# FINANCIAL EQUILIBRIUM, VOTING PROCEDURES, AND COALITION STRUCTURES IN ALLOCATIONAL MECHANISMS

Thesis by
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Finally, I want to acknowledge the Baxter Hall economists' contributions, both by formal instruction and personal example, to my understanding of human behavior. Through the micro-economic paradigm of expected utility maximization I have finally comprehended ultimate truth, that which lies beyond the veil, the jewel within the lotus, the sound of one hand clapping:

**More *Is* Better.**

This realization is so startling, since--as I can only now apprehend--it's something that I'd already known all along.

# ABSTRACT

This thesis is comprised of three chapters, each of which is concerned with properties of allocational mechanisms which include voting procedures as part of their operation. The theme of interaction between economic and political forces recurs in the three chapters, as described below.

Chapter One demonstrates existence of a non-controlling interest share-holders' equilibrium for a stylized one-period stock market economy with fewer securities than states of the world. The economy has two decision mechanisms: Owners vote to change firms' production plans across states, fixing shareholdings; and individuals trade shares and the current production / consumption good, fixing production plans. A shareholders' equilibrium is a production plan profile, and a shares / current good allocation stable for both mechanisms. In equilibrium, no (Kramer direction-restricted) plan revision is supported by a share-weighted majority, and there exists no Pareto superior reallocation.

Chapter Two addresses efficient management of stationary-site, fixed-budget, partisan voter registration drives. Sufficient conditions obtain for unique optimal registrar deployment within contested districts. Each census tract is assigned an expected net plurality return to registration investment index, computed from estimates of registration, partisanship, and turnout. Optimum registration intensity is a logarithmic transformation of a tract's index. These conditions are tested using a merged data set including both census variables and Los Angeles County Registrar data from several 1984 Assembly registration drives. Marginal registration spending benefits, registrar compensation, and the general campaign problem are also discussed.

The last chapter considers social decision procedures at a higher level of abstraction. Chapter Three analyzes the structure of decisive coalition families, given a quasitransitive-valued social decision procedure satisfying the universal domain and IIA axioms. By identifying those alternatives $X^* \subseteq X$ on which the Pareto principle fails, imposition in the social ranking is characterized. Every coaliton is weakly decisive for $X^*$ over $X\sim X^*$, and weakly antidecisive for $X\sim X^*$ over $X^*$; therefore, alternatives in $X\sim X^*$ are never socially ranked above $X^*$. Repeated filtering of alternatives causing Pareto failure shows states in $X^{n*}\sim X^{(n+1)*}$ are never socially ranked above $X^{(n+1)*}$. Limiting results of iterated application of the $*$-operator are also discussed.

# TABLE OF CONTENTS

# Equilibrium In A Stock Market Economy With Shareholder Voting

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This paper demonstrates the existence of a non-controlling interest shareholders' equilibrium for a one period stock market economy in which there may be fewer securities than states of the world. There are two allocational mechanisms in the economy: Firms' owners may vote to change firms' production plans across states while keeping shareholdings fixed, and individuals may trade their shares and stocks of a current production/consumption good while keeping firms' plans fixed. A shareholders' equilibrium is a set of firms' plans, and an allocation of shares and the current good which are stable with respect to both mechanisms. That is no (direction restricted) plan revision is supported by a share weighted majority of firms' owners, and there is no reallocation through trade which will make all individuals better off.

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We would like to thank Professors Kim Border, Robert Forsythe, Edward Green and Richard McKelvey for several illuminating conversations concerning this topic.

## I. INTRODUCTION

In a stock market economy, firms' owners secure (uncertain) future consump - tion opportunities by foregoing current consumption and investing resources in production. Such economies are distinguished from Arrow-Debreu production economies by the absence of explicit markets for contingent claims--in particular, firms' shares cannot be unbundled. Only if there are as many firms with linearly independent production plans as there are future states of the world, it is possible for individuals to assemble packages of firms' shares which are equivalent to Arrow-Debreu securities. Otherwise, the shareholders are typically faced with a non-trivial problem in group decision.

It is well known that if there are fewer firms than states[1], the Fischer [1930] separation principle may fail. That is the objective of the firm may fail to be well defined, and proposals to change firms' production plans may meet with a divided response among the shareholders. An extensive technical literature has evolved concerning such questions as under what conditions might shareholders' opinions be unanimous, or what objective functions firm managers should be given. Nevertheless, in real-world practice shareholders do disagree, and such disputes are at least partially resolved in the corporate form of firm organization by majority rule processes.

The results developed in this paper represent a new contribution to the theory of the firm in incomplete markets--the development of a financial model with an equilibrium solution in which shareholders of publically traded firms make multi-dimensional production decisions by a majority rule process. These results are

1. See, for example, Ekern and Wilson [1974].

genuinely novel; there simply are no other stock market equilibrium existence results similar to those presented here. Furthermore, this paper illuminates the following socially important question: Are the majoritarian decision-making and stock market institutions utilized by publically traded corporations *inherently unstable,* or are frenetic 100 million share days and frequent proxy fights attributable to unpredictable exogenous shocks to the environment? The conclusion suggested by the model developed here is that the institutions are not inherently unstable.

A more detailed summary of the equilibrium results and a literature review are presented in Parts A and B of this section, respectively. Next, sections II and III formally set forth the basic structure of the model, section IV describes stock market trading in detail, and section V is concerned with shareholder voting. The results from these sections are then combined in section VI to give the existence of ***shareholders' equilibrium***. Directions for future research are discussed in section VII.

## A. SUMMARY OF NEW RESULTS

If there are fewer securities than states of the world, managerial objective functions which require only market information may not be well defined for shareholder-owned firms. Under these circumstances, some collective decision making process presumably gives rise to firms' plans. Majority rule mechanisms play an important role in setting real-world firms' production policies. This paper develops a stylized model of a two date financial economy in which shareholders both make production decisions by a share-weighted majority rule mechanism, and also trade their shares on a stock market. Existence of a stockholders' equilibrium is demonstrated, and a class of such equilibria are characterized.

A shareholders' equilibrium is defined here to be an allocation of both the current consumption / production good and firms' shares (together with plans) such that: i) holding firms' plans fixed, there is no other Pareto optimal allocation which is also individually rational, and ii) holding individuals' portfolios fixed, there is no direction restricted revision of any firm's plan which would be approved by a weighted majority of the firms's shareholders.

Firms' technologies are allowed to take a very general form; neither multi - plicative uncertainty nor an exogenously fixed production pattern across states has been imposed. Shareholders are assumed to be myopic when exchanging shares or choosing firms' plans. Non-controlling interest equilibria are guaranteed by the requirement that all individuals maintain positive holdings in every firm; the size of these holdings can be made arbitrarily small as the number of participants in the economy increases. Alternatively, individuals could be directly prevented from holding a controlling interest in any firm.[2]

In order to familiarize the reader with some of the more specialized aspects of the model, the following topics will be discussed in some detail below: The Equilibrium Concept, Myopia and Truthful Voting, Direction Restricted Voting, $\varepsilon$-Majority Rule, Financing Firms' Production Plans; No-Bankruptcy Constraints, Separation of Voting and Trading, Non-Controlling Interest Equilibria, and Optimality.

Equilibrium Concept

Following the technique employed by Dreze [1972], the financial model presented here suppresses the notion of initial endowments for individuals or firms. Instead of an endowment vector, it is assumed there is given some strictly positive aggregate social endowment of a single current production / consumption good. Suppose now that by some unspecified process--perhaps command by a central planner, or by sequential voting and trading--that the social endowment has been divided between production and consumption, firms' shares and stocks of the current good have been allocated to consumers, and production plans (including resource requirements) have been determined.

The equilibrium existence question in this framework can be posed as follows: If each individual in the economy regards his component of such an allocation as a personal endowment which conveys property rights, does there exist an allocation which is also a majority rule shareholders' equilibrium? This type of equilibrium is

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**2.** It is assumed here that any shareholding greater than .5 is a controlling interest. Jordan [1978] defines a "controlling interest" in a given firm to be a shareholding greater than .5 which is sufficient--as specified in the company's bylaws--to determine the firm's production plan. In his construction a controlling interest might, for example, require .9 of a firm's shares.

defined as a portfolio consisting of shares and the consumption good for each individual, and a set of firms' production plans which simultaneously satisfy the following two conditions: i) holding firms' plans fixed, there is no other allocation of shares and the consumption good which is Pareto optimal and is also individually rational with respect to the current allocation, and ii) holding portfolios fixed, there is no direction restricted revision of any firm's current production plan which would pass a share-weighted majority vote of the firm's shareholders and which would not bankrupt any shareholder.

Under standard regularity conditions, it will be demonstrated that such non-controlling interest shareholders' equilibria do in fact exist. Furthermore all such equilibria will be characterized as the set of fixed points of a correspondence constructed in a very natural manner.

## Myopia and Truthful Voting

Each participant in the economy is assumed to behave myopically. In particular, all individuals behave as if they believed every decision making forum, be it a shareholders' meeting or stock market session to be the final allocational event in the economy. This convention is the same as that adopted by Dreze [1974], and is common in both the stock market equilibrium and shareholder unanimity literature.[3]

While this assumption affords significant analytical simplicity, it specifically rules out strategic behavior such as the acquistion of a firm's shares with the intent to gain voting power sufficient to change its production plan. The myopia assumption, in the general form stated above, also implies that during shareholders' meetings

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**3.** For example, Gevers [1974] and Ekern and Wilson [1974] employ similar myopia assumptions.

individuals will vote their true preferences. Given direction restricted voting (to be discussed below) Slutsky [1977], in his general equilibrium model with produced public goods, argues this assumption is a reasonable approximation of real-world voters' behavior.

> It will be assumed here that people vote their true preferences. This is more justifiable in the context of this paper than, for example, in the context of the Lindahl model. Fixed tax shares and majority rule limit the ability of an individual to gain from misrevelation of preference. False voting along a single vector, if the level of goods determined by votes along the other vectors is unchanged, can only hurt the individual. To attain an overall benefit from false voting, the individual would require much information about the preferences of others. Since, in most situations, sufficient information is unavailable, it is at least a good first approximation to neglect this possibility.

The transposition of Slutsky's remarks from his public goods model to the financial framework developed here should be clear.

Direction Restricted Voting

It is well known that unrestricted majority rule processes can give rise to voting cycles and nonexistence of equilibrium. One solution to this problem, which does not not call for unrealistic assumptions concerning either preferences or the pol - icy space, has been proposed by Kramer [1972]. Kramer's mechanism requires that, in a policy space of dimension n, that n linearly independent directions, called ***voting directions***, must be exogenously specifed.

New proposals to depart from the status quo can differ from it along only *one* of the n voting directions. Under these so-called direction restrictions, together with ordinary continuity and quasiconcavity utility assumptions, Kramer demonstrates via a fixed point argument that there exist majority rule stable points (attained by some unspecified process--perhaps, for example, command by a cental planner).

It should be noted Kramer's proof does *not* assert that starting from any given point in the policy space, there exists (much less that there will be spontaneoulsy proposed) some sequence of proposals which will converge to an equilibrium. This much stronger claim, if proved via familiar fixed point theorems, would require demonstrating topological regularity properties of the majority-rule attainability sets associated with arbitrary points in the policy space. These sets certainly do not appear to be convex, while subtler properties such as contactibility or acyclicity depend on the homology groups of the attainabilty sets. This line of investigation does not seem likely to yield results in the near term.

In the corporate context at hand, the exogenously specifed voting directions can be given an interpretation which has considerable intuitive appeal. Since there are J firms and S future states of the world, the space of all firms' plans has dimension J∗S. Thus J∗S linearly independent voting directions must be specifed in order to span the space of firms' plans. While any basis for this space will satisfy the formal requirements for the set of voting directions, a particulary natural choice presents itself in a production economy. The directions have been defined here so that a proposal to revise the status quo production plans may change only the planed output of a single firm in a single future state of the world. Slutsky [1977] comments concerning the interpretation of the voting directions in his public goods model; these remarks can easily be transposed to a financial context.

> The voting vectors can be interpreted in various ways. In a general sense, they serve as a simple proxy for more complicated institutional structures found in actual voting situations. Unrestricted simple majority voting has the demand for public goods determined directly by vote of all individuals with no constraints on feasible votes. In reality, decisions are made by interacting branches of government and legislative bodies. Within any voting body, the results are strongly affected by committee structures, legislative leaders, and rules of procedure, formally or informally specified. These factors are extremely difficult to model in general. The imposition of voting vectors is at least a start in placing some institutional structure on the political process. Although highly simplistic, their use points out how an equilibrium can be created and manipulated through the institutions within which voting operates ..Specifically, the

direction of the vectors can be considered to arise from packaging of commmodities for stategic, administrative, or natural reasons.

The direction restrictions are strong assumptions, particularly since myopic shareholders are prevented from directly proposing production tradeoffs across states.[4] Nevertheless, previous attempts to model majority rule processes in a corporate context (for example, Benninga and Muller [1979], Jordan [1978], and Gevers [1974]) have only provided non-existence results, or else have been restricted to the one-dimensional problem of setting the scale of an exogenously specified production plan.

In assessing the appropriateness of the direction restrictions, it should also be kept in mind that deriving majority rule equilibrium results in a financial setting is a more intricate task than demonstrating existence in a purely political framework. The financial environment is more complex due to interactions between the market and voting processes: i) movement in the policy space of firms' plans in general entails new financing requirements, and ii) shareholders' voting weights may change as a result of trade.

$\varepsilon$Majority Rule

The possibility that voters' weights may change poses no problem in attaining an equilibrium production plan while holding shareholdings fixed. This factor does come into play, however, when searching for equilibrium in both portfolio holdings and firms' plans. In fact, since shareholdings *can* change, the majority rule winner correspondence may fail to be continuous. Consider, for example, a simple case in which,

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4. A coalition of strategic (non-myopic) individuals might accomplish a "diagonal" revision of a firm's plan through a sequence of proposals in the basis directions, even though each step considered on its merits alone would be deemed undesirable.

through a sequence of trading sessions, an individual (or coalition) comes to own a controlling interest in a firm. Imagine that before this individual gained control, the other shareholders have consistently defeated proposals to change the firm's plan. However, after gaining a controlling interest in the firm, he can now change the firm's plan drastically, subject only to the direction and non-bankruptcy constraints.

In order to prevent this type of discontinuity, a modified form of majority rule, termed $\varepsilon$-majority rule, has been constructed here. Under $\varepsilon$majority rule, the distance (in the policy space of firms' plans) from which a winning coalition can depart from the status quo is positively proportional to the size of the coalition's majority.[5] $\varepsilon$majority rule satisfies the continuity properties necessary to demonstrate the existence of equilibrium. In equilibrium the assumption that $\varepsilon$ is greater than zero can be discharged.[6]

Financing Firms' Production Plans; No-Bankruptcy Constraints

After the aggregate social endowment has been allocated to production and consumption, shares have been allocated to individuals, and firms' plans have been chosen, the model has been so devised that each firm's production plan turns out to be fully financed. That is, given a plan vector, each company has a stock of the first date production / consumption good which is just sufficient to undertake its plan. This construction reflects the real-world "fully financed and non-assessable" property of corporate shares. This property limits the liability of the owner with respect to additional contributions to the firm (or to other creditors).

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5. $\varepsilon$, the proportionality factor, must be chosen small enough so that the coalition of the whole is not prevented from undertaking unanimously supported moves in the policy space.

6. This fact was pointed out by K. Border.

However, if the shareholders of some firm vote to revise its plan, in general there will be either a surplus or deficit of the current good required for production, due to the assumed strict monotonicity of each firm's cost function. In the case of a surplus, as modelled here, there will be a refund to the firm's owners on a share-weighted basis. On the other hand, if there is a deficit, the owners will be required to make share-weighted contributions from their personal stocks of the current good. In order that no owner should become bankrupt and thus unable to make required input contributions, the set of legal proposals to revise firms' plans has been appropriately restricted. In addition to the direction restrictions previously discussed, it is also required that any proposed plan revision, if it were to be fully adopted, must not bankrupt any current shareholder of the firm to which the proposal applies.

The assumption that expansion plans must be financed entirely by share-weighted personal contributions from firms' owners lends considerable analytic simplicity to the analysis undertaken here. This is perhaps the simplest financing scheme which can be imagined under which the level of investment is endogenously chosen. However, the construction introduces some difficulties into the model. These problems are twofold in nature: technical and conceptual, and will be discussed in turn.

First of all, an individual with stocks of the current good exactly equal to zero prevents any firm in which he owns a positive share, no matter how small, from increasing production--even if a majority of the other shareholders are in favor of doing so.[7] Furthermore, if such a (weakly) bankrupt shareholder were to trade away his shares in a particular firm, then that firm could suddenly undertake expansion. This

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7. Notice that an individual facing such circumstances imposes an unusual externality on the other firm owners. Also observe that the myopia assumption rules out the possibility that a shareholder with no current assets might hold other owners hostage.

possibility gives rise to a failure of lower hemicontinuity of the firm's legal proposal correspondence. In order to rule out such discontinuity it is necessary that each individual maintain (exogenously set but arbitrarily small) strictly positive holdings of every firm.

Secondly, the financing scenario imperfectly reflects the distinguishing property of common shares, namely that stock ownership should entail no risk of ex post charges due to changes in a firm's fortunes or operations.[8] In the model developed here, of course, share ownership may well require making substantial input contributions to finance plan revisions--indeed, revisions to which some shareholders may be opposed.[9] Of course shares may be traded away after plan revision, but shareholders typically will not agree concerning the effects of plan changes on share value. It would be desirable to retain the notion of limited liability of share ownership in the model.

One possible way to shade the model in this direction would be to permit shareholders who are not liable to be bankrupted by a given plan revision to make input contributions for shareholders who would be bankrupted, in return for an increased share of the firm. Other possibilities would be to permit firms to issue debt or new shares to finance expansion. These possibilities are topics for future research.

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**8.** While real-world shareholders do not bear the risk of being assessed additional charges arising from plan revision, they *do* bear the risk of unwanted revisions. Corporate plan changes are frequently greeted with a divided response among shareholders.

**9.** Individuals are assumed to be myopic with respect to this risk when acquiring shares.

Stylized Trading; The Individually Rational Pareto Correspondence

Consider a simple exchange economy and an aggregate social endowment of J+1 goods, in particular, a current production / consumption good and the shares of J firms with linearly independent production plans.[10] Next define the set of all possible allocations of these goods among the economy's I consumers. Finally, consider the Walras and core correspondences, each of which maps this set of possible allocations back into itself. These correspondences are the natural choices for modelling the process of decentalized reallocation among the consumers in an exchange economy.

Unfortunately, neither the Walras nor the core correspondence is in general connected valued, much less convex or contractible valued. There are well known results which give sufficient conditions for the Walras correspondence to be singleton valued[11]; however, the usual requirement, that all goods be gross substitutes, is not sensible in a stock market economy. Thus the topological hypotheses of the usual fixed point theorems rule out both of these correspondences as analytically tractable models of stock market trading.[12]

However, K. Border and R. McKelvey have suggested that trade in a stock market economy might be sucessfully modelled by a more highly stylized, simpler construction--the individually rational Pareto correspondence. This correspondence

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**10.** In order to insure that firms' plans are linearly independent, each company will be required to produce an arbitrarily small, exogenously determined quantity of a unique good. These ***distinguished goods*** will be discussed in full detail in Section IV.

**11.** See, for example, Arrow and Hahn [1971].

**12.** Of course, the *possiblity* exists that some equilibria may lie in the core or may be Walras equilibria. It is hoped that techniques which appear sufficient to demonstrate the existence of a continuous selection from the individually rational Pareto correspondence can be employed to explore the existence of such equilibria.

associates with a given allocation the set of all reallocations which satisfy: i) no one is worse off after reallocation than under the given allocation, and ii) from the vantage point of the reallocation, there is no other reallocation which would make someone better off without making anyone worse off.[13]

Notice that the individually rational Pareto corresopondence majorizes the core correspondence. That is, some allocations in the individually rational Pareto image of a given state of the economy or "endowment" could, in general, be improved upon by some coalition(s) smaller than the grand coalition. However, such behavior on the part of small coalitions is being ruled out here, presumably due to the cost of coalition formation or the inability of such groups to enforce binding agreements among themselves.

Chipman and Moore [1971] and Zeckhauser and Weinstein [1974] have demonstrated by relatively straightforward intuitive methods, and under ordinary regularity assumptions, that the Pareto correspondence (defined over the set of the economy's possible allocations) takes images which are homeomorphic to the I-1 simplex, where I is the number of consumers.[14] In particular, the Pareto correspondence is contractible valued; this result is extended here to the case of the individually rational Pareto correspondence. Dreze [1972] has demonstrated that the individually rational Pareto correspondence is upper hemicontinuous. These topo - logical results, combined with various properties of the production plan voting

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**13.** Given some allocation, there are in general feasible reallocations which satisfy both, neither, or one or the other of these two conditions.

**14.** Although there is an extensive literature concerning the mathematical theory of Pareto sets, much of this work is considerably less accessible than that of Chipman and Moore and Zeckhauser and Weinstein.

procedure and the set of states of the economy, yield the desired existence result via a fixed point theorem.

Separation of Voting and Trading

The underlying "dynamic" scenario alluded to here in which voting and trading are carried out on an alternating basis, one activity being conducted while the other is held fixed, was first fully developed by Dreze [1974]. As he points out:

> One could, of course, look for a stronger equilibrium concept, ruling out incentives for *simultaneous* adjustments in production plans and portfolios. But no natural definition of this stronger concept seems to present itself.

Slustsky [1977] has also succesfully employed this "two-mechanism" method - ology. He combined Kramer's [1972] notion of direction restricted voting, together with a separate process of simple exchange, to obtain general equilibrium results in an economy with production in the public goods sector.

Non-Controlling Interest Equilibria

As previously discussed, the equilibrium concept defined here is only tenuously connected to a notion of individuals' initial endowments. After firm plans have been chosen and the aggregate social endowment has been allocated, individuals' holdings can (finally) be considered personal endowments. An equilibrium has been defined to be such an allocation from which individuals will not choose to depart, regarding their equilibrium holdings as endowments.

However, a consequence of this construction is an uninteresting class of equilibria in which the stocks of the current good are held entirely by individuals who hold controlling interests in one or more firms. The extreme case of this phenomenon occurs when one individual holds the entire stock of the current good and all the shares of every firm. In equilibrium such an individual will presumably have adjusted each firm's plan to his most preferred choice. Clearly no trade can take place in this situation since no other individual holds any commodities; similarly firm plans are stable since this one individual is the sole owner of every firm. This is indeed an equilibrium, but not an especially interesting one.[15]

The set of stockholders' equilibria for the financial model presented here has been shown to coincide with the set of fixed points of the correspondence $V^{\circ}E_{IP}$, which characterizes individuals' opportunities to reallocate resources. However, fixed point existence results typically do not provide much information about the nature of the fixed points. It appears that without making additional assumptions, it cannot be determined whether or not all the equilibria are controlling interest equilibria.

One possible approach to this problem is to introduce a stronger notion of individual endowments into the model. The initial description of the economy could be expanded to include a set of endowed plans for firms, and endowed portfolios of shares and holdings of the current good for individuals. Suppose the set of all states attainable from this endowment via the repeated action of $V^{\circ}E_{IP}$ were contractible. It would then be possible to apply the Eilenberg-Montgomery theorem directly to demonstrate the existence of equilibria *supported by the endowment.* Even if all such equilibria were controlling interest equilibria, this in itself would be a very interesting result. Unfortunately, the topological properties of the $V^{\circ}E_{IP}$ attainability sets appear to be extremely complex.

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**15.** This problem does not arise in political or models such as Kramer [1972] or Slutsky [1977]. In these models, political contraints keep each voter's weight exogenously fixed. In both models, a one citizen-one vote rule prevents equilibria in which all voting power is concentrated in one individual.

Another strategy is to start from an initial endowment and search for equilibria which are the limits of sequences of reallocations achieved by alternating processes of trade and production planning. In fact, Dreze [1974] employed this technique, but technical considerations prevent its application here.[16] In Dreze's model, firm production plans are realized as Lindahl equilibria of a synthetic public goods economy at alternate steps (interleaved with stock exchange price equilibria) of a so-called mixed tatonnement / non-tatonnement process. In order to obtain convergence results, Dreze imposes an individual rationality condition on the Lindahl equilibria and so can employ individuals' utilities as Lyapunov functions.[17] This condition cannot be expected to hold in any sensible model of a majority rule firm decision-making process; after the ballots have been counted, some individuals may simply end up worse-off than they were under the status quo.

The approach taken here is to rule out controlling interest equilibria by constraining the state space $Z_\zeta$ so that no individual can hold a controlling interest in any firm (through either his own or a central planner's actions). This is accomplished by a strict no short sales requirement: Each individual must always maintain at least a $((50+\varepsilon)/I)\%$ share in every firm; where $\varepsilon > 0$, and I is the number of individuals in the

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**16.** Dreze's equilibrium proof allows that individuals could be characterized by personal initial endowments, but he did not choose to formally introduce personal endowments. The notion of departing from some arbitrary initial point (partially defined by personal holdings of the current good and firms' shares) in the state space and converging to equilibrium is only developed as a technical device in the main existence proof. Before concluding that an equilibrium so obtained is "supported" in some sense by the initial point, it should be kept in mind that although the each inter - mediate Lindahl equilibrium is constrained to be individually rational with respect to the preceding state (attained via exchange) in the sequence, Dreze's definition of Lindahl equilibrium is "consistent with arbitrary transfers of initial resources among consumers."

**17.** Of course, it is not necessary to impose an arbitrary individual rationality condition on the exchange steps of the process.

economy. The required holdings tend not to be large. For example, if there are 101 participants in the economy, it suffices that each individual holds .5% of every firm. Clearly as the number of individuals in the economy increases, the required minimum holdings decrease in size. Another possibility, which could be easily incorporated into the model, would be to simply prohibit any individual from holding fifty per cent or more of any firm's shares.

This minimum holdings requirement does not appear to be an unusually strong or unrealistic assumption. Financial markets in the United States attract large numbers of participants; taking pension plans into consideration, literally millions of individulas own common shares. In an economy of this magnitude, the required holdings defined here are infinitesimal. Furthermore, by means of institutional arrangements such as mutual funds, millions of individuals *do* hold very broadly diversified portfolios. Finally, there are very few publically traded firms in which any one individual holds a controlling interest, and an attempt by any single investor to gain control of several publically traded firms would elicit the most aggressive regulatory scrutiny.

No short sales requirements appear frequently in the financial equilibrium literature. Diamond [1967] assumes strict no short sales, but claims that the condition can be relaxed to weak no short sales. The strict assumption appears in Forsythe and Suchanek [1982]. No short sales restrictions also appear in Hart [1979]. Grossman and Hart [1979] require that each individual's holdings of each firm be bounded below; they comment as follows:

> Before stating our existence theorem, we must deal with one further difficulty which arises in economies with incomplete markets. In order to apply the usual fixed point theorems, we must insure that the economy is bounded. Unfortunately, it turns out that there is no natural bound on shareholdings. In particular, it is possible for one group of consumers to go very long on one share and very short on another, while another group of consumers does the exact opposite, without either group being in any danger of bankruptcy. Moreover, this not only

prevents the use of the standard fixed point theorems, it can also prevent the existence of equilibrium (see Hart [1975]). In order to avoid this problem, we will make the rather unsatisfactory assumption that there is an exogenously determined lower bound on shareholdings (see Radner[1972]).

Similar assumptions occur elsewhere in the stock market literature. Gevers [1974] assumes that $y_{ij}$ "the number of shares of the same firm (j) owned by agent i when the market closes," is strictly greater than zero for all firms and individuals. Jordan [1978] bars short sales and requires that each individual finance investment from his first-date endowment. The strict no short sales assumption also appears in Forsythe and Suchanek [1982].

In conclusion, the existence results derived here assert there is a class of (albeit constrained) non-controlling interest shareholders' equilibria. Notice that these equilibria are indeed non-trivial. Since all individuals hold strictly positive portfolios, trade is eminently possible. Furthermore, no one individual controls any firm so that coalitions of individuals can revise firms' plan--if they so desire and command sufficient power.

Optimality

In general, except under Diamond's [1967] multiplicative conditions or the unanimity conditions of Ekern and Wilson [1974], competitive stock ownership equilibria will fail to be constrained Pareto optimal. The interested reader may wish to consult Forsythe and Suchanek [1982], Stiglitz [1972, 1981], and Jordan [1978] for detailed discussions concerning Pareto optimality in stock market economies. Dreze [1974] also presents an interesting example of an equilibrium in his model which is not optimal. At least holding firms plans fixed, an exchange equilibrium of the stock market is a constrained Pareto optimum--although stockholders typically will not share the same implicit prices for state-contingent consumption.

The above considerations notwithstanding, it should not be surprising that after imposing voting direction restrictions that the equilibria in this model may fail to be constrained Pareto optimal. See Slutsky [1977] and Kramer [1972] regarding the relationship between direction restrictions and optimality. It is interesting to observe that the direction restrictions appear to be related to Grossman's [1977] notion of Social Nash Optimality, in which a planner is constrained to a finite number of allocational activities. This relationship remains a topic for future research.

## B. LITERATURE REVIEW

A very extensive body of literature comprised of over fifty papers in major Economics and Finance journals has evolved regarding both the existence and optimality of stock market equilibria, and the objective of the firm in incomplete financial markets. This review is not an attempt to summarize and critique all of this literature; indeed such a review would represent a monograph in itself. Rather, a number of carefully selected papers, which are especially relevant to the present results, will be reviewed. Each of these papers has generated significant interest, and should be carefully studied before undertaking research in this field. The papers which will be reviewed are: Diamond [1967], Dreze [1974], Gevers [1974], Ekern and Wilson [1974], Jordan [1978], and Benninga and Muller [1979].

### a. P. Diamond, "The Role of a Stock Market in a General Equilibrium Model."

Diamond's [1967] prescient paper was the first work in which firms' production decisions were examined in the context of an incomplete stock market equilibrium. In his extension of the Arrow-Debreu model there is not a complete set of contingent claim contracts. Individuals can only insure incompletely against future uncertainty through share ownership.

Diamond's model employs a generalized form of multiplicative uncertainty, termed stochastic homotheticity. In this framework a firm can never produce a new commodity. Every feasible output proposal for a given firm turns out to be a pattern of state-contingent income which is already available and priced in equilibrium (as a linear combination of existing shares of other firms). That is, every firm production plan leaves unchanged the feasible set of state-contingent incomes available to shareholders in the economy. It turns out that shareholder unanimity regarding production plans is assured in Diamond's model. Unanimity conditions will considered further in the discussion of the Ekern and Wilson [1974] model below.

Equilibrium in Diamond's model satisfies a novel optimality criterion--introduced in the paper--termed "constrained Pareto optimality." The sense of the constraint is that, just as participants in the economy cannot open complete markets, a central planner cannot construct arbitrary patterns of state-contingent incomes from a complete set of Arrow-Debreu securities. Thus the set of feasible centrally planned allocations with which a given equilibrium might be compared is restricted to the subspace available through share ownership.[18] Stiglitz [1982] commented concerning Diamond's optimality criterion as follows:

> In the case of a simple stock market economy with a single commodity . . . the notion of constrained Pareto optimality was introduced by Diamond [1967]. In an economy with a stock market, each individual purchases (his preferred) fraction of each of the firms. In addition, the individual may borrow or lend at the safe rate of interest. Diamond considered an economy in which these were the only securities allowed, and contrasted the market allocation where firms maximized their stock market value, with that where the government allocated all resources, but was severely restricted in its ability to distribute the output of the economy: each individual received a linear function of the output of the different firms. He then showed that, (a) if the technology of each firm exhibited stochastic homotheticity (so that the ratio of the output in any two states was independent of scale); and (b) each firm believed that its market value was proportional to its scale, then the market equilibrium would be a constrained Pareto optimum.

Stiglitz [1982] and Hart [1975] later showed that in general Diamond's result cannot be generalized; the stock market allocation of resources is typically *not* a constrained Pareto optimum when there are two or more outputs.

Near the end of his paper Diamond relaxes the decomposability assumption, so that spanning no longer holds in the model. In a footnote citing Mirrlees, he suggests that a firm manager acting in the interests of final shareholders could achieve

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18. Hereafter, "Pareto optimal" should be understood to mean "constrained Pareto optimal" in Diamond's sense.

a constrained Pareto optimum by maximizing a weighted sum (using post-trade final shareholdings as weights) of shareholders' private valuations of the firm's future production plan, where each shareholder uses his marginal rate of substitution vector between present and future consumption to discount future profits.

Diamond suggests that the manager might simply poll the shareholders to elicit this private information, but does not address the issue of truthful revelation. Concern regarding the incentive thus provided for stockholders to misrepresent their true marginal evaluations is cited by several authors as a prime factor motivating their research. See, for example, Forsythe and Suchanek [1984].

b. J. Dreze, "Investment Under Private Ownership: Optimality, Equilibrium and Stability."

The model developed here was directly inspired by Dreze's [1974] widely cited paper. It was decided to retain the as much of the Dreze framework as possible, except that the Lindahl mechanism for making firm production decisions has been replaced by a majority rule process. It was originally thought that this program would be a relatively straightforward exercise; such was not the case. However, it was possible to keep the basic structure of the Dreze model intact without significant additional assumptions, except for the strict no short sales and distinguished goods requirements.

Dreze mentions that his model represents an attempt to extend the stock market results of Diamond [1967] to a more general form of technological uncertainty. While Dreze employs the simple two-date stock market / production model formulated by Diamond; firms are not restricted to his form of multiplicative uncertainty. Indeed, firms are permitted to have very general production technologies. Production sets must be compact and convex, firms' cost function must be differentiable and strictly monotonic, and positive input is required for positive output.

Perhaps the deepest insight contained in Dreze's paper is the separation of firm production planning and share trading. Another important technical discovery reported in the paper was that the space of allocations available through share ownership may fail to be convex. This non-convexity presents significant technical difficulties.

Dreze defines a so-called ***stockholders' equilibrium*** as a price equilibrium for consumers (holding firms' plans constant) and a Lindahl[19] equilibrium for firms (keeping shareholdings constant). The nature of the Lindahl equilibrium is carefully described in the paper. It is interesting that Dreze does not comment in detail regarding the revelation problem associated wtih such a system of production decision-making. However, in a footnote he mentions Gevers' concern regarding the Condorcet paradox in majority-rule firm planning.

> The definition of a (Lindahl) equilibrium for the firm states that the firm maximizes the present value of its production plan, using the shadow prices ... obtained as weighted averages of individual shadow prices ... reflecting the consumption preferences of the shareholders with the weights given by their respective ownership fractions.* (By theorem) efficient production decisions by the firms imply the existence of such prices.
>
> The definition does not place any restrictions on the allocation among consumers of the adjustments in current consumption required to offset the adjustment in input level ... Alternatively stated, the definition is consistent with arbitrary transfers of initial resources among consumers.
>
> * See the paper by Gevers [1974] on the remote possibility of obtaining these shadow prices, through majority voting ...

Forsythe and Suchanek [1982] comment as follows regarding production financing and the absence of initial endowments in Dreze's model:

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**19.** See Johansen [1963] for a thorough discussion of the Lindahl equilibrium concept.

> Dreze did not consider directly the question of production financing. In his formulation, consumers are not associated with initial endowments or initial shareholdings, nor are they identified with investment commitments that finance firms' production activities. Rather an aggregate social endowment (of a single non-storable production / consumption good) is specified from which firms' inputs are subtracted and the residual distributed among shareholders to yield either a portfolio or Lindahl equilibrium. To prove the existence of a stockholders equilibrium, however, Dreze demonstrates the convergence of a mixed tatonnement-nontatonnement process.

At each step, the process is shown to yield a Pareto result is and constrained to be individually rational; thus individuals' utility functions can be employed to construct a Lyapunov function--which drives the convergence result. However, it should be noted that Dreze gives examples of stockholders' equilibria which fail to be either technologically efficient or constrained Pareto optimal.[20]

c. L. Gevers, "Competitive Equilibrium of the Stock Exchange and Pareto Efficiency."

Gever's [1974] paper analyzes the question, "Under what conditions is the allocation of resources through a competitive stock exchange Pareto-efficient in a suitably restricted sense?" Gevers acknowledges that his model closely resembles that constructed by Dreze, although some modifications to Dreze's myopia assumptions are required since Gevers considers shareholders' prediction of the value of production changes. Perhaps the easiest way to describe Gevers' model is to quote from the introduction to his paper:

> The paper is not concerned with the question of existence. It is easy to show, by means of examples, that the set of competitive allocations that can be sustained by a stock exchange is nonempty. However, no general proof of existence is offered (here). Some counter-examples based on the Condorcet paradox can be found in the appendix.
>
> ... Section II is concerned with technological uncertainty when separate markets for contingent goods are lacking. ... the various outputs of the firms are interpreted as physically identical goods, which are associated with several a priori possible states of the world. Technological choices made at

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20. Additional discussion of Dreze's [1974] paper can be found in the Introduction preceding this section.

period one, and the state of the world that obtains at period two, determine together the amount of output which is actually delivered.

The model rests on two main assumptions. The first one generalizes the customary assumption of free competition; when agents choose their portfolio, they take the price of every share as a datum; moreover, they do not perceive any influence of their portfolio choice on production plans of firms.

The other assumption pertains to the market valuation of a share as it is perceived by the firm's decision-makers. The latter choose a production plan and the number of shares that must be floated in order to finance it. When there is a market for every contingent good, the price of a share must be equal to the value of the dividends and thus must depend on the production plan. When no such markets exist, decision-makers may be assumed to believe that the observed price of a share would change in a definite fashion if they selected another production plan.

This change in market valuation is in the nature of a conjecture; it is less so when short sales are permitted, and there are at least as many firms with linearly independent output vectors as there are states of the world, or when output ratios are fixed. In the latter case, meaningful aggregates can be defined; in the former case, each unit of the contingent good may be priced separately, albeit indirectly.

Except for some unlikely cases ... stockholders generally disagree about the optimal production plan. ... The literature suggests two ways of dealing with this problem (in group decision). Following one trend, one may consider the firm manager as a 'dictator' who selects that production plan which suits him best. The model may then be closed by exogenously assigning a manager to every firm. In a more traditional perspective, one may assume that the manager executes what the stockholders decide through majority voting. Further assumptions, under which these decisions result in an allocation that satisfies the first-order conditions for a constrained Pareto optimum, are described in detail. They are all highly unlikely to be satisfied exactly in reality.

The model presented in section II of Gevers paper is similar to, although not directly comparable with, the model developed in this paper. In particular, he permits the flotation of new shares to finance production, while such financing is not allowed here. His model is also concerned with properties of price equilibria in an incomplete stock market; thus, as mentioned above, some specialized assumptions must be adopted to describe how individuals form opinions about the effect of production plan changes on stock prices.

These differences aside, Gevers *does* consider the possibility of production plan selection by share-weighted majority rule. His chief concern is whether or not a stock market / majority rule equilibrium is constrained Pareto efficient. He concludes that optimality will not obtain except under extremely specialized circumstances.

While Gevers does not directly focus on with the problem of existence, he mentions the "Condorcet Paradox" several times, and gives an example of an economy in which no equilibrium majority rule firm plan can be determined. Gevers' model is sufficiently similar to the model developed here that the reasons for non-existence in his example must be investigated.

In the example, presented in section (3) of the Appendix to his paper, there are three individuals, three states of the world, and one firm. Each individual owns a one-third share of the firms' output, and Gevers supposes that, "stockholders agree on some level of investment $a$, and they are discussing the output mix." The production function in the example is of the form $b_1+b_2+b_3 = a^5$, and so for a fixed level of investment, by appropriate choice of units, the "transformation locus" can be represented by the simplex in $\mathbb{R}^3$.

Gevers goes on to restrict new proposals, in the example, to differ from the status quo only in a direction parallel to one of the edges of the simplex. He points out that each of the three individuals will have a most preferred production point in the simplex. Next, the following construction is suggested: For each vertex of the simplex, draw three lines from the vertex--each starting at the vertex passing through one of the individuals' optima and ending at opposite edge of the simplex. Thus, associated with each vertex is a "median line." Gevers claims, "If the median lines drawn from each vertex do not have a common intersection, ... there is no equilibrium production plan."

Although no proof is offered for this claim, suppose that it is true. The relevant question is then: Can Gever's non-existence example be translated to the model of this paper? Fortunately, such translation is not possible. His example is really an instance of a two-dimensional policy space, with *three* voting directions--clearly these directions cannot be linearly independent. However, the Kramer direction-restricted majority rule mechanism assumed in this paper requires that there be no more voting directions than the dimension of the policy space. Hence the instance of majority rule cycling identified by Gevers cannot occur under the scenario constructed here.

**d.** S. Ekern and R. Wilson, "On the Theory of the Firm in an Economy with Incomplete Markets."

The Eckern and Wilson [1974] paper was the first of the so-called unanimity or spanning papers. Eckern and Wilson extended the "no new goods" intuition inherent in Diamond's earlier work without having to make the same sort of restrictive technological assumptions. In this respect, the Ekern and Wilson program cannot be considered an unqualified success; for although their spanning condition is weaker than Diamond's separability assumption, spanning is still an extremely limiting assumption, which is unlikely to satisfied in any real-world environment. Radner [1974] comments concerning the spanning condition and its consequences, as follows:

> Roughly speaking, one can paraphrase the results of (the Eckern and Wilson paper) as follows. Suppose that the production possibility sets of all producers span a linear subspace of distributions across the states of the world, and that an equilibrium of the stock market and choice of production plans has the property that the equilibrium production plans span that subspace. Then the ex ante (endowed) stockholders in each firm are unanimous in their preferences among alternative production plans, and the ex post (equilibrium) stockholders of each firm are unanimous in their preferences among (local) directions of change from the equilibrium production plan.

The Ekern and Wilson paper gave rise to an entire class of papers collectively known as the "spanning" literature. Forsythe and Suchanek [1981] summarize this literature in the introduction to their paper.

> The general problem concerns firms' choices of production plans and investors' choices of a portfolio of firms' revenue shares in the absence of a complete set of contingent securities markets. ... (One approach to the problem) is to establish conditions under which firms can choose production plans with the unanimous approval of their stockholders. ... Unanimity prevails ... if and only if the Ekern and Wilson [1974] spanning condition is satisfied. Loosely stated, the spanning condition requires that any small adjustment in stockholders' returns achievable by altering firms' production plans must also be achievable by portfolio changes. In short, the set of available state-distributions of returns cannot be affected by firms' decisions. That is, the value of any change in the production plans must equal the cost of making the portfolio change. Since the latter cost is calculated from observable market values, it is the same for all stockholders. Therefore, each firm's manager (assuming he is a stockholder) can use his own preferences when selecting a production plan, and an efficient allocation will be obtained. But the spanning condition is highly restrictive since it is not satisfied in many nonpathological economies.

Eckern and Wilson provide a proof that unanimity necessarily obtains if shareholders are only concerned with the mean and variance of their portfolios. However, this result should be interpreted with care, since mean-variance utility formulations are consistent with expected utility maximization only in the case of random variables with no more than two moments.

It is fair to say that while the spanning literature generated considerable technical interest, by and large the academic finance community came to regard the restrictive conditions required for unanimity results as exceptionally unrealistic. The spanning assumptions of Ekern and Wilson and Radner essentially require that firms cannot produce anything new. Grossman and Stiglitz [1976] have prepared a detailed critique of this assumption.

e. J.S.Jordan, "Investment and Production in the Absence of Contingent Markets I."

Jordan's [1978] paper is devoted to the institution of majority control of firms by investors, and the problem of achieving constrained Pareto optimal allocations through such institutional arrangements. As he states, the paper is a response to the highly restrictive conditions "under which the firm can choose a production plan with the unanimous approval of its shareholders." Jordan constructed a model of great simplicity and generality which admits an example of a stock market economy for which no majority rule equilibrium production plan exists--and so is particularly relevant here. In light of the difficulty in obtaining this mimeo, it should prove convenient to quote at some length from the introduction to the paper.

> There are two periods, present and future, and in the future there are several equiprobable states. There is a single commodity in the present and in each future state. The present commodity is used as an input in the production of the future commodity by a single firm whose technology has constant returns to scale. Thus the alternative production plans can be summarized as a set of activities, where each activity consists of a vector of state-dependent outputs obtained from one unit of input. There are several potential investors, each of whom is described by an endowment of the commodity in the present and in each future state, and a state-independent utility function. Each investor is assumed to know that the future states are equiprobable, so investors are homogeneous with respect to beliefs, but not with respect to endowments and utility functions. The decision problem is to select a single activity and each investor's investment. An investment consists of an amount of the input provided by the investor, which entitles him to the corresponding amount of output in each state determined by the activity. Put somewhat differently, one 'share' in the firm is defined as entitling an investor, in each state, to the output derived frome one unit of input. Since there are constant returns to scale, this definition is unambiguous. In return for each share, an investor contributes one unit of the present commodity. An investment-production plan is Pareto optimal if there is no feasible alternative plan which is strictly preferred by some investor and not less preferred by any other investor. A mechanism is said to be efficient if it selects Pareto optimal plans.
>
> Gevers ... (has) observed that corporate majority rule, in which each investor has one vote per share, is subject to the same Condorcet paradox which arises in the general social decision problem. That is, given any activity y chosen by the firm, there may exist an activity y' which is preferred by a coalition of investors with more than 50% of the shares. However, ... this is not a definitive objection to majority control since it ignores the informational and organizational impediments to the formation of majority coalitions. In order for the activity y to be effectively opposed, a majority coalition of investors must locate one another and agree to support an alternative activity y'. Since the Condorcet paradox depends on the existence of another potential majority coalition which opposes y', such an agreement is extremely problematic.
>
> If the majority consists of a single investor, these informational and organizational difficulties vanish. Accordingly, ... we will refine the concept of majority control to the concept of a 'controlling interest.' Given a mechanism for selecting investment-production plans, a controlling interest is defined as a fraction, strictly between 0.5 and 1.0, such that if a single investor owns more than that fraction of the total stock, the chosen activity must be his most preferred activity. For example, if a corporate charter specifies the election of directors in such a way that an investor who owns 90% of the total stock can nominate and elect the entire board, then .9 could be interpreted as a controlling

interest. However, it will be proved ... that even this extreme refinement of majority control is logically inconsistent with Pareto optimality.

... Since there are no contingent markets, each agent's future consumption must equal his endowment plus his share of the output. Secondly, only one activity can be chosen (by the single firm), so that all investors must invest in the same activity. This technological constraint causes the activity to have a public good aspect. In the absence of this second constraint, each investor could independently invest in his own most preferred activity, and the model would formally reduce to the trivial case of household production. We also assume that the current commodity is not redistributed among the investors. This exclusion of 'sidepayments' means that for any investment production plan, ... the ith investor's 'net trade' ... can be interpreted as a stock market transaction, where the price of one share in terms of the present commodity is one.

The convexity of the activity set and the concavity of the expected utility functions leads immediately to the first-order necessary conditions for optimality ... . Since returns to scale are constant, once the activity is chosen each investor's level of investment should maximize his expected utility.

Jordan points out that the example of the Condorcet paradox presented by Gevers does not apply directly in his model, and then he prodeeds to develop such an applicable example. The example is based on three individuals, three states, and an activity set bounded by the "simplex" in $\mathbb{R}^3$ of all non-negative vectors which sum componentwise to 6. The three investors are constructed to have identical utility functions, but have endowments which differ by a permutation of states. It is finally shown for any given technologically efficient activity that, " ... if each investor is permitted his most preferred level of investment, ... there exists another activity which is preferred by at least two investors who hold a majority of the shares (of the given activity). Jordan remarks:

It should be emphasized that the majority rule paradox depends on each investor's ability to purchase his most preferred number of shares at a price of one unit of the present commodity per share. For example, suppose that investor 1 has historically been the owner and sole investor, so the activity is $y^1$, when investors 2 and 3 arrive and request shares. Investor 1, in order to retain control, could refuse to sell shares or could sell at a price sufficiently high that investors 2 and 3 would demand less than a majority interest. (However) ... both strategems are inconsistent with Pareto optimality.

While Jordan's model differs from that developed here, there are sufficient similarities so that the possibility of the present model admitting a related non-existence example calls for examination. It turns out that there is more "friction," or restriction of individuals' ability to change the state of the economy, built into this model. Such restrictions yield existence results which otherwise might not obtain. These frictions take three forms: i) the direction and non-bankruptcy restrictions imposed on firm plan revisions; ii) the separation of exchange and plan revision, which prohibits simultaneous adjustment of plans and shareholdings; and iii) individuals' myopia during exchange and planning.

Jordan's majority rule counter-example supposes some single constant returns activity (a pattern of production across states), say y, has been identified. Assume that individuals have contributed one unit of the current good for each unit of y,[21] and that each individual has selected his optimal level of investment with respect to the activity, given his endowment and preferences. Then, as Jordan shows, there is some other activity y', such that at the (new) optimal individual contribution levels with respect to y', a majority of shareholders (measured at the share weights associated with y) will prefer y' to y.

Contrast this situation with the current model. Suppose that with respect to an activity (production plan) y an exchange equilibrium has been attained; that is, the aggregate social endowment having been apportioned between current consumption and investment, some Pareto undominated allocation of shares and the current good has been achieved. Next, during the production plan revision process--*which keeps shareholdings and cost shares constant*--shareholders can only propose departures from the status quo which i) lie along one of the exogenously specified voting directions,

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**21.** This notion is well-defined by the constant returns assumption.

and which ii) do not bankrupt any current shareholder due to revised input obligations. Furthermore, voters are myopic in the sense that they do not forsee the possibility of trading their revised shares. Under these restrictions, a state that would not be stable in the Jordan framework might well be a shareholders' equilibrium.

Thus, existence of some other activity y' which would be preferred by a share-weighted majority of current shareholders is not directly relevant regarding the existence of equilibrium in the model presented here. As discussed above, partici - pants in this model cannot change production plans and levels of investment with the same degree of freedom allowed in Jordan's framework--while activity y' and the associated optimal personal levels of investment might appear attractive, this state simply might not be attainable in the present model. These considerations do not accord with the conditions set forth in Jordan's example.

Jordan nexts constructs a generalization of the share-weighted majority concept which he terms a "controlling interest", and it is shown that this mechanism, too, is inconsistent with Pareto optimality. He briefly summarizes these results by asserting, "The controlling interest paradox states that Pareto optimality cannot be generally acheived by any decision mechanism which permits an investor to control the firm if he owns a sufficiently large percentage of the equity." Further consideration of the controlling interest concept is left for the interested reader.

Finally, Jordan observes that (at least in academic literature), "The acquisition of a controlling interest is not the only way in which an investor's influence can increase with his relative shareholding." He points out (as does Dreze) that in Dreze's framework, where production plans are chosen to maximize an average of investors' marginal rates of substitution, equilibria may also exist which are not constrained Pareto optimal.

f. S. Benninga and E. Muller, "Majority Choice and the Objective Function of the Firm Under Uncertainty."

At first glance, Benninga and Muller [1979] might appear to have obtained an equilibrium solution for a general equilibrium production model in which shareholders make multi-dimensional production decisions by majority rule vote. However, upon closer inspection it becomes apparent that the paper essentially fixes firms' current production pattern and restricts the shareholders' problem to the one dimensional question of setting production levels by choosing the level of retained earnings. Furthermore, these authors impose a spanning condition which is even stronger than that employed by Eckern and Wilson.

As Winter [1981] points out in his critique of their paper, the Benninga and Muller results concerning majority rule equilibrium are trivial consequences of Eckern's and Wilson's earlier unanimity results. Furthermore, Winter asserts--and Benninga and Muller [1981] agree in a later rejoinder--that the timing scenario in their model is both more complex and less general than the Arrow-Debreu framework. In particular, resources are *not* committed to production before the resolution of all uncertainty. In conclusion, the Benninga and Muller paper, its provocative title notwithstanding, does not demonstrate equilibrium of a multi-dimensional shareholder majority rule decision problem in the standard stock market / production framework.

## II. THE MODEL

DEFINITION (Stock Market Economy $(X_o, U, C, \zeta)$ )

Let there be given:

- a finite set $I = \{1, \dots, I\}$ of individuals,
- a finite set $J = \{1, \dots, J\}$ of firms[1],
- a two element index set $\{0, 1\}$ of dates, and
- a finite set $S = \{1, \dots, S\}$ of mutually exclusive and collectively exhaustive second date states of nature.

Suppose on the first date that none of the individuals knows with certainty which of the states of nature will be realized on the second date.

Given such an environment, define a stock market economy to be a four-tuple of the form $(X_o, U, C, \zeta)$. The various components of the economy should be interpreted in the following manner:

i) $X_o$ is a non-negative quantity of a non-storable date 0 consumption/investment good; this good will also be referred to as the current good. $X_o$ may be regarded a s an aggregate social endowment.

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1. A fixed number of firms has been assumed to preclude the possibility of innova - tion or the entry of new firms. This is a standard assumption in the stock market equilibrium literature; see, for example, Diamond [1967], Dreze [1974], Ekern and Wilson [1974], and Sharpe [1964]. The assumption is required here for technical feasability. The model is already quite complex. The possibility of obtaining results appears remote if phenomena as intractable as innovation and entry were admitted. In defense of the assumption, it seems entirely reasonable to suppose there are fewer firms than states of the world--and this is really the key issue here.

The endowment $X_o$ can either be consumed today by the economy's participants, or else invested in the J firms' technologies to yield returns of a single second date consumption good across the future states of nature. Investment decisions must be taken, and units of $X_o$ committed, on date 0-- before the second date state of nature is known.

ii) $U = (U_1, \dots, U_I)$ is a $(1 \times I)$ array of utility functions indexed by the set $\{1, \dots, I\}$ of individuals. Each element of this array is a map of the form $U_i: \mathbb{R}^{(1+S+J)}{}_+ \longrightarrow \mathbb{R}_+$. In order to simplify notation, if $q = (q_1, \dots, q_I) \in \times_I(\mathbb{R}^{(1+S+J)}{}_+)$, then $U(q)$ will be written for the $(1 \times I)$ vector $(U_1(q_1), \dots, U_I(q_I))$.

Each individual i makes first date decisions to maximize the utility associated with three distinct consumption activities:

1) consumption of the date-0 or current good, a non-negative 1-vector,

2) contingent consumption of the date-1 produced good across the S future states of nature, a non-negative S-vector, and

3) consumption of J "distinguished" goods indexed by the set of firms, a non-negative J-vector.

Notice $\mathbb{R}^{(1+S+J)}{}_+$ is the domain of each utility function $U_i$. Regarding the second consumption activity, consider that actual physical consumption only occurs in the state of nature which happens to be realized on the second date. Notice, too, that the utility framework which has been adopted here is general enough to include expected utility maximization as a special case. Finally, concerning the third consumption activity, it may be convenient to think of the distinguished goods as being consumed on the second date. Formally, however, the timing of their consumption is of no particular consequence; furthermore the aggregate quantities of the distinguished goods can be arbitrarily small. These goods will be discussed in greater detail in later sections.

iii) $C = (C_1, \ldots, C_J)$ is a $(J \times 1)$ array of convex cost[2] functions indexed by the set $\{1, \ldots, J\}$ of firms. Each component is a map of the form $C_j: \mathbb{R}^S_+ \longrightarrow \mathbb{R}_+$. For convenience, if $b = (b_1, \ldots, b_J) \in \times_J(\mathbb{R}^S_+)$, $C(b)$ will be written for $(C_1(b_1), \ldots, C_J(b_J))$.

If $b_j \in \mathbb{R}^S_+$ denotes a vector of quantities of the second date consumption good across the states of nature, $C_j(b_j)$ is defined to be the minimum quantity of the date 0 production/consumption good required by firm j to produce $b_j$. Notice that $C_j(b_j)$ is always non-negative.

For the purposes of the model to be developed here, a firm's cost function is assumed to constitute a complete description of the firm's technological possibilities. Firms' technologies are assumed to be proprietary and non-transferable. Furthermore, the set of firms and technologies is assumed to be fixed; there is no technological innovation, and no entry of new firms.

iv) $\zeta = (\gamma, \delta)$ is an exogenously set array of parameters. $\gamma$ is an $(I \times J)$ matrix of minimum shareholdings. Thus $\gamma_{ij}$, where $0 \le \gamma_{ij} \le 1$ and $\mathbb{1}_J \cdot \gamma^j = \Sigma_{[i=1,..,I]}\, \gamma_{ij} \le 1$, $\forall\, j \in J$, denotes the minimum (percentage) shareholding individual i is required to maintain in firm j. Notice that the required minimum holdings are non-negative (short sales are prohibited), that no individual is required to hold more than 100% of any firm, and that the sum of the required holdings across individuals is less than 100% for any given firm. The $\gamma$ matrix plays a technical role in the stock market economy model to be developed here; certain results will require that every individual maintain an arbitrarily small, but strictly positive, holding in every firm.

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2. The assumption of convex costs is equivalent to the assumption of non-increasing returns to scale. This assumption is standard in literatrure concerning equilibrium in stock market economies; see, for example, Diamond [1967] and Dreze [1974]. The reader is referred to Arrow and Hahn [1971] for a general discussion of the role of the non-increasing returns assumption in microeconomic production theory.

A firm's shares entitle their owner to both a constant fraction (across the states of nature) of the firm's output of the date-1 good, and to the same fraction of the "distinguished" good distributed by that firm. Individuals can consume the date-1 good and the distinguished goods only by owning shares of firms. The share - holders of a given firm vote for and finance the firm's date-1 production. Additonal characteristics of firms' shares will be discussed in later sections.

$\delta$ is a $(J \times J)$ diagonal matrix of firms' distinguished goods endowments. In particular, $\delta_{jj}$ denotes the quantity of the $j^{\underline{th}}$ distinguished good endowed to the $j^{\underline{th}}$ firm. Notice that since $\delta$ is diagonal, the $j^{\underline{th}}$ distinguished good is endowed only to the $j^{\underline{th}}$ firm. Each distinguished good is unique; the distinguished good associated with the $j^{\underline{th}}$ firm is distinct from that associated with the $k^{\underline{th}}$ firm. Finally, it is important to emphasize that each firm's distinguished good endowment is exogenously determined, and is not a decision variable for the economy's participants. It is not possible for a firm to produce additional quantities of any distinguished good.

The distinguished goods play a technical role in the model; certain results require the assumption that each firm j produce an arbitrarily small, but strictly positive, quantity of the $j^{\underline{th}}$ distinguished good. For convenience, the notation $\delta >> 0$ will be employed to denote that the diagonal elements of $\delta$ are strictly positive.

The stock market economy construct is a highly stylized model of real world allocational institutions characterized by stock market trading and shareholder voting. The next step in the development of the model is to define a convenient representation for a configuration or state of the economy.

DEFINITION (State Space $Z_\zeta$)

Let $(X_o, U, C, \zeta)$ be a stock market economy and define the associated state space $Z_\zeta$ according to:

$Z_\zeta = \{ (x, b, \theta) \in \mathbb{R}^{(I+JS+IJ)}{}_+ \mid$

i) $x \in \mathbb{R}^I{}_+$, where x is an $(I \times 1)$ row vector,

ii) $b \in \mathbb{R}^{JS}{}_+$, where b is a $(J \times S)$ matrix,

iii) $\theta \in \mathsf{X}_J S^{(I-1)}$, and $\theta \geq\geq \gamma$, where $\theta$ and $\gamma$ are $(I \times J)$ matrices; and

iv) $x \cdot \mathbb{1}^I + \mathbb{1}_J \cdot C(b) = X_o. \}$ .

A ***state*** $z = (x, b, \theta) \in Z_\zeta$ should be interpreted as a complete description of the stock market economy $(X_o, U, C, \zeta)$. The notation U(z) will be adopted as an abbreviation for $U(x, \theta \cdot b, \theta \cdot \delta)$ if there is no possibility for confusion.

While a particular state does not explicitly specify each individual's second date consumption, this information is well defined by a given state-- as will be described below. Finally, it should be pointed out that the word "state" will be used here in two contexts: "state z of the economy," and "state s of nature." It seems only a remote possibility that any confusion might arise from this dual usage.

$x = (x_{o1}, \ldots, x_{oi}, \ldots, x_{oI})$ denotes an allocation of the date-0 or current consumption/investment good across individuals. Notice that $x \in \mathbb{R}^I{}_+$; condition i) reflects the assumption that negative quantities of the current good cannot be consumed.

$b = (b_1, \ldots, b_j, \ldots, b_J)^T$ is a $(J \times S)$ matrix of firms' production plans. In the case of firm $j \in J$, for example, $b_j = (b_{j1}, \ldots, b_{js}, \ldots, b_{JS})$ specifies the quantities of the second date consumption good to be produced by firm j, contingent upon the realization of the various possible states of the world. Notice the general form of this technology; in particular a given firm is not restricted to simply decide the scale of some exogenously determined pattern of production across states. Indeed, the production plan of a given firm may represent an aggregate of several distinct productive activites or enterprises conducted by the firm. Observe that, as set forth under condition ii) above, no firm can produce a negative quantity of the second date consumption good in any state of the world.[3]

$\theta$ is an $(I \times J)$ matrix which specifies individuals' shareholdings of the various firms. Short sales are prohibited, the sum of all shares across individuals is 100% for every firm, and the minimum holdings requirements must be maintained.

Condition iv) in the definition above is an accounting identity which specifes that the total quantity, summed across individuals, of the date-0 consumption/production good consumed plus the total quantity, summed across firms, invested in production must equal the aggregate social endowment $X_o$. There is no loss of generality in expressing this condition as an equality, rather than an inequality, since interior states attained by free disposal of the consumption/production good will never be realized when U is componentwise monotonic.

A state specifies each individual's consumption. Recall that three types of consumption are possible: consumption of the current good, contingent consumption of the second date produced good, and consumption of the distinguished goods. The stock of the current good held by each individual is specified directly as the first coordinate of a state. Now, a state specifies both the shareholdings of each individual in each firm, and firms' plans-- and so well defines each individual's contingent con - sumption of the produced good. Finally, since consumption of the distinguished goods

3. Professor Strnad has pointed out that this assumption obviates the chief advantage of the corporate form of firm organization--owners' limited liability.

is proportional to holdings, the shareholdings information specified by a state together with the exogenously specified vector of distinguished goods suffice to determine individuals' distinguished goods consumption.

REMARKS (Mechanisms for Changing the State of the Economy)

The model admits two mechanisms whereby individuals can change the state of the economy on the first date: a market mechanism, and a voting mechanism. The action of these mechanisms is restricted so that one can function only while the other is suspended. The notion of one-on / one-off exclusivity in an economy with dual allocational processes appears to have been first employed by Dreze [1974]. The workings of the two mechanisms will be discussed informally below, in anticipation of formal definitions which follow later in the text.

Let a state z of the economy be given. In the market mechanism, keeping firms plans fixed, individuals may exchange their holdings of shares in the J firms and their stocks of the current good.-- thus arriving at some new state with the same firm plans as the original. No individual will be permitted to hold negative quantities of the current good or to sell shares short. Indeed, every individual must maintain a strictly positive minimum holding in each firm, as specified by the exogenously given matrix $\gamma$. The model of trading employed here is highly stylized: It will be assumed that given a state z, the process of exchange generates some state which is both i) Pareto optimal among all states with the same plan matrix as z, and ii) individually rational, that is no individual is worse off than under z.

