AERODYNAMIC FORCES ON A PROPELLER IN NON-STATIONARY MOTION

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ABSTRACT

The non-stationary lift and moment of an oscillating propeller blade element are determined. The solutions are obtained in the form of double definite integrals which are evaluated for one numerical example in Appendix A.

The three-dimensional nature of the problem is accounted for by determination of the induced velocity field due to an approximate vorticity distribution in the propeller wake. The corresponding blade element circulation is calculated by means of the classical Munk integral theorem. The two dimensional results for non-stationary lift and moment, expressed in terms of the circulation, are then used to obtain the results of this paper. Derivations of the lift and moment equations are included.

The resultant forces on the blade element are resolved into thrust and torque. Also, a qualitative discussion of the effects of compressibility is made based upon the Prandtl-Glauert transformation.

Finally, the results are compared with two dimensional theory and a discussion of the application to problems of flutter and forced oscillations of propellers is made. The discussion is illustrated by means of the numerical example.

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DEFINITION OF SYMBOLS

Numbers in parentheses directly after the symbols give the page on which the symbol is introduced.

a	(18)	Non-dimensional coordinate of the reference station.
$(AR)'$	(39)	Modified ratio of chord to radius.
B	(2)	Number of blades.
c	(2)	Propeller blade chord at reference station.
$G_{0,2}$	(55)	Factor containing circulation integrals.
G_1	(55)	Factor containing apparent mass integrals.
G_2	(69)	Factor containing moment integrals.
I_2 I_3 I_4	(39) (39) (40)	Circulation integrals due to: tip trailing vorticity, root trailing vorticity, shed vorticity.
I_2' I_3' I_4'	(53)	Additional apparent mass integrals.
I_2'' I_3'' I_4''	(68)	Moment integrals.
J	(17)	Advance ratio calculated at blade tip.
J_a	(15)	Advance ratio calculated at reference station.
L	(54)	Total lift(per unit length) on the blade element.
L_o	(9)	Quasi-steady lift, per unit length of blade.
L_1	(9)	Lift due to additional apparent mass, per unit length of blade.
L_2	(9)	Lift correction due to wake, per unit length of blade.

$L^{(T)}$	(54)	Lift on blade element, per unit length of blade, due to translational oscillations.
$L^{(R)}$	(55)	Lift on blade element, per unit length, due to rotational oscillations.
M	(68)	Total moment on blade element about midchord, per unit length of blade.
M_o	(9)	Quasi-steady moment, per unit length of blade.
M_1	(9)	Moment due to additional apparent mass, per unit length of blade.
M_2	(9)	Moment correction due to wake, per unit length of blade.
$M^{(T)}$	(69)	Moment on blade element, per unit length of blade, due to translational oscillations.
$M^{(R)}$	(69)	Moment on blade element, per unit length of blade, due to rotational oscillations.
P_j	(29)	Factor determining the phase of the circulation of the jth propeller blade relative to the other blades.
q_2	(18)	Induced velocity at a point on the blade element due to tip trailing vorticity.
q_3	(30)	Induced velocity at a point on the blade element due to root trailing vorticity.
q_4	(31)	Induced velocity at a point on the blade element due to shed vorticity.
Q	(57)	Resultant torque on the blade element per unit length of blade.
R	(2)	Propeller blade radius.
T	(57)	Resultant thrust of the blade element per unit length of blade.
V	(2)	Translational speed of the propeller.

v_t	(13)	Velocity of translatory oscillations of the blade element.
v_r	(13)	Velocity of rotational oscillations of the blade element.
W	(13)	Velocity resultant of rotational and translational speed of the propeller blade element.
w	(22)	Component of induced velocity perpendicular to the tangent plane of the helical surface at the reference section.
w_1	(35)	"Equivalent" downwash component due to blade element motion.
w_2	(22)	Downwash component due to tip trailing vorticity.
w_3	(30)	Downwash component due to root trailing vorticity.
w_4	(33)	Downwash component due to shed vorticity.
x	(13)	Coordinate along the blade element chord. Origin at midchord.
y, z	(14)	Secondary coordinate system to describe the blade element chord.
r, ϑ, ξ	(14)	Cylindrical coordinate system to describe the wake.
γ_1	(13)	Shed vorticity strength.
γ_2	(12)	Tip and root trailing vorticity strengths.
Γ	(13)	Total circulation per unit length of blade.
Γ_1	(40)	Blade element circulation due to blade motion, per unit length of blade.
Γ_2	(39)	Blade element circulation due to tip trailing vorticity, per unit length of blade.

Γ_3	(39)	Blade element circulation due to root trailing vorticity, per unit length of blade.
Γ_4	(40)	Blade element circulation due to shed vorticity, per unit length of blade.
Υ	(25)	Reduced frequency based upon blade radius.
θ	(13)	Related to coordinate along blade element chord by $x = \frac{c}{2} \cos \theta$.
Λ	(15)	Ratio of rotational to translational speed of propeller.
λ	(15)	Ratio of blade element oscillatory frequency to velocity of translation of the propeller.
ν	(55)	Reduced frequency based upon blade element chord.
ρ, δ	(17)	Geometrical variables in the Biot-Savart relation.
σ_j	(15)	Factor determining the helix associated with the jth propeller blade in the system of B blades.
τ	(25)	Non-dimensional coordinate in the translatory direction.
φ	(17)	Propeller advance angle at the blade tip.
φ_a	(23)	Propeller advance angle at the reference station.
Ω	(2)	Rotational speed of the propeller.
ω	(13)	Frequency of the blade element oscillation.

I. INTRODUCTION

In this paper a method is developed for determining the magnitude and phase of forces exerted upon a propeller operating in a field of non-uniform flow. One can easily suggest situations where such motions will occur.

For example, when a pusher propeller acts in the wake of the wing each propeller blade must pass twice through the boundary layer wake during each propeller revolution. The air in the boundary layer wake will be traveling at a velocity lower than that encountered by the propeller at its vertical position. In fact, there will be a steep gradient in the relative blade velocity as it passes through the position parallel to the wing. More important than the change in magnitude of the relative velocity is the fact that since its rotation speed is constant the blade undergoes a large change in angle of attack as it enters and leaves the regions of diminished translational velocity.

Tractor propellers are also subjected to the same type of motion. If the propeller is located near the leading edge of the wing, as it is in submerged installations, then it is operating in the portion of the flow which is distorted by the wing circulation. That is, when the wing is operating at positive C_L there is an upwash ahead of the wing to prepare the air for passage over the wing. This upwash diminishes rapidly with distance from the wing section in the vertical plane. Exactly in the vertical direction the propeller is, of course, unaffected and flow along the blade is determined by the forward speed, V, of the airplane, and the rotational velocity, Ω , of the propeller. Therefore, in this situation, the propeller also undergoes two complete cycles in angle of attack during each propeller revolution.

The effect on a stationary airfoil in a non-uniform flow field is approximately equivalent to that of an oscillating airfoil in a uniform flow field. This consideration simplifies somewhat the task of describing the motion analytically. Hence, if a solution can be obtained for a propeller blade performing harmonic bending and torsional oscillations then any motion can be solved through superposition by means of harmonic analysis.

Therefore, in this paper the following problem is posed. The given data includes the forward velocity of the airplane, V, and the rotational velocity of the propeller, Ω , assumed to be constant at all times. It is assumed that each propeller blade element is undergoing periodic translational oscillations normal to the chord and periodic rotational oscillations about the mid-chord point. Certain physical characteristics of the propeller are given: for example, the blade radius R, the blade chord c, and the number of blades B. However, for purposes of analysis these values may remain arbitrary and the solution will contain them in the parameters. With these facts in hand the problem is to determine the forces which are exerted upon the propeller from such a motion and to determine the phase difference between the forces and the periodic motions of the blade.

Throughout this discussion it will be assumed that a linearized theory is to be utilized. This assures the validity of the superposition principle and allows the entire analysis to be developed upon the basis that each propeller blade element is a thin, flat airfoil. After the essential results are obtained any effect of thickness and camber can be included by simply superimposing them upon the solution obtained for the flat airfoil. The linear theory also requires that the amplitude of blade element oscillations be small. However, since such a linearized theory will yield the predominant effects, this is not a serious restriction.

Many investigators have worked on problems associated with non-stationary airfoil motion. The most important contributors will be noted here. Efforts to date may be divided into two general categories, those dealing with the two dimensional problems and those with the three dimensional, or finite aspect ratio problems. There is the further subdivision between solutions for an incompressible and a compressible fluid.

Among the earliest contributors who established the fundamentals were Birnbaum[(1)], Wagner[(2)], Glauert[(3)], and Küssner[(4)].

In papers by von Karman[(5)], and Sears[(5,6)], general relations for lift and moment have been determined. These relations have been applied to specific two-dimensional problems such as a rigid airfoil passing through a vertical gust and forces upon an oscillating airfoil. The principal contribution, beyond the fundamental theory, is the presentation of the results in a vectorial graphical form which is extremely convenient and informative. This form of representation of non-stationary forces was also employed by Kassner and Fingado.

Theodorsen[7,8] derived the fundamental equations for lift and moment on an oscillating airfoil previous to the work of von Karman and Sears but upon a less physical basis. Of interest in Theodorsen's work is the correlation between the two dimensional incompressible theory and three dimensional experimental results as applied to the flutter phenomenon.

Biot[9] has developed a simple theory of thin airfoils by means of the acceleration potential which is also applicable to the case of non-stationary motion. In the reference this method is used to obtain the chordwise lift distribution over an airfoil which undergoes vertical translatory oscillations.

Biot[10,11] and his followers at Brown University have also surveyed the problem of two-dimensional non-stationary motion in a compressible fluid. These references are of especial value since they represent a digest of efforts upon this problem made in Europe during the war years. The references include an extensive bibliography a large part of which is either classified or unavailable at the present time. The problem, as dealt with in these papers, is concerned with thin airfoils of infinite aspect ratio in non-stationary motion which is either subsonic or supersonic. The integral equations which are obtained in reference (10) are solved in certain cases and the computations tabulated in reference (11).

When one surveys the work on three dimensional non-stationary motion the material available is scant and inconclusive. Küssner[(12)] has obtained an integral equation which he characterizes as the "most general integral equation of airfoil theory for small disturbances that can be used for computing the pressure jump for a given downwash". He reduces the integral equation to several known forms and gives methods for solving these equations. However, the reference does not contain an attempt to apply the theory to the case of an airfoil of finite span in a non-stationary motion. Inspection of the integral equation indicates that such a solution would be very laborious in that it would involve a large amount of additional analytical development and numerical computation.

Reissner[(13.14)] and Stevens[(14)] have developed a linear theory for the airfoil of finite span in non-stationary motion in an incompressible fluid. There is the restriction that the aspect ratio should not be too low and the authors offer the opinion that their results are valid within the limits of the lifting line theory for a non-oscillating wing. Results are obtained for spanwise lift distribution, moment, and hinge moment on a wing which is undergoing bending, torsion, and aileron deflection. In this method the evaluation of the three dimensional effect also depends upon the solution of an integral equation. The second paper, reference (14), contains numerical data necessary for this solution and the amount of labor represented by that paper is considerable.

Biot[(15)] and Boehnlien[(15)] have developed a theory which utilizes the Biot-Savart law connecting induced downwash with the distribution of vorticity in space. The method closely follows an outline laid down previously by Sears[(16)]. An approximate vorticity distribution is assumed which simplifies the analysis to a large extent. Integrals which are obtained are evaluated only approximately. However, not withstanding the approximations and simplifications in this reference, the results compare very well with the more rigourous but much more complicated method of Reissner and Stevens. The principal objection to the method of reference (15) is the lack of rigor. However, its basic simplicity and close connection with the physical aspects of the problem are favorable and upon these bases it was decided to extend it to the case of the propeller operating in an incompressible flow field in nonstationary motion. Therefore, the principal acknowledgement for the results that are to be obtained in this paper must be made to the authors of references (15) and (16).

A brief outline of the method to be used in solving this problem will now be indicated. Since this approach bears a close resemblance to the lifting line theory for finite wings in uniform flow the similarities will be noted.

To begin, this will be a "strip" theory. That is, a blade element will be considered as though it were a part of an infinite cylinder with certain corrections introduced to account for the three dimensional aspects of the problem. In the lifting line theory it is customary to account for the three dimensional effects by computing the induced downwash at the quarter chord point due to the three dimensional vorticity pattern and then to consider the angle of attack corrected by an amount $\tan^{-1}(w/V)$, w being the downwash and V the velocity of translation. The effect of change in magnitude in the velocity vector is neglected since by reason of linearization of the problem w is necessarily much smaller than V. The situation is similar in the problem at hand with the exception of changes which must be made to account for the fact that the motion is non-stationary. There will be a definite distribution of vorticity in the wake of the propeller. Each blade element can be represented by bound vorticity distributed along the chord. By the fundamental theorem of conservation of vorticity the strengths of the bound vorticity distribution and the wake vorticity can be related analytically. Then if one computes the downwash induced along the chord by the vorticity distributed in the wake this downwash is a function of the bound vorticity distribution.

The analogy with the lifting line theory now appears. An arbitrary blade element at which the downwash has been computed is isolated from the system and treated as a two dimensional problem. For such a two dimensional section the strength of the bound vorticity can be determined as an integral of the downwash distribution over the section. This is the classical Munk integral. So far only the induced downwash has been noted. However, it is clear that the blade element motion itself constitutes an "equivalent" downwash. Thus if the motion of a point on the blade chord has a normal velocity component, $\vec{u}$, this is equivalent to a downwash at that point whose magnitude and direction is given by $-\vec{u}$. Therefore, the total downwash at a point on the blade element chord is the sum of the induced downwash and the "equivalent" downwash at that point. It is the total downwash which is substituted into the Munk integral. Then when the integration is performed an equation is obtained which relates the total bound circulation at that blade element to the motion of the blade element. In particular, in the linear theory, if a periodic motion is assumed the bound vorticity will also be periodic and the relation obtained from the Munk integral will permit the evaluation of the magnitude of the vorticity and its phase with the blade element motion.

However, the problem is not completed as yet since it still remains to determine the lift and moment on the arbitrary blade element. From the two-dimensional results it is possible to write the necessary expressions in terms of the bound vorticity, or blade element circulation. This gives three components for both the lift and moment which are described as the

a) quasi-steady lift and moment, L_o and M_o,

b) lift and moment correction due to wake, L_2 and M_2,

c) and the lift and moment due to the additional apparent mass, L_1 and M_1.

Of these components only the lift and moment due to the wake depend explicitly upon the vorticity distribution in the propeller wake. It is apparent that the quasi-steady lift and moment are in phase with the blade element motion. The apparent mass effect depends upon the blade element acceleration.

With the lift and moment of an arbitrary blade element determined it is a simple matter to resolve these into thrust and torque. Then by considering the entire blade as comprised of a finite number of such blade elements the total thrust and torque are easily obtained by a simple summation.

The following sections contain the detailed calculations as outlined in the above discussion.

II. THE APPROXIMATE VORTICITY DISTRIBUTION

2.1 The propeller wake in non-steady motion. In the case of non-steady motion the circulation at each station along the blade will be a function of the time, t, and the blade station. To provide for the conservation of vorticity each propeller blade in such a flow will shed vorticity along its helical wake. Therefore, the distribution of shed vorticity will also be a function of the time and blade station. In addition, for other than light loading of the propeller there will be a contraction of the wake due to pressure variation within the wake. Since the shed vorticity lies within the wake, its motion relative to the propeller, in the direction of the axis of the propeller hub, is the same as that of the wake of the propeller. From momentum theory the forward velocity and the axial wake velocities are related by,

$$u = \frac{1}{2}(u_1 + V) \qquad (2.1)$$

where u is the velocity at the propeller disc and u_1 the velocity in the wake far behind the propeller. V is the translational speed of the propeller. In order to determine the forces upon the propeller blade it is necessary to calculate the induced velocities at the blade surface due to the vorticity distribution in the wake. It is evident that this will be very complicated if the solution is based upon the vorticity distribution described above. Application of the Biot-Savart law in such a case will involve integration in three variables r, t, and ξ, and determination of distances and angles in an extremely complicated distribution of vorticity. Clearly, the problem in this generality is excessively complicated and it will be necessary to assume a less general, but more tractable distribution of vorticity.

2.2 The assumed vorticity distribution. This hypothecated vorticity distribution will be based upon three principal assumptions.

a) The oscillations of the propeller blades from normal steady operation will be assumed small.

b) The bound vorticity due to the oscillatory motion will be assumed constant along the blades except at the root and tips where it must be zero.

c) All shed vorticity will be assumed to remain fixed in space.

The following inferences are to be drawn from the above assumptions. Assumption (a) implies linearization, hence the principle of superposition will hold. All of the steady forces on the blade can be neglected in the course of this analysis with the understanding that they can be superimposed at any convenient point in the calculations. Also this assumption will permit neglect of the contraction of the wake and the velocity of the wake will be the same far aft of the propeller as at the propeller disc and will have the value,

$$u = u_1 = V \tag{2.2}$$

Assumption (b) is a great simplification but does not violate the physical facts to a large degree since the principal gradient in the circulation with respect to the radius will always occur near the root and tips. This is analogous to the large gradient of circulation near wing tips.

Assumption (c) is commonly made in the two-dimensional theory of non-stationary motion. This assumption is justified so long as the strength of the individual shed vortices is so small as to have a negligible effect upon neighboring vortices.

Based upon the above assumptions one can define the following vorticity distribution of a propeller operating in non-stationary motion. (See figure 2.1).

a) Tip trailing vortices of strength $\gamma_2^{(j)}$ extending along a helical path from each blade tip.

b) A root trailing vortex extending along the hub center line having the combined strength $\sum_j \gamma_2^{(j)}$ of all the bound vortices at the roots of the blades.

c) A ladder of shed vorticity of strength $\gamma_1^{(j)}$ lying in the helical surfaces swept out by the individual propeller blades. The elements of this vorticity are parallel upon separation.to the trailing edges of the blades.

It is convenient at this point to restrict the motion of the individual blade elements. The effect of two periodic motions will be investigated. That is, assume that the blade element is undergoing a translatory oscillation with velocity given by,

$$v_t = A_t We^{i\omega t} \tag{2.3}$$

where v_t is the velocity normal to the chord line, positive downwards. Also, assume that the blade element is undergoing a rotational oscillation about the midchord point with velocity given by,

$$v_r = 2A_r We^{i\omega t} \cos \theta \tag{2.4}$$

W is the resultant of the airplane forward velocity and the propeller rotational velocity, and if x is the coordinate measured along the chord of the blade element with origin at the midchord the variable θ is related to x by $x = \frac{c}{2} \cos \theta$. A_t and A_r are constants which can be chosen so as to give any desired magnitude to v_t and v_r. Under the linearized theory, the bound vorticity will be of the same form and can be written,

$$\Gamma(t) = \Gamma_o e^{i\omega t} \tag{2.5}$$

Following the principle of conservation of vorticity, the wake will be a continuous field of vorticity and each of the elements will be a function of the variation of bound vorticity, or propeller blade circulation. The next step will be to determine the analytical expressions connecting the vorticity elements. But first a coordinate system must be introduced.

Suppose a propeller of B blades fixed in space from which a helical wake travels downstream with translational velocity V feet per second and rotational velocity Ω radians per second. To describe the wake a polar cylindrical coordinate system is used. ξ is the axial distance downstream measured from the trailing edge of the blade. ϑ is the angle between the horizontal and a line in the wake measured positive in the counter clockwise direction, hence opposite to the direction of rotation of a right handed propeller. r measures the radial distance from the ξ axis to a point in the wake. To measure points relative to the blade a rectangular cartesian system is used with origin at the trailing edge. y measures the distance along the blade chord in the plane of the ξ axis positive towards the leading edge z is the distance perpendicular to the horizontal plane being positive in the positive ϑ direction. (See figure 2.2). In these coordinate systems it is apparent that the equation of the helical wake will be,

$$\vartheta - \frac{\Omega \xi}{V} = 0 \text{ or } \pi \tag{2.6}$$

Equation (2.6) applies in the case of a two bladed propeller.

In general, for the jth blade in a system having B blades,

$$\vartheta - \Lambda\xi = \frac{2(j-1)\pi}{B} = \sigma_j \qquad \text{where} \qquad \Lambda = \frac{\Omega}{V} \tag{2.7}$$

As a result of the assumption that all shed and trailing vorticity remains fixed in space, the strength of the tip trailing vortices at any point corresponding to a coordinate ξ along the flight path is,

$$\gamma_2^{(j)} = \Gamma_o^{(j)} e^{i\omega(t - \xi/V)} = \Gamma^{(j)} e^{-i\lambda\xi} \tag{2.8}$$

Here again the superscript refers to the jth blade. And in equation (2.8),

$$\lambda = \omega/V \tag{2.9}$$

In addition, the shed vorticity, $\gamma_1^{(j)}$, is related to the trailing vorticity by,

$$\gamma_1^{(j)} = -\frac{1}{\sqrt{1 + 1/J_a^2}} \frac{d\gamma_2^{(j)}}{d\xi} \qquad \text{where} \qquad \frac{1}{J_a^2} = \Lambda^2 r^2 \tag{2.10}$$

Hence,

$$\gamma_1^{(j)} = \frac{i\lambda\gamma_2^{(j)}}{\sqrt{1 + 1/J_a^2}} = \frac{i\lambda\Gamma^{(j)} e^{-i\lambda\xi}}{\sqrt{1 + 1/J_a^2}} \tag{2.11}$$

Note that $\gamma_1^{(j)}$ is a vorticity strength per unit length.

Finally, the root trailing vortices have the strength,

$$\sum_{j=1}^{B} \gamma_2^{(j)} = \sum_{j=1}^{B} \Gamma^{(j)} e^{-i\lambda\xi} \tag{2.12}$$

The next step in the analysis involves the calculation of the downwash at any blade element induced by the above described distribution of vorticity.

III. CALCULATION OF INDUCED DOWNWASH AT THE PROPELLER BLADES.

3.1 The Biot-Savart Law. The fundamental law relating the induced velocity at a point with the distribution of vorticity in space which induces the motion is due to Biot and Savart. It is of the form,

$$dq = \frac{\Gamma}{4\pi\rho^2} \sin \delta \, d\ell \qquad (3.1)$$

where dq is the induced velocity at a point P in space due to a vortex line of strength Γ. The quantities in equation (3.1) are indicated in figure 3.1. $\overrightarrow{dq}$ is perpendicular to $\vec{\rho}$ and $\overrightarrow{dl}$. Equation (3.1) is derived upon the assumption of an incompressible fluid.

The computations are most easily carried out by considering each of the elements of the wake in turn and computing the downwash due to each separately.

3.2 Downwash due to the tip trailing vorticity. In using the Biot-Savart law for this calculation note that the element of vorticity is geometrically an element of a helical line. To a first order,

$$d\ell = \sqrt{(R \, d\vartheta)^2 + (d\xi)^2} \qquad (3.2)$$

where R is the radius of the propeller blade. Also, the advance angle of the propeller is given by,

$$\varphi = \tan^{-1} \frac{V}{\Omega R} = \tan^{-1} J \qquad (3.3)$$

so that the components of $\vec{dl}$ can also be obtained from figure 3.2 as $\vec{dl} \cos\varphi$ in the rotational direction and $\vec{dl} \sin\varphi$ in the translational direction. It then follows that,

$$\begin{aligned} d\ell \cos\varphi &= R\, d\vartheta \\ d\ell \sin\varphi &= d\xi \end{aligned} \tag{3.4}$$

But previously it has been stated that

$$\vartheta - \Lambda\xi = \sigma_j$$

so that,

$$d\vartheta = \Lambda d\xi$$

Therefore, in the Biot-Savart law it will always be possible to perform the integration with respect to ξ.

Sin δ and ρ in equation (3.1) are functions of ξ, R, ϑ, and the radial location of the arbitrarily chosen blade station which can be expressed non-dimensionally in terms of R. ϑ can be eliminated from these relations by using equation (2.7) so that sin δ and ρ can be expressed in terms of ξ alone and a constant, a, to signify the blade element under consideration.

