Geometric Quantization and Foliation Reduction

Thesis by

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To the memories of
Angela Skerritt,
Patrick Burke,
Bob O'Rourke,
and Jerrold E. Marsden.

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The commutative diagrams in this thesis were produced using Paul Taylor's Diagrams package version 3.94.

Abstract

A standard question in the study of geometric quantization is whether symplectic reduction interacts nicely with the quantized theory, and in particular whether "quantization commutes with reduction." Guillemin and Sternberg first proposed this question, and answered it in the affirmative for the case of a free action of a compact Lie group on a compact Kähler manifold. Subsequent work has focused mainly on extending their proof to non-free actions and non-Kähler manifolds. For realistic physical examples, however, it is desirable to have a proof which also applies to non-compact symplectic manifolds.

In this thesis we give a proof of the quantization-reduction problem for general symplectic manifolds. This is accomplished by working in a particular wavefunction representation, associated with a polarization that is in some sense compatible with reduction. While the polarized sections described by Guillemin and Sternberg are nonzero on a dense subset of the Kähler manifold, the ones considered here are distributional, having support only on regions of the phase space associated with certain quantized, or "admissible", values of momentum.

We first propose a reduction procedure for the prequantum geometric structures that "covers" symplectic reduction, and demonstrate how both symplectic and prequantum reduction can be viewed as examples of foliation reduction. Consistency of prequantum reduction imposes the above-mentioned admissibility conditions on the quantized momenta, which can be seen as analogues of the Bohr-Wilson-Sommerfeld conditions for completely integrable systems.

We then describe our reduction-compatible polarization, and demonstrate a one-to-one correspondence between polarized sections on the unreduced and reduced spaces.

Finally, we describe a factorization of the reduced prequantum bundle, suggested by the structure of the underlying reduced symplectic manifold. This in turn induces a factorization of the space of polarized sections that agrees with its usual decomposition by irreducible representations, and so proves that quantization and reduction do indeed commute in this context.

A significant omission from the proof is the construction of an inner product on the space of polarized sections, and a discussion of its behavior under reduction. In the concluding chapter of the thesis, we suggest some ideas for future work in this direction.

Contents

Chapter 1

Introduction

1.1 The motivation for geometric quantization

Geometric quantization has its conceptual roots in two distinct lines of thought. The first line is the notion of constructing irreducible representations of Lie groups using the tools of complex-analytic and differential geometry, such as complex line bundles, connections etc. This approach began with the Borel-Weil Theorem for compact Lie groups [Ser95], which considers the space of holomorphic sections of a certain homogeneous line bundle, and the natural group action on this space. The technique was later significantly generalized and given a symplectic interpretation by Kirillov, Kostant, and others, leading to the so-called orbit method [Kir04] [Kos70] [AK71].

The second line of thought is the attempt to extend the well-studied canonical quantization of $\mathbb{R}^{2n}$ to more general phase spaces. Segal [Seg60] considered the case of the cotangent bundle of an arbitrary configuration manifold, and introduced a quantization scheme by extending the traditional canonical quantization conditions on $\mathbb{R}^{2n}$ to this case. Ultimately Segal's work was subsumed by that of Kostant and Souriau [Kos70] [Sou97]. Implementing Dirac's assertion that quantization of observables should take Poisson brackets to commutators, they introduced the modern notion of geometric quantization for general symplectic manifolds. In the case when the symplectic manifold is a coadjoint orbit of a Lie group, geometric quantization reduces to the method of orbits described above.

The first step in the geometric scheme is called geometric *pre*quantization.

1.2 Geometric prequantization

We begin with a classical system, described by a symplectic manifold (M, ω), and its corresponding classical observables, described by the algebra of smooth real-valued functions $C^\infty(M, \mathbb{R})$ on M. From these, geometric prequantization aims to construct a Hilbert space $(\mathcal{H}, (\cdot, \cdot))$, and a "quantization" map

$$Q_{\cdot} : C^\infty(M, \mathbb{R}) \to i\mathfrak{u}(\mathcal{H}),$$

where $i\mathfrak{u}(\mathcal{H})$ denotes the algebra of self-adjoint operators on $\mathcal{H}$, and describes the quantum observables of our system. By analogy with canonical quantization, the main properties we would like $Q_{\cdot}$ to satisfy are as follows.

(i) The mapping $f \mapsto Q_f$ is linear.

(ii) $[Q_f, Q_g] = i\hbar Q_{\{f, g\}}$, where $[\cdot, \cdot]$ is the commutator and $\{f, g\}$ is the Poisson bracket on $C^\infty(M)$.

(iii) $Q_{\mathbf{1}} = \mathrm{id}_{\mathcal{H}}$, where $\mathbf{1}$ is the constant function with value 1 on M.

The solution to this problem was proposed independently by Souriau [Sou97] and Kostant [Kos70]. We take a line bundle L, a covariant derivative ∇ on the space of smooth sections $\Gamma(L)$, and a ∇-invariant Hermitian form H on L (see Chapter 3 for full definitions of these terms). The Hermitian form induces an inner product

$$(s, t) = \int_M H(s(x), t(x))\, \omega^n$$

on $\Gamma(L)$, where $n = \frac{1}{2} \dim_{\mathbb{R}} M$. $\mathcal{H}$ is taken to be the completion of $\Gamma(L)$ with respect to this inner product, and the quantization $Q_f : \mathcal{H} \to \mathcal{H}$ of the classical observable f is defined to be

$$Q_f = -i\hbar \nabla_{X_f} + f,$$

where X_f denotes the Hamiltonian vector field for f, defined by the relation $\mathrm{i}_{X_f}\omega = \mathrm{d}f$. In order for $Q_{\cdot}$ to satisfy the condition $[Q_f, Q_g] = i\hbar Q_{\{f, g\}}$, the covariant ∇ must be chosen to have curvature $\frac{i}{\hbar}\omega$, meaning that for any vector fields $X, Y \in \Gamma(TM)$,

$$[\nabla_X, \nabla_Y] - \nabla_{[X, Y]} = \frac{i}{\hbar}\omega(X, Y)\, \mathrm{id}_{\mathcal{H}}.$$

Such an L and ∇ exist if and only if $\frac{\omega}{h}$ integrates to an integer over any closed 2-surface in M.

1.3 Geometric quantization

Applying the geometric prequantization procedure to $\mathbb{R}^{2n}$ with its standard symplectic structure $\omega = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i$, where $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ denote the coordinates on $\mathbb{R}^{2n}$, does not agree with the standard canonical quantization prescription. Roughly speaking, the problem is that sections in $\Gamma(L)$ depend on "too many" coordinates. In the position representation, canonical quantization yields a Hilbert space $L^2(\mathbb{R}^n)$, with wavefunctions depending on the position coordinates q^i only. By contrast, the line-bundle sections in geometric prequantization depend on the full set of coordinates $(q^i,\, p_j)$. In general, we would like the sections of our line bundle to depend on a complete set of Poisson-commuting coordinates (in a local sense at least), and they should be invariant along the complementary directions, which also Poisson commute. It is with this reasoning in mind that the concept of a *polarization* is introduced into the quantization procedure. A polarization is a smooth, involutive Lagrangian subbundle of the complexified tangent bundle $TM^{\mathbb{C}}$, i.e., a distribution F satisfying

$$\omega(F,\, F) = 0, \quad [F,\, F] \subset F, \quad \text{and} \quad \dim_{\mathbb{C}} F = \frac{1}{2} \dim_{\mathbb{R}} M.$$

Instead of the full space of smooth sections of L, we consider instead the space $\Gamma_F(L)$ of covariantly constant sections along F,

$$\Gamma_F(L) = \{s \in \Gamma(L) \,|\, \nabla_X s = 0 \text{ for all } X \in \Gamma(F)\}.$$

The inner product of $\Gamma_F(L)$ must also be modified: $F \cap \overline{F}$ can be written as $D^{\mathbb{C}}$ for some *real* subbundle D of TM, and involutivity of F implies involutivity of D (assuming $\dim_{\mathbb{R}}(F \cap TM)$ is constant). By Frobenius' Theorem, D is integrable, and the collection $\mathcal{D}$ of integral submanifolds of D define a foliation of M. Since the Hermitian form H is ∇-invariant, $H(s,\, s)$ will be constant along the leaves of $\mathcal{D}$, for any $s \in \Gamma_F(L)$,

$$X\,(H(s,\, s)) = H\,(\nabla_X s,\, s) + H\,(s,\, \nabla_X s) = 0 \qquad \text{for any } X \in \Gamma(D).$$

If the $\mathcal{D}$-leaves are noncompact, the inner product $(s,\, s) = \int_M H(s(x),\, s(x))\, \omega^n$ will be infinite due to this constancy. For this reason, an inner product $(s,\, t)$ is defined instead by dropping $H(s,\, t)$ to $M/\mathcal{D}$, and integrating against a suitable defined measure on $M/\mathcal{D}$. Construction of this measure is somewhat involved, and requires the introduction of a *metalinear structure* on M associated with the polarization F. Since the inner product will not be used in this thesis, we refer to [Śni80], [Woo92], [AE05], [Bla77] for a discussion of this part of the theory. Again, $\mathcal{H}$ is defined to be the completion of $\Gamma_F(L)$ with respect to the inner product.

Another issue raised by the restriction to $\Gamma_F(L)$ is the fact that Q_f preserves $\Gamma_F(L)$ if and only if $[X_f,\, F] \subset F$. This limits the classical observables which can be easily quantized. To overcome this issue, a pairing must be defined between spaces $\Gamma_F(L)$ corresponding to different polarizations F. This construction is known as the *Blattner-Kostant-Sternberg pairing*, or BKS pairing for short, and requires the introduction of a *metaplectic structure* on M; again we refer to [Śni80], [Woo92], [AE05], [Bla77] for details.

1.4 Symplectic reduction and its interaction with geometric quantization

Suppose our symplectic manifold (M, ω) has a continuous symmetry, described by a Lie group G and an action of G on M which preserves the symplectic form ω. In favorable circumstances, there exists a *momentum map* $\mathrm{J} : M \to \mathfrak{g}^*$ describing the conserved quantities of associated with the G-action: if $H \in C^\infty(M, \mathbb{R})$ is a G-invariant Hamiltonian on M, then J is constant along the Hamiltonian trajectories corresponding to H, implying that the level sets $\mathrm{J}^{-1}(\mu)$, $\mu \in \mathfrak{g}^*$, are conserved under the Hamiltonian flow due to H. Let G_μ denote the subset of G which also preserves $\mathrm{J}^{-1}(\mu)$. Given certain technical conditions on the G-action on M, the smooth and symplectic structures on M induce corresponding structures on the quotient space $\frac{\mathrm{J}^{-1}(\mu)}{G_\mu}$. This process of constructing a quotient symplectic space, introduced in [MW74], is called *symplectic* or *point reduction*, and the quotient space $\frac{\mathrm{J}^{-1}(\mu)}{G_\mu}$ is referred to as the *symplectic*, *reduced*, or *Marsden-Weinstein quotient*. The physical significance of the quotient is that it factors out the motion of the system associated with the conserved momenta, and contains only the "interesting" dynamics. It is straightforward to reconstruct the full dynamics from that on the quotient.

On the quantum side, a G-symmetry of the Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ with Hamiltonian H corresponds (assuming G is connected) to a unitary representation $U : \widetilde{G} \to \mathrm{U}(\mathcal{H})$ of the universal cover $\widetilde{G}$ of G that commutes with H. If we further assume that G (and hence $\widetilde{G}$) is compact, we can decompose $\mathcal{H}$ into a direct sum of $\widetilde{G}$- and H-invariant subspaces, each of which transforms via a distinct irreducible representation of $\widetilde{G}$. After a choice of maximal torus and positive weights, this decomposition can be expressed in an invariant way as

$$\mathcal{H} \simeq \bigoplus_{\lambda \text{ dominant integral}} (\mathcal{H}^\lambda)^* \otimes \left(\mathcal{H}^\lambda \otimes \mathcal{H}\right)^{\widetilde{G}}, \tag{$\star$}$$

where $\cdot^{\widetilde{G}}$ denotes invariance with respect to the diagonal $\widetilde{G}$-action—see Appendix C for details. This equivalence has the effect of separating the $\widetilde{G}$-action, which acts on the first factor $(\mathcal{H}^\lambda)^*$, and the unitary evolution $\exp\left(-\frac{i}{\hbar} H\, t\right)$, which acts on the second factor $\left(\mathcal{H}^\lambda \otimes \mathcal{H}\right)^{\widetilde{G}}$. The factor $\left(\mathcal{H}^\lambda \otimes \mathcal{H}\right)^{\widetilde{G}}$ contains all the interesting dynamics, and as such is a natural candidate for the "reduced quantum space", the quantum analogue of the symplectic quotient $\frac{\mathrm{J}^{-1}(\mu)}{G_\mu}$.

Assuming the momentum map J is G-equivariant, Kostant showed that the G-action on M can be lifted infinitesimally to L. For connected G, this infinitesimal lift exponentiates to a $\widetilde{G}$-action on L, where $\widetilde{G}$ denotes the universal cover of G—see Section 3.2.4 for details. In turn this gives a $\widetilde{G}$-action on $\Gamma(L)$ and, assuming the polarization F is G-invariant, on $\Gamma_F(L)$. It therefore makes sense to apply quantum reduction to the Hilbert space $\mathcal{H} = \Gamma_F(L)$ obtained in the geometric quantization procedure.

Since geometric quantization is tied so closely with the symplectic structure of (M, ω), it is natural to ask whether geometric quantization interacts "nicely" with reduction. In other words, does the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "\text{quantization}"] \arrow[d, "\text{reduction}"'] & \mathcal{H} \arrow[d, "\text{reduction}"] \\ \dfrac{\mathrm{J}^{-1}(\mu)}{G_\mu} \arrow[r, "\text{quantization}"] & (\mathcal{H}^\lambda \otimes \mathcal{H})^{\widetilde{G}} \end{tikzcd} .$$

To our knowledge, the first work to discuss reduction in the context of geometric quantization is the paper of Reyman and Semenov-Tian-Shansky[1] [RSTS79]. Guillemin and Sternberg [GS82] were the first to explicitly formulate the "quantization commutes with reduction" question, which they proved for the case of a free action of a compact, connected Lie group on a compact Kähler manifold. Subsequent work in this direction has sought to relax the conditions of freeness and Kählerness, with corresponding generalizations of the notion of quantization; for an overview see [Sja96]. More recently, Landsman and his students have sought to further extend the definition of quantization in order to cover noncompact manifolds and groups; see for example [HL08].

Despite the progress on the mathematical aspects of quantization and reduction, applications to systems with physical significance remain sparse. One notable exception is the case of a cotangent bundle of a principal G-bundle, which has two distinct interpretations: (i) for $G = \mathrm{SO}(3)$, as the phase space of the n-body problem [Mon02, Chapter 14], and (ii) as the Kaluza-Klein space for the motion of a particle in a non-Abelian Yang-Mills field, moving according to Wong's equations [Mon02, Chapter 12]. This problem was considered by Gotay [Got86], who imposed the condition that reduction be carried out at invariant values μ of the momentum (satisfying $G_\mu = G$), and by Robson [Rob96], without this condition.

1.5 New results in this thesis

This thesis is heavily inspired by the results of Robert Filippini [Fil95]. Filippini considered the cotangent bundle T^*G of a compact Lie group G with its usual symplectic structure $\omega = -\mathrm{d}\theta$, where θ is the canonical one-form. Taking a trivial line bundle over T^*G with curvature $\frac{i}{\hbar}\omega$, and defining an appropriate polarization, Filippini carried out the geometric quantization of this system, and showed that the resulting Hilbert space is isomorphic to $\bigoplus_\lambda (\mathcal{H}^\lambda)^* \otimes \mathcal{H}^\lambda$, where λ ranges over the set of dominant integral weights. Comparing this to the usual geometric quantization with respect to the vertical polarization, which yields $L^2(G)$, Filippini was able to give a symplectic interpretation

[1] This paper appears however to be little-known in the geometric quantization literature, the majority of its citations instead coming from works related to integrable systems.

of the Peter-Weyl Theorem, which states that

$$L^2(G) \simeq \bigoplus_{\lambda \text{ d.i.}} (\mathcal{H}^\lambda)^* \otimes \mathcal{H}^\lambda.$$

The results of this thesis can first and foremost be seen as an extension of Filippini's construction to arbitrary symplectic manifolds (M, ω), yielding a symplectic interpretation of the decomposition $(\star)$. In order to state explicitly what is proved, we first introduce some notation. We consider a free action of a compact connected Lie group G on (M, ω), preserving ω and admitting a corresponding G-equivariant momentum map $\mathrm{J} : M \to \mathfrak{g}^*$. Let $\pi : M \to M/G$ denote the projection onto the G-orbit space of M, and $\mathcal{O}$ a coadjoint orbit in $\mathfrak{g}^*$. The set $\mathrm{J}^{-1}(\mathcal{O})$ has a dual foliation, by the family of constant momentum surfaces $\{\mathrm{J}^{-1}(\mu) \,|\, \mu \in \mathcal{O}\}$, and by the family of G-orbits $\{\pi^{-1}(a) \,|\, a \in \pi(\mathrm{J}^{-1}(\mathcal{O}))\}$. The restrictions of ω to $\mathrm{J}^{-1}(\mathcal{O})$, $\mathrm{J}^{-1}(\mu)$, and $\pi^{-1}(a)$ define *characteristic foliations*, denoted $\mathcal{R}^{\mathcal{O}}$, $\mathcal{R}^\mu$, and $\mathcal{R}^a$ respectively, which agree on their respective domains. According to the general procedure of *foliation reduction* (Section 2.11), ω induces symplectic forms on the corresponding leaf spaces $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$, and $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$. As a consequence of the fact dual foliations $\{\mathrm{J}^{-1}(\mu) \,|\, \mu \in \mathcal{O}\}$ and $\{\pi^{-1}(a) \,|\, a \in \pi(\mathrm{J}^{-1}(\mathcal{O}))\}$ are symplectic complements of each other, one can define a canonical symplectomorphisms between $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ and $\frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$.

Turning to geometric prequantization, the prequantum geometric structures have an equivalent description, consisting of a principal U(1)-bundle $\dot{L}$ over M and connection one-form α on $\dot{L}$ with curvature $\frac{i}{\hbar}\omega$. In Chapter 4 we apply foliation reduction to the connection form restricted to the U(1)-bundles lying over $\mathrm{J}^{-1}(\mathcal{O})$, $\mathrm{J}^{-1}(\mu)$, and $\pi^{-1}(a)$ respectively. The corresponding characteristic foliations $(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}$, $(\mathcal{R}^{\mathrm{h}})^\mu$, and $(\mathcal{R}^{\mathrm{h}})^a$ turn out to be the horizontal lifts of the characteristic foliations on the base space. In contrast with the base manifold case, prequantum reduction is not always consistent, and only certain regions of the phase space are "admissible" to prequantization reduction. A necessary and sufficient condition for this admissibility is that the leaves of the $(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}} \,/\, (\mathcal{R}^{\mathrm{h}})^\mu \,/\, (\mathcal{R}^{\mathrm{h}})^a$ cover those of $\mathcal{R}^{\mathcal{O}} \,/\, \mathcal{R}^\mu \,/\, \mathcal{R}^a$ injectively. We prove this, and relate it to the common notion of integrality of weights of a representation, which leads to "quantization conditions" on the possible momenta of the quantized theory. These conditions can be seen analogues of the usual Bohr-Wilson-Sommerfeld conditions for completely integrable systems. The prequantum reduction procedure over $\mathrm{J}^{-1}(\mu)$ proposed here agrees with that of [RSTS79] (which was discovered after much of the work in this thesis was completed). The interpretation as foliation reduction is new, however.

In Chapter 5 we then describe how to construct a polarization F on M consistent with the foliation reduction procedure, and show that admissibility is also a necessary criterion for the existence of sections covariantly constant with respect to the polarization. We then demonstrate that the polarization induces polarizations on the reduced spaces, and describe a one to one correspondence between polarized sections on the reduced and unreduced spaces. Even in case of compact Kähler manifolds, the sections so constructed differ from those in [GS82], since they are distributional rather than full sections, having support only on admissible regions of the phase space.

Taken together, the results of Chapters 4 and 5 describe the construction for admissible regions of M of reduced U(1)-bundles $\dot{L}^{\mathcal{O}}_{\mathcal{R}}$, $\dot{L}^{\mu}_{\mathcal{R}}$, and $\dot{L}^{a}_{\mathcal{R}}$, connections $\alpha^{\mathcal{O}}_{\mathcal{R}}$, $\alpha^{\mu}_{\mathcal{R}}$, and $\alpha^{a}_{\mathcal{R}}$, and polarizations $F^{\mathcal{O}}_{\mathcal{R}}$, $F^{\mu}_{\mathcal{R}}$, and $F^{a}_{\mathcal{R}}$. In Chapter 6 we describe a bundle-connection equivalence $\dot{L}^{\mathcal{O}}_{\mathcal{R}} \simeq \dot{L}^{a}_{\mathcal{R}} \boxdot \dot{L}^{\mu}_{\mathcal{R}}$ which covers the canonical symplectomorphism between $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ and $\frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}$. This equivalence induces an isomorphism between the space of covariantly constant sections of the associated complex line bundles, i.e.,

$$\Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}}) \simeq \Gamma_{F^{a}_{\mathcal{R}}}(L^{a}_{\mathcal{R}}) \boxtimes \Gamma_{F^{\mu}_{\mathcal{R}}}(L^{\mu}_{\mathcal{R}}).$$

Employing the Borel-Weil Theorem and Schur's Lemma, we demonstrate how this agrees with the result ($\star$) obtained by quantum reduction, and thus establishes that "quantization commutes with reduction."

1.6 Limitations of the results

In order to define a inner product on our representation space, we must introduce a *metalinear* structure on the space. In addition, most physically interesting Hamiltonians do not preserve the polarization used in the symplectic quantization procedure. To deal with this possibility, the introduction of a *metaplectic* structure is needed. This allows a metalinear structure to be consistently associated with any polarization in the manifold. We do not discuss either of these structures in the thesis, or how they interact with symplectic reduction. As such, none of the quantum representation spaces in the thesis have an inner product. In particular, the isomorphisms from Chapter 5 should be seen as vector space isomorphisms, rather than unitary equivalences between Hilbert spaces. There is also no discussion of physically interesting dynamics. It is hoped these deficiencies can be dealt with in future work.

Chapter 2

Background Material

This introductory chapter is intended to fix notation, definitions, and conventions that will be used throughout the thesis. Much of this material can be found in [AM78], [MMO$^+$07], and particularly, [OR04]. Propositions 2.8.5 (iii), 2.10.1 and 2.10.2 were derived specifically for this thesis since a treatment in the literature could not be found, but are likely well known. In addition, the treatment of orbit reduction is somewhat different to that in the above mentioned references. After completion of this chapter, it was noticed that the approach shares much in common with the original conception of orbit reduction due to Marle [Mar76], [LM87].

2.1 Smooth manifolds and properties of mappings between them

As usual we take a smooth manifold to be a locally Euclidean topological space with a smooth atlas of coordinate charts. Locally Euclidean spaces are automatically T_1, which implies that singleton sets $\{x\}$ are closed. Additionally, all smooth manifolds in this thesis are taken to be *connected* and *finite-dimensional*. Lie groups are automatically Hausdorff, and *connected* Lie groups are automatically second countable.

A smooth map $f : N \to M$ is called

- an ***immersion*** if $T_n f : T_n N \to T_{f(n)} M$ is injective for all $n \in N$;
- a ***submersion*** if $T_n f : T_n N \to T_{f(n)} M$ is surjective for all $n \in N$;
- an ***injective immersion*** if it is both an injection *and* an immersion;
- a ***regular immersion*** if it is an injective immersion satisfying the following condition: for any manifold P, an arbitrary map $g : P \to N$ is smooth if and only if $f \circ g : P \to M$ is smooth;
- an ***embedding*** if it is an injective immersion that is a homeomorphism onto its image $f(N)$ with the subspace topology induced by M.

A subset $N \subset M$ with its own manifold structure is called

- an ***immersed submanifold*** if the inclusion $i_{N,M} : N \hookrightarrow M$ is an immersion;
- an ***initial submanifold*** if $i_{N,M} : N \hookrightarrow M$ is a regular immersion;
- an ***embedded submanifold*** if $i_{N,M} : N \hookrightarrow M$ is an embedding.

Unless otherwise stated, all manifolds and maps discussed in this thesis (including group actions) will be taken to be smooth.

2.2 Lie group and Lie algebra actions

Let M be a manifold, and G a Lie group, with corresponding Lie algebra $\mathfrak{g}$. We suppose there exists a left G-action $\Phi : G \times M \to M$ on M, and use the shorthand $g \cdot x$ for $\Phi(g, x)$. The conditions for this to be a left action are $g \cdot (h \cdot x) = (gh) \cdot x$ and $e \cdot x = x$. We use the same notation to denote the induced left actions on TM and T^*M, i.e., $g \cdot X_x = T_x\Phi_g(X_x) \in T_{g\cdot x}M$ for $X_x \in T_xM$, while $g \cdot \alpha_x = T^*_{g\cdot m}\Phi_{g^{-1}}(\alpha_x) \in T^*_{g\cdot x}M$ for $\alpha_x \in T^*_xM$, where $\Phi_g(x) = \Phi(g, x)$. With these conventions, we clearly have that $g \cdot (h \cdot X_x) = (gh) \cdot X_x$, and similarly for α_x.

Correspondingly there is a infinitesimal action of the Lie algebra $\mathfrak{g}$ on M, given by $\xi \cdot x = T_e\Phi^x(\xi) \in T_xM$ for $\xi \in \mathfrak{g}$, where $\Phi^x(g) = \Phi(g, x)$. The ***infinitesimal generator*** of the left action corresponding to ξ is the vector field $\xi_M \in \Gamma(TM)$, defined by

$$\xi_M(x) = \xi \cdot x.$$

The infinitesimal generators satisfy the property

$$[\xi_M, \zeta_M] = -[\xi, \zeta]_M,$$

where $[\cdot, \cdot]$ on the left and right sides of the equation denote the Lie brackets on M and on $\mathfrak{g}$ respectively. In general a map $\xi \in \mathfrak{g} \mapsto \xi_M \in \Gamma(TM)$ satisfying this property is called a left $\mathfrak{g}$-action on M.

Let $H \subset G$ be an arbitrary subgroup of the Lie group G. It can be shown [Bou89, Chapter 3, §4.5] that H may be given an smooth structure induced by that on G, making H a Lie group and an initial submanifold of G. Further, H is an embedded submanifold of G if and only if it is closed in G [Lee03, Corollary 20.11].

The ***isotropy*** or ***stabilizer group*** G_x of the action Φ at $x \in M$ is the set of groups elements that leave x invariant

$$G_x = \{g \in G \,|\, g \cdot x = x\}.$$

Note that $G_x = (\Phi^x)^{-1}(\{x\})$ is a closed subset of G, and hence an embedded Lie subgroup of G.

The ***symmetry algebra*** $\mathfrak{g}_x$ is the Lie algebra of the isotropy group

$$\mathfrak{g}_x = \{\xi \in \mathfrak{g} \,|\, \xi \cdot x = 0_x \in T_x M\}.$$

Occasionally we will need to consider right actions $\Psi : G \times M \to M$ also. Then we use $x \cdot g$ for $\Psi(g, x)$. The conditions for this to be a right action are $(x \cdot g) \cdot h = x \cdot (gh)$ and $x \cdot e = x$. The obvious analogues of the left action notations apply. In this case, the infinitesimal generators

$$\xi_M(x) = x \cdot \xi$$

satisfy

$$[\xi_M, \zeta_M] = [\xi, \zeta]_M.$$

2.3 Proper group actions

A G-action $\Phi : G \times M \to M$ is called ***free*** if $\Phi(g, x) = x$ for some $g \in G$ and $x \in M$ implies that $g = e$.

We adopt the definition[1] that a map $f : X \to Y$ is ***proper*** if for every sequence (x_n) in X such that $(f(x_n))$ converges in Y, there is a convergent subsequence (x_{n_k}) in X.

The action Φ is ***proper*** if the map $\widetilde{\Phi} : G \times M \to M \times M$, defined by

$$\widetilde{\Phi}(g, x) = (x, \Phi(g, x))$$

is proper. Explicitly, if $((g_n, x_n))$ is a sequence in $G \times M$ such that $((x_n, g_n \cdot x_n))$ converges in $M \times M$, then (g_n) has a convergent subsequence (g_{n_k}).

Properness of an action turns out to be a sufficient condition to ensure nice analytic properties of the quotient space M/G. In particular, we have the following important result.

Proposition 2.3.1. *If G acts freely and properly on M, then M/G is a topological manifold of dimension* $\dim M - \dim G$*, and can be given a smooth structure such that the projection $\pi : M \to M/G$ is a submersion.*

Proof. See for example [Lee03, Theorem 9.16]. □

We state some easily proved consequences of the definition of properness:

- Proper maps are closed.

[1] If X and Y are Hausdorff, Y is first countable, and X is second countable, this is equivalent to the more common definition that the inverse image of any compact set in Y is compact in X.

- If G is compact, then Φ is a proper action.

- Proper actions have compact isotropy groups G_x at each point $x \in M$.

- For linear G-actions on vector spaces (e.g., the coadjoint action discussed next section), the isotropy group of the origin is the entire group G. Hence combining the two previous properties, it follows that linear group actions are proper if and only if G is compact.

- If Φ is proper, then $\Phi^x : G \to M$ is also proper, and hence closed.

- If $H \subset G$ is a subgroup of G, the left H-action $(h, g) \mapsto gh^{-1}$ on G is proper if and only if H is closed.

Since the isotropy group G_x of an action $\Phi : G \times M \to M$ is closed for any $x \in M$, the last result and Proposition 2.3.1 tell us that G/G_x can be given a smooth structure (namely the one making $G \to G/G_x$ a submersion). Using the bijection of G/G_x and the G-orbit $\mathcal{O}_x = G \cdot x$ through x allows this smooth structure to be transferred to $\mathcal{O}_x$. It can be shown [OR04, Proposition 2.3.12] that $\mathcal{O}_x$ equipped with this structure is an initial submanifold of M, is closed if Φ is proper, and is embedded if Φ is proper and M is second countable.

We will require the following result in our discussion of symplectic reduction.

Proposition 2.3.2. *Let $\Phi : G \times M \to M$ be a smooth G-action on a manifold M, H a subgroup of G, and $S \subset M$ a H-invariant initial submanifold of M. Then the restriction $\Phi' : H \times S \to S$ is a smooth H-action on S. If Φ is proper and H is closed in G, Φ' is also proper.*

Proof. We have the identity

$$i_{S,M} \circ \Phi' = \Phi \circ (i_{H,G} \times i_{S,M}),$$

where $i_{A,B} : A \hookrightarrow B$ denotes inclusion. The right hand side, being a composition of smooth maps, is smooth, and so the initial submanifold condition for S implies that $\Phi' : H \times S \to S$ is smooth.

The properness of Φ' follows easily from the definition and the closedness of H. $\square$

2.4 Hamiltonian vector fields, Poisson brackets, and symplectic actions

Now suppose M has the additional structure of a symplectic manifold, with symplectic form ω, i.e., a closed, nondegenerate 2-form on M. To any function $f \in C^\infty(M)$, let $X_f \in \Gamma(TM)$ denote the ***Hamiltonian vector field*** of f, defined by

$$\mathrm{i}_{X_f}\omega = \mathrm{d}f.$$

The ***Poisson bracket*** $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on M is given by

$$\{f, g\} = \omega(X_f, X_g).$$

A ***symplectic*** left G-action on M is a left G-action satisfying

$$\Phi_g^* \omega = \omega.$$

Differentiating this symplectic condition with respect to g, we obtain

$$\mathcal{L}_{\xi_M} \omega = 0.$$

In general, a left $\mathfrak{g}$-action satisfying this property is also called symplectic.

2.5 Adjoint and coadjoint actions

Let $I_g : G \to G$ denote the inner automorphism $I_g(h) = ghg^{-1}$. Then $I_\cdot$ defines a left action on G. I_g preserves the identity e, and so its derivative at e defines the ***adjoint action*** of G on $\mathfrak{g} \simeq T_eG$:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} \qquad \text{given by} \qquad \mathrm{Ad}_g \zeta := T_e I_g \ \zeta.$$

The dual left action on $\mathfrak{g}^*$ is called the ***coadjoint action*** of G on $\mathfrak{g}^*$:

$$\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^* \qquad \text{given by} \qquad \mathrm{Ad}^*_{g^{-1}}(\mu) := \mu \circ \mathrm{Ad}_{g^{-1}}.$$

The ***adjoint action*** $\mathrm{ad}_\cdot$ of $\mathfrak{g}$ on $\mathfrak{g}$ is

$$\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g} \qquad \text{given by} \qquad \mathrm{ad}_\xi(\zeta) := [\xi, \zeta].$$

The adjoint actions of $\mathfrak{g}$ and of G are related by $\mathrm{ad}_\xi = (T_e \mathrm{Ad}_\cdot)\xi$, and so $\mathrm{ad}_\cdot$ is the $\mathfrak{g}$-action induced by the G-action $\mathrm{Ad}_\cdot$, as considered above.

Similarly we have the ***coadjoint action*** $-\mathrm{ad}^*_\cdot$ of $\mathfrak{g}$ on $\mathfrak{g}^*$:

$$-\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^* \qquad \text{given by} \qquad -\mathrm{ad}^*_\xi \mu := -\mu \circ \mathrm{ad}_\xi,$$

and $-\mathrm{ad}^*_\xi = (T_e \mathrm{Ad}^*_\cdot)\xi$.

An easy consequence of the Jacobi identity on $\mathfrak{g}$ is that both the adjoint and coadjoint actions

$\mathrm{ad}_{\cdot} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ and $-\mathrm{ad}^*_{\cdot} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ are Lie algebra homomorphisms,

$$\mathrm{ad}_{[\xi,\zeta]} = \mathrm{ad}_\xi \circ \mathrm{ad}_\zeta - \mathrm{ad}_\zeta \circ \mathrm{ad}_\xi = [\mathrm{ad}_\xi, \mathrm{ad}_\zeta],$$
$$-\mathrm{ad}^*_{[\xi,\zeta]} = (-\mathrm{ad}^*_\xi) \circ (-\mathrm{ad}^*_\zeta) - (-\mathrm{ad}^*_\zeta) \circ (-\mathrm{ad}^*_\xi) = [-\mathrm{ad}^*_\xi, -\mathrm{ad}^*_\zeta].$$

It is worth noting that in terms of the left action $\Phi_g(h) = gh$ and right action $\Psi_g(h) = hg$, we have that

$$\mathrm{Ad}_g \xi = g \cdot \xi \cdot g^{-1},$$
$$\mathrm{Ad}^*_{g^{-1}} \mu = g \cdot \mu \cdot g^{-1}.$$

The tangent space $T_\mu \mathcal{O}$ of the coadjoint orbit $\mathcal{O} = \{\mathrm{Ad}^*_{g^{-1}} \mu : g \in G\}$ at μ is spanned by the vectors $\{-\mathrm{ad}^*_\xi \mu \,|\, \xi \in \mathfrak{g}\}$. The coadjoint orbit possesses two natural symplectic forms $\pm\omega_{\mathcal{O}}$, called the Kostant-Kirillov-Souriau (KKS) forms, defined by

$$(\pm\omega_{\mathcal{O}})_\mu(-\mathrm{ad}^*_\xi \mu, -\mathrm{ad}^*_\eta \mu) = \pm\mu([\xi, \eta]).$$

In terms of the above notation, the isotropy group G_μ and symmetry algebra $\mathfrak{g}_\mu$ of the coadjoint action on $\mathfrak{g}^*$ at μ are

$$G_\mu = \{g \in G \,|\, \mathrm{Ad}^*_{g^{-1}} \mu = \mu\}$$

and

$$\mathfrak{g}_\mu = \{\xi \in \mathfrak{g} \,|\, -\mathrm{ad}^*_\xi \mu = 0\}$$

respectively.

2.6 The momentum map

A vector field $X \in \Gamma(TM)$ is called ***locally Hamiltonian*** if it preserves the symplectic form ω, i.e.,

$$\mathcal{L}_X \omega = 0.$$

Using Cartan's magic formula $\mathcal{L}_X = \mathrm{i}_X \circ \mathrm{d} + \mathrm{d} \circ \mathrm{i}_X$ and the fact that ω is closed, we see this is equivalent to the condition

$$\mathrm{d}(\mathrm{i}_X \omega) = 0,$$

i.e., $\mathrm{i}_X \omega$ is a closed one form. It is a natural question to ask whether $\mathrm{i}_X \omega$ is exact. If this is the case, then

$$\mathrm{i}_X \omega = \mathrm{d}f \qquad \text{for some } f \in C^\infty(M).$$

The definition of the Hamiltonian vector field and the non-degeneracy of ω then imply that $X = X_f$, and so X is Hamiltonian. This explains the terminology "locally Hamiltonian" above.

For a symplectic left G-action Φ, differentiation of the symplectic condition

$$\Phi_g^* \omega = \omega$$

with respect to g demonstrates that for any $\xi \in \mathfrak{g}$, the vector field ξ_M is locally Hamiltonian,

$$\mathcal{L}_{\xi_M} \omega = 0.$$

Suppose in fact ξ_M is Hamiltonian for each $\xi \in \mathfrak{g}$, so that there exist maps $J(\xi) \in C^\infty(M)$ with

$$\xi_M = X_{J(\xi)} \qquad \text{for all } \xi \in \mathfrak{g}.$$

It is easy to arrange $J(\xi)$ to be linear in ξ (just pick a basis $e_1, \ldots, e_r$ of $\mathfrak{g}$, construct maps $J(e_i),\ i = 1, \ldots, r$, and extend by linearity to all of $\mathfrak{g}$). Supposing such a linear map $J : \mathfrak{g} \to C^\infty(M)$ exists, the map $\mathrm{J} : M \to \mathfrak{g}^*$ defined by

$$\langle \mathrm{J}(x), \xi \rangle = J(\xi)(x)$$

is called the ***momentum map*** of the action, where here $\langle \cdot, \cdot \rangle$ denotes the natural pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$. Note that momentum maps, when they exist, are not unique, since one can add any element of $\mathfrak{g}^*$ to J to get another momentum map. For connected symplectic manifolds all momentum maps can be obtained this way.

There is a useful criterion for deciding whether a symplectic action has a momentum map. For convenience the proof of this standard result [OR04, Proposition 4.5.17] is reproduced below.

Proposition 2.6.1. *Let (M, ω) be a symplectic manifold and $\mathfrak{g}$ a Lie algebra acting symplectically on it. There exists a momentum map associated to this action if and only if the linear map*

$$\rho : \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \longrightarrow H^1(M, \mathbb{R}),$$
$$[\xi] \longmapsto [\mathrm{i}_{\xi_M} \omega],$$

is identically zero.

Proof. We must first show that ρ is well-defined. It suffices to show that $\mathrm{i}_{[\xi,\zeta]_M} \omega$ is exact, where ξ, $\zeta \in \mathfrak{g}$. Using the standard identity $\mathrm{i}_{[X,Y]} = \mathcal{L}_X \circ \mathrm{i}_Y - \mathrm{i}_Y \circ \mathcal{L}_X$ and the fact that the $\mathfrak{g}$-action is symplectic,

$$\mathrm{i}_{[\xi,\zeta]_M} \omega = -\mathrm{i}_{[\xi_M, \zeta_M]} \omega$$

$$
\begin{aligned}
&= -(\mathcal{L}_{\xi_M} \circ \mathrm{i}_{\zeta_M} - \mathrm{i}_{\zeta_M} \circ \mathcal{L}_{\xi_M})\omega = -\mathcal{L}_{\xi_M} \circ \mathrm{i}_{\zeta_M}\omega \\
&= -(\mathrm{d} \circ \mathrm{i}_{\xi_M} - \mathrm{i}_{\xi_M} \circ \mathrm{d}) \circ \mathrm{i}_{\zeta_M}\omega \\
&= -\mathrm{d}(\mathrm{i}_{\xi_M}\mathrm{i}_{\zeta_M}\omega),
\end{aligned}
$$

which demonstrates that $\mathrm{i}_{[\xi,\zeta]_M}\omega$ is exact (and that $[\xi,\zeta]_M = X_{\omega(\xi_M,\zeta_M)}$).

Now a momentum map $\mathrm{J} : M \to \mathfrak{g}^*$ exists if and only if for any $\xi \in \mathfrak{g}$ we can write $\mathrm{i}_{\xi_M}\omega = \mathrm{d}(J(\xi))$ for some map $J(\xi) \in C^\infty(M)$. This is equivalent to $[\mathrm{i}_{\xi_M}\omega] = 0$, which in turn can be written as $\rho([\xi]) = 0$ for any $\xi \in \mathfrak{g}$. $\square$

In the cases dealt with in this thesis, $\mathfrak{g}$ will be taken to be semisimple (see discussion next chapter). Then the First Whitehead Lemma for Lie Algebras ([Jac79], [GS84]) says that the first Lie algebra cohomology group $H^1(\mathfrak{g}, \mathbb{R})$ is trivial, or equivalently $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. It follows that map ρ above is trivially zero, and hence a momentum map always exists.

Given a momentum map J, a natural question to ask is whether the map $\xi \mapsto J(\xi)$ defines a Lie algebra homomorphism from $(\mathfrak{g}, [\cdot,\cdot])$ to $(C^\infty(M), \{\cdot,\cdot\})$. That is, whether

$$\{J(\xi), J(\zeta)\} = J([\xi,\zeta]) \qquad \text{for all } \xi, \zeta \in \mathfrak{g}.$$

A straightforward computation shows this is the case if and only if

$$T_x\mathrm{J}(\xi \cdot x) = -\mathrm{ad}_\xi^*\mathrm{J}(x).$$

A momentum map which satisfies this condition is called ***infinitesimally equivariant***. As suggested by the terminology, this is an infinitesimal version of the corresponding property for G-actions: J is said to be ***equivariant*** if it satisfies the property

$$\mathrm{J} \circ \Phi_g = \mathrm{Ad}_{g^{-1}}^* \circ \mathrm{J}.$$

There are several situations in which an equivariant momentum map can be shown to exist. We will simply state the results below, referring to [OR04] for proofs and definitions of relevant concepts.

Proposition 2.6.2. *Let G be a* compact *Lie group acting symplectically on the symplectic manifold (M,ω), with associated momentum map $\mathrm{J} : M \to \mathfrak{g}^*$. Then there exists a momentum map which is equivariant.*

Proposition 2.6.3. *Let G be a Lie group acting symplectically on the* connected *symplectic manifold (M,ω), with associated momentum map $\mathrm{J} : M \to \mathfrak{g}^*$. Define the map $C : G \to \mathfrak{g}^*$ by*

$$C(g) = \mathrm{J}(\Phi_g(x)) - \mathrm{Ad}_{g^{-1}}^*(\mathrm{J}(x)).$$

Then the definition of C is independent of the choice of $x \in M$, C defines a $\mathfrak{g}^*$-valued 1-cocycle on G, and an equivariant momentum map exists if and only if $[C]$ is trivial in $H^1(G, \mathfrak{g}^*)$.

Since we will be dealing with compact Lie groups in this thesis, the first of these propositions is sufficient. However, if G is semisimple (which will also be the case in this thesis), then the Whitehead Lemma for Lie Groups says that $H^1(G, \mathfrak{g}^*) = 0$. So an equivariant momentum map is guaranteed to exist in this case also, provided M is connected.

Proposition 2.6.4. *If G is semisimple and M is connected, then there exists a unique equivariant momentum map.*

Proof. From the previous discussion, we already know that an equivariant momentum map J exists, and we just need to establish uniqueness.

Suppose J′ is another equivariant momentum map. Then for any $\xi \in \mathfrak{g}$, $\mathrm{d}\,(J(\xi) - J'(\xi)) = \mathrm{i}_{\xi_M}\omega - \mathrm{i}_{\xi_M}\omega = 0$. Since M is connected, this implies that $J(\xi) - J'(\xi) = c_\xi$, a constant on M. Since $J(\xi)$ and $J'(\xi)$ are linear in ξ, so is c_ξ, and we can write $\mathrm{J} - \mathrm{J}' = \mu$ for some $\mu \in \mathfrak{g}^*$. Equivariance of J and J′ imply that $\mu = \mathrm{Ad}^*_{g^{-1}}\mu$ for all $g \in G$. Taking the derivative, we get $-\mathrm{ad}^*_\xi\mu = 0$ for all $\xi \in \mathfrak{g}$.

Since $\mathfrak{g}$ is semisimple, the First Whitehead Lemma for Lie Algebras implies that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. So every element of $\mathfrak{g}$ can be written as a linear combination of elements $[\xi, \zeta]$. Since $\mu([\xi, \zeta]) = (\mathrm{ad}^*_\xi\mu)(\zeta) = 0$, μ vanishes on $\mathfrak{g}$. Therefore $\mu = 0$, and $\mathrm{J} = \mathrm{J}'$. □

2.7 Notation for projections, inclusions, and restrictions

Suppose we have a free, proper symplectic G-action on a symplectic manifold M, with corresponding equivariant momentum map J. Define an equivalence relation $\mathcal{R} \subset M \times M$ by

$$(x, y) \in \mathcal{R} \iff y = h \cdot x \text{ for some } h \in G_{\mathrm{J}(x)}.$$

The $\mathcal{R}$-equivalence class containing x is just $G_{\mathrm{J}(x)} \cdot x$. Let

$$\sigma : M \to \frac{M}{\mathcal{R}}$$

denote the corresponding quotient map. So $\sigma(x) = G_{\mathrm{J}(x)} \cdot x$. Also, define

$$\pi : M \to \frac{M}{G}$$

to be the projection onto the G-orbits of M, i.e., $\pi(x) = G \cdot x$.

For

$$\begin{aligned}
&\mu \in \mathrm{J}(M) \subset \mathfrak{g}^*, \\
&a \in \pi(M) = \frac{M}{G}, \\
&\mathcal{O} \in \{\text{coadjoint orbits of } \mathrm{J}(M) \subset \mathfrak{g}^*\},
\end{aligned}$$

introduce the following inclusions

$$\begin{aligned}
&i^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \hookrightarrow M, \\
&i^{\mu} : \mathrm{J}^{-1}(\mu) \hookrightarrow M, \\
&i^{a} : \pi^{-1}(a) \hookrightarrow M,
\end{aligned}$$

along with

$$\begin{aligned}
&i^{\mu,\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \hookrightarrow \mathrm{J}^{-1}(\mathcal{O}), \\
&i^{a,\mathcal{O}} : \pi^{-1}(a) \hookrightarrow \mathrm{J}^{-1}(\mathcal{O}).
\end{aligned}$$

Explicit specification of the codomain will be important when discussing smoothness of the various inclusions.

Additionally, we introduce the following restrictions of $\mathrm{J} : M \to \mathfrak{g}^*$,

$$\begin{aligned}
&\mathrm{J}^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathcal{O}, \\
&\mathrm{J}^{\mu} : \mathrm{J}^{-1}(\mu) \to \mathcal{O} \qquad \text{(which is trivial)}, \\
&\mathrm{J}^{a} : \pi^{-1}(a) \to \mathcal{O},
\end{aligned}$$

and restrictions of $\pi : M \to M/G$,

$$\begin{aligned}
&\pi^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}, \\
&\pi^{\mu} : \mathrm{J}^{-1}(\mu) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}, \\
&\pi^{a} : \pi^{-1}(a) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G} \qquad \text{(trivial)}.
\end{aligned}$$

The equivalence relation $\mathcal{R}$ is restricted similarly,

$$\begin{aligned}
&\mathcal{R}^{\mathcal{O}} := \mathcal{R} \cap \left(\mathrm{J}^{-1}(\mathcal{O}) \times \mathrm{J}^{-1}(\mathcal{O})\right), \\
&\mathcal{R}^{\mu} := \mathcal{R} \cap \left(\mathrm{J}^{-1}(\mu) \times \mathrm{J}^{-1}(\mu)\right),
\end{aligned}$$

$$\mathcal{R}^a := \mathcal{R} \cap \left(\mathrm{J}^{-1}(a) \times \mathrm{J}^{-1}(a)\right).$$

Since $\mathcal{R}$-equivalence classes through points of $\mathrm{J}^{-1}(\mu)$, $\pi^{-1}(a)$, and $\mathrm{J}^{-1}(\mathcal{O})$ are subsets of those sets, the space $M/\mathcal{R}$ of $\mathcal{R}$-equivalence classes restricts to $\mathrm{J}^{-1}(\mu)/\mathcal{R}^\mu$, $\pi^{-1}(a)/\mathcal{R}^a$, and $\mathrm{J}^{-1}(\mathcal{O})/\mathcal{R}^{\mathcal{O}}$ respectively, and the projection map

$$\sigma : M \to \frac{M}{\mathcal{R}}$$

restricts to the projections

$$\begin{aligned}
\sigma^{\mathcal{O}} &: \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}, \\
\sigma^{\mu} &: \mathrm{J}^{-1}(\mu) \to \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}, \\
\sigma^{a} &: \pi^{-1}(a) \to \frac{\pi^{-1}(a)}{\mathcal{R}^{a}}.
\end{aligned}$$

Finally, we will occasionally denote the restrictions of the symplectic form ω to $\mathrm{J}^{-1}(\mathcal{O})$, $\mathrm{J}^{-1}(\mu)$, and $\pi^{-1}(a)$ in the analogous manner,

$$\omega^{\mathcal{O}} := (i^{\mathcal{O}})^*\omega, \qquad \omega^{\mu} := (i^{\mu})^*\omega, \qquad \omega^{a} := (i^{a})^*\omega.$$

2.8 Smooth structures on inverse images and their quotients

In this section we discuss the submanifold properties of the sets $\mathrm{J}^{-1}(\mathcal{O})$, $\mathrm{J}^{-1}(\mu)$, and $\pi^{-1}(a)$ and their various quotients. The main technical result employed is the Transversal Mapping Theorem 2.8.1.

2.8.1 Transversal mappings

The smooth map $f : M \to N$ is said to be ***transversal*** to the immersed submanifold S of N if, for every $x \in f^{-1}(S)$ we have that $(T_x f)(T_x M) + T_{f(x)}S = T_{f(x)}N$.

Proposition 2.8.1 (Transversal Mapping Theorem). *Let $f : M \to N$ be a smooth map transversal to the immersed submanifold S of N. Then:*

(i) There is a smooth manifold structure on $f^{-1}(S)$ with respect to which the inclusion $f^{-1}(S) \hookrightarrow M$ is an immersion and such that the map $f^{-1}(S) \to S$ obtained from f by restriction is a submersion.

(ii) If S is an initial submanifold of N, then $f^{-1}(S)$ is an initial submanifold of M.

(iii) If S is an embedded submanifold of N, then $f^{-1}(S)$ is an embedded submanifold of M.

In all three cases we have that $T_x\left(f^{-1}(S)\right) = (T_x f)^{-1}\left(T_{f(m)}S\right)$ *for all* $x \in f^{-1}(S)$*, implying in particular that the codimension of* $f^{-1}(S)$ *in* M *equals the codimension of* S *in* N*.*

Proof. See [OR04, Theorem 1.1.15] and references therein. □

Corollary 2.8.2. *Let* $f : M \to N$ *be a submersion. Then for every* $n \in N$*,* $f^{-1}(n)$ *is a closed, embedded submanifold of* M*, with* $T_x\left(f^{-1}(n)\right) = \ker T_x f$ *for all* $x \in f^{-1}(n)$*. In particular,* $\dim f^{-1}(n) = \dim M - \dim N$*.*

Proof. Take $S = \{n\}$ in the Transversal Mapping Theorem. Closedness of $f^{-1}(n)$ follows from that fact that $\{n\}$ is closed in the locally Euclidean (and hence T_1) space N. □

2.8.2 Application to the momentum and projection maps

Recall that a smooth function $f : M \to N$ is said to be ***regular*** at $x \in M$ if $T_x f : T_x M \to T_{f(x)}N$ is surjective. If f is regular at every point in M, then it is by definition a submersion.

Lemma 2.8.3. *Suppose we have a left* G*-action on a symplectic manifold, with corresponding (not necessarily equivariant) momentum map* J*. Then* μ *is a regular value of* J *if and only if* $\mathfrak{g}_x = \{0\}$ *for all* $x \in \mathrm{J}^{-1}(\mu)$*. In particular, if the* G*-action is free, then* $\mathrm{J} : M \to \mathfrak{g}^*$ *is a submersion.*

Proof.

$$
\begin{aligned}
\mathrm{J} \text{ is regular } &\Leftrightarrow T_x\mathrm{J} \text{ is surjective} \\
&\Leftrightarrow \{\xi \in \mathfrak{g} \,|\, \langle T_x\mathrm{J}(X_x), \xi\rangle = 0 \;\; \forall X_x \in T_x M\} = \{0\} \\
&\Leftrightarrow \{\xi \in \mathfrak{g} \,|\, \mathrm{d}_x J(\xi)(X_x) = 0 \;\; \forall X_x \in T_x M\} = \{0\} \\
&\Leftrightarrow \{\xi \in \mathfrak{g} \,|\, \omega_x(\xi \cdot x,\, X_x) = 0 \;\; \forall X_x \in T_x M\} = \{0\} \\
&\Leftrightarrow \{\xi \in \mathfrak{g} \,|\, \xi \cdot x = 0\} = \{0\} \quad (\Rightarrow \text{ by nondegeneracy of } \omega) \\
&\Leftrightarrow \mathfrak{g}_x = \{0\}.
\end{aligned}
$$

□

Proposition 2.8.4. *Let* $\mathrm{J} : M \to \mathfrak{g}^*$ *be a equivariant momentum map corresponding to a free* G*-action on* M*. For arbitrary* $\mu \in \mathrm{J}(M) \subset \mathfrak{g}^*$*,* $a \in M/G$*, and coadjoint orbit* $\mathcal{O} \subset \mathfrak{g}^*$*,*

(i) $\mathrm{J}^{-1}(\mu)$ *is a closed, embedded submanifold of* M*;*

(ii) $\pi^{-1}(a)$ *is an initial submanifold of* M*, is closed if the* G*-action is proper, and is embedded if* M *is second countable;*

(iii) $\mathrm{J}^{-1}(\mathcal{O})$ *is an initial submanifold of* M *and* $\mathrm{J}^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathcal{O}$ *is a submersion. Further, if* G *is compact, then* $\mathrm{J}^{-1}(\mathcal{O})$ *is an embedded submanifold of* M*;*

(iv) if the G-action is proper, there exists a smooth structure on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ that makes $\pi^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ a submersion.

Proof. (i) By Lemma 2.8.3, J is a submersion. Then by Corollary 2.8.2, $\mathrm{J}^{-1}(\mu)$ is a closed, embedded submanifold of M.

(ii) Follows from the discussion of Section 2.3.

(iii) For general G-actions, $\mathcal{O}$ is an initial submanifold of $\mathfrak{g}^*$. Since J is a submersion, in particular it is transversal to $\mathcal{O}$ and so Proposition 2.8.1 (ii) tells us that $\mathrm{J}^{-1}(\mathcal{O})$ can be given a smooth structure which makes it an initial submanifold of M, and makes $\mathrm{J}^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathcal{O}$ a submersion.

$\mathcal{O}$ is embedded if the coadjoint action on $\mathfrak{g}^*$ is proper, which occurs if and only if G is compact. So if G is compact, 2.8.1 (iii) tells us that $\mathrm{J}^{-1}(\mathcal{O})$ is an embedded submanifold of M.

(iv) By part (iii) and Proposition 2.3.2, the proper G-action on M restricts to a proper G-action on $\mathrm{J}^{-1}(\mathcal{O})$. Applying Proposition 2.3.1, the result follows.

□

Proposition 2.8.5. *Let $\mu \in \mathrm{J}(M) \subset \mathfrak{g}^*$, $a \in \frac{M}{G}$, and let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. Then:*

(i) There exists a smooth structure on $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ that makes $\sigma^\mu : \mathrm{J}^{-1}(\mu) \to \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ a submersion.