Again, let some state of the economy be given. The second mechanism for change allows shareholders to change firms' production plans, while keeping their ownership fractions of the various firms fixed. Such changes could involve adjusting the quantitiy of the current good committed to production, the output patterns produced by firms across the states of nature--but not the level of investment, or both. Reflecting real world corporate practice, it will be assumed here that changes in a firms' plan must be approved by a share weighted majority of its shareholders. This voting procedure will be constrained by two requirements, one of them a feasibility condition and the other imposed for technical reasons. Both conditions are implemented as restrictions on new proposals to depart from the status quo z.

First of all, changes in a firm's plan will require adjustment of the quantity of the current production / consumption good. At a given state z, each firm possesses exactly a sufficient quantity of this good to undertake its plan as specified by z. If shareholders decide to revise the plan, depending on whether the new plan is more or less ambitious than the original under z, they must make additional share weighted contributions of the current good from their personal stocks (specified under z), or they will recieve share weighted refunds. In order to avoid potential embarassment of other shareholders, no plan to depart from the status quo may be proposed which would require a share weighted contribution sufficient to bankrupt any shareholder of record. Admittedly this restriction rules out various welfare improving schemes such as lending among shareholders, or new equity offerings--but the model is sufficiently delicate that such extensions are better postponed at least until some preliminary results have been achieved.

The second proposed restriction has been adopted purely for technical reasons, in particular to circumvent the tendency for multidimensional majority rule mechanisms to cycle. Perhaps the least artificial device for ensuring the existence of majority rule equilibria is so-called direction restricted majority rule proposed by Kramer [1972]. Under this scheme, a collection of linearly independent voting directions, of rank equal to the dimension of the policy space, is exogenously specified. Proposals for policy change can differ from the staus quo can only along one of these voting directions. In the problem under discussion, a natural collection of voting directions is apparent: A proposal to change firms' plans can suggest revising at most the production of one firm in one state of the world.

It must be admitted that direction restrictions *are* restrictive. However, in the case at hand the departure from real world practice may not be as significant as might appear at first consideration. In particular, one of the kinds of plan revisions ruled out by the direction restrictions in the form adopted here is a coordinated revision of plans by several firms. This would of course require a vote of the combined shareholders of the several firms involved, a corporate event unlikely to meet with approval of antitrust regulators. The real cost of the direction restrictions insofar as accurate reflection of real world practice is concerned would seem to be the prohibition of production tradeoffs across states, and scale adjustment of a production pattern by a single firm. Yet, results cannot be obtained without assumptions; higher resolution stock market models appear to require more powerful majority rule existence theorems than are currently known.[4]

<u>DEFINITION</u> (Exchange Equilibrium for a Stock Market Economy)

An ***exchange equilibrium*** for a stock market economy $(X_o, U, C, \zeta)$ is a state $z = (x, b, \theta) \in Z_\zeta$ such that there is no other state z' which is both: Pareto optimal over the subspace $Z_\zeta|_b$ of all states having production plan b, and also individually rational with respect to z. More formally, there is *no* $z' \in Z_\zeta|_b$ which satisfies both:

i) not $\exists\ z^\dagger \in Z_\zeta|_b$ such that $U(z^\dagger) >> U(z')$, and (Pareto Optimality)

ii) $U(z') \geq\geq U(z)$. (Individual Rationality)

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4. Professor Strnad has suggested that limiting the voting powers of owners and adding management to the model might be a promising direction for future research.

Both of these criteria can be subsumed by the simpler condition:

$$\text{not } \exists\, z' \in Z_\zeta|_b \text{ such that } U(z') >> U(z).$$

DEFINITION (Voting Equilibrium for a Stock Market Economy)

A ***voting equilibrium*** for a stock market economy $(X_o, U, C, \zeta)$ is a state $z = (x, b, \theta)$ $\in Z_\zeta$ which, keeping the stock ownership matrix $\theta$ fixed, cannot be defeated by any other (direction restricted) state $z^\# = (x^\#, b^\#, \theta) \in Z_\zeta|_\theta$ in a share weighted standard majority rule vote by the shareholders of the firm facing the proposed plan revision.[5]

To be more precise let a standard basis $\{e_{11}, \ldots, e_{js}, \ldots, e_{JS}\}$, or set of voting directions, be given for $\mathbb{R}^{JS}$. The $js^{\underline{th}}$ coordinate axis in the space of firms' plans corresponds to the production of firm j in state of nature s. A state $z \in Z_\zeta$ is a voting equilibrium just in case there is no distinct $z^\# = (x^\#, b^\#, \theta) \in Z_\zeta|_\theta$ which satisfies:

i) $b^\# = b + \alpha * e_{js} \geq\geq 0$, for some $j \in J$, $s \in S$, (Direction Restriction)

ii) $x^\# = x + (C(b) - C(b')) \geq\geq 0$, and (Non-Bankruptcy)

iii) $\mathbb{1}_{\{ch\}}(z^\# | z) \cdot \theta^j > .5$. (Share Weighted Majority Rule)

The $(1 \times I)$ vector $\mathbb{1}^*_{\{ch\}}(z^\# | z)$ is an indicator for the set of individuals i who *strictly* prefer $z^\#$ to z, that is for whom: $U_i(x^\#_{oi}, (\theta^\# \cdot b^\#)_i, (\theta^\# \cdot \delta)_i) > U_i(x'_{oi}, (\theta' \cdot b')_i, (\theta' \cdot \delta)_i)$.

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5. The requirement that a plan revision not bankrupt any current shareholders is reflected in the definition of a state of the economy. One of the defining conditions of a state stipulates that no individual can hold a negative quantity of the current good. Only states can be proposed as alternatives to the status quo state, thus capturing the no bankruptcy provision.

Notice that ***standard majority rule*** has been defined so that indifferent voters vote for the status quo, and ties go to the status quo.[6]

DEFINITION (Shareholders' Equilibrium for a Stock Market Economy)

A ***shareholders' equilibrium*** for a stock market economy ($X_o$, U, C, $\zeta$) is a state $z \in Z_\zeta$ which is simultaneously an exchange equilibrium and voting equilibrium for the economy.

DEFINITION (Non-Controlling Interest Shareholders' Equilibrium)

A shareholders' equilibrium $z \in Z_\zeta$ for a stock market economy ($X_o$, U, C, $\zeta$) is called a ***non-controlling interest equilibrium*** just in case there is at least one firm in which no single shareholder has a controlling interest, that is 50% or more of the firm's shares.

6. The existence results developed here obtain whether or not indifferent voters vote for the status quo or the challenger.

PROPOSITION 2.1 (Existence of Non-Controlling Interest Shareholders' Equilibria)

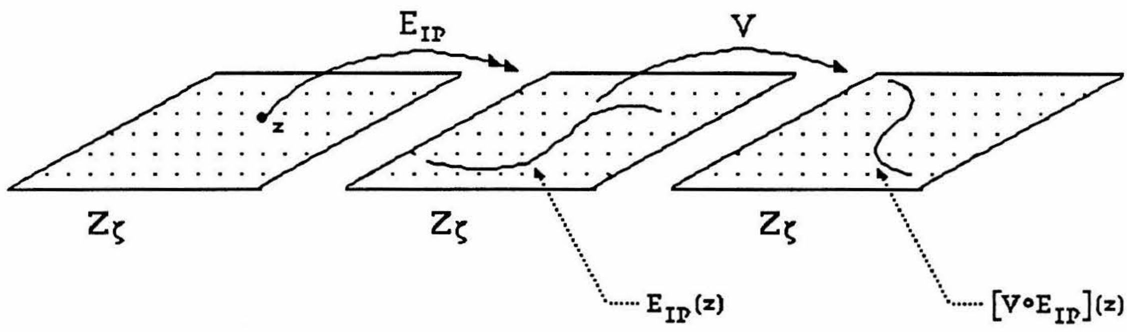


Figure 1.

Let $(X_o, U, C, \zeta)$ be a stock market economy. Suppose that:

i) $X_o \geq 0$.

ii) Each component $U_i$ of U is strictly monotonic, strictly concave, and satisfies the condition $\inf_{[\xi \in \mathbb{R}^{(I+S+J)}{}_+]} U_i(\xi) = 0$.

iii) Each component $C_j$ of C is strictly monotonic, convex, and non-negative.

iv) $\zeta = (\gamma, \delta) >> 0$; the minimum shareholdings matrix and the distinguished goods vector are strictly positive.

v) There is at least one firm in which all individuals must maintain share holdings of no less than .5/I; that is, $\exists\, j \in J\, (\forall\, i \in I\, (\gamma_{ij} > .5/I))$.

Then there exists at least one non-trivial shareholders' equilibrium $z \in Z_\zeta$ for the given economy.

Proof

A detailed proof of the proposition forms the body of this paper; an outline of the proof will be presented here.

If the aggregate social endowment $X_o = 0$, no production is possible, and firms' shares are simply rights to the distinguished goods. In this case the question reduces to the well-understood problem of demonstrating the existence of some state z which is stable with respect to the market process; see Debreu [1957]. Hence suppose $X_o > 0$.

In the diagram above, $E_{IP}$ denotes the individually rational Pareto correspondence on $Z_\zeta$, and $V: Z_\zeta \longrightarrow Z_\zeta$ is a function with the pleasing property that fixed points of V are voting equilibria, and vice versa. It will be demonstrated that the the set of equilibria for the given stock market economy coincides exactly with the set of fixed points for $V \circ E_{IP}$.

Thus the search for equilibrium can be reduced to demonstrating that $V \circ E_{IP}$ does in fact have fixed points in $Z_\zeta$. The desired existence result is obtained by application of the Eilenberg-Montgomery theorem, and so it is necessary to verify the hypotheses of the theorem. It is required to show that $Z_\zeta$ is compact and contractible, and that $V \circ E_{IP}$ is upper hemicontinuous, and non-empty, compact, and contractible valued on $Z_\zeta$.

The proofs that $Z_\zeta$ is compact and contactible are straightforward. To show $V \circ E_{IP}$ is upper hemicontinuous and non-empty, compact valued, it will be argued that $E_{IP}$ exhibits these properties, and that V is a continuous function. By continuity their composition shares these desired properties. Finally, as the 1-1 continuous image of a contractible set is itself contractible, the required contractible valuedness of $V \circ E_{IP}$

will be proved by first showing $E_{IP}$ is contractible valued, and then arguing that V is 1–1 on every set of the form $E_{IP}(z)$. Significant technical digression is necessary to derive some of these intermediate results, in particular that $E_{IP}$ is contractible valued, and that V is continuous.

The contractiblity of $E_{IP}$ is shown by utilizing the fact that the utility image of the Pareto set of an exchange economy is homeomorphic to the simplex $S^{I-1}$; see Chipman and Moore [1971] and Zeckhauser and Weinstein [1974]. These results are extended to cover the case of the individually rational Pareto set in the stock market economy associated with a given state, holding firms' plans fixed. In fact, each such set is either a singleton or else homeomorphic to the simplex $S^{I-1}$. These results are homeomorphically pulled back to the "physical" allocation space of shares and the current good, yielding the desired result.

The proof that V is continuous requires the construction of a modified form of majority rule, called here $\varepsilon$-majority rule. Under $\varepsilon$-majority rule, the distance (measured in the policy space of firms' plans) which a winning coalition can depart from the status quo is positively proportional, $\varepsilon > 0$ being the proportionality factor, to the coalition's margin of victory. The $\varepsilon$-majority rule mechanism also assumes that indifferent voters vote for the challenging proposal.[7] The assumptions that $\varepsilon > 0$ and the assumption favoring challengers are both discharged after the appropriate fixed points have been generated. Finally, as noted previously, the voting mechanism employs the well–known direction restriction technique to obtain existence results.

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7. Notice that under $\varepsilon$-majority rule for $\varepsilon > 0$, ties are awarded to the challenger. This is not a separate tie breaking rule *per se*, but rather a consequence of the requirement that a challenger must obtain a strictly positive margin of victory to overturn the status quo.

# III. THE STATE SPACE

This section shows that the state space $Z_\zeta$ of a stock market economy is both compact and contractible.[1] These results are important since equilibrium states can be characterized as fixed points of an appropriately constructed correspondence defined on $Z_\zeta$. Commonly employed fixed point theorems require domain conditions, typically compactness and some connectivity property such as convexity or contractibility. Compactness of $Z_\zeta$ is a relatively simple matter. Unfortunately, however, $Z_\zeta$ is not convex-- which precludes application of Kakutani's theorem. The problem of non-convexity in stock market economies seems to have been first recognized by Dreze [1974]. Nevertheless, $Z_\zeta$ can be shown to be contractible, which admits the possi - bility of using the Eilenberg-Montgomery theorem to obtain fixed point results.

PROPOSITION 3.1 (Elementary Properties of $Z_\zeta$)

Let $z = (x, b, \theta) \in Z_\zeta$, and suppose $\zeta \geq\geq 0$; then:

i) $Z_\zeta$ is compact.

ii) $Z_\zeta|_b$ is compact.

iii) In general, $Z_\zeta$ is not convex unless each cost function $C_j$ is linear.

iv) $Z_\zeta|_b$, however, *is* convex.

1. Contractibility and related topics are discussed in Willard [1970].

Proof

The proofs of parts i), ii), and iv) are straightforward. Notice compactness obtains from the boundedness of the aggregate social endowment $X_o$, and from continuity and monotonicity of firms' cost functions.

To show iii), let $z = (x, b, \theta)$ and $z' = (x', b', \theta') \in Z_\zeta$. If $Z_\zeta$ is convex then it must be the case that $tz+(1-t)z' \in Z_\zeta$, $\forall\, t \in (0,1)$. In particular, this condition requires that:

$$[tx+(1-t)x']\cdot \mathbb{1}^I = X_o - \mathbb{1}_{\mathcal{J}} C(tb+(1-t)b'). \qquad (*)$$

Now the left hand side of (∗) can be expanded according to:

$$\begin{aligned}[tx+(1-t)x']\cdot \mathbb{1}^I &= t[x\cdot \mathbb{1}^I] + (1-t)[x'\cdot \mathbb{1}^I] \\ &= t[X_o - \mathbb{1}_{\mathcal{J}} C(b)] + (1-t)[X_o - \mathbb{1}_{\mathcal{J}} C(b')] \\ &= X_o - [t\mathbb{1}_{\mathcal{J}} C(b) + (1-t)\mathbb{1}_{\mathcal{J}} C(b')] \\ &= X_o - \mathbb{1}_{\mathcal{J}}(tC(b)+(1-t)C(b')) \qquad (**)\end{aligned}$$

Now, since C has been assumed to be componentwise convex, $C(tb+(1-t)b') \leq\leq tC(b)+(1-t)C(b')$. And so, referring to the right hand side of (∗∗), one can conclude that:

$$[tx+(1-t)x']\cdot \mathbb{1}^I = X_o - [\mathbb{1}_{\mathcal{J}}(tC(b)+(1-t)C(b'))] \leq X_o - \mathbb{1}_{\mathcal{J}} C(tb+(1-t)b'). \qquad (***)$$

Since each component $C_j$ of C is convex, the inequality appearing in (∗∗∗) will be strict for some $t \in (0,1)$ unless every $C_j$ is linear on the interval $(b_j, b'_j)$, $j \in J$. QED.

PROPOSITION 3.2 (Contractibility of $Z_\zeta$)

$Z_\zeta$ is contractible, $\forall\ \zeta \geq\geq 0$.

Proof

Let $z^{\#} = (x^{\#}, b^{\#}, \theta^{\#}) \in Z_\zeta$ be given. Let $k_{z^{\#}}$ and id denote the constant map $Z_\zeta \longrightarrow \{z^{\#}\}$ and the identity on $Z_\zeta$, respectively. It suffices to show that $k_{z^{\#}}$ and id are homotopic, that is $\exists$ a continuous $H: Z_\zeta \times [0,1] \longrightarrow Z_\zeta$ such that $H(\cdot, 0) = k_{z^{\#}}$ and $H(\cdot, 1) = \text{id}$.

For any $z = (x, b, \theta) \in Z_\zeta$ and $t \in [0,1]$ construct H according to:

$$H(z,t) = ((1-t)x^{\#}+tx+y(b,t), (1-t)b^{\#}+tb, (1-t)\theta^{\#}+t\theta),$$

where $y(b,t)$ is a $(1 \times I)$ row vector, each component being identical and specified by:

$$(1/I) * \mathbb{1}_J \cdot \big(((1\text{-}t)C(b^{\#})+tC(b)) - C((1-t)b^{\#}+tb)\big).$$

Recall that $\mathbb{1}_J$ is a $(1 \times J)$ row vector and $C(b)$ is a $(J \times 1)$ column vector, so the multiplicands appearing in the dot product above are conformable. Notice that $y(b,t) \geq\geq 0$; $\forall\ b \in \mathbb{R}^{(JxS)}{}_{+}$, and $t \in [0,1]$ since each component cost function $C_j$ of C is convex. Furthermore, by inspection, $y(b,0) = y(b,1) = 0$.

Clearly H is continuous on $Z_\zeta \times [0,1]$, this property obtaining directly from the continuity of the cost functions $C_j$, $\forall\ j \in J$.

Now,

$$\begin{aligned} H(z,0) &= ((1-0)x^{\#}+0x+y(b,0), (1-1)b^{\#}+1b, (1-0)\theta^{\#}+0\theta) \\ &= (x^{\#}+y(b,0), b^{\#}, \theta^{\#}) \\ &= (x^{\#}, b^{\#}, \theta^{\#}), \forall\ z \in Z_\zeta. \end{aligned}$$

And hence,

$$H(\cdot,0) = k_{z^\#}.$$

Similarly,

$$H(z,1) = ((1-1)x^\#+1x+y(b,1), (1-1)b^\#+1b, (1-1)\theta^\#+1\theta)$$
$$= (x+y(b,1), b, \theta)$$
$$= (x, b, \theta), \forall\, z \in Z_\zeta.$$

$$H(\cdot,1) = \text{id}.$$

It remains to show that each function $H(\cdot,t)$ does, in fact, take values in $Z_\zeta$. It suffices that:

i) $(1-t)x^\#+tx+y(b,t) \geq\geq 0$,

ii) $(1-t)b^\#+tb \geq\geq 0$,

iii) $(1-t)\theta^\#+t\theta \in S_\gamma$, and

iv) $((1-t)x^\#+tx+y(b,t))\cdot \mathbb{1}^I + \mathbb{1}_J C((1-t)b^\#+tb) = X_o$

Notice that i) obtains trivially since $x^\#$, x, $y(b,t) \geq\geq 0$. To show ii) note that since z, $z^\#$ $\in Z_\zeta$, then both $b^\#$, b $\geq\geq 0$. Consider iii) follows directly from the convexity of $S_\gamma$. Finally, to show iv), observe that substituting in the l.h.s. of the expression directly from the definition of y(b,t) gives:

$$((1-t)x^\#+tx)\cdot \mathbb{1}^I + \mathbb{1}_J\big(((1-t)C(b^\#)+tC(b))-C((1-t)b^\#+tb)\big)$$
$$+ \mathbb{1}_J C((1-t)b^\#+tb)$$
$$= ((1-t)x^\#+tx)\cdot \mathbb{1}^I + \mathbb{1}_J((1-t)C(b^\#)+tC(b))$$
$$= (1-t)(x^\#\cdot \mathbb{1}^I+\mathbb{1}_J C(b^\#)) + t\,(x\cdot \mathbb{1}^I+\mathbb{1}_J C(b)), \text{ and recalling } z, z^\# \in Z_\zeta,$$
$$= (1-t)X_o + tX_o,$$
$$= X_o, \text{ as desired.}$$

QED.

## IV. STOCK MARKET TRADING

Stock market trading has been modeled here by the individually rational Pareto correspondence $E_{IP}$, defined on $Z_{\zeta}$. . This section demonstrates that a state z is an exchange equilibrium just in case it is a fixed point of $E_{IP}$. It is also shown that $E_{IP}$ is compact, non-empty valued, upper hemicontinuous, and contractible valued. Note the strict concavity of U and a strictly positive distinguished goods vector are required to show the contractible valuedness of $E_{IP}$.

DEFINITION (Individually Rational Pareto Correspondence $E_{IP}$)

Given any $\zeta \geq\geq 0$, define the correspondence $E^{\zeta}_{IP}: Z_{\zeta} \to\to Z_{\zeta}$ according to:

$E^{\zeta}_{IP}(z) = \{ z' \in Z_{\zeta}|_b \mid$

i) not $\exists$ $z'' \in Z_{\zeta}|_b$ such that $U(x'', \theta''\cdot b, \theta''\cdot\delta) >> U(x', \theta'\cdot b, \theta'\cdot\delta)$, and

ii) $U(x', \theta'\cdot b, \theta'\cdot\delta) \geq\geq U(x, \theta\cdot b, \theta\cdot\delta) \}$, where

$z = (x,b,\theta)$, $z' = (x',b,\theta')$, $z'' = (x'',b,\theta'') \in Z_{\zeta}|_b \subseteq Z_{\zeta}$. If there is no possibility for confusion, $E_{IP}(z)$ will be written for $E^{\zeta}_{IP}(z)$.

The correspondence $E_{IP}$ describes the set of feasible reallocations attainable through stock market trading. Condition i) requires that elements of $E_{IP}(z)$ are ($\gamma$-constrained) Pareto optimal with respect to $Z_{\zeta}|_b$. That is, given an element of $E_{IP}(z)$, there is no other state z" of the economy which would make all participants better off, while not violating the minimum holdings constaint and at the same time keeping firms' plans fixed at the plan b specified by z. The Pareto condition can be expressed here by a strict inequality since U is componentwise continuous and strictly monotonic.

Condition ii) stipulates that points in $E_{IP}(z)$ must be individually rational with respect to z. That is, no individual is worse off under any given element of $E_{IP}(z)$ than under z.

PROPOSITION 4.1 (Elementary Properties of $E_{IP}$ and the Characterization of Exchange Equilibria)

Suppose that U is componentwise continuous.

i) $E_{IP}$ is non-empty valued on $Z_\zeta$.

ii) $E_{IP}$ is compact valued on $Z_\zeta$.

iii) $z \in E_{IP}(z) \Leftrightarrow \{z\} = E_{IP}(z) \Leftrightarrow$ z is an exchange equilibrium, whenever $\delta >> 0$ and U is componentwise strictly quasiconcave.[1]

Proof

To show i) let $z = (x,b,\theta) \in Z_\zeta$ be given. If $z \in E_{IP}(z)$ the problem is trivial, so suppose $z \notin E_{IP}(z)$. Referring to the definition of $E_{IP}$, if z fails to be an element of $E_{IP}(z)$ then $\exists$ some individually rational $z' \in Z_\zeta|_b$ which Pareto dominates z. Since U is continuous and $Z_\zeta|_b$ is compact, there is some such maximal z'--which therefore belongs to $E_{IP}(z)$.

To show ii) let $z = (x,b,\theta) \in Z_\zeta$ be given. Since $E_{IP}(z) \subseteq Z_\zeta$, and $Z_\zeta$ is compact and hence bounded, it suffices to show that $E_{IP}(z)$ is closed. Let $(z^{\#}_n)$ be a sequence taking values in $E_{IP}(z)$, and suppose $(z^{\#}_n) \longrightarrow z^{\#}$; it will be demonstrated that $z^{\#} \in E_{IP}(z)$.

---

1. Even if individuals' preferences for consumption are strictly quasiconcave, their indirect preferences over the space of portfolios composed of firms' shares and the current good may fail to be strictly quasiconcave if firms' plans are linearly dependent. This seems to have been first recognized by Dreze [1972]. The strictly positive vector $\delta$ of firm specific distinguished goods forces the consumption opportunities afforded by share ownership to be linearly independent, regardless of firms' plans.

Recall $E_{IP}(z) \subseteq Z_\zeta|_b$, and that this set is also compact; thus $z^\# \in Z_\zeta|_b$. Suppose however that $z^\# \notin E_{IP}(z)$, then one of two cases must obtain:

a) $\exists\ z'' \in Z_\zeta|_b$ such that $U(x'',\theta''\cdot(b,\delta)) >> U(x^\#,\theta^\#\cdot(b,\delta))$, or

b) $U_i(x^\#,\theta^\#\cdot(b,\delta)) < U_i(x,\theta\cdot(b,\delta))$, for at least one individual $i \in I$.

In case a), since the inequality is strict and U is continuous, there is some real $\eta > 0$ such that the inequality will remain true when $z^\dagger$ is substituted $z^\#$, if $z^\dagger \in N_\eta(z^\#)$. But for n sufficiently large, $z^\#_n = (x^\#_n, b^\#_n{=}b, \theta^\#_n) \in N_\eta(z^\#)$, which implies $\exists\ z'' \in Z_\zeta|_b$ such that $U(x'',\theta''\cdot(b,\delta)) >> U(x^\#_n,\theta^\#_n\cdot(b,\delta))$-- contradicting that $z^\#_n \in E_{IP}(z)$.

On the other hand, should b) be the case, the continuity of $U_i$ and the strictness of the inequality imply there is some real $\xi > 0$ such that the inequality remains true if $z^\ddagger$ is substituted for $z^\#$ whenever $z^\ddagger \in N_\xi(z^\#)$. Thus as $(z^\#_n) \longrightarrow z^\#$, for n sufficiently large, $z^\#_n = (x^\#_n, b^\#_n{=}b, \theta^\#_n) \in N_\xi(z^\#)$, which implies $U_i(x^\#_n,\theta^\#_n\cdot(b,\delta))$ is strictly less than $U_i(x,\theta\cdot(b,\delta))$, for at least one individual $i \in I$-- contradicting that $z^\#_n \in E_{IP}(z)$, and completing the proof of part ii).

Finally, to show iii) suppose that $z \in E_{IP}(z)$ ; it will be demonstrated first that $\{z\} = E_{IP}(z)$. For suppose not, then since $E_{IP}(z) \neq \phi$, $\exists$ some $z' \in E_{IP}(z) \subseteq Z_\zeta|_b$ such that $z' \neq z$. Now, by definition of $E_{IP}(z)$, $U(x',\theta'\cdot(b,\delta)) \geq\geq U(x,\theta\cdot(b,\delta))$; in fact it must be the case that $U(x',\theta'\cdot(b,\delta)) = U(x,\theta\cdot(b,\delta))$. Otherwise, there is at least one individual i for whom $U_i(x',\theta'\cdot(b,\delta)) > U_i(x,\theta\cdot(b,\delta))$; but then the componentwise strict monotonicity and continuity of U, together with the divisibility of $Z_\zeta|_b$ imply that z can be Pareto dominated by some state in $Z_\zeta|_b$--a contradiction. Since $U(x',\theta'\cdot(b,\delta)) = U(x,\theta\cdot(b,\delta))$, and $z' \neq z$ the convexity of $Z_\zeta|_b$ and componentwise strict quasiconcavity of U on $Z_\zeta|_b$ given $\delta >> 0$ imply $\exists$ some $z'' \in Z_\zeta|_b$ such that $U(x'',\theta''\cdot(b,\delta)) >> U(x,\theta\cdot(b,\delta))$ and $U(x',\theta'\cdot(b,\delta))$--contradicting the assumed optimality of z and z'. The remaining steps of the implication cycle are trivial. QED.

PROPOSITION 4.2 (Upper Hemicontinuity of $E_{IP}$)

$E_{IP}$ is upper hemicontinuous on $Z_\zeta$, $\forall\ \zeta \geq\geq 0$.

Proof

Since $E_{IP}$ is compact valued, it suffices to show that the correspondence is closed. Let $(z_n) \longrightarrow z$, $z^\#_n \in E_{IP}(z_n)$, and $(z^\#_n) \longrightarrow z^\#$. Must show that $z^\# \in E_{IP}(z)$ , that is:

i) $z^\# \in Z_\zeta|_b$, where $z = (x,b,\theta)$,

ii) $U(x^\#,\theta^\#\cdot(b,\delta)) \geq\geq U(x,\theta\cdot(b,\delta))$, and

iii) not $\exists\ z'' \in Z_\zeta|_b$ such that $U(x'',\theta''\cdot(b,\delta)) >> U(x^\#,\theta^\#\cdot(b,\delta))$.

In the case of i), notice that $z^\# \in Z_\zeta$, since $Z_\zeta$ is compact; however, it must be verified that $z^\# \in Z_\zeta|_b$, or that $b^\# = b$. Now, $(z_n) \longrightarrow z$, so $(b_n) \longrightarrow b$; as each $z^\#_n \in E_{IP}(z_n)$, then $z^\#_n \in Z_\zeta|_{b_n}$-- which implies $b^\#_n = b_n$. Thus,

$$(z^\#_n) \longrightarrow z^\# \Rightarrow (b^\#_n = b_n) \longrightarrow b^\# \Rightarrow b^\# = b,$$

as desired.

Condition ii) obtains directly form the continuity of each component $U_i$ of U.

Finally, to show iii), suppose not. Then:

$$\exists\ z'' \in Z_\zeta|_b \text{ such that } U(x'',\theta''\cdot(b,\delta)) >> U(x^\#,\theta^\#\cdot(b,\delta)). \qquad (*)$$

Since this inequality is strict, and U is componentwise continuous, $\exists$ open neighborhoods (relative to $Z_\zeta$) $N_\eta(z'')$ and $N_\xi(z^\#)$ of z' and $z^\#$ respectively for which $(*)$ is true. Claim that since both z'' and $z^\# \in Z_\zeta|_b$, $\exists\ \nu > 0$ such that if $z^\ddagger = (x^\ddagger,b^\ddagger,\theta^\ddagger) \in N_\nu(z^\#)$ and $b^\ddagger \leq\leq b$,then $\exists\ z^\dagger = (x^\dagger,b^\ddagger,\theta^\dagger) \in N_\eta(z'')$ and $z^\dagger \in Z_\zeta|_{b^\ddagger}$. Now, if the claim is true, given $\xi$, $\nu > 0$ there is $M \in N$ such that $n > M \Rightarrow z^\#_n \in N_\xi(z^\#) \cap N_\nu(z^\#)$. But then $\exists\ z^\dagger_n \in N_\eta(z'')$ such that $b^\dagger_n = b^\#_n$ and $U(x^\dagger_n,\theta^\dagger_n\cdot(b^\dagger_n,\delta))$ is componentwise strictly greater than $U(x^\#_n,\theta^\#_n\cdot(b^\#_n,\delta))$--contradicting that $z^\#_n \in E_{IP}(z_n)$.

It remains to prove the claim; see the figure below. By continuity of C, there is $\lambda > 0$

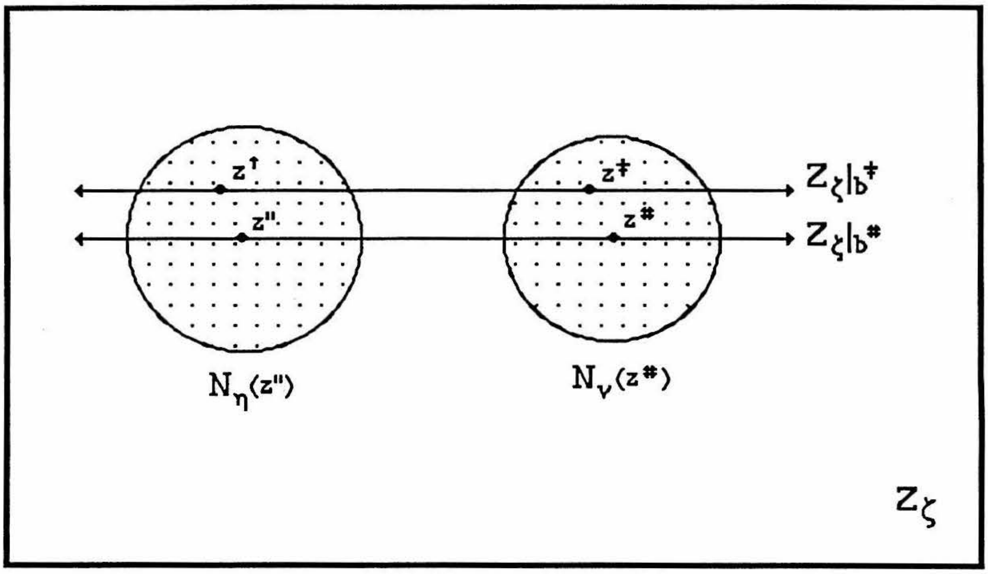


Figure 2.

such that $|\theta''\cdot(C(b^{\#})-C(b^{\ddagger}))| < (2/3)\eta$ whenever $|z^{\#}-z^{\ddagger}| < \lambda$. Choose $\nu < \mathrm{MIN}\{\lambda,\eta/3\}$. Given $z^{\ddagger} \in N_{\nu}(z^{\#})$, construct $z^{\dagger}$ according to: $b^{\dagger} = b^{\ddagger}$, $\theta^{\dagger} = \theta''$, $x^{\dagger} = x''+\theta''\cdot C(b'')-\theta^{\dagger}\cdot C(b^{\dagger}) = x''+\theta''\cdot(C(b'')-C(b^{\ddagger}))$. In order to show $z^{\dagger} \in Z_{\zeta}|_{b^{\ddagger}}$, consider that since $b^{\dagger} = b^{\ddagger}$, $z^{\dagger} \in Z_{\zeta}|_{b^{\ddagger}}$ if $z^{\dagger} \in Z_{\zeta}$. Since $z^{\ddagger} \in Z_{\zeta}$, trivially $b^{\dagger}= b^{\ddagger} \geq\geq 0$; it remains to verify that $x^{\dagger} \geq\geq 0$, and that $x^{\dagger}\cdot \mathbb{1}^{I} = X_o-\mathbb{1}_J\cdot C(b^{\dagger})$.

To show $x^{\dagger} \geq\geq 0$, consider that $x'' \geq\geq 0$ since $z'' \in Z_{\zeta}$; as it has been assumed that $b^{\ddagger} \leq\leq b = b''$, monotonicity of C gives: $C(b'')-C(b^{\dagger}) \geq\geq 0$. Thus one finds $x^{\dagger} = x''+\theta''\cdot(C(b'')-C(b^{\ddagger})) \geq\geq 0$, as desired. Turning now to the issue of $x^{\dagger}\cdot \mathbb{1}^{I}$, observe that $x^{\dagger}\cdot \mathbb{1}^{I}$ can be expressed as:

$$
\begin{aligned}
& (x''+\theta''\cdot C(b'')-\theta^{\dagger}\cdot C(b^{\dagger}))\cdot \mathbb{1}^{I} \\
= \;& x''\cdot \mathbb{1}^{I} + (\theta''\cdot C(b''))\cdot \mathbb{1}^{I} - (\theta^{\dagger}\cdot C(b^{\dagger}))\cdot \mathbb{1}^{I}
\end{aligned}
$$

$$= X_o - \mathbb{1}_{\mathcal{J}} C(b'') + \mathbb{1}_{\mathcal{J}} C(b'') - \mathbb{1}_{\mathcal{J}} C(b^\dagger),$$

which is simply $X_o - \mathbb{1}_{\mathcal{J}} C(b^\dagger)$.

In order to complete the proof of the claim, it is necessary to verify that $|z''-z^\dagger| < \eta$. Observe that

$$|z''-z^\dagger| = |(x'',b'',\theta'')-(x^\dagger,b^\dagger,\theta^\dagger)| \leq |x''-x^\dagger| + |b''-b^\dagger| + |\theta''-\theta^\dagger|. \qquad (**)$$

Now, $b'' = b^\#$, and $b^\dagger = b^\ddagger$; thus, $|b''-b^\dagger| = |b^\# - b^\ddagger| < \nu < \eta/3$, since $|z^\# - z^\ddagger| < \nu$. Notice also that $\theta^\dagger = \theta''$ so that $|\theta''-\theta^\dagger| = 0$. Thus expression (**) can be rewritten as :

$$|z''-z^\dagger| < |x''-x^\dagger| + \eta/3.$$

This implies:

$$|z''-z^\dagger| < |x'' - (x''+\theta''\cdot(C(b'')-C(b^\ddagger)))| + \eta/3 < |\theta''\cdot(C(b'')-C(b^\ddagger))| + \eta/3 < \eta,$$

as desired. QED.

<u>DEFINITION</u> (Constructions for Showing $E_{IP}$ is Contractible Valued)

Given any state $z = (x,b,\theta) \in Z_\zeta$, where $\zeta \geq\geq 0$, let $dh\mathrm{U}[E_{IP}(z)]$ denote the disposable hull of $\mathrm{U}[E_{IP}(z)]$ relative to $\mathbb{R}^I_+$. Clearly $dh\mathrm{U}[E_{IP}(z)] \neq \phi$ since $E_{IP}$ is non-empty valued; in particular $\omega(z) \in dh\mathrm{U}[E_{IP}(z)]$, where $\omega(z)$ denotes $\mathrm{U}(x,\theta\cdot(b,\delta))$. Notice also that $dh\mathrm{U}[E_{IP}(z)] \subseteq \mathbb{R}^I_+$, since each component of U is non-negative valued. Furthermore, $dh\mathrm{U}[E_{IP}(z)]$ is compact since U is continuous and $E_{IP}$ is compact valued on $Z_\zeta$. Define the maps $\varsigma_z$, $\varphi_z$, and $\chi_z$ on $S^{(I-1)}$ as below.[2]

2. Chipman and Moore[1971] employed similar constructions to investigate various topological properties of the Pareto set of a simple exchange economy.

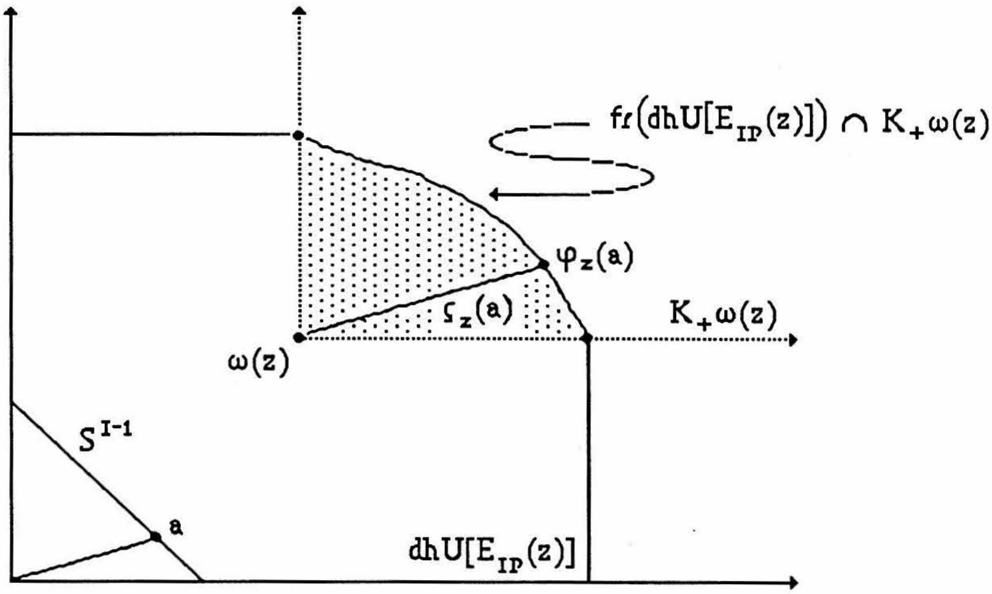


Figure 3.

i) $\varsigma_z$: $S^{(I-1)} \rightarrow\!\rightarrow dh\mathrm{U}[\mathrm{E}_{\mathrm{IP}}(z)] \subseteq \mathbb{R}^I_+$ such that:

$$\varsigma_z(a) = \{\omega' \in dh\mathrm{U}[\mathrm{E}_{\mathrm{IP}}(z)] \mid \exists\, \lambda \geq 0 \text{ s.t. } \omega' = \omega(z)+\lambda * a\}$$

Since $dh\mathrm{U}[\mathrm{E}_{\mathrm{IP}}(z)]$ contains $\omega(z)$ and is compact, $\varsigma_z$ is non-empty, compact, and convex valued by the disposability of $dh\mathrm{U}[\mathrm{E}_{\mathrm{IP}}(z)]$. Therefore $\varsigma_z(a)$ is a line segment in $\mathbb{R}^I_+$ of the form $[\omega(z), \omega(z)+\lambda'' * a]$, for some real $\lambda'' \geq 0$.

ii) $\varphi_z$: $S^{(I-1)} \longrightarrow dh\mathrm{U}[\mathrm{E}_{\mathrm{IP}}(z)] \subseteq \mathbb{R}^I_+$ such that:

$$\varphi_z(a) = \mathrm{ARGMAX}_{[\omega' \in \varsigma_z(a)]}\{\omega' \cdot \mathbb{1}^I\}$$

$\varphi_z(a)$ is the maximum feasible (with respect to $dh\mathrm{U}[\mathrm{E}_{\mathrm{IP}}(z)]$) aggregate utility attainable by traversing utility allocation space in direction $a \in S^{(I-1)}$ away from $\omega(z)$. $\varphi_z$ is a well defined function since the linear optimand $\omega' \cdot \mathbb{1}^I$ is strictly increasing over the interval $\varsigma_z(a)$.

iii) $\chi_z$: $S^{(I-1)} \rightarrow\rightarrow Z_\zeta$ such that:

$$\chi_z(a) = \{z" \in Z_\zeta|_b \mid \omega(z") = \varphi_z(a)\}$$

$\chi_z(a)$ is the set, keeping firms' plans fixed at b, of "physical" allocation pre-images of the utility allocation $\varphi_z(a)$ associated with $a \in S^{(I-1)}$.

PROPOSITION 4.3 (Continuity of $\varsigma_z$, $\varphi_z$, and $\chi_z$)

Suppose U is continuous, then $\forall$ $z \in Z_\zeta$, such that $\zeta \geq\geq 0$:

i) Each correspondence $\varsigma_z$: $S^{(I-1)} \rightarrow\rightarrow dh\text{U}[E_{IP}(z)]$ is continuous.

ii) Each function $\varphi_z$: $S^{(I-1)} \longrightarrow dh\text{U}[E_{IP}(z)]$ is continuous.

iii) Each correspondence $\chi_z$: $S^{(I-1)} \rightarrow\rightarrow Z_\zeta|_b$ is upper hemicontinuous.

Proof

To show upper hemicontinuity of $\varsigma_z$, first observe that since $dh\text{U}[E_{IP}(z)]$ is compact, $\varsigma_z$ is compact valued; thus it suffices to show the correspondence is closed. Let $(a_n)$ be a sequence in $S^{(I-1)}$ such that $(a_n) \longrightarrow a_o$, and suppose $(\omega^\#_n) \longrightarrow \omega^\#_o$, where $\omega^\#_n \in \varsigma_z(a_n)$. Must show $\omega^\#_o \in \varsigma_z(a_o)$. Now, by definition of $\varsigma_z$, each $\omega^\#_n$ is of the form $\omega+\lambda^\#_n*a_n$, for some $\lambda^\#_n \in \mathbb{R}_+$. Since $(\omega^\#_n) = (\omega+\lambda^\#_n*a_n) \longrightarrow \omega^\#_o$, and $(a_n) \longrightarrow a_o$, it follows $\lim(\lambda^\#_n) = \lambda^\#_o$ exists and that $\omega^\#_o = \lim(\omega+\lambda^\#_n*a_n) = \omega+\lim(\lambda^\#_n)*\lim(a_n) = \omega+\lambda^\#_o*a_o$. Notice that $\lambda^\#_o \geq 0$ since each $\lambda^\#_n \geq 0$, by definition of $\varsigma_z$. Thus it only remains to show that $\omega+\lambda^\#_o*a_o \in dh\text{U}[E_{IP}(z)]$ to guarantee membership in $\varsigma_z(z)$. But as each $\omega^\#_n \in dh\text{U}[E_{IP}(z)]$ and this set is compact, $\omega^\#_o = \lim(\omega^\#_n) \in dh\text{U}[E_{IP}(z)]$, as desired.

To show $\varsigma_z$ is lower hemicontinuous, let $(a_n) \longrightarrow a_o$ and suppose $\omega^{\#}_o \in \varsigma_z(a_o)$. It is required to construct $(\omega^{\#}_n) \longrightarrow \omega^{\#}_o$ such that $\omega^{\#}_n \in \varsigma_z(a_n)$. Now, by definition of $\varsigma_z$, $\omega^{\#}_o$ must be of the form $\omega+\lambda^{\#}_o*a_o$, for some $\lambda^{\#}_o \in \mathbb{R}_+$. If $\lambda^{\#}_o = 0$ then $\omega^{\#}_o = \omega$, and simply define $\omega^{\#}_n = \omega$; it is trivial that $\omega \in \varsigma_z(a_n)$. Hence suppose $\lambda^{\#}_o > 0$, and in this case define $\lambda^{\#}_n$ according to:

$$\lambda^{\#}_n = \text{ARGMIN}_{[\lambda''\geq 0]} \ \{|\lambda^{\#}_o-\lambda''| \ \mid \ \omega+\lambda^{\#}_n*a_n) \in dh\text{U}[\text{E}_{\text{IP}}(z)]\}, \ \forall \ n \in \mathbb{N}_+.$$

Clearly $\lambda^{\#}_n$ is well defined since $dh\text{U}[\text{E}_{\text{IP}}(z)]$ is compact. Define the sequence $(\omega^{\#}_n)$ according to:

$$\omega^{\#}_n = \omega+\lambda^{\#}_n*a_n, \ \forall \ n \in \mathbb{N}_+.$$

It is straightforward that $\omega^{\#}_n \in \varsigma_z(a_n)$, from the definition of $\varsigma_z$ and the construction of $\lambda^{\#}_n$. And so it remains to verify that $(\omega^{\#}_n) \longrightarrow \omega^{\#}_o$, and it clearly suffices that $(\lambda^{\#}_n) \longrightarrow \lambda^{\#}_o$. Notice that, by construction of $\lambda^{\#}_n$ and the disposability of dhU$[\text{E}_{\text{IP}}(z)]$, the terms of $(\lambda^{\#}_n)$ are bounded above by $\lambda^{\#}_o$. So suppose $(\lambda^{\#}_n) \not\longrightarrow \lambda^{\#}_o$, then since the terms of the sequence are bounded above, $\exists$ some convergent subsequence $(\lambda^{\#}_m)$ such that:

$$(\lambda^{\#}_m) \longrightarrow \lambda''_o < \lambda^{\#}_o. \tag{$*$}$$

Before proceeding it is necessary to establish the following claim: For any real $\xi$, $0<\xi<\lambda^{\#}_o$, there is $\nu > 0$ such that whenever $|a^{\dagger}-a_o| < \nu$, there is some real $\lambda^{\dagger}$, $0<\lambda^{\dagger}<\lambda^{\#}_o$, such that $|\lambda^{\dagger}-\lambda^{\#}_o| < \xi$ and $\omega+\lambda^{\dagger}*a^{\dagger} \in dh(\omega^{\#}_o) \subseteq dh\text{U}[\text{E}_{\text{IP}}(z)]$. To prove the claim, let $\xi > 0$ be given and recall that $a^{\#}_o \in S^{(I-1)}$ is a $(1 \times I)$ vector of the form $a_o = (a_{o1}, \ldots, a_{oI})$. Define $\alpha = \text{MIN}_{[i\in I]}\{(\xi*a_{oi}) / (\lambda^{\#}_o-\xi)\}$, and choose $\nu$ such that $0 < \nu < \alpha$. Consider that:

$|a^{\dagger}-a_o| < \nu$ which implies $|a^{\dagger}_i-a_{oi}| < \nu < \alpha$, and therefore

$(\lambda^{\#}_o-\xi)(a_{oi}+|a^{\dagger}_i-a_{oi}|) < \lambda^{\#}_o a_{oi}, \ \forall \ i \in I.$

Since this last inequality is strict, it is possible to assert that:

$\exists\,\lambda^\dagger \in (0, \lambda^\#_o)$ such that $|\lambda^\dagger - \lambda^\#_o| < \xi$ and $\lambda^\dagger(a_\alpha + |a^\dagger_i - a_{\alpha i}|) < \lambda^\#_o a_{\alpha i}$, $\forall$ i $\in$ I. Therefore $\lambda^\dagger(a_\alpha + a^\dagger_i - a_{\alpha i}) < \lambda^\#_o a_{\alpha i} \Rightarrow \lambda^\dagger a^\dagger_i < \lambda^\#_o a_{\alpha i}$, $\forall$ i $\in$ I $\Rightarrow \lambda^\dagger a^\dagger$ is componentwise strictly less than $\lambda^\#_o a_o \Rightarrow \omega + \lambda^\dagger a^\dagger << \omega + \lambda^\#_o a_o = \omega^\#_o$. And so, finally:

$$\exists\,\lambda^\dagger \in (0, \lambda^\#_o) \text{ such that } |\lambda^\dagger - \lambda^\#_o| < \xi, \text{ and } \omega + \lambda^\dagger a^\dagger \in dh(\omega^\#_o),$$

which completes the proof of the claim.

It will now be shown that equation (*) contradicts the claim above. Since $(\lambda^\#_m) \longrightarrow \lambda''_o < \lambda^\#_o$, $\lambda''_o$ and $\lambda^\#_o$ can be separated by open neighborhoods of diameter $\xi$. Notice that since $(a_m) \longrightarrow a_o$, for any $\nu > 0\ \exists$ M $\in \mathbb{N}_+$ such that whenever m > M, $|a_m - a_o| < \nu$. Thus regardless of how small $\nu$ is chosen, there can be found some M such that if m > M there is no $\lambda^\dagger_m$ satisfying $|\lambda^\dagger_m - \lambda^\#_o| < \xi$, and $\omega + \lambda^\dagger_m a_m \in dh(\omega^\#_o)$. For the existence of such a $\lambda^\dagger_m$ would contradict the minimality of $\lambda^\#_m$, since $dh(\omega^\#_o) \subseteq dh\mathrm{U}[E_{IP}(z)]$. As the claim has been contradicted, it must be the case that $\lambda^\#_m \longrightarrow \lambda^\#_o$, as desired. Since $\varsigma_z$ has been shown to be both upper and lower hemicontinuous, the correspondence is continuous as claimed.

Recall that $\varphi_z$ is defined by:

$$\varphi_z(a) = \mathrm{ARGMAX}_{[\omega' \in \varsigma_z(a)]}\{\omega' \cdot \mathbb{1}^I\}. \qquad (**)$$

Since $\varsigma_z$ is compact valued and continuous on $S^{(I-1)}$, and the optimand $\omega' \cdot \mathbb{1}^I$ occurring in (**) is a continuous function, the desired continuity of $\varphi_z$ obtains directly from Berge's Maximum Theorem; see Border [1985].

Finally, recall that $\chi_z$ is defined by:

$$\chi_z(a) = \{z'' \in Z_\zeta|_b \mid \omega(z'') = \varphi_z(a)\}.$$

Since $Z_\zeta|_b$ is compact and U is continuous it is straightforward to show that $\chi_z$ is compact valued; hence to demonstrate upper hemicontinuity, it is only necessary to show that $\chi_z$ is closed. Let $(a_n) \longrightarrow a_o$ and $(z_n) \longrightarrow z_o$ where $z_n \in \chi_z(a_n)$; it is required that $z_o \in \chi_z(a_o)$. Clearly $z_o \in Z_\zeta|_b$ by compactness, so turn next to the problem of showing $\omega(z_o) = \varphi_z(a_o)$. Since each $z_n \in \chi_z(a_n)$ it follows $\omega(z_n) = \varphi_z(a_n)$; continuity of U and $\varphi_z$ give that $\lim \omega(z_n) = \lim \varphi_z(a_n) \Rightarrow \omega(\lim z_n) = \varphi_z(\lim a_n) \Rightarrow$ $\omega(z_o) = \varphi_z(a_o)$, as desired. QED.

<u>PROPOSITION 4.4</u> ($\varphi_z$ is a Homeomorphism)

Suppose that U is strictly quasiconcave, then $\forall\ z \in Z_\zeta$ such that $\gamma \geq\geq 0$ and $\delta >> 0$:

i) $U[E_{IP}(z)]$ is the homeomorphic image of $S^{(I-1)}$ under $\varphi_z$, if $\omega(z) \notin U[E_{IP}(z)]$.

ii) Otherwise $U[E_{IP}(z)] = U[\{z\}]$.

<u>Proof</u>

Let $z \in Z_\zeta$, $\gamma \geq\geq 0$, $\delta >> 0$, be given. It has been previously demonstrated that if $\delta >> 0$ and $\omega(z) \in U[E_{IP}(z)]$ then $E_{IP}(z) = \{z\}$, so that trivially $U[E_{IP}(z)] = U[\{z\}]$; hence suppose that $\omega(z) \notin U[E_{IP}(z)]$. In order to show $\varphi_z$ is a homeomorphism it is required to verify that:

a) $\varphi_z$ is a continuous function on $S^{(I-1)}$.

b) $\varphi_z$ is 1-1 on $S^{(I-1)}$.

c) $\varphi_z[S^{(I-1)}] = U[E_{IP}(z)]$.

d) $\varphi_z^{-1}: U[E_{IP}(z)] \longrightarrow S^{(I-1)}$ is continuous on $S^{(I-1}$ .

The continuity of $\varphi_z$ having already been established, consider part b) and suppose $\varphi_z$ is *not* 1-1 on $S^{(I-1)}$. Then $\exists$ distinct $a^\#$, $a^\dagger \in S^{(I-1)}$ such that $\varphi_z(a^\#) = \varphi_z(a^\dagger) \Leftrightarrow$ $\text{ARGMAX}_{[\omega' \in \varsigma_z(a^\#)]}\{\omega' \cdot \mathbb{1}^I\} = \text{ARGMAX}_{[\omega' \in \varsigma_z(a^\dagger)]}\{\omega' \cdot \mathbb{1}^I\} \Leftrightarrow \omega(z)+\lambda^\# * a^\# =$

$\omega(z)+\lambda^{\dagger}*a^{\dagger} \Leftrightarrow \lambda^{\#}*a^{\#} = \lambda^{\dagger}*a^{\dagger}$. Since $a^{\#}$, $a^{\dagger}$ are distinct directions in $\mathbb{R}^I_+$ it must then be the case that $\lambda^{\#} = \lambda^{\dagger} = 0 \Rightarrow \omega(z) \in U[E_{IP}(z)]$--a contradiction.

It is straightforward to show that $\varphi_z[S^{(I-1)}] = \mathrm{fr}(dhU[E_{IP}(z)]) \cap K_{+}\omega(z)$; to prove c) it remains to demonstrate that $\mathrm{fr}(dhU[E_{IP}(z)]) \cap K{+}\omega(z) = U[E_{IP}(z)]$. Recall that the frontier (relative to $\mathbb{R}^I_+$) of a set $X \subseteq \mathbb{R}^I_+$ is defined by $\mathrm{fr}(X) = \mathrm{cl}(x) \cap \mathrm{cl}(\mathbb{R}^I_+ \sim X)$.

To show $U[E_{IP}(z)] \subseteq \mathrm{fr}(dhU[E_{IP}(z)]) \cap K_{+}\omega(z)$, let $z' \in E_{IP}(z)$ and note $(1+\lambda)U(z') \in K_{+}\omega(z)$, $\forall\, \lambda \geq 0$. Choose $\lambda' = \mathrm{lub}\{\, \lambda^{\#} \geq 0 \mid (1+\lambda^{\#})U(z') \in dhU[E_{IP}(z)] \,\}$; $\lambda'$ exists since $dhU[E_{IP}(z)]$ is compact and hence bounded. Notice that $(1+\lambda')U(z') \in \mathrm{fr}(dhU[E_{IP}(z)])$, as can be established directly from the compactness and disposability of $dhU[E_{IP}(z)]$. In fact, $\lambda' = 0$; for suppose not, then let $0 < \lambda'' < \lambda'$, and consider that $(1+\lambda'')U(z') >> U(z')$. Furthermore, $(1+\lambda'')U(z') \in dhU[E_{IP}(z)]$ since $(1+\lambda')U(z') \in \mathrm{fr}(dhU[E_{IP}(z)])$. But by definition of disposable hull, then $\exists\, z^{\dagger} \in E_{IP}(z)$ such that $U(z^{\dagger}) \geq\geq (1+\lambda'')U(z') >> U(z')$-- a contradiction since both $z^{\dagger}$, $z' \in E_{IP}(z)$.

To show $\mathrm{fr}(dhU[E_{IP}(z)]) \cap K_{+}\omega(z) \subseteq U[E_{IP}(z)]$, let $\omega' \in \mathrm{fr}(dhU[E_{IP}(z)]) \cap K{+}\omega(z)$; it suffices that $\exists\, z'' \in E_{IP}(z)$ such that $U(z'') = \omega'$. It has been previously shown that $\exists\, z'' \in E_{IP}(z)$ such that $U(z'') \geq\geq \omega$ . By the disposability of $dhU[E_{IP}(z)]$, WOLOG let $U(z'')$ be of the form $(1+\lambda'')\,\omega'$. Consider that $\mathrm{lub}\{\, \lambda^{\#} \geq 0 \mid (1+\lambda^{\#})\omega' \in dhU[E_{IP}(z)] \}$ $= 0$, for otherwise $\omega' \notin dhU[E_{IP}(z)]$. Since $U(z'') = (1+\lambda'')\,\omega' \in dhU[E_{IP}(z)]$, this means $\lambda'' = 0$ so that $U(z'') = \omega'$, and so completes the proof of c).

Finally, to demonstrate d), observe that whenever $\omega(z) \notin U[E_{IP}(z)]$, $\varphi_z$ has been shown to be a 1-1 continuous function defined on the *compact* domain $S^{(I-1)}$ onto $U[E_{IP}(z)] \subseteq \mathbb{R}^I_+$. But this means the inverse $\varphi_z^{-1}: U[E_{IP}(z)] \longrightarrow S^{(I-1)}$ is necessarily continuous, see 17.14 of Willard [1970]. QED.

PROPOSITION 4.5 ($\chi_z$ is a Homeomorphism)

Suppose that U is strictly quasiconcave, then $\forall$ $z \in Z_\zeta$ such that $\gamma \geq\geq 0$ and $\delta >> 0$:

i) $\chi_z[S^{(I-1)}] = E_{IP}(z)$.

ii) And if $z \notin E_{IP}(z)$ then $E_{IP}(z)$ is the homeomorphic image of $S^{(I-1)}$ under $\chi_z$.

Proof

Let $z \in Z_\zeta$, $\zeta \geq\geq 0$ be given; it has been previously demonstrated that $\chi_z$: $S^{(I-1)} \to\to Z_\zeta|_b$ is upper hemicontinuous. In order to show $\chi_z$ is a homeomorphism under the additional assumption $\delta >> 0$, it is required to verify that:

i) $\chi_z$ is a continuous function on $S^{(I-1)}$.

ii) $\chi_z$ is 1-1 on $S^{(I-1)}$ whenever $z \notin E_{IP}(z)$.

iii) $\chi_z[S^{(I-1)}] = E_{IP}(z)$.

iv) $\chi_z^{-1}$: $[E_{IP}(z) \longrightarrow S^{(I-1)}$ is continuous on $S^{(I-1)}$ whenever $z \notin E_{IP}(z)$ .

To show i), it suffices to demonstrate that $\chi_z$ is single valued. First it will be shown that $\chi_z$ is convex valued. Let $a' \in S^{(I-1)}$, and suppose $z^\#, z^\dagger \in \chi_z(a')$ so that $\omega(z^\#) = \omega(z^\dagger) = \varphi_z(a')$. Since U is strictly quasiconcave, $\omega$ is also quasiconcave on $Z_\zeta|_b$; hence, for $0 \leq t \leq 1$, $\omega(tz^\#+(1-t)z^\dagger) \geq\geq \varphi_z(a')$. By convexity of $Z_\zeta|_b$, $tz^\#+(1-t)z^\dagger \in Z_\zeta|_b$. But then $\omega(tz^\#+(1-t)z^\dagger) = \varphi_z(a')$, for otherwise $tz^\#+(1-t)z^\dagger$ would Pareto dominate $z^\#$ and $z^\dagger$ in $Z_\zeta|_b$, a contradiction. To show $\chi_z$ single valued, suppose not; then for some $a' \in S^{(I-1)}$, $\exists$ distinct $z^\#, z^\dagger \in \chi_z(a')$. From the argument just above $\omega(tz^\#+(1-t)z^\dagger) = \omega(z^\#) = \omega(z^\dagger) = \varphi_z(a')$. However, since U is strictly quasiconcave and $\delta >> 0$, $\omega$ is also strictly quasiconcave on $Z_\zeta|_b$; therefore, $\omega(tz^\#+(1-t)z^\dagger) >>$ both $\omega(z^\#)$ and $\omega(z^\dagger)$--a contradiction.

To show ii), suppose $z \notin E_{IP}(z)$ and that $\chi_z$ is not 1-1; then $\exists$ distinct a', a'' $\in S^{(I-1)}$ such that $\chi_z(a') = \chi_z(a'') \Leftrightarrow \exists\ z', z'' \in Z_\zeta|_b$ such that $\omega(z') = \varphi_z(a') = \omega(z'') = \varphi_z(a'')$ $\Leftrightarrow \omega(z)+(1+\lambda')a' = \omega(z)+(1+\lambda'')a'' \Rightarrow \lambda' = \lambda'' = 0 \Rightarrow \omega(z) = \varphi_z(a') = \varphi_z(a'') \Rightarrow$ $z \in E_{IP}(z)$, a contradiction.

To show iii), notice that if $z \in E_{IP}(z)$ so that $\{z\} = E_{IP}(z)$, the issue is trivial; hence suppose $z \notin E_{IP}(z)$ . Let $z' \in E_{IP}(z)$; since $z \notin E_{IP}(z)$, $\omega'(z') >> \omega(z)$ and so defines a direction $a' \in \mathbb{R}^I_+$ parallel to $(\omega(z), \omega'(z'))$. Direct application of the definiton of $\chi_z$ gives $\chi_z(a') = z'$.

Finally, in the case of iv), recall that a 1-1 continuous map from a compact space to a Hausdorff space has a continuous inverse. QED.

PROPOSITION 4.6 ($E_{IP}$ is Contractible Valued)

Suppose that U is strictly quasiconcave, then $E^\zeta_{IP}: Z_\zeta \rightarrow\rightarrow Z_\zeta$ is a contractible valued correspondence, $\forall\ \gamma \geq\geq 0, \delta >> 0$.

Proof

The issue is trivial in case $z \in E_{IP}(z)$; otherwise, $E_{IP}(z)$ is the homeomorphic image of $S^{(I-1)}$ under $\chi_z$ and so is contractible. Notice the assumption that the distinguished goods vector is strictly positive has been employed previously. QED.

# V. SHAREHOLDER VOTING

This section analyzes shareholder voting in a stock market economy. In particular, the direction restricted, share weighted majority rule equilibria are characterized as fixed points of a so-called ***voting corner function*** V defined on $Z_\zeta$.

The analysis proceeds by first demonstrating continuity of the feasible plan proposal correspondence. Next the $\varepsilon$-majority rule formalism is introduced. Then the correspondences which specifiy the set of potential winning proposals along each voting direction are shown to be continuous, convex-valued and one-sided. Berge's maximum theorem gives the continuity of the functions which distinguish, for each voting direction, the most distant proposal capable of replacing the status quo. The composition of these functions then provides the desired continuous voting corner function. Finally, the set of voting equilibria is shown to coincide with the set of fixed points of the corner function.

