LAGRANGIAN FUNCTIONS WHICH DETERMINE A SYMMETRICAL TENSOR BY SCHRÖDINGER'S RULE

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Summary.

The choice of a Lagrangian function to be used in a variational principle may be limited by the condition that the tensor derived from it by Schrödinger's rule shall be symmetrical. To meet this condition the function must satisfy a certain set of partial differential equations. Particular and general solutions of these equations are found in various cases--according as the function is restricted to depend (A) only on the components of a vector, (B) only on their first derivatives, or (C) on both; and according to the number of dimensions of the vector. Methods of obtaining such solutions, and of proving their independence or of finding the relations between them, are discussed.

This research is an extension of results obtained by Professor H. Bateman.* In order to present the collections of solutions as completely as possible, many are included which may be found in his papers.

* Proc. Nat. Acad. of S., V.13, 5, P.326, May 1927
Proc. Nat. Acad. of S., V.13,11, P.771, Nov.1927
Phys. Rev. V.30, 1, P. 55, July 1927
A Variational Principle in the Theory of Elasticity (Unpublished.)

1. Statement of the Problem.

In the application of a variational principle in studies in theoretical physics, a function, called the Lagrangian function, must be chosen, the variation of whose integral is to vanish. Since in several cases the tensor determined from this Lagrangian function by Schrödinger's rule must be symmetric to give a useful physical interpretation, it has been suggested by Professor H. Bateman that a collection of the possible Lagrangian functions which satisfy this condition is of value. Such functions are of particular interest in three dimensions by reason of their use in modern elasticity theory, and in four dimensions where they are used in electromagnetic theory. However, in formulating the problem of finding these functions, it is evident that they should not be limited dimensionally at the start but only as the mathematical complexities of the investigation require that two, three, or four, rather than n, dimensions be considered.

To state the problem explicitly for the case of n dimensions, let the Lagrangian function, L, depend on the components $(u_1, u_2, \cdots, u_n)$ of an n-vector and on their n^2 derivatives with respect to the n coordinates $(x_1, x_2, \cdots, x_n)$,

(1) $$L = L(u_1, u_2, \cdots, u_n;\ u_{11}, u_{12}, \cdots, u_{nn}),$$

where $u_{ab} = \frac{\partial u_a}{\partial x_b}$. Then the Schrödinger rule may be extended to associate with L the tensor whose components are:

(2) $$T_{ij} = \sum_{\alpha=1}^{\alpha=n} \left\{ u_{i\alpha} \frac{\partial L}{\partial u_{j\alpha}} + u_{\alpha i} \frac{\partial L}{\partial u_{\alpha j}} \right\} + u_i \frac{\partial L}{\partial u_j} - \delta_{ij} L$$

where $i = 1, 2, \cdots, n$ $j = 1, 2, \cdots, n$ δ_{ij} is Kroneker's symbol. The condition for tensor symmetry is that L satisfy the linear homogeneous partial differential equations:

(3) $$T_{ij} = T_{ji} \; .$$

Hence the restated problem is to find solutions, L, (of the form of (1)) of the equations (3).

2. Completeness of the set of equations.

Defining the operators:

(4) $$\Psi_{ij} = \sum_{\alpha=1}^{\alpha=n} \left\{ u_{i\alpha} \frac{\partial}{\partial u_{j\alpha}} + u_{\alpha i} \frac{\partial}{\partial u_{\alpha j}} - u_{j\alpha} \frac{\partial}{\partial u_{i\alpha}} - u_{\alpha j} \frac{\partial}{\partial u_{\alpha i}} \right\} + u_i \frac{\partial}{\partial u_j} - u_j \frac{\partial}{\partial u_i} \;,$$

and

(5) $$\Phi_{abcd} = \Psi_{ab} \left\{ \Psi_{cd} (\;) \right\} - \Psi_{cd} \left\{ \Psi_{ab} (\;) \right\} \;,$$

equations (3) become:

(6) $$\Psi_{ij} (L) = 0 \; .$$

A sufficient condition that this set be complete is that (for each a, b, c, d):

$$\Phi_{abcd} (L) = \Psi_{ij} (L) \text{ for some } i, j \;;$$

namely, that no new equations are added to the set (6) by this operation.

This is proved by direct substitution of (4) in (5) while distinguishing various cases.

