AN ARITHMETICAL THEOREM FOR PARTIALLY ORDERED SETS

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ABSTRACT

The simplest multiplicative systems in which arithmetical ideas can be defined are semigroups. For such systems irreducible (prime) elements can be introduced and conditions under which the fundamental theorem of arithmetic holds have been investigated (Clifford (3)). After identifying associates, the elements of the semigroup form a partially ordered set with respect to the ordinary division relation. This suggests the possibility of an analogous arithmetical result for abstract partially ordered sets. Although nothing corresponding to product exists in a partially ordered set, there is a notion similar to g.c.d. This is the meet operation, defined as greatest lower bound. Thus irreducible elements, namely those elements not expressible as meets of proper divisors can be introduced. The assumption of the ascending chain condition then implies that each element is representable as a reduced meet of irreducibles. The central problem of this thesis is to determine conditions on the structure of the partially ordered set in order that each element have a unique such representation.

Part I contains preliminary results and introduces the principal tools of the investigation. In the second part, basic properties of the lattice of ideals and the connection between its structure and the irreducible decompositions of elements are developed. The proofs of these results are identical with the corresponding ones for the lattice case (Dilworth (2)). The last part contains those results whose proofs are peculiar to partially ordered sets and also contains the proof of the main theorem.

CONTENTS

PART I

DEFINITIONS AND PRELIMINARY RESULTS

A set S is partially ordered by the inclusion relation $\supseteq$ if for all x, y and z in S

(1) $x \supseteq x$

(2) $x \supseteq y$ and $y \supseteq x$ imply $y = x$

(3) $x \supseteq y$ and $y \supseteq z$ imply $x \supseteq z$.

Proper inclusion will be denoted by $\supset$.

<u>Definition 1.1</u> Given a set of elements $a_1, \ldots, a_n$ in S, if an element a exists in S such that

(1) $a_i \supseteq a$ for all i and

(2) $a_i \supseteq b$ for all i implies $a \supseteq b$, then a will be called the <u>meet</u> of the a_i. The meet if it exists is unique and will be written $a = a_1 \wedge \ldots \wedge a_n$ or $a = \bigcap_i a_i$.

<u>Lemma 1.1</u> If $a = \bigcap_i a_i$ and $a_i = \bigcap_j a_{ji}$, then $a = \bigcap_{ji} a_{ji}$.

Clearly $a \subseteq a_{ji}$ for all i and j. If $b \subseteq a_{ji}$ for all i and j, then $b \subseteq a_i$ for all i and hence $b \subseteq a$.

This is a sort of associative law for meets.

<u>Definition 1.2</u> If $a = a_1 \wedge \ldots \wedge a_n$ always implies $a = a_i$ for some i, then a will be called (meet) <u>irreducible</u>.

<u>Theorem 1.1</u> If S satisfies the ascending chain condition, then every element of S is representable as a meet of irreducibles.

A maximal element is already irreducible. Let the theorem hold for all proper divisors of a. If a is irreducible, a is the desired representation; otherwise there exist a_i so $a = \bigcap a_i$, where $a_i \supset a$. Then the induction hypothesis implies irreducibles q_{ij} exist so that $a_i = \bigcap_j q_{ij}$ for each i. By lemma 1.1, $a = \bigcap_{ij} q_{ij}$ and the theorem follows by induction for all a in S.

<u>Definition 1.3</u> A subset A of S will be called an <u>ideal</u> (dual) if

(1) $x \in A$ and $y \geq x$ imply $y \in A$ and

(2) $x = \bigcap_i x_i$ and $x_i \in A$ imply $x \in A$.

For any element x of S the set of elements y such that $y \supseteq x$ is an ideal called the <u>principal</u> ideal generated by x and denoted by (x). The set of ideals of S forms a partially ordered set, L, under the relation <u>$A \supseteq B$ if and only if A is a subset of B</u>. The inclusion relation has been inverted so that $(x) \supseteq (y)$ in L if and only if $x \supseteq y$ in S.

<u>Lemma 1.2</u> L is a complete lattice.

Given a set of ideals A_σ of L, their set meet is evidently an ideal and furnishes their lattice union $\bigcup_\sigma A_\sigma$. L also has a null element, the ideal S itself. Hence (Birkhoff (1) ch. 4, thm.2) L is a complete lattice.

If $A_1 \geq \ldots \supseteq A_\sigma \geq \ldots$, then $\bigcap_\sigma A_\sigma$ is the set join of the A_σ.

<u>Corollary 1.1</u> $(a) = \bigcap_i (a_i)$ if and only if $a = \bigcap_i a_i$.

If $(a) = \bigcap_i (a_i)$, then $(a_i) \supseteq (a)$ and hence $a_i \supseteq a$ for all i. If $b \subseteq a_i$ for all i, then $(b) \subseteq (a_i)$ and $(b) \subseteq \bigcap_i (a_i) = (a)$. Hence $b \subseteq a$ and $a = \bigcap_i a_i$. Similarly $a = \bigcap_i a_i$ implies $(a) = \bigcap_i (a_i)$.

The following lemma provides a constructive definition of ideal meet.

<u>Lemma 1.3</u> $\bigcap_\nu A_\nu$ may be defined inductively as the set union of $S_0, \ldots, S_K, \ldots$, where S_0 is the set union of the A_ν and S_K consists of all elements of S containing meets of elements of S_{K-1}.

By the definition of ideal and ideal inclusion $\bigcap_\nu A_\nu$ must contain at least all the elements mentioned, and these clearly form an ideal.

Arguments will frequently be made using induction on the index k of S_K. Hence the following obvious sharpening of lemma 1.3 will be convenient.

<u>Lemma 1.4</u> If S is a partially ordered set in which every element is expressible as a meet of irreducibles, then $\bigcap_\nu A_\nu$ is the set union $S_0, \ldots, S_K, \ldots$, where S_0 is the set union of the A_ν and S_K consists of all elements of S containing meets of irreducibles of S_{K-1}.

If $a \in S_K$, then $a \supseteq b = \bigcap_i a_i$ for $a_i \in S_{K-1}$. If $a_i = \bigcap_j q_{ij}$ for irreducibles q_{ij}, then the definition of S_{K-1} implies $q_{ij} \in S_{K-1}$ while lemma 1.1 implies $a \supseteq b = \bigcap_{ij} q_{ij}$.

<u>Theorem 1.2</u> If $(a) = \bigcap_i A_i$ then $a = \bigcap_{ij} a_{ij}$ where $a_{ij} \in A_i$.