(ii) There exists a smooth structure on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ that makes $\sigma^a : \pi^{-1}(a) \to \frac{\pi^{-1}(a)}{\mathcal{R}^a}$ a submersion.

(iii) There exists a smooth structure on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ that makes $\sigma^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ a submersion.

Proof. (i) Proposition 2.8.4 (i) implies that $\mathrm{J}^{-1}(\mu)$ is an embedded submanifold of M for every $\mu \in \mathrm{J}(M) \subset \mathfrak{g}^*$. Since the coadjoint isotropy group G_μ is closed for each μ, Proposition 2.3.2 guarantees the existence of a smooth structure on the quotient space $\frac{\mathrm{J}^{-1}(\mu)}{G_\mu} = \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ which makes $\sigma^\mu : \mathrm{J}^{-1}(\mu) \to \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ a submersion.

(ii) Proposition 2.8.4 (ii) says that $\pi^{-1}(a)$ is a closed, initial submanifold of M for any $a \in \frac{M}{G}$. Its smooth structure is derived from that on the Lie group G via the bijection

$$g \in G \longmapsto g \cdot x_0 \in \pi^{-1}(a),$$

where x_0 is any element of $\pi^{-1}(a)$. If $\mathrm{J}(x_0) = \mu$, the $\mathcal{R}$-equivalence class through a point $g \cdot x_0 \in \pi^{-1}(a)$ is

$$G_{\mathrm{J}(g \cdot x_0)} \cdot g \cdot x_0 = G_{\mathrm{Ad}^*_{g^{-1}}\mu} \cdot g \cdot x_0 = gG_\mu \cdot x_0.$$

Since $G \to G/G_\mu$ is a submersion, the quotient space $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ carries a smooth structure making $\sigma^a : \pi^{-1}(a) \to \frac{\pi^{-1}(a)}{\mathcal{R}^a}$ a submersion.

(iii) Let $x \in \mathrm{J}^{-1}(\mathcal{O})$, and write $\mu = \mathrm{J}(x)$. By part (i), $\sigma^\mu : \mathrm{J}^{-1}(\mu) \to \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ is a submersion, so there exists a smooth local section $t : V \subset \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \to \mathrm{J}^{-1}(\mu)$ through $x \in \mathrm{J}^{-1}(\mu)$ [Lee03, Proposition 7.16]. Also $G \to G/G_\mu$ is a submersion, so there exists a smooth local section $s : U \subset G/G_\mu \to G$ through $e \in G$. Define $f : U \times G_\mu \times V \to \mathrm{J}^{-1}(\mathcal{O})$ by

$$f(u, h, v) = s(u) \cdot h \cdot t(v).$$

f is smooth, since it is smooth as a map to M, and $\mathrm{J}^{-1}(\mathcal{O})$ is initial in M.

f is injective: suppose $f(u, h, v) = f(u', h', v')$. Since $h \cdot t(v)$ and $h' \cdot t(v')$ are both in $\mathrm{J}^{-1}(\mu)$, we must have that $s(u') = s(u)l$ for some $l \in G_\mu$. Since $s : U \subset G/G_\mu \to G$ is a section, this is only possible if $l = e$ and $u = u'$. Then $h \cdot t(v) = h' \cdot t(v')$. Since $t : V \subset \frac{\mathrm{J}^{-1}(\mu)}{G_\mu} \to \mathrm{J}^{-1}(\mu)$ is a section, this is only possible if $h = h'$ and $v = v'$.

The point $f(u, h, v)$ has momentum $\mathrm{Ad}^*_{s(u)^{-1}}\mathrm{J}(h \cdot t(v)) = \mathrm{Ad}^*_{s(u)^{-1}}\mu$, with coadjoint stabilizer $G_{\mathrm{Ad}^*_{s(u)^{-1}}\mu} = \mathrm{Ad}_{s(u)}G_\mu$. Hence the $\sigma^{\mathcal{O}}$-fiber through $f(u, h, v)$ is

$$\mathrm{Ad}_{s(u)}G_\mu \cdot s(u) \cdot h \cdot t(v) = s(u) \cdot G_\mu \cdot h \cdot t(v) = s(u) \cdot G_\mu \cdot t(v) = f(u, G_\mu, v),$$

and so f induces a function $f_{\mathcal{R}} : U \times V \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$. By choosing coordinate charts on $U \subset G/G_\mu$, $V \subset \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$, and a neighborhood of the identity in G_μ, and composing their inverses with f and $f_{\mathcal{R}}$, we can define coordinates φ about x and coordinates $\varphi_{\mathcal{R}}$ about $\sigma^{\mathcal{O}}(x)$ with respect to which $\sigma^{\mathcal{O}}$ has the representation

$$(\varphi_{\mathcal{R}} \circ \sigma^{\mathcal{O}} \circ \varphi^{-1})(a, b, c) = (a, c),$$

implying that $\sigma^{\mathcal{O}}$ is regular at x. Since $x \in \mathrm{J}^{-1}(\mathcal{O})$ was arbitrary, $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ can be given a smooth structure such that $\sigma^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ is a submersion. □

2.9 Relationships between the inverse images

From here on we assume we have a free, proper G-action on M, with corresponding equivariant momentum map J.

First note that for $(\mu, a) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, G-equivariance of J implies that $\pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu) \neq \emptyset$.

Proposition 2.9.1. *Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit, and $(\mu, a) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$. Then for any $x \in \pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu)$,*

(i) $T_x(\mathrm{J}^{-1}(\mathcal{O})) = T_x(\pi^{-1}(a)) + T_x(\mathrm{J}^{-1}(\mu))$;

(ii) $\pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu) = G_\mu \cdot x$;

(iii) the submanifolds $\pi^{-1}(a)$ and $\mathrm{J}^{-1}(\mu)$ intersect cleanly, i.e., $T_x(\pi^{-1}(a)) \cap T_x(\mathrm{J}^{-1}(\mu)) = \mathfrak{g}_\mu \cdot x$;

(iv) $T_x(\pi^{-1}(a))^\omega = T_x(\mathrm{J}^{-1}(\mu))$*, where* $\cdot^\omega$ *denotes the symplectic complement;*

(v) $T_x(\mathrm{J}^{-1}(\mathcal{O}))^\omega = \mathfrak{g}_\mu \cdot x$*.*

Proof. (i) By G-equivariance of J we have that $\mathrm{J}^{-1}(\mathcal{O}) = G \cdot \mathrm{J}^{-1}(\mu)$, which proves the inclusion $T_x(\pi^{-1}(a)) + T_x(\mathrm{J}^{-1}(\mu)) \subset T_x(\mathrm{J}^{-1}(\mathcal{O}))$. Conversely, suppose $X_x \in T_x(\mathrm{J}^{-1}(\mathcal{O}))$. Then $X_x \in (T_x\mathrm{J})^{-1}(T_\mu\mathcal{O})$ (by the Transversal Mapping Theorem 2.8.1), i.e., $T_x\mathrm{J}(X_x) \in T_\mu\mathcal{O}$, so $T_x\mathrm{J}(X_x) = -\mathrm{ad}^*_\xi\mu = T_x\mathrm{J}(\xi \cdot x)$ some $\xi \in \mathfrak{g}$. Hence $X_x - \xi \cdot x \in \ker T_x\mathrm{J}$, proving that $X_x = (X_x - \xi \cdot x) + \xi \cdot x \in \ker T_x\mathrm{J} + \ker T_x\pi = T_x(\mathrm{J}^{-1}(\mu)) + T_x(\pi^{-1}(a))$. Hence $T_x(\mathrm{J}^{-1}(\mathcal{O})) \in T_x(\pi^{-1}(a)) + T_x(\mathrm{J}^{-1}(\mu))$.

(ii) $y \in \pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu) \iff y = g \cdot x$ for some $g \in G$ and $\mathrm{Ad}^*_{g^{-1}}\mu = \mathrm{J}(g \cdot x) = \mu \iff y = g \cdot x$ for some $g \in G_\mu$.

(iii) $X_x \in T_x(\pi^{-1}(a)) \cap T_x(\mathrm{J}^{-1}(\mu)) \iff X_x = \xi \cdot x$ for some $x \in \mathfrak{g}$ and $-\mathrm{ad}^*_\xi\mu = T_x\mathrm{J}(\xi \cdot x) = 0 \iff X_x = \xi \cdot x$ for some $x \in \mathfrak{g}_\mu$.

(iv) $\omega_x(\xi \cdot x,\, X_x) = 0$ for all $\xi \in \mathfrak{g} \iff d_x(J(\xi))(X_x) = 0$ for all $\xi \in \mathfrak{g} \iff T_x\mathrm{J}(X_x) = 0 \iff X_x \in T_x(\mathrm{J}^{-1}(\mu))$.

(v) $T_x(\mathrm{J}^{-1}(\mathcal{O}))^\omega = \left[T_x(\pi^{-1}(a)) + T_x(\mathrm{J}^{-1}(\mu))\right]^\omega = T_x(\pi^{-1}(a))^\omega \cap T_x(\mathrm{J}^{-1}(\mu))^\omega = T_x(\mathrm{J}^{-1}(\mu)) \cap T_x(\pi^{-1}(a)) = \mathfrak{g}_\mu \cdot x$.

□

Recall that a p-form β on a manifold S is said to be *degenerate* at $x \in S$ along $X_x \in T_xS$ if $\mathrm{i}_{X_x}\beta_x = 0$.

Corollary 2.9.2. *(i)* $(i^\mu)^*\omega$ *is degenerate at* $x \in \mathrm{J}^{-1}(\mu)$ *along* $\mathfrak{g}_{\mathrm{J}(x)} \cdot x = \mathfrak{g}_\mu \cdot x$*.*

(ii) $(i^a)^*\omega$ *is degenerate at* $x \in \pi^{-1}(a)$ *along* $\mathfrak{g}_{\mathrm{J}(x)} \cdot x$*.*

(iii) $(i^{\mathcal{O}})^*\omega$ *is degenerate at* $x \in \mathrm{J}^{-1}(\mathcal{O})$ *along* $\mathfrak{g}_{\mathrm{J}(x)} \cdot x$*.*

Proof. The restriction of ω to a submanifold S has degeneracy directions at $x \in S$ equal to $T_xS \cap (T_xS)^\omega$. Hence (i), (ii) follow from Proposition 2.9.1 (iii) & (iv), while (iii) follows from Proposition 2.9.1 (v). □

2.10 Preservation of submanifold properties under submersions

2.10.1 Properties of immersions, embeddings, and submersions

We will employ the following results several times in the sequel.

Proposition 2.10.1. *Suppose $S \subset T \subset M$ are manifolds.*

(i) If S is immersed in M, and T is initial in M, then S is immersed in T.

(ii) If S is initial in M, and T is initial in M, then S is initial in T.

(iii) If S is embedded in M, and T is initial in M, then S is embedded in T.

Proof. For $A \subset B$, let $i_{A,B} : A \hookrightarrow B$ denote inclusion.

(i) S immersed in M implies that $i_{T,M} \circ i_{S,T} = i_{S,M} : S \to M$ is smooth. Since T is initial in M, it follows that $i_{S,T} : S \to T$ is smooth. Hence S is immersed in T.

(ii) Let $f : P \to S$ be a map. First suppose f is smooth.Then $i_{T,M} \circ i_{S,T} \circ f = i_{S,M} \circ f : P \to M$ is smooth. Since T is initial in M, $i_{S,T} \circ f : P \to T$ is smooth. Conversely, suppose $i_{S,T} \circ f : P \to T$ is smooth. Then $i_{S,M} \circ f = i_{T,M} \circ i_{S,T} \circ f : P \to M$ is smooth. Since S is initial in M, it follows that $f : P \to S$ is smooth.

(iii) T immersed in M implies that the $\mathcal{T}_T^M \subset \mathcal{T}_T$, while S embedded in M means $\mathcal{T}_S^M = \mathcal{T}_S$. So if $U \in \mathcal{T}_S \subset \mathcal{T}_S^M \implies U = V \cap S$ some $V \in \mathcal{T}_M$. Since $S \subset T$, $U = V \cap S = (V \cap T) \cap S$. $V \cap T \in \mathcal{T}_T^M \subset \mathcal{T}_T$, and so $U \in \mathcal{T}_S^T$. Hence $\mathcal{T}_S \subset \mathcal{T}_S^T$.

However, since $i_{T,M} \circ i_{S,T} = i_{S,M} : S \to M$ is smooth and T is initial in M, it follows that $i_{S,T} : S \to T$ is smooth. So $\mathcal{T}_S^T \subset \mathcal{T}_S$.

□

Proposition 2.10.2. *Let $p : A \to C$ be a surjective submersion, $q : B \to D$ a smooth map, and $F : A \to B$ a smooth map which maps p-fibers into q-fibers, i.e.,*

$$\textit{for all } c \in C \textit{ there exists } d \in D \textit{ such that } F(p^{-1}(c)) \subset q^{-1}(d). \tag{2.1}$$

Then

(i) there exists an smooth map $f : C \to D$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "f"] & D \end{tikzcd} .$$

Suppose in addition F satisfies the stronger condition

$$\textit{for all } c \in C \textit{ there exists } d \in D \textit{ such that } F(p^{-1}(c)) = q^{-1}(d). \tag{2.2}$$

We further have that

(ii) if F is injective, then f is injective.

Suppose in addition $q : B \to D$ is a surjective submersion.
We further have that

(iii) if F is an injective immersion, then f is an injective immersion;

(iv) if F is a regular immersion, then f is a regular immersion;

(v) if F is an embedding, then f is an embedding;

(vi) if F is a submersion, then f is a submersion.

Proof. (i) Define $f(c) = q(F(a))$ where a is an arbitrary element of $p^{-1}(c)$. By condition (2.1) this is well-defined.

To prove smoothness of f use the fact that about $c \in C$ there exists a local smooth section $s : U \subset C \to A$ (see for example [Lee03, Proposition 7.16]). Then $f|_U = q \circ F \circ s$, which being the composition of smooth maps is itself smooth.

(ii) Suppose $f(c_1) = f(c_2)$. So $q(F(a_1)) = q(F(a_2))$, where $a_i \in p^{-1}(c_i)$, i.e., $F(a_1)$ and $F(a_2)$ belong to the same q-fiber in B. Condition (2.2) and the injectivity of F imply that a_1 and a_2 are in the same p-fiber of A. Hence $p(a_1) = p(a_2)$, i.e., $c_1 = c_2$.

(iii) Let $c \in C$, and suppose $T_cf(U_c) = 0$ for some $U_c \in T_cC$. Take $a \in p^{-1}(c)$ and $X_a \in T_aA$ such that $T_ap(X_a) = U_c$. The identity $q \circ F = f \circ p$ differentiates to $T_{F(a)}q \circ T_aF = T_{p(a)}f \circ T_ap$. Applying to X_a gives $T_{F(a)}q(T_aF(X_a)) = 0$, which implies that $T_aF(X_a) \in \ker T_{F(a)}q = T_{F(a)}(q^{-1}(d))$ by the Submersion Theorem, where $d = q(F(a)) = f(c)$. Hence $X_a \in (T_aF)^{-1}(T_{F(a)}(q^{-1}(d)))$, which equals $T_a(p^{-1}(c))$ by condition (2.2) and the injectivity of F. So $X_a \in T_a(p^{-1}(c)) = \ker T_ap$, implying that $U_c = T_ap(X_a) = 0$. So T_cf is injective, proving f is an injective immersion.

(iv) Suppose $g : N \to C$ is a function such that $f \circ g : N \to D$ is smooth. We want to show that g is smooth. For any $n \in N$ pick a smooth section $t : V \to B$ about $f(g(n))$, and define $V' = (f \circ g)^{-1}(V)$. Then $t \circ f \circ g|_{V'} : V' \to B$, being the composition of smooth functions, is smooth, and since $q(F(A)) = f(p(A)) = f(C)$, it has image contained in $F(A)$. Since F is a regular immersion, the map $F^{-1} \circ t \circ f \circ g|_{V'} : V' \to A$ is smooth, hence $p \circ F^{-1} \circ t \circ f \circ g|_{V'} : V' \to C$ is smooth. On $F(A)$, $p \circ F^{-1} = f^{-1} \circ q$ and so this map is $f^{-1} \circ q \circ t \circ f \circ g|_{V'}$, which is just $g|_{V'}$. Hence $g|_{V'}$ is smooth, and so g is smooth.

(v) Let $\mathcal{T}_S$ denote the intrinsic topology on a manifold S, and let $\mathcal{T}_S^T = \{V \cap S \,|\, V \in \mathcal{T}_T\}$ denote the topology induced on S by the larger manifold $T \supset S$. We need to show that $f(\mathcal{T}_C) = \mathcal{T}^D_{f(C)}$.

$\mathcal{T}^D_{f(C)} \subset f(\mathcal{T}_C)$ by the continuity of f (note $f(f^{-1}(V)) = V \cap f(C)$).

Conversely suppose $U \in \mathcal{T}_C$. Then $p^{-1}(U) \in \mathcal{T}_A$. Since F is an embedding, $F(p^{-1}(U)) = O \cap F(A)$ for some $O \in \mathcal{T}_B$. $F(A)$ is saturated by condition (2.2) (i.e., $F(A) = q^{-1}(q(F(A)))$, and so $q(F(p^{-1}(U)))) = q(O \cap F(A)) = q(O) \cap q(F(A))$. Using $q \circ F = f \circ p$, this gives $f(U) = q(O) \cap f(p(A)) = q(O) \cap f(C)$. Since q is a surjective submersion, it is open ([Lee03, Proposition 7.16]), and we have that $f(U) \in \mathcal{T}^D_{f(C)}$.

(vi) Since F is a submersion, so is $q \circ F$. By commutativity of the diagram, $f \circ p$ is a submersion. Then since p is a submersion, f must be a submersion.

□

2.10.2 Application to quotient spaces under the group action

To avoid redundancy in the statement of conditions, from now on it will be assumed that any time $\mu \in \mathfrak{g}^*$ and $\mathcal{O}$ appear in an expression, $\mu \in \mathcal{O}$. Likewise any time $a \in \frac{M}{G}$ and $\mathcal{O}$ appear, $a \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$.

Proposition 2.10.3. *(i)* $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ *is an embedded submanifold of* $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}$.

(ii) $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ *is an initial submanifold of* $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}$*, and embedded if* M *is second countable.*

Proof. (i) Since $\mathrm{J}^{-1}(\mu)$ is embedded in M, and $\mathrm{J}^{-1}(\mathcal{O})$ is initial in M, Proposition 2.10.1 (iii) implies that $i^{\mu,\mathcal{O}} : \mathrm{J}^{-1}(\mu) \hookrightarrow \mathrm{J}^{-1}(\mathcal{O})$ is an embedding. Then Proposition 2.10.2 (v) guarantees the existence of an embedding $i^{\mu,\mathcal{O}}_\mathcal{R} : \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \hookrightarrow \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}$ making the following diagram commute:

$$\begin{tikzcd} \mathrm{J}^{-1}(\mu) \arrow[r, hook, "i^{\mu,\mathcal{O}}"] \arrow[d, "\sigma^\mu"'] & \mathrm{J}^{-1}(\mathcal{O}) \arrow[d, "\sigma^\mathcal{O}"'] \\ \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \arrow[r, hook, "i^{\mu,\mathcal{O}}_\mathcal{R}"] & \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}} \end{tikzcd} .$$

(ii) Since $\pi^{-1}(a)$ is initial in M, and $\mathrm{J}^{-1}(\mathcal{O})$ is initial in M, Proposition 2.10.1 (ii) implies that $i^{a,\mathcal{O}} : \pi^{-1}(a) \hookrightarrow \mathrm{J}^{-1}(\mathcal{O})$ is a regular immersion. Proposition 2.10.2 (iv) guarantees the existence of a regular immersion $i^{a,\mathcal{O}}_\mathcal{R} : \frac{\pi^{-1}(a)}{\mathcal{R}^a} \hookrightarrow \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}$ making the following diagram commute:

$$\begin{tikzcd} \pi^{-1}(a) \arrow[r, hook, "i^{a,\mathcal{O}}"] \arrow[d, "\sigma^a"'] & \mathrm{J}^{-1}(\mathcal{O}) \arrow[d, "\sigma^\mathcal{O}"'] \\ \frac{\pi^{-1}(a)}{\mathcal{R}^a} \arrow[r, hook, "i^{a,\mathcal{O}}_\mathcal{R}"] & \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}} \end{tikzcd} .$$

Mutatis mutandis, the argument in the second countable case is the same.

□

2.11 Foliation reduction

Suppose the coadjoint-action stabilizer groups G_μ are connected. This holds, for example, when G is both compact and connected [DK00, Theorem (3.3.1) (ii)]. In this case, symplectic reduction can be seen as an application of *foliation reduction*, which is described in this section. We adopt this viewpoint since it will be used later to "lift" symplectic reduction to prequantization U(1)-bundles over the symplectic manifold M, and give a quantum analogue of symplectic reduction.

The important features of foliation reduction are given by the following theorem.

Proposition 2.11.1 (The Foliation Reduction Theorem). *Let β be a differential p-form on a manifold S, and define the characteristic distribution N of β by*

$$N_s := \{X_s \in T_sS \,|\, \mathrm{i}_{X_s}\beta_s = 0 \textit{ and } \mathrm{i}_{X_s}(\mathrm{d}_s\beta) = 0\}$$

for any $s \in S$. Assume that N defines a smooth vector subbundle of TM. Then:

(i) N is involutive, and hence has defines a foliation $\mathcal{N}$ on S.

(ii) If the leaf space $S/\mathcal{N}$ can be given a smooth structure such that the map $\pi_\mathcal{N} : S \to S/\mathcal{N}$ is a submersion, then there exists a unique differential p-form $\beta_\mathcal{N}$ on $S/\mathcal{N}$ satisfying

$$(\pi_\mathcal{N})^*\beta_\mathcal{N} = \beta.$$

(iii) If $\mathrm{d}\beta = 0$, then $\mathrm{d}\beta_\mathcal{N} = 0$, and $\beta_\mathcal{N}$ is nondegenerate.

Proof. (i) Let $X, Y \in \Gamma(TN) \subset \Gamma(TS)$. The definition of N and Cartan's magic formula imply that

$$\mathcal{L}_X\beta = \mathrm{d}(\mathrm{i}_X\beta) + \mathrm{i}_X\mathrm{d}\beta = 0,$$

and hence

$$\mathrm{i}_{[X,\,Y]}\beta = [\mathcal{L}_X,\, \mathrm{i}_Y]\beta = 0.$$

It follows that $[X, Y] \in \Gamma(TN)$, and so N is an involutive distribution. The Global Frobenius Theorem [Lee03, Proposition 19.21] then says that the maximal connected integral manifolds of N form a foliation $\mathcal{N}$ of S.

(ii) Define $\beta_\mathcal{N}$ as follows: for a collection $V_u^i \in T_u(S/\mathcal{N})$ $i = 1, \ldots, p$, let $s \in (\pi_\mathcal{N})^{-1}(u)$ and

$Y_s^i \in T_sS$ such that $T_s\pi_{\mathcal{N}}(Y_s^i) = V_u^i$ (such Y^is exist since $\pi_{\mathcal{N}}$ is a submersion). Then

$$(\beta_{\mathcal{N}})_u(V_u^1, \ldots, V_u^p) = \beta_s(Y_s^1, \ldots, Y_s^p).$$

We must show that $\beta_{\mathcal{N}}$ is well-defined. Suppose $t \in (\pi_{\mathcal{N}})^{-1}(u)$ and $T_t\pi_{\mathcal{N}}(Z_t^i) = V_u^i$, $i = 1, \ldots, p$. Then s and t are in the same leaf of the foliation $\mathcal{N}$, and so $t = \varphi(s)$ where φ is a diffeomorphism consisting of a finite composition of flows $\exp(X^n) \circ \exp(X^{n-1}) \circ \ldots \circ \exp(X^1)$ generated by vector fields $X^j \in \Gamma(TN)$. As shown in part (i), each X^j satisfies $\mathcal{L}_{X^j}\beta$, which implies that $\exp(X^j)^*\beta = \beta \; \forall j$ and hence $\varphi^*\beta = \beta$. Also the flow ϕ preserves the leaves of $\mathcal{N}$, and so $\pi_{\mathcal{N}} \circ \varphi = \pi_{\mathcal{N}}$. We then have that

$$\begin{aligned}
\beta_s(Y_s^1, \ldots, Y_s^p) &= (\varphi^*\beta)_s(Y_s^1, \ldots, Y_s^p) \\
&= \beta_{\varphi(s)}(T_s\varphi(Y_s^1), \ldots, T_s\varphi(Y_s^p)) \\
&= \beta_t\left([T_s\varphi(Y_s^1) - Z_t^1] + Z_t^1, \ldots, [T_s\varphi(Y_s^p) - Z_t^p] + Z_t^p\right).
\end{aligned}$$

Using $T_{\varphi(s)}\pi_{\mathcal{N}} \circ T_s\varphi = T_s(\pi_{\mathcal{N}} \circ \varphi) = T_s\pi_{\mathcal{N}}$, the terms in square brackets project to zero under $T_t\pi_{\mathcal{N}}$, and so must lie in N_t, and in particular are directions of degeneracy for β_t. It follows that

$$\beta_s(Y_s^1, \ldots, Y_s^p) = \beta_t(Z_t^1, \ldots, Z_t^p),$$

proving the well-definedness of $\beta_{\mathcal{N}}$. The defining equation for $\beta_{\mathcal{N}}$ can be rewritten as

$$(\beta_{\mathcal{N}})_{\pi_{\mathcal{N}}(s)}(T_s\pi_{\mathcal{N}}(Y_s^1), \ldots, T_s\pi_{\mathcal{N}}(Y_s^p)) = \beta_s(Y_s^1, \ldots, Y_s^p),$$

i.e., $(\pi_{\mathcal{N}})^*\beta_{\mathcal{N}} = \beta$. Since $\pi_{\mathcal{N}}$ is a submersion, this identity implies that $\beta_{\mathcal{N}}$ must be unique.

(iii) $(\pi_{\mathcal{N}})^*\beta_{\mathcal{N}} = \beta \implies (\pi_{\mathcal{N}})^*\mathrm{d}\beta_{\mathcal{N}} = \mathrm{d}(\pi_{\mathcal{N}})^*\beta_{\mathcal{N}} = \mathrm{d}\beta = 0$ by assumption. Since $\pi_{\mathcal{N}}$ is a submersion, it follows that $\mathrm{d}\beta_{\mathcal{N}} = 0$.

Now suppose $V_u \in T_u(S/\mathcal{N})$ is such that $\mathrm{i}_{V_u}(\beta_{\mathcal{N}})_u = 0$. Pick $Y_s \in T_sS$ such that $T_s\pi_{\mathcal{N}}(Y_s) = V_u$. Then

$$\begin{aligned}
\mathrm{i}_{Y_s}\beta_s &= \mathrm{i}_{Y_s}\left((\pi_{\mathcal{N}})^*\beta_{\mathcal{N}}\right)_s \\
&= \mathrm{i}_{Y_s}\left((T_s\pi_{\mathcal{N}})^*(\beta_{\mathcal{N}})_{\pi_{\mathcal{N}}(s)}\right) \\
&= (T_s\pi_{\mathcal{N}})^*\left(\mathrm{i}_{V_u}(\beta_{\mathcal{N}})_u\right) \\
&= 0.
\end{aligned}$$

Also $\mathrm{i}_{Y_s}(\mathrm{d}_s\beta) = 0$ automatically holds since β is closed. Hence $Y_s \in N_s$, and so $V_u =$

$T_s\pi_{\mathcal{N}}(Y_s) = 0$, proving $\beta_{\mathcal{N}}$ is nondegenerate.

□

2.12 The foliation-reduced symplectic manifolds

By the commutativity of exterior derivative and pullback, the restriction $\omega^{\mathcal{O}} = (i^{\mathcal{O}})^*\omega$ of the symplectic form to $\mathrm{J}^{-1}(\mathcal{O})$ is closed, and so its characteristic distribution equals its directions of degeneracy. Corollary 2.9.2 (iii) tells us that at a point $x \in \mathrm{J}^{-1}(\mathcal{O})$ these directions are precisely $\mathfrak{g}_{\mathrm{J}(x)} \cdot x$. Assuming connectedness of the coadjoint-action stabilizer groups $G_{\mathrm{J}(x)}$, the characteristic distribution defines a foliation of $\mathrm{J}^{-1}(\mathcal{O})$ whose connected leaf through x is $G_{\mathrm{J}(x)} \cdot x$, which is the equivalence class of x under the relation $\mathcal{R}^{\mathcal{O}}$. We therefore use the same notation $\mathcal{R}^{\mathcal{O}}$ to also denote the foliation generated by the characteristic distribution. Similarly, since the degeneracy directions of $\omega^{\mu} = (i^{\mu})^*\omega$ and $\omega^{a} = (i^{a})^*\omega$ are 2.9.2 (Corollary 2.9.2 (i), (ii)), the characteristic foliations of these forms are denoted $\mathcal{R}^{\mu}$ and $\mathcal{R}^{a}$.

It will also be useful to introduce a notation for the individual leaves of the foliation $\mathcal{R}^{\mathcal{O}}$. Recall that the leaf through a point $x \in \mathrm{J}^{-1}(\mathcal{O})$ is simply $G_{\mathrm{J}(x)} \cdot x$, and by Proposition 2.9.1 (ii) this is the set $\pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu)$, where $a = \pi(x)$ and $\mu = \mathrm{J}(x)$. So for arbitrary $(\mu, a) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, define

$$\mathcal{R}^{(\mu,a)} = \pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu).$$

Then $\mathcal{R}^{\mathcal{O}} = \{\mathcal{R}^{(\mu,a)} \mid (\mu, a) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}\}$. $\mathcal{R}^{(\mu,a)}$ can be viewed in two ways: as a subset of $\mathrm{J}^{-1}(\mathcal{O})$, or as a point in the leaf space $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$. The intended interpretation will be clear from the context.

The union of $\mathcal{R}^{\mathcal{O}}$ for all coadjoint orbits $\mathcal{O}$ will be denoted $\mathcal{R}$ (again, consistent with the notation for the equivalence relation), and will define a *generalized* foliation of M, in the sense that the dimension of the leaves of $\mathcal{R}$ are not constant.

We have seen (Proposition 2.8.5 (iii)) that $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ can be given a smooth structure which makes $\sigma^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ a submersion. Given this, The Foliation Reduction Theorem (Proposition 2.11.1) tells us that there exists a nondegenerate, closed 2-form $\omega^{\mathcal{O}}_{\mathcal{R}}$ (i.e., a symplectic form) on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ satisfying $\omega^{\mathcal{O}} = (\sigma^{\mathcal{O}})^*\omega^{\mathcal{O}}_{\mathcal{R}}$.

Similar reasoning applies to the 2-forms $\omega^{\mu} = (i^{\mu})^*\omega$ and $\omega^{a} = (i^{a})^*\omega$. In summary, we obtain the following result:

Proposition 2.12.1. *(i) There exists a symplectic form $\omega^{\mathcal{O}}_{\mathcal{R}}$ on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ satisfying $\omega^{\mathcal{O}} = (\sigma^{\mathcal{O}})^*\omega^{\mathcal{O}}_{\mathcal{R}}$.*

(ii) There exists a symplectic form $\omega^{\mu}_{\mathcal{R}}$ on $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}$ satisfying $\omega^{\mu} = (\sigma^{\mu})^*\omega^{\mu}_{\mathcal{R}}$.

(iii) There exists a symplectic form $\omega^{a}_{\mathcal{R}}$ on $\frac{\pi^{-1}(a)}{\mathcal{R}^{a}}$ satisfying $\omega^{a} = (\sigma^{a})^*\omega^{a}_{\mathcal{R}}$.

Note. We point out that since $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} = \frac{\mathrm{J}^{-1}(\mu)}{G_\mu}$, the symplectic manifold $(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}, \omega^\mu_{\mathcal{R}})$ is the usual Marsden-Weinstein quotient (see for example [MMO+07]). Hence foliation reduction on $\mathrm{J}^{-1}(\mu)$ is equivalent to the usual point reduction ([MMO+07]) picture. We will demonstrate in Section 2.15 that foliation reduction on $(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}, \omega^\mathcal{O}_{\mathcal{R}})$ is equivalent to the orbit reduction ([MMO+07]) picture of symplectic reduction.

Since $\omega^\mathcal{O}_{\mathcal{R}}$, $\omega^\mu_{\mathcal{R}}$, and $\omega^a_{\mathcal{R}}$ are obtained by "quotienting out" the same fibers $\mathcal{R}$, the latter two are restrictions of the first. Formally, we have

Corollary 2.12.2. *(i) $\omega^\mu_{\mathcal{R}} = (i^{\mu,\mathcal{O}}_{\mathcal{R}})^*\omega^\mathcal{O}_{\mathcal{R}}$, where $i^{\mu,\mathcal{O}}_{\mathcal{R}}$ is the embedding of $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ into $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}$.*

(ii) $\omega^a_{\mathcal{R}} = (i^{a,\mathcal{O}}_{\mathcal{R}})^*\omega^\mathcal{O}_{\mathcal{R}}$, where $i^{a,\mathcal{O}}_{\mathcal{R}}$ is the regular immersion of $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ into $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}$.

Proof. (i) Applying $(i^{\mu,\mathcal{O}})^*$ to both sides of $(i^\mathcal{O})^*\omega = (\sigma^\mathcal{O})^*\omega^\mathcal{O}_{\mathcal{R}}$, we get

$$(i^\mathcal{O} \circ i^{\mu,\mathcal{O}})^*\omega = (\sigma^\mathcal{O} \circ i^{\mu,\mathcal{O}})^*\omega^\mathcal{O}_{\mathcal{R}}.$$

Now $i^\mathcal{O} \circ i^{\mu,\mathcal{O}} = i^\mu$, and consulting the commutative diagram in Proposition 2.10.3 (i) we recall the relation $\sigma^\mathcal{O} \circ i^{\mu,\mathcal{O}} = i^{\mu,\mathcal{O}}_{\mathcal{R}} \circ \sigma^\mu$. So the above identity becomes

$$(i^\mu)^*\omega = (i^{\mu,\mathcal{O}}_{\mathcal{R}} \circ \sigma^\mu)^*\omega^\mathcal{O}_{\mathcal{R}}.$$

Using $(i^\mu)^*\omega = (\sigma^\mu)^*\omega^\mu_{\mathcal{R}}$ this becomes

$$(\sigma^\mu)^*\omega^\mu_{\mathcal{R}} = (\sigma^\mu)^*(i^{\mu,\mathcal{O}}_{\mathcal{R}})^*\omega^\mathcal{O}_{\mathcal{R}}.$$

σ^μ being a submersion then implies the result.

(ii) Similar, using $\sigma^\mathcal{O} \circ i^{a,\mathcal{O}} = i^{a,\mathcal{O}}_{\mathcal{R}} \circ \sigma^a$ from Proposition 2.10.3 (ii).

□

Hence the reduced symplectic manifold $(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}, \omega^\mathcal{O}_{\mathcal{R}})$ has two distinct foliations by symplectic submanifolds,

$$\left\{ \frac{\pi^{-1}(a)}{\mathcal{R}^a} \,\middle|\, a \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{G} \right\} \qquad \text{and} \qquad \left\{ \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \,\middle|\, \mu \in \mathcal{O} \right\},$$

where the symplectic form on each leaf is just the restriction of $\omega^\mathcal{O}_{\mathcal{R}}$ to each leaf. These two symplectic foliations are dual in the following sense.

Proposition 2.12.3. *Let $(\mu, a) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ for some coadjoint orbit $\mathcal{O}$. Considering $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ and $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ as submanifolds of $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}$, we have that*

(i) $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ and $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ intersect at precisely one point, namely $\mathcal{R}^{(\mu,a)}$;

(ii) the tangent space to $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ *at* $\mathcal{R}^{(\mu,a)}$ *is a direct sum of the tangent spaces to* $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ *and* $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$*, i.e.,*

$$T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}\right)=T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a}\right)\oplus T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}\right);$$

(iii) the factors in the tangent space decomposition in part (ii) are symplectically orthogonal, i.e.,

$$T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a}\right)^{\omega^{\mathcal{O}}_{\mathcal{R}}}=T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}\right).$$

Proof. (i) The set $\pi^{-1}(a)$ is foliated by $\mathcal{R}$-leaves, and the set $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is the collection of these leaves,

$$\frac{\pi^{-1}(a)}{\mathcal{R}^a}=\left\{\mathcal{R}^{(\nu,a)}\;\middle|\;\nu\in\mathcal{O}\right\}.$$

Similarly $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ can be written as

$$\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}=\left\{\mathcal{R}^{(\mu,b)}\;\middle|\;b\in\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}\right\}.$$

These two sets intersect at the point $\mathcal{R}^{(\mu,a)}$.

(ii) Let $x\in\mathrm{J}^{-1}(\mathcal{O})$ be any point of $(\sigma^{\mathcal{O}})^{-1}(\mathcal{R}^{(\mu,a)})$ $(=\mathcal{R}^{(\mu,a)}$ considered as a subset of $\mathrm{J}^{-1}(\mathcal{O}))$. Proposition 2.9.1 (i) says that $T_x(\mathrm{J}^{-1}(\mathcal{O}))=T_x(\pi^{-1}(a))+T_x(\mathrm{J}^{-1}(\mu))$. Applying $T_x\sigma^{\mathcal{O}}$ to both sides yields

$$T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}\right)=T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a}\right)+T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}\right),$$

and the sum is direct by part (i).

(iii) Proposition 2.9.1 (iv) says that $T_x(\pi^{-1}(a))^\omega=T_x(\mathrm{J}^{-1}(\mu))$, and since the latter is a subspace of $T_x(\mathrm{J}^{-1}(\mathcal{O}))$, we can instead write

$$T_x(\pi^{-1}(a))^{\omega^{\mathcal{O}}}=T_x(\mathrm{J}^{-1}(\mu)).$$

The result now follows easily from $\omega^{\mathcal{O}}=(\sigma^{\mathcal{O}})^*\omega^{\mathcal{O}}_{\mathcal{R}}$ and the identities

$$T_x\sigma^{\mathcal{O}}\left[T_x(\pi^{-1}(a))\right]=T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a}\right)\quad\text{and}\quad T_x\sigma^{\mathcal{O}}\left[T_x(\mathrm{J}^{-1}(\mu))\right]=T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}\right).$$

□

2.13 The reduced group action, projection, and momentum map

Equivariance of J says that for $x \in M$, $\mathrm{J}(g \cdot x) = \mathrm{Ad}^*_{g^{-1}}\mathrm{J}(x)$, and hence the coadjoint stabilizer groups corresponding to x and $g \cdot x$ are related by $G_{\mathrm{J}(g\cdot x)} = G_{\mathrm{Ad}^*_{g^{-1}}\mathrm{J}(x)} = \mathrm{Ad}_g G_{\mathrm{J}(x)}$. Applying $g \in G$ to the entire $G_{\mathrm{J}(x)}$-orbit through x, we get

$$g \cdot G_{\mathrm{J}(x)} \cdot x = \left(\mathrm{Ad}_g G_{\mathrm{J}(x)}\right) \cdot g \cdot x = \left(G_{\mathrm{Ad}^*_{g^{-1}}\mathrm{J}(x)}\right) \cdot (g \cdot x).$$

These orbits are precisely the leaves of the foliation $\mathcal{R}$. In other words, the G-action on M preserves $\mathcal{R}$. Restricting to $\mathrm{J}^{-1}(\mathcal{O})$, $\mathcal{R}^{\mathcal{O}}$ is preserved, and so we can drop the G-action to the space $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$. Denoting the G-action on $\mathrm{J}^{-1}(\mathcal{O})$ by $\Phi^{\mathcal{O}}$, we have the following result.

Proposition 2.13.1. *There exists a smooth, symplectic G-action $\Phi^{\mathcal{O}}_{\mathcal{R}}$ on $(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}, \omega^{\mathcal{O}}_{\mathcal{R}})$ making the following diagram commute,*

$$\begin{tikzcd}
G \times \mathrm{J}^{-1}(\mathcal{O}) \arrow[r, "\Phi^{\mathcal{O}}"] \arrow[d, "\mathrm{id}_G \times \sigma^{\mathcal{O}}"'] & \mathrm{J}^{-1}(\mathcal{O}) \arrow[d, "\sigma^{\mathcal{O}}"'] \\
G \times \dfrac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \arrow[r, "\Phi^{\mathcal{O}}_{\mathcal{R}}"] & \dfrac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}
\end{tikzcd}.$$

Proof. As previously discussed, since $\mathrm{J}^{-1}(\mathcal{O})$ is initial in M, the smooth action Φ restricts to a smooth action $\Phi^{\mathcal{O}}$ on $\mathrm{J}^{-1}(\mathcal{O})$. The existence of $\Phi^{\mathcal{O}}_{\mathcal{R}}$ follows from Proposition 2.10.2 (i) and the fact that $\Phi^{\mathcal{O}}$ preserves $\mathcal{R}^{\mathcal{O}}$.

The action properties of $\Phi^{\mathcal{O}}_{\mathcal{R}}$ follow from the action properties of $\Phi^{\mathcal{O}}$, and commutativity of the diagram.

The symplectic property follows from the symplectic property of $\Phi^{\mathcal{O}}$

$$(\Phi^{\mathcal{O}}_g)^* \omega^{\mathcal{O}} = \omega^{\mathcal{O}},$$

the identities $\sigma^{\mathcal{O}} \circ \Phi^{\mathcal{O}}_g = (\Phi^{\mathcal{O}}_{\mathcal{R}})_g \circ \sigma^{\mathcal{O}}$ and $\omega^{\mathcal{O}} = (\sigma^{\mathcal{O}})^* \omega^{\mathcal{O}}_{\mathcal{R}}$, and the fact that $\sigma^{\mathcal{O}}$ is a submersion. □

Note that the diagram in Proposition 2.13.1 expresses the G-equivariance of $\sigma^{\mathcal{O}}$ with respect to the unreduced and reduced G-actions $\Phi^{\mathcal{O}}$ and $\Phi^{\mathcal{O}}_{\mathcal{R}}$. Also note that unlike the unreduced action, the reduced action is not free: any element of $G_{\mathrm{J}(x)}$ will act trivially on the $\mathcal{R}^{\mathcal{O}}$ leaf $G_{\mathrm{J}(x)} \cdot x$ through $x \in \mathrm{J}^{-1}(\mathcal{O})$.

Since we have a smooth G-action on the reduced space $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, it is natural to consider the quotient of $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ by G. Since the “points” of $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ are themselves orbits of a subgroup of G in

$\mathrm{J}^{-1}(\mathcal{O})$, this space agrees with $\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, and we have

Proposition 2.13.2. *The reduced G-action has a projection map $\pi^{\mathcal{O}}_{\mathcal{R}} : \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ making the following diagram commute,*

$$\begin{tikzcd} & \mathrm{J}^{-1}(\mathcal{O}) \arrow[dl, "\sigma^{\mathcal{O}}"'] \arrow[dr, "\pi^{\mathcal{O}}"] & \\ \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \arrow[rr, "\pi^{\mathcal{O}}_{\mathcal{R}}"] & & \frac{\mathrm{J}^{-1}(\mathcal{O})}{G} \end{tikzcd} \quad . \tag{2.3}$$

Further, $\pi^{\mathcal{O}}_{\mathcal{R}}$ is a surjective submersion.

Proof. Proposition 2.8.4 says that $\pi^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ is a submersion. Then applying Proposition 2.10.2 (vi) with $F = \mathrm{id}_{\mathrm{J}^{-1}(\mathcal{O})}$, and noticing that each leaf of $\mathcal{R}^{\mathcal{O}}$ is contained in a G-orbit in $\mathrm{J}^{-1}(\mathcal{O})$, we get the existence of the submersion $\pi^{\mathcal{O}}_{\mathcal{R}}$, which is surjective since $\pi^{\mathcal{O}}$ is surjective.

For $a \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$

$$\begin{aligned} \sigma^{\mathcal{O}}(\pi^{\mathcal{O}^{-1}}(a)) &= \{\sigma^{\mathcal{O}}(x) \,|\, \pi^{\mathcal{O}}(x) = a\} \\ &= \{\sigma^{\mathcal{O}}(x) \,|\, \pi^{\mathcal{O}}_{\mathcal{R}}(\sigma^{\mathcal{O}}(x)) = a\} \\ &= \left\{ y \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \,\middle|\, \pi^{\mathcal{O}}_{\mathcal{R}}(y) = a \right\} \\ &= \pi^{\mathcal{O}^{-1}}_{\mathcal{R}}(a), \end{aligned}$$

the second-to-last line following from the surjectivity of $\sigma^{\mathcal{O}}$. Since $\pi^{\mathcal{O}^{-1}}(a)$ is a G-orbit in $\mathrm{J}^{-1}(\mathcal{O})$, and $\sigma^{\mathcal{O}}$ is G-equivariant, it follows that $\pi^{\mathcal{O}^{-1}}_{\mathcal{R}}(a) = \sigma^{\mathcal{O}}(\pi^{\mathcal{O}^{-1}}(a))$ is a G-orbit in $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$. □

Note. In particular, we have that $\pi^{\mathcal{O}^{-1}}_{\mathcal{R}}(a) = \sigma^{\mathcal{O}}(\pi^{\mathcal{O}^{-1}}(a)) = \sigma^{\mathcal{O}}(\pi^{-1}(a)) = \frac{\pi^{-1}(a)}{\mathcal{R}^{a}}$.

Proposition 2.13.3. *$\pi^{\mathcal{O}}_{\mathcal{R}}$ restricts to a diffeomorphism $\pi^{\mu}_{\mathcal{R}} : \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ which makes the following diagram commute,*

$$\begin{tikzcd} & \mathrm{J}^{-1}(\mu) \arrow[dl, "\sigma^{\mu}"'] \arrow[dr, "\pi^{\mu}"] & \\ \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \arrow[rr, shift left, "\pi^{\mu}_{\mathcal{R}}"] & & \frac{\mathrm{J}^{-1}(\mathcal{O})}{G} \arrow[ll, shift left, "(\pi^{\mu}_{\mathcal{R}})^{-1}"] \end{tikzcd} \quad . \tag{2.4}$$

Proof. By G-equivariance of J, $\mathrm{J}^{-1}(\mathcal{O}) = G \cdot \mathrm{J}^{-1}(\mu)$. So $\pi(\mathrm{J}^{-1}(\mu)) = \pi(\mathrm{J}^{-1}(\mathcal{O})) = \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, demonstrating that π^{μ} is surjective. $\pi^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ is a submersion (Proposition 2.8.4 (iv)), and for any $x \in \mathrm{J}^{-1}(\mu)$, $T_x(\mathrm{J}^{-1}(\mathcal{O})) = T_x(\pi^{-1}(a)) + T_x(\mathrm{J}^{-1}(\mu)) = \ker T_x\pi + T_x(\mathrm{J}^{-1}(\mu))$, where

$a = \pi(x)$ (Proposition 2.9.1 (i)), so $\pi^{\mathcal{O}}$ remains a submersion when restricted to $\mathrm{J}^{-1}(\mu)$, i.e., π^{μ} is a submersion. The fiber of π^{μ} through $x \in \mathrm{J}^{-1}(\mu)$ equals

$$G_{\mu} \cdot x = \pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu) = \mathcal{R}^{(\mu,a)},$$

where $a = \pi(x)$. This agrees with the fiber of $\sigma^{\mu} : \mathrm{J}^{-1}(\mu) \to \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}$ through x. Hence both surjective submersions in Diagram 2.4 have the same fibers. Applying Proposition 2.10.2 (vi) with $F = \mathrm{id}_{\mathrm{J}^{-1}(\mu)}$, in both directions, implies the existence of the submersion $\pi^{\mu}_{\mathcal{R}}$ and its inverse. Hence $\pi^{\mu}_{\mathcal{R}}$ is a diffeomorphism. Comparison with Diagram 2.3 makes it clear that $\pi^{\mu}_{\mathcal{R}}$ is the restriction of $\pi^{\mathcal{O}}_{\mathcal{R}}$ to $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}$. $\square$

Also since we have a G-action on the reduced symplectic manifold $(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}, \omega^{\mathcal{O}}_{\mathcal{R}})$, it is natural to ask whether there is a corresponding momentum map, and whether it is equivariant with respect to the G-action. The answer to both questions is yes.

Proposition 2.13.4. *The reduced G-action has an equivariant momentum map $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}} : \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \to \mathcal{O}$ making the following diagram commute,*

$$\begin{tikzcd}
& \mathrm{J}^{-1}(\mathcal{O}) \arrow[dl, "\sigma^{\mathcal{O}}"'] \arrow[dr, "\mathrm{J}^{\mathcal{O}}"] & \\
\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \arrow[rr, "\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}"] & & \mathcal{O}
\end{tikzcd} \quad . \tag{2.5}$$

Further, $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}$ is a surjective submersion.

Proof. Existence of a submersion $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}$ satisfying the commutative diagram follows from Proposition 2.10.2 (vi) with $F = \mathrm{id}_{\mathrm{J}^{-1}(\mathcal{O})}$, and the fact that each $\mathcal{R}^{\mathcal{O}}$-leaf is contained in some level set $\mathrm{J}^{-1}(\mu)$ of the momentum map. Since $\mathrm{J}^{\mathcal{O}}$ is surjective, so is $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}$.

The G-equivariance of $\sigma^{\mathcal{O}}$ implies that for $\xi \in \mathfrak{g}$, the infinitesimal generators $\xi_{\mathrm{J}^{-1}(\mathcal{O})}$ on $\mathrm{J}^{-1}(\mathcal{O})$ and $\xi_{\mathrm{J}^{-1}(\mathcal{O})/\mathcal{R}^{\mathcal{O}}}$ on $\mathrm{J}^{-1}(\mathcal{O})/\mathcal{R}^{\mathcal{O}}$ are $\sigma^{\mathcal{O}}$-related, i.e.,

$$T\sigma^{\mathcal{O}} \circ \xi_{\mathrm{J}^{-1}(\mathcal{O})} = \xi_{\mathrm{J}^{-1}(\mathcal{O})/\mathcal{R}^{\mathcal{O}}} \circ \sigma^{\mathcal{O}}.$$

Hence we have that

$$\begin{aligned}
(\sigma^{\mathcal{O}})^{*} \left(\mathrm{i}_{\xi_{\mathrm{J}^{-1}(\mathcal{O})/\mathcal{R}^{\mathcal{O}}}} \omega^{\mathcal{O}}_{\mathcal{R}} \right) &= \mathrm{i}_{\xi_{\mathrm{J}^{-1}(\mathcal{O})}} (\sigma^{\mathcal{O}})^{*} \omega^{\mathcal{O}}_{\mathcal{R}} \\
&= \mathrm{i}_{\xi_{\mathrm{J}^{-1}(\mathcal{O})}} \omega^{\mathcal{O}} \\
&= \mathrm{d}\langle J^{\mathcal{O}}, \xi \rangle \\
&= \mathrm{d}\langle J^{\mathcal{O}}_{\mathcal{R}} \circ \sigma^{\mathcal{O}}, \xi \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{d}\left((\sigma^{\mathcal{O}})^* \langle \mathrm{J}^{\mathcal{O}}_{\mathcal{R}}, \xi \rangle\right) \\
&= (\sigma^{\mathcal{O}})^* \mathrm{d} \langle \mathrm{J}^{\mathcal{O}}_{\mathcal{R}}, \xi \rangle.
\end{aligned}
$$

Since $\sigma^{\mathcal{O}}$ is a submersion, it follows that

$$
\mathrm{i}_{\xi_{\mathrm{J}^{-1}(\mathcal{O})/\mathcal{R}^{\mathcal{O}}}} \omega^{\mathcal{O}}_{\mathcal{R}} = \mathrm{d} \langle \mathrm{J}^{\mathcal{O}}_{\mathcal{R}}, \xi \rangle,
$$

implying that $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}$ is the momentum map for the reduced action.

Finally, G-equivariance of $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}$ follows from the G-equivariance of $\mathrm{J}^{\mathcal{O}}$ and $\sigma^{\mathcal{O}}$, and the fact that $\sigma^{\mathcal{O}}$ is a surjection. □

Note. In particular, we have that ${\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}}^{-1}(\mu) = \sigma^{\mathcal{O}}({\mathrm{J}^{\mathcal{O}}}^{-1}(\mu)) = \sigma^{\mathcal{O}}(\mathrm{J}^{-1}(\mu)) = \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}$.

Proposition 2.13.5. $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}$ *restricts to a diffeomorphism* $\mathrm{J}^{a}_{\mathcal{R}} : \frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \to \mathcal{O}$ *making the following diagram commute,*

$$
\begin{tikzcd}
 & \pi^{-1}(a) \arrow[dl, "\sigma^{a}"'] \arrow[dr, "\mathrm{J}^{a}"] & \\
\frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \arrow[rr, shift left, "\mathrm{J}^{a}_{\mathcal{R}}"] & & \mathcal{O} \arrow[ll, shift left, "(\mathrm{J}^{a}_{\mathcal{R}})^{-1}"]
\end{tikzcd}
\quad . \tag{2.6}
$$

Proof. The argument is similar to that in Proposition 2.13.3, and so the details are omitted. □

Corollary 2.13.6. *In terms of the notation for individual* $\mathcal{R}$*-leaves,*

(i) the reduced G*-action satisfies* $g \cdot \mathcal{R}^{(\mu,a)} = \mathcal{R}^{(\mathrm{Ad}^*_{g^{-1}}\mu, a)}$*;*

(ii) the reduced projection satisfies $\pi^{\mathcal{O}}_{\mathcal{R}}(\mathcal{R}^{(\mu,a)}) = a$*;*

(iii) the reduced momentum map satisfies $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}(\mathcal{R}^{(\mu,a)}) = \mu$*.*

Proof. (i) The action of $g \in G$ on the leaf space is

$$
g \cdot \mathcal{R}^{(\mu,a)} = g \cdot (\pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu)) = \pi^{-1}(a) \cap \mathrm{J}^{-1}(\mathrm{Ad}^*_{g^{-1}}\mu) = \mathcal{R}^{(\mathrm{Ad}^*_{g^{-1}}\mu, a)}.
$$

(ii) Considered as a subset of $\mathrm{J}^{-1}(\mathcal{O})$, any point of $x \in \mathcal{R}^{(\mu,a)}$ has $\pi^{\mathcal{O}}(x) = a$. Therefore $\pi^{\mathcal{O}}_{\mathcal{R}}(\mathcal{R}^{(\mu,a)}) = a$.

(iii) Considered as a subset of $\mathrm{J}^{-1}(\mathcal{O})$, any point of $x \in \mathcal{R}^{(\mu,a)}$ has $\mathrm{J}^{\mathcal{O}}(x) = \mu$. Therefore $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}(\mathcal{R}^{(\mu,a)}) = \mu$.

□

2.14 The canonical symplectomorphism

In this section, we give a proof that $(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}, \omega_{\mathcal{R}}^{\mathcal{O}})$ is canonically symplectomorphic to the cross product $(\frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}, \omega_{\mathcal{R}}^{a} \oplus \omega_{\mathcal{R}}^{\mu})$ of any two of its transverse symplectic submanifolds. This is accomplished by demonstrating that both manifolds are symplectomorphic to the manifold $\mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, with a suitably defined symplectic form. We follow this line of proof in order to make contact with the orbit reduction picture of symplectic reduction. Later in Section 6.1.1 we give another "direct" proof, which can then be lifted to the prequantum U(1)-bundles.

Using the diffeomorphism $J_{\mathcal{R}}^{a} : \frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \to \mathcal{O}$, the symplectic form $\omega_{\mathcal{R}}^{a}$ can be pushed forward to a symplectic form on $\mathcal{O}$. Similarly, using $\pi_{\mathcal{R}}^{\mu} : \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, $\omega_{\mathcal{R}}^{\mu}$ can be pushed forward to $\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$. It turns out that the resulting symplectic forms on $\mathcal{O}$ and $\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ are independent of $(\mu, a) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$.

