Ordinary mod p representations of the metaplectic cover of p-adic SL_2

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Abstract

We classify the genuine ordinary mod p representations of the metaplectic group $\widetilde{SL}_2(F)$, where F is a p-adic field, and compute its genuine mod p spherical and Iwahori Hecke algebras. The motivation is an interest in a possible correspondence between genuine mod p representations of $\widetilde{SL}_2(F)$ and mod p representations of the dual group $PGL_2(F)$, so we also compare the two Hecke algebras to the mod p spherical and Iwahori Hecke algebras of $PGL_2(F)$. We show that the genuine mod p spherical Hecke algebra of $\widetilde{SL}_2(F)$ is isomorphic to the mod p spherical Hecke algebra of $PGL_2(F)$, and that one can choose an isomorphism which is compatible with a natural, though partial, correspondence of unramified ordinary representations via the Hecke action on their spherical vectors. We then show that the genuine mod p Iwahori Hecke algebra of $\widetilde{SL}_2(F)$ is a subquotient of the mod p Iwahori Hecke algebra of $PGL_2(F)$, but that the two algebras are not isomorphic. This is in contrast to the situation in characteristic 0, where by work of Savin one can recover the local Shimura correspondence for representations generated by their Iwahori fixed vectors from an isomorphism of Iwahori Hecke algebras.

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Introduction

0.1 Summary of results

The work of this thesis concerns mod p representation theory; that is, the representations are of p-adic groups, and the coefficient field is $\bar{\mathbb{F}}_p$. The main subject is the mod p representation theory of the metaplectic group $\widetilde{SL}_2(F)$, which is the nontrivial central extension of $SL_2(F)$ by $\{\pm 1\}$.

The first chapter contains preliminaries: definitions, well-known results, and some calculations to be used in later chapters. The goal of the second chapter is a classification of the genuine ordinary representations of $\widetilde{SL}_2(F)$. A *genuine* representation of $\widetilde{SL}_2(F)$ is one which does not factor through a representation of $SL_2(F)$, and we define an *ordinary* representation to be a subquotient of a parabolically induced representation. In fact, we show that all of the parabolically induced representations of $\widetilde{SL}_2(F)$ are irreducible and inequivalent.

Let F be a p-adic field with residue field $k = \bar{\mathbb{F}}_q$. In the following theorem and results following from it in later chapters, we assume that $q \equiv 1 \pmod 4$. With respect to a choice of an additive character of $F^\times$, we define (§ 2.3.3) a basic unramified genuine character $\tilde{1}$ of the metaplectic torus $\widetilde{T}$.

Theorem A (Theorem 2.3.5 (1), (2))**.** *1. The irreducible smooth, genuine, ordinary mod p representations of $\widetilde{SL}_2(F)$ are all those of the form $I(\tilde{\chi}) := Ind_{\widetilde{B}}^{\widetilde{SL}_2(F)}\tilde{\chi}$, where Ind is the smooth induction functor and $\tilde{\chi}$ is an arbitrary genuine character of $\widetilde{T}(F)$ (defined with respect to a fixed additive character of F).*

2. *The dimension of* $Hom_{\widetilde{SL}_2(F)}(I(\tilde{\chi}), I(\tilde{\chi'}))$ *is 1 if* $\tilde{\chi} = \tilde{\chi}'$ *and is 0 otherwise, so* $I(\tilde{\chi}) \cong I(\tilde{\chi'})$ *if and only if* $\tilde{\chi} = \tilde{\chi}'$*.*

In addition, we find the invariants of these representations under the compact open subgroups K^*, I^*, and $I(1)^*$ of $\widetilde{SL}_2(F)$. These subgroups are certain lifts to $\widetilde{G}$ of, respectively, the maximal compact subgroup $K = SL_2(\mathcal{O}_F)$, the Iwahori subgroup I, and the pro-p-Iwahori subgroup $I(1)$ in $SL_2(F)$.