Computation of the downwash due to the tip trailing vorticity is simplified if the effects of the rotational and translational components are considered separately. First the velocity induced at an arbitrary blade element due to the translational components of vorticity, $\gamma_{2t}^{(j)}$, will be computed. It is necessary to integrate equation (3.1) in the form,

$$(dq)_{2t}^{(j)} = \frac{\Gamma^{(j)} e^{-i\lambda\xi}}{4\pi(\rho_{2t}^{(j)})^2} (\sin\delta)_{2t}^{(j)} \frac{d\xi}{\sin\varphi} \tag{3.5}$$

The immediate problem is to determine $\rho_{2t}^{(j)}$ and $(\sin\delta)_{2t}^{(j)}$ where $\rho_{2t}^{(j)}$ is the distance between a point on the trailing vorticity line and a point on the arbitrary blade element and $\delta_{2t}^{(j)}$ is the angle between $\rho_{2t}^{(j)}$ and the translational component of trailing vorticity. These quantities are indicated in figure 3.3. From the figure,

$$[\rho_{2t}^{(1)}]^2 = [\xi + y]^2 + R^2[1 + a^2] - 2R^2 a \cos\vartheta$$
$$= R^2\left\{\left[\frac{\xi + y}{R}\right]^2 + [1 + a^2] - 2a\cos\vartheta\right\} \qquad (3.6)$$

The dimension $\rho_{2t}^{(1)}$ applies to the trailing vorticity of the particular blade element at which the downwash is to be computed. The analogous distance from the tip trailing vortex elements of the other blades is given by

$$[\rho_{2t}^{(j)}]^2 = R^2\left\{\left[\frac{\xi + y}{R}\right]^2 + [1 + a^2] - 2a\cos[\vartheta + \sigma_j]\right\} \qquad (3.7)$$

Note that equation (3.7) reduces to (3.6) when j = 1, hence includes that distance also. Also, from equation (2.7),

$$\vartheta = \Lambda\xi$$

when ϑ is measured to the wake of the first blade. Therefore,

$$\cos[\vartheta + \sigma_j] = \cos[\Lambda\xi + \sigma_j] \qquad (3.8)$$

Hence, the expression for $\rho_{2t}^{(j)}$ can be written in the final form,

$$[\rho_{2t}^{(j)}]^2 = R^2[(\frac{\xi + Y}{R})^2 + (1 + a^2) - 2a \cos(\Lambda\xi + \sigma_j)] \quad (3.9)$$

$(\sin \delta)_{2t}^{(j)}$ is also obtained from figure 3.3. In particular,

$$[\sin \delta]_{2t}^{(1)} = \frac{k_1^{(1)}}{\rho_{2t}^{(1)}} = \frac{R\sqrt{(1 + a^2) - 2a \cos\vartheta}}{\rho_{2t}^{(1)}}$$

$$[\sin \delta]_{2t}^{(2)} = \frac{k_1^{(2)}}{\rho_{2t}^{(2)}} = \frac{R\sqrt{(1 + a^2) + 2a \cos\vartheta}}{\rho_{2t}^{(2)}}$$

Or, in general,

$$[\sin \delta]_{2t}^{(j)} = \frac{k_1^{(j)}}{\rho_{2t}^{(j)}} = \frac{R\sqrt{(1 + a^2) - 2a \cos(\Lambda\xi + \sigma_j)}}{\rho_{2t}^{(j)}} \quad (3.10)$$

Before completing the expression for $[dq]_{2t}^{(j)}$ it is necessary to note that $\tan \varphi = \frac{V}{\Omega R}$, so that,

$$\sin \varphi = \frac{V}{\sqrt{V^2 + \Omega^2 R^2}} = \frac{1}{\sqrt{1 + 1/J^2}} \quad (3.11)$$

Then, after making the proper substitutions in equation (3.5),

$$[dq]_{2t}^{(j)} =$$

$$\frac{\Gamma^{(j)} e^{-i\lambda\xi}\sqrt{1 + 1/J^2}\sqrt{(1 + a^2) - 2a \cos(\Lambda\xi + \sigma_j)}\, d\xi}{4\pi R^2[(\frac{\xi + Y}{R})^2 + (1 + a^2) - 2a \cos(\Lambda\xi + \sigma_j)]^{3/2}} \quad (3.12)$$

$[\vec{dq}]_{2t}^{(j)}$ is normal to the plane of $\vec{k}^{(j)}$ and $\vec{\rho}_{2t}^{(j)}$ at $r = aR$. It can be resolved into components which are parallel to the blade chord, parallel to the propeller blade axis, and perpendicular to the blade chord. These components are denoted by u, v, and w respectively. It is necessary to determine which of these components contribute to the forces acting upon the propeller blade element.

It is clear that the component v can create no forces upon the blade if boundary layer effects are neglected. Hence, the component v need not be considered.

To ascertain the effects of the induced velocities u and v one can consider the quasi-steady lift on a blade element subjected to these velocities superimposed upon a resultant velocity of translation W. If the blade element is oscillating about an angle of attack α, the quasi-steady lift per unit length on the blade element is,

$$L_o = a_\infty \alpha \rho \frac{(W + u)^2}{2} c - \frac{a_\infty w}{W} \frac{\rho W^2}{2} c$$

Then, neglecting the component of lift which is due to the steady motion alone one obtains,

$$L_o = \frac{a_\infty \rho c W}{2}(2\alpha u + \frac{\alpha u^2}{W} - w)$$

The lift determined by the term $\frac{\alpha u^2}{2}$ is negligible and one then has,

$$L_o = -\frac{a_\infty \rho c W w}{2}(1 - \frac{2\alpha u}{w})$$

Under normal operation the angle of attack α is of the order of 0.1 radians. Hence, if the lift due to the velocity component u is to be negligible it must be true that

$$u/w << 5$$

This condition is obviously not satisfied if the forward velocity V of the propeller blade element is much less than the rotational speed Ωr. However, under design operating conditions, V and Ωr are generally of the same order when r = R and V is correspondingly greater than Ωr over the inner portions of the blade. The curvature of the vortex wake is such that the direction of the induced downwash at the blade element is determined by the vortex elements which lie near the trailing edge of the blade element and accordingly are nearly in the plane of the blade element chord.

In this analysis, therefore, only the effect of the downwash component w will be considered. It must be borne in mind that these results will, accordingly, not apply to cases when $V << \Omega R$.

It will be necessary to determine the component of $\overrightarrow{[dq]}_{2t}^{(j)}$ which is perpendicular to the helical surface at y, r = a, and $\vartheta = 0$. Precisely, ϑ is a function of y on the blade. However, for blades with radius of the order of six feet and advance angle at the tip of forty-five degrees, the maximum value of ϑ on the blade is approximately three degrees for a tip chord of one foot. Thus, for the purpose of determining the downwash component of induced velocity it is sufficiently accurate to use $\vartheta = 0$ in this calculation. This component of downwash is denoted by $\overrightarrow{[w]}_{2t}^{(j)}$ and is taken positive in the negative z direction. The direction cosines of the unit vector $\overrightarrow{[w]}_{2t}^{(j)} / \left|\overrightarrow{[w]}_{2t}^{(j)}\right|$ are determined as follows. From figure 3.4

it is seen that $\overrightarrow{[w]}_{2t}^{(j)}$ makes the angle $\tan \varphi_a = \frac{V}{\Omega r}$ with the ξ axis direction, and $[\frac{\pi}{2} - \varphi_a]$ with the negative z axis. Denoting l, m, n, as the direction cosines in the r, z, ξ directions respectively, by inspection of the figure it can be seen that they have the values,

$$\ell = 0$$
$$m = - \sin \varphi_a$$
$$n = \cos \varphi_a$$

In addition, the direction cosines of $\overrightarrow{[dq]}_{2t}^{(j)}$ must be determined. This vector is perpendicular to the plane of $\vec{k}^{(j)}$ and $\vec{\rho}_{2t}^{(j)}$ as previously stated. Now refer these lines to an r', z', ξ' coordinate system and write the equation of the plane. From figure 3.5 one can write the coordinates of three points of this plane as,

(1):	0	0	0
(2):	0	0	ξ
(3):	$(\xi + y)$	$R[\cos\vartheta - a]$	$R \sin\vartheta$

Then the equation of the plane is given by,

$$\begin{vmatrix} r' & z' & \xi' & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (\xi + y) & 1 \\ R[\cos\vartheta - a] & R \sin\vartheta & (\xi + y) & 1 \end{vmatrix} = 0 \tag{3.13}$$

Now equation (3.13) is of the form,

$$Ar' + Bz' + C\xi' = 0 \tag{3.14}$$

where,

$$A = - R(\xi + y) \sin\vartheta$$
$$B = R(\xi + y)\,[\cos\vartheta - a]$$
$$C = 0$$

Therefore, the direction cosines of the normal to the plane are given by,

$$\ell^{(1)} = \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}} = \mp \frac{\sin \vartheta}{\sqrt{[1 + a^2] - 2a \cos \vartheta}}$$

$$m^{(1)} = \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}} = \pm \frac{[\cos \vartheta - a]}{\sqrt{[1 + a^2] - 2a \cos \vartheta}} \qquad (3.15)$$

$$n^{(1)} = 0$$

This result can be generalized to give the direction cosines of $\overrightarrow{dq}_{2t}^{(j)}$ by merely substituting $[\Lambda\xi + \sigma_j]$ for ϑ, so that in general the direction cosines are,

$$\ell^{(j)} = \frac{\sin[\Lambda\xi + \sigma_j]}{\sqrt{[1 + a^2] - 2a \cos[\Lambda\xi + \sigma_j]}}$$

$$m^{(j)} = \frac{a - \cos[\Lambda\xi + \sigma_j]}{\sqrt{[1 + a^2] - 2a \cos[\Lambda\xi + \sigma_j]}} \qquad (3.16)$$

$$n^{(j)} = 0$$

By inspection of the case when j = 1 it is seen that the positive sign is to be taken for $\ell^{(j)}$ and the negative sign for $m^{(j)}$. It now follows that the desired component of downwash in the $\overrightarrow{dw}_{2t}^{(j)}$ direction is given by

$$\frac{\overrightarrow{dw}_{2t}^{(j)}}{\left|\overrightarrow{dw}_{2t}^{(j)}\right|} \cdot \overrightarrow{dq}_{2t}^{(j)}$$

or,

$$dw_{2t}^{(j)} =$$

$$\frac{\Gamma^{(j)} e^{-i\lambda\xi}}{4\pi R^2} \sqrt{\frac{1 + 1/J^2}{1 + 1/J_a^2}} \; \frac{\left\{\cos[\Lambda\xi + \sigma_j] - a\right\} \, d\xi}{\left\{[(\xi + y)/R]^2 + [1 + a^2] - 2a \cos[\Lambda\xi + \sigma_j]\right\}^{3/2}} \qquad (3.17)$$

If it is assumed that the motion has been going forward for a sufficient length of time, the wake can be considered as extending to $\xi = \infty$. Then,

$$\overset{(j)}{w}_{2t} = \sum_{j=1}^{B} \frac{\Gamma^{(j)}}{4\pi R} \sqrt{\frac{1 + 1/J^2}{1 + 1/J_a^2}} \int_0^\infty \frac{e^{-i\Upsilon\tau} \left[\cos(\tau/J + \sigma_j) - a \right] d\tau}{\left[(\tau + y/R)^2 + (1 + a^2) - 2a \cos(\tau/J + \sigma_j) \right]^{3/2}} \tag{3.18}$$

where,

$$(\tau = \xi/R) \qquad (\Upsilon = \lambda R) \qquad (J_a = \frac{V}{\Omega r}) \tag{3.19}$$

The next step in the computation is to determine $\vec{w}_{2r}^{(j)}$, the downwash due to the rotational component of the tip trailing vorticity. The appropriate expression for $dq_{2r}^{(j)}$ is,

$$dq_{2r}^{(j)} = \frac{\Gamma^{(j)} e^{-i\lambda\xi} (\sin\delta)_{2r}^{(j)} R\, d\vartheta}{4\pi\, (\rho_{2r}^{(j)})^2 \cos\varphi} \tag{3.20}$$

$\Gamma^{(j)} e^{-i\lambda\xi}$ is the rotational component Υ_2 by equation (2.8). Then from equation (3.4) one can write,

$$dq_{2r}^{(j)} = \frac{\Gamma^{(j)} e^{-i\lambda\xi} \sqrt{1 + 1/J^2}\, (\sin\delta)_{2r}^{(j)}\, d\xi}{4\pi\, (\rho_{2r}^{(j)})^2} \tag{3.21}$$

Values of $(\sin\delta)_{2r}^{(j)}$ and $\rho_{2r}^{(j)}$ are determined from figure 3.6. It is apparent that,

$$[\rho_{2r}^{(j)}]^2 = [\rho_{2t}^{(j)}]^2 = R^2 \left[\left(\frac{\xi + y}{R}\right)^2 + (1 + a^2) - 2a \cos(\Lambda\xi + \sigma_j) \right] \tag{3.22}$$

Also,

$$[h_{2r}^{(j)}]^2 = (\xi + y)^2 + [k_2^{(j)}]^2 = R^2 \left[\left(\frac{\xi + y}{R}\right)^2 + \left\{1 - a \cos(\Lambda\xi + \sigma_j)\right\}^2 \right] \tag{3.23}$$

Then,

$$[\sin \delta]_{2r}^{(j)} = h_{2r}^{(j)} / \rho_{2r}^{(j)}$$

$$= \sqrt{\frac{\left(\frac{\xi + y}{R}\right)^2 + [\,1 - a\ \cos(\Lambda\xi + \sigma_j\,)\,]^2}{\left(\frac{\xi + y}{R}\right)^2 + (1 + a^2) - 2a\ \cos(\Lambda\xi + \sigma_j\,)}} \tag{3.24}$$

And, upon substitution of (3.24) and (3.22) into (3.21),

$$dq_{2r}^{(j)} = \frac{\Gamma^{(j)} e^{-i\lambda\xi}\sqrt{1 + 1/J^2}\sqrt{\left(\frac{\xi + y}{R}\right)^2 + [1 - a\ \cos(\Lambda\xi + \sigma_j)]^2}\ d\xi}{4\pi R^2\ [\left(\frac{\xi + y}{R}\right)^2 + (1 + a^2) - 2a\ \cos(\Lambda\xi + \sigma_j)\,]^{3/2}} \tag{3.25}$$

As before it is necessary to determine the component of $\vec{dq}_{2r}^{(j)}$ in the direction of the unit $\vec{dw}$ vector. $\vec{dq}_{2r}^{(j)}$ is normal to the plane of $\rho_{2r}^{(j)}$ and $h_2^{(j)}$. Thus one can write the general equation of this plane and reduce it to the normal form as before. In the case where j = 1 the coordinates of the points 1, 2, and 3, are,

$$\begin{array}{llll} (1): & 0 & 0 & 0 \\ (2): & R(1/\cos\vartheta - a) & 0 & (\xi + y) \\ (3): & R(\cos\vartheta - a) & R\ \sin\vartheta & (\xi + y) \end{array} \tag{3.26}$$

Then the equation of the plane is given by,

$$\begin{vmatrix} r' & z' & \xi' & 1 \\ 0 & 0 & 0 & 1 \\ R(1/\cos\vartheta - a) & 0 & (\xi + y) & 1 \\ R(\cos\vartheta - a) & R\ \sin\vartheta & (\xi + y) & 1 \end{vmatrix} = 0 \tag{3.27}$$

Hence,

$$\begin{aligned} A &= -R(\xi + y)\ \sin\vartheta \\ B &= R(\xi + y)\ \sin^2\vartheta/\cos\vartheta \\ C &= R^2(1 - a\ \cos\vartheta)\ \sin\vartheta/\cos\vartheta \end{aligned} \tag{3.28}$$

and,

$$\pm\sqrt{A^2 + B^2 + C^2} = \pm R^2 \frac{\sin\vartheta}{\cos\vartheta}\sqrt{\left(\frac{\xi + y}{R}\right)^2 + (1 - a\cos\vartheta)^2} \quad (3.29)$$

Then from equations (3.28) and (3.29) the direction cosines of the normal to the plane are given by,

$$\begin{aligned} \ell_{2r}^{(1)} &= \mp \frac{[(\xi + y)/R]\cos\vartheta}{\sqrt{[(\xi + y)/R]^2 + (1 - a\cos\vartheta)^2}} \\ m_{2r}^{(1)} &= \pm \frac{[(\xi + y)/R]\sin\vartheta}{\sqrt{[(\xi + y)/R]^2 + (1 - a\cos\vartheta)^2}} \\ n_{2r}^{(1)} &= \pm \frac{1 - a\cos\vartheta}{\sqrt{[(\xi + y)/R]^2 + (1 - a\cos\vartheta)^2}} \end{aligned} \quad (3.30)$$

Or, in general, for the jth blade in a system of B blades,

$$\begin{aligned} \ell_{2r}^{(j)} &= - \frac{[(\xi + y)/R]\cos(\Lambda\xi + \sigma_j)}{\sqrt{[(\xi + y)/R]^2 + [1 - a\cos(\Lambda\xi + \sigma_j)]^2}} \\ m_{2r}^{(j)} &= - \frac{[(\xi + y)/R]\sin(\Lambda\xi + \sigma_j)}{\sqrt{[(\xi + y)/R]^2 + [1 - a\cos(\Lambda\xi + \sigma_j)]^2}} \\ n_{2r}^{(j)} &= + \frac{1 - a\cos(\Lambda\xi + \sigma_j)}{\sqrt{[(\xi + y)/R]^2 + [1 - a\cos(\Lambda\xi + \sigma_j)]^2}} \end{aligned} \quad (3.31)$$

By taking the dot product of $\overrightarrow{dq}_{2r}^{(j)}$ and the unit $\overrightarrow{dw}$ vector one obtains,

$$\frac{dw_{2r}^{(j)}}{d\xi} = \frac{\Gamma^{(j)}\sqrt{1 + 1/J^2}\, e^{-i\lambda\xi}}{4\pi R^2} \times \frac{[(\xi + y)/R]\sin\varphi_a \sin(\Lambda\xi + \sigma_j) + [1 - a\cos(\Lambda\xi + \sigma_j)]\cos\varphi_a}{\{[(\xi + y)/R]^2 + (1 + a^2) - 2a\cos(\Lambda\xi + \sigma_j)\}^{3/2}}$$

And, introducing the proper limits,

$$w_{2r}^{(j)} = \sum_{j=1}^{B} \frac{\Gamma^{(j)}}{4\pi R}\sqrt{\frac{1 + 1/J^2}{1 + 1/J_a^2}} \times \int_0^\infty \frac{e^{-i\gamma\tau}\{(\tau + y/R)\sin(\tau/J + \sigma_j) + 1/J_a[1 - a\cos(\tau/J + \sigma_j)]\}}{[(\tau + y/R)^2 + (1 + a^2) - 2a\cos(\tau/J + \sigma_j)]^{3/2}}\, d\tau \quad (3.32)$$

Now, one can superimpose equations (3.18) and (3.32) to obtain the expression for the downwash due to trailing vorticity in the form,

$$w_2^{(j)} = \sum_{j=1}^{B} \frac{\Gamma^{(j)}}{4\pi R} \sqrt{\frac{1 + 1/J^2}{1 + 1/J_a^2}}$$

$$\times \int_0^{\infty} \frac{e^{-i\gamma\tau} \left\{ (\tau + y/R)\sin(\tau/J + \sigma_j) + (1 - a/J_a)\cos(\tau/J + \sigma_j) + \left(\frac{1}{J_a} - a\right) \right\}}{\left[(\tau + y/R)^2 + (1 + a^2) - 2a \cos(\tau/J + \sigma_j) \right]^{3/2}} d\tau \qquad (3.33)$$

The significance of the superscript j should be explained. As mentioned previously, it denotes the jth blade in a system of B blades. $\Gamma^{(j)}$ then infers that the bound vorticity of the several blades is distinct. To obtain the total downwash due to the trailing vorticity one would have to sum the right hand side of equation (3.33) over j. The specific form for the bound vorticity has been assumed to be,

$$\Gamma^{(j)} = \Gamma_o^{(j)} e^{i\omega t}$$

and it will become clear later that this analysis reduces to a homogeneous equation in Γ_o. However, if different values of Γ for each blade are permitted, the problem will consist of a set of simultaneous equations equal in number to the number of blades.

The essential results can be obtained if one assumes that the magnitude of the circulation, Γ_o, of the several blades is the same but that the phases of the circulation differ in some reasonable fashion. One such assumption is that as each of the blades passes through a fixed value of the angular coordinate their circulations at that point are the same. And a more tractable problem is obtained if one assumes that the variation of circulation with time, hence with angular coordinate, is determined by specifying the value of circulation for one blade and imposing that every other blade have the circulation which the one particular blade would have at the positions occupied by the remaining blades.

Thus, in a system of B blades the j^{th} blade will have the circulation

$$\Gamma^{(j)} = \Gamma_0\, e^{i\omega[t + 2(j-1)\pi/B\Omega]} = \Gamma P_j$$

where

$$P_j = e^{i\omega 2(j-1)\pi/B\Omega} = e^{i\lambda\sigma_j/\Lambda}$$

and

$$P_1 = 1$$

The above phase relationship is the most reasonable assumption for application to the problem of cyclic pitch change (see discussion on forced vibrations). However, to treat problems of flutter it may be necessary to consider phase relations of a different nature. For example, if a two bladed propeller is susceptible to flutter in the antisymmetric mode it would be logical to assume that the circulations of the two blades were equal in magnitude but of opposite sign. Or, for flutter in the symmetric mode the circulation of the two blades could be assumed equal both in magnitude and sign. Only the phase relationship given above will be treated in detail here. Then equation (3.33) becomes,

$$w_2 = \frac{\Gamma}{4\pi R}\sqrt{\frac{1 + 1/J^2}{1 + 1/J_a^2}} \sum_{j=1}^{B} P_j \tag{3.34}$$

$$X \int_0^\infty \frac{e^{-i\gamma\tau}\left\{(\tau + \frac{y}{R})\sin(\frac{\tau}{J} + \sigma_j) + (1 - \frac{a}{Ja})\cos(\frac{\tau}{J} + \sigma_j) + (\frac{1}{Ja} - a)\right\}d\tau}{[(\tau + \frac{y}{R})^2 + (1 + a^2) - 2a\cos(\frac{\tau}{J} + \sigma_j)]^{3/2}}$$

3.3 Downwash due to root trailing vorticity. From the assumption of equal bound vorticity for all blades, the simple result is obtained that the strength of the root trailing vorticity is given by $\Gamma e^{-i\lambda\xi}\sum_{j=1}^{B} P_j$. The vortex element lies along a straight line in this case and the downwash is given by,

$$dq_3 = \frac{\Gamma e^{-i\lambda\xi}\sin\delta_3}{4\pi\rho_3^2} \sum_{j=1}^{B} P_j \, d\xi \tag{3.35}$$

The quantities in equation (3.35) are obtained from figure 3.8 as,

$$\rho_3^2 = (\xi + y)^2 + a^2R^2$$

$$\sin\delta_3 = \frac{a}{\sqrt{\left(\frac{\xi + y}{R}\right)^2 + a^2}}$$

so that,

$$q_3 = \frac{\Gamma a}{4\pi R} \sum_{j=1}^{B} P_j \int_0^\infty \frac{e^{-i\gamma\tau}\, d\tau}{[(\tau + y/R)^2 + a^2]^{3/2}} \tag{3.36}$$

Now $\overrightarrow{q_3}$ is in the negative z direction. Therefore when the dot product is taken with the unit $\overrightarrow{dw}$ vector,

$$w_3 = \frac{\Gamma a}{4\pi R\sqrt{1 + 1/J_a^2}} \sum_{j=1}^{B} P_j \int_0^\infty \frac{e^{-i\gamma\tau}\, d\tau}{[(\tau + y/R)^2 + a^2]^{3/2}} \tag{3.37}$$

3.4 Downwash due to the shed vorticity. The strength of the shed vorticity is related to the bound vorticity by equation (2.11) as,

$$\gamma_1^{(j)} = \frac{i\lambda \Gamma e^{-i\lambda\xi} P_j}{\sqrt{1 + 1/J_a^2}} \tag{3.38}$$

It is necessary to calculate the downwash at a point on the blade due to an elemental strip of width dl, where dl is taken in the helical surface; then to integrate the effect of all such strips throughout the entire wake. From equations (3.2) and (3.4),

$$d\ell = \sqrt{1 + 1/J_a^2}\ d\xi \tag{3.39}$$

Therefore,

$$\gamma_1^{(j)}\ d\ell = i\lambda \Gamma P_j\ e^{-i\lambda\xi}\ d\xi \tag{3.40}$$

The downwash due to shed vorticity can now be derived with the aid of figure 3.9 and the simplified Biot-Savart law for a straight vortex element. Taking first the case when j = 1,

$$dq_4^{(1)} = \frac{i\lambda \Gamma P_1 e^{-i\lambda\xi}\ (\cos \nu_1 - \cos \nu_2)\ d\xi}{4\pi h_4^{(1)}} \tag{3.41}$$

Quantities in equation (3.41) are indicated in the figure. Then

$$[h_4^{(1)}]^2 = (\xi + y)^2 + R^2 a^2 \sin^2 \vartheta \tag{3.42}$$

The length of line (1) is,

$$\sqrt{(\xi + y)^2 + R^2 a^2}$$

Hence,

$$\cos \nu_1 = \frac{a \cos \vartheta}{\sqrt{[(\xi + y)/R]^2 + a^2}} \tag{3.43}$$