Proposition 2.14.1. *(i)* $(\mathrm{J}_{\mathcal{R}}^{a})_{*}\omega_{\mathcal{R}}^{a} = (\mathrm{J}_{\mathcal{R}}^{b})_{*}\omega_{\mathcal{R}}^{b}$ *for all* $a, b \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$.

(ii) $(\pi_{\mathcal{R}}^{\mu})_{*}\omega_{\mathcal{R}}^{\mu} = (\pi_{\mathcal{R}}^{\nu})_{*}\omega_{\mathcal{R}}^{\nu}$ *for all* $\mu, \nu \in \mathcal{O}$.

Proof. (i) At any point $\mathcal{R}^{(\nu,a)} \in \frac{\pi^{-1}(a)}{\mathcal{R}^{a}}$, $T_{\mathcal{R}^{(\nu,a)}}\mathrm{J}_{\mathcal{R}}^{a}(\xi \cdot \mathcal{R}^{(\nu,a)}) = -\mathrm{ad}_{\xi}^{*}(\mathrm{J}_{\mathcal{R}}^{a}(\mathcal{R}^{(\nu,a)})) = -\mathrm{ad}_{\xi}^{*}\nu$. Therefore

$$
\begin{aligned}
((\mathrm{J}_{\mathcal{R}}^{a})_{*}\omega_{\mathcal{R}}^{a})_{\nu}\left(-\mathrm{ad}_{\xi}^{*}\nu,\, -\mathrm{ad}_{\zeta}^{*}\nu\right) &= (\omega_{\mathcal{R}}^{a})_{\mathcal{R}^{(\nu,a)}}\left(\xi \cdot \mathcal{R}^{(\nu,a)},\, \zeta \cdot \mathcal{R}^{(\nu,a)}\right) \\
&= (\mathrm{d}\, J_{\mathcal{R}}^{a}(\xi))_{\mathcal{R}^{(\nu,a)}}\left(\zeta \cdot \mathcal{R}^{(\nu,a)}\right) \\
&= \left\langle T_{\mathcal{R}^{(\nu,a)}}\, \mathrm{J}_{\mathcal{R}}^{a}(\zeta \cdot \mathcal{R}^{(\nu,a)}),\, \xi \right\rangle \\
&= \left\langle -\mathrm{ad}_{\zeta}^{*}(\mathrm{J}_{\mathcal{R}}^{a}(\mathcal{R}^{(\nu,a)})),\, \xi \right\rangle \\
&= \langle \nu,\, [\xi,\, \zeta] \rangle \\
&= (\omega_{\mathcal{O}})_{\nu}(-\mathrm{ad}_{\xi}^{*}\nu,\, -\mathrm{ad}_{\zeta}^{*}\nu),
\end{aligned}
$$

where $\omega_{\mathcal{O}}$ is the KKS form, introduced in Section 2.5. So $(\mathrm{J}_{\mathcal{R}}^{a})_{*}\omega_{\mathcal{R}}^{a} = \omega_{\mathcal{O}}$, independent of $a \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$.

(ii) Let $g \in G$ be such that $\nu = \mathrm{Ad}_{g^{-1}}^{*}\mu$. Then $g \cdot \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} = \frac{\mathrm{J}^{-1}(\nu)}{\mathcal{R}^{\nu}}$ and $g \cdot \omega_{\mathcal{R}}^{\mathcal{O}} = \omega_{\mathcal{R}}^{\mathcal{O}}$ together imply that $g \cdot \omega_{\mathcal{R}}^{\mu} = \omega_{\mathcal{R}}^{\nu}$. Applying $(\pi_{\mathcal{R}}^{\nu})_{*}$ to both sides, and using the fact that $\pi_{\mathcal{R}}^{\nu} \circ (\Phi_{\mathcal{R}}^{\mathcal{O}})_{g}\big|_{\mathrm{J}^{-1}(\mu)/\mathcal{R}^{\mu}} = \pi_{\mathcal{R}}^{\mu}$ yields the result.

□

We denote the common pushforward 2-form from 2.14.1 (i) as $\omega_{\mathrm{J}^{-1}(\mathcal{O})/G}$. We have that

- $\mathrm{J}_{\mathcal{R}}^{a}$ is a symplectomorphism from $\left(\frac{\pi^{-1}(a)}{\mathcal{R}^{a}},\, \omega_{\mathcal{R}}^{a}\right)$ to $(\mathcal{O},\, \omega_{\mathcal{O}})$, and
- $\pi_{\mathcal{R}}^{\mu}$ is a symplectomorphism from $\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}},\, \omega_{\mathcal{R}}^{\mu}\right)$ to $\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{G},\, \omega_{\mathrm{J}^{-1}(\mathcal{O})/G}\right)$.

For a Cartesian product $A \times B$, let $p_A : A \times B \to A$ denote projection onto the first factor, and similarly for p_B. Define

$$\omega^a_{\mathcal{R}} \oplus \omega^\mu_{\mathcal{R}} := \left(p_{\frac{\pi^{-1}(a)}{\mathcal{R}^a}}\right)^* \omega^a_{\mathcal{R}} + \left(p_{\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}}\right)^* \omega^\mu_{\mathcal{R}},$$

and similarly for $\omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G}$. Then the two statements above can be combined to say

Corollary 2.14.2. $\mathrm{J}^a_{\mathcal{R}} \times \pi^\mu_{\mathcal{R}}$ *is a symplectomorphism from* $\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu},\, \omega^a_{\mathcal{R}} \oplus \omega^\mu_{\mathcal{R}}\right)$ *to* $\left(\mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G},\, \omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G}\right)$.

We wish to demonstrate that the latter space is symplectomorphic to $\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}},\, \omega^{\mathcal{O}}_{\mathcal{R}}\right)$. To this end, consider the map $\phi^{\mathcal{O}}_{\mathcal{R}}$ defined by

$$\phi^{\mathcal{O}}_{\mathcal{R}} := \left(\mathrm{J}^{\mathcal{O}}_{\mathcal{R}} \times \pi^{\mathcal{O}}_{\mathcal{R}}\right) \circ \Delta^{\mathcal{O}}_{\mathcal{R}} : \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \longrightarrow \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G},$$

where $\Delta^{\mathcal{O}}_{\mathcal{R}} : \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ is the diagonal inclusion $\Delta^{\mathcal{O}}_{\mathcal{R}}(\mathcal{R}^{(\nu,b)}) = (\mathcal{R}^{(\nu,b)},\, \mathcal{R}^{(\nu,b)})$. Being the composition of smooth functions, $\Delta^{\mathcal{O}}_{\mathcal{R}}$ is itself smooth. We will demonstrate that $\phi^{\mathcal{O}}_{\mathcal{R}}$ is the required symplectomorphism.

We first prove the following lemma:

Lemma 2.14.3. *In terms of the decomposition* $T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}\right) = T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\pi^{-1}(b)}{\mathcal{R}^b}\right) \oplus T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\nu)}{\mathcal{R}^\nu}\right)$ *(Proposition 2.12.3 (ii)),*

$$T_{\mathcal{R}^{(\nu,b)}}\, \phi^{\mathcal{O}}_{\mathcal{R}} = \left(T_{\mathcal{R}^{(\nu,b)}} \mathrm{J}^b_{\mathcal{R}}\right) \oplus \left(T_{\mathcal{R}^{(\nu,b)}} \pi^\nu_{\mathcal{R}}\right).$$

Proof. For any curve γ in $\frac{\pi^{-1}(b)}{\mathcal{R}^b}$,

$$\phi^{\mathcal{O}}_{\mathcal{R}}(\gamma(t)) = (\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}(\gamma(t)),\, \pi^{\mathcal{O}}_{\mathcal{R}}(\gamma(t))) = (\mathrm{J}^b_{\mathcal{R}}(\gamma(t)),\, b),$$

which implies that

$$T_{\mathcal{R}^{(\nu,b)}} \phi^{\mathcal{O}}_{\mathcal{R}}\Big|_{T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\pi^{-1}(b)}{\mathcal{R}^b}\right) \oplus \{0_{\mathcal{R}^{(\nu,b)}}\}} = \left(T_{\mathcal{R}^{(\nu,b)}}\, \mathrm{J}^b_{\mathcal{R}}\right) \oplus 0.$$

Similarly

$$T_{\mathcal{R}^{(\nu,b)}} \phi^{\mathcal{O}}_{\mathcal{R}}\Big|_{\{0_{\mathcal{R}^{(\nu,b)}}\} \oplus T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\nu)}{\mathcal{R}^\nu}\right)} = 0 \oplus \left(T_{\mathcal{R}^{(\nu,b)}}\, \pi^\nu_{\mathcal{R}}\right).$$

Combining, we obtain

$$T_{\mathcal{R}^{(\nu,b)}} \phi^{\mathcal{O}}_{\mathcal{R}} = \left(T_{\mathcal{R}^{(\nu,b)}} \mathrm{J}^b_{\mathcal{R}}\right) \oplus \left(T_{\mathcal{R}^{(\nu,b)}} \pi^\nu_{\mathcal{R}}\right).$$

□

Proposition 2.14.4. *(i)* $\phi^{\mathcal{O}}_{\mathcal{R}}$ *is a bijection.*

(ii) $\phi^{\mathcal{O}}_{\mathcal{R}}$ *is a diffeomorphism.*

(iii) $\phi^{\mathcal{O}}_{\mathcal{R}}$ *is a symplectomorphism.*

Proof. (i) $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ consists of points $\mathcal{R}^{(\nu,b)}$, where $(\nu,\, b) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, and we have that

$$\phi^{\mathcal{O}}_{\mathcal{R}}(\mathcal{R}^{(\nu,b)}) = \left(\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}(\mathcal{R}^{(\nu,b)}),\, \pi^{\mathcal{O}}_{\mathcal{R}}(\mathcal{R}^{(\nu,b)})\right) = (\nu,\, b).$$

So $\phi^{\mathcal{O}}$ is a bijection.

(ii) Since $\pi^{\nu}_{\mathcal{R}}$ is a diffeomorphism (Proposition 2.13.3), $T\pi^{\nu}_{\mathcal{R}} : T\left(\frac{\mathrm{J}^{-1}(\nu)}{\mathcal{R}^{\nu}}\right) \to T\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}\right)$ is invertible at every point. Similarly, $\mathrm{J}^b_{\mathcal{R}}$ is a diffeomorphism (Proposition 2.13.5), so $T\mathrm{J}^b_{\mathcal{R}}$: $T\left(\frac{\pi^{-1}(b)}{\mathcal{R}^b}\right) \to T\mathcal{O}$ is invertible at every point. Lemma 2.14.3 says that $T_{\mathcal{R}^{(\nu,b)}}\,\phi^{\mathcal{O}}_{\mathcal{R}}$ can be expressed as

$$\left(T_{\mathcal{R}^{(\nu,b)}}\mathrm{J}^b_{\mathcal{R}}\right) \oplus \left(T_{\mathcal{R}^{(\nu,b)}}\pi^{\nu}_{\mathcal{R}}\right) : T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\pi^{-1}(b)}{\mathcal{R}^b}\right) \oplus T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\nu)}{\mathcal{R}^{\nu}}\right) \longrightarrow T_{\nu}\,\mathcal{O} \oplus T_b\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}\right).$$

It follows that $T\phi^{\mathcal{O}}_{\mathcal{R}}$ is invertible at every point. The Inverse Function Theorem implies that $\phi^{\mathcal{O}}_{\mathcal{R}}$ is a local diffeomorphism, and part (i) implies it is a global diffeomorphism.

(iii) We first note that since $T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}\right) = T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\pi^{-1}(b)}{\mathcal{R}^b}\right) \oplus T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\nu)}{\mathcal{R}^{\nu}}\right)$,

$$(\omega^{\mathcal{O}}_{\mathcal{R}})_{\mathcal{R}^{(\nu,b)}} = (\omega^b_{\mathcal{R}})_{\mathcal{R}^{(\nu,b)}} \oplus (\omega^{\nu}_{\mathcal{R}})_{\mathcal{R}^{(\nu,b)}}.$$

Note, the decomposition is pointwise: $\omega^{\mathcal{O}}_{\mathcal{R}}$ cannot be expressed globally as the direct sum of two symplectic forms. Meanwhile

$$\begin{aligned}\left((\phi^{\mathcal{O}}_{\mathcal{R}})^*(\omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G})\right)_{\mathcal{R}^{(\nu,b)}} &= (\omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G})_{(\nu,\,b)} \circ \left(T_{\mathcal{R}^{(\nu,b)}}(\mathrm{J}^b_{\mathcal{R}}) \oplus T_{\mathcal{R}^{(\nu,b)}}(\pi^{\nu}_{\mathcal{R}})\right)\\ &= \left((\omega_{\mathcal{O}})_{\nu} \circ T_{\mathcal{R}^{(\nu,b)}}(\mathrm{J}^b_{\mathcal{R}})\right) \oplus \left((\omega_{\mathrm{J}^{-1}(\mathcal{O})/G})_b \circ T_{\mathcal{R}^{(\nu,b)}}(\pi^{\nu}_{\mathcal{R}})\right)\\ &= \left((\mathrm{J}^b_{\mathcal{R}})^*\omega_{\mathcal{O}}\right)_{\mathcal{R}^{(\nu,b)}} \oplus \left((\pi^{\nu}_{\mathcal{R}})^*\omega_{\mathrm{J}^{-1}(\mathcal{O})/G}\right)_{\mathcal{R}^{(\nu,b)}}\\ &= (\omega^b_{\mathcal{R}})_{\mathcal{R}^{(\nu,b)}} \oplus (\omega^{\nu}_{\mathcal{R}})_{\mathcal{R}^{(\nu,b)}}\\ &= (\omega^{\mathcal{O}}_{\mathcal{R}})_{\mathcal{R}^{(\nu,b)}}.\end{aligned}$$

Hence

$$(\phi^{\mathcal{O}}_{\mathcal{R}})^*(\omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G}) = \omega^{\mathcal{O}}_{\mathcal{R}},$$

i.e., $\phi^{\mathcal{O}}_{\mathcal{R}} : \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \longrightarrow \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ is a symplectomorphism as claimed.

□

Composing the symplectomorphisms $\mathrm{J}^a_{\mathcal{R}} \times \pi^{\mu}_{\mathcal{R}}$ and ${\phi^{\mathcal{O}}}^{-1}$, we now obtain

Corollary 2.14.5. *For any* $(\mu, a) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, *the map*

$$\phi_{\mathcal{R}}^{\mathcal{O}\,-1} \circ (\mathrm{J}_{\mathcal{R}}^{a} \times \pi_{\mathcal{R}}^{\mu}) : \frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \longrightarrow \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$$

is a (canonical) symplectomorphism.

Proof.

$$\begin{aligned}
\left(\phi_{\mathcal{R}}^{\mathcal{O}\,-1} \circ (\mathrm{J}_{\mathcal{R}}^{a} \times \pi_{\mathcal{R}}^{\mu})\right)^{*} \omega_{\mathcal{R}}^{\mathcal{O}} &= (\mathrm{J}_{\mathcal{R}}^{a} \times \pi_{\mathcal{R}}^{\mu})^{*} \left(\left(\phi_{\mathcal{R}}^{\mathcal{O}\,-1}\right)^{*} \omega_{\mathcal{R}}^{\mathcal{O}}\right) \\
&= (\mathrm{J}_{\mathcal{R}}^{a} \times \pi_{\mathcal{R}}^{\mu})^{*} \left(\omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G}\right) && \text{(Proposition 2.14.4 (iii))} \\
&= ((\mathrm{J}_{\mathcal{R}}^{a})^{*}\omega_{\mathcal{O}}) \oplus \left((\pi_{\mathcal{R}}^{\mu})^{*}\omega_{\mathrm{J}^{-1}(\mathcal{O})/G}\right) \\
&= \omega_{\mathcal{R}}^{a} \oplus \omega_{\mathcal{R}}^{\mu} && \text{(Proposition 2.14.1).}
\end{aligned}$$

□

The correspondence between the three spaces

$$\frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \longleftrightarrow \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G} \longleftrightarrow \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$$

is given by

$$\left(\mathcal{R}^{(\nu,a)}, \mathcal{R}^{(\mu,b)}\right) \xrightarrow[\longleftarrow]{\mathrm{J}_{\mathcal{R}}^{a} \times \pi_{\mathcal{R}}^{\mu}} (\nu, b) \xleftarrow[\longrightarrow]{(\mathrm{J}_{\mathcal{R}}^{\mathcal{O}} \times \pi_{\mathcal{R}}^{\mathcal{O}}) \circ \Delta_{\mathcal{R}}^{\mathcal{O}}} \mathcal{R}^{(\nu,b)}.$$

2.15 Relationship to orbit reduction

In order to describe the relationship to the orbit reduction picture, consider the function

$$\phi^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$$

given by

$$\phi^{\mathcal{O}} = (\mathrm{J}^{\mathcal{O}} \times \pi^{\mathcal{O}}) \circ \Delta^{\mathcal{O}},$$

where $\Delta^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathrm{J}^{-1}(\mathcal{O}) \times \mathrm{J}^{-1}(\mathcal{O})$ is the diagonal map $\Delta^{\mathcal{O}}(x) = (x, x)$. Being the composition of smooth maps, $\phi^{\mathcal{O}}$ is itself smooth, and it is easily seen that $\phi^{\mathcal{O}}$ fits into the following commutative diagram,

$$
\begin{tikzcd}
& \mathrm{J}^{-1}(\mathcal{O}) \arrow[dl, "\sigma^{\mathcal{O}}"'] \arrow[dr, "\phi^{\mathcal{O}}"] & \\
\dfrac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \arrow[rr, shift left, "\phi^{\mathcal{O}}_{\mathcal{R}}"] & & \mathcal{O} \times \dfrac{\mathrm{J}^{-1}(\mathcal{O})}{G} \arrow[ll, shift left, "(\phi^{\mathcal{O}}_{\mathcal{R}})^{-1}"]
\end{tikzcd}
\tag{2.7}
$$

Now

$$
\begin{aligned}
(\phi^{\mathcal{O}})^*(\omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G}) &= (\phi^{\mathcal{O}})^* \left((p_{\mathcal{O}})^*\omega_{\mathcal{O}} + (p_{\mathrm{J}^{-1}(\mathcal{O})/G})^*\omega_{\mathrm{J}^{-1}(\mathcal{O})/G}\right) \\
&= (p_{\mathcal{O}} \circ \phi^{\mathcal{O}})^*\omega_{\mathcal{O}} + (p_{\mathrm{J}^{-1}(\mathcal{O})/G} \circ \phi^{\mathcal{O}})^*\omega_{\mathrm{J}^{-1}(\mathcal{O})/G} \\
&= (\mathrm{J}^{\mathcal{O}})^*\omega_{\mathcal{O}} + (\pi^{\mathcal{O}})^*\omega_{\mathrm{J}^{-1}(\mathcal{O})/G}.
\end{aligned}
$$

Alternatively, since $\phi^{\mathcal{O}} = \phi^{\mathcal{O}}_{\mathcal{R}} \circ \sigma^{\mathcal{O}}$,

$$
\begin{aligned}
(\phi^{\mathcal{O}})^*(\omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G}) &= (\sigma^{\mathcal{O}})^* \left((\phi^{\mathcal{O}}_{\mathcal{R}})^*(\omega_{\mathcal{O}} \oplus \omega_{\mathrm{J}^{-1}(\mathcal{O})/G})\right) \\
&= (\sigma^{\mathcal{O}})^*\omega^{\mathcal{O}}_{\mathcal{R}} \qquad \text{(Proposition 2.14.4 (iii))} \\
&= \omega^{\mathcal{O}} \qquad \text{(Proposition 2.12.1 (i))} \\
&= (i^{\mathcal{O}})^*\omega.
\end{aligned}
$$

Equating these, we obtain the familiar orbit reduction result ([MMO+07, Theorem 1.2.4 (ii)])

Proposition 2.15.1. *There exists a symplectic form $\omega_{\mathrm{J}^{-1}(\mathcal{O})/G}$ on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ such that*

$$
(i^{\mathcal{O}})^*\omega = (\mathrm{J}^{\mathcal{O}})^*\omega_{\mathcal{O}} + (\pi^{\mathcal{O}})^*\omega_{\mathrm{J}^{-1}(\mathcal{O})/G} \ ,
$$

where $\omega_{\mathcal{O}}$ is the positive KKS form on the coadjoint orbit $\mathcal{O}$.

Chapter 3

Prequantization

In this chapter we assume that the G-action on the symplectic manifold (M, ω) has a corresponding *equivariant* momentum map J. Most of the results in this chapter are standard, and derived from [Kos70]. The sole exception is Proposition 3.2.3, which generalizes [Kos70, Theorem 4.5.1] to the case when the group G is not necessarily simply connected.

3.1 Geometric structures on complex vector bundles

3.1.1 Notation for left modules and their quotients

Suppose G and H are Lie groups, S is a smooth left G-module, T is a smooth left H-module, and $\rho : G \to H$ is a amooth group homomorphism. Let $[s, t]_\rho$ denote the equivalence class of $(s, t) \in S \times T$ under the equivalence relation $(s, t) \sim (g \cdot s, \rho(g) \cdot t)$, and left the space of equivalence classes in $S \times T$ be denoted by $S \times_\rho T$,

$$S \times_\rho T = \{[s, t]_\rho \mid s \in S, t \in T\}.$$

Finally, let $C^\infty_\rho(S, T)$ denote the set of smooth G-equivariant maps from S to T,

$$C^\infty_\rho(S, T) = \{f : S \to T \mid f(g \cdot s) = \rho(g) \cdot f(s)\}.$$

3.1.2 Connections and curvature

Let $(\dot{L}, \dot{\tau}, M)$ be a (right) principal U(1)-bundle over M, and let (L, τ, M) be the corresponding associated line bundle. Thinking of $\dot{L}$ as a left U(1)-module under the action $p \mapsto p \cdot w^{-1}$ for $w \in \mathrm{U}(1)$, the associated line bundle is

$$L = \dot{L} \times_{\mathrm{id}_{\mathrm{U}(1)}} \mathbb{C} = \{[p, z]_{\mathrm{id}_{\mathrm{U}(1)}} \mid p \in P, z \in \mathbb{C}\},$$

where $[p,\, z]_{\mathrm{id}_{\mathrm{U}(1)}}$ is the equivalence class of $(p,\, z)$ under the equivalence relation $(p,\, z) \sim (p{\cdot}w^{-1},\, wz)$, and the induced projection $\tau : L \to M$ is simply

$$\tau([p,\, z]_{\mathrm{id}_{\mathrm{U}(1)}}) = \dot{\tau}(p).$$

The fibers $\dot{L}_x$ and L_x above $x \in M$ are defined respectively as $(\dot{\tau})^{-1}(x)$ and $\tau^{-1}(x)$. By definition of a principal U(1)-bundle, $\dot{L}_x$ is homeomorphic to U(1), while L_x has a natural vector structure

$$\lambda_1\, [p,\, z_1]_{\mathrm{id}_{\mathrm{U}(1)}} + \lambda_2\, [p,\, z_2]_{\mathrm{id}_{\mathrm{U}(1)}} = [p,\, \lambda_1 z_1 + \lambda_2 z_2]_{\mathrm{id}_{\mathrm{U}(1)}}\,,$$

and is (non-canonically) isomorphic to $\mathbb{C}$.

A ***connection*** on $\dot{L}$ is a $\mathfrak{u}(1)$-valued 1-form α with the properties that

- $\Psi_w^* \alpha = \alpha$ for all $w \in \mathrm{U}(1)$, and
- $\alpha_p(\varepsilon_{\dot{L}}(p)) = \alpha_p(p \cdot \varepsilon) = \varepsilon$ for all $p \in \dot{L}$, $\varepsilon \in \mathfrak{u}(1)$,

where Ψ is the right U(1)-action on $\dot{L}$, and $\varepsilon_{\dot{L}}$ is the infinitesimal generator of Ψ corresponding to $\varepsilon \in \mathfrak{u}(1)$. Vectors in $\dot{L}$ of the form $p \cdot \varepsilon$, $\varepsilon \in \mathfrak{u}(1)$, are called ***vertical*** vectors, while vectors in the kernel of α are called ***horizontal*** vectors. Any vector in $\dot{L}$ can be uniquely decomposed into a horizontal and vertical part.

Proposition 3.1.1. *There exists a closed, $\mathfrak{u}(1)$-valued 2-form Ω^α on M such that*

$$\mathrm{d}\alpha = (\dot{\tau})^* \Omega^\alpha.$$

Proof. The result is a consequence of two properties.

(i) $\mathbf{i}_{u_p}(\mathbf{d}_p\boldsymbol{\alpha}) = \mathbf{0}$ **for any vertical vector** $\boldsymbol{A_p \in T_p\dot{L}}$**.**
Since A_p is vertical, $A_p = \varepsilon_{\dot{L}}(p)$ for some $\varepsilon \in \mathfrak{u}(1)$. But by Cartan's Magic Formula

$$\mathrm{i}_{\varepsilon_{\dot{L}}} \mathrm{d}\alpha = \mathcal{L}_{\varepsilon_{\dot{L}}} \alpha - \mathrm{d}(\mathrm{i}_{\varepsilon_{\dot{L}}} \alpha) = 0,$$

the first term vanishing since α is U(1)-invariant, and the second term vanishing since $\mathrm{i}_{\varepsilon_{\dot{L}}} \alpha = \alpha(\varepsilon_{\dot{L}}) = \varepsilon = \mathrm{const.}$

(ii) $\mathbf{d}\boldsymbol{\alpha}$ **is U(1)-invariant.**
This follows from the fact that for $w \in \mathrm{U}(1)$

$$\Psi_w^*(\mathrm{d}\alpha) = \mathrm{d}(\Psi_w^* \alpha),$$

and the U(1)-invariance of α.

Then $\Omega^\alpha_x(X_x, Y_x)$ is defined by lifting X_x, Y_x to arbitrary vectors A_p, B_p for $p \in (\dot{\tau})^{-1}(x)$, and calculating $\mathrm{d}_p\alpha\,(A_p, B_p)$. Properties (i) and (ii) ensure that the result is independent of the lift or the point p. Since now we have that $\mathrm{d}\alpha = (\dot{\tau})^*\Omega^\alpha$, apply d to both sides to get $0 = (\dot{\tau})^*\mathrm{d}\Omega^\alpha$. Since $\dot{\tau}$ is a surjective submersion, this implies that $\mathrm{d}\Omega^\alpha = 0$.

□

Ω^α is called the ***curvature*** of the connection α.

The space of ***smooth sections*** of L,

$$\Gamma(L) = \{\text{smooth } s : M \to L \,|\, \tau \circ s = \mathrm{id}_M\},$$

is in one-to-one correspondence with the space

$$C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}, \mathbb{C}) = \{\text{smooth } \dot{s} : \dot{L} \to \mathbb{C} \,|\, \dot{s}(p \cdot w) = w^{-1}\dot{s}(p) \text{ for all } w \in \mathrm{U}(1),\, p \in \dot{L}\}$$

of smooth U(1)-equivariant functions from $\dot{L}$ to $\mathbb{C}$. Explicitly this correspondence is

$$\begin{aligned} &s \in \Gamma(L) \longleftrightarrow \dot{s} \in C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}, \mathbb{C}), \\ &s(x) = [p,\, \dot{s}(p)]_{\mathrm{id}_{\mathrm{U}(1)}} \qquad \text{for any } p \in \dot{L}_x. \end{aligned}$$

Given $X \in \Gamma(TM)$, let X^{h} denote the ***horizontal lift*** of X, i.e., the vector field on $\dot{L}$ satisfying

$$T_p\dot{\tau}(X^{\mathrm{h}}_p) = X_{\dot{\tau}(p)} \qquad \text{and} \qquad \alpha_p(X^{\mathrm{h}}_p) = 0.$$

for all $p \in \dot{L}$. The equivariance of the connection under the right U(1)-action ensures that that X^{h} is invariant under this action. If $s \in \Gamma(L)$ and $\dot{s} \in C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}, \mathbb{C})$ is the corresponding U(1)-equivariant function, it is easily checked that the mapping

$$p \longmapsto X^{\mathrm{h}}_p \dot{s}$$

is also in $C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}, \mathbb{C})$ (under the canonical identification of $T_{\dot{s}(p)}\mathbb{C}$ with $\mathbb{C}$), and so defines a section $\nabla_X s$ of L. In other words

$$(\nabla_X s)(x) = \left[p,\, X^{\mathrm{h}}_p \dot{s}\right]_{\mathrm{id}_{\mathrm{U}(1)}} \qquad \text{for any } p \in \dot{L}_x,$$

or equivalently

$$(\dot{\nabla_X s})(p) = X^{\mathrm{h}}_p \dot{s}.$$

$\nabla_X s$ is called the ***covariant derivative*** of s with respect to X. It is clear from the definition that

$(\nabla_X s)(x)$ depends only on the value of X at x, and the germ of s at x. The following properties of ∇ are easily checked. For $f, g \in C^\infty(M, \mathbb{C})$, $X, Y \in \Gamma(TM)$, $s, t \in \Gamma(L)$,

(i) $\nabla_{fX+gY} s = f\nabla_X s + g\nabla_Y s$;

(ii) $\nabla_X(s+t) = \nabla_X s + \nabla_X t$;

(iii) $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

For $X \in \Gamma(TM)$, we say that $A \in \Gamma(T\dot{L})$ is ***$\dot{\tau}$-related*** to X if

$$T_p\dot{\tau}(A_p) = X_{\dot{\tau}(p)}$$

for all $p \in \dot{L}$, and denote this property by $A \sim_{\dot{\tau}} X$. In particular, $X^{\mathrm{h}} \sim_{\dot{\tau}} X$.

For $\varepsilon \in \mathfrak{u}(1)$, the induced infinitesimal action of $\varepsilon \cdot z = \frac{d}{dt}\exp(t\varepsilon)z\big|_{t=0}$ can be viewed as an element of $\mathbb{C}$ via the canonical isomorphism $T_z\mathbb{C} \simeq \mathbb{C}$. Given this, $\mathfrak{u}(1)$ also has a natural action on L, given by

$$\varepsilon \cdot [p,\, v]_{\mathrm{id}_{\mathrm{U}(1)}} = [p,\, \varepsilon \cdot v]_{\mathrm{id}_{\mathrm{U}(1)}}\,.$$

We are now in a position to prove the following result and its corollary.

Proposition 3.1.2. *For vector fields X, $Y \in \Gamma(TM)$, the following identity holds*[1]*:*

$$[X^{\mathrm{h}},\, Y^{\mathrm{h}}] - [X,\, Y]^{\mathrm{h}} = -\left(\Omega^\alpha(X,\, Y) \circ \dot{\tau}\right)_{\dot{L}}\,.$$

Proof. We have that $X^{\mathrm{h}} \sim_{\dot{\tau}} X$, $Y^{\mathrm{h}} \sim_{\dot{\tau}} Y$, and $[X,\, Y]^{\mathrm{h}} \sim_{\dot{\tau}} [X,\, Y]$, and a standard result (see e.g. [AMR88] Proposition 4.2.25) implies that $[X^{\mathrm{h}},\, Y^{\mathrm{h}}] \sim_{\dot{\tau}} [X,\, Y]$. So

$$T\dot{\tau} \circ \left([X^{\mathrm{h}},\, Y^{\mathrm{h}}] - [X,\, Y]^{\mathrm{h}}\right) = [X,\, Y] \circ \dot{\tau} - [X,\, Y] \circ \dot{\tau} = 0.$$

Hence $[X^{\mathrm{h}},\, Y^{\mathrm{h}}] - [X,\, Y]^{\mathrm{h}}$ is a vertical vector field, and so is given by

$$\begin{aligned}[X^{\mathrm{h}},\, Y^{\mathrm{h}}] - [X,\, Y]^{\mathrm{h}} &= \left(\alpha([X^{\mathrm{h}},\, Y^{\mathrm{h}}] - [X,\, Y]^{\mathrm{h}})\right)_{\dot{L}}\\ &= \left(\alpha([X^{\mathrm{h}},\, Y^{\mathrm{h}}])\right)_{\dot{L}}\,.\end{aligned}$$

[1] Here given $f : \dot{L} \to \mathfrak{u}(1)$, $(f)_{\dot{L}}$ denotes the vector field whose value at $p \in \dot{L}$ is $(f(p))_{\dot{L}}(p)$.

But

$$\begin{aligned}\alpha([X^{\mathrm{h}},\, Y^{\mathrm{h}}]) &= -X^{\mathrm{h}}(\alpha(Y^{\mathrm{h}})) + Y^{\mathrm{h}}(\alpha(X^{\mathrm{h}})) + \alpha([X^{\mathrm{h}},\, Y^{\mathrm{h}}]) \\ &= -d\alpha(X^{\mathrm{h}},\, Y^{\mathrm{h}}) \\ &= -((\dot{\tau})^{*}\Omega^{\alpha})(X^{\mathrm{h}},\, Y^{\mathrm{h}}) \\ &= -\Omega^{\alpha}(X,\, Y) \circ \dot{\tau},\end{aligned}$$

using the fact that X^{h}, Y^{h} are horizontal vector fields. The result follows. □

Corollary 3.1.3. *Given a section $s \in \Gamma(L)$ and vector fields $X, Y \in \Gamma(TM)$,*

$$\left([\nabla_X,\, \nabla_Y] - \nabla_{[X,\, Y]}\right) s = \Omega^{\alpha}(X,\, Y) \cdot s.$$

Proof. The result to be proved is equivalent to the statement that

$$([X^{\mathrm{h}},\, Y^{\mathrm{h}}] - [X,\, Y]^{\mathrm{h}})\, \dot{s} = (\Omega^{\alpha}(X,\, Y) \circ \dot{\tau}) \cdot \dot{s}.$$

By Proposition 3.1.2

$$\begin{aligned}([X^{\mathrm{h}},\, Y^{\mathrm{h}}] - [X,\, Y]^{\mathrm{h}})\, \dot{s} &= -\left(\Omega^{\alpha}(X,\, Y) \circ \dot{\tau}\right)_{\dot{L}}\ \dot{s} \\ &= (\Omega^{\alpha}(X,\, Y) \circ \dot{\tau}) \cdot \dot{s}.\end{aligned}$$

the last equality following from the U(1)-equivariance of $\dot{s}$. □

3.1.3 Connection-invariant Hermitian forms

Let $L \times_M L$ denote the set

$$L \times_M L = \{(r,\, s) \in L \times L \mid \tau(r) = \tau(s)\} = \bigcup_{x \in M} L_x \times L_x.$$

The standard inner product on $\langle z_1,\, z_2\rangle = \overline{z}_1 z_2$ on $\mathbb{C}$ induces a corresponding ***Hermitian structure***

$$H : L \times_M L \longrightarrow \mathbb{C}$$

defined by

$$H([p,\, z_1]_{\mathrm{id}_{\mathrm{U}(1)}},\, [p,\, z_2]_{\mathrm{id}_{\mathrm{U}(1)}}) = \langle z_1,\, z_2\rangle.$$

The U(1)-invariance of $\langle\cdot,\cdot\rangle$ ensures that H is well-defined.

Proposition 3.1.4. *The Hermitian structure H is α-invariant, i.e.,*

$$X\left(H(s,\,t)\right) = H(\nabla_X s,\,t) + H(s,\,\nabla_X t)$$

for all $X \in \Gamma(TM)$, $s, t \in \Gamma(L)$.

Proof. For $x \in M$, let $\sigma : (-\epsilon,\,\epsilon) \to M$ be a curve through x with $\sigma'(0) = X_x$. Then

$$X_x\left(H(s,\,t)\right) = \left.\frac{d}{du}\right|_{u=0} H\left(s(\sigma(u)),\,t(\sigma(u))\right).$$

Then for any path $\gamma : (-\epsilon,\,\epsilon) \to \dot{L}$ through $p \in \dot{L}$ covering σ, $s(\sigma(u)) = [\gamma(u),\,\dot{s}(\gamma(u))]_{\mathrm{id}_{\mathrm{U}(1)}}$ (and similarly for t) and so

$$\begin{aligned}
X_x\left[H(s,\,t)\right] &= \frac{d}{du}\langle \dot{s}(\gamma(u),\,\dot{t}(\gamma(u))\rangle_{u=0} \\
&= \langle U_p\,\dot{s},\,\dot{t}(p)\rangle + \langle \dot{s}(p),\,U_p\,\dot{t}\,\rangle \\
&= H\left([p,\,U_p\,\dot{s}\,]_{\mathrm{id}_{\mathrm{U}(1)}},\,[p,\,\dot{t}(p)]_{\mathrm{id}_{\mathrm{U}(1)}}\right) + H\left([p,\,\dot{s}(p)]_{\mathrm{id}_{\mathrm{U}(1)}},\,[p,\,U_p\,\dot{t}\,]_{\mathrm{id}_{\mathrm{U}(1)}}\right),
\end{aligned}$$

where $U_p = \gamma'(0)$. In particular, by choosing γ to be the horizontal lift of σ through $p \in \dot{L}$, we have that $U_p = X_p^{\mathrm{h}}$, $U_p\,\dot{s} = X_p^{\mathrm{h}}\dot{s} = (\dot{\nabla_X s})(p)$, and so

$$\begin{aligned}
X_x\left(H(s,\,t)\right) &= H\left(\left[p,\,(\dot{\nabla_X s})(p)\right]_{\mathrm{id}_{\mathrm{U}(1)}},\,[p,\,\dot{t}(p)]_{\mathrm{id}_{\mathrm{U}(1)}}\right) + H\left([p,\,\dot{s}(p)]_{\mathrm{id}_{\mathrm{U}(1)}},\,\left[p,\,(\dot{\nabla_X t})(p)\right]_{\mathrm{id}_{\mathrm{U}(1)}}\right) \\
&= H(\nabla_X s(x),\,t(x)) + H(s(x),\,\nabla_X t(x)),
\end{aligned}$$

as claimed. □

3.1.4 Equivalence classes of bundle-connection pairs with given curvature

Given two bundle-connection pairs $(L_1,\,\alpha_1)$ and $(L_2,\,\alpha_2)$ over the same base manifold M, we say they are ***equivalent*** [2] if there exists a U(1)-equivariant diffeomorphism $\dot{F} : L_1 \to L_2$ which covers the identity, i.e., such that

$$\dot{\tau}_2 \circ \dot{F} = \dot{\tau}_1,$$

and for which

$$(\dot{F})^*\alpha_2 = \alpha_1.$$

A criterion for the existence of a bundle-connection pair with specified curvature goes back to Weil [Wei58], and a full characterization of such bundles was given by Kostant [Kos70].

[2] See Appendix A for further discussion of equivalence of bundle-connection pairs.

Proposition 3.1.5. *Let $(\dot{L}, \dot{\tau}, M)$ be a principal* $\mathrm{U}(1)$*-bundle. Let ε_0 denote the positive generator of the kernel of the exponential map* $\exp : \mathfrak{u}(1) \to \mathrm{U}(1)$ *(so $\varepsilon_0 = 2\pi\frac{\partial}{\partial\theta}$, where θ is the usual angular coordinate on* $\mathrm{U}(1)$*). Then a connection on $\dot{L}$ of curvature Ω exists if and only if $[\frac{\Omega}{\varepsilon_0}] \in H^2_{\mathrm{deRham}}(M, \mathbb{R})$ is integral, i.e., lies in the image of the homomorphism $i_\sharp : H^2_{\check{\mathrm{C}}\mathrm{ech}}(M, \mathbb{Z}) \to H^2_{\check{\mathrm{C}}\mathrm{ech}}(M, \mathbb{R}) \simeq H^2_{\mathrm{deRham}}(M, \mathbb{R})$ induced by the natural injection $i : \mathbb{Z} \to \mathbb{R}$. Moreover for M connected, inequivalent bundle-connection pairs $(\dot{L}, \alpha)$ with the same curvature $\Omega^\alpha = \Omega$ are characterized by elements of the character group $\pi_1(M)^* = \mathrm{Hom}(\pi_1(M), \mathrm{U}(1))$ of the fundamental group of M.*

Essentially, every bundle-connection pair can be obtained from a specific one by tensoring with a flat $\mathrm{U}(1)$-bundle, and such bundles are in one-to-one correspondence with the character group of $\pi_1(M)$.

3.2 The prequantization procedure

The ultimate goal of the quantization procedure is to associate with a symplectic manifold (M, ω) a Hilbert space $\mathcal{H}$, and to associate with some subset of the classical observables $C^\infty(M, \mathbb{R})$ a corresponding subset of the quantum observables $\mathrm{Ob}(\mathcal{H})$ (the set of self-adjoint operators on $\mathcal{H}$). The first step in this procedure is ***prequantization***.

3.2.1 The prequantization of classical observables

The setup for geometric prequantization is as follows: starting with a symplectic manifold (M, ω), we take a principal $\mathrm{U}(1)$-bundle $(\dot{\tau}, \dot{L}, M)$ with connection α of curvature $\frac{\varepsilon_0}{h}\omega$ (h being Planck's constant), and corresponding associated line bundle (L, τ, M) and α-invariant Hermitian form H. From Proposition 3.1.5, these structures exist if and only if $[\frac{\omega}{h}]$ is integral. Geometrically this means that ω integrated over any closed 2-surface in M gives an integral multiple of h [Woo92]. The choice of a bundle-connection pair which satisfy these conditions is called the ***prequantum data*** or ***prequantum geometric structures*** over (M, ω).

Consider the space $\Gamma(L)$ of C^∞ sections of L. For any classical observable $f \in C^\infty(M)$, let $Q_f : \Gamma(L) \to \Gamma(L)$ be the operator

$$Q_f = -i\hbar\nabla_{X_f} + f.$$

The motivation for picking a connection on $\dot{L}$ with curvature $\frac{\varepsilon_0}{h}\omega$ is the following theorem:

Proposition 3.2.1. *For any $f, g \in C^\infty(M)$,*

$$[Q_f, Q_g] = i\hbar Q_{\{f, g\}},$$

where $[Q_f,\, Q_g] = Q_f \circ Q_g - Q_g \circ Q_f$.

Proof.

$$\begin{aligned}
[Q_f,\, Q_g] =&(-i\hbar\nabla_{X_f} + f)(-i\hbar\nabla_{X_g} + g) - (-i\hbar\nabla_{X_g} + g)(-i\hbar\nabla_{X_f} + f)\\
=&\left\{(-i\hbar)^2\nabla_{X_f}\nabla_{X_g} - i\hbar(X_f(g) + g\nabla_{X_f}) - i\hbar f\nabla_{X_g} + fg\right\}\\
&-\left\{(-i\hbar)^2\nabla_{X_g}\nabla_{X_f} - i\hbar(X_g(f) + f\nabla_{X_g}) - i\hbar g\nabla_{X_f} + gf\right\}\\
=&(-i\hbar)^2[\nabla_{X_f}, \nabla_{X_g}] - i\hbar X_f(g) + i\hbar X_g(f)\\
=&(-i\hbar)^2\left(\nabla_{[X_f,X_g]} + \omega(X_f, X_g)\,\frac{\varepsilon_0}{h}\,\cdot\right) + 2i\hbar\,\omega(X_f, X_g)\\
=&(-i\hbar)^2\left(-\nabla_{X_{\{f,g\}}} + \omega(X_f, X_g)\frac{i}{\hbar}\right) + 2i\hbar\,\omega(X_f, X_g)\\
=&i\hbar\left(-i\hbar\nabla_{X_{\{f,g\}}} + \{f, g\}\right)\\
=&i\hbar\, Q_{\{f,g\}}.
\end{aligned}$$

□

3.2.2 Geometric interpretation of the prequantized observables

The real function f generates a Hamiltonian flow $\phi^t_f : M \to M$, with generator X_f (unless otherwise stated, we assume that the flow is complete). Let $A_f \in \Gamma(T\dot{L})$ be the vector field

$$A_f = X^{\mathrm{h}}_f - (f \circ \dot{\tau})\left(\frac{\varepsilon_0}{h}\right)_{\dot{L}},$$

and let $\dot{\psi}^t_f : \dot{L} \to \dot{L}$ be the flow[3] generated by A_f. Since A_f is U(1)-invariant, so is $\dot{\psi}^t_f$,

$$\dot{\psi}^t_f(p \cdot w) = \dot{\psi}^t_f(p) \cdot w \qquad \text{for all } w \in \mathrm{U}(1),$$

and clearly $\dot{\psi}^t_f$ covers ϕ^t_f,

$$\dot{\tau} \circ \dot{\psi}^t_f = \phi^t_f.$$

For any section $s \in \Gamma(L)$, let $\dot{s} \in C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}, \mathbb{C})$ be the corresponding equivariant function. Define the operator $U^t_f : C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}, \mathbb{C}) \to C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}, \mathbb{C})$ as

$$\left(U^t_f\, \dot{s}\right)(p) = \dot{s}(\dot{\psi}^{-t}_f(p)).$$

[3] $\dot{\psi}^t_f$ exists for all t if and only if ϕ^t_f exists for all t, and the two flows are related by

$$\dot{\psi}^t_f(p) = (\phi^t_f)^{\mathrm{h}}(p) \cdot \exp\left(-\frac{i}{\hbar}\int_0^t f(\phi^s_f(\dot{\tau}(p)))\, ds\right) = (\phi^t_f)^{\mathrm{h}}(p) \cdot \exp\left(-\frac{i}{\hbar} f(\dot{\tau}(p))t\right) \qquad (\text{using } f \circ \phi^t_f = f),$$

where $(\phi^t_f)^{\mathrm{h}}$ is the horizontal lift of ϕ^t_f to $\dot{L}$.

Then

$$
\begin{aligned}
i\hbar \frac{d}{dt}\left(U_f^t\,\dot{s}\right)\bigg|_{t=0} &= -i\hbar(A_f\dot{s})(p)\\
&= -i\hbar\left(X_f^{\mathrm{h}}\dot{s} - (f\circ\dot{\tau})\left(\frac{\varepsilon_0}{\hbar}\right)_{\dot{L}}\,\dot{s}\right)(p)\\
&= -i\hbar\left(X_f^{\mathrm{h}}\dot{s} + (f\circ\dot{\tau})\,\frac{i}{\hbar}\dot{s}\right)(p) \qquad \text{using the U(1)-equivariance of } \dot{s}\\
&= \left(-i\hbar X_f^{\mathrm{h}}\dot{s} + (f\circ\dot{\tau})\dot{s}\right)(p).
\end{aligned}
$$

In terms of the original section $s \in \Gamma(L)$, the right hand side is just

$$Q_f\,s = -i\hbar\nabla_{X_f}s + fs,$$

and U_f^t corresponds to the exponentiated operator

$$U_f^t\,\dot{s} \longleftrightarrow \exp\left(-\frac{i}{\hbar}Q_f\,t\right)s.$$

The U(1)-equivariance of $\dot{\psi}_f^t$ allows us to define a flow ψ_f^t in L via

$$\psi_f^t([p,\,z]_{\mathrm{id}_{\mathrm{U}(1)}}) = \left[\dot{\psi}_f^t(p),\,z\right]_{\mathrm{id}_{\mathrm{U}(1)}}.$$

In terms of this flow

$$\left[\exp\left(-\frac{i}{\hbar}Q_f\,t\right)s\right](x) = \psi_f^t(s(\phi_f^{-t}(x))).$$

3.2.3 Closedness of the lifted vector fields under the Lie bracket

The family of vector fields $\{A_f \mid f \in C^\infty(M)\}$ have the following nice property.

Proposition 3.2.2. *For $f, g \in C^\infty(M)$*

$$[A_f,\,A_g] = A_{-\{f,\,g\}}.$$

Proof. Since $A_f = X_f^{\mathrm{h}} - (f\circ\dot{\tau})\left(\frac{\varepsilon_0}{\hbar}\right)_{\dot{L}}$,

$$T_p\dot{\tau}\left(A_f(p)\right) = X_f(\dot{\tau}(p)),$$

and so $A_f \sim_{\dot{\tau}} X_f$. We have that ([AMR88] Proposition 4.2.25)

$$A_f \sim_{\dot{\tau}} X_f,\; A_g \sim_{\dot{\tau}} X_g \Longrightarrow [A_f,\,A_g] \sim_{\dot{\tau}} [X_f, X_g] = -X_{\{f,\,g\}},$$

i.e.

$$T_p\dot{\tau}\left([A_f,\, A_g](p)\right) = -X_{\{f,\,g\}}(\dot{\tau}(p)).$$

So the horizontal part of $[A_f,\, A_g]$ is $-X^{\mathrm{h}}_{\{f,\,g\}}$. The vertical part is calculated from

$$\begin{aligned}
\alpha([A_f,\, A_g]) &= -\mathrm{d}\alpha(A_f,\, A_g) + A_f(\alpha(A_g)) - A_g(\alpha(A_f)) \\
&= -(\dot{\tau})^*\left(\frac{\varepsilon_0}{h}\omega\right)(A_f,\, A_g) + A_f\left(-(g\circ\dot{\tau})\frac{\varepsilon_0}{h}\right) - A_g\left(-(f\circ\dot{\tau})\frac{\varepsilon_0}{h}\right) \\
&= -\frac{\varepsilon_0}{h}(\omega\circ\dot{\tau})(T\dot{\tau}(A_f),\, T\dot{\tau}(A_g)) + (T\dot{\tau}(A_f))\left(-\frac{\varepsilon_0}{h}g\right) - (T\dot{\tau}(A_g))\left(-\frac{\varepsilon_0}{h}f\right) \\
&= -\frac{\varepsilon_0}{h}(\omega\circ\dot{\tau})(X_f\circ\dot{\tau},\, X_g\circ\dot{\tau}) - \frac{\varepsilon_0}{h}(X_f\circ\dot{\tau})g + \frac{\varepsilon_0}{h}(X_g\circ\dot{\tau})f \\
&= -\frac{\varepsilon_0}{h}\omega(X_f,\, X_g)\circ\dot{\tau} - \frac{\varepsilon_0}{h}(X_f g)\circ\dot{\tau} + \frac{\varepsilon_0}{h}(X_g f)\circ\dot{\tau} \\
&= -\frac{\varepsilon_0}{h}\{f,\, g\}\circ\dot{\tau} + \frac{\varepsilon_0}{h}\{f,\, g\}\circ\dot{\tau} + \frac{\varepsilon_0}{h}\{f,\, g\}\circ\dot{\tau} \\
&= \frac{\varepsilon_0}{h}\{f,\, g\}\circ\dot{\tau}.
\end{aligned}$$

So the vertical part of $[A_f,\, A_g]$ is $(\{f,\, g\}\circ\dot{\tau})\left(\frac{\varepsilon_0}{h}\right)_{\dot{L}}$. Altogether,

$$[A_f,\, A_g] = -X^{\mathrm{h}}_{\{f,\,g\}} + (\{f,\, g\}\circ\dot{\tau})\left(\frac{\varepsilon_0}{h}\right)_{\dot{L}} = -A_{\{f,\,g\}}.$$

□

The Lie algebra of α-preserving vector fields on $\dot{L}$ can be considered as a $\mathfrak{u}(1)$-central extension of the Lie algebra of Hamiltonian vector fields on M (see Appendix A for notation and details):

$$0 \longrightarrow \mathfrak{u}(1) \xrightarrow{T\Psi_\cdot = (\cdot)_{\dot{L}}} \mathrm{Ham}(\dot{L},\, \alpha) \xrightarrow{\check{}} \mathrm{Ham}(M, \omega) \longrightarrow 0.$$

The $\mathfrak{u}(1)$ freedom in choosing a lift of a Hamiltonian vector field is manifest in the fact that for $b \in \mathbb{R}$, $X_{f+b} = X_f$ but $A_{f+b} = A_f - b\left(\frac{\varepsilon_0}{h}\right)_{\dot{L}}$.

3.2.4 The lifted Lie group and Lie algebra actions

Now consider the case where we have a G-action on M, and a corresponding infinitesimally equivariant momentum map J. Then we have that

$$[A_{J(\xi)},\, A_{J(\zeta)}] = -A_{\{J(\xi),\, J(\zeta)\}} = -A_{J([\xi,\,\zeta])},$$

and so the family of vector fields $\{A_{J(\xi)} \,|\, \xi \in \mathfrak{g}\}$ form a finite-dimensional Lie algebra of vector fields under the Jacobi-Lie bracket. In other words, the left $\mathfrak{g}$-action on M lifts to a left $\mathfrak{g}$-action on $\dot{L}$.

In general, the infinitesimal $\mathfrak{g}$-action on $\dot{L}$ cannot be integrated to a G-action on $\dot{L}$, and we must pass instead to the universal cover $\widetilde{G}$ of G. We will call this the ***lifted $\widetilde{G}$-action***, and denote the action of $\widetilde{g} \in \widetilde{G}$ on $p \in \dot{L}$ by $\widetilde{g} \cdot p$. The infinitesimal generator of the action on $\dot{L}$ is

$$\xi_{\dot{L}} = A_{J(\xi)} = \xi_M^{\mathrm{h}} - (J(\xi) \circ \dot{\tau}) \left(\frac{\varepsilon_0}{h}\right)_{\dot{L}} .$$

U(1)-invariance of the fields A_f ensures that the lifted $\widetilde{G}$-action is compatible with the right U(1)-action on $\dot{L}$, in the sense that

$$\widetilde{g} \cdot (p \cdot w) = (\widetilde{g} \cdot p) \cdot w \qquad \text{for } p \in \dot{L},\, \widetilde{g} \in \tilde{G},\, w \in \mathrm{U}(1),$$

and we can just write $\widetilde{g} \cdot p \cdot w$ without confusion. The corresponding lifted $\widetilde{G}$-action in L is

$$\widetilde{g} \cdot [p,\, z]_{\mathrm{id}_{\mathrm{U}(1)}} = [\widetilde{g} \cdot p,\, z]_{\mathrm{id}_{\mathrm{U}(1)}} .$$

The above mentioned U(1)-equivariance of the $\widetilde{G}$-action on $\dot{L}$ ensures that this is well-defined.

Let $\pi_{\widetilde{G} \to G}$ denote the standard projection of $\widetilde{G}$ onto G. Since the $\widetilde{G}$-action on $\dot{L}$ covers the G-action on M, an element of $\ker(\pi_{\widetilde{G} \to G}) \simeq \pi_1(G)$ maps each fiber $\dot{L}_x$ to itself.

Proposition 3.2.3. *For any $\widetilde{k} \in \ker(\pi_{\widetilde{G} \to G})$ and $p \in \dot{L}$*

$$\widetilde{k} \cdot p = p \cdot \chi(\widetilde{k}),$$

where $\chi : \ker\left(\pi_{\widetilde{G} \to G}\right) \to \mathrm{U}(1)$ is a representation of $\ker\left(\pi_{\widetilde{G} \to G}\right) \simeq \pi_1(G)$.

Proof. In Appendix A it is demonstrated that $\mathcal{L}_{A_f} \alpha = 0$ for any $f \in C^\infty(M)$. In particular, the vector fields $\xi_{\dot{L}} = A_{J(\xi)}$, $\xi \in \mathfrak{g}$, preserve the connection α, and so $\widetilde{g} \cdot \alpha = \alpha$ for any $\widetilde{g} \in \widetilde{G}$. Choosing $\widetilde{k} \in \ker\left(\pi_{\widetilde{G} \to G}\right)$, the map

$$p \longmapsto \widetilde{k} \cdot p$$

is a U(1)-equivariant connection-preserving diffeomorphism which covers the identity map on M. Since M is connected, Proposition A.2.2 implies that this map must be a global right multiplication by an element of U(1), i.e.,

$$\widetilde{k} \cdot p = p \cdot \chi(\widetilde{k})$$

for some $\chi(\widetilde{k}) \in \mathrm{U}(1)$.