Let S be the classes of lemma 1.3 for the meet $\bigcap_i A_i$ and suppose $a \in S_K$ for $k > 1$. Then $a \supseteq \bigcap_i a_i$ for $a_i \in S_{K-1}$. Since each $a_i \in (a)$ it follows that $a = \bigcap_i a_i$. Since $a_i \in S_{K-1}$ there are $b_{ij} \in S_{K-2}$ such that for each i $a_i \supseteq \bigcap_j b_{ij}$. Then $a = \bigcap_{ij} b_{ij} \in S_{K-1}$ since every $b_{ij} \supseteq a$, while if $b \subseteq b_{ij}$ for all i and j then $b \subseteq a_i$ for all i and $b \subseteq \bigcap_i a_i = a$. After k-1 such steps it follows that $a \in S_1$ or $a = \bigcap_{ij} a_{ij}$ with $a_{ij} \in A_i$.

In most applications of theorem 1.2 every element of S will be representable as a meet of irreducibles. In this case the elements a_{ij} may be taken to be irreducible.

Lemma 1.5 (q) is irreducible in L if and only if q is irreducible in S.

If $q = \bigcap_i x_i$ for $x_i \supset q$ then corollary 1.1 implies $(q) = \bigcap_i (x_i)$ for $(x_i) \supsetneq (q)$. Conversely suppose q is irreducible and let $(q) = A \cap B$. Then theorem 1.2 implies there exist $a_i \in A$, $b_i \in B$ such that $q = a_1 \cap \ldots \cap a_k \cap b_1 \cap \ldots \cap b_n$. By irreducibility of q either $q = a_i$ and $(q) \supseteq A$ or $q = b_i$ and $(q) \supseteq B$. Hence $(q) = A$ or $(q) = B$, and (q) is irreducible.

Lemma 1.6 Let every element of S be expressible as a meet of irreducibles. Then $A \supset B$ implies there exists an irreducible q such that $q \in B$ but $q \notin A$.

Let $b \in B$, $b \notin A$ and $b = \bigcap_i q_i$ for irreducibles q_i. Since $b \notin A$, there is an i so $q_i \notin A$, and $q_i \supseteq b$ implies $q_i \in B$.

Lemma 1.7 Let $M_1 \supseteq \ldots \supseteq M_\nu \supseteq \ldots$ be a chain of ideals of L such that every $M_\nu \supset (a)$ and $P = \bigcap_\nu M_\nu$. Then $P \supset (a)$.

For if $P = (a)$, then $a \in M_\tau$ for some τ, contradicting $M_\tau \supset (a)$.

In any partially ordered set x is said to cover y (written $x \succ y$) if $x \supset y$ while no z exists for which $x \supset z \supset y$.

Theorem 1.3 Let $B \supset (a)$ in L. Then there exists an ideal P such that $B \supseteq P \succ (a)$.

This follows by applying lemma 1.7 and the maximum principle to the set of ideals N such that $B \supseteq N \supset (a)$.

<u>Definition 1.4</u> An ideal A will be called <u>upper semi-modular in L</u> if $B \geq A$, $C \succ A$ and $B \not\geq C$ imply $B \cup C \succ B$.

The partially ordered set S will be called <u>upper semi-modular</u> if (x) is upper semi-modular in L for every x in S.

The following four lemmas concern a single upper semi-modular element of a lattice.

<u>Lemma 1.8</u> Let A be an upper semi-modular element of a lattice L and $A_1, \ldots, A_n \succ A$. Then each union independent set of the A_i is contained in a maximal independent set. The union of the A_i in a maximal independent set contains every A_i.

Let $M = (A_1, \ldots, A_r)$ be the given union independent set. Since there is only a finite number of subsets of $A_1 \ldots A_n$ containing $A_1 \ldots A_r$ a maximal independent set containing $A_1 \ldots A_r$ exists.

Let $(A_1, \ldots, A_s)$ be maximal union independent. Suppose $\cup A_i = A_1 \cup \ldots \cup A_s \not\geq A_k$ where $k > s$. Let $A_i' = A_1 \cup \ldots \cup A_{i-1} \cup A_{i+1} \cup \ldots \cup A_s$, $1 \leq i \leq s$. Since $(A_1, \ldots, A_s, A_k)$ is dependent there is an i so $A_i' \cup A_k \supseteq \cup A_i \geq A_i'$. By upper semi-modularity $A_i' \cup A_k \succ A_i'$ and $\cup A_i = A_i' \cup A_i \succ A_i'$. Hence $A_i' \cup A_k = \cup A_i \geq A_k$.

<u>Lemma 1.9</u> Let A be an upper semi-modular element of a lattice L and $A_1, \ldots, A_s \succ A_0$. Then each union independent set of the A_i generates a Boolean algebra.

Let A and B be two subsets of the union independent set $(A_1, \ldots, A_n)$ and ΣA denote the union of the elements of A.

Let $A \cup B$ and $A \cap B$ denote set union and set meet of the sets A and B. Evidently $(\Sigma A) \cup (\Sigma B) = \Sigma(A \cup B)$, while $\Sigma A = \Sigma B$ implies $A = B$ by the independence of the A_i, $o \leq i \leq s$. Next it is shown that

(1) $(\Sigma A) \cap (\Sigma B) = \Sigma(A \cap B)$.

Let $m(A)$ denote the number of elements in A and $n(A) = n - m(A)$. If $n(A \cap B) = 0$, then $m(A \cap B) = n$ and $A = B$. If $n(A \cap B) = 1$, either $A \supseteq B$ or $B \supseteq A$ and (1) holds. Let (1) hold for all A and B such that $n(A \cap B) < \ell$ and suppose $n(A \cap B) = \ell$ for some A and B. Then $m(A \cap B) = n - \ell = r$ so that $A = (A_1, \ldots, A_r, A_{r+1}, \ldots, A_t)$ and $B = (A_1, \ldots A_r, A'_{r+1}, \ldots, A'_t)$. Since (1) is trivial if $B \supseteq A$ it may be assumed that $t > r$. Let $B' = (A_1, \ldots, A_{r+1}, A'_{r+1}, \ldots A'_t)$. Now $m(A \cap B') = r + 1$ and hence $n(A \cap B') = \ell - 1 < \ell$. By the induction assumption $\Sigma(A \cap B') = (\Sigma A) \cap (\Sigma B')$. Hence $\Sigma(A \cap B') = (\Sigma A) \cap (\Sigma B') \supseteq (\Sigma A) \cap (\Sigma B) \supseteq \Sigma(A \cap B)$. Since $A_1, \ldots, A_n$ are independent, $\Sigma(A \cap B) \not\supseteq A_{r+1}$ and hence $\Sigma(A \cap B') \succ \Sigma(A \cap B)$. If $\Sigma(A \cap B') = (\Sigma A) \cap (\Sigma B)$ then $\Sigma B \supseteq A_{r+1}$, contradicting independence of $A_1, \ldots, A_n$. Hence $(\Sigma A) \cap (\Sigma B) = \Sigma(A \cap B)$ and (1) holds for $n(A \cap B) = \ell$. By induction (1) holds for all A and B.