Theorem B (Theorem 2.3.5 (3), (4))**.** *Let* $I(\tilde{\chi})$ *be a genuine ordinary representation of* $\widetilde{SL}_2(F)$*.*

1. *The* $I(1)^*$*-invariant space* $I(\tilde{\chi})^{I(1)^*}$ *is of dimension 2 over* $\bar{\mathbb{F}}_p$*.*

2. *If the restriction of* $\tilde{\chi}$ *to* $\widetilde{T} \cap K^*$ *is not equal to* $\tilde{1}$*, then* $I(\tilde{\chi})$ *has no nontrivial* I^**- or* K^**-invariants. If* $\tilde{\chi}\big|_{\widetilde{T} \cap K^*} = \tilde{1}$*, i.e., if* $\tilde{\chi}$ *is unramified, then* $I(\tilde{\chi})^{I^*} = I(\tilde{\chi})^{I(1)^*}$ *(and so is 2-dimensional), and* $I(\tilde{\chi})^{K^*}$ *is 1-dimensional.*

The third chapter is a study of the genuine mod p spherical Hecke algebra $\mathcal{H}_p^{\epsilon}(\widetilde{SL}_2(F), K^*)$ of $\widetilde{SL}_2(F)$. The results are:

Theorem C (Theorem 3.4.7)**.** 1. *There exists an explicit algebra isomorphism*

$$\mathcal{H}_p^{\epsilon}(\widetilde{SL}_2(F), K^*) \to \mathcal{H}_p(PGL_2(F), K_G)$$

inducing a bijection (which depends on the additive character ψ*) of those irreducible genuine unramified ordinary representations of* $\widetilde{SL}_2(F)$ *associated to characters different from the sign character, with the irreducible unramified ordinary representations of* $PGL_2(F)$*.*

2. *When a genuine character* $\tilde{\chi} = \chi \cdot \gamma_{\psi}$ *is defined with respect to a fixed choice of* ψ *(as in § 2.3.3) and* χ *is a smooth unramified character of* $F^{\times}$ *such that* $\chi^2 \neq 1$*, the irreducible unramified ordinary representation* $I(\tilde{\chi})$ *of* $\widetilde{SL}_2(F)$ *corresponds under the bijection to the irreducible unramified ordinary representation* $I(\chi \otimes \chi^{-1})$ *of* $PGL_2(F)$*.* $I(\tilde{1})$ *corresponds to the trivial representation of* $PGL_2(F)$*.*

3. *The dependence of the bijection on ψ is as follows. For $a \in F^\times/(F^\times)^2$, let χ_a denote the quadratic character of $F^\times$ given by the Hilbert symbol $(-, a)_F$. If $I(\chi \otimes \chi^{-1})$ corresponds to $I(\tilde{\chi})$ when the bijection is defined with respect to a nontrivial additive character ψ, then $I(\chi \otimes \chi^{-1})$ corresponds to $I(\chi_a \cdot \tilde{\chi})$ when the bijection is defined with respect to the character $\psi_a : x \mapsto \psi(ax)$.*

To prove Theorem C (1), we show that the mod p Satake isomorphism for unramified reductive groups can be adapted to define a Satake isomorphism of $\mathcal{H}_p^\epsilon(\widetilde{SL}_2(F), K^*)$ with a subalgebra of the genuine mod p spherical Hecke algebra of the torus $\widetilde{T}$. Both the spherical Hecke algebra of $\widetilde{T}$ and the Satake map can be explicitly described, allowing us to find the action of $\mathcal{H}_p^\epsilon(\widetilde{SL}_2(F), K^*)$ on the K^*-invariant subspaces of the unramified ordinary representations of $\widetilde{SL}_2(F)$. We use these spherical Hecke module structures to define the bijection of Theorem C (2) between the unramified ordinary representations of $\widetilde{SL}_2(F)$ and of $PGL_2(F)$ which are associated to characters $\chi \neq \mathrm{sgn}$ of $F^\times$.