If $\widetilde{k}_1, \widetilde{k}_2 \in \ker\left(\pi_{\widetilde{G} \to G}\right)$, then

$$\begin{aligned} \widetilde{k}_1 \cdot (\widetilde{k}_2 \cdot p) &= \widetilde{k}_1 \cdot (p \cdot \chi(\widetilde{k}_2)) \\ &= (\widetilde{k}_1 \cdot p) \cdot \chi(\widetilde{k}_2) \\ &= (p \cdot \chi(\widetilde{k}_1)) \cdot \chi(\widetilde{k}_2) \end{aligned}$$

$$= p \cdot (\chi(\widetilde{k}_1)\chi(\widetilde{k}_2)),$$

while

$$(\widetilde{k}_1\widetilde{k}_2) \cdot p = p \cdot \chi(\widetilde{k}_1\widetilde{k}_2).$$

The group action property $\widetilde{k}_1 \cdot (\widetilde{k}_2 \cdot p) = (\widetilde{k}_1\widetilde{k}_2) \cdot p$ proves that χ is a representation. $\square$

Chapter 4

Reduced prequantization

In this chapter we discuss reduction of the prequantum structures. This is accomplished by lifting the foliation reduction procedure on $\mathrm{J}^{-1}(\mathcal{O})$ to the part of $\dot{L}$ lying above $\mathrm{J}^{-1}(\mathcal{O})$. In order to do this in a manner consistent with foliation reduction of the classical reduced space $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, and so induce appropriate prequantum structures over $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, certain "quantization" conditions on $\mathcal{O}$ must be satisfied, and we will outline what these are.

The construction of the reduced U(1)-bundle $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ outlined in this chapter agrees with that proposed in [RSTS79]. There, a sufficient condition was given for the existence of a smooth structure on $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$. The condition, suitably generalized to include the case when the G-action on M lifts to a $\widetilde{G}$-action on $\dot{L}$, agrees with the one given here (Proposition 4.3.6). We further demonstrate that this condition is not only sufficient, but necessary (Section 4.3.2), and give an interpretation of reduction of the U(1)-bundle in terms of foliation reduction, which serves to unify reduction of the symplectic manifold (M, ω) and bundle-connection pair $(\dot{L}, \alpha)$ into the same conceptual framework.

4.1 Lifted notations

We lift the superscript conventions to the bundles $\dot{L}$ and L over M as follows: let $\dot{L}^{\mu}$, $\dot{L}^{a}$, and $\dot{L}^{\mathcal{O}}$ denote the subsets of $\dot{L}$ lying above $\mathrm{J}^{-1}(\mu)$, $\pi^{-1}(a)$, and $\mathrm{J}^{-1}(\mathcal{O})$ respectively,

$$\dot{L}^{\mu} := \dot{L}\big|_{\mathrm{J}^{-1}(\mu)} \simeq (i^{\mu})^{*}\dot{L},$$
$$\dot{L}^{a} := \dot{L}\big|_{\pi^{-1}(a)} \simeq (i^{a})^{*}\dot{L},$$
$$\dot{L}^{\mathcal{O}} := \dot{L}\big|_{\mathrm{J}^{-1}(\mathcal{O})} \simeq (i^{\mathcal{O}})^{*}\dot{L},$$

and $\dot{I}^{\mu}$, $\dot{I}^{a}$, and $\dot{I}^{\mathcal{O}}$ the respective inclusion maps,

$$\dot{I}^{\mu} : \dot{L}^{\mu} \hookrightarrow \dot{L},$$
$$\dot{I}^{a} : \dot{L}^{a} \hookrightarrow \dot{L},$$

$$\dot{I}^{\mathcal{O}} : \dot{L}^{\mathcal{O}} \hookrightarrow \dot{L}.$$

For maps on $\dot{L}$, restriction to $\dot{L}^\mu$, $\dot{L}^a$, and $\dot{L}^{\mathcal{O}}$ will also be denoted by the corresponding superscript. So for example,

$$\dot{\tau}^{\mathcal{O}} : \dot{L}^{\mathcal{O}} \to \mathrm{J}^{-1}(\mathcal{O})$$

is the restriction $(\dot{I}^{\mathcal{O}})^*\dot{\tau} = \dot{\tau} \circ \dot{I}^{\mathcal{O}}$ of the U(1)-bundle projection $\dot{\tau}$ to $\dot{L}^{\mathcal{O}}$.

It is not difficult to show that the U(1)-bundle structure on $(\dot{L}, \dot{\tau})$ induces U(1)-bundle structures on $(\dot{L}^\mu, \dot{\tau}^\mu)$, $(\dot{L}^a, \dot{\tau}^a)$, and $(\dot{L}^{\mathcal{O}}, \dot{\tau}^{\mathcal{O}})$, and that moreover with respect to these structures, the maps $\dot{I}^\mu$, $\dot{I}^a$, and $\dot{I}^{\mathcal{O}}$ have the same properties as their base manifold counterparts i^μ, i^a, and $i^{\mathcal{O}}$ (Proposition 2.8.4), namely:

- $\dot{L}^\mu$ is a closed, embedded submanifold of $\dot{L}$;
- $\dot{L}^a$ is initial in $\dot{L}$, closed if the G-action on M is proper, and embedded if M is second-countable;
- $\dot{L}^{\mathcal{O}}$ is initial in $\dot{L}$, and embedded if G is compact.

We use the superscipt notation to denote restrictions of forms also. For example,

$$\alpha^{\mathcal{O}} : T\,\dot{L}^{\mathcal{O}} \to \mathfrak{u}(1)$$

is the restriction $(\dot{I}^{\mathcal{O}})^*\alpha = T^*\dot{I}^{\mathcal{O}} \circ \alpha \circ \dot{I}^{\mathcal{O}}$ of the connection α to $\dot{L}^{\mathcal{O}}$.

Similar conventions apply to the associated line bundle L, with inclusion maps

$$I^\mu : L^\mu \hookrightarrow L, \qquad I^a : L^a \hookrightarrow L, \qquad I^{\mathcal{O}} : L^{\mathcal{O}} \hookrightarrow L.$$

Let $R = T\mathcal{R}$ denote the tangent distribution to the generalized foliation $\mathcal{R} \subset TM$, and $R^{\mathrm{h}} \subset T\dot{L}$ its horizontal lift to $\dot{L}$. If $\xi^{\mathrm{h}} \cdot p$ denotes the horizontal lift of $\xi \cdot x$ to $p \in (\dot{\tau})^{-1}(x)$, then

$$R_x = \mathfrak{g}_{\mathrm{J}(x)} \cdot x \qquad \text{and} \qquad R^{\mathrm{h}}_p = \mathfrak{g}^{\mathrm{h}}_{\mathrm{J}(x)} \cdot p.$$

Following the above convention, let $R^{\mathcal{O}}$ denote the restriction of R to $\mathrm{J}^{-1}(\mathcal{O})$, $(R^{\mathrm{h}})^{\mathcal{O}}$ the restriction of R^{h} to $\dot{L}^{\mathcal{O}}$, and similarly for R^μ, $(R^{\mathrm{h}})^\mu$, R^a, and $(R^{\mathrm{h}})^a$.

4.2 Properties of the universal cover of a compact semisimple Lie group

We recall here some standard properties of compact semisimple Lie groups G and their universal covers $\widetilde{G}$, and indicate where proofs can be found in the literature.

If G is compact and connected, then

- the exponential map $\exp_G : \mathfrak{g} \to G$ is onto [Sep07, Theorem 5.12 (b)];
- the coadjoint stabilizer group G_μ is connected [DK00, Theorem 3.3.1 (ii)].

If in addition G is semisimple, then

- its universal cover $\widetilde{G}$ is compact [Sep07, Corollary 6.33 (a)];
- combining the previous two points, $(\widetilde{G})_\mu$ is connected;
- $\ker(\pi_{\widetilde{G}\to G}) \subset Z(\widetilde{G}) \subset (\widetilde{G})_\mu$ [Sep07, Corollary 6.33 (c), and the fact that $(\widetilde{G})_\mu$ contains a maximal torus];
- $(\pi_{\widetilde{G}\to G})^{-1}(G_\mu) = (\widetilde{G})_\mu \ker(\pi_{\widetilde{G}\to G}) = (\widetilde{G})_\mu$ by the previous point.

4.3 Foliation reduction of the prequantum data

We now apply the foliation reduction technique to each of the bundle-connection pairs $(\dot{L}^\mu,\, \alpha^\mu)$, $(\dot{L}^a,\, \alpha^a)$, and $(\dot{L}^\mathcal{O},\, \alpha^\mathcal{O})$. In the case of admissible momenta/G-orbits/coadjoint orbits, this foliation reduction will cover symplectic reduction on the base spaces, and enable the construction of reduced U(1)-bundles over the spaces $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$, $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$, and $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^\mathcal{O}}$, and corresponding reduced connections.

4.3.1 The characteristic distributions of the restricted connections

Recall the definition of the characteristic distribution of a differential p-form β on a manifold S

$$N_s = \{X_s \in T_sS \,|\, \mathrm{i}_{X_s}\beta_s = 0 \text{ and } \mathrm{i}_{X_s}(\mathrm{d}_s\beta) = 0\}.$$

Proposition 4.3.1. *The characteristic distributions of α^μ, α^a, and $\alpha^\mathcal{O}$ are $(R^\mathrm{h})^\mu$, $(R^\mathrm{h})^a$, and $(R^\mathrm{h})^\mathcal{O}$ respectively.*

Proof. We prove the $\alpha^\mathcal{O}$ case. The other cases are identical.

Let $p \in \dot{L}^\mathcal{O}$. The first condition $i_{X_p}\alpha^\mathcal{O}_p = \alpha^\mathcal{O}_p(X_p) = 0$ tells us that X_p must be a horizontal vector. Using the curvature condition $\mathrm{d}\alpha^\mathcal{O} = (\dot{\tau}^\mathcal{O})^*\omega^\mathcal{O}$, the second condition $\mathrm{i}_{X_p}(\mathrm{d}_p\alpha^\mathcal{O}) = 0$ tells us that $T_p\dot{\tau}^\mathcal{O}(X_p)$ lies in the characteristic distribution of $\omega^\mathcal{O} = (i^\mathcal{O})^*\omega$ at $x = \dot{\tau}(p)$, which is $\mathfrak{g}_{\mathrm{J}(x)} \cdot x = R^\mathcal{O}_x$. So the characteristic distribution of $\alpha^\mathcal{O}$ at a point is the set of horizontal vectors covering $R^\mathcal{O}$, i.e., $(R^\mathrm{h})^\mathcal{O}$. $\square$

By the Foliation Reduction Theorem 2.11.1 (i), the distributions $(R^\mathrm{h})^\mathcal{O}$, $(R^\mathrm{h})^\mu$, and $(R^\mathrm{h})^a$ are all involutive, hence integrable. Consistent with the notation on M, we denote the corresponding foliations $(\mathcal{R}^\mathrm{h})^\mathcal{O}$, $(\mathcal{R}^\mathrm{h})^\mu$ and $(\mathcal{R}^\mathrm{h})^a$, and the generalized foliation of the entirety of $\dot{L}$ as $\mathcal{R}^\mathrm{h}$.

4.3.2 Consistency of foliation reduction and the notion of admissibility

The foliation $\mathcal{R}^\mu$ on $\mathrm{J}^{-1}(\mu)$ consists of G_μ-orbits. Being the horizontal lift of $R^\mu = T\mathcal{R}^\mu$, the involutive distribution $(R^{\mathrm{h}})^\mu$ is generated by the vector fields $\{\xi_L^{\mathrm{h}} \,|\, \xi \in \mathfrak{g}_\mu\}$, with value $\xi_L^{\mathrm{h}}(p) = \xi^{\mathrm{h}} \cdot p$ at $p \in \dot{L}^\mu$. In general, therefore, $(R^{\mathrm{h}})^\mu$ will integrate to a horizontal action of the universal cover $\widetilde{G_\mu}$ of G_μ on $\dot{L}^\mu$. Since this $\widetilde{G_\mu}$-action covers the free G_μ-action on $\mathrm{J}^{-1}(\mu)$, the horizontal $\widetilde{G_\mu}$-action restricts to a $\ker(\pi_{\widetilde{G_\mu}\to G_\mu})$-action on each U(1)-fiber. The Foliation Reduction Theorem 2.11.1 guarantees that α^μ is conserved along $(\mathcal{R}^{\mathrm{h}})^\mu$, and so each element of $\ker(\pi_{\widetilde{G_\mu}\to G_\mu})$ is an α^μ-preserving vertical isomorphism (i.e. an equivalence) on $\dot{L}^\mu$. Proposition A.2.2 demonstrates that such isomorphisms are global right multiplications by elements of U(1), and so each element of $\ker(\pi_{\widetilde{G_\mu}\to G_\mu})$ acts uniformally on the U(1)-fibers of $\dot{L}^\mu$. Hence we can restrict our attention to one U(1)-fiber.

In order for the quotient space $\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ to have a U(1)-bundle structure induced from that on $\dot{L}^\mu$, this action must be discrete, hence finite (by compactness of U(1)), and must equal the action of the nth roots of unity for some n. Equivalently, the leaves of the foliation $(\mathcal{R}^{\mathrm{h}})^\mu$ must intersect a U(1)-fiber at most a finite number of times.

Assuming that the quotient space $\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ can be given a smooth structure such that the quotient map $\dot{\Sigma}^\mu : \dot{L}^\mu \to \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ is a submersion, the Foliation Reduction Theorem 2.11.1 (ii) guarantees the existence of a form $\alpha^\mu_{\mathcal{R}}$ such that

$$\alpha^\mu = (\dot{\Sigma}^\mu)^* \alpha^\mu_{\mathcal{R}}.$$

Suppose now a leaf of $(\mathcal{R}^{\mathrm{h}})^\mu$ intersects a U(1)-fiber at n points. $\dot{\Sigma}^\mu$ restricted to the U(1)-fiber covers the reduced U(1)-fiber n to 1, and so maps the vertical vector $p \cdot \varepsilon$, $\varepsilon \in \mathfrak{u}(1)$, on the fiber to $n\, q \cdot \varepsilon$ on the reduced fiber, where $q = \dot{\Sigma}^\mu(p)$. As a consequence

$$(\alpha^\mu_{\mathcal{R}})_q\,(n\, q \cdot \varepsilon) = (\alpha^\mu_{\mathcal{R}})_q (T_p \dot{\Sigma}(p \cdot \varepsilon)) = \Big((\dot{\Sigma}^\mu_{\mathcal{R}})^* \alpha^\mu_{\mathcal{R}}\Big)_p (p \cdot \varepsilon) = (\alpha^\mu)_p (p \cdot \varepsilon) = \varepsilon,$$

implying that

$$(\alpha^\mu_{\mathcal{R}})_q (q \cdot \varepsilon) = \frac{\varepsilon}{n}.$$

Thus $\alpha^\mu_{\mathcal{R}}$ is not a connection unless $n = 1$. Of course for $n \neq 1$, we could just multiply $\alpha^\mu_{\mathcal{R}}$ by n, but this would imply that the curvature of the reduced connection is $n\frac{\varepsilon_0}{h}\omega^\mu_{\mathcal{R}}$ (see Proposition 4.3.11, and note that its proof does not involve the equivariance of $\dot{\Sigma}^{\mathcal{O}}$), whereas it needs to be $\frac{\varepsilon_0}{h}\omega^\mu_{\mathcal{R}}$ to allow for geometric quantization of the reduced space.

Since each leaf of $(\mathcal{R}^{\mathrm{h}})^\mu$ intersects each U(1)-fiber at most once, $\ker(\pi_{\widetilde{G_\mu}\to G_\mu})$ acts trivially on each U(1)-fiber, and so the horizontal $\widetilde{G_\mu}$-action on $\dot{L}^\mu$ drops to a horizontal $\widetilde{G_\mu}/\ker(\pi_{\widetilde{G_\mu}\to G_\mu}) \simeq G_\mu$-action.

In summary, we have the following theorem.

Proposition 4.3.2. *A necessary condition for the prequantum geometric structures* $(\dot{L},\, \alpha)$ *to induce via foliation reduction appropriate prequantum geometric structures* $\left(\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu},\, \alpha^\mu_{\mathcal{R}}\right)$ *over the reduced space* $\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu},\, \omega^\mu_{\mathcal{R}}\right)$ *is that the leaves of the horizontal foliation* $(\mathcal{R}^{\mathrm{h}})^\mu$ *of* $\dot{L}^\mu$ *intersect each* U(1)*-fiber of* $\dot{L}^\mu$ *at most once. This condition is equivalent to the existence of a horizontal* G_μ*-action everywhere on* $\dot{L}^\mu$*.*

Much of the rest of this chapter will be devoted to showing that this is also a *sufficient* condition for reduction of the prequantum structures. We first point out some properties of the horizontal G_μ-action when it exists, and then explain what conditions existence of the horizontal G_μ action imposes on the possible values of μ.

Proposition 4.3.3. *If a horizontal* G_μ*-action exists on* $\dot{L}^\mu$*, then it acts freely and properly on* $\dot{L}^\mu$*.*

Proof. Freeness of the action follows from the fact that the horizontal G_μ-action on $\dot{L}^\mu$ covers the G_μ-action on $\mathrm{J}^{-1}(\mu)$, and the freeness of the G_μ-action on $\mathrm{J}^{-1}(\mu)$.

Let $((h_n,\, p_n))$ be a sequence in $G_\mu \times \dot{L}^\mu$ such that $((p_n,\, h_n^{\mathrm{h}} \cdot p_n))$ converges in $\dot{L}^\mu \times \dot{L}^\mu$. By continuity and G_μ-equivariance of $\dot{\tau}^\mu$, $((x_n,\, h_n \cdot x_n))$ converges in $\mathrm{J}^{-1}(\mu) \times \mathrm{J}^{-1}(\mu)$, where $x_n = \dot{\tau}^\mu(p_n)$. G_μ is closed in G and $\mathrm{J}^{-1}(\mu)$ is embedded in M, so the G_μ-action on $\mathrm{J}^{-1}(\mu)$ is also proper (Proposition 2.3.2), which then guarantees the existence of a subsequence (g_{n_k}) of (g_n) convergent in G_μ. ☐

Proposition 4.3.4. *If a horizontal* G_μ*-action exists on* $\dot{L}^\mu$*, then a horizontal* G_ν*-action exists on* $\dot{L}^\nu$ *for any* ν *in the coadjoint orbit containing* μ*.*

Proof. Since μ and ν lie in the same coadjoint orbit, there exists $g \in G$ such that $\mathrm{Ad}^*_{g^{-1}}\mu = \nu$. Take $\widetilde{g} \in \widetilde{G}$ such that $\pi_{\widetilde{G}\to G}(\widetilde{g}) = g$. The the map $p \mapsto \widetilde{g} \cdot p$ maps $\dot{L}^\mu$ to $\dot{L}^\nu$. Since the $\widetilde{G}$-action on $\dot{L}$ preserves the connection α, it takes horizontal submanifolds to horizontal submanifolds. Also $p \mapsto \widetilde{g}\cdot p$ covers $g \mapsto g\cdot x$, which maps G_μ-orbits in $\mathrm{J}^{-1}(\mu)$ to G_ν-orbits in $\mathrm{J}^{-1}(\nu)$. Hence the integral submanifolds of $(R^{\mathrm{h}})^\mu$ get mapped to the integral submanifolds of $(R^{\mathrm{h}})^\nu$, and so a horizontal G_ν-action exists on $\dot{L}^\nu$. ☐

In cases where this horizontal action exists on $\dot{L}^{\mathcal{O}}$, we will henceforth refer to it as the ***horizontal*** $\mathbf{G}_{\mathrm{J}}$***-action*** or $\boldsymbol{G}^{\mathbf{h}}_{\mathbf{J}}$***-action***. The foliation $(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}$ defined by the integrable distribution $(R^{\mathrm{h}})^{\mathcal{O}}$ then consists of these horizontal G_{J}-orbits.

Values of μ for which $\dot{L}^\mu$ supports a horizontal G_μ-action are called ***admissible momenta***. Similarly, ***admissible coadjoint orbits*** $\mathcal{O}$ are defined as those for which $\dot{L}^{\mathcal{O}}$ support a horizontal G_{J}-action. In light of the Proposition 4.3.4,

$$\mu \in \mathcal{O} \text{ admissible} \implies \mathcal{O} \text{ admissible}$$

(and the reverse implication is trivially true).

When the horizontal G_{J}-action exists on $\dot{L}^{\mathcal{O}}$, it is compatible with the $\widetilde{G}$-action in the following sense.

Lemma 4.3.5. *For orbits $\mathcal{O}$ such that the horizontal G_{J}-action exists on $\dot{L}^{\mathcal{O}}$, it commutes with the $\widetilde{G}$-action, in the sense that for any $\widetilde{g} \in \widetilde{G}$, $p \in \dot{L}^{\mu}$ and $h \in G_{\mu}$,*

$$\widetilde{g} \cdot (h^{\mathrm{h}} \cdot p) = \big(ghg^{-1}\big)^{\mathrm{h}} \cdot (\widetilde{g} \cdot p),$$

where $g = \pi_{\widetilde{G} \to G}(\widetilde{g})$.

Proof. We prove this by looking at the infinitesimal action, i.e., the horizontal $\mathfrak{g}_{\mathrm{J}}$-action. The result then follows by the connectedness of G_{μ} for each $\mu \in \mathfrak{g}^*$.

Let $\widetilde{g} \in \widetilde{G}$ and $\xi \in \mathfrak{g}_{\mu}$. If $\xi^{\mathrm{h}} \cdot p$ denotes the infinitesimal horizontal G_{μ}-action, then the fact that $\widetilde{g} \cdot \alpha = \alpha$ implies that $\widetilde{g} \cdot (\xi^{\mathrm{h}} \cdot p)$ is horizontal. Since it is a vector at $\widetilde{g} \cdot p$ which covers the vector $g \cdot \xi \cdot x = (\mathrm{Ad}_g\, \xi) \cdot g \cdot x$ where $x = \dot{\tau}(p)$, we get that

$$\widetilde{g} \cdot (\xi^{\mathrm{h}} \cdot p) = (\mathrm{Ad}_g\, \xi)^{\mathrm{h}} \cdot (\widetilde{g} \cdot p).$$

This is the infinitesimal form of the required relation. □

4.3.3 A characterization of admissible momenta

We now give an equivalent criterion for the existence of the horizontal G_{μ}-action on $\dot{L}^{\mu}$ in terms of the value of the momentum μ. First note that the mapping

$$-\frac{\varepsilon_0}{h}\mu : \mathfrak{g}_{\mu} \to \mathfrak{u}(1)$$

is a Lie algebra homomorphism, since

$$-\frac{\varepsilon_0}{h}\langle \mu, [\xi, \zeta]\rangle = -\frac{\varepsilon_0}{h}\langle \mathrm{ad}^*_{\xi}\mu, \zeta\rangle = 0 = \Big[-\frac{\varepsilon_0}{h}\langle \mu, \xi\rangle, -\frac{\varepsilon_0}{h}\langle \mu, \zeta\rangle\Big]$$

for all $\xi, \zeta \in \mathfrak{g}_{\mu}$.

We can now state the criterion.

Proposition 4.3.6. *A horizontal G_{μ}-action on $\dot{L}^{\mu}$ exists if and only the Lie algebra homomorphism $-\frac{\varepsilon_0}{h}\mu : \mathfrak{g}_{\mu} \to \mathfrak{u}(1)$ exponentiates to a character $\chi^{-\frac{i}{\hbar}\mu} : (\widetilde{G})_{\mu} \to \mathrm{U}(1)$ that agrees with $\chi : \ker(\pi_{\widetilde{G}\to G}) \to \mathrm{U}(1)$ on $\ker(\pi_{\widetilde{G}\to G}) \subset (\widetilde{G})_{\mu}$. In this case, the horizontal G_{μ}-action $(h, p) \mapsto h^{\mathrm{h}} \cdot p$ is related to the $\widetilde{G}$-action by*

$$h^{\mathrm{h}} \cdot p = \widetilde{h} \cdot p \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h})^{-1},$$

where $\widetilde{h} \in \widetilde{G}$ *is any element such that* $\pi_{\widetilde{G}\to G}(\widetilde{h}) = h$.

Proof. Suppose first that $-\frac{\varepsilon_0}{h}\mu$ exponentiates as described. Consider the $(\widetilde{G})_\mu$-action

$$(\widetilde{h},\, p) \mapsto \widetilde{h} \cdot p \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h})^{-1}.$$

The action property is easily verified. For $\xi \in \mathfrak{g}_\mu$, the infinitesimal form of this action is

$$\begin{aligned}(\xi,\, p) &\mapsto \xi \cdot p - p \cdot \left(-\frac{\varepsilon_0}{h}\mu(\xi)\right)\\ &= \xi^{\mathrm{h}} \cdot p - \langle \mathrm{J}(\dot{\tau}(p)),\, \xi\rangle \left(\frac{\varepsilon_0}{h}\right)_{\dot{L}}(p) + \langle \mu,\, \xi\rangle \left(\frac{\varepsilon_0}{h}\right)_{\dot{L}}(p)\\ &= \xi^{\mathrm{h}} \cdot p,\end{aligned}$$

so it is a horizontal action. Finally for $\widetilde{k} \in \ker(\pi_{\widetilde{G}\to G})$,

$$\begin{aligned}(\widetilde{k},\, p) &\mapsto \widetilde{k} \cdot p \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{k})^{-1}\\ &= (p \cdot \chi(\widetilde{k})) \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{k})^{-1}\\ &= p,\end{aligned}$$

and so the $(\widetilde{G})_\mu$-action drops to a $(\widetilde{G})_\mu / \ker(\pi_{\widetilde{G}\to G}) = G_\mu$ action.

Conversely, suppose a horizontal G_μ-action $(h,\, p) \mapsto h^{\mathrm{h}} \cdot p$ exists on $\dot{L}^\mu$. Let $\widetilde{h} \in (\widetilde{G})_\mu$, and write $h = \pi_{\widetilde{G}\to G}(\widetilde{h})$. For any $p \in \dot{L}^\mu$, $h^{\mathrm{h}} \cdot p$ and $\widetilde{h} \cdot p$ both cover $h \cdot \dot{\tau}(p)$, and so lie in the same U(1)-fiber of $\dot{L}^\mu$. Write

$$\widetilde{h} \cdot p = h^{\mathrm{h}} \cdot p \cdot \lambda(\widetilde{h},\, p),$$

where $\lambda(\widetilde{h},\, p) \in \mathrm{U}(1)$. Since $(\widetilde{G})_\mu$ is compact and connected, $\exp_{(\widetilde{G})_\mu} : \mathfrak{g}_\mu \to (\widetilde{G})_\mu$ is surjective (Section 4.2), so there exists $\xi \in \mathfrak{g}_\mu$ such that $\exp_{(\widetilde{G})_\mu} \xi = \widetilde{h}$, which implies that $\exp_{G_\mu} \xi = h$. Writing $\widetilde{h}_t = \exp_{(\widetilde{G})_\mu}(t\xi)$, $h_t = \exp_{G_\mu}(t\xi)$, and $\lambda_t = \lambda(\widetilde{h}_t,\, p)$, we have that

$$\widetilde{h}_t \cdot p = (h_t)^{\mathrm{h}} \cdot p \cdot \lambda_t,$$

which upon differentiation gives

$$\begin{aligned}\xi \cdot \widetilde{h}_t \cdot p &= \xi^{\mathrm{h}} \cdot (h_t)^{\mathrm{h}} \cdot p \cdot \lambda_t + (h_t)^{\mathrm{h}} \cdot p \cdot \lambda_t'\\ \implies \xi^{\mathrm{h}} \cdot \widetilde{h}_t \cdot p - \langle \mu,\, \xi\rangle \left(\frac{\varepsilon_0}{h}\right)_{\dot{L}}(\widetilde{h}_t \cdot p) &= \xi^{\mathrm{h}} \cdot (h_t)^{\mathrm{h}} \cdot p \cdot \lambda_t + \left(\lambda_t^{-1}\lambda_t'\right)_{\dot{L}}((h_t)^{\mathrm{h}} \cdot p \cdot \lambda_t),\end{aligned}$$

i.e.,

$$\lambda_t^{-1}\lambda_t' = -\langle \mu,\, \xi\rangle \frac{\varepsilon_0}{h} \in \mathfrak{u}(1).$$

Integrating and using the initial condition $\lambda_0 = 1$,

$$\lambda_t = \exp_{\mathrm{U}(1)}\left(-t\langle\mu,\,\xi\rangle\frac{\varepsilon_0}{h}\right) = \exp\left(-t\frac{i}{\hbar}\langle\mu,\,\xi\rangle\right),$$

and $\lambda(\widetilde{h},\,p) = \lambda_1 = \exp\left(-\frac{i}{\hbar}\langle\mu,\,\xi\rangle\right)$ is in fact independent of p. From

$$\widetilde{h}\cdot p = h^{\mathrm{h}}\cdot p\cdot\lambda(\widetilde{h})$$

it is then straightforward to check that $\lambda : (\widetilde{G})_\mu \to \mathrm{U}(1)$ is a representation, and we have shown above that its derivative at the identity is $-\frac{\varepsilon_0}{h}\mu$. Finally, if $\widetilde{k}\in\ker(\pi_{\widetilde{G}\to G})$, the defining equation for λ says that

$$\widetilde{k}\cdot p = p\cdot\lambda(\widetilde{k}).$$

From $\widetilde{k}\cdot p = p\cdot\chi(\widetilde{k})$ it follows that λ must agree with χ on $\ker(\pi_{\widetilde{G}\to G})$. Hence λ is the required representation $\chi^{-\frac{i}{\hbar}\mu}$ in the statement of the theorem. ☐

Corollary 4.3.7. *If a horizontal G_{J}-action exists on $\dot{L}^a$ for some $a\in\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, then it exists on the entirety of $\dot{L}^{\mathcal{O}}$.*

Proof. Examination of the proof of Proposition 4.3.6 makes it clear that the existence of a horizontal G_μ-action on the bundle $\dot{L}^{(a,\mu)}$ lying over $\mathcal{R}^{(a,\mu)} = \pi^{-1}(a)\cap\mathrm{J}^{-1}(\mu)$ is enough to guarantee the existence of the character $\chi^{-\frac{i}{\hbar}\mu}$, which in turn guarantees the existence of the horizontal action on all of $\mathrm{J}^{-1}(\mu)$. By hypothesis, the action exists on all of $\dot{L}^a$, hence on $\dot{L}^{(a,\mu)}$ for all $\mu\in\mathcal{O}$, hence on $\dot{L}^\mu$ for all $\mu\in\mathcal{O}$, hence on the entirety of $\dot{L}^{\mathcal{O}}$. ☐

We define ***admissible G-orbits*** $a\in\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ to be those for which $\dot{L}^a$ supports a horizontal G_{J}-action. In light of Corollary 4.3.7,

$$a\in\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}\text{ admissible}\implies\mathcal{O}\text{ admissible}$$

(and the reverse implication is trivially true).

Proposition 4.3.6 implies that the set of admissible coadjoint orbits form a discrete set in $\mathfrak{g}^*$. Hence the requirement that the prequantum geometric structures reduce via foliation reduction leads to a quantization condition on the coadjoint orbits.

4.3.4 Foliation reduction of the bundles

The foliation $(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}$ defined by the integrable distribution $(R^{\mathrm{h}})^{\mathcal{O}}$ allows the construction of the quotient space $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, and similarly for $(\mathcal{R}^{\mathrm{h}})^{\mu}$ and $(\mathcal{R}^{\mathrm{h}})^{a}$. In analogy to the submersions

$$\sigma^{\mu} : \mathrm{J}^{-1}(\mu) \to \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}, \qquad \sigma^{a} : \pi^{-1}(a) \to \frac{\pi^{-1}(a)}{\mathcal{R}^{a}}, \qquad \sigma^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}},$$

we define the maps

$$\dot{\Sigma}^{\mu} : \dot{L}^{\mu} \to \frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}, \qquad \dot{\Sigma}^{a} : \dot{L}^{a} \to \frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}}, \qquad \dot{\Sigma}^{\mathcal{O}} : \dot{L}^{\mathcal{O}} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}.$$

In general, the quotient spaces cannot be given a smooth structure. For admissible momenta μ/G-orbits a/coadjoint orbits $\mathcal{O}$, however, they can. The proof of the following theorem mimics closely that of Proposition 2.8.5.

Proposition 4.3.8. *(i) For admissible $\mu \in \mathrm{J}(M) \subset \mathfrak{g}^*$, $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ can be given a smooth structure making the quotient map $\dot{\Sigma}^{\mu} : \dot{L}^{\mu} \to \frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ a surjective submersion.*

(ii) For admissible $a \in \frac{M}{G}$, $\frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}}$ can be given a smooth structure making the quotient map $\dot{\Sigma}^{a}$: $\dot{L}^{a} \to \frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}}$ a sujective submersion.

(iii) For admissible $\mathcal{O} \subset \mathrm{J}(M) \subset \mathfrak{g}^*$, $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ can be given a smooth structure making the quotient map $\dot{\Sigma}^{\mathcal{O}} : \dot{L}^{\mu} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ a surjective submersion.

Proof. (i) For admissible μ we have that

$$\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}} = \frac{\dot{L}^{\mu}}{G^{\mathrm{h}}_{\mu}}.$$

Proposition 4.3.3 guarantees that the horizontal G_{μ}-action on $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ is free and proper, and so by Proposition 2.3.1 the result follows.

(ii) Let $p \in \dot{L}^{a}$, and write $\mu = \mathrm{J}(\dot{\tau}(p))$. Construct a local section $\widetilde{s} : U \to \widetilde{G}$ of $\widetilde{G} \to \widetilde{G}/(\widetilde{G})_{\mu}$ through $\widetilde{e} \in \widetilde{G}$ of the bundle $\widetilde{G} \to \widetilde{G}/(\widetilde{G})_{\mu}$. Define a function $F : U \times G_{\mu} \times \mathrm{U}(1) \to \dot{L}^{a}$ by

$$F(u,\, h,\, w) = \widetilde{s}(u) \cdot h^{\mathrm{h}} \cdot p \cdot w.$$

F is smooth since it is smooth as a function to $\dot{L}$, and $\dot{L}^{a}$ is initial in $\dot{L}$.

F is injective: suppose $F(u,\, h,\, w) = F(u',\, h',\, w')$. $h \cdot p \cdot w$ and $h' \cdot p \cdot w'$ are both in $\dot{L}^{\mu}$, which implies that $\widetilde{s}(u) = \widetilde{s}(u')\widetilde{l}$ for some $\widetilde{l} \in (\widetilde{G})_{\mu}$. Since $\widetilde{s}$ is a section of $\widetilde{G} \to \widetilde{G}/(\widetilde{G})_{\mu}$, this can only occur if $\widetilde{l} = \widetilde{e}$ and hence $u = u'$. Then $h^{\mathrm{h}} \cdot p \cdot w = (h')^{\mathrm{h}} \cdot p \cdot w'$, i.e., $({h'}^{-1}h)^{\mathrm{h}} \cdot p = p \cdot (w'w^{-1})$.

Since each horizontal G_μ-orbit intersects a U(1)-fiber of $\dot{L}^\mu$ precisely once, this implies that $h = h'$ and $w = w'$.

The point $F(u,\, h,\, w)$ lies over $\dot{\tau}^a(F(u,\, h,\, w)) = s(u) \cdot h \cdot x \in \pi^{-1}(a)$, where $x = \dot{\tau}^a(p)$ and $s = \pi_{\widetilde{G}\to G} \circ \widetilde{s} : U \to G$. This point has momentum $\mathrm{J}^a(s(u) \cdot h \cdot p) = \mathrm{Ad}^*_{s(u)^{-1}}\mu$, and so the $\dot{\Sigma}^a$-fiber/horizontal G_{J}-orbit through $F(u,\, h,\, w)$ is

$$\begin{aligned}
\left(G_{\mathrm{Ad}_{s(u)^{-1}}\mu}\right)^{\mathrm{h}} \cdot F(u,\, h,\, w) &= \left(\mathrm{Ad}_{s(u)} G_\mu\right)^{\mathrm{h}} \cdot \widetilde{s}(u) \cdot h^{\mathrm{h}} \cdot p \cdot w \\
&= \widetilde{s}(u) \cdot G^{\mathrm{h}}_\mu \cdot h^{\mathrm{h}} \cdot p \cdot w \qquad \text{by Lemma 4.3.5} \\
&= \widetilde{s}(u) \cdot G^{\mathrm{h}}_\mu \cdot p \cdot w \\
&= F(u,\, G_\mu,\, w).
\end{aligned}$$

Hence F induces a map $F_{\mathcal{R}} : U \times \mathrm{U}(1) \to \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a}$, and as in Proposition 2.8.5 F and $F_{\mathcal{R}}$ can be used to construct local coordinate systems about $p \in \dot{L}^a$ and $\dot{\Sigma}^a(p) \in \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a}$ such that $\dot{\Sigma}^a$ is regular at p, Since p was arbitrary, we obtain a smooth structure on $\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a}$ which makes $\dot{\Sigma}^a : \dot{L}^a \to \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a}$ a submersion.

(iii) Now take $p \in \dot{L}^{\mathcal{O}}$, and write $\mu = \mathrm{J}(\dot{\tau}(p))$ as before. Define $\widetilde{s}$ as in part (ii), and define $t : V \to \dot{L}^\mu$ to be a local section of $\dot{\Sigma}^\mu : \dot{L}^\mu \to \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ through p, which exists by part (i). The function F from part (ii) can be extended to $F : U \times G_\mu \times V \times \mathrm{U}(1) \to \dot{L}^{\mathcal{O}}$, defined by

$$F(u,\, h,\, v,\, w) = \widetilde{s}(u) \cdot h^{\mathrm{h}} \cdot t(v) \cdot w.$$

A similar proof to part (ii) demonstrates that F induces a map $F_{\mathcal{R}} : U \times V \times \mathrm{U}(1) \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, and it follows that $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ can be given a smooth structure which makes $\dot{\Sigma}^{\mathcal{O}} : \dot{L}^{\mathcal{O}} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ a submersion.

□

4.3.5 The bundle structure of the reduced spaces

Since the connection $\alpha^{\mathcal{O}}$ is U(1)-invariant, the foliation $(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}$ is also U(1)-invariant. Also U(1) acts freely on the leaves of $(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}$ by the admissibility condition. It follows that a free U(1)-action can be defined on the reduced space $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ which makes the quotient map $\dot{\Sigma}^{\mathcal{O}} : \dot{L}^{\mathcal{O}} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ U(1)-equivariant. The same holds for the quotient maps $\dot{\Sigma}^\mu$ and $\dot{\Sigma}^a$. This U(1)-action gives $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ the structure of a principle U(1)-bundle, as the following theorem demonstrates.

Proposition 4.3.9. *(i) There exists a map $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}} : \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ making the following diagram* *commute,*

$$\begin{tikzcd}
\dot{L}^{\mathcal{O}} \arrow[r, "\dot{\Sigma}^{\mathcal{O}}"] \arrow[d, "\dot{\tau}^{\mathcal{O}}"'] & \dfrac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \arrow[d, "\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}"] \\
\mathrm{J}^{-1}(\mathcal{O}) \arrow[r, "\sigma^{\mathcal{O}}"'] & \dfrac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}
\end{tikzcd} .$$

(ii) $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ *is smooth.*

(iii) $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ *is a surjective submersion.*

(iv) The fibers of $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ *are precisely the* $\mathrm{U}(1)$*-orbits of the previously defined* $\mathrm{U}(1)$*-action on* $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$.

(v) $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ *has the local triviality property, and so defines* $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ *as a* $\mathrm{U}(1)$*-bundle over* $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$.

Proof. (i) We recall that the fibers of $\dot{\Sigma}^{\mathcal{O}}$ are the $G^{\mathrm{h}}_{\mathrm{J}}$-orbits in $\dot{L}^{\mathcal{O}}$, while the fibers of $\sigma^{\mathcal{O}}$ are the G_{J}-orbits in $\mathrm{J}^{-1}(\mathcal{O})$. $\dot{\tau}^{\mathcal{O}}$ maps the former fibers to the latter, which implies the existence of a map $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ making the diagram commute. Explicitly, for $p \in \dot{L}^{\mathcal{O}}$ and $x = \dot{\tau}^{\mathcal{O}}(p)$,

$$\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}\left(\dot{\Sigma}^{\mathcal{O}}(p)\right) = \sigma^{\mathcal{O}}(x), \qquad \text{or} \qquad \dot{\tau}^{\mathcal{O}}_{\mathcal{R}}\left(G^{\mathrm{h}}_{\mathrm{J}(x)} \cdot p\right) = G_{\mathrm{J}(x)} \cdot x.$$

(ii) Given a point in $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, construct a local section $t : U \to \dot{L}^{\mathcal{O}}$ of $\dot{\Sigma}^{\mathcal{O}}$ through the point. Then $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}\big|_U = \sigma^{\mathcal{O}} \circ \dot{\tau}^{\mathcal{O}} \circ s$, which being the composition of smooth maps is itself smooth. So $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ is smooth at the point, hence smooth.

(iii) Let $\mathcal{R}^{(\nu,b)} \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, and pick $x \in \mathcal{R}^{(\nu,b)} \subset \mathrm{J}^{-1}(\mathcal{O})$ and $p \in (\dot{\tau}^{\mathcal{O}})^{-1}(x)$. Then

$$\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}(\dot{\Sigma}^{\mathcal{O}}(p)) = \sigma^{\mathcal{O}}(\dot{\tau}^{\mathcal{O}}(p)) = \sigma^{\mathcal{O}}(x) = \mathcal{R}^{(\nu,b)},$$

so $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ is surjective.

Differentiating the commutivity relation $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}} \circ \dot{\Sigma}^{\mathcal{O}} = \sigma^{\mathcal{O}} \circ \dot{\tau}^{\mathcal{O}}$ yields

$$T\dot{\tau}^{\mathcal{O}}_{\mathcal{R}} \circ T\dot{\Sigma}^{\mathcal{O}} = T\sigma^{\mathcal{O}} \circ T\dot{\tau}^{\mathcal{O}},$$

from which it follows easily that $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ is a submersion.

(iv) Suppose we have two points $q_1, q_2 \in \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ such that $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}(q_1) = \dot{\tau}^{\mathcal{O}}_{\mathcal{R}}(q_2)$. Take p_1, p_2 in $\dot{L}^{\mathcal{O}}$

such that $q_1 = \dot{\Sigma}^{\mathcal{O}}(p_1)$, $q_2 = \dot{\Sigma}^{\mathcal{O}}(p_2)$. Then

$$\begin{aligned}
&\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}(\dot{\Sigma}^{\mathcal{O}}(p_1)) = \dot{\tau}^{\mathcal{O}}_{\mathcal{R}}(\dot{\Sigma}^{\mathcal{O}}(p_2)) \\
\Longrightarrow\ &\sigma^{\mathcal{O}}(\dot{\tau}^{\mathcal{O}}(p_1)) = \sigma^{\mathcal{O}}(\dot{\tau}^{\mathcal{O}}(p_2)) && \text{by commutativity of the diagram} \\
\Longrightarrow\ &\dot{\tau}^{\mathcal{O}}(p_1) = h \cdot \dot{\tau}^{\mathcal{O}}(p_2) = \dot{\tau}^{\mathcal{O}}(h^{\mathrm{h}} \cdot p_2) && \text{for some } h \in G_{\mathrm{J}(\dot{\tau}(p_1))} \\
\Longrightarrow\ &p_1 = h^{\mathrm{h}} \cdot p_2 \cdot w && \text{for some } w \in \mathrm{U}(1) \\
\Longrightarrow\ &q_1 = \dot{\Sigma}^{\mathcal{O}}(p_1) = \dot{\Sigma}^{\mathcal{O}}(h^{\mathrm{h}} \cdot p_2 \cdot w) = \dot{\Sigma}^{\mathcal{O}}(p_2) \cdot w = q_2 \cdot w,
\end{aligned}$$

using U(1)-equivariance of $\dot{\Sigma}^{\mathcal{O}}$ in the last identity. Hence q_1, q_2 lie in the same U(1)-orbit, so the fibers of $\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}$ lie in the U(1)-orbits. The reverse implication essentially reverses the above argument.

(v) To prove local triviality about a point of $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, construct a section $t : V \to \mathrm{J}^{-1}(\mathcal{O})$ of $\sigma^{\mathcal{O}}$ about this point. By taking V smaller if necessary, there exists an open set $V' \subset \mathrm{J}^{-1}(\mathcal{O})$ containing $t(V)$ and a local trivialisation $\psi : (\dot{\tau}^{\mathcal{O}})^{-1}(V') \to V' \times \mathrm{U}(1)$ of $\dot{L}^{\mathcal{O}}$. Define $\lambda : V \times \mathrm{U}(1) \to (\dot{\tau}^{\mathcal{O}}_{\mathcal{R}})^{-1}(V)$ by

$$\lambda(v,\, w) = \dot{\Sigma}^{\mathcal{O}}(\psi^{-1}(t(v),\, w)).$$

Being a composition of smooth maps, λ is smooth. Since ψ^{-1} and $\dot{\Sigma}^{\mathcal{O}}$ are U(1)-equivariant, λ is itself U(1)-equivariant. Also

$$\begin{aligned}
\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}(\lambda(v, w)) &= (\dot{\tau}^{\mathcal{O}}_{\mathcal{R}} \circ \dot{\Sigma}^{\mathcal{O}})(\psi^{-1}(t(v),\, w)) \\
&= (\sigma^{\mathcal{O}} \circ \dot{\tau}^{\mathcal{O}})(\psi^{-1}(t(v),\, w)) \\
&= \sigma^{\mathcal{O}}(t(v)) && \text{since } \psi \text{ is a local trivialization} \\
&= v && \text{since } t \text{ is a section of } \sigma^{\mathcal{O}}.
\end{aligned}$$

This property combined with U(1)-equivariance of $\dot{\Sigma}^{\mathcal{O}}$ is enough to prove bijectivity of λ. $\lambda^{-1} : (\dot{\tau}^{\mathcal{O}}_{\mathcal{R}})^{-1}(V) \to V \times \mathrm{U}(1)$ works as the required local trivialization.

□

By restricting to either $\dot{L}^{\mu}$ or $\dot{L}^{a}$ in $\dot{L}^{\mathcal{O}}$ we of course obtain the corresponding commutative diagrams,

$$
\begin{tikzcd}
\dot{L}^{\mu} \arrow[r, "\dot{\Sigma}^{\mu}"] \arrow[d, "\dot{\tau}^{\mu}"'] & \dfrac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}} \arrow[d, "\dot{\tau}^{\mu}_{\mathcal{R}}"] \\
\mathrm{J}^{-1}(\mu) \arrow[r, "\sigma^{\mu}"'] & \dfrac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\dot{L}^{a} \arrow[r, "\dot{\Sigma}^{a}"] \arrow[d, "\dot{\tau}^{a}"'] & \dfrac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}} \arrow[d, "\dot{\tau}^{a}_{\mathcal{R}}"] \\
\pi^{-1}(a) \arrow[r, "\sigma^{a}"'] & \dfrac{\pi^{-1}(a)}{\mathcal{R}^{a}}
\end{tikzcd}
.
$$

4.3.6 The reduced connections and their curvatures

Now that we have demonstrated the quotient map $\dot{\Sigma}^{\mathcal{O}}$ is a submersion, the Foliation Reduction Theorem 2.11.1 guarantees the existence of a reduced 1-form $\alpha^{\mathcal{O}}_{\mathcal{R}} : T\left(\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}\right) \to \mathfrak{u}(1)$ on the quotient space satisfying

$$\alpha^{\mathcal{O}} = (\dot{\Sigma}^{\mathcal{O}})^{*}\alpha^{\mathcal{O}}_{\mathcal{R}}.$$

Proposition 4.3.10. *$\alpha^{\mathcal{O}}_{\mathcal{R}}$ defines a connection on the reduced space $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$.*

Proof. We must verify the two defining conditions of a connection (see Section 3.1.2).

- Since $\alpha^{\mathcal{O}}$ is U(1)-invariant, and $\dot{\Sigma}^{\mathcal{O}}$ is U(1)-equivariant, $\alpha^{\mathcal{O}} = (\dot{\Sigma}^{\mathcal{O}})^{*}\alpha^{\mathcal{O}}_{\mathcal{R}}$ implies that $\alpha^{\mathcal{O}}_{\mathcal{R}}$ is also U(1)-invariant.

- The U(1)-equivariance of $\dot{\Sigma}^{\mathcal{O}}$ implies that $T_p\dot{\Sigma}^{\mathcal{O}}(p \cdot \varepsilon) = \dot{\Sigma}^{\mathcal{O}}(p) \cdot \varepsilon$ for $\varepsilon \in \mathfrak{u}(1)$. From $\alpha^{\mathcal{O}} = (\dot{\Sigma}^{\mathcal{O}})^{*}\alpha^{\mathcal{O}}_{\mathcal{R}}$ it follows that $(\alpha^{\mathcal{O}}_{\mathcal{R}})_q(q \cdot \varepsilon) = \varepsilon$ for any $q \in \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$. Hence $\alpha^{\mathcal{O}}_{\mathcal{R}}$ defines a connection on the U(1)-bundle $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$.

□

The curvature of the reduced connection $\alpha^{\mathcal{O}}_{\mathcal{R}}$ is related to the reduced symplectic form $\omega^{\mathcal{O}}_{\mathcal{R}}$ in the required way.

Proposition 4.3.11. *The curvature of $\alpha^{\mathcal{O}}_{\mathcal{R}}$ is $\frac{\varepsilon_0}{h}\,\omega^{\mathcal{O}}_{\mathcal{R}}$, i.e,*

$$\mathrm{d}\,\alpha^{\mathcal{R}}_{\mathcal{O}} = (\dot{\tau}^{\mathcal{O}}_{\mathcal{R}})^{*}\left(\frac{\varepsilon_0}{h}\,\omega^{\mathcal{O}}_{\mathcal{R}}\right).$$

Proof.

$$
\begin{aligned}
(\dot\Sigma^{\mathcal{O}})^* \mathrm{d}\,\alpha^{\mathcal{R}}_{\mathcal{O}} &= \mathrm{d}\left((\dot\Sigma^{\mathcal{O}})^*\alpha^{\mathcal{O}}_{\mathcal{R}}\right) \\
&= \mathrm{d}\,\alpha^{\mathcal{O}} && \text{using } \alpha^{\mathcal{O}} = (\dot\Sigma^{\mathcal{O}})^*\alpha^{\mathcal{O}}_{\mathcal{R}} \\
&= (\dot\tau^{\mathcal{O}})^*\left(\frac{\varepsilon_0}{h}\,\omega^{\mathcal{O}}\right) && \text{using the fact that the curvature of } \alpha \text{ is } \frac{\varepsilon_0}{h}\,\omega \\
&= \frac{\varepsilon_0}{h}(\dot\tau^{\mathcal{O}})^*\left((\sigma^{\mathcal{O}})^*\omega^{\mathcal{O}}_{\mathcal{R}}\right) && \text{using } \omega^{\mathcal{O}} = (\sigma^{\mathcal{O}})^*\omega^{\mathcal{O}}_{\mathcal{R}} \\
&= \frac{\varepsilon_0}{h}\,(\dot\tau^{\mathcal{O}}_{\mathcal{R}} \circ \dot\Sigma^{\mathcal{O}})^*\omega^{\mathcal{O}}_{\mathcal{R}} && \text{using } \sigma^{\mathcal{O}} \circ \dot\tau^{\mathcal{O}} = \dot\tau^{\mathcal{O}}_{\mathcal{R}} \circ \dot\Sigma^{\mathcal{O}} \\
&= (\dot\Sigma^{\mathcal{O}})^*\left((\dot\tau^{\mathcal{O}}_{\mathcal{R}})^*\left(\frac{\varepsilon_0}{h}\,\omega^{\mathcal{O}}_{\mathcal{R}}\right)\right).
\end{aligned}
$$

Since $\dot\Sigma^{\mathcal{O}}$ is a submersion, this gives the required relation $\mathrm{d}\,\alpha^{\mathcal{O}}_{\mathcal{R}} = (\dot\tau^{\mathcal{O}}_{\mathcal{R}})^*\left(\frac{\varepsilon_0}{h}\,\omega^{\mathcal{O}}_{\mathcal{R}}\right)$. $\square$

The same arguments show that $\alpha^{\mu}_{\mathcal{R}}$ and $\alpha^{a}_{\mathcal{R}}$ are reduced connections on $\frac{\dot L^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ and $\frac{\dot L^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}}$ respectively, and that

$$
\mathrm{d}\,\alpha^{\mu}_{\mathcal{R}} = (\dot\tau^{\mu}_{\mathcal{R}})^*\left(\frac{\varepsilon_0}{h}\,\omega^{\mu}_{\mathcal{R}}\right) \qquad \text{and} \qquad \mathrm{d}\,\alpha^{a}_{\mathcal{R}} = (\dot\tau^{a}_{\mathcal{R}})^*\left(\frac{\varepsilon_0}{h}\,\omega^{a}_{\mathcal{R}}\right).
$$

4.4 The reduced lifted group action

We have seen that the G-action on $\mathrm{J}^{-1}(\mathcal{O})$ induces a reduced G-action on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ (Section 2.13). In a similar manner, the $\widetilde{G}$-action on $\dot L^{\mathcal{O}}$ induces a reduced $\widetilde{G}$-action on $\frac{\dot L^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$. This is a consequence of the fact that

- the $\widetilde{G}$-action on $\dot L^{\mathcal{O}}$ preserves the connection $\alpha^{\mathcal{O}}$, and hence the horizontal distribution on $\dot L^{\mathcal{O}}$;
- the $\widetilde{G}$-action covers the G-action on $\mathrm{J}^{-1}(\mathcal{O})$;
- the G-action on $\mathrm{J}^{-1}(\mathcal{O})$ preserves the foliation $\mathcal{R}^{\mathcal{O}}$.

Combining these three properties, we see that the $\widetilde{G}$-action preserves the lifted foliation $\left(\mathcal{R}^{\mathrm{h}}\right)^{\mathcal{O}}$. It therefore drops to the reduced space $\frac{\dot L^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, and we have an analogous result to Proposition 2.13.1. Denoting the $\widetilde{G}$-action on $\dot L$ by $\dot{\mathbf{\Phi}} : \widetilde{G} \times \dot L \to \dot L$,

Proposition 4.4.1. *There exists a smooth, $\alpha^{\mathcal{O}}_{\mathcal{R}}$-preserving $\widetilde{G}$-action $\dot{\mathbf{\Phi}}^{\mathcal{O}}_{\mathcal{R}}$ on $\left(\frac{\dot L^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}, \alpha^{\mathcal{O}}_{\mathcal{R}}\right)$ making the following diagram commute,*

$$
\begin{tikzcd}
\widetilde{G} \times \dot L^{\mathcal{O}} \arrow[r, "\dot{\mathbf{\Phi}}^{\mathcal{O}}"] \arrow[d, "\mathrm{id}_{\widetilde{G}} \times \dot\Sigma^{\mathcal{O}}"'] & \dot L^{\mathcal{O}} \arrow[d, "\dot\Sigma^{\mathcal{O}}"'] \\
\widetilde{G} \times \dfrac{\dot L^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \arrow[r, "\dot{\mathbf{\Phi}}^{\mathcal{O}}_{\mathcal{R}}"] & \dfrac{\dot L^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}
\end{tikzcd}.
$$

Note. The commutative diagram in Proposition 4.4.1 expresses the fact that $\dot{\Sigma}^{\mathcal{O}}$ is $\widetilde{G}$-equivariant with respect to the unreduced and reduced $\widetilde{G}$-actions on $\dot{L}^{\mathcal{O}}$ and $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ respectively.

Similar results hold for the $\widetilde{G}$-invariant bundle $\dot{L}^{a}$ and its reduced space $\frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}}$.

We also have the analogous result to Proposition 2.10.3.

Proposition 4.4.2. *(i) $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ is an embedded submanifold of $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$.*

(ii) $\frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}}$ is an initial submanifold of $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, and embedded if M is second countable.

Proof. (i) $\dot{L}^{\mu}$ is embedded in M, and $\dot{L}^{\mathcal{O}}$ is initial in M. So Proposition 2.10.1 (iii) implies that $\dot{L}^{\mu}$ is embedded in $\dot{L}^{\mathcal{O}}$. The result now follows from Proposition 2.10.2 (v).

