AN ARITHMETICAL THEOREM
FOR PARTIALLY ORDERED SETS

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ABSTRACT

The simplest multiplicative systems in which arithmetical ideas can be defined are semigroups. For such systems irreducible (prime) elements can be introduced and conditions under which the fundamental theorem of arithmetic holds have been investigated (Clifford (3)). After identifying associates, the elements of the semigroup form a partially ordered set with respect to the ordinary division relation. This suggests the possibility of an analogous arithmetical result for abstract partially ordered sets. Although nothing corresponding to product exists in a partially ordered set, there is a notion similar to g.c.d.

This is the meet operation, defined as greatest lower bound. Thus irreducible elements, namely those elements not expressible as meets of proper divisors can be introduced. The assumption of the ascending chain condition then implies that each element is representable as a reduced meet of irreducibles. The central problem of this thesis is to determine conditions on the structure of the partially ordered set in order that each element have a unique such representation.

Part I contains preliminary results and introduces the principal tools of the investigation. In the second part, basic properties of the lattice of ideals and the connection between its structure and the irreducible decompositions of elements are developed. The proofs of these results are identical with the corresponding ones for the lattice case (Dilworth (2)). The last part contains those results whose proofs are peculiar to partially ordered sets and also contains the proof of the main theorem.
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PART I
DEFINITIONS AND PRELIMINARY RESULTS

A set $S$ is partially ordered by the inclusion relation $\subseteq$ if for all $x$, $y$ and $z$ in $S$

1. $x \subseteq x$
2. $x \subseteq y$ and $y \subseteq x$ imply $y = x$
3. $x \subseteq y$ and $y \subseteq z$ imply $x \subseteq z$.

Proper inclusion will be denoted by $\subsetneq$.

Definition 1.1 Given a set of elements $a_1, \ldots, a_n$ in $S$, if an element $a$ exists in $S$ such that

1. $a_i \subseteq a$ for all $i$ and
2. $a_i \subseteq b$ for all $i$ implies $a \subseteq b$, then $a$ will be called the meet of the $a_i$. The meet if it exists is unique and will be written $a = \wedge a_i$ or $a = \bigwedge a_i$.

Lemma 1.1 If $a = \bigwedge a_i$ and $a_i = \bigwedge a_i'$, then $a = \bigwedge a_i'$.

Clearly $a \subseteq a_i$ for all $i$ and $j$. If $b \subseteq a_i$ for all $i$ and $j$, then $b \subseteq a_i$ for all $i$ and hence $b \subseteq a$.

This is a sort of associative law for meets.

Definition 1.2 If $a = \bigwedge a_i, \ldots, a_n$ always implies $a = a_i$ for some $i$, then $a$ will be called (meet) irreducible.

Theorem 1.1 If $S$ satisfies the ascending chain condition, then every element of $S$ is representable as a meet of irreducibles.

A maximal element is already irreducible. Let the theorem hold for all proper divisors of $a$. If $a$ is irreducible, $a$ is
the desired representation; otherwise there exist \( a_i \) so \( a = \bigwedge a_i \), where \( a_i \supset a \). Then the induction hypothesis implies irreducibles \( q_{i,j} \) exist so that \( a_i = \bigcap q_{i,j} \) for each \( i \). By lemma 1.1, \( a = \bigcap q_{i,j} \) and the theorem follows by induction for all \( a \) in \( S \).

**Definition 1.3** A subset \( A \) of \( S \) will be called an **ideal** (dual) if

1. \( x \in A \) and \( y \supseteq x \) imply \( y \in A \) and
2. \( x = \bigcap x_i \) and \( x_i \in A \) imply \( x \in A \).

For any element \( x \) of \( S \) the set of elements \( y \) such that \( y \supseteq x \) is an ideal called the **principal** ideal generated by \( x \) and denoted by \( (x) \). The set of ideals of \( S \) forms a partially ordered set, \( L \), under the relation \( A \supseteq B \) if and only if \( A \) is a subset of \( B \). The inclusion relation has been inverted so that \( (x) \supseteq (y) \) in \( L \) if and only if \( x \supseteq y \) in \( S \).

**Lemma 1.2** \( L \) is a complete lattice.

Given a set of ideals \( A_\sigma \) of \( L \), their set meet is evidently an ideal and furnishes their lattice union \( \bigcup A_\sigma \). \( L \) also has a null element, the ideal \( S \) itself. Hence (Birkhoff (1) ch. 4, thm. 2) \( L \) is a complete lattice.

If \( A_1 \supseteq \ldots \supseteq A_\sigma \supseteq \ldots \), then \( \bigcap A_\sigma \) is the set join of the \( A_\sigma \).

**Corollary 1.1** \( (a) = \bigcap (a_\sigma) \) if and only if \( a = \bigwedge a_\sigma \).

If \( (a) = \bigcap (a_\sigma) \), then \( (a_\sigma) \supseteq (a) \) and hence \( a_\sigma \supseteq a \) for all \( i \).

If \( b \subset a_\sigma \) for all \( i \), then \( (b) \subset (a_\sigma) \) and \( (b) \subset \bigcap (a_\sigma) = (a) \). Hence \( b \subset a \) and \( a = \bigcap a_\sigma \). Similarly \( a = \bigcap a_\sigma \) implies \( (a) = \bigcap (a_\sigma) \).

The following lemma provides a constructive definition of ideal meet.
Lemma 1.3 \( \bigcap A_\nu \) may be defined inductively as the set union of \( S_0, \ldots, S_k, \ldots \), where \( S_0 \) is the set union of the \( A_\nu \) and \( S_1 \) consists of all elements of \( S \) containing meets of elements of \( S_{k-1} \).

By the definition of ideal and ideal inclusion \( \bigcap A_\nu \) must contain at least all the elements mentioned, and these clearly form an ideal.

Arguments will frequently be made using induction on the index \( k \) of \( S_k \). Hence the following obvious sharpening of lemma 1.3 will be convenient.

Lemma 1.4 If \( S \) is a partially ordered set in which every element is expressible as a meet of irreducibles, then \( \bigcap A_\nu \) is the set union \( S_0, \ldots, S_k, \ldots \), where \( S_0 \) is the set union of the \( A_\nu \) and \( S_1 \) consists of all elements of \( S \) containing meets of irreducibles of \( S_{k-1} \).

