is strictly convex if every one-dimensional flat which does not meet the interior of B meets B in only one point.

Let M be a subspace of H_n and let $x \in H_n$. The projection of x onto M by α is the element $y \in M$ which is closest to x, i.e., such that

$$\alpha(y-x) = \inf\{\alpha(z-x) \mid z \in M\} . \tag{1}$$

The existence and uniqueness of the projection are established by the following theorem (see also Meinardus p. 2, Köthe p. 347).

Theorem 2.1 Let α be a strictly convex norm on H_n, let M be a subspace of H_n, and let $x \in H_n$. Then there is a unique $y \in M$ which satisfies (1).

Proof. Consider the set $M' = \{z \in M \mid \alpha(z) > 2\alpha(x)\}$. For all $z \in M'$, $\alpha(x)+\alpha(z-x) \geqslant \alpha(z) > 2\alpha(x)$; hence $\alpha(z-x) > \alpha(x) = \alpha(0-x)$, and the infimum in (1) is not approached on M'. Since $M-M'$ is compact, the infimum in (1) is attained for some $y \in M$.

Now let $y' \in M$ be such that $\alpha(y'-x) = \alpha(y-x)$. Then for all scalars c, $cy+(1-c)y' \in M$ and hence

$$\begin{aligned}\alpha(\, c(y-x)+(1-c)(y'-x)\,) &= \alpha(\, cy+(1-c)y'-x\,)\\ &\geqslant \alpha(y-x) = \alpha(y'-x).\end{aligned}$$

Since α is strictly convex, $y - x = y' - x$ and $y' = y$. ∎

The function $P_{M,\alpha}$ which carries x into its projection on M by α is called the projection function. Henceforth, whenever the notation $P_{M,\alpha}$ is used, it is presumed that α is strictly convex and M is a subspace of H_n or other appropriate Hilbert space. Where no confusion results, the shorter forms P_M and P may be used.

CHAPTER III

BASIC PROPERTIES OF THE PROJECTION FUNCTION

The following theorem shows that the projection function is "almost linear".

Theorem 3.1 The function $P_{M,\alpha}$ is homogeneous, and $P(x+y) = P(x)+y$ if $y \in M$.

Proof. Assume, for purpose of contradiction, that P is discontinuous at x. Then there will be a sequence $\{x_i\}$ such that

$$\lim_{i \to \infty} x_i = x$$

and

$$\lim_{i \to \infty} P(x_i) = z \neq P(x),$$

or else $\{P(x_i)\}$ is unbounded. However, the boundedness of $\{P(x_i)\}$ is implicit in the proof of Theorem 2.1.

By the definition of P, we have

$$\alpha(P(x_i)-x_i) \leq \alpha(P(x)-x_i)$$

for every i. Take limits as $i \to \infty$ to obtain

$$\alpha(z-x) \leq \alpha(P(x)-x),$$

which implies that $z = P(x)$, a contradiction.

Now let $P(x) = u$. Then $\alpha(u-x) \leq \alpha(s-x)$ for all $s \in M$, and for any nonzero scalar c,

$$\alpha(cu-cx) = |c|\ \alpha(u-x) \leq |c|\ \alpha(s-x) = \alpha(cs-cx),$$

which implies that $\alpha(cu-cx) \leq \alpha(s-cx)$ for all $s \in M$. Hence we have $P(cx) = cu$. For $c = 0$ this result is trivial.

Now let $y \in M$. We have

$$\alpha(P(x)-x) \leqslant \alpha(s-x),$$

$$\alpha(P(x)+y-(x+y)) \leqslant \alpha(s+y-(x+y))$$

for all $s \in M$. Since $s+y$ also runs over all of M, the last inequality implies that $P(x+y) = P(x)+y$. ∎

Theorem 3.2 Every $x \in H_n$ can be expressed uniquely as $x = y+z$, where $P_M(y) = 0$ and $z \in M$. Furthermore, $z = P_M(x)$.

Proof. We have $x = (x-P(x))+P(x)$, and by Theorem 3.1, $P(x-P(x)) = P(x)-P(x) = 0$, so the representation exists. If we have $x = y+z$, then $P(x) = P(y)+z = z$, so the representation is unique. ∎

Theorem 3.3 For every projection function $\|P_{M,\alpha}\|_{\alpha\alpha} < 2$.

Proof. Let x be such that $\|P\|_{\alpha\alpha} = \alpha(P(x))$ and $\alpha(x) = 1$. If $P(x) = 0$, the result is trivial. If not, then

$$1 = \alpha(x) = \alpha(0-x) > \alpha(P(x)-x)$$

and $$\|P\|_{\alpha\alpha} = \alpha(P(x)) \leqslant \alpha(P(x)-x) + \alpha(x),$$

which imply that $\|P\|_{\alpha\alpha} < 2$. ∎

It will be shown later (Chapter 5) that this inequality is the best possible.

CHAPTER IV

LINEARITY OF THE PROJECTION FUNCTION

The projection function is not linear for every norm and subspace. For example, consider the norm α on real H_3 given by

$$\alpha(x_1, x_2, x_3) = (x_1^4 + x_2^4 + x_3^4)^{\frac{1}{4}}$$

and determine kernel $P_{M,\alpha}$, where M is the one-dimensional subspace spanned by $y = (y_1, y_2, y_3)$. Minimizing $\alpha(x - cy)$ over all real c by ordinary variational techniques gives the result

$$\text{kernel } P_{M,\alpha} = \left\{(x_1, x_2, x_3) \mid y_1x_1^3 + y_2x_2^3 + y_3x_3^3 = 0\right\},$$

which is not a subspace for all y. Hence $P_{M,\alpha}$ is not linear for all y.

This example suggests the following characterization of linearity.