Case I (a) $a = b$; $\Psi_{ab} = 0$; $\Phi_{abcd} (L) = 0$

(b) $c = d$; $\Psi_{cd} = 0$; $\Phi_{abcd} (L) = 0$

Case II a, b, c, d *distinct* ; $\Phi_{abcd} (L) = 0$

Case III (a) $a = c$ a, b, d *distinct* ; $\Phi_{abcd} (L) = \Psi_{db} (L)$

(b) $b = d$ a, b, c *distinct* ; $\Phi_{abcd} (L) = \Psi_{ca} (L)$

Case IV (a) $a=d$ a, b, c *distinct* ; $\Phi_{abca}(L) = \Psi_{bc}(L)$

(b) $b=c$ a, b, d *distinct* ; $\Phi_{abbd}(L) = \Psi_{ad}(L)$

Case V (a) $a=c$ $b=d$; $\Phi_{abab}(L) = 0$

(b) $a=d$ $b=c$; $\Phi_{abba}(L) = 0$

Note that Case II occurs only when $n \geqq 4$, and Cases III, IV only when $n \geqq 3$. Zero is always a member of the set $\Psi_{ij}(L)$; in fact, $\Psi_{ii} = 0$. Since the five cases are exhaustive, it is established that equations (6) form a complete set.

3. Classification of cases.

For convenient reference the various cases to be discussed are symbolized by:

A, when L depends only on the vector components;

B, when L depends only on the first derivatives of the vector components;

C, when L depends both on the vector components and their first derivatives.

A subscript on A, B or C indicates the number of dimensions considered. Note that equations (6) reduce to simpler form in cases A and B.

4. Number of equations.

The set (6) contains n^2 equations, but it is necessary to choose from these a set of linearly independent equations, as few in number as possible--

that is, a fundamental set in terms of which the others may be linearly expressed.

Case A. Choose the set:

$$\Psi_{ij}(L) = 0 \quad i = 1 \quad j = 2, 3, \cdots, n \qquad n-1 \text{ equations}.$$

Cases B, C. Choose the set:

$$\Psi_{ij}(L) = 0 \quad i = 1, 2, \cdots, j-1 \quad j = 2, 3, \cdots, n .$$

These are linearly independent by inspection, since no term involving the derivative of a vector component (such as $u_{ab}\frac{\partial L}{\partial u_{cd}}$) occurs in more than one of them; while they are sufficient in number, since all others are either identically true, such as $\Psi_{ii}(L) = 0$, or derivable from the chosen set by means of the relation $\Psi_{ij}(L) = -\Psi_{ji}(L)$. There are then $\frac{n(n-1)}{2}$ equations in the required set.

5. <u>Number of solutions.</u>

In each case the general solution of equations (6) is of the form:

$$L = F(L_1, L_2, \cdots, L_p),$$

where F is an arbitrary function and $L_1, L_2, \cdots, L_p$ are particular solutions forming an independent set. Therefore p equals the number of variables minus the number of equations.

Case A_n. $p = n - (n-1) = 1$.

Case B_n. $p = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Case C_n. $p = (n^2+n) - \frac{n(n-1)}{2} = \frac{n(n+3)}{2}$, but the solutions of A_n and B_n are automatically solutions of C_n;

hence the number of further solutions (involving both the u_a and u_{ab}) is only:

$$p' = \frac{n(n+3)}{2} - \frac{n(n+1)}{2} - 1 = n - 1 .$$

6. Independence of solutions.

In order that the solutions form an independent set, there must be no relation between them such as:

$$G(L_1, L_2, \cdots, L_p) = 0 .$$

This can be tested for the simpler sets by inspection or by showing that a Jacobian involving the p functions and p of the variables is non-vanishing. However, for the more complex sets a condensed method may be used since the functions are all polynomials.

To demonstrate the independence of p polynomials $(L_1, L_2, \cdots, L_p)$ in m variables $(y_1, y_2, \cdots, y_p)$ consider the matrix:

$$\left(\left(\begin{matrix} \frac{\partial L_1}{\partial y_1} & \cdots & \frac{\partial L_p}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial L_1}{\partial y_m} & \cdots & \frac{\partial L_p}{\partial y_m} \end{matrix}\right)\right)$$

These variables may be either the u_a , u_{ab} , or any set of independent linear functions of them. From the matrix select a p-row determinant and rearrange its columns and rows. Select some of the variables to be zero, so that the elements of the determinant are simplified. The desired result is a determinant which equals the product of the elements of its major diagonal and hence non-vanishing. This implies that the value of the determinant was not identically zero before simplifying, therefore that the functions are independent.