If \( a \in S_k \), then \( a \geq b = \bigwedge q_{i,j} \) for \( a_{i,j} \in S_{k-1} \). If \( a_{i,j} = \bigwedge q_{i,j} \) for irreducibles \( q_{i,j} \), then the definition of \( S_{k-1} \) implies \( a_{i,j} \in S_{k-1} \) while lemma 1.1 implies \( a \geq b = \bigwedge q_{i,j} \).

Theorem 1.2 If \( (a) = \bigcap A_{\alpha} \) then \( a = \bigwedge a_{i,j} \) where \( a_{i,j} \in A_{\alpha} \).

Let \( S \) be the classes of lemma 1.3 for the meet \( \bigcap A_{\alpha} \) and suppose \( a \in S_k \) for \( k > 1 \). Then \( a \geq \bigwedge a_{i,j} \) for \( a_{i,j} \in S_{k-1} \). Since each \( a_{i,j} \in (a) \) it follows that \( a = \bigwedge a_{i,j} \). Since \( a_{i,j} \in S_{k-1} \), there are \( b_{i,j} \in S_{k-2} \) such that for each \( i \) \( a_{i,j} \geq \bigwedge b_{i,j} \). Then \( a = \bigwedge b_{i,j} \in S_{k-1} \) since every \( b_{i,j} \geq a \), while if \( b \leq b_{i,j} \) for all \( i \) and \( j \) then \( b \leq a_{i,j} \) for all \( i \) and \( b \leq \bigwedge a_{i,j} = a \). After \( k-1 \) such steps it follows that \( a \in S_i \) or \( a = \bigwedge a_{i,j} \) with \( a_{i,j} \in A_{\alpha} \).
In most applications of theorem 1.2 every element of $S$ will be representable as a meet of irreducibles. In this case the elements $a_{i,j}$ may be taken to be irreducible.

**Lemma 1.5** $(q)$ is irreducible in $L$ if and only if $q$ is irreducible in $S$.

If $q = \bigcap x_i$ for $x_i \supseteq q$ then corollary 1.1 implies $(q) = \bigcap (x_i)$ for $(x_i) \supseteq (q)$. Conversely suppose $q$ is irreducible and let $(q) = A \cap B$. Then theorem 1.2 implies there exist $a_i \in A$, $b_i \in B$ such that $q = a_i \cdots a_n b_i \cdots b_n$. By irreducibility of $q$ either $q = a_i$ and $(q) \supseteq A$ or $q = b_i$ and $(q) \supseteq B$. Hence $(q) = A$ or $(q) = B$, and $(q)$ is irreducible.

**Lemma 1.6** Let every element of $S$ be expressible as a meet of irreducibles. Then $A \supseteq B$ implies there exists an irreducible $q$ such that $q \in B$ but $q \notin A$.

Let $b \in B$, $b \notin A$ and $b = \bigcap q_i$ for irreducibles $q_i$. Since $b \notin A$, there is an $i$ so $q_i \notin A$, and $q_i \supseteq b$ implies $q_i \in B$.

**Lemma 1.7** Let $M_1 \supseteq \ldots \supseteq M_{\nu} \supseteq \ldots$ be a chain of ideals of $L$ such that every $M_{\nu} \supseteq (a)$ and $P = \bigcap M_{\nu}$. Then $P \supseteq (a)$.

For if $P = (a)$, then $a \in M_{\nu}$ for some $\nu$, contradicting $M_{\nu} \supseteq (a)$.

In any partially ordered set $x$ is said to cover $y$ (written $x \triangleright y$) if $x \supset y$ while no $z$ exists for which $x \supset z \supset y$.

**Theorem 1.3** Let $B \supseteq (a)$ in $L$. Then there exists an ideal $P$ such that $B \supseteq P \supset (a)$.
This follows by applying lemma 1.7 and the maximum principle to the set of ideals \( N \) such that \( B \supseteq N \Rightarrow (a) \).

**Definition 1.4** An ideal \( A \) will be called **upper semi-modular** in \( L \) if \( B \supseteq A, C \supset A \) and \( B \not\supset C \) imply \( B \cup C \supset B \).

The partially ordered set \( S \) will be called **upper semi-modular** if \((x)\) is upper semi-modular in \( L \) for every \( x \) in \( S \).

The following four lemmas concern a single upper semi-modular element of a lattice.

**Lemma 1.8** Let \( A \) be an upper semi-modular element of a lattice \( L \) and \( A_1, \ldots, A_n \supset A \). Then each union independent set of the \( A_i \) is contained in a maximal independent set. The union of the \( A_i \) in a maximal independent set contains every \( A_i \).

Let \( M = (A_1, \ldots, A_r) \) be the given union independent set. Since there is only a finite number of subsets of \( A_1 \ldots A_n \) containing \( A_1 \ldots A_r \) a maximal independent set containing \( A_1 \ldots A_r \) exists.

Let \( (A_1, \ldots, A_s) \) be maximal union independent. Suppose \( U A_i = A_1 \cup \ldots \cup A_s \not\supset A_k \) where \( k > s \). Let \( A_i = A_1 \cup \ldots \cup A_{i-1} A_{i+1} \cup \ldots \cup A_s \), \( 1 \leq i \leq s \). Since \((A_1, \ldots, A_s, A_k)\) is dependent there is an \( i \) so \( A_i \cup A_k \subseteq U A_i \supseteq A_i \). By upper semi-modularity \( A_i \cup A_k \supset A_i \) and \( U A_i = A_i \cup A_k \supset A_i \). Hence \( A_i \cup A_k = U A_i \supseteq A_k \).

**Lemma 1.9** Let \( A \) be an upper semi-modular element of a lattice \( L \) and \( A_1, \ldots, A_s \supset A \). Then each union independent set of the \( A_i \) generates a Boolean algebra.

Let \( A \) and \( B \) be two subsets of the union independent set \((A_1, \ldots, A_n)\) and \( \sum_{A} \) denote the union of the elements of \( A \).
Let \( A \cup B \) and \( A \cap B \) denote set union and set meet of the sets \( A \) and \( B \). Evidently \( (\Sigma A) \cup (\Sigma B) = \Sigma (A \cup B) \), while \( \Sigma A = \Sigma B \) implies \( A = B \) by the independence of the \( A_i \), \( 0 \leq i \leq s \). Next it is shown that

\[
(1) \quad (\Sigma A) \cap (\Sigma B) = \Sigma (A \cap B).
\]

Let \( m(A) \) denote the number of elements in \( A \) and \( n(A) = n - m(A) \).