DEFINITION (Feasible Proposal Correspondences $\beta$ and $\beta_{js}$)

Let $\{e_1, \ldots, e_{js}, \ldots, e_{JS}\}$ denote the standard basis for the space $\mathbb{R}^{JS}_+$ of firms' production plans . Suppose that $\zeta \geq\geq 0$, and let $z = (x, b, \theta) \in Z_\zeta$.

i) Define the ***feasible proposal correspondence*** $\beta: Z_\zeta \rightarrow\rightarrow Z_\zeta$ according to:

$$\beta(z) = \{ z^\# = (x^\#, b^\#, \theta^\#) \in Z_\zeta \mid$$

$$\text{a)} \quad x^\# = x + \theta \cdot [C(b) - C(b^\#)] \geq\geq 0,$$

$$\text{b)} \quad b^\# = b + \alpha e_{js}, \text{ for some direction } e_{js} \text{ and } \alpha \in \mathbb{R}, \text{ and}$$

$$\text{c)} \quad \theta^\# = \theta \geq\geq \gamma >> 0 \}.$$[1]

ii) Given a direction $e_{js}$ define the correspondence $\beta_{js} : Z_\zeta \rightarrow\rightarrow Z_\zeta$ according to:

$$\beta_{js}(z) = \beta(z) \cap \{ z^\# = (x^\#, b^\#, \theta^\#) \in Z_\zeta \mid b^\# = b + \alpha e_{js}, \alpha \in \mathbb{R} \}.$$

$\beta(z)$ should be interpreted as the set of feasible proposals which can be made by shareholders to depart from the status quo z. The three conditions in the definition above embody fundamental assumptions regarding shareholder voting and the financing of new production plans. $\beta_{js}(z) \subseteq \beta(z)$ is the set of feasible proposals to change the production plan of a single firm j in a single future state of the world s.

Condition a) in the definition of $\beta$ states that the current consumption specified under any proposal must be obtained from the status quo current consumption vector by making appropriate share weighted refunds or assessing additional charges to current shareholders, reflecting changed production costs under the new proposal. Furthermore, the current consumption vector under any new proposal must be non-negative across individuals. This is a non-bankruptcy condition. Note that changes in production plans must be financed *exclusively* by firms' current shareholders from their stocks of the date zero consumption / production good. No proposal can suggest production changes which would require (share weighted) contributions which would

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**1.** In the definition of $\beta(z)$, it may have been noticed that x is a $(1 \times I)$ vector while the expression $x + \theta \cdot [C(b) - C(b')]$ is $(I \times 1)$. Thus their sum is technically not well defined since the summands are not conformable. In the interest of notational simplicity, the reader is asked to make the obvious transposistion.

bankrupt any shareholder. Firms cannot issue debt or new equity, and neither can individuals borrow from one another.

Condition b) specifies that any proposal to change the status quo production plan b can only differ from b along one of the basis directions for the space $\mathbb{R}^{JS}{}_{+}$ of firms' production plans. That is, insofar as production plans are concerned, a proposal can only suggest modifying the production of a single firm in a single futute state of the world. This condition has been adopted to utilize the direction restricted majority rule equilibrium existence results first proved by Kramer [1972]. It is well known that, in general, majority rule mechanisms may cycle over multi-dimensional choice spaces. Direction restrictions resolve the problem of non-existence of majority rule voting equilibrium without imposing stringent conditions on voters' preferences.

Finally, condition c) states that a proposal cannot change the pattern of ownership of firms' shares from the status quo situation. This is an instance of the fundamental separation property of the stock market economy. There are two basic mechanisms for changing the state of the economy, voting and stock market trading. However, these mechanisms cannot operate simultaneously. During trade, firms' plans must remain fixed; and during production plan voting, no shares can be exchanged.

An important technical complication arises in the model because $\beta$ may fail to be lower hemicontinuous on $Z_\zeta$ unless every individual holds some (arbitrarily small) positive fraction of every firm. This is the only reason that the the assumption $\gamma >> 0$ appears at various junctures in the development of equilibrium existence results.[2]

PROPOSITION 5.1 (Upper Hemicontinuity of the Feasible Proposal Correspondences)

Suppose that **C** is continuous, then $\forall\ \zeta \geq\geq 0$:

i) $\beta: Z_\zeta \longrightarrow Z_\zeta$ is upper hemicontinuous.

ii) Each correspondence $\beta_{js}: Z_\zeta \longrightarrow Z_\zeta$ is upper hemicontinuous.

Proof

Part i) will be demonstrated here, the proof of the second part being straightforward given the first. Let a sequence $(z_n)$ in $Z_\zeta$ be given such that $(z_n) \longrightarrow z_o$. Since $\beta$ is compact valued on $Z_\zeta$, $\forall\ \zeta \geq\geq 0$, it suffices that if $(z^{\#}_n) \longrightarrow z^{\#}_o$, where $z^{\#}_n \in \beta(z_n)$, then $z^{\#}_o \in \beta(z_o)$. Thus, recalling the definition of of $\beta$:

$$\beta(z) = \{\ z' = (x',b',\theta') \in Z_\zeta \mid x' = x+\theta\cdot[C(b)-C(b')],\ b' = b+\alpha e_{js},\ \theta' = \theta\},$$

it is required to show:

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2. Failure of lower hemicontinuity can occur as follows. Consider a sequence of points in $Z_\zeta$, identical except for the shareholding matrices. At each term in the sequence, let some individual i have, exactly 0 units of the current good. Suppose individual i is always a shareholder of firm j, and always the only (weakly) bankrupt shareholder. Since i is bankrupt, no proposal to increase j's production is feasible. Now let i's holdings vanish in the limit. Suddenly-- and discontinuously-- the remaining non-bankrupt shareholders are free to make expansionary proposals.

This discontinuity resembles the situation in exchange economies in which consumers' demand correspondences may fail to be lower hemicontinuous at the minimum wealth level which will permit consumption in their consumption sets. Similarly, the solution tends to take the form of minimum wealth constraints; see Border [1985].

a) $x^{\#}_{o} = x_o + \theta_o \cdot [C(b_o) - C(b^{\#}_{o})] \geq\geq 0$,

b) $x^{\#}_{o} \geq\geq 0$,

c) $b^{\#}_{o} \geq\geq 0$,

d) $b^{\#}_{o} = b_o + \alpha e_{js}$, for some $\alpha \in \mathbb{R}$ and some voting direction $e_{js}$, and

e) $\theta^{\#}_{o} = \theta_o$.

Recall that the set of voting directions $\{e_1, \ldots, e_{js}, \ldots e_{JS}\}$ is the standard (orthogonal, unit length) basis for $\mathbb{R}^{JS}_{+}$, the space of firms' production plans.

Since $z^{\#}_{n} \in \beta(z_n)$, each $z^{\#}_{n}$ is of the form $x_n + \theta_n \cdot [C(b_n) - C(b^{\#}_{n})]$. Thus a) obtains from the continuity of each component $C_j$ of C, since both $(z_n) \longrightarrow z_o$ and $(z^{\#}_{n}) \longrightarrow z^{\#}_{o}$. To show b) consider that $x^{\#}_{n} \geq\geq 0$ since each $z^{\#}_{n} \in \beta(z_n)$; thus $(x^{\#}_{n}) \longrightarrow x^{\#}_{o} \Rightarrow x^{\#}_{o} \geq\geq 0$. Part c) obtains by a similar argument. In the case of d), since each $z^{\#}_{n} \in \beta(z_n)$ then $\lim(b^{\#}_{n}) = \lim(b_o + \alpha_n e_n) = b_o + \lim(\alpha_n e_n)$, where $\alpha_n \in \mathbb{R}$ and $e_n \in \{e_1, \ldots, e_{JS}\}$. (Notice that, in general, $e_n$ may depend on the sequence index n.) If $b^{\#}_{o} = b_o$, the problem is vacuous; therefore, suppose $b^{\#}_{o} \neq b_o$. In this case both $\lim(\alpha_n)$ and $\lim(e_n)$ exist are non-zero, so it suffices that $\lim(e_n) = e_{js}$, for some standard basis vector. Claim that as $(e_n)$ converges, eventually $e_n$ = some one particular basis vector $e_{js}$. For otherwise, given any $N \in \mathbb{N}_{+}$, $\exists$ n,m > N such that $e_n \neq e_m$. Now since $(e_n)$ is Cauchy, given any $\xi > 0$-- and in particular, let $\xi < 2^{(1/2)}$-- $\exists$ $M \in \mathbb{N}_{+}$ such that whenever n,m > M then $|e_n - e_m| < \xi$. But, as above, n,m > M can be chosen such that $e_n \neq e_m$; since both are standard unit basis vectors, $|e_n - e_m| = 2^{(1/2)} > \xi$, a contradiction which proves $\lim(e_n) = e_{js}$ and so concludes the demonstration of e). Finally, e) obtains since each $z^{\#}_{o} \in \beta(z_n)$ implies $\theta^{\#}_{n} = \theta_n$; given that $(z_n) \longrightarrow z_o$, then $\theta^{\#}_{o} = \lim(\theta^{\#}_{n}) = \lim(\theta_n) = \theta_o$, as desired.

Notice that the argument just given is consistent with $\zeta = 0$; in particular individuals need not maintain strictly positive holdings of every firm in order to demonstrate the upper hemicontinuity of $\beta$. QED.

PROPOSITION 5.2 (Lower Hemicontinuity of the Feasible Proposal Correspondences)

Suppose that C is non-negative, continuous, and strictly monotinc; then $\forall\ \gamma >> 0$, $\forall\ \delta \geq\geq 0$:

i) $\beta: Z_\zeta \longrightarrow Z_\zeta$ is lower hemicontinuous.

ii) Each $\beta_{js}: Z_\zeta \longrightarrow Z_\zeta$ is lower hemicontinuous.

Proof

Part i) will be demonstrated here; the argument for part ii) is similar. Let $(z_n)$ be a sequence in (the compact set) $Z_\zeta$ such that $(z_n) \longrightarrow z_o$, and choose $z^\#_o \in \beta(z_o)$. Will construct $(z^\#_n) \longrightarrow z^\#_o$, where $z^\#_n \in \beta(z_n)$. Temporarily adopt the notation $(z_n) = ((x_n, b_n, \theta_n))$. Notice that since $z^\#_o \in \beta(z_o)$ and $z_o \in Z_\zeta$, then $z^\#_o$ must be of the form $(x^\#_o, b^\#_o, \theta^\#_o)$, where:

a) $x^\#_o = x_o + \theta_o \cdot [C(b_o) - C(b_o + \alpha_o e_{js})]$ for some $\alpha_o \in \mathbb{R}$ and planning (basis) vector $e_{js}$, and

b) $b^\#_o = b_o + \alpha_o e_{js} \geq\geq 0$, and iii) $\theta^\#_o = \theta_o \geq\geq \gamma >> 0$.

Define $\alpha_n = \mathrm{ARGMIN}_{[\alpha' \in \mathbb{R}]} \{ |\alpha' - \alpha_o| \ \big|\ (x_n + \theta_n \cdot [C(b_n) - C(b_n + \alpha' e_{js})], b_n + \alpha' e_{js}, \theta_n) \in Z_\zeta$, and $\alpha' \in [-|\alpha_o|, |\alpha_o|] \}$. Equivalently, this definition specifies $\alpha_n$ as the unique solution to:

$$\mathrm{MIN}_{[\alpha' \in \mathbb{R}]} |\alpha' - \alpha_o|$$

S.T. 1) $x_n + \theta_n \cdot [C(b_n) - C(b_n + \alpha' e_{js})] \geq\geq 0$,

2) $b_n + \alpha' e_{js} \geq\geq 0$, and

3) $\alpha' \in [-|\alpha_o|, |\alpha_o|]$.

Claim that $\alpha_n$ is well defined. Clearly $\alpha' = \alpha_o$ is the unique solution to the problem if neither constraint 1) nor 2) fails. Suppose, however, that 1) is violated in which case at least one component of the left hand side must be negative. Recall that the vector C of cost functions is non-negative and strictly monotonically increasing in each of its components (each defined over $\mathbb{R}^S_+$) and $z_n \in Z_\zeta$ so that $x_n \geq\geq 0$ and $\theta_n \geq\geq \gamma >> 0$. Thus a violation of 1) can occcur when choosing $\alpha' = \alpha_o$ only if $\alpha_o > 0$, and is sufficiently large that the expansionist plan $b_n+\alpha_o e_{js}$ would bankrupt some shareholder of firm j. In this case, the unique optimal choice of $\alpha'$ must lie in $[0,\alpha_o]$. For if $\alpha' = \alpha_o$ violates 1), $\alpha' > \alpha_o$ surely will also; furthermore, $\alpha'$ is feasible since $z_n \in Z_\zeta$, while choosing $\alpha'$ negative always results in a larger value for the minimand. Uniqueness of the solution, $\alpha'$, in $[0,\alpha_o]$ obtains since each component of the left hand side of 1) increases monotonically as the choice of $\alpha'$ is reduced from $\alpha_o$ toward 0. Note 2) is satisfied by any $\alpha' \in [0,\alpha_o]$, for $\alpha_o > 0$.

On the other hand, if 2) fails when $\alpha' = \alpha_o$ then since both $b_n$, $e_{js} \geq\geq 0$, $\alpha_o$ must be negative-- corresponding to a reduction of firm j's production in future state s. (Failure of 2) can occur only if the norm of the contemplated reduction $\alpha_o e_{js}$ exceeds the total planned production on dimension js under $b_n$. The constraint says that the firm cannot produce negative quantities in any state.) In this case, the optimal solution must lie in $[\alpha_o,0]$. For if $\alpha' = \alpha_o < 0$ violates 2), so will $\alpha' < \alpha_o$. Notice also that $\alpha' = 0$ is feas - ible since $z_n \in Z_\zeta$, while positive choice of $\alpha'$ can only result in a larger value of the minimand than $\alpha' = 0$. Uniqueness of $\alpha'$ obtains from the uniqueness of the solutions to $b_n + \alpha' e_{js} = 0$ (restricting attention to the $j^{\text{th}}$ component) as $\alpha'$ increases from $\alpha_o$ toward 0. Thus $\alpha_n$ has been shown to be well defined.

Now construct the sequence $(z^\#_n)$ according to $z^\#_n = \big(x_n+\theta_n\cdot[C(b_n)-C(b_n+\alpha_n e_{js})]$, $b_n+\alpha' e_{js}$, $\theta_n\big)$. Clearly $z^\#_n \in Z_\zeta$, because both $z_n \in Z_\zeta$ and by definition of $\alpha_n$. Furthermore, inspection of the definitions of $z_n$ and $\beta$ will reveal that, as constructed,

$z^{\#}_n \in \beta(z_n)$ since $z^{\#}_n$ is obtained from $z_n$ by: i) changing the production plan of a single firm in a single future state, ii) making the required adjustments in shareholders' current accounts without causing bankruptcy, and iii) while holding $\theta_n$ fixed.

Claim that if $(\alpha_n) \longrightarrow \alpha_o$ then since $(z_n) = (x_n, b_n, \theta_n) \longrightarrow z_o = (x_o, b_o, \theta_o)$ then $(z^{\#}_n) = (x^{\#}_n, b^{\#}_n, \theta^{\#}_n) \longrightarrow z^{\#}_o = (x^{\#}_o, b^{\#}_o, \theta^{\#}_o)$. It suffices to prove the claim componentwise. Trivially, $(\theta^{\#}_n) \longrightarrow \theta^{\#}_o$ since $\theta^{\#}_n = \theta_n$ and $\theta^{\#}_o = \theta_o$. To show that $(b^{\#}_n) \longrightarrow b^{\#}_o$, observe that $(b_n) \longrightarrow b_o$, and $b^{\#}_o = b_o+\alpha_o e_{js}$, while by definition $(b^{\#}_n) = (b_n+ \alpha_n e_{js})$. Thus $(b^{\#}_n) \longrightarrow \lim(b_n+\alpha_n e_{js}) = \lim(b_n) + \lim(\alpha_n)e_{js} = b_o+\alpha_o e_{js}$, since it was assumed $(\alpha_n) \longrightarrow \alpha_o$. Finally, to show $(x^{\#}_n) \longrightarrow x^{\#}_o$, notice that: i) $(x_n) \longrightarrow x_o$, ii) also recall that $x^{\#}_o = x_o+\theta_o \cdot [C(b_o)-C(b_o+\alpha_o e_{js})]$, and iii) observe by definition $(x^{\#}_n) = \big(x_n+\theta_n \cdot [C(b_n)-C(b_n+\alpha_n e_{js})]\big)$. Therefore, $(x^{\#}_n) \longrightarrow \lim\big(x_n + \theta_n \cdot [C(b_n)-C(b_n+\alpha_n e_{js})]\big)$ $= \lim(x_n)+\lim(\theta_n)\cdot[\lim C(b_n)-\lim C(b_n+\alpha_n e_{js})]$, which by the continuity of C equals $x_o+\theta_o \cdot [C(\lim(b_n))-C(\lim(b_n)+\lim(\alpha_n)e_{js})] = x_o+\theta_o \cdot [C(b_o)-C(b_o+\alpha_o e_{js})] = x^{\#}_o$, completing the proof of the claim.

It remains to show that $(\alpha_n) \longrightarrow \alpha_o$. Consider the three cases $\alpha_o = 0$, $\alpha_o < 0$, and $\alpha_o > 0$. If $\alpha_o = 0$, each $\alpha_n = 0$, directly from their definition, and convergence is trivial. If $\alpha_o < 0$, there is no possibility that the plan $b_n + \alpha_o e_{js}$ could bankrupt any individual. However, some $\alpha_n$ might have to be chosen larger (less negative) than $\alpha_o$ in the interval $[\alpha_o, 0]$ in order to insure that the expression $b_n+\alpha_n e_{js}$ is non-negative in its $js^{\underline{th}}$ component, whenever the corresponding component of of $b_n$ is less than that of $b_o$. Nevertheless, as $(b_n) \longrightarrow b_o$, $\alpha_n$ can clearly be chosen closer to $\alpha_o$ without making the $js^{\underline{th}}$ component of $b_n+\alpha_n e_{js}$ negative, since the corresponding component of $b_o+\alpha_o e_{js}$ is non-negative. Hence $(\alpha_n) \longrightarrow \alpha_o$ in this case.

Finally, turn to the case $\alpha_o > 0$, and suppose $(\alpha_n) \not\longrightarrow \alpha_o$. Then since each $\alpha_n \in [0,\alpha_o]$, there exists some convergent subsequence $(\alpha_m) \longrightarrow \alpha'' \neq \alpha_o$. In particular, notice that $\alpha'' < \alpha_o$ since it must be the case that $\alpha'' \in [0,\alpha_o)$. Define $x'' = x_o+\theta_o\cdot[C(b_o)-C(b_o+\alpha'' e_{js})]$; claim that at least one component of x" must be non-positive. For otherwise, that is if x" is componentwise strictly positive, continuity of C and the convergence $(\alpha_m) \longrightarrow \alpha''$ imply that within any neighborhood of x" there are eventually terms $x^{\#}_m = x_m+\theta_m\cdot[C(b_m)-C(b_m+\alpha_m e_{js})]$ of $(x^{\#}_m)$ which are componentwise strictly positive. Consider that since $(\alpha_m) \longrightarrow \alpha'' \neq \alpha_o$, for m sufficiently large, the terms $\alpha_m$ must lie outside some neighborhood of $\alpha_o$. However, since $\alpha_o > 0$, optimally chosen $\alpha_m$ can be different from $\alpha_o$ only if the constraint $x^{\#}_m \geq\geq 0$ is binding with respect to some component. Such cannot be the case if $x^{\#}_m$ is strictly positive, contradicting the constructed minimality of $\alpha_m$, and proving the claim. Now, notice that since some component of x" *is* non-positive then at least one component of $x^{\#}_o = x_o+\theta_o\cdot[C(b_o)-C(b_o+\alpha_o e_{js})]$ must be strictly negative since $\alpha_o > \alpha''$ and $\theta_o \geq\geq \gamma >> 0$. But this contradicts the assumption that $z^{\#}_o = (x^{\#}_o, b^{\#}_o, \theta^{\#}_o) \in Z_\zeta \subseteq \mathbb{R}^{(I+JS+IJ)}{}_+$, and so concludes the demonstration of the last case. Notice that the strict no short sales assumption, $\theta_o \geq\geq \gamma >> 0$, was required above. QED.

<u>DEFINITION</u> (Preference Indicator Vectors)

Let $z, z^{\#} \in Z_\zeta$, and $\zeta \geq\geq 0$.

i) $ch(z^{\#}|z) = \{\, i \in I \mid U_i(x^{\#}_{oi}, (\theta^{\#}\cdot b^{\#})_i, (\theta^{\#}\cdot\delta)_i) \geq U_i(x_{oi}, (\theta\cdot b)_i, (\theta\cdot\delta)_i) \,\}$. This is the set of all individuals who weakly prefer (consumption available in) state $z^{\#}$ to state z.[3]

ii) $ch^*(z^{\#}|z)$ denotes the set of all individuals who strictly prefer $z^{\#}$ to z.

iii) $\mathbb{1}_{\{ch\}}(z^{\#}|z)$ is the $(1 \times I)$ indicator row vector for the set $ch(z^{\#}|z)$.

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3. In the context of shareholder voting, $z^{\#}$ is typically a challenger which has been proposed against some status quo z.

iv) $\mathbb{1}^*_{\{ch\}}(z^\# | z)$ is the indicator vector for $ch^*(z^\# | z)$.

<u>DEFINITION</u> ($\varepsilon$-Majority Rule Direction Restricted Winner Correspondences $v^\varepsilon_{js}$)

Let $b^* = \mathrm{MAX}\ \{\ |b - b^\#|\ \mid \exists\ z = (x,b,\theta),\ z^\# = (x^\#,b^\#,\theta^\#) \in Z_\zeta\}$.[4] For any $\varepsilon > 0$ satisfying the condition:

$$\varepsilon < .5 / b^*,$$

and any direction $e_{js}$ for $\mathbb{R}^{JS}_+$, the space of firms' production plans, define the ***winner correspondences*** $v^\varepsilon_{js}$: $Z_\zeta \to\to Z_\zeta$ according to:

$$v^\varepsilon_{js}(z) = \Big\{\ z^\# = (x^\#,b^\#,\theta^\#) \in Z_\zeta\ \Big|$$

i) $z^\# \in \beta(z)$

ii) $b^\# = b + \alpha e_{js}$; for some $\alpha \in \mathbb{R}$ , and for direction $e_{js}$

iii) $\mathbb{1}_{\{ch\}}(z^\# | z) \cdot \theta^j \geq .5 + \varepsilon |b - b^\#|\ \Big\}$

Whenever there is no possibility for confusion, $v_{js}$ will be written for $v^\varepsilon_{js}$.

$v_{js}(z)$ should be interpreted as the set of non-bankrupting proposals to change the production of firm j in state s, while making required share weighted adjustments in current consumption, *which would defeat the status quo* z under share weighted, $\varepsilon$-majority rule.

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4. Notice that $b^*$ is well defined by the compactness of $Z_\zeta$, and that $b^*$ does not depend on the choice of $\zeta$.

<u>REMARKS</u> (Properties of the Winner Correspondences)

A few comments may help to clarify the rather involved definition above. Observe that the $\varepsilon$-majority rule voting mechanism is defined by specifying the associated winner correspondences.

$\varepsilon$-majority rule is a technical device which facilitates the derivation of continuity properties of shareholder voting. As will be seen, $\varepsilon$-majority rule can eventually be dispensed with. An equilibrium of a stock market economy with $\varepsilon$-majority rule is also an equilibrium of the same economy with standard ($\varepsilon = 0$) majority rule. Notice that $\varepsilon$ is a parameter of the model, and is not a matter of choice for the participants.

$\varepsilon$-majority rule is a modification of standard majority rule which *eliminates the possibility of tie votes* and thus renders weighted majority rule continuous in voters' weights. The mechanism stipulates that the distance in policy space which a majority coalition can depart from the status quo is positively proportional to the size of the coalition's margin of victory or ***mandate***, $\varepsilon$ being the proportionality factor. In the present instance, the policy space is the space of firms' plans. The condition that $\varepsilon < .5 / b^*$ insures that the $\varepsilon$-majority rule mechanism satisfies the Pareto principle; that is, keeping in mind the direction restrictions, the coaliton of the whole always possesses a sufficiently large mandate to attain any point in the policy space.

Condition i), namely that $z^\# \in \beta(z)$, entails the various requirements set forth in the definiton of $\beta$. These are, first of all, that $\theta^\# = \theta$; in other words, the process of shareholder voting leaves share ownership unchanged. Secondly, the change from z to $z^\#$ cannot have bankrupted any shareholders. Thirdly, there can have been no loss of resources or costs of voting in moving from $z^\#$ to z; any changes in current consumption arising from the move must be due solely to changes in production costs.

Finally, membership in $\beta(z)$ requires that the production plan $b^{\#}$ specified under $z^{\#}$ can differ from b only along one of basis directions for $\mathbb{R}^{JS}{}_{+}$. This means $b^{\#}$ can differ with respect to b only in the production of one firm in one future state; the importance of such direction restrictions has been discussed earlier.

Condition ii), that $b^{\#} = b+\alpha e_{js}$, for some $\alpha \in \mathbb{R}$, simply specifies the single dimension along which $b^{\#}$ and b can differ, namely the $js^{\underline{th}}$ dimension. Changes in the production plan matrix along dimension js should be regarded as changes in the planned production of one firm j in one future state s. Notice that corresponding to each planning direction there is an associated winner correspondence.

Condition iii), that $\mathbb{1}_{\{ch\}}(z^{\#}|z)\cdot\theta^{j} \geq .5 + \varepsilon|b^{\#}-b|$, is the fundamental characteristic of ε-majority rule. $\mathbb{1}_{\{ch\}}(z^{\#}|z)$ is the indicator vector for all individuals *weakly* preferring challenger $z^{\#}$ to the status quo z. It is assumed that indifferent voters *vote for the challenger;* while perhaps unusual, this assumption, too, will ultimately be relaxed. $\theta^{j} \in S^{I}$ is an $(I\times 1)$ collumn vector which specifies, in percentage terms, individuals' holdings in firm j. Thus the dot product of $\mathbb{1}_{\{ch\}}(z^{\#}|z)$ with $\theta^{j}$ is the share weighted vote of all those individuals supporting $z^{\#}$ over z. Notice that individuals who do not hold shares in firm j have weight zero under $\theta^{j}$. In order for the coalition indicated by $\mathbb{1}_{\{ch\}}(z^{\#}|z)$ to prevail, their weighted vote must be at least $.5+\varepsilon|b^{\#}-b|$; consider that $|b^{\#}-b|$ is magnitude of the proposed change in firm plans. The summand $\varepsilon|b^{\#}-b|$ should be interpreted as the size of the mandate required by ε–majority rule for the adoption of the challenger.

PROPOSITION 5.3 (Upper Hemicontinuity of the Winner Correspondences)

Suppose that U is continuous, and that the assumptions for Proposition 5.1 hold.

i) Each $v_{js}: Z_\zeta \longrightarrow Z_\zeta$ is upper hemicontinuous whenever $\varepsilon > 0$, $\forall\ \zeta \geq\geq 0$.

ii) If it is further assumed that tie votes are awarded to challenging proposals, the proposition holds for $\varepsilon = 0$.

Proof

Let $(z_n)$ be a sequence in $Z_\zeta$ such that $(z_n) \longrightarrow z$, and suppose $(z^\#_n) \longrightarrow z^\#$, where $z^\#_n \in v_{js}(z_n)$. Since $v_{js}$ is compact valued on $Z_\zeta$, it suffices to show that $z^\# \in v_{js}(z)$.

The requirements for $z^\#$ to be a member of $z^\# \in v_{js}(z)$ are that keeping $\theta$ fixed, $z^\#$ must be a non-bankrupting, js$^{\underline{th}}$ direction, $\varepsilon$-majority rule winning proposal with respect to z. These requirements can be expressed more formally, in order, as follows:

a) $\theta^\# = \theta$,

b) $x^\# = x + \theta \cdot [C(b) - C(b^\#)] \geq\geq 0$,

c) $b^\#$ is of the form $b + \alpha e_{js} \geq\geq 0$, and

d) $\mathbb{1}_{\{ch\}}(z^\# | z) \cdot \theta^j \geq .5 + \varepsilon | b^\# - b |$.

Condition a) obtains trivially since $z^\#_n \in v_{js}(z_n) \Rightarrow \theta^\#_n = \theta_n$. Notice that b) follows from the previously demonstrated upper hemicontinuity of $\beta$. For observe $z^\#_n \in v_{js}(z_n)$ $\Rightarrow z^\#_n \in \beta(z_n)$; thus, $z^\# \in \beta(z)$ which entails b) directly from the definition of $\beta$. Requirement c) obtains since each $b^\#_n$ must be of the form $b_n + \alpha_n e_{js} \geq\geq 0$ and the given facts that $\lim(b^\#_n)$ exists while $\lim(b_n) = b$.

It remains to show d), namely that $z^\#$ is, in fact, an $\varepsilon$-majority rule winning proposal with respect to z. Consider that since indifferent voters have been assumed to cast their votes for the challenging proposal, $\exists$ N $\in$ $\mathbb{N}_+$ such that $\mathbb{1}_{\{ch\}}(z^{\#}|z)$ is componentwise (weakly) greater than $\mathbb{1}_{\{ch\}}(z^{\#}_n|z_n)$ whenever n > N. For suppose not; then since there are only finitely many voters, there must be some individual i and subsequences $(z^{\#}_m) \longrightarrow z^{\#}$ and $(z_m) \longrightarrow z$ such that $U_i(z^{\#}) \equiv U_i(x^{\#}_{\alpha i}, \theta^{\#}\cdot(b^{\#},\delta)_i) <$ $U_i(x_{\alpha i}, \theta\cdot(b,\delta)_i) \equiv U_i(z)$, while $U_i(z^{\#}_m) \geq U_i(z_m)$, $\forall$ m. However, continuity of $U_i$ then requires $U_i(z^{\#}) \geq U_i(z)$, a contradiction. Thus for n > M:

$$\mathbb{1}_{\{ch\}}(z^{\#}|z)\cdot\theta^j_n \geq \mathbb{1}_{\{ch\}}(z^{\#}_n|z_n)\cdot\theta^j_n \geq .5+|b^{\#}_n-b_n|,$$
since $z^{\#}_n \in v_{js}(z_n)$

$$\Rightarrow \quad \lim_{[n\to\inf]} \mathbb{1}_{\{ch\}}(z^{\#}|z)\cdot\theta^j_n \geq \lim_{[n\to\inf]} .5+|b^{\#}_n-b_n|$$

$$\Rightarrow \quad \mathbb{1}_{\{ch\}}(z^{\#}|z)\cdot\theta^j \geq .5+|b^{\#}-b|, \text{ as desired.} \tag{*}$$

Note however that if $\varepsilon$ = 0, the assumption that tie votes are awarded to the challenger is required. In line (*) above if $\varepsilon$ = 0, $z^{\#} \neq z$, and it happens that $\mathbb{1}_{\{ch\}}(z^{\#}|z)\cdot\theta^j = .5$, then $z^{\#} \notin v_{js}(z)$ unless ties go to the challenger. When $\varepsilon > 0$, the question of tie votes is moot since a vote of .5 is never sufficiently large to depart from the status quo. Finally it is interesting to observe that if $\varepsilon = 0$ and ties are awarded to the challenger, $v_{js}$ will fail to be *lower* hemicontinuous. QED.

PROPOSITION 5.4 (Lemma 1 for the Lower Hemicontinuity of $v_{js}$)

Suppose that U is strictly quasiconcave and $\beta$ is continuous. Let $z \in Z_\zeta$, where $\gamma >> 0$, and $\delta \geq\geq 0$. Assume $z^{\#} \in v_{js}(z)$, and suppose there is some voter $i \in ch(z^{\#}|z)$. Given any real $\eta > 0$, $\exists$ $\mu_i \in (0,\eta)$ such that for any $z' \in N_{\mu_i}(z)$ there exists $z^{\dagger}_i \in N_\eta(z)$, perhaps depending on z', which satisfies:

i) $i \in ch(z^{\dagger}_i | z)$,

ii) $z^{\dagger}_i \in \beta_{js}(z')$, and

iii) $|z' - z^{\dagger}_i| < |z - z^{\#}|$.

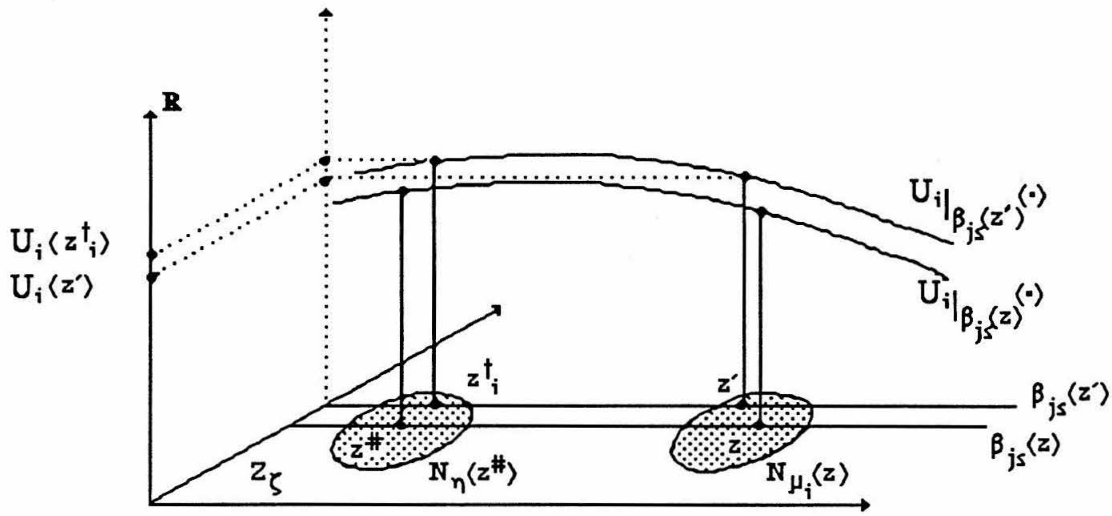



Figure 4.

<u>Proof</u>

These results obtain from the strict quasiconcavity of $U_i$ and the continuity of $\beta$. Recall that $z^{\#} \in v_{js}(z) \Rightarrow z^{\#} \in \beta_{js}(z)$. Since $z^{\#} \in v_{js}(z)$, necessarily $z^{\#} \in \beta_{js}(z)$. Claim $\exists\, \mu_i^1 < 0$, such that $z^{\#} \in N_{\mu_i^1}(z) \Rightarrow \beta_{js}(z^{\#}) \cap N_\eta(z) \neq \phi$. Suppose the claim is false, and consider a decreasing sequence of possible choices for $\mu_i^1$ converging to zero. Then there is a sequence $(z_n') \longrightarrow z$ such that $\beta_{js}(z_n') \cap N_\eta(z^{\#}) = \phi$. Since $\beta_{js}$ is lower hemicontinuous on $Z_\zeta$, while $(z_n') \longrightarrow z$ and $z^{\#} \in \beta(z)$, then $\exists$ a sequence $(z^{\dagger}_n) \longrightarrow z^{\#}$, where $z^{\dagger}_n \in \beta(z^{\dagger}_n)$ -- a contradiction since $\beta(z^{\dagger}_n)$ was assumed to be bounded outside $N_\eta(z^{\#})$. WOLOG let $\mu_i^1 < \eta$.

By continuity of $U_i$, given $\eta > 0$ there exists $\mu_i^2 \leq \mu_i^1$ such that $z' \in N_{\mu_i^2}(z) \Rightarrow \exists\ z^{\dagger} \in N_{\eta}(z^{\#}) \cap \beta_{js}(z)$ and $U_i(z^{\dagger}) \geq U_i(z')$. Since $U_i(z^{\dagger}) \geq U_i(z')$, if $z^{\dagger}_i \in N_{\eta}(z^{\#}) \cap \beta_{js}(z)$ such that $|z^{\dagger} - z'| \geq |z^{\dagger}_i - z'|$ then strict quasiconcavity of $U_i$ implies $U_i(z^{\dagger}_i) \geq U_i(z')$.

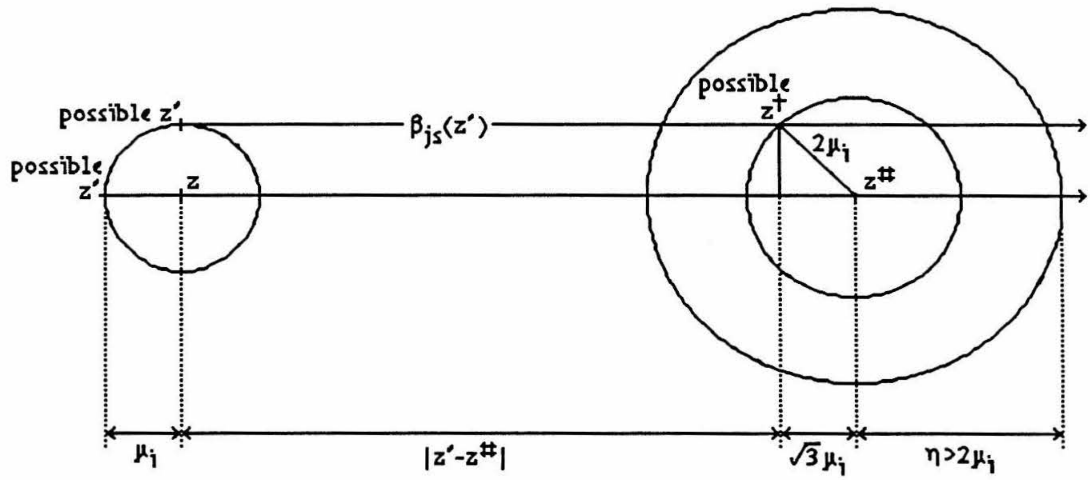
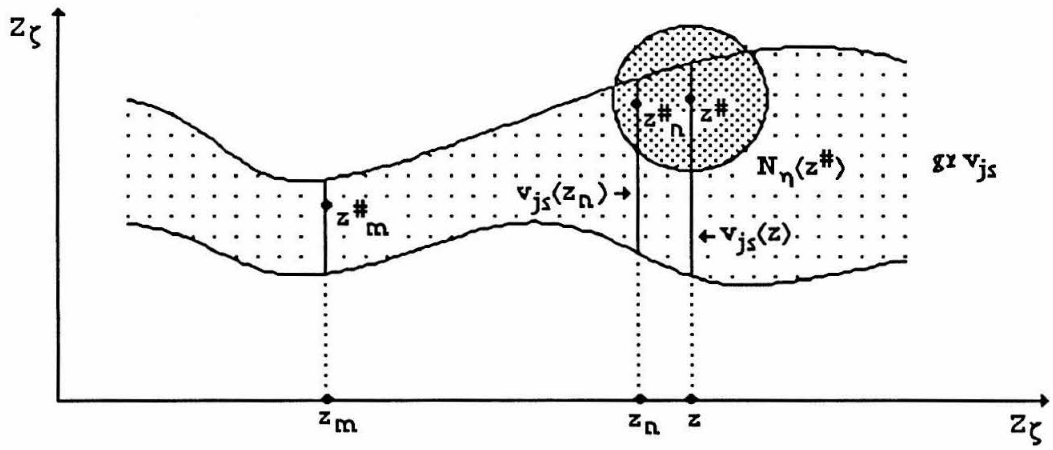



Figure 5.

Finally, it remains to show that $z^{\dagger}_i$ can be chosen such that condition iii) is satisfied. Choose $\mu_i < \text{MIN}\{\mu_i^2, \eta/2\}$ and let $z^{\dagger}_i$ be the point of $cl(N_{2\mu_i}(z)) \cap \beta_{js}(z')$ which is closest to z'. In this case, $|z'-z^{\dagger}| < |z-z^{\#}| - \sqrt{3}\mu_i + \mu_i < |z-z^{\#}|$, as desired. QED.

<u>PROPOSITION 5.5</u> (Lemma 2 for the Lower Hemicontinuity of $v_{js}$)

Let $(z_n)$ be a sequence taking values in $Z_{\zeta}$, $\zeta \geq 0$, such that $(z_n) \longrightarrow z$, and.suppose $z^{\#} \in v_{js}(z)$. Then given any real $\eta > 0$, $\exists\ M \in N_+$ such that whenever $n > M$, $\exists\ z^{\#}_n \in N_{\eta}(z^{\#}) \cap v_{js}(z_n)$.




<u>Proof</u>

Let $\eta > 0$ be given. By previous lemma, corresponding to each individual $i \in ch(z^\#|z)$ there is some $\mu_i$, where $0 < \mu_i < \eta$, such that if $z' \in N_{\mu_i}(z)$, $\exists\ z^\dagger_i \in N_\eta(z^\#)$ satisfying:

i) $i \in ch(z^\dagger_i | z')$,

ii) $z^\dagger_i \in \beta_{js}(z')$, and

iii) $|z'-z^\dagger_i| < |z-z^\#|$.

Define $\mu = \text{MIN}\{\mu_i \mid i \in ch(z^\#|z)\}$ and for any $z' \in N_\mu(z)$ let,

$$z^\dagger = \text{ARGMIN}\{\,|z^\dagger_i - z'| \mid i \in ch(z^\#|z)\}.$$

Then claim for any $z' \in N_\mu(z)$, $\exists\ z^\dagger_i \in N_\eta(z^\#)$ which satisfies:

iv) $\mathbb{1}_{\{ch\}}(z^\dagger|z') \geq\geq \mathbb{1}_{\{ch\}}(z^\#|z)$,

v) $z^\dagger \in \beta_{js}(z')$, and

vi) $|z'-z^\dagger| < |z-z^\#|$.

Existence of such $z^\dagger$ obtains since each $z^\dagger_i \in N_\eta(z^\#)$. Inequality iv) follows from the quasiconcavity of $U_i$; if $i \in ch(z^\dagger_i | z')$ and $z^\dagger \in [z', z^\dagger_i]$, then $i \in ch(z^\dagger | z')$. Conditions v) and vi) are simply instances of ii) and iii) respectively, since $z^\dagger$ is chosen from the $z^\dagger_i$s.

Now, since $(z_n) \longrightarrow z$, there is some $M' \in \mathbb{N}_+$ such that $n > M' \Rightarrow z_n \in N_\mu(z)$. For each such $z_n$, recalling the discussion of the previous paragraph, $\exists\, z^\#_n \in N_\eta(z^\#)$ such that:

vii) $\mathbb{1}_{\{ch\}}(z^\#_n | z_n) \geq\geq \mathbb{1}_{\{ch\}}(z^\# | z)$,

viii) $z^\#_n \in \beta_{js}(z_n)$, and

ix) $|z_n - z^\#_n| < |z - z^\#|$.

Hence,

$$\mathbb{1}_{\{ch\}}(z^\#_n | z_n) \cdot \theta^j \geq\geq \mathbb{1}_{\{ch\}}(z^\# | z) \cdot \theta^j \geq .5 + \varepsilon | b^\# - b | > .5 + \varepsilon | b_n - b^\#_n |; \qquad (*)$$

Counting the occurrences of inequality symbols in the expression above from the left, the first follows from vii) by taking the dot product of each side by $\theta^j$, the second obtains since $z^\#$ is an $\varepsilon$-majority rule winner with respect to z, and the third is a consequence of ix).[5]

Finally since $(\theta^j_n) \longrightarrow \theta^j \neq 0$ and the last inequality in $(*)$ is strict (notice this strictness obtains from the assumption that $\varepsilon > 0$), $\exists\, M \geq M'$ such that $n > M \Rightarrow$

$$\mathbb{1}_{\{ch\}}(z^\#_n | z_n) \cdot \theta^j_n > .5 + | b^\#_n - b_n |,$$

or equivalently that $z^\#_n \in v_{js}(z_n)$, as desired. QED.

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5. Recall that $\theta^j$ is an $(I \times 1)$ column vector which specifies shareholdings across individuals in firm j. The lemma assumes $z^\# \in v_{js}(z)$ , that is that $z^\#$ is a proposal to change the production plan of firm j in state s, and furthermore that $z^\#$ is an $\varepsilon$-majority rule winner with respect to z.

PROPOSITION 5.6 (Lower Hemicontinuity of $v_{js}$)

Suppose that U is strictly quasiconcave and $\beta$ is continuous.

i) $v_{js}: Z_\zeta \longrightarrow Z_\zeta$ is lower hemicontinuous whenever $\varepsilon > 0$, $\forall\ \gamma >> 0$, $\delta \geq\geq 0$.

ii) If it is further assumed that tie votes are awarded to status quo proposals, the proposition holds in case $\varepsilon = 0$.

Proof

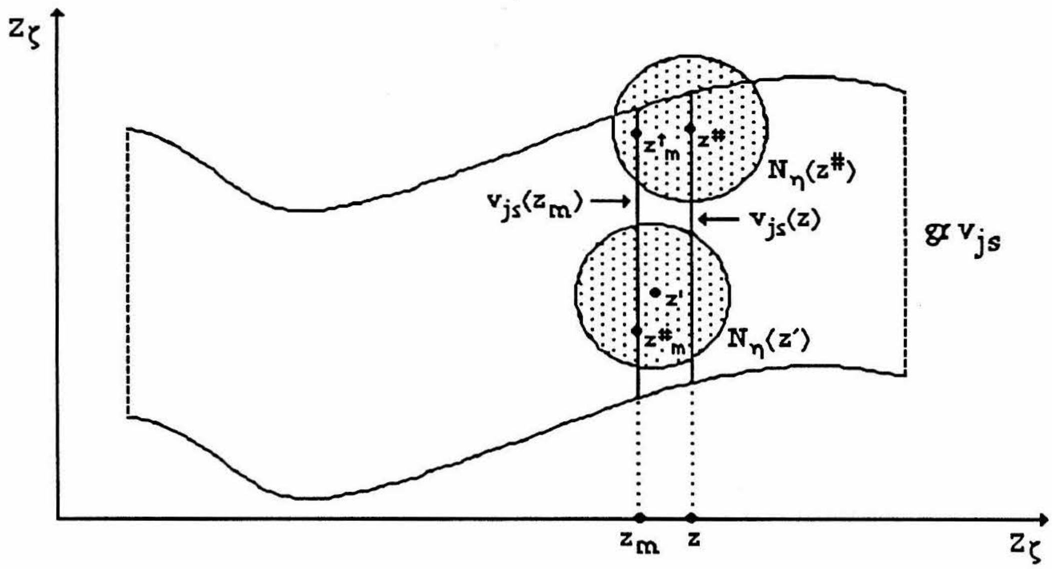


Figure 6.

Let $(z_n) \longrightarrow z$ be a convergent sequence in $Z_\zeta$, and suppose $z^\# \in v_{js}(z)$. It is required to exhibit a convergent sequence $(z^\#_n) \longrightarrow z^\#$, such that $z^\#_n \in v_{js}(z_n)$. Choose $z^\#_n \in \text{ARGMIN}\{\,|z''-z^\#| \mid z'' \in v_{js}(z_n)\}$, this set being non-empty since $v_{js}$ is compact valued. Suppose $(z^\#_n) \not\longrightarrow z^\#$; by compactness of $Z_\zeta$, $\exists$ a convergent subsequence $(z^\#_m) \longrightarrow z' \neq z^\#$. Let $\eta > 0$ such that $N_\eta(z') \cap N_\eta(z^\#) = \phi$. Clearly there is some

$M_1 \in \mathbb{N}_+$ such that whenever $m > M_1$, $z^{\#}_m \in N_\eta(z')$, in which case $|z^{\#}_m - z^{\#}| > \eta$. Since $(z_m) \longrightarrow z$ and $z^{\#} \in v_{js}(z)$, by previous lemma[6] $\exists\, M_2$ such that $m > M_2 \Rightarrow \exists\, z^{\dagger}_m \in N_\eta(z^{\#}) \cap v_{js}(z_m)$, in which case $|z^{\dagger}_m - z^{\#}| < \eta$. Let $M = \text{MAX}\{M_1, M_2\}$, and note that for $m > M$, $|z^{\#}_m - z^{\#}| > \eta > |z^{\dagger}_m - z^{\#}|$; however, since both $z^{\dagger}_m$, $z^{\#}_m \in v_{js}(z_m)$, this inequality contradicts the constructed minimality of the distance from $z^{\#}_m$ to $z^{\#}$ among the members of $v_{js}(z_m)$. Hence, $(z^{\#}_n)$ must converge to $z^{\#}$, as desired. Regarding part ii), notice that when $\varepsilon = 0$, $v_{js}$ may fail to be lower hemicontinuous unless ties are awarded to the status quo. It is interesting to note the opposite situation prevails concerning the upper hemicontinuity of $v_{js}$. QED.

PROPOSITION 5.7 (Additional Properties of the Winner Correspondences)

Suppose that $\varepsilon > 0$, and U is componentwise continuous on $\beta_{js}(z)$, then :

i) $v_{js}$ is nonempty valued.

ii) $v_{js}$ is compact valued.

If, in addition, U is componentwise quasiconcave on $\beta_{js}(z)$ then :

iii) $v_{js}$ is convex valued.

iv) $v_{js}$ is one-sided valued; that is, the projection of $v_{js}(z)$ onto the space of firms' plans $\mathbb{R}^{JS}_+$ is an interval in $js^{\underline{th}}$ coordinate axis, and z itself is one endpoint of the interval.

Proof

Let $z \in Z_\zeta$ be given. Since indifferent voters vote for the challenger, weighted vote for $z \in v_{js}(z)$. This same assumption together with the continuity of U give compactness

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6. Recall that the strict no short sales assumption $\gamma >> 0$ was employed in the proof of Lemma 1, upon which Lemma 2 depends.

of $v_{js}(z)$. Convexity and one sidedness are direct consequences of quasiconcavity.[7] QED.

DEFINITION (Star Arm Endpoint Function $v^*_{js}$)

Suppose that U is componentwise quasiconcave on $\beta_{js}(z)$, $\forall\ z \in Z_\zeta$. Let $\varepsilon \geq 0$ be given, and define the function $v^*_{js}: Z_\zeta \longrightarrow Z_\zeta$ according to:

$$v^*_{js}(z) = \text{ARGMAX } \{ |z'-z| \mid z' \in v^\varepsilon_{js}(z) \}.$$

It is not anticipated that it will be necessary to distinguish $v^*_{js}$ by $\varepsilon$, and so this notation has not been introduced. Notice that $v^*_{js}$ is well defined as a function, that is the ARGMAX in the defining equation exists uniquely, because of the one sidedness of $v^\varepsilon_{js}(z)$. Unless U is quasiconcave, $v^*_{js}$ may be multi-valued.

PROPOSITION 5.8 (Continuity of the Star Arm Endpoint Function)

Suppose that U is componentwise quasiconcave on $\beta_{js}(z)$; $\forall\ z \in Z_\zeta$. Then given any $\varepsilon > 0$, $v^*_{js}: Z_\zeta \longrightarrow Z_\zeta$ is continuous.

Proof

By direct application of Berge's Maximum Theorem, and the previous results that $v^\varepsilon_{js}$ is both upper and lower hemicontinuous on $Z_\zeta$ whenever $\varepsilon > 0$. QED.

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7. McKelvey [1986] also discusses the one sidedness of majority rule winner correspondences.

DEFINITION (Voting Corner Function)

Suppose U is componentwise quasiconcave on $\beta_{js}(z)$; $\forall\ z \in Z_\zeta, \forall\ js \in \{1, \ldots, JS\}$.
Let $V_\varepsilon$: $Z_\zeta \longrightarrow Z_\zeta$ be defined by:

$$V_\varepsilon(z) = (v^*_{11} \circ \cdots \circ v^*_{js} \circ \cdots \circ v^*_{JS})(z).$$

As before, the quasiconcavity of U is required so that each component $v^*_{js}$ and hence $V_\varepsilon$ is well defined as a function. Notice that $V_\varepsilon$ is continuous whenever $\varepsilon > 0$.

PROPOSITION 5.9 (Characterization of Voting Equilibria)

i) The set of direction restricted $\varepsilon$-majority rule voting equilibria on $Z_\zeta$ is exactly the set of fixed points of $V_\varepsilon$ on $Z_\zeta$.

ii) The set of fixed points of $V_\varepsilon$ is *independent of the order of composition* of the functions $v^*_{js}$.

iii) Every $V_\varepsilon$ equilibrium is also a $V_0$ equilibrium (that is, a direction restricted majority rule equilibrium with $\varepsilon = 0$), if ties are awarded to the status quo.

iv) Every $V_\varepsilon$ equilibrium is also an $\varepsilon$-majority rule equilibrium with indifferent voters choosing the status quo.

Proof

Parts i) and ii) are straightforward consequences of the definition of $V_\varepsilon$. If iii) is false, some $z^\#$ is a fixed point of $V_\varepsilon$, but not of $V_0$. Then $\exists$ a voting direction $e_{js}$, a state $z' \in \beta_{js}(z)$, and $\alpha' \in \mathbb{R}$ such that $b' = b^\# + \alpha' e_{js}$ and $\mathbb{1}_{\{ch\}}(z' \mid z^\#) \cdot \theta^j > .5$. Strictness of this last inequality obtains from the assumption that ties are awarded to the status quo.

Let $\xi = \mathbb{1}_{\{ch\}}(z' \mid z^{\#})\cdot\theta^j - .5$; clearly $\xi > 0$. Notice that $\forall$ z between $z^{\#}$ and z', $\mathbb{1}_{\{ch\}}(z \mid z^{\#})\cdot\theta^j \geq \mathbb{1}_{\{ch\}}(z' \mid z^{\#})\cdot\theta^j = \xi + .5$. Let $0 < \alpha^{\dagger} < \xi/\varepsilon$ such that both $z^{\dagger} = (x^{\dagger}, b^{\dagger}, \theta^{\#}) \in \beta_{js}(z)$, where $b^{\dagger} = b^{\#} + \alpha^{\dagger} e_{js}$, and also $x^{\dagger} = \theta^{\#}\cdot[C(b^{\#}) - C(b^{\#} + \alpha^{\dagger} e_{js})]$; clearly such choice is feasible.

Since $z^{\dagger}$ lies between $z^{\#}$ and z' along $\beta_{js}(z)$, the quasiconcavity of U implies $\mathbb{1}_{\{ch\}}(z^{\dagger} \mid z^{\#})\cdot\theta^j \geq .5 + \xi > .5 + \varepsilon * \alpha^{\dagger} = .5 + \varepsilon * \mid b^{\dagger} - b^{\#} \mid$. But this means $z^{\dagger}$ is an $\varepsilon$–majority winner over $z^{\#}$, that is $z^{\dagger} \in v_{js}(z^{\#})$--contradicting that $z^{\#}$ is a fixed point of $V_{\varepsilon}$, which completes the proof of iii). To show iv) consider $\forall$ z,z' that $\mathbb{1}_{\{ch\}}(z' \mid z)\cdot\theta^j \geq \mathbb{1}^*_{\{ch\}}(z' \mid z)\cdot\theta^j$, where $\mathbb{1}^*_{\{ch\}}(z' \mid z)\cdot\theta^j$ denotes the share weighted vote for z' when indifferent voters vote for the status quo. QED.

# VI. EXISTENCE OF EQUILIBRIUM

This section employs previous results concerning exchange and voting equilibria to demonstrate the existence of shareholders' equilibria for a given stock market economy. The analysis proceeds by showing $V_\varepsilon{}^oE_{IP}$ satisfies the hypotheses of the Eilenberg-Montgomery theorem, in particular that the correspondence is contractible valued. The contractible valued argument requires that $V_\varepsilon$ is 1–1 on the individually rational Pareto sets. Finally, it is argued that fixed points of $V_\varepsilon{}^oE_{IP}$ are indeed shareholders' equilibria for the given economy.

PROPOSITION 6.1 ($V_\varepsilon$ is 1-1)

Let $z = (x, b, \theta) \in Z_\zeta$, $\zeta \geq 0$, and suppose $\varepsilon \geq 0$.

i) $V_\varepsilon$ is a 1-1 function on every set of the form $E^\zeta{}_{IP}(z)$.

ii) In fact the proposition is true for *any* subset of $Z_\zeta|_b$.

Proof

By induction on $Q = J*S$, the number of planning directions. For notational simplicity, renumber the set of index directions according to $(1,1) \longrightarrow 1, \ldots, (j,s) \longrightarrow q, \ldots$, and $(J,S) \longrightarrow Q$. Thus denote (temporarily) $v^*_{11}$ by $v^*_1, \ldots$, and, $\ldots$, $v^*_{JS}$ by $v^*_Q$. The proposition will be demonstrated for $E^\zeta{}_{IP}(z)$, the extension to ii) being entirely straightforward. Let $z \in Z_\zeta$ be given.

<u>$n = 1$</u>

Claim that $v^*_1$ is 1-1 on $E_{IP}(z)$. For suppose not, then $\exists$ distinct $z^\#$, $z' \in E_{IP}(z)$ such that $v^*_1(z^\#) = v^*_1(z') = z^\dagger$. Since $E_{IP}$ fixes firms' plans, it must be the case that all elements of $E_{IP}(z)$ share the same firm plan with $z = (x, b, \theta)$; in particular, :

$$b^\# = b' = b. \tag{$*$}$$

Similarly, since $v^*_1$ fixes shareholdings, it must also be the case that the third components of the $v^*_1$-preimages of $z^\dagger$, namely $z^\#$ and $z'$, are identical and equal to $\theta^\dagger$, that is:

$$\theta^\# = \theta' = \theta^\dagger. \tag{$**$}$$

However, as $z^\#$ and $z'$ have been assumed distinct, it neccesarily follows that $x^\# \neq x'$. Now, since $v^*_1$ maps both $z^\#$ and $z'$ to $z^\dagger$, the first components $x^\#$ and $x'$ (resp.) must both be mapped to the corresponding component of $z^\dagger$, namely $x^\dagger$. Recall the first component action of $v^*_1$ is restricted not to alter $x^\#$ and $x'$ except insofar as the vector of share weighted production costs of $b^\#$ and $b'$ (resp.) differ from $b^\dagger$, which is to say:

$$v^*_1|_x: x^\# \longrightarrow x^\dagger \Rightarrow x^\dagger = x^\# + \theta^\# \cdot [C(b^\#) - C(b^\dagger)], \text{ and}$$
$$v^*_1|_x: x' \longrightarrow x^\dagger \Rightarrow x^\dagger = x' + \theta' \cdot [C(b') - C(b^\dagger)].$$

Setting equal the r.h.s. of the above equations gives:

$$x^\# + \theta^\# \cdot [C(b^\#) - C(b^\dagger)] = x' + \theta' \cdot [C(b') - C(b^\dagger)].$$

Substituting for $b^\#$ and $b'$ from ($*$), and for $\theta^\#$ and $\theta'$ from ($**$) one obtains:

$$x^\# + \theta^\dagger \cdot [C(b) - C(b^\dagger)] = x' + \theta^\dagger \cdot [C(b) - C(b^\dagger)],$$

which implies $x^\# = x'$, a contradiction.

<u>$n = q$</u>

Suppose as an inductive hypothesis that $v^*_q \circ \cdots \circ v^*_1$ is 1-1on $E_{IP}(z)$.

<u>$n = q \Rightarrow n = q + 1$</u>

Claim that $v^*_{q+1} \circ \cdots \circ v^*_1$ is 1-1 on $E_{IP}(z)$; notice it suffices that $v^*_{q+1}$ is 1-1 on the set $(v^*_q \circ \cdots \circ v^*_1)[E_{IP}(z)]$. Suppose the claim is false, then $\exists$ distinct $z^\#$, $z' \in$

$(v^*_q \circ \cdots \circ v^*_1)[E_{IP}(z)]$ such that $v^*_{q+1}(z^\#) = v^*_{q+1}(z') = z^\dagger$. Since the third component action of $E_{IP}$ can alter firms' plans on at most the q+1$^{\underline{st}}$ planning direction, $z^\#$ and z' can have identical images under $v^*_{q+1}$ only if $b^\#_{)q+1(} = b'_{)q+1(}$. Furthermore, $b^\#_{q+1} = b'_{q+1}$, because: i) both $z^\#$ and z' have $(v^*_q \circ \cdots \circ v^*_1)$-preimages in $E_{IP}(z)$ characterized by identical firm plans on all directions, and in particular on direction q+1; and ii) none of the maps $v^*_q, \ldots, v^*_1$ could have changed firms' plans except on directions q, ... , 1 (resp). Thus one can assert:

$$b^\# = b' = \text{(say) } b''. \tag{***}$$

As before, since $v^*_{q+1}$ fixes shareholdings, it is necessary that:

$$\theta^\# = \theta' = \theta^\dagger. \tag{****}$$

Since $z^\#$ and z' have been assumed distinct they must differ with respect to some component, which requires by process of elimination that $x^\# \neq x'$. But in this case:

$$x^\dagger = x^\# + \theta^\# \cdot [C(b^\#) - C(b^\dagger)] = x' + \theta' \cdot [C(b') - C(b^\dagger)],$$

and substituting from (***) and (****) gives:

$$x^\# + \theta^\dagger \cdot [C(b'') - C(b^\dagger)] = x' + \theta^\dagger \cdot [C(b'') - C(b^\dagger)],$$

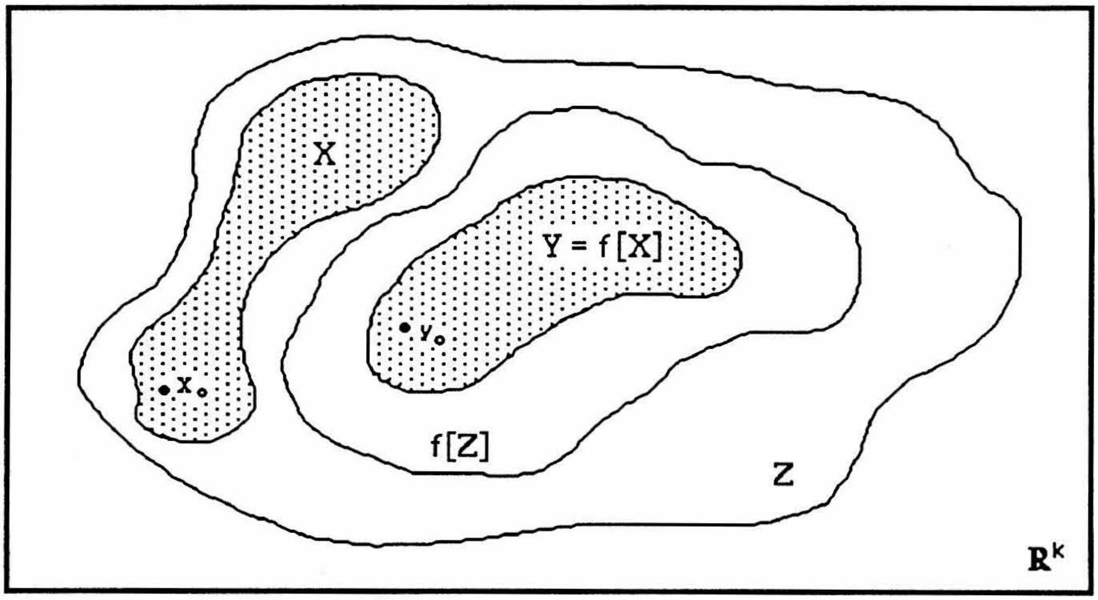
which implies $x^\# = x'$, a contradiction.