In the fourth chapter, we compute a presentation for the genuine mod p Iwahori Hecke algebra of $\widetilde{SL}_2(F)$:

Theorem D (Theorem 4.3.7)**.** *The genuine mod p Iwahori Hecke algebra of $\widetilde{SL}_2(F)$ has the following presentation as a noncommutative polynomial algebra:*

$$\mathcal{H}_p^\epsilon(\widetilde{SL}_2(F), I^*) = \bar{\mathbb{F}}_p\langle x,\, y\rangle/(x^2 + x, y^2).$$

We then compare this algebra to the mod p Iwahori Hecke algebra of $PGL_2(F)$ computed by Barthel and Livné [3]. The motivation is to understand whether the partial correspondence of unramified prinicipal series representations between $\widetilde{SL}_2(F)$ and $PGL_2(F)$ can be extended in a natural way, e.g., via a map of Hecke modules, to representations which are generated by their Iwahori-fixed vectors. In characteristic 0, Savin [23] proved that the genuine Iwahori Hecke algebra of a covering group is isomorphic to the Iwahori Hecke algebra of its reductive dual group and that this induces an equivalence of categories of representations generated by their Iwahori-fixed vectors. However, we show that there is no such isomorphism for the mod p genuine Iwahori Hecke algebra of $\widetilde{SL}_2(F)$ and the mod p Iwahori Hecke algebra of $PGL_2(F)$:

Theorem E (Corollary 4.3.8)**.** *The genuine Iwahori Hecke algebra $\mathcal{H}_p^{\epsilon}(\widetilde{SL}_2(F), I^*)$ is not isomorphic to $\mathcal{H}_p(PGL_2(F), I_G)$. In fact, their abelianizations are not isomorphic.*

Finally, we note the genuine mod p Iwahori Hecke algebra is a subquotient of the mod p Iwahori Hecke algebra of $PGL_2(F)$, namely a quotient of the image of the mod p Iwahori Hecke algebra of $SL_2(F)$ by the square of one of its two generators.

0.2 Perspective

These results are motivated by an interest in the mod p analogue of the local Shimura correspondence, and in particular aim towards an explicit understanding of the relationship between mod p representations of the metaplectic group $\widetilde{SL}_2(F)$ and of its dual group $PGL_2(F)$.

The global Shimura correspondence, which relates cusp forms of weight $k + \frac{1}{2}$ to those of weight $2k$, was given a representation-theoretic interpretation by Waldspurger and others. In this formulation, which made it possible to use the Weil representation to relate Fourier coefficients to twists of L-values, a genuine automorphic representation of $\widetilde{SL}_2(\mathbb{A})$ corresponds to an automorphic representation of $PGL_2(\mathbb{A})$ (satisfying some local conditions). Locally at p, this becomes a correspondence between genuine irreducible representations of $\widetilde{SL}(F)$ and irreducible representations of $PGL_2(F)$, where F is a finite extension of $\mathbb{Q}_p$. This, the local Shimura correspondence, is usually described in terms of theta lifting via the Weil representation. An alternative interpretation by Savin [22] views the correspondence, in the classical case as well as more generally, as an isomorphism between the genuine Iwahori Hecke algebra of a metaplectic group and the usual Iwahori Hecke algebra of its dual group.

The work in this thesis uses Savin's point of view to relate the mod p representations of $\widetilde{SL}_2(F)$ and $PGL_2(F)$ via their Hecke algebras. The strategy is to analyze the $\bar{\mathbb{F}}_p$-valued Hecke algebras of the two groups, and precisely understand what the modules over the Hecke algebras of $\widetilde{SL}_2(F)$ may say about its mod p representations. The results of Chapter 3 show that the picture given by the mod p spherical Hecke algebras is quite similar to what is seen in characteristic 0: though we cannot translate the isomorphism of spherical Hecke algebras into a natural correspondence of all unramified ordinary representations, we can define a bijection which is compatible with the Hecke isomorphism for all but one representation on each side. On the other hand, the results of Chapter 4 show that the situation is quite different for the Iwahori Hecke algebra.