And the length of line (2) is,

$$\sqrt{[h_4^{(1)}]^2 + R^2(1 - a \cos \vartheta)^2}$$

so that,

$$\cos \nu_2 = \frac{a \cos \vartheta - 1}{\sqrt{[(\xi + y)/R]^2 + (1 + a^2) - 2a \cos \vartheta}} \tag{3.44}$$

Equations (3.44), (3.43), and (3.42) are now substituted into (3.41) and immediately written in terms of the jth blade, thus,

$$dq_4^{(j)} = \frac{i\lambda\Gamma e^{-i\lambda\xi} p_j}{4\pi R^2 \sqrt{[(\xi + y)/R]^2 + a^2 \sin^2(\Lambda\xi + \sigma_j)}} \left\{ \frac{a \cos(\Lambda\xi + \sigma_j)}{\sqrt{[(\xi + y)/R]^2 + a^2}} + \frac{1 - a \cos(\Lambda\xi + \sigma_j)}{\sqrt{[(\xi + y)/R]^2 + (1 + a^2) - 2a \cos(\Lambda\xi + \sigma_j)}} \right\} d\xi \tag{3.45}$$

Again the component of $\overrightarrow{dq}$ in the direction of $\overrightarrow{dw}$ must be obtained. The coordinates of points 1, 2, and 3 are,

$$\begin{array}{llll} (1): & 0 & 0 & 0 \\ (2): & -Ra & 0 & (\xi + y) \\ (3): & R(\cos \vartheta - a) & R \sin \vartheta & (\xi + y) \end{array} \tag{3.46}$$

Hence the equation of the plane is,

$$\begin{vmatrix} r' & z' & \xi' & 1 \\ 0 & 0 & 0 & 1 \\ -Ra & 0 & (\xi + y) & 1 \\ R(\cos \vartheta - a) & R \sin \vartheta & (\xi + y) & 1 \end{vmatrix} = 0 \tag{3.47}$$

so that when the equation of the plane is written in the form,

$$Ar' + Bz' + C\xi' = 0 \tag{3.48}$$

the coefficients are,

$$\begin{aligned} A &= -R(\xi + y)\sin\vartheta \\ B &= R(\xi + y)\cos\vartheta \\ C &= -R^2 a\sin\vartheta \end{aligned} \tag{3.49}$$

The direction cosines of the normal to the plane are accordingly,

$$\begin{aligned} \ell_4^{(j)} &= \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}} = +\frac{\frac{\xi + y}{R}\sin(\Lambda\xi + \sigma_j)}{\sqrt{\left(\frac{\xi + y}{R}\right)^2 + a^2\sin^2(\Lambda\xi + \sigma_j)}} \\ m_4^{(j)} &= \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}} = -\frac{\frac{\xi + y}{R}\cos(\Lambda\xi + \sigma_j)}{\sqrt{\left(\frac{\xi + y}{R}\right)^2 + a^2\sin^2(\Lambda\xi + \sigma_j)}} \\ n_4^{(j)} &= \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}} = +\frac{a\sin(\Lambda\xi + \sigma_j)}{\sqrt{\left(\frac{\xi + y}{R}\right)^2 + a^2\sin^2(\Lambda\xi + \sigma_j)}} \end{aligned} \tag{3.50}$$

And, taking the dot product with the unit $\overrightarrow{dw}$ vector,

$$\begin{aligned} w_4 = \frac{i\lambda\Gamma}{4\pi\sqrt{1 + 1/J_a^2}} \sum_{j=1}^{B} P_j \int_0^\infty \Bigg\{ & \frac{1 - a\cos(\frac{\tau}{J} + \sigma_j)}{\sqrt{(\tau + \frac{y}{R})^2 + (1 + a^2) - 2a\cos(\frac{\tau}{J} + \sigma_j)}} \\ & + \frac{a\cos(\frac{\tau}{J} + \sigma_j)}{\sqrt{(\tau + \frac{y}{R})^2 + a^2}} \Bigg\} \left\{ \frac{e^{-i\gamma\tau}\left[(\tau + \frac{y}{R})\cos(\frac{\tau}{J} + \sigma_j) + \frac{a}{J_a}\sin(\frac{\tau}{J} + \sigma_j)\right]}{\left[(\tau + \frac{y}{R})^2 + a^2\sin^2(\frac{\tau}{J} + \sigma_j)\right]} \right\} d\tau \end{aligned} \tag{3.51}$$

3.5 Blade element oscillatory motion expressed as an equivalent downwash. To complete the description of the velocity component which acts normal to the blade element, one must take into account the oscillatory motion of the element itself. From the principle of relative motion it follows that a pure translatory velocity of the blade element in the positive z direction can also be expressed as a positive downwash since $\vec{w}$ is taken to be positive in the negative z direction. A similar argument holds when the blade element is undergoing a pure rotational oscillation but here the downwash is a function of the position along the chord.

In section 2.2 it was assumed that the blade element was undergoing a translatory oscillation with velocity given by,

$$v_t = A_t W e^{i\omega t}$$

and a rotational oscillation about the midchord point with the velocity given by,

$$v_r = 2A_r W e^{i\omega t} \cos\theta$$

Accordingly, the downwash will be,

$$w_1 = -(v_t + v_r - W\alpha) \tag{3.52}$$

where,

$$\alpha = -\frac{2A_r W e^{i\omega t}}{i\omega c} \tag{3.53}$$

The last term in equation (3.52) arises from the fact that the rotational oscillation of the blade element causes an angle of attack change with respect to the relative wind. Hence for the blade element located at r/R = a, the expression for the downwash from this cause becomes,

$$w_1 = - [A_t W e^{i\omega t} + A_r W e^{i\omega t} (2 \cos \theta - \frac{4i}{\gamma (AR)'})] \qquad (3.54)$$

IV. BLADE ELEMENT CIRCULATION CALCULATIONS.

4.1 The Munk Integral Theorem. This theorem relates the circulation about a two dimensional section with the downwash at that section. In applying it to this problem two assumptions are made.

a. There is no component of the downwash in the direction of $\vec{W}$ in the r'-z plane.

b. The radial velocity component is very small with respect to $\vec{w}$ and has a negligible effect upon forces produced by the blade element.

These are the assumptions commonly made in determining the spanwise lift distribution corresponding to a given wing shape. Accordingly, the only effective velocities at the blade section are the rectilinear velocity $\vec{W}$ and the downwash $\vec{w}$. The force on the blade element is assumed to be the same as if it were a section of an infinite cylinder in the r direction in a two-dimensional flow given by $\vec{W}$ and $\vec{w}$.

Suppose that the two dimensional thin airfoil lies along the ξ axis in a z-ξ plane (say). Then if the airfoil is oscillating a fluid particle passing over the airfoil will follow a path given by $F(\xi, t)$. This is the same as though the section had an instantaneous camber given by $z = F(\xi)$.

Now the classical Munk integral for a cambered thin airfoil at zero angle of attack is,

$$C_L = \frac{8}{c^2} \int_{-c/2}^{c/2} \frac{F(\xi)\, d\xi}{(1 - 2\xi/c)\sqrt{1 - (2\xi/c)^2}} \tag{4.1}$$

When written in terms of the circulation equation (4.1) is,

$$\Gamma = \frac{4W}{c} \int_{-c/2}^{c/2} \frac{F(\xi)\, d\xi}{(1 - 2\xi/c)\sqrt{1 - (2\xi/c)^2}} \tag{4.2}$$

The aim is to associate some factor in this equation with the downwash at the blade section. From physical reasoning it is apparent that at the airfoil the flow must be parallel to the blade surface. Also, since this is a linearized theory, $\vec{w} << \vec{W}$ so that the absolute magnitude of the resultant of $\vec{w}$ and $\vec{W}$ is very nearly equal to $|\vec{W}|$. Then,

$$\frac{dF}{d\xi} = -\frac{w}{W} \tag{4.3}$$

This leads to an integration by parts in equation (4.2). But, first, the following change in variable is introduced.

$$\xi = \frac{c}{2} \cos\theta \tag{4.4}$$

Equation (4.2) then is,

$$\Gamma = -2W \int_0^{\pi} \frac{F(\frac{c}{2}\cos\theta)\, d\theta}{1 - \cos\theta} \tag{4.5}$$

Integration by parts yields,

$$\Gamma = -2W\left[-F\left(\frac{c}{2}\cos\theta\right)\frac{1+\cos\theta}{\sin\theta}\Big|_0^{\pi} - \frac{c}{2}\int_0^{\pi}\frac{dF}{d\xi}(1+\cos\theta)d\theta\right] \tag{4.6}$$

At the leading edge $\theta = \pi$ and,

$$\lim_{\theta\to\pi}\left[\frac{1+\cos\theta}{\sin\theta}\right] = 0$$

and since the Kutta condition must hold at the trailing edge, the first term in equation (4.6) vanishes. Therefore,

$$\Gamma = Wc\int_0^{\pi}\frac{dF}{d\xi}(1+\cos\theta)\,d\theta \tag{4.7}$$

Then when equation (4.3) is substituted into (4.7), the relation for Γ is obtained in the form,

$$\Gamma = -c\int_0^{\pi} w(1+\cos\theta)\,d\theta \tag{4.8}$$

The next step in the analysis consists in writing down the expressions for the circulation due to trailing vorticity, shed vorticity, and the section oscillation.

4.2 Blade element circulation due to:

(a) tip trailing vorticity. One more relation between variables must be pointed out. In deriving the relations for the downwash a coordinate y parallel to ξ was taken to describe points along the chord of the blade section. Actually, due to the twist of the blade y does not lie parallel to the chord but rather it is as shown in figure 4.2. Since the chord is parallel to the resultant velocity $\vec{W}$ the coordinates y and θ are related by,

$$\frac{y}{R} = \frac{c}{2R} \frac{1 - \cos\theta}{\sqrt{1 + 1/J_a^2}} = (AR)' \sin^2(\theta/2) \tag{4.9}$$

Substituting equation (4.9) into (3.34) and then that equation into (4.8) yields,

$$\Gamma_2 = -\frac{\Gamma (AR)' \sqrt{1 + 1/J^2}}{\pi} \sum_{j=1}^{B} P_j \int_0^{\pi/2} \int_0^{\infty} e^{-i\gamma\tau} \cos^2\theta$$

$$\times \frac{[(\tau + (AR)'\sin^2\theta)\sin(\frac{\tau}{J} + \sigma_j) + (1 - \frac{a}{J_a})\cos(\frac{\tau}{J} + \sigma_j) + (\frac{1}{J_a} - a)]}{[(\tau + (AR)'\sin^2\theta)^2 + (1 + a^2) - 2a\cos(\frac{\tau}{J} + \sigma_j)]^{3/2}} d\tau d\theta \tag{4.10}$$

where,

$$\frac{(AR)}{\sqrt{1 + 1/J_a^2}} = (AR)'$$

Equation (4.10) can accordingly be written in the form,

$$\Gamma_2 = -\frac{\Gamma (AR)' \sqrt{1 + 1/J^2}}{\pi} I_2 \tag{4.11}$$

Evaluation of this, and the following circulation integrals is performed in Appendix A.

(b) Root trailing vorticity. w_3, the downwash due to root trailing vorticity, is given in equation (3.37). The corresponding circulation integral is,

$$\Gamma_3 = -\frac{\Gamma a (AR)'}{\pi} \sum_{j=1}^{B} P_j \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-i\gamma\tau} \cos^2\theta \, d\tau d\theta}{[(\tau + (AR)'\sin^2\theta)^2 + a^2]^{3/2}} \tag{4.12}$$

or,

$$\Gamma_3 = -\frac{\Gamma a (AR)'}{\pi} I_3$$

(c) Shed vorticity. w_4, the downwash due to shed vorticity is given in equation (3.51). The corresponding circulation integral is,

$$\Gamma_4 = -\frac{i\Gamma(AR)'\Upsilon}{\pi}\sum_{j=1}^{B} P_j \int_0^{\pi/2}\int_0^{\infty}\left\{\frac{a\cos(\frac{\tau}{J}+\sigma_j)}{\sqrt{[\tau+(AR)'\sin^2\theta]^2+a^2}}\right.$$

$$\left.+\frac{1-a\cos(\frac{\tau}{J}+\sigma_j)}{\sqrt{[\tau+(AR)'\sin^2\theta]^2+(1+a^2)-2a\cos(\frac{\tau}{J}+\sigma_j)}}\right\} \quad (4.13)$$

$$X\ \frac{e^{-i\Upsilon\tau}\left\{[\tau+(AR)'\sin^2\theta]\cos(\frac{\tau}{J}+\sigma_j)+\frac{a}{J_a}\sin(\frac{\tau}{J}+\sigma_j)\right\}}{[\tau+(AR)'\sin^2\theta]^2+a^2\sin^2(\frac{\tau}{J}+\sigma_j)}\cos^2\theta\, d\tau d\theta$$

or,

$$\Gamma_4 = -\frac{i\Gamma(AR)'\Upsilon}{\pi} I_4$$

(d) Blade element oscillation. w_1, the downwash equivalent to the blade element oscillation is given in equation (3.54). The corresponding circulation integral is,

$$\Gamma_1 = c\int_0^{\pi}[A_r We^{i\omega t}(2\cos\theta - \frac{4i}{\Upsilon(AR)'}) + A_t We^{i\omega t}](1+\cos\theta)d\theta$$

which integrates immediately to,

$$\Gamma_1 = c\pi[A_t We^{i\omega t} + A_r We^{i\omega t}(1-\frac{4i}{\Upsilon(AR)'})] \quad (4.14)$$

4.3 Resultant blade element circulation. The circulation at an arbitrary blade element station for any blade in a system of B blades has been expressed in terms of four components: namely, the circulation due to,

a) tip trailing vorticity, Γ_2,

b) root trailing vorticity, Γ_3,

c) shed vorticity, Γ_4, and

d) blade element motion, Γ_1.

The resultant circulation has been denoted by Γ. The resultant, and each component of circulation, is periodic. Then, since the total blade element circulation is the resultant of the components of circulation, and since each of the components has been expressed in terms of the resultant circulation with the exception of Γ_1, one can write an equation for Γ. It is of the form,

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 = -\frac{\Gamma(AR)'}{\pi}\sqrt{1 + 1/J^2}\, I_2 - \frac{\Gamma a(AR)'}{\pi} I_3 - \frac{i\Gamma(AR)'\Upsilon}{\pi} I_4 + c\pi W e^{i\omega t}\left[A_r\left(1 - \frac{4i}{\Upsilon(AR)'}\right) + A_t\right] \tag{4.15}$$

Therefore, the instantaneous blade element circulation is given by,

$$\Gamma = \frac{c\pi\left[A_t W e^{i\omega t} + A_r W e^{i\omega t}\left(1 - \frac{4i}{\Upsilon(AR)'}\right)\right]}{1 + \frac{(AR)'}{\pi}\left[\sqrt{1 + 1/J^2}\, I_2 + aI_3 + i\Upsilon I_4\right]} \tag{4.16}$$

Having determined the circulation it is now possible to determine the blade element lift and moment, hence the thrust and torque on each element. To accomplish this it is necessary to obtain these quantities in terms of the instantaneous circulation. The necessary equations are derived in the next section.

V. DERIVATION OF THE LIFT EQUATION FOR A THIN AIRFOIL IN TWO-DIMENSIONAL NON-STATIONARY MOTION.

5.1 Impulse of a vortex system. The lift on a thin airfoil in two-dimensional non-stationary motion can be determined by utilizing the following general theorem from incompressible fluid flow theory.

> "The impulse of the flow produced by a closed, plane vortex element has only a normal component and is proportional to the strength of the element and the area enclosed by it."

In proving the above theorem it is not necessary to impose that the vortex distribution be plane and a much more general result can be obtained. The essentials of the proof to be given below are presented in reference 19.

Consider a vortex system contained in a closed surface S in an infinite, incompressible fluid which is at rest at infinity. It is desired to find a distribution of impulsive forces, per unit mass of fluid, X', Y', Z', defined to be in the x, y, z, directions respectively, which will generate the actual u, v, w, components of velocity in the fluid. That is, a volume of fluid S is subjected to a system of dirac forces,

$$F' = \lim_{\tau \to 0} \int_0^{\tau} F \, dt \tag{5.1}$$

an infinite force acting for an infinitesimal time and then one studies the development of fluid velocities and determines the ensuing impulse resultants arising from the vortex system.

Denote by S the surface containing the vorticity and imbed the origin of an x, y, z, coordinate system in S. The impulsive forces X'_i acting on the volume of fluid S will, in general, give rise to a vortex motion and an irrotational motion defining a finite potential at all points within S. The vortex motion within S in turn induces a potential motion outside of S, the potential of which will be single valued since the domain outside of S is free of singularities. Denote by Φ the single valued velocity potential outside of S and by Φ_1 the velocity potential inside of S, which must have the following properties,

a) Φ_1 finite within S.

b) Φ_1 satisfies Laplace's equation $\nabla^2\Phi_1 = 0$ within S.

c) Φ and Φ_1 must be continuous on S. That is, $\Phi_1 = \Phi$ on S.

Then Φ_1 is the velocity potential of the motion which would be produced within S by a system of impulsive pressures $\rho\Phi$ acting over the surface. Impulsive pressures are defined by

$$\varpi = \underset{\tau \to 0}{\text{Limit}} \int_0^{\tau} \tau' \, dt$$

By using the momentum equation one can show that $\rho\Phi$ is a function of the impulsive pressures on the fluid and that any irrotational motion can be created by impulsive pressures.

Accordingly, assume that the impulsive forces are of the form,

$$\left.\begin{aligned} X' &= u + \frac{\partial\Phi_1}{\partial x} \\ Y' &= v + \frac{\partial\Phi_1}{\partial y} \\ Z' &= w + \frac{\partial\Phi_1}{\partial z} \end{aligned}\right\} \text{within } S \qquad X' = Y' = Z' = 0 \quad \text{outside } S \tag{5.2}$$

Note that from the definition of impulsive force, equation (5.1), that X', Y', Z' have the dimensions of momentum.

The distribution of impulsive pressures is given by $\rho\Phi_1$ within S and $\rho\Phi$ outside of S.

Now it can be shown from considerations of conservation of momentum that the motion of a fluid element that was originally at rest arising from application of impulsive forces will be given by

$$\begin{aligned} u &= X' - \frac{\partial\varpi}{\partial x}\frac{1}{\rho} \\ v &= Y' - \frac{\partial\varpi}{\partial y}\frac{1}{\rho} \\ w &= Z' - \frac{\partial\varpi}{\partial z}\frac{1}{\rho} \end{aligned} \tag{5.3}$$

where $-\frac{1}{\rho}\frac{\partial\varpi}{\partial x}$ is the resultant impulsive pressure in the direction of the impulsive force X' and so forth. Then since,

$$\varpi = \rho\Phi_1 \tag{5.4}$$

within S, and

$$\varpi = \rho\Phi \tag{5.5}$$

outside S, it is seen that the proposed system of impulsive forces in equation (5.2) will give rise to a motion in the fluid possessing velocity components u, v, w.

Also, from equations (5.2) it is clear that the forces X'_i are discontinuous on S. However, this discontinuity must occur only in the normal component of the X'_i forces since $\Phi_1 = \Phi$ on S. This is equivalent to stating that the only forces acting on the surface S must be impulsive pressures. One, therefore, can derive a series of relations which must hold for the distribution of X'_i at the surface S.

Denote by l, m, n, the direction cosines of the inward normal $\vec{n}$ to the surface S and let $\vec{i}$, $\vec{j}$, and $\vec{k}$ be unit vectors in the x, y, z, directions respectively. At the surface S it has been shown that the impulsive force resultant must be in the $\vec{n}$ direction. Writing

$$\vec{F} = X'\vec{i} + Y'\vec{j} + Z'\vec{k} \tag{5.6}$$

$\vec{F}$ and $\vec{n}$ will be parallel if their vector product vanishes, that is, if $\vec{F} \times \vec{n} = 0$. Performing this operation one obtains

$$\vec{F} \times \vec{n} = (mZ' - nY')\vec{i} + (nX' - \ell Z')\vec{j} + (\ell Y' - mX')\vec{k} \tag{5.7}$$

which can only vanish if each of the terms in brackets vanishes.

Therefore on S, the impulsive forces are related by,

$$
\begin{aligned}
mZ' - nY' &= 0 \\
nX' - \ell Z' &= 0 \\
\ell Y' - mX' &= 0
\end{aligned}
\tag{5.8}
$$

Now recall that the vorticity in an incompressible fluid is defined by

$$
\begin{aligned}
2\xi &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\
2\eta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\
2\zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
\end{aligned}
\tag{5.9}
$$

and form a surface integral of the sort,

$$
2\iiint_V (y\zeta - z\eta)\,dV = \iiint_V [y(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) - z(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})]\,dV \tag{5.10}
$$

Equation (5.10) can be expressed in terms of the impulsive forces by using equations (5.2). These give,

$$
\begin{aligned}
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= \frac{\partial Y'}{\partial x} - \frac{\partial X'}{\partial y} \\
\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} &= \frac{\partial X'}{\partial z} - \frac{\partial Z'}{\partial x}
\end{aligned}
\tag{5.11}
$$

so that

$$
\begin{aligned}
2\iiint_V (y\zeta - z\eta)\,dV &= \iiint_V [y(\frac{\partial Y'}{\partial x} - \frac{\partial X'}{\partial y}) - z(\frac{\partial X'}{\partial z} - \frac{\partial Z'}{\partial x})]\,dV \\
&= \iiint_V [\frac{\partial(yY')}{\partial x} - \frac{\partial(yX')}{\partial y}]\,dV - \iiint_V [\frac{\partial(zX')}{\partial z} - \frac{\partial(zZ')}{\partial x}]\,dV \\
&\quad + 2\iiint_V X'\,dV
\end{aligned}
\tag{5.12}
$$

Now one can apply the Gauss theorem in the form,

$$\iiint_V \nabla \times \vec{F}\, dV = \iint_S \vec{n} \times \vec{F}\, da \tag{5.13}$$

In equation (5.13) let,

$$\vec{F}' = yX'\,\vec{i} + yY'\vec{j} + yZ'\,\vec{k}$$

Then,

$$\iiint_V \nabla \times \vec{F}'\, dV = \iiint_V \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (yX') & (yY') & (yZ') \end{vmatrix} dV \tag{5.14}$$

and,

$$\iint_S \vec{n} \times \vec{F}'\, da = \iint_S \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \ell & m & n \\ (yX') & (yY') & (yZ') \end{vmatrix} da \tag{5.15}$$

So, if the $\vec{k}$ components of the vector equation (5.13) are equated, one obtains,

$$\iiint_V \left[\frac{\partial(yY')}{\partial x} - \frac{\partial(yX')}{\partial y}\right] dV = \iint_S y(\ell Y' - mX')\, da \tag{5.16}$$

And, if in a similar fashion one writes

$$\vec{F}'' = zX'\,\vec{i} + zY'\vec{j} + zZ'\,\vec{k}$$

then,

$$\iiint_V \left[\frac{\partial(zX')}{\partial z} - \frac{\partial(zZ')}{\partial x}\right] dV = \iint_S z(nX' - \ell Z')\, da \tag{5.17}$$

Therefore, substituting equations (5.16) and (5.17) into equation (5.12),

$$2 \iiint_V (y\zeta - z\eta)\,dV = \iint_S [y(\ell Y' - mX') - z(nX' - \ell Z')]\,da + 2 \iiint_V X'\,dV \tag{5.18}$$

and the surface integral vanishes by equations (5.8) so that the remaining expression becomes

$$\iiint_V (y\zeta - z\eta)\,dV = \iiint_V X'\,dV \tag{5.19}$$

In a similar fashion,

$$\begin{aligned} \iiint_V (z\xi - x\zeta)\,dV &= \iiint_V Y'\,dV \\ \iiint_V (x\eta - y\xi)\,dV &= \iiint_V Z'\,dV \end{aligned} \tag{5.20}$$

Introducing the fluid density ρ and denoting the resultant force-resultants of the impulse of the vortex system by P, Q, R in the x, y, z, directions respectively,

$$\begin{aligned} P &= \rho \iiint_V (y\zeta - z\eta)\,dV \\ Q &= \rho \iiint_V (z\xi - x\zeta)\,dV \\ R &= \rho \iiint_V (x\eta - y\xi)\,dV \end{aligned} \tag{5.21}$$

Therefore, it is immediately seen that if the vortex system is planar (lying in the x-y plane) with vorticity components ξ, η in the x,y directions respectively, the impulse is in the z direction and is given by

$$R = \rho \iint_S (x\eta - y\xi)\, da \tag{5.22}$$

where da = dx dy and S is the portion of the x-y plane which contains the vorticity. The theorem has accordingly been proven.