Theorem 4.1 The function $P_{M,\alpha}$ is linear if and only if kernel $P_{M,\alpha}$ is a subspace.

Proof. The necessity of the stated condition is obvious. Let $x, y \in H_n$ and decompose them as $x = x' + P(x)$, $y = y' + P(y)$, according to Theorem 3.2. Then by Theorem 3.1,

$$P(x+y) = P(x' + y' + P(x) + P(y)) = P(x' + y') + P(x) + P(y).$$

Since $x', y' \in$ kernel P, $x' + y' \in$ kernel P, and $P(x+y) = P(x) + P(y)$. Hence P is linear. ∎

Theorem 4.2 If $\dim M = n-1$, then $P_{M,\alpha}$ is linear.

Proof. Let $x \in H_n - M$, and let $y = x - P(x)$. Then $y \neq 0$ and by Theorem 3.2, $y \in$ kernel P. Let L be the subspace of all scalar multiples of y. Then it is clear that $L \subset$ kernel P, since by

Theorem 3.1 $P(cy) = cP(y) = 0$ for all scalars c.

Now let $z \in$ kernel P. Since dim $M = n-1$, H_n is a direct sum of M and L, and $z = z_1 + z_2$, where $z_1 \in L$ and $z_2 \in M$. By Theorem 3.1, $P(z) = P(z_1) + z_2 = z_2$. But $P(z) = 0$; hence $z_2 = 0$ and $z = z_1 \in L$. Therefore L = kernel P and P is linear by Theorem 4.1. ∎

Theorem 4.3 If $P_{M,\alpha}$ is linear for all r-dimensional subspaces M, where $r \geq 1$, then it is linear for all subspaces of higher dimension.

Proof. Let N be a subspace with dim $N > r$, and assume, for purpose of contradiction, that P_N is not linear. Then by Theorem 4.1 its kernel is not closed under addition, i.e., there exist two elements x_1, x_2 of kernel P_N such that $P_N(x_1 + x_2) = y \neq 0$.

Now choose an r-dimensional subspace M of N which contains y. Then $x_1, x_2 \in$ kernel P_M, but $P_M(x_1 + x_2) = y \neq 0$, which violates the hypothesis. ∎

CHAPTER V

THE PROJECTIVE BOUND

The real number $\varrho(\alpha)$ defined by

$$\varrho(\alpha) = \sup_{M} \| P_{M,\alpha} \|_{\alpha\alpha}$$

is called the projective bound of α. The following theorem shows that the supremum is finite, and that for $1 \leq k \leq n-1$ the supremum is attained for some k-dimensional subspace M.

Theorem 5.1 The sets of real numbers

$$S_k = \left\{ \alpha(P_{M,\alpha}(x)) \mid \alpha(x) = 1, \text{ M is k-dimensional} \right\}$$

for $k = 1, 2, \ldots, n-1$ are identical. Furthermore, S_k is bounded and contains its supremum $\varrho(\alpha)$.

Proof. Suppose $c \in S_k$; we must show that $c \in S_j$ for any $j = 1, 2, \ldots, n-1$. For some k-dimensional subspace M and some $x \in H_n$,

$$c = \alpha(y),$$

$$y = P_M(x),$$

$$\alpha(x) = 1.$$

If $y = x$, then $c = 1$ and $c \in S_j$ is easily shown.

If $y \neq x$, then $\alpha(z - x) \geq \alpha(y - x)$ for all $z \in M$, i.e., we have $\alpha(z) \geq \alpha(y - x)$ for all $z \in y - x + M$. By Theorem 1.1 there is a support hyperplane $y - x + N$ for the equilibrated convex body $\{z \mid \alpha(z) \leq \alpha(y-x)\}$ at $y - x$, such that $y - x + M \subset y - x + N$, that is, $M \subset N$. Let L be any j-dimensional subspace of N which contains y. Then $\alpha(z) \geq \alpha(y-x)$ for all $z \in y - x + L$, that is, $\alpha(z - x) \geq \alpha(y - x)$ for all $z \in L$. Hence $y = P_L(x)$ and $c \in S_j$.

Therefore, all the S_k are equal, and we need to prove the second assertion only for S_1.

Theorem 3.3 shows that S_1 is bounded. Then either S_1 contains its supremum or there would be two sequences $\{x_i\}$ and $\{y_i\}$ such that $\alpha(x_i) = \alpha(y_i) = 1$ for all i, and

$$\lim_{i \to \infty} \alpha(P_{L_i}(x_i)) = \varrho(\alpha), \tag{1}$$

where L_i is the subspace spanned by y_i. By taking appropriate subsequences, we can also require that

$$\lim_{i \to \infty} x_i = x, \tag{2}$$

$$\lim_{i \to \infty} y_i = y,$$

$$\lim_{i \to \infty} P_{L_i}(x_i) = w. \tag{3}$$

If L is the subspace spanned by y, then clearly $w \in L$.

Now let $z \in L$; then $z = cy$ for some scalar c. By the definition of P_{L_i},

$$\alpha(x_i - cy_i) \geq \alpha(x_i - P_{L_i}(x_i))$$

for every i. Taking limits as $i \to \infty$, we have $\alpha(x-z) \geq \alpha(x-w)$. Since $z \in L$ was arbitrary, $w = P_L(x)$. From (2), $\alpha(x) = 1$; from (1) and (3), $\alpha(w) = \varrho(\alpha)$; hence $\varrho(\alpha) \in S_1$. ∎

From Theorem 3.3, we have

<u>Corollary 5.2</u> For any strictly convex norm α,

$$1 \leq \varrho(\alpha) < 2.$$

The upper limit is approached for strictly convex norms which approximate the "maximum norm" α on real H_2 given by

$$\alpha(x_1, x_2) = \max(|x_1|, |x_2|).$$

(See Fig. 2.) The lower limit is attained by the norms described in the next chapter.