If \( n(A \cap B) = 0 \), then \( m(A \cap B) = n \) and \( A = B \). If \( n(A \cap B) = 1 \), either \( A \supseteq B \) or \( B \supseteq A \) and \( (1) \) holds. Let \( (1) \) hold for all \( A \) and \( B \) such that \( n(A \cap B) < l \) and suppose \( n(A \cap B) = l \) for some \( A \) and \( B \). Then \( m(A \cap B) = n - l = r \) so that \( A = (A_1, \ldots, A_r, A_{r+1}, \ldots, A_s) \) and \( B = (A_1, \ldots, A_r, A'_r, \ldots, A'_s) \). Since \( (1) \) is trivial if \( B \supseteq A \) it may be assumed that \( t > r \). Let \( B' = (A_1, \ldots, A_{r+t}, A'_r, \ldots, A'_r) \).

Now \( m(A \cap B') = r + 1 \) and hence \( n(A \cap B') = l - 1 < l \). By the induction assumption \( \Sigma (A \cap B') = (\Sigma A) \cap (\Sigma B') \). Hence \( \Sigma (A \cap B') = (\Sigma A) \cap (\Sigma B') \supseteq (\Sigma A) \cap (\Sigma B) \supseteq \Sigma (A \cap B) \). Since \( A_1, \ldots, A_n \) are independent, \( \Sigma (A \cap B) \not\supseteq A_{r+1} \) and hence \( \Sigma (A \cap B') \not\supseteq \Sigma (A \cap B) \). If \( \Sigma (A \cap B') = (\Sigma A) \cap (\Sigma B) \) then \( \Sigma B \supseteq A_{r+1} \), contradicting independence of \( A_1, \ldots, A_n \).

Hence \( (\Sigma A) \cap (\Sigma B) = \Sigma (A \cap B) \) and \( (1) \) holds for \( n(A \cap B) = l \). By induction \( (1) \) holds for all \( A \) and \( B \).

Thus the elements expressible as joins of \( A_1, \ldots, A_n \) are isomorphic to the subsets of \( (A_1, \ldots, A_n) \) under meet and join and \( A_1, \ldots, A_n \) generate a Boolean algebra.

**Lemma 1.10** Let \( A \) be an upper semi-modular element of a lattice \( L \) and \( A_1, \ldots, A_n \uparrow A \). Then any two maximal union independent sets of the \( A_i \) have the same number of elements and any element of one set may be replaced by a suitably chosen element of the other without altering the maximal property.
By lemma 1.8 if $A'_1, \ldots, A'_k$ and $A'', \ldots, A''_\alpha$ are two maximal independent sets of the $A_i$ then $\bigcup A'_i = \bigcup A''_i$. If every $A''_i \leq B = \bigcup A'_j$, then $B \supseteq \bigcup A''_j \supset A''_i$ contradicts independence of $A''_i$. Hence there exists an $A''_j$ such that $B \nsubseteq A''_j$. Then $\bigcup A'_i \supseteq B \cup A''_j \supset B$ by upper semi-modularity of $A_i$ and $B \cup A''_j = \bigcup A''_i$. Hence $A'_1, \ldots, A'_i, A''_j, \ldots, A'_k, A''_\alpha$, if independent are maximal independent. If not independent, since $B \supseteq A''_j$ there exists an $A_i$ such that $C \cup A''_i \subseteq A_i$ where $C = \bigcup_{h \neq j} A'_h$. Then $\bigcup A'_i \supseteq C \cup A''_i \supset C$ contradicts $\bigcup A'_i \supseteq \bigcup A'_h \cup \bigcup A''_i$.

The replacement property implies $k = l$. For if $k < l$, after replacing every $A'_i$ by an $A''_i$ there would remain some $A''_i$ divisible by the union of the $A''_i$ that replaced the $A'_i$, contradicting the independence of the $A''_i$. Similarly $l \geq k$, and $l = k$.

**Lemma 1.11** Let $A$ be an upper semi-modular element of a lattice and $A_1, \ldots, A_\alpha \geq A$. Then any chain joining $\bigcup A_i$ to $A$ has not more than $k + 1$ distinct members.

Clearly $A_1, \ldots, A_\alpha$ may be supposed independent. Let $A = B_0 \leq B_1 \leq \ldots \leq B_\alpha = \bigcup A_i$ and suppose $l > k$. By upper semi-modularity $B_0 \leq B_0 \cup A_1 \leq \ldots \leq B_0 \cup A_1 \cup \ldots \cup A_{k-1} \leq A_1 \cup \ldots \cup A_\alpha$. Suppose it has been shown that $B_0 \leq B_1 \leq \ldots \leq B_\alpha \cup A_1 \leq \ldots \leq B_\alpha \cup A_1 \cup \ldots \cup A_{k-1} \leq A_1 \cup \ldots \cup A_\alpha$ where $k_i \leq k - i$ and $i < k$. Consider the chain $B_0 \leq B_1 \leq \ldots \leq B_{\alpha+1} \leq B_{\alpha+1} \cup A_1 \leq \ldots \leq B_{\alpha+1} \cup A_1 \cup \ldots \cup A_{k-1} \leq A_1 \cup \ldots \cup A_\alpha$ and assume all its members are distinct. If $B_i \cup A_1 \cup \ldots \cup A_{k-1} \geq B_{\alpha+1}$, then $A_1 \cup \ldots \cup A_{k-1} \geq B_{\alpha+1} \cup A_1 \cup \ldots \cup A_{k-1} \geq B_{\alpha+1} \cup A_1 \cup \ldots \cup A_{k-1}$. But $A_1 \cup \ldots \cup A_{k-1} \geq B_{\alpha+1} \cup A_1 \cup \ldots \cup A_{k-1}$ and hence $A_1 \cup \ldots \cup A_{k-1} = B_{\alpha+1} \cup A_1 \cup \ldots \cup A_{k-1}$ contrary to assumption. Thus $B_i \cup A_1 \cup \ldots \cup A_{k-1} \geq B_{\alpha+1}$. Continuing in this way, eventually $B_i \cup A_1 \supseteq B_{\alpha+1}$. But then $B_i \cup A_1 \supseteq B_{\alpha+1} \supset B_i$ and $B_i \cup A_1 \supseteq B_i$ imply $B_i \cup A_1 = B_i = B_{\alpha+1} \cup A_1$, contradicting the assump-
tion. Hence at least two members of the chain considered are equal. Thus (renumbering the \( A_i \) if necessary) \( B_0 \subset B_1 \subset \cdots \subset B_{i-1} \subset B_i \subset \cdots \subset B_{i+1} \subset B_i \subset \cdots \subset B_{i+1} \subset B_{i+1} \subset \cdots \subset B_{i+1} \subset \cdots \), where \( k_{i+1} \leq k_i - 1 \leq k - (i + 1) \). By induction \( B_0 \subset \cdots \subset B_{r-1} \subset A_1 \subset \cdots \subset A_k \) where \( r \leq k \). But then \( B_r = A_1 \subset \cdots \subset A_k \) and hence \( r = l \), contradicting \( l > k \). Thus \( l \leq k \) and the lemma follows.
PART II
UPPER SEMI-MODULAR PARTIALLY ORDERED SETS

This section contains some general theorems relating the irreducible decompositions of elements of \( S \) to structural properties of the lattice of ideals \( L \). All the proofs of these theorems are those of Dilworth (2).