5.2 The lift equation. The results of part 5.1 can now be applied to compute the lift on a thin airfoil in two dimensional non-stationary motion.

Let the airfoil be placed in the x-y plane with the origin of coordinates at the midchord. Let the chord of the airfoil be c. Denote the velocity of the airfoil by W. The airfoil element is represented by vorticity lying on the x axis between $x = \pm c/2$. The strength per unit length of this vorticity is denoted by $\gamma_y(x)$. The integrated strength of this vorticity is Γ which is a function of time. All changes in Γ appear as free vorticity in the wake assumed to lie on the x-axis at the point in space at which it was created. The vorticity strength per unit length in the wake is denoted by $\gamma_\eta(x)$.

Under the assumption of two dimensional motion there are no vortex filaments parallel to the x-axis. Therefore, in this case, equation (5.22) for the impulsive force on the fluid becomes,

$$R = \rho\left[\int_{-c/2}^{c/2} x\gamma_y(x)\,dx + \int_{c/2}^{\infty} x\gamma_\eta(x)\,dx\right] \tag{5.23}$$

where it is assumed that the motion has been going on for a sufficient length of time such that the wake can be considered to extend to infinity.

Now the instantaneous lift on the wing is the negative time derivative of R. Therefore,

$$L = -\frac{dR}{dt} = -\rho \frac{d}{dt}\int_{-c/2}^{+c/2} x\gamma_y(x)dx - \rho\frac{d}{dt}\int_{+c/2}^{\infty} x\gamma_\eta(x)dx \tag{5.24}$$

where

$$\frac{d}{dt}\int_{c/2}^{\infty} x\gamma_\eta(x)dx = W\int_{c/2}^{\infty}\gamma_\eta(x)dx + \frac{Wc}{2}\gamma_\eta\left(\frac{c}{2}\right) \tag{5.25}$$

As has been stated before, the change in the total circulation, Γ, about the airfoil appears as free vorticity with location at the instantaneous coordinate of the trailing edge. It is therefore possible to write

$$\gamma_\eta\left(\frac{c}{2}\right) = \gamma_y\left(\frac{c}{2}\right) = -\frac{1}{W}\frac{d\Gamma}{dt} \tag{5.26}$$

Also, the combined vortex strength in the space must remain unaltered so that

$$\int_{c/2}^{\infty}\gamma_\eta(x)dx = -\Gamma \tag{5.27}$$

Therefore, if equations (5.27), (5.26), and (5.25) are introduced into equation (5.24), one obtains the equation for the lift in the form,

$$L = -\rho \frac{d}{dt} \int_{-c/2}^{c/2} x\gamma_y(x)dx + \rho W\Gamma + \frac{\rho c}{2}\frac{d\Gamma}{dt} \tag{5.28}$$

The first term on the right hand side of equation (5.28) is the contribution of the additional apparent mass to the lift. In the integral, the vorticity $\gamma_y(x)$ can be written in terms of the downwash, w, at the airfoil by methods outlined in reference 17.

5.3 An alternate form of the expression for the lift due to additional apparent mass. In the previous section, equation (5.28), the lift on a two-dimensional thin airfoil in non-stationary motion contributed by the additional apparent mass was obtained in the form,

$$L_1 = -\rho \frac{d}{dt} \int_{-c/2}^{c/2} x\gamma_y(x)dx \tag{5.29}$$

In reference 17 equations are developed for the moment upon a thin two-dimensional airfoil. The following identity is established.

$$\int_{-c/2}^{c/2} x\gamma_y(x)dx = \frac{c^2 W}{4} \int_0^{2\pi} \frac{dF(x)}{dx} \sin^2\theta \, d\theta \tag{5.30}$$

where F(x) is the instantaneous "camber" of the oscillating airfoil as described in part 4.1 of this paper. From equation (4.3),

$$\frac{dF(x)}{dx} = -\frac{w(x)}{W} \tag{5.31}$$

W, as before, is the velocity of translation of the airfoil. Therefore,

$$\int_{-c/2}^{c/2} x\gamma_y(x)dx = \frac{c^2}{4}\int_0^{2\pi} w(x)\ \sin^2\theta\ d\theta \tag{5.32}$$

Recall that the variables x and θ are related by $x = \frac{c}{2}\cos\theta$.

Thus w is even in θ about $\theta = \pi$ so that the expression for the additional apparent mass lift can be written as,

$$L_1 = -\frac{\rho c^2}{2}\frac{d}{dt}\int_0^{\pi} w(x)\ \sin^2\theta\ d\theta \tag{5.33}$$

Equation (5.33) can be transformed further into a form which is convenient for computational purposes. Writing,

$$\sin^2\theta = (1 + \cos\theta)(1 - \cos\theta) \tag{5.34}$$

and recalling from equation (4.8) that,

$$\Gamma = -c\int_0^{\pi} w\ (1 + \cos\theta)\ d\theta \tag{5.35}$$

equation (5.33) can then be written as,

$$L_1 = \rho c\frac{d\Gamma}{dt} + 4\rho c^2\frac{d}{dt}\int_0^{\pi/2} [w(2\theta)\ \cos^2\theta]\ \cos^2\theta\ d\theta \tag{5.36}$$

VI. LIFT ON A PROPELLER BLADE ELEMENT IN NON-STATIONARY MOTION.

6.1 The total lift equation. Applying the expression for the circulation about a propeller blade element operating in non-stationary motion to the two dimensional lift equation as derived in section V, one can now proceed to write down an expression for the lift on a propeller blade element in non-stationary motion.

First, equation (5.36) for the lift on the element due to additional apparent mass will be written in another form. As in the integrals for the circulation, Γ_1, w in equation (5.36) is the total downwash at the airfoil comprised of the equivalent downwash due to the airfoil motion itself plus the downwash induced at the airfoil by the wake. Note that these integrals will vary from the circulation integrals only by a factor of $\cos^2\theta$ multiplying the entire integrand in each case. The induced circulation integrals have been denoted by I_2, I_3, and I_4. The corresponding apparent mass integrals will be denoted by I'_2, I'_3, and I'_4. Then, introducing equation (3.54) for w_1, integrating and collecting terms, the expression for L_1 becomes,

$$L_1 = -\pi\rho c^2 i\omega \left[\tfrac{3}{4} A_t W e^{i\omega t} + A_r W e^{i\omega t}\left(1 - \frac{3i}{(AR)'Y}\right)\right] + \rho c i\omega\Gamma\left[1 + \frac{(AR)'}{\pi}\left(\sqrt{1 + 1/J^2}\, I'_2 + a I'_3 + iY I'_4\right)\right] \qquad (6.1)$$

Therefore, if equation (6.1) for the lift due to additional apparent mass is introduced into equation (5.28) one obtains the following expression for the lift per unit length on a propeller blade element which is undergoing translational and rotational oscillations.

$$L = -\pi\rho c^2 i\omega\left[\frac{3}{4}A_t We^{i\omega t} + A_r We^{i\omega t}\left(1 - \frac{3i}{(AR)'\Upsilon}\right)\right] + \rho c i\omega\Gamma\left[1 + \frac{(AR)'}{\pi}\left(\sqrt{1 + 1/J^2}\, I'_2 + aI'_3 + i\Upsilon I'_4\right)\right] + \rho\Gamma W\left(1 + \frac{i}{2}\Upsilon(AR)'\right) \tag{6.2}$$

where, from equation (4.16),

$$\Gamma = \frac{c\pi\left[A_t We^{i\omega t} + A_r We^{i\omega t}\left(1 - \frac{4i}{\Upsilon(AR)'}\right)\right]}{1 + \frac{(AR)'}{\pi}\left[\sqrt{1 + 1/J^2}\, I_2 + aI_3 + i\Upsilon I_4\right]} \tag{6.3}$$

6.2 Lift due to translational and rotational oscillations.

For purposes of comparison with the two dimensional theory it is convenient to express the lift as two components arising from the translational and rotational modes of oscillation. Denoting the translational component by $L^{(T)}$ and the rotational component by $L^{(R)}$, from equations (6.2) and (6.3) one obtains

$$\frac{L^{(T)}}{\frac{1}{2}\rho W^2 e^{i\omega t} cA_t} = C_L^{(T)} = 2\pi\left[\frac{1 + \frac{i}{2}\nu}{1 + G_{o,2}} + i\nu\left(\frac{1 + G_1}{1 + G_{o,2}} - \frac{3}{4}\right)\right] \tag{6.4}$$

and,

$$\frac{L^{(R)}}{\frac{1}{2}\rho W^2 e^{i\omega t} cA_r} = C_L^{(R)} = 2\pi\left\{\frac{(1 + \frac{i}{2}\nu)(1 - \frac{4i}{\nu})}{1 + G_{o,2}} + i\nu\left[(1 - \frac{4i}{\nu})\frac{1 + G_1}{1 + G_{o,2}} - (1 - \frac{3i}{\nu})\right]\right\} \qquad (6.5)$$

ν is the reduced frequency based on the blade element chord and is given by,

$$\nu = c\omega/W$$

and

$$1 + G_{o,2} = 1 + \frac{(AR)'}{\pi}[\sqrt{1 + 1/J^2}\, I_2 + aI_3 + i\Upsilon I_4]$$
$$1 + G_1 = 1 + \frac{(AR)'}{\pi}[\sqrt{1 + 1/J^2}\, I'_2 + aI'_3 + i\Upsilon I'_4] \qquad (6.6)$$

Note that in equations (6.4) and (6.5) the first terms on the right hand side of the equations are the net effect of the quasi-steady lift and the induced lift. The remaining terms on the right hand side of these equations contribute the lift due to the additional apparent mass.

6.3 Corresponding two-dimensional lift equations. The equations given in section 6.2 above, are analogous to the following equations obtained from reference 5 for the lift on a two-dimensional airfoil undergoing translatory and rotational oscillations. The notation of the reference has been changed slightly to facilitate the comparison. Thus, for translational oscillations,

$$\frac{L^{(T)}}{\frac{1}{2}\rho W^2 e^{i\omega t} A_t c} = 2\pi \left[\frac{K_1(\frac{i\nu}{2})}{K_o(\frac{i\nu}{2}) + K_1(\frac{i\nu}{2})} + \frac{i\nu}{4} \right] \tag{6.7}$$

and for rotational oscillations,

$$\frac{L^{(R)}}{\frac{1}{2}\rho W^2 e^{i\omega t} A_r c} = 2\pi \left[\frac{K_1(\frac{i\nu}{2})}{K_o(\frac{i\nu}{2}) + K_1(\frac{i\nu}{2})}(1 - \frac{4i}{\nu}) + 1 \right] \tag{6.8}$$

where K_o and K_1 are modified Bessel functions of the second kind of order zero and one respectively.

VII. THRUST AND TORQUE ON A PROPELLER BLADE ELEMENT IN NON-STATIONARY MOTION.

7.1 Resolution of the instantaneous lift vector. As developed in section VI, the instantaneous lift vector acts perpendicular to the resultant direction of the blade element in steady motion. That is, it acts perpendicular to the velocity vector $\vec{W}$ whose magnitude is given by $|\vec{W}| = \sqrt{V^2 + (\Omega r)^2}$.

The resolution of this lift vector into thrust and torque is obtained from figure 7.1. From there it is seen that,

$$T = L \cos \varphi = \frac{L}{\sqrt{1 + J_a^2}} \tag{7.1}$$

and

$$Q = rL \sin \varphi = \frac{Lr\ J_a}{\sqrt{1 + J_a^2}} \tag{7.2}$$

where T is the thrust, and Q is the torque, per unit length of blade.

7.2 Existence of induced drag in oscillating systems. No account has been taken of possible induced drag upon the blade element. In deriving the expression for the instantaneous lift on a two dimensional blade element it was shown that the force exerted by a planar system of vorticity acts perpendicular to the plane of the vorticity. From this general result an explicit formula for the lift on a two-dimensional airfoil in non-stationary motion was obtained in terms of the circulation about the airfoil and then in terms of the downwash at the airfoil. By this means an expression was obtained which gave only the lift component acting upon the airfoil.

In three dimensional stationary wing theory the induced drag is generally computed in the following way. One calculates the induced downwash at a particular section of the wing due to the wake vorticity distribution created by the wing. Then the characteristics of this section are assumed to be those of a two dimensional section subjected to the velocity of translation of the wing on which is superimposed the induced downwash at the section. The induced downwash, w, at the section has the effect of rotating the velocity vector at the section, hence rotates the force vector and gives rise to a component of the force vector in the direction of the motion. This is designated as the induced drag. See Figure 7.2. Using the differential Kutta-Joukowski law, one finds that the forces per unit length on the airfoil element are given in this case by,

$$\begin{aligned} \text{Force} &= F' = \rho q \Gamma \\ \text{Lift} &= L' = \rho W \Gamma \\ \text{Drag} &= D' = \rho w \Gamma \end{aligned} \qquad (7.3)$$

It is clear that under the condition that $w \ll W$ one will have $|\vec{W}|$ and $|\vec{q}|$ very nearly equal and that this will yield F' and L' equal to within the same approximation. With this assumption, the expression for D' can be written,

$$D' = \frac{w\ F'}{q} = \frac{w\ L'}{W} \tag{7.4}$$

Expressions for the instantaneous lift on an airfoil element in three dimensional non-stationary motion have been developed in the preceding sections. Thus one could, to a first approximation, assume that the force distribution depicted in figure 7.2 is the instantaneous force distribution associated with the non-stationary motion and utilize equation (7.4) to calculate the induced drag on the lifting element. However, a difficulty arises in attempting to determine the proper point on the airfoil at which to calculate w. In non-stationary motion w varies greatly over the airfoil chord. It is known that the three dimensional effect is equivalent to the introduction of an effective camber of the airfoil element. From two-dimensional theory it can be shown that for a two dimensional thin airfoil having a camber line in the form of a parabola or a circular arc, the effective angle of attack of the element is the slope at the three quarter chord point. With this fact as a guide it is reasonable to compute w at the three quarter chord point and to use this value in equation (7.4) to determine the instantaneous induced drag. Two points should be kept in mind. First, the value of the lift as used in the equation for the drag takes into account the distribution of w over the entire chord. Second, the selection of the three quarter chord point as a coordinate at which to compute w has been made arbitrarily and is not justified in any rigorous manner.

For most problems in which the non-stationary forces are required it happens that the drag force is of little importance. The drag force is directed against the greatest axis of inertia of the airfoil section and, therefore, is of negligible importance in flutter calculations. In computing the non-stationary forces on the propeller, the drag force has its principal component directed in the direction of the torque and, therefore, will effect the thrust of the propeller very little. Also, the induced drag force is proportional to the product of the induced velocity and the instantaneous lift, which itself is linearly related to the induced velocity. Therefore, the instantaneous drag force is of the order of the square of a small quantity. Finally, at the high speeds at which propeller blade elements operate, the induced drag is much smaller than the profile drag of an actual propeller blade element.

From all of these considerations, it is reasonable to neglect any consideration of the induced drag so that equations (7.1) and (7.2) give the instantaneous thrust and torque upon the blade element per unit length.

VIII. DERIVATION OF THE MOMENT EQUATION FOR A THIN AIRFOIL IN TWO DIMENSIONAL NON-STATIONARY MOTION.

8.1 Moment of momentum of a vortex system. Following an analysis analogous to that of section 5.1 one can derive the expressions for the moment of momentum in an incompressible fluid, at rest at infinity, about a fixed origin arising from the creation of an arbitrary distribution of vorticity in the space.

As in section 5.1, let ξ, η, ζ, be the components of vorticity in the x,y,z directions respectively and denote the corresponding moments of moment a by $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$. It is then possible to show (reference 19) that,

$$\begin{aligned} \mathcal{L} &= -\frac{\rho}{2}\int_\tau (y^2 + z^2)\xi \, d\tau \\ \mathcal{M} &= -\frac{\rho}{2}\int_\tau (z^2 + x^2)\eta \, d\tau \\ \mathcal{N} &= -\frac{\rho}{2}\int_\tau (x^2 + y^2)\zeta \, d\tau \end{aligned} \tag{8.1}$$

where τ is that portion of the space containing the vorticity.

In the case of a two-dimensional thin airfoil in non-stationary motion translating in the direction of the x-axis, the only component of vorticity is in the y-direction, the motion of the airfoil being in the x-y plane. Then equations (8.1) reduce to a single equation expressing the moment of momentum about the y-axis. It is,

$$\mathcal{M} = -\frac{\rho}{2}\int x^2 \eta(x) \, dx \tag{8.2}$$

8.2 The moment equation. Following the notation of section 5.2, the equation for the moment of momentum on the fluid arising from the non-stationary motion of a two dimensional thin airfoil can be written as,

$$\mathfrak{M} = -\frac{\rho}{2}\left[\int_{-c/2}^{+c/2}(x+p)^2\gamma_y(x)dx + \int_{+c/2}^{\infty}(x+p)^2\gamma_\eta(x)dx\right] \quad (8.3)$$

where p is the distance of the midpoint of the airfoil from a fixed origin on the x axis. It then follows that the moment on the airfoil about the midchord is given by $-\frac{d\mathfrak{M}}{dt}$ evaluated at p = 0. It is to be noted that $\frac{dp}{dt} = -W$ where W is the velocity of translation of the airfoil. Thus,

$$M = \frac{\rho}{2}\frac{d}{dt}\left[\int_{-c/2}^{+c/2}(x+p)^2\gamma_y(x)dx + \int_{+c/2}^{\infty}(x+p)^2\gamma_\eta(x)dx\right]_{p=0} \quad (8.4)$$

Then, performing the indicated differentiation one obtains,

$$M = \frac{\rho}{2}\frac{d}{dt}\left[\int_{-c/2}^{+c/2}x^2\gamma_y dx + \int_{+c/2}^{\infty}x^2\gamma_\eta dx\right] - \rho W\left[\int_{-c/2}^{+c/2}x\gamma_y dx + \int_{+c/2}^{\infty}x\gamma_\eta dx\right] \quad (8.5)$$

The sign convention has been selected such that a diving moment is positive.

Equation (8.5) can be simplified as follows. Performing the differentiation of the second term one obtains,

$$\frac{d}{dt}\int_{+c/2}^{\infty}x^2\gamma_\eta dx = 2W\int_{+c/2}^{\infty}x\gamma_\eta dx + \frac{Wc^3}{8}\gamma_\eta(c/2) \quad (8.6)$$

And, since, from equation (5.26),

$$\gamma_\eta(c/2) = \gamma_y(c/2) = -\frac{1}{W}\frac{d\Gamma}{dt} \tag{8.7}$$

it follows that the equation for the moment can be written,

$$M = \frac{\rho}{2}\frac{d}{dt}\int_{-c/2}^{+c/2} x^2\gamma_y dx - \rho W\int_{-c/2}^{+c/2} x\gamma_y dx - \frac{\rho c^2}{8}\frac{d\Gamma}{dt} \tag{8.8}$$

The second integral in equation (8.8) also arose in determining the lift due to additional apparent mass. It has been expressed in terms of the downwash along the airfoil chord and is given in that form in equation (5.32). It still remains to express the first term of equation (8.8) in terms of the downwash along the chord.

The distribution of vorticity along the chord can be related to the downwash there by means of a relation obtained in reference 17. In deriving the Munk Integrals there for the lift and moment on a thin airfoil it is shown that the vorticity can be expressed by,

$$\gamma_y(x) = \frac{W}{\pi \sin\theta}\int_0^{2\pi} \frac{dF(x)}{dx}\cot\frac{\theta-\tau}{2}\sin\tau\, d\tau + \frac{2\Gamma}{\pi\sin\theta} \tag{8.9}$$

where $x = \frac{c}{2}\cos\theta$. The "instantaneous" camber has been expressed in terms of the downwash in equation (4.3) as $\frac{dF(x)}{dx} = -\frac{w(x)}{W}$. Therefore, denoting the first term in equation (8.8) by I, it can be written,

$$I = \frac{\rho c^3}{16\pi}\frac{d}{dt}\int_0^{\pi}[-\int_0^{2\pi} w(x)\cot\frac{\theta - \tau}{2}\sin\tau\, d\tau + \frac{2\Gamma}{c}]\cos^2\theta\, d\theta \tag{8.10}$$

The second term in equation (8.10) can be integrated immediately and the expression for I becomes,

$$I = -\frac{\rho c^3}{16\pi}\frac{d}{dt}\int_0^{\pi}[\int_0^{2\pi} w(x)\cot\frac{\theta - \tau}{2}\sin\tau\, d\tau]\cos^2\theta\, d\theta + \frac{\rho c^2}{16}\frac{d\Gamma}{dt} \tag{8.11}$$

To simplify the first term one can utilize the identity,

$$\cot\frac{\theta - \tau}{2} = \frac{2\sin\tau}{\cos\tau - \cos\theta} \tag{8.12}$$

If equation (8.12) is substituted in equation (8.11) and the order of integration interchanged, I can be expressed by,

$$I = \frac{\rho c^3}{8\pi}\frac{d}{dt}\int_0^{\pi}[\int_0^{\pi}\frac{\cos^2\theta\, d\theta}{\cos\theta - \cos\tau}]w(x)\sin^2\tau\, d\tau + \frac{\rho c^2}{16}\frac{d\Gamma}{dt} \tag{8.13}$$

The integral in terms of θ, here denoted by J, is an improper integral as written above. It can be evaluated quite simply by means of contour integration. Denote $\cos\tau = d$ and introduce the complex variable $z = e^{i\theta}$.

Then J can be evaluated about the unit circle as contour in the complex plane. In terms of z, J is given by,

$$J = \frac{1}{4i}\oint\frac{(z^2 + 1)^2 dz}{z^2(z - z_1)(z - z_2)} \tag{8.14}$$

where z_1 and z_2 are the roots of the quadratic equation

$z^2 - 2dz + 1 = 0$. They are given by $z_{1,2} = e^{\pm i\tau}$. It follows that the singularities of the integrand are a double pole at the origin and simple poles at $z = z_1$ and $z = z_2$. It is to be noted that the singularities at z_1 and z_2 lie on the contour of integration. However, the contour can be indented at those two points and a theorem from complex integration utilized which states that the contribution to a line integral at such an indentation about a simple pole is $2\pi i$ times the residue of the integrand at such a point times the angle turned through in going about the indentation, divided by 2π. Upon calculating the contribution from the simple poles in this manner it is found that they are equal and of opposite sign and hence cancel identically. Therefore, the sole contribution to the integral comes from the double pole at the origin. The residue there is calculated by means of the formula,

$$R_{z=0} = \mathrm{Res}\left[\frac{(z^2+1)^2}{z^2(z-z_1)(z-z_2)}\right]_{z=0} = \lim_{z\to 0}\frac{d}{dz}\left[\frac{(z^2+1)^2}{(z-z_1)(z-z_2)}\right] \qquad (8.15)$$

and this yields,

$$R_{z=0} = e^{i\tau} + e^{-i\tau} = 2\cos\tau \qquad (8.16)$$

Then, since the value of a contour integral is $2\pi i$ times the sum of the residues of the integrand, it follows that J has the value,

$$J = \pi\cos\tau \qquad (8.17)$$

so that the expression for I can finally be written,

$$I = \frac{\rho c^3}{8} \frac{d}{dt} \int_0^{\pi} w(x) \sin^2\theta \cos^2\theta \, d\theta + \frac{\rho c^2}{16} \frac{d\Gamma}{dt} \tag{8.18}$$

Now if equation (5.32) and equation (8.18) are substituted in the expression for the moment on the airfoil, equation (8.8), one obtains,

$$\begin{aligned} M = -\frac{\rho c^2}{16} \frac{d\Gamma}{dt} - \frac{\rho W c^2}{2} \int_0^{\pi} w(\tfrac{c}{2}\cos\theta) \sin^2\theta \, d\theta \\ + \frac{\rho c^3}{8} \frac{d}{dt} \int_0^{\pi} w(\tfrac{c}{2}\cos\theta) \sin^2\theta \cos\theta \, d\theta \end{aligned} \tag{8.19}$$

Equation (8.19) then expresses the instantaneous moment on a thin, two-dimensional airfoil in non-stationary motion in terms of the instantaneous downwash along the airfoil chord.

IX. MOMENT ON A PROPELLER BLADE ELEMENT IN NON-STATIONARY MOTION.

9.1 The total moment equation. Here, as in the development of the total lift equation, section 6.1, it is assumed that a strip theory is applicable. Just as in first order three dimensional stationary wing theory, the moment on a propeller blade element in non-stationary motion is assumed to be that of a two dimensional section in a resultant flow modified by the three dimensional distribution of induced velocities. Thus the moment on the propeller blade element is expressed by equation (8.19) if one substitutes for w and Γ the values given in sections III and IV respectively. To perform these substitutions, it is convenient to express the moment in another form which will also be simpler from the computational standpoint.