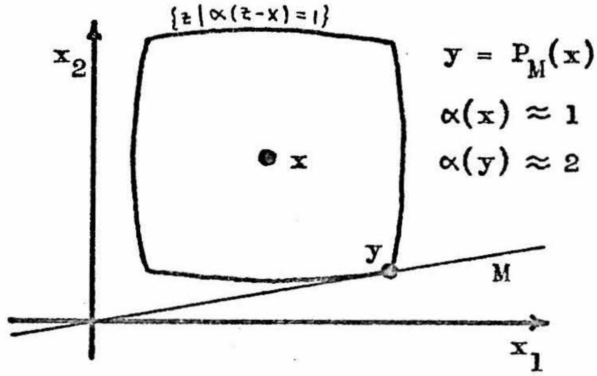

Fig. 2. A strictly convex norm α for which $\varrho(\alpha)$ is close to 2.

CHAPTER VI

PROJECTIVE NORMS

A strictly convex norm α on H_n for which $\varrho(\alpha) = 1$ is called a projective norm. The inner product norm $(x,x)^{\frac{1}{2}}$ is projective, and so are the "elliptical" norms $(x,T(x))^{\frac{1}{2}}$, where T is a positive-definite self-adjoint linear transformation of H_n. Later we shall give examples of non-elliptical projective norms on real H_2. For spaces of dimension three or higher, all projective norms are elliptical, both in the real case (Kakutani) and in the complex case (Bohnenblust).

Lemma 6.1 Suppose α is a projective norm, $P_{M,\alpha}$ is linear, and

$$N = \text{kernel } P_{M,\alpha} \quad ;$$

then

$$M = \text{kernel } P_{N,\alpha} \quad .$$

Proof. Suppose $x \in M$ and $y \in N$. By Theorem 3.1, $P_M(x+y) = x$. The definitions of $\|P_M\|_{\alpha\alpha}$ and $\varrho(\alpha)$ and the first hypothesis give

$$\alpha(x) \leq \|P_M\|_{\alpha\alpha}\ \alpha(x+y) \leq \varrho(\alpha)\ \alpha(x+y) = \alpha(x+y)$$

for all $y \in N$. Hence $P_N(x) = 0$, that is, $x \in \text{kernel } P_N$.

Now suppose $x \in \text{kernel } P_N$ and write $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in N$. By Theorem 3.1,

$$0 = P_N(x) = P_N(x_1) + x_2 \ .$$

By the previous paragraph, $P_N(x_1) = 0$. Hence $x_2 = 0$ and $x = x_1 \in M$.

Lemma 6.2 If α is a projective norm, then $P_{M,\alpha}$ is linear for all subspaces M of H_n.

Proof. By Theorem 4.3, it is sufficient to prove P_M is linear if M is one-dimensional.

Let u be a nonzero element of M, and let $u + N$ be a support hyperplane for the equilibrated convex body $\{x \in H_n \mid \alpha(x) \leq \alpha(u)\}$ at u. Then $\alpha(x) \geq \alpha(u)$ for every $x \in u+N$, that is, $\alpha(x+u) \geq \alpha(u)$ for every $x \in N$. Hence $u \in \text{kernel } P_N$. By Theorem 4.2, P_N is linear. Since $\dim N = n-1$, $\dim \text{kernel } P_N = 1$, and therefore $M = \text{kernel } P_N$. By Lemma 6.1, $N = \text{kernel } P_M$ and hence P_M is linear by an application of Theorem 4.1. ∎

Lemma 6.3 Suppose $P_{M,\alpha}$ is linear and $N = \text{kernel } P_{M,\alpha}$. Then

$$M = \text{kernel } P_{N,\alpha} \iff P_{M,\alpha} + P_{N,\alpha} = I.$$

Proof. First assume $M = \text{kernel } P_N$, let $x \in H_n$ be arbitrary and express it as $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in N$. Then the application of both sides of $P_M + P_N = I$ to $x_1 + x_2$ gives an identity.

Now assume $P_M + P_N = I$. If $x \in M$, then $P_M(x) + P_N(x) = x + P_N(x) = x$, so $x \in \text{kernel } P_N$. On the other hand, if we have $x \in \text{kernel } P_N$, then $P_M(x) = x$ and $x \in M$. ∎

Theorem 6.4 Let α be a projective norm. Then $P_{M,\alpha}$ is linear for every subspace M of H_n, and if $N = \text{kernel } P_{M,\alpha}$, then

$$M = \text{kernel } P_{N,\alpha}$$

and

$$P_{M,\alpha} + P_{N,\alpha} = I.$$

Proof. This follows directly from Lemmas 6.1, 6.2 and 6.3. ∎

Theorem 6.5 Suppose $1 \leq k \leq n-1$, and for every k-dimensional subspace M of H_n, $P_{M,\alpha}$ is linear and either

$$M = \text{kernel } P_{N,\alpha} \text{ or, equivalently, } P_{M,\alpha} + P_{N,\alpha} = I, \tag{1}$$

where $N = \text{kernel } P_{M,\alpha}$. Then α is projective.