Since the above-mentioned rearrangements and selections are not systematic but depend on the computer's foresight, an example is appended. In case B_4 (which see in tables) the functions $\theta, \varphi, \psi, \omega, \chi, \alpha, \beta, \gamma_1, \gamma_2, \delta$ in the variables $a, b, c, d, f, g, h, i, j, k, \xi, \eta, \zeta, \lambda, \mu, \nu$ are proved independent by the following determinant in which all terms in $a, c, h, i, j, k, \eta, \zeta, \lambda, \nu$ have been omitted.

$$\begin{array}{c|cccccccccc|}
 & \omega & \gamma_2 & \beta_1 & \varphi & \theta & \chi & \gamma_1 & \delta_1 & \alpha & \psi \\
d & -bg^2 & b\mu^2 & \xi^2 & b & 1 & 0 & 0 & -f^2\xi^2-f^2\mu^2 & 0 & -f^2-g^2 \\
j & 0 & -2f\mu\xi-2j\mu^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\zeta & 0 & -2bg\xi & 2g\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f & 0 & -2f\xi^2-2j\mu\xi & 0 & -2f & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & b\xi^2 & \mu^2 & b & 1 & 0 & 0 & 0 & 0 & 0 \\
\eta & 0 & 2fg\xi & 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 & 0 & 0 & \xi & 2fg\mu & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 2g\mu\xi & -2bg\xi\mu & 0 & 0 \\
\xi & 0 & -2f^2\xi & 0 & 0 & 0 & 0 & 0 & -2bg^2\xi & 2\xi & 0 \\
b & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -g^2\xi^2 & 0 & -g^2
\end{array}$$

TABLES OF SOLUTIONS, for Cases $A, B_2, B_3, B_4, B_n, C_2, C_3$

with definitions of variables, complete independent sets, relations between solutions, etc.

Starred solutions are given in Bateman's papers.

Case A $\quad p=1 \quad m=n$ Solution is:

$$A_n = \sum_{a=1}^{a=n} u_a^2$$

Case B_2 $\quad p=3 \quad m=4$ Solutions are:

$$\begin{cases} S_1 = \sum_{a=1}^{a=2} u_{aa} & = R_1 \\ S_2 = \sum_{a=1}^{a=2} \sum_{b=1}^{b=2} u_{ab} \cdot u_{ba} & R_2 = \sum_{a=1}^{a=2} \sum_{a=1}^{a=2} u_{ba} \cdot u_{ba} \end{cases}$$

$$\begin{cases} \theta = a + b & \Theta = A + B \\ \varphi = ab - h^2 & \Phi = AB - H^2 \\ \alpha = \zeta^2 & \end{cases}$$

where:

$$a = u_{11} \quad b = u_{22} \quad 2h = u_{12} + u_{21} \quad 2\zeta = u_{12} - u_{21}$$

$$A = u_{11}^2 + u_{21}^2 \quad B = u_{12}^2 + u_{22}^2 \quad H = u_{11} u_{12} + u_{21} u_{22}$$

or:

$$A = u_{11}^2 + u_{12}^2 \quad B = u_{21}^2 + u_{22}^2 \quad H = u_{11} u_{21} + u_{12} u_{22}$$

Independent sets: (1) S_1, S_2, R_2 (2) θ, φ, α (3) Θ, Φ, α

Relations between solutions:

$$\begin{cases} \theta = S_1 = R_1 & S_1 = R_1 = \theta \\ \varphi = \frac{1}{2} S_1^2 - \frac{1}{4} S_2 - \frac{1}{4} R_2 & S_2 = -2\alpha - 2\varphi + \theta^2 \\ \alpha = \frac{1}{4} R_2 - \frac{1}{4} S_2 & R_2 = 2\alpha - 2\varphi + \theta^2 \end{cases}$$

$$\begin{cases} \Theta = 2\alpha - 2\varphi + \theta^2 \\ \Phi = (\varphi + \alpha)^2 \end{cases}$$