Comparing equations (8.19) and (5.33) and noting that differentiation of the circulation integrals with respect to t merely introduces a multiplicative factor, $i\omega$, it is seen that the second term of equation (8.19) can be expressed in terms of the lift due to additional apparent mass. This then can be expressed in terms of the airfoil motion and the apparent mass integrals by means of equation (6.1). The relation between the second term in the equation for the moment and L_1 is,

$$-\frac{\rho W c^2}{2}\int_0^\pi w(x)\sin^2\theta\, d\theta = \frac{W}{i\omega}L_1 \qquad (9.1)$$

Simplification of the first term in equation (8.19) is not accomplished quite so easily. However, by means of elementary trigonometric identities and the use of equations (5.35) and (5.36) for Γ and L_1, this term, here denoted by K, can be reduced to,

$$K = -\frac{\rho c^2 i\omega\Gamma}{2} + \frac{3cL_1}{4} - 2\rho c^3 \frac{d}{dt}\int_0^{\pi/2} w(2\theta)\cos^6\theta \, d\theta \qquad (9.2)$$

Note that the remaining integral is in the same form as the additional apparent mass integrals except that the integrand is multiplied by an additional $\cos^2\theta$. Where these integrals appear in the resulting expressions for the moment they will be designated by I''.

Upon substituting equations (9.1) and (9.2) into (8.19), the moment upon the propeller blade element is given by,

$$M = -\frac{9}{16}\rho c^2 i\omega\Gamma + \left(\frac{3c}{4} + \frac{W}{i\omega}\right)L_1 - 2\rho c^3 \frac{d}{dt}\int_0^{\pi/2} w(2\theta)\cos^6\theta \, d\theta \qquad (9.3)$$

Next, the explicit integrals for the downwash components, as given in section III can be substituted into the above relation. After integrating the term involving the "equivalent" downwash due to the blade element motion itself, the moment is expressable as,

$$M = \frac{15\ \rho c^3\pi i\omega}{32}\left[\frac{2}{3}A_t W e^{i\omega t} + A_r W e^{i\omega t}\left(1 - \frac{81}{3Y(AR)^T}\right)\right] + \left(\frac{3c}{4} + \frac{W}{i\omega}\right)L_1 - \frac{9}{16}\rho c^2 i\omega\Gamma(1 + G_2) \qquad (9.4)$$

In equation (9.4), the value of L_1 to be used is given as equation (6.1), and the value of Γ to be used is given as equation (6.3). The quantity $(1 + G_2)$ is,

$$1 + G_2 = 1 + \frac{8(AR)'}{9\pi}[\sqrt{1 + 1/J^2}\, I''_2 + aI''_3 + iYI''_4] \qquad (9.5)$$

9.2 Moment due to translational and rotational oscillations.

The expression (9.4) for the total moment could be simplified somewhat but it is more convenient to express the moment in two components, namely, that due to translational oscillations, denoted by $M^{(T)}$, and that due to rotational oscillations, denoted by $M^{(R)}$. Splitting equation (9.4) in this manner yields,

$$\frac{M^{(T)}}{\frac{1}{2}\rho W^2 e^{i\omega t} c^2 A_t} = C_M^{(T)} = \frac{\pi}{2}\Big[-(3 + i\nu) - \frac{9i\nu}{4}\frac{1 + G_2}{1 + G_{0,2}} + (3i\nu + 4)\frac{1 + G_1}{1 + G_{0,2}}\Big] \qquad (9.6)$$

and,

$$\frac{M^{(R)}}{\frac{1}{2}\rho W^2 e^{i\omega t} c^2 A_r} = C_M^{(R)} = \frac{\pi}{2}\Big[\Big(16 + 3i\nu - \frac{16i}{\nu}\Big)\frac{1 + G_1}{1 + G_{0,2}} - 9\Big(1 + \frac{i\nu}{4}\Big)\frac{1 + G_2}{1 + G_{0,2}} + \Big(-\frac{1}{2} + \frac{12i}{\nu} - \frac{9i\nu}{8}\Big)\Big] \qquad (9.7)$$

The expressions for $(1 + G_{0,2})$ and $(1 + G_1)$ are given as equations (6.6).

9.3 Corresponding two-dimensional moment equations.

The following equations, obtained from reference (5), refer to the moment on a two dimensional thin-airfoil in non-stationary motion. The notation of the reference has been changed to show the correspondence with the three dimensional equations, (9.6) and (9.7), above. For translatinnal oscillations,

$$\frac{M^{(T)}}{\frac{1}{2}\rho W^2 e^{i\omega t} c^2 A_t} = -\frac{\pi}{2}\left[\frac{K_1(\frac{i\nu}{2})}{K_o(\frac{i\nu}{2}) + K_1(\frac{i\nu}{2})}\right] \qquad (9.8)$$

and for rotational oscillations,

$$\frac{M^{(R)}}{\frac{1}{2}\rho W^2 e^{i\omega t} c^2 A_r} = \frac{\pi}{2}\left[\frac{K_o(\frac{i\nu}{2})}{K_o(\frac{i\nu}{2}) + K_1(\frac{i\nu}{2})}\left(1 + \frac{4i}{\nu}\right) + \frac{i\nu}{8}\right] \qquad (9.9)$$

K_0 and K_1 are modified Bessel functions of the second kind of order zero and one respectively.

X. THE EFFECT OF COMPRESSIBILITY.

10.1 The Prandtl-Glauert rule. The major portion of a propeller blade operates in a velocity field which is high subsonic. It, therefore, is important to appraise any aerodynamic theory of the propeller with the view towards corrections for the effect of compressibility. A successful, approximate means for doing this is afforded by the Prandtl-Glauert rule which, although proven inaccurate for the higher subsonic Mach numbers, gives a very simple means of determining the qualitative effect of compressibility.

The Prandtl-Glauert rule is formulated by considering the combined continuity momentum equation for the steady flow of a compressible fluid in linearized form. Thus, if u, v, and w are the perturbation velocities superimposed upon a uniform flow of velocity U and Mach number M from infinity, the linearized equation of continuity-momentum is,

$$\frac{\partial u}{\partial x} + \frac{1}{\beta^2}\frac{\partial v}{\partial y} + \frac{1}{\beta^2}\frac{\partial w}{\partial z} = 0 \qquad (10.1)$$

where

$$\beta^2 = 1 - M^2$$

Or, in terms of a perturbation potential defined by,

$$\vec{q} = \nabla\Phi \qquad (10.2)$$

where $\vec{q}$ is the resultant perturbation velocity, the governing differential equation is,

$$\Phi_{xx} + \frac{1}{\beta^2}\Phi_{yy} + \frac{1}{\beta^2}\Phi_{zz} = 0 \qquad (10.3)$$

This differential equation can be reduced to the LaPlace's equation, governing the velocity potential of an incompressible flow, by means of the affine transformation,

$$\begin{aligned} \xi_1 &= x \\ \xi_2 &= \beta y \\ \xi_3 &= \beta z \end{aligned} \tag{10.4}$$

This transformation then yields the differential equation,

$$\Phi_{\xi_1 \xi_1} + \Phi_{\xi_2 \xi_2} + \Phi_{\xi_3 \xi_3} = 0 \tag{10.5}$$

If the same transformation is applied to the boundary conditions, one has managed to transform the problem from the compressible form where solution is difficult to the incompressible form where the extensive knowledge of harmonic functions can be utilized.

10.2 <u>The Biot-Savart relation for compressible flow.</u> The greatest utility of the Prandtl-Glauert rule, for application in this paper, lies in the fact that it enables one to write the Biot-Savart relation for a compressible fluid. The possibility of writing the Biot-Savart relation in this form was recognized by Tsien and Lees and is published in reference (20). There it is noted that since the Prandtl-Glauert rule transforms the compressible problem to the incompressible problem, one has merely to write the Biot-Savart relation in the usual form. Then, upon inverting the Prandtl-Glauert transformation, and denoting the velocity components in the compressible problem by q', one obtains,

$$\overrightarrow{dq'} = \frac{\Gamma}{4\pi\rho^3}\, \vec{\rho} \times \overrightarrow{d\ell} \tag{10.6}$$

where,

$$\rho = \sqrt{(x - \bar{x})^2 + \beta^2[(y - \bar{y})^2 + (z - \bar{z})^2]}$$

$$\vec{d\ell} = \vec{i}\, d\bar{x} + \vec{j} d\bar{y} + \vec{k}\, d\bar{z} \qquad (10.7)$$

Equation (10.6) then is the Biot-Savart relation for a single vortex filament in a compressible subsonic motion. Γ is unchanged by the transformation and has the same value in both the compressible and incompressible motion.

In applying (10.6) it must be remembered that the velocity components are also transformed by the Prandtl-Glauert transformation. Thus the velocity components u', v', w', in the compressible motion are given by,

$$du'(x, y, z) = \frac{\beta^2}{4\pi} \frac{\Gamma[(z - \bar{z})d\bar{y} - (y - \bar{y})d\bar{z}]}{\rho^3}$$

$$dv'(x, y, z) = \frac{\beta^2}{4\pi} \frac{\Gamma[(x - \bar{x})d\bar{z} - (z - \bar{z})d\bar{x}]}{\rho^3} \qquad (10.8)$$

$$dw'(x, y, z) = \frac{\beta^2}{4\pi} \frac{\Gamma[(y - \bar{y})d\bar{x} - (x - \bar{x})d\bar{y}]}{\rho^3}$$

10.3 Application of the Biot-Savart relation for compressible, subsonic, flow to a lightly loaded propeller. In reference (20), the authors apply the Biot-Savart relation for compressible, subsonic, flow to the problem of calculating the induced velocities due to the vorticity field of a lightly loaded propeller. They find that the expression for the axial velocity u' has one term which is independent of Mach number effect. This term is just the quantity for the change in axial velocity through a propeller with an infinite number of blades. The conclusion is, therefore, that the momentum change and hence the thrust of a lightly loaded propeller having an infinite number of blades is unaffected by compressibility in the first order.

The remaining terms in the expression for u' are modified by the Mach number factor, β^2, and represent the compressibility effect upon the correction due to the number of blades. These terms are mathematically complicated and are presented by the authors in representation form. Only an estimate can be made of the effect of compressibility upon these terms and it is seen that the correction for number of blades must be increased by a factor somewhat larger than $1/\beta$.

The important fact is that the Mach number correction is, itself, applied to a correction. In figure 62, page 264, of reference (21) the optimum loading distribution, $F(\Gamma)$, of a propeller is plotted as a function of the advance ratio along the blades and is reproduced in this paper as figure 10.1. This figure gives an immediate picture of the correction due to a finite number of blades. It is seen that when the advance ratio, $\frac{V}{\Omega R} = J$, is large this correction is negligible. In the case used as an example in this paper J has been given the value $1/\pi$. Referring to figure 10.1 it is seen that for a two-bladed propeller the effect of the number of blades is of the order of 10%. For a four bladed propeller the correction is much less.

To a first order, therefore, the results obtained herein also apply to the case of a propeller in non-stationary compressible subsonic flow. Of course, as the operating speed of the propeller blade element approaches near to $M = 1$ these results no longer apply. However, since the compressibility correction only appears in the blade number correction it is felt that the results can be applied with confidence up to about $M = 0.8$. This is merely speculation based upon the above qualitative discussion.

XI. RESULTS AND DISCUSSION.

The results of this theory can be applied to at least two distinct problems. These problems fall in the general categories of forced vibrations and flutter. Since the application to the forced vibration problem was the original aim of this paper that problem will be discussed first.

Forced vibrations; cyclic pitch change.

In the particular type of forced vibration of interest here it is assumed that changes in incidence during the blade cycle arise from non-uniformity of the flow. Several instances in which such a motion will occur have been described in the introduction. It must be clearly understood that there is a distinction between the problem of non-uniform flow on the one hand and non-stationary motion on the other. The results of this paper have been obtained upon the basis of non-stationary motion, that is, the problem of an oscillating propeller blade in a uniform flow field has been treated. Now, however, it is desired to apply these results to the case of a non-oscillating propeller blade in a non-uniform flow field. This latter problem is one in which potential theory is not applicable and, accordingly, is quite difficult to treat. But, the effect of a stationary airfoil in a non-uniform flow field is approximately equivalent to that of an oscillating airfoil in a uniform flow field, if the non-uniformity of the flow is slight such that one can assume small oscillations of the airfoil. Under such conditions the results of this theory will be applied directly to the forced vibration problem without further qualification.

In general, the frequency of the forcing function will be a multiple of the rotational speed of the propeller measured in cycles per second. For the large propellers the lowest multiple frequencies are dangerously near the natural frequencies of the blades. This suggests that some means be attempted to eliminate the forcing function entirely. Since it is not possible to eliminate the non-uniformity of the flow two courses remain. Either the propeller installation can be designed so as to operate in a region where the non-uniformity is negligible or, since this is usually not possible, cyclic pitch change can be used to compensate for the non-uniformity of flow.

If cyclic pitch change is used there arises the problem of determining the required magnitudes of change of angle of attack and the point in the cycle at which these changes should be made, that is, the proper phase between the motion and the proposed angle change must be known. The results of this paper afford a means for determining this phase angle as a function of the reduced frequency of the forcing function. For example, in the particular example computed here it is found that the forcing function lags the rotational motion of the propeller by about 12.5 degrees.

Two dimensional theory yields an angle of lag, in this case, of only seven degrees. These numbers have been taken from figure A.1. For other values of reduced frequency the phase angle will be quite different. The general relationship between phase angle and reduced frequency as obtained from two dimensional theory is illustrated in reference (5). See figure A.4.

Equations (6.4) and (6.5) for the lift due to translational and rotational oscillations have been written in a form such that the bracketed terms on the right hand side of these equations represent the amplitude of the angle of attack. For a given non-uniformity of the flow it is possible, therefore, to compute the necessary angle of attack change to compensate for the non-uniformity of flow. This information, together with the knowledge of the phase angle, permits one to treat the cyclic pitch change problem completely.

Flutter.

To date, virtually all propeller flutter calculations have been made by replacing the blade with a two dimensional section. Representative values of the blade stiffness in torsion and bending are assigned to this section and two dimensional aerodynamic characteristics for non-stationary motion are used. Such calculations always yield a flutter speed which is higher than that actually observed.

With three dimensional aerodynamic theory now available it is possible, in principle, to extend the usual methods of flutter calculation to account for the fact that the problem is actually a three-dimensional one. Given sufficient computational facilities such a project could be undertaken. However, the most practical procedure is to continue to assume a two dimensional section mechanically and to replace the two dimensional aerodynamic characteristics by the finite aspect ratio results obtained in this paper. Such a course would entail a computational project to determine the G functions in equations (6.4), (6.5), (9.8), and (9.9) for the lift and moment for a sufficient range of the variables. Since only mechanical integrations are involved this would not be difficult although their evaluation by means of the ordinary calculating machine is quite lengthy. Once a sufficient range of these variables is obtained the flutter calculations will be no more involved than Theodorsen's procedure, reference (7), since the functions G merely replace the aerodynamic variable $Y = F + i G$ in the Theodorsen method. The G of this paper and that of Theodorsen are unrelated.

The principal value to be obtained from the use of the three dimensional aerodynamic characteristics is that the calculated flutter speed will be nearer the true flutter speed. Referring to figure A-3 it is seen that the aerodynamic force lags the motion in a larger amount as predicted by the three dimensional theory over that given by the two dimensional theory. The three dimensional theory will therefore lead to a more accurate determination of the flutter speed. The extent of this improved accuracy can only be determined by the calculation of an actual case which is then checked by experiment.

APPENDIX A

Evaluation of the Circulation Integrals of Part IV

In order to obtain the results of this theory it is necessary to evaluate the integrals set forth in Part IV. The integrals are complicated and difficult to express in closed form. It is, therefore, necessary to resort to numerical integration. The procedure will be outlined here along with limiting values of the integrals which can be determined analytically. Since numerical integration requires that values of the parameters be assumed, each such evaluation represents a given problem and a series of such calculations will be required in order to determine the behavior of the results with respect to the various parameters. One such example will be carried out in detail. Based upon design specifications given by a propeller rpm of 1260, an airplane velocity of 300 miles per hour, a propeller radius of 10 feet, and a forcing frequency which is twice the propeller rpm, the following values are assumed for the parameters.

a. $J = V/\Omega R = 1/\pi$

b. $I = \omega R/V = 2\pi$

c. $a = r/R = 0.6$

d. No. of blades = B = 2

e. c/R = blade chord-radius ratio = 0.1
= constant along the entire blade.

Then, with the above values assumed for the fundamental parameters, one can compute the following:

f. $J_a = 1/0.6\pi = V/\Omega r$

g. $AR' = c/R\sqrt{1+1/J_a^2} = 0.1/\sqrt{1 + 0.36\pi^2}$

h. $\sqrt{1 + 1/J^2} = \sqrt{1 + \pi^2}$

i. $\sigma_1 = 0 \qquad \sigma_2 = \pi$

j. $P_1 = 1 \qquad P_2 = 1$

A-1. Calculation of Integrals I_2 and I_2'.

For a two bladed propeller the integral I_2 becomes,

$$\int_0^{\pi/2}\int_0^{\infty} e^{-i\gamma\tau}\left\{\frac{[\tau + (AR)'\sin^2\theta]\sin\frac{\tau}{J} + (1 - \frac{a}{J_a})\cos\frac{\tau}{J} + \frac{1}{J_a} - a}{\{[\tau + (AR)'\sin^2\theta]^2 + (1 + a^2) - 2a\cos\frac{\tau}{J}\}^{3/2}}\right. \tag{a-1}$$

$$\left. + \frac{-[\tau + (AR)'\sin^2\theta]\sin\frac{\tau}{J} - (1 - \frac{a}{J_a})\cos\frac{\tau}{J} + \frac{1}{J_a} - a}{\{[\tau + (AR)'\sin^2\theta]^2 + (1 + a^2) + 2a\cos\frac{\tau}{J}\}^{3/2}}\right\}\cos^2\theta \, d\tau d\theta$$

and the corresponding I'_2 integral is

$$\int_0^{\pi/2}\int_0^{\infty} e^{-i\gamma\tau}\left\{\frac{[\tau + (AR)'\sin^2\theta]\sin\frac{\tau}{J} + (1 - \frac{a}{J_a})\cos\frac{\tau}{J} + \frac{1}{J_a} - a}{\{[\tau + (AR)'\sin^2\theta]^2 + (1 + a^2) - 2a\cos\frac{\tau}{J}\}^{3/2}}\right. \tag{a-2}$$

$$\left. + \frac{-[\tau + (AR)'\sin^2\theta]\sin\frac{\tau}{J} - (1 - \frac{a}{J_a})\cos\frac{\tau}{J} + \frac{1}{J_a} - a}{\{[\tau + (AR)'\sin^2\theta]^2 + (1 + a^2) + 2a\cos\frac{\tau}{J}\}^{3/2}}\right\}\cos^4\theta \, d\tau d\theta$$

Now for large values of τ, equation (a-1) reduces to,

$$K\int_0^{\pi/2}\int_{\delta}^{\infty}\frac{e^{-i\gamma\tau}}{\tau^3}\cos^2\theta \, d\tau d\theta \qquad K = \text{constant} \tag{a-3}$$

where the lower limit δ of the integration with respect to τ must be determined such that the integrand of equation (a-1) is closely represented by the approximation indicated in equation (a-3) when τ is large. The integrands of equations (a-1) and (a-3) agree within 0.6% if the lower limit δ is taken to be 20. Therefore the integral of equation (a-3) will be evaluated in the form,

$$K \int_0^{\pi/2} \cos^2\theta \int_{20}^{\infty} \frac{e^{-i\gamma\tau}}{\tau^3} \, d\tau d\theta \tag{a-4}$$

Equation (a-4) integrates to,

$$K \int_0^{\pi/2} \left\{ \frac{e^{-40i\pi}}{800} - \frac{i\pi e^{-40i\pi}}{20} + 2\pi^2 \left[Ci(40\pi) + i\left\{\frac{\pi}{2} - Si(40\pi)\right\} \right] \right\} \cos^2\theta \, d\theta \tag{a-5}$$

where Ci and Si are the cosine and sine integrals* as defined by,

$$Si(x) = \int_0^x \frac{\sin t}{t} \, dt \qquad Ci(x) = \int_{\infty}^x \frac{\cos t}{t} \, dt \tag{a-6}$$

In the range above x = 10 the asymptotic formulae for these functions are sufficiently accurate.

$$Si(x) \sim \frac{\pi}{2} - P(x)\cos x - Q(x)\sin x$$

$$Ci(x) \sim P(x)\sin x - Q(x)\cos x \tag{a-7}$$

where

$$P(x) = \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \frac{6!}{x^7} + \ldots\ldots$$

$$Q(x) = \frac{1}{x^2} - \frac{3!}{x^4} + \frac{5!}{x^6} - \frac{7!}{x^8} + \ldots\ldots \tag{a-8}$$

And, since in an asymptotic expansion the error committed in using n terms of the series is given by the (n + 1)st term, it is seen to be sufficient to take no more than two terms of P(x) and Q(x) when $\delta = 20$.

Then,

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* Tabulated in WPA Tables of Sine, Cosine and Exponential Integrals. Vol. I, Fed. Works Agency, WPA for City of N.Y., 1940

$$Si(40\pi) \approx \frac{\pi}{2} - \frac{1}{40\pi} + \frac{2!}{(40\pi)^3} \; : \; Ci(40\pi) \approx -\frac{1}{(40\pi)^2} + \frac{3!}{(40\pi)^4} \tag{9.9}$$

so, equation (a.5) becomes,

$$K \int_0^{\pi/2} \left[\frac{1}{800} - \frac{i\pi}{20} - \frac{1}{800} + \frac{i\pi}{20} + \frac{3!}{800(40\pi)^2} - \frac{2!}{800(40\pi)}\right]\cos^2\theta \, d\theta \tag{9.10}$$

Hence, the contribution to I_2 and I'_2 is negligible in this case for all $\delta \geqq 20$. These integrals will accordingly be evaluated numerically with the limits in τ taken to be zero and twenty. With all constants computed to five significant figures these integrals become,

$$I_2 = \int_0^{\pi/2} \int_0^{20} e^{-2\pi i\tau} \cos^2\theta$$

$$\times \left\{ \frac{[(\tau + .046865\sin^2\theta)\sin\pi\tau - .13097\cos\pi\tau + 1.2850)}{[(\tau + .046865\sin^2\theta)^2 + 1.3600 - 1.2000\cos\pi\tau]^{3/2}} - \frac{[(\tau + .046865\sin^2\theta)\sin\pi\tau - .13097\cos\pi\tau - 1.2850]}{[(\tau + .046865\sin^2\theta)^2 + 1.3600 + 1.2000\cos\pi\tau]^{3/2}} \right\} d\tau d\theta \tag{a-11}$$

$$I'_2 = \int_0^{\pi/2} \int_0^{20} e^{-2\pi i\tau} \cos^4\theta$$

$$\times \left\{ \frac{[(\tau + .046865\sin^2\theta)\sin\pi\tau - .13097\cos\pi\tau + 1.2850]}{[(\tau + .046865\sin^2\theta)^2 + 1.3600 - 1.2000\cos\pi\tau]^{3/2}} - \frac{[(\tau + .046865\sin^2\theta)\sin\pi\tau - .13097\cos\pi\tau - 1.2850]}{[(\tau + .046865\sin^2\theta)^2 + 1.3600 + 1.2000\cos\pi\tau]^{3/2}} \right\} d\tau d\theta$$

The numerical method used in evaluating the above integrals and the other integrals contained in this theory is outlined in Appendix B. After carrying out the integration it is found that

$$I_2 = 1.331 - 1.121\, i$$
$$I'_2 = 1.131 - 0.8299\, i \qquad \text{(a-12)}$$

A-2. Calculation of Integrals I_3 and I'_3.