<u>n = Q</u>

Having demonstrated the inductive step, $V_\varepsilon = v^*_Q \circ \cdots \circ v^*_{q+1} \circ v^*_q \circ \cdots \circ v^*_1$ can now be asserted to be 1-1 on $E_{IP}(z)$, as desired. QED.

PROPOSITION 6.2 (Contractibility Lemma)

Let $Z \subseteq \mathbb{R}^K$ and $X \subseteq Z$, such that X is contractible. Suppose that $f: Z \longrightarrow Z$ is continuous, and that $f|_X$ is 1-1. Then $Y = f[X]$ is contractible.




Proof

Since X is contractible, $\exists$ a homotopy H between $id_X: X \longrightarrow X$ and some constant map $k_{x_o}: X \longrightarrow X$ such that $k_{x_o}: (x) = x_o \in X, \forall x \in X$. In particular, $H: X \times [0,1] \longrightarrow X$ is continuous and satisfies: $H(x,0) = x$, and $H(x,1) = x_o, \forall x \in X$.

To show $Y = f[X]$ is contractible, it suffices to construct a continuous function $G: Y \times [0,1] \longrightarrow Y$ such that $G(y,0) = y$ and $G(y,1) = y_o$, for some $y_o \in Y$. That is, G is required to be a homotopy between $id_Y: Y \longrightarrow Y$ and $k_{y_o}: Y \longrightarrow Y$. Choose $y_o = f(x_o)$; trivially $y_o \in f[X] = Y$. Construct G according to:

$$G(y,\alpha) = f(H(x,\alpha)); \forall y \in Y, \forall \alpha \in [0,1], \text{ where } y = f(x).$$

Notice G is well defined since f is 1-1on X; continuity obtains directly form the continuity of f and H. Finally, observe that $G(y,0) = f(H(x,0)) = f(x) = y$, and $G(y,1) = f(H(x,1)) = f(x_o) = y_o$, as desired. QED.

PROPOSITION 6.3 (Properties of $V_\varepsilon{}^oE_{IP}$)

Let $\zeta >> 0$, and $\varepsilon > 0$.

i) $V_\varepsilon{}^oE_{IP}$ is upper hemicontinuous on $Z_\zeta$.

ii) $V_\varepsilon{}^oE_{IP}$ is non-empty and compact valued on $Z_\zeta$.

iii) $V_\varepsilon{}^oE_{IP}$ is contractible valued on $Z_\zeta$.

Proof

To show part i), recall that under the stated assumptions $E_{IP}$ has been shown to be an upper hemicontinuous correspondence, and $V_\varepsilon$ is a continuous function. Hence the composition is upper hemicontinuous. In the case of ii), recall also that $E_{IP}$ has been shown to be non-empty and compact valued. Hence $V_\varepsilon{}^oE_{IP}$ is non-empty valued, and compactness follows from the continuity of $V_\varepsilon$. Finally, to prove iii), recall that $E_{IP}$ has been shown to be contractible valued. Since $V_\varepsilon$ is 1−1 on every set of the form $E_{IP}(z)$, the contractibility lemma presented just above yields the desired result. QED.

PROPOSITION 6.4 (Fixed Points of $V_\varepsilon{}^{o}E_{IP}$)

Let $\zeta >> 0$, and suppose $\varepsilon > 0$.

i) There exists at least one fixed point of $V_\varepsilon{}^{o}E_{IP}$ in $Z_\zeta$.

ii) In particular there is at least one fixed point of $V_\varepsilon{}^{o}E_{IP}$ in $Z_\zeta$, given some firm j in which every individual i holds at least a $\gamma_{ij} > .5/I$ share.

Proof

It has already been demonstrated that $Z_\zeta$ is both compact and contractible. The proposition immediately above asserts that $V_\varepsilon{}^{o}E_{IP}$ satisfies the hypotheses of the Eilenberg-Montgomery theorem; therefore, the correspondence possesses a fixed point on $Z_\zeta$. Since the proposition is true for any strictly positive minimum shareholdings matrix $\gamma$, it is true in particular for a choice of $\gamma$ which precludes any individual from accumulating a controlling interest in some firm j. QED.

It is now possible to state and prove the main result--the existence of non-controlling interest shareholders' equilibria. Fortunately the difficult work has already been done; all that remains is to assemble the pieces.

PROPOSITION 6.5 (Existence of Shareholders' Equilibria)

Let $\zeta >> 0$ be given.

i) The set of fixed points of $V_\varepsilon{}^{o}E_{IP}$ , $\varepsilon > 0$, coincides exactly with the set of shareholders' equilibria for a stock market economy with $\varepsilon$-majority rule voting.

ii) If there is at least one firm j such that $\gamma_{ij} > .5/I$ for all individuals i, then every fixed point of $V_\varepsilon{}^\circ E_{IP}$ is a non-controlling interest shareholders' equilibrium.

iii) Every shareholders' equilibrium of a stock market economy with $\varepsilon$–majority rule voting is a shareholders' equilibrium of the same stock market economy with standard majority rule voting.

Proof

To show i) suppose some fixed point $z = (x, b, \theta)$ of $V_\varepsilon{}^\circ E_{IP}$ is not an $\varepsilon$-majority rule shareholders' equilibrium. Then either there is some distinct $z' = (x', b, \theta') \in Z_\zeta|_b$ such that $U(z') >> U(z) \Rightarrow z' \in E_{IP}(z)$ , or else there is some distinct $z'' = (x'', b'', \theta)$ $\in \beta(z)$ preferred by a share weighted $\varepsilon$-majority to z. Now if z' strictly dominates z and b' = b, then monotonicity of preferences forces $\theta' \neq \theta$; the utility improvement achieved by the move to z' must have resulted from the exchange of both shares and the current good. Furthermore since z' strictly dominates z, $z \notin E_{IP}(z)$. Since $V_\varepsilon$ fixes the third coordinates of its arguments, there is no element of $E_{IP}(z)$ which can be the $V_\varepsilon$-preimage of z, contradicting the assumption that z was a fixed point of $V_\varepsilon{}^\circ E_{IP}$. On the other hand, suppose some $\varepsilon$-majority favors z" to z. Claim that if z is a fixed point of $V_\varepsilon{}^\circ E_{IP}$, then $z \in E_{IP}(z)$. But if $z \in E_{IP}(z)$ then $z = E_{IP}(z)$, and so if z is a fixed point of $V_\varepsilon{}^\circ E_{IP}$, then $z = V_\varepsilon(z)$ which implies z" cannot be preferred to z.

Next it must be shown that every $\varepsilon$-majority shareholders' equilibrium is a fixed point of $V_\varepsilon{}^\circ E_{IP}$. If no $z' = (x', b, \theta') \in Z_\zeta|_b$ dominates z, then $z = E_{IP}(z)$, and if no $\varepsilon$-majority prefers any z" to z, then $V_\varepsilon(z) = z$-- which establishes that z is a fixed point of $V_\varepsilon{}^\circ E_{IP}$, as desired.

Part ii) follows from the observation that if no $\varepsilon$-majority chooses to depart from z, no simple majority will either. Finally, regarding part iii), notice that as the number of individuals I grows large, the minimum holdings sufficient to prevent a controlling interest becomes small. QED.

# VII. DIRECTIONS FOR FUTURE RESEARCH

The purpose of this section is to call attention to several unanswered questions in the theory of financial equilibrium, which this paper has only begun to explore.

- This research indicates there is exists a continuous selection from the Pareto correspondence, that is a continuous function on $Z_\zeta$ which always takes a value in the individually rational Pareto set of its argument. It would be particularly interesting if the methods employed to show the existence of such a selection could be extended to show the existence of a selection from the core correspondence.

- The results obtained here could be strengthened if the problem of bankruptcy were modelled in a less stylized manner. In the current version of the model, no proposal can be made which, if passed, would bankrupt any shareholder. Several more realistic financing schemes can be imagined; for example, distressed shareholders might be bought out by other shareholders in return for a greater share of the firm. Adjustments to the financing scenario might also make it possible to eliminate the role of the minimum holdings requirement in guaranteeing continuity of the non-bankrupting proposal correspondence.

- It appears that a reasonable degree of effort would allow the distinguished goods to be dispensed with. Presumably, given a sequence of distinguished goods vectors converging to zero, the limits of the associated sequence of equilibria would be equilibria of the limit economy.

- Perhaps the most significant generalization of this model would be the introduction of individual initial endowments, and a demonstration of some relationship between endowments and equilibrium. The prototypical example of such a relationship can be found in simple exchange economies, in which equilibria are supported by initial endowments. In translating this concept to stock market economies, several technical and conceptual problems arise from production financing, and the possibility that individuals may be made worse off by production plan voting.

- Another direction for possible future research is suggested by the observation that the voting direction restrictions assumed here appear to be related to Grossman's [1977] notion of Social Nash Optimality, in which a planner is constrained to a finite number of allocational activities.

# NOTATION INDEX

| | |
|---|---|
| $\mathbb{N}_+$ | the non-negative integers |
| $\mathbb{R}^n{}_+$ | the non-negative orthant of n-dimensional Euclidean space |
| $e_{js}$ | the $js^{th}$ standard orthogonal unit basis vector for $\mathbb{R}^{JS}{}_+$ |
| $S^{(n-1)}$ | the n-1 dimensional simplex in $\mathbb{R}^n{}_+$ |
| a | an arbitrary point on the simplex |
| I | $I = \{1, \dots, i, \dots, I\}$, a finite set of individuals |
| i | an arbitrary individual |
| J | $J = \{1, \dots, j, \dots, J\}$, a finite set of firms |
| j | an arbitrary firm |
| S | $S = \{1, \dots, s, \dots, S\}$, a finite set of second date states of nature |
| s | an arbitrary state |
| $Z_\zeta$ | state space, $Z_\zeta \subseteq \mathbb{R}^{(I \times JS \times IJ)}{}_+$ |
| $z = (x, b, \theta)$ | an arbitrary state in $Z_\zeta$ |
| $(z_n)$ | $= (x_n, b_n, \theta_n)$ , a sequence of states in $Z_\zeta$ |
| $Z_\zeta\|_b$ | $\{z' = (x', b', \theta') \in Z_\zeta \mid b' = b$, where $z = (x, b, \theta) \in Z_\zeta\}$ |
| $Z_\zeta\|_\theta$ | $\{z' = (x', b', \theta') \in Z_\zeta \mid \theta' = \theta$, where $z = (x, b, \theta) \in Z_\zeta\}$ |
| $x = (x_{o1}, \dots, x_{oI})$ | current consumption vector (1 x I), an element of $\mathbb{R}^I{}_+$ |
| $x_o$ | typically $\sum_{[i=1,..,I]} x_{oi}$, occasionally the limit of a sequence $(x_n)$ |
| $X_o$ | strictly positive aggregate social endowment |
| $x_{si}$ | individual i's consumption in state s |
| b | (J x S) production plan matrix, $b_{js}$ is firm j's production in state s |
| $b_j$ | (1 x S) row vector, firm j's planned production across states |
| $\theta$ | (I x J) shareholding matrix |

| | |
|---|---|
| $\theta_{ij}$ | individual i's share of firm j, $\theta_{ij} \geq \delta_{ij} > 0$ |
| $\theta_i$ | (1 x J) row vector, individual i's stock portfolio |
| $\theta^j$ | (I x 1) column vector, firm j's shareholders of record |
| $(\theta \cdot b)_i$ | (1 x I) row vector, $i^{th}$ row of $\theta \cdot b$, i's consumption across states |
| $\gamma$ | (I x J) minimum shareholding matrix, typically $\gamma >> 0$ |
| $\gamma^j$ | $j^{th}$ column of $\gamma$, individuals' required minimum holdings of firm j |
| $S^{(I-1)}{}_{\gamma j}$ | $\gamma^j$ "bordered" $S^{(I-1)}$ simplex |
| $S_\gamma$ | $S^{(I-1)}{}_{\gamma 1} \times \cdots \times S^{(I-1)}{}_{\gamma j} \times \cdots \times S^{(I-1)}{}_{\gamma J}$ |
| $\delta$ | (J x J) diagonal matrix of firms' distinguished goods production |
| $\delta_{jj}$ | firm j's production of distinguished good j; $\delta_{jk} = 0$, $j \neq k$ |
| $\delta_j$ | $= \delta_{jj}$, typically $\delta_j > 0$ |
| $(\theta \cdot \delta)_i$ | (1 x I) row vector, $i^{th}$ row of $\theta \cdot \delta$, i's distinguished goods |
| $d_{ij}$ | individual i's consumption of distinguished good j |
| $\zeta$ | $= (\gamma, \delta)$, shareholdings and distinguished good parameters |
| $U_i$ | individual i's utility function over $\mathbb{R}^{(1+S+J)}{}_+$ |
| $U(z)$ | $= U(x, \theta \cdot b, \theta \cdot \delta) = (U_1(x_{01}, (\theta \cdot b)_1, (\theta \cdot \delta)_1), \ldots, U_I(x_{0I}, (\theta \cdot b)_I, (\theta \cdot \delta)_I))$ |
| $C_j$ | cost function for firm j |
| $C_j(b_j)$ | firm j's cost of producing plan $b_j$ |
| $C$ | (J x 1) column vector of firms' production functions |
| $C(b)$ | $= (C_1(b_1), \ldots, C_j(b_j), \ldots, C_J(b_J))^T$ |
| $(X_o, U, C, \zeta)$, | a stock market economy |
| $\omega(z)$ | $= U(z)$ |
| $K_+\omega(z)$ | $= \{ \omega' \in \mathbb{R}^I{}_+ \mid \omega' \geq\geq \omega(z) \}$ |
| $E^\zeta{}_{IP}$ or $E_{IP}$ | the individually rational Pareto correspondence on $Z_\zeta$ |
| $dh\mathrm{U}[E_{IP}(z)]$ | the disposable hull of $U[E_{IP}(z)]$ relative to $\mathbb{R}^I{}_+$ |
| $\mathrm{fr}(dh\mathrm{U}[E_{IP}(z)])$ | frontier of $dh\mathrm{U}[E_{IP}(z)]$ |
| $\varsigma_z$ | a correspondence $S^{(I-1)} \rightarrow\rightarrow dh\mathrm{U}[E_{IP}(z)] \subseteq \mathbb{R}^I{}_+$ |

| | |
|---|---|
| $\varsigma_z(a)$ | $= \{\omega' \in dh\mathrm{U}[E_{IP}(z)] \mid \exists \lambda \geq 0 \text{ s.t. } \omega' = \omega(z)+\lambda * a\}$ |
| $\varphi_z$ | a homeomorphism $S^{(I-1)} \longrightarrow dh\mathrm{U}[E_{IP}(z)] \subseteq \mathbb{R}^I_+$ |
| $\varphi_z(a)$ | $= \mathrm{ARGMAX}_{[\omega' \in \varsigma_z(a)]}\{\omega' \cdot \mathbb{1}^I\}$ |
| $\chi_z$ | a correspondence $S^{(I-1)} \rightarrow\rightarrow Z_\zeta$ |
| $\chi_z(a)$ | $= \{z'' \in Z_\zeta\vert_b \mid \omega(z'') = \varphi_z(a)\}$ |
| $\{e_{11}, \dots, e_{JS}\}$, | set of unit orthogonal voting (basis) directions for $\mathbb{R}^{JS}_+$ |
| $\beta$ | feasible proposal correspondence, $\beta$: $Z_\zeta \rightarrow\rightarrow Z_\zeta$ |
| $\beta(z)$ | set of feasible proposals given status quo $z \in Z_\zeta$ |
| $\beta_{js}$ | $js^{\underline{th}}$ direction feasible proposal correspondence, $\beta_{js}$: $Z_\zeta \rightarrow\rightarrow Z_\zeta$ |
| $\beta_{js}(z)$ | $= \{z' = (x', b', \theta') \in \beta(z) \mid b' = b + \lambda e_{js}, \lambda \in \mathbb{R}\}$ |
| $ch(z' \vert z)$ | the set of voters weakly preferring z' to z |
| $ch^*(z' \vert z)$ | the set of voters strictly preferring z' to z |
| $\mathbb{1}_{\{ch\}}(z' \vert z)$ | = (1 x I) indicator vector for voters weakly preferring z' to z |
| $\mathbb{1}^*_{\{ch\}}(z' \vert z)$ | = (1 x I) indicator vector for voters strictly preferring z' to z |
| $v_{js}$ | the $js^{\underline{th}}$ direction $\varepsilon$-majority rule winner correspondence |
| $v_{js}(z)$ | set of $js^{\underline{th}}$ direction $\varepsilon$-majority rule winners given status quo z |
| $v^*_{js}(z)$ | the element of $v_{js}(z)$ most distant from z |
| $v^*_{js}$ | the correspondence on $Z_\zeta$ defined above |
| $V_\varepsilon$ or V | $= v^*_{11} \circ \cdots \circ v^*_{js} \circ \cdots \circ v^*_{JS}$, the voting corner function |
| $V \circ E_{IP}$ | equilibrium characterization correspondence |

<u>REMARKS</u> (Notation)

On occasion in continuity proofs, the notation $(z_n) = ((x_n, b_n, \theta_n)) \longrightarrow z_o = (x_o, b_o, \theta_o)$ denotes that the sequence $(z_n)$ in $Z_\zeta$ converges to $z_o$. In these instances, $x_n$ should be interpreted as the (1 x I) vector of individuals' holdings of the current good associated with the $n^{\underline{th}}$ term $z_n$ of the sequence. Similarly $x_o$ should be interpreted as the (1 x I) vector of individuals' holdings of the current good associated with the limit $z_o$ of the sequence and not the aggregate quantity of the current good held by individuals in state $z = (x, b, \theta)$. In the same circumstances $b_n$ denotes the $(J \times S)$ production plan matrix associated with $z_n$, and not the production plan of firm n. Finally, in such cases, $\theta_n$ denotes the $(I \times J)$ shareholding matrix associated with $z_n$, and not the stock portfolio of individual n. It is not expected that this usage will give rise to any confusion.

The notation $\delta >> 0$ denotes that the *diagonal* elements of the diagonal matrix $\delta$ are strictly positive, and *not* that every every element of the matrix is strictly positive. Otherwise ">>" should be interpreted in the usual manner.

It may be may noticed that expressions of the form $x + \theta \cdot C(b)$, where x is $(1 \times I)$, $\theta$ is $(I \times J)$, and $C(b)$ is $(J \times 1)$, are non-conformable. Since $\theta \cdot C(b)$ is $(I \times 1)$, strictly speaking the sum is not well defined. In the interest of notational simplicity, the reader is asked to make the obvious transposition.

| | |
|---|---|
| $\mathbb{1}^J$ | (J x 1) column vetor of 1s |
| $\mathbb{1}_I$ | (1 x I) row vector of 1s |
| $M^T$ | transpose of matrix or vector M |
| $\cdot$ | dot product |
| $*$ | multiplication |
| $\circ$ | composition, $(f \circ g)(x) = f(g(x))$ |
| $>>$ | componentwise greater than |
| $\geq\geq$ | componentwise greater than or equal |
| $\geq>$ | componentwise greater than or equal, but not identical |
| $\rightarrow\rightarrow$ | correspondence arrow |
| $\mathbf{X}$ | Cartesian product |
| $\mathbf{X}_I S$ | I-fold Cartesian product of set S |
| cl | topological closure |
| $\phi$ | empty set |
| $N_\eta(z)$ | open $\eta$-neighborhood of z in the Euclidean metric topology |

# BIBLIOGRAPHY


1. K.J. ARROW and F.H. HAHN, *General Competitive Analysis*. San Francisco: Holden-Day, 1971.

2. S. BENNINGA and E. MULLER, "Majority Choice and the Objective Function of the Firm Under Uncertainty." *The Bell Journal of Economics*, 10 (1979), 670-682.

3. ------, "Majority Choice and the Objective Function of the Firm Under Uncertainty: Reply." *The Bell Journal of Economics*, 12 (1981), 338-339.

4. K.C. BORDER, *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge: Cambridge University Press, 1985.

5. J.S. CHIPMAN and J.C. MOORE, "The Compensation Principle in Welfare Economics," in *Papers in Quantitative Economics 2*, ed. by Avrid M. Zarley. Lawrence: The University Press of Kansas, 1971.

6. G. DEBREU, *The Theory of Value*. New York: John Wiley, 1959.

7. P.A. DIAMOND, "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty." *American Economic Review*, 57 (1967), 759-776.

8. J.H. DREZE, "Investment Under Private Ownership: Optimality, Equilibrium, and Stability," in *Allocation Under Uncertainty: Equilibrium and Optimality*, ed. by Jacques Dreze. New York: Macmillan, 1974.

9. S. EKERN and R. WILSON, "On the Theory of the Firm in an Economy with Incomplete Markets." *The Bell Journal of Economics*, 5 (1974), 171-180.

10. I. FISHER, *The Theory of Interest*. New York: Macmillan, 1930.

11. R. FORSYTHE and G.L. SUCHANEK, "The Impossibility of Efficient Decision Rules for Firms in Competitive Stock Market Economies." *Journal of Financial and Quantitative Analysis*, 17 (1982), 555-574.

12. ------, "Collective Decision Mechanisms and Efficient Stock Market Allocations: Existence of a Participation Equilibrium." *International Economic Review*, 25 (1984), 1-21.

13. L. GEVERS, "Competitive Equilibrium of the Stock Exchange and Pareto Efficiency," in *Allocation Under Uncertainty: Equilibrium and Optimality,* ed. by Jacques Dreze. New York: Macmillan, 1974.

14. S.J. GROSSMAN, "A Characterization of the Optimality of Equilibrium in Incomplete Markets." *Journal of Economic Theory*, 15 (1977), 1-15.

15. S.J. GROSSMAN and O.D. HART, "A Theory of Competitive Equilibrium in Stock Market Economies." *Econometrica*, 47 (1979), 293-329.

16. O.D. HART, "On the Optimality of Equilibrium when the Market Structure is Incomplete." *Journal of Economic Theory*, 11 (1975), 418-443.

17. L. JOHANSEN, "Some Notes on the Lindahl Theory of Determination of Public Expenditures." *International Economic Review* , 4 (1963), 346-358.

18. ------, "On Shareholder Unanimity in Large Stock Market Economies." *Econometrica,* 47 (1979), 1057-1082.

19. J.S. JORDAN, "Investment and Production in the Absence of Contingent Markets I: The Controlling Interest Paradox." Mimeo, Graduate School of Business, University of Minnesota, 1978.

20. G.H. KRAMER, "Sophisticated Voting Over Multidimensional Choice Spaces." *Journal of Mathematical Sociology*, 2 (1972), 165-180.

21. R.D. McKELVEY, "Covering, Dominance, and Institution Free Properties of Social Choice." Mimeo, Division of the Humanities and Social Sciences, California Institute of Technology, 1986.

22. R. RADNER, "Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets." *Econometrica,* 40 (1972), 289-303.

23. ------, "A Note on Unanimity of Stockholders' Preferences Among Alternative Production Plans: A Reformulation of the Ekern-Wilson Model." *The Bell Journal of Economics,* 5 (1974), 181-184.

24. S. SLUTSKY, "A Voting Model for the Allocation of Public Goods: Existence of an Equilibrium." *Journal of Economic Theory*, 14 (1977), 299-325.

25. J.E. STIGLITZ, "On the Optimality of the Stock Market Allocation of Investment." *Quarterly Journal of Economics,* 86 (1972), 25-60.

26. ------, "Pareto Optimality and Competition." *Journal of Finance,* 36 (1981), 235-251.

27 S. WILLARD, *General Topology*. Menlo Park: Addison-Wesley, 1970.

28. R.A.WINTER, "Majority Choice and the Objective Function of the Firm Under Uncertainty: Note." *The Bell Journal of Economics* , 12 (1981), 335-337.

29. R.J. ZECKHAUSER and M.C. WEINSTEIN, "The Topology of Pareto-Optimal Regions with Public Goods." *Econometrica,* 42 (1974)643-667.

# THE EFFICIENT ALLOCATION OF RESOURCES IN VOTER REGISTRATION DRIVES

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This paper analyzes the efficient management of Aggregately Targeted, Individually Selective (ATIS) partisan voter registration drives with fixed operating budgets. Sufficient conditions are derived for unique optimal deployment of mobile registrars within contested districts. To implement these conditions, each census tract is assigned an index of expected plurality return to registration investment; these indices are computed from estimates of registration, partisanship, and turnout parameters. The optimum registration intensity equals the logarithm of a tract's index plus a constant term. Using Los Angeles County Registrar 1984 registration data, registration drives conducted in several Assembly districts are tested for consistency with these conditions. The marginal benefit of additional registration spending, compensation plans for mobile registrars, and the Nash equilibrium solution for the general campaign problem are also discussed.

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# I. INTRODUCTION

Political parties and civic organizations devote a great deal of money and effort to registration drives. During the half century since the New Deal, the Democratic Party has relied heavily upon minority communities, low-income clienteles, and the youngest eligible age groups for electoral support. Individuals with these characteristics tend to exhibit relatively low rates political of participation[1]; this consideration, together with the formidable barrier registration poses to voting[2], have long made registration drives a sine qua non for the Democrats.

Shifting political circumstances have forced the Republicans to think differently about the value of registrion efforts. In the past Republican strategists have tended to assume registration efforts were redundant, as GOP supporters would presumably self-register even in the absence of registration efforts. However, recent research[3] indicates voter participation is significantly diminished by residential mobility and the attendant necessity for re-registration. Given the unabated post-war trend toward increased geographic mobility, voter registration drives now represent an important new factor in the Republicans' strategic calculus.

Furthermore, there have been recent demographic changes in Republican support. In particular, Ronald Reagan's second Presidential campaign was the first instance in the post-war period of a Republican Presidential candidate being favored by a higher percentage of voters under 30 than over 30. The party also has shown it

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1. See Verba and Nie [1972], and Wolfinger and Rosenstone [1980].

2. Wolfinger and Rosenstone [1978], and Erikson [1981] discuss the inhibatory effects of registration requirements on political participation.

3. See Glass, Squire, and Wolfinger [1984].

believes it can make significant inroads into the growing Hispanic electorate. The lower participation rates of these groups compared with the party's traditional supporters would appear to make registration efforts more rewarding than was formerly the case.

Thus registration drives have become an important instrument in the parties' strategic arsenals. For example, in California during the period between the 1984 primary and general elections, 720,000 Republican registrations were recorded statewide, while the Democrats' registration total was 660,000 registrations. Overall, the number of new California voters, including self-registrants and group-registrants by all groups, rose by 1.5 million during this period.[4] Cain and McCue [1985a], based on a sample of Assembly districts in Los Angeles County, report that nearly two-thirds of all registrations for the two major parties between 1982 and 1984 were group registrations. Extrapolating this two-thirds group-registration percentage to the statewide context, the significant social impact of voter registration drives, and the importance to partisan organizations of efficient registration drive management become apparent.[5]

New sources of data, particularly identification codes on registered voter tapes, now allow the determination of the group (if any) which registered each voter, and the new registrants' turnout propensity. When merged with indicators of partisanship and participation such as census data and past election returns, these group-registration data are particularly valuable for developing targeting criteria for registration drives, and for examining whether parties are currently conducting

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4. These figures are reported in Cain and McCue [1985].

5. The same authors have also reported [1985b] that 58% of the 108,653 individuals in Los Angeles County who registered to vote in the period after the 1980 election and up to 29 days before the 1982 November election were group registered.

registration drives as efficiently as possible. An implication of this research, then, is that parties could, by utilizing similar techniques, improve the effectiveness of their registration drives.

In conclusion, this paper has six major goals:

i) To develop and test a model, based on a set of elementary principles, which will describe the registration activities of political parties.

ii) To advise partisan voter registration drive managers regarding the efficient allocation of scarce registration resources, given demographic and past participation information for the census tracts which comprise contested districts.

iii) To determine the registration drive manager's optimal response, given a fixed budget, to registration efforts of other political organizations, both partisan and non-partisan.

iv) To estimate the registration costs of mis-managed registration efforts.

v) To determine the optimal compensation plan a partisan organization should offer its mobile registrars.

vi) To discuss the Nash equilibrium solution for the simultaneous move game which determines parties' registration expenditures.

A literature review follows in section II, next section III develops the registration drive manager's problem from basic assumptions to optimal strategy, and section IV is concerned with describing the data set and comparing realized versus potential registration performance. Finally, section V indicates directions for future research.

## II. LITERATURE REVIEW

Four branches of the political science literature are especially relevant to the analysis of the optimal management of voter registration drives:

A) The literature concerning the application of optimization techniques to practical problems encountered in political campaigns.

B) A second group of papers relating registration requirements and political participation.

C) A third body of work which addresses the broader problems of predicting political participation on the basis of demographic variables.

D) Recent research which has indicated some of the qualitative properties of registration production technology.

Papers from the first category are relevant to the registration drive problem primarily because they emphasize that political campaign problems are amenable to mathematical programming methods. The second group of papers helps to answer the question: Why should parties be concerned with conducting registration drives at all, much less with conducting them optimally? The third type of literature proves useful for putting the theoretically optimal registration strategy into real world operation. Finally, the fourth body of work undertakes a detailed investigation of the efficacy of different types of registration drives; these results can be of value when tailoring a registration drive to a specific target population.

## A. OPTIMIZATION TECHNIQUES IN POLITICAL CAMPAIGNS

The process of formally articulating goals, and assessing available resources, information and technology, and then exploiting classical optimization techniques has proven to be successful in business, engineering and military contexts.Pioneering research by Kramer suggested that similar systematic methods could be brought to bear on the problems of campaign management with great advantage.

- G.H. KRAMER, "A Decision Theoretic Analysis of a Problem in Political Campaigning," 1966.
- -----, "The Effects of Precinct-Level Canvassing on Voter Behavior," 1970.

## B. REGISTRATION REQUIREMENTS AND PARTICIPATION

These papers highlight the formidable barriers to voting posed by registration requirements, and hence demonstrate the importance of registration drives to parties in terms of increased expected plurality.

- R.S. ERIKSON, "Why Do People Vote? Because They Are Registered," 1981.
- D. GLASS, P. SQUIRE and R. WOLFINGER, "Residential Mobility and Voter Turnout," 1984.
- S. KELLEY, R.E. AYRES and W.C. BOWEN, "Registration and Voting: Putting First Things First," 1967.
- R.E. WOLFINGER and S.J. ROSENSTONE, "The Effect of Registration Laws on Voter Turnout," 1978.

## C. PREDICTING POLITICAL PARTICIPATION

In order to implement an efficient registration effort, the manager must estimate several parameters which describe political participation across the census tracts which comprise his district. The estimation of such parameters from backround characteristics is a classical problem in political science.

- S. VERBA and N.H. NIE, *Participation in America*, 1972.
- R.E. WOLFINGER and S.J. ROSENSTONE, *Who Votes,* 1980.

## D. REGISTRATION PRODUCTION TECHNOLOGY

This research develops a fourfold taxonomy of registration efforts, and investigates political differences among the registrants yielded by different types of drives. Of particular interest is the identification of the problems of self-registrant cannibalization, and the inverse relationship between registrants' interest in politics and the degree of assistance provided them in the registration process. This line of research may lead to innovations in registration technology, such as compensating mobile registrars based on demographic attributes of the registrants they produce.

- B.E. CAIN and K.F. McCUE, "Do Registration Drives Matter: The Realities of Partisan Dreams," 1985.
- ------, "The Efficacy of Registration Drives," 1986.

# III. THE MODEL

The model is a stylized representation of the resource allocation problem faced by a partisan voter registration drive manager over one campaign season. The manager has at his disposal an exogenously specified fixed budget[1], and a single registration production technology-- the posting of mobile registrars in public places. His fundamental objective is to maximize the expected contribution made by the registration drive to his party's election day plurality.

In operational terms, the manager's task is to determine the most efficient deployment pattern of registration workers across the census tracts which comprise the contested district.[2] He has at his disposal demographic information and past voting records, and thus can estimate productivity indices for the various tracts. His decision problem may be further complicated by registration drives conducted by a second political party and by a non-partisan organization.

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**1.** The manager may seek to expand the scale of the registration drive by additional fund raising, or by requesting approval for further expenditures of existing campaign funds. This presents an instance of the general campaign problem: What is the optimal allocation of campaign resources among various political activities such as: fund raising, canvassing, registration, direct mailings, and public appearances?

The general problem is particularly refractory to formal analysis since the benefits of campaign activities are extremely difficult to estimate. In the interests of anlytic tractability and solving a well-defined practical problem, attention will be focused on managing a fixed-budget registration drive.

**2.** Another important aspect of the manager's job is the selection of the best sites within tracts at which to post mobile registrars. Some locations can be expected to be more heavily trafficked, or to attract a more desireable demographic mix of potential registrants, than others. However, in the analysis to be developed here, this facet of the manager's problem will not be explicitly modelled.

## III.A. BASIC ASSUMPTIONS

Modelling the registration drive manager's problem at first appears to be a relatively transparent undertaking. However, analagous to the development cycle encountered in many software engineering tasks, as the registration problem is formalized, unanticipated difficulties arise, what were originally thought to be minor complications turn out to be serious inconsistencies, and so on. This pattern will be familiar to anyone who has ever written a single line of program code.

In particular, a complete specification of the registration task and the production technology requires a suprising degree of attention to detail and a number of simplifying assumptions. Nevertheless, the assumptions adopted here seem to accord well with current political practices in California. Ultimately the problem formulation allows a satisfying cancellation of complexity-- yielding a straightforward solution to the registration drive manager's problem.

The purpose of this section is to familiarize the reader with the fundamental constructs of the model-- the participants and their environment, their goals, and the timing scenarios. With this broad perspective in mind, the role of additional specialized assumptions can be better appreciated as they are presented in later sections.

### III.A.1 Participants in the Registration Contest and their Environment

There are two classes of participants in the model: individuals and organizations. Consider the individuals' defining properties first. Individuals will be modelled quite simply here; the detailed specifications of consumption preferences and information sets charactersitic of microeconomic models are not relevant in the present context.

Suppose there are a total of Q ***qualified*** individuals, out of a total population P, residing in a political jurisdiction or ***district.***[3] An individual is said to be qualified if he is already registered to vote or satisfies the citizenship, residency, and age requirements for registration. Let an individual be called ***eligible*** if he is qualified but not yet registered. The district is comprised of $I = \{1, \ldots, i, \ldots, I\}$ neighborhoods or ***census tracts***; approximately 5,000 individuals reside in a typical census tract. A given individual can be registered at most once, and must be registered in the tract in which he resides. Denote by $Q_i$ and $E_i$ the qualified and eligible population of tract i respectively. Suppose that each $Q_i$ remains fixed for the registration period.

In addition to registration and residence status, there are three other parameters required to completely describe an individual:

- His partisanship.
- Whether or not he will register to vote on his own initiative (self-register).
- Whether or not he will actually turn out to vote.

Notice that while a total of only five parameters define an individual, the values these characteristics take among the population of a given tract are typically not directly observable by the registration drive manager. In general he will find it necessary to

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**3.** No particular assumptions will be formally adopted concerning the type of district or its size. However, the reader may wish to keep in mind a California Assembly district as an appropriate example; the total population of these districts is approximately 295,000 and there are on average 295,000 / 5000 = 59 census tracts per district. Empirical anlaysis based on the model will employ data from Assembly districts in Los Angeles County .

estimate the parameters which describe population of a tract on the basis of census data, current registrations, and past elections returns.

Several assumptions concerning individuals and the political environment are formally set forth below. The central theme running through this group of assumptions is that, with the exception of registration status, the defining properties of individuals and the demographic characteristics of the district do not change during the registration period-- or as a result of registration drive itself. In other words, the registration manager need not attempt to hit a moving target.

Individuals' Parameters

**I1)** Qualified individuals' self-registration parameters remain constant during the registration period. Furthermore, turnout and partisanship partisanship parameters remain fixed at least until election day.[4]

**I2)** Registered partisans who turn out vote for their own parties' candidates.[5]

**I3)** All individuals who self-register do so on the second date. (See the timing assumptions below.)

**I3a)** The pool of individuals who will self-register decays linearly to zero over the course of the registration drive.

**4.** Evidence will be presented that conversion during registration drives is a relatively rare phenonenon.

**5.** This assumption can be relaxed, given estimates for the probabilty of cross over voting in the various tracts.

Political Characteristics of the District Population

**P1)** Partisanship proportions are the same among the registered and qualified groups in each tract.[6] (These proportions may vary across tracts, however.)

**P2)** The expected turnout proportions among the ex ante (before the registration drive) and ex post (after the registration drive) groups of registered partisans in each tract are identical. (These proportions may also vary across tracts, however.)

Migration Across Boundaries

**B1)** Each individual resides in exactly one tract, and can register at most once in this single tract.

**B2)** No individual is contacted or registered by mobile registrars outside his home tract.

**B3)** There is neither in or out migration, nor change in other demographic characteristics in any tract during the registration period.

Notice that assumption **I1)**, the fixed individual parameter assumption, entails several important consequences. By assuming that individuals' participation parameters remain fixed, it follows that:

- There is no self registration sensitization of group-only registerable partisans of one party resulting from contact with registrars representing another party.
- There is no change in turnout rates resulting from multiple contacts with mobile registrars.
- Cross-registered individuals would have eventually registered themselves had they not been group-registered first.

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**6.** This assumption is supported by Wolfinger's [1980] analysis of ANES survey results. Professor McKelvey has observed that **P1)** entails that there are equal percentages (among qualified partisans) of unregistered Democrats and Republicans in each tract.

There are three organizations conducting group registration efforts in the district: two political parties, ***Party A*** and ***Party B***; and a single non-partisan organization, *N*. Let $Q^A_i$ and $E^A_i$ denote respectively the number of qualified and eligible A-partisans residing in tract i. Denote by $\pi^A_i$ the proportion of A–partisans among $E_i$, by $\sigma^A_i$ the proportion of $E^A_i$ expected to self register in the absence of any registration drive, and by $\tau^A_i$ the proportion of A-partisans expected to turn out and vote among all eligible group-only registerable A-partisans.

Significant technical simplification results if it is assumed that both parties can perfectly predict the registration efforts of the non-partisan organization N,[7] and that this group generates all of its registrations instantaneously on the first date (see the timing assumptions below)[8]. The following assumptions describe these organizations and their objectives:

Registration Groups

**G1)** There are three registration groups: two political parties, A and B, and a non-partisan organization N.

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7. It seems plausible that both parties should be able to make good decisions concerning non-partisan registration efforts; indeed, a truly non-partisan organization might well announce its registration intentions. Furthermore, since both parties are concerned with maximizing their expected plurality, they need only estimate the expected non-partisan registration yield.

8. Actually, non-partisan groups most likely conduct their drives concurrently with those of the political parties, but formally introducing a concurrent non-partisan drive into the model significantly complicates the parties' objective functions and the associated systems of first order conditions. Since the non-partisan registration yield is relatively small compared to that of the parties, the instantaneous yield approximation would seem to be of no great consequence.

G2) Once having set a registration budget, each political parties' registration drive objective is to maximize the contribution--attributable exclusively to its registration drive-- to its own expected election-day plurality.

G3) Both parties can perfectly predict the registration yield of group N; furthermore this non-partisan production occurs instantaneously at time 0.

G4) The non-partisan organization registers members of various sub-populations in proportion to these groups' relative frequencies among the eligible populations of each tract.

G5) Both parties have access to the same demographic and past elections data for the district; and both possess the same analytic skill.

## III.A.2 Timing Scenarios

Timing assumptions often play a crucial role in social science models, particularly if conflicts of interest are the focus of the analysis. To a large extent, a model's timing scenario determines the participants' ability to update their information and to dynamically revise their stategic decisions.

The model employs a straightforward two-date, one-period timing scenario. The first date represents the beginning of the period in which registrations can be generated, and the second is the last day to register for the next election.[9] At first glance, this scenario may appear somewhat artificial since voter registration does take

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9. While no particular dates or registration period have been assumed, the 59th day before election day is one natural date to regard as the middle of the group registration season in Los Angeles County. Shortly after this date the registrar of voters publishes the so-called 59-day close, the official registration roll for the county as of the 59th day before election day. Publication of the 29-day close marks the end of the registration season; registrants filing later cannot vote in the upcoming election.

place throughout the year. However, group-registration drives are strongly seasonal, with the greatest intensity of effort occurring shortly before the last day to register.[10]

Each registration drive manager, partisan and non-partisan, must take decisions and irrevocably allocate his force of mobile registrars on the first date. In the case of more than one organization conducting registration drives within the district, it will be assumed that they all move simultaneously on the first date.

The distinguishing simplifying feature of these assumptions is that the managers are unable to dynamically update their deployment strategies as reports come back from the field. Once a manager selects his registration drive strategy, it cannot be revised or updated. Whether or not managers would want to update their strategies is a separate issue which will be taken up later.[11]

Timing Assumptions

**T1)** There are two dates: 0 and 1, and a single period which will be termed the ***registration period.***

**T2)** All political groups mobilize their registration forces simultaneously and irrevocably on the first date.

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**10.** Several factors contribute to this seasonality. One such factor is geographic mobility and the concomitant need to re-register a fraction of those registered early in the campaign cycle. Another factor is that the act of registration itself seems to increase the salience of politics for registrants. Since this effect decays across time, early registration drives generate a relatively smaller turnout, other considerations being equal.

**11.** Notice that it was not necessary to specify the available registration production method(s) in order for these timing assumptions to be well-defined. Furthermore, no assumptions made so far restrict drive managers to a single type of registration activity.

## III.A.3 The Registration Drive Objective

Each partisan organization will be assumed to conduct its registration efforts so as to maximize the exclusively attributable or ***marginal*** contribution to its own expected plurality.[12] Notice that choice of this objective presupposes that the party explicitly recognizes the possiblity of redundantly registering self-registrants. The non-partisan organization's allocation problem is not the focus of this analysis; therefore, the non-partisan objective will not be defined here; only the non-partisan registration yield-- but not the objective function-- formally appears in the model.[13]

### a. The Manager's Problem: Maximize Expected Plurality Given a Fixed Budget

Since each partisan organization has a well defined registration drive objective, it is convenient to imagine that each registration effort is conducted by a single individual, or ***manager***. A registration drive manager will be modelled as maximizing the expected number of net own party registrations attributable solely to the drive,

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12. The assumptions that there is only one district, and that partisans of a given party vote for all the party's candidates are relevant when asserting the existence of a well defined party registration objective. Suppose that party A instructs its manager to conduct a registration drive to benefit two of its candidates running in two different districts; how should the manager split the budget between the districts? If there is a possibility of significant cross-over voting, two candidates from the same party running for different offices with overlapping district boundaries may even disagree concerning whether or not to launch registration efforts. A related class of problems arises in the case of simultaneous races for different offices with overlapping district boundaries: candidates from a given party may select sub-optimal levels of registration investment due to the free rider problem.

**13.** Indeed, it is not transparent that given a plausible definition of "non-partisan" that an objective consistent with the definition can be defined.

given an exogenously specified budget constraint. Observe that both the manager's objective and the budget constraint must be expressed in expected terms, since the number of registrations generated and hence the costs of a registration drive are a random variable. It will be assumed that the manager pursues the goal of expected plurality maximization assiduously, without shirking; a manager does not pose an incentive compatability problem for the party which employs him.

**b.** Expected Plurality and the Probability of Winning

Presumably political candidates have only an indirect interest in expected plurality; the probability of winning office would appear to be candidates' prime consideration when making campaign decisions. However, as Kramer [1966] points out:

> Just as there are many ways of running a compaign, so also there is a variety of possible goals which a candidate may be pursuing. ... [I]f a general analysis is to proceed we must concentrate upon the major and most tangible of the goals. For most candidates in most contests this goal is clearly to win. ... With the usual electoral arrangements, winning is normally closely related to the size of the candidate's plurality.

Kramer goes on to point out that maximizing the probability of winning and maximizing expected plurality will tend to yield different decisions when a candidate considers strategies which are both very risky and very productive--an unlikely scenario in the case of registration drives. On balance he finds that the analytic simplicity afforded by the expected plurality formulation justifies its adoption over the probability of winning:

> [The probability of winning] formulation is computationally quite difficult to work with; in practical applications one would have to resort to simulation techniques which are expensive and often cumbersome. The expected plurality criterion is much simpler in this respect, and possesses the convenient property that if we can evaluate the candidate's expected plurality in each of several subunits (e.g., precincts) in his constituency, then his overall plurality can be obtained by simple summation. Clearly this is not true of the probalistic criterion. Moreover, the expected plurality criterion is more easily comprehended and communicated, since campaigners traditionally think in terms of so many votes gained or lost, and the criterion translates directly into these terms. Either formulation provides us with a reasonable, quantitative value criterion; however, in subsequent discussion we shall employ the expected plurality criterion.

## III.B. THE REGISTRATION PRODUCTION TECHNOLOGY

Voter registrations are generated by goal-seeking partisan organizations, analagous to the production of other goods and services by profit-maximizing firms. Like economic enterprises, political parties face two kinds of constraints in determining their optimal registration policies: technological constraints, and market constraints. Technological constraints reflect the sheer physical and informational feasibility of a registration production plan. On the other hand, market constraints concern the effects of the actions of other agents; for example, the suppliers of registration services may only accept certain prices for their inputs, or voters may resist changing their political affiliation.

When the registration drive manager determines his optimal actions, he must take into account both sorts of constraints. The market constraints faced by the manager are straightforward; it will be assumed that he acts as a price taker in both the labor and political markets.[14] Thus the primary focus in this section will be on the technological constraints present in a registration effort. In order to analyze the efficient management of registration drives, it is necessary to develop a convenient formal description of a partisan organization's registration production possibilites--that is, which combinations of inputs and outputs are feasible.

Before deriving such a detailed specification of the most commonly employed registration production technology, the taxonomic problem of classifying different registration techniques will be considered briefly in the next section.

## III.B.1 Classification of Registration Techniques

Registration techniques vary with respect to the methods by which individuals are contacted, filtered, and registered. The particular method by which voters are contacted, the possibility of focusing registration efforts on individuals of a desired type--either before or after they are contacted, and the regulatory environment in which registration contacts occur are all prime strategic considerations for the registration dirve manager. This section will provide an overview of registration tactics by examining two schemes for the classification of registration efforts: the active / passive criterion advanced by Cain and McCue [1985], and classification according to filtering capability.

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**14.** Presumably the registration manager is not charged with the responsibilty of shaping the party's appeal so as to attract the maximum number of voters. He will be assumed to take the party's posistions and the partisanship of the electorate as givens. Recall that it has been previously assumed that individuals' partisanship, self-registration, and turnout parameters are fixed. These assumptions areall the more reasonable in light of the fact that registration dirves are typically conducted relatively late in the campaign cycle.

a. Active and Passive Registration Efforts

Political parties can contact individuals and encourage them to register in a variety of different ways, for example by: blind door-to-door canvassing, selective[15] door-to-door canvassing, work place visits by registration teams, media campaigns of every sort, mailings, and by other methods to be found in the campaign manuals of various political parties.[16]

Cain and McCue [1985] have classified various registration production techniques along a so-called active/passive dimension. An ***active*** registration effort reduces the cost of registration[17] to registrants by actively locating potential voters, providing them with registration forms, assisting them in preparing the forms, and returning the completed forms to the registrar of voters.

For example, a door-to-door registration drive in which the canvassers returned registrants' affidavits would be classified as very active along the active / passive dimension. On the other hand, a program which simply made affidavits available at public locations would be considered ***passive***-- as registrants would bear all the transactions costs of completing and mailing their registration forms. A

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**15.** A variety of different selection criteria might be employed to select the most promising households for registration workers to contact. For example, one possible strategy would be to visit only those households which have fewer than some ceiling number of registered voters.

**16.** See, for example, *The Democratic Campaign Manual.* Democratic National Committee: Washington, D.C., 1964.

**17.** The costs of registration include: the opportunity cost of time lost from other activities, search costs entailed in locating registration forms, the costs of travel to pick up the forms, the psychic costs of filling them out, and the cost of mailing the forms to the registrar of voters.

registration effort consisting solely of a media campaign exhorting citizens to register would be classified even further along the passive direction.

### b. Classification of Registration Techniques by Filtering Capability

The effectiveness of a partisan registration effort and the optimal strategy to be pursued by a registration drive manager both depend critically on the ability of the party's mobile registrars to select or ***filter*** contacts and differentially register them in the party's best interest. This filtering process carried out by registration workers can be further refined into two distinct activities, what will be called here ***contact filtering***, and ***registration filtering***. Contact filtering refers to the restriction of exposure of the registration effort to selected subgroups of the population. Registration filtering refers to the capability of mobile registrars to refuse to register members of selected subgroups-- presumably partisans of other parties-- after such individuals have been contacted.

Different registration methods can be usefully classified according to the contact filtering and registration filtering opportunities each method admits with respect to the dimensions which define politically relevant subgroups of the population. Five such dimensions have been identified here as particularly important regarding the efficient conduct of registration drives:

i) Qualified /Unqualified,

ii) Eligible /Registered,

iii) Partisanship, and

iv) Self–Registrant /Group–Registrant.

v) Will Turnout /Won't Turnout.

The registration benefits of contact filtering and registration filtering along each of these dimensions will be discsussed in the next four sections.

## i. Qualified / Unqualified Filtering

Recall that an individual is said to be qualified if he is registered or eligible to register, that is: he satisfies the citizenship, age and residency requirements for registration. Since no unqualified individuals can be legally registered, every registration effort must conduct registration filtering with respect to this dimension. Indeed, the registrar of voters verifies registration filtering of unqualified individuals.

Certain registration methods may operate more efficiently when accompanied by Qualified /Unqualified contact filtering. Such filtering should be particularly important in case the registration organization bears some direct cost per contact. For example, if registrations are being generated by sending teams of registrars to workplaces, the cost effectiveness of sending such teams to firms in industries known to employ undocumented workers would have to be carefully weighed against the registration benefits. Similarly, it would not make sense to place advertisements exhorting citizens to vote in childrens' media.

## ii. Registered / Eligible Filtering

Presumably, individuals who are already registered do not seek to re-register unless they have moved, are changing their party status or have been purged from the registration rolls for non-participation[18]. It is therefore reasonable to suppose that individuals self select along this dimension insofar as registration filtration is concerned. In practice, registration filtering along this dimension presents a problem to parties which pay registration workers on the basis of own party registrations generated. There is no benefit to the party from re-registering an already registered partisan.

If an organization bears a direct cost per contact in conjunction with the registration tactic being employed, it may well make sense to filter out contact with already registered individuals-- especially if conversion is unlikely, or if opposistion partisans may be sensitized by such contact. For example, a door-to-door registration effort could employ current registration rolls to select target addresses which do not show any (or some threshold number of) registered residents. Direct-mail campaigns urging registration might also exploit similar contact selection techniques along the Registered /Eligible dimension.

### iii. Partisan Filtering

Partisan contact filtering is the restriction of exposure of a registration effort to partisans of a given party. Partisan registration filtering is the practice of discriminating in the supply of registration services among the contacted population on the basis of partisanship. The presence or absence of partisan filtering is a crucial distinguishing attribute of any registration effort. The inability to conduct partisan

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**18.** Los Angeles County practices a relatively generous purge policy. As long as an individual can still receive mail at his registered address he will remain registered, regardless of past voting history.

filtering significantly complicates a partisan registration manager's problem by raising the spectre of registering members of opposistion parties.

In Los Angeles County, prior to 1977, all registration workers were agents of the registrar of voters, and were specifically charged not to practice partisan filtering. Presumably these government employees did not seek to differentially contact various parisan sub-populations concerning registration matters. Furthermore, as pre-1977 registration affidavits could not be self-completed, mobile registrars could not decline to assist individuals in preparing their registration forms on the basis of partisan considerations.

With the advent of the mail-in registration form, the registrars of voters in California counties cashiered their registration workers. Registration forms, which could now be self-prepared by registrants, were distributed at various government offices, including post offices. Civic and partisan organizations were also permitted to distribute, prepare, and return registration forms. The transfer of the voter registration function out of the registrar's office made registration drives an instrument of party competition, and also made effective partisan filtering a high relief feature of California's political landscape.

Of course, before 1977 parties could conduct passive campaigns encouraging registration, and these appeals could be targeted to differentially contact the appropriate partisans. After 1977 active registration techniques also became available to parties. Registration managers could now practice contact filtering in active drives by, for example, selecting the neighborhoods to be visited by mobile registrars. Furthermore, registration filtering could now be practiced in the context of active, partisan registration efforts. While registration workers were charged to provide forms to anyone who requesting them-- regardless of partisanship, mobile registrars were not required to assist opposition partisans in preparing or returning the forms.

### iv. Self-Registrant / Group-Registrant Filtering

Clearly no advantage is gained by directing scarce registration resources to individuals ultimately expected to register themselves. To the extent that the propensity to self-register is correlated with demographic variables, registration efforts can be tailored-- depending on the particular registration technique employed-- to reduce the costs of contacting sub-populations likely to self-register. It would not appear that any active method of generating registrations is likely to be sucessful at culling out self-registrants from the contacted population ex post, so as to avoid the deadweight loss of registering would be self-registrants.

## III.B.2 The ATIS Registration Technique

Perhaps the most frequently employed active registration technique is what will be referred to here as the ***Aggregately Targeted, Individually Selective*** (ATIS) method.[19] Drives of this type are sometimes referred to as "site" registration drives. This is certainly the most commonly employed registration drive tactic in Los Angeles county, and the focus of this paper will be restricted to analyzing registration drives employing the ATIS strategy.

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**19.** I am not aware of any formal research concerning the frequency with which parties employ different registration techniques. However, several seasoned California campaign professionals have agreed that the ATIS technique is by far the most frequently employed registration production method in Los Angeles county.

## a. ATIS Contacts

In the ATIS method of registration production a partisan organization assigns its mobile registrars to station themselves in specifically targeted public places, such as shopping malls or convenience stores, and to inquire of passing citizens whether or not they are registered to vote. Individuals who respond negatively (and match the registrar's partisanship in the partisan filtering case) are then encouraged to register. Presumably they agree to do so since registration is essentially costless to registrants.[20]

The ability of the registration drive manager to choose the intensity of registration effort within each tract is reflected in the phrase "aggregately targeted," while the phrase "individually selective" refers to the ability of partisan mobile registrars to discriminate among potential registrants on the basis of partisanship. The selection capabilities of mobile registrars will be discussed in detail in the next section.

Contacting citizens is the first step of an active registration effort, and formulating a precise definition of an ATIS contact is the first step in determining optimal ATIS registration strategies. An ATIS contact between a registration worker posted at his station and a passing inidvidual is defined to be the pyhsical arrival of the individual within "***conversational distance***" of the registrar's station. Notice that actual conversation need not have taken place for the contact or exposure to have occurred. "Conversational distance" is a primitive concept which will not be formally defined here.

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**20.** Indeed, it will be assumed here that all unregistered individuals encouraged to register by mobile regisitrars will in fact do so.

b. The ATIS Arrival Process

The hallmark feature of the ATIS registration method is the arrival of individuals at mobile registrars' stations. In order to specify a functional form relating the method's inputs, registrar-hours, to the outputs, contacts and registrations, the nature of this arrival process must be specified. A set of plausible regularity assumptions greatly simplifies the analysis, without seeming to represent a significant departure from the workings of real-world ATIS registration drives. In particular, the arrival of individuals will be modelled as a Poisson process. Arrivees may be thought of as having been culled at random from the entire qualified population of a tract. Having once arrived at a station, the arrivee is returned to qualified population from which he may be selected to arrive again.

A ***stochastic process*** is an indexed family of random variables. An elementary example of a stochastic process known as a ***Poisson process*** arises naturally in many models of queueing phenomena. Karlin and Taylor [1975] present an informal definition of a Poisson process, which has been paraphrased below to cover the context at hand.

> Let X(t) denote the number of ATIS contacts (hereafter, simply *contacts*)at at mobile registrar's station in the time interval [0,t]. Suppose that the number of potential contacts is very large, or that being contacted once does not affect the chance of a a given individual being contacted again. Suppose that as many contacts are likely to occur during one interval of time as another. And finally, suppose that the probability of two contacts occurring simultaneously is vanishingly small. Under these ideal conditions, the process $\{X(t) \mid t \geq 0\}$ is a Poisson process.
>
> This example serves to point up the Markov property (the chance of a contact does not depend upon the number which have already occurred) and the "no premium for waiting" property, which is the most distinctive property possessed by the Poisson process.

The arrival of clients for service is a classical application of the Poisson process formalism, and so the interested reader is referred to Karlin and Taylor for a detailed discussion of the properties of Poisson processes. The following well known proposition is the most important consequence of the Poisson assumptions for understanding the results of this paper:

- If $\mu$ is the probability of at least one contact occurring in a unit of time, $\mu * t$ is the expected number of contacts in a time interval of length t.

Three regularity assumptions concerning the arrival of ATIS contacts are listed below:

ATIS Arrival Assumptions

**A1)** The arrival of qualified individuals at each partisan mobile registrar's station within a given tract is a Poisson process, characterized by a mean arrival rate of rate $\mu$ individuals per hour per station, *randomly selected with replacement from the qualified population of the tract as a whole.*[21]

**A2)** The arrival rate $\mu$ is constant across tracts.

**A3)** Service times at mobile registrars' stations are zero; there is no queueing or crowding.[22]

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**21.** The random selection assumption in the context of the arrival process is related to a notion of perfect mixing of individuals within each tract. Individuals might be said to be perfectly mixed if at any given moment the likelihood some one individual occupies a specified location in the tract does not depend on his previous locational history.

**22.** The reader is invited to speculate rearding the following question: If there are no conversion, self-selection, or crowding phenomena, and if perfect partisan filtering is possible, why do mobile registrars bother to post partisan signals at their stations?

Notice that the random arrival assumption **A1)** entails the following consequences:

- The same registration results can be expected from posting n registrars for 1 hour in a given tract as from posting 1 registrar for n hours.
- Arrivees' demographic, political, and registration characteristics reflect relative frequencies in the qualified population.
- In particular, the self-registration propensity among individuals contacted by mobile registrars is the same as that among those not contacted.
- Registered or previously contacted individuals do not self-select to avoid contact with mobile registrars.
- Registrars cannot a priori filter arrivals by any crterion.

### c. Filtering Capabilities of the ATIS Technique

The filtering capabilites of the ATIS registration method are presented in the table below. The ATIS method is an active method of registration; it reduces the costs of registration to the opportunity cost of a sixty-second conversation with a mobile registrar. The ATIS method allows effective partisan registration (post contact) filtering by essentially making the method a relatively passive one from the perspective of opposition partisans.

**FILTERING CAPABILITY OF THE ATIS REGISTRATION METHOD**

| FILTER DIMENSION \ | Contact Filtering | Registration Filtering |
|---|---|---|
| Qualified /Unqualified | N/A; all contacts assumed qualified. | N/A; all contacts assumed qualified. |
| Eligible /Registered | Tract targeting. | Perfect; verified by mobile registrar after contact. |
| Partisanship | Tract targeting. | Assistance refusal; perfect for group-only reg's. Ineffective for self-registrants. |
| Self–Reg /Group–Reg | Tract targeting. | None. |
| Turnout | Tract targeting. | None. |

Table 1.

### III.B.3 A Prototypical ATIS Service Cycle with Partisan Filtering

This section presents a prototypical arrivee service cycle at a mobile registrar's station. While not every feature of this scenario is a formal assumption of the model, the example is consistent with the general assumptions that have been made. The flowchart in Figure 1 depicts the interaction between an arrivee and an ATIS registration worker.

Imagine an individual arriving at-- or contacting-- the registrar's station, that is he has approached within ***"conversational distance"*** of the registrar. The individual has two choices at this point: he can pause at the station, or continue walking past. If the individual appears likely to leave the vicinity of the station, the registrar also has two choices: he can opt to do nothing while the arrivee departs, or he can initiate a conversation with him.

Now, if the individual is already registered, presumably he has no reason to communicate with the registrar and will therefore proceed past without stopping. The registrar may choose to inquire after him, but before discussing this branch of the diagram, consider the other possibility at the first decision juncture-- namely that the individual is not registered. The unregistered individual will observe the partisan literature posted at the station. If the the registrar is a representative of the arrivee's party, it is more likely that the individual may decide to pause at the station, depending on of his degree of interest in politics. On the other hand, if the registrar is not a representative of the arrivee's party, it is plausible that only a strongly enthusiastic partisan--indeed, a self registrant--would pause at the station.

THE ATIS SERVICE CYCLE

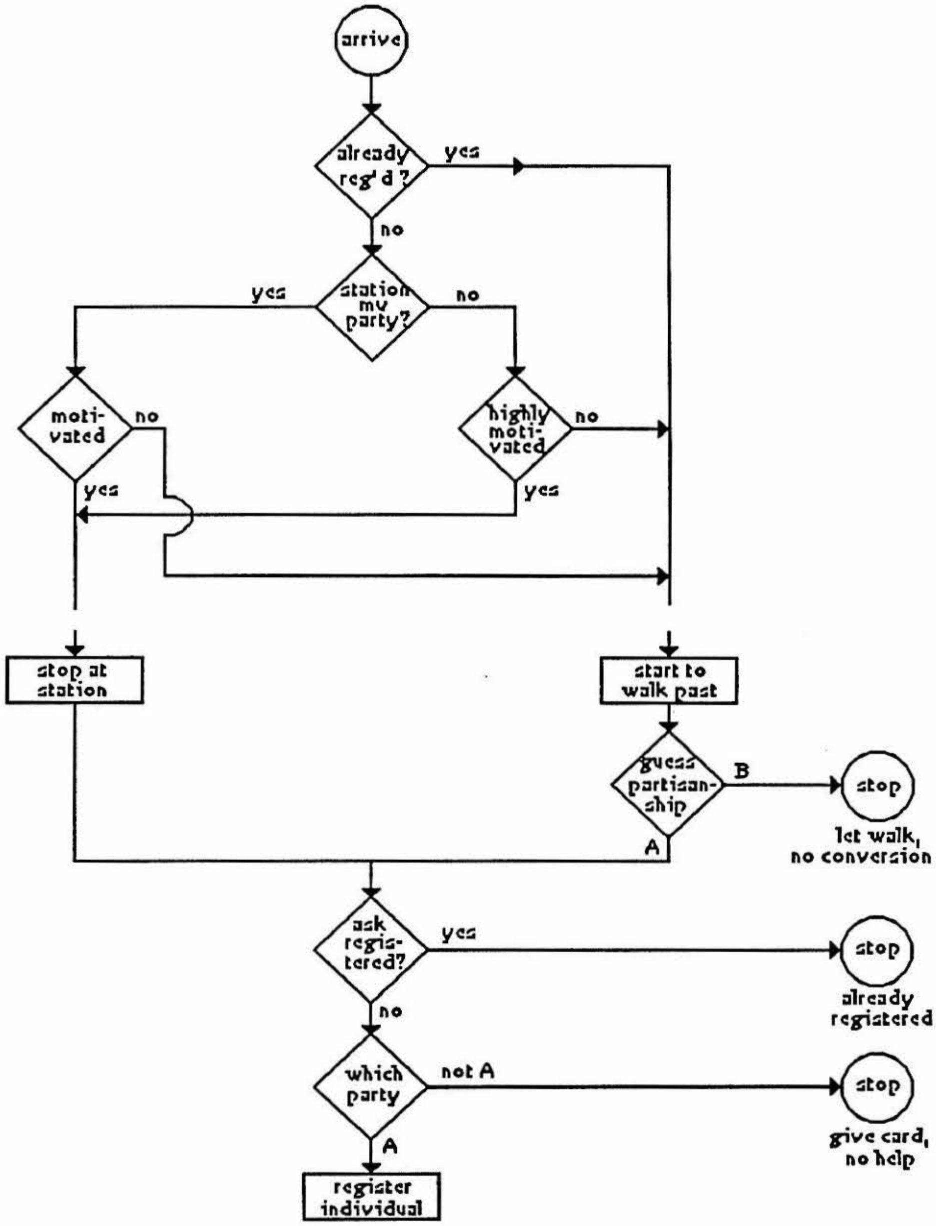


Figure 1.

