ELASTIC INSTABILITY OF CANTILEVER STRUTS UNDER COMBINED AXIAL AND TRANSVERSE FORCES AT THE FREE END

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ABSTRACT

This investigation considers the elastic instability of cantilever struts under applied axial and transverse forces at the free end. Fig. 1 shows the general case of such a strut.

First the strut of uniform depth and without sweep is studied. This is shown in Fig. 2. A derivation is given for the governing differential equation and boundary conditions. These are then solved for the minimum coupled eigenvalues, which correspond to the critical load combinations. Fig. 10 is a plot of these calculated critical loadings.

Next an experimental investigation, whose main purpose was to provide a check on the above theoretical calculations, is presented. Various difficulties are discussed in addition to the techniques finally adopted. Experimental values are shown to check theory within several per cent. See Fig. 16. Also Southwell's experimental procedure for determining instability loading is shown to apply to this case of coupled loading.

The theory is then extended to include the problem of the tapered strut. Equations and boundary conditions are given for the arbitrary taper case and a solution presented for the limiting strut having complete taper. These results are given in Fig. 24.

In the concluding Part some of the more important unsolved problems are discussed in detail. These include the strut with arbitrary taper, the swept strut, and the strut which buckles inelastically.

The Appendix derives the differential equation for the non-tapered strut by variational procedure.

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PART I

INTRODUCTION

Struts of the general nature indicated in Fig. 1 are used to support aerodynamic models in wind tunnel testing. Prediction of the loading which will cause an instability type failure is therefore a matter of interest to wind tunnel laboratories.

Either of the forces shown in Fig. 1 is capable of causing strut instability. Acting together these forces couple and, thus united, may likewise bring on buckling of the strut. It is this coupled action in producing instability which is of primary interest in this investigation.

In setting out to theoretically examine such a problem it is advisable to first study the simplest possible case. For the strut shown in Fig. 1 this occurs when the geometric complications of taper and sweep are avoided. Consequently the uniform strut of Fig. 2 represents the initial problem to be analyzed. In the detailed investigation of this untapered and unswept strut, the differential equation is shown to be of the fourth order in the lateral displacement and to have variable coefficients. Together with the four appropriate boundary conditions, the mathematical formulation leads to a linear eigenvalue problem.

A method is suggested for evaluating the minimum critical load combinations. An infinity of such combinations exist for the first buckled mode; these are shown to be representable by means of a single non-dimensional interaction curve. Such a curve is also particularly convenient for design purposes and hence has been selected as the means for presenting solutions to this problem.

Since not all actual support struts will require sweep, but in most instances will possess taper, the effect of this parameter is next introduced into the analysis. The resulting increase in complexity of the governing differential equation is noted and the changes occurring in the statement of the boundary conditions pointed out. Since the first solution was for the untapered strut, the other limiting case of complete taper is therefore solved in detail. Results are again presented in the form of an interaction curve.

It should be emphasized that the above solutions are for the problem of elastic instability only. This is in keeping with the vast majority of instability solutions and, insofar as the strut is concerned, this restriction is not necessarily a severe one. This is discussed further in Part II.

The degenerate or special cases of this problem occur when either load component (P_1 or P_2 in Fig. 2) vanishes. The problem then reduces to either Euler's problem of the column, or Prandtl's problem in the lateral instability of the deep beam. Before discussing these special cases further, it is useful to first consider some of the considerations involved in the use of the strut for supporting wind tunnel models.

It scarcely seems necessary to point out that such a support strut should exert a minimum influence on the tunnel air flow. To achieve this requirement of minimum disturbance, the strut should possess a properly proportioned cross-section of minimum area. A symmetrical cross-section as shown in Fig. 1 would ordinarily be used, and the disturbing effect on the air flow becomes a minimum as the thickness ratio (t/h in Fig. 1) decreases.

Another important effect of the low thickness ratio is to raise the strut critical Mach number. In high speed tests it is important that the formation of shock waves at the support strut be delayed as long as possible. Otherwise the aerodynamic behavior of the model under test will be unduly influenced by the presence of the strut. Probably the two most powerful means for raising the strut critical Mach number is by decreasing the section thickness ratio and by eventually employing sweepforward.

As a direct result of these considerations a strut possessing a deep but thin cross-section $(h \gg t)$ appears most favorable. This is important in itself for several reasons. First it indicates one of the main assumptions to be made in making the theoretical approach to the problem. Second, this assumption is the same as that made by Prandtl in investigating the lateral instability of the deep beam. Consequently, Prandtl's problem becomes a special case of this present analysis. And finally, the necessity for a minimum cross-section makes elastic instability the probable case in practice.

Euler's work on the elastic column in 1744 represented the first theoretical solution of a problem in elastic instability. The result of this analysis was an expression for the critical or buckling load. This equation, popularly known as Euler's equation for the elastic column, is as well known as any in the entire field of structures. Although seldom clearly pointed out in texts on the subject, several facts of fundamental significance are implicit in Euler's analysis on the column. These, therefore, will be discussed further in Part II.

Prandtl considered his problem of the deep beam in a doctoral dissertation at Nuremberg in 1899. Under the action of load P_1 only (see Fig. 2), the beam becomes subject to a torsional type of instability. Prandtl showed that the behavior of this instability is described by a Bessel equation of one-fourth order. The analysis, as well as the expression for the critical load as given by Prandtl, may be found on p. 248 of Ref. 1.

Returning now to the general problem being considered here, the only previous investigation which the author could discover was some unpublished results by A. Richardson of the Southern California Cooperative Wind Tunnel. The work by Richardson on this problem has been reported in the form of a CWT Report and is listed under Ref. 6. This work will be commented on further a little later. However, in order to check the previous history of the present problem a careful search of the literature was first carried out. When this failed to reveal any previous treatment correspondence was entered into with Professors S. P. Timoshenko and I. S. Sokolnikoff. Neither of these well known men in the field of elasticity were aware of any previous work on the problem. Later, when the first portion (Part III) of the analysis was complete, the writer had an opportunity to discuss the problem with Mr. Paul Kuhn of the NACA. He too was unaware of any earlier treatment. These inquiries indicated that no other treatment of the problem had been reported. Later Dr. E. E. Sechler of Caltech forwarded Richardson's calculations to the present writer.

The work by Richardson differs from the analysis given herein in several significant respects. First Richardson does not obtain his solution from the differential equation but uses an approximate Rayleigh-

Ritz approach to the problem. Consequently he devotes a good deal of discussion in Ref. 6 to the proper choice of a deflection function. However, by applying this procedure Richardson does calculate a critical loading curve corresponding to that given in Fig. 10 of this report.

As might have been anticipated the Rayleigh-Ritz solution, which will generally represent an energy state somewhat larger than the true minimum condition, gives critical values slightly greater than those obtained directly from the differential equation. The maximum deviation between the results of this present analysis as given in Fig. 10 and Richardson's values as given in Fig. 4 of Ref. 6 is about 7%. This comparison is useful in that it may be of help in deciding what refinement in the selection of the deflection function is necessary for possible future applications of the energy method.

In this thesis the arbitrarily tapered strut is also considered and a detailed solution presented for the limiting case of complete taper. Also it is shown that although taper does seriously complicate the problem no comparable difficulty arises from introducing the geometric parameter of sweep. Richardson does not take up the tapered problem nor the swept strut in his analysis.

It should be pointed out that in this investigation critical loads are calculated only for compressive P_2 . Since in an actual wind tunnel test P_2 may also be tensile, this part of the interaction curve should probably also be calculated in detail. Sections 2 and 3 of Part VI discuss this problem in detail. It is shown that the same expressions and methods as previously used for the case of compressive P_2 , may likewise be used to compute critical values when P_2 is tensile. Hence this aspect of the problem is reduced to one of numerical calculation. Fig. 25 indicates how the complete interaction curve would appear. For the uniform strut Richardson does give this additional portion of the interaction curve. (See Fig. 4 of Ref. 6.)

For the case in which P_2 is tensile it is clear that the resultant force at which instability occurs must be considerably greater than for the case in which P_2 is compressive. A tensile P_2 tends to stabilize the strut and hence delays the occurrence of buckling. These conclusions are also demonstrated by the shape of the complete interaction curve as drawn in Fig. 25.

Actually, in any aerodynamic test, it is reasonable to suppose that the maximum and negative model lifts plus the corresponding drags will be at least approximately known. With this information available it is possible to decide whether the compressive or tensile P_2 branch of the interaction curve is apt to be critical for the design of the strut. On this basis a suitable strut may be selected which will be safe for both positive and negative model lifts; i.e., P_2 compressive or tensile.

There is still one more consideration involved in the application of such struts to wind tunnel testing which deserves mention. Due to slight imperfections in alignment, initial straightness, etc., of the strut, and due to model side forces, it is realistic to suppose that the strut itself will behave as an airfoil. Consequently lift and drag forces will develop on the strut, which must then support these in addition to the forces P_1 and P_2 already discussed. An aeroelastic problem therefore arises which is much broader in scope than the elastic instability problem as initially posed. Although no analysis is attempted on the aeroelastic problem it is given some discussion in Part II.

PART II

SOME REMARKS ON STRUCTURAL INSTABILITY

This section of the thesis will present some of the underlying ideas concerning instability behavior and the calculation of critical loads. These remarks will attempt to point out the essential differences between the various problems which may arise. Although non-mathematical in content, the ideas developed will have direct bearing on the theoretical procedures which follow.

Instability problems are conveniently separated into several classes. First, the broad categories of elastic and inelastic cases may be established. Euler or long columns belong to the former, while short columns are an example of the latter. Elastic instability assumes that the critical loading is reached without violating Hooke's Law at any point in the member. On the other hand, this restriction does not apply to structures which buckle inelastically.

The elastic instability problem may be discussed by using constant values for E and G. Furthermore, the critical loading may be calculated from a linear differential equation. This is rather remarkable in view of the fact that instability is intrinsically a non-linear phenomenon. The inelastic problem, on the other hand, is at once mathematically non-linear since constant E and G no longer exist. When the material in the structure becomes inelastic E and G become functions of the strain which, in turn, varies with the applied loading.

It is also possible to have a mathematically non-linear, elastic, instability problem. A long column, for example, may be so slender that it will suffer large bending deflections before becoming unstable. The differential equation of bending must then contain the exact, rather than approximate curvature expression. This at once results in a non-linear equation and the integration, in the case of the column, leads to an elliptic integral. It should be observed, however, that the non-linearity for this case arises from geometric considerations; for the inelastic case it arises from the significant change which takes place in the column material.

It is not inconceivable that the inelastic column might also suffer deflections which would necessitate the use of the non-linear curvature expression. As a matter of fact, this would probably be the more likely condition in an actual engineering structure. The mathematical difficulties arising from such a situation can hardly be overstated. Present knowledge of the elastic-plastic behavior of materials, or of treating non-linear differential equations, is generally inadequate to cope with problems of this degree of difficulty.

As a result the vast majority of theoretical analyses have been restricted to those problems which are governed by linear differential equations. The basic assumptions are that Hooke's Law remains valid up to the critical loading and that approximate, linear, curvature expressions may be used. The latter implies small deflections.

Since the wind tunnel strut must be slender for reasons already stated, elastic instability becomes the more likely case in practice. Hence from the designer's point of view, the restrictive assumption that buckling be elastic is not necessarily a severe one.

Euler investigated the elastic column by using the linear differential equation of bending. By so doing he was able to calculate the critical load and show that the deflection curve is a sine wave. However, he was unable to calculate the amplitude of the sine wave and thus his linearized treatment was incapable of yielding a complete solution to the problem.

Several important questions may now be raised concerning analyses of this type. First, is it necessary that the column deflect before buckling? Such deflection is usually assumed when deriving the conventional second order, linear, differential equation from which the critical Euler load is ordinarily calculated. Also experimental evidence indicates that practical columns invariably bend to a lesser or greater degree prior to buckling. Fig. 108, p. 174 of Ref. 1, indicates the results of various experimental attempts to bring a column to the buckling load without previous bending deflection. The various imperfections in experimental procedure, which makes some bending deflection unavoidable even under the best of laboratory conditions, are discussed on p. 173 of this same reference.

The work of Durup and Weisenberg as reported in Ref. 8 is instructive in further illustrating the instability behavior of actual structures. These investigators tested tapered struts similar to those being examined here but with only transverse load P_1 acting. The interesting and useful feature of their work was that the eccentricity of the load P_1 could be mechanically controlled. If desired, the direction of P_1 could be continuously adjusted during a given test in such a manner that no strut deflections occurred prior to collapse. (This would be extremely difficult to achieve in a column test but is not at all impossible when dealing with the cantilevered strut under transverse load.) In addition to tests in which no deflection prior to collapse was permitted, numerous experiments were conducted in which prescribed eccentricity of loading was established. The results of many tests of this type showed that the critical load was independent of the eccentricity and therefore independent of deflections prior to buckling. Elastic behavior was maintained throughout these tests by Durup and Weisenberg.

The evidence of these tests, and for columns as previously indicated, probably reveals why the critical load may be calculated from a linear differential equation. The linear equation is valid for those tests where experimental conditions are such that very little deflection occurs prior to buckling. Therefore the instability loading can be calculated from the linear equation for these cases. However, as experimentally observed, the critical loading remains independent of any deflections occurring up to buckling. Consequently a critical load calculation from the linear equations is valid no matter what the condition prior to elastic buckling may be. The theoretical analysis by Lagrange on the non-linear column problem verifies this conclusion. (Lagrange's work is discussed in more detail later on.)

The question now arises as to what takes place within a structure, such as a column, when the critical loading is reached. Apparently a decisive change occurs for the equilibrium state undergoes a transition from a stable to an unstable condition.

To investigate this it is convenient to examine the energy state in a column by applying the principle of virtual work. (This principle is clearly stated in Ref. 4 on p. 125. Application to a column is given on p. 77 of Ref. 1.) This principle is well known in elasticity and therefore need not be elaborated on here. If a structure is in equilibrium any small virtual displacement from this equilibrium position* will result in a change $\delta(U - W)$ of the total potential energy of the system. Here U is the strain energy and W the external work due to applied forces. Then the principle of virtual work may be shown (as in Ref. 1) to lead to the following basic statements concerning the critical condition: namely, if

$$\delta(U - W) > 0 \quad , \quad P < P_{crit.}$$

$$\delta(U - W) = 0 \quad , \quad P = P_{crit.}$$

$$\delta(U - W) < 0 \quad , \quad P > P_{crit.}$$

where P = applied load.

Hence a criterion for determining the critical state exists and is given by $\delta(U - W) = 0$.

Application of this method to the column leads directly to a linear, differential equation.** Out of this equation one may calculate the usual Euler load. (It is interesting to note that the variational method as outlined above leads to a fourth order differential equation for the column, rather than to the customary second order equation. In fact, the fourth order equation is easily shown to be the second order equation differentiated twice. The significance of this has recently been pointed out by H. Lurie in a PhD Thesis at the California Institute of Technology.)

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* Such a virtual displacement to be compatible with the structural material and consistent with the constraints on the structure.

** Linear since linear terms are assumed in writing the strain energy U .

Any other structural member subject to instability failure may be similarly discussed. The important point is that the elastic instability condition may be calculated without regard for deflections prior to buckling. Hence Euler's critical column load has practical significance and both theory and experiment bear this out. The non-linearity in the differential equation arising from the large deflection curvature expression will throw further light on this.

Some years after Euler's original work on the column, the problem was again taken up and this time by the French mathematician, Lagrange. Lagrange assumed elastic behavior but did not restrict himself to the linear expression for curvature. By using the precise differential equation of bending, he was able to demonstrate two important facts: (1) that the non-linear equation yielded the same value for the critical load as obtained by Euler; (2) that the amplitude factor on the sinusoidal deflection curve could now be calculated from the non-linear treatment.

These results did not come without effort. Lagrange, in solving the non-linear bending equation, was forced to evaluate an elliptic integral of the first kind. Such integrals were manageable in Lagrange's day, whereas they were only imperfectly understood during Euler's time. (The great contributions of Abel, Jacobi, and Legendre on elliptic integrals still lay in the future when Euler did his work on the column.)

Some basic conclusions may be drawn from these analytical investigations by Euler and Lagrange. First, as already seen, it is sufficient to consider the linear equation when calculating the critical loading. If deflection magnitudes are required one must resort to the non-linear formulation of the problem. Furthermore, by eliminating the non-linear aspects of the problem, certain physical quantities are no longer fully described; i.e., they are not calculable from the linearized statement of the problem. (This is also true of non-linear problems in fields other than elasticity and indicates that one may possibly lose the very quantity of interest when a linearized version is investigated.) Nevertheless, although only incomplete results may come from the linear equation, as in the case of the column, these may be of the greatest practical significance.

Fortunately the designer and engineer is much more concerned with critical loads than with deflection magnitudes. Hence, there is seldom any practical need for considering the precise equation. If a non-linear treatment were to be undertaken, it would be of much greater practical value to consider the case of inelastic buckling than to investigate the large deflection problem. As already noted instability is primarily a function of the nature and geometry of the unstrained structure and its material. Thus the instability load for a column is not changed if the column simultaneously acts as a beam. Deflections may be neglected and as a result are of little interest to the designer. Furthermore the imperfections which would determine deflections are so complex as to escape actual detection and measurement.

The calculation of the instability loading from a linearized differential equation is the cornerstone on which the present analysis rests.

A fundamental problem, different from those discussed above, occurs when the model support strut is placed in the wind tunnel air stream. Fig. 3 shows the two possible cases. In 3-a it is assumed that the strut assumes a zero angle of attack to the wind throughout the duration of the model test. In 3-b it is assumed that such perfect alignment is impossible and that an angle of attack will actually exist. The strut then acts as a wing and generates lift and drag forces as shown in 3-b.

The generally unstable nature of the strut air forces are also shown in Fig. 3-b. The strut lift and drag tend to increase the strut angle of attack. This in turn increases the magnitude of the strut lift and drag, assuming a constant wind velocity. This action continues until the strut achieves equilibrium between aerodynamic and elastic forces, or until strut failure occurs.

Failure of this type is commonly termed torsional divergence. It will occur when the dynamic pressure, q , reaches a certain critical value at which the elastic resisting forces in the strut become overpowered.

Torsional divergence although not a serious problem for the strut with sweepback is extremely serious for the swept-forward strut. In this latter instance the strut will experience an increase in angle of attack toward the free end (wash-in toward the tip) due to bending deflections. The resulting build-up in air forces will quickly bring on torsional divergence.

The inherent difficulties in analyzing such a problem are great indeed. First, one must be able to calculate the strut deflections under any of the possible loadings. Second, the aerodynamic forces on the strut, due to any angle of attack distribution, must be calculable. Finally, these two problems must be properly combined so that the resulting equilibrium state, for any given initial conditions of strut attitude and tunnel flow, may be determined.

Such calculations have been carried out with some success on airplane wings. These, however, have not been subject to the forces P_1 and P_2 which occur here. As a result of the effect of these forces, the strut aeroelastic problem is considerably more difficult than that of the wing. It is not the purpose of this present investigation to examine the aeroelastic problem. From the designer's viewpoint, however, the possibility of torsional divergence should be kept in mind. Such divergence is likewise an instability phenomenon.

PART III

BASIC PROBLEM - STRUT OF UNIFORM DEPTH

1. Derivation of Differential Equation

Two views of the strut with zero taper are shown in Fig. 4. The basic dimensions and loads are given in Fig. 4a, while the deflected shape, in which both sidesway and twist are present, is shown in Fig. 4b. Bending deflections parallel to P_1 are assumed to be of second order importance and are made negligible in the analysis by requiring that $h >> t$.

There are two ways in which to derive the differential equation for such a system. The first is to consider the deflected beam and to write equilibrium equations for any arbitrary station along its length. The second method is to minimize the potential energy of the system by an application of the indirect method of the calculus of variations. Since the former method gives a better picture of the physical problem it will be employed here. The second method is taken up in the Appendix. Both methods, although quite different in their approach to the problem, yield the same basic equation.

Assume a deflected shape as shown in Fig. 5. The origin of the coordinate system is most conveniently taken at the centroid of the free end. The angle of twist is measured from the fixed vertical direction (parallel to OZ) at the wall as shown in Fig. 5c. Axes OY' and OZ' are principal axes of the cross-section. The beam bends in the xy plane as shown in Fig. 5b. This bending is represented by the curvature of $y = y(x)$ in Fig. 5b. Similarly the twist is given by

$\theta = \theta(x)$ and this occurs simultaneously with the aforementioned bending.

Since the beam is assumed to remain elastic, θ and y are everywhere small, as are their derivatives with respect to x. Hence only first order terms will be retained. This will result in a linear theory which, however, is capable of yielding the desired critical loads.

To obtain the differential equation it is necessary to consider the bending and twisting effects of P_1 and P_2 about some arbitrary section as A in Fig. 5b.

1a). Twisting Effect of P_1. It is seen from Fig. 5b that the twisting moment of P_1 about A will be P_1 times the distance between the force P_1 and a tangent to $y = y(x)$ at A. This can be more clearly seen from Fig. 6, which is an enlargement of Fig. 5b. Line OC is drawn perpendicular to tangent AD. From the sketch

$$(M_t)_1 = P_1 \, (OC)$$

where $(M_t)_1$ is the twisting moment due to P_1.

But,

$$OB = y \quad , \quad AB = x$$

$$DB = x \tan \alpha_A = x \left(\frac{dy}{dx}\right)_A$$

$$= x \frac{dy}{dx} \quad \text{(since point A is arbitrary)}$$

therefore,

$$OD = OB - DB$$

$$= y - x \frac{dy}{dx}$$

However, it is also seen from the sketch that,

$$OC = OD \cos \alpha_A = OD \quad (\cos \alpha_A = 1 \text{ since } \alpha_A \text{ is small})$$

Or,

$$(M_t)_1 = P_1 \left(y - x \frac{dy}{dx}\right) \qquad (1)$$

From torsion theory it is known that θ and $(M_t)_1$ are related by,

$$d\theta = - \frac{(M_t)_1 \, dx}{GJ}$$

where, G = modulus of rigidity in torsion = $\frac{E}{2(1 + \nu)}$

J = torsional rigidity factor (for rectangle $= \frac{1}{3} ht^3$ when $h >> t$)

(-) sign indicates that θ decreases as x increases

GJ = torsional stiffness factor = C.

Substituting from (1),

$$\frac{d\theta}{dx} = \frac{P_1}{GJ} \left(x \frac{dy}{dx} - y\right) = \frac{P_1}{C} \left(x \frac{dy}{dx} - y\right) \qquad (2)$$

1b). Twisting Effect of P_2. It is assumed that P_1 and P_2 remain parallel to their initial pointings as the strut deflects. Therefore P_2 will continue to be directed along OX. Since no sag takes place, P_2 cannot cause twist about arbitrary section A. Hence, $(M_t)_2 = 0$.

1c). Bending Effect of P_1. The moment due to P_1 at station A is simply $P_1 x$. It is convenient to represent this vectorially as in Fig. 7. The component causing lateral bending is then seen to be,

$$M_1 = P_1 x \sin \theta_A = P_1 x\, \theta_A \quad (\text{for } \theta_A \text{ small, } \sin\theta_A = \theta_A)$$
$$= P_1 x\, \theta \quad (\text{again } A \text{ is any station on the strut}) \tag{3}$$

1d). Bending Effect of P_2. The force P_2 does contribute to the lateral bending (deflections along OY). The magnitude of the moment is simply $P_2 y$ acting as shown in Fig. 8.

The component of interest is

$$M_2 = P_2 y \cos \theta_A = P_2 y \tag{4}$$

The sense of $P_2 y$ is the same as that of $P_1 x\, \theta$.

Complete Bending Expression. Since the linear differential equation of bending is

$$EI \frac{d^2y}{dx^2} = - M$$

where EI = flexural stiffness factor = B ,

substituting from (3) and (4) gives,

$$EI \frac{d^2y}{dx^2} = B \frac{d^2y}{dx^2} = - (P_1\, \theta x + P_2 y) \tag{5}$$

where $I = I_{z'} = ht^3/12$ for rectangle.

1e). Combined Bending and Torsion. Equation (2) contains the torsional effect and Equation (5) the bending action for the strut. It is seen at once that they contain both θ and y as dependent variables. Either one of these may be eliminated. Prandtl eliminates y and obtains an equation in θ and x for the case when $P_2 = 0$. If y is eliminated between (2) and (5) a second order equation in φ results where $\varphi = \frac{d\theta}{dx}$. This equation proves to be actually less convenient to deal with than the fourth order equation in y and x .

Hence θ will be eliminated between (2) and (5). Differentiating (2) once gives,

$$\frac{d^2\theta}{dx^2} = \frac{P_1}{C} (x \frac{d^2y}{dx^2})$$

and successive differentiations of (5) gives,

$$B \frac{d^3y}{dx^3} = - P_1\theta - P_1 x \frac{d\theta}{dx} - P_2 \frac{dy}{dx}$$

$$B \frac{d^4y}{dx^4} = - 2 P_1 \frac{d\theta}{dx} - P_1 x \frac{d^2\theta}{dx^2} - P_2 \frac{d^2y}{dx^2}$$

Substituting from (2) and the first two of the above expressions into the last and simplifying gives,

$$\frac{d^4y}{dx^4} + (\frac{{P_1}^2}{BC} x^2 + \frac{P_2}{B}) \frac{d^2y}{dx^2} + 2 \frac{{P_1}^2}{BC} x \frac{dy}{dx} - 2 \frac{{P_1}^2}{BC} y = 0 \qquad (6)$$

This will be considered as the basic differential equation for the beam of zero taper. Putting P_1 equal to zero immediately yields the simple linear column bending equation--although here it appears differentiated twice. Likewise if P_2 is put equal to zero and the dependent variable changed to θ , the result is the equivalent Bessel equation as discussed by Prandtl.

A review of the literature has failed to reveal a previous treatment of an equation similar to (6). The collection of differential equations by Kamke (Ref. 3) does not include the type given in (6) above. It is an interesting differential equation since its degenerate forms are essentially the simple harmonic motion equation and the Bessel equation of one-fourth order.

2. Formulation of the Boundary Conditions

The solution to be determined $y = y(x)$ must satisfy four boundary conditions in addition to equation (6). In seeking these conditions it is useful to first write all the reasonable constraints on the strut. These are,

(i) $y = 0$ at $x = 0$ (no deflection at the free end where the origin is located)

(ii) $\frac{d^2y}{dx^2} = 0$ at $x = 0$ (no bending moment at the free end)

(iii) $\frac{d\theta}{dx} = 0$ at $x = 0$ (no twisting moment at the free end)

(iv) $\theta = 0$ at $x = \ell$ (no twist at the fixed end)

(v) $\frac{dy}{dx} = 0$ at $x = \ell$ (zero slope at the fixed end)

These conditions are one too many and hence cannot all be independent. The indeterminancy can be resolved by writing the previous equation (2) as,

$$\left.\frac{d\theta}{dx}\right|_{x=0} = \left.\frac{P_1}{C}\left(x\frac{dy}{dx} - y\right)\right|_{x=0}$$

or,

$$\frac{d\theta}{dx}\text{ (at free end)} = 0$$

Hence (iii) is actually a restatement of (i) and may be dropped.

As given above the boundary conditions are not in their most useful form. It will prove expedient later on to have (iv) and (v) rewritten. This is done below.

2a). Restatement of (iv). From equation (5),

$$\frac{d^2y}{dx^2} = -\frac{P_1}{B}\theta x - \frac{P_2}{B}y$$

and this gives,

$$\left(\frac{d^2y}{dx^2} + \frac{P_2 y}{B}\right)\Bigg|_{x=\ell} = -\frac{P_1}{B}\theta x\Bigg|_{x=\ell}$$

Now let,

$$y = y_{max.} = y_o \quad \text{at} \quad x = \ell \tag{7}$$

The above then becomes,

$$\left(\frac{d^2y}{dx^2} + \frac{P_2}{B}y\right)\Bigg|_{x=\ell} = 0 \quad \text{by (iv)} .$$

Therefore,

$$\frac{d^2y}{dx^2} = -\frac{P_2}{B}y_o \quad \text{at} \quad x = \ell \tag{8}$$

Equation (8) is now a restatement of (iv).

2b). Restatement of (v). Differentiating equation (5) once has already been seen to give,

$$B\frac{d^3y}{dx^3} = -P_1\theta - P_1 x\frac{d\theta}{dx} - P_2\frac{dy}{dx}$$

from which,

$$\left[\frac{d^3y}{dx^3} + \frac{P_1}{B}\left(\theta + x\frac{d\theta}{dx}\right)\right]_{x=\ell} = -\frac{P_2}{B}\frac{dy}{dx}\Bigg|_{x=\ell}$$

$$= 0 \quad \text{by (v)} .$$

Now from equation (2),

$$\left.\frac{d\theta}{dx}\right|_{x=\ell} = \frac{P_1}{C}\left(x\frac{dy}{dx} - y\right)\Bigg|_{x=\ell}$$

$$= \frac{P_1}{C}(\ell \cdot 0 - y_o) = -\frac{P_1 y_o}{C}$$

Hence,

$$\frac{d^3y}{dx^3} + \frac{P_1}{B}\left\{0 + \ell\left(-\frac{P_1 y_o}{C}\right)\right\} = 0 \quad \text{at} \quad x = \ell$$

Or,

$$\frac{d^3y}{dx^3} = \frac{P_1^{\,2}}{BC} y_o \quad \text{at} \quad x = \ell \tag{9}$$

and equation (9) is now a restatement of boundary condition (v).

3. <u>Dimensionless Form of Differential Equation and Boundary Conditions</u>

Before seeking a solution to the mathematical problem--as represented by equation (6) and boundary conditions (i), (ii), (8), and (9)--it is advantageous to put these into non-dimensional form. This may be accomplished quite readily by introducing the dimensionless variables,

$$\bar{x} = x/\ell \quad , \quad \bar{y} = y/h \quad , \quad \bar{y}_o = y_o/h \tag{10}$$

where $y_o = y_{max.} = y$ at $x = \ell$.

Calculation of the various derivatives proceeds as follows:

$$\frac{dy}{dx} = \frac{d(h\bar{y})}{d(\ell\bar{x})} = \frac{h}{\ell}\frac{d\bar{y}}{d\bar{x}}$$

Also,

$$\frac{dy}{dx} = \frac{dy}{d\bar{x}} \frac{d\bar{x}}{dx} = \frac{1}{\ell} \frac{dy}{d\bar{x}}$$

or,

$$\frac{d}{dx} \equiv \frac{1}{\ell} \frac{d}{d\bar{x}}$$

hence,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx}\right) = \frac{1}{\ell} \frac{d}{d\bar{x}} \left(\frac{h}{\ell} \frac{d\bar{y}}{d\bar{x}}\right) = \frac{h}{\ell^2} \frac{d^2\bar{y}}{d\bar{x}^2}$$

Likewise,

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2}\right) = \frac{1}{\ell} \frac{d}{d\bar{x}} \left(\frac{h}{\ell^2} \frac{d^2\bar{y}}{d\bar{x}^2}\right) = \frac{h}{\ell^3} \frac{d^3\bar{y}}{d\bar{x}^3}$$

$$\frac{d^4y}{dx^4} = \frac{h}{\ell^4} \frac{d^4\bar{y}}{d\bar{x}^4}$$

Substituting into the differential equation (6) and the appropriate boundary conditions gives,

$$\frac{d^4\bar{y}}{d\bar{x}^4} + \frac{P_1^2 \ell^4}{BC} \bar{x}^2 + \frac{P_2 \ell^2}{B} \frac{d^2\bar{y}}{d\bar{x}^2} + 2 \frac{P_1^2 \ell^4}{BC} \left(\bar{x} \frac{d\bar{y}}{d\bar{x}} - \bar{y}\right) = 0$$

$$\bar{y} = 0 \quad \text{at} \quad \bar{x} = 0$$

$$\frac{d^2\bar{y}}{d\bar{x}^2} = 0 \quad \text{at} \quad \bar{x} = 0$$

$$\frac{d^2\bar{y}}{d\bar{x}^2} = - \frac{P_2 \ell^2}{B} \bar{y}_o \quad \text{at} \quad \bar{x} = 1$$

$$\frac{d^3\bar{y}}{d\bar{x}^3} = \frac{P_1^2 \ell^4}{BC} \bar{y}_o \quad \text{at} \quad \bar{x} = 1$$

From the above dimensionless forms it is seen at once that the significant parameters of the problem are,

$$\frac{P_1^{\,2}\ell^4}{EIGJ} = \frac{P_1^{\,2}\ell^4}{BC} = k_1^{\,2} \qquad (11)$$

$$\frac{P_2\ell^2}{EI} = \frac{P_2\ell^2}{B} = k_2 \qquad (12)$$

With this simplification the final dimensionless form of the boundary conditions and differential equation becomes,

$$\frac{d^4\bar{y}}{d\bar{x}^4} + (k_1^{\,2}\,\bar{x}^2 + k_2)\frac{d^2\bar{y}}{d\bar{x}^2} + 2k_1^{\,2}\left(\bar{x}\frac{d\bar{y}}{d\bar{x}} - \bar{y}\right) = 0 \qquad (13)$$

$$\begin{aligned} \bar{y} &= 0 \quad \text{at} \quad \bar{x} = 0 \\ \frac{d^2\bar{y}}{d\bar{x}^2} &= 0 \quad \text{at} \quad \bar{x} = 0 \\ \frac{d^2\bar{y}}{d\bar{x}^2} &= -k_2\,\bar{y}_o \quad \text{at} \quad \bar{x} = 1 \\ \frac{d^3\bar{y}}{d\bar{x}^3} &= k_1^{\,2}\,\bar{y}_o \quad \text{at} \quad \bar{x} = 1 \end{aligned} \qquad (14)$$

This is now the system which must be examined for the buckling characteristics of the strut.