Let \( U_\alpha \) denote the union of all ideals covering \((a)\) in \( L \) and let \( L_\alpha \) denote the quotient lattice \( U_\alpha/(a) \). Lemma 1.11 than implies.

**Lemma 2.1** Let \( S \) be upper semi-modular. Then if \( P_1, \ldots, P_K \) is a maximal independent set of point ideals of \( L_\alpha \) the length of any chain of \( L_\alpha \) is at most \( K \).

According to lemma 2.1, \( L_\alpha \) is Archimedean if and only if \( U_\alpha \) is the join of a finite number of point ideals of \( L_\alpha \).

**Theorem 2.1** Let \( S \) be upper semi-modular and every element be expressible as a meet of irreducibles. Then \( L_\alpha \) is Archimedean if and only if the number of components in the reduced irreducible decompositions of \( a \) is bounded.

(A) If \( L_\alpha \) is not Archimedean let \( n \) be any integer. Then lemma 2.1 implies there exist \( n \) union independent \( P_i \) covering \((a)\). By lemma 1.9 these generate a Boolean algebra. Let \( A_\alpha = \bigcup P_j \), then \((a) = \bigcap A_\alpha \) is a reduced representation \((a \in \) no meet of fewer \( A_\alpha \)\). By theorem 1.2 there are irreducibles \( q_{ij} \) in \( A_j \) so \( a = \bigcap q_{ij} \). After eliminating superfluous factors a reduced representation having at least \( n \) irreducible components is had,
since unless every $A_j$ is represented $a$ would be in a meet of fewer $A_j$. Hence the number of components in representations of $a$ is unbounded.

(B) Let the number of components be unbounded so that for each $k$ there exists a reduced decomposition into irreducibles, $a = q_1 \land \ldots \land q_n$ with $n \geq k$. Then by lemma 1.5 and corollary 1.1 $(a) = (q_1) \land \ldots \land (q_n)$ is a reduced decomposition into irreducibles in $L$. Let $Q'_a = \bigcap_{j \neq i} (q_j)$. Then $Q'_a \triangleright (a)$ and theorem 1.3 implies there are $P_i$ so $Q'_a \subseteq P_i \triangleright (a)$. If $\bigcup_{j \neq i} P_j \supseteq P_i$, then since $(q_i) \supseteq Q'_j$ for all $j \neq i$ it follows that $(q_i) \supseteq P_j$ for all $j \neq i$ and hence $(a) = (q_i) \land Q'_a \subseteq P_i \triangleright (a)$, a contradiction. Hence the $P_i$ are independent, for every $k$ there exist at least $k$ independent points of $L_a$ and $L_a$ is not Archimedean.

**Theorem 2.2** Let $S$ be upper semi-modular. Then if $L_a$ is Archimedean it is complemented and every ideal of $L_a$ is expressible as a meet of maximal ideals.

Let $A$ be in $L_a$ and $P_1, \ldots P_k$ be a maximal independent set of points of $L_a$ divisible by $A$. Extend $P_1, \ldots P_k$ to a maximal independent set $P_1, \ldots P_n$. Let $A' = P_{k+1} \lor \ldots \lor P_n$. Then $A \lor A' = \bigcup_{a}$ and $A \lor A' = \bigcup_{a}$. Suppose $A \land A' \neq (a)$. Then by theorem 1.3, there is a $P$ so $A \land A' \supseteq P \triangleright (a)$. Since $A \supseteq P$ the maximal property of $P_1, \ldots P_k$ implies $P_{k+1} \lor \ldots \lor P_k \supseteq P$ by lemma 1.8. Since by lemma 1.9 the $P_1, \ldots P_n$ generate a Boolean algebra, $(a) = (P_{k+1} \lor \ldots \lor P_k) \land A' \supseteq P \triangleright (a)$, a contradiction. Hence $A \land A' = (a)$ and $L_a$ is complemented.

Now let $Q$ be irreducible in $L_a$. Let $P_1, \ldots P_k$ be a maximal independent set of points of $L_a$ divisible by $Q$ and let this set
be extended to the maximal independent set \( P_1 \ldots P_n \). Then \( Q \not\leq P_{k+1}, \ldots, P_n \) and hence by upper semi-modularity \( Q \cdot P_i \rhd Q \) for \( i = k + 1, \ldots, n \). Since \( Q \) is irreducible \( Q \cdot P_{k+1} = \ldots = Q \cdot P_n \); hence \( U_a = Q \cdot U_a = Q \cdot P_{k+1} \ldots \cdot P_n = Q \cdot P_{k+1} \cdot Q \) and \( Q \) is maximal in \( L_a \). Since \( L_a \) is Archimedean every ideal is expressible as a meet of irreducibles of \( L_a \) and hence as a meet of maximal ideals of \( L_a \).

If \( L_a \) is Archimedean an arbitrary complement of \( A \) in \( L_a \) will be denoted by \( A' \).

In the following a connection is established between the irreducible representations of \( a \) in \( S \) and certain representations of \( (a) \) in \( L_a \).

**Definition 2.1** An ideal \( C \not\geq U_a \) of \( L_a \) is called **characteristic** if there is an irreducible \( q \) of \( S \) dividing exactly the same point ideals of \( L_a \) as \( C \).

**Theorem 2.3** An element \( a \) of \( S \) has a reduced representation \( a = \bigcap q_i \) for irreducibles \( q_i \) if and only if \( (a) \) has a reduced representation \( (a) = \bigcap C_i \), where \( C_i \) are characteristic ideals of \( L_a \) such that \( q_i \in C_i \).