If an individual appears about to pass by the station, the registrar must decide whether or not to initiate a conversation with him. Presumably the registrar estimates the arrivee's partisanship based on some spectrum of observable characteristics, and decides to address him only if he conjectures their partisanship matches. Otherwise, the registrar will simply do nothing while the individual departs.[23]

Progressing down the diagram, suppose the juncture has been reached at which an individual has stopped at the registration station, either on his own accord or because the registrar has asked him to.[24] The registrar's first question in either case is to inquire whether the individual is registered or not.[25] Obviously, if the individual is registered, no further interaction will occur; and the arrivee departs while the

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**23.** It is unlikely the registrar will let very many arrivees simply walk past. While he clearly does not want to waste any effort serving cross-partisans, the extent of his obligation to such individuals is the provision of registration affidavits. He can decline to assist processing the forms. On the other hand, if the registrar mistakenly lets his party's partisans bypass his station, he will loose potential registration commissions.

Furthermore, if the model is to be consistent with this scenario, the registrar must make accurate partisanship guesses. Otherwise, eligible individuals with a low level of interest in politics will fail to be registered at their first contact with a registrar of their party-- as the model predicts.

**24.** Of course an individual is free to ignore the registrar's summons. However, under assumptions R1) – R3) which will be formally presented in the next section, an individual will refuse to pause when requested only if: i) he is already registered, or ii) he is a group-only registerable partisan who believes the registrar represents an opposition party. An arrivee might form such a belief after inspecting partisan literature posted at the registrar's station. In the absence of such partisan signals, R1) – R3) require that an unregistered individual stop at the station.

**25.** Individuals who stop of their own accord presumably are not already registered, but it seems reasonable to suppose the registrar will ask about their registration status just to ascertain there has been no mis-understanding. Among those arrivees stopped by the registrar himself, some will be already registered, and some not; in these cases the determination of registration status is a necessity.

registrar prepares for the next arrival. On the other hand, if the individual is not registered the registrar's next move will be to determine the arrivee's partisanship.

If the individual asks to register for a party other than the registrar's, the registrar must, as prescribed by the California elections code, provide the arrivee with an affidavit before dispatching him.[26] However, the registrar is not obligated to assist the individual in filling out the form or returning it. On the other hand, if the arrivee's declared party affiliation matches the registrar's, the registrar will aid in the completion and processing of the form.

The ATIS arrival service cycle has now been completed, and the registrar resets and awaits the next arrival. Presumably the length of time required to serve an arrival is relatively short compared to the mean length of time between arrivals. Indeed, in the interests of analytic simplicity it will be assumed that the service cycle is length zero, that is the processing of an arrivee is instantaneous. This will obviate the need to formally introduce the possibility of queueing in the model, which would be mathematically complex but would not seem to be an important feature of the ATIS registration process.

### III.B.4 The ATIS Production Function

The purpose of this section is to express the expected number of marginal registrations produced by a partisan ATIS registration drive as a function of the

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**26.** Recall it has been assumed that self-registrants postpone self registration until the second date. Thus an A-party self-registrant, even though he had just received an affidavit from a B-party registrar would not return the form until the second date. Such an individual could still be contacted by A-party mobile registrars during the registration period and group registered.

drive's ***contact intensities*** $\lambda_i$ across tracts. A ***marginal*** registration is defined to be that of an individual who would not have registered in the absence of the drive.[27] The contact intensity of a registration drive in tract i is defined by: $\lambda_i = \gamma_i/Q_i$, the number of contacts $\gamma_i$ generated by the drive as percentage of the qualified tract population $Q_i$. These contact intensities should be regarded as the inputs to the ATIS production process; the production function developed in this section describes the feasible combinations of contact intensities and expected marginal registrations.

Two cases will be considered: first that mobile registrars can practice partisan filtering defined by assumptions **R1) – R3)** below, and secondly that they cannot. The filtering case is applicable to the current politcal environment in California--while the no-filtering case, analyzed in the appendix, is mainly of historical interest. Prior to 1977 mobile registrars were non-partisan deputies of the registrar of voters and were prohibited from engaging in partisan filtering.[28]

Three assumptions have been made below regarding ATIS registration pro - duction. Notice particulary that these assumptions determine the partisan registra - tion (post-contact) filtering capability of ATIS mobile registrars.

**R1)** Eligible self-registrants will obtain and return registration affidavits at first contact with any registration group.

**R2)** Eligible group-only registrants will become registered if and only if con - tacted by an own-party or non-partisan group. These individuals will be registered at the first such contact, but never cross-register.

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**27.** Notice that all marginal registrations must be generated from the pool of eligible group-only registerable individuals not registered by the non-partisan organization.

**28.** Presumably the no-filtering case is also more appropriate for describing registration drives conducted by non-partisan organizations.

**R3)** Partisan mobile registrars must provide affidavits on demand to cross-registrants, but need not assist in preparing or returning the forms.

These assumptions determine the registration filtering capability of mobile registrars as follows: (WOLOG, consider the case of an A-party registrar repre - senting party A.) By previous assumptions, each individual is either a self-registrant or a group-only registrant, and either an A-partisan or not. After contacting an eligible individual, an A-party mobile registrar will immediately verify the contact's partisanship. Although the registrar cannot refuse to distribute an affidavit to a non-A-partisan, he can decline to assist such an individual in preparing and returning the form. The registrar will, however, provide preparation assistance to an A-partisan.

This is an effective filtering strategy. For consider that self-registrants, regardless of partisanship, will obtain and return registration forms one way or another-- nothing is lost by providing an affidavit to a self-registrant. On the other hand, group only registrants will return their forms only if assisted--restricting registration assistance solely to A-partisans prevents the registration of opposistion partisans.[29] Notice also that **R1) – R3)** imply that individuals cross-registered by party A *would have self-registered anyway*; therefore, cross-registrations should *not* be subtracted from own-party registrations when computing party A 's marginal plurality

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**29.** Registration filtering losses may occur in the case of group-only registerable individuals who decline to state a party preference (GORDS). Under R1)-R3), A-party registrars would not assist such individuals. Some would vote for party A if registered, and so represent a filtering loss. However, the fewer GORDS individuals, the smaller the losses. The weaker GORDS individuals' political interest and the lower their turnout probability, the less important the registration loss. Finally, while some GORDS registrants would turn out for party A, some would also turn out for the opposistion.

yield. In the partisan filtering case, expected marginal registration plurality is equal to expected (own-party) marginal registrations.

The first step in computing the expected number of marginal registrations due to an A-party registration effort is to determine the expected number of eligible A-partisans in each tract of the district. This is the pool of individuals from which the marginal registrations will be generated. It has been previously assumed in **G3)** that both parties can perfectly predict the registration yield of group N, and that this non-partisan production occurs instantaneously at time 0. In particular, suppose that group N registers $r^N_i$ individuals in tract i leaving a remaining eligible population of $E_i - r^N_i$ at time 0.[30] Recall that $\pi^A_i$ denotes the proportion of A-party partisans among $E_i$. According to assumption **G4)**, the non-partisan group-registers sub-populations of a tract in proportion to their relative frequencies; therefore, the expected number of A-partisans remaining after the initial non-partisan effort can be expressed as:

$$\pi^A_i(E_i - r^N_i). \qquad \text{(eq 1)}$$

It remains to determine a) the percentage of this sub-population expected to be

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**30.** Assumption **G3)**, that the non-partisan registration yield $r^N_i$ is obtained instantaneously on the first date, has been made for technical convenience. Of course if the non partisan group conducts its eforts concurrently with the parties over the course of the registration period, it is not strictly correct to assume that there are $E_i - r^N_i$ eligible individuals remaining at time 0.

The expected number of A-party registrations produced by party A under the concurrent scenario *can* be calculated; however, the resulting first order system is quite complex and appears unlikely to afford desired closed form solutions for the optimal contact intensities. Since the expected number of A–party registrations obtained under **G3)** approximates that obtained under the concurrent scenario, and since the non-partisan effort is relatively minor, there is no advantage in analyzing the more complex case.

contacted and registered by an effort of a given intensity, and b) the percentage of registrations which are attributable soley to the registration effort.

a. <u>An Occupancy Problem: Numbers of Individuals Contacted at Least Once</u>

This purpose of this section is to determine the percentage of eligible A-partisans contacted at least once-- and hence registered-- during an A-party registration effort of a given intensity. Since ATIS mobile registrars are unable to distinguish a priori between registered and eligible individuals, it is likely that during a registration drive some individuals will be contacted several times by registration worker(s)[31]. This phenonenon gives rise to an interesting combinatorial problem: Suppose a registration effort in tract i results in $\gamma_i$ contacts from a qualified population of $Q_i$. What is the expected number of individuals contacted *at least once* during this drive of intensity $\lambda_i$?

This problem can be interpreted as an instance of a well-known schema in probability theory--the occupancy paradigm--based on the notion of balls falling into compartmentalized boxes.[32] Let $\gamma_i$ balls be dropped randomly into a box containing $Q_i$ identically sized compartments, $D_i$ of which have been distinguished or specified in

31. Recall self-registration or turnout sensitization effects due to multiple contacts have been assumed away in **I1)**,and will not be analyzed further here.

32. Occupancy problems occur frequently in the quantam mechanics and thermo-dynamics literature. Perusal of a journal such as *Biometrika* will reveal considerable research interest in similar problems, particularly on the part of epidemiologists. However, in the epidemiological literature, immunization workers-- who play a role analagous to that of registration workers here-- are not typically modelled as being unable to distinguish a priori between immunized and unimmunized members of the population.

advance. A compartment is said to be ***occupied*** (after the drop) if it contains one or more balls. How many of the $D_i$ distinguished compartments can be expected to be occupied after $\gamma_i$ balls have been dropped?

Dropping one of the $\gamma_i$ balls into a compartment corresponds to the registration worker contacting a member of the qualified population $Q_i$, and the first drop into each of the $D_i$ distinguished boxes represents a successful registration in the distinguished sub-population. The possibility for a compartment to contain more than one ball reflects the possibility that a given individual may be contacted redundantly several times during the registration drive.

By means of a suprisingly tedious argument, David & Barton [1962] demon - strate that the expected number of occupied compartments among the $D_i$ is given by:

$$D_i[1 - (1-1/Q_i)^{\gamma_i}],$$

where $\lambda_i = \gamma_i/Q_i$,is the contact intensity[33]. This expectation can be quite closely ap - proximated by the simpler expression:

$$D_i(1-e^{-\lambda_i}).$$

The interested reader may verify that this approximation affords non-trivial

33. The reason this approximation works is that $\ln(1+x) \approx x$, for x small-- as can be verified by direct application of Taylor's theorem. Given this result, notice that $\ln(1-1/Q_i)^{\gamma_i}] = \gamma_i \ln(1-1/Q_i) \approx \gamma_i(-1/Q_i) = -\gamma_i/Q_i = -\lambda_i$, the approximation improving as $Q_i$ increases. Now as $\ln(1-1/Q_i)^{\gamma_i} \approx -\lambda_i$, taking exp of both sides gives $(1-1/Q_i)^{\gamma_i} \approx e^{-\lambda_i}$. Finally, substituting $e^{-\lambda_i}$ for $(1-1/Q_i)^{\gamma_i}$ gives $D_i[1-(1-1/Q_i)^{\gamma_i}] \approx D_i(1-e^{-\lambda_i})$, as desired.

computational advantages when manipulating first-order equation systems. The excellent quality of the approximation, particularly for values of $\gamma_i$ and $Q_i$ likely to be encountered in real-world registration drives, can be appreciated by inspecting the table which appears in the appendix.

In conclusion, the expected number of marginal registrations produced by an ATIS registration drive conducted at a contact intensity $\lambda_i$ among a qualified population $Q_i$,of whom $(E_i-r^N_i)$ are eligible to be registered and a fraction $\pi^A_i$ are a-partisans, can be closely approximated by choosing $D_i = \pi^A_i(E_i-r^N_i)$ to obtain:

$$\pi^A_i(E_i-r^N_i)(1-e^{-\lambda_i}). \qquad \text{(eq 2)}$$

b. <u>Factoring Out the Self-Registrants</u>

It would not be correct to regard the expectation (eq 2) as the marginal registration impact of the A-party effort in tract i; some of these registrations almost certainly correspond to individuals who *would have self-registered anyway.* This section will analyze the problem of eliminating the self-registrants from the marginal expected registration count.

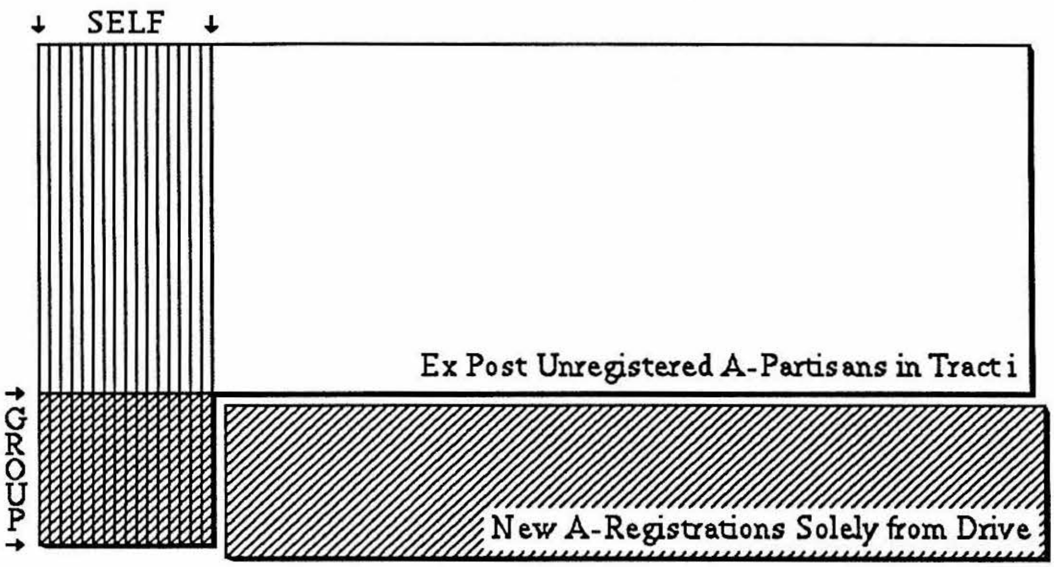


Figure 2.

Let the entire region depicted in the figure above represent all the eligible A-partisans remaining in tract i immediately after the non-partisan effort has been completed. Again, as there are a total of $E_i$ eligible individuals from both parties, and a fraction $\pi^A_i$ of these are A-partisans, the total number of unregistered A-partisans remaining after the non-partisan drive is $\pi^A_i(E_i-r^N_i)$.

Recalling the earlier discussion of the occupancy problem, the expected number of group-registrations produced by an A-party registration drive conducted in the tract at intensity $\lambda_i$ is given by:

$$\pi^A_i(E_i-r^N_i)(1-e^{-\lambda_i}).$$

These group-registered individuals are represented by the "GROUP" band running along the southern edge of figure 2. Notice that, in accordance with **R1) – R3)**, no non-A-partisans can have been group-registered in the A-party effort.

The registration drive manager should not assume, however, that all of these group-registrations can be attributed to the drive. Indeed, some of the individuals who were group-registered would have self-registered anyway. Suppose it is expected that a fraction $\sigma^A_i \geq 0$ of the unregistered A-partisans in tract i would self-register, unless contacted first by a registration worker. Then the number of unregistered A-partisans who could be expected to self-register in the absence of any A-party registration effort is:

$$\sigma^A_i \pi^A_i (E_i - r^N_i)(1 - e^{-\lambda_i}).$$

This group of individuals is represented by the "SELF" band which appears along the west edge of figure 2.

Assumption **A1)** guarantees the propensity to self-register is the same among those contacted by registration workers in tract i as among those not contacted. Therefore, a fraction $\sigma^A_i$ of the gross group-registrations reported by the drive should be discounted by the manager when evaluating performance. The expected number of potential self-registrants who were group-registered instead is given by:

$$\sigma^A_i \pi^A_i (E_i - r^N_i)(1 - e^{-\lambda_i}).$$

And this group is represented in figure 2 by the double crosshatched region where the "GROUP" and "SELF" bands intersect in the southwest corner of the diagram.

Finally, how many of the group registrations can be exclusively attributed to the registration drive? The expected number of such group-only registrations is obtained by simply subtracting the expected number of group registrants who would have self-registered from the ranks of all group-registrations-- thus netting out the redundant group-registrations, as below:

$$\pi^{A}_{i}(E_i - r^{N}_{i})(1-e^{-\lambda_i}) - \sigma^{A}_{i}\pi^{A}_{i}(E_i - r^{N}_{i})(1-e^{-\lambda_i})$$
$$= (1-\sigma^{A}_{i})\pi^{A}_{i}(E_i - r^{N}_{i})(1-e^{-\lambda_i}). \qquad \text{(eq 3)}$$

The set of group-only registerable A-partisans is represented by the diagonally shaded region in the southeast corner of figure 2.

c. <u>Expected ATIS Marginal Registrations</u>

Summing the expectation (eq 3) across the I tracts which comprise the district gives the district wide total expected number of group-only registerable A-partisans registered by A-party mobile registrars

$$\Sigma_I\, (1-\sigma^{A}_{i})\pi^{A}_{i}(E_i - r^{N}_{i})(1-e^{-\lambda_i}). \qquad \text{(eq 4)}$$

In the absence of any A-party registration drive, these individuals *would not otherwise have been registered*, and thus represent the marginal increase in A-party registrations attributable to the effort in tract i. As per **R1)** - **R3)**, no group-only registerable A-partisans were registered by any opposition party, and vice versa. Furthermore, the self-registrant opposition partisans registered by party A have not been charged to party A's marginal yield in tract i. Notice also that the expression above (eq 4) does not depend upon assumption **I3)**, namely that self-registrants delay registration until the final date, or upon the alternate assumption **I3a)** of linear decay.

## III.B.5 Strategy Revision During Fixed-Budget ATIS Drives

This section is concerned with the import of assumption **T2)**, that all groups mobilize their registration forces simultaneously and irrevocably on the first date. Given that the parties' managers' budgets are fixed, and supposing the managers make good estimates of both the non-partisan registration yield and the political and demographic charactersistics of the tracts, would either manager *want* to revise his registration strategy after observing the other party's efforts? If not, then assumption **T2)** could be regarded as relatively innocuous in the context of the model's other assumptions. Furthermore, such results would simplify the analysis of party competition during registration drives, since neither party would choose to dynamically update its strategy in response to the other party's efforts.

The problem reduces naturally to two cases: partisan filtering, and no partisan filtering. The second case is analyzed in the appendix, and yields the interesting conclusions that with no partisan filtering, neither manager would want to revise his strategy, and that the two parties will never conduct concurrent registration efforts in the same tracts. The results in the partisan filtering case are presented in the following proposition.

PROPOSITION (Redundancy of Assumption **T2** with Partisan Filtering)

Given the other assumptions of the model, assumption **T2)** is redundant; that is neither partisan manager would want to revise his strategy during the course of the registration drive.[34]

Proof

Observe that from a partisan manager's point of view, the marginal productivity of a given tract depends only on the concentration of group-only registerable own-party eligibles. Since these individuals will never be registered by the opposition party, neither manager can benefit from the other party's efforts.[35] Additionally, as there are no conversion effects, a registration drive by the other party does not decrease the

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**34.** Professor McKelvey has pointed out that the import of this proposition is that registration drive managers, as modelled here, are involved in a trivial game situation—since each player's payoff depends only on his strategy, but not his opponent's. The parties play a non-trivial game when choosing registration effort levels, but this competition is not the focus of the current analysis.

**35.** Notice assumption **I3)**, that all self-registration takes place on the second date, prevents a rather complex interaction between concurrent registration drives conducted by parties A and B in the same census tract. Consider that group registration efforts by party B will inevitably result in the contact of some eligible A-party self registrants. If these individuals, having obtained affidavits from B-party registrars, were to immediately self register, they would of course no longer be eligible if contacted subsequently by A-party mobile registrars. This would represent a benefit to party A due to the actions of party B; since the group registration of an own-party self-registrant is a deadweight loss to party A-- because a payment is entailed for registering an individual who would have eventually registered himself. However, with **I3)** in force, an A-party self registrant-- even though he has been previously contacted by B-party registrars-- will remain eligible during the registration period, and therefore can be group-registered at first contact with A-party registrars. See section III.C.1.c for additional discussion of this topic.

number of own-party eligibles. Given that each faces a fixed budget, the opposition party's efforts are therefore irrelevant to both managers. But this means knowledge of such efforts would similarly be of no tangible benefit. Thus both managers will find their date 0 optimal strategies, computed in the absence of information about the other party's strategy, will appear just as effective after the other party's strategy has been observed. QED.

### III.B.6 Compensation Schemes for Mobile Registrars

Political parties typically compensate registration workers under the terms of incentive contracts which specify that the worker be paid a constant dollar amount for each *own-party* registration produced.[36] Notice that even though registration workers are not paid directly on the basis of contact intensity, managers cannot set arbitrarily high contact intensities. Mobile registrars confront a problem of decreasing returns; as more own-party eligibles are registered, the next such individual becomes more difficult to find. Presumably, parties must pay registration workers their opportunity costs of foregoing other employment, at least on an average across the whole registration force. It must be possible for registrars to generate a sufficient number of successes to recoup their opportunity costs-- otherwise they will simply decline to continue employment with the party.

Thus managers may find it useful to regard registrars' payments as the cost of *contacting* own-party eligibles-- even though these payments appear to be the cost of generating registrations. That registrars are commonly paid according to own-party registrations might well be an artifact of the difficulty of supervising a geographically

**36.** I am not aware of any formal research concerning the incidence of parties' choice among various contract types for the compensation of registration forces.

dispersed labor force producing a product from inputs, registrar effort and the random arrival of eligibles, which are costly for managers to observe. If more efficient-- indeed, any-- supervision technologies were available to managers, it would not be suprising to observe parties employing different compensation schemes for their registration forces.

The design of compensation schemes for mobile registrars represents an area for potential improvements in registration efficiency, particularly as regards the turnout and loyalty qualities of the registrations produced. This topic will be considered further in section V.

## III.C. THE MANAGER'S OPTIMAL STRATEGY

The registration drive manager has been modelled as a rational, that is goal-seeking, individual intent upon solving a well-defined optimization problem.[37] Given a fixed budget, his problem is to allocate ATIS mobile registrars across the census tracts comprising a contested district so as to maximize his party's expected plurality. In this section the registration drive manager's problem will be foramalized and sufficient conditions for its unique solution will be derived.

It will be demonstrated that the manager can determine his optimal strategy by first ranking the tracts according to a measure of their expected return to registration effort. In practice, several arguments of this return index must be estimated, for example: the numbers of unregistered partisans per tract, the propensity of such

37. The problem of providing incentives to the registration drive manager will be omitted from consideration here. He will be assumed to take his assigned objective function as a given, and will diligently search for the optimal solution.

individuals to self-register, and the likelihood that group-registered individuals will turn out. Finally, the desired ATIS contact intensities for each tract are obtained as the natural log of each tract's productivity index plus a constant term. The interested may consult the appendix for a parallel treatment of the no partisan filtering case.

## III.C.1 The Manager's Maximization Problem

The statement of a formal optimization problem is comprised of three parts: the set of choice variables or strategy space, the objective function, and the constraints. It is suprising that a construct of this simplicitcy should exhibit such profound explanatory power in the social sciences. In any event, the three components of the registration manager's problem will now be discussed in turn.

An ATIS registration drive manager can control the contact intensity of the effort in each of the census tracts which make up the district. That is he can direct an exact number of mobile registrar-hours to each tract during the course of the drive. In particular, his force of mobile registrars is assumed to take direction, to attend specified locations for the assigned number of hours, and not to stray to other tracts--as set forth below:

### The Registration Manager

**M1)** The registration drive manager is able to control expected contact inten - sities across tracts by posting mobile registrars at targeted locations for specified time intervals.

Notice that the migration assumptions are also ultimately assumptions which bear upon the nature of the manager's control variables. Assumption **B2)**, that individuals stay within their tracts entails that when a manager allocates some number of mobile registrar-hours to a given tract, the targeted population will in fact be present. And the random arrival within tracts assumption **A1)** admits the conclusion that mobile registrars' hours are additive in the following sense: The expected number of individuals in a given tract contacted at least once by two mobile registrars, each stationed one hour in the tract, is the same as the expected number generated by one mobile registrar working two hours.[38]

~~For technical convenience~~ the manager's choice variables have been expressed in terms of contact intensities $\lambda_i = \gamma_i / Q_i$, that is, the expected number of contacts in a tract $\gamma_i$ divided by the total qualified population in the tract $Q_i$. However, the final results would be no different if the manager had been modelled as selecting the expected contacts per tract, or the number of mobile registrar-hours per tract.

As discussed previously, the registration drive manager's goal is to maximize the contribution to his party's[39] election day plurality attributable to the drive, given a fixed budget constraint and the ATIS registration technology.

The Registration Manager, Additional Assumptions

**M2)** Partisan registration managers' objectives match those of their respective parties; managers pose no incentive compatability problems to the parties.

**M3)** The ATIS registration drive is the sole registration production method.

**M4)** The manager assumes his registration budget is exogenously fixed.

**M5)** Registration managers act as price takers in labor markets.

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**38.** An example of how this assumption could fail is two registrars who have set up their stations right next to each other.

**39.** In an election with several different races, all voters presumably vote the straight party ticket.

The expected number of registrations produced in a tract by an ATIS drive of a given intensity has been derived in section III.B.5. It remains to consider the role of the turnout factor in order to finish the construction of his objective function. Presumably, the manager's ultimate purpose is not simply to deliver new net registrations for his party, but rather new net votes on election day. Recall that $\tau^A_i$ denotes the proportion of A-partisans expected to actually vote if registered among all eligible A-partisans not expected to self-register; $\tau^A_i$ is the expected turnout percentage for group only registered A-partisans.

Recalling expression (eq 4) developed previously for the expected number of marginal registrations, the desired expression for the expected marginal increase in party–A election day plurality as a result of the drive at contact intensity $\lambda_i$ in tract i can now be obtained. The expected number of individuals group-registered for party A who would not have registered otherwise--and who will *also turn out* for party A is given by:

$$(1-\sigma^A_i)\pi^A_i\tau^A_i(E_i-r^N_i)(1-e^{-\lambda_i})$$

And so the ***manager's objective function*** is simply the sum, across all the tracts, of the expected marginal plurality in each tract:

$$\Sigma_I \left[(1-\sigma^A_i)\pi^A_i\tau^A_i(E_i-r^N_i)(1-e^{-\lambda_i})\right]. \qquad \text{(eq 5)}$$

As discussed at the beginning of section III.B, a partisan registration drive manager (affiliated, say, with Party A) faces not only technological limitations, but also resource and market constraints. Formally, these additional constraints are threefold:

i) Non-Negativity Constraints.

ii) The Budget Constraint.

iii) Opportunity Cost Constraints.

The non-negativity constraints are additional technological conditions; it is simply not possible for the registration drive manager to select negative values for the $\lambda_i$. Doing so would correspond to de-registering individuals in the district[40].

The budget constaint reflects the reality that registration is not costless, and the assumption the manager has an exogenously determined finite budget with which to finance the drive. It will be assumed that the expected number of registrations chargeable by the A–Party registration force times the cost per registration must not exceed the manager's budget constraint.

In order to determine the expected number of registrations for which the party must pay, consider that the registration force generates registrations corresponding to two types of individuals: A-party group-only registrants, and A-party self-registrants. Given perfect partisan filtering, so that no A-party group-only registrants will be registered by opposition parties, the expected number of group-only A-party registrations generated by a drive conducted at contact intensity $\lambda_i$ has already been shown to be $(1-\sigma^A_i)\pi^A_i(E_i-r^N_i)(1-e^{-\lambda_i})$. Under, assumption **I3)** all self registrants--

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**40.** It might be possible to include some notion of de-registration in a model of voter registration drives which allowed for the possibiltiy of conversion effects during registration. However, such effects have been disallowed here, and in any event conversion phenomena are generally thought to be insignificant during registration drives; see Cain and McCue [1985].

and in particular A-party self registrants— postpone self registration until the second date. Therefore all A-party self registrants contacted at least once by an own-party registrar will be group registered. After subtracting the initial non-partisan group registrations, the expected number of eligible A-party self-registrants on date 0 is $\sigma^A_i \pi^A_i(E_i - r^N_i)$; therefore, the expected number of such individuals contacted at least once-- and hence group registered by Party A--must be :

$$\sigma^A_i \pi^A_i(E_i - r^N_i)(1 - e^{-\lambda_i}). \qquad \text{(eq 6)}$$

And so the desired expected number of registrations in tract i for which the party must pay its registration force is obtained as the sum below:

$$(1 - \sigma^A_i)\pi^A_i(E_i - r^N_i)(1 - e^{-\lambda_i}) + \sigma^A_i \pi^A_i(E_i - r^N_i)(1 - e^{-\lambda_i})$$
$$= \pi^A_i(E_i - r^N_i)(1 - e^{-\lambda_i}). \qquad \text{(eq 7)}$$

Summing the expression above across tracts and multiplying by the unit registration cost c, the A Party ***manager's budget constraint*** can be expressed as:

$$c * \Sigma_I \left( \pi^A_i(E_i - r^N_i)(1 - e^{-\lambda_i}) \right) \leq \mathbf{C}. \qquad \text{(eq 8)}$$

The third constraint, the opportunity cost constraint, reflects the market reality that mobile registrars will not provide their services unless registration commissions cover their opportunity costs. In practice this constraint turns out to bound the model; without it managers would have an incentive to select arbitrarily high contact intensities. Since the manager pays the registration force on a per registration basis, if he does not take the opportunity cost constraint into account, he will continue to assign registrars to exhausted tracts where expected waiting times between sucessful registrations are long--and commission charges per hour quite low. Of course such tactics simply aren't feasible, and this consideration must be built into the model.[41]

Mobile Registrar Compensation

**M6)** Mobile registrars are compensated by a constant dollar payment for each own-party registration generated. These payments represent the only cost of an ATIS registration effort.

**M7)** The expected total payments to mobile registrars must at least equal the registrars' total opportunity cost.

Summing expression (eq 7), the tract i expected number of chargeable registra - tions, across tracts and multiplying the result by the unit registration commission c gives the expected total compensation paid to mobile registrars. Comparing this expresion to the opportunity cost borne by the registration force yields the ***opportunity cost constaint*** below:

$$c*\sum_I\left( \pi^A_i(E_i - r^N_i)(1-e^{-\lambda_i}) \right) \geq M*\sum_I t_i, \qquad \text{(eq 9)}$$

where M denotes a mobile registrar's hourly opportunity cost, and $t_i$ is the total number of hours worked by A-party mobile registrars in tract i.

**41.** Another way to look at the opportunity cost constraint is that the party is really paying the mobile registrars for contact hours; however, given the geographic dispersion of the registration force and the technological uncertainty inherent in the registration production process, it is very difficult for the party to monitor mobile registrar performance. The constant payment per registration compensation package is an incentive scheme designed to encourage the registration force to police itself.

<u>DEFINITION</u> (Productivity Index $\beta_i$)

For notational and conceptual convenience, let $\beta_i$ be defined as below:

$$\beta_i = [(1-\sigma^A_i)\pi^A_i\tau^A_i](E_i-r^N_i)/Q_i. \qquad \text{(eq 10)}$$

$\beta_i$ can be interpreted as a tract's ***registration productivity index*** from the A–party registration drive manager's point of view. $\beta_i$, where $0 \leq \beta_i \leq 1$, is the percentage of group-only registerable A-eligibles (net of those registered by the non-partisan group) who will turn out if registered, among all qualified individuals in tract i.[42] Re-writing the manager's objective function (eq 5) in terms of $\beta_i$, one obtains:

$$\Sigma_I \beta_i Q_i(1-e^{-\lambda_i})$$

Finally, having discussed its three components, the desired formal statement of the A–Party manager's ***ATIS registration problem*** can now be presented:

$$\mathbf{MAX}_\lambda\ \Sigma_I \beta_i Q_i(1-e^{-\lambda_i}) \qquad \text{(eq 11)}$$

**S.T.** **1)** $\lambda_i \geq 0$, for $i = 1, \ldots, I$. (Non-Negativity Constraint)

**2)** $c*\Sigma_I(\pi^A_i(E_i - r^N_i)(1-e^{-\lambda_i})) \leq \mathbf{C}$. (Budget Constraint)

**3)** $c*\Sigma_I(\pi^A_i(E_i - r^N_i)(1-e^{-\lambda_i})) \geq M*\Sigma_I t_i$. (Opportunity Cost Constraint)

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**42.** Notice that $\beta_i$ is an index of concentration, and thus measures a tract's productivity per unit of registration effort, rather than in absolute terms.

## III.C.2 Solving the Manager's Optimization Problem

The optimization problem (eq 11) can be manipulated to yield an analytically more tractable form. The resulting problem is equivalent to the original in the sense that both problems admit exactly the same set of solution(s). The desired simplification will be accomplished by eliminating and combining the constraints of the original problem. Next, the associated first-order system will be solved. Finally, it will be demonstrated that this solution represents the unique constrained optimum for the manager's problem.

### a. A Simplified Equivalent Problem

In order to eliminate constraint 1); it will be assumed that at an optimum, none of the choice variables $\lambda_i$ is negative. Intuitively this should be the case, since a negative choice for $\lambda_i$ makes the $i^{\underline{th}}$ summand of the objective function negative, a situation which hardly seems likely to be optimal.[43]

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**43.** Of course, it is formally possible--although it makes no practical sense--that a negative choice for some $\lambda_i$ given a relatively small $\beta_i$ (and a correspondingly small registration penalty associated with a negative choice of $\lambda_i$) might feed back through the budget constraint 2) and "create" resources thereby permitting the choice of an artificially large positve $\lambda_j$ given a relatively large $\beta_j$.

Should one of the non-negativity constraints in fact be binding, first-order conditions derived under the assumption that all such constraints were *not* binding would not correctly express the optimality conditon that the gradient of the objective function is a linear combination of the gradients of the binding constraints. However, this possi - bility appears remote in the present instance.

Constraint 2), the budget constraint, requires that the registration drive manager choose the $\lambda_i$ so that the expected payments to mobile registrars do not exceed his budget C.[44] In fact this constraint must *hold with equality* at an optimum, since--assuming perfect partisan filtering--the manager can always increase the expected number group-only registrations for his party by conducting a more intensive drive. Thus any allocation which leaves a positive expected budget residue must perforce be suboptimal; therefore, the manager will continue spending until the budget is exhausted.

Finally, constraint 3) stipulates that the total expected payments to mobile registrars must at least equal their opportunity costs. It will be assumed that in labor market equilibrium, the price c paid per A-party registration will adjust so that the third constraint will also *hold with equality.* Presumably the registration force will not agree to work for less than its opportunity cost, while the party need not pay more.

Notice that the left-hand sides of constraints 2) and 3) appearing in (eq 11) are identical. Since, as argued above, both of these constraints must hold with equality, the right hand sides can be set equal to yield a *single* equality constraint:

$$\mathbf{C} = M * \textstyle\sum_I t_i. \qquad \text{(eq 12)}$$

Recall that the contact intensity $\lambda_i$ is defined as $\gamma_i / Q_i$, where $\gamma_i$ is the total number of contacts made in tract i. If the expected number of arrivals at each mobile registrar's

44. The number of A party registrations which will be actualy produced by the mobile registrars is, of course, a random variable. Thus the manager cannot make a perfect prediction concerning the cost of his registration activities. If the realized cost of the registration drive exceeds the budget, presumably the difference can be made up budget surpluses realized from other campaign activities, or from additional fund raising conducted after the campaign.

station in tract i is $\mu$ per hour, the expected total number of registrars' hours in tract i required to generate $\gamma_i$ contacts is $\mu * t_i$. Thus $t_i$ and $\lambda_i$ are related according to:

$$\gamma_i = \mu t_i \Leftrightarrow (\gamma_i/Q_i) * Q_i = \mu t_i \Leftrightarrow \lambda_i Q_i = \mu t_i$$
$$\Leftrightarrow t_i = \lambda_i Q_i / \mu, \text{ for } \mu \neq 0.$$

And so, substituting for $t_i$ in (eq 12), the manager's maximization problem can be reformulated as:

$$\textbf{MAX}_\lambda \ \Sigma_I \ \beta_i Q_i (1 - e^{-\lambda_i}) \tag{eq 13}$$
$$\textbf{S.T. 1a)} \ M * \Sigma_I \ \lambda_i Q_i / \mu = \mathbf{C}.$$

Rearranging constraint 1a) above and attaching it to the objective function with a Lagrange multiplier[45] $\zeta$ yields the equivalent *unconstrained* problem:

$$\text{MAX}_\lambda \ \Sigma_I \ \beta_i Q_i (1 - e^{-\lambda_i}) - \zeta * (\Sigma_I \ \lambda_i Q_i - \mathrm{C}\mu/\mathrm{M}). \tag{eq 14}$$

## b. First-Order Conditions

Differentiating (eq 14) with respect to $\lambda$ and $\zeta$ yields the following system of I+1 first-order conditions:

$$\partial/\partial\lambda_i \ : \ \beta_i Q_i e^{-\lambda_i} - \zeta Q_i = 0 \ ; \ \text{for } i = 1, \ldots, I, \text{ and} \tag{eq 15}$$
$$\partial/\partial\zeta \ : \ \Sigma_I \ \lambda_i Q_i - \mathrm{C}\mu/\mathrm{M} = 0. \tag{eq 16}$$

---

**45.** Recall that a Lagrange multiplier can be interpreted as the percentage increase in the value of the objective function (at an optimum) if the associated constraint were relaxed by 1%. See Varian [1984].

First consider the I equations of type (eq 15). Rearranging gives:

$$e^{-\lambda_i} = \zeta/\beta_i \Leftrightarrow \ln(e^{-\lambda_i}) = \ln(\zeta/\beta_i)$$
$$\Leftrightarrow \lambda_i = \ln(\beta_i) - \ln(\zeta)\ ; \text{ for } i = 1, \ldots, I. \qquad \text{(eq 17)}$$

It remains to express $\ln(\zeta)$ in terms of the parameters of the model. To this end, multiply each equation immediately above by $Q_i$ and so obtain:

$$\lambda_i Q_i = Q_i(\ln(\beta_i) - \ln(\zeta))\ ; \text{ for } i = 1, \ldots, I.$$

And summing these equations across I gives:

$$\Sigma_I \lambda_i Q_i = \Sigma_I Q_i(\ln(\beta_i) - \ln(\zeta)).$$

Rearranging the first-order condition (eq 16) and setting its right-hand side equal to that above yields:

$$C\mu/M = \Sigma_I Q_i(\ln(\beta_i) - \ln(\zeta)) = \Sigma_I Q_i \ln(\beta_i) - \ln(\zeta) * Q$$
$$\Leftrightarrow \ln(\zeta) = (\Sigma_I Q_i \ln(\beta_i) - C\mu/M)/Q. \qquad \text{(eq 18)}$$

Finally, substituting from (eq 18) for $\ln(\zeta)$ in (eq 17), the desired ***first-order system*** obtains as:

$$\lambda_i = \ln(\beta_i) - (\Sigma_I Q_i \ln(\beta_i) - C\mu/M)/Q; \text{ for } i = 1, \ldots, I. \qquad \text{(eq 19)}$$

It is interesting to note that the expressions (eq 17, 18, 19) above can also be manipulated to yield:

$$\ln(\zeta) = (\Sigma_I Q_i \ln(\beta_i) - \Sigma_I \lambda_i Q_i)/Q, \text{ and}$$
$$\lambda_i = \ln(\beta_i) - (\Sigma_I Q_i \ln(\beta_i) - \Sigma_I \lambda_i Q_i)/Q; \text{ for } i = 1, \ldots, I.$$

Thus it can be further concluded that $C\mu/M = \Sigma_I \lambda_i Q_i$.

Again, recall that $\beta_i$, the productivity index for tract i, is the percentage among all qualified individuals in tract i of group-only registerable A-partisans who will turn out if registered. Recall also that and $\mu$ denotes the arrival rate[46], C and M are the budget and registrars' opportunity cost respectively[47] , and Q denotes the entire qualified population of the district.

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**46.** Notice assumption **A2)** , that the arrival rate is constant across, facilitates the the derivation of these pointblank first order conditions. If arrival rates $\mu_i \neq 0$ are *different* across tracts, the interested reader can verify the manager's optimization problem takes the form:

$$\mathrm{MAX}_\lambda \; \Sigma_I \beta_i Q_i(1-e^{-\lambda_i}) - \zeta * (\Sigma_I (\lambda_i Q_i/\mu_i) - C/M).$$

The corresponding first order conditions are $\lambda_i = \ln(\mu_i) + \ln(\beta_i) - \ln(\zeta)$, where the expression for $\ln(\zeta)$ is somewhat more complex than that which appears in (eq #). In particular:

$$\ln(\zeta) = \left(\Sigma_I (Q_i/\mu_i)[\ln(\mu_i) + \ln(\beta_i)] - (C/M)\right) / \Sigma_I (Q_i/\mu_i)$$

Notice the occurence of $\mu_i$ in the equation for $\lambda_i$.

**47.** Observe that the parameters **C** and **M** appearing in the constant term are estimable. Presumably M is just the minimum wage, while **C** is the party's registration drive budget. Since the total number of registrations produced and the cost per registration are known, **C** can be estimated if it is assumed that payments to mobile registrars are the only costs of the registration drive.

c. Sufficiency and Uniqueness

First of all, notice that each expression $(1\text{-}e^{-\lambda_i})$ is both strictly concave and strictly monotonically increasing in $\lambda_i$.[48] Referring to the definition of $\beta_i$ above, since $\sigma^A{}_i$, $\pi^A{}_i$, and $\tau^A{}_i$ are strictly bounded in the interval (0,1), then $\beta_i$ must be strictly positive--so that each of the functions $\beta_i(1\text{-}e^{-\lambda_i})$ must also be strictly concave and increasing in $\lambda_i$. Since the sum of such functions shares these properties, the objective function $\sum_I \beta_i Q_i(1\text{-}e^{-\lambda_i})$ is also strictly concave increasing in $\lambda$. Observe that the constraint 1a) appearing in (eq 13) is a linear, and hence convex, function of the choice variables $\lambda_i$. It is well known (see Varian [1984]) that any solution to the associated first-order system of such a nicely behaved problem must be an optimum. And indeed, inspection of the first-order system (eq 19) of the manager's problem will reveal that it admits a single, unique solution.

d. Benefits of Additional Registration Spending

This section answers the question: What is the expected registration increase if an optimally conducted registration effort is allocated an additional dollar?

The value of the Lagrange multiplier $\zeta$ at the optimum determined by the first-order system (eq 19) gives the percentage increase in the value of the objective function as the budget constraint in (eq 13) is relaxed. $\zeta$ is functionally related to the optimal values of the contact intensities $\lambda_i$ acording to:

$$\lambda_i = \ln(\beta_i) - \ln(\zeta);\ \text{for } i = 1, \ldots, I, \qquad \text{(eq 20)}$$

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48. The function $(1\text{-}e^{-\lambda_i})$ takes the value 0 when $\lambda_i = 0$, and increases aymptotically toward 1 as the values of $\lambda_i$ increase.

and also to the equation:

$$\ln(\zeta) = \left( \Sigma_I\, Q_i \ln(\beta_i) - C\mu/M \right) / \, Q. \qquad \text{(eq 21)}$$

Either of these relationships should premit estimation of the registration benefit of relaxing the budget constraint. The second equation above is especially interesting becuase it does not contain direct instances of optimal values of the choice variables $\lambda_i$.

# IV. EMPIRICAL ANALYSIS

This section examines the extent to which the ATIS model explains the allocation of effort in registration drives conducted across several Southern California Assembly districts. The analysis focuses on registration activity during the nine–week period of intense effort preceeding the last day to register for the 1984 general elections.[1] While some features of Los Angeles County Assembly races are obviously unique, it is reasonable to suppose these contests are representative of political campaigns in general--and so represent a convenient design for testing the ATIS model. Furthermore the demographic and political diversity of Los Angeles county afford an opportunity to gauge the parametric sensitivity of the model.

Certain ancillary issues can also be illuminated by empirical analysis. For example, evidence has been marshalled to support the assumption that cross–registrants (partisans group-registered by the opposistion party) would have self–registered except for prior contact by registration workers. The degree to which mobile registrars can practice partisan filtering has also been investigated.

Empirical results concerning the ATIS model also indicate directions for improving the efficiency of future registration drives. To the extent that i) the assumptions of the model reflect current registration practices, and ii) managers were able to make ex ante estimates of the required parameters, systematic deviations from the model's optimality conditions represent avenues for improved registration efforts.

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**1.** The 1984 general election presents an attractive opportunity for empirical analysis in part because Census data gathered in 1980 is still relatively current.

In conclusion, the empirical section of this paper consists of five major subsections:

Part A) describes the data set and provides summary statistics concerning the Assembly districts chosen for detailed scrutiny.

Part B) presents results concerning the effectiveness of parties' partisan filtering, and lends support for the perfect filtering assumptions **R1)-R3)**.

Part C) discusses methods employed to estimate both contact intensity and the model's parameters, and reports tests of the ATIS first-order conditions.

Part D) advances and tests an alternative model of registration drive managers' decision making.

Part E) analyzes potential improvements of registration drive efficiency.

## IV.A. DESCRIPTION OF THE DATA SET

The data analyzed in this study consist of observations of voter registrations from several different Los Angeles County Assembly districts during the nine-week period from 1st August 1984 until the twenty-ninth day prior[2] to the November 1984 general election. Hereafter this time interval will be referred to as the *registration period.* Cain and McCue [1985a,b] originally assembled the data utilized here, and have commented in detail concerning its unique attributes:

> The data for our analysis comes from Los Angeles County. In addition to being one of the cental areas of voter registration in the state, Los Angeles is uniquely suited for an empirical study of this sort. Voter registration records are detailed and in readily accessible computerized form. The creation of the data set employed here required three pieces of information: the registration number assigned to the application form of each newly registered voter, the registered voter file, and the purge file.[3]
>
> A group undertaking registration drives in LA County is issued registration forms with affidavit numbers that fall within a given range. The affidavit number on the completed registration form identifies how the individual was registered and by whom. By matching this number with the registered voter file, it is possible to know the voter's age, marital status, sex, time of registration, surname and party registration. A comparison of the newly registered list with the purge list of nonvoters then generates the names of those who were newly registered but did not vote. Since the data set includes information about whether an individual was registered by a group or self-registered, we can compare the characteristics of the two kinds of registrants as well as their voting rates.

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**2.** This is the last day to register in order to vote in the November election. Group registration activity declines precipitously after this date.

**3.** Cain and McCue [1985b] report that in Los Angeles County each voter failing to vote in a general election is sent a postcard to be returned to the registrar of voters, thus confirming the validity of the voter's registered address. Voters who neglect to return the card are finally purged from the registration rolls.

> [Even though] the registration tapes do not include such crucial information as the income, education, or race of the registrant, by matching addresses on the registration tapes to those on the Census Dime files, it is possible to merge census data with them. [Thus it is possible to create] a combined data set of registration information for each newly registered voter, census variables that describe the demographic characteristics of the bloc the individual lives in, a record of whether the person voted in the 1984 election, the political behavior of the precinct the individual lives in, and the group that registered him.

The data just described are characterized by two distinct degrees of resolution: the registration and voting data are available at the individual level, while census data is only available at the census bloc or tract level. Approximately 5000 individuals reside in a typical Los Angeles county census tract. Since a merged data set can only be as fine as its coarsest component, when merging the registration and census data it was necessary to aggregate the individual level observations of registration and voting variables to obtain tract totals. This procedure yields the census tract as the basic unit of analysis or case. A representative Assembly district is comprised of approximately 60 census tracts.[4]

Notice that although the registration affidavit for a group-registered individual indicates the registration group and the registrant's address (and hence his census tract), the location of the registration *contact* is not specified. This aspect of the data

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**4.** Not suprisingly, the geographic boundaries of Assembly districts and census tracts do not coincide exactly. An Assembly district is typically ringed by split tracts which straddle the boundary. This phenomenon was taken account of here by restricting attention only to those tracts of a given district containing at least 1000 qualified district residents.

Counting all split tracts, AD63 contains 66 tracts, AD41 has 73, while AD39 is comprised of 82 tracts. Removal of split tracts reduces these totals approximately 15%; however the associated *population* loss is only a few thousand individuals out of an average qualified population of 295,000.

set highlights the importance of assumption **B2)**, that registrants are contacted in their home tracts.[5]

Several different registration groups, including political parties, committees formed by individual candidates, and a variety of civic organizations, conducted regis - tration drives in Los Angeles County Assembly districts during 1984 . In order to test the ATIS model, these various organizations were classified as partisan or non-partisan, and if partisan, as Republican, Democrat, or other.[6] To the degree that organizations classified as representing the same party coordinate their registration efforts, it is reasonable to model their efforts as being orchestrated by a single manager. The involvement of the Speaker's office in financing and coordinating registration efforts in recent Democratic campaigns for the Assembly lends further plausibility to the single manager scenario.

It would be unwieldy to try to analyze all new registrations in Los Angeles County during the peak registration season prior to the 1984 election. Creation of the requisite merged data sets for both parties in each of dozens of Assembly districs would simply represent too great a computational burden. Following Cain and McCue [1985a] the sample has been narrowed to three selected Assembly districts:

- the 63rd, near Downey, a marginal district which changed hands between the 1982 and 1984 elections,

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**5.** Potentially very interesting research concerning registration drives might be conducted by arranging to have mobile registrars log the locations of their registration successes. These locations could then be compared with registrants' home addresses to develop a chart of population circulation within the district.

**6.** The classification scheme adopted by Cain and McCue [1985a,b] was employed here.

- the 41st, which includes parts of Pasdena, Altadena and Glendale, a safely Republican district, and
- the 39th, just north of downtown Los Angeles, excluding Hollywood, which is safely Democratic.

The numbers of individuals registered in these three districts during the nine–week registration period are reported in the table below. The total new registration column shows new registrations by party generated from all sources. The discrepancy between the sum of the Democratic and Republican totals and the figure reported in the TOTAL row is accounted for by third party and decline-to-state (DCL) registrations. Own party group-registrations are reported separately; this column shows Democratic registrations produced by Democratic groups, and Republican registrations produced by Republican groups. The third column reports self-registrations by party. Notice that for a given party, the difference between the party new registration total and the sum of own party group-registrations plus self-registrations for the party is due to the efforts of non-partisan registration organizations and cross-registrations by other parties.[7]

7. The totals reported were computed directly from individual level registration data.

NEW REGISTRATIONS SUMMARY

| | Total New Reg | Own Group-reg | Self-reg |
|---|---|---|---|
| AD 63 | | | |
| Dem | 7419 | 5531 | 1136 |
| Rep | 4412 | 1126 | 1757 |
| TOTAL | 12,815 | | |
| AD 41 | | | |
| Dem | 5838 | 2761 | 1805 |
| Rep | 7264 | 2756 | 2564 |
| TOTAL | 14,825 | | |
| AD 39 | | | |
| Dem | 7005 | 4519 | 1299 |
| Rep | 7200 | 4332 | 1029 |
| TOTAL | 15,440 | | |

Table 2.

It is interesting to note the strong group-registration success achieved by the Democrats in the marginal 63rd district. In the other two districts, both parties were about equally successful in terms of absolute numbers of group-registrations produced.

## IV.B. TESTS OF PARTISAN FILTERING

The ability to conduct successful partisan registrant filtering is a critical determinant of parties' optimal registration strategy. (The theoretical importance of such filtering can be appreciated by comparing the predictions of the ATIS model with the no-filtering model first-order conditions presented in the Appendix.) The available data permit at least two tests of the extent to which partisan ATIS mobile registrars can successfully conduct partisan filtering. One test is based on the incidence of cross-registration, while the other examines the correlation of the parties' filtering success with filtering difficulty.

### B.1 Incidence of Cross-registration

Recall that an individual is said to have been *cross-registered* if he has been group-registered by a registrar representing an opposition party. If parties could *not* practice partisan filtering, the partisanship ratios among group-registrations generated by opposing partisan organizations should tend to be the same in a given district. Further, these partisanship ratios should tend to resemble those among the unregistered population as a whole.[8]

In fact, the partisanship ratios of registrations generated by different parties are dramatically different. The numbers of Democratic registrations produced by Republican groups, and vice versa are shown in the table below. The figures in square brackets are the ratio of cross-registrations to own-party group-registrations plus cross-registrations. These percentages point out the very low incidence of cross-

8. The reader is referred again to Wolfinger [1980] for support of this somewhat suprising claim.

registration; the filtering errror would be even smaller if group-produced DCL and third-party registrations were included in the denominator. Finally, it is interesting to note that the Republicans appear to exhibit a somewhat better filtering performance than do the Democrats.

INCIDENCE OF CROSS-REGISTRATION

| | Rep by Dem | Dem by Rep |
|---|---|---|
| AD 63 | 904 [14%] | 174 [13%] |
| AD 41 | 704 [11%] | 268 [9%] |
| AD 39 | 580 [11%] | 279 [6%] |

Table 3.

## B.2 Filtering Efficiency and Filtering Difficulty

It might be argued that aggregate measures of filtering efficiency do not necessarily lend strong support to the claim that parties possess powerful filtering capability. Indeed, seeming filtering ability might simply be an artifact of astute tract targeting. One possible reply to this contention is that if a party *can* filter perfectly then its filtering errors across tracts should be unrelated to the difficulty of the filtering task. This relationship was analyzed for AD63. Unfortunately, the existing data do not permit testing this relationship for AD39 and AD41.

Suppose, in the case of the Democrats, that the ratio of Republicans registered by Democrats to all Democratic-produced group-registrations is adopted as a measure of tract filtering error; denote this variable by:

$$\text{DFLTERR} = \text{REP BY DEM} / \text{DEM-PRODUCED}$$

And let the difficulty of the Democrats' filtering task be measured by:

$$\text{REPCON} = \pi^{\text{REP}} * (E_i - r^{N}_{i}) / Q_i,$$

the ex ante concentration of eligible Republicans among all qualified tract residents. Define analagous measures RFLTERR and REPCON for the Republicans. The relationship between these error and difficulty variables can be better understood by examining the following scattergrams and regression results.

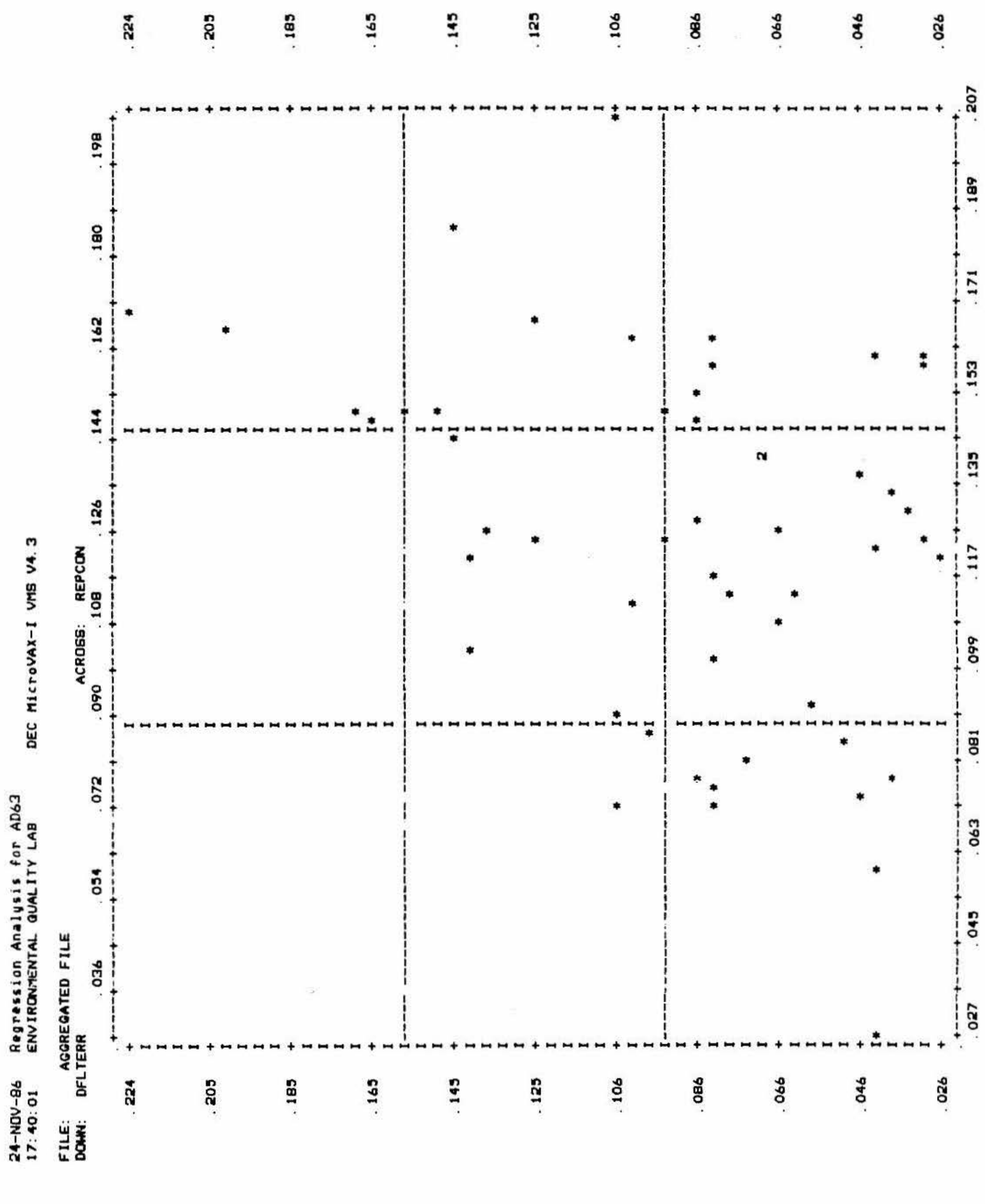
24-NOV-86 Regression Analysis for AD63
17:40:01 ENVIRONMENTAL QUALITY LAB
DEC MicroVAX-I VMS V4.3
FILE: AGGREGATED FILE
DOWN: DFLTERR
ACROSS: REPCON

Figure 3.

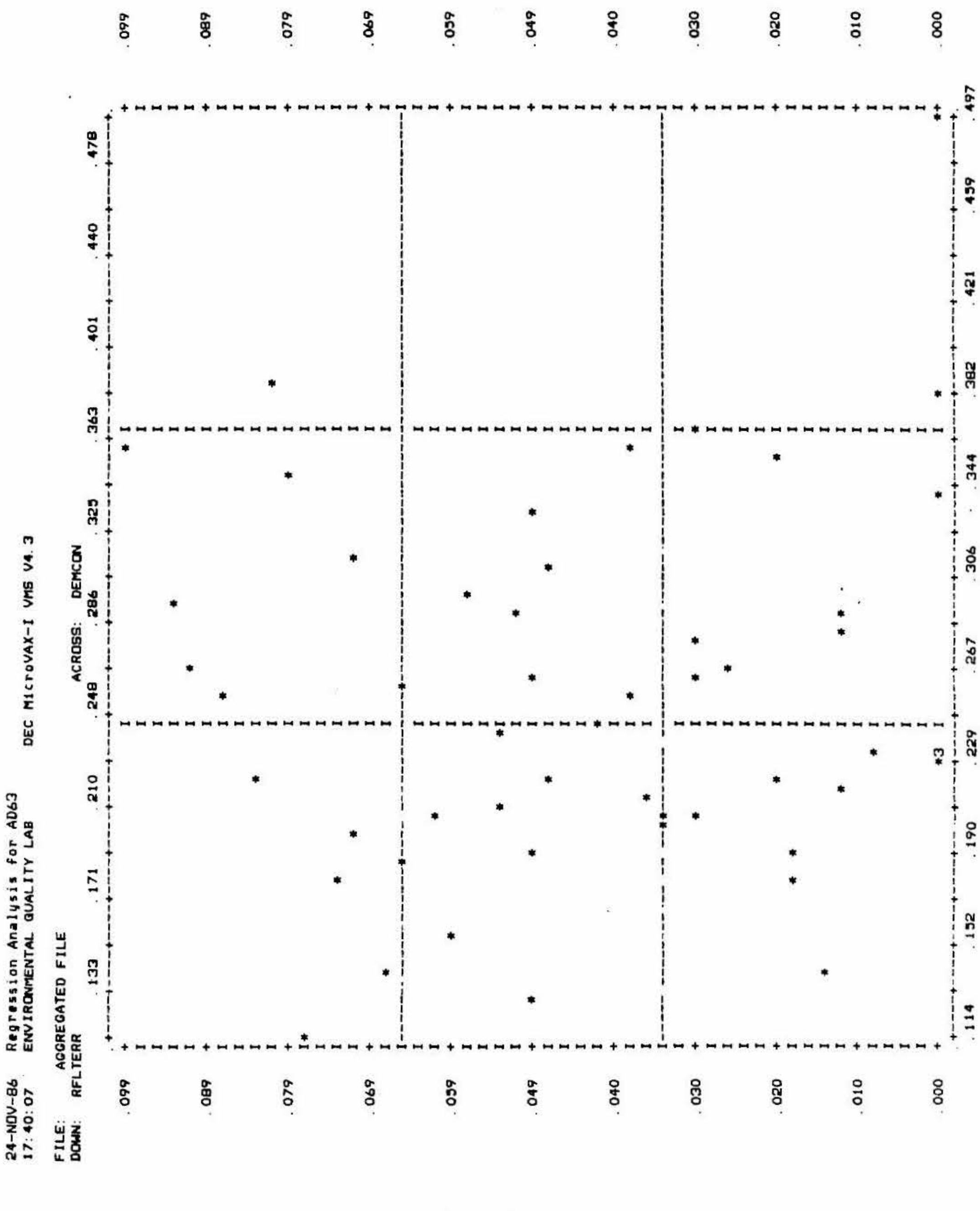
24-NOV-86 17:40:07
Regression Analysis for AD63
ENVIRONMENTAL QUALITY LAB
DEC MicroVAX-I VMS V4.3
FILE: AGGREGATED FILE
DOWN: RFLTERR
ACROSS: DEMCON

Figure 4.

REGRESSION OF DFLTERR ON REPCON, AD63

| COEF ⇒ | REPCON | CONST | $r^2$ | F | n |
|---|---|---|---|---|---|
| | .476** | .030 | .138 | 8.195 | 53 |
| | .166 | .022 | | | |

$\rho$(DFLTERR,REPCON) = .372  $\mu$(DFLTERR) = .089  $\mu$(REPCON) = .124.