4. Remarks Concerning the Mathematical Problem

Equations (13) and (14) represent the most convenient form of the mathematical problem describing the physical behavior of the strut. Inspection reveals that the equation and boundary conditions are linear--as they should be since only linear bending and twisting effects were assumed. Actually (13) and (14) constitute a linear eigenvalue problem.

An obvious solution is $\bar{y}(\bar{x}) \equiv 0$. This would represent the zeroth or unbuckled mode and in this analysis will be referred to as the trivial solution. The mathematical problem, in looking for a non-trivial solution, will be to determine the minimum eigenvalues which will satisfy (13) and (14).

From the physical nature of buckling it is clear that an infinite set of buckled modes is possible. For the simple column this consists of n-looped sine waves where n takes on all integral values from one to infinity. Each buckled mode corresponds to a given critical value of the loading. The loads increase in magnitude as the number of waves in the buckled structure increases. Hence the critical modes are infinite in number and each is associated with a definite loading. The various critical loads are distinct in value since the various buckled shapes require distinctly different energy levels.

For each buckled mode the mathematical problem contains an eigenvalue--in this problem a set of values for k_1^2 and k_2 which will satisfy the differential equation (13) and the boundary conditions (14) simultaneously. Thus an infinite set of eigenvalues must exist in order that the physical problem be completely described. Only the minimum set is of practical interest, since these will represent the critical combinations of P_1 and P_2 which will cause buckling of the strut. The higher buckled modes can be readily calculated if these should be of interest. This will be pointed out in some detail later on, although solutions will only be carried out for the minimum values.

It has already been pointed out that equation (13) has as degenerate forms the simple harmonic motion equation and an equivalent form of

Bessel's equation. These are both treatable by a power series type solution. As a matter of fact such a procedure is necessary in investigating Bessel's equation. Hence a solution of (13) in closed form is probably impossible of attainment. Furthermore a power series development can be made to lend itself readily to a calculation of the eigenvalues. For these reasons a power series solution will be used here.

5. Recursion Formula from Power Series

Expand $\bar{y}(\bar{x})$ in an infinite series as follows:

$$\bar{y}(\bar{x}) = \sum_{s=0}^{\infty} a_s \bar{x}^s = a_o + a_1\bar{x} + a_2 \bar{x}^2 + \cdots \qquad (15)$$

Derivatives may be written at once as,

$$\frac{d\bar{y}}{d\bar{x}} = \bar{y}' (\bar{x}) = \sum_{s=0}^{\infty} a_s \, s \, \bar{x}^{s-1} \qquad (16)$$

$$\frac{d^2\bar{y}}{d\bar{x}^2} = \bar{y}'' (\bar{x}) = \sum_{s=0}^{\infty} a_s \, s(s-1) \, \bar{x}^{s-2} \qquad (17)$$

$$\frac{d^3\bar{y}}{d\bar{x}^3} = \bar{y}''' (\bar{x}) = \sum_{s=0}^{\infty} a_s \, s(s-1)(s-2) \, \bar{x}^{s-3} \qquad (18)$$

$$\frac{d^4\bar{y}}{d\bar{x}^4} = \bar{y}^{iv} (\bar{x}) = \sum_{s=0}^{\infty} a_s \, s(s-1)(s-2)(s-3) \, \bar{x}^{s-4} \qquad (19)$$

Substituting into equation (13) gives,

$$\sum_{s=0}^{\infty} \Big[a_s \, s(s-1)(s-2)(s-3) \, \bar{x}^{s-4} + a_s \, s(s-1) \, k_2 \, \bar{x}^{s-2} + a_s \left\{ s(s-1) \, k_1^2 + 2s \, k_1^2 - 2k_1^2 \right\} \bar{x}^s \Big] = 0$$

This equation must hold for all $\bar{x}$. This is only possible if the coefficient of each power of $\bar{x}$ vanishes identically. To insure that this condition be fulfilled, select any power of $\bar{x}$ and equate its coefficient to zero. Such a general term will be given, for example, by $\bar{x}^{-s-4}$. Collecting all terms in $\bar{x}^{-s-4}$ (and remembering that $\bar{x}^{-s-2}$ leads $\bar{x}^{-s-4}$ by two terms in the expansion process, etc.),

$$\Big[a_s\, s(s-1)(s-2)(s-3) + a_{s-2}\, (s-2)(s-3)\, k_2 + a_{s-4} \Big\{ (s-4)(s-5)\, k_1^{\,2} + 2(s-4)\, k_1^{\,2} - 2\, k_1^{\,2} \Big\} \Big] \bar{x}^{-s-4} = 0$$

In the above the bracketed term must vanish since in general $\bar{x}^{-s-4} \neq 0$. Hence the following recursion formula is obtained:

$$a_s\, s(s-1)(s-2)(s-3) + a_{s-2}\, (s-2)(s-3)\, k_2 + a_{s-4} \Big\{ (s-4)(s-5)\, k_1^{\,2} + 2\,(s-4)\, k_1^{\,2} - 2\, k_1^{\,2} \Big\} = 0 \qquad (20)$$

The usefulness of the recursion formula lies in the fact that by means of it all the coefficients in the expansion (15) may be calculated.

6. Calculating the Power Series Coefficients

It should be kept in mind from (15) that the power series is only defined for values of $s \geq 0$. Now since (20) is general, and must therefore hold for all values of s, it becomes useful to substitute actual values of s into (20) and observe the results. Or,

$\underline{s = 0}$ $\quad a_o(0)$ + undefined terms = 0 ; hence $a_o \neq 0$

$\underline{s = 1}$ $\quad a_1(0)$ + undefined terms = 0 ; hence $a_1 \neq 0$

$\underline{s = 2}$ $\quad a_2(0) + a_o(0)$ + undefined terms = 0 ; hence $a_2 \neq 0$

$\underline{s = 3}$ $\quad a_3(0) + a_1(0)$ + undefined terms = 0 ; hence $a_3 \neq 0$

s = 4 $\quad 1\cdot2\cdot3\cdot4\ a_4 + 1\cdot2\ k_2a_2 + a_o\left\{0 + 0 - 2\ k_1^{\ 2}\right\} = 0$

$$\text{or,} \quad a_4 = \frac{k_1^{\ 2}\ a_o - k_2a_2}{3\cdot4}$$

s = 5 $\quad 2\cdot3\cdot4\cdot5\ a_5 + 2\cdot3\ k_2a_3 + a_1(0) = 0$

$$\text{or,} \quad a_5 = \frac{-k_2a_3}{4\cdot5}$$

s = 6 $\quad 3\cdot4\cdot5\cdot6\ a_6 + 3\cdot4\ k_2a_4 + 4\ k_1^{\ 2}\ a_2 = 0$

$$\text{or,} \quad a_6 = -\frac{k_1^{\ 2}\ k_2a_o - (k_2^{\ 2} - 4\ k_1^{\ 2})a_2}{3\cdot4\cdot5\cdot6}$$

s = 7 $\quad 4\cdot5\cdot6\cdot7\ a_7 + 4\cdot5\ k_2a_5 + a_3\left\{3\cdot2\ k_1^{\ 2} + 2\cdot3\ k_1^{\ 2} - 2\ k_1^{\ 2}\right\} = 0$

$$\text{or,} \quad a_7 = \frac{k_2^{\ 2} - 10\ k_1^{\ 2}}{4\cdot5\cdot6\cdot7}\ a_3$$

s = 8 $\quad 5\cdot6\cdot7\cdot8\ a_8 + 5\cdot6\ k_2a_6 + a_4\left\{4\cdot3\ k_1^{\ 2} + 2\cdot4\ k_1^{\ 2} - 2\ k_1^{\ 2}\right\} = 0$

$$\text{or,} \quad a_8 = \frac{(k_1^2k_2^2 - 18k_1^4)a_o + (22k_1^2k_2 - k_2^3)a_2}{3\cdot4\cdot5\cdot6\cdot7\cdot8}$$

This procedure can be carried out for any number of terms. In this particular problem it is necessary to consider a relatively large number of terms in order that all quantities of a given type be completely collected (as all terms in $k_1^{\ 2}k_2$, $k_1^{\ 2}k_2^{\ 2}$ . . . etc., etc.). Furthermore it can be seen from the above calculations that all even coefficients (as a_4 , a_6 . . . etc.) depend only on a_o and a_2 . Using equations (15) and (17) the first two boundary conditions of (14) become,

$$\bar{y} = a_o + a_1\bar{x} + a_2\bar{x}^2 + a_3\bar{x}^3 + a_4\bar{x}^4 + \cdots$$

$$= 0 \quad \text{at} \quad \bar{x} = 0 \quad \text{if} \quad a_o = 0 \qquad (21)$$

$$\bar{y}'' = 2\,a_2 + 6\,a_3\bar{x} + 12\,a_4\bar{x}^2 + \cdots$$

$$= 0 \quad \text{at} \quad \bar{x} = 0 \quad \text{if} \quad a_2 = 0 \qquad (22)$$

Hence, since $a_o = a_2 = 0$, all even coefficients in (15) must vanish. This, then, makes it necessary to continue the calculations for odd coefficients only. All the odd terms up to a_{19} are listed below.

$$a_1 \neq 0$$

$$a_3 \neq 0$$

$$a_5 = -\frac{k_2}{4\cdot 5}\,a_3 = -\frac{k_2}{5!/3!}\,a_3$$

$$a_7 = \frac{k_2^2 - 10k_1^2}{4\cdot 5\cdot 6\cdot 7}\,a_3 = \frac{k_2^2 - 10k_1^2}{7!/3!}\,a_3$$

$$a_9 = -\frac{k_2^3 - 38\,k_2k_1^2}{9!/3!}\,a_3 \qquad (23)$$

$$a_{11} = \frac{k_2^4 - 92\,k_2^2k_1^2 + 540\,k_1^4}{11!/3!}\,a_3$$

$$a_{13} = -\frac{k_2^5 - 180\,k_2^3k_1^2 + 3884\,k_2k_1^4}{13!/3!}\,a_3$$

$$a_{15} = \frac{k_2^6 - 310\,k_2^4k_1^2 + 15844\,k_2^2k_1^4 - 70200\,k_1^6}{15!/3!}\,a_3$$

(Equation (23) is continued on p. 32.)

$$a_{17} = -\frac{k_2^{\,7} - 490\, k_2^{\,5}k_1^{\,2} + 48244\, k_2^{\,2}k_1^{\,4} - 769{,}320\, k_2k_1^{\,6}}{17!/3!}\, a_3$$

$$a_{19} = \frac{k_2^{\,8} - 728k_2^{\,6}k_1^{\,2} + 122{,}024k_2^{\,4}k_1^{\,4} - 4{,}540{,}192k_2^{\,2}k_1^{\,6} + 16{,}707{,}600k_1^{\,8}}{19!/3!}\, a_3 \tag{23}$$

For the accuracy maintained in the solution which follows only coefficients up through a_{15} need be determined.

7. Remaining Two Boundary Conditions

On substituting the coefficients (up through a_{15}) into the expansion (15) there results

$$\begin{aligned}\bar{y} = a_1\bar{x} + a_3 \Bigg[& \bar{x}^3 - \frac{k_2}{5!/3!}\bar{x}^5 + \frac{k_2^{\,2} - 10k_1^{\,2}}{7!/3!}\bar{x}^7 \\ & - \frac{k_2^{\,3} - 38k_2k_1^{\,2}}{9!/3!}\bar{x}^9 + \frac{k_2^{\,4} - 92k_2^{\,2}k_1^{\,2} + 540k_1^{\,4}}{11!/3!}\bar{x}^{11} \\ & - \frac{k_2^{\,5} - 180k_2^{\,3}k_1^{\,2} + 3884k_2k_1^{\,4}}{13!/3!}\bar{x}^{13} \\ & + \frac{k_2^{\,6} - 310k_2^{\,4}k_1^{\,2} + 15{,}844k_2^{\,2}k_1^{\,4} - 70{,}200k_1^{\,6}}{15!/3!}\bar{x}^{15} \\ & + \cdot \cdot \cdot \cdot \cdot \cdot \Bigg]\end{aligned} \tag{24}$$

Using this last expression the two remaining boundary conditions of equation (14) may be imposed. Since the conditions are,

$$\begin{aligned}\bar{y}'' &= -k_2\,\bar{y}_o \quad \text{at} \quad \bar{x} = 1 \\ \bar{y}''' &= k_1^{\,2}\,\bar{y}_o \quad \text{at} \quad \bar{x} = 1\end{aligned} \tag{25}$$

it becomes necessary to calculate the second and third derivatives of equation (24). These may be written as,

$$\bar{y}'' = 6\,a_3\Big[\bar{x} - \frac{k_2}{3!}\bar{x}^3 + \frac{k_2^2 - 10k_1^2}{5!}\bar{x}^5 - \frac{k_2^3 - 38k_2k_1^2}{7!}\bar{x}^7$$

$$+ \frac{k_2^4 - 92k_2^2k_1^2 + 540k_1^4}{9!}\bar{x}^9 - \frac{k_2^5 - 180k_2^3k_1^2 + 3884k_2k_1^4}{11!}\bar{x}^{11}$$

$$+ \frac{k_2^6 - 310k_2^4k_1^2 + 15{,}844k_2^2k_1^4 - 70{,}200k_1^6}{13!}\bar{x}^{13} + \cdots\Big]$$

and

$$\bar{y}''' = 6\,a_3\Big[1 - \frac{k_2}{2!}\bar{x}^2 + \frac{k_2^2 - 10k_1^2}{4!}\bar{x}^4 - \frac{k_2^3 - 38k_2k_1^2}{6!}\bar{x}^6$$

$$+ \frac{k_2^4 - 92k_2^2k_1^2 + 540k_1^4}{8!}\bar{x}^8 - \frac{k_2^5 - 180k_2^3k_1^2 + 3884k_1^4}{10!}\bar{x}^{10}$$

$$+ \frac{k_2^6 - 310k_2^4k_1^2 + 15{,}844k_2^2k_1^4 - 70{,}200k_1^6}{12!}\bar{x}^{12} \cdots\Big]$$

Requiring that the boundary conditions of (25) hold then gives,

$$\bar{y}'' = -k_2\bar{y}_o = 6a_3\Big[1 - \frac{k_2}{3!} + \frac{k_2^2 - 10k_1^2}{5!} - \frac{k_2^3 - 38k_2k_1^2}{7!} + \cdots \text{etc.}\Big]$$

$$\bar{y}''' = k_1^2\bar{y}_o = 6a_3\Big[1 - \frac{k_2}{2!} + \frac{k_2^2 - 10k_1^2}{4!} - \frac{k_2^3 - 38k_2k_1^2}{6!} + \cdots \text{etc.}\Big]$$

These can now be combined into a single expression. Multiply the first by k_1^2 and the second by k_2 and add the results. Recalling from equation (23) that $a_3 \neq 0$ this leads to the following expression:

$$k_1^2 - \frac{10}{5!} k_1^4 + \frac{540}{9!} k_1^6 - \frac{70,200}{13!} k_1^8 + \cdots\cdots$$

$$+ k_2 - \frac{k_2^2}{2!} + \frac{k_2^3}{4!} - \frac{k_2^4}{6!} + \frac{k_2^5}{8!} - \frac{k_2^6}{10!} + \frac{k_2^7}{12!} - \cdots\cdots$$

$$- (\frac{1}{3!} + \frac{10}{4!}) k_1^2 k_2 + (\frac{1}{5!} + \frac{38}{6!}) k_1^2 k_2^2 - (\frac{1}{7!} + \frac{92}{8!}) k_1^2 k_2^3$$

$$+ (\frac{1}{9!} + \frac{180}{10!}) k_1^2 k_2^4 - (\frac{1}{11!} + \frac{310}{12!}) k_1^2 k_2^5 + \cdots\cdots$$

$$+ (\frac{38}{7!} + \frac{540}{8!}) k_1^4 k_2 - (\frac{92}{9!} + \frac{3884}{10!}) k_1^4 k_2^2$$

$$+ (\frac{180}{11!} + \frac{15,844}{12!}) k_1^4 k_2^3 + \cdots\cdots$$

$$- (\frac{3884}{11!} + \frac{70,200}{12!}) k_1^6 k_2 + \cdots\cdots = 0 \qquad (26)$$

The values of k_1^2 and k_2 which satisfy (26) will now satisfy both the differential equation (13) and the four boundary conditions of equations (14). At the same time the deflection function $\bar{y}(\bar{x}) \not\equiv 0$, since $a_3 \neq 0$. Hence solutions to (26) will represent the eigenvalues of the mathematical problem and the buckling load combinations for the physical problem.

Inspection of (26) immediately reveals that k_1^2 and k_2 are coupled. Calculating the critical values will therefore be considerably more complicated than for the usual instability problem where such coupling is not present. Before attempting the general problem, the two important degenerate cases will be discussed. Thus the solutions to Euler's column problem and Prandtl's beam problem must be contained in (26) as special cases.

8. Degenerate Cases of the General Problem

Case A (The Euler Column)

When $P_1 = 0$ the original strut problem reduces to that of a column and the axial load is P_2. Since by equation (11) when $P_1 = 0$, $k_1^2 = 0$ also; it is clear that (26) reduces to,

$$k_2 - \frac{k_2^2}{2!} + \frac{k_2^3}{4!} - \frac{k_2^4}{6!} + \ldots\ldots = 0$$

Since $P_2 \neq 0$, it follows from (12) that $k_2 \neq 0$. Hence the above may be written as,

$$1 - \frac{k_2}{2!} + \frac{k_2^2}{4!} - \frac{k_2^3}{6!} + \ldots\ldots = 0 \qquad (27)$$

Values of k_2 which satisfy this equation are eigenvalues. They also are critical load values for the physical problem. Since the above is the cosine expansion for $\sqrt{k_2}$ one can write,

$$\cos \sqrt{k_2} = 0$$

or

$$\sqrt{(k_2)_{crit.}} = n \frac{\pi}{2}$$

which gives, on using (12),

$$(k_2)_{crit.} = \frac{(P_2)_{crit.} \ell^2}{EI} = n^2 \frac{\pi^2}{4}$$

or,

$$(P_2)_{crit.} = \frac{n^2 \pi^2 EI}{4\ell^2} = \frac{\pi^2 EI}{4\ell^2} \qquad \text{for } n = 1$$

and this is precisely the Euler value for the cantilever column of length ℓ . (Ref. 1, p. 66, equa. 59)

Case B (The Prandtl Problem)

Prandtl's well known problem is obtained by putting $k_2 = 0$ in equation (26). This leaves,

$$k_1^2 - \frac{10}{5!} k_1^4 + \frac{540}{9!} k_1^6 - \frac{70,200}{13!} k_1^8 + \ldots\ldots = 0$$

Since $k_1 \neq 0$ because $P_1 \neq 0$, it is possible to reduce the above expression to,

$$1 - \frac{10}{5!} k_1^2 + \frac{540}{9!} k_1^4 - \frac{70,200}{13!} k_1^6 + \ldots\ldots = 0$$

This latter expression may be further simplified to read,

$$1 - \frac{k_1^2}{3\cdot4} + \frac{k_1^4}{3\cdot4\cdot7\cdot8} - \frac{k_1^6}{3\cdot4\cdot7\cdot8\cdot11\cdot12} + \ldots\ldots = 0 \qquad (28)$$

There is no need to calculate the minimum value of k_1 for this equation. On p. 247 of Ref. 1, equation (g), this same expression appears as the solution to Prandtl's problem. Hence the minimum root is as given in this same reference and is,

$$(k_1)^2_{crit.} = (4.013)^2$$

or,

$$\frac{(P_1)^2_{crit.}\ell^4}{EIGJ} = (4.013)^2$$

which comes on substituting from equation (11). This last result agrees with equation (146), p. 248 of Ref. 1.

9. Numerical Evaluation of Minimum Eigenvalues

There is need for a practical method for determining the minimum root of an infinite expression such as equation (28). Such calculations are not given in the texts on elasticity; in fact, the writer has never seen any reference as to how the critical value of such equations is actually calculated. Hence a suggestion, which works very well, will be made here concerning such calculations.

The proposed method hinges on two facts: one, that the roots of (28) represent the critical loads for the various buckled modes, all of which differ by appreciable finite amounts; second, that by a change of variable the maximum root (corresponding to the minimum or critical root by the change of variable) may be easily calculated using Graeffe's root squaring method. (See Ref. 2, p. 194, for an account of this method.)

Thus to apply Graeffe's method for finding the maximum root let,

$$k_1^2 = \frac{1}{z_1^2} \tag{29}$$

so that,

$$(k_1)_{min.} \quad \text{corresponds to} \quad (z_1)_{max.}$$

Substituting (29) into (28),

$$1 - \frac{1}{3\cdot4}\frac{1}{z_1^2} + \frac{1}{3\cdot4\cdot7\cdot8}\frac{1}{z_1^4} - \frac{1}{3\cdot4\cdot7\cdot8\cdot11\cdot12}\frac{1}{z_1^6} + \ldots = 0 \tag{30}$$

Using only the first three terms of (30),

$$z_1^4 - \frac{z_1^2}{3\cdot4} + \frac{1}{3\cdot4\cdot7\cdot8} = 0$$

which now may be used to write,

$$f(z_1^2) = (z_1^2)^2 - \frac{z_1^2}{3\cdot 4} + \frac{1}{3\cdot 4\cdot 7\cdot 8}$$

$$f(-z_1^2) = (z_1^2)^2 + \frac{z_1^2}{3\cdot 4} + \frac{1}{3\cdot 4\cdot 7\cdot 8}$$

$$f(z_1^2)\cdot f(-z_1^2) = (z_1^2)^4 - \frac{(z_1^2)^2}{252} + \frac{1}{(3\cdot 4\cdot 7\cdot 8)^2}$$

Graeffe's method then states that maximum z_1^2 is given by,

$$(z_1^2)_{max.} = \sqrt{1/252} = 1/15.9$$

But,

$$(k_1^2)_{min.} = \frac{1}{(z_1^2)_{max.}} = 15.9 \qquad \text{(first approximation)}$$

And from (11),

$$(k_1^2)_{min.} = \frac{(P_1^2)_{crit.}\ell^4}{EIGJ} = 15.9$$

or,

$$(P_1)_{crit.} = \frac{3.99}{\ell^2}\sqrt{EIGJ}$$

This result agrees with the previously stated correct result within 1%. The great utility and ease of Graeffe's method for calculations of this type will be taken advantage of in attacking the coupled problem.

From these results it is now clear that the general expression equation (26) contains the problems of Euler and Prandtl as special cases.

10. Procedure for the Coupled Problem

When both k_1^2 and k_2 are different from zero the solution of equation (26) for the critical values becomes more difficult. A system for calculating the minimum coupled eigenvalues needs to be devised. This is done below.

First select a definite resultant load by making

$$\frac{k_1^2}{k_2} = \alpha \tag{31}$$

For any given value of the parameter α , the ratio of the critical loads is therefore fixed. By selecting different values of α , various directions of the resultant load are obtained. Since, when $k_1^2 = 0$, $\alpha = 0$ and when $k_2 = 0$, $\alpha = \infty$, it is clear that α may range between 0 and ∞ ; hence an infinity of directions for the resultant load is possible. This corresponds to the actual state of affairs for the strut.

In equation (31) write $k_1^2 = \alpha k_2$ and substitute this into equation (26). The result will be an equation in α and k_2 . For any selected value of α , a critical k_2 may be calculated and the corresponding critical k_1^2 is obtained from equation (31).

Graphically a plot of (31) is a parabola as shown in Fig. 9. Here $\alpha_1 < \alpha_2 < \alpha_3$ etc. Corresponding to each α , a critical k_2 is determined from equation (26). Then, for example, when $\alpha = \alpha_3$ the critical k_2 calculated from (26) corresponds to point D as shown in the sketch. Actually one selects α (as $\alpha = \alpha_3$) , calculates critical k_2 , then obtains the critical k_1^2 from equation (31). This locates point D without requiring that the parabola be constructed.

It is clear from (31) that when α is small, $\frac{k_1^2}{k_2}$

is also small and the critical loading is predominantly P_2 . On the other hand, when α is large, $k_1^2 > k_2$, and P_1 predominates. Hence for $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ etc., the critical points C , D , E , and F are spaced as shown. At point A all the loading is P_1 , or the Prandtl problem exists. When only P_2 is present, Euler's problem results and the critical condition is given by point B . Hence the solution naturally leads to the interaction type curve ACDEFB .

Each point on this curve represents a critical combination of k_1^2 and k_2 . As required by the physical problem an infinity of such combinations exist. All points within the curve and the coordinate axis represent a non-critical state; all points outside this region represent a buckled state. Hence for zero margin of safety in design the loads P_1 and P_2 should yield values of k_1^2 and k_2 which just fall on the curve.

11. Calculations for the Critical Coupled Values

In this section a few sample calculations for the determination of points on the interaction curve will be given.

Substituting equation (31) into (26) gives,

$$(1+\alpha) - \left[\frac{1}{2!} + \left(\frac{1}{3!} + \frac{10}{4!}\right)\alpha + \frac{10}{5!}\alpha^2\right] k^2$$

$$+ \left[\frac{1}{4!} + \left(\frac{1}{5!} + \frac{38}{6!}\right)\alpha + \left(\frac{38}{7!} + \frac{540}{8!}\right)\alpha^2 + \frac{540}{9!}\alpha^3\right] k_2^2$$

$$- \left[\frac{1}{6!} + \left(\frac{1}{7!} + \frac{81}{8!}\right)\alpha + \left(\frac{81}{9!} + \frac{3884}{11!}\right)\alpha^2\right.$$

$$\left. + \left(\frac{3884}{11!} + \frac{70200}{11!}\right)\alpha^3 + \frac{70200}{13!}\alpha^4\right] k_2^3 + \ldots = 0$$

which reduces to,

$$(1 + \alpha) - (0.5 + 0.583,333\,\alpha + 0.083,333\,\alpha^2)\,k_2$$

$$+ (0.041,667 + 0.061,111\,\alpha + 0.020,933\,\alpha^2$$

$$+ 0.001,488\,\alpha^3)\,k_2^2$$

$$- (0.001,388 + 0.002,480\,\alpha + 0.001,324\,\alpha^2$$

$$+ 0.000,233,9\,\alpha^3 + 0.000,011,3\,\alpha^4)\,k_2^3$$

$$+ \ldots\ldots = 0 \qquad (32)$$

Now consider special values of α . <u>$\alpha = 1/2$</u> Equation (32) reduces to,

$$1 - 0.541,667\,k_2 + 0.051,761\,k_2^2 - 0.002,006\,k_2^3$$

$$+ \ldots\ldots = 0 \qquad (33)$$

From this equation k_2 critical or minimum can be calculated. For calculating minimum k_2 a substitution similar to that used for minimum k_1^2 will be used. (See equation (29).) Thus let

$$k_2 = \frac{1}{z_2} \qquad (34)$$

so that minimum k_2 corresponds to maximum z_2. Then substituting equation (34) into (33) gives,

$$z_2^3 - 0.541,667\,z_2^2 + 0.051,761\,z_2 - 0.002,006 + \ldots = 0$$

Applying Graeffe's method as before,

$$f(z_2) = z_2^3 - 0.541,667\ z_2^2 + 0.051,761\ z_2 - 0.002,006$$

$$f\ (-z_2) = -z_2^3 - 0.541,667\ z_2^2 - 0.051,761\ z_2 - 0.002,006$$

$$f\ (z_2)\cdot f(-z_2) = -\ z_2^6 + 0.189,881\ z_2^4 - 0.000,505\ z_2^2 +$$

$$+\ 0,000,004$$

and from this last result,

$$(z_2)_{max.} = \sqrt{0.189,881} = 0.435 \quad \text{(first approximation)}$$

The second approximation is made by reapplying Graeffe's Rule. This should be done to note the divergence between the first and second approximations. From the above it can be written at once that,

$$f(z_2^2) = -\ (z_2^2)^3 + 0.189,881\ (z_2^2)^2 - 0.000,505\ (z_2^2)$$

$$+\ 0.000,004$$

$$f(-z_2^2) = +\ (z_2^2)^3 + 0.189,881\ (z_2^2)^2 + 0.000,505\ (z_2^2)$$

$$+\ 0.000,004$$

So that,

$$f(z_2^2)\cdot f(-z_2^2) = -\ z_2^{12} + 0.035,051\ z_2^8 + 0.000,001\ z_2^4$$

$$+\ \cdot\ \cdot\ \cdot\ \cdot$$

and hence,

$$(z_2)_{max.} \pm \sqrt[4]{0.035,051} = \sqrt{0.1873} = 0.433 \quad \text{(second}$$

approximation)

The change is quite small indicating that the first approximation is very accurate. (Incidentally, the first approximation could have been calculated from a quadratic in z_2. This, however, would not have permitted an accurate calculation for the second approximation. Hence, a cubic was used from the first.)

Returning to equation (34) it follows that,

$$(k_2)_{min.} = \frac{1}{(z_2)_{max.}} = \frac{1}{0.433} = 2.31$$

But from equation (12),

$$(k_2)_{min.} = \frac{(P_2)_{crit.}\ell^2}{EI}$$

or,

$$(P_2)_{crit.} = \frac{2.31}{\ell^2}\ EI \tag{35}$$

At the same time k_1^2 is related to k_2 by equation (31). Consequently,

$$(k_1^2)_{min.} = \alpha\,(k_2)_{min.} \qquad \text{where} \quad \alpha = 1/2\ .$$

Or,

$$(k_1^2)_{min.} = \frac{1}{2}\,(2.31) = 1.155$$

and by equation (11) the critical value of P_1 must be,

$$(k_1^2)_{min.} = \frac{(P_1^2)_{crit.}}{\ell^4}\,BC = 1.155$$

or,

$$(P_1)_{crit.} = \frac{1.075}{\ell^2} \sqrt{BC} \tag{36}$$

These calculations may now be repeated for various values of α .

As α increases, numerical accuracy by the above procedure becomes more difficult. It then becomes advisable to let,

$$\frac{k_2}{k_1^2} = \beta \tag{37}$$

Putting this into equation (26) gives,

$$\begin{aligned}(1+\beta) &- \left[\frac{10}{5!} + \left(\frac{1}{3!} + \frac{10}{4!}\right)\beta + \frac{1}{2!}\beta^2\right] k_1^2 \\ &+ \left[\frac{540}{9!} + \left(\frac{38}{7!} + \frac{540}{8!}\right)\beta + \left(\frac{1}{5!} + \frac{38}{6!}\right)\beta^2 + \frac{1}{4!}\beta^3\right] k_1^4 \\ &+ \left[\frac{70200}{13!} + \left(\frac{3884}{11!} + \frac{70200}{12!}\right)\beta + \left(\frac{92}{9!} + \frac{3884}{11!}\right)\beta^2 \right. \\ &\left. + \left(\frac{1}{7!} + \frac{92}{8!}\right)\beta^3 + \frac{1}{6!}\beta^4\right] k_1^6 \\ &+ \ldots\ldots = 0\end{aligned}$$

which can be simplified to,

$$\begin{aligned}(1+\beta) &- (0.083,333 + 0.583,333\beta + 0.5\beta^2) k_1^2 \\ &+ (0.001,488 + 0.020,933\beta + 0.061,111\beta^2 \\ &+ 0.041,667\beta^3) k_1^4 \\ &- (0.000,011,3 + 0.000,233,9\beta + 0.001,324\beta^2 \\ &+ 0.002,480\beta^3 + 0.001,388\beta^4) k_1^6 \\ &+ \ldots\ldots = 0\end{aligned} \tag{38}$$

The value of β may now be specified in equation (38) and the corresponding value of $(k_1^2)_{min.}$ calculated. The procedure is similar to that already used in determining $(k_2)_{min.}$ when α is given.

A summary of calculated results is given in Table I. These have been used in plotting the interaction curve (Fig. 10). Each point on the curve represents a critical or buckling combination of P_1 and P_2. The solution is for the first buckled mode. Higher modes could be calculated readily from the expressions already developed. These are seldom of importance in actual design problems.