(A) Let \( a = \bigcap q_i \) be a reduced representation with irreducibles \( q_i \). If any \( q_i \in U_a \) then \( Q_i \cap (q_i) \geq (a) \) by corollary 1.1 since the representation is reduced. Hence there exists a \( P_i \) such that \( Q_i \geq P_i \cdot (a) \) and \( (a) = q_i \cap Q_i \geq P_i \cdot (a) \) is a contradiction. So \( q_i \not\in U_a \). Let \( C_i \subseteq (q_i) \) be a characteristic ideal of \( L_a \) associated with \( q_i \). One is always to hand since the union of those points of \( L_a \) divisible by \( (q_i) \) will serve
and is different from \( U_\alpha \) since \( q_\alpha \notin U_\alpha \). Then \( (a) = \bigcap (q_\alpha) \supseteq \bigcap C_\alpha \supseteq (a) \) implies \( (a) = \bigcap C_\alpha \). If this is not a reduced representation, \( C_\alpha \supseteq \bigcap_\beta C_\beta \) and \( Q_\alpha \supseteq P_\alpha \supseteq (a) \) together with the definition of characteristic ideal imply \( (a) = \bigcap C_\alpha \supseteq P_\alpha \supseteq (a) \), a contradiction. Hence the representation is reduced.

(B) Let \( (a) = \bigcap C_\alpha \) be a reduced representation with associated irreducibles \( q_\alpha \in C_\alpha \). If \( \bigcap (q_\alpha) \neq (a) \) there's a \( P \) so \( \bigcap (q_\alpha) \supseteq P \supseteq (a) \) and hence \( (a) = \bigcap C_\alpha \supseteq P \supseteq (a) \), a contradiction. Hence \( a = \bigcap q_\alpha \). If this is not reduced, part A of the proof implies \( \bigcap C_\alpha \) is not reduced.

The next theorem gives a characterization of characteristic ideals in terms of the structure of \( L_\alpha \).

**Theorem 2.4** Let \( S \) be upper semi-modular and each element expressible as a meet of irreducibles. Then if \( L_\alpha \) is Archimedean \( C \) is characteristic if and only if there exists an ideal \( R \) of \( L \) such that \( R \supseteq C, C' \circ R \supseteq R \) and \( C' \cap R = (a) \) for every \( C' \).

(A) Let such an ideal exist. Then \( U_\circ R = C' \circ R = C' \circ R \).

By lemma 1.6 there's an irreducible \( q \) so \( q \in R, q \notin C' \circ R = U_\circ R \).

Since \( (q) \supseteq R \supseteq C \), \( (q) \) divides every point ideal of \( L_\alpha \) that \( C \) does.

On the other hand let \( (q) \supseteq P \supseteq (a) \). Then if \( R \neq P \), \( C' \circ R = U_\circ R \supseteq P \circ R \supseteq R \). Hence \( C' \circ R = P \circ R \) and \( (q) \supseteq P \circ R = C' \circ R = U_\circ R \), contradicting the choice of \( q \). Hence \( R \supseteq P \). If \( C \neq P \), then \( C' \supseteq P \) for some \( C' \), and \( (a) = C' \cap R \supseteq P \supseteq (a) \), which is impossible. Hence \( (q) \supseteq P \) implies \( C \supseteq P \) and \( C \) is characteristic.

(B) Let \( C \) be characteristic and \( q \) an associated irreducible; \( (q) \) will be shown to have the properties required of \( R \). Since
C \neq U_a, C' \neq (a) and hence P exists so \( C' \geq P \triangleright (a) \). Since \( q \) is irreducible (lemma 1.5), \((q) \triangleright P = (q) \triangleright U_a\). Hence \((q) \triangleright U_a = (q) \triangleright C' = (q) \triangleright P \triangleright (q)\). If \( C' \cap (q) \neq (a) \) there is a \( P \) so \( C' \cap (q) \geq P \triangleright (a) \) and hence \( C' \cap C \geq P \triangleright (a) \) since \( C \) is characteristic associated with \( P \). This is impossible; hence \( C' \cap (q) = (a) \).

**Corollary 2.1** Each maximal ideal of \( L_a \) is characteristic.

For the \( R \) of theorem 2.4 the maximal ideal itself may be taken.

**Theorem 2.5** Let \( S \) be upper semi-modular and every element be expressible as a meet of irreducibles. Then if \( L_a \) is Archimedean each characteristic ideal \( C \) of \( L_a \) occurs in a reduced representation \((a) = C \cap C_1 \cap ... \cap C_k\), where \( k \) is the number of maximal independent point ideals of \( L_a \) divisible by \( C \) and \( C_i \) are characteristic ideals of \( L_a \).

Let \( P_1 \ldots P_k \) be a maximal independent set of point ideals divisible by \( C \) and imbed them in a maximal independent set \( P_1 \ldots P_n \). Let \( C_i = \bigcup P_i \) for \( i = 1, \ldots, k \). If \( C \cap C_1 \cap ... \cap C_k \neq (a) \), there is a \( P \) so \( C \cap C_1 \cap ... \cap C_k \supseteq P \triangleright (a) \) and \( C \supseteq P \) implies \( P_1 \cap ... \cap P_k \supseteq P \) by lemma 1.8. Since \( P_1 \ldots P_n \) generate a Boolean algebra \((a) = (P_1 \cap ... \cap P_k) \cap C_1 \cap ... \cap C_k \supseteq P \triangleright (a)\), a contradiction. Hence \((a) = C \cap C_1 \cap ... \cap C_k\). Since \( C \cap C_1 \cap ... \cap C_i < C_{i+1} \cap ... \cap C_k \supseteq P_1 > (a) \) the representation is reduced. Since \( C_i \) are maximal ideals of \( L_a \) they are characteristic.

**Corollary 2.2** Let \( S \) be upper semi-modular and every element be expressible as a meet of irreducibles. Then if \( L_a \) is Archi-
medean of length $k$, $a$ has a reduced decomposition into irreducibles with $k$ components.

By lemma 1.9, theorem 2.5, and theorem 2.4.
PART III

PARTIALLY ORDERED SETS WITH UNIQUE DECOMPOSITIONS

The object of this section is to characterise those partially ordered sets for which every element has a unique irreducible decomposition. The main result is

**Theorem 3.1.** Let $S$ satisfy the ascending chain condition. Then each element of $S$ has a unique representation as a reduced meet of irreducibles if and only if $S$ is upper semi-modular and $L_\alpha$ is a Boolean algebra for each $\alpha$.