REGRESSION OF RFLTERR ON DEMCON, AD63

| COEF ⇒ | DEMCON | CONST | $r^2$ | F | n |
|---|---|---|---|---|---|
| | −.033 | .050** | .007 | .381 | 53 |
| | .059 | .014 | | | |

$\rho$(RFLTERR,DEMCON) = −.086  $\mu$(RFLTERR) = .042  $\mu$(DEMCON) = .250.

"**" indicates significance measured by a t test at the .01 level; standard errors appear below the regression coefficients. n = number of cases.

Table 4.

These results suggest that it is reasonable to assume that partisan registration organizations can conduct effective partisan filtering. There appears to be no significant relationship between filtering error and difficulty in the case of the Republicans, and the relationship is quite tenuous for the Democrats.

Some additional evidence which supports the perfect filtering assumption can be found in the discussion of the no-filtering model the Appendix. In particular, if parties *cannot* filter, a tract separation theorem obtains. Ignoring self-registration and turnout effects, under no-filtering conditions no party will conduct a registration drive in a tract in which opposition eligibles outnumber own-party eligibles. Thus at most *one* party will launch registration efforts in a given tract. However, in the three Assembly districts under consideration, both parties registered individuals from nearly every tract.

B.3 Cross-Registrant Self-Registration Propensity

Having examined the marked extent to which parties are able to avoid registering opposition partisans, it is natural to inquire how this phenomenon comes about. The ATIS model presumes parties accomplish filtering by declining to provide registration assistance to opposition partisans. Although, some individuals *are* cross-registered[9], it has been suggested here they do not represent a plurality cost to the cross-registering organization--the sense of the claim being that such individuals would have self-registered had they not first encountered a mobile registrar .

Confidence in this proposition should be strengthened if it can be demonstrated that cross-registrants tend to resemble self-registrants more closely than group-registrants insofar as the salience of politics is concerned. One important measure of individual political initiative is electoral participation. The following table shows that cross-registrants voted nearly as often as self-registrants, and considerably more often than did group-registrants. These results are consistent with the ATIS assumption that cross-registrants would have otherwise self-registered.

9. The ATIS model assumes such individuals complete and return their affidavits without registrar assistance.

NEW REGISTRANT PARTICIPATION PERCENTAGE BY REGISTRATION STATUS, 1984 GENERAL ELECTION

| | Self | Cross | Own Pty |
|---|---|---|---|
| AD 63 | | | |
| Dem | 81% | 81% | 67% |
| Rep | 88% | 81% | 81% |
| AD 41 | | | |
| Dem | 86% | 83% | 75% |
| Rep | 89% | 84% | 79% |
| AD 39 | | | |
| Dem | 81% | 84% | 70% |
| Rep | 85% | 80% | 79% |

Table 5.

The available data will support other tests of the similarity between cross - registrants and selfregistrants, based on census variables such as age, ethnicity, and income proxies. However, further analysis of this nature belongs in a study specifi - cally devoted to the crossregistration phenonemon.

It is interesting to compare the results for the 1984 general election presented in Table 5 above with similar participation percentages computed by Cain and McCue [1985a] for the November 1982 election. Their analysis was based on all the 108,653 new registrations filed in Los Angeles County after the 1980 general election up until 54 days before the 1982 election.

NEW REGISTRANT PARTICIPATION PERCENTAGE BY REGISTRATION STATUS, 1982 GENERAL ELECTION

| Party | Self | All Groups |
|---|---|---|
| Dem | 58% | 38% |
| Rep | 63% | 53% |
| DCL | 41% | 34% |

Table 6.

As in 1984, there were sharp differences between the participation rates of selfregistered and group-registered individuals. The generally lower levels of participation in 1982 are not atypical for a non-Presidential election.

## IV.C. TESTS OF OPTIMAL ATIS CONTACT ALLOCATION

This section examines the ATIS model's ability to predict parties' allocation of registration effort across tracts. The model's fundamental behavioral premise is that registation managers choose contact intensities which maximize expected group only registerable (GOR) plurality. If this is the case, observed registration efforts should satisfy the ATIS first-order conditions derived previously.

One possible initial test of the model involves examining the correlation between the observed contact intensities $\lambda_i$, and logs of the tract productivity indices $\ln(\beta_i)$. An additional test entails regression of $\lambda$ on $\ln(\beta)$, or alternately on the logs of the multiplicative components of $\beta$.

Before such analysis can be undertaken, however, estimates of several intermediate expressions appearing in $\lambda_i$ and $\beta_i$ must be developed. These estimation tasks have been grouped into two classes: 1) the estimation of demographic and political parameters which are more or less directly observable given the available data, and 2) the estimation of several variables which must be inferred on the basis of specialized assumptions.

### C.1 Estimating Demographic and Political Parameters

The implicit tract contact intensities $\lambda_i$ must be computed from observed registrations, by invoking the inverse of the registration production function. This production relationship depends directly on both:

- the numbers of qualified and eligible individuals in each tract, and
- the tract productivity indices $\beta_i$.

In turn the productivity indices further depend on:

- turn out probability among new tract registrants, and
- partisanship of unregistered tract residents.

This section describes the estimation of tract qualified and eligible populations, numbers of self and group-registrants, and turnout and partisanship parameters from the available census and past elections data.

**a.** Qualified and Eligible Tract Populations, $Q_i$ and $E_i$

The number of qualified individuals per tract has been estimated here as the number of tract residents over the age of 18, based on 1980 Federal Census data. The number of aliens counted as members of tract populations is an important proviso in Latino tracts. Unfortunately, available census data do not provide sufficient information to control for the alien population. However, an adjustment to the nominal tract populations has been made here by assuming that .3 of tract Latino populations over 18 are not qualified to register.

Recall that so-called eligible individuals satisfy the legal requirements for registration, but remain unregistered at the start of the registration period. The number of eligibles in a given tract has been estimated as the number of qualifed individuals net of those registered, as reported by the registrar of voters 29 days before the elections, and corrected for the number of individuals newly registered during the registration period itself. That is:

$$E_i = \text{(EX ANTE) ELIGIBLES} = \text{QUALIFIEDS} - \text{TOTAL (EX POST) REGISTERED} + \text{NEW REGISTRANTS}$$

b. Self-registrants and Group-registrants

One of the singular advantages of the data set being employed here is that registrants can be classified as group-registered or self-registered according to the registration group codes appearing on each affidavit.[10]

All individuals who obtained and returned registration forms, in the absence of any interaction with a registrar, have been counted as self-registrants.[11] On the other hand, all individuals whose registration codes indicate contact with a mobile registrar have been classified as group-registered--except for cross-registrants, for reasons previously discussed. Group-registrants have been further distinguished as having been registered by partisan or non-partisan groups. For technical convenience it has been assumed in **G3)** that non-partisan group-registrations occur at the start of the registration period. Notice that according to the taxonomy proposed here, it is not possible for non-partisan group-registrants to be classified as cross-registrants.

c. Tract Turnout Probability $\tau_i$

It is interesting that there is not as much variation in turnout across tracts as one might initially expect. Indeed, turnout considered as a percentage of the entire

---

**10.** This part describes the method by which a simple count of tract group-registrants and self-registrants was obtained. The more complex task of estimating tract residents' propensity to self register will discussed in a following section.

**11.** It may well be the case that, as a group, individuals who chance to obtain affidavits at convenient locations such as shopping malls or banks have not demonstrated the same degree of self-registration propensity as others who, for example, have visited the registrar of voters for the express purpose of registering. Nevertheless, the self-registrant category has not been refined on the basis of such considerations.

population of tracts varies a great deal, whereas turnout among the *registered* population does not show nearly as much variation.

Two types of estimates for tract turnout probability $\tau_i$ suggest themselves: turnout of the previously registered individuals in the tract, and ex post turnout of the new registrants. The first type of estimate implicitly supposes that the registration manager estimates turnout based on previous elections data. The second presumes that on the basis of past experience, the manager estimates turnout probabilities correctly. Estimates of the second type will be employed here, under what amounts to a perfect foresight assumption on the part of registration managers. Turnout esti - mates have been computed as:

$$\tau^A{}_i = (\text{\#NEWLY REG'D. A}\underline{\text{S}}\text{ WHO VOTE}) \, / \, (\text{\#GROUP-REG'D. A}\underline{\text{S}}\text{ BY PARTY A})$$

A table of turnout percentages among new registrants by district, party, and registration category has already been presented.

## d. Tract Partisanship Percentages $\pi_i$

A useful basic premise for developing estimates of partisanship percentages is that a tract's partisanship is not related to the geographical location of the tract per se, but rather to some more fundamental set of charactersistics which tend to be shared by tract residents. Estimates of partisanship percentages can therefore be based on the percentages corresponding to the registered population of a tract, corrected for systematic differences in backround variables between the registered and eligible populations in a tract.

However, to simplify the analysis, tract patisanship percentages have been estimated by the partisan proportions among the registered individuals in each tract at the start of the registration period. That is:

$$\pi^{A}{}_{i} = (\#\text{EX POST REG'D } A\underline{S} - \text{NEW REG'D } A\underline{S}) \,/\, (\#\text{ ALL EX POST REG'D} - \text{ALL NEW REG'D}).$$

It is not unreasonable to suppose that registration managers can estimate these parameters in a similar fashion.

## C.2 Estimating Unobserved Variables

The empirical analysis also entails estimating three variables which are not directly obsevable from the available data:

a) the propensity to self-register,

b) the tract contact intensities, and

c) the constant term appearing in the first-order conditions.

Furthermore, the expression for this constant term contains instances of three other unobservable parameters: the registration budget **C**, the arrival rate $\mu$, and the typical mobile registrar's opportunity cost M. To avoid circularity, the estimates of all the unobservable parameters should not be based on the assumption that the parties are behaving optimally, since these estimates will be employed to test this condition.

### a. Self-registration Propensity $\sigma^{A}{}_{i}$

Explicit recognition of the significance of the self-registration propensity for efficient registration drives is a hallmark feature of the ATIS model. Presumably registrars should seek to avoid the costly redundant registration of self-registrants, and focus their efforts on the group only registerable population. Recall that $\sigma^A_i$ denotes the proportion of eligible A-partisans expected to self-register in the absence of any registration efforts. Although, this important parameter of the model is not directly observable[12]; however, it *can* be estimated from observable self-registration variables. Consider the following figure:

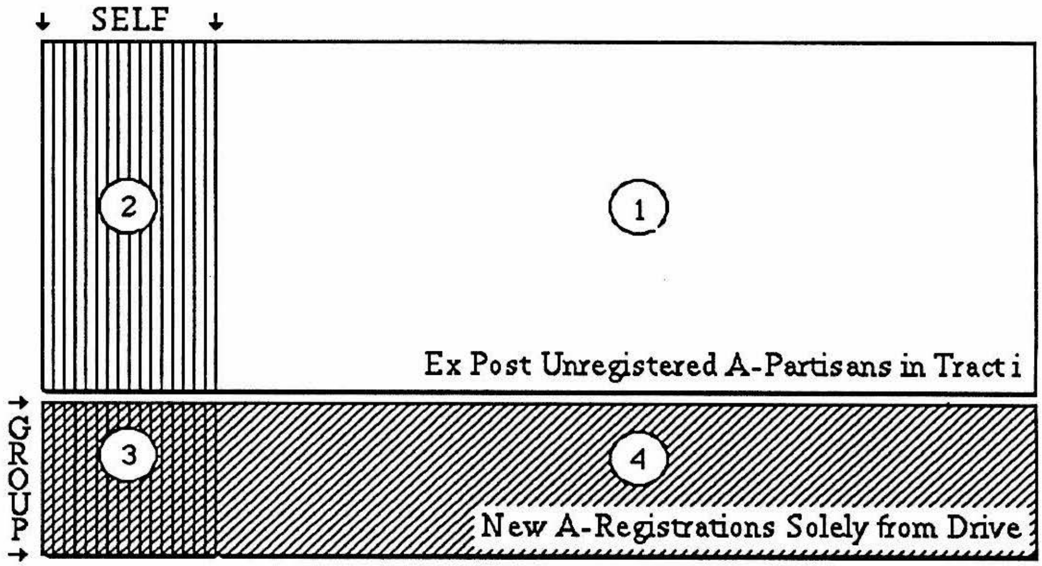


Figure 5.

Suppose the entire rectangular region depicted in the figure, that is, quadrants 1 through 4, represents all eligible A-partisans in tract i, including those who will be registered by non-partisan groups. The task at hand is to estimate the area of the

---

**12.** Not only is the self registration propensity not directly observable from the data under consideration here, but only an expensive controlled study of group registration across demographically and politically similar tracts would seem capable of providing direct observations of this parameter.

"SELF" band, consisting of quadrants 2 and 3 along the west edge of the figure, relative to the area of the entire figure. Three pieces of information can be brought to bear on this problem:

- Reliable estimates can be made of the size of the entire eligible poulation $E_i$ from census and registration data.
- The total number of group-registrations for party A can be observed. This number corresponds to the "GROUP" band consisting of quadrants 3 and 4 along the south edge of the figure..
- The total number of self-registrations for party A can also be observed. This number corresponds to quadrant 2.

The key datum which would solve the problem--if it were directly observable--is the number of individuals represented by quadrant 3 alone, that is the number of self-registrants redundantly group-registered. If this were known, the area of the "SELF" band could be determined, and the desired estimate of $\sigma^A{}_i$ could be obtained as the ratio of the"SELF" band to the entire diagram.

Since the entire qualified population $E_i$ can be estimated, and quadrants 2 and (3 + 4) can be observed, the remainder-- quadrant 1 can be estimated by simply subtracting 2 and (3 + 4) from $E_i$. Having done this, the area of quadrant 3 can be estimated-- if it is assumed that the ratio of quadrant 3 to 4 equals that of quadrant 2 to 1, as specified below.[13]

---

**13.** One way to think about this assumption runs as follows: Consider the ex ante A-eligible population as having been divided into two groups: those contacted (and hence registered) by mobile registrars, and those not contacted. There would appear to be no a priori reason to suppose the self registration propensity among these two groups is different. In the absence of any reason to think otherwise, its reasonable to assume the two groups' self registration propensities are identical.

- The percentage of eligible A-partisans who would self-register is the same among A-party group-registrants, as among the remaining A-eligible population.

This assumption together with the following two auxiliary technical assump - tions suffice to yield the desired estimate of $\sigma^A_i$.

- Party-A group-registrants who were cross-registered would have self-registered otherwise.
- The percentage of individuals who would have self-registered among eligible A-partisans group-registered by A-party registrars is the same as that among those registered by non-partisan organizations.

Hence $\sigma^A_i$ can be estimated according to the equation:

$$\sigma^A_i = \left\{ \text{A SELF-REG} + [(\text{A SELF-REG} / (\text{A ELG} - \text{A BY A})) * \text{A BY A}] \right\} / \text{A ELG},$$

where:

- A SELF-REG

  denotes the number of individuals who self-registered for Party A in tract i, plus the number in tract i who were group-registered for party A by Party B.

- A ELG

  is the number of ex ante eligible A-partisans in tract i, net of those individuals group-registered for party A by non-partisan organizations.

  That is $\text{A ELG} = \pi^A_i * (E_i - \text{A by NONPART})$. (The estimation of the tract i ex ante A-party partisan percentage $\pi^A_i$ and the eligible population $E_i$ have been previously discussed.)

- A by A

is the number of own-party registrations produced by Party A in tract i.

Examination of this definition, together with the three assumptions set forth earlier, will show that $\sigma^A_i$ so estimated is indeed the ratio of quadrants (2 + 3) to quadrants (1 + 4), as depicted in Figure 5.

b. <u>Tract Contact Intensity</u> $\lambda_i$

Although the contact intensities $\lambda_i$[14] are not directly observable, they can be inferred from the numbers of registrations observed in each tract by applying the inverse of the registration production function . Recall from the earlier discussion of the ATIS registration production function that the expected *total* (not just group-only) number of A-Party registrations generated by a registration drive conducted in tract i at contact intensity $\lambda_i$ is given by the expression:

$$\pi^A_i(E_i - r^N_i)(1 - e^{-\lambda_i}).$$

Thus the unobservable variable $\lambda_i$ can be estimated according to the relationship below, where $r^A_i$ is A BY A, the observed total number of tract i A-Party group regis - trations produced by party A:

14. $\lambda_i$ will be written for $\lambda^A_i$ whenever there is no possibility for confusion.

$$\begin{aligned} r^A_i &= \pi^A_i(E_i - r^N_i)(1 - e^{-\lambda_i}) \\ \Leftrightarrow\ e^{-\lambda_i} &= (\pi^A_i(E_i - r^N_i) - r^A_i)/((E_i - r^N_i)) \\ \Leftrightarrow\ \lambda_i &= \ln(\pi^A_i(E_i - r^N_i)) - \ln(\pi^A_i(E_i - r^N_i) - r^A_i) \\ &= \ln(\text{A ELG}) - \ln(\text{A ELG} - \text{A byA}) \end{aligned} \qquad \text{(eq 22)}$$

Estimates of this form for $\lambda_i$ possess several desirable properties, which are especially fortunate given the key role played by contact intensities in the model.

- Notice that this estimate of $\lambda_i$ *does not* depend on the assumption that parties are behaving optimally.
- Given the perfect mixing assumption, $\lambda_i$ can be estimated on the basis of own-party group-registrations, $r^A_i$. DCL-registrants, cross-registrants, and regis - tration efforts by other parties will not bias the estimate.
- The quality of the estimate does not depend on the equal arrival rate assumption.[15]
- Since $\lambda_i$ is inferred from own-party vote production, the estimate does not depend on the validity of the perfect filtering assumption.

Finally, notice that $\lambda_i$ is an expression of the form ln(x)–ln(x-h), and so can be approximated as h/x. For the three Assembly districts under consideration,

$$r^A_i \,/\, \pi^A_i(E_i - r^N_i) = \text{A byA} \,/\, \text{A ELG}$$

---

15. Suppose two tracts, identical except for different arrival rates, have yielded the same number of A-party registrations, then the contact intensity estimates for the two tracts will also coincide. However, the low arrival rate tract will require relatively more registrar hours to generate this intensity.

provides an excellent approximation to $\lambda_i$ as specified in (eq 22). Adopt the notation[16]

$$\mu^A_i = r^A_i / \pi^A_i (E_i - r^N_i).$$

In each of the three Assembly districts and for each party, the correlation coefficient $\rho(\mu^A_i, \lambda^A_i)$ is always greater than .989, and the slope of this linear relationship is very nearly unity. $\mu^A_i$, and therefore $\lambda^A_i$, can be interpreted as the percentage of A-party eligibles group-registered by party A. While the definition of $\lambda^A_i$ might seem rather obscure, the notion of the percentage of own-party group-registration success is quite intuitive. A scattergram has been included to illustrate this relationship for the case of the Democratic party in the 63rd AD.

**16.** The symbol $\mu$ has been employed earlier to denote the mean arrival rate of qualified individuals at a mobile registrar's station. The intended meaning should be clear from the context.

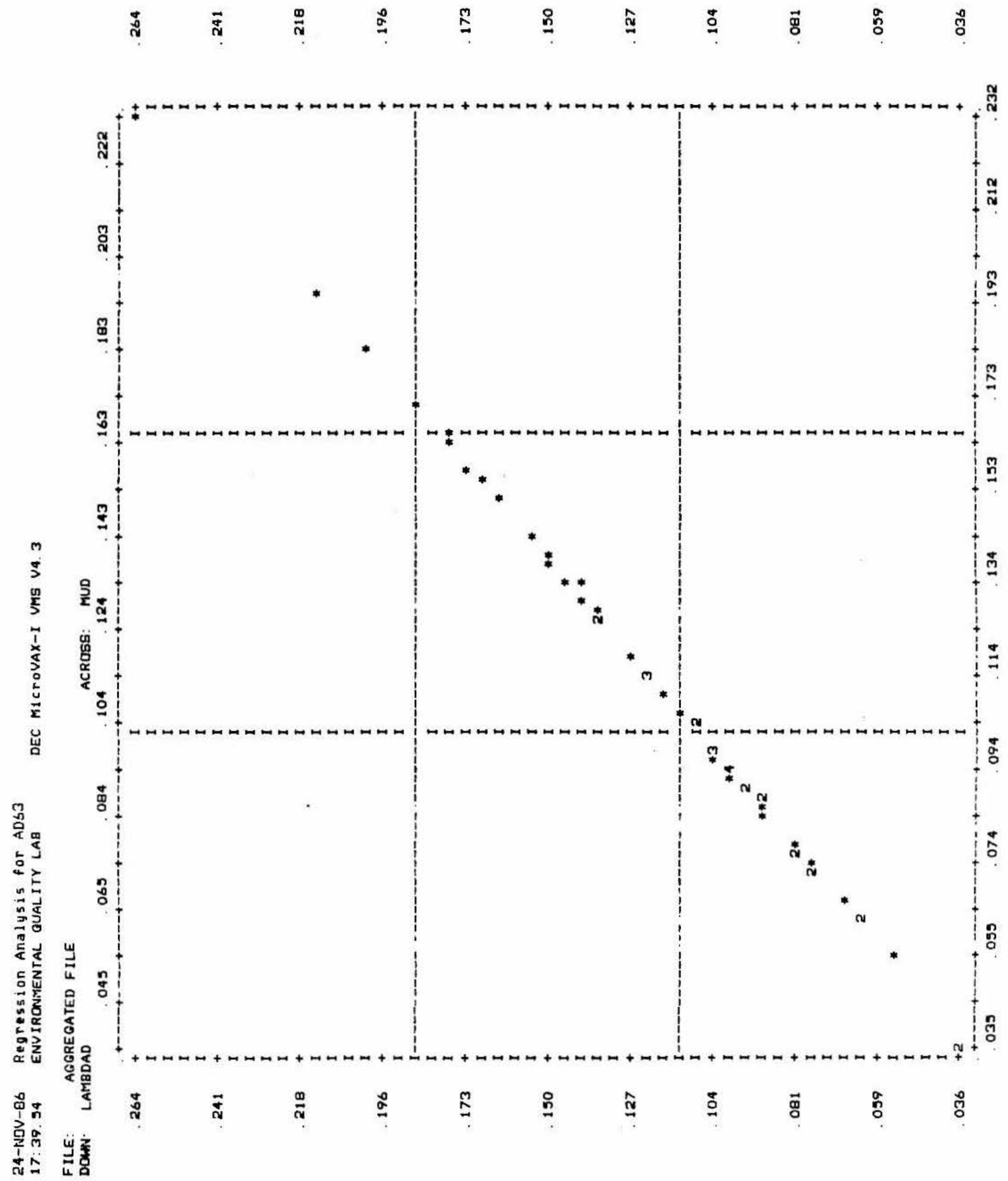
24-NOV-86 17:39.54
Regression Analysis for AD63
ENVIRONMENTAL QUALITY LAB
DEC MicroVAX-I VMS V4.3
FILE: AGGREGATED FILE
DOWN: LAMBDAD
ACROSS: MUD

Figure 6.

c. The Linear Decay of Self-Registrants

The above estimate of $\lambda_i$ is based on the earlier assumption that all self-registrants, even those who obtain forms from opposition registrars during the registration period, postpone registration until the final date unless contacted by an own-party registrar. An alternative assumption is that the number of self-registrants remaining unregistered decays linearly from the begining of the registration period.

Each self-registrant can be thought of as being indexed by a point in time during the registration period at which he will self-register unless he has been previously contacted by an own-party registrar. In the interests of mathematical tractability, it should also be assumed that contact by an opposition registrar will not result in a self-registrant registering prior to his indexed decay time.[17] Under this scenario, one finds:

$$r^A_i = \sigma^A_i \pi^A_i (E_i - r^N_i)[1-(1-e^{-\lambda_i})/\lambda_i] + (1-\sigma^A_i)\pi^A_i(E_i - r^N_i)(1-e^{-\lambda_i}) \qquad \text{(eq 23)}$$

Since this expression is monotonic in $\lambda_i$, a unique estimate can be computed given an observation of $r^A_i$. In more practical terms, for values of $\lambda_i$ between 0 and 1, the expression $1-(1-e^{-\lambda_i})/\lambda_i$ is roughly $(1-e^{-\lambda_i})/2$, as can be verified by direct computation.[18] Thus the expression above can be written as:

---

**17.** Analagous to the earlier scenario, the contacted individual will obtain an affidavit from the opposition registrar, but no other registration assistance. The individual will delay returning the form until his scheduled decay time, or until he encounters an own-party registrar.

**18.** For example, $1-(1-e^{-.2})/.2 = .094$, and $(1-e^{-.2}) = .181$; $.094/.181 = .519$. Similarly, $1-(1-e^{-.6})/.6 = .248$, and $(1-e^{-.6}) = .451$; $.248/.251 = .550$.

$$r^A_i \approx \sigma^A_i \pi^A_i (E_i - r^N_i)(1-e^{-\lambda_i})/2 + (1-\sigma^A_i)\pi^A_i(E_i - r^N_i)(1-e^{-\lambda_i})$$
$$= (1-\sigma^A_i/2)\pi^A_i(E_i - r^N_i)(1-e^{-\lambda_i})$$
$$\Leftrightarrow \lambda_i \approx \ln((1-\sigma^A_i/2)\pi^A_i(E_i - r^N_i)) - \ln((1-\sigma^A_i/2)\pi^A_i(E_i - r^N_i) - r^A_i). \qquad \text{(eq 24)}$$

Initial results will be reported using the first method (eq 22) of estimating $\lambda_i$. Further analysis based on the linear decay method of estimating $\lambda_i$ will be undertaken if the possibility for significant explanatory improvement is apparent.

### d. Estimating the Constant Term

Finally, recall that the expression $(\Sigma_I Q_i \ln(\beta_i) - C\mu/M) / Q$ appears in the first-order conditions for the optimal contact intensities. This expression has been estimated in the following manner. First of all, to estimate the unobserved arrival rate $\mu$ recall that:

$$t_i \text{ hours} = (\lambda_i \text{ contacts/person} * Q_i \text{ persons}) / \mu \text{ contacts/hour, for } \mu \neq 0.$$

Trivially then $\lambda_i Q_i = t_i \mu$, and summing this expression across the I tracts gives:

$$\Sigma_I \lambda_i Q_i = \mu * \Sigma_I t_i = \mu * \text{TOTAL HOURS},$$

where TOTAL HOURS denotes the expected number of total hours spent by the A–party registration force across the district. Suppose that i) payments to the registration force represent the only costs of Party A's registration effort, and ii) that the entire budget will be spent. Then TOTAL HOURS equals $C/M$, the total budget divided by the average hourly opportunity cost. Therefore:

$$\mu = (\Sigma_I \lambda_i Q_i) \,/\, (\text{TOTAL HOURS}) \Leftrightarrow \mu = (\Sigma_I \lambda_i Q_i) \,/\, (C/M).$$

Notice that the estimate above for the arrival rate $\mu$ *does not* depend on the observed $\lambda_i$ being optimal. Substituting this expression for $\mu$ into the constant term gives the estimate:

$$\text{CONSTANT} = \Sigma_I Q_i(\ln(\beta_i) - \lambda_i) \,/\, Q.$$

## C.3 Summary Measures and Correlations of Tract Parameters

This section first reports the means and standard deviations for the contact instensity $\lambda$, and the components $\sigma$, $\tau$, $\pi$, and CON of the productivity index $\beta$, where the unit of analysis is the census tract. The discussion then turns to the correlation of registration effort with eligibility concentration, and of contact intensity with tract productivity.

### a. Contact Intensity and Registration Productivity Variables

Consider the following table of means for tract contact intensity and productivity variables. Notice that the entries in the Democratic (resp. Republican) rows of the table correspond to the values associated with the Democratic (resp. Republican) registration efforts. Thus, for example, the entry .607 in the Democratic row for the 63rd Assembly District under the $E(\tau)$ column indicates that on average 60.7% of the eligible individuals in each tract of the district at the beginning of the registration period were Democratic partisans. The E(CON) column shows tract means for the (non-partisan) concentration of eligible individuals, where:

$$CON = (E_i - r^N{}_i) / Q_i.$$

The number of observations reflects the omission of split tracts from the sample.

SUMMARY MEASURES OF CONTACT INTENSITY AND REGISTRATION PRODUCTIVITY

| | E(λ) | E(β) | E(σ) | E(τ) | E(π) | E(CON) | # OBS |
|---|---|---|---|---|---|---|---|
| AD 63 | | | | | | .413 | |
| Dem | .117 | .164 | .045 | .680 | .607 | | 53 |
| Rep | .047 | .082 | .183 | .800 | .293 | | 52 |
| AD 41 | | | | | | .353 | |
| Dem | .083 | .105 | .105 | .793 | .390 | | 56 |
| Rep | .078 | .120 | .136 | .795 | .474 | | 56 |
| AD 39 | | | | | | .429 | |
| Dem | .094 | .174 | .055 | .719 | .588 | | 66 |
| Rep | .0182 | .084 | .140 | .784 | .271 | | 66 |

Table 7.

**b.** Negative Correlation Between Registration Effort and Eligibility Concentration

One of the most interesting aspects of this data set is a pervasive negative correlation between measures of partisan registration effort such as $\lambda$ or $\mu$ and measures of eligibility concentration such as CON or its partisan analogs DEMCON = $\pi^{DEM}$*CON and REPCON = $\pi^{REP}$*CON. This negative relationship is somewhat paradoxical; intuitively, tracts exhibiting heavy concentrations of eligibles should attract relatively high contact intensities and yield sizeable numbers of successful registrations. However, as the tables below indicate, intuition does not always accord well with empiry.

CORRELATIONS BETWEEN REGISTRATION EFFORT AND ELIGIBILITY CONCENTRATION

| | $\rho(\mu,\text{CON})$ | $\rho(\mu,\text{PTYCON})$ | $\rho(\lambda,\text{CON})$ | $\rho(\lambda,\text{PTYCON})$ |
|---|---|---|---|---|
| AD 63 | | | | |
| Dem | -.497** | -.231* | -.494** | -.227 |
| Rep | -.449** | -.503** | -.443** | -.504** |
| AD 41 | | | | |
| Dem | -.471** | -.135 | -.449** | -.114 |
| Rep | -.589** | -.618** | -.576** | -.614** |
| AD 39 | | | | |
| Dem | -.401** | -.314** | -.394** | -.395** |
| Rep | -.662** | -.137 | -.608** | -.142 |

"**" and "*" indicate significance measured by a one tailed test at the .01 and .05 levels respectively.

Table 8.

Scattergrams of $\mu^{DEM}$ and $\mu^{REP}$ (resp.) with CON and DEMCON and REPCON (resp.) for AD63 have been included to further illustrate this pattern of negative correlation . An explanation will be suggested for this ***"Paradox of Registration"*** after the ATIS model has been tested.

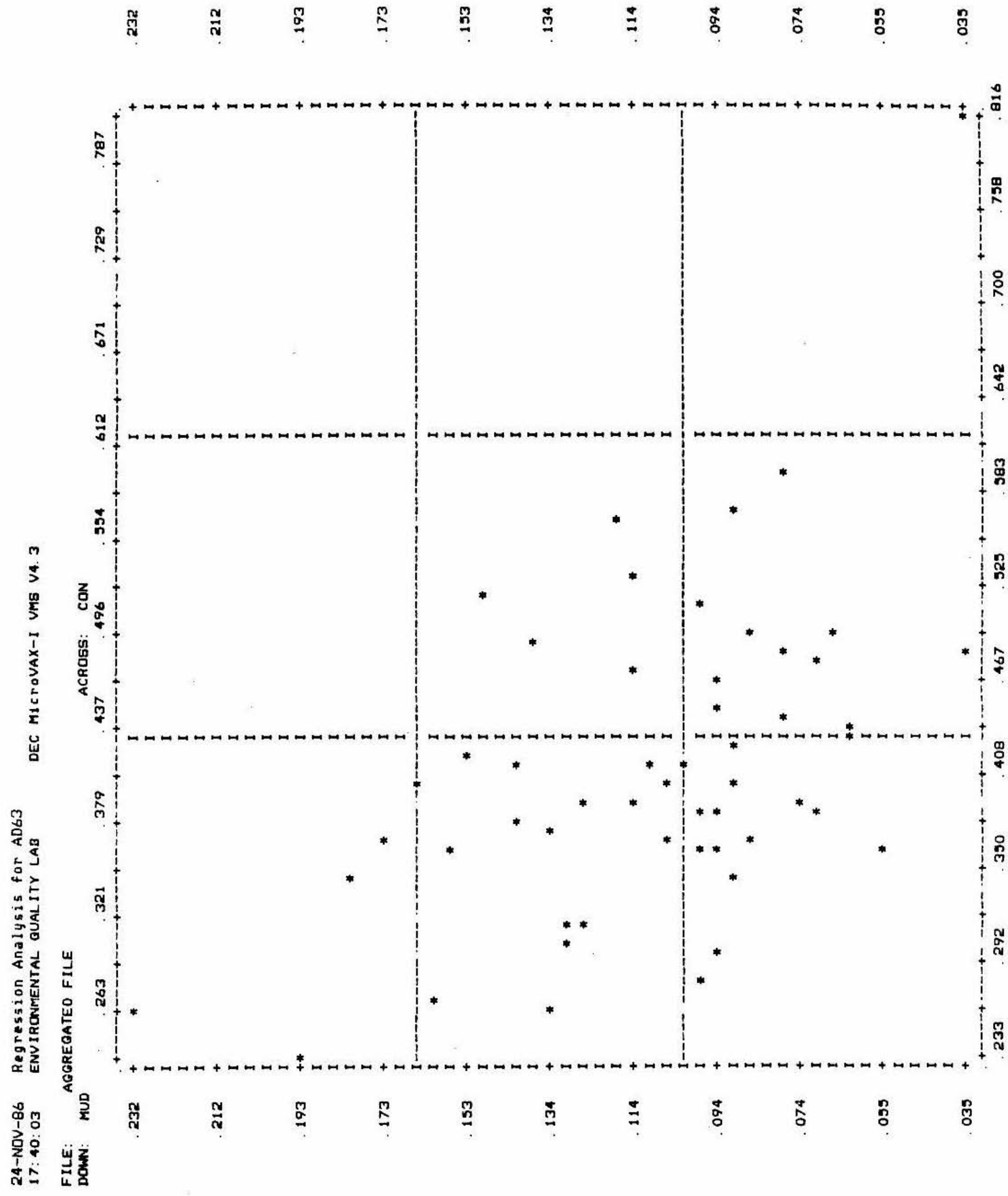
24-NOV-86
17:40:03
Regression Analysis for AD63
ENVIRONMENTAL QUALITY LAB
DEC MicroVAX-I VMS V4.3
FILE: AGGREGATED FILE
DOWN: MUD
ACROSS: CON

Figure 7.

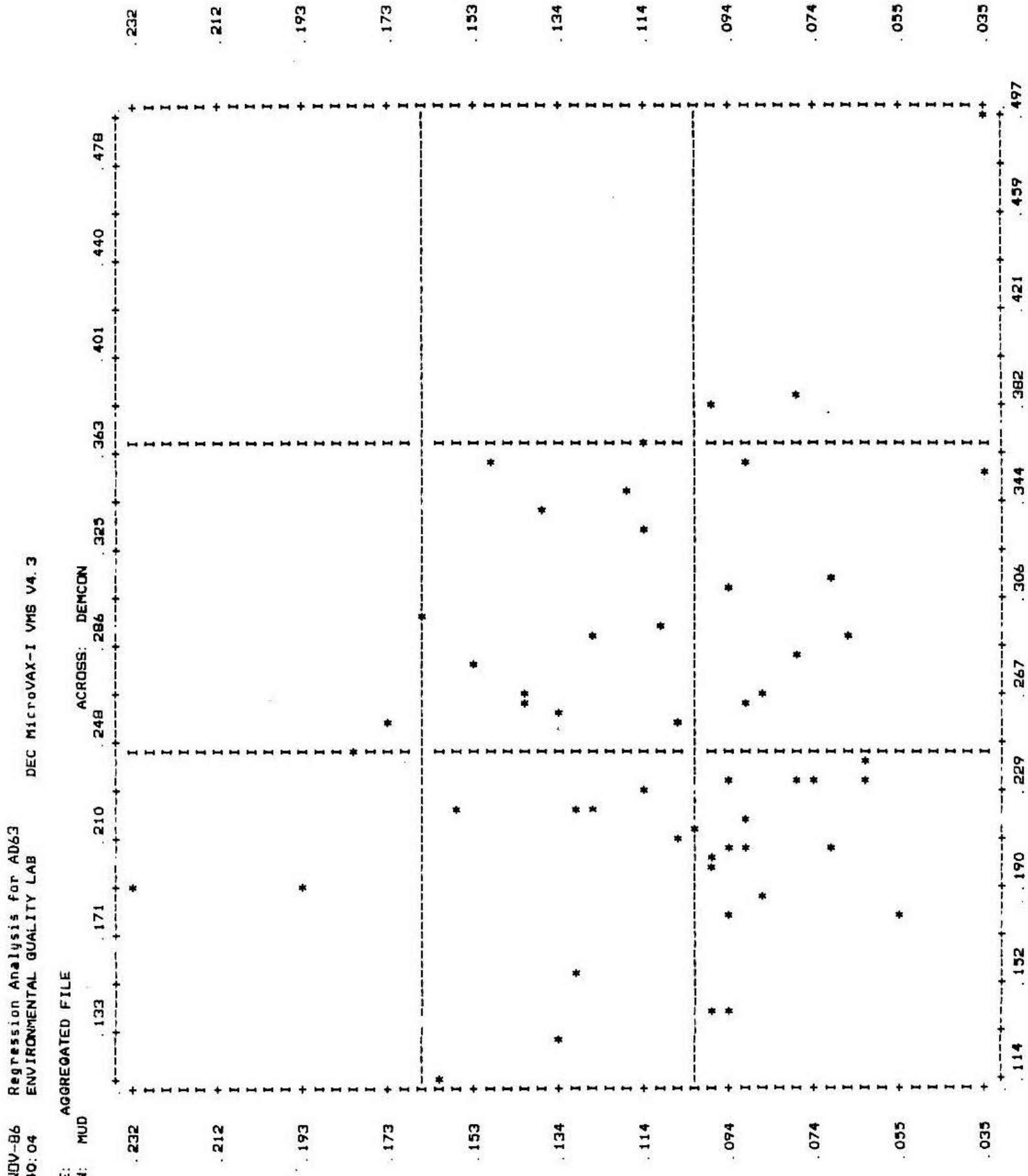
24-NOV-86 Regression Analysis for AD63 DEC MicroVAX-I VMS V4.3
17:40:04 ENVIRONMENTAL QUALITY LAB
FILE: AGGREGATED FILE
DOWN: MUD
ACROSS: DEMCON

Figure 8.

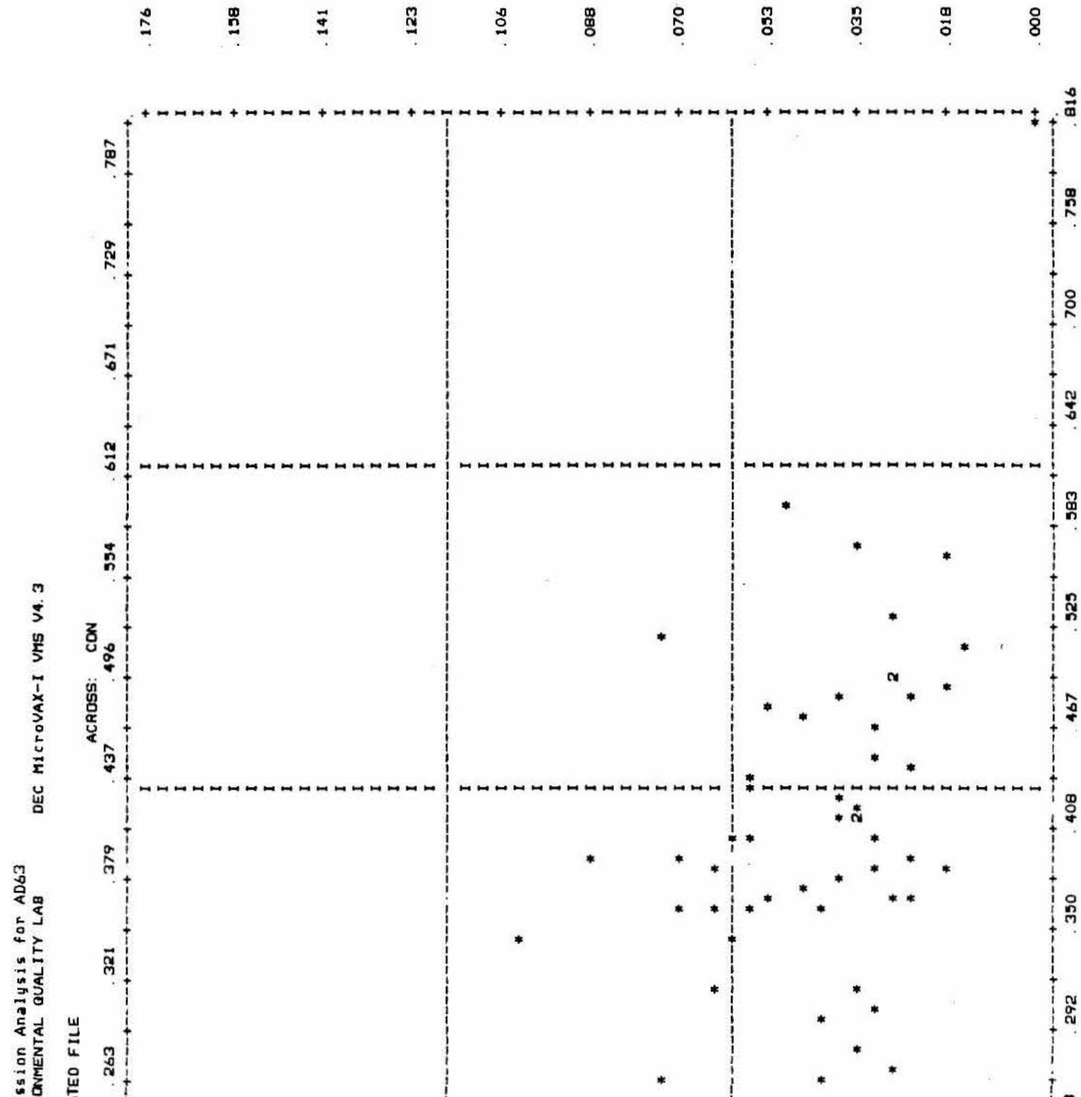
24-NOV-86 17:40:09
Regression Analysis for AD63
ENVIRONMENTAL QUALITY LAB
DEC MicroVAX-I VMS V4.3
FILE: AGGREGATED FILE
DOWN: MUR
ACROSS: CON
.263 .321 .379 .437 .496 .554 .612 .671 .729 .787
.233 .292 .350 .408 .467 .525 .583 .642 .700 .758 .816
.176 .158 .141 .123 .106 .088 .070 .053 .035 .018 .000

Figure 9.

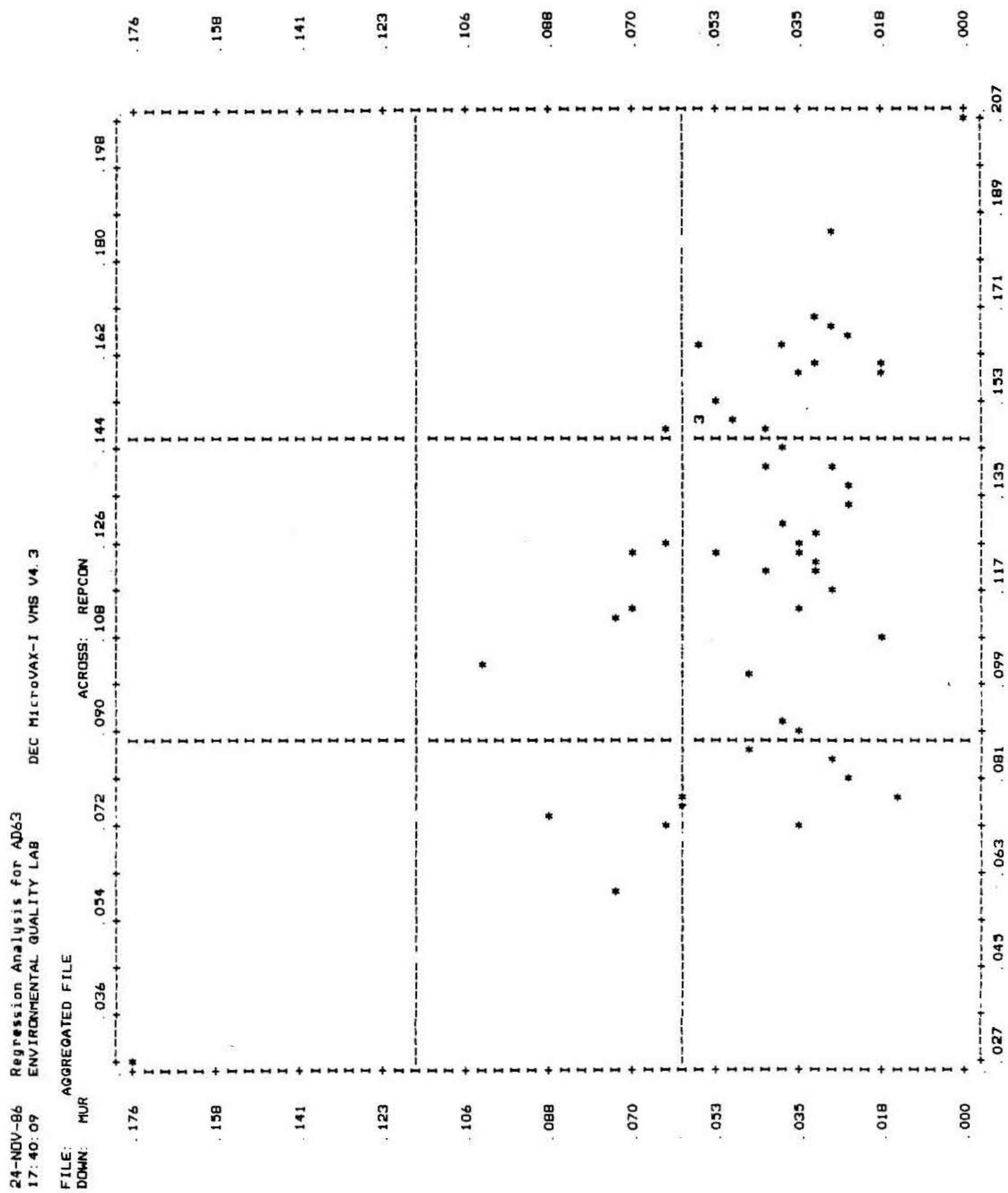
24-NOV-86
17:40:09
Regression Analysis for AD63
ENVIRONMENTAL QUALITY LAB
DEC MicroVAX-I VMS V4.3
FILE: AGGREGATED FILE
DOWN: MUR
ACROSS: REPCON

Figure 10.

c. <u>Correlations of $\lambda$ with $\ln(\beta)$ and</u> $\pi^{PTY}$

As an initial step in the evaluation of the ATIS model, the correlation coefficient between $\lambda_i$ and $\ln(\beta_i)$ was estimated. Since the model predicts the affine relationship:

$$\lambda_i = \ln(\beta_i) - (\Sigma_I Q_i \ln(\beta_i) - C\mu/M)/Q \text{ ; for } i = 1, \dots, I,$$

$\lambda_i$ and $\ln(\beta_i)$ should be highly correlated as they are linearly related by the equation above, except for an additive term which is constant across tracts. The correlations are reported for the three districts by party in the following table. In light of these rather disappointing results,and in order to gain some perspective, the correlation coefficient of $\lambda$ with the own-party ex ante registration percentage, $\pi^{PTY}$, has also been reported. Recall that this registration percentage has been estimated according to:

$$\pi^A{}_i = (\text{\#EX POST REG'D } A^{\underline{S}} - \text{NEW REG'D } A^{\underline{S}}) \big/ (\text{\# ALL EX POST REG'D} - \text{ALL NEW REG'D}).$$

The relatively high correlations observed between $\lambda$ and $\pi^{PTY}$ highlight the importance of partisanship percentage in tract targeting decisions.

CORRELATIONS OF CONTACT INTENSITY WITH TRACT PRODUCTIVITY AND PARTISANSHIP PARAMETERS

| | $\rho(\lambda, \ln\beta)$ | $\rho(\lambda, \pi^{PTY})$ |
|---|---|---|
| AD 63 | | |
| Dem | -.294 | .429 |
| Rep | -.048 | -.271 |
| AD 41 | | |
| Dem | -.118 | .478 |
| Rep | -.304 | .028 |
| AD 39 | | |
| Dem | -.396 | -.098 |
| Rep | .113 | .282 |

Table 9.

One possible explanation for the relatively strong positive correlation between the variables $\lambda$ and $\pi^{PTY}{}_{REG}$ in the heavily Republican 41st district is the geographic concentration of black voters in this district, and the relative ease of locating them. Furthermore, the correlation coefficient between the percentage of black residents and the percentage of Democratic registrants across the tracts in the 41st is .649. Thus it would seem that the parties' registration efforts $\lambda$ could be precisely targeted toward productive tracts, yielding the high observed correlation.

## C.4 Regression Analysis of the ATIS Model

The form of the ATIS first-order conditions suggest the model can be tested by by hypothesizing an additive error term and regressing $\lambda_i$ on $\ln(\beta_i)$.[19] If the model is not to be rejected, the coefficient of $\ln(\beta_i)$ should not be significantly different from unity, and the constant term not different from $\sum_I Q_i(\ln(\beta_i)-\lambda_i) / Q$. Notice that even though the system of equations:

$$\lambda_i = \ln(\beta_i) - \sum_I Q_i(\ln(\beta_i)-\lambda_i) / Q + \varepsilon_i; \text{ for } i = 1, \ldots, I$$

determines $\lambda_i$ based in part on all the remaining $\lambda_j$, this simultaneity is confined to the the second summand on the right-hand side of each equation. Since this term is constant across all the equations, the model can be estimated by standard OLS techniques. The regression results are reported in the following table:

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**19.** The reader may be concerned that the split tract selection process may have eliminated some tracts in which partisan registration effort was zero, or that some tracts remaining in the sample may be characterized by zero effort by one or both parties. In any such a case the non-negativity constraints would be binding. However, it can be verified that the Kuhn-Tucker first order conditions imply that the suggested regression equation should hold among the tracts for which the non-negativity constraints are not binding. In any event, after selecting out the split tracts there is only a single tract in the Republican 63rd in which no own group registrations were recorded.

REGRESSION OF λ ON ln(β)

| COEF ⇒ | ln(β) | CONST | $r^2$ | F | n |
|---|---|---|---|---|---|
| AD 63 | | | | | |
| Dem | -.294* | .165** | .103 | 5.871 | 53 |
| | .121 | .021 | | | |
| Rep | -.379** | .079** | .158 | 9.357 | 52 |
| | .124 | .011 | | | |
| AD 41 | | | | | |
| Dem | -.325 | .117** | .058 | 3.336 | 56 |
| | .178 | .021 | | | |
| Rep | -.429** | .129** | .376 | 32.547 | 56 |
| | .075 | .0097 | | | |
| AD 39 | | | | | |
| Dem | -.334** | .152** | .131 | 9.678 | 66 |
| | .107 | .0197 | | | |
| Rep | -.543 | .228** | .046 | 3.091 | 66 |
| | .309 | .028 | | | |

"**" and "*" indicate significance measured by a two tailed t test at the .01 and .05 levels respectively. Standard errors are reported below the regression coefficients.

Table 10.

In light of these rather disappointing results, one possible avenue for further analysis is the regression of λ on the decomposed form of ln(β).[20] Recall that β is a multiplicative expression comprised of terms corresponding to: self-registration propensity, partisanship percentage, turnout probability, and the concentration of

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20. Professor McKelvey suggested this approach.

eligible individuals among the qualified population. Thus $\ln(\beta)$ can be expressed as the sum of the logs of the multiplicands of $\beta$. When $\lambda$ is regressed on this set of explanatory variables, it is possible their individual effects can be better understood.

REGRESSION OF $\lambda$ ON THE COMPONENTS OF $\ln(\beta)$

| COEF ⇒ | $\ln(1-\sigma)$ | $\ln(\tau)$ | $\ln(\pi)$ | ln(CON) | CONST | $r^2$ | F | n |
|---|---|---|---|---|---|---|---|---|
| AD 63 | | | | | | | | |
| Dem | .384 | -.055 | .140** | -.149** | .047 | .575 | 16.222 | 53 |
| | .347 | .043 | .032 | .026 | .034 | | | |
| Rep | -.116* | -.0089 | -.0102 | -.037 | -.026 | .462 | 10.075 | 52 |
| | .047 | .019 | .016 | .022 | .028 | | | |
| AD 41 | | | | | | | | |
| Dem | -.597** | -.095* | .157** | -.090** | .039 | .742 | 36.590 | 56 |
| | .138 | .036 | .019 | .019 | .022 | | | |
| Rep | -.319** | -.0067 | -7.76E-04 | -.012 | .0148 | .576 | 17.289 | 56 |
| | .078 | .039 | .012 | .014 | .015 | | | |
| AD 39 | | | | | | | | |
| Dem | -1.173** | -.0039 | .078* | -.019 | .049 | .534 | 17.509 | 66 |
| | .185 | .039 | .036 | .029 | .032 | | | |
| Rep | -.513** | -.115 | .0028 | -.125** | -.031 | .561 | 19.483 | 66 |
| | .123 | .076 | .017 | .043 | .047 | | | |

"**" and "*" indicate significance measured by a two tailed t test at the .01 and .05 levels respectively.
Standard errors are reported below the regression coefficients.

Table 11.

These results suggest that registration managers may be ignoring turnout effects and self-registration propensity when selecting contact intensities. Indeed, suppose that managers are rewarded simply on the basis of the number of own-party registrations produced. In this case the manager's problem assumes the following form, where $\pi^A{}_i * \text{CON} = \pi^A{}_i * (E_i - r^N{}_i)/Q_i$:

$$\text{MAX}_\lambda \, \Sigma_I \, (\pi^A{}_i * \text{CON}_i) Q_i (1 - e^{-\lambda_i}) - \zeta * (\Sigma_I \, \lambda_i Q_i - C\mu/M).$$

It can be easily verified that this problem yields first-order conditions analagous to those already developed for the ATIS problem, except that every instance of $\beta_i$ in the equations is replaced by $\pi^A{}_i * \text{CON}_i$. Thus this model can be tested by regressing $\lambda$ on $\ln(\pi^A)$ and $\ln(\text{CON})$. Notice the regression coefficients are again predicted to be unity. This model may well better reflect the incentives faced by registration managers. Nevertheless, preliminary investigation indicates that while $\ln(\pi^A)$ and $\ln(\text{CON})$ explain nearly as much of the variance in $\lambda$ as did the variables in the previous regressions, the coefficients do not exhibit the desired qualitative properties.

As an exercise in empirical investigation, $\lambda$ was also regressed on $\pi^A$ alone. It is not implausible that a registration manager might employ this variable as an index of tract registration productivity, rather than the more complex measure $\ln(\beta)$. Registration drive managers may not take into account turnout effects, self-registration propensity, or the efforts of non-partisan registration groups. Furthermore they may discount the diminishing marginal returns to registration effort. In this case the simpler productivity index $\pi^A$ would be an appropriate return measure.

While no claim has been made concerning the sign or magnitude of the coefficient of $\pi^A$ in the following regression, it is interesting to note the percentage of the variance in $\lambda$ explained by this variable alone. These regression results are reported in the next table.

REGRESSION OF $\lambda$ ON $\pi^{PTY}$

| COEF ⇒ | %PTY REG | CONST | $r^2$ | F |
|---|---|---|---|---|
| AD 63 | | | | |
| Dem | .218** | -.017 | .184 | 11.50 |
| | .064 | .040 | | |
| Rep | -.086* | .073** | .073 | 4.04 |
| | .043 | .014 | | |
| AD 39 | | | | |
| Dem | -.050 | .124 | .009 | .621 |
| Rep | .213 | .117 | .079 | 5.52 |
| AD 41 | | | | |
| Dem | .261** | -.024 | .228 | 16.01 |
| | .065 | .028 | | |
| Rep | .007 | .074** | .001 | .044 |
| | .033 | .018 | | |

"**" and "*" indicate significance measured by a two tailed t test at the .01 and .05 levels respectively. Standard errors are reported below the regression coefficients.

Table 12.

## C.4 Systematic Deviations from the ATIS Model

Since the ATIS model predicts $\lambda_i^{OPT} = \ln(\beta_i) + \text{CONSTANT}$, if the deviations of the observed contact intensities from those predicted, $\lambda_i^{OBS} - \ln(\beta_i)$ ignoring the constant term, are regressed on an appropriate set backround variables, it might be possible to discover some distinguishing characteristics of the sub-optimally targeted tracts.

A potential problem with this method of analysis is an embarassment of riches afforded by the large number of possible explanatory variables contained in census, past elections, and registration data. Literally hundreds of different demographic and political variables are available to describe each census tract. Some method of data reduction was required to help determine which variables could best illuminate the deviations between the predicted $\lambda_i$s and the observed registration contact intensities.

One such method is factor analysis.[21] In the present instance factor analysis tended to indicate three factors were particularly important in explaining these deviations. Inspection of the components of the factors suggested that the first factor was related to an individual's degree of social atomization or the strength of his relationships with other members of his community, as measured by variables such as age, marital status, and household size. The second factor clearly was associated with economic well-being, measured by variables such as median rent, and the number of persons living per room. Finally, the third factor could clearly be seen to be related to an individual's posistion on the liberal-conservative spectrum, measured by

21. Factor analysis, for well known reasons, is employed relatively infrequently by economists. However, the method has achieved a noteworthy degree of acceptance in both theoretical and empirical work concerning securities prices. The Arbitrage Pricing Theory of Ross and Roll, and Rosenberg's contributions to the prediction of stock price volatility come to mind.

variables such as partisanship and voting performance with respect to various indicator initiative measures. The factor analysis for the Democratic registration effort in the 63rd Assembly District can be found in the Appendix.

In light of these observations, it was deemed appropriate to regress the deviations between the predicted and observed $\lambda_i^s$ on the following set of variables:

- VARIABLE 1 = % party reg = the ex ante % of registered tract residents registered for <party>.
- VARIABLE 2 = % single = the % of tract residents who are single, separated, or divorced.
- VARIABLE 3 = census variable x279 = median tract dwelling price, hereafter called HOUSING.

The results of this empirical exploration are reported in the following table. Notice that the relatively small magnitude of the coefficients for census variable x279 is due to a scale factor; the other two variables are percentages, while x279 measures home prices in dollars.

REGRESSION OF DEVIATIONS ON BACKROUND VARIABLES

| COEF ⇒ | % REG | % SINGLE | HOUSING | INTRCPT | ADJ $r^2$ | F |
|---|---|---|---|---|---|---|
| <u>AD 63</u> | | | | | | |
| Dem | -1.533* | -.018* | -1.353E-06 | 1.530* | .330 | 9.552 |
| Rep | -6.904* | .020 | -9.185E-06* | 6.707* | .701 | 40.881 |
| <u>AD 41</u> | | | | | | |
| Dem | -.981 | -.034* | 5.302E-06* | 1.357* | .587 | 27.079 |
| Rep | -12.164* | -.051* | 1.243E-05* | 8.217* | .668 | 37.878 |
| <u>AD 39</u> | | | | | | |
| Dem | -1.61* | -.009 | 5.743E-06* | .640 | .547 | 27.154 |
| Rep | -12.008* | -.005 | 1.061E-05* | 5.748* | .828 | 105.449 |

"*" indicates significance measured by a t test at the.05 level.

Table 13.

Since the dependent variable in these regressions is of the form $\lambda^{OBS}$ - $\lambda^{OPT}$, (ignoring the constant term from the first-order conditions) a positive coefficient for one of the explanatory variables indicates that as the magnitude of that variable increases, the deviation between $\lambda^{OBS}$ and $\lambda^{OPT}$ increases--indicating sub-optimal over-registration. Of course, negative coefficients suggest the opposite relationship.

Lastly, given the relative success of the explanation of the deviations from the model predictions, the observed contact intensities $\lambda$ were also regressed on this same set of variables. These results are reported in the next table.

REGRESSION OF λ ON BACKROUND VARIABLES

| COEF ⇒ | % REG | % SINGLE | HOUSING | INTRCPT | ADJ $r^2$ | F |
|---|---|---|---|---|---|---|
| AD 63 | | | | | | |
| Dem | .155* | -.001 | -3.375E-07 | -1.901* | .216 | 5.781 |
| Rep | -.131 | -7.441E-04 | 1.835E-07 | .090 | .105 | 3.003 |
| AD 41 | | | | | | |
| Dem | .437* | -.001 | 6.984E-07* | -2.132* | .337 | 10.337 |
| Rep | -.082 | -.002* | 2.473E-07 | .143* | .143 | 4.049 |
| AD 39 | | | | | | |
| Dem | -.025 | -.002 | 3.401E-07 | -1.993* | .019 | 1.429 |
| Rep | -.044 | .002 | 1.427E-06* | -.003 | .210 | 6.781 |

"*" indicates significance measured by a t test at the .05 level.

Table 14.

## IV.D. DIFFERENTIAL REGISTRATION COSTS; A MODIFIED ATIS MODEL

One possible explanation for the modest performance of the ATIS model lies with the assumptions that all individuals within a given tract are equally likely to encounter a mobile registrar, and that unregistered individuals will register upon first contact with an own-party registrar. This conjecture is reinforced by the observation that both contact intensity and registration success tend to be negatively correlated with the concentration of eligibles and also with the concentration of group-only registerable eligibles. Why should parties tend to avoid neighborhoods with high concentrations of group-only registerable partisans? This so-called ***paradox of registration*** may be due to the fact that such individuals pose an onerous burden to the registration force.

Suppose that an individual's likelihood of registering when contacted by an own-party registrar *increases monotonically with his degree of political interest.* At one extreme, for example, political participation is sufficiently important to a self-registrant that only 0 contacts are required to register him. An individual for whom politics is less salient might not self-register, but would agree to register when first contacted. However, those individuals even less interested in the political process might require several contacts by registrars before finally becoming sufficiently sensitized to agree to register.

The tract partisan self-registration propensity $\sigma^A_i$ would appear to provide a reasonable index of the general level of political interest among tract residents. Furthermore, this variable is unquestionably related to the ease of registration operations; if the tract group-registration propensity is unity, a trivial registration effort will register all eligible partisans. Similarly, it is not unreasonable to suppose that low levels of tract self-registration propensity might be associated with individuals' relative unwillingness to register.

The most rigorous way to investigate the relationship between self-registration propensity and registration production would be to derive and estimate a relationship between tract self-registration propensity and the distribution of the number of contacts required to register tract residents. Having done this, further study of the occupancy paradigm could presumably yield the expected number of registrations for a given contact intensity.

However, this method would entail solving for the expected number of boxes which contain at least k balls when m are dropped for k = 1, 2, 3, . . . Next these results would have to be combined with the tract population distribution for the number of contacts required for registration, in order to finally obtain a registration production function which takes contact intensity as its argument. This is a seemingly hopeless computational task, since the expected number of boxes containing at least k balls is a very complex expression except in the special case k = 1; see David and Barton [1962]. Even if the desired production function could be obtained, deriving and solving the associated first-order system for the manager's problem would likely be even more difficult. Clearly a more heuristic approach is called for.