Case B_3 $\quad p = 6 \quad m = 9$ Solutions are:

$$\begin{cases} S_1 = \sum_{a=1}^{a=3} u_{aa} & = R_1 \\ S_2 = \sum_{a=1}^{a=3} \sum_{b=1}^{b=3} u_{ab} \cdot u_{ba} & R_2 = \sum_{a=1}^{a=3} \sum_{b=1}^{b=3} u_{ba} \cdot u_{ba} \\ S_3 = \sum_{a=1}^{a=3} \sum_{b=1}^{b=3} \sum_{c=1}^{c=3} u_{ab} \cdot u_{bc} \cdot u_{ca} & R_3 = \sum_{a=1}^{a=3} \sum_{b=1}^{b=3} \sum_{c=1}^{c=3} u_{ba} \cdot u_{bc} \cdot u_{ca} \end{cases}$$

$$* \begin{cases} \theta = a + b + c \\ \varphi = bc - f^2 + ca - g^2 + ab - h^2 \\ \psi = abc + 2fgh - af^2 - bg^2 - ch^2 \\ \alpha = \xi^2 + \eta^2 + \zeta^2 \\ \beta = a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta \\ \gamma = (bc - f^2)\xi^2 + (ca - g^2)\eta^2 + (ab - h^2)\zeta^2 \\ \qquad + 2(gh - af)\eta\zeta + 2(hf - bg)\zeta\xi + 2(fg - ch)\xi\eta \end{cases}$$

$$* \begin{cases} \Theta = A + B + C \\ \Phi = BC - F^2 + CA - G^2 + AB - H^2 \\ \Psi = ABC + 2FGH - AF^2 - BG^2 - CH^2 \end{cases}$$

$$* \begin{cases} \Gamma = A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta \\ \Delta = u_{11}u_{22}u_{33} - u_{11}u_{23}u_{32} + u_{12}u_{23}u_{31} - u_{13}u_{22}u_{31} + u_{13}u_{21}u_{32} - u_{12}u_{21}u_{33} \\ \Lambda = u_{22}u_{33} - u_{23}u_{32} + u_{33}u_{11} - u_{31}u_{13} + u_{11}u_{22} - u_{12}u_{21} \end{cases}$$

where:

$$\begin{cases} a = u_{11} & b = u_{22} & c = u_{33} \\ 2f = u_{23} + u_{32} & 2g = u_{31} + u_{13} & 2h = u_{12} + u_{21} \\ 2\xi = u_{23} - u_{32} & 2\eta = u_{31} - u_{13} & 2\zeta = u_{12} - u_{21}. \end{cases}$$

Either:

$$\begin{cases} A = u_{11}^2 + u_{21}^2 + u_{31}^2 & F = u_{12}u_{13} + u_{22}u_{23} + u_{32}u_{33} \\ B = u_{12}^2 + u_{22}^2 + u_{32}^2 & G = u_{13}u_{11} + u_{23}u_{21} + u_{33}u_{31} \\ C = u_{13}^2 + u_{23}^2 + u_{33}^2 & H = u_{11}u_{12} + u_{21}u_{22} + u_{31}u_{32} \end{cases}$$

Case B_3 (cont.)

or:

$$
\begin{cases}
A = u_{11}^2 + u_{12}^2 + u_{13}^2 & F = u_{21}u_{31} + u_{22}u_{32} + u_{23}u_{33} \\
B = u_{21}^2 + u_{22}^2 + u_{23}^2 & G = u_{31}u_{11} + u_{32}u_{12} + u_{33}u_{13} \\
C = u_{31}^2 + u_{32}^2 + u_{33}^2 & H = u_{11}u_{21} + u_{12}u_{22} + u_{13}u_{23}
\end{cases}
$$

Independent sets: (1) $\theta, \varphi, \psi, \alpha, \beta, \gamma$ (2) $\Theta, \Phi, \Psi, \alpha, \beta, \gamma$ (3) $S_1, S_2, S_3, R_2, R_3, \gamma$

Relations between solutions:

$$
\begin{cases}
S_1 = R_1 = \theta \\
S_2 = -2\alpha - 2\varphi + \theta^2 \\
S_3 = 3\beta + 3\psi - 3\theta\alpha - 3\theta\varphi + \theta^3 \\
R_2 = 2\alpha - 2\varphi + \theta^2 \\
R_3 = -\beta + 3\psi + \theta\alpha - 3\theta\varphi + \theta^3
\end{cases}
$$