PART IV

EXPERIMENTAL INVESTIGATION

1. General Comments

The basic purpose of the experimental program is to verify the theoretical derivations and calculations. To do this the critical curve of Fig. 10 will be determined experimentally. Agreement of the two curves will be considered as verification of the theory.

In undertaking such an experimental study the usefulness of the basic differential equation (equation 13 of Part III) should be noted. The fundamental dimensionless parameters k_1 and k_2 appear in this equation, and are also the coordinates of the interaction curve of Fig. 10. Hence, even though a solution of the differential equation may be lacking, the non-dimensional form of the equation nevertheless gives the most useful means for representing the experimental data.

Any attempt to measure critical loadings for a typical strut will soon be concerned with two problems: (1) how to apply the theoretically assumed loadings, and (2) how to determine when the critical loading is reached.

2. Method for Applying Load

The two force components P_1 and P_2 must be applied at the free end of the strut in such a manner that no constraints are developed. In other words, these forces must be "freely floating".

A first attempt at securing these conditions is shown in Fig. 11. Points A , B , C , and D all lie in a single plane. By varying the location of B , the resultant load R in AB can be given any desired pointing. A pin at A carries the force R into the strut through direct bearing action. By making $L >> \ell$, any deflection of the strut prior to buckling will result in only a very small component of R acting normal to the initial plane ABCD . Hence, although a constraining force is developed, its magnitude can be made very small.

However, experimental data showed that for any reasonable ratio of L/ℓ , the constraint developed was sufficient to appreciably raise the critical loading. For this reason the basic idea contained in the method of Fig. 11 was abandoned.

Various solutions to this loading difficulty suggest themselves and the simplest was found to be entirely satisfactory. Since any actual test specimen will suffer a certain amount of deflection before buckling, it is necessary that such deflection be permitted. By utilizing an image strut as shown in Fig. 12, point C was found to deflect precisely as A deflected. Consequently points A , C , and B always formed a vertical plane and no constraining component of R ever developed.

A symmetrical specimen was therefore securely clamped at D (Fig. 12) by clamping between rigid structural steel sections. The loading pins at A and C were placed at the elastic axis of the strut. Dead weights (W) were used to supply the loading. Lateral deflections were measured in the vicinity of the free end with a dial gauge. In all tests the strut was permitted to deflect away--rather than toward--the dial gauge. In that way the spring force acting on the spindle of the dial gauge was not introduced as a constraining force on the strut.

The image method with the above refinements gave very satisfactory results. Its simplicity and ease of operation are too apparent to need further discussion.

3. Determining the Critical Loading

This is an important consideration since there may be times when the attainment of the critical condition is not altogether obvious. For example, assume that a certain test specimen is not initially flat and that a certain amount of eccentricity has occurred in the loading; i.e., the bearing pressure at pin A in Fig. 12 is such that the resultant pressure is eccentric to the elastic axis. In such a case considerable deflection of the strut may take place before instability occurs. Any casual observation of these deflections, as loading increases, may fail to detect the instability condition.

As a result it is necessary that a consistent method be devised, whereby the occurrence of the critical condition may be definitely recognized. It should, however, be kept in mind that for some struts the critical loading is accompanied by an unmistakable deflection. The experimental work of Ref. 8 illustrates this quite well.

Ref. 8 lists the results of a professional degree thesis at Galcit*, 1947, in which certain strut instability measurements were made. These tests were all for the Prandtl case in which only load component P_1 was different from zero. However, struts with varying degrees of taper were tested, in addition to those of uniform section throughout. In these tests the dead weight vertical loading was applied to the beam

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* Guggenheim Aeronautical Laboratory, California Institute of Technology.

through a mechanical unit which could be adjusted to give any desired eccentricity of loading.

By properly adjusting the line of action of the applied load it thereby became possible to load the strut without causing any lateral deflection to take place. In Ref. 8 such a test was called the "critical run". At the instability loading the beam would suffer a sudden collapse. The "critical run" loading was taken to be the critical load for the beam.

Other tests were then conducted at varying eccentricities of loading. In all instances a plot of P vs δ gave a curve which tended to become asymptotic to $P_{crit.}$ (Here δ is the torsional deflection of the beam. However, as shown in the derivation of the differential equation in Part III, either the angle of twist or the transverse bending deflection may be selected as the dependent variable. In Ref. 8 the torsional deflection was measured by mounting a long, slender pointer on the free end and noting its movement relative to a fixed, graduated scale.)

Fig. 13 gives a sketch showing the conclusions mentioned above. These are typical of the results given in Ref. 8.

Another conclusion reached in Ref. 8 is that Southwell's method* for experimentally determining critical loads may be used for these cases. Southwell's method consists essentially of plotting P (ordinates) vs P/δ (abscissa). Since this must be a linear relationship for elastic structures (by Hooke's Law) it is merely necessary to determine several values of P and the corresponding deflections,

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* See Ref. 1, p. 177, for a detailed discussion.

δ , for the given structure. It is advisable to take these fairly close to the critical condition. Then plotting these results, using P and P/δ coordinates, and extending the resulting straight line to $P/\delta = 0$, will yield the critical value of P . This is illustrated in Fig. 14 for an actual Prandtl type test specimen.

The reasonableness of this procedure is simple enough to understand. Since δ tends toward very large values, as P approaches $P_{crit.}$, in the limit as P becomes $P_{crit.}$ the value of P/δ will become zero. Ref. 8 shows this to be very nearly true for the Prandtl cases investigated. Southwell showed that it must hold for a column.

There is still some question as to how such a plot will work out when bending, rather than torsional deflections, are measured for the Prandtl type beam. To examine this a typical case was studied in this present investigation. The beam tested had the specifications shown on Fig. 14. Test data gave results which have also been plotted on Fig. 14. Calculations for the critical load of this strut, using Prandtl's equation, are given below.

$$I = ht^3/12 = 437 \times 10^{-6} \text{ in.}^4$$

$$J = ht^3/3 = 1748 \times 10^{-6} \text{ in.}^4$$

$$EI = B = 4630 \text{ lbs.-in.}^2$$

$$GJ = C = 7120 \text{ lbs.-in.}^2$$

$$BC = 5750 \text{ lbs.-in.}^2$$

$$(P_1)_{crit.} = \frac{4.013\sqrt{BC}}{\ell^2} = \frac{4.013\,(5750)}{1020} = 22.6 \text{ lbs.}$$

Or the calculated critical load is 22.6 lbs. It should be noted that slide rule accuracy is generally sufficient for these calculations. However, since the thickness appears as a cube in the final result, it should be measured with the aid of a micrometer.

Fig. 14 shows that, for this example, Southwell's method gives an excellent value for the critical loading. The curved line is shown to fall off asymptotically to $(P_1)_{crit.}$ as loading is increased.

As a result of these preliminary investigations either curve of Fig. 14 shows promise of indicating the critical condition for the general problem. However, Southwell's method has not as yet been shown to apply when instability is due to the coupled effect of P_1 and P_2.

4. Typical Test - General Case

It is helpful to examine in some detail a typical test as performed for the case in which both P_1 and P_2 are present. In this section only those plots corresponding to Fig. 14 for the Prandtl case will be discussed. (The example selected here is more fully discussed in the next section as Specimen A. The angle of the resultant load is at $\varphi = 58° 30'$.)

Table II gives the experimental data from which Fig. 15 was plotted. The load in this case is simply the total weight W of Fig. 12. Since instability occurs at a definite value for W, it is sufficient to be able to determine $W_{crit.}$ in order to know when the critical condition is reached. When $W_{crit.}$ is known, $(P_1)_{crit.}$ and $(P_2)_{crit.}$ may be calculated from the geometry of Fig. 12.

For this strut definite instability occurred at a load of 65.31 lbs. Hence the critical value on Fig. 15 is known. A further glance at Fig. 15 will show how the curve representing the plot of P vs δ has tended to become asymptotic to $P_{crit.}$ This indicates that the critical point cannot lie far beyond the last plotted point on the P vs δ curve. Finally Southwell's method--constructed through the three points of maximum loading--is shown to closely approximate the actual instability loading.

These results are very useful. First they indicate how the critical loading for a test strut may be determined experimentally. Then they also indicate that under the coupled action of the applied forces, the instability loading may be predicted by using Southwell's suggested procedure.

On the basis of the above results the experimental determinations desired in this present investigation were obtained.

5. Test Results

Two different struts were tested at various critical combinations of P_1 and P_2. The method for performing the tests and for determining the critical loading is as discussed in the previous sections. Table III gives pertinent data on these specimens while Fig. 12 shows the general arrangements, etc.

From equations (11) and (12) of Part III the non-dimensional critical load parameters are,

$$(k_1)_{crit.} = \frac{(P_1)_{crit.}\ell^2}{\sqrt{BC}}$$

$$(k_2)_{crit.} = \frac{(P_2)_{crit.}\ell^2}{B}$$

Hence when the loads are known experimentally, the critical dimensionless parameters may be determined from the above equations. Their proximity to the curve of Fig. 10 will then indicate the agreement between theory and experiment.

In the sketch of the specimen (Fig. 12) the angle φ between P_2 and R is shown. This is convenient in visualizing the point of R for the various tests.

The experimental results in the form of $(P_1)_{crit.}$ and $(P_2)_{crit.}$ are summarized in Tables IV - A and B. Corresponding values of the parameters $(k_1)_{crit.}$ and $(k_2)_{crit.}$ are also given. These latter values are plotted on Fig. 16. The critical curve of Fig. 10 is also given on Fig. 16, thereby indicating the agreement attained between theoretical and experimental results. The check is sufficiently good to justify the theoretical calculations.

Tables V - A and B contain the complete load and deflection data as recorded for the two test specimens. Figures 17 - 22 plot these data in a form similar to that already discussed with reference to Fig. 14. Hence these remaining plots contain the detailed information which is used to obtain the critical loads for Tables IV - A and B and in which the experimental critical values of Fig. 16 are based. All general conclusions previously reached in regard to the experimental program and test results are seen to also apply to these graphs (Figs. 17 - 22).

PART V

ANALYSIS OF THE TAPERED STRUT

1. Introductory Comments

The analysis on the untapered strut plus the subsequent experimental verification gave encouragement for further pursuit of the theoretical work. Consequently the next logical step, that of introducing taper into the problem, will be discussed in this section.

It will be seen that for arbitrary taper the mathematical problem becomes considerably complicated relative to the first problem. Attempts at achieving a solution soon exposed serious difficulties. As a matter of fact no solution could be located in the literature for either of the degenerate problems (P_1 or P_2 equal to zero) when the taper was arbitrary and the sides of the strut straight lines.

Consequently the problem is set up for the case of arbitrary taper. Then the limiting case of complete taper is considered in complete detail. The solution thus obtained, together with the previous results, then sets the limits on the effect of taper. All arbitrary tapers must yield results lying between these limiting cases.

An example of the influence of taper on the buckling strength may be of interest. Suppose P_1 and P_2 to be equal so that their resultant acts at $45°$ to either. Then at the critical condition, the resultant for the fully tapered strut will be 41 1/2 per cent less than for the strut without taper. As a result it is not advisable to neglect the effect of taper when designing these struts.

2. Derivation of the Differential Equation

The immediate objective will be to derive the equation for the tapered strut, corresponding to equation (6) of Part III for the untapered strut.

To do this the selection of coordinates, terms, and so on in Fig. 23 will be made. The loading is the same as before and therefore is not shown on the figure.

From Fig. 23,

$$h = \frac{h_o}{d + \ell}(d + x) \tag{39}$$

Also for struts in which the thickness of cross-sections is independent of x (as each section a rectangle),

$$I = I_o \frac{d + x}{d + \ell} \tag{40}$$

$$J = J_o \frac{d + x}{d + \ell} \tag{41}$$

where I_o and J_o occur at the fixed end, $x = \ell$, and

I and J are minimum values at any cross-section.

At first glance it might appear sufficient to put the variable I and J into equation (6) directly. This, however, is not permissible since in deriving (6) certain differentiations were performed and these assumed I and J as constants. Hence it is necessary to go back to basic equations (2) and (5) of Part III and introduce the variable inertia terms there. Hence rewriting (2) and (5) from Part III with the new inertia terms,

$$\frac{d\theta}{dx} = \frac{P_1}{GJ_o} \frac{d + \ell}{d + x} (xy' - y) = \frac{P_1}{C_o} \frac{d + \ell}{d + x} (xy' - y)$$

$$EI_o \frac{d + x}{d + \ell} \frac{d^2y}{dx^2} = B_o \frac{d + x}{d + \ell} \frac{d^2y}{dx^2} = - (P_1 \theta x + P_2 y)$$

$$B_o \frac{d + x}{d + \ell} \frac{d^2y}{dx^2} = - (P_1 \theta x + P_2 y)$$

or more conveniently,

$$(d + x) \theta' = \frac{P_1 (d + \ell)}{C_o} (xy' - y) \tag{42}$$

$$(d + x) y'' = - \frac{d + \ell}{B_o} (P_1 \theta x - P_2 y) \tag{43}$$

Now differentiate (42) once with respect to x ,

$$(d + x) \theta'' + \theta' = \frac{P_1 (d + \ell)}{C_o} (xy'' + y' - y')$$

Multiplying by $(d + x)$,

$$(d + x)^2 \theta'' + (d + x) \theta' = \frac{P_1 (d + \ell)}{C_o} (d + x) xy'' \tag{44}$$

Subtracting (42) from (44),

$$(d + x)^2 \theta'' = \frac{P_1 (d + \ell)}{C_o} \Big[(d + x) xy'' - xy' + y\Big] \tag{45}$$

Likewise differentiate (43),

$$(d + x) y''' + y'' = - \frac{d + \ell}{B_o} (P_1\theta + P_1x\theta' + P_2 y')$$

differentiating again,

$$(d + x) y^{IV} + y''' + y''' = - \frac{d + \ell}{B_o} (P_1\theta' + P_1\theta' + P_1x\theta'' + P_2y'')$$

$$(d + x) y^{IV} + 2 y''' = - \frac{d + \ell}{B_o} (2 P_1\theta' + P_1 x\theta'' + P_2 y'')$$

Multiply this last equation by $(d + x)$,

$$(d + x)^2\, y^{iv} + 2\,(d + x)\, y''' = - \frac{d + \ell}{B_o}\left[2P_1(d + x)\theta' + P_1 x(d + x)\theta'' + P_2(d + x)y''\right] \qquad (46)$$

Now substitute from (42) and (45) into (46), thereby eliminating θ as a dependent variable. Or,

$$(d + x)^2\, y^{iv} + 2(d + x)y''' = - \frac{d + \ell}{B_o}\left[2P_1 \frac{P_1(d + \ell)}{C_o}(xy' - y) + P_1 x \frac{P_1(d + \ell)}{C_o}\left\{ xy'' + \frac{y - xy'}{d + x}\right\} + P_2\,(d + x)y''\right]$$

which may be simplified in several steps as indicated below,

$$(d + x)^2\, y^{iv} + 2(d + x)y''' = -\left[\frac{P_1{}^2(d + \ell)^2}{B_o C_o}\, x^2 + \frac{P_2(d+\ell)}{B_o}(d+x)\right] y'' - \left[2\,\frac{P_1{}^2(d+\ell)^2}{B_o C_o} - \frac{P_1{}^2(d+\ell)^2}{B_o C_o}\,\frac{x}{d+x}\right](xy' - y)$$

hence,

$$(d+x)^2\, y^{iv} + 2(d+x)y''' + \left[\frac{P_1{}^2(d+\ell)^2}{B_o C_o}\, x^2 + \frac{P_2(d+\ell)}{B_o}\,(d+x)\right] y'' + \frac{P_1{}^2(d+\ell)^2}{B_o C_o}\left[2 - \frac{x}{d+x}\right](xy' - y) = 0 \qquad (47)$$

Now introduce dimensionless variables. These should be consistent with the definitions made previously for the untapered strut. Or by analogy with equations (10) of Part III

$$\bar{x} = \frac{x}{d + \ell}\ , \quad \bar{y} = \frac{y}{h_o}\ , \quad \bar{y}_o = \frac{y_o}{h_o} \qquad (48)$$

where, $y_o = y_{max.} = y$ at $x = \ell$.

It is convenient to write,

$$d + \ell = L \tag{49}$$

so that,

$$\bar{x} = \frac{x}{L} \tag{50}$$

By analogy with the calculations following equation (10) of Part III the derivatives of equation (47) are:

$$\frac{dy}{dx} = y' = \frac{h_o}{L} \bar{y}'$$

$$\frac{d^2y}{dx^2} = y'' = \frac{h_o}{L^2} \bar{y}''$$

$$\frac{d^3y}{dx^3} = y''' = \frac{h_o}{L^3} \bar{y}'''$$

$$\frac{d^4y}{dx^4} = y^{iv} = \frac{h_o}{L^4} \bar{y}^{iv}$$

Substituting into (47),

$$(d+L\bar{x})^2 \frac{h_o}{L^4} \bar{y}^{iv} + 2(d+L\bar{x}) \frac{h_o}{L^3} \bar{y}''' + \left[\frac{P_1^2 L^2}{B_o C_o} L^2 \bar{x}^2 + \frac{P_2 L}{B_o} (d+L\bar{x})\right] \frac{h_o}{L^2} \bar{y}''$$

$$+ \frac{P_1^2 L^2}{B_o C_o} \left[2 - \frac{L\bar{x}}{d+L\bar{x}}\right] \left(L\bar{x} \frac{h_o}{L} \bar{y}' - h_o \bar{y}\right) = 0$$

Since $h_o \neq 0$ it may be divided out of the above equation. The above equation may now be written as,

$$\left(\frac{d}{L} + \bar{x}\right)^2 \bar{y}^{iv} + 2\left(\frac{d}{L} + \bar{x}\right)\bar{y}''' + \left[\frac{P_1^2 L^4}{B_o C_o}\,\bar{x}^2 + \frac{P_2\,L^2}{B_o}\left(\frac{d}{L} + \bar{x}\right)\right]\bar{y}''$$

$$+ \frac{P_1^2 L^2}{B_o C_o}\left[2L^2 - \frac{L^2\bar{x}}{\frac{d}{L} + \bar{x}}\right](\bar{x}\,\bar{y}' - \bar{y}) = 0 \qquad (51)$$

which is the <u>equation for the tapered beam</u>. It can be written in more compact form by letting,

$$\lambda = \frac{d}{L} = \frac{d}{d + \ell} = \frac{1}{1 + \frac{\ell}{d}} \qquad (52)$$

In (52) λ is the taper ratio. When $d = 0$, $\lambda = 0$ and the fully tapered strut results. If $d = \infty$, $\lambda = 1$ but equation (51) does not degenerate to the corresponding equation in Part III. This can be expected since the complexities due to variable I and J appear in the above equation. Introducing (52) into (51) gives,

$$(\lambda + \bar{x})^2\,\bar{y}^{iv} + 2(\lambda + \bar{x})\bar{y}''' + \left[\frac{P_1^2 L^4}{B_o C_o}\,\bar{x}^2 + \frac{P_2 L^2}{B_o}(\lambda + \bar{x})\right]\bar{y}''$$

$$+ \frac{P_1^2 L^4}{B_o C_o}\left(2 - \frac{\bar{x}}{\lambda + \bar{x}}\right)(\bar{x}\,\bar{y}' - \bar{y}) = 0 \qquad (53)$$

The above equation for arbitrary taper is difficult to solve as already mentioned. For complete taper $d = 0$, $\lambda = 0$, $L = \ell$; hence the above becomes for this special case,

$$\bar{x}^2\,\bar{y}^{iv} + 2\,\bar{x}\,\bar{y}''' + \left[\frac{P_1^2\ell^4}{B_o C_o}\,\bar{x}^2 + \frac{P_2\ell^2}{B_o}\,\bar{x}\right]\bar{y}''$$

$$+ \frac{P_1^2\ell^4}{B_o C_o}\,\bar{x}\,\bar{y}' - \frac{P_1^2\ell^4}{B_o C_o}\,\bar{y} = 0 \qquad (54)$$

Corresponding to equations (11) and (12) of Part III the significant dimensionless parameters now are,

$$k_1^2 = \frac{P_1^2 \ell^4}{EI_o \, GJ_o} = \frac{P_1^2 \ell^4}{B_o \, C_o} \tag{55}$$

$$k_2 = \frac{P_2 \ell^2}{EI_o} = \frac{P_2 \ell^2}{B_o} \tag{56}$$

so that (54) may be written,

$$\bar{x}^2 \bar{y}^{iv} + 2\bar{x}\,\bar{y}''' + (k_1^2 \bar{x}^2 + k_2 \bar{x})\bar{y}'' + k_1^2 \bar{x}\,\bar{y}' - k_1^2 \bar{y} = 0 \tag{57}$$

This is the <u>equation which will be solved in this section</u>. Comparison with (13) of Part III reveals the complexities added by putting taper into the problem. The third derivative term now appears and the $k_2 \bar{y}''$ term becomes multiplied by $\bar{x}$. Also the fourth derivative is now multiplied by x^2. The last two terms in (57) do not have the factor 2 which appears in (13) of Part III. Altogether a more complicated mathematical problem is posed by (57); however it will yield to a power series solution as did the former case.

It should be noted that the dimensionless parameters should have L in place of ℓ for the arbitrary taper equation (53).

3. The Boundary Conditions

The differential equation was seen to undergo considerable change due to taper effects. Hence it can be expected that the derivatives in the boundary conditions may appear differently than for the untapered problem.

Examining the various conditions which the strut must satisfy gives the boundary conditions. These are developed in detail below.

a). <u>No deflection at free end</u>

$y = 0$ at $x = 0$ is required condition,

or in dimensionless form,

$$h_o \bar{y}_o = 0 \quad \text{at} \quad \bar{x} L = 0$$

since $h_o \neq 0$, $L \neq 0$,

$$\bar{y}_o = 0 \quad \text{at} \quad \bar{x} = 0 \tag{58}$$

is the required boundary condition.

b). <u>No bending moment at free end</u>

$$EI \frac{d^2y}{dx^2} = B\, y'' = 0 \quad \text{at} \quad x = 0 \quad \text{is required condition,}$$

or,

$$B_o \frac{d + x}{d + \ell} y'' \Big|_{x=0} = 0$$

since $\dfrac{B_o}{d + \ell} \neq 0$ at $x = 0$,

$$(d + x)\, y'' \Big|_{x=0} = 0$$

or in dimensionless form,

$$(d + L \bar{x}) \frac{h_o}{L^2} \bar{y}'' \Big|_{\bar{x}=0} = 0$$

$$(\frac{d}{L} + \bar{x}) \frac{h_o}{L} \bar{y}'' \Big|_{\bar{x}=0} = 0$$

and here $\frac{h_o}{L} \neq 0$ for $\bar{x} = 0$ and hence,

$$(\lambda + \bar{x})\, \bar{y}'' \Big|_{\bar{x}=0} = 0 \qquad (59)$$

This is the <u>general form</u> of the boundary condition. When $\lambda = 0$,

$$\bar{x}\, \bar{y}'' \Big|_{\bar{x}=0} = 0 \qquad (60)$$

and it is obvious that this is <u>automatically assured</u> since $\bar{x}$ multiplies $\bar{y}''$. Hence a simplification occurs for the completely tapered strut.

c). <u>No twist at fixed end</u>

To examine the twist θ at the fixed end consider the equation immediately preceding (42). This is,

$$B_o \frac{d + x}{d + \ell}\, y'' = -\,(P_1\, x\, \theta + P_2\, y) \qquad (61)$$

from which,

$$P_1\, x\, \theta = -\frac{B_o}{L}(d + x)\, y'' - P_2\, y$$

Now at $x = \ell$, $\theta = 0$, $y = y_o$, and $y'' \neq 0$. Or,

$$0 = -\left[\frac{B_o}{L}(d + x)\, y'' + P_2\, y\right]_{x=\ell}$$

which is,

$$y'' = -\frac{P_2\, y_o}{B_o} \quad \text{at} \quad x = \ell$$

In dimensionless terms,

$$\frac{h_o}{L^2}\,\bar{y}'' = -\,\frac{P_2\,h_o\,\bar{y}_o}{B_o} \qquad \text{at} \qquad \bar{x} = \frac{\ell}{L}$$

$$\bar{y}'' = -\,\frac{P_2\,L^2}{B_o}\,\bar{y}_o \qquad \text{at} \qquad \bar{x} = \frac{\ell}{L} \tag{62}$$

This is again the general statement of the boundary condition. For complete taper $L = \ell$ and,

$$\bar{y}'' = -\,\frac{P_2\,\ell^2}{B_o}\,\bar{y}_o = -\,k_2\,\bar{y}_o \qquad \text{at} \qquad \bar{x} = 1 \tag{63}$$

and this is the form useful in the analysis which follows.

d). <u>Zero slope to elastic curve at fixed end</u>

The actual condition to be imposed is,

$$y' = 0 \qquad \text{at} \qquad x = \ell$$

This implies a clamped edge.

To find y' differentiate (61) once to get,

$$\frac{B_o}{L}\left[(d + x)y''' + y''\right] = -\,(P_1\,x\,\theta' + P_1\,\theta + P_2\,y')$$

Or the slope is given by,

$$P_2\,y' = -\,\frac{B_o}{L}\left[(d + x)y''' + y''\right] - P_1\,x\,\theta' - P_1\,\theta$$

The boundary condition may therefore be represented by,

$$\frac{B_o}{L}\left[(d + x)y''' + y''\right] + P_1\,x\,\theta' + P_1\,\theta = 0 \qquad \text{at} \qquad x = \ell$$

To find $x\,\theta'$ multiply the second equation above (42) by x to get,

$$x\,\theta' = \frac{P_1\,(d+\ell)}{C_o}\;\frac{x}{d+x}\,(x\,y' - y)$$

Putting this into the above equation and letting $x = \ell$,

$$B_o\,y''' + \frac{B_o}{L}\,y'' - \frac{P_1{}^2\ell}{C_o}\,y_o = 0 \quad \text{at} \quad x = \ell$$

In obtaining this last result use has been made of the fact that $y' = 0$, $\theta = 0$, $y = y_o$ at $x = \ell$. The last equation may also be written as,

$$y''' + \frac{1}{L}\,y'' - \frac{P_1{}^2\ell}{B_o\,C_o}\,y_o = 0 \quad \text{at} \quad x = \ell$$

and since by the second equation above (62), $y'' = -\dfrac{P_2\,y_o}{EI_o}$ at $x = \ell$,

$$y''' - \frac{P_2\,y_o}{LB_o} - \frac{P_1{}^2}{B_o\,C_o}\,y_o = 0 \quad \text{at} \quad x = \ell$$

It is now convenient to introduce non-dimensional quantities,

$$\frac{h_o}{L^3}\,\bar{y}''' - \frac{1}{L}\,\frac{P_2\,h_o\,\bar{y}_o}{B_o} - \frac{P_1{}^2\ell}{B_o\,C_o}\,h_o\,\bar{y}_o = 0 \quad \text{at } \bar{x} = \frac{\ell}{L}$$

or,

$$\bar{y}''' = \frac{P_1{}^2\,\ell\,L^3}{B_o\,C_o}\,\bar{y}_o + \frac{P_2\,L^2}{B_o}\,\bar{y}_o \quad \text{at} \quad \bar{x} = \frac{\ell}{L} \qquad (64)$$

which is the general form for the boundary condition. When $d = 0$, $L = \ell$, $\bar{x} = 1$ the above reduces to,

$$\bar{y}''' = \frac{P_1{}^2\,\ell^4}{B_o\,C_o}\,\bar{y}_o + \frac{P_2\,\ell^2}{B_o}\,\bar{y}_o \quad \text{at} \quad \bar{x} = 1$$

Introducing the parameters of (55) and (56),

$$y''' = (k_1{}^2 + k_2)\,\bar{y}_o \quad \text{at} \quad \bar{x} = 1 \qquad (65)$$

This is the form which will be used in the analysis which follows.

4. <u>Mathematical Problem</u>

For the completely tapered beam the mathematical statement governing the physical problem is then (from (57), (58), (63) and (65)),

$$\bar{x}^2\bar{y}^{iv} + 2\,\bar{x}\,\bar{y}''' + (k_1^2\,\bar{x}^2 + k_2\bar{x})\bar{y}'' + k_1^2(\bar{x}\,\bar{y}' - \bar{y}) = 0 \qquad (66)$$

$$\left.\begin{array}{lll} \bar{y} = 0 & \text{at} & \bar{x} = 0 \\ \bar{x}\,\bar{y}'' = 0 & \text{at} & \bar{x} = 0 \\ \bar{y}'' = -\,k_2\,\bar{y}_o & \text{at} & \bar{x} = 1 \\ \bar{y}''' = (k_1^2 + k_2)\,\bar{y}_o & \text{at} & \bar{x} = 1 \end{array}\right\} \begin{array}{l} a \\ b \\ c \\ d \end{array} \qquad (67)$$

The analogous mathematical problem for the untapered strut is grouped under equations (13) and (14) of Part III.

A solution for (66) and (67) will again be sought in the form of an infinite power series.

5. <u>Recursion Formula from Power Series</u>

As in Part III let,

$$\begin{aligned} \bar{y}\,(\bar{x}) &= \sum_{s=0}^{\infty} a_s\,\bar{x}^{s} \\ \bar{y}'(\bar{x}) &= \sum_{s=0}^{\infty} a_s\,s\,\bar{x}^{s-1} \\ \bar{y}''(\bar{x}) &= \sum_{s=0}^{\infty} a_s\,s\,(s-1)\,\bar{x}^{s-2} \\ \bar{y}'''(\bar{x}) &= \sum_{s=0}^{\infty} a_s\,s\,(s-1)(s-2)\,\bar{x}^{s-3} \\ \bar{y}^{iv}(\bar{x}) &= \sum_{s=0}^{\infty} a_s\,s\,(s-1)(s-2)(s-3)\,\bar{x}^{s-4} \end{aligned} \qquad (68)$$

Substituting into the differential equation (66),

$$\bar{x}^2 \sum_{s=0}^{\infty} a_s\, s(s-1)(s-2)(s-3)\bar{x}^{s-4} + 2\bar{x} \sum_{s=0}^{\infty} a_s\, s(s-1)(s-2)\bar{x}^{s-3}$$

$$+ (k_1{}^2\, \bar{x}^2 + k_2\bar{x}) \sum_{s=0}^{\infty} a_s\, s(s-1)\bar{x}^{s-2}$$

$$+ k_1{}^2\, \bar{x} \sum_{s=0}^{\infty} a_s\, s\, \bar{x}^{s-1} - k_1{}^2 \sum_{s=0}^{\infty} a_s\, \bar{x}^{s} = 0$$

or,

$$\sum_{s=0}^{\infty} \Big[\Big\{ a_s\, s(s-1)(s-2)(s-3) + 2a_s\, s(s-1)(s-2) \Big\}\, \bar{x}^{s-2}$$

$$+ k_2\, a_s\, s(s-1)\bar{x}^{s-1} + k_1{}^2\, a_s \Big\{ s(s-1) + s-1 \Big\}\, \bar{x}^{s} \Big] = 0$$

Now collect a general term of the expansion--as $\bar{x}^{s-2}$. Then,

$$\Big[a_s\, s(s-1)(s-2)(s-3) + 2a_s\, s(s-1)(s-2) + k_2\, a_{s-1}\, (s-1)(s-2)$$

$$+ k_1{}^2\, a_{s-2} \Big\{ (s-2)(s-3) + s-2 + 1 \Big\} \Big]\, \bar{x}^{s-2} = 0$$

In general $\bar{x}^{s-2} \neq 0$; hence the bracketed term must vanish. Doing this and simplifying leads to the following recurrence relation:

$$a_s\, s(s-1)^2(s-2) + a_{s-1}\, k_2\, (s-1)(s-2) + a_{s-2}\, k_1{}^2\, (s-1)(s-3) = 0 \quad (69)$$

Out of this equation all the coefficients may be calculated by assigning values to s . Again note that $0 \leq s \leq \infty$; that is, s is restricted to being a positive integer.