Proof. Assume, for purpose of contradiction, that $\varrho(\alpha) > 1$. Then there will be a k-dimensional subspace M and $x, y \in H_n$ such that

$$y = P_M(x),$$

$$\alpha(y) = \|P_M\|_{\alpha\alpha}\, \alpha(x) = \varrho(\alpha)\, \alpha(x) > \alpha(x). \tag{2}$$

Clearly $y \neq x$. Let $N = \text{kernel } P_M$. By Theorem 3.2 and (2),

$$0 \neq y - x \in N, \quad \alpha(x) = \alpha(y - (y-x)) < \alpha(y) \tag{3}$$

By Lemma 6.3 the two conditions in (1) are equivalent. We use the latter condition and apply both sides of it to y, obtaining the relation $y + P_N(y) = y$, or $P_N(y) = 0$, which contradicts (3). ▯

We can now exhibit examples of non-elliptical projective norms on real H_2. Consider the norm

$$\alpha(x_1, x_2) = \begin{cases} (\,|x_1|^p + |x_2|^p\,)^{1/p} & \text{if } x_1 x_2 \geq 0, \\ (\,|x_1|^q + |x_2|^q\,)^{1/q} & \text{if } x_1 x_2 < 0, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

We shall show that α satisfies the hypotheses of Theorem 6.5 with $k = 1$. By Theorem 4.2, $P_{M,\alpha}$ is linear for all one-dimensional subspaces M.

Following the notation of Theorem 6.5, we let u span M and let v span $N = \text{kernel } P_M$. By examination of the unit ball (see Fig. 3), we see that if $u = (1,0)$ or $(0,1)$, then the hypotheses of Theorem 6.5 are satisfied. In other cases, u and v are in adjacent quadrants, and we can also demand, without loss of generality, that $u_1 = v_1 = 1$.

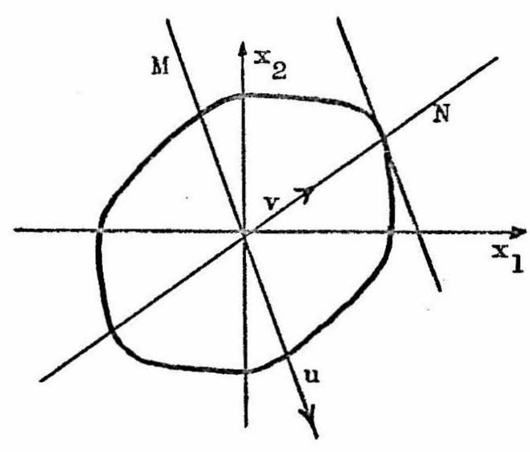

Fig. 3. Example of a non-elliptical projective norm.

Since $v \in$ kernel P_M, $\alpha(v+cu)$ is minimal for $c=0$. For sufficiently small c, $v+cu$ is in the same quadrant with v. Hence if v is in the first quadrant,

$$\alpha(v+cu) = (|1+c|^p + |v_2 + cu_2|^p)^{1/p}. \tag{4}$$

(The same argument can be used, mutatis mutandis, if v is in the fourth quadrant.)

We can minimize (4) by differentiating $(\alpha(v+cu))^p$ with respect to c. Since the minimum occurs at $c=0$, this gives

$$p + p|u_2|\,|v_2| \text{ sgn } u_2 \text{ sgn } v_2 = 0$$

Since u and v are in adjacent quadrants, sgn u_2 sgn $v_2 = -1$, and hence

$$|v_2| = \left(\frac{1}{|u_2|}\right)^{\frac{1}{p-1}} .$$

This gives v, which spans $N =$ kernel P_M. Similarly, we find the $w = (1, w_2)$ which spans kernel P_N. We compute (note $p+q = pq$)

$$|w_2| = \left(\frac{1}{|v_2|}\right)^{\frac{1}{q-1}} = \left(|u_2|^{\frac{1}{p-1}}\right)^{\frac{1}{q-1}} = |u_2| ,$$

which shows that $M =$ kernel P_N.

CHAPTER VII

DUAL NORMS AND PROJECTION

If α is a norm on H_n, the dual norm α_D is defined by

$$\alpha_D(x) = \sup_{y\neq 0} \frac{|(x,y)|}{\alpha(y)} .$$

If $u \neq 0$ and v is such that $\{ x \mid (v,x) = 1 \}$ is a support hyperplane for the equilibrated convex body $\{ x \mid \alpha(x) \leq \alpha(u) \}$ at u, then v is called a <u>dual</u> of u with respect to α. Corollary 1.2 shows that each nonzero u has at least one dual.

We shall need some elementary results about the dual norm and the dual.

<u>Proposition 7.1</u> If $u \neq 0$ and v is a dual of u with respect to α, then $\alpha_D(v) = 1/\alpha(u)$ and u is a dual of v with respect to α_D.

<u>Proof</u>. For $(v,x) \neq 0$, $(v, x/(v,x)) = 1$. Therefore, by hypothesis, $\alpha(x/(v,x)) = \alpha(x)/ |(v,x)| \geq \alpha(u)$, that is,

$$\frac{1}{\alpha(u)} \geq \frac{|(v,x)|}{\alpha(x)} ,$$

which holds even for $(v,x) = 0$, and with equality for $x = u$. Therefore $\alpha_D(v) = 1/\alpha(u)$.

Now assume $(u,x) = 1$. Then

$$\alpha_D(x) = \sup_{y\neq 0} \frac{|(x,y)|}{\alpha(y)} \geq \frac{|(x,u)|}{\alpha(u)} = \frac{1}{\alpha(u)} = \alpha_D(v) ,$$

and hence u is a dual of v with respect to α_D. ▮

<u>Corollary 7.2</u> For all $x \in H_n$, $\alpha_{DD}(x) = \alpha(x)$.

__Proof.__ Let y be a dual of x with respect to α (if $x \neq 0$). Then $\alpha(x) = 1/\alpha_D(y)$. But x is also a dual of y with respect to α_D, and hence $\alpha_{DD}(x) = 1/\alpha_D(y)$. ▮

We can also apply Proposition 7.1 to show that if u and v are duals, then they give equality in the generalized Hölder inequality $|\alpha(u)\alpha_D(v)| \leq 1$. This fact is sometimes used to define duals.