For a two bladed propeller the integrals I_3 and I'_3 are,

$$I_3 = \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-i\Upsilon\tau} \cos^2\theta}{\{[\tau + (AR)' \sin^2\theta]^2 + a^2\}^{3/2}} d\tau d\theta$$
$$I'_3 = \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-i\Upsilon\tau} \cos^4\theta}{\{[\tau + (AR)' \sin^2\theta]^2 + a^2\}^{3/2}} d\tau d\theta \qquad \text{(a-13)}$$

For large values of τ I_3 becomes

$$\int_0^{\pi/2} \int_{\delta}^{\infty} \frac{e^{-i\Upsilon\tau} \cos^2\theta}{\tau^3} d\tau d\theta \qquad \text{(a-14)}$$

This equation is then in the form of equation (a-4) which was shown to be negligible for $\delta \geqq 20$. These integrals will accordingly be evaluated numerically with the limits in τ taken to be zero and twenty. With all constants computed to five significant figures the integrals I_3 and I'_3 become,

$$I_3 = \int_0^{\pi/2} \int_0^{20} \frac{e^{-2\pi i\tau} \cos^2\theta \, d\tau d\theta}{[(\tau + .046865 \sin^2\theta)^2 + .36000]^{3/2}} \qquad \text{(a-15)}$$
$$I'_3 = \int_0^{\pi/2} \int_0^{20} \frac{e^{-2\pi i\tau} \cos^4\theta \, d\tau d\theta}{[(\tau + .046865 \sin^2\theta)^2 + .36000]^{3/2}}$$

After carrying out the integration it is found that

$$I_3 = 0.1377 - 0.6546\, i$$
$$I'_3 = 0.06263 - 0.5161\, i \qquad \text{(a-16)}$$

A-3. Calculation of Integrals I_4 and I'_4.

For a two bladed propeller the integrals I_4 and I'_4 become,

$$I_4 = \int_0^{\pi/2}\int_0^{\infty}\left[\frac{e^{-i\tau}\left\{[\tau + (AR)'\sin^2\theta]\cos\frac{\tau}{J} + \frac{a}{J_a}\sin\frac{\tau}{J}\right\}}{[\tau + (AR)'\sin^2\theta]^2 + a^2\sin^2(\tau/J)}\right.$$
$$\times\left\{\frac{1 - a\cos(\tau/J)}{\sqrt{[\tau + (AR)'\sin^2\theta]^2 + (1 + a^2) - 2a\cos(\tau/J)}}\right.$$
$$- \frac{1 + a\cos(\tau/J)}{\sqrt{[\tau + (AR)'\sin^2\theta]^2 + (1 + a^2) + 2a\cos(\tau/J)}} \qquad \text{(a-17)}$$
$$\left.\left. + \frac{2a\cos(\tau/J)}{\sqrt{[\tau + (AR)'\sin^2\theta]^2 + a^2}}\right\}\right]\cos^2\theta\, d\tau d\theta$$

$$I'_4 = \int_0^{\pi/2}\int_0^{\infty}\left[\text{Same as terms in } I_4\right]\cos^4\theta\, d\tau d\theta$$

Note that for large values of τ, the terms in the brackets reduce to,

$$\frac{2a\cos(\tau/J)}{\tau} + \frac{1 - a\cos(\tau/J)}{\tau} - \frac{1 + a\cos(\tau/J)}{\tau} = 0 \qquad \text{(a-18)}$$

Because the integrand is singular at the origin, the value of the integrals for small values of the variables will be determined analytically. For $\theta \leqq \varepsilon$ and $\tau \leqq \eta$, where both ε and η are small compared with one, the integrals for I_4 and I'_4 both become,

$$2\int_0^{\varepsilon}\int_0^{\eta} \frac{[\tau + (AR)'\,\theta^2] + a\tau/J_a J}{[\tau + (AR)'\,\theta^2]^2 + a\tau^2/J_a J}\, d\tau d\theta$$

$$- 1/2\int_0^{\varepsilon}\int_0^{\eta} \frac{\tau\,[\tau + (AR)'\,\theta^2] + a\tau/J_a J}{[\tau + (AR)'\,\theta^2]^2 + a\tau^2/J_a J}\, d\tau d\theta \qquad \text{(a-19)}$$

Denoting the first integral in (a-19) by R and the second by S, the integration in R with respect to τ can be carried out immediately to give,

$$R = \frac{1}{2}\int_0^{\varepsilon} \ell og\left\{[\eta + (AR)'\theta^2]^2 + \frac{a\eta^2}{J_a J}\right\} d\theta - \int_0^{\varepsilon} \ell og[(AR)'\theta^2]\, d\theta \qquad \text{(a-20)}$$

The integrations in equation (a-20) will be carried out next. Consider,

$$T = \int_0^{\psi} \ell og(ax^4 + bx^2 + c)\, dx \qquad \text{(a-22)}$$

which after an integration by parts can be written,

$$T = \psi\,\ell og(a\psi^4 + b\psi^2 + c) - 4\psi + \int_0^{\psi} \frac{2bx^2 + 4c}{ax^4 + bx^2 + c}\, dx \qquad \text{(a-23)}$$

The remaining integral in equation (a-23) can be evaluated with sufficient accuracy by expanding the denominator of the integrand as follows,

$$\frac{1}{ax^4 + bx^2 + c} = \frac{1}{c}[1 - \frac{b}{c}x^2 + (\frac{b^2}{c^2} - \frac{a}{c})x^4$$
$$+ (\frac{2ba}{c^2} - \frac{b^3}{c^3})x^6 + (\frac{a^2}{c^2} - \frac{3b^2a}{c^3} + \frac{b^4}{c^4})x^8 + \ldots] \qquad \text{(a-24)}$$

For small values of x the expansion can be terminated after only a few terms and the integral, equation (a-23), can then be written as,

$$\int_0^{\psi} \frac{2bx^2 + 4c}{c}\left[1 - \frac{b}{c}x^2 + \left(\frac{b^2}{c^2} - \frac{a}{c}\right)x^4\right] dx \qquad \text{(a-25)}$$

Equation (a-25) then integrates to,

$$\frac{1}{c}\left[4c\psi - \frac{2b}{3}\psi^3 + \left(\frac{2b^2}{c} - 4a\right)\frac{\psi^5}{5} + \left(\frac{2b^3}{c^2} - \frac{2ab}{c}\right)\frac{\psi^7}{7}\right] \qquad \text{(a-26)}$$

The integrals in R can now be completely evaluated by applying equations (a-22) and (a-26) where,

$$a = (AR')^2$$

$$b = 2\,\eta\,AR'$$

$$c = \eta^2(1 + a/J_a J)$$

$$\psi = \varepsilon$$

The double integral R then becomes

$$R = \varepsilon \log\left\{\frac{[\eta + (AR)'\varepsilon^2]^2 + \frac{a\eta^2}{J_a J}}{(AR)'\varepsilon^2}\right\} + 2\varepsilon - \frac{2(AR)'}{\eta(1 + \frac{a}{J_a J})}\frac{\varepsilon^3}{3}$$

$$+ \frac{2(AR)'^2(1 - \frac{a}{J_a J})}{\eta^2(1 + \frac{a}{J_a J})^2}\frac{\varepsilon^5}{5} + \frac{2(AR)'^3(3 - \frac{a}{J_a J})}{\eta^3(1 + \frac{a}{J_a J})^3}\frac{\varepsilon^7}{7} \qquad \text{(a-27)}$$

The integral in equation (a-19) which has been denoted by S will be evaluated next. The integrand is an improper fraction. However, after applying long division and collecting terms, the integral S becomes,

$$S = \int_0^{\varepsilon}\left\{\eta - \int_0^{\eta} \frac{(AR)'\theta^2\tau + (AR)'^2\theta^4}{(1+\frac{a}{J_aJ})\tau^2 + 2(AR)'\theta^2\tau + (AR)'^2\theta^4}d\tau\right\} d\theta \qquad \text{(a-28)}$$

Now if F(x) and f(x) are two polynomials in x, with f(x) of higher degree than F(x) and having only simple roots, a, then the proper fraction F(x)/f(x) can be resolved into partial fractions as follows,

$$\frac{F(x)}{f(x)} = \sum_a \frac{F(a)}{\Phi(a)}\frac{1}{x-a} \quad \text{where} \quad \Phi(a) = \left[\frac{f(x)}{x-a}\right]_{x=a} \qquad \text{(a-29)}$$

Therefore, if the integrand of the integral with respect to τ in equation (a-28) is denoted by $F(\tau)/f(\tau)$ and the roots of $f(\tau)$ by τ_1 and τ_2, these roots are given by,

$$\tau_1 = \frac{-(AR)'\theta^2}{1+\frac{a}{J_aJ}}\left(1 - i\sqrt{\frac{a}{J_aJ}}\right); \quad \tau_2 = \frac{-(AR)'\theta^2}{1+\frac{a}{J_aJ}}\left(1 + i\sqrt{\frac{a}{J_aJ}}\right) \qquad \text{(a-30)}$$

Equation (a-29) can then be applied to the integration in (a-28) with respect to τ to give,

$$S = \int_0^{\varepsilon}\left\{\eta - \frac{(AR)'\tau_1\theta^2 + (AR)'\theta^2}{\tau_1 - \tau_2}\log\frac{\tau_1 - \eta}{\tau_1} - \frac{(AR)'\theta^2 + (AR)'\theta^2\tau_2}{\tau_2 - \tau_1}\log\frac{\tau_2 - \eta}{\tau_2}\right\} d\theta \qquad \text{(a-31)}$$

Equation (a-31) can be further simplified to,

$$S = \int_0^{\varepsilon}\left\{\eta - \frac{(1+\frac{a}{J_aJ})}{2i\sqrt{\frac{a}{J_aJ}}}\left[[\tau_1 + (AR)'\theta^2]\log\frac{\tau_1 - \eta}{\tau_1} - [\tau_2 + (AR)'\theta^2]\log\frac{\tau_2 - \eta}{\tau_2}\right]\right\} d\theta \qquad \text{(a-32)}$$

Only the real part of equation (a-32) is required. This is determined by usual methods to be,

$$S = \int_0^{\varepsilon}\left\{\eta - \frac{1}{2}(AR)'\theta^2 \ell og\frac{[\eta + (AR)'\theta^2]^2 + \frac{a}{J_aJ}\eta^2}{(AR)'^2\theta^4} + (AR)'(\frac{a}{J_aJ})^{3/2}\frac{\theta^4}{\theta^2 + \eta(1 + a/J_aJ)/(AR)'}\right\}d\theta \tag{a-33}$$

It accordingly becomes necessary to carry out the indicated integrations in equation (a-33). Denoting the second integral in equation (a-33) by T and the last by U, they can be evaluated as follows. After an integration by parts, the integral T is reduced to,

$$T = \frac{\varepsilon^3}{3}\ \ell og\left\{[\eta + (AR)'\varepsilon^2]^2 + \frac{a}{J_aJ}\ \eta^2\right\} - \frac{4(AR)'}{3}\int_0^{\varepsilon}\frac{[\eta + (AR)'\theta^2]\ \theta^4}{[\eta + (AR)'\theta^2]^2 + a\eta^2/J_aJ}d\theta \tag{a-34}$$

The denominator of the integrand of the remaining integral in equation (a-34) is now expanded in the following form.

$$\frac{1}{[\eta + (AR)'\theta^2]^2 + \frac{a}{J_aJ}\eta^2} = \frac{1}{(1 + \frac{a}{J_aJ})\eta^2}[1 - \frac{2(AR)'}{(1 + \frac{a}{J_aJ})\eta}\theta^2 + \frac{(4\eta - 1 - a/J_aJ)(AR)'^2}{(1 + a/J_aJ)^2\ \eta^2}\theta^4 + \dots] \tag{a-35}$$

This expansion can then be substituted in the equation (a-34) for T and the integration carried out to give,

$$T = \frac{\varepsilon^3}{3}\ell og\left\{[\eta + (AR)'\varepsilon^2]^2 + \frac{a}{J_aJ}\eta^2\right\} - \frac{4(AR)'}{3(1 + \frac{a}{J_aJ})\eta^2}[\eta\ \frac{\varepsilon^5}{5} - (AR)'\frac{1 - a/J_aJ}{1 + a/J_aJ}\ \frac{\varepsilon^7}{7} - \frac{(AR)'^2(3 + 3a/J_aJ - 4\eta)}{(1 + a/J_aJ)^2\eta}\ \frac{\varepsilon^9}{9}] \tag{a-36}$$

The integrand of the integral denoted by U in equation (a-33) is an improper fraction. After applying long division the integration can be made directly and yields,

$$U = \frac{\varepsilon^3}{3} - \frac{\eta(1 + \frac{a}{J_a J})}{(AR)'}\varepsilon + [\frac{\eta(1 + \frac{a}{J_a J})}{(AR)'}]^{3/2} \tan^{-1} \frac{\varepsilon}{\sqrt{\frac{\eta(1 + \frac{a}{J_a J})}{(AR)'}}} \qquad \text{(a-37)}$$

The evaluation of the integral is now complete. Substituting equations (a-36) and (a-37) into equation (a-33), and reducing to simplest terms, the expression for S is,

$$S = \eta\varepsilon + (AR)'(a/J_a J)^{3/2} \frac{\varepsilon^3}{3} - \frac{(AR)'}{2}\Big[\frac{\varepsilon^3}{3} \log \{[\eta + (AR)'\varepsilon^2]^2 + a\eta^2/J_a J\} - \frac{4(AR)'}{3(1 + a/J_a J)\eta^2}[\frac{\eta\varepsilon^5}{5} - (AR)'\frac{1 - a/J_a J}{1 + a/J_a J}\frac{\varepsilon^7}{7} - \frac{(AR)'^2(3 + 3a/J_a J - 4\eta)}{(1 + a/J_a J)^2\eta}\frac{\varepsilon^9}{9}]\Big] \qquad \text{(a-38)}$$

The calculation of the integrals I_4 and I'_4 for the range of the variables τ and θ near the origin has now been completed and it is found that both of these integrals are given by,

$$I_4 = I'_4 = 2(R - i\Upsilon S) \qquad (\tau \text{ and } \theta \ll 1) \qquad \text{(a-39)}$$

where R is given in equation (a-27) and S in equation (a-38).

It is necessary to select values of η and ε which are compatible with the approximations made to obtain equation (a-39) for I_4 and I'_4. It was assumed that $\sin \Upsilon\tau \cong \Upsilon\tau$ and $\cos \Upsilon\tau \cong 1$. In this numerical example $\Upsilon = 2\pi$. Hence the approximation requires that $\sin 2\pi\tau \cong 2\pi\tau$ and $\cos 2\pi\tau \cong 1$. If $\tau = .05/\pi$, the approximation $\sin 2\pi\tau = 2\pi\tau$ is in error by one part in one thousand and the approximation $\cos 2\pi\tau = 1$ is in error by five parts in one thousand. The approximation was also made that $\sin\theta = \theta$ and $\cos\theta = 1$. If $\theta = 3.75$ degrees this approximation is better than that for τ. Finally it was assumed that

$$\frac{1}{\sqrt{[\tau + (AR)'\theta^2]^2 + \xi^2}} = \frac{1}{\xi}$$

With $\tau = .05/\pi$ and $\theta = 3.75$ degrees $= .065445$ radians,

$$[\tau + (AR)'\theta^2]^2 = 0.0002538$$

In the numerical example, $\xi^2 = .16$, $.36$, and 2.56. Taking the smallest value,

$$\frac{1}{\sqrt{[\tau + (AR)'\theta^2]^2 + \xi^2}} = 2.498$$

The value of this factor has been taken to be $1/0.4 = 2.500$ so that the approximation is in error by two parts in 2500. Therefore, it is seen that sufficient accuracy is obtained by taking $\eta = .05/\pi$ and $\varepsilon = 3.75$ degrees.

Introducing the above values for the limits into equation (a-39), and using the numerical values of the parameters chosen for this example equation (a-39) yields,

$$I_4 = I'_4 = 0.93352 - 0.21726\, i \qquad \text{(a-40)}$$

Equations (a-17) for I_4 and I'_4 have now been integrated over the range of variables near the origin. It was previously shown that the contribution to the integrals from the region in which τ and θ are large is negligible. It therefore remains to evaluate the intermediate region in order to complete the integration for the purposes of the numerical example. With the numerical values of the limits introduced and the integrands suppressed for the time being, one can write the integrals which are to be evaluated numerically in the following way.

$$\int_0^{3.75^\circ}\int_{.05/\pi}^{20}[\quad]d\tau d\theta + \int_{3.75^\circ}^{\pi/2}\int_0^{20}[\quad]d\tau d\theta \tag{a-41}$$

The integrand, indicated in the above by square brackets, and with the numerical values of the parameters introduced is, in the case of I_4,

$$e^{-2\pi i\gamma\tau}\,\frac{(\tau + .046865\sin^2\theta)\cos\pi\tau + .36\pi\sin\pi\tau}{(\tau + .046865\sin^2\theta)^2 + .36\sin^2\pi\tau}$$
$$\times\left\{\frac{1 - .6\cos\pi\tau}{\sqrt{(\tau + .046865\sin^2\theta)^2 + 1.36 - 1.2\cos\pi\tau}} - \frac{1 + .6\cos\pi\tau}{\sqrt{(\tau + .046865\sin^2\theta)^2 + 1.36 + 1.2\cos\pi\tau}} + \frac{1.2\cos\pi\tau}{\sqrt{(\tau + .046865\sin^2\theta)^2 + .36}}\right\}\cos^2\theta \tag{a-42}$$

and in the case of I'_4,

$$e^{-2\pi i\gamma\tau}\,\frac{(\tau + .046865\sin^2\theta)\cos\pi\tau + .36\sin\pi\tau}{(\tau + .046865\sin^2\theta)^2 + .36\sin^2\pi\tau}$$
$$\times\left\{\frac{1 - .6\cos\pi\tau}{\sqrt{(\tau + .046865\sin^2\theta)^2 + 1.36 - 1.2\cos\pi\tau}} - \frac{1 + .6\cos\pi\tau}{\sqrt{(\tau + .046865\sin^2\theta)^2 + 1.36 + 1.2\cos\pi\tau}} + \frac{1.2\cos\pi\tau}{\sqrt{(\tau + .046865\sin^2\theta)^2 + .36}}\right\}\cos^4\theta \tag{a-43}$$

Then, after carrying out the integrations outlined in equation (a-41) and adding the values in (a-40) it is found that,

$$I_4 = 7.1065 - 1.9752\, i$$
$$I'_4 = 6.0391 - 1.6968\, i \qquad \text{(a-44)}$$

A-4 Summary of results for the numerical example.

With the numerical integrations completed it is now possible to substitute the values into the equations for the lift and to form comparisons with the two dimensional theory. Thus, for this particular case, the terms in equations (6.4) and (6.5) have the following values,

$$1 + i\nu/2 = 1.000 + 0.1472\, i$$
$$1 + G_{0,2} = 1.253 + 0.5993\, i$$
$$1 + G_1 = 1.216 + 0.5160\, i$$
$$\nu = 0.29446$$

and in equations (6.7) and (6.8), the ratio of the Bessel functions has the value

$$\frac{K_1(i\nu/2)}{K_0(i\nu/2) + K_1(i\nu/2)} = 0.775 - 0.186\, i$$

Using the above values, the results can be tabulated in the form below. L_o, L_2, and L_1 refer to the quasi-steady, induced, and apparent mass lift as described on page 9.

		Two Dimensions	Three Dimensions
$\frac{1}{2\pi}[C_{L_0}^{(T)} + C_{L_2}^{(T)}]$	=	.775 - .186 i	.6952 - .2150 i
$\frac{1}{2\pi}[C_{L_1}^{(T)}]$	=	.07362 I	.01251 + .05886 i
$\frac{1}{2\pi}[C_{L_{total}}^{(T)}]$	=	.775 - .1124 i	.7077 - .1561 i
$\frac{1}{2\pi}[C_{L_0}^{(R)} + C_{L_2}^{(R)}]$	=	− 1.752 - 10.71 i	-2.225 - 9.659 i
$\frac{1}{2\pi}[C_{L_1}^{(R)}]$	=	1.000	.8118 + .1847 i
$\frac{1}{2\pi}[C_{L_{total}}^{(R)}]$	=	-.752 - 10.71 i	-1.413 - 9.474 i

The quasi-steady component of the lift, L_o, is in phase with the motion. Therefore, it is convenient to express the components L_1 and L_2 in ratio with L_o. For the two dimensional case,

$$\frac{1}{2\pi} C_{L_0}^{(T)} = 1.000$$

$$\frac{1}{2\pi} C_{L_0}^{(R)} = 1 - \frac{4i}{\nu} \tag{a-45}$$

Therefore, the desired ratios have the following values for this numerical example.

		Two Dimensions	Three Dimensions
$[C_{L_0}^{(T)} + C_{L_2}^{(T)}] / C_{L_0}^{(T)}$	=	.775 - .186 i	.6952 - .2150 i
$[C_{L_1}^{(T)}] / C_{L_0}^{(T)}$	=	.07362 i	.01251 + .05886 i
$[C_{L_{total}}^{(T)}] / C_{L_0}^{(T)}$	=	.775 - .1124 i	.7077 - .1561 i
$[C_{L_0}^{(R)} + C_{L_2}^{(R)}] / C_{L_0}^{(R)}$	=	.775 - .186 i	.6952 - .2150 i

		Two Dimensions	Three Dimensions
$[\, C_{L_1}^{(R)} \,] / C_{L_0}^{(R)}$	=	.00542 + .07361 i	-.00920 + .06076 i
$[\, C_{L_{total}}^{(R)} \,] / C_{L_0}^{(R)}$	=	.7804 - .1024 i	.6860 - .1542 i

The above results are indicated in figures A-1 and A-2 in graphical form. The angle which the vectors make with the horizontal axis is the phase angle between the particular lift component and the blade element motion.

APPENDIX B

MECHANICAL QUADRATURE

In applying the methods of numerical integration to a double definite integral it is found to be unnecessary to develop a special method, or formula, for handling such a problem and that the rules of Simpson and Weddle are adequate.

The analysis for reaching the above conclusion is given in reference (18) and will be outlined here.

Suppose that one wishes to integrate a function, $z = f(x,y)$ of the two independent variables x and y over a rectangular domain. It is first possible to develop a double interpolation formula for application in this problem. It is clear that the development of a double interpolation formula must be preceded by a tractable definition of double differences. To this end let $z=f(x,y)$ be any function and let $z_{rs} = f(x_r, y_s)$. The z_{rs} then represent the elements of a rectangular array each element of which corresponds to a preassigned value of x and a preassigned value of y. Or, graphically, the z_{rs} are the elevations of the function surface described by the function $z = f(x,y)$ above the x-y plane. Every regular surface can then be described as closely as one pleases by taking a sufficient number of the elements z_{rs}. Double differences in the array can then be <u>defined</u> by,

$$\Delta^{1+0}z_{rs} = \Delta_x z_{rs} = z_{r+1,s} - z_{rs}$$

$$\Delta^{0+1}z_{rs} = \Delta_y z_{rs} = z_{r,s+1} - z_{rs}$$

$$\Delta^{2+0}z_{rs} = \Delta_x^2 z_{rs} = z_{r+2,s} - 2z_{r+1,s} + z_{rs}$$

$$\Delta^{0+2}z_{rs} = \Delta_y^2 z_{rs} = z_{r,s+2} - 2z_{r,s+1} + z_{rs}$$

$$\Delta^{2+1}z_{rs} = \Delta^{2+0}z_{r,s+1} - \Delta^{2+0}z_{rs}$$

$$\Delta^{1+2}z_{rs} = \Delta^{0+2}z_{r+1,s} - \Delta^{0+2}z_{rs}$$

$$\Delta^{3+0}z_{rs} = \Delta_x^3 z_{rs} = z_{r+3,s} - 3z_{r+2,s} + 3z_{r+1,s} - z_{rs}$$

$$\Delta^{0+3}z_{rs} = \Delta_y^3 z_{rs} = z_{r,s+3} - 3z_{r,s+2} + 3z_{r,s+1} - z_{rs}$$

$$\Delta^{3+1}z_{rs} = \Delta^{3+0}z_{r,s+1} - \Delta^{3+0}z_{rs}$$

$$\Delta^{1+3}z_{rs} = \Delta^{0+3}z_{r+1,s} - \Delta^{0+3}z_{rs}$$

$$\Delta^{4+0}z_{rs}=\Delta_x^4 z_{rs}=z_{r+4,s} - 4z_{r+3,s} + 6z_{r+2,s} - 4z_{r+1,s} + z_{rs}$$

$$\Delta^{0+4}z_{rs}=\Delta_y^4 z_{rs}=z_{r,s+4} - 4z_{r,s+3} + 6z_{r,s+2} - 4z_{r,s+1} + z_{rs}$$

$$\begin{aligned}\Delta^{2+2}z_{rs} &= \Delta^{2+0}z_{r,s+2} - 2\Delta^{2+0}z_{r,s+1} + \Delta^{2+0}z_{rs} \\ &= \Delta^{0+2}z_{r+2,s} - 2\Delta^{0+2}z_{r+1,s} + \Delta^{0+2}z_{rs}\end{aligned}$$

And, in general,

$$\begin{aligned}\Delta^{m+n}z_{rs} &= \Delta^{m+0}z_{r,n} - n\Delta^{m+0}z_{r,n-1} + \frac{n(n-1)}{2}\Delta^{m+0}z_{r,n-2} \\ &\quad +\ldots\ldots+ \Delta^{m+0}z_{rs} \\ &= \Delta^{0+n}z_{m,s} - m\Delta^{0+n}z_{m-1,s} + \frac{m(m-1)}{2}\Delta^{0+n}z_{m-2,s} \\ &\quad +\ldots\ldots+ \Delta^{0+n}z_{rs}\end{aligned}$$

The significance of the above double differences is best seen by analogy with single differences. For example, if y_o, y_1, ..., y_n denote a set of values of any function $y = f(x)$ then the single diagonal differences are constructed as follows. By definition, $y_1 - y_o$, $y_2 - y_1$, ..., $y_n - y_{n-1}$ are called the first differences of the function $y = f(x)$. These first differences are then denoted in general by,

$$\Delta y_r = y_{r+1} - y_r$$

The second differences of the function y are formed by taking the difference of the first differences. Thus,

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r = y_{r+2} - 2y_{r+1} + y_r$$

And, similarly, the third differences are

$$\Delta^3 y_r = \Delta^2 y_{r+1} - \Delta^2 y_r = y_{r+3} - 3y_{r+2} + 3y_{r+1} - y_r$$

and so forth.