6. Coefficients of the Power Series

The details for calculating the coefficients appearing in the expansion of $\bar{y}(\bar{x})$ --as given in (68)--are calculated the same as in Part III, Section 6. Therefore, with similar details, the results may be written at once as,

$$
\begin{aligned}
a_0 &\neq 0 && \text{from } s = 0 \\
a_1 &\neq 0 && \text{from } s = 1 \\
a_2 &= 0 && \text{from } s = 2 \\
a_3 &= -\frac{k_2}{3!} a_2 && \text{from } s = 3 \\
a_4 &= \frac{k_2^2 - 3k_1^2}{\frac{1}{2}(3!)(4!)} a_2 && \text{from } s = 4 \\
a_5 &= -\frac{k_2^3 - 11k_2k_1^2}{\frac{1}{2}(4!)(5!)} a_2 && \text{from } s = 5 \\
& && \text{(etc.)} \\
a_6 &= \frac{k_2^4 - 26k_2^2k_1^2 + 45k_1^4}{\frac{1}{2}(5!)(6!)} a_2 \\
a_7 &= -\frac{k_2^5 - 50k_2^3k_1^2 + 309k_2k_1^4}{\frac{1}{2}(6!)(7!)} a_2 \\
a_8 &= \frac{k_2^6 - 85k_2^4k_1^2 + 1219k_2^2k_1^4 - 1575k_1^6}{\frac{1}{2}(7!)(8!)} a_2 \\
a_9 &= -\frac{k_2^7 - 133k_2^5k_1^2 + 3619k_2^3k_1^4 - 16{,}407k_2k_1^6}{\frac{1}{2}(8!)(9!)} a_2
\end{aligned}
\tag{70}
$$

$$a_{10} = \frac{k_2^8 - 196k_2^6k_1^2 + 8974k_2^4k_1^4 - 93{,}204k_2^2k_1^6 + 99{,}225k_1^8}{\frac{1}{2}(9!)(10!)}\, a_2$$

$$a_{11} = -\frac{k_2^9 - 276k_2^7k_1^2 + 19{,}614k_2^5k_1^4 - 382{,}724k_2^3k_1^6 + 1{,}411{,}785k_2k_1^8}{\frac{1}{2}(10!)(11!)}\, a_2$$

$$a_{12} = \frac{k_2^{10} - 375k_2^8k_1^2 + 39{,}018k_2^6k_1^4 - 1{,}271{,}150k_2^4k_1^6 + 10{,}638{,}981k_2^2k_1^8 - 9{,}823{,}275k_1^{10}}{\frac{1}{2}(11!)(12!)}\, a_2$$

$$a_{13} = -\frac{k_2^{11} - 495k_2^9k_1^2 + 72{,}138k_2^7k_1^4 - 3{,}624{,}830k_2^5k_1^6 + 56{,}565{,}861k_2^3k_1^8 - 179{,}237{,}475k_2k_1^{10}}{\frac{1}{2}(12!)(13!)}\, a_2$$

$$a_{14} = \frac{k_2^{12} - 638k_2^{10}k_1^2 + 125{,}763k_2^8k_1^4 - 9{,}204{,}404k_2^6k_1^6 + 238{,}340{,}311k_2^4k_1^8 - 1{,}700{,}611{,}758k_2^2k_1^{10} + 1{,}404{,}728{,}325k_1^{12}}{\frac{1}{2}(13!)(14!)}\, a_2$$

7. Imposing the Boundary Conditions

The coefficients of equation (70) may now be substituted back into the series for $\bar{y}\,(\bar{x})$. Then on this expression the boundary conditions (67) should be imposed.

Since $a_0 = 0$ it follows that the first boundary condition is already satisfied. Furthermore the second is likewise assured since $\bar{x}\,\bar{y}'' = 0$ at $\bar{x} = 0$, independent of $\bar{y}''$. Hence the last two conditions of (67) must now be guaranteed.

Expanding the series for $\bar{y}(\bar{x})$ gives,

$$\bar{y}(\bar{x}) = a_1\bar{x} + a_2\bar{x}^2 + a_3\bar{x}^3 + a_4\bar{x}^4 + a_5\bar{x}^5 + a_6\bar{x}^6 + \ldots + a_{14}\bar{x}^{14}$$

It is now convenient to use the notation,

$$a_i = f_i\,(k_1\,,\,k_2)\,a_2 = f_i\,a_2 \tag{71}$$

for example,

$$a_3 = f_3\,a_2$$

$$a_4 = f_4\,a_2$$

etc., etc.

Hence,

$$y(\bar{x}) = a_1\bar{x}+a_2\bar{x}^2+f_3a_2\bar{x}^3+f_4a_2\bar{x}^4+f_5a_2\bar{x}^5+f_6a_2\bar{x}^6+\ldots+f_{14}a_2\bar{x}^{14}$$

Differentiating,

$$\bar{y}''(\bar{x}) = \Big[2+6f_3\bar{x}+12f_4\bar{x}^2+20f_5\bar{x}^3+30f_6\bar{x}^4+42f_7\bar{x}^5+56f_8\bar{x}^6+72f_9\bar{x}^7$$

$$+90f_{10}\bar{x}^8+110f_{11}\bar{x}^9+132f_{12}\bar{x}^{10}+156f_{13}\bar{x}^{11}$$

$$+182f_{14}\bar{x}^{12}+\ldots\Big]\,a_2 \tag{72}$$

$$\bar{y}'''(\bar{x}) = \Big[6f_3+24f_4\bar{x}+60f_5\bar{x}^2+120f_6\bar{x}^3+210f_7\bar{x}^4+336f_8\bar{x}^5$$

$$+504f_9\bar{x}^6+720f_{10}\bar{x}^7+990f_{11}\bar{x}^8+1320f_{12}\bar{x}^9$$

$$+1716f_{13}\bar{x}^{10}+1944f_{14}\bar{x}^{11}+\ldots \Big] a_2 \qquad (73)$$

The last two boundary conditions of (67) may be written as,

$$\bar{y}'' + k_2\,\bar{y}_o = 0 \qquad \text{at} \qquad \bar{x} = 1$$

$$\bar{y}''' - (k_1^2 + k_2)\,\bar{y}_o = 0 \qquad \text{at} \qquad \bar{x} = 1$$

Multiplying the first by $k_1^2 + k_2$, the second by k_2 , and adding,

$$(k_1^2 + k_2)\bar{y}'' + k_2\bar{y}''' = 0 \qquad \text{at} \qquad \bar{x} = 1 \qquad (74)$$

Substituting from (72) and (73) into (74) gives,

$$(k_1^2+k_2)\Big[2+6f_3\bar{x}+12f_4\bar{x}^2+\ldots+182f_{14}\bar{x}^{12}+\ldots\Big]a_2$$

$$+k_2\Big[6f_3+24f_4\bar{x}+60f_5\bar{x}^2+\ldots$$

$$+1944f_{14}\bar{x}^{14}+\ldots\Big]a_2 = 0$$

Letting $\bar{x} = 1$ and since $a_2 \neq 0$ by (70),

$$(k_1^2+k_2)(2+6f_3+12f_4+20f_5+\ldots+182f_{14})$$

$$+k_2(6f_3+24f_4+60f_5+\ldots+1944f_{14}) = 0 \qquad (75)$$

This is now the expression out of which the eigenvalues must be calculated. All the f_i terms are functions of k_1 and k_2 as indicated in (71). Substituting from the a_i into (75) and writing out completely gives,

$$2k_1^2 - 6\frac{k_1^2k_2}{3!} + 12\frac{k_1^2k_2^2 - 3k_1^4}{\frac{1}{2}(3!)(4!)} - 20\frac{k_1^2k_2^3 - 11k_1^4k_2}{\frac{1}{2}(4!)(5!)} + 30\frac{k_1^2k_2^4 - 26k_1^4k_2^2 + 45k_1^6}{\frac{1}{2}(5!)(6!)}$$

$$-42\frac{k_1^2k_2^5 - 50k_1^4k_2^3 + 309k_1^6k_2}{\frac{1}{2}(6!)(7!)} + 56\frac{k_1^2k_2^6 - 85k_1^4k_2^4 + 1219k_1^6k_2^2 - 1575k_1^8}{\frac{1}{2}(7!)(8!)}$$

$$-72\frac{k_1^2k_2^7 - 133k_1^4k_2^5 + 3619k_1^6k_2^3 - 16{,}407k_1^8k_2}{\frac{1}{2}(8!)(9!)}$$

$$+90\frac{k_1^2k_2^8 - 196k_1^4k_2^6 + 8974k_1^6k_2^4 - 93{,}204k_1^8k_2^2 + 99{,}225k_1^{10}}{\frac{1}{2}(9!)(10!)}$$

$$-110\frac{k_1^2k_2^9 - 276k_1^4k_2^7 + 19{,}614k_1^6k_2^5 - 382{,}724k_1^8k_2^3 - 1{,}411{,}785k_1^{10}k_2}{\frac{1}{2}(10!)(11!)}$$

$$+132\frac{k_1^2k_2^{10} - 375k_1^4k_2^8 + 39{,}018k_1^6k_2^6 - 1{,}271{,}150k_1^8k_2^4 + 10{,}638{,}981k_1^{10}k_2^2 - 9{,}823{,}275k_1^{12}}{\frac{1}{2}(11!)(12!)}$$

$$-156\frac{k_1^2k_2^{11} - 495k_1^4k_2^9 + 72{,}138k_1^6k_2^7 - 3{,}624{,}830k_1^8k_2^5 + 56{,}565{,}861k_1^{10}k_2^3 - 179{,}237{,}475k_1^{12}k_2}{\frac{1}{2}(12!)(13!)}$$

+ (continued)

$$+182\frac{k_1^2k_2^{12}-638k_1^4k_2^{10}+125{,}763k_1^6k_2^8-9{,}204{,}404k_1^8k_2^6+238{,}340{,}311k_1^{10}k_2^4-1{,}700{,}611{,}758k_1^{12}k_2^2+1{,}404{,}728{,}325k_1^{14}}{\frac{1}{2}(13!)(14!)}$$

$$+2k_2-6\,\frac{k_2^2}{3!}+12\,\frac{k_2^3-3k_1^2k_2}{\frac{1}{2}(3!)(4!)}-20\,\frac{k_2^4-11k_1^2k_2^2}{\frac{1}{2}(4!)(5!)}+30\frac{k_2^5-26k_1^2k_2^3+45k_1^4k_2}{\frac{1}{2}(5!)(6!)}$$

$$-42\frac{k_2^6-50k_1^2k_2^4+309k_1^4k_2^2}{\frac{1}{2}(6!)(7!)}+56\frac{k_2^7-85k_1^2k_2^5+1219k_1^4k_2^3-1575k_1^6k_2}{\frac{1}{2}(7!)(8!)}$$

$$-72\frac{k_2^8-133k_1^2k_2^6+3619k_1^4k_2^4-16{,}407k_1^6k_2^2}{\frac{1}{2}(8!)(9!)}$$

$$+90\frac{k_2^9-196k_1^2k_2^7+8974k_1^4k_2^5-93{,}204k_1^6k_2^3+99{,}225k_1^8k_2}{\frac{1}{2}(9!)(10!)}$$

$$-110\frac{k_2^{10}-276k_1^2k_2^8+19{,}614k_1^4k_2^6-382{,}724k_1^6k_2^4+1{,}411{,}785k_1^8k_2^2}{\frac{1}{2}(10!)(11!)}$$

$$+132\,\frac{k_2^{11}-375k_1^2k_2^9+39{,}018k_1^4k_2^7-1{,}271{,}150k_1^6k_2^5+10{,}638{,}981{,}k_1^8k_2^3-9{,}823{,}275k_1^{10}k_2}{\frac{1}{2}(11!)(12!)}$$

$$-156\frac{k_2^{12}-495k_1^2k_2^{10}+72{,}138k_1^4k_2^8-3{,}624{,}830k_1^6k_2^6+56{,}565{,}861k_1^6k_2^4-179{,}237{,}475k_1^{10}k_2^2}{\frac{1}{2}(12!)(13!)}$$

+ (continued)

$$+182\,\frac{k_2^{13}-638k_1^2k_2^{11}+125{,}763k_1^4k_2^9-9{,}204{,}404k_1^6k_2^7+238{,}340{,}311k_1^8k_2^5-1{,}700{,}611{,}758k_1^{10}k_2^3+1{,}404{,}728{,}325k_1^{12}k_2}{\frac{1}{2}\,(13!)(14!)}$$

$$-6\,\frac{k_2^2}{3!}+24\,\frac{k_2^3-3k_1^2k_2}{\frac{1}{2}\,(3!)(4!)}-60\,\frac{k_2^4-11k_1^2k_2^2}{\frac{1}{2}\,(4!)(5!)}+120\,\frac{k_2^5-26k_1^2k_2^3+45k_1^4k_2}{\frac{1}{2}\,(5!)(6!)}$$

$$-210\,\frac{k_2^6-50k_1^2k_2^4+309k_1^4k_2^2}{\frac{1}{2}\,(6!)(7!)}+336\,\frac{k_2^7-85k_1^2k_2^5+1219k_1^4k_2^3-1575k_1^6k_2}{\frac{1}{2}\,(7!)(8!)}$$

$$-504\,\frac{k_2^8-133k_1^2k_2^6+3619k_1^4k_2^4-16{,}407k_1^6k_2^2}{\frac{1}{2}\,(8!)(9!)}$$

$$+720\,\frac{k_2^9-196k_1^2k_2^7+8974k_1^4k_2^5-93{,}204k_1^6k_2^3+99{,}225k_1^8k_2}{\frac{1}{2}\,(9!)(10!)}$$

$$-990\,\frac{k_2^{10}-276k_1^2k_2^8+19{,}614k_1^4k_2^6-382{,}724k_1^6k_2^4+1{,}411{,}785k_1^8k_2^2}{\frac{1}{2}\,(10!)(11!)}$$

$$+1320\,\frac{k_2^{11}-375k_1^2k_2^9+39{,}018k_1^4k_2^7-1{,}271{,}150k_1^6k_2^5+10{,}638{,}981k_1^8k_2^3-9{,}823{,}275k_1^{10}k_2}{\frac{1}{2}\,(11!)(12!)}$$

– (continued)

$$-1716\frac{k_2^{12}-495k_1^2k_2^{10}+72,138k_1^4k_2^8-3,624,830k_1^6k_2^6+56,565,861k_1^8k_2^4-179,237,475k_1^{10}k_2^2}{\frac{1}{2}(12!)(13!)}$$

$$+2184\frac{k_2^{13}-638k_1^2k_2^{11}+125,763k_1^4k_2^9-9,204,404k_1^6k_2^7+238,340,311k_1^8k_2^5-1,700,611,758k_1^{10}k_2^3+1,404,728,325k_1^{12}k_2}{\frac{1}{2}(13!)(14!)}$$

$$+ \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; = 0$$

This last expression is lengthy and complicated. In order to make it easier to handle collect terms. This results in,

$$k_1^2 - \frac{3k_1^4}{2!3!} + \frac{45k_1^6}{4!5!} - \frac{1575k_1^8}{6!7!} + \frac{99,225k_1^{10}}{8!9!} - \frac{9,823,275k_1^{12}}{10!11!}$$

$$+\frac{1,404,728,325k_1^{14}}{12!\,13!} + \; . \; . \; . \; . \; .$$

$$+k_2 - \left(\frac{1}{2!}+\frac{1}{2!}\right)k_2^2 + \left(\frac{1}{1!3!}+\frac{1}{2!3!}\right)k_2^3 - \left(\frac{1}{2!4!}+\frac{1}{3!4!}\right)k_2^4$$

$$+\left(\frac{1}{3!5!}+\frac{1}{4!5!}\right)k_2^5 - \left(\frac{1}{4!6!}+\frac{1}{5!6!}\right)k_2^6 + \left(\frac{1}{5!7!}+\frac{1}{6!7!}\right)k_2^7 - \left(\frac{1}{6!8!}+\frac{1}{7!8!}\right)k_2^8$$

$$+\left(\frac{1}{7!9!}+\frac{1}{8!9!}\right)k_2^9 - \left(\frac{1}{8!10!}+\frac{1}{9!10!}\right)k_2^{10} + \left(\frac{1}{9!11!}+\frac{1}{10!11!}\right)k_2^{11} - \text{(continued)}$$

$$- \left(\frac{1}{10!12!} + \frac{1}{11!12!}\right)k_2^{12} + \left(\frac{1}{11!13!} + \frac{1}{12!13!}\right)k_2^{13} + \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$- \left(\frac{1}{2!} + \frac{3}{1!3!} + \frac{3}{2!3!}\right)k_1^2 k_2 \quad + \quad \left(\frac{1}{2!3!} + \frac{11}{2!4!} + \frac{11}{3!4!}\right)k_1^2 k_2^2$$

$$- \left(\frac{1}{3!4!} + \frac{26}{3!5!} + \frac{26}{4!5!}\right)k_1^2 k_2^3 + \left(\frac{1}{4!5!} + \frac{50}{4!6!} + \frac{50}{5!6!}\right)k_1^2 k_2^4$$

$$- \left(\frac{1}{5!6!} + \frac{85}{5!7!} + \frac{85}{6!7!}\right)k_1^2 k_2^5 + \left(\frac{1}{6!7!} + \frac{133}{6!8!} + \frac{133}{7!8!}\right)k_1^2 k_2^6$$

$$- \left(\frac{1}{7!8!} + \frac{196}{7!9!} + \frac{196}{8!9!}\right)k_1^2 k_2^7 + \left(\frac{1}{8!9!} + \frac{276}{8!10!} + \frac{276}{9!10!}\right)k_1^2 k_2^8$$

$$- \left(\frac{1}{9!10!} + \frac{375}{9!11!} + \frac{375}{10!11!}\right)k_1^2 k_2^9 + \left(\frac{1}{10!11!} + \frac{495}{10!12!} + \frac{495}{11!12!}\right)k_1^2 k_2^{10}$$

$$- \left(\frac{1}{11!12!} + \frac{638}{11!13!} + \frac{638}{12!13!}\right)k_1^2 k_2^{11} + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ \left(\frac{11}{3!4!} + \frac{45}{3!5!} + \frac{45}{4!5!}\right)k_1^4 k_2 - \left(\frac{26}{4!5!} + \frac{309}{4!6!} + \frac{309}{5!6!}\right)k_1^4 k_2^2$$

$$+ \left(\frac{50}{5!6!} + \frac{1219}{5!7!} + \frac{1219}{6!7!}\right)k_1^4 k_2^3 - \left(\frac{85}{6!7!} + \frac{3619}{6!8!} + \frac{3619}{7!8!}\right)k_1^4 k_2^4$$

$$+ \left(\frac{133}{7!8!} + \frac{8974}{7!9!} + \frac{8974}{8!9!}\right)k_1^4 k_2^5 - \left(\frac{196}{8!9!} + \frac{19{,}614}{8!\,10!} + \frac{19{,}614}{9!10!}\right)k_1^4 k_2^6$$

+ (continued)

$$+ \left(\frac{276}{9!10!} + \frac{39{,}018}{9!11!} + \frac{39{,}018}{10!11!}\right)k_1^4 k_2^7 - \left(\frac{375}{10!11!} + \frac{72{,}138}{10!12!} + \frac{72{,}138}{11!12!}\right)k_1^4 k_2^8$$

$$+ \left(\frac{495}{11!12!} + \frac{125{,}763}{11!13!} + \frac{125{,}763}{12!13!}\right)k_1^4 k_2^9 + \cdots$$

$$- \left(\frac{309}{5!6!} + \frac{1575}{5!7!} + \frac{1575}{6!7!}\right)k_1^6 k_2 + \left(\frac{1219}{6!7!} + \frac{16{,}407}{6!\,8!} + \frac{16{,}407}{7!8!}\right)k_1^6 k_2^2$$

$$- \left(\frac{3619}{7!8!} + \frac{93{,}204}{7!9!} + \frac{93{,}204}{8!9!}\right)k_1^6 k_2^3 + \left(\frac{8974}{8!9!} + \frac{382{,}724}{8!10!} + \frac{382{,}724}{9!10!}\right)k_1^6 k_2^4$$

$$- \left(\frac{19{,}614}{9!10!} + \frac{1{,}271{,}150}{9!11!} + \frac{1{,}271{,}150}{10!11!}\right)k_1^6 k_2^5$$

$$+ \left(\frac{39{,}018}{10!11!} + \frac{3{,}624{,}830}{10!12!} + \frac{3{,}624{,}830}{11!12!}\right)k_1^6 k_2^6$$

$$- \left(\frac{72{,}138}{11!12!} + \frac{9{,}204{,}404}{11!13!} + \frac{9{,}204{,}404}{12!13!}\right)k_1^6 k_2^7 + \cdots$$

$$+ \left(\frac{16{,}407}{7!8!} + \frac{99{,}225}{7!9!} + \frac{99{,}225}{8!9!}\right)k_1^8 k_2 - \left(\frac{93{,}204}{8!9!} + \frac{1{,}411{,}785}{8!10!} + \frac{1{,}411{,}785}{9!10!}\right)k_1^8 k_2^2$$

$$+ \left(\frac{382{,}724}{9!10!} + \frac{10{,}638{,}981}{9!11!} + \frac{10{,}638{,}981}{10!11!}\right)k_1^8 k_2^3$$

$$- \left(\frac{1{,}271{,}150}{10!11!} + \frac{56{,}565{,}861}{10!12!} + \frac{56{,}565{,}861}{11!12!}\right)k_1^8 k_2^4 \quad + \text{(continued)}$$

$$+ \left(\frac{3,624,830}{11!12!} + \frac{238,340,311}{11!13!} + \frac{238,340,311}{12!13!}\right)k_1^{8}k_2^{5} \quad . \quad . \quad . \quad .$$

$$- \left(\frac{1,411,785}{9!10!} + \frac{9,823,275}{9!11!} + \frac{9,823,275}{10!11!}\right)k_1^{10}k_2$$

$$+ \left(\frac{10,638,981}{10!11!} + \frac{179,237,475}{10!12!} + \frac{179,237,475}{11!12!}\right)k_1^{10}k_2^{2}$$

$$- \left(\frac{56,565,861}{11!12!} + \frac{1,700,611,758}{11!13!} + \frac{1,700,611,758}{12!13!}\right)k_1^{10}k_2^{3} + \quad . \quad . \quad .$$

$$+ \left(\frac{1,700,611,758}{11!12!} + \frac{1,404,728,325}{11!13!} + \frac{1,404,728,325}{12!13!}\right)k_1^{12}k_2 + \quad . \quad . \quad .$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = \quad 0 \qquad (76)$$

This long expression corresponds to equation (26) of Part III. The tremendous complication, due to complete taper, is obvious on comparing the two equations.

Values of k_1^2 and k_2 which satisfy (76) will likewise satisfy the differential equation and boundary conditions for this fully tapered beam. At the same time we know that the deflection is not zero since $a_2 \neq 0$. As a result solutions of (76), for minimum coupled k_1^2 and k_2 , will represent solutions to the physical problem. The critical loads P_1 and P_2 are given by $(k_1^2)_{min.}$ and $(k_2)_{min.}$ from equations (55) and (56).

8. Degenerate Cases

Case A (The Completely Tapered Column)

It is convenient to study the calculation of the critical load for this case since the exact solution is known in terms of Bessel Functions. (See Ref. 9, p. 94.) Thus the correct result is,

$$(P_2)_{crit.} = \frac{1.446 \ B_o}{\ell^2} = \frac{1.446 \ EI_o}{\ell^2} \tag{77}$$

Case A occurs when $k_1 = 0$ in equation (76). Since $k_2 \neq 0$ the resulting equation may easily be seen to reduce to,

$$1 - k_2 + \frac{k_2^2}{(2!)^2} - \frac{k_2^3}{(3!)^2} + \frac{k_2^4}{(4!)^2} - \frac{k_2^5}{(5!)^2} + \frac{k_2^6}{(6!)^2} - \ldots = 0 \tag{78}$$

The minimum value of k_2 given by this expression will then correspond to $(P_2)_{crit.}$ for the fully tapered column.

To calculate numerically this critical condition it is again convenient to use Graeffe's Method. The proper manner of applying Graeffe's method will therefore be sought by carrying out some detailed evaluations of $(P_2)_{crit.}$

First Approximation to $(P_2)_{crit.}$ $(P_1 = 0)$

Let, $z_z = \frac{1}{k_2}$

so that,

$$(z_z)_{max.} \text{ corresponds to } (k_2)_{min.}$$

Substituting into (78) gives,

$$1 - \frac{1}{z_2} + \frac{1}{(2!)^2}\frac{1}{z_2^2} - \frac{1}{(3!)^2}\frac{1}{z_2^3} + \frac{1}{(4!)^2}\frac{1}{z_2^4} - \ldots = 0 \tag{79}$$

Use the first three terms only so that,

$$1 - \frac{1}{z_2} + \frac{1}{(2!)^2}\frac{1}{z_2^2} = 0$$

Or,

$$f(z_2) = z_2^{\,2} - z_2 + \frac{1}{4}$$

$$f(-z_2) = z_2^{\,2} + z_2 + \frac{1}{4}$$

and,

$$f(z_2)\cdot f(-z_2) = z_2^{\,4} - \frac{1}{2}\,z_2^{\,2} + \frac{1}{16}$$

Graeffe's method then states that,

$$(z_2)_{max.} \simeq \sqrt{1/2}$$

Hence,

$$(k_2)_{min.} = \frac{1}{(z_2)_{max.}} = \sqrt{2}$$

Applying equation (56) which relates k_2 to P_2 ,

$$(k_2)_{min.} = \frac{(P_2)_{crit.}\ell^{\,2}}{EI_o} = \sqrt{2}$$

or,

$$(P_2)_{crit.} = \frac{1.414\ EI_o}{\ell^{\,2}}$$

This is the first zpproximation to $(P_2)_{crit.}$ It is too small by approximately 2%.

Second Approximation to $(P_2)_{crit.}$ $(P_1 = 0)$

Ordinarily a second approximation is secured by a reapplication of Graeffe's method. This will not give results different from the first approximation, however, unless additional terms in (79) are used. Furthermore, using additional terms in (79) and applying Graeffe's method once more will yield a result no different from the first approximation. Hence to get a significant second approximation, take one additional term in (79) and carry out Graeffe's method twice. The details are shown below.

Select from (79),

$$z_2^3 - z_2^2 + \frac{1}{4} z_2 - \frac{1}{36} = 0$$

from which,

$$f(z_2) = z_2^3 - z_2^2 + \frac{1}{4} z_2 - \frac{1}{36}$$

$$f(-z_2) = -z_2^3 - z_2^2 - \frac{1}{4} z_2 - \frac{1}{36}$$

and

$$f(z_2) \cdot f(-z_2) = - z_2^6 + \frac{1}{2} z_2^4 - \frac{1}{144} z_2^2 + \frac{1}{(36)^2}$$

As before

$$(z_2)_{max.} = \sqrt{1/2}$$

Now apply the method a second time. Write from the previous expression,

$$f(z_2^2) = -(z_2^2)^3 + \frac{1}{2}(z_2^2)^2 - \frac{1}{144}(z_2^2) + \frac{1}{(36)^2}$$

$$f(-z_2^2) = +(z_2^2)^3 + \frac{1}{2}(z_2^2)^2 + \frac{1}{144}(z_2^2) + \frac{1}{(36)^2}$$

so that,

$$f(z_2^2)\cdot f(-z_2^2) = -(z_2^2)^6 + \frac{17}{72}(z_2^2)^4 + \frac{15}{16(36)^2}(z_2^2)^2 + \frac{1}{(36)^4}$$

Graeffe's method now gives as the second approximation for $(z_2)_{max.}$,

$$(z_2)_{max.} \cong \sqrt[4]{\frac{17}{72}} = \frac{1}{1.435} \quad \text{(to slide rule accuracy)}$$

Consequently,

$$(P_2)_{crit.} = \frac{1.435\ EI_o}{\ell^2}$$

The error is now less than 1%.

Third Approximation to $(P_2)_{crit.}$ $(P_1 = 0)$

A third application of Graeffe's method would not be advisable unless an additional term is used in equation (79). Thus select all the terms shown in (79) and apply Graeffe's method three times. This will give a sufficiently accurate value for $(P_2)_{crit.}$ (with $P_1 = 0$) so that it may be used as an end point for the interaction curve. Thus,

$$f(z_2) = z_2^4 - z_2^3 + \frac{1}{4}z_2^2 - \frac{1}{36}z_2 + \frac{1}{576}$$

$$f(-z\ ) = z_2^4 + z_2^3 + \frac{1}{4}z_2^2 + \frac{1}{36}z_2 + \frac{1}{576}$$

so that,

$$f(z_2)\cdot f(-z_2) = z_2^8 - \frac{1}{2} z_2^6 + \frac{1}{96} z_2^4 + \frac{1}{8(36)^2} z_2^2 + \frac{1}{(576)^2}$$

It is obvious that taking these additional terms and applying the method only once leads to a result no different from that obtained from the much simpler first approximation. Therefore apply the method a second time.

$$f(z_2^2) = (z_2^2)^4 - \frac{1}{2}(z_2^2)^3 + \frac{1}{96}(z_2^2)^2 + \frac{1}{8(36)^2}(z_2^2)^1 + \frac{1}{(576)^2}$$

$$f(-z_2^2) = (z_2^2)^4 + \frac{1}{2}(z_2^2)^3 + \frac{1}{96}(z_2^2)^2 - \frac{1}{8(36)^2}(z_2^2)^1 + \frac{1}{(576)^2}$$

and hence,

$$f(z_2^2)\cdot f(-z_2^2) = (z_2^2)^8 - \frac{11}{48}(z_2^2)^6 + \frac{33}{256(36)^2}(z_2^2)^4$$

$$- \frac{191}{192}\frac{1}{(8)^2}\frac{1}{(36)^2}(z_2^2)^2 + \frac{1}{(576)^4}$$

The approximation to $(z_2)_{max.}$ is now,

$$(z_2)_{max.} \cong \sqrt[4]{11/48} = \frac{1}{1.445} \quad \text{(to slide rule accuracy)}$$

This compares favorably with $\frac{1}{1.434}$ from the previous approximation.

However it is reasonable to carry the method out for a third step this time. Therefore from the product of $f(z_2^2)\cdot f(-z_2^2)$ form the expression for $f(z_2^4)$ as follows:

$$f(z_2{}^4) = (z_2{}^4)^4 - \frac{11}{48}(z_2{}^4)^3 + \frac{33}{256(36)^2}(z_2{}^4)^2$$

$$- \frac{191}{192}\frac{1}{(8)^2(36)^2}(z_2{}^4)^1 + \frac{1}{(576)^4}$$

$$f(-z_2{}^4) = (z_2{}^4)^4 + \frac{11}{48}(z_2{}^4)^3 + \frac{33}{256(36)^2}(z_2{}^4)^2$$

$$+ \frac{191}{192}\frac{1}{(8)^2(36)^2}(z_2{}^4)^1 + \frac{1}{(576)^4}$$

Now the third term in the product of $f(z_2{}^4) \cdot f(-z_2{}^4)$ is the one of interest. This will be the term in $(z_2{}^4)^6$. Hence the complete product need not be carried out and it is sufficient to write,

$$f(z_2{}^4) \cdot f(-z_2{}^4) = (z_2{}^4)^8 - \frac{11(263)}{24(6)^2(8)^2}(z_2{}^4)^6 + \ldots$$

so that,

$$(z_2)_{max.} \cong \sqrt[8]{\frac{11(263)}{24(6)^2(8)^2}} = \sqrt[8]{\frac{1}{19.1106}} = \frac{1}{1.44}$$

also,

$$(z_2)_{max.} \cong \sqrt[8]{\frac{11(263)}{24(6)^2(8)^2}} = \sqrt[8]{\frac{2893}{55296}} = \sqrt[8]{\frac{1}{19.1}} = \frac{1}{1.445}$$

and from this result,

$$(P_2)_{crit.} = \frac{1.445\ EI_o}{2} \quad \text{(to slide rule accuracy)} \qquad (80)$$

Hence the approximation is now very accurate. This final result may be used in the plotting of one of the end points on the interaction curve.