Thus the elements expressible as joins of $A_1, \ldots, A_n$ are isomorphic to the subsets of $(A_1, \ldots, A_n)$ under meet and join and $A_1, \ldots, A_n$ generate a Boolean algebra.

<u>Lemma 1.10</u> Let A be an upper semi-modular element of a lattice L and $A_1, \ldots, A_n \succ A$. Then any two maximal union independent sets of the A_i have the same number of elements andany element of one set may be replaced by a suitably chosen element of the other without altering the maximal property.

By lemma 1.8 if $A'_1 \ldots A'_k$ and $A''_1 \ldots A''_\ell$ are two maximal independent sets of the A_i then $\bigcup A'_i = \bigcup A''_j$. If every $A''_j \subseteq B = \bigcup_{h \neq i} A'_h$, then $B \supseteq \bigcup A''_j \supseteq A'_i$ contradicts independence of A'_i. Hence there exists an A''_j such that $B \not\supseteq A''_j$. Then $\bigcup A'_i \supseteq B \cup A''_j \succ B$ by upper semi-modularity of A, and $B \cup A''_j = \bigcup A'_i$. Hence $A'_1, \ldots, A'_{i-1}, A'_{i+1}, \ldots, A'_k, A''_j$, if independent are maximal independent. If not independent, since $B \not\supseteq A''_j$ there exists an A'_r such that $C \cup A''_j \supseteq A'_r$ where $C = \bigcup_{h \neq r,i} A'_h$. Then $\bigcup A'_j = B \cup A''_j = C \cup A''_j \succeq C$ contradicts $\bigcup A'_j \succ \bigcup_{h \neq i} A'_h \succ \bigcup_{h \neq i,r} A'_h$.

The replacement property implies $k = \ell$. For if $k < \ell$, after replacing every A'_i by an A''_j there would remain some A''_t divisible by the union of the A''_j that replaced the A'_i, contradicting the independence of the A''_j. Similarly $\ell \geq k$, and $\ell = k$.

<u>Lemma 1.11</u> Let A be an upper semi-modular element of a lattice and $A_1 \ldots A_k \succ A$. Then any chain joining $\bigcup A_i$ to A has not more than $k + 1$ distinct members.

Clearly $A_1 \ldots A_k$ may be supposed independent. Let $A = B_0 \subset B_1 \subset \ldots \subset B_\ell = \bigcup A_i$ and suppose $\ell > k$. By upper semi-modularity $B_0 \prec B_0 \cup A_1 \prec \ldots \prec B_0 \cup A_1 \cup \ldots \cup A_{k-1} \prec A_1 \cup \ldots \cup A_k$. Suppose it has been shown that $B_0 \subset B_1 \subset \ldots \subset B_i \prec B_i \cup A_1 \prec \ldots \prec B_i \cup A_1 \cup \ldots \cup A_{k_i - 1} \prec A_1 \cup \ldots \cup A_k$ where $k_i \leq k-i$ and $i < k$. Consider the chain $B_0 \subset B_1 \subset \ldots \subset B_{i+1} \subseteq B_{i+1} \cup A_1 \subseteq \ldots \subseteq B_{i+1} \cup A_1 \cup \ldots \cup A_{k_i - 1} \subseteq A_1 \cup \ldots \cup A_k$ and assume all its members are distinct. If $B_i \cup A_1 \cup \ldots \cup A_{k_i - 1} \not\supseteq B_{i+1}$, then $A_1 \cup \ldots \cup A_k \supseteq B_{i+1} \cup A_1 \cup \ldots \cup A_{k_i - 1} \supset B_i \cup A_1 \cup \ldots \cup A_{k_i - 1}$. But $A_1 \cup \ldots \cup A_k \succ B_i \cup A_1 \cup \ldots \cup A_{k_i - 1}$ and hence $A_1 \cup \ldots \cup A_k = B_{i+1} \cup A_1 \cup \ldots \cup A_{k_i - 1}$ contrary to assumption. Thus $B_i \cup A_1 \cup \ldots \cup A_{k_i - 1} \supseteq B_{i+1}$. Continuing in this way, eventually $B_i \cup A_1 \supseteq B_{i+1}$. But then $B_i \cup A_1 \supseteq B_{i+1} \supset B_i$ and $B_i \cup A_1 \succ B_i$ imply $B_i \cup A_1 = B_{i+1} = B_{i+1} \cup A_1$, contradicting the assumption. Hence at least two members of the chain considered are equal. Thus (renumbering the A_i if necessary) $B_0 \subset B_1 \subseteq \cdots \subset B_i \subset B_{i+1} \prec B_{i+1} \cup A_1 \prec \cdots \prec B_{i+1} \cup A_1 \cup \cdots \cup A_{k_{i+1}-1} \prec A_1 \cup \cdots \cup A_k$, where $k_{i+1} \leq k_i - 1 \leq k - (i + 1)$. By induction $B_0 \subset \cdots \subset B_{r-1} \prec A_1 \cup \cdots \cup A_k$ where $r \leq k$. But then $B_r = A_1 \cup \cdots \cup A_k$ and hence $r = \ell$, contradicting $\ell > k$. Thus $\ell \leq k$ and the lemma follows.

PART II

UPPER SEMI-MODULAR PARTIALLY ORDERED SETS

This section contains some general theorems relating the irreducible decompositions of elements of S to structural properties of the lattice of ideals L. All the proofs of these theorems are those of Dilworth (2).

Let U_a denote the union of all ideals covering (a) in L and let L_a denote the quotient lattice $U_a/(a)$. Lemma 1.11 than implies.

Lemma 2.1 Let S be upper semi-modular. Then if $P_1 \dots P_k$ is a maximal independent set of point ideals of L_a the length of any chain of L_a is at most k.

According to lemma 2.1, L_a is Archimedean if and only if U_a is the join of a finite number of point ideals of L_a.