__Proposition 7.3__ If α_D is strictly convex, then every nonzero u has a unique dual with respect to α.

__Proof.__ Let v_1 and v_2 be duals of u with respect to α. Then $(v_1,u) = (v_2,u) = 1$ and for any scalar c,

$$\alpha_D(\ cv_1 + (1-c)v_2\ ) \geq \frac{|\ c(v_1,u) + (1-c)\ (v_2,u)\ |}{\alpha(u)}$$

$$= \frac{1}{\alpha(u)} = \alpha_D(v_1) = \alpha_D(v_2).$$

Since α_D is strictly convex, $v_1 = v_2$. ▮

__Theorem 7.4__ Suppose $u \neq 0$. Then $u \in \text{kernel } P_{M,\alpha}$ if and only if there is a dual v of u with respect to α such that $v \in M^\perp$.

__Proof.__ First assume $v \in M^\perp$ and v is a dual of u with respect to α. Then for any $x \in M$, $(v,u-x) = (v,u) - (v,x) = 1$; hence $\alpha(u-x) \geq \alpha(u)$ and $u \in \text{kernel } P$.

Now assume $u \in \text{kernel } P$. Then $\alpha(u+x) \geq \alpha(u)$ for all $x \in M$. By Theorem 1.1, there is a support hyperplane $\{x \mid (v,x) = 1\}$ for the body $\{x \mid \alpha(x) \leq \alpha(u)\}$ at u which contains the flat $u + M$. By definition, v is a

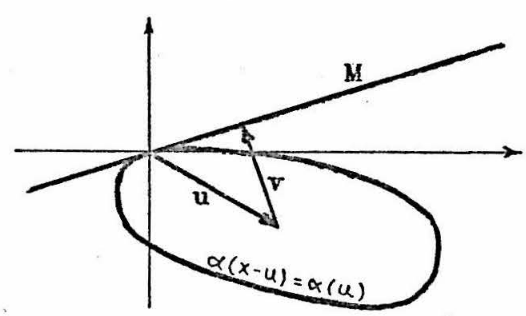

Fig. 4. Illustration of the proof of Theorem 7.4.

dual of u with respect to α. Also, for every $x \in M$, $u + x$ is on the hyperplane, that is, $(v,u) + (v,x) = 1$. Since $(v,u) = 1$, $(v,x) = 0$ and $v \in M^{\perp}$. ∎

<u>Theorem 7.5</u> Suppose α and α_D are both strictly convex, $P_{M,\alpha}$ is linear, and

$$N = \text{kernel } P_{M,\alpha} \quad ;$$

then

$$M^{\perp} = \text{kernel } P_{N^{\perp},\alpha_D} \quad , \tag{1}$$

and hence $P_{N^{\perp},\alpha_D}$ is linear.

<u>Proof.</u> By Proposition 7.4, for every $v \in H_n$ there is a unique dual u with respect to α_D, and v is the unique dual of u with respect to α. By Theorem 7.4,

$$v \in \text{kernel } P_{N^{\perp},\alpha_D} \iff u \in N.$$

By a second application of Theorem 7.4, $u \in N \iff v \in M^{\perp}$, since $N = \text{kernel } P_{M,\alpha}$. These two equivalences prove (1). ∎

<u>Theorem 7.6</u> Suppose α and α_D are both strictly convex, $P_{M,\alpha}$ is linear, and

$$N = \text{kernel } P_{M,\alpha} \quad ;$$

then

$$P_{N^{\perp},\alpha_D} = (P_{M,\alpha})^H$$

<u>Proof.</u> By Theorem 7.5, $P_{N^{\perp},\alpha_D}$ is linear. It is sufficient to show that for arbitrary $x, y \in H_n$,

$$0 = \Big(x , \quad P_{N^{\perp},\alpha_D}(y) - (P_{M,\alpha})^H(y)\Big)$$

$$= \Big(x\ ,\ P_{N^{\perp},\alpha_D}(y) \Big) - \Big(P_{M,\alpha}(x)\ ,\ y \Big) . \qquad (2)$$

Let $x = x_1 + x_2$, where $x_1 \in M$, $x_2 \in N$, and let $y = y_1 + y_2$, where $y_1 \in N^{\perp}$, $y_2 \in M^{\perp}$. Then, by using Theorem 7.5, (2) can be demonstrated easily. ▮

<u>Theorem 7.7</u> If α and α_D are both strictly convex, $\varrho(\alpha) = \varrho(\alpha_D)$.

<u>Proof.</u> Let M be an $(n-1)$-dimensional subspace of H_n such that $\varrho(\alpha) = \|P_{M,\alpha}\|_{\alpha\alpha}$, and let $N =$ kernel $P_{M,\alpha}$. Then by Theorem 4.2, $P_{M,\alpha}$ is linear, and by Theorem 7.6, $P_{N^{\perp},\alpha_D} = (P_{M,\alpha})^H$. Hence

$$\varrho(\alpha_D) \geq \|P_{N^{\perp},\alpha_D}\|_{\alpha_D\alpha_D} = \|(P_{M,\alpha})^H\|_{\alpha_D\alpha_D}$$

$$= \sup_{x\neq 0} \frac{\alpha_D(\ (P_{M,\alpha})^H(x)\ )}{\alpha_D(x)}$$

$$= \sup_{x,y\neq 0} \frac{|(x,\ P_{M,\alpha}(y))|}{\alpha_D(x)\,\alpha(y)}$$

$$= \sup_{y\neq 0} \left(\frac{1}{\alpha(y)} \sup_{x\neq 0} \frac{|(x,\ P_{M,\alpha}(y))|}{\alpha_D(x)} \right)$$

$$= \sup_{y\neq 0} \frac{\alpha(P_{M,\alpha}(y))}{\alpha(y)} = \|P_{M,\alpha}\|_{\alpha\alpha} = \varrho(\alpha).$$

To establish the reverse inequality $\varrho(\alpha) \geq \varrho(\alpha_D)$, interchange the roles of α and α_D. ▮

<u>Theorem 7.8</u> If α is a projective norm, then α_D is strictly convex.