(ii) $\dot{L}^{a}$ is initial in M, embedded if M is second countable, and $\dot{L}^{\mathcal{O}}$ is initial in M. So Proposition 2.10.1 (ii) implies that $\dot{L}^{a}$ is initial in $\dot{L}^{\mathcal{O}}$, and embedded in $\dot{L}^{\mathcal{O}}$ if M is second countable. The result now follows from Proposition 2.10.2 (iv) and (v).

□

We recall that in the case of the reduced G-action on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, G_μ acts trivially on points in $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \subset \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$. The $(\widetilde{G})_\mu$-action on $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$, however, is non-trivial:

Proposition 4.4.3. *For $\widetilde{h} \in (\widetilde{G})_\mu$ and $q \in \frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$,*

$$\widetilde{h} \cdot q = q \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h}).$$

Proof. Let $p \in (\dot{\Sigma}^{\mathcal{O}})^{-1}(q) \subset \dot{L}^{\mathcal{O}}$. Proposition 4.3.6 says that

$$\widetilde{h} \cdot p = h^{\mathrm{h}} \cdot p \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h}),$$

where $h = \pi_{\widetilde{G}\to G}(\widetilde{h})$. Applying $\dot{\Sigma}^{\mathcal{O}}$ to both sides of the above equation, and using the fact that $\dot{\Sigma}^{\mathcal{O}}$ is both U(1)- and $\widetilde{G}$-equivariant, we get

$$\begin{aligned} &\widetilde{h} \cdot \dot{\Sigma}^{\mathcal{O}}(p) = \dot{\Sigma}^{\mathcal{O}}(h^{\mathrm{h}} \cdot p) \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h}) \\ \Longrightarrow\, &\widetilde{h} \cdot q = q \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h}). \end{aligned}$$

□

Chapter 5

Quantization

The space of sections obtained in prequantization is too large to be useful. Roughly, we would like our wavefunctions to depend on a maximal set of Poisson-commuting coordinates in the symplectic manifold (M, ω). This is accomplished by the introduction of a *polarization* F on (M, ω), and by only considering sections of L which are covariantly constant along directions in F. This places restrictions on the class of classical observables which can be quantized via $f \mapsto Q_f$, since such observables must now preserve the space of covariantly constant sections of L.

The novel part of this chapter is contained in Sections 5.4, 5.6, and 5.7, where we construct a polarization on M which is "compatible" with the reduction of the bundle $\dot{L}$ discussed in the previous section. This compatibility allows the polarization to be dropped to the reduced space, and guarantees a one-to-one correspondence between covariantly constant sections of the unreduced and reduced spaces.

5.1 Polarizations

A ***polarization*** F on M is a subbundle of the complexified tangent bundle $TM^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ of M with the following properties:

(i) F is isotropic i.e. $\omega_x(Y_x, Z_x) = 0 \quad \forall Y_x, Z_x \in F_x$.

(ii) $\dim_{\mathbb{C}} F = \frac{1}{2} \dim M$.

(iii) F is involutive, i.e., $[F, F] \subset F$.

(iv) $\dim_{\mathbb{R}}(F \cap TM)$ is constant.

Let D and E be the real distributions such that $D^{\mathbb{C}} = F \cap \overline{F}$ and $E^{\mathbb{C}} = F + \overline{F}$ (conjugation taken with respect to the natural complex structure on $TM^{\mathbb{C}}$). Note that condition (iv) says that $\dim_{\mathbb{R}} D = \text{constant}$, and hence $\dim_{\mathbb{R}} E = \text{constant}$, while condition (iii) implies that D is integrable. In addition to the above conditions, we usually require that:

(v) $F + \overline{F}$ is involutive (and hence E is integrable).

F is said to be ***totally complex*** if $D = \{0\}$. In particular this implies that, $TM^{\mathbb{C}} = F \oplus \overline{F} = E^{\mathbb{C}}$.

5.2 The use of polarizations in quantization

Let F be a polarization on (M, ω), and $(\dot{L}, \alpha)$ a U(1)-bundle over M with curvature $\frac{\omega}{\hbar}\varepsilon_0$, and associated Hermitian line bundle (L, H). We denote the space of sections of L covariantly constant along directions in F by $\Gamma_F(L)$

$$\Gamma_F(L) = \{s \in \Gamma(L) \,|\, \nabla_X\, s = 0 \text{ for all } X \in \Gamma(F)\}.$$

For an observable $f \in C^\infty(M)$, we define the corresponding quantum operator as before

$$Q_f = -i\hbar\nabla_{X_f} + f.$$

Now, however, we require that Q_f preserves the space $\Gamma_F(L)$, i.e., if $\nabla_X s = 0\ \forall X \in \Gamma(F)$, then $\nabla_X(Q_f\, s) = 0\ \forall X \in \Gamma(F)$. Using Corollary 3.1.3 (with $\Omega^\alpha = \frac{\varepsilon_0}{\hbar}\omega$), we have that for any vector field X in $\Gamma(F)$ and section $s \in \Gamma_F(L)$

$$\begin{aligned}
\nabla_X(Q_f s) &= -i\hbar\nabla_X(\nabla_{X_f} s) + \nabla_X(f\, s) \\
&= -i\hbar\left(\nabla_{X_f}(\nabla_X s) + \nabla_{[X,\, X_f]} s + \frac{i}{\hbar}\omega(X,\, X_f)s\right) + (Xf)s + f\nabla_X s \\
&= -i\hbar\nabla_{[X,\, X_f]} s - df(X)s + (Xf)s \\
&= -i\hbar\nabla_{[X,\, X_f]} s.
\end{aligned}$$

The condition that this be zero for all sections s implies that f must satisfy

$$\mathcal{L}_{X_f} X = [X_f,\, X] \in \Gamma(F)$$

for all vector fields $X \in \Gamma(F)$, or equivalently

$$(\phi_f^t)_* F = F,$$

where $\phi_f^t : M \to M$ is the Hamiltonian flow generated by X_f.

5.3 A polarization on coadjoint orbits

In this section we construct a totally complex polarization which exists on the coadjoint orbits of all semisimple compact Lie groups.

5.3.1 The structure of simple Lie algebras

In order to define the above mentioned polarization on the coadjoint orbits of a semisimple Lie group G, a brief review of the structure theory of complex semisimple Lie algebras is useful. A standard reference for the following material is [Hum72].

A ***simple*** Lie algebra is a Lie algebra which possesses no non-trivial ideals. A ***semisimple*** Lie algebra is a Lie algebra which possesses no abelian ideals. It can be shown that a semisimple Lie algebra can be written as the (Lie algebra) direct sum of simple Lie algebras, so it entails no loss of generality to restrict our attention in this subsection to simple Lie algebras. The group G is called simple if $\mathfrak{g}$ is simple, and similarly for semisimple.

Let $\mathfrak{k}$ be a complex simple Lie algebra. A ***Cartan subalgebra*** $\mathfrak{h}$ of $\mathfrak{k}$ is a nilpotent subalgebra of $\mathfrak{k}$ which equals its normalizer in $\mathfrak{k}$. For semisimple (and hence simple) $\mathfrak{k}$, this characterization is equivalent to saying that $\mathfrak{h}$ is a maximal commutative subspace of $\mathfrak{k}$ such that $\mathrm{ad}_\eta : \mathfrak{k} \to \mathfrak{k}$ is diagonalizable for all $\eta \in \mathfrak{h}$.

Define the ***Killing form*** $\kappa : \mathfrak{k} \times \mathfrak{k} \to \mathbb{C}$ by

$$\kappa(\xi, \zeta) = \mathrm{Trace}_{\mathfrak{k}}\,(\mathrm{ad}_\xi \circ \mathrm{ad}_\zeta).$$

The homomorphism property of ad. gives that

$$\kappa(\mathrm{ad}_\xi(\eta), \zeta) = -\kappa(\eta, \mathrm{ad}_\xi(\zeta)).$$

Since ad_η is diagonalizable for each $\eta \in \mathfrak{h}$, and $[\mathrm{ad}_{\eta_1}, \mathrm{ad}_{\eta_2}] = \mathrm{ad}_{[\eta_1,\eta_2]} = 0\ \forall \eta_1, \eta_2 \in \mathfrak{h}$, we see that $\mathfrak{k}$ can be decomposed into simultaneous eigenspaces of the adjoint actions $\mathrm{ad}_\eta : \mathfrak{k} \to \mathfrak{k},\ \eta \in \mathfrak{h}$. For a vector in such a simultaneous eigenspace we can write

$$\mathrm{ad}_\eta(\xi) = \alpha(\eta)\,\xi$$

for some $\alpha \in \mathfrak{h}^*$. The nonzero αs are called the ***roots*** of the Lie algebra, and the collection of roots is denoted Δ. Notice that the eigenspace corresponding to $\alpha = 0$ is just $\mathfrak{h}$ itself. Then $\mathfrak{k}$ has the following ***root space decomposition***

$$\mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{j}_\alpha,$$

where $\mathfrak{j}_\alpha$ is the α-eigenspace.

The following facts are standard.

- $\dim_{\mathbb{C}} \mathfrak{j}_\alpha = 1 \quad \text{for all } \alpha \in \Delta$.
- $\operatorname{span}_{\mathbb{C}} \Delta = \mathfrak{h}^*$.
- κ and $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ are nondegenerate.
- $\kappa(\mathfrak{h}, \mathfrak{j}_\alpha) = 0$, and $\kappa(\mathfrak{j}_\alpha, \mathfrak{j}_\beta) = 0$ unless $\alpha = -\beta$.

Since κ and $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ are nondegenerate, they define isomorphisms $\sharp : \mathfrak{k}^* \to \mathfrak{k}$ and $\sharp_{\mathfrak{h}} : \mathfrak{h}^* \to \mathfrak{h}$ via

$$\kappa(\nu^\sharp, \xi) = \nu(\xi) \quad \forall \xi \in \mathfrak{k}$$

and

$$\kappa(\alpha^{\sharp_{\mathfrak{h}}}, \eta) = \alpha(\eta) \quad \forall \eta \in \mathfrak{h}.$$

$\sharp_{\mathfrak{h}}$ induces an bilinear form $(\cdot,\cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$, given by

$$(\alpha, \beta) = \kappa(\alpha^{\sharp_{\mathfrak{h}}}, \beta^{\sharp_{\mathfrak{h}}}).$$

$(\cdot,\cdot)$ is real-valued and positive definite on $\operatorname{span}_{\mathbb{R}} \Delta \subset \mathfrak{h}^*$, and so defines an inner product on this space.

Let $\text{ad-stab}_{\mathfrak{k}}(\zeta)$ denote the stabilizer of $\zeta \in \mathfrak{k}$ under the adjoint action,

$$\text{ad-stab}_{\mathfrak{k}}(\zeta) = \{\, \xi \in \mathfrak{k} \mid \mathrm{ad}_\xi(\zeta) = 0 \,\}$$

(also called the centralizer of ζ in $\mathfrak{k}$), and $\text{ad}^*\text{-stab}_{\mathfrak{k}}(\mu)$ denote the stabilizer of $\mu \in \mathfrak{k}^*$ under the coadjoint action,

$$\text{ad}^*\text{-stab}_{\mathfrak{k}}(\mu) = \{\, \xi \in \mathfrak{k} \mid -\mathrm{ad}^*_\xi(\mu) = 0 \,\}.$$

Given $\alpha \in \mathfrak{k}^*$, the easily proved identity

$$-\mathrm{ad}_{\alpha^\sharp} \xi = (-\mathrm{ad}^*_\xi \alpha)^\sharp$$

makes it clear that

$$\text{ad-stab}_{\mathfrak{k}}(\alpha^\sharp) = \text{ad}^*\text{-stab}_{\mathfrak{k}}(\alpha).$$

5.3.2 Construction of the polarization

Now restrict to the case where G is compact and semisimple (so G has a discrete, hence finite, center). Since G is compact, there exist Ad-invariant inner products on $\mathfrak{g}$ (e.g., let $\langle\cdot,\cdot\rangle$ be an arbitrary inner product, and define $\langle\langle\cdot,\cdot\rangle\rangle = \int_G \langle \mathrm{Ad}_g(\cdot), \mathrm{Ad}_g(\cdot)\rangle dg$, where dg is the Haar measure on G). Therefore ad_ξ, $\xi \in \mathfrak{g}$, is skew-symmetric with respect to some inner product on $\mathfrak{g}$, and so can be represented in an orthogonal basis by a skew-symmetric matrix A_ξ. It follows that the Killing form is negative-definite on $\mathfrak{g}$, since

$$-\kappa(\xi,\xi) = -\mathrm{Tr}_{\mathfrak{g}}\left(\mathrm{ad}_\xi \circ \mathrm{ad}_\xi\right) = -\mathrm{Tr}(A_\xi A_\xi) = \mathrm{Tr}(A_\xi^t A_\xi) = \sum_{ij}((A_\xi)_{ij})^2 \geq 0 \qquad \text{for all } \xi \in \mathfrak{t},$$

and in fact $-\kappa$ provides an example of an Ad-invariant inner product on $\mathfrak{g}$.

Let $\mathfrak{g}^{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. The bilinear inner product $-\kappa$ on $\mathfrak{g}$ induces a Hermitian inner product on $\mathfrak{g}^{\mathbb{C}}$ in the usual way. $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ extends to a skew-Hermitian linear map on $\mathfrak{g}^{\mathbb{C}}$. It follows that $\mathrm{ad}_\xi : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, $\xi \in \mathfrak{g}$, is diagonalizable, with pure imaginary eigenvalues.

Let $\mu \in \mathfrak{g}^*$, and $G_\mu \subset G$ be the stabilizer group of μ under the coadjoint action, defined as

$$G_\mu = \{\, g \in G \mid \mathrm{Ad}^*_{g^{-1}}\mu = 0\}.$$

Denote the Lie algebra of G_μ by $\mathfrak{g}_\mu$. Explicitly, $\mathfrak{g}_\mu$ is given by

$$\mathfrak{g}_\mu = \{\, \xi \in \mathfrak{g} \,|\, -\mathrm{ad}^*_\xi \mu = 0\}.$$

As mentioned above, $-\mathrm{ad}_{\mu^\sharp} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ has pure imaginary eigenvalues. The result from the previous section

$$\text{ad-stab}_{\mathfrak{g}^{\mathbb{C}}}(\mu^\sharp) = \text{ad}^*\text{-stab}_{\mathfrak{g}^{\mathbb{C}}}(\mu)$$

tells us that the 0-eigenspace is just $\mathfrak{g}_\mu^{\mathbb{C}}$. Let $\mathfrak{n}_\mu^+$ be the (direct) sum of the eigenspaces corresponding to eigenvalues along the strictly positive imaginary axis, and $\mathfrak{n}_\mu^-$ likewise for the strictly negative axis. We have the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{n}_\mu^- \oplus \mathfrak{g}_\mu^{\mathbb{C}} \oplus \mathfrak{n}_\mu^+.$$

The Jacobi identity implies that

$$-\mathrm{ad}_{\mu^\#}[\xi,\,\zeta] = [-\mathrm{ad}_{\mu^\#}\xi,\,\zeta] + [\xi, -\mathrm{ad}_{\mu^\#}\zeta].$$

From this it is easily seen that both $\mathfrak{n}_\mu^-$ and $\mathfrak{n}_\mu^+$ are subalgebras of $\mathfrak{g}^{\mathbb{C}}$, as are the ***parabolic subalgebras*** $\mathfrak{p}_\mu^- = \mathfrak{g}_\mu^{\mathbb{C}} \oplus \mathfrak{n}_\mu^-$ and $\mathfrak{p}_\mu^+ = \mathfrak{g}_\mu^{\mathbb{C}} \oplus \mathfrak{n}_\mu^+$ associated to $\mu \in \mathfrak{g}^*$.

The polarization on $\mathcal{O}$ at μ is then given by

$$P_\mu = \{-\mathrm{ad}^*_\xi \mu \,|\, \xi \in \mathfrak{n}_\mu^+\}.$$

Clearly $\dim_{\mathbb{C}} P = \frac{1}{2}\dim_{\mathbb{R}} T\mathcal{O}$ and $P \cap T\mathcal{O} = \{0\} \implies \dim_{\mathbb{R}}(P \cap T\mathcal{O}) = 0 = \text{constant}$.

Proposition 5.3.1. *P is Ad^*-invariant, smooth, and isotropic on $(\mathcal{O}, \omega_{\mathcal{O}})$.*

Proof.

- **Ad*-invariance.**

  If $\xi \in \mathfrak{n}_\mu^+$, it is easily shown that $\mathrm{Ad}_g\xi \in \mathfrak{n}^+_{\mathrm{Ad}^*_{g^{-1}}\mu}$ (with the same eigenvalue as $\xi \in \mathfrak{n}_\mu^+$), and so

  $$-\mathrm{ad}^*_{\mathrm{Ad}_g\xi}(\mathrm{Ad}^*_{g^{-1}}\mu) \in P_{\mathrm{Ad}^*_{g^{-1}}\mu}.$$

  But $-\mathrm{ad}^*_{\mathrm{Ad}_g\xi}(\mathrm{Ad}^*_{g^{-1}}\mu) = T_\mu \mathrm{Ad}^*_{g^{-1}}(-\mathrm{ad}^*_\xi\mu) \simeq \mathrm{Ad}^*_{g^{-1}}(-\mathrm{ad}^*_\xi\mu)$, demonstrating that P is Ad^*-invariant.

- **smoothness on $\mathcal{O}$.**

  Ad^* is smooth on $\mathfrak{g}^*$, and hence on $\mathcal{O}$, since $\mathcal{O}$ is an initial submanifold. The result then follows from the Ad^*-invariance of P.

- **isotropy.**

  Let $\xi \in \mathfrak{n}_\mu^+$, and apply μ to both sides of the eigenvector equation

  $$-\mathrm{ad}_{\mu^\sharp}\xi = i\lambda\,\xi, \qquad \lambda \in (0, \infty).$$

  We get

  $$i\lambda\,\mu(\xi) = \mu(-\mathrm{ad}_{\mu^\sharp}\,\xi) = \kappa(\mu^\sharp, -\mathrm{ad}_{\mu^\sharp}\,\xi) = \kappa(\mathrm{ad}_{\mu^\sharp}(\mu^\sharp), \xi) = 0.$$

  Since $\lambda \neq 0$ this tells us that $\mu|_{\mathfrak{n}_\mu^+} = 0$. It follows that for $\xi, \zeta \in \mathfrak{n}_\mu^+$,

  $$(\omega_{\mathcal{O}})_\mu(-\mathrm{ad}^*_\xi\mu, -\mathrm{ad}^*_\zeta\mu) = \mu([\xi, \zeta]) = 0,$$

  since $[\xi, \zeta] \in \mathfrak{n}_\mu^+$. Hence P is isotropic.

$\square$

The only property of a polarization left to demonstrate is involutivity. To do this, we will need the following lemma.

Lemma 5.3.2. *Let $\pi : N \to B$ a submersion. Let S be a complex distribution on B, and define $R = (T\pi)^{-1}(S)$. Then S is involutive if and only if R is involutive.*

Proof. Suppose S is involutive. For $p_0 \in N$, pick a section $s : U \to N$ about $\pi(p_0)$ and a basis of vector fields X^i for R along $s(U)$. Define the (linearly dependent) vector fields Y^i on U by

$$Y^i(b) = T_{s(p)}\pi\left(X^i(s(p))\right).$$

As a consequence of the Local Onto Theorem ([AMR88, Theorem 3.5.2]), the X^i can be extended to a linearly independent set in a neighborhood of $p_0 \in s(U)$ which are π-related to the Y^i, and hence form a basis of R. It follows that $[X^i, X^j]$ and $[Y^i, Y^j]$ are π-related, i.e.,

$$T_p\pi\left([X^i, X^j]_p\right) = [Y^i, Y^j]_{\pi(p)}.$$

Since S is involutive, $[Y^i, Y^j]_{\pi(p)} \in S_{\pi(p)}$, so from $R = (T\pi)^{-1}(S)$ we see that $[X^i, X^j]_p \in R_p$. Since the space of vector fields in R on $\pi^{-1}(U)$ form a $C^\infty(\pi^{-1}(U))$-module, with basis X^i, it follows from properties of the Jacobi-Lie bracket that R is involutive.

Conversely, suppose R is involutive, and let Y, Y' be vector fields in S on some neighborhood of B. For similar reasons to above, Y and Y' can be lifted to vector fields X and X' in a neighborhood of N in such a way that $X \sim_\pi Y$ and $X' \sim_\pi Y'$. It follows that

$$[X, X'] \sim_\pi [Y, Y'].$$

Since $R = (T\pi)^{-1}(S)$, X and X' are sections of R, and so $[X, X'] \in \Gamma(R)$ since R is involutive. Hence $[Y, Y'] \in \Gamma(S) = \Gamma(T\pi(R))$. So S is involutive.

$\square$

Proposition 5.3.3. *P is involutive.*

Proof. For $a \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, $\mathrm{J}^a : \pi^{-1}(a) \to \mathcal{O}$ is a submersion. Take $x_0 \in \pi^{-1}(a)$ such that $\mathrm{J}(x_0) = \mu$. The fiber of J^a through and arbitrary point $g \cdot x_0$ of $\pi^{-1}(a)$ is $G_{\mathrm{J}(g\cdot x_0)} \cdot g \cdot x_0 = G_{\mathrm{Ad}^*_{g^{-1}}\mu} \cdot g \cdot x_0 = g \cdot G_\mu \cdot x_0$. Recalling that $P_\nu = \{-\mathrm{ad}^*_\xi \nu \,|\, \xi \in \mathfrak{n}^+_\nu\}$, it is clear that

$$(T_{g\cdot x_0}\mathrm{J}^a)^{-1}(P_{\mathrm{J}(g\cdot x_0)}) = \mathfrak{p}^+_{\mathrm{J}(g\cdot x_0)} \cdot (g \cdot x_0) = g \cdot \mathfrak{p}^+_\mu \cdot x_0,$$

where $\mathfrak{p}^+_\nu = \mathfrak{g}^{\mathbb{C}}_\nu \oplus \mathfrak{n}^+_\nu$ is the positive parabolic subalgebra of $\mathfrak{g}$ associated to $\nu \in \mathfrak{g}^*$. This lifted polarization is spanned by global vector fields of the form $Y^\xi(g \cdot x_0) = g \cdot \xi \cdot x_0$, $\xi \in \mathfrak{p}^+_\mu$. Since $[Y^\xi, Y^\zeta] = Y^{[\xi,\zeta]}$, and $\mathfrak{p}^+_\mu$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$, it is integrable. It follows from Lemma 5.3.2 that P is involutive.

$\square$

5.3.3 Connection with the root space decomposition

The eigenvectors of $-\mathrm{ad}_{\mu^\sharp}$ may be described in terms of a root space decomposition of $\mathfrak{g}^{\mathbb{C}}$, giving an alternative construction of P. We explain the connection for the case where $\mathfrak{g}$ is simple. The semisimple case is easily extrapolated by decomposing $\mathfrak{g}$ into its simple ideals.

If T is a ***maximal torus*** of G (i.e., a maximal commutative connected subgroup of G), with corresponding Lie algebra $\mathfrak{t}$, then $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k} = \mathfrak{g}^{\mathbb{C}}$, and the corresponding root space decomposition is

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{j}_\alpha.$$

The fact that the eigenvalues of ad_η, $\eta \in \mathfrak{t}$, are pure imaginary means that the roots Δ take pure imaginary values on $\mathfrak{t}$. It is a standard theorem that every element of G is contained in some maximal torus T, implying that every element of $\mathfrak{g}$ is contained in some maximal commutative (real) subalgebra $\mathfrak{t}$.

For a point μ in the coadjoint orbit $\mathcal{O} \subset \mathfrak{g}$, extend μ linearly to an element of $(\mathfrak{g}^{\mathbb{C}})^*$. Let $\mathfrak{t}$ be a maximal commutative subalgebra containing $\mu^\sharp$ (note that $\mu^\sharp \in \mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$, since μ is real valued on $\mathfrak{g}$). By the general theory in Section 5.3.1, $\frac{i}{\hbar}\mu|_{\mathfrak{t}^{\mathbb{C}}} \in \mathrm{span}_{\mathbb{R}}\Delta$.

In the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$, $-\mathrm{ad}_{\mu^\sharp}$ acts as

$$-\mathrm{ad}_{\mu^\sharp}(\xi) = \begin{cases} 0 & \xi \in \mathfrak{t}^{\mathbb{C}} \\ -\alpha(\mu^\sharp)\,\xi = i\hbar\,\left(\frac{i}{\hbar}\mu, \alpha\right)\,\xi & \xi \in \mathfrak{j}_\alpha \end{cases},$$

where for simplicity, we write $\frac{i}{\hbar}\mu$ instead of $\frac{i}{\hbar}\mu|_{\mathfrak{t}^{\mathbb{C}}}$ in the second case. Hence the zero eigenspace, which we recall is just $\mathfrak{g}^{\mathbb{C}}_\mu$, is given by

$$\mathfrak{g}^{\mathbb{C}}_\mu = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\left(\frac{i}{\hbar}\mu, \alpha\right)=0} \mathfrak{j}_\alpha,$$

while the negative and positive eigenspaces $\mathfrak{n}^-_\mu$ and $\mathfrak{n}^+_\mu$ defined in the previous section are given by

$$\mathfrak{n}^-_\mu = \bigoplus_{\left(\frac{i}{\hbar}\mu, \alpha\right)<0} \mathfrak{j}_\alpha \qquad \text{and} \qquad \mathfrak{n}^+_\mu = \bigoplus_{\left(\frac{i}{\hbar}\mu, \alpha\right)>0} \mathfrak{j}_\alpha.$$

The form $\frac{i}{\hbar}\mu$ defines a set of positive roots

$$\Delta^+ = \left\{ \alpha \in \Delta \,\middle|\, \left(\frac{i}{\hbar}\mu,\, \alpha\right) \geq 0 \right\},$$

with respect to which it is dominant.

5.4 A polarization compatible with foliation reduction

We now describe how induce a *generalized* polarization F on M which is compatible with foliation reduction, and allows reduction of the covariantly constant sections on $\dot{L}^{\mathcal{O}}$ to the reduced bundle $\dot{L}^{\mathcal{O}}_{\mathcal{R}}$

when $\mathcal{O}$ is admissible. F is a generalized polarization in the sense that it is not necessarily smooth, nor does it necessarily satisfy $\dim_{\mathbb{R}}(TM \cap F) =$ constant. However it does obey these properties when restricted to each $\mathrm{J}^{-1}(\mathcal{O})$, which is sufficient for later purposes.

For a given coadjoint orbit $\mathcal{O}$, let $Q_{\mathrm{J}^{-1}(\mathcal{O})/G}$ be *any* polarization on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, and let P be the polarization on $\mathcal{O}$ described in the previous section. We recall the definition of $\phi^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$,

$$\phi^{\mathcal{O}} = (\mathrm{J}^{\mathcal{O}} \times \pi^{\mathcal{O}}) \circ \Delta^{\mathcal{O}},$$

where $\Delta^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathrm{J}^{-1}(\mathcal{O}) \times \mathrm{J}^{-1}(\mathcal{O})$ is the diagonal map $x \mapsto (x, x)$. We define the a polarization $F^{\mathcal{O}}$ on $\mathrm{J}^{-1}(\mathcal{O})$ by

$$F^{\mathcal{O}}_x = (T_x\phi^{\mathcal{O}})^{-1}(P_{\mathrm{J}(x)} \oplus Q_{\pi(x)}) \qquad \text{for } x \in \mathrm{J}^{-1}(\mathcal{O}).$$

Note that $F^{\mathcal{O}}_x \subset T_x(\mathrm{J}^{-1}(\mathcal{O}))^{\mathbb{C}}$ by its definition.

By picking a polarization Q on each symplectic manifold $\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ and constructing $F^{\mathcal{O}}$ on each initial submanifold $\mathrm{J}^{-1}(\mathcal{O})$, we build up a generalized distribution F on the entire symplectic manifold M.

Proposition 5.4.1. *F is a generalized G-invariant polarization on M.*

Proof.

- **F is smooth and involutive on $\mathrm{J}^{-1}(\mathcal{O})$.**

  This follows from the fact that $P \oplus Q$ is smooth and involutive on $\mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, the fact that $\phi^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ is a submersion, and Lemma 5.3.2.

- **F is isotropic.**

  Let $X_x, X'_x \in F_x$. Then $T_x\mathrm{J}^{\mathcal{O}}(X_x), T_x\mathrm{J}^{\mathcal{O}}(X'_x) \in P_{\mathrm{J}(x)}$, so

  $$\left(\omega_{\mathrm{J}^{-1}(\mathcal{O})/G}\right)_{\mathrm{J}(x)} (T_x\mathrm{J}^{\mathcal{O}}(X_x), T_x\mathrm{J}^{\mathcal{O}}(X'_x)) = 0,$$

  while $T_x\pi^{\mathcal{O}}(X_x), T_x\pi^{\mathcal{O}}(X'_x) \in Q_{\pi(x)}$, so

  $$(\omega_{\mathcal{O}})_{\pi(x)}(T_x\pi^{\mathcal{O}}(X_x), T_x\pi^{\mathcal{O}}(X'_x)) = 0.$$

  The result then follows from $(i^{\mathcal{O}})^*\omega = (\mathrm{J}^{\mathcal{O}})^*\omega_{\mathcal{O}} + (\pi^{\mathcal{O}})^*\omega_{\mathrm{J}^{-1}(\mathcal{O})/G}$ (Proposition 2.15.1).

- **$\dim_{\mathbb{C}} F = \frac{1}{2}\dim_{\mathbb{R}} M$.**

  Let $\dim_{\mathbb{R}}M = 2n,\ \dim_{\mathbb{R}}G = g,\ \dim_{\mathbb{R}}G_\mu = g_\mu$. Since $\phi^{\mathcal{O}}$ defines a G_μ-fibration,

  $$\begin{aligned}\dim_{\mathbb{C}}F &= \dim_{\mathbb{C}}P + \dim_{\mathbb{C}}Q + \dim_{\mathbb{R}}G_\mu \\ &= \frac{1}{2}\dim_{\mathbb{R}}\mathcal{O} + \frac{1}{2}\dim_{\mathbb{R}}\frac{\mathrm{J}^{-1}(\mathcal{O})}{G} + \dim_{\mathbb{R}}G_\mu \\ &= \frac{1}{2}(g - g_\mu) + \frac{1}{2}(2n - g_\mu - g) + g_\mu\end{aligned}$$

$$= n$$
$$= \frac{1}{2}\dim_{\mathbb{R}} M.$$

- ***F* is *G*-invariant**

  Let $X_x \in F_x$. Then $T_x \mathrm{J}^{\mathcal{O}}(X_x) \in P_{\mathrm{J}(x)}$, so

$$\begin{aligned} T_{g\cdot x}\mathrm{J}^{\mathcal{O}}(g\cdot X_x) &= T_{g\cdot x}\mathrm{J}^{\mathcal{O}}\left(T_x(\Phi^{\mathcal{O}})_g(X_x)\right) = T_x(\mathrm{J}^{\mathcal{O}}\circ(\Phi^{\mathcal{O}})_g)(X_x) \\ &= T_x(\mathrm{Ad}^*_{g^{-1}}\circ \mathrm{J}^{\mathcal{O}})(X_x) = T_{\mathrm{J}(x)}\mathrm{Ad}_{g^{-1}}(T_x\mathrm{J}^{\mathcal{O}}(X_x)) \\ &\in T_{\mathrm{J}(x)}\mathrm{Ad}_{g^{-1}}\left(P_{\mathrm{J}(x)}\right) = P_{\mathrm{Ad}^*_{g^{-1}}\mathrm{J}(x)} = P_{\mathrm{J}(g\cdot x)}. \end{aligned}$$

  Also $T_x\pi^{\mathcal{O}}(X_x) \in Q_{\pi(x)}$, so

$$\begin{aligned} T_{g\cdot x}\pi^{\mathcal{O}}(g\cdot X_x) &= T_{g\cdot x}\pi^{\mathcal{O}}\left(T_x(\Phi^{\mathcal{O}})_g(X_x)\right) = T_x(\pi^{\mathcal{O}}\circ(\Phi^{\mathcal{O}})_g)(X_x) \\ &= T_x\pi^{\mathcal{O}}(X_x) \in Q_{\pi(x)} = Q_{\pi(g\cdot x)}. \end{aligned}$$

  Overall,

$$T_{g\cdot x}\phi^{\mathcal{O}}(g\cdot X_x) = (T_{g\cdot x}\mathrm{J}^{\mathcal{O}}(g\cdot X_x),\, T_{g\cdot x}\pi^{\mathcal{O}}(g\cdot X_x)) \in P_{\mathrm{J}(g\cdot x)} \oplus Q_{\pi(g\cdot x)}.$$

  It follows that $g\cdot X_x \in F_{g\cdot x} = (T_{g\cdot x}\phi^{\mathcal{O}})^{-1}\left(P_{\mathrm{J}(g\cdot x)} \oplus Q_{\pi(g\cdot x)}\right)$, from which we conclude that $g\cdot F_x = F_{g\cdot x}$.

$\square$

For $a \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, the *intersection* of $F^{\mathcal{O}}$ with $T(\pi^{-1}(a))^{\mathbb{C}}$ will be denoted F^a,

$$F^a_x := F_x \cap T_x(\pi^{-1}(a))^{\mathbb{C}} \qquad \text{for all } x \in \pi^{-1}(a),$$

while for $\mu \in \mathcal{O}$, its intersection with $T(\mathrm{J}^{-1}(\mu))^{\mathbb{C}}$ will be denoted F^μ,

$$F^\mu_x := F_x \cap T_x(\mathrm{J}^{-1}(\mu))^{\mathbb{C}} \qquad \text{for all } x \in \mathrm{J}^{-1}(\mu).$$

From $F = (T\phi^{\mathcal{O}})^{-1}(P\oplus Q)$ and the definition of $\phi^{\mathcal{O}}$ it is not difficult to show that

$$F^a = (T\mathrm{J}^a)^{-1}(P) \qquad \text{and} \qquad F^\mu = (T\pi^\mu)^{-1}(Q).$$

Proposition 5.4.2. *$R^{\mathbb{C}} \subset F$, where $R = T\mathcal{R}$ is the tangent distribution to the generalized foliation $\mathcal{R}$, and $R^{\mathbb{C}}$ denotes its complexification.*

Proof. As previously discussed, the fibers of $\phi^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ are the foliation $\mathcal{R}$. Since $F = (T\phi^{\mathcal{O}})^{-1}(P \oplus Q) \subset T(\mathrm{J}^{-1}(\mathcal{O}))^{\mathbb{C}}$, this implies that $R^{\mathbb{C}} \subset F$. $\square$

Combining $F^a = (T\mathrm{J}^a)^{-1}(P)$, the definition of P, the equivariance of J, and the previous theorem gives the following explicit expression for F^a:

$$F_x^a = (\mathfrak{g}_{\mathrm{J}(x)}^{\mathbb{C}} \oplus \mathfrak{n}_{\mathrm{J}(x)}^{+}) \cdot x = \mathfrak{p}_{\mathrm{J}(x)}^{+} \cdot x.$$

5.5 Admissibility and covariantly constant sections

Proposition 5.5.1. *For $\dot{L}^{\mathcal{O}}$ to support sections which are covariantly constant along $F^{\mathcal{O}}$, $\mathcal{O}$ must be admissible.*

Proof. Let $s \in \Gamma_{F^{\mathcal{O}}}(L^{\mathcal{O}})$, with $\dot{s} \in C^{\infty}_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^{\mathcal{O}}, \mathbb{C})$ its corresponding U(1)-equivariant function. Since $R \subset F$, it follows that s must be covariantly constant along the leaves of the foliation $\mathcal{R}^{\mathcal{O}}$. So over each $\mathrm{J}^{-1}(\mu) \subset \mathrm{J}^{-1}(\mathcal{O})$,

$$\nabla_{\xi_M} s = 0 \quad \text{for all } \xi \in \mathfrak{g}_\mu \ ,$$

or equivalently,

$$\xi_M^{\mathrm{h}}\, \dot{s} = 0 \quad \text{for all } \xi \in \mathfrak{g}_\mu \ .$$

The latter condition implies that $\dot{s}$ is constant along the leaves of $(\mathcal{R}^{\mathrm{h}})^{\mu}$. Suppose such a leaf intersected a U(1)-fiber of $\dot{L}^{\mu}$ at two distinct points, $p \neq p' \implies p' = p \cdot w$ for some $w \in \mathrm{U}(1)$ with $w \neq 1$. Since $\dot{s}$ is constant along the submanifold, we have that

$$\dot{s}(p) = \dot{s}(p') = \dot{s}(p \cdot w) = w^{-1}\dot{s}(p),$$

the latter equality a consequence of the U(1)-equivariance of $\dot{s}$. Since $w \neq 1$, the only way this is possible is if $\dot{s}$ is zero along the submanifold. Therefore for each $\mu \in \mathcal{O}$, the leaves of $(\mathcal{R}^{\mathrm{h}})^{\mu}$ must intersect each U(1)-fiber at most once. By definition this means that $\mathcal{O}$ is admissible. $\square$

The admissible coadjoint orbits are quantized. Hence the relevant representation space for quantization is

$$\bigoplus_{\mathcal{O} \text{ admissible}} \Gamma_{F^{\mathcal{O}}}\left(L^{\mathcal{O}}\right).$$

5.6 The reduced polarizations

The final structures required for geometric quantization of the reduced spaces are the polarizations. Define the distributions

$$F_{\mathcal{R}}^{\mathcal{O}} := T\sigma^{\mathcal{O}}\left(F^{\mathcal{O}}\right),$$
$$F_{\mathcal{R}}^{\mu} := T\sigma^{\mu}\left(F^{\mu}\right),$$
$$F_{\mathcal{R}}^{a} := T\sigma^{a}\left(F^{a}\right).$$

Lemma 5.6.1. *(i) $F_{\mathcal{R}}^{\mathcal{O}}$ is a polarization on $\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}, \omega_{\mathcal{R}}^{\mathcal{O}}\right)$.*

(ii) $F_{\mathcal{R}}^{\mu}$ is a polarization on $\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}, \omega_{\mathcal{R}}^{\mu}\right)$.

(iii) $F_{\mathcal{R}}^{a}$ is a polarization on $\left(\frac{\pi^{-1}(a)}{\mathcal{R}^{a}}, \omega_{\mathcal{R}}^{a}\right)$.

Proof. We will just prove part (i), the other parts being similar.

- **Smoothness:** follows from the smoothness of $F^{\mathcal{O}}$ and the fact that $\sigma^{\mathcal{O}}$ is a submersion.

- **Isotropy:** follows from isotropy of $F^{\mathcal{O}}$ and the identity $\omega^{\mathcal{O}} = (\sigma^{\mathcal{O}})^{*}\omega_{\mathcal{R}}^{\mathcal{O}}$.

- **Involutivity:** Since the tangent spaces to the fibers of $\sigma^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathrm{J}^{-1}(\mathcal{O})/\mathcal{R}^{\mathcal{O}}$ are contained in $F^{\mathcal{O}}$, we have that $F^{\mathcal{O}} = [T\sigma^{\mathcal{O}}]^{-1}F_{\mathcal{R}}^{\mathcal{O}}$. Then involutivity of $F_{\mathcal{R}}^{\mathcal{O}}$ follows from involutivity of $F^{\mathcal{O}}$ and Lemma 5.3.2.

- **Dimensionality:** let $\dim_{\mathbb{R}} M = 2n$ and $g_{\mu} = \dim_{\mathbb{R}} G_{\mu}$ for any $\mu \in \mathcal{O}$. By the Transversal Mapping Theorem 2.8.1, $\operatorname{codim}_{\mathbb{R}} \mathrm{J}^{-1}(\mathcal{O}) = \operatorname{codim}_{\mathbb{R}} \mathcal{O} = g_{\mu}$. Hence $\dim_{\mathbb{R}} \mathrm{J}^{-1}(\mathcal{O}) = 2n - g_{\mu}$, and $\dim_{\mathbb{R}} \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} = 2n - 2g_{\mu}$. On the other hand, $\dim_{\mathbb{C}} F^{\mathcal{O}} = \frac{1}{2}\dim_{\mathbb{R}} M = n$. Since the tangent space to the fibers of $\sigma^{\mathcal{O}}$ are contained in $F^{\mathcal{O}}$, $\dim_{\mathbb{C}} F^{\mathcal{O}} = n - g_{\mu} = \frac{1}{2}\dim_{\mathbb{R}} \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, as required.

Parts (ii) and (iii) follow similarly, using $\dim_{\mathbb{C}} F^{\mu} = g_{\mu} + \frac{1}{2}\dim_{\mathbb{R}} \frac{\mathrm{J}^{-1}(\mathcal{O})}{G} = n - \frac{1}{2}(g - g_{\mu})$ and $\dim_{\mathbb{C}} F^{a} = g_{\mu} + \frac{1}{2}\dim_{\mathbb{R}} \mathcal{O} = \frac{1}{2}(g + g_{\mu})$, where $g = \dim_{\mathbb{R}} G$. $\square$

Combining the result $F_x^a = (\mathfrak{g}_{\mathrm{J}(x)}^{\mathbb{C}} \oplus \mathfrak{n}_{\mathrm{J}(x)}^{+}) \cdot x$ of Section 5.4 and the definition $F_{\mathcal{R}}^{a} = T\sigma^{a}(F^{a})$ gives the following explicit expression for $F_{\mathcal{R}}^{a}$:

$$(F_{\mathcal{R}}^{a})_{\mathcal{R}^{(\mu,a)}} = \mathfrak{n}_{\mu}^{+} \cdot \mathcal{R}^{(\mu,a)}.$$

5.7 The relationship between covariant sections on the reduced and unreduced bundles

Let us employ the compact notation $\dot{L}^{\mathcal{O}}_{\mathcal{R}}$ for $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, and denote the line bundle associated to $\dot{L}^{\mathcal{O}}_{\mathcal{R}}$ by $L^{\mathcal{O}}_{\mathcal{R}}$. The connection $\alpha^{\mathcal{O}}_{\mathcal{R}}$ on $\dot{L}^{\mathcal{O}}_{\mathcal{R}}$ induces a covariant derivative $\nabla^{\mathcal{O}}_{\mathcal{R}}$ on $L^{\mathcal{O}}_{\mathcal{R}}$.

The set of sections of $L^{\mathcal{O}}$ which are covariantly constant with respect to $F^{\mathcal{O}}$ is denoted $\Gamma_{F^{\mathcal{O}}}(L^{\mathcal{O}})$,

$$\Gamma_{F^{\mathcal{O}}}(L^{\mathcal{O}}) = \left\{ s \in \Gamma(L^{\mathcal{O}}) \,|\, (\nabla^{\mathcal{O}})_X s = 0 \text{ for all } X \in \Gamma\left(F^{\mathcal{O}}\right) \right\},$$

and the covariantly constant sections on the reduced bundle are denoted likewise,

$$\Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}}) = \left\{ t \in \Gamma(L^{\mathcal{O}}_{\mathcal{R}}) \,|\, (\nabla^{\mathcal{O}}_{\mathcal{R}})_Y t = 0 \text{ for all } Y \in \Gamma\left(F^{\mathcal{O}}_{\mathcal{R}}\right) \right\}.$$

Proposition 5.7.1. *$\Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}})$ is canonically isomorphic to $\Gamma_{F^{\mathcal{O}}}(L^{\mathcal{O}})$.*

Proof. Recall that a section $s \in \Gamma(L^{\mathcal{O}})$ is equivalent to a U(1)-equivariant complex function $\dot{s} \in C^{\infty}_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^{\mathcal{O}}, \mathbb{C})$, and under this equivalence, the covariant derivative $(\nabla^{\mathcal{O}})_X s \in \Gamma(L^{\mathcal{O}})$ corresponds to the function $X^{\mathrm{h}}\dot{s} \in C^{\infty}_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^{\mathcal{O}}, \mathbb{C})$.

Suppose $s \in \Gamma_{F^{\mathcal{O}}}(L^{\mathcal{O}})$. Then in particular $(\nabla^{\mathcal{O}})_X s = 0$ for X a vector field in the characteristic distribution $R^{\mathcal{O}} = T\mathcal{R}^{\mathcal{O}} \subset F^{\mathcal{O}}$. Hence $X^{\mathrm{h}}\dot{s} = 0$ for such X, implying that $\dot{s}$ is constant along $(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}$. This is precisely the fiber of the submersion $\dot{\Sigma}^{\mathcal{O}} : \dot{L}^{\mathcal{O}} \to \dot{L}^{\mathcal{O}}_{\mathcal{R}}$, and so $\dot{s} : \dot{L}^{\mathcal{O}} \to \mathbb{C}$ descends to a smooth function $\dot{s}_{\mathcal{R}} : \dot{L}^{\mathcal{O}}_{\mathcal{R}} \to \mathbb{C}$ (satisfying $\dot{s} = \dot{s}_{\mathcal{R}} \circ \dot{\Sigma}^{\mathcal{O}}$). Since $\dot{s}$ and $\dot{\Sigma}^{\mathcal{O}}$ are both U(1)-equivariant, and $\dot{\Sigma}^{\mathcal{O}}$ is surjective, $\dot{s}_{\mathcal{R}}$ is also U(1)-equivariant, implying that it corresponds to a section of $L^{\mathcal{O}}_{\mathcal{R}}$. Since $\dot{s}$ is constant along the horizontal lift of $F^{\mathcal{O}}$, and

$$F^{\mathcal{O}}_{\mathcal{R}} = T\sigma^{\mathcal{O}}\left(F^{\mathcal{O}}\right) \text{ and } \alpha^{\mathcal{O}} = (\dot{\Sigma}^{\mathcal{O}})^{*}\alpha^{\mathcal{O}}_{\mathcal{R}} \implies F^{\mathcal{O}\mathrm{h}}_{\mathcal{R}} = T\dot{\Sigma}^{\mathcal{O}}\left(F^{\mathcal{O}\mathrm{h}}\right),$$

$\dot{s}_{\mathcal{R}}$ is constant along the horizontal lift of $F^{\mathcal{O}}_{\mathcal{R}}$, and so corresponds to an element of $\Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}})$.

Conversely, suppose $t \in \Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}})$. Then $\dot{t} \circ \dot{\Sigma}^{\mathcal{O}}$ satisfies all the properties to correspond to an element of $\Gamma_{F^{\mathcal{O}}}(L^{\mathcal{O}})$, and $\dot{t} \mapsto \dot{t} \circ \dot{\Sigma}^{\mathcal{O}}$ is an inverse to the map $\dot{s} \mapsto \dot{s}_{\mathcal{R}}$ described above. $\square$

Employing identical arguments, we likewise have canonical isomorphisms

$$\Gamma_{F^{a}_{\mathcal{R}}}(L^{a}_{\mathcal{R}}) \simeq \Gamma_{F^{a}}(L^{a}) \qquad \text{and} \qquad \Gamma_{F^{\mu}_{\mathcal{R}}}(L^{\mu}_{\mathcal{R}}) \simeq \Gamma_{F^{\mu}}(L^{\mu}).$$

5.8 Connection with the Borel-Weil Theorem

The left $\widetilde{G}$-actions on $\dot{L}^{a}$ and $\dot{L}^{a}_{\mathcal{R}}$ induce corresponding $\widetilde{G}$-representations on the isomorphic spaces $\Gamma_{F^{a}}(L^{a})$ and $\Gamma_{F^{a}_{\mathcal{R}}}(L^{a}_{\mathcal{R}})$ of covariantly constant sections. In this section, we demonstrate that these spaces form an irreducible $\widetilde{G}$-representation, characterized by the coadjoint orbit $\mathrm{J}(\pi^{-1}(a))$. The proof of this result is essentially the classical Borel-Weil Theorem adapted to the constructions in this thesis.

5.8.1 Equivalence of the unreduced and reduced representations

We first demonstrate that the $\widetilde{G}$-representations on the spaces of sections $\Gamma_{F^a}(L^a)$ and $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ are equivalent. Interpreting such sections as U(1)-equivariant functions, the $\widetilde{G}$-representations are simply

$$(U_{\widetilde{g}}\,\dot{s})(p) = \dot{s}(\widetilde{g}^{-1}\cdot p) \quad \text{and} \quad (V_{\widetilde{g}}\dot{t})(q) = \dot{t}(\widetilde{g}^{-1}\cdot q),$$

for $p \in \dot{L}^a$, $q \in \dot{L}^a_{\mathcal{R}}$. If $\dot{s}$ and $\dot{t}$ are equivalent sections under the isomorphism $\Gamma_{F^a}(L^a) \simeq \Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ discussed in Section 5.7 (so $\dot{s} = \dot{t}\circ\dot{\Sigma}^a$), we see that

$$\begin{aligned} U_{\widetilde{g}}\dot{s}(p) &= \dot{s}(\widetilde{g}^{-1}\cdot p) \\ &= \dot{t}(\dot{\Sigma}^a(\widetilde{g}^{-1}\cdot p)) \\ &= \dot{t}(\widetilde{g}^{-1}\cdot\dot{\Sigma}^a(p)) \quad \text{by } \widetilde{G}\text{-equivariance of } \dot{\Sigma}^a \\ &= (V_{\widetilde{g}}\dot{t})(\dot{\Sigma}^a(p)), \end{aligned}$$

i.e., $U_{\widetilde{g}}(\dot{t}\circ\dot{\Sigma}^a) = (V_{\widetilde{g}}\dot{t})\circ\dot{\Sigma}^a$, and so the isomorphism $\Gamma_{F^a}(L^a) \simeq \Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ intertwines the two $\widetilde{G}$-representations U and V. Hence they are equivalent.

5.8.2 The correspondence between polarized sections and functions on the group

We intend to demonstrate that the $\widetilde{G}$-representations on $\Gamma_{F^a}(L^a)$ and $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ are irreducible. The demonstrated equivalence of the previous section means we need only consider one of the spaces; we will concentrate on $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$, although the proofs below apply with appropriate modifications for $\Gamma_{F^a}(L^a)$ also.

The space $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ of covariantly constant sections is equivalent to the subset of $C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^a_{\mathcal{R}},\mathbb{C})$ which satisfy $Y^{\mathrm{h}}\dot{t} = 0$ for all $Y \in \Gamma(F^a_{\mathcal{R}})$. Let q_0 be an arbitrary element of $\dot{L}^a_{\mathcal{R}}$, and let $\dot{\tau}^a_{\mathcal{R}}(q_0) = \mathcal{R}^{(\mu,a)}$. We have the following result.

Proposition 5.8.1. *(i) There is a one-to-one correspondence, dependent on q_0, between the sets $C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^a_{\mathcal{R}},\mathbb{C})$ and*

$$C^\infty_{\chi^{-\frac{i}{\hbar}\mu}}(\widetilde{G},\mathbb{C}) = \Big\{\dot{r} : \widetilde{G}\to\mathbb{C}\,\Big|\,\dot{r}(\widetilde{g}\widetilde{h}^{-1}) = \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h})\,\dot{r}(\widetilde{g}) \textit{ for all } \widetilde{h}\in(\widetilde{G})_\mu\Big\}.$$

(ii) There is a one-to-one correspondence, dependent on q_0, between the sets

$$\left\{\dot{t} \in C^{\infty}_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^a_{\mathcal{R}}, \mathbb{C}) \,\middle|\, Y^{\mathrm{h}}\dot{t} = 0 \text{ for all } Y \in \Gamma(F^a_{\mathcal{R}})\right\}$$

and

$$\left\{\dot{r} \in C^{\infty}_{\chi^{-\frac{i}{\hbar}\mu}}(\widetilde{G}, \mathbb{C}) \,\middle|\, (\widetilde{g} \cdot \xi)\dot{r} = 0 \text{ for all } \xi \in \mathfrak{n}^+_{\mu},\, \widetilde{g} \in \widetilde{G}\right\}.$$

Proof. (i) Given $\dot{t} \in C^{\infty}_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^a_{\mathcal{R}}, \mathbb{C})$, define $\dot{r} : \widetilde{G} \to \mathbb{C}$ by $\dot{r}(\widetilde{g}) = \dot{t}(\widetilde{g} \cdot q_0)$. Using Proposition 4.4.3, we see that

$$\dot{r}(\widetilde{g}\widetilde{h}^{-1}) = \dot{t}(\widetilde{g} \cdot (\widetilde{h}^{-1} \cdot q_0)) = \dot{t}(\widetilde{g} \cdot q_0 \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h}^{-1})) = \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h})\, \dot{t}(\widetilde{g} \cdot q_0) = \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h})\, \dot{r}(\widetilde{g}),$$

so $\dot{r} \in C^{\infty}_{\chi^{-\frac{i}{\hbar}\mu}}(\widetilde{G}, \mathbb{C})$.

Conversely, suppose $\dot{r} \in C^{\infty}_{\chi^{-\frac{i}{\hbar}\mu}}(\widetilde{G}, \mathbb{C})$, and define $\dot{t}(\widetilde{g} \cdot q_0) = \dot{r}(\widetilde{g})$. $\dot{t}$ will extend to a U(1)-equivariant function on $\dot{L}^a_{\mathcal{R}}$ *provided* $\dot{t}(\widetilde{g}' \cdot q_0) = w^{-1}\dot{t}(\widetilde{g} \cdot q_0)$ when $\widetilde{g}' \cdot q_0 = \widetilde{g} \cdot q_0 \cdot w$. The latter condition occurs if and only if $\widetilde{g}^{-1}\widetilde{g}' = \widetilde{h} \in (\widetilde{G})_{\mu}$. Proposition 4.4.3 then implies that $w = \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h})$, and using the $(\widetilde{G})_{\mu}$-equivariance of $\dot{r}$,

$$\dot{t}(\widetilde{g}' \cdot q_0) = \dot{r}(\widetilde{g}') = \dot{r}(\widetilde{g}\widetilde{h}) = \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h}^{-1})\, \dot{r}(\widetilde{g}) = \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h})^{-1}\, \dot{t}(\widetilde{g} \cdot q_0).$$

The above described maps between $C^{\infty}_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^a_{\mathcal{R}}, \mathbb{C})$ and $C^{\infty}_{\chi^{-\frac{i}{\hbar}\mu}}(\widetilde{G}, \mathbb{C})$ are clearly inverses of one another, and so define a one-to-one correspondence as claimed.

(ii) The reduced polarization at a general point $g\mathcal{R}^{(\mu,a)}$ of $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is (Section 5.6)

$$(F^a_{\mathcal{R}})_{g \cdot \mathcal{R}^{(\mu,a)}} = g \cdot (F^a_{\mathcal{R}})_{\mathcal{R}^{(\mu,a)}} = g \cdot \mathfrak{n}^+_{\mu} \cdot \mathcal{R}^{(\mu,a)}.$$

Hence the polarized sections of $C^{\infty}_{\chi^{-\frac{i}{\hbar}\mu}}(\dot{L}^a_{\mathcal{R}}, \mathbb{C})$ are those satisfying $(g \cdot \xi \cdot \mathcal{R}^{(\mu,a)})^{\mathrm{h}}\dot{t} = 0$ for all $\xi \in \mathfrak{n}^+_{\mu}$, $g \in G$. The relationship $\zeta_{\dot{L}^a_{\mathcal{R}}} = \zeta^{\mathrm{h}}_{\frac{\pi^{-1}(a)}{\mathcal{R}^a}} - (J^a_{\mathcal{R}}(\xi) \circ \dot{\tau}^a_{\mathcal{R}})\left(\frac{\varepsilon_0}{h}\right)_{\dot{L}^a_{\mathcal{R}}}$ coupled with the fact that $\mu\big|_{\mathfrak{n}^+_{\mu}} = 0$ (isotropy in Proposition 5.3.1), tells us that $(g \cdot \xi \cdot \mathcal{R}^{(\mu,a)})^{\mathrm{h}}_{\widetilde{g} \cdot q_0} = \widetilde{g} \cdot \xi \cdot q_0$ for $\xi \in \mathfrak{n}^+_{\mu}$, from which it follows that that the space of polarized sections is

$$\left\{\dot{t} \in C^{\infty}_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^a_{\mathcal{R}}, \mathbb{C}) \,\middle|\, (\widetilde{g} \cdot \xi \cdot q_0)\dot{t} = 0 \text{ for all } \xi \in \mathfrak{n}^+_{\mu},\, \widetilde{g} \in \widetilde{G}\right\}.$$

Via the correspondence from part (i), this clearly translates into

$$\left\{\dot{r} \in C^{\infty}_{\chi^{-\frac{i}{\hbar}\mu}}(\widetilde{G}, \mathbb{C}) \,\middle|\, (\widetilde{g} \cdot \xi)\dot{r} = 0 \text{ for all } \xi \in \mathfrak{n}^+_{\mu},\, \widetilde{g} \in \widetilde{G}\right\}.$$

□

Note. Given $p_0 \in \dot{L}^a$, and $q_0 = \dot{\Sigma}^a(p_0) \in \dot{L}^a_{\mathcal{R}}$, we have the p_0- and q_0-dependent bundle isomorphisms

$$\dot{L}^a \simeq_{p_0} \widetilde{G} \times_\chi \mathrm{U}(1) \quad \text{and} \quad \dot{L}^a_{\mathcal{R}} \simeq_{q_0} \widetilde{G} \times_{\chi^{-\frac{i}{\hbar}\mu}} \mathrm{U}(1).$$

Proposition 5.8.1 (i) can also be seen as a consequence of the second isomorphism, since then

$$L^a_{\mathcal{R}} = \dot{L}^a_{\mathcal{R}} \times_{\mathrm{id}_{\mathrm{U}(1)}} \mathbb{C} \simeq_{q_0} \left(\widetilde{G} \times_{\chi^{-\frac{i}{\hbar}\mu}} \mathrm{U}(1)\right) \times_{\mathrm{id}_{\mathrm{U}(1)}} \mathbb{C} \simeq \widetilde{G} \times_{\chi^{-\frac{i}{\hbar}\mu}} \mathbb{C}.$$

5.8.3 Highest Weight Theory and the Peter-Weyl Theorem

Since $\Gamma(L^a_{\mathcal{R}})$ as complex-valued functions on $\widetilde{G}$, we can invoke some standard results from the representation theory of compact Lie groups. We summarize these results here for convenience.