<u>Case B</u> (The Fully Tapered Beam Problem)

This also has a known solution. (See Ref. 10 and also table on p. 250 of Ref. 1.) The solution is,

$$(P_1)_{crit.} = \frac{2.405\sqrt{B_o C_o}}{\ell^2} = \frac{2.405\sqrt{EI_o\ GJ_o}}{\ell^2}$$

Putting $k_2 = 0$ in equation (76) and remembering that for this case $k_1^2 \neq 0$,

$$1 - \frac{k_1^2}{(2!)^2} + \frac{3^2 k_1^4}{(4!)^2} - \frac{3^2 \cdot 5^2 k_1^6}{(6!)^2} + \frac{3^2 \cdot 5^2 \cdot 7^2 k_1^8}{(8!)^2} - \frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 k_1^{10}}{(10!)^2} + \ldots = 0 \qquad (81)$$

As before the problem now is reduced to that of calculating $(k_1^2)_{min.}$ from the above equation. Thus proceed as in previous examples and let,

$$(z_1)_{max.} = \frac{1}{(k_1^2)_{min.}} \qquad (82)$$

which substituted into (81) gives,

$$1 - \frac{1}{(2!)^2}\frac{1}{z_1} + \frac{3^2}{(4!)^2}\frac{1}{z_1^2} - \frac{3^2 \cdot 5^2}{(6!)^2}\frac{1}{z_1^3} + \frac{3^2 \cdot 5^2 \cdot 7^2}{(8!)^2}\frac{1}{z_1^4} + \ldots = 0$$

By experience with Case A only terms up to z_1^4 will be retained. A second application of Graeffe's method on this equation will give sufficiently accurate results. Thus,

$$f(z_1) = z_1^4 - \frac{1}{4} z_1^3 + \frac{1}{64} z_1^2 - \frac{1}{2304} z_1 + \frac{1}{147{,}456} = 0$$

and

$$f(-z_1) = z_1^4 + \frac{1}{4}z_1^3 + \frac{1}{64}z_1^2 + \frac{1}{2304}z_1 + \frac{1}{147,456} = 0$$

and hence,

$$f(z_1) \cdot f(-z_1) = z_1^8 - \frac{1}{32} z_1^6 + \frac{1}{24576} z_1^4 - \ldots$$

so that,

$$f(z_1^2) = (z_1^2)^4 - \frac{1}{32} (z_1^2)^3 + \frac{1}{24576} (z_1^2)^2 - \ldots$$

$$f(-z_1^2) = (z_1^2)^4 + \frac{1}{32} (z_1^2)^3 + \frac{1}{24576} (z_1^2)^2 + \ldots$$

and

$$f(z_1^2) \cdot f(-z_1^2) = (z_1^2)^8 - \frac{11}{12,288} (z_1^2)^6 + \ldots$$

It is unnecessary to carry additional terms since only the first or lowest buckled mode is being considered. From this last expression,

$$(z_1)_{max.} \cong \sqrt[4]{\frac{11}{12,288}}$$

$$(k_1^2)_{min.} = \frac{1}{(z_1)_{max.}} = \sqrt[4]{\frac{12,288}{11}} = \sqrt[4]{1117}$$

$$(k_1^2)_{min.} = 5.78 \quad \text{(to slide rule accuracy)}$$

Consequently from equation (55) critical P_1 , when $P_2 = 0$, is given by,

$$P_1 = \frac{k_1\sqrt{EI_o \, GJ_o}}{\ell^2} = \frac{2.40\sqrt{EI_o \, GJ_o}}{\ell^2}$$

This agrees remarkably well with the results previously attributed to Federhofer.

9. The Coupled Problem

This, in principle, is the same as the earlier calculation made for the untapered strut. Consequently the procedure adopted there will be also used for this present case of the tapered strut.

Corresponding to equation (31) of Part III the following may therefore be written:

$$\frac{k_1^2}{k_2} = \alpha \qquad (83)$$

This may now be substituted in equation (76) to give the following equation in k_2 :

$$\alpha k_2 - 0.25 (\alpha k_2)^2 + 0.015,625,0 (\alpha k_2)^3 - 0.000,434,0 (\alpha k_2)^4$$

$$+ 0.000,006,8 (\alpha k_2)^5 - 0.000,000,1 (\alpha k_2)^6 + \ldots$$

$$k_2 - k_2^2 + 0.25 k_2^3 - 0.027,777,8 k_2^4 + 0.001,736,1 k_2^5$$

$$- 0.000,069,4 k_2^6 + 0.000,001,9 k_2^7 \ldots$$

$$- 1.25 \alpha k_2^2 + 0.388,888,9 \alpha k_2^3 - 0.052,083,2 \alpha k_2^4$$

$$+ 0.003,819,4 \alpha k_2^5 - 0.000,175,4 \alpha k_2^6 + \text{(continued)}$$

$$+ 0.000,005,5\ \alpha k_2^{7} - 0.000,000,1\ \alpha k_2^{8} + \cdots$$

$$+ 0.154,513,8\ \alpha^2 k_2^{3} - 0.030,486,0\ \alpha^2 k_2^{4}$$

$$+ 0.002,930,1\ \alpha^2 k_2^{5} - 0.000,165,8\ \alpha^2 k_2^{6}$$

$$+ 0.000,006,2\ \alpha^2 k_2^{7} - 0.000,000,1\ \alpha^2 k_2^{8} + \cdots$$

$$- 0.006,614,4\ \alpha^3 k_2^{4} + 0.000,981,8\ \alpha^3 k_2^{5}$$

$$- 0,000,075,1\ \alpha^3 k_2^{6} + 0.000,003,5\ \alpha^3 k_2^{7}$$

$$- 0.000,000,1\ \alpha^3 k_2^{8} + \cdots$$

$$+ 0.000,141,7\ \alpha^4 k_2^{5} - 0.000,017,0\ \alpha^4 k_2^{6}$$

$$+ 0.000,001,1\ \alpha^4 k_2^{7} - \cdots$$

$$- 0.000,001,9\ \alpha^5 k_2^{6} + 0.000,000,1\ \alpha^5 k_2^{7} - \cdots = 0 \qquad (84)$$

When $\alpha = 0$ this becomes equivalent to equation (78) for the degenerate case of the column. However, the above expression may now be solved for $(k_2)_{min.}$ for various values of the parameter α. Then the corresponding value of $(k_1^2)_{min.}$ is obtained by substitution into (83).

Solutions have been calculated for α = 1/10, 1/4, 1/2, and 1. For larger values of α it becomes desirable to re-define the parameter relating k_1^2 and k_2 as,

$$\frac{k_2}{k_1^2} = \beta \tag{85}$$

The numerical work is improved by following such a procedure. Substituting (85) into (76) gives,

$$k_1^2 - 0.25k_1^4 + 0.015,625,0k_1^6 - 0.000,434,0k_1^8 + 0.000,006,8k_1^{10} - 0.000,000,1k_1^{12} + \ldots$$

$$+\beta k_1^2 - \beta^2 k_1^4 + 0.25\,\beta^3 k_1^6 - 0.027,777,8\,\beta^4 k_1^8 + 0.001,736,1\,\beta^5 k_1^{10} - 0,000,069,4\,\beta^6 k_1^{12} + 0.000,001,9\,\beta^7 k_1^{14} - \ldots$$

$$- 1.25\beta k_1^4 + 0.388,888,9\,\beta^2 k_1^6 - 0.052,083,2\,\beta^3 k_1^8 + 0.003,819,4\,\beta^4 k_1^{10} - 0.000,175,4\,\beta^5 k_1^{12} + 0.000,001,5\,\beta^6 k_1^{14} - \ldots.$$

$$+ 0.154,513,8\,\beta k_1^6 - 0.030,486,0\,\beta^2 k_1^8 + 0.002,930,1\,\beta^3 k_1^{10} - 0.000,165,8\,\beta^4 k_1^{12} + 0.000,006,2\,\beta^5 k_1^{14} - \ldots$$

$$- 0.006,614,4\,\beta k_1^8 + 0.000,981,8\,\beta^2 k_1^{10} - 0.000,075,1\,\beta^3 k_1^{12} + 0.000,003,5\beta^4 k_1^{14} - \ldots$$

(continued)

$$+ 0.000,141,7 \beta k_1^{10} + 0.000,017,0 \beta^2 k_1^{12} - 0.000,001,1 \beta^3 k_1^{14} + \dots$$

$$- 0.000,001,9 \beta k_1^{12} + 0.000,000,1 \beta^2 k_1^{14} - \dots$$

$$+ 0.000,000,0 \beta k_1^{14} - \dots\dots = 0 \qquad (86)$$

When $\beta = 0$ the above degenerates to equation (81) for the beam problem. Solutions have been calculated for equation (86) to supplement the previous results from equation (84). Values of β used were 0.5 , 0.3 , 0.2 , and 0.1 .

To insure decent numerical accuracy equations (84) and (86) were taken with seven terms for each value of the parameter. Calculations were then made for $(k_1)_{min.}$ and $(k_2)_{min.}$ by carrying out Graeffe's method through a second approximation. The numerical details have been amply illustrated in previous calculations and therefore need not be repeated here. Results are listed in Table VI. A plot of these values gives the design curve of Fig. 24.

PART VI

PROBLEMS FOR FURTHER INVESTIGATION

1. Introductory Comments

Since there are a number of unsolved problems connected with this research and since these are of importance to the designer, they will be considered in some detail in this section. Wherever possible the governing differential equation and boundary conditions will be given.

Those cases previously considered have been restricted to a compressive axial load component. Actually this axial load could also be tension. In fact, the aerodynamic model may produce either effect depending on how it is mounted and what the angle of attack happens to be. Consequently it is necessary that this possibility be recognized. Sections 2 and 3 discuss this aspect of the problem.

In section 4 a brief discussion is given of the possible influence of axial stresses on the torsional rigidity of the strut. This is then related to the shape of the cross-section.

Section 5 takes up the general problem of the tapered strut with sweepback. A reasonable picture of the physical behavior of such a strut is given. From this discussion it follows that sweepback does not complicate the problem to the same extent that taper does. Both axial compression or tension are permitted for the swept strut.

The final section, 6, deals with the inelastic problem. No theoretical analysis is attempted. It is suggested that an experimental study be made, much as for the case of inelastic (i.e., short) columns.

2. Untapered Strut - Axial Tension (P_2) + Transverse Component (P_1)

Dependent on the attitude of the aerodynamic model being tested, the axial component P_2 may be either compressive or tensile in nature. In the analysis of Part III a compressive component was assumed.

When P_2 is tension, rather than compression, it tends to delay buckling. Consequently the resultant load needed to produce strut instability can be expected to increase greatly as P_2 becomes stabilizing in nature.

A complete analysis should admit the possibility of P_2 being a tensile component. This can be done by reconsidering the analysis in Part III.

Since P_2 produces only bending at an arbitrary section along the strut (see Sections 1b and 1d of Part III) it is merely necessary to correct its sign in equation (5) of Part III. Instead of being negative, P_2 now becomes positive.

This correction automatically carries through the remaining equations. Thus the new equations are:

(Equation (6) of Part III becomes),

$$\frac{d^4y}{dx^4} + \left(\frac{{P_1}^2}{BC} x^2 - \frac{P_2}{B}\right) \frac{d^2y}{dx^2} + 2 \frac{{P_1}^2}{BC} x \frac{dy}{dx} - 2 \frac{{P_1}^2}{BC} y = 0 \qquad (87)$$

This is the basic differential equation for this problem.

Establishing non-dimensional variables and coefficients as before (see equations (10), (11), and (12) of Part III) the non-dimensional form of the preceding differential equation appears as,

$$\frac{d^4\bar{y}}{d\bar{x}^4} + (k_1^2\,\bar{x}^2 - k_2)\,\frac{d^2\bar{y}}{d\bar{x}^2} + 2\,k_1^2\,(\bar{x}\,\frac{d\bar{y}}{d\bar{x}} - \bar{y}) = 0 \qquad (88)$$

The corresponding equation in Part III is (13).

The boundary conditions also undergo some change. Thus equation (8) of Part III represents a boundary condition which depends on equation (5), also of Part III. Therefore, a change in (8) may be expected. As before, it is only necessary to change the sign of P_2, a fact which is easily demonstrated by carrying through the details analogous to those of 2a in Part III. Eventually the boundary conditions (14) of III become ,

$$\begin{aligned}
&\bar{y} = 0 && \text{at} \quad \bar{x} = 0 \quad \text{(unchanged)}\\
&\frac{d^2\bar{y}}{d\bar{x}^2} = 0 && \text{at} \quad \bar{x} = 0 \quad \text{(unchanged)}\\
&\frac{d^2\bar{y}}{d\bar{x}^2} = k_2\,\bar{y}_o && \text{at} \quad \bar{x} = 1 \quad \text{(change in sign)}\\
&\frac{d^3\bar{y}}{d\bar{x}^3} = k_1^2\,(\bar{y}_o\,) && \text{at} \quad \bar{x} = 1 \quad \text{(unchanged)}
\end{aligned} \qquad (89)$$

In spite of these changes in the boundary conditions, detailed calculations show that the problem reduces to equation (26) of Part III with $- k_2$ everywhere substituted for $+ k_2$. Or the critical loads must satisfy,

$$k_1^2 - \frac{10}{5!}\,k_1^4 + \frac{540}{9!}\,k_1^6 - \frac{70,200}{13!}\,k_1^8 + \ldots .$$

(continued)

$$-k_2 - \frac{k_2^2}{2!} - \frac{k_2^3}{4!} - \frac{k_2^4}{6!} - \frac{k_2^5}{8!} - \frac{k_2^6}{10!} - \frac{k_2^7}{12!} - \ldots.$$

$$+\left(\frac{1}{3!} + \frac{10}{4!}\right)k_1^2k_2 + \left(\frac{1}{5!} + \frac{38}{6!}\right)k_1^2k_2^2 + \left(\frac{1}{7!} + \frac{92}{8!}\right)k_1^2k_2^3$$

$$+\left(\frac{1}{9!} + \frac{180}{10!}\right)k_1^2k_2^4 + \left(\frac{1}{11!} + \frac{310}{12!}\right)k_1^2k_2^5 + \ldots$$

$$-\left(\frac{38}{7!} + \frac{540}{8!}\right)k_1^4k_2 - \left(\frac{92}{9!} + \frac{3884}{10!}\right)k_1^4k_2^2 - \left(\frac{180}{11!} + \frac{15,844}{12!}\right)k_1^4k_2^3 + \ldots$$

$$+\left(\frac{3884}{11!} + \frac{70,200}{12!}\right)k_1^6k_2 \quad \ldots\ldots = 0 \qquad (90)$$

This equation differs significantly from (26) of Part III. Thus when $k_1 = 0$ the above degenerates to,

$$-k_2\left(1 + \frac{k_2}{2!} + \frac{k_2^2}{4!} + \frac{k_2^3}{6!} + \frac{k_2^4}{8!} + \frac{k_2^6}{10!} + \ldots\right) = 0$$

The portion in parenthesis can have no real positive roots. Hence the only real solution in this case is for $k_2 = 0$. Or instability is impossible unless P_1 is acting. This is already known from the physical problem.

The minimum eigenvalues of (90) have not been calculated out in detail. To do so one would proceed as in Part III. The plotted results would have the appearance of the dotted curve in Fig. 25.

The angle of the resultant load may be fixed by φ as in Fig. 26. In terms of the dimensionless parameters k_1 and k_2 this angle is given as follows:

$$\tan \varphi = \frac{P_1}{P_2} = \frac{k_1}{k_2}\sqrt{\frac{C}{B}} = \frac{k_1}{k_2}\sqrt{\frac{GJ}{BC}} \tag{91}$$

where P_1 and P_2 are as given by equations (11) and (12) of Part III.

Suppose $\nu = 1/3$ so that $G = \frac{E}{2(1+\nu)} = \frac{3E}{8}$. Also if the cross-section is a rectangle with $h >> t$, then $J = 4I$, where I is I minimum. For this case,

$$\tan \varphi = \frac{k_1}{k_2}\sqrt{3/2}$$

or

$$\varphi = \arctan 1.226 \frac{k_1}{k_2}$$

Finally if $\varphi = \frac{\pi}{2}$ in (91), then $k_2 = k_1 \frac{\sqrt{\frac{GJ}{EI}}}{\infty} = 0$.

3. <u>Tapered Strut - Axial Tension (P_2) + Transverse Component (P_1)</u>

Assuming again that P_1 is unchanged but that P_2 is now a tensile force, the basic equations of Part V undergo change.

Detailed consideration of the influence of this change in P_2, indicates that both the differential equation and boundary conditions of V become altered. Furthermore the change which takes place can, in every instance, be correctly obtained by changing k_2 as in V to $-k_2$.

For complete taper the dimensionless parameters are given by (55) and (56) of V. These are,

$$k_1^2 = \frac{P_1^2 \ell^4}{EI_o \, GJ_o} = \frac{P_1^2 \ell^4}{B_o \, C_o} \quad , \quad k_2 = \frac{P_2 \ell^2}{EI_o} = \frac{P_2 \ell^2}{B_o}$$

As mentioned in the second paragraph below (57) - V, for arbitrary taper L should replace ℓ in the above expressions. Or,

$$(k_1')^2 = \frac{P_1^2 L^4}{EI_o \, GJ_o} = \frac{P_1^2 L^4}{B_o \, C_o} \quad , \quad k_2' = \frac{P_2 L^2}{EI_o} = \frac{P_2 L^2}{B_o} \tag{92}$$

With this change in notation equation (53) - V may be rewritten to apply when P_2 is tensile, or,

$$(\lambda + \bar{x})^2 \bar{y}^{IV} + 2(\lambda + \bar{x}) \bar{y}''' + \left[(k_1')^2 \bar{x}^2 - k_2' (\lambda + \bar{x})\right] \bar{y}'' + (k_1')^2 \left(2 - \frac{\bar{x}}{\lambda + \bar{x}}\right)(\bar{x} \bar{y}' - \bar{y}) = 0 \tag{93}$$

At the same time the boundary conditions change to,

$$\begin{aligned}
&\bar{y}_o = 0 && \text{at} \quad \bar{x} = 0 \\
&(\lambda + \bar{x}) \bar{y}'' = 0 && \text{at} \quad \bar{x} = 0 \\
&\bar{y}'' = \frac{P_2 L^2}{B_o} \bar{y}_o = k_2' \bar{y}_o && \text{at} \quad \bar{x} = \frac{\ell}{L} \\
&\bar{y}''' = \left(\frac{P_1^2 L^3 \ell}{B_o \, C_o} - \frac{P_2 L^2}{B_o} \right) \bar{y}_o = \left[(k_1')^2 \frac{\ell}{L} - k_2' \right] \bar{y}_o && \text{at} \quad \bar{x} = \frac{\ell}{L}
\end{aligned} \tag{94}$$

For the special case of complete taper the differential equation and boundary conditions are given by (66), (67) - V. These again apply when P_2 is compressive. When P_2 is tensile these become,

$$\bar{x}^2 \bar{y}^{iv} + 2\,\bar{x}\,\bar{y}''' + (k_1^2 \bar{x}^2 - k_2 \bar{x})\,\bar{y}'' + k_1^2 (\bar{x}\,\bar{y}' - \bar{y}) = 0 \qquad (95)$$

$$\begin{aligned}
&\bar{y}_o = 0 && \text{at} \quad \bar{x} = 0 \\
&\bar{x}\,\bar{y}'' = 0 && \text{at} \quad \bar{x} = 0 \\
&\bar{y}'' = k_2\,\bar{y}_o && \text{at} \quad \bar{x} = 1 \\
&\bar{y}''' = (k_1^2 - k_2)\,\bar{y}_o && \text{at} \quad \bar{x} = 1
\end{aligned} \qquad (96)$$

The solution to the above set of equations must satisfy (76) - V with $+k_2$ everywhere replaced by $-k_2$. Since this expression is so very lengthy it will not be written out again here.

However, if only P_2 acts on the strut it is physically obvious that instability cannot occur. Then (76) - V becomes (remembering to replace k_2 by $-k_2$),

$$-k_2 - \left(\frac{1}{2!} + \frac{1}{2!}\right)k_2^2 - \left(\frac{1}{1!3!} + \frac{1}{2!3!}\right)k_2^3 - \left(\frac{1}{2!4!} + \frac{1}{3!4!}\right)k_2^4$$

$$- \left(\frac{1}{3!5!} + \frac{1}{4!5!}\right)k_2^5 - \; . \; . \; . \; . \; = 0$$

or,

$$-k_2 \left[1 + k_2 + \frac{k_2^2}{(2!)^2} + \frac{k_2^3}{(3!)^2} + \frac{k_2^4}{(4!)^2} + \frac{k_2^5}{(5!)^2} + \; . \; . \; . \right] = 0 \qquad (97)$$

Again the portion inside brackets can have no real, positive root. Hence $k_2 = 0$ is the only real solution.

4. Shape of the Cross-section

So far in this present analysis very little mention has been made as to the actual cross-sectional shapes which may be permitted. As a matter of fact the only stipulations have been that the depth be considerably greater than the thickness $(h >> t)$ and that the sections have two axes of symmetry. One of these axes of symmetry (that of minimum moment of inertia) must be parallel to the transverse load P_1.

It is an interesting fact that the torsional rigidity of a member is a function of the internal axial stresses in the member. Thus consider two strips of the same material, identical in size and shape, and possessing deep but thin rectangular cross-sections. Further suppose that one of these strips has been subjected to a rolling operation such that surface tensile stresses have been locked into the member. Under these conditions the beams will look alike in every respect, will exhibit identical bending stiffnesses, yet the strip which has been pre-stressed will have a noticeably greater torsional rigidity. This fact is of some interest in discussing the effect of cross-sectional shape on the instability analysis of this present investigation.

In conventional torsion theory it is customary to assume that distances between cross-sections remain unchanged as twisting takes place. For most materials (as the usual structural metals) and for many cross-sectional shapes, this is essentially true. However, for some sections--as the thin rectangular section--this assumption cannot be made even for metal parts. Twisting of such members causes the cross-sections to approach one another, thereby inducing axial stresses.

It is possible to investigate these axial stresses in some detail. They are distributed over the cross-sections in such a manner that the net axial force vanishes. Furthermore they have a component which causes a moment about the axis of twist. This moment increases as the cube of the angle of twist.

Thus the induced axial stresses serve to stiffen the section against torsional deflection, producing an effect analogous to increasing the torsional rigidity of the member. This is the reason that, in the example mentioned earlier, the strip with locked in tensile stresses had more torsional stiffness than the strip which had not been pre-stressed.

Effects similar to these may occur for struts of the type being investigated here. For example, assume deflections to occur prior to the instability loading. Bending as well as torsion will then cause axial stresses which may influence the torsional rigidity. As a result the use of $C = GJ$ for the stiffness in torsion may have to be reexamined and modified.

Professor Goodier of Stanford University has recently carried out some investigations (unpublished) to determine the scope and magnitude of such torsional effects. There is evidence that when the cross-section has two axes of symmetry no torsional stiffening due to bending is possible. Furthermore, in so far as the present problem is concerned, there is also reason to feel that this phenomenon cannot influence the actual critical loading. As pointed out previously, the instability condition is basically an energy condition and hence does not depend on previous deflections for its attainment. In this sense, when the critical loading is applied the energy state is such that $\delta (U - W) = 0$ and by definition instability is imminent. Nevertheless the possible influence of cross-sectional shape might be given further consideration. The purpose here is merely to point out these rather interesting possibilities.

Actually this discussion can be carried much further. For example, thin, open-section columns are subject to a torsional type instability under axial compressive loading. This may be viewed as being caused by a decrease in torsional rigidity due to the compressive stress. At the critical loading the compressive stress will have reached a magnitude such that the torsional stiffness due to G and J is nullified by the opposing influence due to the twisting components of the axial stresses. This case then is the opposite of that in which axial tension stiffened the member torsionally. (Open section columns were specified above since these are the most apt to fail by twisting before general Euler column instability occurs.)

5. The Swept and Tapered Strut

In the general case the wind tunnel model support strut may be both tapered and swept. The difficulties encountered in calculating the critical loadings for the tapered but unswept strut, serve to emphasize the probable greater complexity of the general case. It will be shown, however, that the strut with sweepforward can be reduced to the same problem as the strut without sweep.

As in the previous cases already discussed, buckling will again be manifested by a combined twisting and bending action, the strut essentially displacing normal to its own plane as it deflects. This at once suggests the difficult problem of the swept, cantilever plate, loaded normal to its own plane. The solution to this problem has not as yet been made known.

However, it should be kept clearly in mind that the model support strut has been considered as a beam throughout this analysis. In fact in the derivations of Parts III and IV, the differential equation of beam bending was explicitly used. Consequenly there is little justification for introducing the plate problem into the case of the swept strut.

Physically the swept strut will be assumed to possess an elastic axis. Sections normal to this axis (as B - A - C in Fig. 27) will then displace without rotating due to loads concentrated along this axis. Similarly such sections will only rotate when a torsional moment acts along the elastic axis.

In other words, by postulating the existence of an elastic axis the behavior of the swept strut is essentially reduced to a consideration of bending and twisting, these actions now being calculable by the conventional methods as used previously for the simpler unswept cases.

For the symmetry properties already noted as being probable for such struts, 0ξ of Fig. 27 will represent the elastic axis. Normal sections as B - A - C will then bend or twist with respect to this axis. These effects are separable.

Some assumptions are, of course, inherent in such a simplification. Most important is that the aspect ratio, AR, be large. Calculating the AR as though the strut were a wing half-span,

$$AR = \frac{(\text{span})^2}{\text{total area}} = \frac{(2\ell)^2}{2\left[\frac{1}{2}(h_o\ell + h_1\ell)\right]} = \frac{4\ell}{h_o + h_1} \tag{98}$$

Experiments on actual swept airplane wings indicate that the assumption of an elastic axis is quite accurate if this ratio, as given by (98), is of the order of eight or nine. In keeping with the aerodynamic requirements and the condition that the slenderness ratio be sufficient to give elastic buckling, it appears reasonable to adopt the elastic axis concept into the analysis.

It is also true that the strut will suffer collapse outboard of the clamped edge neighborhood. Or at the critical section the use of an elastic axis is reasonable.

A strut having ℓ = 30 in., h_o = 10 in., h_1 = 5 in., will have an AR of eight.

Since the strut is now reduced to a consideration of behavior with respect to the elastic axis, the load components parallel and normal to this axis will be calculated (Fig. 28). Or,

$$(\text{along } 0\xi) \quad P_\xi = -P_1 \sin\psi + P_2 \cos\psi \tag{99}$$

$$\text{(along } O\eta\text{)} \quad P_\eta = -P_1 \cos\Psi - P_2 \sin\Psi \qquad (100)$$

where Ψ is the sweep angle as defined in Fig. 28 and P_2 has been assumed compressive.

These are now the axial and transverse loads respectively, acting on the elastic axis of the swept strut. The trigonometric terms are constants for any given sweep angle. As before, the origin O moves with the strut free end and P_1 and P_2 remain parallel to their initial directions.

The problem may now be approached in several ways. In many respects the simplest will be to deal directly with cross-sections normal to the elastic axis. Although these may be related to the cross-sections in the stream direction (h_o, h_1, etc. in Fig. 26) it is not necessarily helpful to do so. Hence the problem will be formulated from Fig. 28.

As shown in Fig. 28 the elastic axis defines sections such as B - A - C. For these sections I' and J' may be calculated. Then the problem becomes identical to that considered in Part V. The loads are now P_ξ and P_η. Hence all the equations previously developed, including those where P_ξ is tensile, may be used directly.

As a result of this interpretation of the problem it is seen that sweepforward does not introduce complications comparable to those caused by taper.

6. Buckling of the Inelastic Strut

Inelastic buckling occurs when, under the applied loading, Hooke's Law no longer holds throughout the strut.

When this occurs the strut problem becomes closely analogous to that of the short column. As in the case of the column a completely theoretical approach is not yet feasible from a design point of view.

Nevertheless certain empirical facts, whose accuracy in the case of the column is well understood, may be reasonably extended to the problem of the strut. Chief amongst these is the fact that if the tangent modulus $(\overline{E})$ is substituted for the elastic modulus (E) in Euler's equation, the result is an equation which quite closely represents inelastic buckling. (See Ref. 11, pages 158-163 and Ref. 12, pages 1-22 to 1-24.)

The basic idea behind such a procedure seems quite reasonable. In the Euler equation E is the only material constant which appears; hence when the column material properties undergo change due to inelastic action setting in, it would necessarily mean that E must change. Such a statement while not entirely comprehensive in its depicting of elastic-plastic behavior, yet has been found to agree well with a large number of test results.

In view of this an attempt will be made to extend this concept to the case of the strut. This will not only give a means for estimating inelastic buckling loads but will also indicate how experimental results might best be represented.

It is clear that in the strut problem, changes in G, as well as in E, must be considered. That is, when the proportional limit is surpassed and $\overline{E}$ is used in place of E, then some rational means for likewise modifying G must be found.

Elastically E and G are related by,

$$G = \frac{E}{2(1+\nu)} \tag{101}$$

where ν is Poisson's Ratio. Since the maximum column stress is essentially the yield point*, the value of ν over the range between the proportional limit and the yield stress will be substantially constant. Hence to a reasonable first approximation for this problem,

$$\bar{G} = \frac{\bar{E}}{2(1 + \nu)} \qquad (102)$$

Hence the change over from elastic to inelastic behavior will, on the above basis, be assumed to occur simultaneously in bending and twisting. Physically this also appears correct, since the strut cannot bend without twisting and vice versa; hence, yielding in one type of action will quickly precipitate yielding in the other.

With this in mind equations (11) and (12) of Part III can now be examined. The critical elastic loads in bending and torsion are respectively,

$$P_2 = \frac{k_2 B}{\ell^2} \qquad (12)$$

$$P_1 = \frac{k_1 \sqrt{BC}}{\ell^2} \qquad (11)$$

Since $B = EI$ and $C = GJ$, let $\bar{B} = \bar{E}I$ and $\bar{C} = \bar{G}J$ so that for inelastic buckling,

$$\bar{P}_2 = \frac{k_2 \bar{B}}{\ell^2} \qquad (103)$$

$$\bar{P}_1 = \frac{k_1 \sqrt{\overline{BC}}}{\ell^2} \qquad (104)$$

---

* See Ref. 12, page 1-23.

Thus the inelastic loads are obtained by modifying B and C as indicated and leaving k_1 and k_2 unchanged. This is also in agreement with column procedure, where the end fixity factor (usually written as C in column theory) does not depend on anything other than the end supports. Likewise k_1 and k_2 are functions of the end conditions and will behave inelastically as they do for the elastic cases.

A first graphical representation may now be made. Consider Fig. 29 in which the elastic column curve is represented by AB , which is an Euler curve. The corresponding elastic torsional curve is shown by curve DE and is drawn by noting the mathematical analogy between equations (11) and (12) as just written. Combinations of P_1 and P_2 will cause elastic failure described by the interaction curve EB . Hence to the left of plane EOB the strut behavior is completely elastic.

When inelastic behavior sets in the typical column tangent modulus curve is shown as BC on Fig. 29. Comparison between equations (104) and (103) show that when P_1 alone is acting, a similar type inelastic curve must occur. This is shown as EF .

Surface EFCB then defines coupled loads capable of buckling the "short" strut. It corresponds to the short column curve. The coupled behavior is essentially dependent on the curve relating k_1 and k_2 (as Fig. 10 of Part III). As already pointed out this will remain unchanged during the transition to the inelastic regime.

Such a picture of what appears to be a likely representation of the strut behavior should be checked experimentally. The techniques already developed should prove sufficient for this purpose. Actually non-dimensional curves similar to those used on columns ought to be used instead of the dimensional curves of Fig. 29. However, the procedure for doing this is so well established that it need not be elaborated on here. The important point would be to check the anticipated behavior as outlined in the above discussion, and for this purpose an experimental program is necessary. The usefulness is apparent, since for some applications inelastic behavior may be unavoidable.

APPENDIX

The equation which is to be derived here by variational methods is that previously given for the uniform strut. This equation, (6) of Part III, is repeated below for easy reference.

$$\frac{d^4y}{dx^4} + \left(\frac{{P_1}^2}{BC} x^2 + \frac{P_2}{B}\right) \frac{d^2y}{dx^2} + 2 \frac{{P_2}^2}{BC} \left(x \frac{dy}{dx} - y\right) = 0 \qquad (6)$$

In order to derive this expression by variational procedure the following well known minimum principle from elasticity will be employed; namely, that equilibrium is established when the variation of the total potential energy of the system vanishes. Symbolically this may be represented as,

$$\delta (U - W) = 0 \qquad (105)$$

where, U = total strain energy

W = total external work done by applied forces

δ = conventional variation symbol of Calculus of Variations

This Principle is well known in Elasticity, and hence need not be elaborated on here.

To calculate the strain energy, U, it is convenient to separately consider bending and twisting effects. Since the strain energies for these separate effects are known for beams from strength of materials they may be set down at once as,

$$U_{bending} = \int_0^L \frac{B}{2} (y'')^2 \, dx$$

$$U_{twisting} = \int_0^L \frac{C}{2} (\theta')^2 \, dx$$

where, B = bending stiffness of strut

C = torsional stiffness of strut

$$y'' = \frac{d^2y}{dx^2}$$

$$\theta' = \frac{d\theta}{dx}$$

Hence the total strain energy is,

$$U = \int_0^L \left[\frac{B}{2} (y'')^2 + \frac{C}{2} (\theta')^2 \right] dx \qquad (106)$$

The work term, W , is somewhat more difficult to calculate. It may be first expressed as,

$$W = P_1 \, \delta_1 + P_2 \, \delta_2 \qquad (107)$$

where, δ_1 = deflection of centroid of strut free end in direction of P_1

δ_2 = deflection of same point in direction of P_2

The problem then becomes one of determining δ_1 and δ_2.

<u>Deflection δ_1</u>. Initially it might seem that, due to the assumptions made, δ_1 should be zero. Thus $h \gg t$ was required in deriving (6) so that bending deflections in the xz or vertical plane (see Fig. 29a) could be neglected. Nevertheless, it will be

shown that due to bending in the xy or horizontal plane, plus twisting about the neutral axis, that a deflection δ_1 does exist.