This theorem will follow from a series of lemmas the first of which proves the necessity of these conditions.

**Lemma 3.1** Let $S$ satisfy the ascending chain condition and each element of $S$ have a unique representation as a reduced meet of irreducibles. Then $S$ is upper semi-modular and $L_\alpha$ is a Boolean algebra for each $\alpha$.

(A) Let $B \supset (a)$, $C \supset (a)$, $C \not\supset B$ and suppose there is a $D$ so $B \supset C \supset D \supset C$. By lemma 1.6 there are irreducibles $q_c$ and $q_d$ in $S$ such that $q_c \in C$, $q_c \not\in D$, $q_d \in D$ and $q_d \not\in B \cup C$ (hence $q_d \not\in B$).

Now $B \supset B \cap (q_c) \supset (a)$. If $B = B \cap (q_c)$ then $q_c \in B$, and $q_c \in B \cup C \supset D$ implies $q_c \in D$, a contradiction. Hence $(a) = B \cap (q_c)$. Also $B \supset B \cap (q_d) \supset (a)$ and if $B = B \cap (q_d)$ then $q_d \in B$, a contradiction; so $(a) = B \cap (q_d)$.

By theorem 1.2 there are irreducibles $b_1$, $b'_1$ in $B$ and $c_1$, $d_1$ in $(q_c)$, $(q_d)$ respectively such that $a = b_1 \cdots b_m \cap c_1 \cdots \cap c_n = b'_1 \cdots b'_k \cap d_1 \cdots \cap d_l$. If $y \leq b_1$ for all $i$ and $y \leq q_c$ then $y \leq a$. Hence $a = b_1 \cap \cdots \cap b_m \cap q_c$. Similarly
Both representations may be assumed reduced. Unless \( q_c \) actually occurs after converting to a reduced representation it will follow that \( a = \bigwedge b_x \in B > (a) \), which is impossible. Similarly \( q_d \) actually occurs in the second representation. If \( q_c = q_d \) then \( q_c \) is in \( D \), while if \( q_c = b_x \), then \( a \in B > (a) \), a contradiction in either case. Hence the two representations are distinct. This contradicts uniqueness; hence \( B \cup C < C \), and \( S \) is upper semi-modular.

The necessity of upper semi-modularity in the lattice case was first noticed by Morgan Ward.

(B) Since the decomposition is unique the number of components is bounded for any \( a \) and \( L_a \) is Archimedean by Theorem 2.1. Let \( P_1 \ldots P_\kappa \) be a maximal independent set of point ideals of \( L_a \). By lemma 1.9 they generate a Boolean algebra having maximal ideals \( M_1 \ldots M_\kappa \). Since \( \bigcup a \bigcup P_\xi \bigcup M_\xi \), the \( M_\xi \) are also maximal ideals of \( L_a \) and hence are characteristic by corollary 2.1. Since \( (a) = \bigwedge M_\xi \), \( a \) has a reduced decomposition into irreducibles, \( a = \bigwedge q_\xi \), with \( q_\xi \in M_\xi \). Let \( M \) be any other maximal ideal of \( L_a \). Then there is an irreducible \( q \in M \) so that \( q \notin \bigcup a \). \( M \) is a characteristic ideal associated with the irreducible \( q \). By theorems 2.5 and 2.3 \( q \) is a component in some decomposition of \( a \). Since the decompositions are assumed unique \( q = q_\xi \) for some \( i \), and \( q \in M \cup M_\xi = \bigcup a \) is a contradiction unless \( M = M_\xi \). Hence \( M_\xi \) are all the maximal ideals of \( L_a \), and by theorem 2.2 all the elements of \( L_a \) are in the Boolean algebra generated by the \( P_\xi \).

**Lemma 3.2** If \( L_a \) is a Boolean algebra it is Archimedean.
If $L_\alpha$ has an infinite number of point ideals let $P_1, \ldots, P_n$ be a denumerable sequence of them and define $P'_n = \bigcap_{i \neq n} P_i$. Since $L_\alpha$ is a Boolean algebra, $(a) = \bigcap P'_i$. Because an infinite meet of ideals consists of all elements in finite meets $(a) = P'_n \cdots \bigcap P'_k$ for some $k$. Then $(a) \not\supseteq P_{k+1} \supseteq (a)$, a contradiction; hence $L_\alpha$ has only a finite number of point ideals and is Archimedean by lemma 2.1.

**Lemma 3.3** Let $S$ be upper semi-modular and satisfy the ascending chain condition. Let every three ideals covering a principal ideal generate a Boolean algebra and let $q$ be an irreducible of $S$ such that $q \supseteq a$, $B$ and $C \supseteq (a)$ and $B \neq C$. Then either $q \in B$ or $q \in C$.

The lemma is proved by a double induction, the first on the element $a$ using the ascending chain condition. The lemma holds vacuously if $a$ is maximal. Assume it holds for all proper divisors of $a$ in $S$. Since $B \wedge C = (a)$, the irreducible $q$ is distinct from $a$. Hence by theorem 1.3 there exists an ideal $A$ such that $(q) \supseteq A \supseteq (a)$. If $A \neq B, C$ the hypothesis of the lemma implies $A = (A \vee B) \wedge (A \vee C)$, while if $A = B$ or $A = C$ the conclusion is trivial.

It is convenient to prove the lemma for all irreducibles $q'$ in $A$. Let $S_0 \ldots S_k \ldots$ be the classes of lemma 1.4 for the meet of $A \vee B$ and $A \vee C$. The second induction will be on the index $k$ of $S_k$. If $q' \in S_k$, then $q' \in A \vee B \supseteq B$ or $q' \in A \vee C \supseteq C$ and the lemma holds. Now suppose the lemma is true for all $q' \in S_k$ for $k < n$ and let $q' \in S_n$, $q' \notin S_{n-1}$. By lemma 1.4 there
exist irreducibles $q_\gamma \in S_{\kappa-1}$ such that $q' \geq y = q_\gamma \ldots \cdot q_\kappa$, where the representation of $y$ is reduced.

If $l = 1$ then $q' \in S_{\kappa-1}$, hence $l > 1$. Then each $q_\gamma \in A$ and the induction hypothesis on $S_{\kappa}$ implies that for each $i$ either $q_\gamma \in B$ or $q_\gamma \in C$. If $(y) \cup C = (y) \cup B$, then $q_\gamma \in B \cup C$ for all $i$ and hence $(q') \supseteq (y) \supseteq B \cup C$ gives the lemma for $q'$. If $(y) \cup C \neq (y) \cup B$, by upper semi-modularity both $B' = (y) \cup B$ and $C' = (y) \cup C$ cover $(y)$. Since $(q') \supseteq (y) \supseteq A \bowtie (a)$ $y$ is a proper divisor of $a$. Hence by the first induction $q' \in B' \supseteq B$ or $q' \in C' \supseteq C$.