Let $\widetilde{T}$ be a maximal torus of $\widetilde{G}$ contained in $(\widetilde{G})_\mu$, $\mathfrak{t}$ its Lie algebra, Δ the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$, and $\Delta^+ = \{\alpha \in \Delta \,|\, (\frac{i}{\hbar}\mu, \alpha) \geq 0\}$ the choice of positive roots (cf. Section 5.3.3). A representation of $\widetilde{G}$ on a Hilbert space $\mathcal{H}$ can be decomposed into a direct sum of joint eigenspaces of $\mathfrak{t}^{\mathbb{C}}$. Such a joint eigenvector v satisfies $\eta \cdot v = \lambda(\eta)v$ (where $\cdot$ denotes the representation action) for all $\eta \in \mathfrak{t}^{\mathbb{C}}$ and some $\lambda \in (\mathfrak{t}^{\mathbb{C}})^*$. The element λ is called a *weight* of the representation, and the corresponding eigenspace is denoted $E_\lambda(\mathcal{H})$. The compactness of $\widetilde{G}$ allows us to put a $\widetilde{G}$-invariant inner product on $\mathcal{H}$ that makes the representation unitary, which in particular implies that weights λ are pure imaginary on $\mathfrak{t}$ (so $\lambda \in i\mathfrak{t}^*$). From the identity $[\eta, \xi] = \alpha(\eta)\xi$ for $\xi \in \mathfrak{j}_\alpha$, $\eta \in \mathfrak{t}$, it follows that $\xi \cdot E_\lambda(\mathcal{H}) \subset E_{\lambda+\alpha}(\mathcal{H})$. A *highest weight* λ is one for which $\xi \cdot E_\lambda(\mathcal{H}) = \{0\}$ for all $\xi \in \oplus_{\alpha \in \Delta^+} \mathfrak{j}_\alpha$. For an irreducible representation the highest weight λ is unique, and $\dim_{\mathbb{C}} E_\lambda(\mathcal{H}) = 1$.

Recall that an element $\lambda \in i\mathfrak{t}^*$ is said to be *dominant* if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta^+$, and *integral*[1] if $\lambda : \mathfrak{t} \to i\mathbb{R}$ exponentiates to a character $\chi^\lambda : \widetilde{T} \to \mathrm{U}(1)$. The Highest Weight Theorem ([Sep07, Theorem 7.3]) asserts that the irreducible representations of $\widetilde{G}$ are in one-to-one correspondence with the dominant, integral elements of $i\mathfrak{t}^*$, which occur as highest weights for the representation. We denote[2] the irreducible $\widetilde{G}$-representation corresponding to highest weight $\lambda \in i\mathfrak{t}^*$ by $\mathcal{H}^\lambda$.

The Peter-Weyl Theorem ([Sep07, Corollary 3.26]) asserts that

$$L^2(\widetilde{G}) \simeq \bigoplus_{\lambda \text{ dominant integral}} (\mathcal{H}^\lambda)^* \otimes \mathcal{H}^\lambda.$$

Under the equivalence, $\alpha \otimes v \in (\mathcal{H}^\lambda)^* \otimes \mathcal{H}^\lambda$ corresponds to the map $\widetilde{g} \mapsto \alpha(\widetilde{g} \cdot v)$ in $C^\infty(\widetilde{G}, \mathbb{C}) \subset L^2(\widetilde{G})$.

[1] Since $\widetilde{G}$ is simply connected, the concepts of algebraic and analytic integrality agree, and are not distinguished here.

[2] This notation is however imperfect, since it does not make reference to the choice of maximal torus T containing $\mu^\sharp$.

5.8.4 Irreducibility of the representation

We are now in a position to demonstrate the irreducibility of the $\widetilde{G}$-representation on $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$. From Proposition 5.8.1 (i) it is clear that the $\widetilde{G}$-representation on $\Gamma(L^a_{\mathcal{R}})$ is equivalent to the $\widetilde{G}$-action on $C^\infty_{\chi^{-\frac{i}{\hbar}\mu}}(\widetilde{G}, \mathbb{C})$ induced by left multiplication on $\widetilde{G}$,

$$(\widetilde{g} \cdot \dot{r})(\widetilde{g}') = \dot{r}(\widetilde{g}^{-1}\widetilde{g}').$$

Under the Peter-Weyl correspondence, the $(\widetilde{G})_\mu$-equivariance condition $\dot{r}(\widetilde{g}\widetilde{h}) = \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h}^{-1})\, \dot{r}(\widetilde{g}) = \chi^{\frac{i}{\hbar}\mu}(\widetilde{h})\, \dot{r}(\widetilde{g})$ tells us that $\dot{r}$ corresponds to an element of

$$\bigoplus_{\lambda \text{ d.i.}} (\mathcal{H}^\lambda)^* \otimes E_{\frac{i}{\hbar}\mu}(\mathcal{H}^\lambda),$$

(i.e., $\alpha \otimes v \in \bigoplus_{\lambda \text{ d.i.}} (\mathcal{H}^\lambda)^* \otimes E_{\frac{i}{\hbar}\mu}(\mathcal{H}^\lambda)$ implies that $\alpha(\widetilde{g}\widetilde{h} \cdot v) = \chi^{\frac{i}{\hbar}\mu}(\widetilde{h})\, \alpha(\widetilde{g} \cdot v)$). Restricting now to $\dot{r} \in \Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$, we see that the corresponding function on $\widetilde{G}$ satisfies $(\widetilde{g} \cdot \xi)\dot{r} = 0$ for all $\xi \in \mathfrak{n}^+_\mu$, $\widetilde{g} \in \widetilde{G}$. Recall that $\mathfrak{n}^+_\mu = \bigoplus_{(\frac{i}{\hbar}\mu, \alpha) > 0} \mathfrak{j}_\alpha$ (Section 5.3.3). Also, for $\alpha \in \Delta$ satisfying $(\frac{i}{\hbar}\mu, \alpha) = 0$, $\mu(\zeta) = 0$ for $\zeta \in \mathfrak{j}_\alpha$ (Section 5.3.1), implying that $\chi^{\frac{i}{\hbar}\mu}(\exp_{\widetilde{G}}(\zeta)) = 1$ and $\dot{r}(\widetilde{g} \exp_{\widetilde{G}}(\zeta)) = \dot{r}(\widetilde{g})$. Overall,

$$(\widetilde{g} \cdot \xi)\dot{r} = 0 \quad \text{for all} \quad \xi \in \bigoplus_{(\frac{i}{\hbar}\mu, \alpha) \geq 0} \mathfrak{j}_\alpha,$$

which tells us that $\frac{i}{\hbar}\mu$ is a highest weight, and so under the Peter-Weyl correspondence, $\dot{r}$ corresponds to an element of $(\mathcal{H}^{\frac{i}{\hbar}\mu}) \otimes \{v\}$, where v is a highest weight vector in $\mathcal{H}^{\frac{i}{\hbar}\mu}$. We finally conclude that as a $\widetilde{G}$-representation,

$$\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}}) \simeq_{q_0} (\mathcal{H}^{\frac{i}{\hbar}\mu})^*,$$

and hence is irreducible[3].

Using the polarization $F^a_{\mathcal{R}}$, we can give the reduced space $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ the structure of a complex manifold. The space of sections $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ can then be interpretated as the space of *antiholomorphic* sections of $L^a_{\mathcal{R}}$. See Appendix D for details.

5.8.5 Application to the cotangent bundle of a Lie group

As an application of this construction, consider the case of $M = T^*G$ with the usual left G-action, and symplectic form $\omega = -\mathrm{d}\theta$, where θ is the canonical 1-form $\theta_{\alpha_g}(X_{\alpha_g}) = \alpha\left(T_\alpha \pi(X_\alpha)\right)$ for $\pi : T^*G \to G$ the natural projection. One possible U(1)-bundle-connection pair over T^*G is the trivial U(1)-bundle $\dot{L} = T^*G \times \mathrm{U}(1)$ with connection $\alpha = -\frac{\varepsilon_0}{h}\dot{\tau}^*\theta + \pi^*_{\mathrm{U}(1)}\Theta^{\mathrm{U}(1)}_{\mathrm{MC}}$, where $\Theta^{\mathrm{U}(1)}_{\mathrm{MC}}$ denotes the

[3] The dual $\widetilde{G}$-representation on $(\mathcal{H}^{\frac{i}{\hbar}\mu})^*$ is irreducible with highest weight $w_0(-\frac{i}{\hbar}\mu)$, where w_0 denotes the longest element of the Weyl group—see [Sep07, Lemma 7.5] for details.

Maurer-Cartan form on U(1). The lifted $\widetilde{G}$-action on $T^*G \times \mathrm{U}(1)$ is simply $\widetilde{g} \cdot (\alpha, z) = (\pi_{\widetilde{G}\to G}(\widetilde{g}) \cdot \alpha, z)$, and $\chi(\widetilde{k}) = 1$ for all $\widetilde{k} \in \ker(\pi_{\widetilde{G}\to G})$. Admissible momentum $\mu \in \mathfrak{g}^*$ are those which the Lie algebra homomorphism $-\frac{\varepsilon_0}{h}\mu : \mathfrak{g} \to \mathfrak{u}(1)$ exponentiates to a Lie group homomorphism $\chi^{-\frac{i}{\hbar}\mu}$: $(\widetilde{G})_\mu \to \mathrm{U}(1)$ that agrees with $\chi : \ker(\pi_{\widetilde{G}\to G}) \to \mathrm{U}(1)$ on $\ker(\pi_{\widetilde{G}\to G})$ (Proposition 4.3.6). Since in this case χ is trivial, $\chi^{-\frac{i}{\hbar}\mu}$ factors to a Lie group homomorphism $\chi'^{-\frac{i}{\hbar}\mu} : G_\mu \to \mathrm{U}(1)$.

Proposition 3.1.5 tells us that all other compatible bundles are characterized by elements of the character group of the first fundamental group $\pi_1(T^*G)^* \simeq \pi_1(G)^* \simeq \pi_1(\ker(\pi_{\widetilde{G}\to G}))$. Hence all possible holonomies of $\ker(\pi_{\widetilde{G}\to G})$-action can be achieved by different choices of bundle.

In particular, for the familiar case $G = \mathrm{SO}(3)$, $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$. The trivial bundle corresponds to the integral spin representations, while the $\mathbb{Z}_2$-twisted bundle corresponds to the half-integral spin representations. Either integral or half-integral representations occur in the geometric quantization of $(T^*\mathrm{SO}(3), -\mathrm{d}\theta)$, but not both.

Chapter 6

Factorization of the reduced representation space

A Lie group of symmetries G acting on a quantum system, described by a Hilbert space $\mathcal{H}$ and Hamiltonian H, corresponds to a representation U of the universal cover $\widetilde{G}$ of G on $\mathcal{H}$ which commutes with H. Given such a group of symmetries, it is natural to decompose $\mathcal{H}$ into U- and H-invariant subspaces which transform via the irreducible representations $(\rho^\lambda)^*$ of $\widetilde{G}$. This is accomplished using the operators $P^\lambda = d^\lambda \int_{\widetilde{G}} \mathrm{Tr}\left[\rho^\lambda(\widetilde{g})\right]\ U(\widetilde{g})\ d\mu(\widetilde{g})$ (see Appendix C for details). Each of the reduced spaces can be further factorized as $(\mathcal{H}^\lambda)^* \otimes \left(\mathcal{H}^\lambda \otimes \mathcal{H}\right)^{\widetilde{G}}$ in such a way that U acts on the first factor, and H on the second. Following [TI00], we propose this as a natural definition of quantum reduction. To goal of this chapter is to show how this factorization can be accomplished at the symplectic level, and so demonstrate that "quantization commutes with reduction".

To our knowledge, all of the results in this chapter are new.

6.1 An isomorphism of the reduced bundle-connection pairs

For admissable $\mathcal{O}$, the bundle $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ has a connection $\alpha^{\mathcal{O}}_{\mathcal{R}}$ with curvature $\omega^{\mathcal{O}}_{\mathcal{R}}$, while the bundle[1] $\frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}} \boxdot \frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ has a connection $\alpha^{a}_{\mathcal{R}} \boxplus \alpha^{\mu}_{\mathcal{R}}$ with curvature $\omega^{a}_{\mathcal{R}} \oplus \omega^{\mu}_{\mathcal{R}}$. Corollary 2.14.5 establishes that the symplectic manifolds

$$\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}, \omega^a_{\mathcal{R}} \oplus \omega^\mu_{\mathcal{R}}\right) \qquad \text{and} \qquad \left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}, \omega^{\mathcal{O}}_{\mathcal{R}}\right)$$

are symplectomorphic (via the canonical symplectomorphism ${\phi^{\mathcal{O}}_{\mathcal{R}}}^{-1} \circ (\mathrm{J}^a_{\mathcal{R}} \times \pi^\mu_{\mathcal{R}}) : \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$). Using this symplectomorphism to pull $\left(\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}, \alpha^{\mathcal{O}}_{\mathcal{R}}\right)$ back to $\frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$, we have two U(1)-bundle-connection pairs over $\frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ with the same curvature $\omega^a_{\mathcal{R}} \oplus \omega^\mu_{\mathcal{R}}$. Proposition 3.1.5 establishes that these two bundle-connection pairs must be equivalent up to a flat bundle-

[1]See Appendix B for definition of the bundle product $\boxdot$ and associated connection.

connection pair, and so $\left(\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu},\, \alpha^a_{\mathcal{R}} \boxplus \alpha^\mu_{\mathcal{R}}\right)$ and $\left(\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}},\, \alpha^{\mathcal{O}}_{\mathcal{R}}\right)$ are isomorphic[2] up to a flat bundle-connection pair. In fact, these two are isomorphic, and we establish this fact in this section.

6.1.1 The symplectomorphism between the reduced spaces revisited

We give here a different proof of Corollary 2.14.5, which will then be 'lifted' to the reduced U(1)-bundles to establish the corresponding result on these bundles. To establish this proof, we reinterpret the canonical symplectomorphism as follows: recall the notation for points in $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ (i.e., G_{J}-orbits)

$$\mathcal{R}^{(\mu,a)} = \pi^{-1}(a) \cap \mathrm{J}^{-1}(\mu).$$

For $(\mu,\, a) \in \mathcal{O} \times \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$, the canonical symplectomorphism

$$\phi^{\mathcal{O}\,-1}_{\mathcal{R}} \circ (\mathrm{J}^a_{\mathcal{R}} \times \pi^\mu_{\mathcal{R}}) : \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \longrightarrow \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$$

satisfies

$$\left(\phi^{\mathcal{O}\,-1}_{\mathcal{R}} \circ (\mathrm{J}^a_{\mathcal{R}} \times \pi^\mu_{\mathcal{R}})\right)\left(\mathcal{R}^{(\nu,a)},\, \mathcal{R}^{(\mu,b)}\right) = \phi^{\mathcal{O}\,-1}_{\mathcal{R}}(\nu,\, b) = \mathcal{R}^{(\nu,b)}$$

(Section 2.14). As described in Section 2.13, the (free) G-action on M drops to a (non-free) G-action on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, and $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is the G-orbit of $\mathcal{R}^{(\mu,a)}$ under this action. Recalling that $g \cdot \mathcal{R}^{(\mu,b)} = \mathcal{R}^{(\mathrm{Ad}^*_{g^{-1}}\mu,b)}$ (Corollary 2.13.6 (i)), the canonical symplectomorphism can now be expressed as

$$\left(\phi^{\mathcal{O}\,-1}_{\mathcal{R}} \circ (\mathrm{J}^a_{\mathcal{R}} \times \pi^\mu_{\mathcal{R}})\right)\left(g \cdot \mathcal{R}^{(\mu,a)},\, \mathcal{R}^{(\mu,b)}\right) = g \cdot \mathcal{R}^{(\mu,b)}.$$

Let

$$\Phi^a_{\mathcal{R}} : G \times \frac{\pi^{-1}(a)}{\mathcal{R}^a} \to \frac{\pi^{-1}(a)}{\mathcal{R}^a} \qquad \text{and} \qquad \Phi^\mu_{\mathcal{R}} : G \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$$

denote the obvious restrictions of the (smooth) reduced G-action $\Phi^{\mathcal{O}}_{\mathcal{R}}$ on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$. The maps $\Phi^a_{\mathcal{R}}$ and $\Phi^\mu_{\mathcal{R}}$ are smooth since $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ and $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ are initial and embedded in $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ respectively (Proposition 2.10.3 (ii), (i)). The above expression for the canonical symplectomorphism can now be written

$$\left(\phi^{\mathcal{O}\,-1}_{\mathcal{R}} \circ (\mathrm{J}^a_{\mathcal{R}} \times \pi^\mu_{\mathcal{R}})\right)\left((\Phi^a_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}}(g),\, \mathcal{R}^{(\mu,b)}\right) = \Phi^\mu_{\mathcal{R}}(g,\, \mathcal{R}^{(\mu,b)}),$$

where we recall that $(\Phi^a_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}}(g) = \Phi^a_{\mathcal{R}}(g,\, \mathcal{R}^{(\mu,a)}) = g \cdot \mathcal{R}^{(\mu,a)}$. This fact motivates the following construction of the canonical symplectomorphism, which will be "lifted" to a U(1)-bundle-connection isomorphism in Section 6.1.2.

Proposition 6.1.1. *(i)* $(\Phi^a_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}} \times \mathrm{id}_{\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}} : G \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \longrightarrow \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ *is a surjective submersion.*

[2]See Appendix A for definition of bundle-connection isomorphism.

(ii) $\Phi^{\mu}_{\mathcal{R}} : G \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ *is a surjective submersion.*

(iii) There exists a diffeomorphism $e^{a,\mu} : \frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ *making the following diagram commute:*

$$\begin{tikzcd} & G \times \dfrac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \arrow[dl, "(\Phi^{a}_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}} \times \mathrm{id}_{\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}}"'] \arrow[dr, "\Phi^{\mu}_{\mathcal{R}}"] & \\ \dfrac{\pi^{-1}(a)}{\mathcal{R}^{a}} \times \dfrac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \arrow[rr, shift left, "e^{a,\mu}"] & & \dfrac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \arrow[ll, shift left, "(e^{a,\mu})^{-1}"] \end{tikzcd} .$$

Explicitly, the diffeomorphism $e^{a,\mu}$ *can be expressed as*

$$e^{a,\mu}\left(g \cdot \mathcal{R}^{(\mu,a)},\, \mathcal{R}^{(\mu,b)}\right) = g \cdot \mathcal{R}^{(\mu,b)}.$$

(iv) $e^{a,\mu}$ *is a symplectomorphism, i.e.,* $(e^{a,\mu})^{*}\omega^{\mathcal{O}}_{\mathcal{R}} = \omega^{a}_{\mathcal{R}} \oplus \omega^{\mu}_{\mathcal{R}}$.

Proof. (i) Since $G \cdot \mathcal{R}^{(\mu,a)} = \frac{\pi^{-1}(a)}{\mathcal{R}^{a}}$, $(\Phi^{a}_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}}$ is clearly surjective, and hence so is $(\Phi^{a}_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}} \times \mathrm{id}_{\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}}$.

Proposition 2.8.5 (ii), together with the correspondence of the smooth structures on $\pi^{-1}(a)$ and G, implies that $(\Phi^{a}_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}}$ is a submersion, and hence so is $(\Phi^{a}_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}} \times \mathrm{id}_{\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}}$.

(ii) Since $G \cdot \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} = \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, $\Phi^{\mu}_{\mathcal{R}}$ is clearly surjective.

Proposition 2.12.3 (ii) says that for any $\mathcal{R}^{(\nu,b)} \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$,

$$T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}\right) = T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\pi^{-1}(b)}{\mathcal{R}^{b}}\right) \oplus T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\nu)}{\mathcal{R}^{\nu}}\right).$$

So any vector in $T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}\right)$ can be written as $\xi \cdot \mathcal{R}^{(\nu,b)} + Y_{\mathcal{R}^{(\nu,b)}}$ for some $\xi \cdot \mathcal{R}^{(\nu,b)} \in T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\pi^{-1}(b)}{\mathcal{R}^{b}}\right)$ and $Y_{\mathcal{R}^{(\nu,b)}} \in T_{\mathcal{R}^{(\nu,b)}}\left(\frac{\mathrm{J}^{-1}(\nu)}{\mathcal{R}^{\nu}}\right)$. Taking $g \in G$ such that $\mathrm{Ad}^{*}_{g^{-1}}\mu = \nu$, we get that $g^{-1} \cdot Y_{\mathcal{R}^{(\nu,b)}} \in T_{\mathcal{R}^{(\mu,b)}}\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}\right)$. Since by definition $\Phi^{\mu}_{\mathcal{R}}(g, \mathcal{R}^{(\mu,b)}) = g \cdot \mathcal{R}^{(\mu,b)}$, we have

$$T_{\left(g,\, \mathcal{R}^{(\mu,b)}\right)}\Phi^{\mu}_{\mathcal{R}}(\xi \cdot g,\, g^{-1} \cdot Y_{\mathcal{R}^{(\nu,b)}}) = \xi \cdot g \cdot \mathcal{R}^{(\mu,b)} + g \cdot \left(g^{-1} \cdot Y_{\mathcal{R}^{(\nu,b)}}\right) = \xi \cdot \mathcal{R}^{(\nu,b)} + Y_{\mathcal{R}^{(\nu,b)}}.$$

So $\Phi^{\mu}_{\mathcal{R}}$ is a submersion.

(iii) The fibers of $(\Phi^{a}_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}} \times \mathrm{id}_{\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}}$ and $\Phi^{\mu}_{\mathcal{R}}$ through $(g, \mathcal{R}^{(\mu,b)}) \in G \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}$ agree, both being $gG_{\mu} \times \{\mathcal{R}^{(\mu,b)}\}$. Applying Proposition 2.10.2 (v) with $F = \mathrm{id}_{G \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}}$ in both directions implies the existence of smooth $e^{a,\mu}$ and $(e^{a,\mu})^{-1}$ satisfying the commutative diagram. This further implies that $e^{a,\mu}$ is a diffeomorphism.

The explicit expression for $e^{a,\mu}$ follows by noting that, for example,

$$\left((\Phi^a_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}} \times \mathrm{id}\right)\left(g,\, \mathcal{R}^{(\mu,b)}\right) = \left(g \cdot \mathcal{R}^{(\mu,a)},\, \mathcal{R}^{(\mu,b)}\right).$$

Hence commutativity of the diagram implies

$$e^{a,\mu}\left(g \cdot \mathcal{R}^{(\mu,a)},\, \mathcal{R}^{(\mu,b)}\right) = \Phi^\mu_{\mathcal{R}}(g,\, \mathcal{R}^{(\mu,b)}) = g \cdot \mathcal{R}^{(\mu,b)}.$$

(iv) Differentiating the identity $e^{a,\mu} \circ \left((\Phi^a_{\mathcal{R}})^{\mathcal{R}^{(\mu,a)}} \times \mathrm{id}_{\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}}\right) = \Phi^\mu_{\mathcal{R}}$ at $(g,\, \mathcal{R}^{(\mu,b)})$ in the direction $(\xi \cdot g,\, Y_{\mathcal{R}^{(\mu,b)}}) \in T_g G \oplus T_{\mathcal{R}^{(\mu,b)}}\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}\right)$ implies

$$T_{(g\cdot\mathcal{R}^{(\mu,a)},\, \mathcal{R}^{(\mu,b)})}\, e^{a,\mu}\left(\xi \cdot g \cdot \mathcal{R}^{(\mu,a)},\, Y_{\mathcal{R}^{(\mu,b)}}\right) = \xi \cdot g \cdot \mathcal{R}^{(\mu,b)} + g \cdot Y_{\mathcal{R}^{(\mu,b)}}.$$

Taking any other vector pair $\left(\xi' \cdot g \cdot \mathcal{R}^{(\mu,a)},\, Y'_{\mathcal{R}^{(\mu,b)}}\right) \in T_{g\cdot\mathcal{R}^{(\mu,a)}}\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a}\right) \oplus T_{\mathcal{R}^{(\mu,b)}}\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}\right)$, we get

$$\begin{aligned}
&\left((e^{a,\mu})^*\omega^{\mathcal{O}}_{\mathcal{R}}\right)_{(g\cdot\mathcal{R}^{(\mu,a)},\, \mathcal{R}^{(\mu,b)})}\left(\left(\xi \cdot g \cdot \mathcal{R}^{(\mu,a)},\, Y_{\mathcal{R}^{(\mu,b)}}\right),\, \left(\xi' \cdot g \cdot \mathcal{R}^{(\mu,a)},\, Y'_{\mathcal{R}^{(\mu,b)}}\right)\right)\\
&= \left(\omega^{\mathcal{O}}_{\mathcal{R}}\right)_{g\cdot\mathcal{R}^{(\mu,b)}}\left(\xi \cdot g \cdot \mathcal{R}^{(\mu,b)} + g \cdot Y_{\mathcal{R}^{(\mu,b)}},\, \xi' \cdot g \cdot \mathcal{R}^{(\mu,b)} + g \cdot Y'_{\mathcal{R}^{(\mu,b)}}\right)\\
&= \left(\omega^{\mathcal{O}}_{\mathcal{R}}\right)_{g\cdot\mathcal{R}^{(\mu,b)}}\left(\xi \cdot g \cdot \mathcal{R}^{(\mu,b)},\, \xi' \cdot g \cdot \mathcal{R}^{(\mu,b)}\right) + \left(\omega^{\mathcal{O}}_{\mathcal{R}}\right)_{g\cdot\mathcal{R}^{(\mu,b)}}\left(g \cdot Y_{\mathcal{R}^{(\mu,b)}},\, g \cdot Y'_{\mathcal{R}^{(\mu,b)}}\right)\\
&= \left\langle \mathrm{Ad}^*_{g^{-1}}\mu,\, [\xi,\, \xi']\right\rangle + \left(\omega^{\mathcal{O}}_{\mathcal{R}}\right)_{\mathcal{R}^{(\mu,b)}}\left(Y_{\mathcal{R}^{(\mu,b)}},\, Y'_{\mathcal{R}^{(\mu,b)}}\right)\\
&= (\omega^a_{\mathcal{R}})_{g\cdot\mathcal{R}^{(\mu,a)}}\left(\xi \cdot g \cdot \mathcal{R}^{(\mu,a)},\, \xi' \cdot g \cdot \mathcal{R}^{(\mu,a)}\right) + (\omega^\mu_{\mathcal{R}})_{\mathcal{R}^{(\mu,b)}}\left(Y_{\mathcal{R}^{(\mu,b)}},\, Y'_{\mathcal{R}^{(\mu,b)}}\right)\\
&= (\omega^a_{\mathcal{R}} \oplus \omega^\mu_{\mathcal{R}})_{(g\cdot\mathcal{R}^{(\mu,a)},\, \mathcal{R}^{(\mu,b)})}\left(\left(\xi \cdot g \cdot \mathcal{R}^{(\mu,a)},\, Y_{\mathcal{R}^{(\mu,b)}}\right),\, \left(\xi' \cdot g \cdot \mathcal{R}^{(\mu,a)},\, Y'_{\mathcal{R}^{(\mu,b)}}\right)\right),
\end{aligned}$$

the second equality following from Proposition 2.12.3 (iii), and the third from properties of the momentum map $\mathrm{J}^{\mathcal{O}}_{\mathcal{R}}$ for the first term, and G-invariance of $\omega^{\mathcal{O}}_{\mathcal{R}}$ for the second term. Therefore $(e^{a,\mu})^*\omega^{\mathcal{O}}_{\mathcal{R}} = \omega^a_{\mathcal{R}} \oplus \omega^\mu_{\mathcal{R}}$, as claimed.

□

6.1.2 The lifted construction

We define now a bundle-connection isomorphism from $\left(\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu},\, \alpha^a_{\mathcal{R}} \boxplus \alpha^\mu_{\mathcal{R}}\right)$ to $\left(\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}},\, \alpha^{\mathcal{O}}_{\mathcal{R}}\right)$ which covers the symplectomorphism $e^{a,\mu} : \left(\frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu},\, \omega^a_{\mathcal{R}} \oplus \omega^\mu_{\mathcal{R}}\right) \to \left(\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}},\, \omega^{\mathcal{O}}_{\mathcal{R}}\right)$. Note that for any bundle-connection isomorphism, composition with right multiplication by an element of U(1) also yields an isomorphism. So we do not expect our construction of an isomorphism to be canonical, as $e^{a,\mu}$ is. However, it will be canonical up to global U(1)-phase.

The submanifolds $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ and $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$ intersect at the point $\mathcal{R}^{(\mu,a)}$ (Proposition 2.12.3 (i)), and

$\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a}$ and $\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ intersect in the U(1)-fiber lying over $\mathcal{R}^{(\mu,a)}$. Let q_0 be any point in this fiber. The arbitrary choice of q_0 will correspond to the U(1)-arbitrariness of the isomorphism mentioned in the previous paragraph.

As in Section 4.4, let $\dot{\mathbf{\Phi}}^{\mathcal{O}}_{\mathcal{R}} : \widetilde{G} \times \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ denote the reduced $\widetilde{G}$-action on $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$. In addition, let

$$\dot{\mathbf{\Phi}}^{a}_{\mathcal{R}} : \widetilde{G} \times \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \to \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \qquad \text{and} \qquad \dot{\mathbf{\Phi}}^{\mu}_{\mathcal{R}} : \widetilde{G} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$$

denote the obvious restrictions of this action. Since $\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a}$ and $\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ are initial and embedded in $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ respectively (Proposition 4.4.2 (ii), (i)), these restrictions are also smooth. We are now ready to state the lifted version of 6.1.1:

Proposition 6.1.2. *(i)* $(\dot{\mathbf{\Phi}}^{a}_{\mathcal{R}})^{q_0} \boxdot \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}} : \widetilde{G} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} \to \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ *is a surjective submersion.*

(ii) $\dot{\mathbf{\Phi}}^{\mu}_{\mathcal{R}} : \widetilde{G} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} \to \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ *is a surjective submersion.*

(iii) There exists a diffeomorphism $\dot{E}^{a,\mu}_{q_0} : \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ *making the following diagram commute,*

$$\begin{tikzcd}[column sep=large] & \widetilde{G} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} \arrow[dl, "(\dot{\mathbf{\Phi}}^{a}_{\mathcal{R}})^{q_0} \boxdot \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}"'] \arrow[dr, "\dot{\mathbf{\Phi}}^{\mu}_{\mathcal{R}}"] & \\ \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} \arrow[rr, shift left, "\dot{E}^{a,\mu}_{q_0}"] & & \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \arrow[ll, shift left, "(\dot{E}^{a,\mu}_{q_0})^{-1}"] \end{tikzcd} ,$$

where $\left((\dot{\mathbf{\Phi}}^{a}_{\mathcal{R}})^{q_0} \boxdot \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}\right)(\widetilde{g}, q) = (\widetilde{g} \cdot q_0) \boxdot q$. *The diffeomorphism* $\dot{E}^{a,\mu}_{q_0}$ *can be expressed explicitly as*

$$\dot{E}^{a,\mu}_{q_0}\left((\widetilde{g} \cdot q_0) \boxdot q\right) = \widetilde{g} \cdot q.$$

(iv) $\dot{E}^{a,\mu}_{q_0}$ *is a bundle-connection isomorphism, i.e., is* U(1)*-equivariant and satisfies* $(\dot{E}^{a,\mu}_{q_0})^* \alpha^{\mathcal{O}}_{\mathcal{R}} = \alpha^{a}_{\mathcal{R}} \boxplus \alpha^{\mu}_{\mathcal{R}}$.

Proof. (i) We have $\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} = \widetilde{G} \cdot q_0 \cdot \mathrm{U}(1)$. By absorbing the U(1) factor into $\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$, we see that an arbitrary element of $\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ can be written (in general non-uniquely) as $(\widetilde{g} \cdot q_0) \boxdot q$ for some $\widetilde{g} \in \widetilde{G}$ and $q \in \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$. This is just $\left((\dot{\mathbf{\Phi}}^{a}_{\mathcal{R}})^{q_0} \boxdot \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}\right)(\widetilde{g}, q)$, and hence $(\dot{\mathbf{\Phi}}^{a}_{\mathcal{R}})^{q_0} \boxdot \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}$ is surjective.

By the same reasoning, an arbitrary element of $T_{(\widetilde{g} \cdot q_0) \boxdot q}\left(\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}\right)$ can be written as $(\xi \cdot \widetilde{g} \cdot q_0) \boxplus V_q$, where $\xi \in \mathfrak{g}$ and $V_q \in T_q\left(\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}\right)$. Since this equals $T_{(\widetilde{g} \cdot q_0) \boxdot q}\left((\dot{\mathbf{\Phi}}^{a}_{\mathcal{R}})^{q_0} \boxdot \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}\right)(\xi \cdot$

$\widetilde{g},\, V_q)$, $(\dot{\mathbf{\Phi}}^a_{\mathcal{R}})^{q_0} \boxdot \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}$ is a submersion.

(ii) Since $\widetilde{G} \cdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} = \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, $\dot{\mathbf{\Phi}}^\mu_{\mathcal{R}}$ is clearly surjective.

The tangent space $T_q\left(\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}\right)$ equals $\mathfrak{g} \cdot q + T_q\left(\frac{\dot{L}^\nu}{(\mathcal{R}^{\mathrm{h}})^\nu}\right)$, where $\nu = \mathrm{J}^{\mathcal{O}}(\dot{\tau}^{\mathcal{O}}(q))$. So any vector in $T_q\left(\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}\right)$ can be written as $\xi \cdot q + V_q$, where $\xi \in \mathfrak{g}$ and $V_q \in T_q\left(\frac{\dot{L}^\nu}{(\mathcal{R}^{\mathrm{h}})^\nu}\right)$. Let $\widetilde{g} \in \widetilde{G}$ be such that $\mathrm{Ad}^*_{\widetilde{g}^{-1}}\mu = \nu$. Then $\left(\xi \cdot \widetilde{g},\, \widetilde{g}^{-1} \cdot V_q\right) \in T_{(\widetilde{g},\, \widetilde{g}^{-1} \cdot q)}\left(\widetilde{G} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}\right)$, and

$$T_{(\widetilde{g},\, \widetilde{g}^{-1} \cdot q)} \dot{\mathbf{\Phi}}^\mu_{\mathcal{R}}\left(\xi \cdot \widetilde{g},\, \widetilde{g}^{-1} \cdot V_q\right) = \xi \cdot \widetilde{g} \cdot (\widetilde{g}^{-1} \cdot q) + \widetilde{g} \cdot (\widetilde{g}^{-1} \cdot V_q) = \xi \cdot q + V_q.$$

Hence $\dot{\mathbf{\Phi}}^\mu_{\mathcal{R}}$ is a submersion.

(iii) From Proposition 4.4.3, we see that the maps $\dot{\mathbf{\Phi}}^\mu_{\mathcal{R}} \times \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}$ and $\dot{\mathbf{\Phi}}^a_{\mathcal{R}}$ have the same fiber through $(\widetilde{g},\, q) \in \widetilde{G} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$, namely $\left\{\left(\widetilde{g}\widetilde{h},\, q \cdot \chi^{-\frac{i}{\hbar}\mu}(\widetilde{h}^{-1})\right) \,\middle|\, \widetilde{h} \in (\widetilde{G})_\mu\right\}$. Applying Proposition 2.10.2 (v) with $F = \mathrm{id}_{\widetilde{G} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}$ in both directions implies the existence of smooth $\dot{E}^{a,\mu}_{q_0}$ and $(\dot{E}^{a,\mu}_{q_0})^{-1}$ satisfying the commutative diagram. This further implies that $\dot{E}^{a,\mu}_{q_0}$ is a diffeomorphism.

An arbitarary $(\widetilde{g} \cdot q_0) \boxdot q \in \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ is the image of, for example, $(\widetilde{g},\, q) \in \widetilde{G} \times \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ under $\dot{\mathbf{\Phi}}^\mu_{\mathcal{R}} \times \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}$. Hence by commutativity of the diagram

$$\dot{E}^{a,\mu}_{q_0}((\widetilde{g} \cdot q_0) \boxdot q) = \dot{\mathbf{\Phi}}^\mu_{\mathcal{R}}(\widetilde{g},\, q) = \widetilde{g} \cdot q.$$

(iv) For arbitrary $(\widetilde{g} \cdot q_0) \boxdot q \in \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$ and $w \in \mathrm{U}(1)$,

$$\dot{E}^{a,\mu}_{q_0}\left(((\widetilde{g} \cdot q_0) \boxdot q) \cdot w\right) = \dot{E}^{a,\mu}_{q_0}\left((\widetilde{g} \cdot q_0) \boxdot (q \cdot w)\right) = \widetilde{g} \cdot (q \cdot w) = (\widetilde{g} \cdot q) \cdot w = \dot{E}^{a,\mu}_{q_0}((\widetilde{g} \cdot q_0) \boxdot q) \cdot w.$$

Hence $\dot{E}^{a,\mu}_{q_0}$ is U(1)-equivariant.

Differentiating the identity $\dot{E}^{a,\mu}_{q_0} \circ \left((\dot{\mathbf{\Phi}}^a_{\mathcal{R}})^{q_0} \boxdot \mathrm{id}_{\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}}\right) = \dot{\mathbf{\Phi}}^\mu_{\mathcal{R}}$ at $(\widetilde{g},\, q)$ in the direction $(\xi \cdot \widetilde{g},\, V_q) \in T_{\widetilde{g}}\widetilde{G} \oplus T_q\left(\frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}\right)$ implies

$$T_{(\widetilde{g} \cdot q_0) \boxdot q}(\dot{E}^{a,\mu}_{q_0})\left((\xi \cdot \widetilde{g} \cdot q_0) \boxplus V_q\right) = \xi \cdot \widetilde{g} \cdot q + \widetilde{g} \cdot V_q.$$

Hence for arbitrary vector $(\xi \cdot \widetilde{g} \cdot q_0) \boxplus V_q \in T_{(\widetilde{g} \cdot q_0) \boxdot q}\left(\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}\right)$, and taking $g = \pi_{\widetilde{G} \to G}(\widetilde{g})$ and $y = \dot{\tau}^{\mathcal{O}}(q)$, $y_0 = \dot{\tau}^{\mathcal{O}}(q_0)$,

$$\begin{aligned}
\left((\dot{E}^{a,\mu}_{q_0})^* \alpha^{\mathcal{O}}_{\mathcal{R}}\right)_{(\widetilde{g} \cdot q_0) \boxdot q}\left((\xi \cdot \widetilde{g} \cdot q_0) \boxplus V_q\right) &= (\alpha^{\mathcal{O}}_{\mathcal{R}})_{\widetilde{g} \cdot q}(\xi \cdot \widetilde{g} \cdot q + \widetilde{g} \cdot V_q) \\
&= (\alpha^{\mathcal{O}}_{\mathcal{R}})_{\widetilde{g} \cdot q}(\xi \cdot \widetilde{g} \cdot q) + (\alpha^{\mathcal{O}}_{\mathcal{R}})_{\widetilde{g} \cdot q}(\widetilde{g} \cdot V_q) \\
&= -\langle \mathrm{J}(g \cdot y),\, \xi \rangle + (\alpha^{\mathcal{O}}_{\mathcal{R}})_q(V_q) \qquad \text{by } \widetilde{G}\text{-invariance of } \alpha^{\mathcal{O}}_{\mathcal{R}}
\end{aligned}$$

$$= -\langle \mathrm{J}(g \cdot y_0),\, \xi \rangle + (\alpha^a_{\mathcal{R}})_q(V_q) \qquad \text{since } \mathrm{J}(y) = \mathrm{J}(y_0) = \mu$$
$$= (\alpha^a_{\mathcal{R}})_{\widetilde{g} \cdot q_0}(\xi \cdot \widetilde{g} \cdot q_0) + (\alpha^\mu_{\mathcal{R}})_q(V_q)$$
$$= (\alpha^a_{\mathcal{R}} \boxplus \alpha^\mu_{\mathcal{R}})((\xi \cdot \widetilde{g} \cdot q_0) \boxplus V_q).$$

This proves that $(\dot{E}^{a,\mu}_{q_0})^* \alpha^{\mathcal{O}}_{\mathcal{R}} = \alpha^\mu_{\mathcal{R}} \boxplus \alpha^a_{\mathcal{R}}$.

□

Note. From its definition, $\dot{E}^{a,\mu}_{q_0} : \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ clearly covers $e^{a,\mu} : \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$.

6.1.3 The polarization isomorphism

As might be expected, the canonical symplectomorphism

$$e^{a,\mu} = \phi^{\mathcal{O}\,-1}_{\mathcal{R}} \circ (\mathrm{J}^a_{\mathcal{R}} \times \pi^\mu_{\mathcal{R}}) : \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$$

also relates the reduced polarizations to one another.

Lemma 6.1.3. $T\left(\phi^{\mathcal{O}\,-1}_{\mathcal{R}} \circ (\mathrm{J}^a_{\mathcal{R}} \times \pi^\mu_{\mathcal{R}})\right)(F^a_{\mathcal{R}} \oplus F^\mu_{\mathcal{R}}) = F^{\mathcal{O}}_{\mathcal{R}}$.

Proof. Recall the relations (Section 5.4)

$$F^{\mathcal{O}} = (T\phi^{\mathcal{O}})^{-1}(P \oplus Q), \qquad F^a = (T\mathrm{J}^a)^{-1}(P), \qquad F^\mu = (T\pi^\mu)^{-1}(Q).$$

Since $\phi^{\mathcal{O}}$, J^a, and π^μ are submersions, these imply

$$T\phi^{\mathcal{O}}\left(F^{\mathcal{O}}\right) = P \oplus Q, \qquad T\mathrm{J}^a\left(F^a\right) = P, \qquad T\pi^\mu\left(F^\mu\right) = Q.$$

Referring to Diagrams 2.7, 2.6, and 2.4,

$$\phi^{\mathcal{O}} = \phi^{\mathcal{O}}_{\mathcal{R}} \circ \sigma^{\mathcal{O}}, \qquad \mathrm{J}^a = \mathrm{J}^a_{\mathcal{R}} \circ \sigma^a, \qquad \pi^\mu = \pi^\mu_{\mathcal{R}} \circ \sigma^\mu,$$

and so

$$\begin{aligned} P \oplus Q &= T(\phi^{\mathcal{O}}_{\mathcal{R}} \circ \sigma^{\mathcal{O}})\left(F^{\mathcal{O}}\right) \\ &= T\phi^{\mathcal{O}}_{\mathcal{R}}\left(F^{\mathcal{O}}_{\mathcal{R}})\right) \\ \Longrightarrow\ F^{\mathcal{O}}_{\mathcal{R}} &= (T\phi^{\mathcal{O}}_{\mathcal{R}})^{-1}\left(P \oplus Q\right) \\ &= T(\phi^{\mathcal{O}\,-1}_{\mathcal{R}})\left(P \oplus Q\right). \end{aligned}$$

□

Similarly

$$T\mathrm{J}^a_{\mathcal{R}}\left(F^a_{\mathcal{R}}\right) = P \qquad \text{and} \qquad T\pi^\mu_{\mathcal{R}}\left(F^\mu_{\mathcal{R}}\right) = Q.$$

Combining, we get

$$T\left(\phi^{\mathcal{O}\,-1}_{\mathcal{R}} \circ (\mathrm{J}^a_{\mathcal{R}} \times \pi^a_{\mathcal{R}})\right)(F^a_{\mathcal{R}} \oplus F^\mu_{\mathcal{R}}) = T(\phi^{\mathcal{O}\,-1}_{\mathcal{R}})\,(T\mathrm{J}^a_{\mathcal{R}}(F^a_{\mathcal{R}}) \oplus T\pi^\mu_{\mathcal{R}}(F^\mu_{\mathcal{R}})) = T(\phi^{\mathcal{O}\,-1}_{\mathcal{R}})(P \oplus Q) = F^{\mathcal{O}}_{\mathcal{R}}.$$

6.2 The lifted dynamics and group action under the decomposition

Dynamics under a G-invariant Hamiltonian preserves the level sets of the momentum, and so drops to the reduced spaces $\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$. We describe here how this dynamics appears under the decomposition $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \simeq \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$, and how the lifted dynamics appear under the decomposition $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \simeq \frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu}$.

6.2.1 The relation between unreduced and reduced flows

Suppose $H \in C^\infty(M, \mathbb{R})$ is a G-invariant function, serving as the Hamiltonian of the system. Let $H^{\mathcal{O}}$ denote its restriction to $\mathrm{J}^{-1}(\mathcal{O})$. The G-invariance of $H^{\mathcal{O}}$ guarantees the existence of a reduced Hamiltonian $H^{\mathcal{O}}_{\mathcal{R}}$ on $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$, characterized by the commutative diagram

$$\begin{tikzcd} & \mathrm{J}^{-1}(\mathcal{O}) \arrow[dl, "\sigma^{\mathcal{O}}"'] \arrow[dr, "H^{\mathcal{O}}"] & \\ \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \arrow[rr, "H^{\mathcal{O}}_{\mathcal{R}}"] & & \mathbb{R} \end{tikzcd} ,$$

and as usual by Proposition 2.10.2 (i), $H^{\mathcal{O}}_{\mathcal{R}}$ is smooth.

Let $X^{\mathcal{O}}_H$ be the restriction of the Hamiltonian vector field X_H to $\mathrm{J}^{-1}(\mathcal{O})$. Then clearly

$$\mathrm{i}_{X^{\mathcal{O}}_H}\omega^{\mathcal{O}} = \mathrm{d}H^{\mathcal{O}}.$$

On the reduced space $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ we have a Hamiltonian vector field $X_{H^{\mathcal{O}}_{\mathcal{R}}}$ corresponding to $H^{\mathcal{O}}_{\mathcal{R}}$

$$\mathrm{i}_{X_{H^{\mathcal{O}}_{\mathcal{R}}}}\omega^{\mathcal{O}}_{\mathcal{R}} = \mathrm{d}H^{\mathcal{O}}_{\mathcal{R}}.$$

From the relation $\omega^{\mathcal{O}} = (\sigma^{\mathcal{O}})^*\omega^{\mathcal{O}}_{\mathcal{R}}$ it is straightforward to check that $X^{\mathcal{O}}_H$ and $X_{H^{\mathcal{O}}_{\mathcal{R}}}$ are $\sigma^{\mathcal{O}}$-related

$$X^{\mathcal{O}}_H \sim_{\sigma^{\mathcal{O}}} X_{H^{\mathcal{O}}_{\mathcal{R}}} \qquad \text{or} \qquad T\sigma^{\mathcal{O}} \circ X^{\mathcal{O}}_H = X_{H^{\mathcal{O}}_{\mathcal{R}}} \circ \sigma^{\mathcal{O}}.$$

This implies that the restricted Hamiltonian flow $(\phi_H^t)^{\mathcal{O}}$ and reduced Hamiltonian flow $\phi^t_{H^{\mathcal{O}}_{\mathcal{R}}}$ are related by

$$\sigma^{\mathcal{O}} \circ (\phi_H^t)^{\mathcal{O}} = \phi^t_{H^{\mathcal{O}}_{\mathcal{R}}} \circ \sigma^{\mathcal{O}}.$$

Since the Hamiltonian flow ϕ_H^t preserves $\mathrm{J}^{-1}(\mu)$, similar remarks apply to the restriction X_H^μ of X_H to $\mathrm{J}^{-1}(\mu)$

$$X_H^\mu \sim_{\sigma^\mu} X_{H^\mu_{\mathcal{R}}} \qquad \text{and} \qquad \sigma^\mu \circ (\phi_H^t)^\mu = \phi^t_{H^\mu_{\mathcal{R}}} \circ \sigma^\mu.$$

6.2.2 Group invariance and decomposition

G-invariance of both $H^{\mathcal{O}}$ and $H^{\mathcal{O}}_{\mathcal{R}}$ (under the unreduced and reduced G-actions respectively—see Section 2.13) and symplecticity of the unreduced and reduced G-actions (Proposition 2.13.1) implies G-invariance of the respective vector fields $X_H^{\mathcal{O}}$ and $X_{H^{\mathcal{O}}_{\mathcal{R}}}$,

$$X_H^{\mathcal{O}}(g \cdot x) = g \cdot X_H^{\mathcal{O}}(x), \qquad\qquad X_{H^{\mathcal{O}}_{\mathcal{R}}}(g \cdot y) = g \cdot X_{H^{\mathcal{O}}_{\mathcal{R}}}(y),$$

and G-equivariance of their flows $(\phi_H^t)^{\mathcal{O}} : \mathrm{J}^{-1}(\mathcal{O}) \to \mathrm{J}^{-1}(\mathcal{O})$ and $\phi^t_{H^{\mathcal{O}}_{\mathcal{R}}} : \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$,

$$(\phi_H^t)^{\mathcal{O}}(g \cdot x) = g \cdot (\phi_H^t)^{\mathcal{O}}(x), \qquad\qquad \phi^t_{H^{\mathcal{O}}_{\mathcal{R}}}(g \cdot y) = g \cdot \phi^t_{H^{\mathcal{O}}_{\mathcal{R}}}(y).$$

The canonical symplectomorphism $e^{a,\mu} : \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \to \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ is given by

$$e^{a,\mu}\left(g \cdot \mathcal{R}^{(\mu,a)}, \mathcal{R}^{(\mu,b)}\right) = g \cdot \mathcal{R}^{(\mu,b)}.$$

As a consequence

$$\begin{aligned}
e^{a,\mu}\left(g \cdot \mathcal{R}^{(\mu,a)}, \phi^t_{H^\mu_{\mathcal{R}}}\left(\mathcal{R}^{(\mu,b)}\right)\right) &= g \cdot \phi^t_{H^\mu_{\mathcal{R}}}\left(\mathcal{R}^{(\mu,b)}\right) \\
&= g \cdot \phi^t_{H^{\mathcal{O}}_{\mathcal{R}}}\left(\mathcal{R}^{(\mu,b)}\right) \\
&= \phi^t_{H^{\mathcal{O}}_{\mathcal{R}}}\left(g \cdot \mathcal{R}^{(\mu,b)}\right) \qquad \text{by } G\text{-equivariance of the flow} \\
&= \phi^t_{H^{\mathcal{O}}_{\mathcal{R}}}\left(e^{a,\mu}\left(g \cdot \mathcal{R}^{(\mu,a)}, \mathcal{R}^{(\mu,b)}\right)\right),
\end{aligned}$$

that is,

$$e^{a,\mu} \circ \left(\mathrm{id}_{\frac{\pi^{-1}(a)}{\mathcal{R}^a}} \times \phi^t_{H^\mu_{\mathcal{R}}}\right) = \phi^t_{H^{\mathcal{O}}_{\mathcal{R}}} \circ e^{a,\mu}.$$

We see that the flow in $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}}$ appears as a flow solely along the second factor under the decomposition $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \simeq \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu}$.

6.2.3 The lifted dynamics under the decomposition

Recall the discussion from Section 3.2.2: the lifted flow $(\dot{\psi}_H^t)^{\mathcal{O}}$ on $\dot{L}^{\mathcal{O}}$ is generated by the vector field $A_H^{\mathcal{O}}$. The reduced Hamiltonian $H_{\mathcal{R}}^{\mathcal{O}}$ generates a reduced vector field $A_{H_{\mathcal{R}}^{\mathcal{O}}}$ and reduced flow $\dot{\psi}^t_{H_{\mathcal{R}}^{\mathcal{O}}}$ on $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, which is related to those on $\dot{L}^{\mathcal{O}}$ via $\dot{\Sigma}^{\mathcal{O}}$,

$$A_H^{\mathcal{O}} \sim_{\dot{\Sigma}^{\mathcal{O}}} A_{H_{\mathcal{R}}^{\mathcal{O}}} \qquad \text{and} \qquad \dot{\Sigma}^{\mathcal{O}} \circ (\dot{\psi}_H^t)^{\mathcal{O}} = \dot{\psi}^t_{H_{\mathcal{R}}^{\mathcal{O}}} \circ \dot{\Sigma}^{\mathcal{O}}.$$

$A_H^{\mathcal{O}}$ and $A_{H_{\mathcal{R}}^{\mathcal{O}}}$ both being (right) U(1)-invariant, $(\dot{\psi}_H^t)^{\mathcal{O}}$ and $\dot{\psi}^t_{H_{\mathcal{R}}^{\mathcal{O}}}$ are (right) U(1)-equivariant.

Again all of the above also applies for the flow $\dot{\psi}^t_{H_{\mathcal{R}}^{\mu}}$ on $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$.