A general formula for double interpolation is derived in O. Biermann's, Mathematische Naherungsmethoden, and in terms of the above definitions of the double differences is,

$$z = f(x\ y) = z_{00} + \frac{x-x_0}{h}\Delta^{1+0} z_{00} + \frac{y-y_0}{k}\Delta^{0+1} z_{00}$$

$$+ \frac{1}{2!}\left[\frac{(x-x_0)(x-x_1)}{h^2}\Delta^{2+0} z_{00} + \frac{2(x-x_0)(y-y_0)}{hk}\Delta^{1+1} z_{00} +\right.$$

$$+ \frac{(y-y_0)(y-y_1)}{k^2}\Delta^{0+2}z_{00}] + \cdots + \frac{1}{m!}[\frac{(x-x_0)(x-x_1)\cdots(x-x_{m-1})}{h^m}\Delta^{m+0}z_{00}$$

$$+ \frac{m(x-x_0)(x-x_1)\cdots(x-x_{m-2})(y-y_0)}{h^{m-1}k}\Delta^{(m-1)+1}z_{00}$$

$$+ \frac{m(m-1)(x-x_0)(x-x_1)\cdots(x-x_{m-3})(y-y_0)(y-y_1)}{h^{m-2}k^2}\Delta^{(m-2)+2}z_{00}$$

$$+ \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{m-1})}{k^m}\Delta^{0+m}z_{00}] + R(x_0,y_0)$$

where h and k are the equidistant intervals in x and y respectively and $R(x_0,y_0)$ is the remainder term.

The above equation can be written more simply by making the following changes of variable. Let,

$$u = \frac{x-x_0}{h} \qquad\qquad v = \frac{y-y_0}{k}$$

From these definitions of u and v it follows that,

$$\frac{x-x_{m-1}}{h} = u-(m-1) \qquad\qquad \frac{y-y_{m-1}}{k} = v-(m-1)$$

Then, in terms of the new variables u and v, the formula for double interpolation becomes,

$$z = f(x,y) = f(x_0+hu, y_0+kv) = z_{00} + u\Delta^{1+0}z_{00}$$
$$+ v\Delta^{0+1}z_{00} + \frac{1}{2!}[u(u-1)\Delta^{2+0}z_{00} + 2uv\Delta^{1+1}z_{00}$$
$$+ v(v-1)\Delta^{0+2}z_{00}] + \frac{1}{3!}[u(u-1)(u-2)\Delta^{3+0}z_{00}$$
$$+ 3u(u-1)v\Delta^{2+1}z_{00} + 3uv(v-1)\Delta^{1+2}z_{00}$$
$$+ v(v-1)(v-2)\Delta^{0+3}z_{00}] + \frac{1}{4!}[u(u-1)(u-2)(u-3)\Delta^{4+0}z_{00}$$
$$+ 4u(u-1)(u-2)v\Delta^{3+1}z_{00} + 6u(u-1)v(v-1)\Delta^{2+2}z_{00}$$
$$+ 4uv(v-1)(v-2)\Delta^{1+3}z_{00} + v(v-1)(v-2)(v-3)\Delta^{0+4}z_{00}]$$
$$+ \cdots\cdots$$

Now the result that is to be obtained, as indicated in the first paragraph of this appendix, is made evident if one integrates the above interpolation formula over two intervals in the x direction and two in the y direction. Thus,

$$I = \int_{x_0}^{x_0+2h} \int_{y_0}^{y_0+2k} f(x,y)\,dx\,dy = hk \int_0^2 \int_0^2 f(u,v)\,du\,dv$$

It is seen that, over the above limited ranges of integration, the terms involving the differences $\Delta^{3+0}, \Delta^{0+3}, \Delta^{4+0}, \Delta^{3+1}$, ... will not enter. Then,

$$I = \frac{hk}{9}[36z_{00} + 36\Delta^{1+0}z_{00} + 36\Delta^{0+1}z_{00} + 36\Delta^{1+1}z_{00} + 6\Delta^{2+0}z_{00} + 6\Delta^{0+2}z_{00} + 6\Delta^{2+1}z_{00} + 6\Delta^{1+2}z_{00} + \Delta^{2+2}z_{00}]$$

which, upon substituting the values for the differences becomes,

$$I = \frac{hk}{9}[z_{00} + z_{02} + z_{22} + z_{20} + 4(z_{01} + z_{12} + z_{21} + z_{10}) + 16z_{11}]$$

The similarity between this expression and the Simpson's rule for a function of a single independent variable should be noted. Also, note that the formula for I can be written in either of the following ways,

$$I = \frac{h}{3}[\frac{k}{3}(z_{00} + 4z_{01} + z_{02}) + \frac{4k}{3}(z_{10} + 4z_{11} + z_{12}) + \frac{k}{3}(z_{20} + 4z_{21} + z_{22})]$$

or,

$$I = \frac{k}{3}[\frac{h}{3}(z_{00} + 4z_{10} + z_{20}) + \frac{4h}{3}(z_{01} + 4z_{11} + z_{21}) + \frac{h}{3}(z_{02} + 4z_{12} + z_{22})]$$

Then it is apparent that $\frac{k}{3}(z_{00} + 4z_{01} + z_{02})$ is Simpson's rule applied to the first column of the array, $\frac{h}{3}(z_{00} + 4z_{10} + z_{20})$ is

Simpson's rule applied to the first row of the array, and similarly for the other such terms in the expressions for I. Therefore, if one designates the results of applying Simpson's rule to the columns by A_0, A_1, A_2; and the results of applying the rule to the rows by B_0, B_1, B_2, it is clear that the expressions for I can be written,

$$I = \frac{h}{3}[A_0 + 4A_1 + A_2]$$

or,

$$I = \frac{k}{3}[B_0 + 4B_1 + B_2]$$

Since the above analysis can be extended to rectangular arrays of an arbitrary number of elements, the conclusion is that a function of two independent variables can be numerically integrated over a rectangular domain by means of the formulae,

$$\int_{x_0}^{x_0+nh}\int_{y_0}^{y_0+nk} f(x,y)dxdy = \frac{h}{3}[A_0 + 4A_1 + 2A_2 + A_3]$$

or,

$$\int_{x_0}^{x_0+nh}\int_{y_0}^{y_0+nk} f(x,y)dxdy = \frac{k}{3}[B_0 + 4B_1 + 2B_2 + B_3]$$

where,

$$A_0 = \frac{k}{3}[z_{00} + 4(z_{01} + z_{03} +\dots+ z_{0,m-1}) + 2(z_{02} + z_{04} +\dots+ z_{0,m-2}) + z_{0m}]$$

$$A_1 = \frac{k}{3}\{[z_{10} + 4(z_{11} + z_{13} + \dots + z_{1,m-1}) + 2(z_{12} + z_{14} + \dots + z_{1,m-2}) + z_{1m}] + [z_{30} + 4(z_{31} + z_{33} + \dots + z_{3,m-1}) + 2(z_{32} + z_{34} + \dots + z_{3,m-2}) + z_{3m}] + \dots + [z_{n-1,0} + 4(z_{n-1,1} + z_{n-1,3} + \dots + z_{n-1,m-1}) + 2(z_{n-1,2} + z_{n-1,4} + \dots + z_{n-1,m-2}) + z_{n-1,m}]\}$$

$$A_2 = \frac{k}{3}\{[z_{20} + 4(z_{21} + z_{23} + \dots + z_{2,m-1}) + 2(z_{22} + z_{24} + \dots + z_{2,m-2}) + z_{2m}] + [z_{40} + 4(z_{41} + z_{43} + \dots + z_{4,m-1}) + 2(z_{42} + z_{44} + \dots + z_{4,m-2}) + z_{4m}] + \dots + [z_{n-2,0} + 4(z_{n-2,1} + z_{n-2,3} + \dots + z_{n-2,m-1}) + 2(z_{n-2,2} + z_{n-2,4} + \dots + z_{n-2,m-2}) + z_{n-2,m}]\}$$

$$A_3 = \frac{k}{3}\{z_{n0} + 4(z_{n1} + z_{n3} + \dots + z_{n,m-1}) + 2(z_{n2} + z_{n4} + \dots + z_{n,m-2}) + z_{nm}\}$$

The formulae for B_0, B_1, B_2, and B_3 are completely analogous so that the integration can be carried out in either order first by columns or first by rows.

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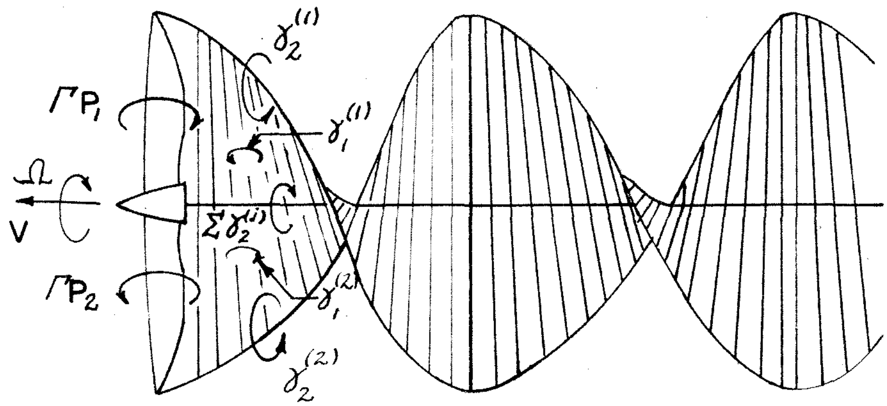


FIG. 2.1
APPROXIMATE VORTEX DISTRIBUTION OF A TWO BLADED PROPELLER IN NON-STATIONARY MOTION

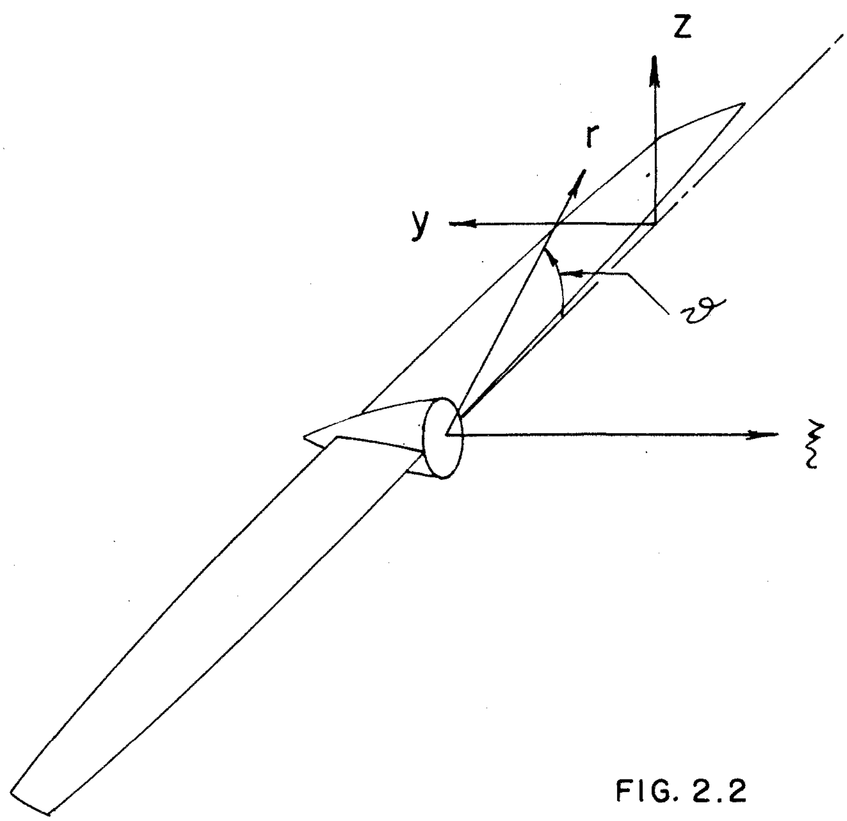


FIG. 2.2
COORDINATE SYSTEMS

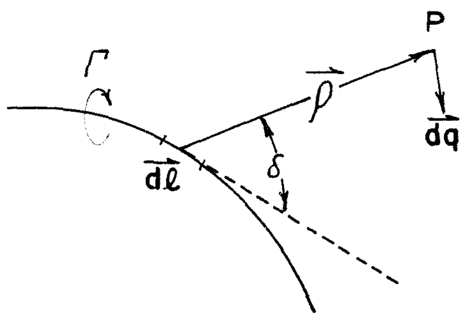


FIG. 3.1
QUANTITIES IN THE
BIOT-SAVART RELATION

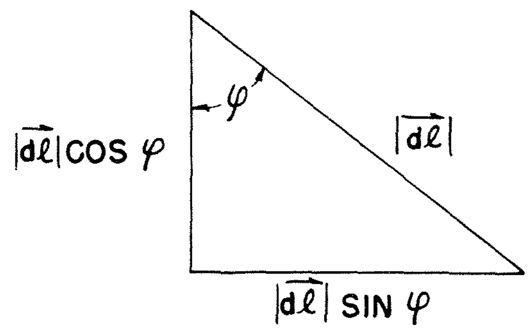


FIG. 3.2

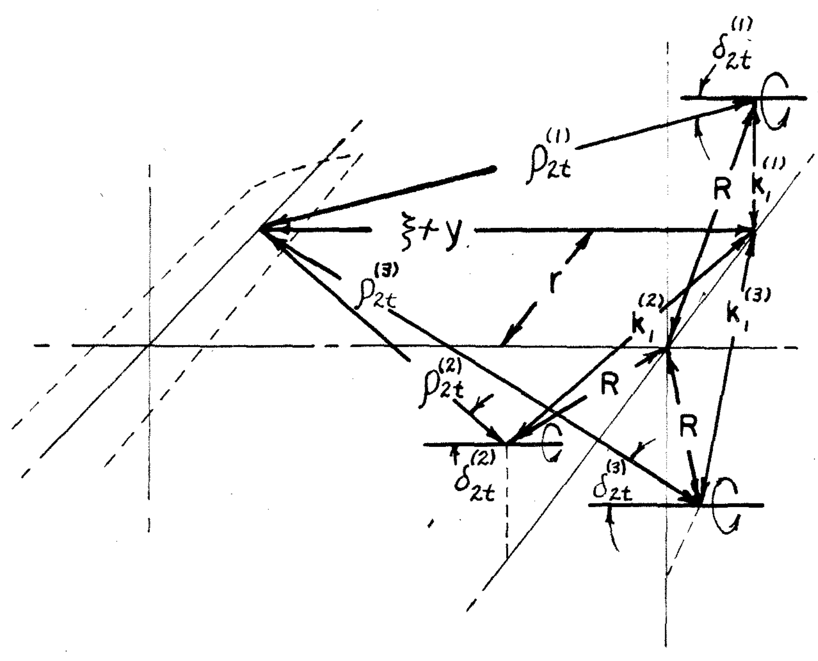


FIG. 3.3

GEOMETRY OF INDUCED DOWNWASH
DUE TO TRAILING VORTICITY

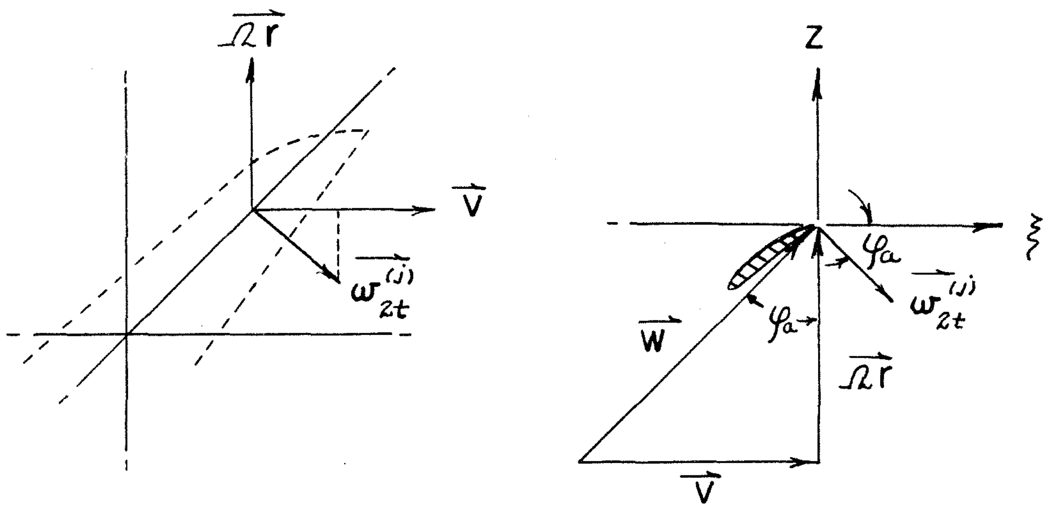


FIG. 3.4
DIRECTION COSINES OF $\overrightarrow{\omega_{zt}^{(j)}}$

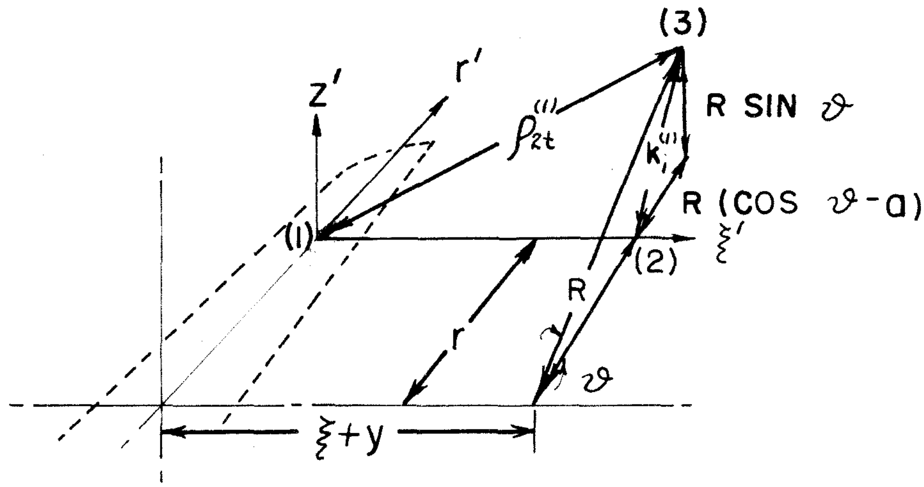


FIG. 3.5
DIRECTION COSINES OF $\overrightarrow{q_{zt}^{(i)}}$

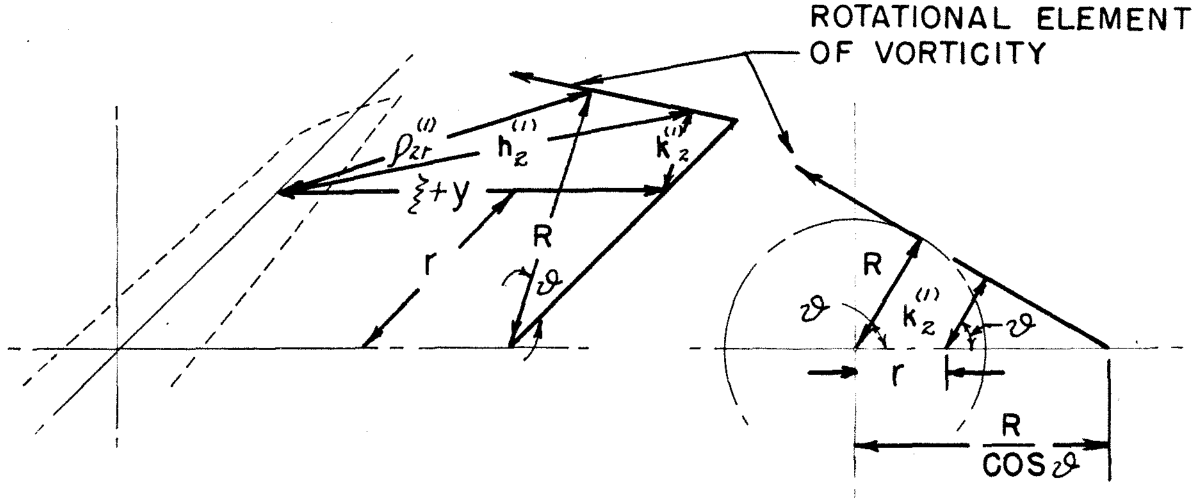


FIG. 3.6
GEOMETRY OF INDUCED DOWNWASH DUE TO ROTATIONAL COMPONENT OF TRAILING VORTICITY

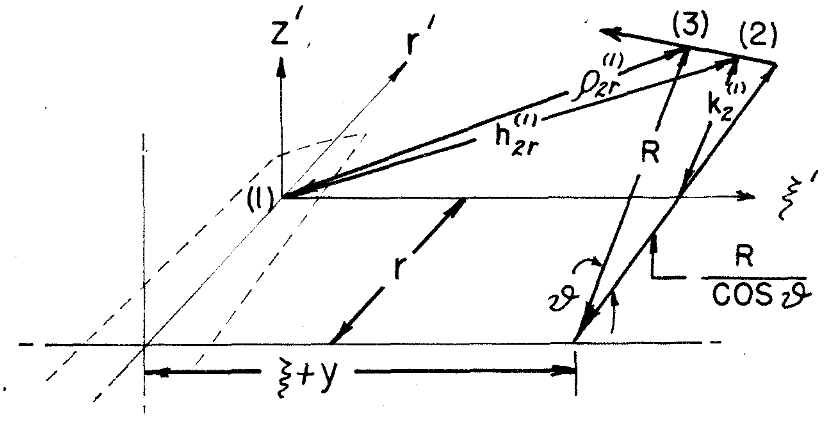


FIG. 3.7
DIRECTION COSINES OF $\overline{q_{2r}^{(j)}}$

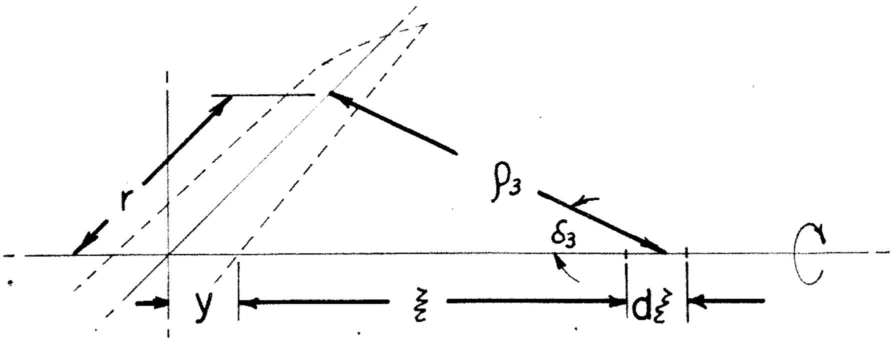


FIG. 3.8
GEOMETRY OF THE ROOT
TRAILING VORTICITY

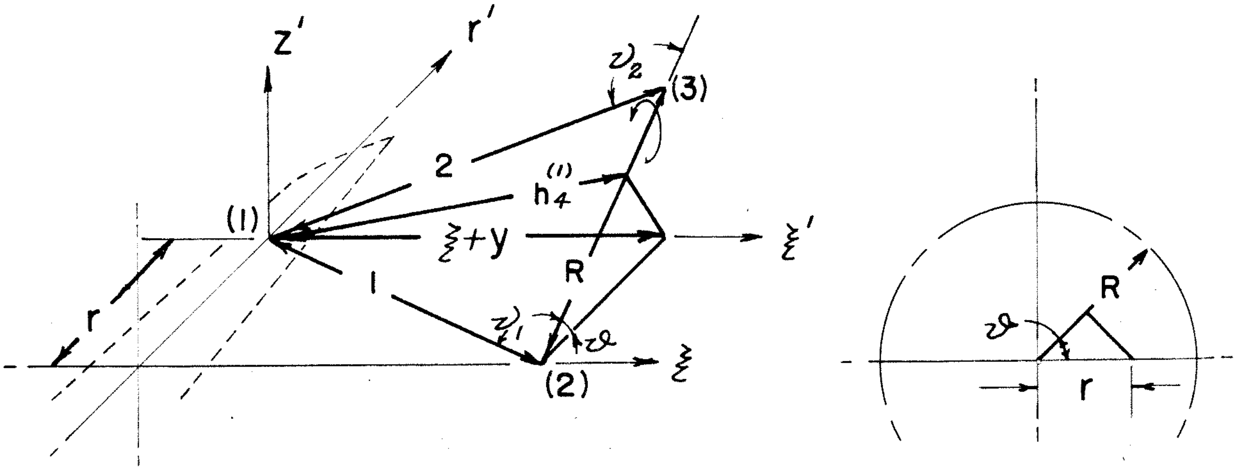


FIG. 3.9

GEOMETRY OF THE INDUCED DOWNWASH
DUE TO SHED VORTICITY.
DIRECTION COSINES OF $\overrightarrow{q_4^{(j)}}$