$$
\begin{cases}
\theta = S_1 = R_1 \\
\varphi = \frac{1}{2}S_1^2 - \frac{1}{4}S_2 - \frac{1}{4}R_2 \\
\psi = \frac{1}{6}S_1 - \frac{1}{4}S_1R_2 - \frac{1}{4}S_1S_2 + \frac{1}{12}S_3 + \frac{1}{4}R_3 \\
\alpha = \frac{1}{4}R_2 - \frac{1}{4}S_2 \\
\beta = \frac{1}{4}S_1R_2 - \frac{1}{4}S_1S_2 + \frac{1}{4}S_3 - \frac{1}{4}R_3
\end{cases}
$$

$$
\begin{cases}
\Theta = 2\alpha - 2\varphi + \theta^2 \\
\Phi = (\alpha - \varphi)^2 + 4\gamma + 2\theta(\beta - \psi) \\
\Psi = \Delta^2 \\
\Gamma = \gamma + \theta\beta - \varphi\alpha \\
\Delta = \beta + \psi \\
\Lambda = \varphi + \alpha
\end{cases}
$$

<u>Case B_4</u> $p = 10$ $m = 16$ Solutions are:

$$\begin{cases} S_1 = \sum_{a=1}^{a=4} u_{aa} & = R_1 \\ S_2 = \sum_{a=1}^{a=4} \sum_{b=1}^{b=4} u_{ab} u_{ba} & R_2 = \sum_{a=1}^{a=4} \sum_{b=1}^{b=4} u_{ba} u_{ba} \\ S_3 = \sum_{a=1}^{a=4} \sum_{b=1}^{b=4} \sum_{c=1}^{c=4} u_{ab} u_{bc} u_{ca} & R_3 = \sum_{a=1}^{a=4} \sum_{b=1}^{b=4} \sum_{c=1}^{c=4} u_{ba} u_{bc} u_{ca} \\ S_4 = \sum_{a=1}^{a=4} \sum_{b=1}^{b=4} \sum_{c=1}^{c=4} \sum_{d=1}^{d=4} u_{ab} u_{bc} u_{cd} u_{da} & R_4 = \sum_{a=1}^{a=4} \sum_{b=1}^{b=4} \sum_{c=1}^{c=4} \sum_{d=1}^{d=4} u_{ba} u_{bc} u_{cd} u_{da} \end{cases}$$

$$* \begin{cases} \theta = a + b + c + d \\ \varphi = bc - f^2 + ca - g^2 + ab - h^2 + ad - i^2 + bd - j^2 + cd - k^2 \\ \psi = \begin{vmatrix} b & f & j \\ f & c & k \\ j & k & d \end{vmatrix} + \begin{vmatrix} a & g & i \\ g & c & k \\ i & k & d \end{vmatrix} + \begin{vmatrix} a & h & i \\ h & b & j \\ i & j & d \end{vmatrix} + \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \\ \omega = \begin{vmatrix} a & h & g & i \\ h & b & f & j \\ g & f & c & k \\ i & j & k & d \end{vmatrix} \end{cases}$$

$$* \begin{cases} \Theta = A + B + C + D \\ \Phi = BC - F^2 + CA - G^2 + AB - H^2 + AD - I^2 + BD - J^2 + CD - K^2 \\ \Psi = \begin{vmatrix} B & F & J \\ F & C & K \\ J & K & D \end{vmatrix} + \begin{vmatrix} A & G & I \\ G & C & K \\ I & K & D \end{vmatrix} + \begin{vmatrix} A & H & I \\ H & B & J \\ I & J & D \end{vmatrix} + \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \\ \Omega = \begin{vmatrix} A & H & G & I \\ H & B & F & J \\ G & F & C & K \\ I & J & K & D \end{vmatrix} \end{cases}$$

$$* \begin{cases} \chi = \xi\lambda + \eta\mu + \zeta\nu \\ \alpha = \xi^2 + \eta^2 + \zeta^2 + \lambda^2 + \mu^2 + \nu^2 \end{cases}$$

$$\Delta = \begin{vmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{vmatrix}$$

Case B_4 (cont.)