A very exaggerated picture of the deflected strut (Fig. 30) brings this out rather vividly. The vertical displacement of origin O as the beam deflects, can be visualized from such a sketch.

Fig. 30b shows δ_1' as the deflection of the origin measured in the horizontal xy-plane. This deflection is readily calculated by using the so-called Second Moment Area Theorem from Strength of Materials. Thus if the contribution of a length dx of beam at arbitrary point x is considered (Fig. 31a), then by this theorem,

$$d\delta_1' = \frac{M}{EI} x\, dx = \frac{M}{B} x\, dx = y'' x\, dx$$

This is merely an application of conventional beam theory.

Now consider the vertical deflection suffered by $d\delta_1'$ at the free end as a twist θ is imported to the beam at arbitrary point x. Due to this twist the cross-section at x rotates through an angle θ. Furthermore, the element $d\delta_1'$ can be thought of as being rigidly connected to this cross-section at x. Hence it will deflect downward by an angle θ as shown in Fig. 31b. The vertical component $d\delta_1$ due to twisting and bending effects at station x is then,

$$d\delta_1 = \theta y'' x\, dx$$

This effect at a point can now be extended to include the entire strut by integration,

$$\delta_1 = \int_0^L \theta y'' x\, dx \qquad (108)$$

Deflection δ_2 . This represents the decrease in distance from the fixed edge to the centroid of the free end, measured parallel to ox , which takes place as the strut deflects. It may be calculated (see Ref. 1, p. 27) by keeping in mind that the actual length of the neutral axis does not change as deflection takes place. The result as given on p. 28 of Ref. 1 is the well known expression,

$$\delta_2 = \frac{1}{2} \int_0^L (y')^2 \, dx \qquad (109)$$

Substituting these last two equations into (107) gives,

$$W = \int_0^L \left[P_1 \, \theta \, y'' \, x + \frac{1}{2} P_2 \, (y')^2 \right] dx \qquad (110)$$

Equations (106) and (110) may now be substituted into (105) to yield the statement of the variational problem,

$$\delta \left\{ \int_0^L \left[\frac{B}{2} (y'')^2 + \frac{C}{2} (\theta')^2 - P_1 \, \theta \, y'' x - \frac{P_2}{2} (y')^2 \right] dx \right\} = 0 \qquad (111)$$

In dealing with this expression it is not necessary to apply the detailed methods of the variational calculus. Thus on p. 290 of Ref. 5 it is given that if,

$$I = \int_{xo}^{x_1} F\,(x,\ q,\ q',\ q'',\ \ldots\ q^{(n)}\,)\, dx = \text{a min.}$$

then the corresponding differential equation to be satisfied is,

$$F_q - \frac{d}{dx} F_{q'} + \frac{d^2}{dx^2} F_{q''} - \ . \ . \ . + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0 \quad (112)$$

The identity of this with the problem posed by equation (111) is evident on inspection.

Equation (111) contains the two dependent variables y and θ; hence (112) will be applied to each separately.

When $q = y$ equation (111) substituted into (112) gives,

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0$$

or,

$$0 - \frac{d}{dx} (-P_2 \ y') + \frac{d^2}{dx^2} \left[B \ y'' - P_1 \ \theta \ x \right] = 0$$

i.e.,

$$B \ y^{(iv)} - P_1 \ x \ \theta'' + P_2 \ y'' - 2 \ P_1 \ \theta' = 0 \quad (113)$$

When $q = \theta$ the result is,

$$F_\theta - \frac{d}{dx} F_{\theta'} = 0$$

or

$$-P_1 \ y'' \ x - \frac{d}{dx} (C \ \theta') = 0$$

$$P_1 \ x \ y'' + C \ \theta'' = 0 \quad (114)$$

This last equation may be integrated once to give,

$$C \ \theta' + P_1 \ (xy' - y) = 0$$

where the left hand side is zero since at $x = y = 0$, the value of $C\theta'$ (which is also the twisting moment) is likewise zero. Substituting the above into (113),

$$B\, y^{(iv)} - P_1\, x \left(\frac{-P_1\, xy''}{C}\right) + P_2\, y'' - 2P_1 \left[- \frac{P_1}{C}(xy' - y)\right] = 0$$

or

$$y^{(iv)} + \left(\frac{P_1^{\,2}}{BC}\, x^2 + \frac{P_2}{B}\right) y'' + 2\, \frac{P_1^{\,2}}{BC}(xy' - y) = 0$$

which is again equation (6).

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TABLE I

CRITICAL LOADS - UNTAPERED STRUT

ITEM	α						β			
	0	1/10	1/4	1/2	1	2	0	1/50	1/10	1/4
$(k_1)_{min.}$	0	0.494	.733	1.075	1.475	1.971	4.013	3.805	3.182	2.521
$(k_2)_{min.}$	2.467	2.436	2.389	2.311	2.174	1.942	0	0.290	1.012	1.588

In the above table: $(P_1)_{crit.} = \dfrac{(k_1)_{min.}\sqrt{BC}}{\ell^2}$

$$(P_2)_{crit.} = \frac{(k_2)_{min.}\ B}{\ell^2}$$

TABLE II

TYPICAL EXPERIMENTAL DATA - UNIFORM STRUT

W lbs.	δ in.	W/δ
0	0	
27.31	.015	
31.31	.022	
35.31	.026	
39.31	.036	
43.31	.045	
47.31	.053	
51.31	.070	
55.31	.095	
57.31	.124	
58.31	.138	
59.31	.149	398
60.31	.164	368
61.31	.183	335
62.31	.207	301
63.31	.264	240
64.31	.447	143.8
65.31		0

TABLE III

SECTION AND MATERIAL PROPERTIES

FOR TEST SPECIMENS

(See Fig. 12 for typical sketch.)

SPECIMEN	ℓ In.	h In.	t In.	I In.^{4}	J In.^{4}	EI = B	GJ = C	$\sqrt{BC}$
A	18	2	.127	341×10^{-6}	1364×10^{-6}	3620	5550	4480
B	19	3	.126	512×10^{-6}	2048×10^{-6}	5270	8120	6540

In the above Table:

B = Flexural stiffness

C = Torsional stiffness

$E = 10.6 \times 10^6$ psi

$\nu = 0.3$ (Poisson's Ratio)

$G = E/2\,(1 + \nu) = 4.07 \times 10^6$ psi

TABLE IV-A

TEST RESULTS - SPECIMEN A

TEST NUMBER	φ	$(P_1)_{crit.}$	$(P_2)_{crit.}$	k_1	k_2
A-1	19° 32'	9.59	27.00	.69	2.42
A-2	36° 12'	17.98	24.55	1.30	2.20
A-3	58° 30'	32.15	19.69	2.32	1.76
A-4	71° 6'	41.25	14.10	2.98	1.26

Note:

φ is measured from P_2 to the resultant load. See Fig. 26.

$$k_1 = (P_1)_{crit.} \frac{\ell^2}{\sqrt{BC}} = \frac{(P_1)_{crit.}}{13.82}$$

$$k_2 = (P_2)_{crit.} \frac{\ell^2}{B} = \frac{(P_2)_{crit.}}{11.18}$$

TABLE IV-B

TEST RESULTS - SPECIMEN B

TEST NUMBER	φ	$(P_1)_{crit.}$	$(P_2)_{crit.}$	k_1	k_2
B-1	31° 15'	20.15	33.15	1.20	2.28
B-2	56° 30'	38.65	25.60	2.14	1.76

Note: See comment following Table IV-A concerning angle φ .

$$k_1 = \frac{(P_1)_{crit.}}{18.05}$$

$$k_2 = \frac{(P_2)_{crit.}}{14.57}$$

TABLE - V-A

SPECIMEN A - COMPLETE TEST DATA

TEST A-1		TEST A-2		TEST A-3		TEST A-4	
Load W lbs.	Deflec. In.	W lbs.	In.	W lbs.	In.	W lbs.	In.
0	0	0	0	0	0	0	0
10.31	.032	10.31	.016	27.31	.015	27.31	.014
12.31	.075	18.31	.038	31.31	.022	35.31	.022
14.31	.115	22.31	.058	35.31	.026	43.31	.031
15.31	.144	26.31	.094	39.31	.036	51.31	.043
16.31	.183	28.31	.114	43.31	.045	59.31	.057
16.75	.208	30.31	.141	47.31	.053	64.31	.072
17.19	.237	32.31	.225	51.31	.070	66.31	.080
17.63	.282	33.31	.250	55.31	.095	68.31	.094
17.85	.313	34.31	.307	57.31	.124	70.31	.105
18.07	.343	35.31	.437	58.31	.138	72.31	.119
18.29	.377	35.97	.689	59.31	.149	73.31	.123
18.51	.439	36.63	buckled	60.31	.164	74.31	.137
18.73	.517			61.31	.183	75.31	.149
18.84	.561			62.31	.207	76.31	.166
18.95	.655			63.31	.264	77.31	.180
19.06	.746			64.31	.447	78.31	.197
19.18	.848			65.31	buckled	79.31	.230
						80.31	.265
						80.97	.294
						81.63	.328
						82.07	.364
						82.51	.439
						82.95	buckled

TABLE - V-B

SPECIMEN B - COMPLETE TEST DATA

TEST B-1		TEST B-2	
Load W lbs.	Deflec. In.	W lbs.	In.
0	0	0	0
10.31	.006	27.31	.029
18.31	.015	35.31	.043
23.31	.029	43.31	.062
27.31	.051	47.31	.073
29.31	.063	51.31	.087
30.31	.070	55.31	.103
31.31	.079	59.31	.133
32.31	.089	61.31	.150
33.31	.103	62.31	.160
34.31	.123	63.31	.170
35.31	.145	64.31	.180
36.31	.176	65.31	.197
37.31	.210	66.31	.209
38.31	.278	67.31	.227
39.31	.384	68.31	.248
40.31	.555	69.31	.268
		70.31	.287
		71.31	.314
		72.31	.343
		73.31	.391
		74.31	.448
		75.31	.500
		76.31	.609
		77.31	.794

TABLE VI

CRITICAL LOADS - COMPLETELY TAPERED STRUT

		$(k_1)_{min.}$	$(k_2)_{min.}$
α	0	0	1.445
	1/10	.3761	1.4148
	1/4	.5853	1.3705
	1/2	.8068	1.3017
	1	1.0868	1.1811
β	0	2.4041	0
	1/10	2.0589	.4239
	1/5	1.8118	.6565
	.30	1.6475	.8143
	1/2	1.4033	.9847

In the above Table:

$$(P_1)_{crit.} = (k_1)_{min.} \frac{\sqrt{B_o C_o}}{\ell^2}$$

$$(P_2)_{crit.} = (k_2)_{min.} \frac{B_o}{\ell^2}$$

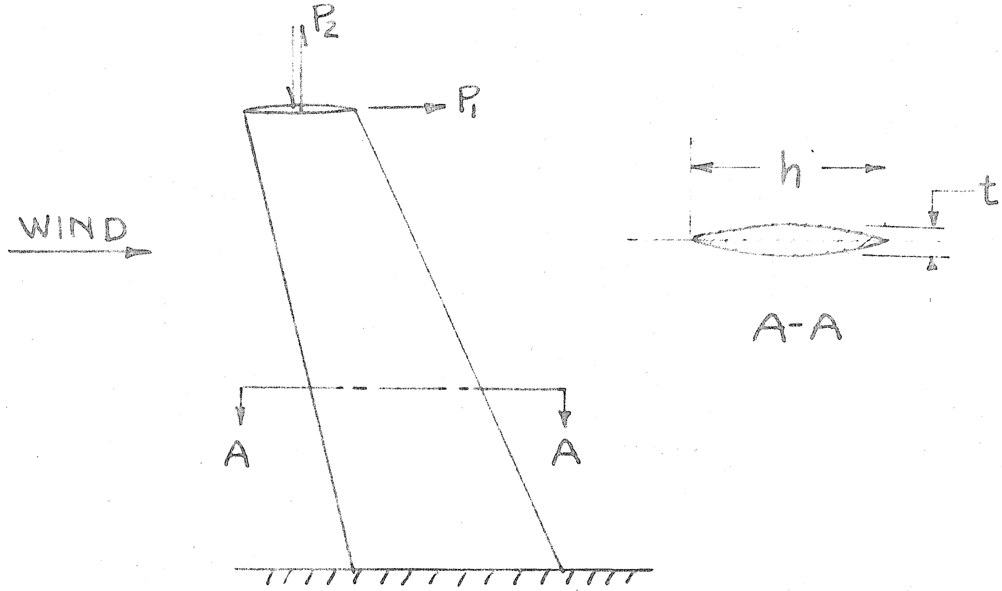


FIG. 1

TYPICAL WIND TUNNEL MODEL SUPPORT STRUT

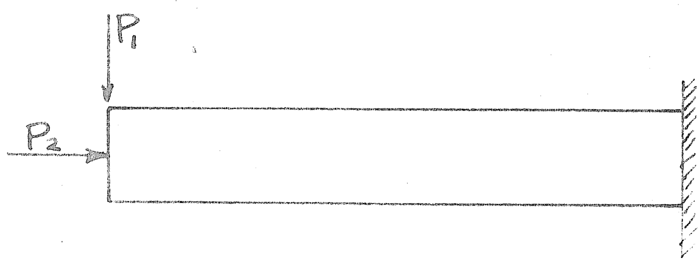
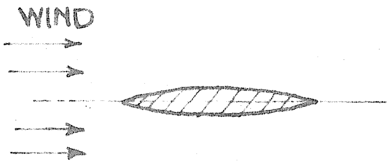