Theorem 2.1 Let S be upper semi-modular and every element be expressible as a meet of irreducibles. Then L_a is Archimedean if and only if the number of components in the reduced irreducible decompositions of a is bounded.

(A) If L_a is not Archimedean let n be any integer. Then lemma 2.1 implies there exist n union independent P_i covering (a). By lemma 1.9 these generate a Boolean algebra. Let $A_i = \bigcup_{j \neq i} P_j$, then $(a) = \bigcap_i A_i$ is a reduced representation (a in no meet of fewer A_i). By theorem 1.2 there are irreducibles q_{ij} in A_j so $a = \bigcap_{i,j} q_{ij}$. After eliminating superfluous factors a reduced representation having at least n irreducible components is had, since unless every A_i is represented a would be in a meet of fewer A_i. Hence the number of components in representations of a is unbounded.

(B) Let the number of components be unbounded so that for each k there exists a reduced decomposition into irreducibles, $a = q_1 \cap \dots \cap q_n$ with $n \geq k$. Then by lemma 1.5 and corollary 1.1 $(a) = (q_1) \cap \dots \cap (q_n)$ is a reduced decomposition into irreducibles in L. Let $Q_i' = \bigcap_{j \neq i} (q_j)$. Then $Q_i' \supset (a)$ and theorem 1.3 implies there are P_i so $Q_i' \geq P_i \succ (a)$. If $\bigcup_{j \neq i} P_j \geq P_i$, then since $(q_i) \geq Q_j''$ for all $j \neq i$ it follows that $(q_i) \geq P_j$ for all $j \neq i$ and hence $(a) = (q_i) \cap Q_i' \geq P_i \succ (a)$, a contradiction. Hence the P_i are independent, for every k there exist at least k independent points of L_a and L_a is not Archimedean.

<u>Theorem 2.2</u> Let S be upper semi-modular. Then if L_a is Archimedean it is complemented and every ideal of L_a is expressible as a meet of maximal ideals.

Let A be in L_a and $P_1 \dots P_k$ be a maximal independent set of points of L_a divisible by A. Extend $P_1 \dots P_k$ to a maximal independent set $P_1 \dots P_n$. Let $A' = P_{k+1} \cup \dots \cup P_n$. Then $A \cup A' \geq U_a$ and $A \cup A' = U_a$. Suppose $A \cap A' \neq (a)$. Then by theorem 1.3, there is a P so $A \cap A' \geq P \succ (a)$. Since $A \geq P$ the maximal property of $P_1 \dots P_k$ implies $P_1 \cup \dots \cup P_k \geq P$ by lemma 1.8. Since by lemma 1.9 the $P_1 \dots P_n$ generate a Boolean algebra, $(a) = (P_1 \cup \dots \cup P_k) \cap A' \geq P \succ (a)$, a contradiction. Hence $A \cap A' = (a)$ and L_a is complemented.

Now let Q be irreducible in L_a. Let $P_1 \dots P_k$ be a maximal independent set of points of L_a divisible by Q and let this set be extended to the maximal independent set $P_1 \dots P_n$. Then $Q \ngeq P_{k+1}, \dots, P_n$ and hence by upper semi-modularity $Q \cup P_i \succ Q$ for $i = k + 1, \dots, n$. Since Q is irreducible $Q \cup P_{k+1} = \dots = Q \cup P_n$; hence $U_a = Q \cup U_a = Q \cup P_{k+1} \cup \dots \cup P_n = Q \cup P_{k+1} \succ Q$ and Q is maximal in L_a. Since L_a is Archimedean every ideal is expressible as a meet of irreducibles of L_a and hence as a meet of maximal ideals of L_a.

If L_a is Archimedean an arbitrary complement of A in L_a will be denoted by A'.

In the following a connection is established between the irreducible representations of a in S and certain representations of (a) in L_a.

<u>Definition 2.1</u> An ideal $C \neq U_a$ of L_a is called <u>characteristic</u> if there is an irreducible q of S dividing exactly the same point ideals of L_a as C.

<u>Theorem 2.3</u> An element a of S has a reduced representation $a = \bigcap_i q_i$ for irreducibles q_i if and only if (a) has a reduced representation $(a) = \bigcap_i C_i$, where C_i are characteristic ideals of L_a such that $q_i \in C_i$.

(A) Let $a = \bigcap_i q_i$ be a reduced representation with irreducibles q_i. If any $q_i \in U_a$ then $Q_i' = \bigcap_{j \neq i} (q_j) \supset (a)$ by corollary 1.1 since the representation is reduced. Hence there exists a P_i such that $Q_i' \geq P_i \succ (a)$ and $(a) = (q_i) \cap Q_i' \geq P_i \succ (a)$ is a contradiction. So $q_i \notin U_a$. Let $C_i \subseteq (q_i)$ be a characteristic ideal of L_a associated with q_i. One is always to hand since the union of those points of L_a divisible by (q_i) will serve and is different from U_a since $q_i \notin U_\alpha$. Then $(a) = \bigcap_i (q_i) \geq \bigcap_i C_i \geq (a)$ implies $(a) = \bigcap_i C_i$. If this is not a reduced representation, $C_i \geq \bigcap_{j \neq i} C_j$ and $Q_i' \geq P_i \succ (a)$ together with the definition of characteristic ideal imply $(a) = \bigcap_{j \neq i} C_j \geq P_i \succ (a)$, a contradiction. Hence the representation is reduced.

(B) Let $(a) = \bigcap_i C_i$ be a reduced representation with associated irreducibles $q_i \in C_i$. If $\bigcap_i (q_i) \neq (a)$ there's a P so $\bigcap_i (q_i) \geq P \succ (a)$ and hence $(a) = \bigcap_i C_i \geq P \succ (a)$, a contradiction. Hence $a = \bigcap_i q_i$. If this is not reduced, part A of the proof implies $\bigcap_i C_i$ is not reduced.

The next theorem gives a characterization of characteristic ideals in terms of the structure of L_a.

<u>Theorem 2.4</u> Let S be upper semi-modular and each element expressible as a meet of irreducibles. Then if L_a is Archimedean C is characteristic if and only if there exists an ideal R of L such that $R \geq C$, $C' \cup R \succ R$ and $C' \cap R = (a)$ for every C'.