$$\beta_1 = a(\xi^2+\mu^2+\nu^2) + b(\eta^2+\nu^2+\lambda^2) + c(\zeta^2+\lambda^2+\mu^2) + d(\xi^2+\eta^2+\zeta^2)$$
$$+2f(\eta\zeta-\mu\nu) + 2g(\zeta\xi-\nu\lambda) + 2h(\xi\eta-\lambda\mu)$$
$$+2i(\zeta\mu-\eta\nu) + 2j(\xi\nu-\zeta\lambda) + 2k(\eta\lambda-\xi\mu)$$

$$\beta_2 = a(\lambda^2+\eta^2+\zeta^2) + b(\mu^2+\zeta^2+\xi^2) + c(\nu^2+\xi^2+\eta^2) + d(\lambda^2+\mu^2+\nu^2)$$
$$+2f(\mu\nu-\eta\zeta) + 2g(\nu\lambda-\zeta\xi) + 2h(\lambda\mu-\xi\eta)$$
$$+2i(\eta\nu-\zeta\mu) + 2j(\zeta\lambda-\xi\nu) + 2k(\xi\mu-\eta\lambda)$$

$$\gamma_1 = \xi^2(ad-i^2)+\xi\eta(hd-ij)+\xi\zeta(gd-ik)+\xi\lambda(hk-gj)+\xi\mu(gi-ak)+\xi\nu(aj-hi)$$
$$+\eta\xi(hd-ji)+\eta^2(bd-j^2)+\eta\zeta(fd-jk)+\eta\lambda(bk-fj)+\eta\mu(fi-hk)+\eta\nu(hj-bi)$$
$$+\zeta\xi(gd-ki)+\zeta\eta(fd-kj)+\zeta^2(cd-k^2)+\zeta\lambda(fk-cj)+\zeta\mu(ci-gk)+\zeta\nu(gj-fi)$$
$$+\lambda\xi(hk-jg)+\lambda\eta(bk-jf)+\lambda\zeta(fk-jc)+\lambda^2(bc-f^2)+\lambda\mu(fg-hc)+\lambda\nu(hf-bg)$$
$$+\mu\xi(gi-ka)+\mu\eta(fi-kh)+\mu\zeta(ci-kg)+\mu\lambda(fg-ch)+\mu^2(ca-g^2)+\mu\nu(gh-fa)$$
$$+\nu\xi(aj-ih)+\nu\eta(hj-ib)+\nu\zeta(gj-if)+\nu\lambda(hf-gb)+\nu\mu(gh-af)+\nu^2(ab-h^2)$$

$$\gamma_2 = \lambda^2(ad-i^2)+\lambda\mu(hd-ij)+\lambda\nu(gd-ik)+\lambda\xi(hk-gj)+\lambda\eta(gi-ak)+\lambda\zeta(aj-hi)$$
$$+\mu\lambda(hd-ji)+\mu^2(bd-j^2)+\mu\nu(fd-jk)+\mu\xi(bk-fj)+\mu\eta(fi-hk)+\mu\zeta(hj-bi)$$
$$+\nu\lambda(gd-ki)+\nu\mu(fd-kj)+\nu^2(cd-k^2)+\mu\xi(fk-cj)+\mu\eta(ci-gk)+\mu\zeta(gj-fi)$$
$$+\xi\lambda(hk-jg)+\xi\mu(bk-jf)+\xi\nu(fk-jc)+\xi^2(bc-f^2)+\xi\eta(fg-hc)+\xi\zeta(hf-bg)$$
$$+\eta\lambda(gi-ka)+\eta\mu(fi-kh)+\eta\nu(ci-kg)+\eta\xi(fg-ch)+\eta^2(ca-g^2)+\eta\zeta(gh-fa)$$
$$+\zeta\lambda(aj-ih)+\zeta\mu(hj-ib)+\zeta\nu(gj-if)+\zeta\xi(hf-gb)+\zeta\eta(gh-af)+\zeta^2(ab-h^2)$$

$$\delta_1 = \begin{vmatrix} b & f & j \\ f & c & k \\ j & k & d \end{vmatrix}(\xi^2+\mu^2+\nu^2) + \begin{vmatrix} a & g & i \\ g & c & k \\ i & k & d \end{vmatrix}(\eta^2+\nu^2+\lambda^2) + \begin{vmatrix} a & h & i \\ h & b & j \\ i & j & d \end{vmatrix}(\zeta^2+\lambda^2+\mu^2) + \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}(\xi^2+\eta^2+\zeta^2)$$
$$-2\begin{vmatrix} a & h & i \\ g & f & k \\ i & j & d \end{vmatrix}(\eta\zeta-\mu\nu) + 2\begin{vmatrix} h & b & j \\ g & f & k \\ i & j & d \end{vmatrix}(\zeta\xi-\nu\lambda) - 2\begin{vmatrix} h & f & j \\ g & c & k \\ i & k & d \end{vmatrix}(\xi\eta-\lambda\mu)$$
$$-2\begin{vmatrix} h & b & f \\ g & f & c \\ i & j & k \end{vmatrix}(\zeta\mu-\eta\nu) + 2\begin{vmatrix} a & h & g \\ g & f & c \\ i & j & k \end{vmatrix}(\xi\nu-\zeta\lambda) - 2\begin{vmatrix} a & h & g \\ h & b & f \\ i & j & k \end{vmatrix}(\eta\lambda-\xi\mu)$$