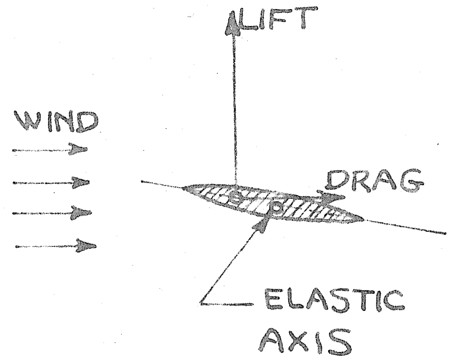
FIG. 2

UNTAPERED AND UNSWEPT STRUT



3a

UNDEFLECTED STRUT

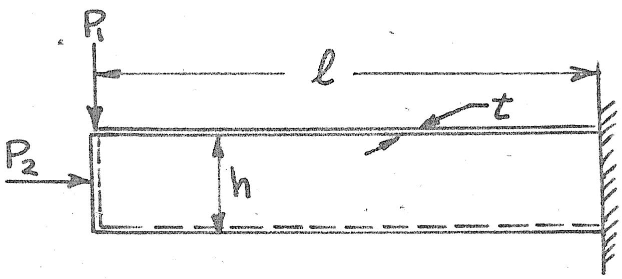


3b

DEFLECTED STRUT

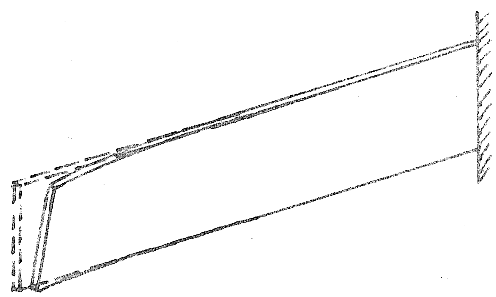
FIG. 3

STRUT AEROELASTIC BEHAVIOR



4a

LOADS AND DIMENSIONS



4b

DEFLECTED SHAPE

FIG. 4

STRUT FOR INITIAL THEORETICAL PROBLEM

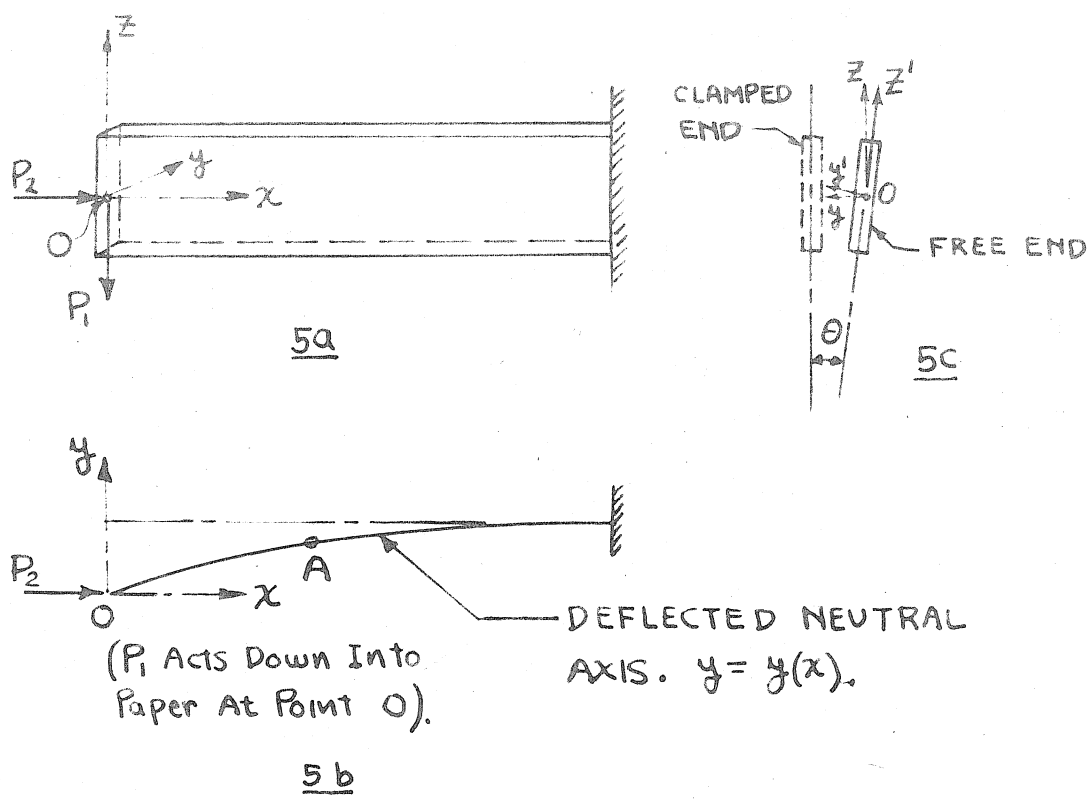


FIG. 5

COORDINATE SYSTEM AND DEFLECTED POSITION – INITIAL PROBLEM

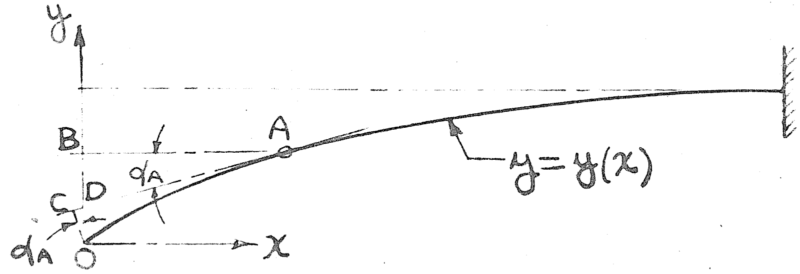


FIG. 6

ENLARGED PLAN VIEW DEFLECTED NEUTRAL AXIS

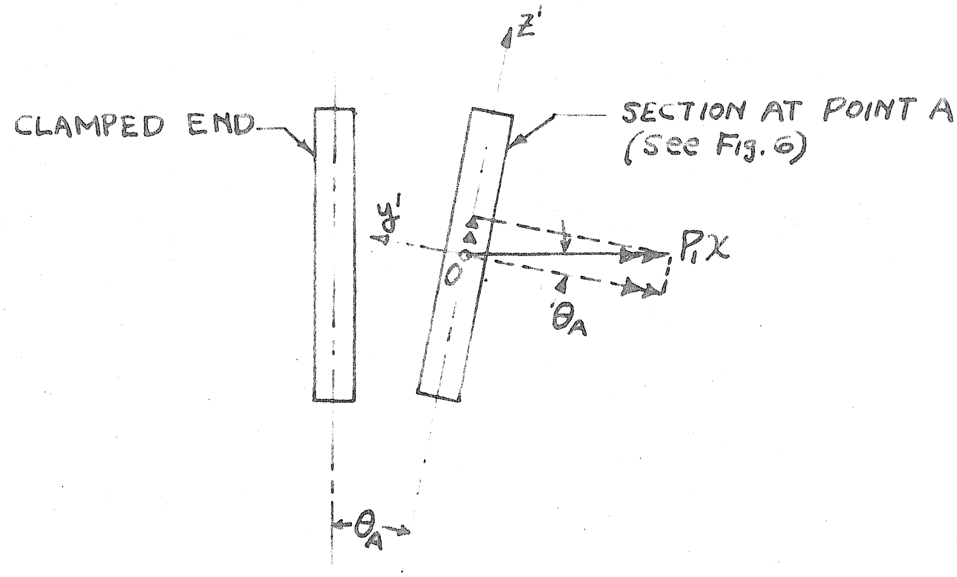


FIG. 7

MOMENT VECTOR - LOAD P_1

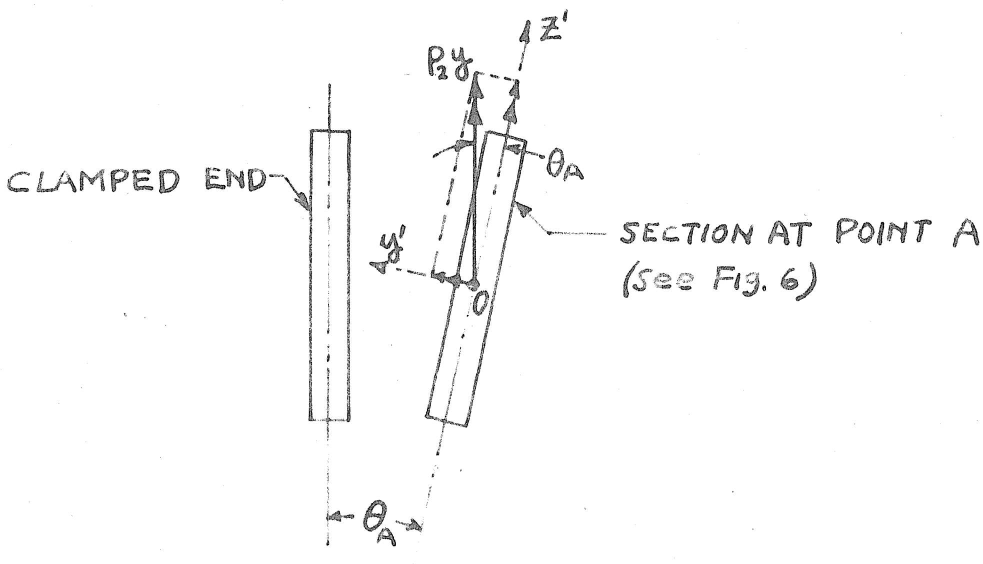


FIG. 8

MOMENT VECTOR - LOAD P_2

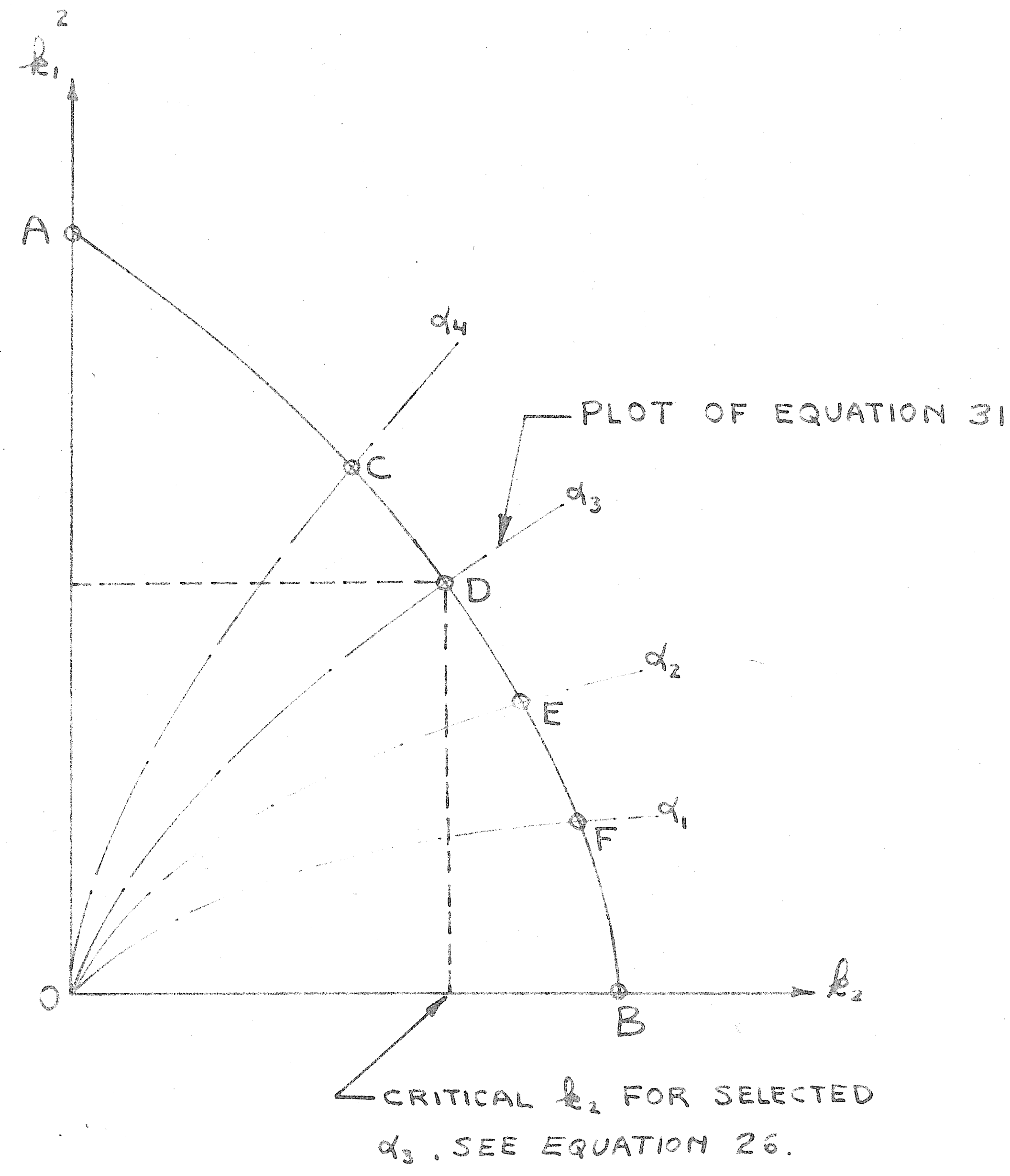


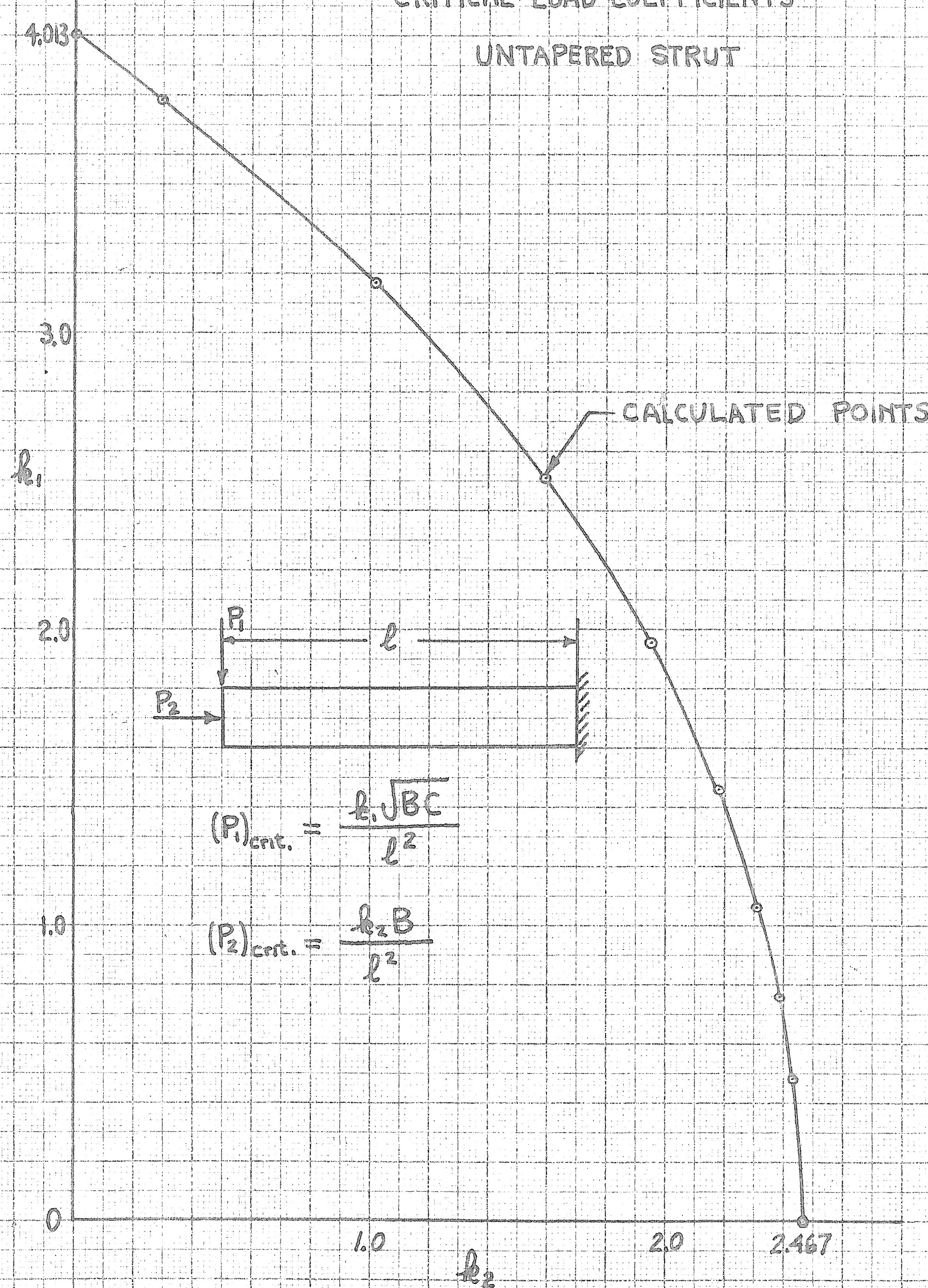
FIG. 9

SCHEME FOR PLOTTING CRITICAL VALUES

FIG. 10

CRITICAL LOAD COEFFICIENTS

UNTAPERED STRUT



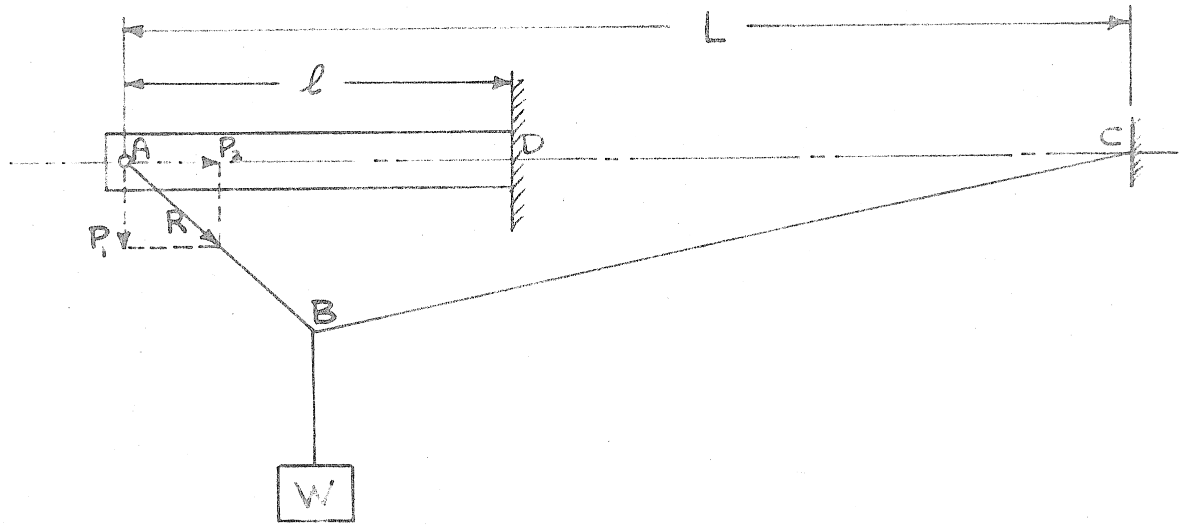


FIG. 11

FIRST METHOD OF LOADING THE STRUT

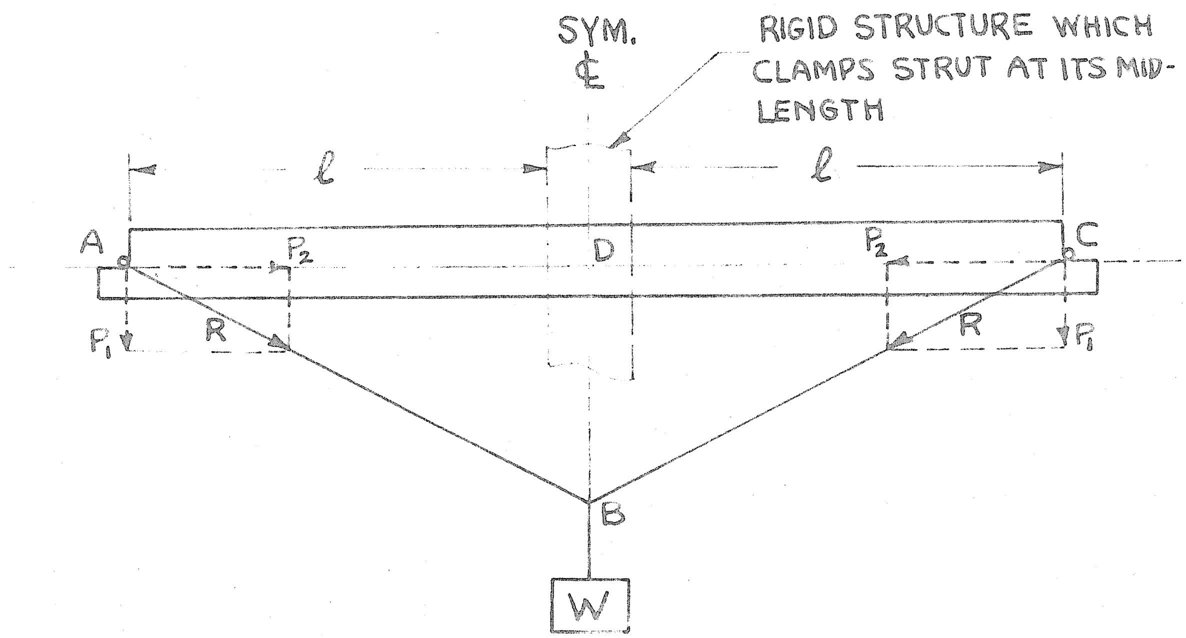


FIG. 12

IMAGE METHOD OF LOADING STRUT

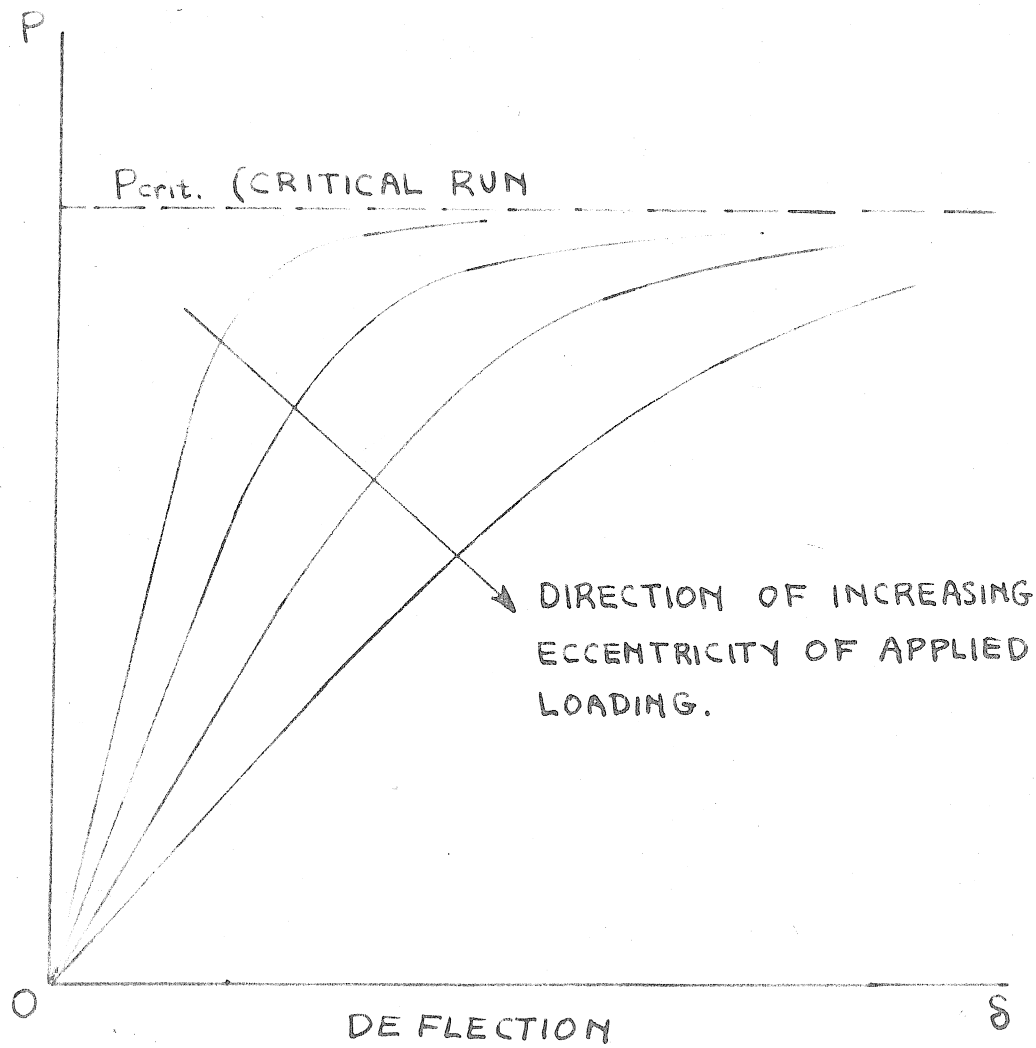


FIG. 13

CURVES TYPICAL OF TEST RESULTS AS GIVEN BY REF. 8 - PRANDTL TYPE TAPERED STRUTS

FIG. 14

EXPERIMENTAL CRITICAL LOAD
TYPICAL PRANDTL STRUT

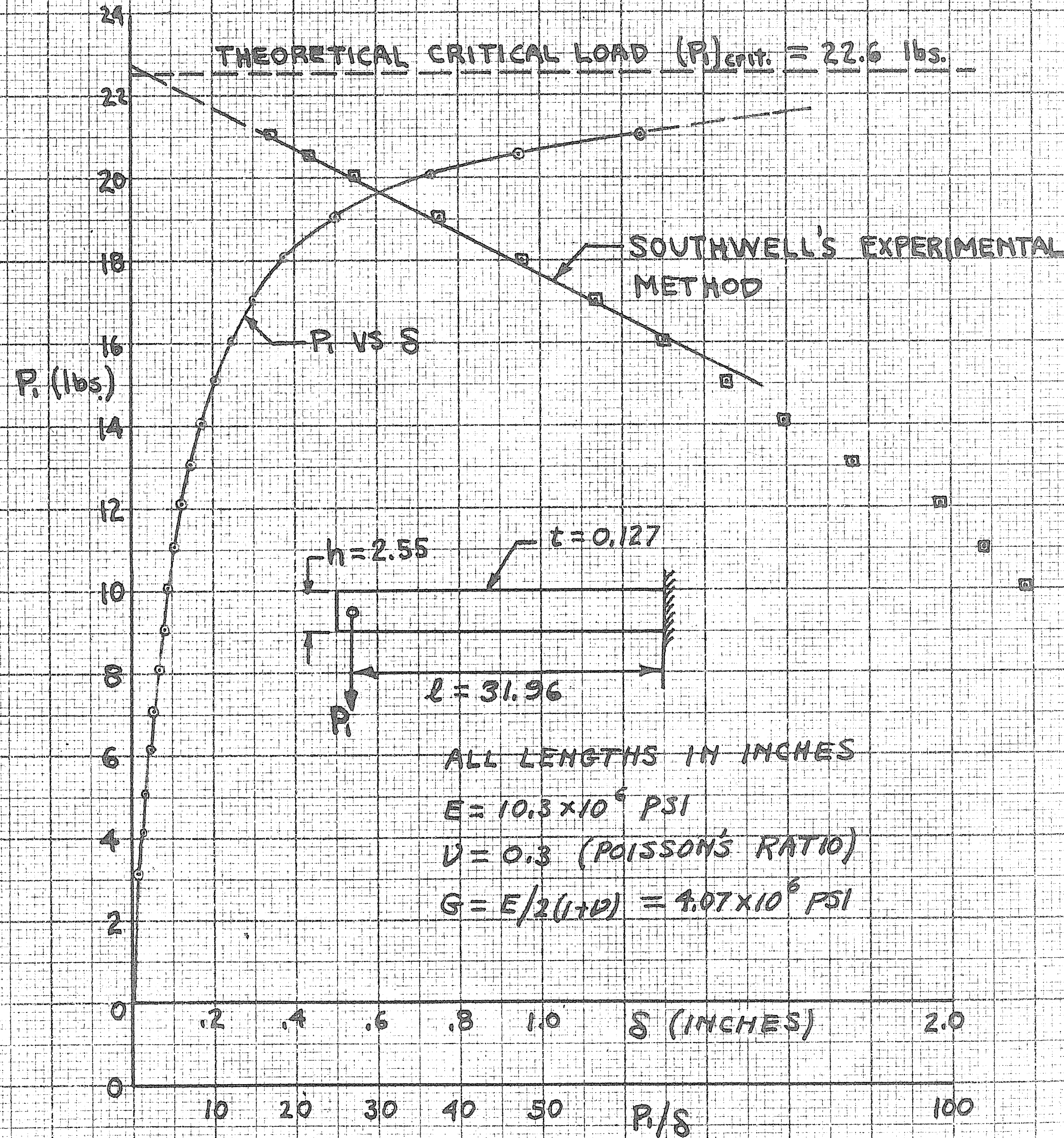


FIG. 15

TYPICAL TEST RESULTS
FOR
STRUT UNDER COMBINED LOADING

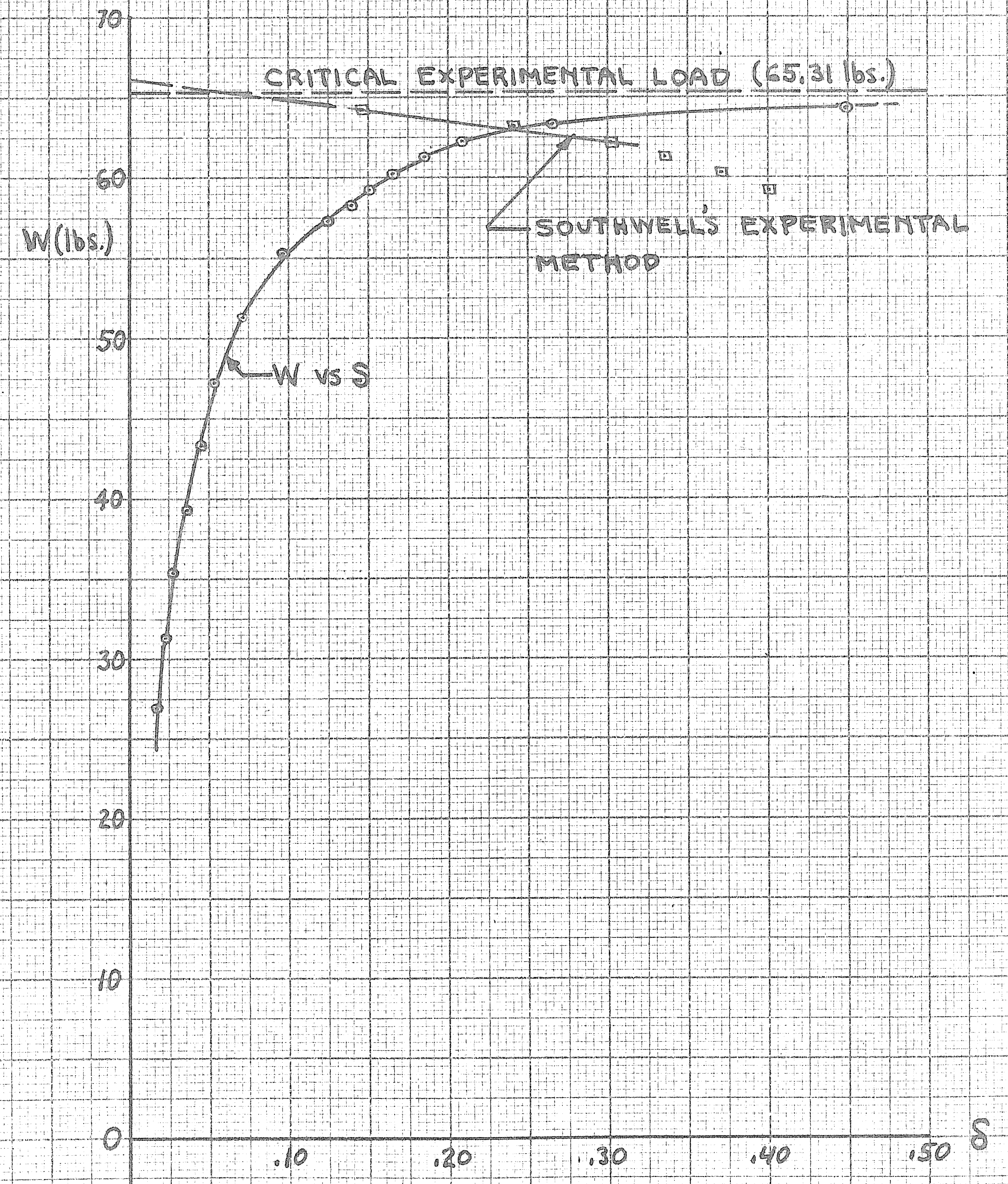


FIG. 16

COMPARISON OF EXPERIMENTAL AND THEORETICAL RESULTS

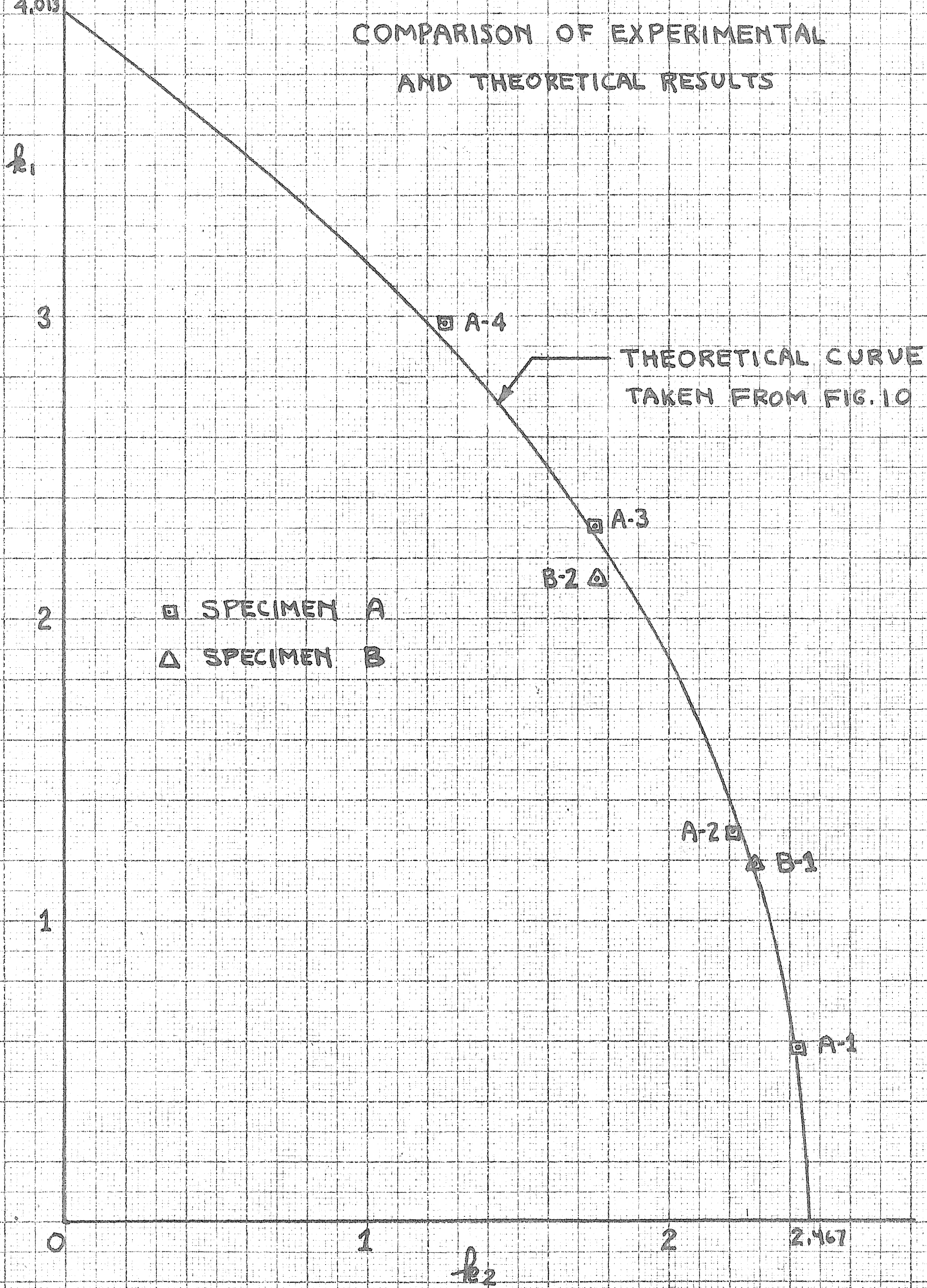


FIG. 17

TEST RESULTS - SPECIMEN A-1

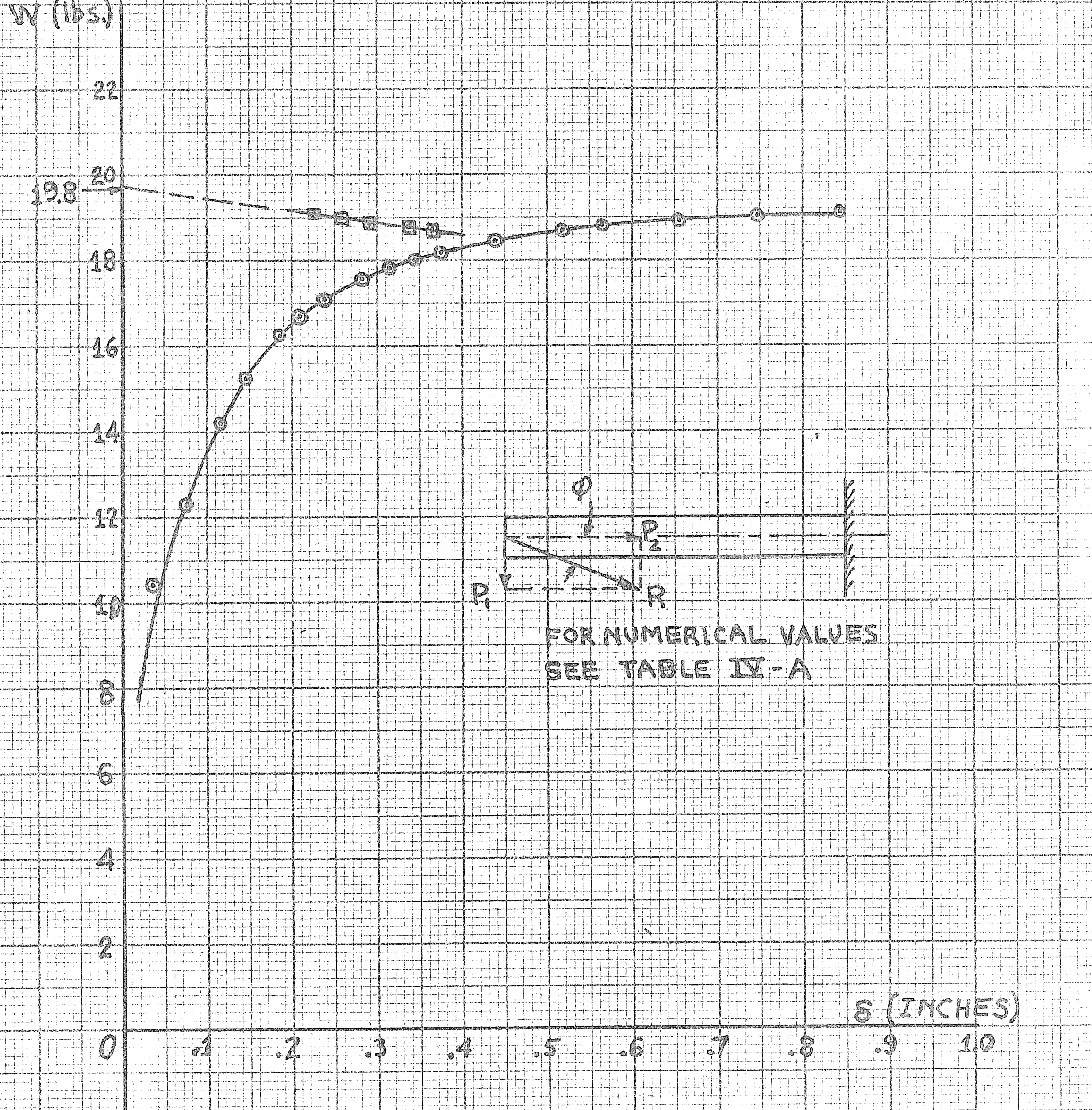


FIG. 18

TEST RESULTS - SPECIMEN A-2

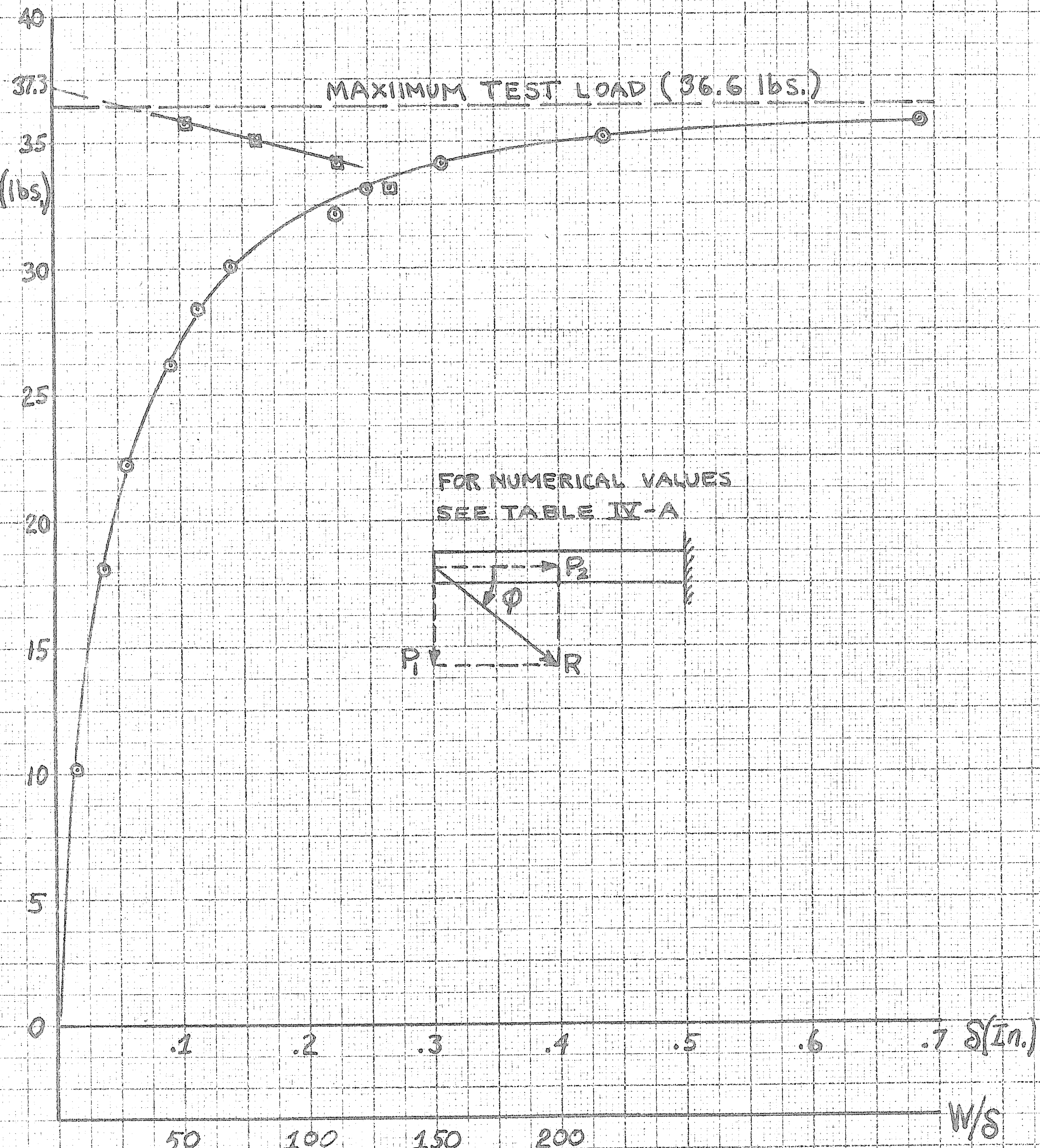
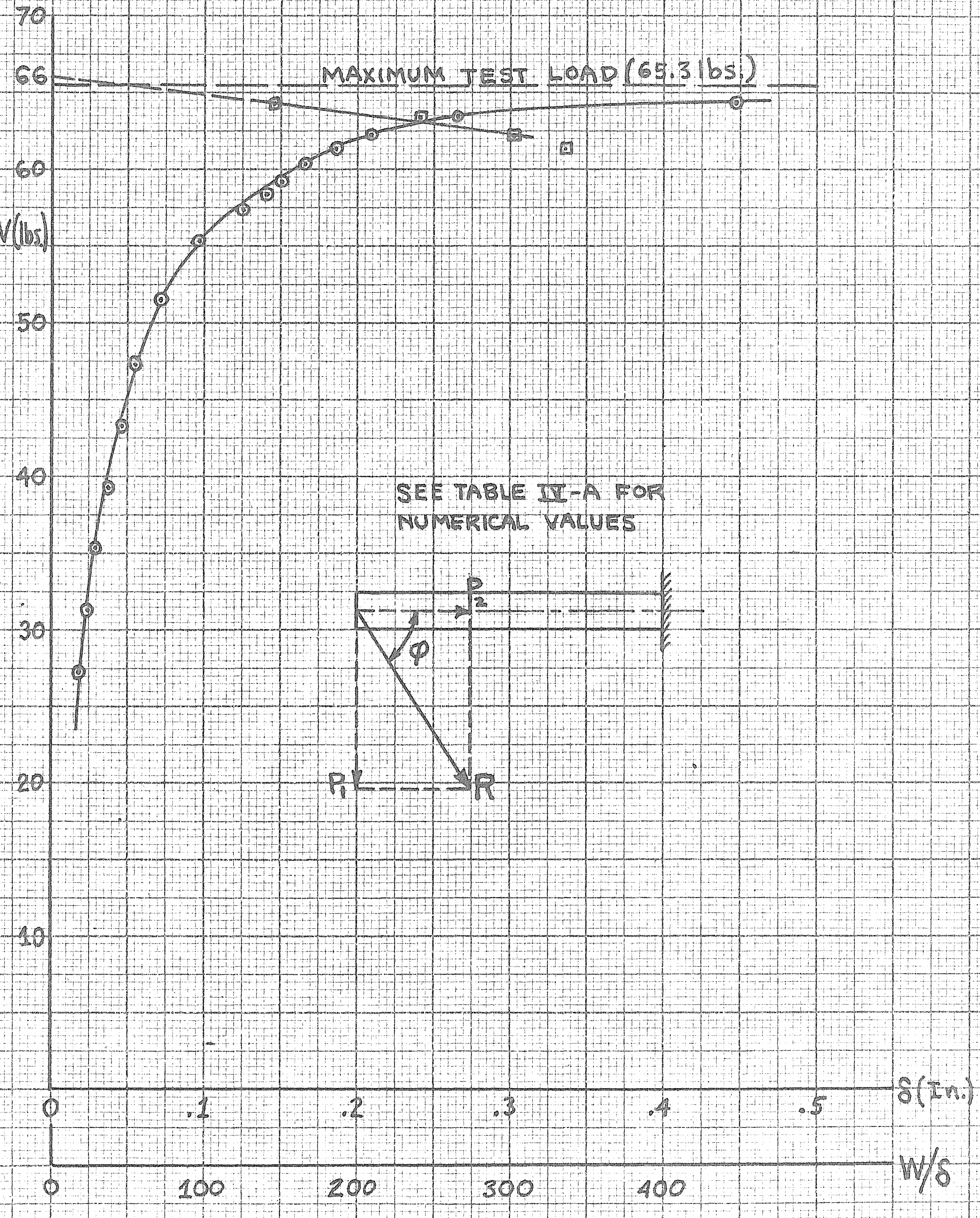