(A) Let such an ideal exist. Then $U_a \cup R = C \cup C' \cup R = C' \cup R$. By lemma 1.6 there's an irreducible q so $q \in R$, $q \notin C' \cup R = U_a \cup R$. Since $(q) \geq R \geq C$, (q) divides every point ideal of L_a that C does. On the other hand let $(q) \geq P \succ (a)$. Then if $R \ngeq P$, $C' \cup R = U_a \cup R \geq P \cup R \supset R$. Hence $C' \cup R = P \cup R$ and $(q) \geq P \cup R = C' \cup R = U_a \cup R$, contradicting the choice of q. Hence $R \geq P$. If $C \ngeq P$, then $C' \geq P$ for some C', and $(a) = C' \cap R \geq P \succ (a)$, which is impossible. Hence $(q) \geq P$ implies $C \geq P$ and C is characteristic.

(B) Let C be characteristic and q an associated irreducible; (q) will be shown to have the properties required of R. Since

$C \not\geq U_a$, $C' \neq (a)$ and hence P exists so $C' \geq P \succ (a)$. Since q is irreducible (lemma 1.5), $(q) \cup P = (q) \cup U_a$. Hence $(q) \cup U_a = (q) \cup C' = (q) \cup P \succ (q)$. If $C' \cap (q) \neq (a)$ there is a P so $C' \cap (q) \geq P \succ (a)$ and hence $C' \cap C \geq P \succ (a)$ since C is characteristic associated with P. This is impossible; hence $C' \cap (q) = (a)$.

Corollary 2.1 Each maximal ideal of L_a is characteristic.

For the R of theorem 2.4 the maximal ideal itself may be taken.

Theorem 2.5 Let S be upper semi-modular and every element be expressible as a meet of irreducibles. Then if L_a is Archimedean each characteristic ideal C of L_a occurs in a reduced representation $(a) = C \cap C_1 \cap \ldots \cap C_k$, where k is the number of maximal independent point ideals of L_a divisible by C and C_i are characteristic ideals of L_a.

Let $P_1 \ldots P_k$ be a maximal independent set of point ideals divisible by C and imbed them in a maximal independent set $P_1 \ldots P_n$. Let $C_i = \bigcup_{j \neq i} P_j$ for $i = 1, \ldots, k$. If $C \cap C_1 \cap \ldots \cap C_k \neq (a)$, there is a P so $C \cap C_1 \cap \ldots \cap C_k \geq P \succ (a)$ and $C \geq P$ implies $P_1 \cup \ldots \cup P_k \geq P$ by lemma 1.8. Since $P_1 \ldots P_n$ generate a Boolean algebra $(a) = (P_1 \cup \ldots \cup P_k) \cap C_1 \cap \ldots \cap C_k \geq P \succ (a)$, a contradiction. Hence $(a) = C \cap C_1 \cap \ldots \cap C_k$. Since $C \cap C_1 \cap \ldots \cap C_{i-1} \cap C_{i+1} \cap \ldots \cap C_k \geq P_i \succ (a)$ the representation is reduced. Since C_i are maximal ideals of L_a they are characteristic.

Corollary 2.2 Let S be upper semi-modular and every element be expressible as a meet of irreducibles. Then if L_a is Archimedean of length k, a has a reduced decomposition into irreducibles with k components.

By lemma 1.9, theorem 2.5, and theorem 2.4.

PART III

PARTIALLY ORDERED SETS WITH UNIQUE DECOMPOSITIONS

The object of this section is to characterise those partially ordered sets for which every element has a unique irreducible decomposition. The main result is

Theorem 3.1 . Let S satisfy the ascending chain condition. Then each element of S has a unique representation as a reduced meet of irreducibles if and only if S is upper semi-modular and L_a is a Boolean algebra for each a.

This theorem will follow from a series of lemmas the first of which proves the necessity of these conditions.

Lemma 3.1 Let S satisfy the ascending chain condition and each element of S have a unique representation as a reduced meet of irreducibles. Then S is upper semi-modular and L_a is a Boolean algebra for each a.

(A) Let $B \succ (a)$, $C \geq (a)$, $C \ngeq B$ and suppose there is a D so $B \cup C \supset D \supset C$. By lemma 1.6 there are irreducibles q_c and q_D in S such that $q_c \in C$, $q_c \notin D$, $q_D \in D$ and $q_D \notin B \cup C$ (hence $q_D \notin B$). Now $B \geq B \cap (q_c) \geq (a)$. If $B = B \cap (q_c)$ then $q_c \in B$, and $q_c \in B \cup C \supset D$ implies $q_c \in D$, a contradiction. Hence $(a) = B \cap (q_c)$. Also $B \supset B \cap (q_D) \geq (a)$ and if $B = B \cap (q_D)$ then $q_D \in B$, a contradiction; so $(a) = B \cap (q_D)$.

By theorem 1.2 there are irreducibles b_i, b_i' in B and c_i, d_i in (q_c), (q_D) respectively such that $a = b_1 \cap \dots \cap b_m \cap c_1 \cap \dots \cap c_n = b_1' \cap \dots \cap b_k' \cap d_1 \cap \dots \cap d_\ell$. If $y \leq b_i$ for all i and $y \leq q_c$ then $y \leq a$. Hence $a = b_1 \cap \dots \cap b_m \cap q_c$. Similarly

$a = b_1' \wedge \dots \wedge b_k' \wedge q_D$. Both representations may be assumed reduced. Unless q_C actually occurs after converting to a reduced representation it will follow that $a = \bigcap_i b_i \in B \succ (a)$, which is impossible. Similarly q_D actually occurs in the second representation. If $q_C = q_D$ then q_C is in D, while if $q_C = b_i'$, then $a \in B \succ (a)$, a contradiction in either case. Hence the two representations are distinct. This contradicts uniqueness; hence $B \vee C \succ C$, and S is upper semi-modular.

The necessity of upper semi-modularity in the lattice case was first noticed by Morgan Ward.

(B) Since the decomposition is unique the number of components is bounded for any a and L_a is Archimedean by Theorem 2.1. Let $P_1 \dots P_k$ be a maximal independent set of point ideals of L_a. By lemma 1.9 they generate a Boolean algebra having maximal ideals $M_1 \dots M_k$. Since $U_a = \bigcup_i P_i \succ M_i$, the M_i are also maximal ideals of L_a and hence are characteristic by corollary 2.1. Since $(a) = \bigcap_i M_i$, a has a reduced decomposition into irreducibles, $a = \bigcap_i q_i$, with $q_i \in M_i$. Let M be any other maximal ideal of L_a. Then there is an irreducible $q \in M$ so that $q \notin U_a$. M is a characteristic ideal associated with the irreducible q. By theorems 2.5 and 2.3 q is a component in some decomposition of a. Since the decompositions are assumed unique $q = q_i$ for some i, and $q \in M \vee M_i = U_a$ is a contradiction unless $M = M_i$. Hence M_i are all the maximal ideals of L_a, and by theorem 2.2 all the elements of L_a are in the Boolean algebra generated by the P_i.