<u>Proof.</u> Let v_1 and v_2 be two vectors such that

$$\alpha_D(\ cv_1 + (1-c)v_2\ ) \geq \alpha_D(v_1) = \alpha_D(v_2) \qquad (3)$$

for all scalars c. We must prove that $v_1 = v_2$. If $v_1 = v_2 = 0$, this result is trivial; therefore we assume $v_1 \neq 0 \neq v_2$.

The set $K = \{cv_1 + (1-c)v_2 \mid c \text{ is a scalar}\}$ is a flat containing v_1 and v_2 and such that $x \in K \Rightarrow \alpha_D(x) \geq \alpha_D(v_1)$ by (3). By Theorem 1.1, there is a support hyperplane $\{x \mid (u,x) = 1\}$ containing K.

Then u is a dual of both v_1 and v_2 with respect to α_D, and hence v_1 and v_2 are both duals of u with respect to α. Let $M = \{v_1\}^{\perp}$ and $N = \{v_2\}^{\perp}$; then by Theorem 7.4, $u \in \text{kernel } P_{M,\alpha}$ and $u \in \text{kernel } P_{N,\alpha}$. Since the kernels are one-dimensional, both are equal to the subspace L spanned by u. By Theorem 6.4, $M = N = \text{kernel } P_{L,\alpha}$. Hence v_1 and v_2 are linearly dependent, and $v_1 = bv_2$ for some scalar b. We substitute into (3) to obtain

$$\alpha_D\Big(((b-1)c + 1)v_2\Big) \geq \alpha_D(v_2).$$

Hence $|(b-1)c + 1| \geq 1$ for all scalars c, which implies that $b = 1$ and $v_1 = v_2$. Therefore α_D is strictly convex. ∎

Theorems 7.7 and 7.8 together give

<u>Corollary 7.9</u> If α is a projective norm, so is α_D.

CHAPTER VIII

THE GENERALIZED RECIPROCAL

Suppose that throughout this chapter

α is a strictly convex norm on H_n,

β is a strictly convex norm on H_m,

$F : H_m \to H_n$ is a linear function,

R = range F,

K = kernel F.

Then let

$x \in H_n$ be arbitrary,

$y \in H_m$ be such that $F(y) = P_{R,\alpha}(x)$,

$x' = y - P_{K,\beta}(y)$, that is, the point on the flat $y + K$ which is closest to the origin.

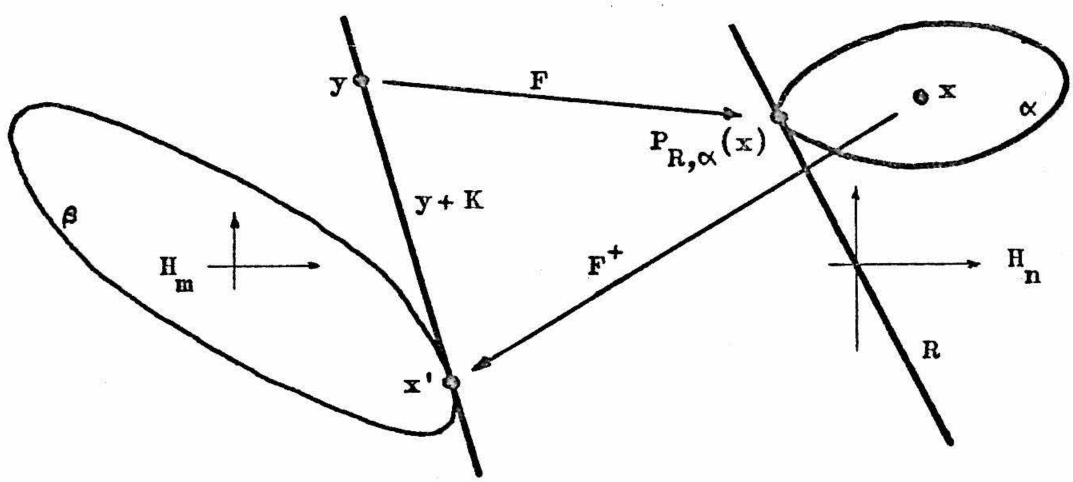

Fig. 5. The geometric definition of the generalized reciprocal of a linear function.

Now x' is independent of the choice of y, for a different choice of y would merely give another point on the flat $y+K$. Therefore, there is a well-defined function $F^+_{\alpha\beta}$ which carries x into x'. It is called the generalized reciprocal of F with respect to α and β. Where the norms are understood, the notation F^+ will be used instead.

If α and β are the inner product norms, then F^+ is the Moore-Penrose reciprocal of F. Many properties of the Moore-Penrose reciprocal are specializations of the properties which we are about to derive. (See also Ben-Israel.)

The following properties of F^+ are consequences of the definition:

(i) F^+ is a homogeneous function,

(ii) if $P_{R,\alpha}$ and $P_{K,\beta}$ are linear, so is F^+,

(iii) Kernel $F^+ = \text{kernel } P_{R,\alpha}$,

(iv) range $F^+ = \text{kernel } P_{K,\beta}$,

(v) $F^+F = I - P_{K,\alpha}$,

(vi) $FF^+ = P_{R,\beta}$,

(vii) if F^+ is linear, rank F^+ = rank F,

(viii) if F is nonsingular, $F^+ = F^{-1}$,

(ix) for nonzero scalars c, $(cF)^+ = \frac{1}{c}F^+$.