This proves the lemma for $S_{\kappa}$ and the induction is complete.

**Lemma 3.4** Let $S$ be upper semi-modular and satisfy the ascending chain condition, and let every three ideals covering a principal ideal generate a Boolean algebra. If $a = q_1 \cdot \ldots \cdot q_\kappa$ is a reduced decomposition into irreducibles, then $L_a$ is Archimedean of length $k$ and each $(q_\gamma)$ divides a maximal ideal of $L_a$.

Let $A_\gamma$ be the union of the point ideals of $L_a$ contained in $(q_\gamma)$. Then $A_\gamma$ is a characteristic ideal associated with the irreducible $q_\gamma$ (theorem 2.3). Since $A_\gamma \neq U_a$, there is a point $P$ such that $A_\gamma \not\supset P$. By lemma 3.3 $A_\gamma \supset P'$ for every point $P'$ of $L_a$ that is different from $P$. Hence $A_\gamma \cup P = U_a$ and $A_\gamma$ is maximal by upper semi-modularity.

Let $B_0 = U_a$ and $B_\gamma$ be the union of points of $L_{a_{\gamma}}$ divisible by $(q_1), \ldots, (q_\gamma)$. Then $B_1 = A_\gamma$ and $B_0 \supset B_1$. Evidently $B_{\gamma-1} \supset B_\gamma$. If $B_{\gamma-1} = B_\gamma$ then $\bigcap_{j \neq \gamma} (q_j) \supseteq P_\gamma \bowtie (a)$ since representation is reduced. Then $(q_1) \cap \ldots \cap (q_{\gamma-1}) \supseteq P_\gamma$ implies $B_\gamma \supseteq B_{\gamma-1} \supseteq P_\gamma$. This implies $(q_\gamma) \supseteq P_\gamma$ and hence $(a) = \bigcap (q_\gamma) \supseteq P_\gamma \bowtie (a)$, a contradiction. Hence $B_{\gamma-1} \neq B_\gamma$. Let $P$ and $P'$ be two point ideals of $L_a$.
divisible by $B_{k-1}$. Since $q_k$ is irreducible $q_k \in P$ or $q_k \in P'$ by lemma 3.3. Then $B_k \geq P$ or $B_k \geq P'$. Hence $B_{k-1} = B_k \vee P$ (say) and $B_{k-1} > B_k$ by upper semi-modularity. Hence the chain $\bigcup_0 \ldots B_k$ has length $k$ and by lemma 2.1 $L_\alpha$ is Archimedean of length $k$.

**Lemma 3.5** Let $S$ be upper semi-modular satisfying the ascending chain condition and let every three ideals covering a principal ideal generate a Boolean algebra. If $q$ and $q'$ are irreducibles dividing $a$, while $P$ is a point of $L_\alpha$ such that neither $(q)$ nor $(q')$ divides $P$, then $q = q'$.

The proof is by induction on $a$. The lemma holds vacuously if $a$ is maximal. Let it hold for all proper divisors of $a$.

If $P$ is the only point of $L_\alpha$, then $q \gg a$ implies $(q) \geq P \nearrow (a)$ by theorem 1.3, a contradiction; thus $q = a$. Similarly $q' = a$, and the lemma holds for $a$ in this case. Otherwise there exists a point $P'$ in $L_\alpha$ so $P' \neq P$. Then lemma 3.3 implies $q, q' \in P'$.

Define $A = (q) \cap (q')$, and let $S_\kappa$ be the classes of lemma 1.4 for this meet. It will next be shown that $q$ and $q'$ are the only irreducibles of $A$ not in $P$. Suppose $r \in A$ and $r \notin P$. Then if $r \in S_\kappa$, $r \geq y = s_1 \ldots s_n$ for $s_i$ irreducibles of $S_{\kappa-1}$. Since $(y) \notin P$, $(y) \vee P$ is a point of $L_\alpha$ and there is an $s_\alpha$ so $s_\alpha \notin (y) \vee P$. Since $(y) \ni A \ni P' \nearrow (a)$, $y \gg a$ and the induction hypothesis implies $r = s_\alpha \in S_{\kappa-1}$. Repeating this argument, eventually $r \in S_\alpha$. If $r \in (q)$ and $r \neq q$ then $Q$ exists such that $(r) \ni Q \nearrow (q)$. By upper semi-modularity $(q) \vee P \nearrow (q)$, while $(r) \not\ni P$ implies $(q) \vee P \not\ni Q$. Thus $(q) = Q \cap (P \vee (q))$, contradicting irreducibility. Hence $r = q$. Similarly $r \in (q')$ implies $r = q'$. 
Now define $A_0 = (q)$, $B_0 = (q')$ and by induction

$A_K = A_{K-1} \cap (P \cup B_{K-1})$, $B_K = B_{K-1} \cap (P \cup A_{K-1})$. Note that $A_{K+1} = (P \cup B_K) \cap A_K = (P \cup B_K) \cap (P \cup B_{K-1}) \cap A_{K-1} = \ldots = (P \cup B_K) \ldots (P \cup B_0) \cap (q) = (P \cup B_K) \cap (q)$. Let $S_K$ be as above. It will next be shown that if $x \in P \cup A$ and $x \in S$, then $x \in A_K$ or $x \in B_K$. Trivially if $x \in S_0$, then $x \in A_0$ or $x \in B_0$. Suppose the statement holds for all elements of $S_{K-1}$ and let $x \in S_K$. Then $x \in y = r_1 \ldots r_m$ for irreducibles $r_i \in S_{K-1}$. Two possibilities exist. If $y \in P$, then all $r_i \in P \cup A$ and by the induction hypothesis on $S_{K-1}$ each $r_i$ is in $A_{K-1} \cup P$ or $B_{K-1} \cup P$. Hence every $r_i \in (A_{K-1} \cup P) \cap (B_{K-1} \cup P) \supseteq A_K$ and therefore $x \in A_K$. If $y \notin P$, then $(y) \cup P \supset (y)$. Then the truth of the lemma for $y \supset a$ implies there is exactly one $r_i$, say $r_1$, so that $r_1 \notin P \cup (y)$ while $r_2, \ldots, r_m \in P$. Then the result for $S_{K-1}$ implies $r_2, \ldots, r_m \in (A_{K-1} \cup P) \cap (B_{K-1} \cup P)$ while the preceding paragraph gives $r_1 = q \in A_0 \supseteq A_{K-1}$ or $r_1 = q \in B_0 \supseteq B_{K-1}$. Thus $x \in A_{K-1} \cup (P \cup B_{K-1}) = A_K$ or $x \in B_{K-1} \cup (P \cup A_{K-1}) = B_K$ and the statement follows for $x \in S_K$.