The strategy which will developed here is the use of the *tract self-registration propensity as a deflator for contact intensity*, the idea being that nominal contact intensity will not yield the numbers of registrations predicted by the ATIS production function. The underlying justification for deflating the nominal contact intensity is to take account of the fact that not all contacts of own-party eligibles result in registration success.

Consider the division of the tract i A-party elegible population into two components, the self-registrants $\sigma^{A}_{i}\pi^{A}_{i}(E_{i}-r^{N}_{i})$, and the group-only registrants $(1-\sigma^{A}_{i})\pi^{A}_{i}(E_{i}-r^{N}_{i})$. Since the self-registrants will presumably register on first contact, the previously derived production relationship will apply to them, that is if party-A conducts a drive with contact intenstiy $\lambda_i$ it can expect to register $\sigma^{A}_{i}\pi^{A}_{i}(E_{i}-r^{N}_{i})(1-e^{-\lambda i})$ self-registrants. On the other hand, the drive will not be so effective as regards the group-only registerable population--since not all these indiviudals will register at first contact.

Suppose the "true" contact intensity among the group-only registerables is only $\sigma^{A}_{i}*\lambda_{i}$. In this case the party can only expect to register $(1-\sigma^{A}_{i})\pi^{A}_{i}(E_{i}-r^{N}_{i})(1-\exp(-\sigma^{A}_{i}\lambda_{i}))$ self-registrants. Thus the total expected registra - tions $r^{A}_{i}$ which can be expected from the drive is given by the following sum, where $ELG^{A}_{i}$ denotes $\pi^{A}_{i}(E_{i}-r^{N}_{i})$:

$$\begin{aligned} r^{A}_{i} &= \sigma^{A}_{i}\pi^{A}_{i}(E_{i}-r^{N}_{i})(1-\exp(-\lambda_{i})) + (1-\sigma^{A}_{i})\pi^{A}_{i}(E_{i}-r^{N}_{i})(1-\exp(-\sigma^{A}_{i}\lambda_{i})) \\ &= \sigma^{A}_{i}ELG^{A}_{i}(1-\exp(-\lambda_{i})) + (1-\sigma^{A}_{i})ELG^{A}_{i}(1-\exp(-\sigma^{A}_{i}\lambda_{i})) \\ &= ELG^{A}_{i} * [\sigma^{A}_{i}(1-\exp(-\lambda_{i})) + (1-\sigma^{A}_{i})(1-\exp(-\sigma^{A}_{i}\lambda_{i}))]. \end{aligned}$$

As before, such a production relationship can be inverted to give an estimate of contact intensity based on the number of observed group registrations. In order to avoid notational confusion with contact intensities estimated by the ATIS production function, estimates based on the equation above will be denoted by LMBD2. Again letting $\pi^{A}_{i}(E_{i}-r^{N}_{i}) = ELG^{A}_{i}$, straightforward manipulation gives the following expression for LMBD2:

$$LMBD2^{A}_{i} = \ln((1-\sigma^{A}_{i})*ELG^{A}_{i}) + \ln(\sigma^{A}_{i}*ELG^{A}_{i}) - \ln(ELG^{A}_{i}-r^{A}_{i}) \\ / (1+\sigma^{A}_{i}).$$

Suppose that, given the production relationship above, the registration manager simply attempts to maximize the number of own-party registrations produced--without taking into consideration registrants' self-registration propensity or their turnout probability. His problem is then to:

$$\mathrm{MAX}_{\lambda} \sum_I [\sigma^A{}_i \pi^A{}_i (E_i - r^N{}_i)(1-\exp(-\lambda_i)) + (1-\sigma^A{}_i)\pi^A{}_i(E_i - r^N{}_i)(1-\exp(-\sigma^A{}_i \lambda_i))]$$
$$- \zeta * (\textstyle\sum_I \lambda_i Q_i - C\mu/M).$$

Consider that the earlier arguments concerning the budget constraint are still valid in this new context. After manipulation, Kuhn-Tucker ***first-order conditions*** for the tracts in which the non-negativity constraints are not binding can be obtained as:

$$\lambda^A{}_i = \ln\{\sigma^A{}_i[1+(1-\sigma^A{}_i)\exp(\sigma^A{}_i)]\} + \ln(\pi^A{}_i) + \ln((E_i - r^N{}_i)/Q_i) - \ln(\zeta),$$
$$\text{for tracts i such that } \lambda^A{}_i > 0. \qquad (*)$$

This equation can be tested by regression analysis, restricting the sample to those tracts with non-zero registration effort if necessary. To simplify notation let:

$$\mathrm{SELF2}_i = \sigma^A{}_i[1+(1-\sigma^A{}_i)\exp(\sigma^A{}_i)], \qquad (**)$$

and denote $(E_i - r^N{}_i)/Q_i$ by $\mathrm{CON}_i$. SELF2, considered as an explanatory variable for registration effort in the context of a regression analysis, may be thought of as the tract self-registration propensity $\sigma^A{}_i$ grossed-up or inflated by the term $(1-\sigma^A{}_i)\exp(\sigma^A{}_i)$

which takes account of the relative ease of registering individuals in tracts characterized by a high self-registration propensity.[22]

Recalling that contact intenstiy must now be estimated as $LMBD2_i$, the following regression equation is obtained:

$$LMBD2^A_i = \ln(SELF2^A_i) + \ln(\pi^A_i) + \ln(CON_i) + \varepsilon_i.$$

Once again, the coefficients are all predicted to be unity.[23] The regression results are reported in the following table:

**22.** This interpretation of SELF2 can be further reinforced by noting that, for small values of $\sigma^A_i$, $\exp(\sigma^A_i)$ can be closely approximated by $1+\sigma^A_i$. Substituting this approximation into (**) gives $SELF2_i \approx 2\sigma^A_i - (\sigma^A_i)^3 \approx 2\sigma^A_i$, since the cubed term can be ignored for real-world values of $\sigma^A_i$. Comparing the first-order conditons (*) with those of the original ATIS model indicates the relative benefits of targeting tracts with high self-registration propensity.

**23.** The constant term in the regresssion results corresponds to the expression $\ln(\zeta)$ appearing in the first-order conditions (*).

REGRESSION OF LMBD2 ON PRODUCTIVITY VARIABLES

| COEF ⇒ | ln(SELF2) | ln(π) | ln(CON) | CONST | $r^2$ | F | n |
|---|---|---|---|---|---|---|---|
| <u>AD 63</u> | | | | | | | |
| Dem | 1.037†† | 1.069†† | .436† | 7.016** | .628 | 27.545 | 53 |
| | .136 | .305 | .261 | .567 | | | |
| Rep | .450 | 1.286†† | .471† | 5.983** | .551 | 20.003 | 53 |
| | .159 | .181 | .251 | .548 | | | |
| <u>AD 41</u> | | | | | | | |
| Dem | .707† | 1.027†† | 1.058†† | 6.752** | .748 | 51.409 | 56 |
| | .127 | .149 | .167 | .356 | | | |
| Rep | .581† | .922†† | 1.147†† | 6.491** | .586 | 24.508 | 56 |
| | .164 | .122 | .194 | .489 | | | |
| <u>AD 39</u> | | | | | | | |
| Dem | .862†† | .777†† | 1.355†† | 6.919** | .488 | 19.699 | 66 |
| | .134 | .322 | .239 | .487 | | | |
| Rep | .226 | .931†† | .854†† | 5.504** | .668 | 41.593 | 66 |
| | .142 | .087 | .242 | .438 | | | |

"††" indicates that the coefficient is *not* significantly different from 1 at either the .05 or .01 confidence levels, while "†" indicates that the coefficient is significantly different from 1 at the .05 level but *not* at the .01 level, as measured by a two tailed t test.[24] "**" and "*" indicate significant difference from 0 measured by a two tailed t test at the .01 and .05 levels respectively. Standard errors are reported below the corresponding regression terms.

Table 15.

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**24.** The t test results measure whether or not the regression coefficients are significantly different from 1; the constant terms are tested for difference from zero. To test whether or not a coefficient is different from 1, form the statistic: (COEF– 1) / STD ERROR ~ t(n–k); where n is the number of observations, and k is the number of regressors.

These results are considerably more satisfying than those obtained from the original ATIS model. In particular the signs of the regression coefficients accord with the modified model's predictions, tend to be highly significant, and several are quite close to unity--as predicted.

In order to further characterize the modified contact intensities LMBD2, a table of correlations with own-party group registration success $\mu^{PTY}$ is presented below.

CORRELATION BETWEEN LMBD2 AND $\mu$

| | $\rho(\mu,\text{LMBD2})$ |
|---|---|
| AD 63 | |
| <u>Dem</u> | .206 |
| <u>Rep</u> | -.214 |
| AD 41 | |
| <u>Dem</u> | .234* |
| <u>Rep</u> | -.208 |
| AD 39 | |
| <u>Dem</u> | .191 |
| <u>Rep</u> | .113 |

"**" and "*" indicate significance measured by a one tailed test at the .01 and .05 levels respectively.

Table 16.

Suppose now that the registration manager faces the same so-called deflated production relationship defined above but that his optimand is the same as that developed in the original ATIS model--namely to maximize registrations of group-only registerable individuals who will turn out on election day. Recall that under the "deflated" scenario a registration effort of intensity $\lambda_i$ can be expected to produce $(1-\sigma^A_i)\pi^A_i(E_i-r^N_i)(1-\exp(-\sigma^A_i\lambda_i)) = (1-\sigma^A_i)\mathrm{ELG}^A_i(1-\exp(-\sigma^A_i\lambda_i))$ group-only register - able successes. Thus taking turnout into account, the manager's problem can be expressed as:

$$\mathrm{MAX}_\lambda \textstyle\sum_I [(1-\sigma^A_i)\mathrm{ELG}^A_i(1-\exp(-\sigma^A_i\lambda_i))] - \zeta*(\sum_I \lambda_i Q_i - C\mu/M).$$

This problem yields Kuhn-Tucker first-order conditions for those tracts i satisfying the non-negativity constaints for contact intensity, as below:

$$\lambda^A_i*\sigma^A_i = \ln[\sigma^A_i(1-\sigma^A_i)] + \ln(\tau^A_i) + \ln(\pi^A_i) + \ln(\mathrm{CON}_i) - \ln(\zeta).$$
$$\Leftrightarrow\ \lambda^A_i = \ln[\sigma^A_i(1-\sigma^A_i)]/\sigma^A_i + \ln(\tau^A_i)/\sigma^A_i + \ln(\pi^A_i)/\sigma^A_i + \ln(\mathrm{CON}_i)/\sigma^A_i - \ln(\zeta)/\sigma^A_i.$$

To simplify notation let $\sigma^A_i(1-\sigma^A_i)$ be denoted by $\mathrm{SELF3}_i$, $(E_i-r^N_i)/Q_i$ by $\mathrm{CON}_i$ and notice that in this situation $\lambda^A_i$ should again be estimated by $\mathrm{LMBD2}_i$. Thus, hypothesizing an additive error term $\varepsilon_i$, the following regression equation is obtained:

$$\mathrm{LMBD2}^A_i = \ln(\mathrm{SELF3}^A_i)/\sigma^A_i + \ln(\tau^A_i)/\sigma^A_i + \ln(\pi^A_i)/\sigma^A_i + \ln(\mathrm{CON}_i)/\sigma^A_i + \mathrm{COEF}*1/\sigma^A_i + \varepsilon_i,$$
$$\text{for tracts i such that } \mathrm{LMBD2}^A_i > 0.$$

Observe the model predicts that coefficients for all the regressors except $1/\sigma^A_i$ should necessarily be unity. Actual regression results for this variant of the ATIS model have been left as a topic for future research.

## IV.E. POTENTIAL IMPROVEMENT OF SUBOPTIMAL DRIVES

To the extent that the assumptions of the ATIS model are deemed appropriate, deviations from the first-order conditions represent opportunities to improve the efficiency of registration efforts. Kramer [1966] has remarked:

> In the past two decades, the use of quantitative methods as aids for decision-making has become common in many fields, particularly those involving military and industrial operations. More recently, efforts have been made to apply these efforts to other govenmental activities. By and large, however, these efforts have not been made by political scientists, nor have the methods employed, despite their increasing sophistication and power, had great impact upon the discipline.

The potential improvements in registration yield for registration drives in Los Angeles County during the 1984 campaign season can be computed by estimating the registration budget for the district in question, estimating the parameter values required by the model, computing the optimal contact intensities predicted by the model, computing the expected registration yields associated with these intensities, and comparing the optimal yields with the yields which were in fact obtained.

Recall the optimal contact intensities $\lambda_i$ specified by the ATIS model are specified by the Kuhn-Tucker first-order system:

$$\lambda_i = \ln(\beta_i) - \ln(t)\text{; for some } t \geq 0\text{, whenever } \lambda_i > 0,$$

$$\beta_j Q_j = tQ_j - t_j\text{; for some } t_j \geq 0\text{, whenever } \lambda_j = 0\text{, and}$$

$$\textstyle\sum_I \lambda_i Q_i = C\mu/M \quad \text{(budget constraint).}$$

In attempting to solve this system, notice that since $0 < \beta_i < 1$ then $\ln(\beta_i) < 0$; hence $\lambda_i > 0$ iff $0 < t < \beta_i < 1$. And observe that the condition $\lambda_j = 0$ unless $t < \beta_j$ guarantees the existence of K-T multipliers $\beta_j$ associated with the binding non-negativity constraints. Thus the system can be solved by the following iterative method: i) choose some

$t \in [0,\mathrm{MAX}\{\beta_i\}]$; ii) if $t < \beta_i$ then choose $\lambda_i = \ln(\beta_i) - \ln(t)$, otherwise set $\lambda_i = 0$; and iii) if $\Sigma_I \lambda_i Q_i$ is sufficiently close to $C\mu/M$, then stop. Otherwise, if $\Sigma_I \lambda_i Q_i > C\mu/M$ then choose t larger; else if $\Sigma_I \lambda_i Q_i < C\mu/M$ choose t smaller. This process converges since $\Sigma_I \lambda_i Q_i$ increases monotonically as t decreases.

As mentioned above, once the optimal contact intensities have been computed, the registration production function can be employed to determine the difference between potential and observed registration yields. Of course, any policy recommen - dations based on this technique must be qualified by the analyst's degree of confidence in the ATIS model.

# V. DIRECTIONS FOR FUTURE RESEARCH

## A. Better Estimates of the Registration Production Function

The model developed here has the potential for application in real-world registration drives. Recently, in fact, a serious effort was undertaken to incorporate several aspects of the ATIS model in a registration drive conducted during March and April 1987 in the 63rd Assembly District, . The most important application of the model in this drive was to point out to both the registration drive manager, and to the manager's manager, that the objective of a registration drive ought to be the maximization of the expected number of *net votes attributable to the drive*--not simply the maximization of the number of new registrations.

It is hoped that the ATIS model can be sufficiently refined to generate actual tract targeting schedules for the registration force. To accomplish this goal it will be necessary to verify the production assumptions adopted here. There is much to be learned about eligible individuals' inter-district flows, the apparent tenedency for people to register in their home tracts, the number of contacts required to register an individual, the propensity to self-register, and the sensitization of opposition partisans by registration drives.

Such empirical questions have not yet been well-studied, and probably can only be investigated by field research in conjunction with actual registration drives. A substantial quantity of registration production information was gathered during the recent drive in AD63, and this data will form the raw material for future research.

## B. Optimizing Across Several Campaign Activities

The so-called general campaign problem concerns the optimal allocation of resources across *all* campaign activities from canvassing and mailings to registration efforts and fundraising--while the opposition party is simultaneously solving the same problem. It appears that this problem is simply too complex to analyze given the current capabilities of quantitative social science.

The multi-activity problem is intractable for several reasons: it is very difficult to specify a party's objective function, except in the broadest terms. In particular great uncertainty attaches to assessments of the contribution to expected plurality (or to the expected probability of winning) made by intangible campaign activities. Furthermore, the possibility of raising additional campaign funds violates the specification of a classical constrained optimization problem. Finally, adding the dimension of party competition further complicates a difficult problem.

An important first step towards better understanding the campaign problem would be to solve the relatively simple problem of setting the magnitude of the registration budget in competition with a second party, assuming that both registration efforts will be conducted optimally.

## C. Unravelling the Paradox of Registration

The phenomenon of new registrations being disproportionately produced from areas in which registration is already high has been discussed in some detail. This puzzling situation represents an interesting area for further research.

One possible approach is to refine and test the model of managerial decision making. It has been suggested that the so-called deflated production relationship, which represents an attempt to model differential registration costs, discussed in section IV.D be tested under the assumption that the manager is atempting to maximize registrations of group-only registerable individuals who will also turn out.

# APPENDIX 1.

## NOTATION and DEFINITIONS SUMMARY

| | |
|---|---|
| $I$ | $I = \{1, \ldots, I\}$, the index set of Census tracts which comprise the district. |
| $P_i$ | the population of tract i. |
| $Q$ | the number of qualified individuals in the district; a qualifed individual is either registered or eligible to register. |
| $Q_i$ | total number of qualifed individuals in tract i. |
| $E$ | total number of eligible individuals in the district; an eligible individual is qualified, but unregistered. |
| $E_i$ | total number of eligible individuals residing in tract i. |
| $E_i/Q_i$ | the proportion of eligible voters among all qualified voters in tract i. |
| $A$ | one political party. |
| $B$ | the second political party. |
| $N$ | a non-partisan organization |
| $Q^A_i$ | the number of qualifed A-partisans in tract i. |
| $E^A_i$ | the number of eligible (qualified but unregistered) A-partisans in tract i. |
| $\pi^A_i$ | proportion of A-partisans among $E_i$. The symbol "$\pi$" is mnemonic for "***p***artisan." |

$\sigma^A_i$ proportion of eligible A-partisans in tract i expected to self register in the absence of any registration drive.[1] The symbol "$\sigma$" is mnemonic for "*s*elf."

$\tau^A_i$ the proportion of A-partisans expected to actually vote if registered among all eligible group-only registerable A-partisans.[2] The symbol "$\tau$" is mnemonic for "*t*urnout."

$\alpha_i$ $= [(1-\sigma^A_i)\pi^A_i\tau^A_i - (1-\sigma^B_i)(1-\pi^A_i)\tau^B_i ] * (E_i - r^N_i)/Q_i$.

$\beta_i$ $= [(1-\sigma^A_i)\pi^A_i\tau^A_i ] * (E_i - r^N_i)/Q_i$; the percentage, among all qualified individuals in tract i, of individuals who are eligible (net of those previously registered by non-partisan groups) and who are also expected: i) not to self register, ii) to be A-partisans, and iii) to turn out if registered .

$\gamma_i$. total number of all exposures to or contacts by A-party registrars among qualified individuals in tract i.[3]

$\lambda_i$ $= \gamma_i/Q_i$, the A-party registration ***contact intensity*** in tract i. Note the absence of a superscript "A" in the interests of notational simplicity.

$\lambda^B_i$ the B-party contact intensity in tract i.

$\lambda$ $= \{\lambda_1, \ldots, \lambda_i, \ldots, \lambda_I\}$

$t_i$ the total number of hours worked by A-party mobile registrars in tract i.

t the total number of hours worked by A-party mobile registrars district wide.

M a mobile registrar's hourly opportunity cost.

$\mu$ mean hourly arrival rate of qualified individuals at a mobile registrar's station.

---

**1.** Note that $\sigma^A_i$ need not equal the proportion of self-registrants among the already registered A-partisans in tract i.

**2.** Note that $\tau^A_i$ need not equal the proportion of A-partisans among the already registered individuals tract i.

**3.** Notice that a given individual may be contacted more than once during the course of a registration drive. The symbol "$\gamma$" is mnemonic for "*c*ontatct."

$X_i(t_i)$ a random variable; the number of arrivals in tract i in the time interval $[0,t_i]$.

$X(t)$ a random variable; the number of arrivals district wide in the time interval $[0,t]$.

c constant unit cost per A-party registration paid by party A to mobile registrars.

C exogenously specified registration drive budget constraint for party A.

$r^A{}_i$ total number of registrations produced by A-partisan organizations in tract i.

$r^B{}_i$ total number of registrations produced by B-partisan organizations in tract i.

$r^N{}_i$ total number of registrations produced by non-partisan organizations in tract i.

$r_i$ total number of registrations produced by all groups in tract i.

r total number of registrations produced by all groups districtwide.

$e^x$ = exp(x).

* multiplication.

• dot product.

^ exponentiation.

$X^T$ transpose of matrix X.

MDAS order of arithmetic operations: multiplication, division, addition, subtraction.

Group-register An individual is said to have been ***group-registered*** if he obtains his registration affidavit from a mobile registrar affiliated with a political or civic organization, and the group returns the completed affadavit to the registrar of voters. Group registration, as defined here, does not allow for the possibility that a registrant may return his affidavit himself.

| | |
|---|---|
| Self-register | An individual is said to have been ***self-registered*** if he obtains his registration affidavit from the registrar of voters, the post office, or other government agency, and the completed affadavit is accepted by the registrar of voters. |
| Cross-register | An individual is said to have been ***cross-registered*** if he obtains his registration affidavit from a mobile registrar affiliated with one political party, but registers for a different party. |
| GOR | An acronym for *G*roup *O*nly*R*egisterable. Such an individual is eligible to register, but will do so only if group-registered. |
| Qualified | An individual is said to be ***qualified*** if he is already registered or satisfies the citizenship, residency, and age requirements for registration. |
| Eligible | An individual is called ***eligible*** if he is qualified but not yet registered. |
| Contact | A ***contact*** of a qualified individual by a mobile registrar, or equivalently an ***exposure*** to the registrar, is defined to be the physical arrival of the individual within "conversational distance" of the registrar's station.[4] |
| Marginal | A ***marginal*** registration produced by a registration drive corresponds to a group only registerable individual who would not have registered in the absence of the drive. |
| ATIS | An acronym for *A*ggregately *T*argeted, *I*ndividually *S*elective. An ATIS registration drive deploys mobile registrars stationed in selected heavily trafficed public places to search for unregistered partisans. |

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**4.** As defined here, actual conversation need not take place for the contact to be said to have occurred. "Conversational distance" is a primitive concept which will not be formally defined.

# APPENDIX 2.

## ASSUMPTION INDEX

### TIMING SCENARIO

**T1)** There are two dates: 0 and 1, and a single period which will be termed the ***registration period.***

**T2)** All political groups mobilize their registration forces simultaneously and irrevocably on first date.

### INDIVIDUALS' PARAMETERS

**I1)** Qualified individuals'self-registration parameters remain constant during the registration period. Furthermore, their turnout and partisanship partisanship parameters remain fixed at least until election day.

**I2)** Registered partisans who turn out vote for their own parties' candidates.

**I3)** All individuals who self-register do so on the second date.

**I3a)** The pool of individuals who will self-register decays linearly to zero over the course of the registration drive.

### THE ENVIRONMENT

#### Political Characteristics of the District Population

**P1)** Partisanship proportions are the same among the registered and qualified groups in each tract. (However these proportions may vary across tracts.)

**P2)** The expected turnout proportions among the ex ante and ex post groups of registered partisans in each tract are identical. (However these propor - tions may also vary across tracts.)

Migration Across Boundaries

**B1)** Each individual resides in exactly one tract, and can register at most once in this single tract.

**B2)** No individual is contacted or registered by mobile registrars outside his home tract.

**B3)** There is neither in or out migration, nor change in other demographic characteristics in any tract during the registration period.

REGISTRATION GROUPS

**G1)** There are three registration groups: two political parties A and B, and a non-partisan organization N.

**G2)** Once having set a registration budget, each political parties' registration drive objective is to maximize the contribution--attributable exclusively to its registration drive-- to its own expected election day plurality.

**G3)** Both parties can perfectly predict the registration yield of group N; furthermore this non-partisan production occurs instantaneously at time 0.

**G4)** The non-partisan organization registers members of various sub-popula - tions in proportion to these groups' relative frequencies among the eligible populations of each tract.

**G5)** Both parties have access to the same demographic and past elections data for the district; and both possess the same analytic skill.

## REGISTRATION TECHNOLOGY

### ATIS Registration Production

**R1)** Eligible self-registrants will obtain and return registration affidavits at first contact with any registration group.

**R2)** Eligible group only registrants will become registered if and only if con - tacted by an own-party or non-partisan group. These individuals will be registered at the first such contact, but never cross-register.

**R3)** Partisan mobile registrars must provide affidavits on demand to cross-registrants, but need not assist in preparing or returning the forms.

### The ATIS Arrival Process

**A1)** The arrival of qualified individuals at each partisan mobile registrar's station within a given tract is a Poisson process, characterized by a mean arrival rate of rate $\mu$ individuals per hour per station, *randomly selected with replacement from the qualified population of the tract as a whole.*

**A2)** The arrival rate $\mu$ isconstant across tracts.

**A3)** Service times at mobile registrars' stations are zero; there is no queueing or crowding.

The Manager's Optimization Problem

**M1)** The registration drive manager is able to control expected contact intensities across tracts by posting mobile registrars at targeted locations for specified time intervals.

**M2)** Partisan registration managers' objectives match those of their respective parties; managers pose no incentive compatability problems to the parties.

**M3)** The ATIS registration drive is the sole registration production method.

**M4)** The manager assumes his registration budget is exogenously fixed.

**M5)** Mobile registrars are compensated by a constant dollar payment for each own-party registration generated. These payments represent the only cost of an ATIS registration effort.

**M6)** The expected total payments to mobile registrars must at least equal the registrars' total opportunity cost.

# APPENDIX 3.

## THE APPROXIMATION OF $(1-1/Q_i)^{\gamma_i}$ BY $e^{-(\gamma_i/Q_i)}$

**t**rue = $(1-1/Q_i)^{\gamma_i}$, **a**pproximation = $e^{-(\gamma_i/Q_i)}$

| $Q_i \setminus \gamma_i$ | 10 | 100 | 500 | 1000 | 5000 | 10,000 |
|---|---|---|---|---|---|---|
| **10** | **t** .3487 | **t** .00003 | **t** 1.3220-23 | **t** 1.7478-46 | **t** 0 | **t** 0 |
| | **a** .3679 | **a** .00005 | **a** 1.9287-22 | **a** 3.7200-44 | **a** 0 | **a** 0 |
| **100** | **t** .9044 | **t** .3660 | **t** .0066 | **t** .00004 | **t** 0 | **t** 0 |
| | **a** .9048 | **a** .3679 | **a** .0067 | **a** .00005 | **a** 1.9287-22 | **a** 3.7200-44 |
| **1000** | **t** .9900 | **t** .9048 | **t** .6064 | **t** .3677 | **t** .0067 | **t** .00005 |
| | **a** .9900 | **a** .9048 | **a** .6065 | **a** .3679 | **a** .0067 | **a** .00005 |
| **5000** | **t** .9980 | **t** .9802 | **t** .9048 | **t** .8187 | **t** .3678 | **t** .1353 |
| | **a** .9980 | **a** .9802 | **a** .9048 | **a** .8187 | **a** .3679 | **a** .1353 |
| **10,000** | **t** .9990 | **t** .990 | **t** .9512 | **t** .9048 | **t** .6065 | **t** .3679 |
| | **t** .9990 | **t** .990 | **t** .9512 | **t** .9048 | **t** .6065 | **t** .3679 |
| **25,000** | **t** .9996 | **t** .9960 | **t** .9802 | **t** .9608 | **t** .8187 | **t** .6703 |
| | **a** .9996 | **a** .9960 | **a** .9802 | **a** .9608 | **a** .8187 | **a** .6703 |

Table 1.

# APPENDIX 4.

## ATIS Registration Drives *Without* Partisan Filtering

ATIS registration drives have been analyzed in the perfect partisan filtering case under the assumption that only self registrants would ever be cross-registered. It was assumed that partisan mobile registrars would decline to assist cross-registrants in the preparation of registration affidavits, and that only self-registrants would be possessed of sufficient initiative to return the forms without assistance. In this appendix the polar case will be considered, namely that mobile registrars cannot choose to supply or withold their assistance on the basis of a registrant's partisanship.

The *partisan* registration drive manager's problem will be analyzed in the no filtering case. This analysis will highlight the importance of mobile registrars' ability to filter registrants. Furthermore, the no filtering scenario is better suited for the analysis registration efforts in jurisdictions which require the same level of service be provided to all contacts, and may also be more appropriate in the case of non-partisan drives.

The assumptions adopted previously will be retained, with a few modifications. For simplicity,it will be assumed that there are two parties but no non-partisan registration group, and that every individual is a partisan of one of the parties. Assumptions **G3)** and **G4)**, and **R2)** and **R3)** pertain to the actions of the non-partisan group, and to the filtering process and so will no longer be required. Modified versions of three of the earlier assumptions appear below:

**I1')** Qualified individuals'self-registration, turnout, and partisanship parameters remain constant during the registration period. Every individual is a partisan of either party A or party B.

**G1')** There are two registration groups, political parties A and B.

**R1')** Eligible individuals will register at first contact with any registration group.

In the no-filtering case, ATIS registration workers encountering individuals at random in public places such as shopping malls and convenience stores cannot determine a priori the partisanship of the contacts, and cannot discriminate on the basis of partisanship in providing registration services. In such an environment registration workers for party A will inevitably contact and register some B-partisans during the course of the registration drive.

Suppose that party A conducts a registration drive at contact intensity $\lambda_i \geq 0$ in each tract i of the district. Notice that $\pi^B{}_i$, the fraction of $E_i$ composed of B-partisans, is equal to $1-\pi^A{}_i$. Let $\sigma^B{}_i$ denote the fraction of unregistered, qualified B-partisans in tract i who are expected to self-register. Thus, arguing by methods employed in the body of the text, the expected number of B-partisans cross-registered as a consequence of party A's registration drive is given by the expression:

$$(1-\sigma^B{}_i)\pi^B{}_i E_i(1-e^{-\lambda_i})$$

Hence the expected number of individuals group-registered for party A who would not have registered otherwise, net of the number of individuals group-registered for party B who would not have registered otherwise is given by:

$$(1-\sigma^A{}_i)\pi^A{}_i E_i(1-e^{-\lambda_i}) - (1-\sigma^B{}_i)\pi^B{}_i E_i(1-e^{-\lambda_i}).$$

Suppose that the registration drive manager's goal is to maximize the contribution to his party's election day plurality attributable to the drive, given an inflexible budget constraint and a single registration technology. It remains to consider the role of the turnout factor in order to finish the construction of the manager's objective function. Let $\tau^A_i$ and $\tau^B_i$ denote repectively the expected turnout percentage for group-only registered A and B-partisans. The expected number of individuals group-registered in tract i for party A who would not have registered otherwise-- and who will also turn out for party A, net of the number of individuals group-registered for party B who would not have registered otherwise-- and who will also turn out for party B is given by:

$$(1-\sigma^A_i)\pi^A_i\tau^A_iE_i(1-e^{-\lambda_i}) - (1-\sigma^B_i)\pi^B_i\tau^B_iE_i(1-e^{-\lambda_i}).$$

And so the manager's objective function is simply the sum, across all the tracts, of the expected marginal plurality in each tract:

$$\Sigma_I \left[(1-\sigma^A_i)\pi^A_i\tau^A_iE_i(1-e^{-\lambda_i}) - (1-\sigma^B_i)\pi^B_i\tau^B_iE_i(1-e^{-\lambda_i})\right].$$

As before, the control variables which can be manipulated by the registration drive manager are the contact intensities $\lambda_i$ for each tract i of the district. Thus, assuming the non-negativity constraints $\lambda_i \geq 0$ are redundant and equating the right hand sides of the budget and opportunity cost constraints as in the perfect filtering case, the manager's voter registration drive resource allocation problem can be formulated as:

$$\mathbf{MAX}_\lambda \left[\Sigma_I \left[(1-\sigma^A_i)\pi^A_i\tau^A_i - (1-\sigma^B_i)(1-\pi^A_i)\tau^B_i\right]E_i(1-e^{-\lambda_i})\right]$$

**S.T. 1a)** $M*\Sigma_I \lambda_i Q_i / \mu = C.$

Notice that at an optimum the manager will set $\lambda_i = 0$ in tracts which for $(1-\sigma^A_i)\pi^A_i\tau^A_i < (1-\sigma^B_i)(1-\pi^A_i)\tau^B_i]E_i(1-e^{-\lambda_i})$; in these tracts, positive contact intensity will produce more opposition than own-party group-only registerable votes.[5,6]

Renumber the productive tracts as $1^*, \ldots, I^*$ and let $\alpha_i$ be defined as:

$$\alpha_i = [(1-\sigma^A_i)\pi^A_i\tau^A_i - (1-\sigma^B_i)(1-\pi^A_i)\tau^B_i] * E_i/Q_i.$$

$\alpha_i$ can be interpreted as the net percentage of group-only registerable eligible individuals who would turn out, among all the qualified individuals in tract i. Then employing $\xi$ as a Lagrange multiplier for the budget / opportunity cost constraint, the maximization problem above can formulated equivalently as:

$$\mathbf{MAX}_\lambda \left[\Sigma_{1^*} \alpha_i Q_i(1-e^{-\lambda_i}) - \xi(M * \Sigma_I \lambda_i Q_i/\mu - C)\right].$$

After manipulation parallel to that in the perfect filtering case, the following system of first-order conditions is obtained:

$$\lambda_i = \ln(\alpha_i) - \left[\left(\Sigma_I Q_i \ln(\alpha_i) - C\mu/M\right) / \left(\Sigma_{1^*}(Q_i)\right)\right]; \forall\, i \in \{1^*, \ldots, I^*\}, \text{ and}$$

$$\lambda_j = 0, \forall\, j \notin \{1^*, \ldots, I^*\}.$$

5. This result simplifies the problem of modelling party competition in registration drives: Two parties will not both choose to conduct registration drives in the same census tracts. Furthermore, given a fixed budget and having once selected the optimal pattern of registration efforts, a given party will not revise its strategy in response to opposistion efforts.

6. Obviously this "tract separation" prediction of the no-filtering model can be tested quite easily. In the three Assembly districts for which data was available, nearly every census tract reported group-registrations by both parties.

# APPENDIX 5.

## FACTOR ANANYSIS

8-OCT-86 Analysis of tract data
23:32:07 ENVIRONMENTAL QUALITY LAB DEC MicroVAX-I VMS V4.3

FILE: AGGREGATED FILE

------------------------- F A C T O R   A N A L Y S I S ----------

FACTOR MATRIX:

| | FACTOR 1 | FACTOR 2 | FACTOR 3 | FACTOR 4 | FACTOR 5 | FACTOR 6 |
|---|---|---|---|---|---|---|
| PAGE1824 | .88153 | | | | | |
| PX133 | .81726 | | .38395 | | | |
| PX128 | .76697 | | .51259 | | | |
| PPR15YE | -.74304 | | .38300 | | .41729 | |
| PX132 | -.73797 | .50631 | -.37782 | | | |
| PAGE3544 | -.72768 | | .56736 | | | |
| PRENTERS | .72281 | | | -.54858 | | |
| PX129 | .71606 | .49278 | | | | |
| PX127 | -.71553 | .54735 | -.36314 | | | |
| PX130 | .70599 | .31655 | .55964 | | | |
| PPR11YE | -.68865 | | .54905 | | | |
| PX126 | .68370 | | .53838 | .39964 | | |
| PX135 | .67602 | .45315 | | | .33723 | |
| X258 | -.61091 | | .45311 | | | .50413 |
| PX131 | .60685 | | .52334 | .35618 | | |
| PDEMREG | .58177 | -.54203 | -.48032 | | | |
| X279 | -.57454 | | .43796 | -.32238 | | .48876 |
| ERROR | -.55235 | .55003 | -.36612 | | | |
| PPROOM | | -.92304 | | | | |
| PQUALIFY | | .89517 | .36595 | | | |
| PAGE5564 | | .86475 | | | | |
| PGOVDEM | .30085 | -.86241 | | | | |
| PAGE64PL | .37255 | .80514 | | | | |
| PAGE4554 | | .73955 | | .51229 | | |
| PPR12YE | | -.73708 | | | .56827 | |
| PX134 | .43026 | .67765 | | | | |
| PHISP | .42132 | -.66244 | -.48026 | | | |
| PPR14YE | -.54865 | .46089 | .35826 | | | |
| PPR13YE | -.38182 | -.64701 | | | .42398 | |
| PBLACK | -.33637 | | .79931 | | | |
| LAMBDA | | | -.66778 | | | .34325 |
| PAGE2534 | | -.37837 | .65974 | | | |

6-OCT-86 Analysis of tract data
23:32:04 ENVIRONMENTAL QUALITY LAB DEC MicroVAX-I VMS V4.3

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ANALYSIS NUMBER 1 LISTWISE DELETION OF CASES WITH MISSING VALUES

EXTRACTION 1 FOR ANALYSIS 1, PRINCIPAL-COMPONENTS ANALYSIS (PC)

INITIAL STATISTICS:

| VARIABLE | COMMUNALITY | * | FACTOR | EIGENVALUE | PCT OF VAR | CUM PCT |
|---|---|---|---|---|---|---|
| | | * | | | | |
| LAMBDA | 1.00000 | * | 1 | 10.16033 | 31.8 | 31.8 |
| ERROR | 1.00000 | * | 2 | 8.71717 | 27.2 | 59.0 |
| X258 | 1.00000 | * | 3 | 5.41113 | 16.9 | 75.9 |
| X279 | 1.00000 | * | 4 | 1.70199 | 5.3 | 81.2 |
| PQUALIFY | 1.00000 | * | 5 | 1.25321 | 3.9 | 85.1 |
| PGOVDEM | 1.00000 | * | 6 | 1.06009 | 3.3 | 88.4 |
| PDEMREG | 1.00000 | * | 7 | .97663 | 3.1 | 91.5 |
| PPROOM | 1.00000 | * | 8 | .59849 | 1.9 | 93.4 |
| PRENTERS | 1.00000 | * | 9 | .54192 | 1.7 | 95.1 |
| PX126 | 1.00000 | * | 10 | .31032 | 1.0 | 96.0 |
| PX127 | 1.00000 | * | 11 | .19657 | .6 | 96.6 |
| PX128 | 1.00000 | * | 12 | .15256 | .5 | 97.1 |
| PX129 | 1.00000 | * | 13 | .14376 | .4 | 97.6 |
| PX130 | 1.00000 | * | 14 | .11385 | .4 | 97.9 |
| PX131 | 1.00000 | * | 15 | .11214 | .4 | 98.3 |
| PX132 | 1.00000 | * | 16 | .09089 | .3 | 98.6 |
| PX133 | 1.00000 | * | 17 | .08119 | .3 | 98.8 |
| PX134 | 1.00000 | * | 18 | .07629 | .2 | 99.1 |
| PX135 | 1.00000 | * | 19 | .06447 | .2 | 99.3 |
| PBLACK | 1.00000 | * | 20 | .05724 | .2 | 99.4 |
| PHISP | 1.00000 | * | 21 | .04210 | .1 | 99.6 |
| PPR11YE | 1.00000 | * | 22 | .03538 | .1 | 99.7 |
| PPR12YE | 1.00000 | * | 23 | .02934 | .1 | 99.8 |
| PPR13YE | 1.00000 | * | 24 | .01853 | .1 | 99.8 |
| PPR14YE | 1.00000 | * | 25 | .01529 | .0 | 99.9 |
| PPR15YE | 1.00000 | * | 26 | .01241 | .0 | 99.9 |
| PAGE1824 | 1.00000 | * | 27 | .01044 | .0 | 99.9 |
| PAGE2534 | 1.00000 | * | 28 | .00712 | .0 | 100.0 |
| PAGE3544 | 1.00000 | * | 29 | .00555 | .0 | 100.0 |
| PAGE4554 | 1.00000 | * | 30 | .00240 | .0 | 100.0 |
| PAGE5564 | 1.00000 | * | 31 | .00085 | .0 | 100.0 |
| PAGE64PL | 1.00000 | * | 32 | .00036 | .0 | 100.0 |

PC EXTRACTED 6 FACTORS.

23 32.08 ENVIRONMENTAL QUALITY LAB DEC MicroVAX-I VMS V4.3

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ROTATED FACTOR MATRIX:

| | FACTOR 1 | FACTOR 2 | FACTOR 3 | FACTOR 4 | FACTOR 5 | FACTOR 6 |
|---|---|---|---|---|---|---|
| PX126 | .94042 | | | | | |
| PX128 | .91430 | | | | | |
| PX131 | .86706 | | | | | |
| PX130 | .86102 | | | .33497 | | |
| PX133 | .80203 | | | | | |
| PX132 | -.78497 | | .46451 | | | |
| PX127 | -.75799 | .31363 | .49629 | | | |
| PX129 | .59277 | | .43822 | .31254 | | |
| PAGE2534 | .59101 | | -.48023 | | | |
| | | | | | | |
| PDEMREG | | -.87572 | | | | |
| PPR11YE | | .84650 | | | | |
| PPR14YE | | .83309 | .36286 | | | |
| PHISP | | -.75136 | -.36762 | | | -.31295 |
| PPR15YE | | .72294 | | | .56897 | |
| PAGE3544 | | .71859 | | -.49114 | | |
| LAMBDA | | -.63904 | .49069 | | | |
| PGOVDEM | | -.62035 | -.46244 | | .52569 | |
| PBLACK | .57285 | .57753 | | -.45301 | | .30354 |
| PAGE1824 | .53923 | -.54793 | | .48218 | | |
| | | | | | | |
| PAGE4554 | | | .89511 | | | |
| PAGE5564 | | | .87832 | | -.33951 | |
| PPROOM | | -.31704 | -.70997 | -.46472 | | |
| ERROR | -.57720 | | .69809 | | | |
| PQUALIFY | .33199 | .46107 | .68144 | | -.30020 | |
| PAGE64PL | | | .60160 | .57053 | | -.34304 |
| | | | | | | |
| PRENTERS | .38241 | | | .83510 | | |
| PX135 | .56066 | | | .71480 | | |
| PX134 | | | .51380 | .64485 | | -.34549 |
| | | | | | | |
| PPR12YE | | | -.32705 | | .87920 | |
| PPR13YE | | | -.34178 | | .77136 | |
| | | | | | | |
| X258 | | .44683 | | | | .80982 |
| X279 | | .40834 | | | | .80822 |

6-OCT-86 Analysis of tract data
23:32:07 ENVIRONMENTAL QUALITY LAB DEC MicroVAX-I VMS V4.3

FILE: AGGREGATED FILE

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FINAL STATISTICS:

| VARIABLE | COMMUNALITY | * | FACTOR | EIGENVALUE | PCT OF VAR | CUM PCT |
|---|---|---|---|---|---|---|
| | | * | | | | |
| LAMBDA | .73471 | * | 1 | 10.16033 | 31.8 | 31.8 |
| ERROR | .87577 | * | 2 | 8.71717 | 27.2 | 59.0 |
| X258 | .89580 | * | 3 | 5.41113 | 16.9 | 75.9 |
| X279 | .86548 | * | 4 | 1.70199 | 5.3 | 81.2 |
| PQUALIFY | .96691 | * | 5 | 1.25321 | 3.9 | 85.1 |
| PGOVDEM | .95103 | * | 6 | 1.06009 | 3.3 | 88.4 |
| PDEMREG | .90993 | * | | | | |
| PPROOM | .90335 | * | | | | |
| PRENTERS | .93312 | * | | | | |
| PX126 | .94378 | * | | | | |
| PX127 | .94684 | * | | | | |
| PX128 | .90031 | * | | | | |
| PX129 | .80782 | * | | | | |
| PX130 | .91463 | * | | | | |
| PX131 | .80105 | * | | | | |
| PX132 | .95001 | * | | | | |
| PX133 | .85956 | * | | | | |
| PX134 | .69871 | * | | | | |
| PX135 | .90879 | * | | | | |
| PBLACK | .87979 | * | | | | |
| PHISP | .86087 | * | | | | |
| PPR11YE | .84774 | * | | | | |
| PPR12YE | .93855 | * | | | | |
| PPR13YE | .81416 | * | | | | |
| PPR14YE | .91042 | * | | | | |
| PPR15YE | .95100 | * | | | | |
| PAGE1824 | .85449 | * | | | | |
| PAGE2534 | .71050 | * | | | | |
| PAGE3544 | .91086 | * | | | | |
| PAGE4554 | .87347 | * | | | | |
| PAGE5564 | .93809 | * | | | | |
| PAGE64PL | .89936 | * | | | | |

VARIMAX ROTATION 1 FOR EXTRACTION 1 IN ANALYSIS 1 - KAISER

VARIMAX CONVERGED IN 14 ITERATIONS.

# APPENDIX 6

## ATIS REGISTRATION MATERIALS

NORWALK DEMOCRATIC
VOTER REGISTRATION PROJECT -- 1987

JOB: Register Democrats to vote at busy locations in the Norwalk area.

PAY: $1.50 for each valid registered Democrat in the 63rd Assembly District.

HOURS: Choose your own hours and work sites

PAY PERIOD: Payday is every Friday for work done from Thursday thru Wednesday.

---

The "Rap" on site:

"Hi, have you registered to vote, yet?
Have you moved recently?
Great, we're registering Democrats, you're a Democrat
aren't you?" (Assist the voter in filling out the form)...
"Thanks, have a good day."

AFFIDAVIT OF REGISTRATION

STATE OF CALIFORNIA
COUNTY OF LOS ANGELES

PRINT IN INK

## For U.S. Citizens Only

**1** Name (First Middle Last) Optional ☐ Mr. ☐ Mrs. ☐ Miss ☐ Ms.

**2** Residence (Number - Street - Apartment No.)

City County ZIP Code

**3** If no street address, describe location of residence: (cross streets, route, box, section, township, range, etc.)

**4** Mailing Address (if different from residence)

City State ZIP Code

**5** Date of Birth

(Month – Day – Year)

**6** Birthplace (Name of U.S. State or Foreign Country)

**7** Political Party (check one)

- ☐ American Independent Party
- ☐ Democratic Party
- ☐ Libertarian Party
- ☐ Peace and Freedom Party
- ☐ Republican Party
- ☐ Decline to State
- ☐ Other (Specify) ____________

**8** Occupation

**9** Telephone (Optional)
Area Code
(   )

**10** Not applicable in this County

**11** PRIOR REGISTRATION Have you ever been registered to vote? Yes ☐ No ☐

If yes, complete this section to the best of your knowledge concerning your **most recent** registration.

Name (as registered)

Former Address

City County State

Political Party

READ THIS STATEMENT AND WARNING PRIOR TO SIGNING

**I am a citizen of the United States** and will be at least 18 years of age at the time of the next election. I am not imprisoned or on parole for the conviction of a felony. I certify under **penalty of perjury** under the laws of the State of California, that the information on this affidavit is true and correct.

WARNING

Perjury is punishable by imprisonment in state prison for two, three or four years. §126 Penal Code

**12** Signature ▶ Date ▶

**13** Signature of person assisting (if any)

19 ZY 261372

# BIBLIOGRAPHY


1. B.E. CAIN and K.F. McCUE, "Do Registration Drives Matter: The Realities of Partisan Dreams." Presented to the American Polictical Science Association, New Orleans, 1985a.

2. ------, "The Efficacy of Registration Drives." *Journal of Politics,* 47 (1985b), 1221-1230.

3. F.N. DAVID and D.E. BARTON, *Combinatorial Chance*. Hafner: New York, 1962.

4. R.S. ERIKSON, "Why Do People Vote? Because They Are Registered." *American Politics Quarterly,* 9 (1981), 259-276.

5. D. GLASS, P. SQUIRE and R. WOLFINGER, "Residential Mobility and Voter Turnout." Delivered for Conference at the University of Chicago, April 26-28, 1984.

6. S. KARLIN and H.M. TAYLOR, *A First Course in Stochastic Processes, Second Edition*. Academic Press: New York, 1975.

7. S. KELLEY, R.E. AYRES and W.C. BOWEN, Registration and Voting: Putting First Things First." *American Political Science Review,* 61 (1967), 359-379.

8. G.H. KRAMER, "A Decision Theoretic Analysis of a Problem in Political Campaigning," in *Mathematical Applications in Political Science,* II, ed. by J.L. Bernd. Dallas: Southern Methodist University, the Arnold Foundation, 1966.

9. ------, "The Effects of Precinct-Level Canvassing on Voter Behavior." *Public Opinion Quarterly,* 34 (1970), 560-572.

10. H.R. VARIAN, *Microeconomic Analysis, Second Edition.* Norton: New York, 1984.

11. S. VERBA and N.H. NIE, *Participation in America.* Harper & Row: New York, 1972.

12. R.E. WOLFINGER and S.J. ROSENSTONE, "The Effect of Registration Laws on Voter Turnout." *American Political Science Review,* 72 (1978), 22-45.

13. ------, *Who Votes.* Yale University Press: New Haven, 1980.

# QUASITRANSITIVE SOCIAL CHOICE WITHOUT THE PARETO PRINCIPLE

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Division of the Humanities and Social Sciences
California Institute of Technology

Summer 1986

This paper analyzes the structure of families of decisive coalitions, given an underlying quasitransitive valued social decision procedure which satisfies the universal domain and independence of irrelevant alternatives axioms. By identifying the collection of alternatives $X^* \subseteq X$ on which the Pareto principle fails, the extent of imposition in the social ranking can be characterized. In particular, *every* coalition is weakly decisive for $X^*$ over $X \sim X^*$, and weakly antidecisive for $X \sim X^*$ over $X^*$. Therefore alternatives in $X \sim X^*$ can never be socially ranked above those in $X^*$. Further results can be obtained by repeatedly refining $X^*$ so as to remove alternatives on which the Pareto principle fails.

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I would like to thank Professor John Ferejohn for suggesting this problem and for several helpful conversations.

# I. INTRODUCTION

Ever since Arrow's [1951] results first appeared, the central problem of the theory of social choice has been to classify the properties of families of decisive coalitions entailed by ethical and regularity restrictions on social welfare functions. This paper is an attempt to clarify some of these properties assuming only the principle of the independence of irrelevant alternatives, universal domain and quasitransitive valuedness of the social welfare function. In particular, the analysis does not require the Pareto principle.

If a social welfare function fails to satisfy the Pareto principle, the collection X of alternatives must contain at least one element x which is unbeatable-- regardless of the preferences of any coalition-- against at least one other $y \in X$. That is there is no coalition which can enforce the strict preference of its members for x over y.

Thus X can be partitioned into a set $X^*$ of alternatives each of which is unbeatable with respect to some member of X and the complement $X \sim X^*$ on which the Pareto axiom holds. It is then instructive to characterize the decisive and blocking coalitions for:

- $X \sim X^*$ against $X \sim X^*$,
- $X \sim X^*$ against $X^*$,
- $X^*$ against $X \sim X^*$, and
- $X^*$ against $X^*$.

This line of reasoning may be pursued further: $X^*$ itself may contain elements which are unbeatable with respect to alternatives in $X^*$-- this is to say that the Pareto axiom fails again, locally on $X^*$. In this case $X^*$ may be partitioned into $(X^*)^* = X^{2*}$

and $X^* \sim X^{2*}$, and the same analysis applied in the case of the partition $(X^*, X\sim X^*)$ can be employed again. This process can be iterated until $X^{n*} = \phi$ or $X^{n*} = X^{(n+1)*}$, for some n.

The main result of this paper is that the elements of $X\sim X^*$ can never be socially ranked above any of the elements of $X^*$. In a similar vein, it has been established that every coalition is weakly antidecisive with respect to members of $X\sim X^*$ over elements of $X^*$. It is hoped that a characterization or classification scheme can be developed for sets of alternatives which are invariant under the $*$-operator ; such sets are exactly those in which every element of the set is unbeatable against at least one other member of the set.

In what follows, section II is a brief literature review, section III sets forth the social choice model employed here, and sections IV through VII present specialized results concerning the families of decisive coalitions for $(X\sim X^*, X\sim X^*)$, $(X^*, X\sim X^*)$, $(X\sim X^*, X^*)$, and $(X^*, X^*)$ respectively. Section VIII indicates some directions for future research. Finally, a notation and definitons summary can be found in the appendix.

# II. LITERATURE REVIEW

This paper is related to two different areas of research in generalizing Arrow's Theorem: i) relaxing the requirement that the social ranking of alternatives must be transitive; and ii) relaxing the Pareto principle, the requirement that an unanimously preferred alternative be socially preferred.

In his 1977 paper, Sen reviews impossibility results of the first variety. Each of the propositions discussed below depends upon some form of relaxation of the requirement that a social welfare function yield a transitive social preference relation.

PROPOSITION (Acyclic and Quasitransitive Social Choice )

i) (Sen) Let X be a finite set of alternatives. $\exists$ an (acyclic[1], complete, reflexive)-valued social welfare function satisfying the UNIVERSAL DOMAIN, IIA, PARETO and NON-DICTATORSHIP conditions.

ii) (Gibbard) A social welfare function is said to be ***oligarchic*** if there is "a unique group of persons in the community such that if any one of them strictly prefers any x to any y, society must regard x to be at least as good as y, and if all members of the group strictly prefer x to y, then society must strictly prefer x to y. Each person in the oligarchy has a veto."

Any (complete, reflexive)-valued social welfare function which is also quasitransitive-valued and satisfies the UNIVERSAL DOMAIN, IIA, and PARETO conditions must be oligarchic.

---

**1.** Recall that a relation R is said to be ***acyclic*** if $x_1Px_2$, $x_2Px_3$, ... , $x_{n-1}Px_n \Rightarrow \neg x_nPx_1$. Definitions of other social choice terminology can be found in section III. of this paper.

iii) (Brown) If the set of alternatives is infinite and individuals' preferences are acyclic, any social welfare function which generates acyclic preferences and satisfies Arrow's conditions will fail to produce an oligrachy only if there is a subset of individuals, the ***collegium***, who must prefer x to y for social preference for x against y. Their unanimous assent is thus a necessay condition, though not sufficient.

Proof

i) See Sen [1969, 1977]. The proof is by example; construct f such that given any profile R, $xR_fy$ unless y is Pareto preferred to x. The social welfare function thus obtained in fact satisfies the stated conditions.

ii) Again, the interested reader is referred to Sen's 1977 paper and to Sen [1970]. He states this result was proved by Gibbard in an unpublished 1969 paper. The proposition is also a consequence of Hansson's result, which will be reviewed in the following pages, that the family of decisive coalitions under these assumptions forms a filter. The oligarchy claimed in ii) can be obtained by taking the intersection of all coalitions in this decisive filter.

iii) See Brown [1973].

Wilson [1972] reports a generalization of Arrow's theorem along a different direction-- the relaxation of the Pareto principle. Let X denote the set of alternatives, and $\mathbb{R}$ the collection of all complete, transitive binary preference profiles on X; if $R \in \mathbb{R}$, let $R_f$ denote the social order of X generated by f on R. Define the relation $R_f^*$ on X, called ***weak imposition***, according to $xR_f^*y$ iff $\forall R \in \mathbb{R} (xR_fy)$; and denote by $P_f^*$ and $I_f^*$ the strict and indifferent components of $R_f^*$ respectively. A social welfare function is said to be ***null*** on a collection of alternatives $V \subseteq X$ just in case $\forall R \in \mathbb{R}, \forall x,y \in V (xI_fy)$; that is a null social welfare function generates social indifference among the alternatives in V from every profile. Finally, a social welfare function is called ***two-way dictatorial*** on $V \subseteq X$ in case there is some individual $i \in N$ for whom:

$$(\forall\, R\in \mathbb{R},\ \forall\, x,y \in V\ (xR_f y \Rightarrow xR_i y)) \vee (\forall\, R\in \mathbb{R},\ \forall\, x,y \in V\ (xR_f y \Rightarrow yR_i x)).$$

The assent, either direct or inverse, of such an individual is necessary before any one alternative can be socially ranked over another.

PROPOSITION (Relaxing the Pareto Principle)

(Wilson) Suppose that a transitive valued social welfare function f satisfies the IIA and UNIVERSAL DOMAIN conditions.

i) $R_f{}^*$ is an equivalence relation on the set X of alternatives.

ii) Let $I_f{}^*[x]$ denote the $R_f{}^*$-equivalence class to which $x \in X$ belongs. If $xI_f{}^*y$ then either $I_f{}^*[x] = \{x,y\}$, or else f is null or two–way dictatorial on $I_f{}^*[x]$.

## III. THE MODEL

Let X be the set of all possible social states or ***alternatives***; X need not necessarily be finite, but must contain at least three elements. The elements of X should be regarded as being mutually exclusive and collectively exhaustive; only one social state can prevail. Society is composed of a finite set $N = \{1, \ldots, i, \ldots, n\}$ of ***individuals***. Each individual i is characterized by a complete, transitive, reflexive ***preference realtion*** $R_i \subseteq X \times X$. Denote by $P_i$ and $I_i$ the asymmetric and symmetric parts of $R_i$. That is (x,y) belongs to the asymmetric part of $R_i$ which implies $xP_iy$, iff $xR_iy$ and $\neg yR_ix$; similarly (x,y) belongs to the symmetric part of $R_i$ so that $xI_iy$ iff both $xR_iy$ and $yR_ix$. An n-tuple of preference relations, one corresponding to each individual, is called a ***profile***; let $\mathbb{R}$ denote the set of all profiles which are complete, transitive, and reflexive over X.

A ***social welfare function*** is a map

$$f: \mathbb{R} \longrightarrow \{\text{COMPLETE BINARY RELATIONS ON } X\}.$$

The f-image of a profile R will be denoted $R_f$, and let $P_f$ and $I_f$ denote the asymmetric and symmetric parts of $R_f$ respectively. Any social welfare function defined over the entire domain $\mathbb{R}$ is said to satisfy the ***universal domain*** condition (UDOM); this paper will be exclusively concerned with such maps.

A binary relation, say $R_i$, is said to be ***quasitransitive*** on X just in case its asymmetric component is transitive, that is $\forall x,y,z \in X\ (xP_iy \wedge yP_iz \Rightarrow xP_iz)$. Notice that indifference need not be transitive in the case of a quasitransitive relation. Let **Q** denote the set of all quasitransitive relations on X. The results of this paper require that social welfare functions take values in **Q**; such functions will be called quasitransitive (QUASI). Let F denote the set of all quasitransitive social welfare functions on $\mathbb{R}$.

A social welfare function is said to satisfy the so-called principle of ***independence of irrelevant alternatives*** (IIA) just in case:

$$\forall R,R' \in \mathcal{R}, \forall x,y \in X \left(R\big|_{\{x,y\}} = R'\big|_{\{x,y\}} \Rightarrow R_f\big|_{\{x,y\}} = R'_f\big|_{\{x,y\}}\right).$$

This principle stipulates that if two profiles R and R' coincide on some collection of alternatives $V \subseteq X$, then the social rankings $R'_f$ and $R_f$ generated by f must agree on V. The IIA principle will be required of all social welfare functions considered here.

The ***Pareto*** principle (PARETO) requires that a social welfare function f satisfy the condition:

$$\forall R \in \mathcal{R}, \forall x, y \in X\ (xP_Ny \Rightarrow xP_fy);$$

that is, if every individual strictly prefers x to y then x must be socially ranked above y. Unless otherwise specified, the results of this paper do not require the Pareto principle. There may be instances in which the Pareto principle holds only on a subset of the set of alternatives; if $V \subseteq X$, the Pareto principle is said to hold locally on V with respect to f, written $\text{PARETO}_f[V]$, just in case:

$$\text{PARETO}_f[V] \Leftrightarrow (\forall R \in \mathcal{R}, \forall x, y \in V(xP_Ny \Rightarrow xP_fy)). \qquad (*)$$

If there is no possibility for confusion, the subscript "f" will be omitted. Observe that the defining conditions must obtain regardless of the particular preference profile characterizing the individuals in N. Notice also that condition (∗) only requires that alternatives in V satisfy the Pareto principle with respect to other alternatives in V, not necessarily with respect to all the alternatives in X.

A subset of individuals $C \subseteq N$ is called a ***coalition***. Let $x,y \in X$, if every member of coalition C at least weakly prefers x to y, that is:

$$\forall i \in C\ (xR_iy),$$

then write $xR_cy$. In this way a coalitional preference relation $R_c$ is defined over X.

Similarly define the ***strict coalitional preference*** relation[1] $P_c$ according to:

$$\forall C \subseteq N, \forall x, y \in X \, (xP_C y \Leftrightarrow \forall i \in C \, (xP_i y)).$$

Let $P_c$ and $I_c$ denote denote the asymmetric and symmetric parts of $R_c$ respectively. Let $U, V \subseteq X$ be collections of alternatives, a coalition C is said to be ***decisive*** for U over V with respect to a social welfare function f just in case:

$$\forall R \in \mathcal{R}, \forall u \in U, \forall v \in V \, (uP_c v \Rightarrow uP_f v).$$

Notice that the power of a decisive coalition does not depend upon the preferences of its members. Let $D_f$ (U,V) denote the family of all f-decisive coalitions for U over V. Coalition C is said to be ***weakly decisive*** for U over V with repsect to f iff:

$$\forall R \in \mathcal{R}, \forall u \in U, \forall v \in V \, (uP_c v \Rightarrow \neg\neg vP_f u).$$

And denote by $WD_f$ (U,V) the set of all weakly f-decisive coalitions for U over V.

So-called anti-decisive coalitions turn out to play an important role in the theory of social choice without the Pareto principle. A coalition C is said to be anti-decisive for U over V with respect to f if the members' strict preferences are exactly inverted in the social preference relation generated by f, that is:

$$\forall R \in \mathcal{R}, \forall u \in U, \forall v \in V \, (uP_c v \Rightarrow vP_f u).$$

Let $A_f$ (U,V) denote the set of all f-anti-decisive coalitions for U over V. Mirroring the definitions of decisiveness, coalition C is said to be weakly ***anti-decisive*** for U over V with respect to f just in case:

$$\forall R \in \mathcal{R}, \forall u \in U, \forall v \in V \, (uP_c v \Rightarrow \neg\neg uP_f v).$$

---

**1.** I would like to thank my Committee for pointing out that $P_c$, as defined here, is *not* simply the asymmetric part of $R_c$. For example, consider a two member coalition $C = \{1,2\}$, and suppose that $xP_1y$, while $xI_2y$. Referring to the definitions of $P_c$ and $R_c$ above, notice that $xR_cy$ and $\neg yR_cx$ which implies (x,y) is an element of the asymmetric part of $R_c$--but it is not the case that $xP_cy$. Note that, for a given non-empty coalition C and corresponding preference profile, $P_c$ *is* an asymmetric relation. This asymmetry property is used repeatedly; however, no results derived here depend upon $P_c$ being the asymmetric part of $R_c$.

While the members of an anti-decisive coalition cannot hope to find their strict preferences for U over V embodied in the social preference relation generated by f, they will not necessarily have to endure the strict opposite of their wishes. The collection of all weakly f-anti-decisive coalitions for U over V is denoted by $WA_f(U,V)$.