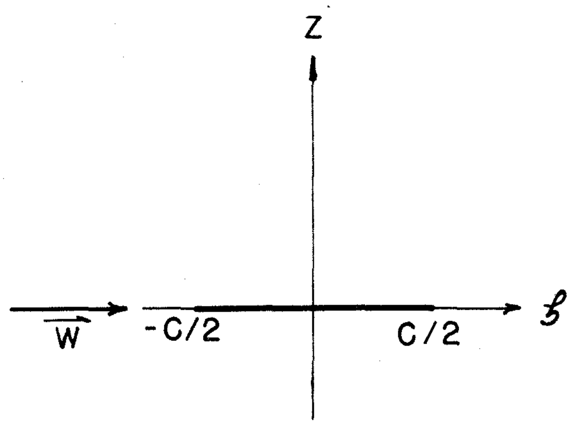


FIG. 4.1

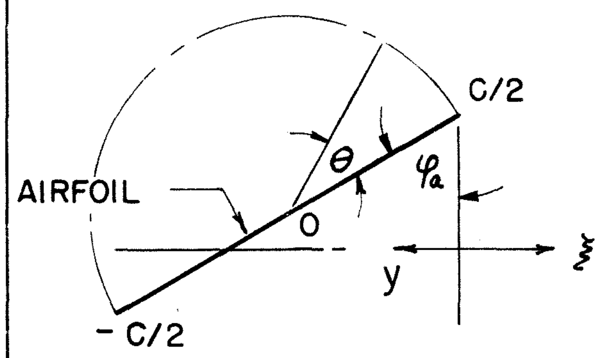


FIG. 4.2
RELATION BETWEEN
COORDINATES y AND θ

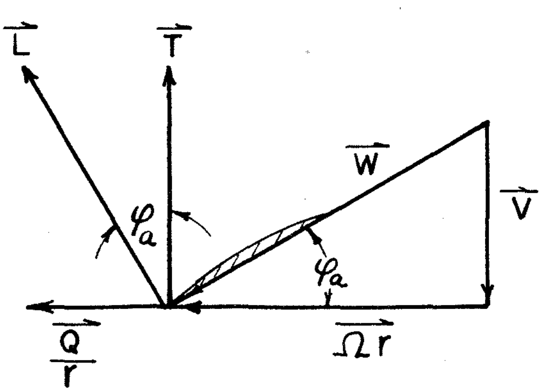


FIG. 7.1
RESOLUTION OF LIFT
INTO THRUST AND TORQUE

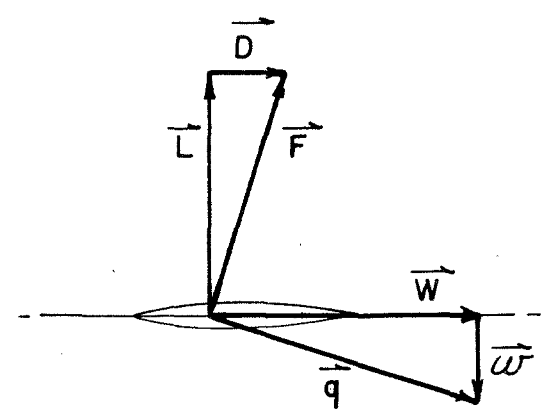


FIG. 7.2
FORCES ON AN AIRFOIL
ELEMENT IN THREE
DIMENSIONAL MOTION

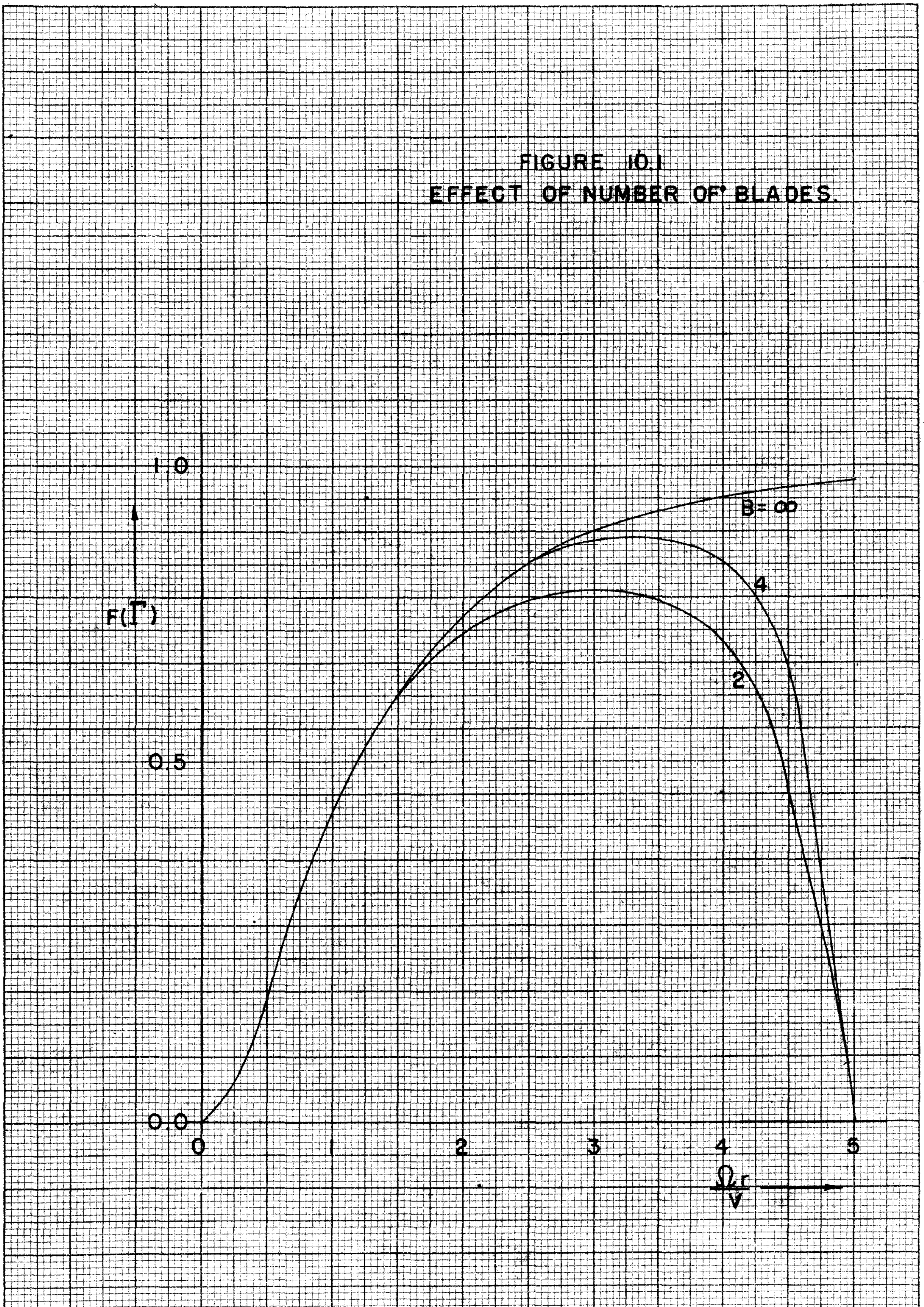
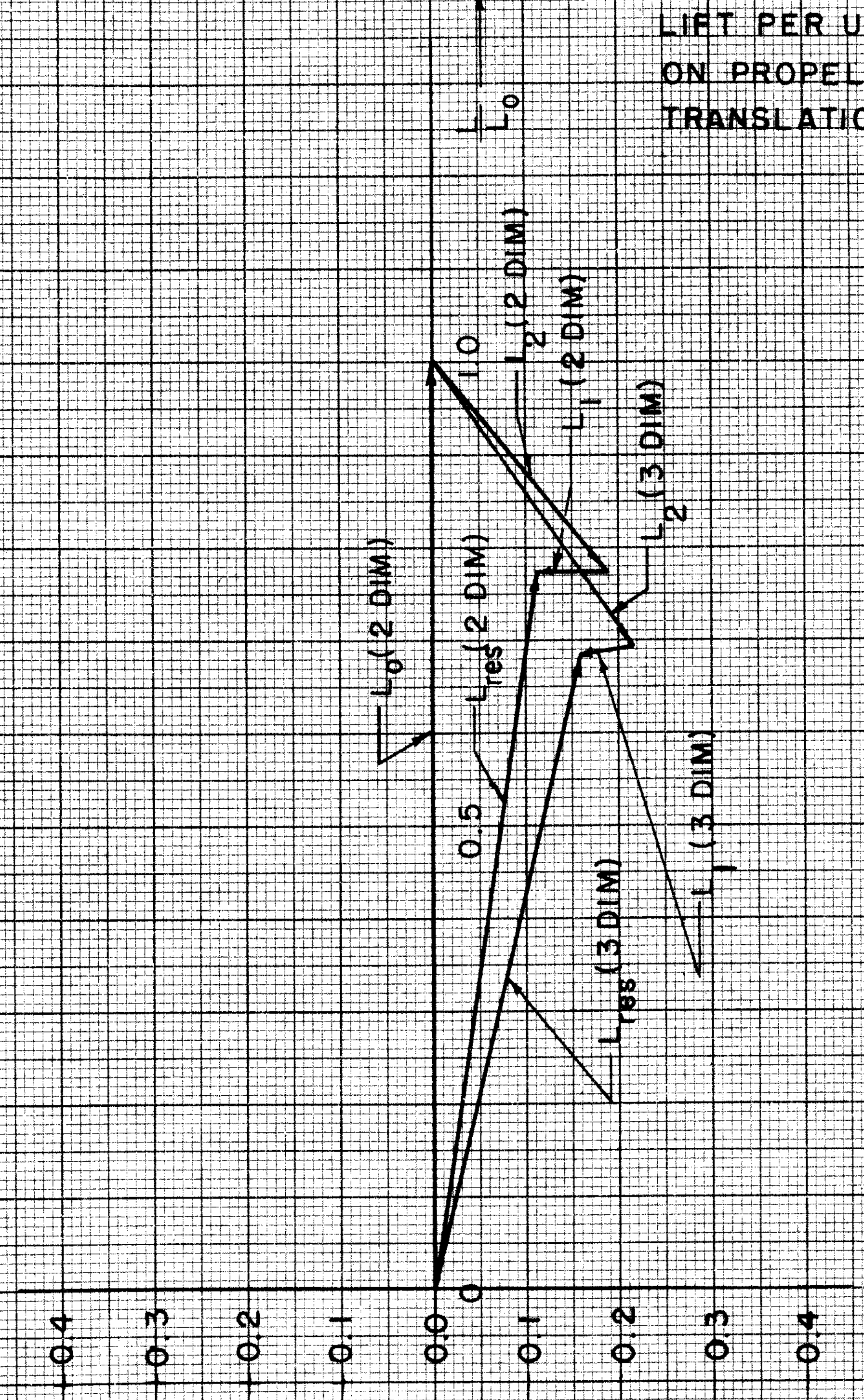


FIGURE 10.1
EFFECT OF NUMBER OF BLADES.
1.0
0.5
0.0
F(Γ)
B = ∞
4
2
0
1
2
3
4
5
Ωr/V

FIGURE A.1
LIFT PER UNIT LENGTH
ON PROPELLER BLADE ELEMENT.
TRANSLATIONAL OSCILLATIONS.



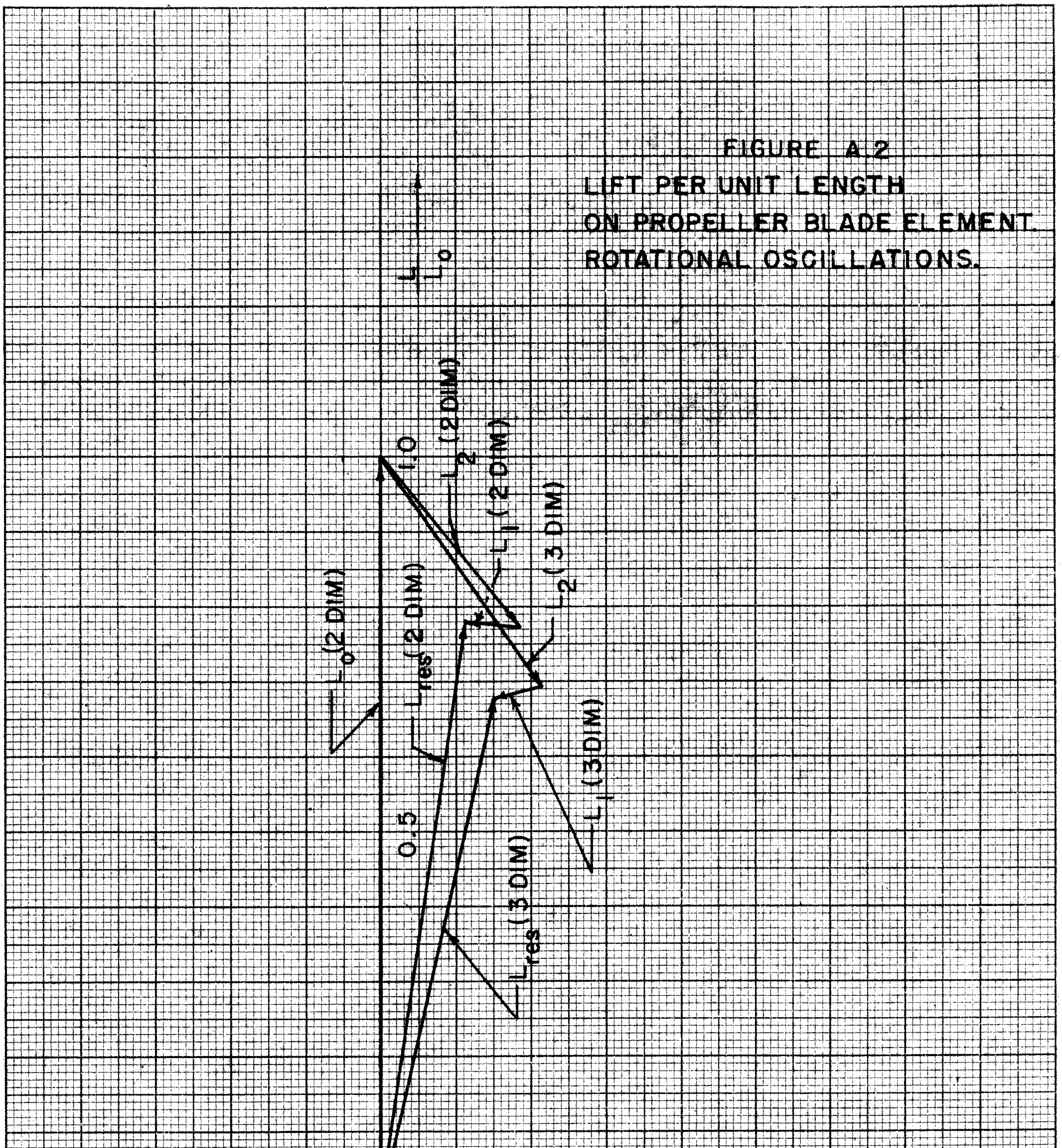


FIGURE A.2
LIFT PER UNIT LENGTH
ON PROPELLER BLADE ELEMENT
ROTATIONAL OSCILLATIONS.

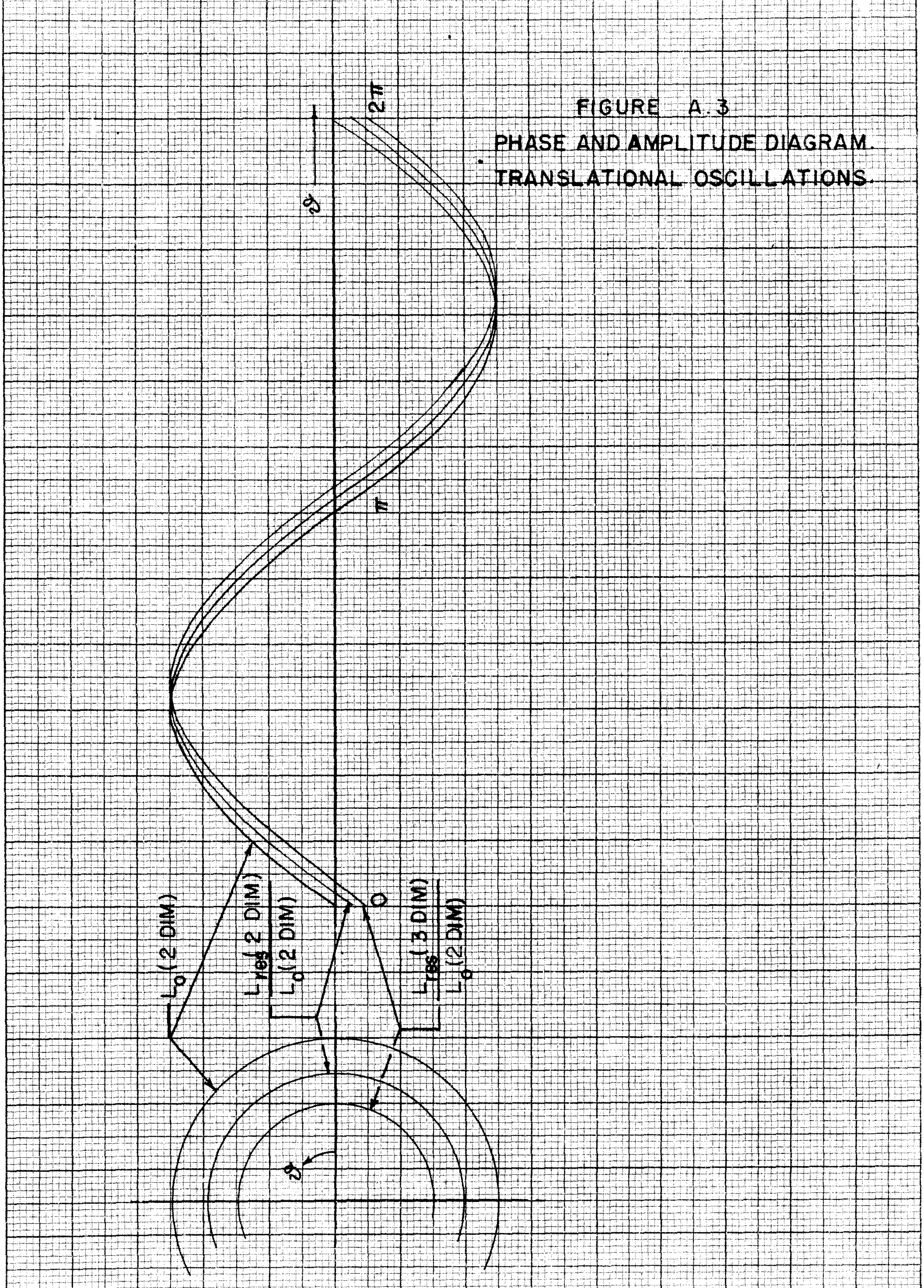


FIGURE A.3
PHASE AND AMPLITUDE DIAGRAM.
TRANSLATIONAL OSCILLATIONS.

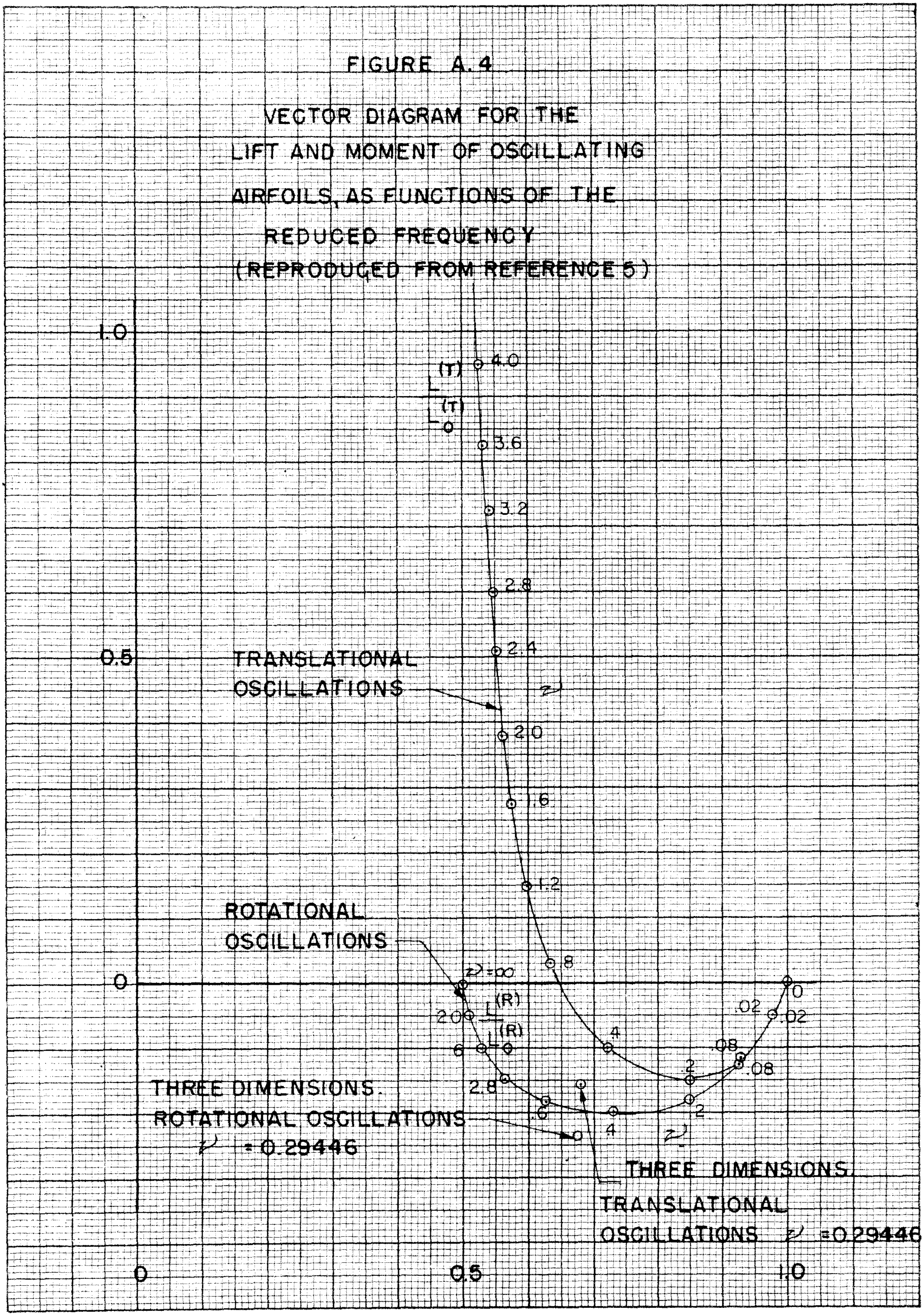


FIGURE A.4
VECTOR DIAGRAM FOR THE
LIFT AND MOMENT OF OSCILLATING
AIRFOILS, AS FUNCTIONS OF THE
REDUCED FREQUENCY
(REPRODUCED FROM REFERENCE 5)
1.0
0.5
0
4.0
3.6
3.2
2.8
2.4
2.0
1.6
1.2
.8
.4
.2
.08
.02
TRANSLATIONAL
OSCILLATIONS
ROTATIONAL
OSCILLATIONS
THREE DIMENSIONS.
ROTATIONAL OSCILLATIONS
= 0.29446
THREE DIMENSIONS.
TRANSLATIONAL
OSCILLATIONS = 0.29446
0
0.5
1.0