<u>Lemma 3.2</u> If L_a is a Boolean algebra it is Archimedean.

If L_a has an infinite number of point ideals let $P_1 \ldots P_n \ldots$ be a denumerable sequence of them and define $P_i' = \bigcap_{j \neq i} P_i$. Since L_a is a Boolean algebra, $(a) = \bigwedge_i P_i'$. Because an infinite meet of ideals consists of all elements in finite meets $(a) = P_1' \wedge \ldots \wedge P_k'$ for some k. Then $(a) \geq P_{k+1} \succ (a)$, a contradiction; hence L_a has only a finite number of point ideals and is Archimedean by lemma 2.1.

<u>Lemma 3.3</u> Let S be upper semi-modular and satisfy the ascending chain condition. Let every three ideals covering a principal ideal generate a Boolean algebra and let q be an irreducible of S such that $q \geq a$, B and $C \succ (a)$ and $B \neq C$. Then either $q \in B$ or $q \in C$.

The lemma is proved by a double induction, the first on the element a using the ascending chain condition. The lemma holds vacuously if a is maximal. Assume it holds for all proper divisors of a in S. Since $B \wedge C = (a)$, the irreducible q is distinct from a. Hence by theorem 1.3 there exists an ideal A such that $(q) \geq A \succ (a)$. If $A \neq B$, C the hypothesis of the lemma implies $A = (A \vee B) \wedge (A \vee C)$, while if $A = B$ or $A = C$ the conclusion is trivial.

It is convenient to prove the lemma for all irreducibles q' in A. Let $S_0 \ldots S_k \ldots$ be the classes of lemma 1.4 for the meet of $A \vee B$ and $A \vee C$. The second induction will be on the index k of S_k. If $q' \in S_0$, then $q' \in A \vee B \geq B$ or $q' \in A \vee C \geq C$ and the lemma holds. Now suppose the lemma is true for all $q' \in S_k$ for $k < n$ and let $q' \in S_n$, $q' \notin S_{n-1}$. By lemma 1.4 there exist irreducibles $q_i \in S_{n-1}$ such that $q' \geq y = q_1 \wedge \dots \wedge q_\ell$, where the representation of y is reduced.

If $\ell = 1$ then $q' \in S_{n-1}$, hence $\ell > 1$. Then each $q_i \in A$ and the induction hypothesis on S_k implies that for each i either $q_i \in B$ or $q_i \in C$. If $(y) \vee C = (y) \vee B$, then $q_i \in B \vee C$ for all i and hence $(q') \geq (y) \geq B \vee C$ gives the lemma for q'. If $(y) \vee C \neq (y) \vee B$, by upper semi-modularity both $B' = (y) \vee B$ and $C' = (y) \vee C$ cover (y). Since $(q') \geq (y) \geq A \succ (a)$ y is a proper divisor of a. Hence by the first induction $q' \in B' \geq B$ or $q' \in C' \geq C$. This proves the lemma for S_n and the induction is complete.

<u>Lemma 3.4</u> Let S be upper semi-modular and satisfy the ascending chain condition, and let every three ideals covering a principal ideal generate a Boolean algebra. If $a = q_1 \wedge \dots \wedge q_k$ is a reduced decomposition into irreducibles, then L_a is Archimedean of length k and each (q_i) divides a maximal ideal of L_a.

Let A_i be the union of the point ideals of L_a contained in (q_i). Then A_i is a characteristic ideal associated with the irreducible q_i (theorem 2.3). Since $A_i \neq U_a$ there is a point P such that $A_i \not\geq P$. By lemma 3.3 $A_i \geq P'$ for every point P' of L_a that is different from P. Hence $A_i \vee P = U_a$ and A_i is maximal by upper semi-modularity.

Let $B_0 = U_a$ and B_i be the union of points of L_a divisible by $(q_1), \dots, (q_i)$. Then $B_1 = A_1$ and $B_0 \succ B_1$. Evidently $B_{\ell-1} \geq B_\ell$. If $B_{\ell-1} = B_\ell$ then $\bigcap_{j \neq \ell} (q_j) \geq P_\ell \succ (a)$ since representation is reduced. Then $(q_1) \wedge \dots \wedge (q_{\ell-1}) \geq P_\ell$ implies $B_\ell = B_{\ell-1} \geq P_\ell$. This implies $(q_\ell) \geq P_\ell$ and hence $(a) = \bigcap_i (q_i) \geq P_\ell \succ (a)$, a contradiction. Hence $B_{\ell-1} \neq B_\ell$. Let P and P' be two point ideals of L_a

divisible by $B_{\ell-1}$. Since q_ℓ is irreducible $q_\ell \in P$ or $q_\ell \in P'$ by lemma 3.3. Then $B_\ell \supseteq P$ or $B_\ell \supseteq P'$. Hence $B_{\ell-1} = B_\ell \vee P$ (say) and $B_{\ell-1} \succ B_\ell$ by upper semi-modularity. Hence the chain $U_a \succ B_1 \succ \ldots \succ B_k = (a)$ has length k and by lemma 2.1 L_a is Archimedean of length k.

<u>Lemma 3.5</u> Let S be upper semi-modular satisfying the ascending chain condition and let every three ideals covering a principal ideal generate a Boolean algebra. If q and q' are irreducibles dividing a, while P is a point of L_a such that neither (q) nor (q') divides P, then q = q'.

The proof is by induction on a. The lemma holds vacuously if a is maximal. Let it hold for all proper divisors of a.