$$\delta_2 = \begin{vmatrix} b & f & j \\ f & c & k \\ j & k & d \end{vmatrix}(\lambda^2+\eta^2+\zeta^2) + \begin{vmatrix} a & g & i \\ g & c & k \\ i & k & d \end{vmatrix}(\mu^2+\zeta^2+\xi^2) + \begin{vmatrix} a & h & i \\ h & b & j \\ i & j & d \end{vmatrix}(\nu^2+\xi^2+\eta^2) + \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}(\lambda^2+\mu^2+\nu^2)$$
$$-2\begin{vmatrix} a & h & i \\ g & f & k \\ i & j & d \end{vmatrix}(\mu\nu-\eta\zeta) + 2\begin{vmatrix} h & b & j \\ g & f & k \\ i & j & d \end{vmatrix}(\nu\lambda-\zeta\xi) - 2\begin{vmatrix} h & f & j \\ g & c & k \\ i & k & d \end{vmatrix}(\lambda\mu-\xi\eta)$$
$$-2\begin{vmatrix} h & b & f \\ g & f & c \\ i & j & k \end{vmatrix}(\eta\nu-\zeta\mu) + 2\begin{vmatrix} a & h & g \\ g & f & c \\ i & j & k \end{vmatrix}(\zeta\lambda-\xi\nu) - 2\begin{vmatrix} a & h & g \\ h & b & f \\ i & j & k \end{vmatrix}(\xi\mu-\eta\lambda)$$

Case B_4 (cont.)

where:

$$\begin{cases} a = u_{11} \quad b = u_{22} \quad c = u_{33} \quad d = u_{44} \\ 2f = u_{23}+u_{32} \quad 2g = u_{31}+u_{13} \quad 2h = u_{12}+u_{21} \quad 2i = u_{14}+u_{41} \quad 2j = u_{24}+u_{42} \quad 2k = u_{34}+u_{43} \\ 2\xi = u_{23}-u_{32} \quad 2\eta = u_{31}-u_{13} \quad 2\zeta = u_{12}-u_{21} \quad 2\lambda = u_{14}-u_{41} \quad 2\mu = u_{24}-u_{42} \quad 2\nu = u_{34}-u_{43} \end{cases}$$

Either:

$$\begin{cases} A = u_{11}^2 + u_{21}^2 + u_{31}^2 + u_{41}^2 & B = u_{12}^2 + u_{22}^2 + u_{32}^2 + u_{42}^2 \\ C = u_{13}^2 + u_{23}^2 + u_{33}^2 + u_{43}^2 & D = u_{14}^2 + u_{24}^2 + u_{34}^2 + u_{44}^2 \\ F = u_{12}u_{13} + u_{22}u_{23} + u_{32}u_{33} + u_{42}u_{43} & I = u_{11}u_{14} + u_{21}u_{24} + u_{31}u_{34} + u_{41}u_{44} \\ G = u_{13}u_{11} + u_{23}u_{21} + u_{33}u_{31} + u_{43}u_{41} & J = u_{12}u_{14} + u_{22}u_{24} + u_{32}u_{34} + u_{42}u_{44} \\ H = u_{11}u_{12} + u_{21}u_{22} + u_{31}u_{32} + u_{41}u_{42} & K = u_{13}u_{14} + u_{23}u_{24} + u_{33}u_{34} + u_{43}u_{44} \end{cases}$$

or:

$$\begin{cases} A = u_{11}^2 + u_{12}^2 + u_{13}^2 + u_{14}^2 & B = u_{21}^2 + u_{22}^2 + u_{23}^2 + u_{24}^2 \\ C = u_{31}^2 + u_{32}^2 + u_{33}^2 + u_{34}^2 & D = u_{41}^2 + u_{42}^2 + u_{43}^2 + u_{44}^2 \\ F = u_{21}u_{31} + u_{22}u_{32} + u_{23}u_{33} + u_{24}u_{34} & I = u_{11}u_{41} + u_{12}u_{42} + u_{13}u_{43} + u_{14}u_{44} \\ G = u_{31}u_{11} + u_{32}u_{12} + u_{33}u_{13} + u_{34}u_{14} & J = u_{21}u_{41} + u_{22}u_{42} + u_{23}u_{43} + u_{24}u_{44} \\ H = u_{11}u_{21} + u_{12}u_{22} + u_{13}u_{23} + u_{14}u_{24} & K = u_{41}u_{11} + u_{42}u_{12} + u_{43}u_{13} + u_{44}u_{14} \end{cases}$$