Since for any $\xi \in \mathfrak{g}$,

$$[A_H^{\mathcal{O}}, \xi_{\dot{L}^{\mathcal{O}}}] = [A_H^{\mathcal{O}}, A_{J(\xi)}^{\mathcal{O}}] = -A^{\mathcal{O}}_{\{H, J(\xi)\}} = -A_0^{\mathcal{O}} = 0,$$

the vector field $A_H^{\mathcal{O}}$ is $\widetilde{G}$-invariant,

$$A_H^{\mathcal{O}}(\widetilde{g} \cdot p) = \widetilde{g} \cdot A_H^{\mathcal{O}}(p),$$

and $(\dot{\psi}_H^t)^{\mathcal{O}}$ is $\widetilde{G}$-equivariant,

$$(\dot{\psi}_H^t)^{\mathcal{O}}(\widetilde{g} \cdot p) = \widetilde{g} \cdot (\dot{\psi}_H^t)^{\mathcal{O}}(p).$$

Similarly $A_{H_{\mathcal{R}}^{\mathcal{O}}}$ is $\widetilde{G}$-invariant (with respect to the reduced $\widetilde{G}$-action—see Section 4.4),

$$A_{H_{\mathcal{R}}^{\mathcal{O}}}(\widetilde{g} \cdot q) = \widetilde{g} \cdot A_{H_{\mathcal{R}}^{\mathcal{O}}}(q),$$

and $\dot{\psi}^t_{H_{\mathcal{R}}^{\mathcal{O}}}$ is $\widetilde{G}$-equivariant,

$$\dot{\psi}^t_{H_{\mathcal{R}}^{\mathcal{O}}}(\widetilde{g} \cdot q) = \widetilde{g} \cdot \dot{\psi}^t_{H_{\mathcal{R}}^{\mathcal{O}}}(q).$$

The isomorphism $\dot{E}^{a,\mu}_{q_0} : \frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}} \boxdot \frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}} \to \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ is

$$\dot{E}^{a,\mu}_{q_0}\left((\widetilde{g} \cdot q_0) \boxdot q\right) = \widetilde{g} \cdot q.$$

Composition with the lifted flow yields

$$\begin{aligned}
\dot{E}^{a,\mu}_{q_0}\left((\widetilde{g} \cdot q_0) \boxdot \dot{\psi}^t_{H_{\mathcal{R}}^{\mu}}(q)\right) &= \widetilde{g} \cdot \dot{\psi}^t_{H_{\mathcal{R}}^{\mu}}(q) \\
&= \widetilde{g} \cdot \dot{\psi}^t_{H_{\mathcal{R}}^{\mathcal{O}}}(q) \\
&= \psi^t_{H_{\mathcal{R}}^{\mathcal{O}}}(\widetilde{g} \cdot q) \qquad \text{by } \widetilde{G}\text{-equivariance of the lifted flow} \\
&= \psi^t_{H_{\mathcal{R}}^{\mathcal{O}}}\left(\dot{E}^{a,\mu}_{q_0}((\widetilde{g} \cdot q_0) \boxdot q)\right),
\end{aligned}$$

that is,

$$\dot{E}^{a,\mu}_{q_0} \circ \left(\mathrm{id}_{\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a}} \boxdot \dot{\psi}^t_{H^\mu_{\mathcal{R}}} \right) = \dot{\psi}^t_{H^{\mathcal{O}}_{\mathcal{R}}} \circ \dot{E}^{a,\mu}_{q_0}.$$

The lifted flow on $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ decomposes into the identity times a flow on $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ under the decomposition $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \simeq \frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}} \boxdot \frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$.

6.2.4 The lifted group action under the decomposition

By constrast, the $\widetilde{G}$-action on $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ restricts to a $\widetilde{G}$-action on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$. We have that

$$\begin{aligned}
\dot{E}^{a,\mu}_{q_0}\left((\widetilde{g}' \cdot (\widetilde{g} \cdot q_0)) \boxdot q\right) &= \dot{E}^{a,\mu}_{q_0}\left((\widetilde{g}'\widetilde{g} \cdot q_0) \boxdot q\right) \\
&= (\widetilde{g}'\widetilde{g}) \cdot q \\
&= \widetilde{g}' \cdot (\widetilde{g} \cdot q) \\
&= \widetilde{g}' \cdot \dot{E}^{a,\mu}_{q_0}((\widetilde{g} \cdot q_0) \boxdot q),
\end{aligned}$$

and so the left $\widetilde{G}$-action on $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$ translates into a left $\widetilde{G}$-action on $\frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}}$ and an identity map on $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ under the decomposition $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \simeq \frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}} \boxdot \frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$.

6.3 Decomposition of the space of covariantly constant sections

We have demonstrated (Section 6.1.2) the existence of a bundle-connection isomorphism $\dot{E}^{a,\mu}_{q_0}$ covering $e^{a,\mu}$,

$$\begin{tikzcd}
\frac{\dot{L}^a}{(\mathcal{R}^{\mathrm{h}})^a} \boxdot \frac{\dot{L}^\mu}{(\mathcal{R}^{\mathrm{h}})^\mu} \arrow[r, shift left, "\dot{E}^{a,\mu}_{q_0}"] \arrow[d, "\dot{\tau}^a_{\mathcal{R}} \boxdot \dot{\tau}^\mu_{\mathcal{R}}"'] & \frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}} \arrow[l, shift left, "(\dot{E}^{a,\mu}_{q_0})^{-1}"] \arrow[d, "\dot{\tau}^{\mathcal{O}}_{\mathcal{R}}"] \\
\frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^\mu} \arrow[r, shift left, "e^{a,\mu}"] & \frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \arrow[l, shift left, "(e^{a,\mu})^{-1}"]
\end{tikzcd},$$

with

$$(\dot{E}^{a,\mu}_{q_0})^* \alpha^{\mathcal{O}}_{\mathcal{R}} = \alpha^a_{\mathcal{R}} \boxplus \alpha^\mu_{\mathcal{R}}$$

covering

$$(e^{a,\mu})^* \omega^{\mathcal{O}}_{\mathcal{R}} = \omega^a_{\mathcal{R}} \oplus \omega^\mu_{\mathcal{R}}.$$

Again denote the U(1)-bundles $\frac{\dot{L}^{\mathcal{O}}}{(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}}$, $\frac{\dot{L}^{a}}{(\mathcal{R}^{\mathrm{h}})^{a}}$, and $\frac{\dot{L}^{\mu}}{(\mathcal{R}^{\mathrm{h}})^{\mu}}$ by $\dot{L}^{\mathcal{O}}_{\mathcal{R}}$, $\dot{L}^{a}_{\mathcal{R}}$, and $\dot{L}^{\mu}_{\mathcal{R}}$ respectively, and their associated bundles by $L^{\mathcal{O}}_{\mathcal{R}}$, $L^{a}_{\mathcal{R}}$, and $L^{\mu}_{\mathcal{R}}$ respectively. Using the natural identification $(\dot{L}_1 \boxdot \dot{L}_2) \times_{\mathrm{U}(1)} \mathbb{C} \simeq L_1 \boxtimes L_2$ (see Appendix B), $\dot{E}^{a,\mu}_{q_0}$ induces a line bundle isomorphism $E^{a,\mu}_{q_0}$ of the associated line bundles mapping the corresponding induced covariant derivatives to each other:

$$\begin{tikzcd}[column sep=huge]
L^{a}_{\mathcal{R}} \boxtimes L^{\mu}_{\mathcal{R}} \arrow[r, shift left, "E^{a,\mu}_{q_0}"] \arrow[d, "\tau^{a}_{\mathcal{R}} \boxtimes \tau^{\mu}_{\mathcal{R}}"'] & L^{\mathcal{O}}_{\mathcal{R}} \arrow[l, shift left, "(E^{a,\mu}_{q_0})^{-1}"] \arrow[d, "\tau^{\mathcal{O}}_{\mathcal{R}}"] \\
\dfrac{\pi^{-1}(a)}{\mathcal{R}^{a}} \times \dfrac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}} \arrow[r, shift left, "e^{a,\mu}"] & \dfrac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \arrow[l, shift left, "(e^{a,\mu})^{-1}"]
\end{tikzcd} .$$

We denote the corresponding isomorphism of sections using the same symbol for convenience

$$E^{a,\mu}_{q_0} : \Gamma\left(L^{a}_{\mathcal{R}} \boxtimes L^{\mu}_{\mathcal{R}}\right) \longrightarrow \Gamma\left(L^{\mathcal{O}}_{\mathcal{R}}\right).$$

Proposition 6.3.1. *$E^{a,\mu}_{q_0}$ preserves the covariant derivatives, i.e, for $s \boxtimes t \in \Gamma\left(L^{a}_{\mathcal{R}} \boxtimes L^{\mu}_{\mathcal{R}}\right) \simeq \Gamma\left(L^{a}_{\mathcal{R}}\right) \boxtimes \Gamma\left(L^{\mu}_{\mathcal{R}}\right)$ and $(X, Y) \in T\left(\frac{\pi^{-1}(a)}{\mathcal{R}^{a}}\right) \times T\left(\frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}\right)$*

$$\left(\nabla^{\mathcal{O}}_{\mathcal{R}}\right)_{Te^{a,\mu}(X,\,Y)} \left\{E^{a,\mu}_{q_0}(s \boxtimes t)\right\} = \left(\nabla^{a}_{\mathcal{R}} \boxplus \nabla^{\mu}_{\mathcal{R}}\right)_{(X,\,Y)} \left\{s \boxtimes t\right\}.$$

Proof. This is essentially an unpacking of definitions:

$$\begin{aligned}
(E^{a,\mu}_{q_0}(s \boxtimes t))(\mathcal{R}^{(\nu,a)},\, \mathcal{R}^{(\mu,b)}) &= E^{a,\mu}_{q_0}((s \boxtimes t)(\mathcal{R}^{(\nu,a)},\, \mathcal{R}^{(\mu,b)})) \\
&= E^{a,\mu}_{q_0}\left(\left[p \boxdot q,\, (\dot{s} \boxdot \dot{t})(p \boxdot q)\right]_{\mathrm{id}_{\mathrm{U}(1)}}\right) \\
&= \left[\dot{E}^{a,\mu}_{q_0}(p \boxdot q),\, (\dot{s} \boxdot \dot{t})(p \boxdot q)\right]_{\mathrm{id}_{\mathrm{U}(1)}} \\
&= \left[r,\, (\dot{s} \boxdot \dot{t})((\dot{E}^{a,\mu}_{q_0})^{-1}(r))\right]_{\mathrm{id}_{\mathrm{U}(1)}},
\end{aligned}$$

where p, q, and r are any elements of the U(1)-fibers over $\mathcal{R}^{(\nu,a)}$, $\mathcal{R}^{(\mu,b)}$, and $\mathcal{R}^{(\nu,b)}$ respectively. Thus the U(1)-equivariant function corresponding to $E^{a,\mu}_{q_0}(s \boxtimes t) \in \Gamma(L^{\mathcal{O}}_{\mathcal{R}})$ is $(\dot{s} \boxdot \dot{t}) \circ (\dot{E}^{a,\mu}_{q_0})^{-1}$. This implies that the U(1)-equivariant function corresponding to $\left(\nabla^{\mathcal{O}}_{\mathcal{R}}\right)_{Te^{a,\mu}(X,\,Y)} \left\{E^{a,\mu}_{q_0}(s \boxtimes t)\right\}$ is

$$\begin{aligned}
(Te^{a,\mu}(X,\,Y))^{\mathrm{h}} \left\{(\dot{s} \boxdot \dot{t}) \circ (\dot{E}^{a,\mu}_{q_0})^{-1}\right\} &= T\dot{E}^{a,\mu}_{q_0}\left((X,\,Y)^{\mathrm{h}}\right) \left\{(\dot{s} \boxdot \dot{t}) \circ (\dot{E}^{a,\mu}_{q_0})^{-1}\right\} \\
&\qquad \text{since } T\dot{E}^{a,\mu}_{q_0} \text{ preserves horizontal vectors} \\
&= (X^{\mathrm{h}} \boxplus Y^{\mathrm{h}})\{\dot{s} \boxdot \dot{t}\} \\
&= (X^{\mathrm{h}}\dot{s}) \boxdot \dot{t} + \dot{s} \boxdot (Y^{\mathrm{h}}\dot{t}).
\end{aligned}$$

The last expression is the U(1)-equivariant function corresponding to

$$\{(\nabla^a_{\mathcal{R}})_X\, s\} \boxtimes t + s \boxtimes \{(\nabla^\mu_{\mathcal{R}})_Y\, t\} = (\nabla^a_{\mathcal{R}} \boxplus \nabla^\mu_{\mathcal{R}})_{(X,\,Y)}\, \{s \boxtimes t\}\,,$$

so we are done.

$\square$

In addition, we have shown (Lemma 6.1.3) that $Te^{a,\mu}$ is an isomorphism of reduced polarizations,

$$Te^{a,\mu}\,(F^a_{\mathcal{R}} \oplus F^\mu_{\mathcal{R}}) = F^{\mathcal{O}}_{\mathcal{R}}.$$

Combining everything, we have an isomorphism

$$E^{a,\mu}_{q_0} : \Gamma_{F^a_{\mathcal{R}} \oplus F^\mu_{\mathcal{R}}}\,(L^a_{\mathcal{R}} \boxtimes L^\mu_{\mathcal{R}}) \longrightarrow \Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}}),$$

or, using $\Gamma_{F_1 \oplus F_2}\,(L_1 \boxtimes L_2) = \Gamma_{F_1}\,(L_1) \boxtimes \Gamma_{F_2}\,(L_2)$ (see Appendix B),

$$E^{a,\mu}_{q_0} : \Gamma_{F^a_{\mathcal{R}}}\,(L^a_{\mathcal{R}}) \boxtimes \Gamma_{F^\mu_{\mathcal{R}}}\,(L^\mu_{\mathcal{R}}) \longrightarrow \Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}\left(L^{\mathcal{O}}_{\mathcal{R}}\right).$$

The discussion of the previous section tells us that through the decomposition $(\dot{E}^{a,\mu}_{q_0})^{-1}$, the $\widetilde{G}$-action and lifted flow $\dot{\psi}^t_{H^{\mathcal{O}}_{\mathcal{R}}}$ act separately on the spaces $\dot{L}^a_{\mathcal{R}}$ and $\dot{L}^\mu_{\mathcal{R}}$ respectively, and these induce corresponding actions on the spaces of sections $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ and $\Gamma_{F^\mu_{\mathcal{R}}}(L^\mu_{\mathcal{R}})$. Our discussion of the Borel-Weil theorem (Section 5.8) tells us that $\Gamma_{F^a_{\mathcal{R}}}\,(L^a_{\mathcal{R}})$ is an irreducible representation of $\widetilde{G}$. Hence we have separated the space of sections $\Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}})$ into a part on which the dynamics acts trivially, but which transforms under the $\widetilde{G}$-action via an irreducible representation, and a part where all the dynamics takes place, but which transforms trivially under the $\widetilde{G}$-action.

6.4 Commutativity of quantization and reduction

Recalling that the overall representation space is $\bigoplus_{\mathcal{O}} \Gamma_{F^{\mathcal{O}}}(L^{\mathcal{O}}) \simeq \bigoplus_{\mathcal{O}} \Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}})$, where the sum is over all admissible orbits $\mathcal{O}$, the results of the previous section tell us that

$$\bigoplus_{\mathcal{O}\text{ admissible}} \Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}}) \simeq \bigoplus_{\mathcal{O}\text{ admissible}} \Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}}) \boxtimes \Gamma_{F^\mu_{\mathcal{R}}}(L^\mu_{\mathcal{R}}),$$

where in the second sum, we pick one (arbitrary) representative $a \in \pi(\mathrm{J}^{-1}(\mathcal{O}))$ and $\mu \in \mathcal{O}$ for each admissible coadjoint orbit $\mathcal{O}$. Our final task is to show that the above isomorphism has the form

$$\mathcal{H} \simeq \bigoplus_{\lambda\text{ dominant integral}} (\mathcal{H}^\lambda)^* \otimes \left(\mathcal{H}^\lambda \otimes \mathcal{H}\right)^{\widetilde{G}},$$

this being the decomposition of the vector space $\mathcal{H}$ obtained by quantum reduction (see Appendix C). Doing so will prove that "quantization commutes with reduction."

First fix a maximal torus $\widetilde{T} \subset \widetilde{G}$ and corresponding set of positive roots, and choose μ in each admissible coadjoint orbit $\mathcal{O}$ such that $\frac{i}{\hbar}\mu$ is dominant in $i\mathfrak{t}^*$—this can always be arranged as a consequence of standard theorems on maximal tori in compact simple Lie groups. The discussion of the Borel-Weil Theorem in Section 5.8 tells us that $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}}) \simeq_{q_0} (\mathcal{H}^{\frac{i}{\hbar}\mu})^*$ (where $q_0 \in \dot{L}^a_{\mathcal{R}}$ lies above $\mathcal{R}^{(\mu,a)}$). Define $\mathcal{H}_\lambda$ by

$$\mathcal{H}_\lambda = \begin{cases} \Gamma_{F^\mu_{\mathcal{R}}}(L^\mu_{\mathcal{R}}) & \text{if } \lambda = \frac{i}{\hbar}\mu \text{ for some admissible } \mu \\ \{0\} & \text{otherwise} \end{cases} .$$

Then our representation space is

$$\mathcal{H} \simeq \bigoplus_{\mathcal{O}\text{ admissible}} \Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}(L^{\mathcal{O}}_{\mathcal{R}}) \simeq \bigoplus_{\substack{\mu \text{ admissible} \\ \frac{i}{\hbar}\mu \text{ dominant}}} (\mathcal{H}^{\frac{i}{\hbar}\mu})^* \otimes \Gamma_{F^\mu_{\mathcal{R}}}(L^\mu_{\mathcal{R}}) = \bigoplus_{\lambda \text{ d.i.}} (\mathcal{H}^\lambda)^* \otimes \mathcal{H}_\lambda.$$

The following proposition completes the proof.

Proposition 6.4.1. *Let $\mathcal{H}$ be a vector space carrying a representation of $\widetilde{G}$, and suppose there exists an isomorphism*

$$\mathcal{H} \simeq \bigoplus_{\lambda \text{ d.i.}} (\mathcal{H}^\lambda)^* \otimes \mathcal{H}_\lambda,$$

such that through the isomorphism, the representation acts via the irreducible representation $(\mathcal{H}^\lambda)^*$ on the first factor, and trivially on the second factor. Then

$$\mathcal{H}_\lambda \simeq (\mathcal{H}^\lambda \otimes \mathcal{H})^{\widetilde{G}}.$$

Proof. Given the hypothesis, we can say

$$\left(\mathcal{H}^\lambda \otimes \mathcal{H}\right)^{\widetilde{G}} \simeq \left(\mathcal{H}^\lambda \otimes \left(\bigoplus_{\lambda' \text{ d.i.}} (\mathcal{H}^{\lambda'})^* \otimes \mathcal{H}_{\lambda'}\right)\right)^{\widetilde{G}} \simeq \bigoplus_{\lambda' \text{ d.i}} \left(\mathcal{H}^\lambda \otimes (\mathcal{H}^{\lambda'})^*\right)^{\widetilde{G}} \otimes \mathcal{H}_{\lambda'}.$$

Via the canonical homomorphism $V^* \otimes W \simeq \mathrm{Hom}(V, W)$, the space of linear maps from V to W, we have

$$\left(\mathcal{H}^\lambda \otimes (\mathcal{H}^{\lambda'})^*\right)^{\widetilde{G}} \simeq \mathrm{Hom}_{\widetilde{G}}\left((\mathcal{H}^\lambda)^*, (\mathcal{H}^{\lambda'})^*\right)$$

where $\mathrm{Hom}_{\widetilde{G}}\left((\mathcal{H}^\lambda)^*, (\mathcal{H}^{\lambda'})^*\right)$ is the space of $\widetilde{G}$-equivariant linear maps, or *intertwiners*, between the irreducible representations $(\mathcal{H}^\lambda)^*$ and $(\mathcal{H}^{\lambda'})^*$. Schur's Lemma tells that this is nonzero if and only if $\lambda' = \lambda$, in which case it is one dimensional,

$$\mathrm{Hom}_{\widetilde{G}}\left((\mathcal{H}^{\lambda})^*,\, (\mathcal{H}^{\lambda'})^*\right) = \delta^{\lambda\lambda'}\mathbb{C}\{\mathrm{id}_{(\mathcal{H}^{\lambda})^*}\},$$

and we can write

$$\left(\mathcal{H}^{\lambda} \otimes \mathcal{H}\right)^{\widetilde{G}} \simeq \bigoplus_{\lambda' \text{ d.i.}} \delta^{\lambda\lambda'}\mathbb{C}\{\mathrm{id}_{(\mathcal{H}^{\lambda})^*}\} \otimes \mathcal{H}^{\lambda'} \simeq \mathcal{H}^{\lambda},$$

proving the assertion. □

For completeness, a "symplectic" proof that $\mathrm{Hom}_{\widetilde{G}}\left((\mathcal{H}^{\lambda})^*,\, (\mathcal{H}^{\lambda'})^*\right) = \delta^{\lambda\lambda'}\mathbb{C}\{\mathrm{id}_{(\mathcal{H}^{\lambda})^*}\}$ is provided in Appendix D, following ideas from [GS82].

Chapter 7

Conclusion

Previous discussions of geometric quantization and its interaction with symplectic reduction have tended to focus on the complex line bundle/covariant derivative picture. This can obscure the geometric significance of various constructions. By contrast, in this thesis we deal mainly with the U(1)-bundle/connection prequantum structures, and have proposed a notion of "prequantum reduction" of these structures. It is hoped that the advantages of this approach are now apparent. To summarize:

(i) Prequantum reduction and symplectic reduction can be given a satisfying unification within the framework of foliation reduction.

(ii) The quantization conditions on "admissible" momenta appear at the prequantum stage as consistency relations on the reduced prequantum structures, without reference to a polarization.

(iii) These quantization conditions can be given a geometric interpretation, namely that the leaves of the lifted foliation $(\mathcal{R}^{\mathrm{h}})^{\mathcal{O}}$ injectively cover those of $\mathcal{R}^{\mathcal{O}}$.

(iv) The factorization of the space of polarized sections coming from quantum reduction (Appendix C) is seen to be induced by a corresponding factorization on the reduced U(1)-bundles, which in turn covers the canonical symplectomorphism $\frac{\mathrm{J}^{-1}(\mathcal{O})}{\mathcal{R}^{\mathcal{O}}} \simeq \frac{\pi^{-1}(a)}{\mathcal{R}^{a}} \times \frac{\mathrm{J}^{-1}(\mu)}{\mathcal{R}^{\mu}}$ of the base manifold.

As stated in the introduction, one glaring omission in the treatment of geometric quantization in this thesis is the definition of an inner product on the space of polarized sections, and corresponding discussion of its behavior under reduction. More generally, in order to quantize those classical observables whose flows do not preserve the polarization, one must construct a nondegenerate pairing (the so-called *Blattner-Kostant-Sternberg* or *BKS* pairing) between spaces of sections covariantly constant with respect to different polarizations. Construction of this pairing can be achieved using a metaplectic structure on the symplectic manifold—see [Bla77] or [GS90] for a discussion. For many classical systems, the obvious polarizations are not preserved by most physically interesting Hamiltonians (typically those quadratic in momentum), and so this omission is significant. It is surprising that after over forty years of geometric quantization, no complete treatment of as basic an example as the symmetric rigid body exists[1]. The phase space for the symmetric rigid body is $T^*\mathrm{SO}(3)$ with its usual symplectic form, and Hamiltonian

$$H(\mathbf{\Pi}) = \frac{1}{2I_1}\mathbf{\Pi}^2 - \frac{I_3 - I_1}{2I_1 I_3}\,\Pi_3^2,$$

where I_1 and I_3 are the principal moments of inertia, and $\mathbf{\Pi}$ is the angular momentum in the body frame. The Marsden-Weinstein quotient of the system is symplectomorphic to a coadjoint orbit in $\mathfrak{so}(3)^*$. In the standard treatment of this system (see for example [LL77, Section 103]), the quantum mechanical eigenstates of this system agree with the (body) angular momentum eigenstates. It would seem like a useful exercise to reproduce this result in the geometric quantization framework. At a point $\mu \cdot g \in T^*\mathrm{SO}(3)$, the polarization employed by Filippini in [Fil95] (and generalized in this thesis) is

$$F_{\mu\cdot g} = (\mathfrak{n}_\mu^+ \cdot \mu \cdot g) \oplus (\mathfrak{g}_\mu^{\mathbb{C}} \cdot \mu \cdot g) \oplus (\mu \cdot \mathfrak{n}_\mu^- \cdot g).$$

The Hamiltonian flow due to H does not preserve this polarization, and so consideration of an appropriate metaplectic structure on $T^*\mathrm{SO}(3)$ and its behavior under symplectic reduction is crucial. The topic of metaplectic reduction appears to be little explored in the literature; particularly relevant to the presentation in this thesis is the work of Robinson [Rob92]. The applicability of Robinson's results remains unclear. Perhaps a consideration of metaplectic reduction within the context of foliation reduction could provide some clues to an appropriate treatment.

[1] Robson [Rob96] considers cotangent bundles in general, but the quantized Hamiltonian is constructed by considerations outside geometric quantization.

Appendix A

Prequantization and central extensions

The results of this appendix are adapted from [Kos70]. We remind that we are restricting to the case where the manifold M is connected.

A.1 Definitions

Let $(\dot{L}_1, \dot{\tau}_1, M_1)$, $(\dot{L}_2, \dot{\tau}_2, M_2)$ be two (right) principal U(1)-bundles with respective connections α_i. Denote the corresponding associated line bundles by L_i, and α_i-invariant Hermitian structures by H_i. An ***isomorphism*** of $(\dot{L}_1,\, \alpha_1)$ and $(\dot{L}_2,\, \alpha_2)$ is a U(1)-equivariant diffeomorphism $\dot{F} : \dot{L}_1 \to \dot{L}_2$ such that

$$(\dot{F})^* \alpha_2 = \alpha_1.$$

The U(1)-equivariance ensures that $\dot{F}$ maps fibers to fibers, and so there is a map $\check{F} : M_1 \to M_2$ which makes the following diagram commute,

$$\begin{array}{ccc} \dot{L}_1 & \xrightarrow{\dot{F}} & \dot{L}_2 \\ \downarrow{\scriptstyle \dot{\tau}_1} & & \downarrow{\scriptstyle \dot{\tau}_2} \\ M_1 & \xrightarrow{\check{F}} & M_2 \end{array} \quad .$$

$\dot{F}$ is said to ***cover*** $\check{F}$. $\dot{F}$ induces a corresponding isomorphism $F : L_1 \to L_2$ (also covering $\check{F}$),

$$F\left([p,\, z]_{\mathrm{id}_{\mathrm{U}(1)}}\right) = \left[\dot{F}(p),\, z\right]_{\mathrm{id}_{\mathrm{U}(1)}}.$$

The U(1)-equivariance of $\dot{F}$ ensures that F is well-defined. An immediate consequence of the definition is that for $x \in M_1$, F maps $(L_1)_x$ isomorphically to $(L_2)_{\check{F}(x)}$, and it is easily checked that

$F^*H_2 = H_1$. So F defines an isometry between these fibers. If $M_1 = M_2 = M$, say, an isomorphism of $(\dot{L}_1,\, \alpha_1)$ and $(\dot{L}_2,\, \alpha_2)$ which covers the identity map $\mathrm{id}_M : M \to M$ is said to be a ***vertical isomorphism***, and $(\dot{L}_1,\, \alpha_1)$ and $(\dot{L}_2,\, \alpha_2)$ are said to be ***vertically isomorphic*** or ***equivalent***.

Given a bundle $\dot{L}$ over M and a diffeomorphism $\rho : M \to M$, we can define the ***pullback bundle*** $\rho^*\dot{L}$ to be the set

$$\{(x, p) \,|\, x \in M,\, p \in \dot{L},\, \rho(x) = \dot{\tau}(p)\},$$

with projection $\dot{\tau}_\rho(x, p) = x$. In other words, $\rho^*\dot{L}$ is the bundle over M with fiber $\dot{L}_{\rho(x)}$ over the point $x \in M$. We can define a diffeomorphism $\dot{\Theta}_\rho : \rho^*\dot{L} \to \dot{L}$ by

$$\dot{\Theta}_\rho(x, p) = p.$$

$\dot{\Theta}_\rho$ is then an isomorphism of $(\rho^*\dot{L},\, (\dot{T}_\rho)^*\alpha)$ and $(\dot{L},\, \alpha)$ which covers ρ. The Hermitian forms associated with $(\rho^*\dot{L},\, (\dot{\Theta}_\rho)^*\alpha)$ and $(\dot{L},\, \alpha)$ are related by

$$H_\rho = (\Theta_\rho)^* H,$$

as is easily checked.

We define the ***isotropy group*** $\mathbf{Isot}(\dot{\boldsymbol{L}},\, \boldsymbol{\alpha})$ of $(\dot{L},\, \alpha)$ to be the group of diffeomorphisms $\rho : M \to M$ such that $(\rho^*\dot{L},\, (\dot{\Theta}_\rho)^*\alpha)$ and $(\dot{L},\, \alpha)$ are equivalent, i.e., related by a vertical isomorphism. We define the ***isomorphism group*** $\mathbf{Isom}(\dot{\boldsymbol{L}},\, \boldsymbol{\alpha})$ of $(\dot{L},\, \alpha)$ to be the group of isomorphisms of $(\dot{L},\, \alpha)$ with itself.

A.2 The isomorphism group as a Lie group central extension of the isotropy group

Proposition A.2.1. *The map* $\check{} : \dot{E} \mapsto \check{E}$ *maps* $\mathrm{Isom}(\dot{L},\, \alpha)$ *into* $\mathrm{Isot}(\dot{L},\, \alpha)$.

Proof. Let $\dot{E} \in \mathrm{Isom}(\dot{L},\, \alpha)$. Then we have the following commutative diagram,

$$\begin{tikzcd}
\dot{L} \arrow[r, "\dot{E}"] \arrow[d, "\dot{\tau}"'] & \dot{L} \arrow[d, "\dot{\tau}"'] & (\check{E})^*\dot{L} \arrow[l, "\dot{\Theta}_{\check{E}}"'] \arrow[d, "\dot{\tau}_{\check{E}}"'] \\
M \arrow[r, "\check{E}"] & M & M \arrow[l, "\check{E}"']
\end{tikzcd} .$$

The diffeomorphism $(\dot{E})^{-1} \circ \dot{\Theta}_{\check{E}} : (\check{E})^{*}\dot{L} \to \dot{L}$ covers id_M, and

$$((\dot{E})^{-1} \circ \dot{\Theta}_{\check{E}})^{*}\alpha = (\dot{\Theta}_{\check{E}})^{*}(\dot{E}^{-1})^{*}\alpha = (\dot{\Theta}_{\check{E}})^{*}\alpha,$$

since $\dot{E} \in \mathrm{Isom}(\dot{L},\, \alpha)$. It follows that $((\check{E})^{*}\dot{L},\, (\dot{\Theta}_{\check{E}})^{*}\alpha)$ and $(\dot{L},\, \alpha)$ are vertically isomorphic, i.e., $\check{E} \in \mathrm{Isot}(\dot{L},\, \alpha)$. □

Proposition A.2.2. *The sequence*

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\Psi.} \mathrm{Isom}(\dot{L},\, \alpha) \xrightarrow{\check{\dot{\ }}} \mathrm{Isot}(\dot{L},\, \alpha) \longrightarrow 1$$

is exact, making $\mathrm{Isom}(\dot{L},\, \alpha)$ *a* $\mathrm{U}(1)$*-central extension of* $\mathrm{Isot}(\dot{L},\, \alpha)$ *(as before,* $\Psi.$ *is the right* $\mathrm{U}(1)$*-action on* $\dot{L}$*).*

Proof. $\Psi.$ is clearly injective. To show surjectivity of $\check{\dot{\ }}$, let $\rho \in \mathrm{Isot}(\dot{L},\, \alpha)$, and let $\dot{F} : \dot{L} \to \rho^{*}\dot{L}$ be a vertical isomorphism between $(\dot{L},\, \alpha)$ and $(\rho^{*}\dot{L},\, (\dot{\Theta}_{\rho})^{*}\alpha)$. Then we have the commutative diagram

$$\begin{tikzcd}
\dot{L} \arrow[r, "\dot{F}"] \arrow[d, "\dot{\tau}"'] & \rho^{*}\dot{L} \arrow[r, "\dot{\Theta}_{\rho}"] \arrow[d, "\dot{\tau}_{\rho}"'] & \dot{L} \arrow[d, "\dot{\tau}"'] \\
M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\rho"] & M
\end{tikzcd} .$$

The diffeomorphism $\dot{\Theta}_{\rho} \circ \dot{F} : \dot{L} \to \dot{L}$ covers $\rho : M \to M$ and

$$(\dot{\Theta}_{\rho} \circ \dot{F})^{*}\alpha = (\dot{F})^{*}(\dot{\Theta}_{\rho})^{*}\alpha = \alpha,$$

so $\dot{\Theta}_{\rho} \circ \dot{F}$ is an element of $\mathrm{Isom}(\dot{L},\, \alpha)$ covering ρ, and hence maps to ρ under $\check{\dot{\ }}$.

It is clear that $\mathrm{Im}(\Psi.) \subset \ker[\check{\dot{\ }}]$.

Finally, suppose $\dot{G} \in \ker[\check{\dot{\ }}]$. Then $\check{G} = \mathrm{id}_M$, so $\dot{G}$ must be of the form

$$\dot{G}(p) = p \cdot w(p) \qquad\qquad \text{for all } p \in \dot{L},$$

for some $w : \dot{L} \to \mathrm{U}(1)$. $\mathrm{U}(1)$-equivariance of $\dot{G}$ implies that $w(p)$ is constant along each fiber, i.e, $w(p) = \check{w}(\dot{\tau}(p))$ for some $\check{w} : M \to \mathrm{U}(1)$. It is then easily checked that

$$(\dot{G})^{*}\alpha = \alpha + \frac{1}{w}\, dw.$$

Since $\dot{G} \in \mathrm{Isom}(\dot{L},\, \alpha)$, $(\dot{G})^{*}\alpha = \alpha$, so $dw = 0$. Connectedness of M implies that $w = \text{constant on } \dot{L}$.

So $\dot{G} = \Psi_w \in \mathrm{Im}(\Psi.)$.

$\square$

A.3 The lifted Hamiltonian vector fields as a Lie algebra central extension of the Hamiltonian vector fields

In the case when $(M,\, \omega)$ is a symplectic manifold, and $(\dot{L},\, \alpha)$ is a U(1)-bundle over M with curvature $\Omega^\alpha = \frac{\varepsilon_0}{h}\omega$, proposition A.2.2 has a corresponding infinitesimal version. Let $\mathrm{Ham}(M,\omega)$ denote the set of Hamiltonian vector fields on M, and let $\mathrm{Ham}(\dot{L},\, \alpha)$ denote the set of lifted Hamiltonian vector fields, defined by

$$\mathrm{Ham}(\dot{L},\, \alpha) = \left\{ A_f \;= X_f^{\mathrm{h}} - (f \circ \dot{\tau}) \left(\frac{\varepsilon_0}{h}\right)_{\dot{L}} \,\middle|\, f \in C^\infty(M) \right\}.$$

The vector fields in $\mathrm{Ham}(\dot{L},\, \alpha)$ preserve the connection α (see proof of Proposition A.4.1 (i)).

The covering projection $\check{\cdot}$ induces an analogous projection for U(1)-invariant vector fields on $\dot{L}$.

Proposition A.3.1. *The sequence*

$$0 \longrightarrow \mathfrak{u}(1) \xrightarrow{T\Psi. = (\cdot)_{\dot{L}}} \mathrm{Ham}(\dot{L},\, \alpha) \xrightarrow{\check{\cdot}} \mathrm{Ham}(M,\omega) \longrightarrow 0$$

is exact.

Proof. $T\Psi.$ is clearly injective, and $\check{\cdot}$ is clearly surjective. If $T\dot{\tau}(A_f) = 0$, then $X_f = 0$, implying that $f = \text{constant}$. So $A_f = -\text{constant} \times \left(\frac{\varepsilon_0}{h}\right)_{\dot{L}}$, which is true if and only if $A_f \in \mathrm{Im}(T\Psi.)$. $\square$

A.4 Relationship between the central extensions

We now demonstrate why Proposition A.3.1 is the infinitesimal version of Proposition A.2.2.

Proposition A.4.1. *(i) The Lie algebra of* $\mathrm{Isom}(\dot{L},\, \alpha)$ *is* $\mathrm{Ham}(\dot{L},\, \alpha)$.

(ii) The Lie algebra of $\mathrm{Isot}(\dot{L},\, \alpha)$ *is* $\mathrm{Ham}(M,\omega)$.

Note on Proposition A.4.1: here we are totally ignoring the technicalities associated with infinite-dimensional Lie groups. If necessary, the obvious modified proposition can be taken to hold for arbitrary finite-dimensional subgroups of $\mathrm{Isot}(\dot{L},\, \alpha)$ and $\mathrm{Isom}(\dot{L},\, \alpha)$.

Proof. (i) $A_f \in \mathrm{Ham}(\dot{L},\, \alpha)$ is U(1)-invariant, so it generates a one-parameter group of U(1)-

equivariant diffeomorphisms of $\dot{L}$. Also

$$\begin{aligned}
\mathcal{L}_{A_f}\alpha &= \mathrm{d}\left(\mathrm{i}_{A_f}\alpha\right) + \mathrm{i}_{A_f}\mathrm{d}\alpha \\
&= \mathrm{d}\left(-(f\circ\dot{\tau})\frac{\varepsilon_0}{h}\right) + \mathrm{i}_{A_f}(\dot{\tau})^*\left(\frac{\varepsilon_0}{h}\omega\right) \\
&= -\frac{\varepsilon_0}{h}(\dot{\tau})^*\mathrm{d}f + \frac{\varepsilon_0}{h}(\dot{\tau})^*(\mathrm{i}_{X_f}\omega) \\
&= 0,
\end{aligned}$$

so the diffeomorphisms preserve α. Hence, the one-parameter subgroup lies in $\mathrm{Isom}(\dot{L},\,\alpha)$. Conversely, let $\dot{F}_t$ be a one-parameter subgroup of $\mathrm{Isom}(\dot{L},\,\alpha)$. Let A be the U(1)-invariant vector field generated by $\dot{F}_t$, i.e, $A_p = \frac{d}{dt}\dot{F}_t(p)|_{t=0}$. The U(1)-invariance properties of A and α tell us that $\alpha(A)$ is constant along the fibers of $\dot{L}$. Since α is $\mathfrak{u}(1)$-valued we can write

$$\alpha(A) = -(f\circ\dot{\tau})\frac{\varepsilon_0}{h}$$

for some $f \in C^\infty(M)$. Finally

$$(\dot{F}_t)^*\alpha = \alpha \implies \mathcal{L}_A\alpha = 0.$$

Using Cartan's magic formula $\mathcal{L}_A = \mathrm{d}\circ\mathrm{i}_A + \mathrm{i}_A\circ\mathrm{d}$ this becomes

$$\begin{aligned}
&\mathrm{d}(\alpha(A)) + \mathrm{i}_A\mathrm{d}\alpha = 0 \\
\implies &\mathrm{d}\left(-(f\circ\dot{\tau})\frac{\varepsilon_0}{h}\right) + \mathrm{i}_A(\dot{\tau})^*\left(\frac{\varepsilon_0}{h}\omega\right) = 0 \\
\implies &(\dot{\tau})^*\mathrm{d}f = (\dot{\tau})^*\left(\mathrm{i}_{\check{A}}\omega\right).
\end{aligned}$$

Since $\dot{\tau}$ is a surjective submersion, this tells us that

$$\mathrm{d}f = \mathrm{i}_{\check{A}}\omega,$$

i.e. , $\check{A} = X_f$. Then

$$\begin{aligned}
A &= (\check{A})^{\mathrm{h}} + (\alpha(A))_{\dot{L}} \\
&= X_f^{\mathrm{h}} - (f\circ\dot{\tau})\left(\frac{\varepsilon_0}{h}\right)_{\dot{L}} \\
&= A_f,
\end{aligned}$$

i.e. , $A \in \mathrm{Ham}(\dot{L},\,\alpha)$.

(ii) Let $X \in \mathrm{Ham}(M,\omega)$. Then $X = X_f$ for some $f \in C^\infty(M)$. For this f, part (i) demonstrates that A_f generates a one-parameter subgroup of $\mathrm{Isom}(\dot{L},\, \alpha)$. By Proposition A.2.1, this projects to a one-parameter subgroup of $\mathrm{Isot}(\dot{L},\, \alpha)$. This is precisely the one-parameter subgroup generated by $X = X_f$ (i.e., the Hamiltonian flow of ϕ_f^t of f).

Conversely, let ρ_t be a one-parameter subgroup of $\mathrm{Isot}(\dot{L},\, \alpha)$. By Proposition A.2.2, for each t there is a $\dot{E}_t \in \mathrm{Isom}(\dot{L},\, \alpha)$ covering ρ_t. We can arrange for $\dot{E}_t$ to depend smoothly on t. $\dot{E}_{t_1} \circ \dot{E}_{t_2} \circ (\dot{E}_{t_1+t_2})^{-1}$ is an element of $\mathrm{Isom}(\dot{L},\, \alpha)$ covering id_M, so again by Proposition A.2.2, for each $t_1,\, t_2 \in \mathbb{R}$ there is a $\xi(t_1,\, t_2) \in \mathfrak{u}(1)$ such that

$$\dot{E}_{t_1} \circ \dot{E}_{t_2} = \dot{E}_{t_1+t_2} \cdot e^{\xi(t_1,\, t_2)}.$$

Associativity of function composition and equivariance of the $\dot{E}_t$ impose the following (cocycle) condition on ξ

$$\xi(t_1,\, t_2) + \xi(t_1+t_2,\, t_3) = \xi(t_2,\, t_3) + \xi(t_1,\, t_2+t_3) \qquad \forall t_1,\, t_2,\, t_3 \in \mathbb{R}.$$

Let $\chi(t) = -\xi(t, 0) + \int_0^t \xi_2(s,\, 0)\, ds$ (where ξ_2 denotes derivative with respect to the second variable), and let $\dot{F}_t = \dot{E}_t \cdot e^{\chi(t)} \in \mathrm{Isom}(\dot{L},\, \alpha)$. Then

$$\dot{F}_{t_1} \circ \dot{F}_{t_2} = \dot{F}_{t_1+t_2} \cdot e^{\xi(t_1,\, t_2) - (\chi(t_1+t_2) - \chi(t_1) - \chi(t_2))}.$$

But

$$\begin{aligned} \chi(t_1+t_2) - \chi(t_1) - \chi(t_2) &= -\xi(t_1+t_2, 0) + \xi(t_1, 0) + \xi(t_2, 0) \\ &\quad + \int_{t_1}^{t_1+t_2} \xi_2(s,\, 0)\, ds - \int_0^{t_2} \xi_2(s,\, 0)\, ds \\ &= -\xi(t_1+t_2, 0) + \xi(t_1, 0) + \xi(t_2, 0) \\ &\quad + \int_0^{t_2} \{\xi_2(u+t_1,\, 0) - \xi_2(u,\, 0)\}\, du. \end{aligned}$$

Differentiating the cocycle condition with respect to t_3, and evaluating at $t_3 = 0$, gives

$$\xi_2(t_1+t_2,\, 0) = \xi_2(t_2,\, 0) + \xi_2(t_1,\, t_2),$$

so the above integral becomes

$$\int_0^{t_2} \xi_2(t_1,\, u)\, du = \xi(t_1,\, t_2) - \xi(t_1, 0).$$

It follows that

$$\begin{aligned}\chi(t_1+t_2)-\chi(t_1)-\chi(t_2) &= -\xi(t_1+t_2,0)+\xi(t_2,0)+\xi(t_1,t_2)\\ &= \xi(t_1,t_2) \qquad\qquad \text{by the cocycle condition with } t_3=0,\end{aligned}$$

and so

$$\dot{F}_{t_1}\circ\dot{F}_{t_2}=\dot{F}_{t_1+t_2}.\quad [1]$$

We now have a one-parameter subgroup of $\mathrm{Isom}(\dot{L},\,\alpha)$ which, by part (i), is generated by a vector field A_f for some $f\in C^\infty(M)$. Then $\rho_t=\check{F}_t$ is generated by $\check{A}_f=X_f\in \mathrm{Ham}(M,\omega)$.

□

[1]This result can be summarized by saying that the second group cohomology of $\mathbb{R}$ is trivial—see for example [dAI95].

Appendix B

The exterior products of circle bundles and their associated line bundles

The aim of this appendix is to introduce the exterior product $\boxdot$ on principal U(1)-bundles and its relation to the exterior tensor product $\boxtimes$ on complex line bundles, and to prove that if L_1 and L_2 are complex line bundles over M_1 and M_2 respectively, and F_1 and F_2 are distributions on M_1 and M_2, then the following identity holds

$$\Gamma_{F_1 \oplus F_2}(L_1 \boxtimes L_2) = \Gamma_{F_1}(L_1) \boxtimes \Gamma_{F_2}(L_2).$$

B.1 The exterior product on circle bundles

Suppose that $\dot{L}_1$ is a (right) U(1)-bundle over M_1, and $\dot{L}_2$ is a U(1)-bundle over M_2, with corresponding projections $\dot{\tau}_1$, $\dot{\tau}_2$, where $M_1 \neq M_2$ in general. Defining the equivalence relation

$$(p_1,\, p_2) \sim_{\boxdot} (p_1 \cdot w,\, p_2 \cdot w^{-1}) \qquad \text{for all } w \in \mathrm{U}(1),$$

on $\dot{L}_1 \times \dot{L}_2$, we define the ***exterior product*** $\dot{L}_1 \boxdot \dot{L}_2$ to be the space of equivalence classes of $\sim_{\boxdot}$,

$$\dot{L}_1 \boxdot \dot{L}_2 = \{\, [p_1,\, p_2]_{\sim_{\boxdot}} \,|\, p_1 \in \dot{L}_1,\, p_2 \in \dot{L}_2 \,\}.$$

and denote the projection by $\boxdot : \dot{L}_1 \times \dot{L}_2 \to \dot{L}_1 \boxdot \dot{L}_2$,

$$p_1 \boxdot p_2 := [p_1,\, p_2]_{\sim_{\boxdot}}.$$

Then $\dot{L}_1 \boxdot \dot{L}_2$ is a right principle U(1)-bundle, with right U(1)-action

$$(p_1 \boxdot p_2) \cdot w := (p_1 \cdot w) \boxdot p_2 = p_1 \boxdot (p_2 \cdot w),$$

and projection $\dot{\tau}_1 \boxdot \dot{\tau}_2 : \dot{L}_1 \boxdot \dot{L}_2 \to M_1 \times M_2$,

$$\dot{\tau}_1 \boxdot \dot{\tau}_2 \, (p_1 \boxdot p_2) := (\dot{\tau}_1(p_1),\, \dot{\tau}_2(p_2)).$$

B.2 The exterior tensor product on line bundles

For the associated line bundles L_1, L_2, their standard ***exterior tensor product*** $L_1 \boxtimes L_2$ can be constructed as the quotient of $L_1 \times L_2$ under the equivalence relation

$$(u_1,\, u_2) \sim_{\boxtimes} (c \cdot u_1,\, c^{-1} \cdot u_2) \qquad \text{for all } c \in \mathbb{C} - \{0\},$$

with quotient map $\boxtimes : L_1 \times L_2 \to L_1 \boxtimes L_2$,

$$u_1 \boxtimes u_2 = [u_1,\, u_2]_{\sim_{\boxtimes}}.$$

$L_1 \boxtimes L_2$ has a left $(\mathbb{C} - \{0\})$-action,

$$c \cdot (u_1 \boxtimes u_2) := (c \cdot u_1) \boxtimes u_2 = u_1 \boxtimes (c \cdot u_2),$$

and a projection $\tau_1 \boxtimes \tau_2 : L_1 \boxtimes L_2 \to M_1 \times M_2$, given by

$$\tau_1 \boxtimes \tau_2 \, (u_1 \boxtimes u_2) := (\tau_1(u_1),\, \tau_2(u_2)).$$

$L_1 \boxtimes L_2$ can be thought of as the line bundle associated to $\dot{L}_1 \boxdot \dot{L}_2$ via the natural isomorphism $I_{\boxtimes} : L_1 \boxtimes L_2 \to (\dot{L}_1 \boxdot \dot{L}_2) \times_{\mathrm{id}_{\mathrm{U}(1)}} \mathbb{C}$ given by

$$I_{\boxtimes} \left([p_1,\, z_1]_{\mathrm{id}_{\mathrm{U}(1)}} \boxtimes [p_2,\, z_2]_{\mathrm{id}_{\mathrm{U}(1)}} \right) = [p_1 \boxdot p_2,\, z_1 z_2]_{\mathrm{id}_{\mathrm{U}(1)}}.$$

With this identification, the induced Hermitian forms

$$\begin{aligned} H_{1\boxtimes 2} &: (L_1 \boxtimes L_2) \times_{(M_1 \times M_2)} (L_1 \boxtimes L_2) \to \mathbb{C}, \\ H_1 &: L_1 \times_{M_1} L_2 \to \mathbb{C}, \\ H_2 &: L_2 \times_{M_2} L_2 \to \mathbb{C}, \end{aligned}$$

on $L_1 \boxtimes L_2$, L_1 and L_2 are related as follows: given pairs $(u_1 \boxtimes u_2, v_1 \boxtimes v_2) \in (L_1 \boxtimes L_2) \times_{(M_1 \times M_2)} (L_1 \boxtimes L_2)$, write the elements of L_1 as

$$u_1 = [p_1,\, y_1]_{\mathrm{id}_{\mathrm{U}(1)}}\,,\ v_1 = [p_1,\, z_1]_{\mathrm{id}_{\mathrm{U}(1)}}\,,$$

and the elements of L_2 as

$$u_2 = [p_2,\, y_2]_{\mathrm{id}_{\mathrm{U}(1)}}\,,\ v_2 = [p_2,\, z_2]_{\mathrm{id}_{\mathrm{U}(1)}}\,.$$

Then

$$\begin{aligned} H_{1\boxtimes 2}\,(u_1 \boxtimes u_2,\, v_1 \boxtimes v_2) &= H_{1\boxtimes 2}\left([p_1 \boxdot p_2,\, y_1 y_2]_{\mathrm{id}_{\mathrm{U}(1)}}\,,\, [p_1 \boxdot p_2,\, z_1 z_2]_{\mathrm{id}_{\mathrm{U}(1)}}\right) \\ &= \overline{y_1 y_2} z_1 z_2 \\ &= (\overline{y_1}\, z_1)(\overline{y_2}\, z_2) \\ &= H_1\left([p_1,\, y_1]_{\mathrm{id}_{\mathrm{U}(1)}}\,,\, [p_1,\, z_1]_{\mathrm{id}_{\mathrm{U}(1)}}\right) H_2\left([p_2,\, y_2]_{\mathrm{id}_{\mathrm{U}(1)}}\,,\, [p_2,\, z_2]_{\mathrm{id}_{\mathrm{U}(1)}}\right) \\ &= H_1(u_1,\, v_1)\, H_2(u_2,\, v_2). \end{aligned}$$

B.3 The connection on the exterior product and its curvature

Given $U_1 \in T_{p_1}\dot{L}_1$ and $U_2 \in T_{p_2}\dot{L}_2$, we define a vector $U_1 \boxplus U_2 \in T_{p_1 \boxdot p_2}\left(\dot{L}_1 \boxdot \dot{L}_2\right)$ as follows: let $\gamma_i : (-\epsilon,\, \epsilon) \to \dot{L}_i$, $i = 1, 2$, be such that $\gamma_i'(0) = U_i$. Define

$$U_1 \boxplus U_2 := (\gamma_1 \boxdot \gamma_2)'(0).$$

Since every curve $\gamma : (\epsilon,\, \epsilon) \to \dot{L}_1 \boxdot \dot{L}_2$ has a representation $\gamma = \gamma_1 \boxdot \gamma_2$, every vector on $\dot{L}_1 \boxdot \dot{L}_2$ can be written in this form. From the definition it follows that

$$U_1 \boxplus U_2 + V_1 \boxplus V_2 = (U_1 + V_1) \boxplus (U_2 + V_2)\,.$$

If $U_1 \boxplus U_2 = 0_{p_1} \boxplus 0_{p_2}$, then $(\gamma_1 \boxdot \gamma_2)(t) = \gamma_1(t) \boxdot \gamma_2(t) = p_1 \boxdot p_2$, so $\gamma_1(t) = p \cdot w(t)$, $\gamma_2(t) = p_2 \cdot w(t)^{-1}$ for some $w : (-\epsilon,\, \epsilon) \to \mathrm{U}(1)$, and so

$$U_1 = p_1 \cdot \varepsilon, \qquad U_2 = -p_2 \cdot \varepsilon,$$

where $\varepsilon = w'(0)$.

Given connections α_1, α_2 on $\dot{L}_1$, $\dot{L}_2$, we define a connection $\alpha_1 \boxplus \alpha_2$ on $\dot{L}_1 \boxdot \dot{L}_2$ by

$$(\alpha_1 \boxplus \alpha_2)_{p_1 \boxdot p_2}(U_1 \boxplus U_2) = (\alpha_1)_{p_1}(U_1) + (\alpha_2)_{p_2}(U_2).$$

Since

$$(\alpha_1 \boxplus \alpha_2)_{p_1 \boxdot p_2}(p_1 \cdot \varepsilon \boxplus (-p_2 \cdot \varepsilon)) = (\alpha_1)_{p_1}(p_1 \cdot \varepsilon) + (\alpha_2)_{p_2}(-p_2 \cdot \varepsilon) = \varepsilon + (-\varepsilon) = 0,$$

the connection $\alpha_1 \boxplus \alpha_2$ is well-defined.