If P is the only point of L_a, then $q \supset a$ implies $(q) \supseteq P \succ (a)$ by theorem 1.3, a contradiction; thus q = a. Similarly q' = a, and the lemma holds for a in this case. Otherwise there exists a point P' in L_a so $P' \neq P$. Then lemma 3.3 implies $q, q' \in P'$. Define $A = (q) \cap (q')$, and let S_k be the classes of lemma 1.4 for this meet. It will next be shown that q and q' are the only irreducibles of A not in P. Suppose $r \in A$ and $r \notin P$. Then if $r \in S_k$, $r \supseteq y = s_1 \cap \ldots \cap s_n$ for s_i irreducibles of S_{k-1}. Since $(y) \not\supseteq P$, $(y) \vee P$ is a point of L_y and there is an s_i so $s_i \notin (y) \vee P$. Since $(y) \supseteq A \supseteq P' \succ (a)$, $y \supset a$ and the induction hypothesis implies $r = s_i \in S_{k-1}$. Repeating this argument, eventually $r \in S_0$. If $r \in (q)$ and $r \neq q$ then Q exists such that $(r) \supseteq Q \succ (q)$. By upper semi-modularity $(q) \vee P \succ (q)$, while $(r) \not\supseteq P$ implies $(q) \vee P \neq Q$. Thus $(q) = Q \cap (P \vee (q))$, contradicting irreducibility. Hence r = q. Similarly $r \in (q')$ implies r = q'.

Now define $A_0 = (q)$, $B_0 = (q')$ and by induction $A_K = A_{K-1} \wedge (P \vee B_{K-1})$, $B_K = B_{K-1} \wedge (P \vee A_{K-1})$. Note that $A_{K+1} = (P \vee B_K) \wedge A_K = (P \vee B_K) \wedge (P \vee B_{K-1}) \wedge A_{K-1} = \ldots = (P \vee B_K) \ldots (P \vee B_0) \wedge (q) = (P \vee B_K) \wedge (q)$. Let S_K be as above. It will next be shown that if $x \in P \vee A$ and $x \in S$, then $x \in A_K$ or $x \in B_K$. Trivially if $x \in S_0$, then $x \in A_0$ or $x \in B_0$. Suppose the statement holds for all elements of S_{K-1} and let $x \in S_K$. Then $x \geq y = r_1 \wedge \ldots\ r_m$ for irreducibles $r_i \in S_{K-1}$. Two possibilities exist. If $y \in P$, then all $r_i \in P \vee A$ and by the induction hypothesis on S_{K-1} each r_i is in $A_{K-1} \vee P$ or $B_{K-1} \vee P$. Hence every $r_i \in (A_{K-1} \vee P) \wedge (B_{K-1} \vee P) \geq A_K$ and therefor $x \in A_K$. If $y \notin P$, then $(y) \vee P \succ (y)$. Then the truth of the lemma for $y \supset a$ implies there is exactly one r_i, say r_1, so that $r_1 \notin P \vee (y)$ while $r_2, \ldots, r_m \in P$. Then the result for S_{K-1} implies $r_2, \ldots, r_m \in (A_{K-1} \vee P) \wedge (B_{K-1} \vee P)$ while the preceding paragraph gives $r_1 = q \in A_0 \geq A_{K-1}$ or $r_1 = q' \in B_0 \geq B_{K-1}$. Thus $x \in A_{K-1} \wedge (P \vee B_{K-1}) = A_K$ or $x \in B_{K-1} \wedge (P \vee A_{K-1}) = B_K$ and the statement follows for $x \in S_K$.

Let $C = \bigcap_i A_i$. Since $A_0 \geq A_1 \geq \ldots$ this is the set union of the A_i. Now $P \vee A \geq C$. For if $x \in P \vee A$ and $x \in S_K$, then either $x \in A_K \geq C$ or $x \in B_K \vee P \geq A_{K+1} \geq C$. Also $C \geq A$. For clearly A_0, $B_0 \geq A$ while if A_K, $B_K \geq A$ then $A_{K+1} = (B_K \vee P) \wedge A_K \geq A$ and $B_{K+1} \geq A$. By upper semi-modularity $P \vee A \succ A$ and hence $P \vee A = C$ or $C = A$. If $P \vee A = C$ then $(q) = A_0 \geq C \geq P$ contradicts $q \notin P$. Therefor $A = C = \bigcap_i A_i$.

Since $q' \in A = C$ there is a k such that $q' \in A_{K+1} = (q) \wedge (P \vee B_K)$. Let T_i be the classes of lemma 1.4 for the meet of (q) and $P \vee B_K$. If $q' \in T_K$ then $q' \geq y = r_1 \wedge \ldots \wedge r_h$ for irreducibles

$r_i \in T_{k-1}$. Then $q' \notin P$ implies $y \notin P$ and hence there is an $r_h \notin P$. Since q', $r_h \notin P \cup (y) \succ (y)$ the induction hypothesis on $y \supset a$ implies $q' = r_h \in T_{k-1}$. Hence after k steps it follows that $q' \in T_0$. Now $q' \notin P \cup B_k$ since $q' \notin P$; hence $q' \in (q)$. If $q' \supset q$ then R exists so $(q') \supseteq R \succ (q)$. Then $q' \notin P$ implies $R \neq (q) \cup P$ and hence $(q) = R \cap (P \cup (q))$, contradicting irreducibility of (q). Hence $q = q'$.

<u>Lemma 3.6</u> Let S be upper semi-modular satisfying the ascending chain condition and let every three ideals covering a principal ideal generate a Boolean algebra. Then every element of S has a unique representation $a = q_1 \cap \dots \cap q_k$ as a reduced meet of irreducibles, L_a is a Boolean algebra of order 2^k and each (q_i) divides a maximal element of L_a.

By theorem 1.1 such a representation exists. By lemma 3.4 each (q_i) divides a maximal ideal M_i belonging to L_a and (q_i) fails to divide exactly one point P_i of L_a. Furthermore for no two q_i is this P_i the same. For suppose $(q_1) \not\supseteq P$ and $(q_2) \not\supseteq P$. Then since $\bigcap_i q_i = a$ is reduced, Q exists so $(q_2) \cap \dots \cap (q_k) \geq Q \succ (a)$. Then lemma 3.3 implies $(q_1) \geq Q$ and hence $(a) = \bigcap_i (q_i) \geq Q \succ (a)$, a contradiction. Since $a = \bigcap_i q_i$, each point of L_a fails to be contained in some (q_i). Hence there is a (1-1) correspondence between irreducibles q_i and the points P_i of L_a. If $a = q_1' \cap \dots \cap q_t'$ is another representation, then by the above t = k and each (q_i') fails to divide exactly one point of L_a, which may be taken to be P_i. Then lemma 3.5 implies $q_i = q_i'$ for all i, and uniqueness holds. Then lemma 3.1 implies that L_a is a Boolean algebra of order 2^k.

Lemmas 3.1 and 3.6 give theorem 3.1.

From lemma 4.6 the following characterization of partially ordered sets with unique decompositions also follows.

<u>Theorem 4.2</u> Let S satisfy the ascending chain condition. Then each element of S has a unique reduced decomposition into irreducibles if and only if S is upper semi-modular and every three ideals covering a principal ideal are independent.