Members of ***blocking*** coalitions--as defined here--lack sufficient power to enforce their strict preference in the social preference relation, but can secure exact social indifference between any pair of coalitionally strictly ordered alternatives.[2] More formally, coalition C is said to be a blocking coalition for U over V with respect to f iff:

$$\forall R \in \mathcal{R}, \forall u \in U, \forall v \in V \ (uP_c v \Rightarrow uI_f v).$$

Denote the set of all f-blocking coalitions for U over V by $B_f(U,V)$. Finally, the set $SWD_f(U,V)$ of ***strictly weakly decisive*** coalitions for U over V with respect to f is defined by:

$$SWD_f(U,V) = WD_f(U,V) \sim D_f(U,V) \sim B_f(U,V).$$

The members of such a coalition find they can enforce their strict coalitional preference as strict social preference for some pair(s) of alternatives (u,v), but that for some other pair(s) of alternatives (u',v') their strict preference results only in social indifference. Similarly, the set of ***strictly weakly anti-decisive*** coalitions for U over V with respect to f is defined by:

$$SWA_f(U,V) = WA_f(U,V) \sim A_f(U,V) \sim B_f(U,V).$$

---

**2.** This construction is somewhat non-standard, but has been adopted here to permit greater precision. More typically in social choice literature, a blocking coalition can prevent the strict social ordering of any two states contrary to the members' unanimous strict preference. Decisive, weakly decisive, and blocking coalitions--as defined here--are *all* "blocking" in this sense.

PROPOSITION 1. (Elementary Consequences of the Definitions)

Let $f \in F$ be quasitransitive and $U,V,W \subseteq X$.

i) $\forall R \in \mathcal{R}, \forall x,y \in X\ (xP_\phi y)$.

Given any preference profile, every member of the empty coalition strictly prefers x to y.

ii) $\forall x \in X\ (\phi \notin D_f(x,x))$.

The empty coalition is not f-decisive for any alternative x over x itself.

iii) $\forall x \in X\ (D_f(x,x) = 2^N \sim \phi)$.

Every non-empty coalition is decisive for an alternative x over x itself.

iv) $\mathrm{PARETO}_f[\phi]$.

f satisfies the Pareto principle trivially over an empty collection of alternatives.

v) $\forall x \in X\ (\mathrm{PARETO}_f[x])$.

f satisfies the Pareto principle trivially over every singleton collection of alternatives.

vi) $D_f(\phi,\phi) = D_f(\phi,V) = D_f(V,\phi) = 2^N$, for all $V \subseteq X$.

Every coalition is f-decisive for one collection of alternatives over another, if one of the collections is empty.

vii) Let $U \neq \phi$; then $\phi \notin D_f(U,U)$.[3]

The empty coalition is never f-decisive for a non-empty set of alternatives.

---

**3.** Note, however, that if $U,V \subseteq X$ such that $U \cap V = \phi$, then it is possible that $\phi \in D_f(U,V)$. In this case $D_f(U,V) = \{\phi\}$, and every element of U is strictly imposed over every element of V.

viii) $D_f(U,V) = \bigcap_{\{u\in U \wedge v\in V\}} D_f(u,v) = \bigcap_{\{u\in U\}} D_f(u,V) = \bigcap_{\{v\in V\}} D_f(U,v)$.

Every f-decisive coalition for U over V is decisive for each element u of U over V, and for U over each element v of V.

ix) Let $U \subseteq V$; then $D_f(V,W) \subseteq D_f(U,W)$, and $D_f(W,V) \subseteq D_f(W,U)$.

Every coalition f-decisive for V over W is decisive for every subset of V over W; furthermore, every coalition decisive for W over V is decisive for W over every subset of V.

x) $D_f(U,V) \cap D_f(V,W) \subseteq D_f(U,W)$.

Decisiveness of a coalition is a transitive property over the family of all subsets of alternatives, in the following sense: If a coalition is decisive for U over V and for V over W, then the coalition is also decisive for U over W.

xi) $D_f(U,V)$ and $B_f(U,V)$ are closed under superset in N.

Adding members to a decisive or blocking coalition does not dilute the coalition's decisive or blocking power.

## IV. ALTERNATIVES IN X~X* vs. ALTERNATIVES IN X~X*

Given a social welfare function f which is not known a priori to satisfy the Pareto principle over the set X of alternatives, one may reason that either f satisfies the Pareto principle-- $PARETO_f[X]$-- or else that f does not. If not there must be at least one alternative $x \in X$ at which the principle fails. That is for some distinct alternative y all individuals strictly prefer y to x, that is $yP_Nx$, but x is strictly ranked above y in the social ordering generated by f, $xP_fy$.

Suppose that all the alternatives at which the Pareto principle fails are culled into a distinguished collection X*. Then f satisfies the Pareto principle at each of the alternatives in the remainder X~X*, and the local behavior of f on this set can be analyzed under the assumption that the Pareto principle holds.

DEFINITION (The Pareto Failure Set X*)

Let $f \in F$ be a social welfare function, and $V \subseteq X$. An alternative $x \in X$ is said to be ***f–unbeatable in V*** just in case there is at least one profile $R \in \mathbb{R}$ and at least one distinct alternative $y \in V$ (which may depend on the profile R) such that $yP_Nx$ but $xP_fy$. Define the Pareto failure set ***X**** to be the set of all alternatives which are f-unbeatable in X.

The next four claims are direct consequences of the definition of X*. The fifth is Hansson's [1972] elegant social choice result, and is included here for completeness. Notice that for the same reasons $D_f(X{\sim}X^*, X{\sim}X^*)$ is a filter, $A_f(X{\sim}X^*, X{\sim}X^*) = \phi$.

PROPOSITION 2. (Consequences of the Definition of $X^*$, Hansson's Arrow Result)

i) $\mathrm{PARETO}_f[X]$ iff $X^* = \phi$.

f satisfies the Pareto principle on X just in case the Pareto failure set $X^*$ is empty.

ii) $\mathrm{PARETO}_f[X \sim X^*]$.

f satisfies the Pareto principle on the portion of X remaining after the failure set $X^*$ has been excised.

iii) $X \sim X^* \in \mathrm{MAX}_{\{Y \subseteq X\}}\{Y \mid \mathrm{PARETO}_f[Y]\}$.

$X \sim X^*$ is a maximal element of the family of subsets of X on which f satisfies the Pareto principle.

iv) If $X = X^*$, then $\forall n \in \mathbb{N}\ (X = X^{n*})$.

If the Pareto principle fails at every alternative in X, then repeated attempts to isolate those alternatives at which f satifies the principle will prove fruitless.

v) (Hansson) If $\#\{X \sim X^*\} \geq 3$, then $D_f(X \sim X^*, X \sim X^*)$ is a filter--and so contains an ***oligarchy*** obtained by intersecting all elements of the filter, even though the social ordering is only required to be quasitransitive.

If f also generates transitive social orderings, then $D_f(X \sim X^*, X \sim X^*)$ is an ultrafilter[1]-- and so contains a singleton decisive coalition or ***dictator***.

---

1. A ***filter*** F on a set S is a non-empty family of non-empty subsets of S such that: i) the family is closed under intersection, if $f_1, f_2 \in F$ then $f_1 \cap f_2 \in F$, and ii) the family is closed under taking supersets in S, if $f \in F$ and $f \subseteq f' \subseteq S$ then $f' \in F$. An ***ultrafilter*** U on S is a filter which is not strictly contained in any other filter on S, that is U is maximally fine. It can be easily shown that U is an ultrafilter iff for every $T \subseteq S$ either $T \in F$ or $S \sim T \in F$. Every ultrafilter on a finite set contains a singleton element. See Willard [1970].

REMARKS (Semigroups and $D_f(X{\sim}X^*, X{\sim}X^*)$)

As a topic for further research, consider the following problem. Let u,v,w,y be distinct elements of $X{\sim}X^*$; is it the case that $D_f(u,v) = D_f(w,y)$? In case $D_f(X{\sim}X^*, X{\sim}X^*)$ is an ultrafilter one can certainly answer affirmatively. However, if only *quasitransitive* range is assumed, $D_f(X{\sim}X^*, X{\sim}X^*)$ is merely a filter; and the problem is more complex.

One might consider the functions $D_f(\bullet,x)\colon X{\sim}X^* \longrightarrow 2^{\wedge\wedge}(2^N)$ such that: $y \longrightarrow D_f(y,x)$, and define the relation $\equiv$ on $X{\sim}X^*$ according to: $x \equiv y$ iff:

$$D_f(\bullet,x)\,\big|_{(X\sim X^*)\sim\{x,y\}} = D_f(\bullet,y)\,\big|_{(X\sim X^*)\sim\{x,y\}}.$$

Notice the relation $\equiv$ would partition $X{\sim}X^*$ into a finite number of equivalence classes, each of the form $[x]$ where $x \in X$. Define an associative, binary operation $*$ on $[X{\sim}X^*]_\equiv$ according to $[x] * [y] = [z]$ iff:

$$D_f(\bullet,\{x,y\})\,\big|_{(X\sim X^*)\sim\{x,y,z\}} = D_f(\bullet,z)\,\big|_{(X\sim X^*)\sim\{x,y,z\}}.$$

A potential problem arises in that $[X{\sim}X^*]_\equiv$ may fail to be $*$-complete; there simply may not be an alternative z which is equal with respect to unbeatability at the set of alternatives $\{x,y\}$. However, $X{\sim}X^*$ could be augmented so as to make $[X{\sim}X^*]_\equiv$ $*$-complete. Then $([X{\sim}X^*]_\equiv, *)$ would form a semigroup. Insofar as $D_f(X{\sim}X^*,X{\sim}X^*)$ could be said to have been encoded into its associated semigroup, the sizeable body of theorems concerning the taxonomy, characterization, and complexity of finite semigroups could be brought to bear on the nature of $D_f(X{\sim}X^*, X{\sim}X^*)$. Notice, for example, that if $D_f(X{\sim}X^*, X{\sim}X^*)$ is an ultrafilter, then $[X{\sim}X^*]_\equiv$ is just a single equivalence class. As another example, in a context more general than $X{\sim}X^*$, if there is an alternative $x \in X$ which is always strictly imposed so that $D_f(x,X) = 2^N$, then $[x]$ will act as an identity in the semigroup $([x],*)$.

## V. ALTERNATIVES IN $X^*$ vs. ALTERNATIVES IN $X{\sim}X^*$

$D_f(X^*,X{\sim}X^*)$ can be expected to be a large family; intuitively, even relatively powerless coalitions ought to be able to carry the unbeatable alternatives in $X^*$ over the beatable alternatives in $X{\sim}X^*$. In this section, unless otherwise stated, $X^*$ and $X{\sim}X^*$ are both assumed to be non-empty.

The next proposition points out that once it has been established $D_f(X^*,X{\sim}X^*) \neq \phi$, closure under superset forces that $A_f(X^*,X{\sim}X^*)$ and $B_f(X^*,X{\sim}X^*)$ are both empty. Notice also that the Pareto principle must hold between $X^*$ and $X{\sim}X^*$ in the sense that strict preference of the grand coalition for an alternative in $X^*$ over one in $X{\sim}X^*$ will be duplicated in the social ranking of these alternatives.

PROPOSITION 3. (Elementary Properties of $D_f(X^*,X{\sim}X^*)$, $A_f(X^*,X{\sim}X^*)$, and $B_f(X^*,X{\sim}X^*)$)

i) $D_f(X^*,X{\sim}X^*)$, $A_f(X^*,X{\sim}X^*)$, and $B_f(X^*,X{\sim}X^*)$ are each closed under superset in N.

ii) $D_f(X^*,X{\sim}X^*)$, $A_f(X^*,X{\sim}X^*)$, and $B_f(X^*,X{\sim}X^*)$ are mutually disjoint.

A given coalition can belong to at most one of the decisive, anti-decisive, and blocking families for $X^*$ over $X{\sim}X^*$.[1]

iii) $D_f(X^*,X{\sim}X^*) \neq \phi$, and $\forall x \in X^*, \forall y \in X{\sim}X^* \, (D_f(x,y) \neq \phi)$. In particular, $N \in D_f(X^*,X{\sim}X^*)$.

There exists at least one coalition which is decisive for the Pareto failure set $X^*$ over the purged set of alternatives $X{\sim}X^*$. Since this decisive family is closed under superset, it must contain the coalition of the whole.

---

1. Indeed, this claim applies to any two collections of alternatives $U,V \subseteq X$.

iv) $A_f(X^*,X\sim X^*) = B_f(X^*,X\sim X^*) = \phi$. In fact, $\forall x \in X^*, \forall y \in X\sim X^* \; (A_f(x,y) = B_f(x,y) = \phi)$.

No coalition is either antidecisive or blocking for the failure set $X^*$ over $X\sim X^*$.

Proof

Verification of i), ii), and iv) are left for the interested reader. In the case of iii), suppose not. Then for some $x \in X^*$, and $y \in X\sim X^*$, $D_f(x,y) = \phi$; but this gives $y \in X^*$, a contradiction. The local claim comprising the second half of iii) is now obvious. QED.

The next proposition is directed to the question: How large is the family $D_f(X^*,X\sim X^*)$ of f-decisive coalitions for $X^*$ over $X\sim X^*$? The three parts of the proposi - tion each assert that $D_f(X^*,X\sim X^*)$ must contain coalitions of a certain kind and therefore must be at least as large as these families.

PROPOSITION 4. (Lower Bounds for $D_f(X^*,X\sim X^*)$)

Suppose that $X^* \neq \phi$ and $\#\{X\sim X^*\} \geq 2$.

i) Let $c \in D_f(X^*,X\sim X^*)$ and $d \in D_f(X^*,X\sim X^*)$, then $c \cap d \in D_f(X^*,X\sim X^*)$.

The family of coalitions f-decisive for $X^*$ over $X\sim X^*$ is closed under intersection.

ii) $D_f(X\sim X^*,X\sim X^*) \subseteq D_f(X^*,X\sim X^*)$.

Every coalition which is decisive for $X\sim X^*$-- in particular the oligarchy $\cap D_f(X\sim X^*,X\sim X^*)$-- is also decisive for $X^*$ over $X\sim X^*$.

iii) In fact, a sharper but perhaps more obscure local result can be obtained. Suppose $v \in X^*$ and distinct $y, z \in X\sim X^*$. Then $D_f(z,y) \subseteq D_f(v,z)$.

Proof

In the case of i), choose $v \in X^*$ and distinct $y,z \in X{\sim}X^*$, as depicted in figure 1 below.

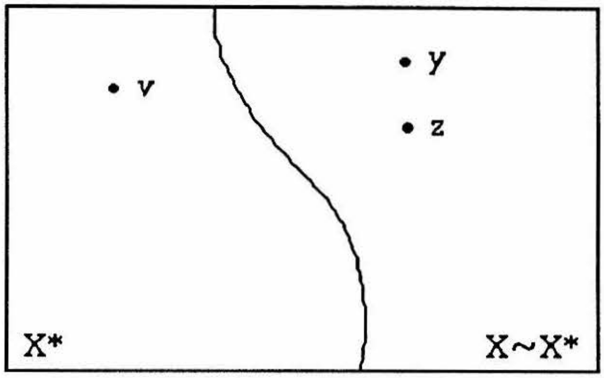


Figure 1 and 2

Let $R \in \mathbb{R}$ be given s.t. $vP_{c\cap d}y$; it is required to show that $vP_f y$ obtains. Construct R' such that $R\,|_{\{v,y\}} = R'\,|_{\{v,y\}}$, according to table 1 below:

| R | | | R' | | |
|---|---|---|---|---|---|
| c~d | c∩d | d~c | c~d | c∩d | d~c |
| [v,y] | v | [v,y] | $[v,y]_R$ | v | z |
| | y | | z | z | $[v,y]_R$ |
| | | | | y | |

Table 1

The notation [v,y] means the relation between v and y under R among the members of the coalition c ~ d is arbitrary. Notice also that the subscript R indicates that the relation between v and y under the profile R' among the members of the coalition shown at the column heading agrees with R.

Now $vP_c'z$ so that $vP'_fz$ since $c \in D_f(X^*,X\sim X^*)$ and $zP'_dy$ since $d \in D_f(X\sim X^*,X\sim X^*)$. Thus quasitransitivity gives $vP'_fy$, and $vP_fy$ obtains by IIA. Hence $c \cap d \in D_f(X\sim X^*,X\sim X^*)$, as desired.

In the case of ii), referring to the proof of i) above, choose $C = N$ (recall $N \in D_f(X^*,X\sim X^*)$) and let d range over the members of $D_f(X\sim X^*,X\sim X^*)$.

Finally, in the case of iii), refer to figure 2 and simply apply the technique employed to prove ii) immediately above. QED.

A stronger condition than was employed in the previous claim gives $D_f(X^*,X^*)$ as another lower bound for $D_f(X^*,X\sim X^*)$.

<u>PROPOSITION 5.</u> ($D_f(X^*,X^*)$ Bounds $D_f(X^*,X\sim X^*)$ from Below)

Let $\#\{X^*\} \geq 2$ and $X\sim X^* \neq \phi$. Suppose that there is at least one alternative $w \in X^*$ such that $D_f(X^*,w) \neq \phi$.

i) If $c \in D_f(X^*,X^*)$ and $d \in D_f(X^*,X\sim X^*)$, then $c \cap d \in D_f(X^*,X\sim X^*)$.

The intersection of an f-decisive coalition for X* with a coalition decisive for X* over X~X* is itself decisive for X* over X~X*.

ii) $D_f(X^*,X^*) \subseteq D_f(X^*,X\sim X^*)$.

Every coalition which is f-decisive for X* is also decisive for X* over X~X*.

iii) Unless every member of $D_f(X^*,X^*)$ has non-empty intersection with every member of $D_f(X^*,X\sim X^*)$, then $D_f(X^*,X\sim X^*) = 2^N$.

Note the condition $\exists\, w \in X^*\ (D_f(X^*,w) \neq \phi)$ obtains if PARETO$_f$[X*], or if $X^* \neq (X^*)^*$.

Proof

In the case of i), let $v,w \in X$ and $x \in X \sim X^*$, as depicted in figure 3 below.

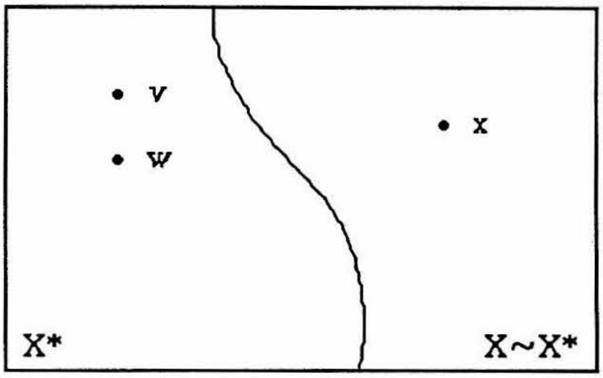


Figure 3

Suppose $R \in \mathcal{R}$ is given such that $vP_{c \cap d}x$. Construct $R' \in \mathcal{R}$ such that $R\big|_{\{v,y\}} = R'\big|_{\{v,y\}}$, and so that the requirements of table 2 below are also satisfied.

| R | | | R' | | |
|---|---|---|---|---|---|
| $c \sim d$ | $c \cap d$ | $d \sim c$ | $c \sim d$ | $c \cap d$ | $d \sim c$ |
| [v,x] | v | [v,x] | $[v,x]_R$ | v | w |
| | x | | | w | $[v,x]_R$ |
| | | | w | x | |

Table 2

Thus $vP'_c w$ so $vP'_f w$ since assumed $c \in D_f(X^*,X^*)$, and $wP'_d x$ so $wP'_f x$ since assumed $d \in D_f(X^*,X^*)$. $vP'_f x$ follows by quasitransitivity, and IIA gives $vP_f x$. Hence $c \cap d \in D_f(v,x)$ and arbitrainess of v,x allow $c \cap d \in D_f(X^*,X \sim X^*)$, as desired.

Turning now to ii), to show the first part, choose d = N and let c range over the members of $D_f$ (X*,X*).

Finally to show part iii), notice that if $c \cap d = \phi \in D_f$ (X*,X~X*), then closure under superset gives that $D_f$ (X*,X~X*) = $2^N$. QED.

REMARKS (Weakly Decisive and Strictly Weakly Decisive Coalitions)

Let U,V $\subseteq$ X; recall that a coalition C $\subseteq$ N is said to be weakly decisive for U over V just in case $\forall R \in \mathcal{R}$, $\forall u \in U$, $\forall v \in V$ ($uP_cv \Rightarrow uP_fv \vee uI_fv$). Recall also that C is said to be strictly weakly decisive for U over V iff C $\in WD_f(U,V) \sim D_f(U,V) \sim B_f(U,V)$. Thus a coalition C $\in SWD_f(U,V)$ will never suffer the imposition of any v $\in$ V over any u $\in$ U contrary to the coalition's strict preference. However, for at least one choice of u $\in$ U, v $\in$ V, and R $\in \mathcal{R}$ it must be the case that $uP_cv$ but $uI_fv$ so that c is not decisive-- C $\notin D_f(U,V)$; and also, for at least one choice of u' $\in$ U, v' $\in$ V, and R' $\in \mathcal{R}$ , $u'P'_cv'$ but $u'P_f'v'$ so that C is not merely blocking-- C $\notin B_f(U,V)$.

The next proposition represents another assessment of the size of the family of coalitions with decisive power for X* over X~X*. By weakening the definition of decisiveness somewhat it is possible to dispense with the specialized assumption of the previous proposition. Notice Proposition 6 implies that the alternatives in X~X* can *never* be socially ranked by f above those in X*, regardless of individuals' preferences. By identifying the collection of alternatives at which the Pareto principle fails, the ***structure of imposition*** in social preference can be illuminated.

PROPOSITION 6. (Imposition of X* Over X~X*)

Suppose that $X^* \neq \phi$ and $\#\{X{\sim}X^*\} \geq 2$; then elements of X~X* are never strictly socially ranked over elements of X*. That is X* is ***imposed*** over X~X*, in the following sense:

i) If $C \notin D_f(X^*,X{\sim}X^*)$, then $C \in SWD_f(X^*,X{\sim}X^*)$. Note that coalition C need not be non-empty.

ii) $D_f(X^*,X{\sim}X^*) \cup SWD_f(X^*,X{\sim}X^*) = 2^N$.

Proof

To show i), recall it was previously shown that $N \in D_f(X^*,X{\sim}X^*)$. Let $C \subseteq N$ such that $C \in D_f(X^*,X{\sim}X^*)$ be given. Then $\exists R \in \mathcal{R}$, $\exists x \in X^*$, $\exists y \in X{\sim}X^*$ $(xP_Cy \wedge \neg xP_fy)$. Now $\neg xP_fy$ allows for two cases: a) $xI_fy$ or b) $yP_fx$. Observing that $B_f(X^*,X{\sim}X^*) = \phi$, if a) obtains, nothing remains to prove. Hence suppose b), $yP_fx$, is the case. Since it has been assumed that $\#\{X{\sim}X^*\} \geq 2$, choose $w \in X{\sim}X^*$ distinct from y above. Construct $R' \in \mathcal{R}$ such that:

1) $R'|_{\{x,y\}} = R|_{\{x,y\}}$,

2) $wP'_Ny$, and

3) $wP'_{C^c}x$.

These conditions are depicted in table 3 and figure 4, as follows:

| R | | R' | |
|---|---|---|---|
| $\underline{C}$ | $\underline{C}^c$ | $\underline{C}$ | $\underline{C}^c$ |
| x | [x,y] | [x,w] | w |
| y | | y | $[x,y]_R$ |

Table 3

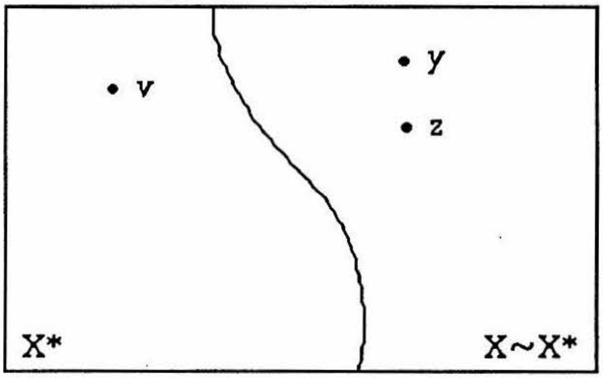


Figure 4

Now since $yP_fx$, IIA yields $yP'_fx$. Notice next that $w,y \in X{\sim}X^*$, and as previously demonstrated Pareto holds on this set. Thus $xP'_Ny$ gives $wP'_fx$. It will now be shown that $C^c \in D_f(w,x)$, which is a contradiction since $w \in X{\sim}X^*$, $x \in X^*$ and so as shown earlier $D_f(w,x) = \phi$. Let $R'' \in \mathbb{R}$ be given such that $wP''_{C^c}x$. Observe that both $wP''_{C^c}x$ and $wP'_{C^c}x$, and also that the relation between x and w under R' was left unspecified with respect to the members of C. Thus R' can be constructed to satisfy not only 1), 2) and 3) above but also the additional requirement: $R'|_{\{x,y\}} = R''|_{\{x,y\}}$. Then having established $wP'_fx$, IIA gives $wP''_fx$. Hence $C^c \in D_f(w,x)$, which is the desired contradiction. Notice that ii) is simply a restatement of part i). QED.

REMARKS (Properties of $SWD_f(X^*,X{\sim}X^*)$)

Notice that $SWD_f(X^*,X{\sim}X^*)$ is contained within an ideal[2] on N. This result obtains since $D_f(X{\sim}X^*,X{\sim}X^*)$ is both a filter and is also contained within $D_f(X^*,X{\sim}X^*)$. Notice that $N \notin SWD_f(X^*,X{\sim}X^*)$; furthermore, if $D_f(X^*,X{\sim}X^*) \neq 2^N$, then $\phi \in SWD_f(X^*,X{\sim}X^*)$. However, unless $D_f(X^*,X{\sim}X^*)$ is also a filter, the closure of $SWD_f(X^*,X{\sim}X^*)$ under union will fail. Hence, in general, it can only be asserted that $SWD_f(X^*,X{\sim}X^*)$ is contained within an ideal, not that $SWD_f(X^*,X{\sim}X^*)$ is itself an ideal.

Consider the following conjecture. Let Y be some set of alternatives. If given $D_f(Y,Y)$ and $SWD_f(Y,Y)$ which are in fact dual, it is possible to solve the backwards problem: Does the social welfare function f which generated $D_f(Y,Y)$ and $SWD_f(Y,Y)$ satisfy transitive range, quasitransitive range, Pareto, and IIA?

**2.** Recall that an ideal on N is a family of subsets of N satisfying the requirements: N itself is not a member of the family; the empty set is a member; and the family is closed under union.

## VI. ALTERNATIVES IN $X \sim X^*$ vs. ALTERNATIVES IN $X^*$

$D_f(X \sim X^*, X^*)$ can be expected to be a small family; few if any coalitions ought to be able to defeat an unbeatable element of $X^*$ with a beatable alternative in $X \sim X^*$. On the other hand, it is plausible that one or the other of $A_f(X \sim X^*, X^*)$ or $B_f(X \sim X^*, X^*)$ is large.

The sense of the next claim is that if an alternative lies in $X^*$, it is unbeatable against at least one alternative in $X \sim X^*$. The alternatives in $X^*$ are not simply unbeatable with respect to one another, but rather are also stable with respect to the alternatives contained in $X \sim X^*$.

PROPOSITION 7. (No Coalition is Decisive for $X \sim X^*$ Over $X^*$)

Suppose $X \sim X^*, X^* \neq \phi$ and let $y \in X^*$.

i) $\exists x \in X \sim X^*$ such that $D_f(x,y) = \phi$.

ii) $\forall x \in X \sim X^*, \forall y \in X^* \; (D_f(x,y) = \phi)$.

iii) $D_f(X \sim X^*, X^*) = \phi$.

Proof

Notice that i) is trivial if $X^*$ is a singleton. Hence suppose $\#\{X \sim X^*\} \geq 2$. If the claim is false, then $\forall x \in X \sim X^* \; (D_f(x,y) \neq \phi)$. Now since $y \in X^*$ by definition $\exists z \in X \; (D_f(z,y) = \phi \wedge (y \neq z))$. Consider any $v \in X \sim X^*$ and observe $D_f(z,v) \neq \phi$ for otherwise $v \in X^*$. Since $v \in X \sim X^*$, assuming the claim false gives that $D_f(v,y) \neq \phi$. Closure of decisive families under superset allows in particular that $N \in D_f(z,v)$, $N \in D_f(v,y)$. Thus $N \in D_f(z,v) \cap D_f(v,y) \subseteq D_f(z,y) = \phi$, a contradiction.

To show ii), suppose not. As above, since $y \in X^*$, $\exists v \in X{\sim}X^*$ such that $D_f(v,y) = \phi$. The negation of the claim allows some $x \in X{\sim}X^*$ such that $D_f(x,y) \neq \phi$. Since $v,x \in X{\sim}X^*$ and PARETO[$X{\sim}X^*$], $D_f(v,x) \neq \phi$. In particular $N \notin D_f(v,x) \cap D_f(x,y) \subseteq D_f(v,y) = \phi$, a contradiction.

Finally, to show iii) observe that $Df(X{\sim}X^*, X^*) = \bigcap_{\{x \in X{\sim}X^*, y \in X^*\}} D_f(x,y) = \bigcap_{\{x \in X{\sim}X^*, y \in X^*\}} \phi = \phi$, as desired. QED.

<u>REMARKS</u> ($B_f(X{\sim}X^*,X^*)$ and $A_f(X{\sim}X^*,X^*)$)

In what follows, suppose that $X^* \neq \phi$, and $\#\{X{\sim}X^*\} \geq 2$ We now seek to characterize $B_f(X{\sim}X^*,X^*)$ and $A_f(X{\sim}X^*,X^*)$. An important unresolved problem is to develop interesting conditions which guarantee $N \in B_f(X{\sim}X^*,X^*)$ or $N \in A_f(X{\sim}X^*,X^*)$, as these two sets are disjoint.

<u>PROPOSITION 8.</u> (Properties of $B_f(X{\sim}X^*,X^*)$ and $A_f(X{\sim}X^*,X^*)$)

i) $B_f(X{\sim}X^*,X^*) \cap A_f(X{\sim}X^*,X^*) = \phi$.

No coalition is both blocking and antidecisive for X~X* over X*.

ii) $\forall v,w \in X \; (A_f(v,w) = \phi \Rightarrow (N \in D_f(v,w) \vee N \in B_f(v,w)))$.

iii) $A_f(X{\sim}X^*,X^*) = \phi \Rightarrow N \in B_f(X{\sim}X^*,X^*)$.

If no coalition is antidecisive for X~X* over X*, then the grand coalition is blocking for X~X* over X*.

Proof

Part i) is straightforward. To show ii), suppose not. Then $A_f(v,w) = \phi$,but $N \notin D_f(v,w)$ and $N \in B_f(v,w)$. Now $A_f(v,w) = \phi \Rightarrow N \notin A_f(v,w) \Rightarrow \forall R \in \mathcal{R}\ (vP_Nw \wedge \neg wP_fv)$, where the last implication obtains from IIA. $N \notin D_f(v,w)$ and $N \notin B_f(v,w)$ give, again by IIA, that $\forall R \in \mathcal{R}$ ,$(vP_Nw \wedge \neg vP_fw \wedge \neg vI_fw)$. But this violates completeness of social preference.

Finally, to show iii), recall that $D_f(X{\sim}X^*,X^*) = \phi$, and apply part ii) above. QED.

The reader may wish to compare the next proposition with proposition 6, which established similar results concerning coalitions decisive for X* over X~X*. Notice below that since the empty coalition is at least antidecisive for X~X* over X*, alternatives in X~X* can never be socially ranked strictly above those in X*.

PROPOSITION 9. (Every Coalition is Weakly Antidecisive for X~X* Over X*)

Let $C \subseteq N$, if $C \notin A_f(X{\sim}X^*,X^*)$; then $C \in WA_f(X{\sim}X^*,X^*)$.

Proof

If $C \notin A_f(X{\sim}X^*,X^*)$ then $\exists R \in \mathcal{R}$, $\exists x \in X^*$, $\exists y \in X{\sim}X^*$ $(yP_cx \wedge \neg xP_fy)$; please see figure 5 below.

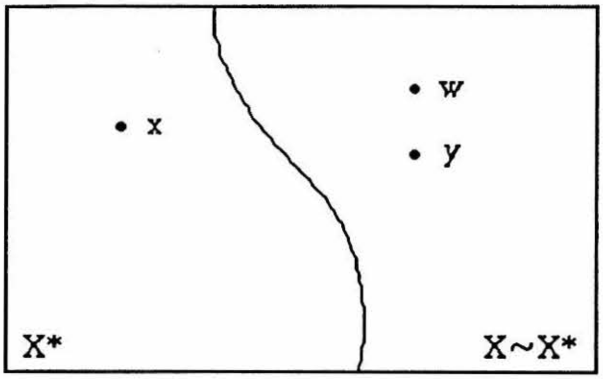


Figure 5

Thus either a) $xI_fy$, or b) $yP_fx$. Should a) obtain, there is nothing to prove; so suppose b) is the case. Construct $R' \in \mathcal{R}$ such that $R|_{\{x,y\}} = R'|_{\{x,y\}}$, and also:

| R | | R' | |
|---|---|---|---|
| $\underline{c}$ | $\underline{c}^c$ | $\underline{c}$ | $\underline{c}^c$ |
| y | [x,y] | w | w |
| x | | y | $[x,y]_R$ |
| | | x | |

Since $yP_fx$, IIA gives $yP'_fx$. Recall that Pareto holds on $X{\sim}X^*$ so that $wP'_fy$. By quasitransitivity then $wP'_fx$. Now $xP'_Nx$, so by IIA it follows that $\forall R'' \in R$, $(wP_N''x \Rightarrow xP_f''x)$. Thus $N \in D_f(w,x)$-- which is a contradiction since it has been previously shown that $w \in X{\sim}X^*$ and $x \in X^*$ require $D_f(w,x) = \phi$. QED.

REMARKS ($B_f(X \sim X^*, X^*)$)

The preceeding development will allow the following (albeit somewhat weak) characterization of $B_f(X \sim X^*, X^*)$. Since$B_f(X \sim X^*, X^*)$ is *not* in general a filter in the quasitransitive case, sharper results would not seem attainable.

PROPOSITION 10. (Characterization of $B_f(X \sim X^*, X^*)$)

Let $c \subseteq N$. If $c \in B_f(X \sim X^*, X^*)$, then $c^c \in WA_f(X \sim X^*, X^*)$ and $c^c \in SWD_f(X^*, X^* \sim X)$.

If coalition C is blocking for $X \sim X^*$ over $X^*$, then the coalition consisting of all those individuals *not* members of C is both weakly antidecisive for $X \sim X^*$ over $X^*$, and strictly weakly decisive for $X^*$ over $X \sim X^*$.

Proof

Recall that the following results have been shown previously:

i) If $c \in B_f(X \sim X^*, X^*)$, then $c^c \notin A_f(X \sim X^*, X^*)$; and

ii) If $c^c \notin A_f(X \sim X^*, X^*)$, then $c^c$ $WA_f(X \sim X^*, X^*)$;

$\therefore$ If $c \in B_f(X \sim X^*, X^*)$, then $c^c \in WA_f(X \sim X^*, X^*)$. (*)

And it has also been demonstrated that::

iii) If $c \in B_f(X \sim X^*, X^*)$, then $c^c \notin Df(X \sim X^*, X^*)$; and

iv) If $c^c \notin Df(X \sim X^*, X^*)$, then $c^c \in SWDf(X \sim X^*, X^*)$;

$\therefore$ If $c \in B_f(X \sim X^*, X^*)$, then $c \in SWDf(X \sim X^*, X^*)$. (**)

The two conclusions (*) and (**) above form the conjunction claimed. QED.

REMARKS ($A_f(X{\sim}X^*,X^*)$)

The following claims bear upon the characterization of $A_f(X{\sim}X^*,X^*)$. Note since $X{\sim}X^* \cap X^* = \phi$, there is no necessary contradiction if $\phi \in A_f(X{\sim}X^*,X^*)$.

PROPOSITION 11. (Closure Properties of $A_f(X{\sim}X^*,X^*)$)

Suppose $\#\{X{\sim}X^*\} \geq 2$ and $X^* \neq \phi$.

i) $A_f(X{\sim}X^*,X^*)$ is closed under intersection.

ii) $A_f(X{\sim}X^*,X^*)$ is closed w.r.t. complementation in N.

Proof

Notice that i) is trivial if $A_f(X{\sim}X^*,X^*) = \phi$, so suppose this set is not empty. Let $c,d \in A_f(X{\sim}X^*,X^*)$, $x \in X{\sim}X^*$, $y \in X^*$ and $R \in \mathcal{R}$ such that $xP_{(c \cap d)}y$; see figure 6. Then construct $R' \in \mathcal{R}$ in accordance with figure 6 and table 4 below:

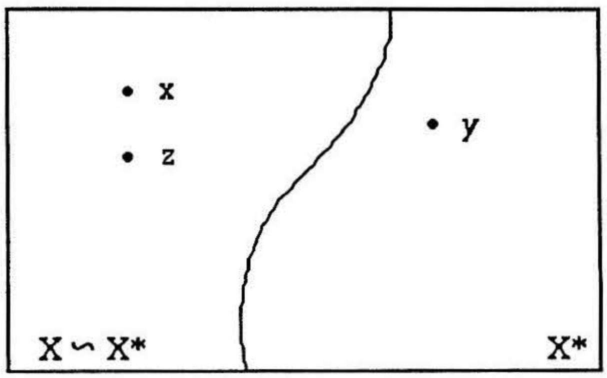


Figure 6

| R | | | R' | | |
|---|---|---|---|---|---|
| $\underline{c\sim d}$ | $\underline{c \cap d}$ | $\underline{d\sim c}$ | $\underline{c\sim d}$ | $\underline{c \cap d}$ | $\underline{d\sim c}$ |
| [x,y] | x | [x,y] | z | z | z |
| | y | | $[x,y]_R$ | x | $[x,y]_R$ |
| | | | | y | |

Table 4

Now $zP'_c y$ so $yP'_f z$ since $c \in A_f(X\sim X^*,X^*)$; and $zP'_N x$ so $zP_f x$ since PARETO[X~X*]. Quasitransitivity gives therefore $yP'_f x$, and by IIA $yP_f x$ obtains. Thus $c \cap d \in A_f(X\sim X^*,X^*)$, which completes the proof of i).

The proof of ii) is similarly trivial if $A_f(X\sim X^*,X^*)= \phi$; again suppose this set is nonempty. Referring to figure 6 above, let $c \in A_f(X\sim X^*,X^*)$, $x \in X\sim X^*$, $y \in X^*$, and $R \in \mathcal{R}$ such that $xP_x cy$. Then construct $R'' \in \mathcal{R}$ such that:

| R | | R" | |
|---|---|---|---|
| $\underline{c}$ | $\underline{c}^c$ | $\underline{c}$ | $\underline{c}^c$ |
| [x,y] | x | z | z |
| | y | $[x,y]_R$ | x |
| | | | y |

Table 5

Now $zP''_c y$ so $yP''_f z$ since $c \in A_f(X\sim X^*,X^*)$; $zP''_N x$ so $zP''_f x$ since PARETO[X~X*]. Quasitransitivity gives $yP''_f x$, and IIA allows $yP_f x$. Thus $c^c \in A_f(X\sim X^*,X^*)$, and so completes the proof of ii). QED.

PROPOSITION 12 (Triviality of $A_f(X\sim X^*,X^*)$)

$A_f(X\sim X^*,X^*) = \phi$ or $2^N$, when $\#\{X\sim X^*\} \geq 2$ and $X^* \neq \phi$.

Either no coalition is antidecisive for $X\sim X^*$ over $X^*$, or else every coalition is.

Proof

If $A_f(X\sim X^*,X^*) \neq \phi$, then closure under superset in N gives $N \in A_f(X\sim X^*,X^*)$. Closure under complementation gives $\phi \in A_f(X\sim X^*,X^*)$; and so closure under superset gives $A_f(X\sim X^*,X^*) = 2^N$. QED.

## VII. ALTERNATIVES IN X* vs. ALTERNATIVES IN X*

Having gathered the unbeatable alternatives into $X^*$, one can consider this set not only as a part of X but also as a collection of alternatives in its own right. This viewpoint leads naturally to the questions: Does the social welfare function f satisfy Pareto principle over the members of $X^*$, and what properties characterize $A_f$, $B_f$ and $D_f$ on $X^*$? To answer these questions the $*$ operator can be applied to $X^*$, thereby partitioning the set into $(X^*)^* = X^{2^*}$ and $X^*{\sim}X^{2^*}$. The analysis of the previous sections can then be applied; nevertheless, it will still remain to analyze the decisive, blocking, and antidecisive coalitions on $X^{2^*}$. However $X^{2^*}$ itself can also be partitioned by $*$ ... This section investigates the outcome of this process.

PROPOSITION 13. (Properties of the $*$ Operator)

i) $X \supseteq X^* \supseteq X^{2^*} \supseteq \ldots \supseteq X^{n^*} \supseteq \cdots$.

ii) $\mathrm{PARETO}_f[X^{n^*}] \Rightarrow x^{(n+1)^*} = \phi$.

iii) $\#\{X^{n^*}\} = 1 \Rightarrow x^{(n+1)^*} = \phi$.

iv) Suppose $X^{n^*} = X^{(n+1)^*}$, then $\forall m \in \mathbb{N}(m \geq n \Rightarrow X^{N^*} = X^{m^*})$.

v) $(X^{n*}{\sim}X^{(n+1)*}) \cap (X^{(n+1)*}{\sim}X^{(n+2)*}) = \phi$, $\forall\ n \geq 0$. See figure 7 below.

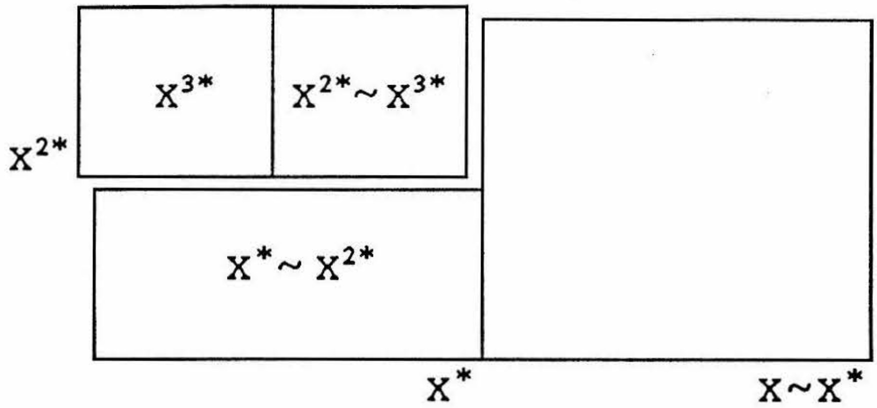


Figure 7

vi) Both $D_f(X^{n*},X^{n*}) \subsetneq D_f(X^{n*}\sim X^{(n+1)*},X^{n*}\sim X^{(n+1)*})$, and $D_f(X^{n*},X^{n*}) \subsetneq D_f(X^{(n+1)*},X^{(n+1)*})$, $\forall\ n \geq 0$.

vii) Suppose $\#\{X^{n*}\sim X^{(n+1)*}\} \geq 2$ and $X^{(n+1)*} \neq \phi$, then:

$$D_f(X^{n*}\sim X^{(n+1)*},X^{n*}\sim X^{(n+1)*}) \subseteq D_f(X^{(n+1)*},X^{n*}\sim X^{(n+1)*}).$$

Furthermore, if $X^{n*}\sim X^{(n+1)*} \neq \phi$ and $\exists w \in X^{(n+1)*}$ such that $D_f(X^{(n+1)*},w) \neq \phi$, it follows that:

$$D_f(X^{(n+1)*},x^{(n+1)*}) \subseteq D_f(X^{(n+1)*},X^{n*} \sim X^{(n+1)*}).$$

Proof

The proofs of parts i), ii), v), and vi) are straightforward. To show iii), let $x^{n*} = \{x\}$, and observe that not $\exists\ y \in x$ such that $D_f(y,x) = \phi$. In the case of iv), observe that $X^{(n+2)*} = (X^{(n+1)*})^* = (X^{n*})^* = X^{(n+1)*} = X^{n*}$, and proceed by induction. Finally, to show vii) recall it has been previously demonstrated that $D_f(X\sim X^*,X\sim X^*) \subseteq D_f(X^*,X\sim X^*)$, and again proceed by induction. QED.

REMARKS (Extending Proposition 6)

It seems likely that Proposition 6 can be generalized in the following manner. Suppose $X^{n*} \neq \phi$, and $\#\{X^{n*} \sim X^{(n+1)*}\} \geq 2$, then every coalition is either decisive or strictly weakly decisive for $X^{(n+1)*}$ over $X^{n*} \sim X^{(n+1)*}$. In this case it would be possible to assert that the alternatives in $X^{n*} \sim X^{(n+1)*}$ could never be socially ranked above those in $X^{(n+1)*}$, thus further characterizing the structure of imposition in the social order generated by f.

REMARKS (Iteration of the $*$-Operator on X)

Consider what happens as * is iterated on X; suppose nonempty $X^{n*}$ has been generated. The next action by * will purge $X^{n*}$ of those alternatives which are unbeatable only against some alternative(s) outside of $X^{n*}$, in $X^{(n-1)}$. The elements surviving the purge are exactly those which are unbeatable against some alternative which is itself a member of $X^{n*}$. This new elite collection forms $X^{(n+1)*}$.

PROPOSITION (PARETO Fails Forwards and Holds Backwards Over the $*$-Chain)

i) $D_f(X^{n*},X^{(n+1)*}) = \phi$, and $N \in D_f(X^{(n+1)*},X^{n*})$.

ii) Let i, $n \in \mathbb{N}$ and $i \geq 1$, then $N \in D_f(X^{(n+i)*},X^{n*})$.

Proof

The proof of part i) is exactly analogous to the case n= 0. The first equation holds since every member of $X^{(n+1)*}$ is unbeatable against some member $X^{n*}$, and so $X^{(n+1)*} \subseteq X^{n*}$. In case of the second statement, let $x \in X^{(n+1)*}$ and $y \in X^{n*}$; if $D_f(x,y) = \phi$, then $y \in X^{(n+1)*}$-- a contradiction. Since x and y were arbitrary, $N \in D^f(X^{(n+1)*},X^{n*})$.

In the case of ii), notice that it has been shown in part i) above that the claim holds for i = 1. Suppose as an inductive hypothesis that the result is true for i = m. Consider the consequences if the claim fails for i = m+1: $N \notin D_f(X^{(n+m+1)*},X^{n*}) \Rightarrow \exists x \in X^{(n+m+1)*}$, $\exists y \in X^{n*}$ $(N \notin D_f(x,y))$. Closure under superset of decisive families then requires $D_f(x,y) = \phi$. By definition, $X^{(n+m-1)*} \subseteq X^{(n+m)*}$ so that $x \in X^{(n+m)*}$, but then $N \in D_f(X^{(n+m)*},X^{n*})$ which requires $N \in D_f(x,y)$, a contradiction. QED.

REMARKS (Repeated Iteration of the $*$-Operator)

As the $*$-operator is repeatedly iterated, one of the following scenarios must eventually occur:

1) $\text{PARETO}_f[X^{n*}]$ for some n, so that $X^{(n+1)*} = X^{(n+2)*} = \ldots = \phi$, where $\#\{X^{n*}\} \geq 2$.

2) For some $n \geq 1$, $X^{n*} = \{x\}$ so that $X^{(n+1)*} = \phi$.

3) $X^{n*} = X^{(n+1)*} = \ldots$, where $X^{n*}$ is nonempty.

4) $X^{n*} \supsetneq X^{(n+1)*} \supsetneq X^{(n+2)*} \supsetneq \ldots$, where each containment is strict. In this case PARETO never holds, and every element of the chain is non-empty.

A moment's reflection serves to establish that these four cases above are in fact exhaustive. For consider, by definiton of the $*$-operator, the elements of the chain $X \supsetneq X^* \supsetneq X^{2*} \supsetneq X^{3*} \ldots$ decrease monotonically in cardinality. Thus the empty set must appear and terminate the chain as in cases **1)** and **2)**, two consecutive elements must be equal which implies that all succeeding elements are identical as in case **3)**, or else the chain must continue nontrivially forever as in case **4)**. The four different possibilities regarding the tail of $X \supsetneq X^* \supsetneq X^{2*} \supsetneq X^{3*} \ldots$ provide a framework for the analysis of the coalition structures on the elements $X^{n*}$.

PROPOSITION 14 (Trivial Decisive Coalitions if PARETO Fails Everywhere)

In cases **3)** and **4)**, PARETO never holds on any $X^{n*}$, $n \geq 0$; thus $D_f(X,X) = D_f(X^{n*},X^{n*}) = \phi$, for all $n \geq 0$.

Proof

The proof appears as part of the remarks below. QED.

REMARKS (Usefulness of the $*$ Construction)

Case **3)** bounds the explanatory power of the $*$ construction. For in this case X contains a set of elements each of which is unbeatable against some other alternative in the set. The most dramatic version of this scenario occurs if every member of $X^{n*}$ is unbeatable against every other member of $X^{n*}$. These appear to be quite complex situations, and no analysis of them has been undertaken here. Case **2)** is actually just the degenerate instance of **3)**. The reason that $X^{(n+1)*} = \phi$ instead of $X^{(n+1)} = X^{n*}$ as in **2)** is that unbeatability of an alternative, as defined here, requires unbeatablility against a distinct second alternative. Hence a singleton cannot contain any unbeatable alternatives.

Recall earlier that the conditions $\#\{X^*\} \geq 2$ and $\exists w \in X^* \ (D_f(X^*,w) \neq \phi)$ were employed to show $D_f(X^*,X^*) \subseteq D_f(X^*,X \sim X^*)$ where $X \sim X^* \neq \phi$. Trivially in case **3)** and **4)** $D_f(X^*,X^*) = \phi$, so the claim obtains regardless of the conditions. In case 2), where $X^{n*} = \{x\}$, if $n > 1$, then $D_f(X^*,X^*) = \phi$, and the claim again holds trivially; if $n = 1$, $X^* = \{x\}$ so $D_f(X^*,X^*) = 2^N \sim \phi$ and it cannot be asserted from the previous development that $2^N \sim \phi \subseteq D_f(X^*,X \sim X^*)$. In case **1)** there is a collection of two or more alternatives in $X^*$ such that Pareto holds among them. Then, just as it was shown $D_f(X^*,X^*) \subseteq D_f(X^*,X \sim X^*)$, it can be shown that $D_f(X^{n*},X^{n*}) \subseteq D_f(X^*,X \sim X^*)$. If $\#\{X^{n*}\} \geq 3$ then $D_f(X^{n*},X^{n*})$ is a filter; and the oligarchy composed of its generating set can not only dictate social preference on $X^{n*}$ but also between $X^*$ and $X \sim X^*$. In fact $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X)$, which will give that $D_f(X^{n*},X^{n*}) = D_f(X^{n*},X)$, as will be shown below.

PROPOSITION 15. (Decisive Coalitions if PARETO Eventually Holds)

Suppose that case **1)** holds; that is for some n, $\#\{X^{n*}\} \geq 2$ and PARETOf$[X^{n*}]$.

i) $D_f(X^{n*},X^{n*}) \subseteq D_f\,(X^{*},X{\sim}X^{*}) \subseteq D_f\,(X^{n*},X{\sim}X^{*})$.

ii) $D_f(X^{n*},X^{n*}) = D_f(X^{n*},X^{*})$.

iii) $D_f(X^{n*},X^{n*}) = D_f(X^{n*},X)$.

Proof

To show i), recall the first containment has been shown for n = 1; the extension to Pareto holding on $X^{n*}$ instead is transparent. The second containment holds since $X^{n*} \subseteq X^{*}$.

To show ii), notice that $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^{n*})$ since $X^{n*} \subseteq X^{n*}$. Next, $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^{n*})$ will be shown by induction. First it will be established that $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^{(n-1)*})$.

Choose any $C \in D_f(X^{n*},X^{n*})$ and let $x \in X^{n*}$, $z \in X^{(n-1)*}$. Trivially $C \in D_f(x,z)$ is $z \in X^{n*}$ so take $z \in X^{(n-1)*} \sim X^{n*}$. Let $R \in \mathcal{R}$ be given such that $xP_cz$, and construct $R' \in \mathcal{R}$ such that:

| R | | R' | |
|---|---|---|---|
| C | $C^c$ | C | $C^c$ |
| x | [x,z] | x | y |
| z | | y | $[x,z]_R$ |
| | | z | |

Table 6

Now $xP'_{c}y$ so that $xP'_{f}y$ since $C \in D_f(X^{n*},X^{n*})$ by assumption, and $yP'_{N}z$ so that $yP'_{f}z$ since $N \in D_f(X^{n*},X^{(n-1)*})$, as previously shown. Therefore, by quasitransitivity, $xP'_{f}z$ and by IIA, $C \in D_f(x,z)$. Arbitrariness then gives $C \in D_f(X^{n*},X^{(n-1)*})$. Suppose, as an inductive hypothesis, that $C \in D_f(X^{n*},X^{(n-m)*})$, for $m < n$. It is required to show that $C \in D_f(X^{n*},X^{(n-m-1)*})$. To this end let $x \in X^{n*}$, $v \in X^{(n-m-1)*}$, and $u \in X^{(n-m)*}$. If $v \in X^{(n-m)*}$, $C \in D_f(x,v)$ directly from the inductive hypothesis. Hence, suppose that $v \in X^{(n-m-1)*} \sim X^{(n-m)*}$. Let $\boldsymbol{R} \in \mathbb{R}$ be given s.t. $x\boldsymbol{P}_{c}v$ and construct $\boldsymbol{R}' \in \mathbb{R}$ such that:

| ***R*** | | ***R'*** | |
|---|---|---|---|
| $\underline{C}$ | $\underline{C}^c$ | $\underline{C}$ | $\underline{C}^c$ |
| x | [x,v] | x | u |
| v | | u | $[x,v]_{\boldsymbol{R}}$ |
| | | v | |

Table 7

Now $x\boldsymbol{P}'_{c}u$ gives that $x\boldsymbol{P}'_{f}u$ since $C \in D_f(X^{n*},X^{(n-m)*})$ by the inductive hypothesis. $u\boldsymbol{P}'_{N}v$ gives that $u\boldsymbol{P}'_{f}v$ since $N \in D_f(X^{(n-m)*},X^{(n-m-1)*})$, as demonstrated earlier. By quasitransitivity $x\boldsymbol{P}'_{f}v$ obtains, and IIA gives $x\boldsymbol{P}'_{f}v$ so that $C \in D_f(x,v)$; and again arbitrariness of x,v allows $C \in D_f(X^{n*},X^{(n-m-1)*})$. Since C was arbitrary, the conclusion can be universalized to give $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^{(n-m-1)*})$, where $m < n$. In particular, this is true for $m = n-2$, in which case $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^{*})$.

Finally to show iii), proceed as above but now choose $m = n-1$, and recall that $X = X^{0*}$. QED.

To complete the investigation of the properties of $X^*$, the families of antidecisive and blocking coalitions on $X^*$ will be partially characterized.

PROPOSITION 16. (Antidecisive and Blocking Coalitions if PARETO Eventually Holds)

Suppose that case **1)** holds; that is for some n, $\#\{X^{n*}\} \geq 2$ and $PARETO_f[X^{n*}]$. Then $A_f(X^*,X^*) = B_f(X^*,X^*) = \phi$.

Proof

Suppose $A_f(X^*,X^*) \neq \phi$, then $N \in A_f(X^*,X^*)$. Let $x,y \in X^{n*}$, then $\forall R \in R$, $xP_Ny \Rightarrow yP_fx$, which is a contradiction. However, $N \in D_f(X^{n*},X^{n*}) \Rightarrow N \in D_f(x,y) \Rightarrow (xP_Ny \Rightarrow xP_fy)$, a contradiction. The argument for the case of $B_f(X^*,X^*)$ is similar. QED.

# VIII. DIRECTIONS FOR FUTURE RESEARCH

It would appear that the next task in the analysis of the decisive coalition structure on $X^*$ is to investigate $B_f(X^{n*},X^{n*})$, and $A_f(X^*,X^*)$ in case **3**). It seems easiest to start with the simplest version of the case, namely $\forall x,y \in X^{n*}(D_f(x,y) = \phi)$. In this instance $\forall x,y \in X^{n*}(N \in A_f(x,y) \vee N \in B_f(x,y))$. Thus the set of ordered pairs $X^{n*} \mathbf{X} X^{n*}$ can be partitioned according as $N \in A_f(x,y)$ or $N \in B_f(x,y)$ -- both cannot occur. This will hardly be an equivalence relation in the quasitransitive case; nevertheless, some regularity properties do obtain. The A half of the relation is its own transitive closure, since $N \in A_f(w,x)$ and $A_f(x,z)$ imply $N \in A_f(w,z)$. Unfortunately the situation is not so straightforward regarding symmetry or transitivity. Wilson [1972] proceeds by analyzing the structure of a similar relation-- which is an equivalence relation; however, unless social welfare functions are transitive valued, the analysis of "Social Choice without Pareto Principle" is less penetrating.

# NOTATION and DEFINITIONS APPENDIX

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| $N$ | a finite set of individuals, $N \subseteq \mathbb{N}$. |
| $2^N$ | the set of all subsets of N. |
| $C$ | a coalition, $C \subseteq N$. |
| $C^c$ | the complement of C in N. |
| $X$ | a set of alternatives. |
| $R_i$ | individual i's complete, reflexive, transitive preference relation on X. |
| $I_i$, $P_i$ | indifference and strict preference for individual i; obtained from $R_i$. |
| $\mathbb{R}$ | set of all complete, reflexive, transitive profiles for N on X. |
| $R$ | some one profile; a profile is an N-tuple of preference relations. |
| $Q$ | set of all complete, reflexive, quasitransitive relations on X. Recall that a relation $R_i$ is said to be quasitransitive on X iff $\forall x,y,z \in X\ (xP_iy \wedge yP_iz \Rightarrow xP_iz)$. |
| $f$ | a social welfare function; $\mathbb{R} : \longrightarrow Q$; notice a social welfare function f must be defined over the *entire* domain $\mathbb{R}$. |
| $F$ | the family of all such social welfare functions as above. |
| $R_f$ | social preference corresponding to $R \in \mathbb{R}$ under f; $R_f = f(R)$. |
| $I_f$, $P_f$ | social indifference and social strict preference relations associated with $R_f$. |
| $P_c$, $R_c$, $I_c$ | $\forall R \in \mathbb{R}, \forall x,y \in X, \forall C \subseteq N\ \big(xP_cy \Leftrightarrow (\forall i \in \mathbb{N}\ (i \in C \Rightarrow xP_iy))\big)$; $R_c$ and $I_c$ are defined similarly. |

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| $D_f(U,V)$ | $\forall f \in F, \forall U,V \subseteq X, \forall C \subseteq N \left(C \in D_f(U,V) \Leftrightarrow (\forall R \in \mathcal{R}, \forall u \in U, \forall v \in V\ [uP_cv \Rightarrow uP_fv])\right)$. This is the set of coalitions decisive for U over V. |
| $A_f(U,V)$ = | the set of anti-decisive coalitions for U over V; defined exactly as $D_f(U,V)$ above, except the expression in square brackets should be replaced by $[uP_cv \Rightarrow vP_fu]$. |
| $B_f(U,V)$ | the set of blocking coalitions for U over V. Defined as above but with $[uP_cv \Rightarrow uI_fv]$. |
| $WD_f(U,V)$ | family of coalitions weakly decisive for U over V; defined as above except for making the replacement: $[uP_cv \Rightarrow \neg vP_fu]$. |
| $WA_f(U,V)$ | weakly antidecisive coalitions; as before, but with: $[uP_cv \Rightarrow \neg uP_fv]$. |
| $SWD_f(U,V)$ | the family of strictly weakly decisive coalitions for U over V; $SWD_f(U,V) = WD_f(U,V) \sim D_f(U,V) \sim B_f(U,V)$. |
| $SWA_f(U,V)$ | the family of strictly weakly anti-decisive coalitions for U over V; $SWA_f(U,V) = WA_f(U,V) \sim A_f(U,V) \sim B_f(U,V)$. |
| $D_f(x,y)$ | If $x,y \in X$, $D_f(x,y)$ will be written for for $D_f(\{x\},\{y\})$. The same convention will adopted for $A_f(x,y)$ and $B_f(x,y)$. |
| $X_f^*$ | $\{x \in X \mid \exists y\, X(x \neq y \wedge D_f(y,x) = \phi)\}$. Unless there is risk of confusion the subscript "f" will be omitted. This is the set of all alternatives in X which are unbeatable with respect to f against at least one (different than itself) alternative in X. |
| $X_f^{(n+1)*}$ | $\{x \in X_f^{n*} \mid \exists y \in X_f^{n*}\ (x \neq y \wedge D_f(y,x) = \phi)\}$. Again the subscript will be suppressed; note also that $X^{1*} = X^*$, and $X^{0*} = X$. |
| $PARETO_f[V]$ | $\forall f, \forall V \subseteq X \left(PARETO_f[V] \Leftrightarrow (\forall R \in \mathcal{R}, \forall x, y \in V(xP_Ny \Rightarrow xP_fy))\right)$. If there is no possibility confusion the subscript "f" will be omitted. |
| PARETO | A social welfare function $f \in F$ is said to satisfy PARETO iff $PARETO_f[X]$. |

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|---|---|
| IIA | A social welfare function $f \in F$ is said to satisfy IIA just in case: $\forall R,R' \in \mathbb{R}, \forall x,y \in X\ (R\mid_{\{x,y\}} = R'\mid_{\{x,y\}} \Rightarrow R_f\mid_{\{x,y\}} = R'_f\mid_{\{x,y\}})$. |
| QUASI | A social welfare function $f \in F$ is said to be QUASITRANSITIVE iff: $\forall R \in \mathbb{R}, \forall x,y,z \in X\ (xP_fy \wedge yP_fz \Rightarrow xP_fz)$. |
| UDOM | A social welfare function f is said to satisfy UDOM iff f is defined over all of $\mathbb{R}$; notice that all members of $\mathcal{F}$ satisfy this property. |
| {V} | the cardinality of the set V. |
| $\wedge$ | logical "and." |
| $\vee$ | logical "or." |
| $\neg$ | logical "not." |
| $\phi$ | empty set. |
| ^^ | exponentiation; a^^b = $a^b$. |
| ~ | set theoretic minus. |

# BIBLIOGRAPHY


1. K.J. ARROW, *Social Choice and Individual Values*. New York: Wiley, 1951.

2. B. HANSSON, "The Existence of Group Preferences." Working Paper No. 3, The Mattias Fremling Society, Lund, Sweden, 1972.

3. J.S. KELLEY, *Arrow Impossibility Theorems*. New York: Academic Press, 1978.

4. A.K. SEN, "Quasi-transitivity, Rational Choice and Collective Decisions." *Review of Economic Studies,* 36 (1969), 381-394.

5. ------, *Collective Choice and Social Welfare*. San Francisco: Holden-Day, 1970.

6. ------, "Social Choice Theory: A Re-Examination." *Econometrica,* 45 (1977), 53-89.

7. S. WILLARD, *General Topology*. Menlo Park: Addison-Wesley, 1970.

8. R.B. WILSON, "Social Choice Theory without the Pareto Principle." *Journal of Economic Theory,* 5 (1972), 478-486.