From the definition of $\alpha_1 \boxplus \alpha_2$ it is not difficult to show that it satisfies the two defining properties of a U(1)-bundle connection, namely

$$\begin{aligned} (\alpha_1 \boxplus \alpha_2)_{p_1 \boxdot p_2}((p_1 \boxdot p_2) \cdot \varepsilon) &= \varepsilon \qquad \text{for } \varepsilon \in \mathfrak{u}(1),\, p_1 \boxdot p_2 \in \dot{L}_1 \boxdot \dot{L}_2, \text{ and} \\ \Psi_w^*(\alpha_1 \boxplus \alpha_2) &= \alpha_1 \boxplus \alpha_2 \qquad \text{for } w \in \mathrm{U}(1). \end{aligned}$$

Denote the curvature of connection α_i by Ω_i, i.e.,

$$\mathrm{d}\alpha_1 = (\dot{\tau}_1)^* \Omega_1 \qquad \text{and} \qquad \mathrm{d}\alpha_2 = (\dot{\tau}_2)^* \Omega_2.$$

Using the obvious extension of $\boxplus$ to $\mathfrak{u}(1)$-valued 2-forms, we have that

$$\begin{aligned} \mathrm{d}(\alpha_1 \boxplus \alpha_2) &= \mathrm{d}\alpha_1 \boxplus \mathrm{d}\alpha_2 \\ &= (\dot{\tau}_1)^* \Omega_1 \boxplus (\dot{\tau}_2)^* \Omega_2 \\ &= (\dot{\tau}_1 \boxdot \dot{\tau}_2)^* \left((p_{M_1})^* \Omega_1 + (p_{M_2})^* \Omega_2 \right), \end{aligned}$$

the latter equality following from $T_{p_1 \boxdot p_2}(\dot{\tau}_1 \boxdot \dot{\tau}_2)(U_1 \boxplus U_2) = (T_{p_1}\dot{\tau}_1(U_1),\, T_{p_2}\dot{\tau}_2(U_2))$. Hence

$$\text{curvature of } \alpha_1 \boxplus \alpha_2 = (p_{M_1})^* \Omega_1 + (p_{M_2})^* \Omega_2 = \Omega_1 \oplus \Omega_2.$$

B.4 The induced covariant derivative on sections of the exterior tensor product of line bundles

For sections s_1, s_2 of L_1, L_2, define the section $s_1 \boxtimes s_2$ of $L_1 \boxtimes L_2$ by

$$(s_1 \boxtimes s_2)(x_1,\, x_2) = s_1(x_1) \boxtimes s_2(x_2).$$

We wish to derive an expression for the covariant derivative of $s_1 \boxtimes s_2$ along a vector field $(Z_1,\, Z_2) \in \Gamma(T(M_1 \times M_2)) = \Gamma(TM_1) \times \Gamma(TM_2)$. Under the identification of $L_1 \boxtimes L_2$ with $(\dot{L}_1 \boxdot \dot{L}_2) \times_{\mathrm{id}_{\mathrm{U}(1)}} \mathbb{C}$ discussed in Section B.2,

$$\begin{aligned}(s_1 \boxtimes s_2)(x) &= s_1(x) \boxtimes s_2(x) \\ &= [p_1,\, \dot{s}_1(p_1)]_{\mathrm{id}_{\mathrm{U}(1)}} \boxtimes [p_2,\, \dot{s}_2(p_2)]_{\mathrm{id}_{\mathrm{U}(1)}} \\ &= [p_1 \boxdot p_2,\, \dot{s}_1(p_1)\dot{s}_2(p_2)]_{\mathrm{id}_{\mathrm{U}(1)}}\,,\end{aligned}$$

and therefore the complex-valued equivariant function on $\dot{L}_1 \boxdot \dot{L}_2$ corresponding to $s_1 \boxtimes s_2$ is

$$(s_1 \dot{\boxtimes} s_2)(p_1 \boxdot p_2) = \dot{s}_1(p_1)\dot{s}_2(p_2).$$

Suppose $X_1 \in T_{x_1}M_1$, $X_2 \in T_{x_2}M_2$. From the definition of $\alpha_1 \boxplus \alpha_2$, the horizontal lift of $(X_1,\, X_2)$ to $\dot{L}_1 \boxdot \dot{L}_2$ is

$$(X_1,\, X_2)^{\mathrm{h}_1 \boxdot_2}(p_1 \boxdot p_2) = X_1^{\mathrm{h}_1}(p_1) \boxplus X_2^{\mathrm{h}_2}(p_2),$$

for $p_1 \in (\dot{\tau}_1)^{-1}(x_1)$, $p_2 \in (\dot{\tau}_2)^{-1}(x_2)$. If γ_i is a curve corresponding to $X_i^{\mathrm{h}_i}$, then

$$\begin{aligned}(X_1,\, X_2)^{\mathrm{h}_1 \boxdot_2}(p_1 \boxdot p_2)\,(s_1 \dot{\boxtimes} s_2) &= \frac{d}{dt}\left[(s_1 \dot{\boxtimes} s_2)\left((\gamma_1 \boxdot \gamma_2)(t)\right)\right]_{t=0} \\ &= \frac{d}{dt}\left[\dot{s}_1(\gamma_1(t))\, \dot{s}_2\left((\gamma_2(t))\right)\right]_{t=0} \\ &= \left(X_1^{\mathrm{h}_1}(p_1)\, \dot{s}_1\right) \dot{s}_2(p_2) + \dot{s}_1(p_1) \left(X_2^{\mathrm{h}_2}(p_2)\, \dot{s}_2\right).\end{aligned}$$

Hence the induced covariant derivative in the direction $(X_1,\, X_2)$ is given by

$$\begin{aligned}&\left[p_1 \boxdot p_2,\, (X_1,\, X_2)^{\mathrm{h}_1 \boxdot_2}(p_1 \boxdot p_2)\,(s_1 \dot{\boxtimes} s_2)\right]_{\mathrm{id}_{\mathrm{U}(1)}} \\ &= \left[p_1 \boxdot p_2,\, \left(X_1^{\mathrm{h}_1}(p_1)\, \dot{s}_1\right) \dot{s}_2(p_2) + \dot{s}_1(p_1) \left(X_2^{\mathrm{h}_2}(p_2)\, \dot{s}_2\right)\right]_{\mathrm{id}_{\mathrm{U}(1)}} \\ &= \left[p_1,\, X_1^{\mathrm{h}_1}(p_1)\, \dot{s}_1\right]_{\mathrm{id}_{\mathrm{U}(1)}} \boxtimes [p_2,\, \dot{s}_2(p_2)]_{\mathrm{id}_{\mathrm{U}(1)}} + [p_1,\, \dot{s}_1(p_1)]_{\mathrm{id}_{\mathrm{U}(1)}} \boxtimes \left[p_2,\, X_2^{\mathrm{h}_2}(p_2)\, \dot{s}_2\right]_{\mathrm{id}_{\mathrm{U}(1)}} \\ &= \left(\nabla^1_{X_1} s_1\right)(x_1) \boxtimes s_2(x_2) + s_1(x_1) \boxtimes \left(\nabla^2_{X_2} s_2\right)(x_2) \\ &= \left(\left(\nabla^1_{X_1} s_1\right) \boxtimes s_2 + s_1 \boxtimes \left(\nabla^2_{X_2} s_2\right)\right)(x_1,\, x_2).\end{aligned}$$

We denote this covariant derivative by $\nabla^1 \boxplus \nabla^2$, i.e.,

$$(\nabla^1 \boxplus \nabla^2)_{(X_1,\, X_2)}(s_1 \boxtimes s_2) = \left(\nabla^1_{X_1} s_1\right) \boxtimes s_2 + s_1 \boxtimes \left(\nabla^2_{X_2} s_2\right).$$

This demonstrates in particular that if F_1 and F_2 are distributions on M_1 and M_2 respectively, then

$$\Gamma_{F_1 \oplus F_2}(L_1 \boxtimes L_2) = \Gamma_{F_1}(L_1) \boxtimes \Gamma_{F_2}(L_2).$$

Appendix C

Quantum reduction

The aim of this appendix is to demonstrate the isomorphism

$$\mathcal{H} \simeq \bigoplus_{\lambda \text{ dominant integral}} (\mathcal{H}^\lambda)^* \otimes \left(\mathcal{H}^\lambda \otimes \mathcal{H}\right)^{\widetilde{G}},$$

which serves to separate the symmetry and dynamical actions on the Hilbert space $\mathcal{H}$. The discussion here is derived from [TI00].

Let G be a compact, connected, semisimple Lie group which acts as a symmetry on the Hilbert space $\mathcal{H}$, i.e., G maps rays to rays, and commutes with the Hamiltonian $H \in i\mathfrak{u}(\mathcal{H})$ of the system. The symmetry defines a projective representation of G on $\mathcal{H}$, which can be lifted to a proper representation $U : \widetilde{G} \to \mathrm{U}(\mathcal{H})$ of the universal $\widetilde{G}$ of G. $\widetilde{G}$ is also compact (Section 4.2).

Picking a maximal torus $\widetilde{T}$ in $\widetilde{G}$, the Highest Weight Theorem labels the irreducible representations of $\widetilde{G}$ by elements of $i\mathfrak{t}^*$ that are both dominant (with respect to some choice of positive roots) and integral—see Section 5.8.3 for a discussion. For a given dominant integral weight λ, let $\rho^\lambda : \widetilde{G} \to \mathrm{U}(\mathcal{H}^\lambda)$ denote the corresponding unitary irreducible representation, $(\rho^\lambda)^* : \widetilde{G} \to \mathrm{U}((\mathcal{H}^\lambda)^*)$ its dual representation, and d^λ the dimension of $\mathcal{H}^\lambda$. Define operators $P^\lambda : \mathcal{H} \to \mathcal{H}$ by

$$P^\lambda = d^\lambda \int_{\widetilde{G}} \mathrm{Tr}\left[\rho^\lambda(\widetilde{g})\right] U(\widetilde{g})\,\mathrm{d}\mu(\widetilde{g}),$$

where $\mathrm{d}\mu$ denotes the Haar measure on $\widetilde{G}$, normalized so that $\int_{\widetilde{G}} \mathrm{d}\mu(\widetilde{g}) = 1$. Let $\{e_i \,|\, i = 1, \ldots, d^\lambda\}$ be an orthonormal basis for $\mathcal{H}^\lambda$, and $\{\rho^\lambda_{ij} \,|\, i, j = 1, \ldots, d^\lambda\}$ the matrix elements of ρ^λ with respect to this basis. The Schur orthogonality relations are

$$\int_{\widetilde{G}} \rho^\lambda_{ji}(\widetilde{g}^{-1}) \rho^{\lambda'}_{mn}(\widetilde{g})\,\mathrm{d}\mu(\widetilde{g}) = \int_{\widetilde{G}} \overline{\rho^\lambda_{ij}(\widetilde{g})} \rho^{\lambda'}_{mn}(\widetilde{g})\,\mathrm{d}\mu(\widetilde{g}) = \frac{1}{d^\lambda} \delta^{\lambda\lambda'} \delta_{im} \delta_{jn}.$$

Using these, and invariance of the Haar measure, it can be shown that

$$P^\lambda P^{\lambda'} = \delta^{\lambda\lambda'} P^\lambda \qquad \text{and} \qquad \left(P^\lambda\right)^\dagger = P^\lambda,$$

i.e., the P^λ are orthogonal projections mapping to mutually orthogonal subspaces of $\mathcal{H}$.

H commutes with the representation U, and hence with the P^λ. U is also easily seen to commute with the P^λ. Therefore each $\mathrm{Im}P^\lambda$ is both U- and H-invariant.

We can factorize the subspace $\mathrm{Im}P^\lambda$ in a manner which explicitly separates the U- and H-actions. Let $L^\lambda : \mathcal{H} \to \mathrm{U}(\mathcal{H}^\lambda) \otimes \mathcal{H}$ be the mapping

$$L^\lambda = \sqrt{d^\lambda} \int_{\widetilde{G}} \rho^\lambda(\widetilde{g}) \otimes U(\widetilde{g})\, \mathrm{d}\mu(\widetilde{g}).$$

Via the canonical isomorphism $\mathfrak{gl}(V) \simeq V^* \otimes V$, L^λ can be considered instead as an operator $K^\lambda : \mathcal{H} \to (\mathcal{H}^\lambda)^* \otimes \mathcal{H}^\lambda \otimes \mathcal{H}$. Explicitly

$$K^\lambda = \sqrt{d^\lambda} \int_{\widetilde{G}} \sum_{ij} \rho^\lambda_{ji}(\widetilde{g})\, e^i \otimes e_j \otimes U(\widetilde{g})\, \mathrm{d}\mu(\widetilde{g}),$$

where $\{e^i \mid i = 1, \ldots, d^\lambda\}$ is the dual basis in $(\mathcal{H}^\lambda)^*$ to $\{e_i \mid i = 1, \ldots, d^\lambda\}$. The expression is of course independent of the choice of the e_i.

Invariance of the Haar measure shows that $\big(\rho^\lambda(\widetilde{g}) \otimes U(\widetilde{g})\big)\, L^\lambda = L^\lambda$, and so in fact

$$K^\lambda : \mathcal{H} \to (\mathcal{H}^\lambda)^* \otimes \big(\mathcal{H}^\lambda \otimes \mathcal{H}\big)^{\widetilde{G}},$$

where $(V_1 \otimes V_2)^{\widetilde{G}}$ denotes the invariant part of the diagonal representation on $V_1 \otimes V_2$.

It can be checked that

$$K^\lambda P^{\lambda'} = \delta^{\lambda\lambda'} K^\lambda,$$

and that K^λ is an isometry from $\mathrm{Im}P^\lambda$ to $(\mathcal{H}^\lambda)^* \otimes \big(\mathcal{H}^\lambda \otimes \mathcal{H}\big)^{\widetilde{G}}$.

Since H commutes with U,

$$K^\lambda H = (\mathrm{id}_{\mathcal{H}^{\lambda *}} \otimes \mathrm{id}_{\mathcal{H}^\lambda} \otimes H)\, K^\lambda.$$

So through the isometry K^λ, the Hamiltonian H acts on the second factor only.

It can also be verified from the expression for K^λ that

$$K^\lambda U(\widetilde{g}) = \Big((\rho^\lambda)^*(\widetilde{g}) \otimes \mathrm{id}_{(\mathcal{H}^\lambda \otimes \mathcal{H})^{\widetilde{G}}}\Big)\, U(\widetilde{g}),$$

and so through K^λ, the representation U acts on the first factor (as the irreducible representation $(\rho^\lambda)^*$).

Overall

$$\bigoplus_{\lambda \text{ d.i.}} K^\lambda : \mathcal{H} \longrightarrow \bigoplus_{\lambda \text{ d.i.}} (\mathcal{H}^\lambda)^* \otimes \big(\mathcal{H}^\lambda \otimes \mathcal{H}\big)^{\widetilde{G}},$$

provides a isometry of the Hilbert space $\mathcal{H}$ which separates out the U- and H-actions on $\mathcal{H}$.

Note. The isometry $K^\lambda : \mathrm{Im} P^\lambda \longrightarrow (\mathcal{H}^\lambda)^* \otimes (\mathcal{H}^\lambda \otimes \mathcal{H})^{\widetilde{G}}$ implicitly depends on the choice of maximal torus $\widetilde{H} \subset \widetilde{G}$, mimicking the q_0-dependence of the decomposition $(E^{a,\mu}_{q_0})^{-1} : \Gamma_{F^{\mathcal{O}}_{\mathcal{R}}}\left(L^{\mathcal{O}}_{\mathcal{R}}\right) \longrightarrow \Gamma_{F^a_{\mathcal{R}}}\left(L^a_{\mathcal{R}}\right) \boxtimes \Gamma_{F^\mu_{\mathcal{R}}}\left(L^\mu_{\mathcal{R}}\right)$ (Section 6.3).

Appendix D

A symplectic proof of Schur's Lemma

In this appendix we present a "symplectic" proof that $\mathrm{Hom}_{\widetilde{G}}\left((\mathcal{H}^{\lambda})^*, (\mathcal{H}^{\lambda'})^*\right) \simeq \left(\mathcal{H}^{\lambda} \otimes (\mathcal{H}^{\lambda'})^*\right)^{\widetilde{G}} = \delta^{\lambda\lambda'}\mathbb{C}\{\mathrm{id}_{\mathcal{H}^{\lambda}}\}$. We do this by using the fact that $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}}) \simeq_{\mathcal{R}^{(\mu,a)}} (\mathcal{H}^{\frac{i}{\hbar}\mu})^*$, and considering the space of $\widetilde{G}$-invariant covariantly constant sections of $(L^b_{\mathcal{R}})^* \boxtimes L^a_{\mathcal{R}}$. The ideas in the proof developed here come from [GS82].

D.1 Complex structures

Let (M, ω) be a symplectic manifold. An ***almost complex structure*** is a field of linear maps $\mathcal{J}_x : T_xM \to T_xM$ such that for every $x \in M$

(i) $\mathcal{J}_x^2 = -\mathrm{id}_{T_xM}$;

(ii) $\omega_x\left(\mathcal{J}_x(X_x), \mathcal{J}_x(Y_x)\right) = \omega_x(X_x, Y_x)$ for any $X_x, Y_x \in T_xM$.

Let F be a totally complex polarization on M, so $TM^{\mathbb{C}} = F \oplus \overline{F}$. Define $\mathcal{J}_F^{\mathbb{C}} : TM^{\mathbb{C}} \to TM^{\mathbb{C}}$ by

$$\mathcal{J}_F^{\mathbb{C}}(Y) = \begin{cases} iY & Y \in F \\ -iY & Y \in \overline{F} \end{cases},$$

and extended by linearity to all of $TM^{\mathbb{C}}$. Since $\mathcal{J}_F^{\mathbb{C}}$ commutes with complex conjugation, it restricts to a linear map $\mathcal{J}_F : TM \to TM$. Then F can be written as

$$F = \{X - i\mathcal{J}_F X \mid X \in TM\}.$$

Since $(\mathcal{J}_F^{\mathbb{C}})^2 = -\mathrm{id}_{TM^{\mathbb{C}}}$, it follows that $\mathcal{J}_F^2 = -\mathrm{id}_{TM}$. Also, F is Lagrangian, and so for real vectors X, Y at a point of M, $\omega(X - i\mathcal{J}_F X, Y - i\mathcal{J}_F Y) = 0$. The real part of this condition then implies that $\omega(\mathcal{J}_F X, \mathcal{J}_F Y) = \omega(X, Y)$. Therefore $\mathcal{J}_F$ is an almost complex structure.

The involutivity of F implies M can be given the structure of a complex manifold in such a way that $\mathcal{J}_F$ agrees with holomorphic complex structure, described in any system of complex coordinates $z^\alpha = x^\alpha + iy^\alpha$ as

$$\mathcal{J}_{\mathrm{hol}}\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial}{\partial y^\alpha}, \qquad \mathcal{J}_{\mathrm{hol}}\left(\frac{\partial}{\partial y^\alpha}\right) = -\frac{\partial}{\partial x^\alpha},$$

(see for example [KN96, Chapter IX Theorem 2.5]). In this case, $\mathcal{J}_F$ is called a ***complex structure***, and $F = \mathrm{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z^\alpha}\right\}$.

Since F is maximally isotropic and totally complex, $-i\,\omega(Z,\,\overline{Z}) \neq 0$ for all nonzero $Z \in F$, and so is either always positive or always negative. Suppose we have prequantum data $(\dot{L},\,\alpha)$ over $(M,\,\omega)$. If M is compact, sections of the associated line bundle L covariantly constant with respect to F exist only if

$$-i\,\omega(Z,\,\overline{Z}) > 0 \qquad \text{for nonzero } Z \in F$$

(see for example [GH94, Chapter 1, Section 2]). Such a polarization is referred to as a ***strictly positive*** polarization. Using the characterization $F = \{X - i\mathcal{J}_F X,\,|\,X \in TM\}$ of the polarization, strict positivity of F implies that

$$\omega(X,\,\mathcal{J}_F X) > 0 \qquad \text{for all nonzero } X \in TM,$$

and correspondingly we say that $\mathcal{J}_F$ is strictly positive. For strictly positive complex structures, the real-valued nondegenerate bilinear form $g(X,\,Y) := \omega(X,\,\mathcal{J}_F Y)$ (which is easily shown to be symmetric from properties of the complex structure) defines a Riemannian metric, called the ***Kähler metric*** on $(M,\,\omega)$ corresponding to $\mathcal{J}_F$.

D.2 The complexified group action

Suppose $(M,\,\omega)$ is a symplectic manifold with a (not necessarily free) symplectic G-action, corresponding equivariant momentum map J, and totally complex G-invariant polarization F.

Since G is compact and semisimple, there exists a unique (up to isomorphism) group, called the ***complexification*** $G^{\mathbb{C}}$ of G, whose Lie algebra is $\mathfrak{g} \oplus i\mathfrak{g}$ and which contains G as a maximal compact subgroup. Moreover, $\ker(\pi_{\widetilde{G^{\mathbb{C}}} \to G^{\mathbb{C}}}) = \ker(\pi_{\widetilde{G} \to G}) = K$, say, implying that $G^{\mathbb{C}} \simeq \widetilde{G^{\mathbb{C}}}/K$ just as $G \simeq \widetilde{G}/K$. Also, $(\widetilde{G})^{\mathbb{C}} = \widetilde{G^{\mathbb{C}}}$, since they are both simply connected Lie groups with the same Lie algebra $\mathfrak{g} \oplus i\mathfrak{g}$.

We first demonstrate that the G-action on M can be extended to an F-preserving $G^{\mathbb{C}}$-action by means of the complex structure $\mathcal{J}_F$. For a vector field $X \in \Gamma(TM)$,

$$X \text{ preserves } F \iff \mathcal{L}_X F \subset F$$

$$\iff [X,\, Y - i\mathcal{J}_F Y] \in \Gamma(F) \text{ for all } Y \in \Gamma(TM)$$
$$\iff [X,\, \mathcal{J}_F Y] = \mathcal{J}_F[X,\, Y] \text{ for all } Y \in \Gamma(TM).$$

Since F is involutive, $[X - i\mathcal{J}_F X,\, Y - i\mathcal{J}_F Y] \in \Gamma(F)$ for all $X,\, Y \in \Gamma(F)$, implying that

$$[X,\, \mathcal{J}_F Y] + [\mathcal{J}_F X,\, Y] = \mathcal{J}_F\left([X,\, Y] - [\mathcal{J}_F X,\, \mathcal{J}_F Y]\right).$$

If X preserves F, the previous result tells us that $[X,\, \mathcal{J}_F Y] = \mathcal{J}_F[X,\, Y]$, and so

$$[\mathcal{J}_F X,\, Y] = -\mathcal{J}_F[\mathcal{J}_F X,\, \mathcal{J}_F Y] \qquad \text{or} \qquad [\mathcal{J}_F X,\, \mathcal{J}_F Y] = \mathcal{J}_F[\mathcal{J}_F X,\, Y].$$

So $\mathcal{J}_F X$ also preserves F. Finally if *both* X and Y preserve F, then

$$[\mathcal{J}_F X,\, \mathcal{J}_F Y] = \mathcal{J}_F^2[X,\, Y] = -[X,\, Y].$$

For $f \in C^\infty(M,\, \mathbb{R})$, define $Y_f = \mathcal{J}_f X_f$, where X_f is the Hamiltonian field corresponding to f. If X_f preserves F, so does Y_f. If X_g is another Hamiltonian vector field preserving F, we get

$$\begin{aligned}
[X_f,\, X_g] &= -X_{\{f,\, g\}}, \\
[X_f,\, Y_g] &= \mathcal{J}_F[X_f,\, X_g] = -\mathcal{J}_F X_{\{f,\, g\}} = -Y_{\{f,\, g\}}, \\
[Y_f,\, Y_g] &= -[X_f,\, X_g] = X_{\{f,\, g\}}.
\end{aligned}$$

For $i\xi \in i\mathfrak{g}$, define the vector field $(i\xi)_M$ to be $Y_{J(\xi)} = \mathcal{J}_F \xi_M$. F is G-invariant, so ξ_M preserves F, implying that $(i\xi)_M$ does also. Taking $f = J(\xi)$, $g = J(\zeta)$, the above results give

$$\begin{aligned}
[\xi_M,\, \zeta_M] &= -[\xi,\, \zeta]_M, \\
[\xi_M,\, (i\zeta)_M] &= -(i[\xi,\, \zeta])_M = -[\xi,\, i\zeta]_M, \\
[(i\xi)_M,\, (i\zeta)_M] &= [\xi,\, \zeta]_M = -[i\xi,\, i\zeta]_M.
\end{aligned}$$

This defines the structure of an infinitesimal $\mathfrak{g} \oplus i\mathfrak{g}$-action on M, which exponentiates to a $\widetilde{G^{\mathbb{C}}}$-action. Since it restricts to the G-action, $K = \ker\left(\pi_{\widetilde{G} \to G}\right)$ acts trivially, and from $G^{\mathbb{C}} \simeq \widetilde{G^{\mathbb{C}}}/K$ we see that the $\widetilde{G^{\mathbb{C}}}$-action drops to a $G^{\mathbb{C}}$-action on M.

D.3 Lifting of the complexified action

Now let $(\dot{L},\, \alpha)$ be a prequantum structure over $(M,\, \omega)$. In order to lift the $G^{\mathbb{C}}$-action on M, we need to construct the"complexified" bundle $\dot{L}^{\mathbb{C}} = \dot{L} \times_{\mathrm{id}_{\mathrm{U}(1)}} \mathbb{C}^\times$, where $\mathbb{C}^\times = \mathbb{C} - \{0\} \simeq \mathrm{U}(1)^{\mathbb{C}}$, considered as an abelian group. The injection $\dot{I}^{\mathbb{C}} : \dot{L} \hookrightarrow \dot{L}^{\mathbb{C}}$ given by $\dot{I}^{\mathbb{C}}(p) = [p,\, 1]_{\mathrm{id}_{\mathrm{U}(1)}}$ induces a line bundle isomorphism $I^{\mathbb{C}} : \dot{L} \times_{\mathrm{id}_{\mathrm{U}(1)}} \mathbb{C} \to \dot{L}^{\mathbb{C}} \times_{\mathrm{id}_{\mathbb{C}^\times}} \mathbb{C}$, given explictly by

$$I^{\mathbb{C}}\left([p,\, z]_{\mathrm{id}_{\mathrm{U}(1)}}\right) = \left[\dot{I}^{\mathbb{C}}(p),\, z\right]_{\mathrm{id}_{\mathbb{C}^\times}} = \left[[p,\, 1]_{\mathrm{id}_{\mathrm{U}(1)}},\, z\right]_{\mathrm{id}_{\mathbb{C}^\times}}.$$

$\alpha = (\dot{I}^{\mathbb{C}})^* \alpha^{\mathbb{C}}$ uniquely defines a $T_1\mathbb{C}^\times \simeq \mathbb{C}$-valued 1-form $\alpha^{\mathbb{C}}$ on $\dot{L}^{\mathbb{C}}$, whose curvature is also ω, and $\dot{\tau}^{\mathbb{C}}\left([p,\, z]_{\mathrm{id}_{\mathbb{C}^\times}}\right) = \dot{\tau}(p)$ is the projection on $\dot{L}^{\mathbb{C}}$.

Additionally, $\dot{I}^{\mathbb{C}}$ induces and isomorphism between the set $C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L},\, \mathbb{C})$ of equivariant functions on $\dot{L}$ and the set $C^\infty_{\mathrm{id}_{\mathbb{C}^\times}}(\dot{L}^{\mathbb{C}},\, \mathbb{C})$ of equivariant functions on $\dot{L}^{\mathbb{C}}$ (by equivariant extension in one direction, restriction in the other), and clearly covariantly constant sections of L go over to covariantly constant sectios of $L^{\mathbb{C}}$.

For $f \in C^\infty(M,\, \mathbb{R})$, define the vector fields

$$\begin{aligned} A_f &= X_f^{\mathrm{h}} - f \circ \dot{\tau}^{\mathbb{C}} \left(\frac{\varepsilon_0}{h}\right)_{\dot{L}^{\mathbb{C}}} && \text{(as in } \dot{L}\text{)}, \\ B_f &= (\mathcal{J}_F X_f)^{\mathrm{h}} - f \circ \dot{\tau}^{\mathbb{C}} \left(\mathcal{J}_{\mathbb{C}^\times} \frac{\varepsilon_0}{h}\right)_{\dot{L}^{\mathbb{C}}}, \end{aligned}$$

where $\mathcal{J}_{\mathbb{C}^\times}$ denotes the usual complex structure on $\mathbb{C}^\times$ (corresponding to multiplication by i).

By some tedious but straightforward computations, we can check that when X_f and X_g preserve F, the analogous structure to that on M holds, namely

$$\begin{aligned} [A_f,\, A_g] &= -A_{\{f,\, g\}}, \\ [A_f,\, B_g] &= -B_{\{f,\, g\}}, \\ [B_f,\, B_g] &= A_{\{f,\, g\}}, \end{aligned}$$

and defining $(i\xi)_{\dot{L}^{\mathbb{C}}} = B_{J(\xi)}$,

$$\begin{aligned} [\xi_{\dot{L}^{\mathbb{C}}},\, \zeta_{\dot{L}^{\mathbb{C}}}] &= -[\xi,\, \zeta]_{\dot{L}^{\mathbb{C}}}, \\ [\xi_{\dot{L}^{\mathbb{C}}},\, (i\zeta)_{\dot{L}^{\mathbb{C}}}] &= -[\xi,\, i\zeta]_{\dot{L}^{\mathbb{C}}}, \\ [(i\xi)_{\dot{L}^{\mathbb{C}}},\, (i\zeta)_{\dot{L}^{\mathbb{C}}}] &= -[i\xi,\, i\zeta]_{\dot{L}^{\mathbb{C}}}. \end{aligned}$$

Again we have an infinitesimal $\mathfrak{g} \oplus i\mathfrak{g}$-action, which exponentiates to a $(\widetilde{G})^{\mathbb{C}} = \widetilde{G^{\mathbb{C}}}$-action on $\dot{L}^{\mathbb{C}}$.

Now suppose s is a section of $L = \dot{L}^{\mathbb{C}} \times_{\mathbb{C}^\times} \mathbb{C}$ covariantly constant with respect to F, and $\dot{s} \in C^\infty_{\mathrm{id}_{\mathbb{C}}}(\dot{L}^{\mathbb{C}},\, \mathbb{C}^\times)$ is the corresponding equivariant function. Then for any Hamiltonian vector field X_f we have that $(X_f - i\mathcal{J}_F X_f)^{\mathrm{h}} \dot{s} = 0$, or equivalently $(\mathcal{J}_F X_f)^{\mathrm{h}} \dot{s} = i(X_f^{\mathrm{h}} \dot{s})$. Hence

$$B_f \dot{s} = i(X_f^{\mathrm{h}} \dot{s}) - \frac{1}{\hbar}(f \circ \dot{\tau}^{\mathbb{C}})\dot{s} = i\left(X_f^{\mathrm{h}} - f \circ \dot{\tau}^{\mathbb{C}} \left(\frac{\varepsilon_0}{h}\right)_{\dot{L}^{\mathbb{C}}}\right)\dot{s} = iA_f \dot{s}.$$

Taking $f = J(\xi)$ for $\xi \in \mathfrak{g}$, we obtain in particular that

$$(i\xi)^{\mathrm{h}}_M \dot{s} = i(\xi^{\mathrm{h}}_M \dot{s}) \qquad \text{and} \qquad (i\xi)_{\dot{L}^{\mathbb{C}}} \dot{s} = i(\xi_{\dot{L}^{\mathbb{C}}} \dot{s}) = 0.$$

The second condition implies that a $\widetilde{G}$-invariant covariantly constant section is automatically $\widetilde{G}^{\mathbb{C}}$-invariant. Since the $\widetilde{G}^{\mathbb{C}}$-action covers the $G^{\mathbb{C}}$-action on M, it is completely determined over the $G^{\mathbb{C}}$-orbit of a point $x \in M$ by its values on the fiber $(\dot{\tau}^{\mathbb{C}})^{-1}(x)$, and due to $\mathbb{C}^{\times}$-equivariance along the fiber this is characterized by an element of $\mathbb{C}$.

Note. In general, a line bundle L over *connected* M supports nonvanishing $\widetilde{G}$-invariant sections only if the $\widetilde{G}$-action on $\dot{L}$ reduces to a G-action. To see this, let s be such a section, $\dot{s}$ the associated U(1)-equivariant functions, and $x \in M$ a point at which $s(x) \neq 0$. Then for any $p \in \dot{L}_x = (\dot{\tau})^{-1}(x)$ and $\widetilde{k} \in \ker(\pi_{\widetilde{G}\to G})$,

$$\dot{s}(p) = \dot{s}(\widetilde{k} \cdot p) = \dot{s}(p \cdot \chi(\widetilde{k})) = \chi(\widetilde{k})^{-1} \dot{s}(p).$$

Since $\dot{s}(p) \neq 0$, it follows that $\chi(\widetilde{k}) = 1$ for all $\widetilde{k} \in \ker(\pi_{\widetilde{G}\to G})$. Since M is connected, $\ker(\pi_{\widetilde{G}\to G})$ acts uniformly on $\dot{L}$ (Proposition 3.2.3), implying that the $\widetilde{G}$-action on $\dot{L}$ reduces to a $\widetilde{G}/\ker(\pi_{\widetilde{G}\to G}) \simeq G$-action.

We now state an important necessary condition on the existence of nonzero $\widetilde{G}$-invariant covariantly constant sections of L over *compact* manifolds M.

Proposition D.3.1. *Suppose M is compact, and $s \in \Gamma_F(L)$ is nonzero and $\widetilde{G}$-invariant. Then the Hermitian norm $\langle s, s\rangle$ of s achieves its maximum on the set $\mathrm{J}^{-1}(0)$. In particular, 0 is in the image of the momentum map.*

Proof. Let s be such a section, and $x \in M$ a point at which $\langle s,\, s\rangle$ achieves its maximum. The condition of $\widetilde{G}$-invariance says that $\nabla_{\xi_M} s = -\frac{i}{\hbar} J(\xi)\, s$ for all $\xi \in \mathfrak{g}$. It follows that

$$\begin{aligned}
(i\xi)_M \langle s, s\rangle &= \langle \nabla_{(i\xi)_M} s,\, s\rangle + \langle s,\, \nabla_{(i\xi)_M} s\rangle && \text{by } \alpha\text{-invariance of } \langle\cdot,\cdot\rangle \\
&= \langle i\nabla_{\xi_M} s,\, s\rangle + \langle s,\, i\nabla_{\xi_M} s\rangle && \text{by the covariantly constant condition} \\
&= \frac{2}{\hbar} J(\xi)\, \langle s,\, s\rangle && \text{by } \widetilde{G}\text{-invariance of } s.
\end{aligned}$$

Evaluating this identity at x, and using the fact that $(i\xi)_M(x)\langle s, s\rangle = 0$, we have that $J(\xi)(x) = 0$ for all $\xi \in \mathfrak{g}$, i.e., $\mathrm{J}(x) = 0$. $\square$

D.4 The complex structure on the reduced group orbit

We now consider the reduced G-orbit $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$, and the complex structure on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ associated with $F^a_{\mathcal{R}}$. First recall the definition of $F^a_{\mathcal{R}}$:

$$(F^a_{\mathcal{R}})_{\mathcal{R}^{(\mu,a)}} = \mathfrak{n}^+_\mu \cdot \mathcal{R}^{(\mu,a)},$$

where $\mathfrak{n}^+_\mu$ is the span of the eigenvectors of $-\mathrm{ad}_{\mu^\sharp} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ corresponding to eigenvalues $i\lambda$ for $\lambda \in (0, \infty)$. Similarly, $\mathfrak{n}^-_\mu$ is the span of the eigenvectors of $-\mathrm{ad}_{\mu^\sharp}$ with eigenvalues lying on the negative imaginary axis. Since $\mu^\sharp \in \mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$, $\mathfrak{n}^+_\mu$ and $\mathfrak{n}^-_\mu$ are complex conjugates of one another. It follows that

$$F^a_{\mathcal{R}} \cap \overline{F^a_{\mathcal{R}}} = \{0\},$$

i.e., $F^a_{\mathcal{R}}$ is a totally complex polarization. Since $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is compact, a necessary condition for the existence of nonzero covariantly constant sections of $L^a_{\mathcal{R}}$ is that $F^a_{\mathcal{R}}$ is a strictly positive polarization. We first verify that this is the case.

Proposition D.4.1. *$F^a_{\mathcal{R}}$ is a strictly positive polarization, i.e., $-i\,\omega^a_{\mathcal{R}}(Z, \overline{Z}) > 0$ for all nonzero $Z \in F^a_{\mathcal{R}}$.*

Proof. For arbitrary $\xi \cdot \mathcal{R}^{(\mu,a)}, \zeta \cdot \mathcal{R}^{(\mu,a)} \in T_{\mathcal{R}^{(\mu,a)}}\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a}\right)$ (where $\xi, \zeta \in \mathfrak{g}$),

$$(\omega^a_{\mathcal{R}})_{\mathcal{R}^{(\mu,a)}}(\xi \cdot \mathcal{R}^{(\mu,a)}, \zeta \cdot \mathcal{R}^{(\mu,a)}) = J^a_{\mathcal{R}}(\mathcal{R}^{(\mu,a)})([\xi, \zeta]) = \mu([\xi,\zeta]),$$

and by complex extension this holds for $\xi, \zeta \in \mathfrak{g}^{\mathbb{C}}$ also.

Now take nonzero $\xi \in \mathfrak{n}^+_\mu$, so $\xi \cdot \mathcal{R}^{(\mu,a)} \in (F^a_{\mathcal{R}})_{\mathcal{R}^{(\mu,a)}}$. By definition, ξ satisfies $-\mathrm{ad}_{\mu^\sharp}\xi = i\lambda\,\xi$ for some $\lambda \in (0,\infty)$. Then

$$\begin{aligned}
-i(\omega^a_{\mathcal{R}})_{\mathcal{R}^{(\mu,a)}}(\xi \cdot \mathcal{R}^{(\mu,a)}, \overline{\xi \cdot \mathcal{R}^{(\mu,a)}}) &= -i\,\mu([\xi, \overline{\xi}]) \\
&= -i\,\kappa(\mu^\sharp, [\xi, \overline{\xi}]) \qquad \text{by the definition of } \mu^\sharp \\
&= i\,\kappa([\xi, \mu^\sharp], \overline{\xi}) \qquad \text{by properties of the Killing form} \\
&= i\,\kappa(i\lambda\xi, \overline{\xi}) \\
&= -\lambda\,\kappa(\xi, \overline{\xi}),
\end{aligned}$$

and the latter expression is strictly positive, since $\lambda > 0$ and the Killing form is negative definite on $\mathfrak{g}$. □

Since $F^a_{\mathcal{R}}$ is strictly positive, it induces an almost complex structure $\mathcal{J}_{F^a_{\mathcal{R}}}$ on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$. Since $F^a_{\mathcal{R}}$ is integrable, $\mathcal{J}_{F^a_{\mathcal{R}}}$ is a complex structure, and allows the construction of a holomorphic atlas of complex coordinates on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$, with respect to which $\Gamma(F^a_{\mathcal{R}})$ is the set of antiholomorphic vector fields.

Now let $\mathcal{R}^{(\mu,a)}$ be an arbitrary point in $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$. The stabilizer group of the (reduced) G-action on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is G_μ, and there is a smooth diffeomorphism between G/G_μ and $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ (indeed, the smooth structure on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ was defined by transferring the smooth structure on G/G_μ via this mapping—see Proposition 2.8.5 (ii)). We have

$$\frac{\pi^{-1}(a)}{\mathcal{R}^a} \simeq_{\mathcal{R}^{(\mu,a)}} \frac{G}{G_\mu}.$$

Now consider the embedding $G \hookrightarrow G^{\mathbb{C}}$. This embedding induces a mapping $G/G_\mu \to G^{\mathbb{C}}/P^-_\mu$, where P^-_μ is the Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra equal to the parabolic subalgebra $\mathfrak{p}^-_\mu = \mathfrak{g}^{\mathbb{C}}_\mu \oplus \mathfrak{n}^-_\mu$. It can be shown ([Sep07, Theorem 7.50]) that $G^{\mathbb{C}}/P^-_\mu$ possesses a differentiable structure which makes this mapping a diffeomorphism. An arbitrary element of $T_{gG_\mu}(G/G_\mu)$ has the form $g\cdot(\xi+\overline{\xi})\cdot G_\mu$, where $\xi \in \mathfrak{n}^+_\mu$. The derivative of $G/G\mu \to G^{\mathbb{C}}/P^-_\mu$ maps this vector to $g\cdot\xi\cdot P^-_\mu \in T_{gP^-_\mu}(G^{\mathbb{C}}/P^-_\mu)$. Note here that $g\cdot\xi\cdot P^-_\mu$ is an element of the *real* tangent space to $G^{\mathbb{C}}/P^-_\mu$. Overall we have

$$\frac{\pi^{-1}(a)}{\mathcal{R}^a} \simeq_{\mathcal{R}^{(\mu,a)}} \frac{G}{G_\mu} \simeq \frac{G^{\mathbb{C}}}{P^-_\mu}.$$

Since both $G^{\mathbb{C}}$ and P^-_μ are complex manifolds, their quotient $G^{\mathbb{C}}/P^-_\mu$ possesses a complex structure, which just corresponds to multiplication by i,

$$g\cdot\xi\cdot P^-_\mu \longmapsto g\cdot(i\xi)\cdot P^-_\mu$$

(where again $g\cdot(i\xi)\cdot P^-_\mu$ is a element of the *real* tangent space $T_{gP^-_\mu}(G^{\mathbb{C}}/P^-_\mu))$. Taking $g \in G \subset G^{\mathbb{C}}$, the corresponding element in $T_{g\cdot\mathcal{R}^{(\mu,a)}}\left(\frac{\pi^{-1}(a)}{\mathcal{R}^a}\right)$ is

$$g\cdot(i\,\xi+\overline{i\,\xi})\cdot\mathcal{R}^{(\mu,a)} = i\,g\cdot\xi\cdot\mathcal{R}^{(\mu,a)} - i\,g\cdot\overline{\xi}\cdot\mathcal{R}^{(\mu,a)} = (\mathcal{J}_{F^a_{\mathcal{R}}})_{g\cdot\mathcal{R}^{(\mu,a)}}\left(g\cdot\xi\cdot\mathcal{R}^{(\mu,a)} - g\cdot\overline{\xi}\cdot\mathcal{R}^{(\mu,a)}\right),$$

the second equality a consequence of $g\cdot\xi\cdot\mathcal{R}^{(\mu,a)} \in (F^a_{\mathcal{R}})_{g\cdot\mathcal{R}^{(\mu,a)}}$ and $g\cdot\overline{\xi}\cdot\mathcal{R}^{(\mu,a)} \in (\overline{F^a_{\mathcal{R}}})_{g\cdot\mathcal{R}^{(\mu,a)}}$ for $\xi \in \mathfrak{n}^+_\mu$. Hence the complex structure $\mathcal{J}_{F^a_{\mathcal{R}}}$ on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is just the one induced by the usual complex structure on $G^{\mathbb{C}}/P^-_\mu$ under the diffeomorphism $\frac{\pi^{-1}(a)}{\mathcal{R}^a} \simeq_{\mathcal{R}^{(\mu,a)}} \frac{G^{\mathbb{C}}}{P^-_\mu}$. Looking back at the definition of the $G^{\mathbb{C}}$-action on $\frac{\pi^{-1}(a)}{\mathcal{R}^a}$, we see that it simply corresponds to the natural left $G^{\mathbb{C}}$-action on $G^{\mathbb{C}}/P^-_\mu$ under the above diffeomorphism.

D.5 Schur's Lemma

We now apply the results of the previous sections to give a "symplectic" proof of Schur's Lemma. Recall first the result of Section 5.8.4: for admissable $a \in \frac{M}{G}$, the natural $\widetilde{G}$-representation on the space of sections $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ is irreducible. Choosing $q_a \in \dot{L}^a_{\mathcal{R}}$ over $\mathcal{R}^{(\mu,a)} \in \frac{\pi^{-1}(a)}{\mathcal{R}^a}$, T a maximal torus in G_μ, and a choice Δ^+ of positive roots determined by $\frac{i}{\hbar}\mu$, the irreducible $\widetilde{G}$-representation $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$ is the dual of that with highest weight $\frac{i}{\hbar}\mu$,

$$\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}}) \simeq_{q_a} (\mathcal{H}^{\frac{i}{\hbar}\mu})^*.$$

We also apply the geometric quantization procedure to the symplectic manifold $(\frac{\pi^{-1}(b)}{\mathcal{R}^b}, -\omega^b_{\mathcal{R}})$, where b is an admissible element of $\frac{M}{G}$. The momentum map corresponding to the reduced G-actions is now $-\mathrm{J}^b_{\mathcal{R}}$. The polarization $\overline{F^b_{\mathcal{R}}}$ is a totally complex and strictly positive, and determines a complex structure $\mathcal{J}_{\overline{F^b_{\mathcal{R}}}} = -\mathcal{J}_{F^b_{\mathcal{R}}}$ on $\frac{\pi^{-1}(b)}{\mathcal{R}^b}$. This complex structure is the one induced by the series of diffeomorphisms

$$\frac{\pi^{-1}(b)}{\mathcal{R}^b} \simeq_{\mathcal{R}^{(\nu,b)}} \frac{G}{G_{-\nu}} \simeq \frac{G^{\mathbb{C}}}{P^-_{-\nu}} = \frac{G^{\mathbb{C}}}{P^+_{\nu}},$$

(recall, the momentum of $\mathcal{R}^{(\nu,b)}$ is now $-\nu$), where P^+_ν is the positive parabolic subgroup of $G^{\mathbb{C}}$, which has Lie algebra $\mathfrak{p}^+_\nu = \mathfrak{g}^{\mathbb{C}}_\nu \oplus \mathfrak{n}^+_\nu$. There exists a U(1)-bundle over $\frac{\pi^{-1}(b)}{\mathcal{R}^b}$ with connection of curvature $-\frac{\varepsilon_0}{h}\omega^a_{\mathcal{R}}$. The associated line bundle is $(L^b_{\mathcal{R}})^*$, the dual to the usual one; this can be seen by the Chern-Weil correspondence, since the extra minus in the symplectic form corresponds to transition functions which are inverses of those on $L^b_{\mathcal{R}}$. Taking q^*_b a point in this bundle over $\mathcal{R}^{(\nu,b)} \in \frac{\pi^{-1}(b)}{\mathcal{R}^b}$ and following through the Borel-Weil argument[1], we obtain

$$\Gamma_{\overline{F^b_{\mathcal{R}}}}\left((L^b_{\mathcal{R}})^*\right) \simeq_{q^*_b} \mathcal{H}^{\frac{i}{\hbar}\nu}.$$

We are interested in calculating $\mathrm{Hom}_{\widetilde{G}}\left(\Gamma_{F^b_{\mathcal{R}}}(L^b_{\mathcal{R}}),\, \Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})\right)$, the set of $\widetilde{G}$-intertwiners between the irreducible representation spaces $\Gamma_{F^b_{\mathcal{R}}}(L^b_{\mathcal{R}})$ and $\Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})$. Under the canonical isomorphism $\mathrm{Hom}(V_1, V_2) \simeq V_1^* \otimes V_2$, this is the same as

$$\left(\Gamma_{\overline{F^b_{\mathcal{R}}}}\left((L^b_{\mathcal{R}})^*\right) \boxtimes \Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})\right)^{\widetilde{G}} = \Gamma_{\overline{F^b_{\mathcal{R}}} \oplus F^a_{\mathcal{R}}}\left((L^b_{\mathcal{R}})^* \boxtimes L^a_{\mathcal{R}}\right)^{\widetilde{G}},$$

the space of $\widetilde{G}$-invariant sections of $(L^b_{\mathcal{R}})^* \boxtimes L^a_{\mathcal{R}}$ which are covariantly constant with respect to the polarization $\overline{F^b_{\mathcal{R}}} \oplus F^a_{\mathcal{R}}$. The space $\frac{\pi^{-1}(b)}{\mathcal{R}^b} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is compact, and so using Proposition D.3.1 we can straight away express a necessary condition for the existence of nonzero $\widetilde{G}$-invariant sections.

Corollary D.5.1. *The space of $\widetilde{G}$-invariant sections $\Gamma_{\overline{F^b_{\mathcal{R}}} \oplus F^a_{\mathcal{R}}}\left((L^b_{\mathcal{R}})^* \boxtimes L^a_{\mathcal{R}}\right)^{\widetilde{G}}$ is nonzero only if the G-orbits a and b both belong to $\frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$ for some coadjoint orbit $\mathcal{O}$.*

Proof. The momentum map on $\frac{\pi^{-1}(b)}{\mathcal{R}^b} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is $J^b_{\mathcal{R}} \circ p_{\frac{\pi^{-1}(b)}{\mathcal{R}^b}} - J^a_{\mathcal{R}} \circ p_{\frac{\pi^{-1}(a)}{\mathcal{R}^a}}$. For nonzero $\widetilde{G}$-invariant

[1] Here it is convenient to take the Peter-Weyl equivalence as "$\alpha \otimes v \in (\mathcal{H}^\lambda)^* \otimes \mathcal{H}^\lambda$ corresponds to the map $\widetilde{g} \mapsto \alpha(\widetilde{g}^{-1} \cdot v)$ in $C^\infty(\widetilde{G}, \mathbb{C}) \subset L^2(\widetilde{G})$." The convention from Section 5.8.3 yields $\Gamma_{\overline{F^b_{\mathcal{R}}}}\left((L^b_{\mathcal{R}})^*\right) \simeq_{q^*_0} \left(\mathcal{H}^{w_0(-\frac{i}{\hbar}\nu)}\right)^*$, where w_0 is the longest element of the Weyl group, though the latter space is isomorphic to $\mathcal{H}^{\frac{i}{\hbar}\nu}$.

sections to exist, the image of this map must contain 0, implying that $\frac{\pi^{-1}(b)}{\mathcal{R}^b} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a}$ contains a point of the form $(\mathcal{R}^{(\mu,a)}, \mathcal{R}^{(\mu,b)})$ for some $\mu \in \mathfrak{g}^*$. The G-orbits a and b both intersect $\mathrm{J}^{-1}(\mu)$ in the unreduced space M, and therefore are subsets of $\mathrm{J}^{-1}(\mathcal{O})$, where $\mathcal{O}$ is the coadjoint orbit through μ. □

Now restrict to the case $a, b \in \frac{\mathrm{J}^{-1}(\mathcal{O})}{G}$. So take points $\mathcal{R}^{(\mu,a)} \in \frac{\pi^{-1}(a)}{\mathcal{R}^a}$ and $\mathcal{R}^{(\mu,b)} \in \frac{\pi^{-1}(b)}{\mathcal{R}^b}$ (note these are labeled by the same $\mu \in \mathfrak{g}^*$), and combine the previously discussed diffeomorphisms,

$$\frac{\pi^{-1}(b)}{\mathcal{R}^b} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a} \simeq_{(\mathcal{R}^{(\mu,b)}, \mathcal{R}^{(\mu,a)})} \frac{G^{\mathbb{C}}}{P_\mu^+} \times \frac{G^{\mathbb{C}}}{P_\mu^-}.$$

We are interested in the orbits of the diagonal $G^{\mathbb{C}}$-action in this space. A general $G^{\mathbb{C}}$orbit can be written as

$$G^{\mathbb{C}} \cdot (eP_\mu^+, gP_\mu^-)$$

for some $g \in G^{\mathbb{C}}$. To find explicitly what these orbits look like, we employ the *generalized Bruhat decomposition* (see for example [BL00, Chapter 1]). This characterizes the orbits of the space $G^{\mathbb{C}}/P_\mu^-$ under the natural left P_μ^+-action, the so-called *Schubert cells*. Choosing a maximal torus $T \subset G_\mu$ (implying $T^{\mathbb{C}} \subset P_\mu^{\pm}$), the decomposition says that within each orbit P_μ^+-orbit, there exists a point nP_μ^-, where $n \in G^{\mathbb{C}}$ is an element of the normalizer of $T^{\mathbb{C}}$. In particular, $P_\mu^+ g P_\mu^-$ contains a point $n_g P_\mu^-$, where n_g normalizes $T^{\mathbb{C}}$, and so gives a corresponding element w_g of the Weyl group $W = N_{G^{\mathbb{C}}}(T^{\mathbb{C}})/T^{\mathbb{C}} = N_G(T)/T$. In general, for the case when G_μ is larger than the maximal torus T, there are several n_g's and several w_g's in the P_μ^+-orbit through gP_μ^-—see [BL00] for a discussion. The orbits

$$G^{\mathbb{C}} \cdot (eP_\mu^+, gP_\mu^-) \quad \text{and} \quad G^{\mathbb{C}} \cdot (eP_\mu^+, n_g P_\mu^-)$$

agree, since P_μ^+ acts trivially on the eP_μ^+. The stabilizer group of the diagonal $G^{\mathbb{C}}$-action at the point $(eP_\mu^+, n_g P_\mu^-)$ is

$$P_\mu^+ \cap (n_g P_\mu^- n_g^{-1}) = P_\mu^+ \cap P^-_{\mathrm{Ad}^*_{n_g^{-1}} \mu} = P_\mu^+ \cap P^-_{w_g \cdot \mu}.$$

This stabilizer group is of smallest dimension when $w_g = \mathrm{id}$, corresponding to $g = e$. In that case, we get $P_\mu^+ \cap P_\mu^- = G_\mu^{\mathbb{C}}$, and the corresponding orbit $G^{\mathbb{C}} \cdot (eP_\mu^+, eP_\mu^-) \subset G^{\mathbb{C}}/P_\mu^+ \times G^{\mathbb{C}}/P_\mu^-$ is largest. The real dimension of the orbit is

$$\dim_{\mathbb{R}} \frac{G^{\mathbb{C}}}{G_\mu^{\mathbb{C}}} = 2 \dim_{\mathbb{R}} \frac{G}{G_\mu},$$

which is the dimension of $\frac{\pi^{-1}(b)}{\mathcal{R}^b} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a} \simeq_{(\mathcal{R}^{(\mu,b)}, \mathcal{R}^{(\mu,a)})} \frac{G^{\mathbb{C}}}{P_\mu^+} \times \frac{G^{\mathbb{C}}}{P_\mu^-}$ itself. Hence the $G^{\mathbb{C}}$-orbit is through $(\mathcal{R}^{(\mu,b)}, \mathcal{R}^{(\mu,a)})$ is open in $\frac{\pi^{-1}(b)}{\mathcal{R}^b} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a}$.

It can be shown that the other $G^{\mathbb{C}}$-orbits have complex codimension ≥ 1 in $\frac{\pi^{-1}(b)}{\mathcal{R}^b} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a}$. This is particularly clear for the case when G_μ is a torus T, since then any nontrivial element w of the Weyl group will cause the initially completely disjoint root spaces of $\mathfrak{p}_\mu^+ = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{n}_\mu^+$ and $\mathfrak{p}_\mu^- = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{n}_\mu^-$ to overlap between $\mathfrak{p}_\mu^+$ and $\mathfrak{p}_{w\cdot\mu}^+$ in at least one root space, which has complex dimension 1. Since this represents the stabilizer of the $G^{\mathbb{C}}$-orbit, the statement follows.

A $\widetilde{G}^{\mathbb{C}}$-invariant section s over the orbit $G^{\mathbb{C}} \cdot (\mathcal{R}^{(\mu,b)}, \mathcal{R}^{(\mu,a)}) \subset \frac{\pi^{-1}(a)}{\mathcal{R}^a} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a}$ is determined by its value at $(\mathcal{R}^{(\mu,b)}, \mathcal{R}^{(\mu,a)})$. Proposition D.3.1 demonstrates that s is bounded on $G^{\mathbb{C}} \cdot (\mathcal{R}^{(\mu,b)}, \mathcal{R}^{(\mu,a)})$, and so it can be extended uniquely to the rest of the $\frac{\pi^{-1}(b)}{\mathcal{R}^b} \times \frac{\pi^{-1}(a)}{\mathcal{R}^a}$, since the union of the remaining orbits has complex codimension ≥ 1. Hence the space of sections $\Gamma_{\overline{F^b_{\mathcal{R}}} \oplus F^a_{\mathcal{R}}} \left((L^b_{\mathcal{R}})^* \boxtimes L^a_{\mathcal{R}}\right)^{\widetilde{G}}$ is one complex dimensional.

Tracing through the use of the Peter-Weyl and Borel-Weil theorems, we can say that a general section in $\Gamma_{\overline{F^b_{\mathcal{R}}} \oplus F^a_{\mathcal{R}}} \left((L^b_{\mathcal{R}})^* \boxtimes L^a_{\mathcal{R}}\right)$, expressed in terms of the corresponding U(1)-equivariant function, is a complex linear combination of sections of the form

$$\dot{k}(\widetilde{g}_a \cdot q_a^* \boxdot \widetilde{g}_b \cdot q_b) = \alpha_\downarrow(\widetilde{g}_a^{-1} \cdot v)\, \alpha(\widetilde{g}_b \cdot v^\uparrow)$$

(extended by U(1)-equivariance), where $\alpha_\downarrow$ is a lowest weight vector in $(\mathcal{H}^{\frac{i}{\hbar}\mu})^*$ (of weight $-\frac{i}{\hbar}\mu$), $v^\uparrow$ is a highest weight vector in $\mathcal{H}^{\frac{i}{\hbar}\mu}$, and $\alpha \in (\mathcal{H}^{\frac{i}{\hbar}\mu})^*$, $v \in \mathcal{H}^{\frac{i}{\hbar}\mu}$ are arbitrary.

$$\dot{k}(\widetilde{g}_a \cdot q_a^* \boxdot \widetilde{g}_b \cdot q_b) = \sum_i \alpha_\downarrow(\widetilde{g}_a^{-1} \cdot e_i)\, e^i(\widetilde{g}_b \cdot v^\uparrow) = \alpha_\downarrow(\widetilde{g}_a^{-1} \cdot \widetilde{g}_b \cdot v^\uparrow),$$

where e_i is a basis for $\mathcal{H}^{\frac{i}{\hbar}\mu}$, and e^i is the corresponding dual basis in $(\mathcal{H}^{\frac{i}{\hbar}\mu})^*$.

Returning to the form $\mathrm{Hom}_{\widetilde{G}}\left(\Gamma_{F^b_{\mathcal{R}}}(L^b_{\mathcal{R}}),\, \Gamma_{F^a_{\mathcal{R}}}(L^a_{\mathcal{R}})\right)$, we get the space of all complex multiples of the following map: given $q_b \in (\dot{\tau}^b_{\mathcal{R}})^{-1}(\mathcal{R}^{(\mu,b)})$, $q_a \in (\dot{\tau}^a_{\mathcal{R}})^{-1}(\mathcal{R}^{(\mu,a)})$, and a section $\dot{t}^b_{\mathcal{R}} \in C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^b_{\mathcal{R}}, \mathbb{C})$, the image section $\dot{t}^a_{\mathcal{R}} \in C^\infty_{\mathrm{id}_{\mathrm{U}(1)}}(\dot{L}^a_{\mathcal{R}}, \mathbb{C})$ is defined by

$$\dot{t}^a_{\mathcal{R}}(\widetilde{g} \cdot q_a \cdot w) := \dot{t}^b_{\mathcal{R}}(\widetilde{g} \cdot q_b \cdot w)$$

for all $\widetilde{g} \in \widetilde{G}$, and $w \in \mathrm{U}(1)$. In other words, it is the map induced by the bundle-connection isomorphism $\dot{L}^b_{\mathcal{R}} \simeq \dot{L}^a_{\mathcal{R}}$, dependent on q_b and q_a.

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