FIG. 19

TEST RESULTS - SPECIMEN A-3



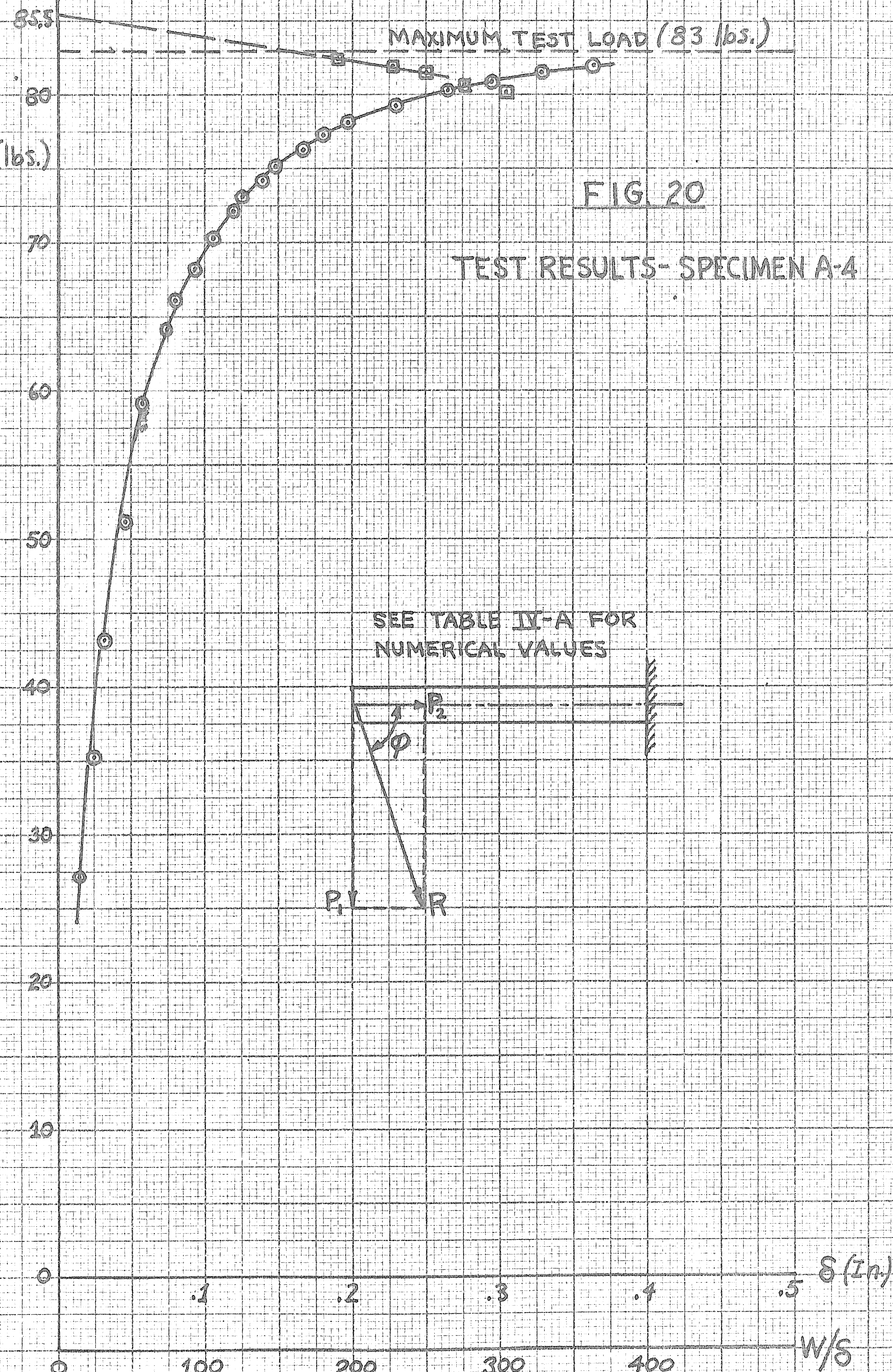


FIG. 20

TEST RESULTS - SPECIMEN A-4

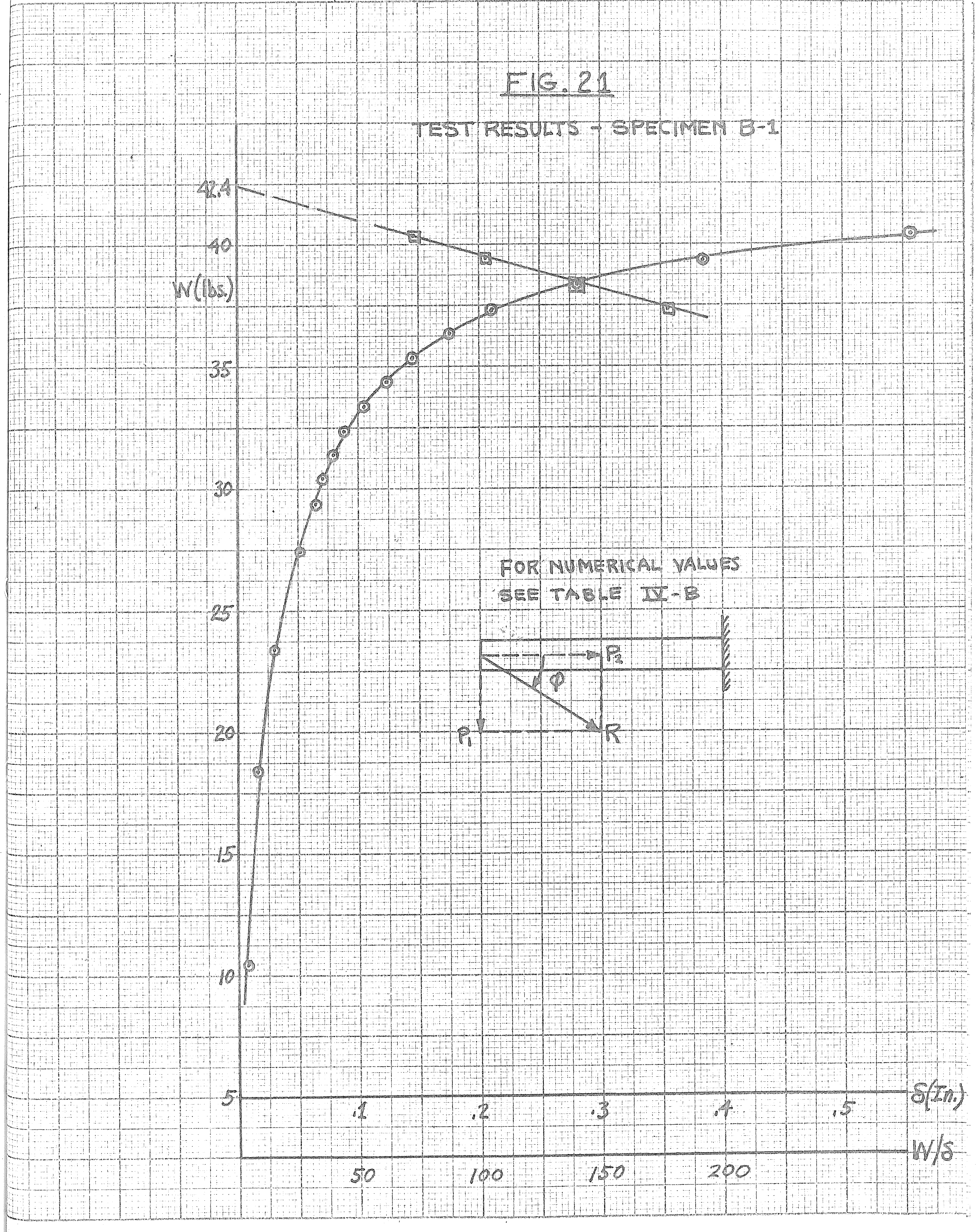
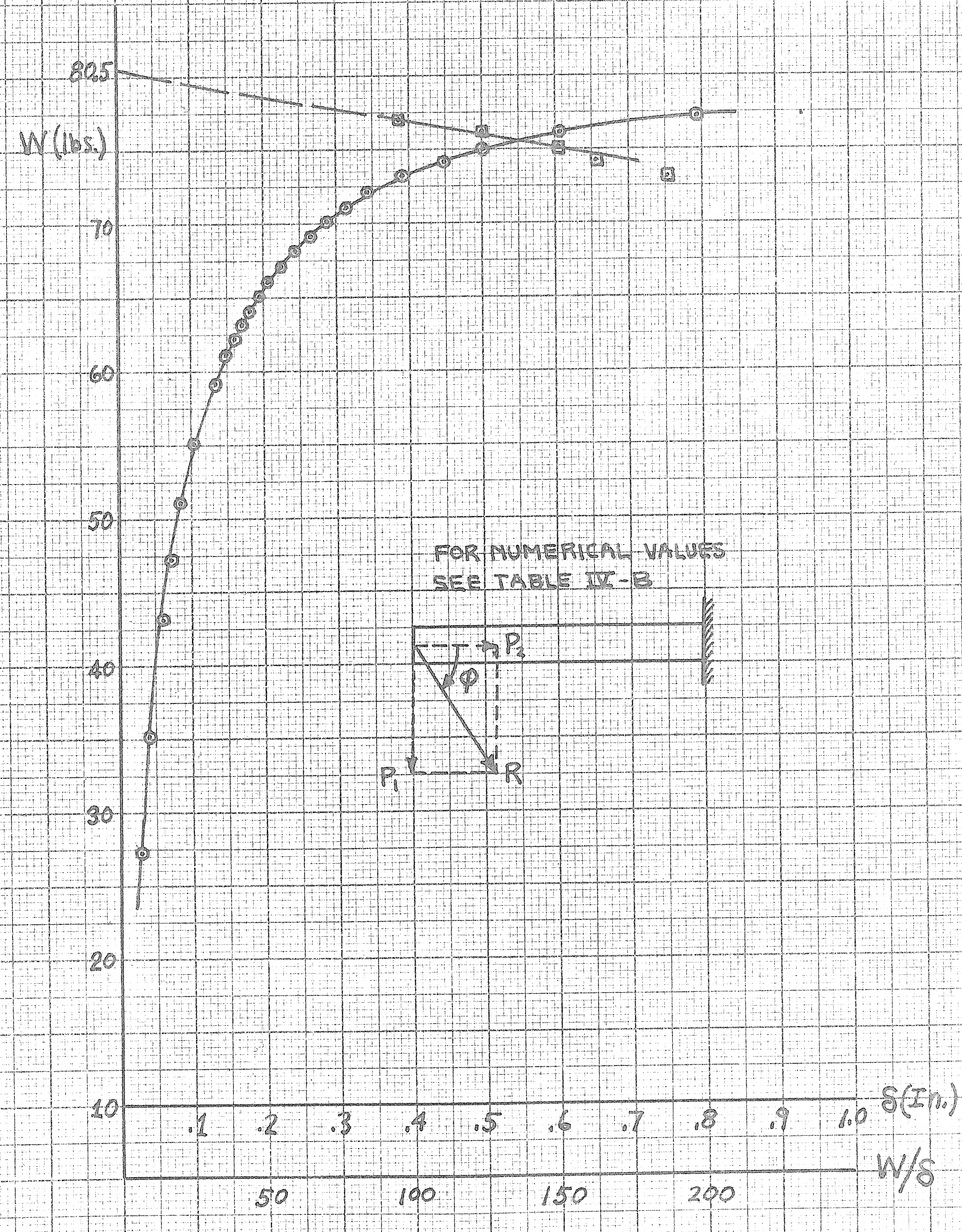


FIG. 21
TEST RESULTS - SPECIMEN B-1
42.4
40
W(lbs.)
35
30
25
20
15
10
5
FOR NUMERICAL VALUES
SEE TABLE IV-B
P2
φ
P1
R
.1
.2
.3
.4
.5
δ(In.)
W/δ
50
100
150
200

FIG. 22

TEST RESULTS - SPECIMEN B-2



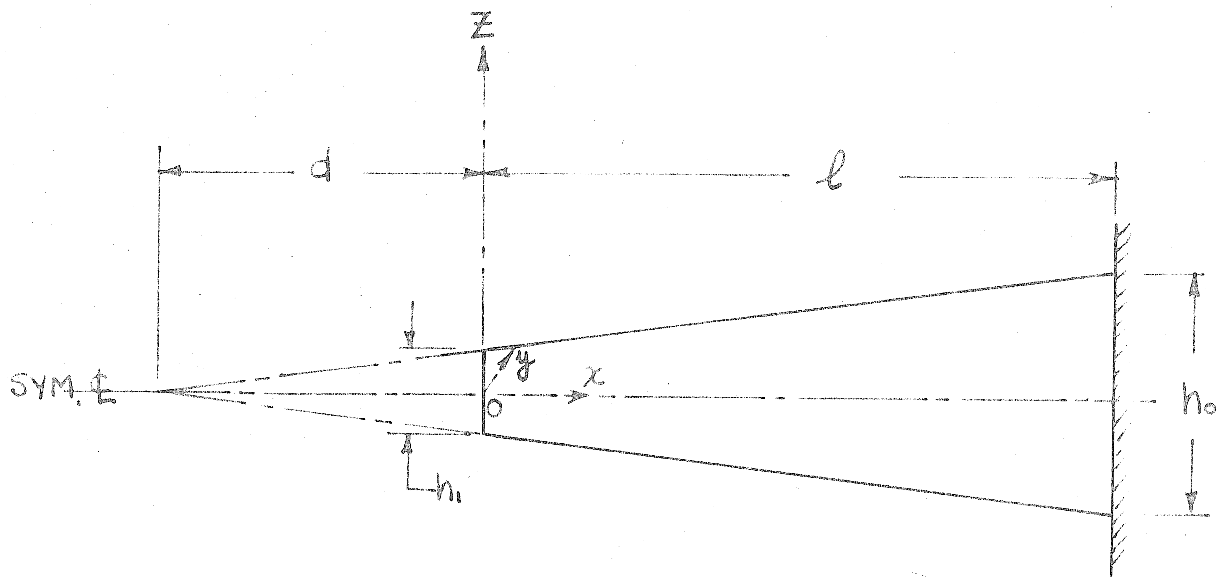


FIG. 23

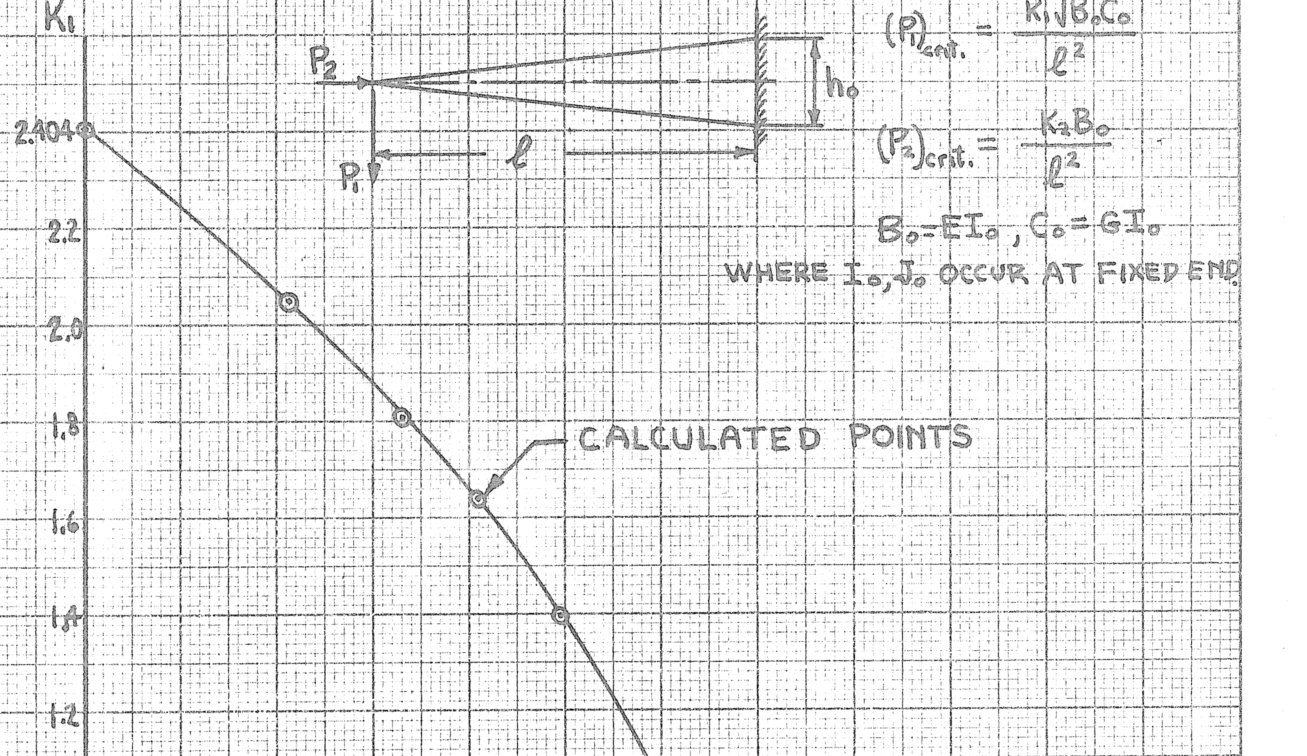
COORDINATE SYSTEM AND DIMENSIONS

TAPERED STRUT

FIG. 24

CRITICAL LOAD COEFFICIENTS

FULLY TAPERED STRUT



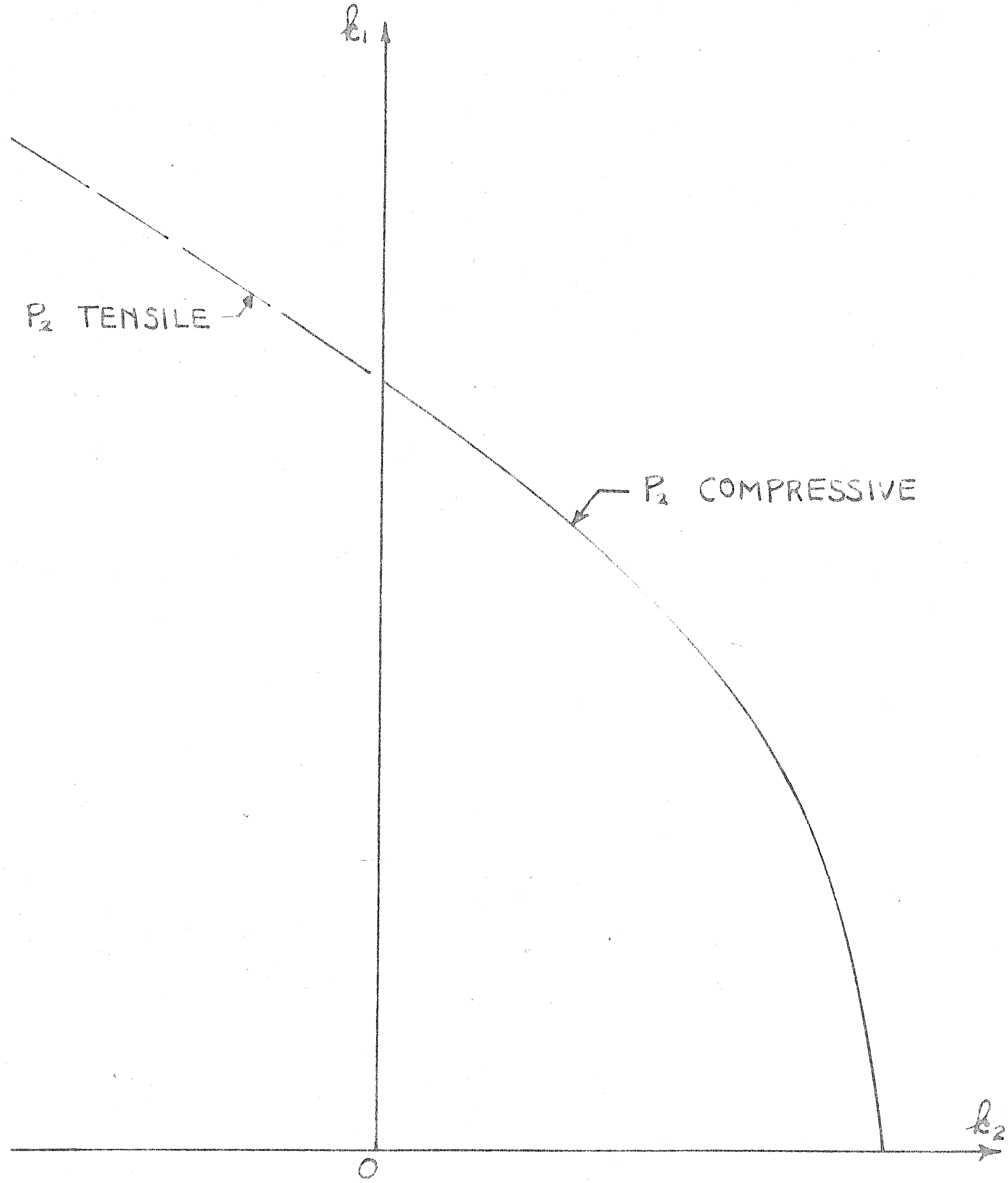


FIG. 25

FORM OF COMPLETE INTERACTION CURVE

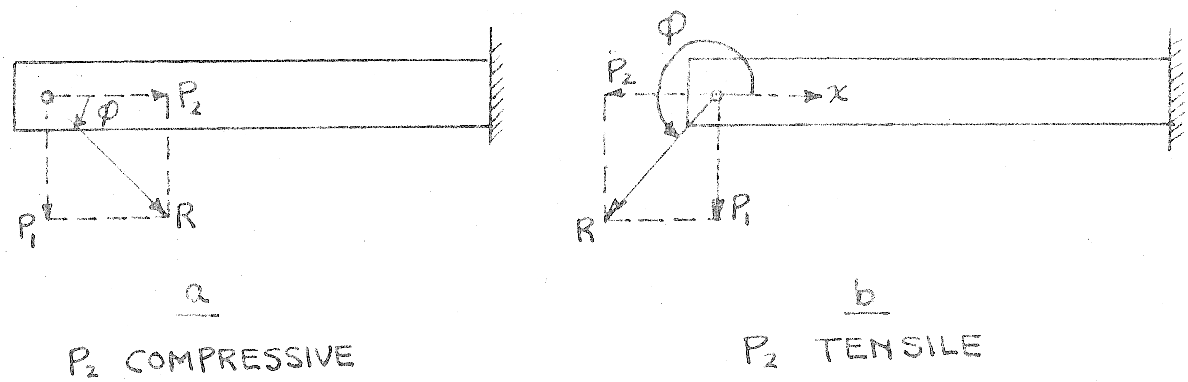


FIG. 26

SPECIFICATION OF
RESULTANT LOAD DIRECTION

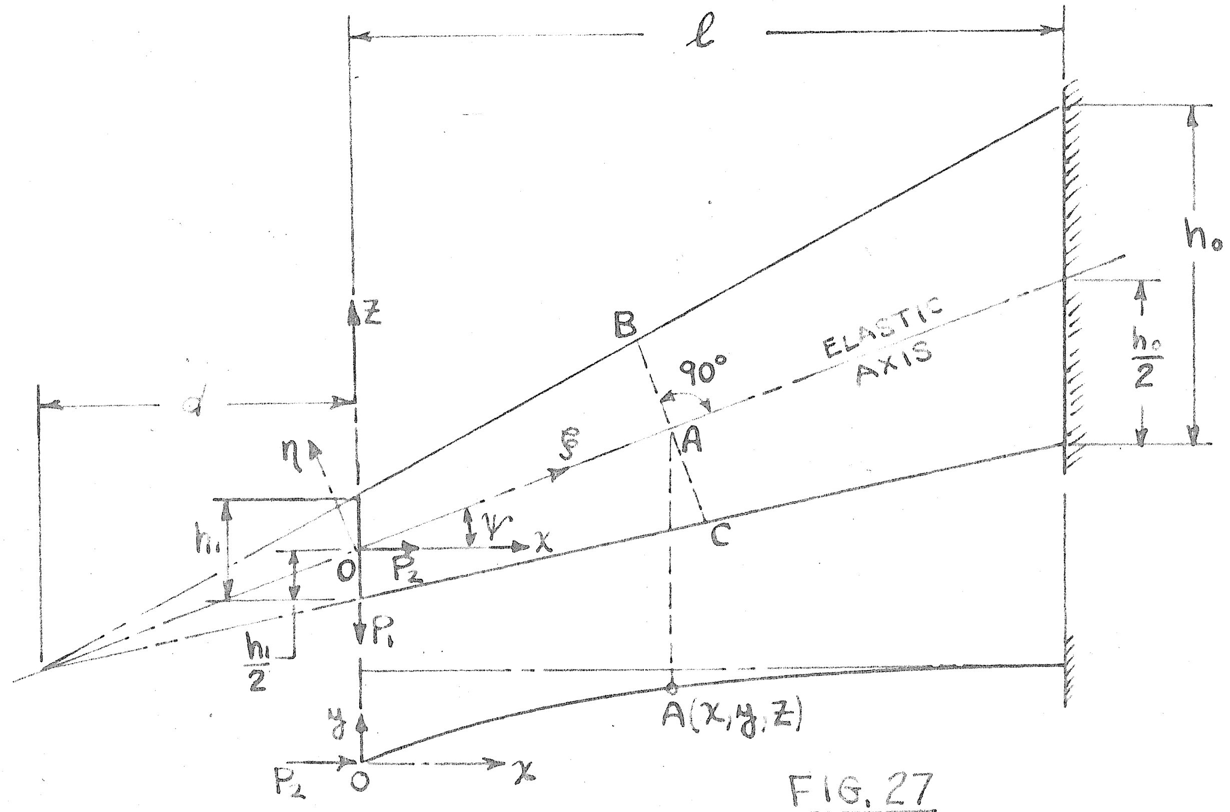


FIG. 27

TAPERED AND SWEPT STRUT

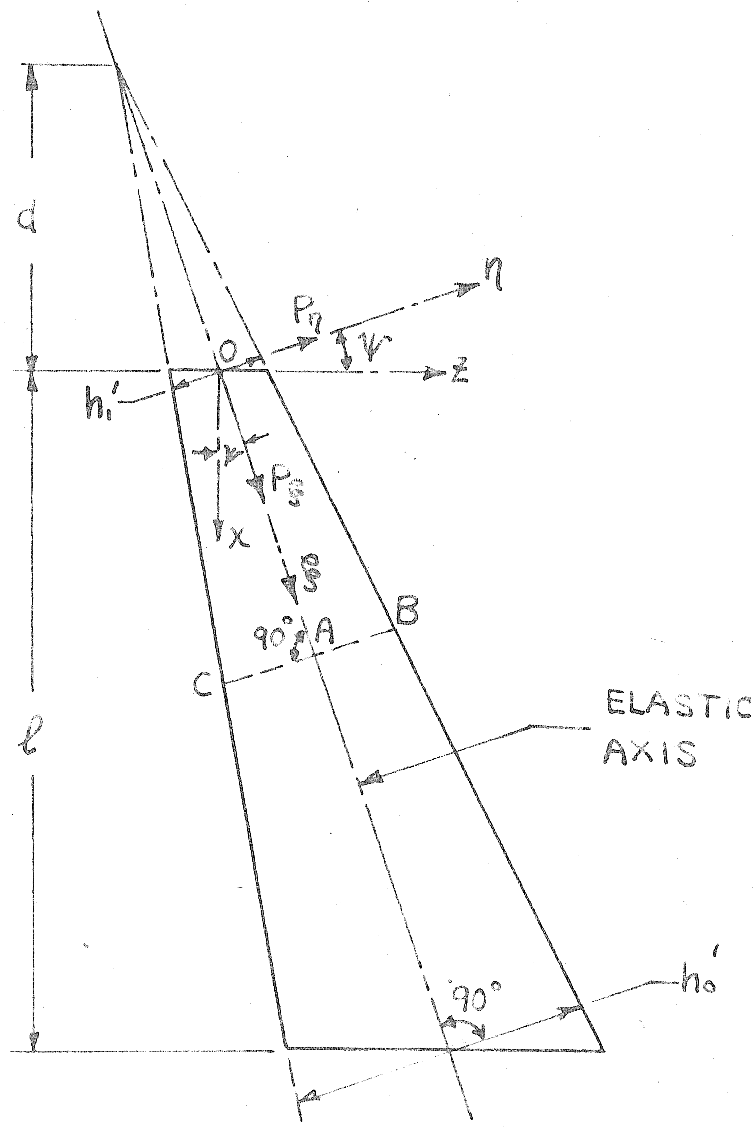


FIG. 28

COORDINATES, LOADS, AND DIMENSIONS WITH RESPECT TO THE ELASTIC AXIS FOR THE SWEPT STRUT

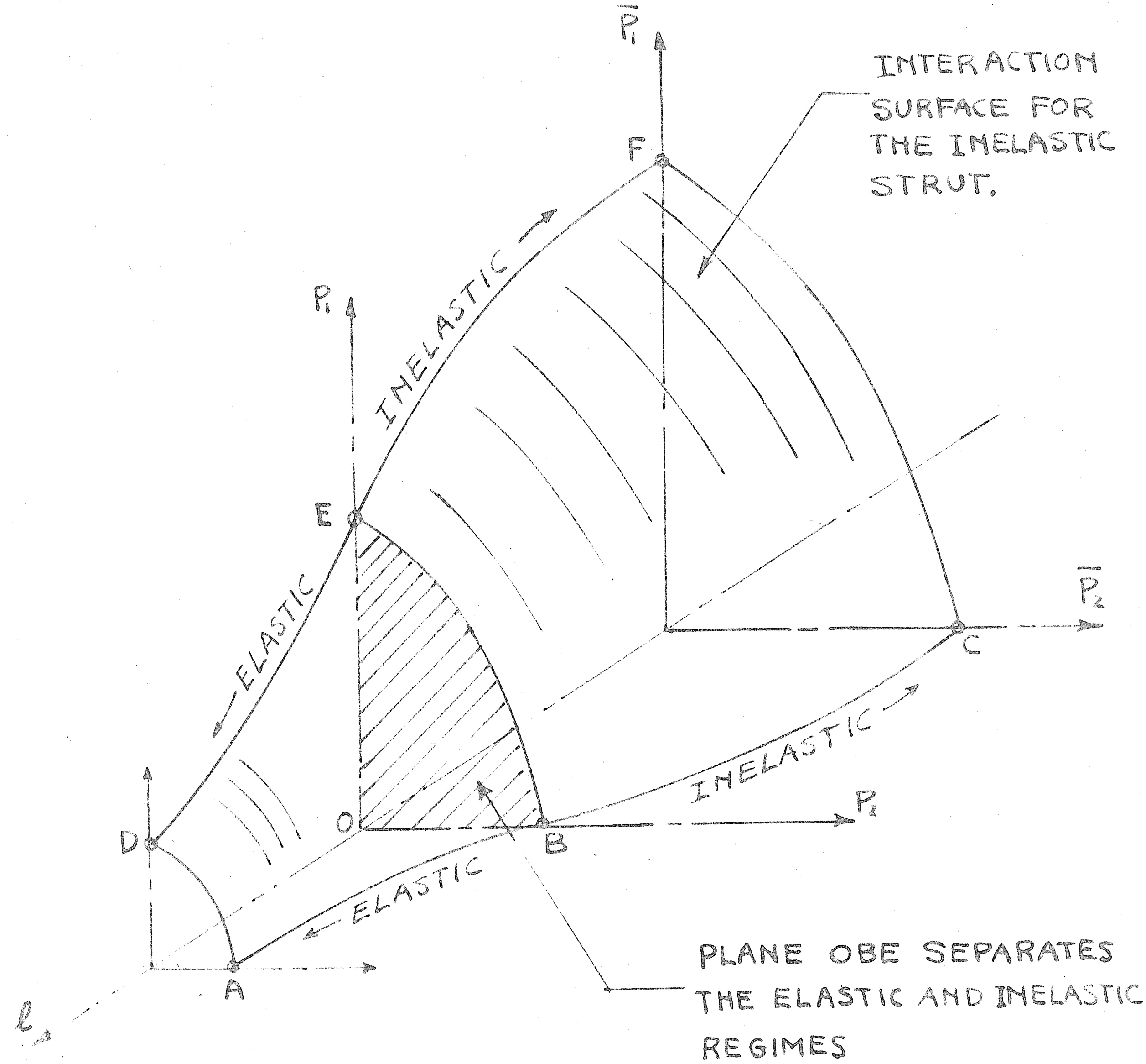
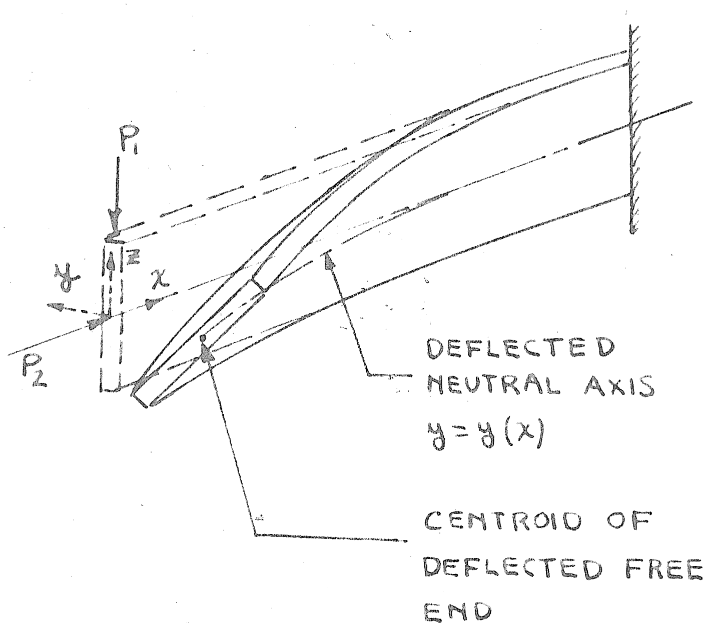


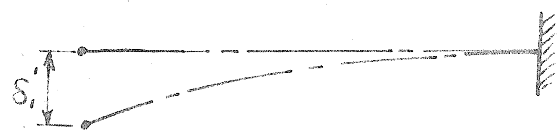
FIG. 29

SUGGESTED METHOD FOR REPRESENTING THE CRITICAL LOADS - INELASTIC STRUT



a

DEFLECTED STRUT ILLUSTRATING VERTICAL DISPLACEMENT OF CENTROID OF FREE END.

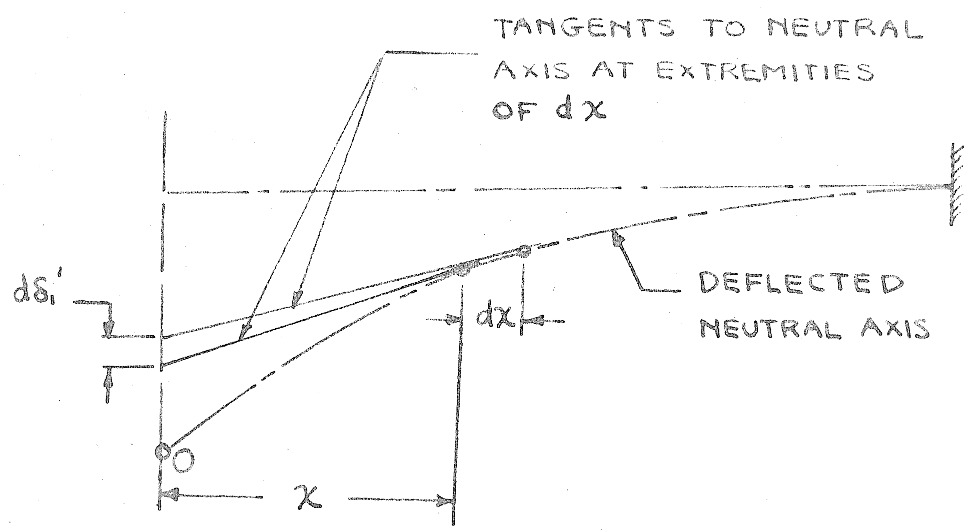


b

HORIZONTAL DEFLECTION AT STRUT FREE END

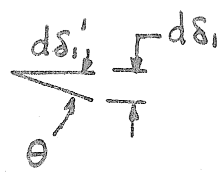
FIG. 30

DEFLECTED STRUT



a

CONTRIBUTION OF ELEMENT dx TO THE HORIZONTAL DEFLECTION OF THE STRUT FREE END CENTROID



b

VERTICAL COMPONENT OF DEFLECTION. DUE TO $d\delta_1'$ AND TWIST θ AT STA. x

---

FIG. 31

VERTICAL DEFLECTION AT STRUT FREE END