Properties (i) and (ii) are obvious.

If $x \in \text{kernel } P_{R,\alpha}$, then $x' = 0$ (following our previous notation). If $x \notin \text{kernel } P_{R,\alpha}$, then the flat $y+K$ does not contain 0, and $x' \neq 0$. Hence we have (iii).

By Theorem 3.1, $P_{K,\beta}(x') = P_{K,\beta}(y - P_{K,\beta}(y)) = 0$. If $P_{K,\beta}(z) = 0$ then $z = y - P_{K,\beta}(y)$ for some y, and $z = F^+(F(y))$. Hence we have property (iv).

To evaluate $F^+(F(z))$ we can use $y = z$. Then $F^+(F(z)) = z - P_{K,\beta}(z)$ and we have (v).

By the definition, $F(F^+(x)) = F(x') = F(y - P_{K,\beta}(y)) = F(y) = P_{R,\alpha}(x)$, and we have (vi).

If F^+ is linear, then by (v) $P_{K,\beta}$ is linear. By (iv) we have rank F^+ = dim kernel $P_{K,\beta} = m - \dim K$ = rank F, which proves (vii).

Properties (viii) and (ix) are obvious.

Properties (v), (vi) and (vii) can be used as an alternate definition of F^+ in some cases, as the following theorem shows.

<u>Theorem 8.1</u> If $G : H_n \to H_m$ is a linear function such that

$$GF = I - P_{K,\beta}, \tag{1}$$

$$FG = P_{R,\alpha}, \tag{2}$$

$$\text{rank } G = \text{rank } F,$$

then $G = F^+$.

<u>Proof.</u> Let $x \in R$; then $x = F(y)$ for some $y \in H_m$, and

$$G(x) = G(F(y)) = (I - P_{K,\beta})(y) = F^+(F(y)) = F^+(x), \tag{3}$$

by (1) and property (v).

For any $w \in H_n$,

$$G(P_{R,\alpha}(w)) = F^+(P_{R,\alpha}(w)) = F^+(w),$$

by (3) with $x = P_{R,\alpha}(w)$ and the definition of F^+. Therefore, we have range $F^+ \subset$ range G. The projection functions in (1) and (2) are linear, and hence F^+ is linear by property (ii). Then by property (vii) and hypothesis, $\operatorname{rank} F^+ = \operatorname{rank} F = \operatorname{rank} G$. Therefore $\operatorname{range} F^+ = \operatorname{range} G$.

Now let $x \in \operatorname{kernel} P_{R,\alpha}$. Then $F^+(x) = 0$ by property (iii), $G(x) \in \operatorname{range} G = \operatorname{range} F^+$, and by (2),

$$F(G(x)) = P_{R,\alpha}(x) = 0.$$

Hence $G(x) \in K$ also. Since by property (iv) and Theorem 3.2, range F^+ and K have only 0 in common, $G(x) = 0 = F^+(x)$.

Since R and kernel $P_{R,\alpha}$ together span H_n, the fact that we have $G(x) = F^+(x)$ for x on these two sets shows that $G = F^+$. ∎

Lemma 8.2 If M and N are subspaces of H_n and $\dim N > \dim M$, then there is at least one nonzero $x \in N \cap \operatorname{kernel} P_{M,\alpha}$.

Proof. Let S be the sphere $\{x \in N \mid (x,x) = 1\}$. Then P gives a continuous mapping from S to M. By the Borsuk-Ulam Theorem (see Spanier, p. 266), there is an $x \in S$ such that $P(x) = P(-x)$. Since P is a homogeneous function, $P(x) = 0$. ∎

Theorem 8.3 If $F : H_m \to H_n$ is a linear function of positive rank, then

$$\frac{1}{\|F^+\|_{\alpha\beta}} \leq \inf \left\{ \|G\|_{\alpha\beta} \;\middle|\; \operatorname{rank}(F+G) < \operatorname{rank} F \right\}$$

$$\leq \frac{b}{\|F^+\|_{\alpha\beta}} < \frac{2}{\|F^+\|_{\alpha\beta}}, \tag{4}$$

where $b = 1$ if $\operatorname{rank} F = n$, and $b = \varrho(\alpha)$ otherwise.

Proof. Suppose $\operatorname{rank}(F+G) < \operatorname{rank} F$; then

$$\dim \operatorname{kernel}(F+G) > \dim K.$$

By Lemma 8.2, there is a nonzero $x \in \operatorname{kernel}(F+G)$, i.e.,

$$F(x) + G(x) = 0,$$
$$-F^+(F(x)) = F^+(G(x)), \tag{5}$$

such that $x \in \text{kernel } P_{K,\alpha}$. Then by property (v), $F^+(F(x)) = x$, and (5) becomes

$$-x = F^+(G(x)),$$

which yields

$$\beta(x) = \beta(F^+(G(x))) \leq \|F^+\|_{\beta\alpha}\alpha(G(x)) \leq \|F^+\|_{\beta\alpha}\|G\|_{\alpha\beta}\,\beta(x),$$

and establishes the first inequality in (4).