This condition is evidently easier to apply to examples.

Another method of investigating arithmetical properties of a partially ordered set would be to imbed it in a lattice and then apply the known theory for the lattice case. The two most well known imbeddings are those which imbed S in either the lattice of ideals or in the lattice of normally closed subsets of A. A subset of S is normally closed if it is identical with the set of upper bounds to the set of its lower bounds. The two examples that follow show that neither of these imbeddings can yield the results obtained above. In each example the partially ordered set has unique decompositions. In the first the lattice of normally closed subsets contains an element not uniquely representable as a meet of irreducibles. In the second the lattice of ideals contains an element not uniquely representable.

In figure 1, L is the lattice of closed subsets of S. The set A = (a,b,c) is closed while, for example, the set (a,b) is not since the set of upper bounds to the set (r,s,z) of its lower bounds is (a,b,c). In L the element $A = a \wedge b = a \wedge c = b \wedge c$ does not have a unique representation as a reduced meet of irreducibles.

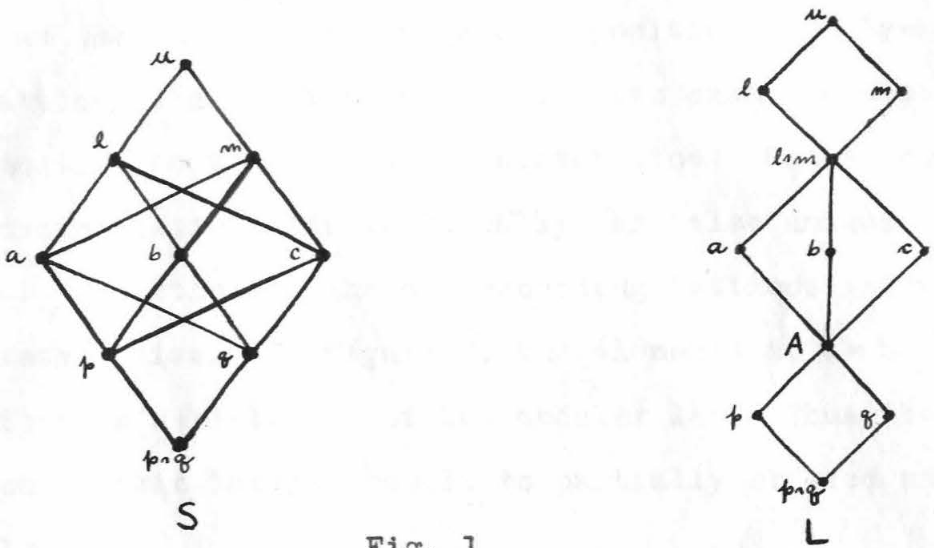

Fig. 1

In figure 2, L is the lattice of ideals of S. Circles indicate non-principal ideals. The ideal $A = a \wedge b \wedge d = a \wedge c \wedge d$ is not uniquely representable as a meet of irreducibles.

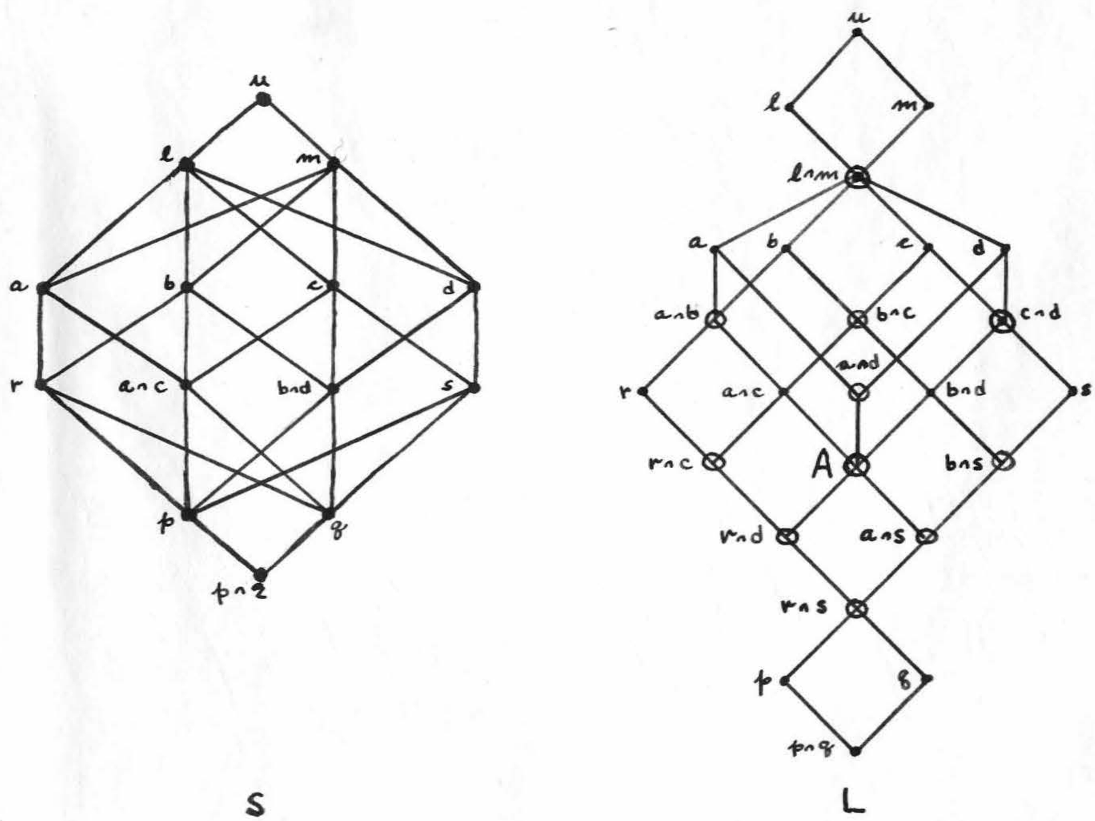

Fig. 2

In a finite dimensional lattice the uniqueness of both the meet and the join irreducible decompositions of elements implies the lattice is distributive. In the two examples above, there is symmetry about a horizontal center line. Hence join irreducible decompositions (defined dually) are also unique in these examples. Nevertheless the corresponding lattices in each case are not distributive. In figure 2, the elements d, $a \frown b$ and $a \frown c$ of L furnish a violation of the modular law. Thus the simple extension of this lattice result to partially ordered sets fails to hold.

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