Let $C = \bigcap A_\ast$. Since $A_0 \supseteq A_1 \supseteq \ldots$ this is the set union of the $A_\ast$. Now $P \cup A \supseteq C$. For if $x \in P \cup A$ and $x \in S_K$, then either $x \in A_K \supseteq C$ or $x \in B_K \supseteq P \supseteq A_{K+1} \supseteq C$. Also $C \supseteq A$. For clearly $A_0$, $B_0 \supseteq A$ while if $A_K$, $B_K \supseteq A$ then $A_{K+1} = (B_K \cup P) \cap A_K \supseteq A$ and $B_{K+1} \supseteq A$. By upper semi-modularity $P \cup A \supseteq A$ and hence $P \cup A = C$ or $C = A$. If $P \cup A = C$ then $(q) = A_0 \supseteq C \supseteq P$ contradicts $q \notin P$. Therefore $A = C = \bigcap A_\ast$.

Since $q \in A = C$ there is a $k$ such that $q \in A_{K+1} = (q) \cap (P \cup B_K)$. Let $T_\ast$ be the classes of lemma 1.4 for the meet of $(q)$ and $P \cup B_K$. If $q \in T_K$ then $q \supseteq y = r_1 \ldots r_m$ for irreducibles
Then \( q' \notin P \) implies \( y \notin P \) and hence there is an \( r_i \notin P \). Since \( q' \), \( r_i \notin P \cup (y) \supset (y) \) the induction hypothesis on \( y \supset \) a implies \( q' = r_i \in T_{k-1} \). Hence after \( k \) steps it follows that \( q' \in T_0 \). Now \( q' \notin P \cup B \) since \( q' \notin P \); hence \( q' \in (q) \). If \( q' \supset q \) then \( R \) exists so \( (q') \supset R \supset (q) \). Then \( q' \notin P \) implies \( R \neq (q) \cup P \) and hence \( (q) = R \cap (P \cup (q)) \), contradicting irreducibility of \((q)\). Hence \( q = q' \).

**Lemma 3.6** Let \( S \) be upper semi-modular satisfying the ascending chain condition and let every three ideals covering a principal ideal generate a Boolean algebra. Then every element of \( S \) has a unique representation \( a = q_i \land \ldots \land q_k \) as a reduced meet of irreducibles, \( L_\alpha \) is a Boolean algebra of order \( 2^\kappa \) and each \((q_i)\) divides a maximal element of \( L_\alpha \).

By theorem 1.1 such a representation exists. By lemma 3.4 each \((q_i)\) divides a maximal ideal \( M_i \) belonging to \( L_\alpha \) and \((q_i)\) fails to divide exactly one point \( P_i \) of \( L_\alpha \). Furthermore for no two \( q_i \) is this \( P_i \) the same. For suppose \((q_1) \notin P \) and \((q_2) \notin P \). Then since \( \bigcap q_i = a \) is reduced, \( Q \) exists so \((q_2) \land \ldots \land (q_k) \supset Q \supset (a) \). Then lemma 3.3 implies \((q_1) \supseteq Q \) and hence \((a) = \bigcap (q_i) \supseteq Q \supset (a) \), a contradiction. Since \( a = \bigcap q_i \), each point of \( L_\kappa \) fails to be contained in some \((q_i)\). Hence there is a \((1-1)\) correspondence between irreducibles \( q_i \) and the points \( P_i \) of \( L_\alpha \). If \( a = q_i' \land \ldots \land q_i' \) is another representation, then by the above \( t = k \) and each \((q_i')\) fails to divide exactly one point of \( L_\alpha \), which may be taken to be \( P_i \). Then lemma 3.5 implies \( q_i = q_i' \) for all \( i \), and uniqueness holds. Then lemma 3.1 implies that \( L_\alpha \) is a Boolean algebra of order \( 2^\kappa \).
Lemmas 3.1 and 3.6 give theorem 3.1.

From lemma 4.6 the following characterization of partially ordered sets with unique decompositions also follows.

**Theorem 4.2** Let $S$ satisfy the ascending chain condition. Then each element of $S$ has a unique reduced decomposition into irreducibles if and only if $S$ is upper semi-modular and every three ideals covering a principal ideal are independent.

This condition is evidently easier to apply to examples.

Another method of investigating arithmetical properties of a partially ordered set would be to imbed it in a lattice and then apply the known theory for the lattice case. The two most well known imbeddings are those which imbed $S$ in either the lattice of ideals or in the lattice of normally closed subsets of $A$. A subset of $S$ is normally closed if it is identical with the set of upper bounds to the set of its lower bounds. The two examples that follow show that neither of these imbeddings can yield the results obtained above. In each example the partially ordered set has unique decompositions. In the first the lattice of normally closed subsets contains an element not uniquely representable as a meet of irreducibles. In the second the lattice of ideals contains an element not uniquely representable.

In figure 1, $L$ is the lattice of closed subsets of $S$. The set $A = (a,b,c)$ is closed while, for example, the set $(a,b)$ is not since the set of upper bounds to the set $(r,s,z)$ of its lower bounds is $(a,b,c)$. In $L$ the element $A = a \land b = a \land c = b \land c$ does not have a unique representation as a reduced meet
of irreducibles.

In figure 2, L is the lattice of ideals of S. Circles indicate non-principal ideals. The ideal $A = a \land b \land d = a \land c \land d$ is not uniquely representable as a meet of irreducibles.
In a finite dimensional lattice the uniqueness of both the meet and the join irreducible decompositions of elements implies the lattice is distributive. In the two examples above, there is symmetry about a horizontal center line. Hence join irreducible decompositions (defined dually) are also unique in these examples. Nevertheless the corresponding lattices in each case are not distributive. In figure 2, the elements d, a \wedge b and a \vee c of L furnish a violation of the modular law. Thus the simple extension of this lattice result to partially ordered sets fails to hold.
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