Now let $y \neq 0$ be such that

$$\beta(F^+(y)) = \|F^+\|_{\beta\alpha}\,\alpha(y), \tag{6}$$

and let
$$z = P_{R,\alpha}(y). \tag{7}$$

From the definition of F^+ we have

$$F^+(z) = F^+(y). \tag{8}$$

Also, $\alpha(z) \leq \|P_{R,\alpha}\|_{\alpha\alpha}\alpha(y) \leq \varrho(\alpha)\,\alpha(y)$. In the case where rank $F = n$ we have $R = H_n$, $z = y$ and $\alpha(z) = \alpha(y)$. Hence

$$\alpha(z) \leq b\,\alpha(y), \tag{9}$$

where b is as in (4). Then (6), (8) and (9) together yield

$$\beta(F^+(z)) \geq \frac{1}{b}\|F^+\|_{\beta\alpha}\,\alpha(z) \tag{10}$$

Now let $w_1 = F^+(z)$; clearly $w_1 \neq 0$. Then by property (iv) $w_1 \in \text{range } F^+ = \text{kernel } P_{K,\beta}$. Hence $P_{K,\beta}(w_1) = 0$, that is, $\beta(w_1 + v) \geq \beta(w_1)$ for all $v \in K$, that is, $\beta(w) \geq \beta(w_1)$ for all w in the flat $w_1 + K$. By Theorem 1.1, there is a support hyperplane for the equilibrated convex body $\{w \in H_m \mid \beta(w) \leq \beta(w_1)\}$ at w_1 which contains $w_1 + K$. This hyperplane can be written as

$$\left\{ w_1 + \sum_{i=2}^{m} c_i w_i \;\middle|\; c_2, c_3, \ldots, c_m \text{ scalars} \right\}, \tag{11}$$

where $w_2, w_3, \ldots, w_m$ span a subspace which includes K.

Now let $G : H_m \to H_n$ be the linear function for which

$$G(w_1) = G(F^+(z)) = -z, \tag{12}$$

$$G(w_i) = 0, \quad i = 2, 3, \ldots, m. \tag{13}$$

Since $w_2, w_3, \ldots, w_m$ span all of K, (13) implies that $(F+G)(w) = 0$ for all $w \in K$. Also, for w_1, which does <u>not</u> belong to K, we have, by (12), property (vi) and (7),

$$(F+G)(w_1) = F(w_1) - z = F(F^+(z)) - z = P_{R,\alpha}(z) - z = 0.$$

Therefore, $$\operatorname{rank}(F+G) < \operatorname{rank} F . \tag{14}$$

For some nonzero $w \in H_m$, which we can write as $w = \sum_{i=1}^{m} d_i w_i$, we have

$$\|G\|_{\alpha\beta} = \frac{\alpha(G(w))}{\beta(w)} = \frac{\alpha(-d_1 z)}{\beta\left(\sum_{i=1}^{m} d_i w_i\right)} .$$

If $d_1 = 0$ we would have an absurdity, since $G \neq 0$ by (12). Therefore, we can divide by $|d_1|$ to obtain

$$\|G\|_{\alpha\beta} = \frac{\alpha(z)}{\beta\left(w_1 + \sum_{i=2}^{m} (d_i/d_1) w_i\right)} .$$

The denominator is the norm of a point on the support hyperplane (11); Hence by (10)

$$\|G\|_{\alpha\beta} \leq \frac{\alpha(z)}{\beta(w_1)} = \frac{\alpha(z)}{\beta(F^+(z))} \leq \frac{b}{\|F^+\|_{\beta\alpha}} . \tag{15}$$

The second inequality in (4) then follows from (14) and (15).

The third inequality in (4) follows from Corollary 5.2. ∎

In the case where F is nonsingular, (4) reduces to

$$\inf \left\{ \|G\|_{\alpha\beta} \;\middle|\; F+G \ \text{singular} \right\} = \frac{1}{\|F^{-1}\|_{\beta\alpha}}$$

This result has also been proved by others (see also Franck p. 1297, Kahan p. 775, and Maitre p. 910).

Theorem 8.4 A necessary and sufficient condition that $(F^+)^+ = F$ for every F is that α and β be projective norms.

Proof. If α and β are projective norms, then all projections are linear by Theorem 6.4, and by property (ii), so is F^+. Let $R' =$ range F^+ and $K' =$ kernel F^+. Then by Theorem 6.4 and properties (iii), (iv), (v) and (vi),

$$F^+F = I - P_{K,\beta} = P_{R',\beta} ,$$

$$FF^+ = P_{R,\alpha} = I - P_{K',\alpha} .$$

Also, rank $F^+ =$ rank F by property (vii); hence $F = (F^+)^+$ by Theorem 8.1.

Conversely, if $(F^+)^+ = F$, then

$$I - P_{K,\beta} = F^+F = F^+(F^+)^+ = P_{R',\beta} ,$$

$$P_{R,\alpha} = FF^+ = (F^+)^+F = I - P_{K',\alpha} ,$$

and α and β are projective norms by Theorem 6.5. ∎

Theorem 8.5 If α and β are projective norms, then

$$(F^H)^+_{\beta_D\alpha_D} = (F^+_{\alpha\beta})^H . \qquad (16)$$

<u>Proof.</u> By properties (vi) and (v), and by Theorem 6.4,

$$FF^+ = P_{R,\alpha} = I - P_{M,\alpha}, \qquad M = \text{kernel } P_{R,\alpha},$$

$$F^+F = I - P_{K,\beta} = P_{N,\beta}, \qquad N = \text{kernel } P_{K,\beta},$$

By Theorem 6.4, $R = \text{kernel } P_{M,\alpha}$ and $K = \text{kernel } P_{N,\beta}$. By Theorem 7.8, α_D and β_D are strictly convex. Then by Theorem 7.5,

$$FF^+ = I - (P_{R^\perp,\alpha_D})^H ,$$

$$F^+F = (P_{K^\perp,\beta_D})^H ,$$

and

$$(F^+)^H F^H = I - P_{R^\perp,\alpha_D} ,$$

$$F^H (F^+)^H = P_{K^\perp,\beta_D} .$$

Since $R^\perp = \text{kernel } F^H$ and $K^\perp = \text{range } F^H$, (16) follows by Theorem 8.1, with G replaced by $(F^+)^H$ and F replaced by F^H. ▮

It is conjectured that (16) is true even if α and β are not projective norms.

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