Independent set: $\theta, \varphi, \psi, \omega, \chi, \alpha, \beta_1$ or β_2, γ_1, γ_2, δ_1 or δ_2

Relations between solutions:

$$\begin{cases} \beta_1 + \beta_2 = \alpha\theta \\ \delta_1 + \delta_2 = \alpha\varphi \end{cases}$$

$$\begin{cases} \Theta = 2\alpha - 2\varphi + \theta^2 \\ \Phi = (\alpha - \varphi)^2 + 2\theta(\beta_1 - \psi) + 2(2\gamma_2 - \gamma_1 + \chi^2 + \omega) \\ \Psi = \psi^2 + \beta_1^2 - \varphi\omega - 2\varphi\chi^2 \\ \Omega = \Delta^2 \end{cases}$$

$$\Delta = \omega + \gamma_1 + \chi^2$$

$S_1 = R_1 = \theta$, S_2, S_3, R_2, R_3 as in Case B_3

<u>Case B_n</u> $p = \frac{n(n+1)}{2}$ $m = n^2$ Solutions are:

$$\begin{cases} S_q = \sum_{a_1=1}^{a_1=n} \sum_{a_2=1}^{a_2=n} \sum_{a_3=1}^{a_3=n} \cdots \sum_{a_q=1}^{a_q=n} u_{a_1 a_2} \cdot u_{a_2 a_3} \cdots u_{a_{q-1} a_q} \cdot u_{a_q a_1} & q = 1, 2, \cdots \\ R_q = \sum_{a_1=1}^{a_1=n} \sum_{a_2=1}^{a_2=n} \sum_{a_3=1}^{a_3=n} \cdots \sum_{a_q=1}^{a_q=n} (u_{a_2 a_1}) u_{a_2 a_3} \cdot u_{a_3 a_4} \cdots u_{a_{q-1} a_q} \cdot u_{a_q a_1} & q = 1, 2, \cdots \end{cases}$$

The only linear solution is: $\theta = S_1 = R_1$.

The only quadratic solutions are: S_2, R_2, and combinations of S_2, R_2, S_1^2

(except χ when $n = 4$)

<u>Case C_2</u> $p = 5$ $p' = 1$ $m = 6$ Solutions are:

A_2 (from Case A_2), θ, φ, α (from Case B_2), and

$$\begin{cases} U_1 = u_1^2 u_{11} + u_2^2 u_{22} + u_1 u_2 (u_{12} + u_{21}) \\ U_2 = u_1^2 u_{22} + u_2^2 u_{11} - u_1 u_2 (u_{12} + u_{21}) \\ U_3 = u_1^2 u_{12} - u_2^2 u_{21} - u_1 u_2 (u_{11} - u_{22}) \\ U_4 = u_1^2 u_{21} - u_2^2 u_{12} - u_1 u_2 (u_{11} - u_{22}) \end{cases}$$

Independent set: $A_2, \theta, \varphi, \alpha$ and one of U_1, U_2, U_3, U_4

Relations between solutions

$$\begin{cases} U_1 + U_2 = A_2 \theta \\ U_3 - U_4 = A_2 \sqrt{4\alpha} \end{cases}$$

<u>Case C_3</u> $p = 9$ $p' = 2$ $m = 12$ Solutions are:

A_3 (from Case A_3), $\theta, \varphi, \psi, \alpha, \beta, \gamma$ (from Case B_3), and

$$\begin{cases} V = u\xi + v\eta + w\zeta \\ W = u(a\xi + h\eta + g\zeta) + v(h\xi + b\eta + f\zeta) + w(g\xi + f\eta + c\zeta) \end{cases}$$

where:

$u = u_1$ $v = u_2$ $w = u_3$ and other letters as in Case B_3

*Independent set: $A_3, \theta, \varphi, \psi, \alpha, \beta, \gamma, V, W$