In future work, we hope to see how the failure of isomorphism of Hecke algebras appears in the relationship between representations of $\widetilde{SL}_2(F)$ and of $PGL_2(F)$ which are generated by Iwahori-fixed vectors. And as there are many genuine mod p representations of $\widetilde{SL}_2(F)$ which do not have Iwahori-fixed vectors, the spherical and Iwahori Hecke algebras considered in this thesis cannot be expected to give the full picture. However, since every mod p representation of a pro-p group has an invariant vector, it will be interesting to compare the genuine pro-p Iwahori Hecke algebra of $\widetilde{SL}_2(F)$ with the pro-p Iwahori Hecke algebra of $PGL_2(F)$. We outline some further questions for future work in the final section of Chapter 4.

Chapter 1

Basic constructions and known results

The first section of this chapter reviews the structure of $GL_2(F)$ and $SL_2(F)$ when F is a p-adic field, and then gives a concrete description of the nontrivial degree-2 central extension $\widetilde{SL}_2(F)$ of $SL_2(F)$. The second section of the chapter is devoted to describing the Bruhat-Tits tree of $SL_2(F)$ and actions of $SL_2(F)$ and $PGL_2(F)$ on it. The tree will be very useful in later calculations, as $SL_2(F)$ and $PGL_2(F)$ act on parts of it exactly as they do on their double cosets with respect to certain compact open subgroups.

Notation

Let F denote a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ the ring of integers in F, π a fixed uniformizer of $\mathcal{O}_F$, v the discrete valuation on F normalized so that $v(\pi) = 1$, and k the residue field of $\mathcal{O}_F$. The order of k will be denoted by q. In this chapter alone, we will make an effort to distinguish between a group scheme and its F-points: the name of a group scheme, for example $\mathbf{SL}_2$, will be written in boldface, while its F-points will be written in plain type, e.g., $SL_2(F)$. As SL_2 and GL_2 are mentioned equally often in this chapter, their full names will be written out, but the reader should note that $SL_2(F)$ (respectively, $\mathbf{SL}_2$) will be denoted by G (respectively, $\mathbf{G}$) in later chapters.

In this chapter, a tilde over the name of a group denotes the metaplectic double cover of that group: for example, $\widetilde{SL}_2(F)$ is the twofold metaplectic cover of $SL_2(F)$. Note in some references, such as [11], this cover is represented by $\widehat{SL}_2(F)$ or by $\overline{SL}_2(F)$ while $\widetilde{SL}_2(F)$

denotes the universal central extension of $SL_2(F)$ by K_2.

1.1 Structure of GL_2 and SL_2: Root data and distinguished elements

1.1.1 Root datum of SL_2

Let $\mathbf{T}$ be the split maximal torus of $\mathbf{SL_2}$. The character group $X^*(\mathbf{T})$ of $\mathbf{T}$ is isomorphic to $\mathbb{Z}$ via the map

$$\left(\chi : \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mapsto x^k\right) \mapsto k \in \mathbb{Z},$$

and the cocharacter group $X_*(\mathbf{T})$ is also isomorphic to $\mathbb{Z}$ via

$$\left(x \mapsto \begin{pmatrix} x^k & 0 \\ 0 & x^{-k} \end{pmatrix}\right) \mapsto k \in \mathbb{Z}.$$

The natural pairing $X^*(\mathbf{T}) \times X_*(\mathbf{T}) \to \mathbb{Z}$ is just multiplication: if $k \in X^*(T)$ and $\ell \in X_*(T)$, then $\langle k, \ell\rangle(x) = x^{k\ell}$ for all $x \in F^\times$.

Let $t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathbf{T}(F)$. The Lie algebra of $\mathbf{SL_2}$ is the trace-0 subspace of the matrix algebra M_2, and the adjoint action of t on decomposes into (1) the trivial action on the diagonal, (2) the character $x \mapsto x^2$ on the subspace $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \subset M_2$, and (3) the character $x \mapsto x^{-2}$ on the subspace $\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$. Hence the roots of $(\mathbf{SL_2}, \mathbf{T})$ are $\{2k : k \in \mathbb{Z}\}$, and we choose $\alpha = 2$ to be the positive simple root. The corresponding Borel subgroup $\mathbf{B}$, which is the stabilizer of the space $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ on which the conjugation action of $\mathbf{T}$ is given by α, is realized as the subgroup of upper triangular matrices, and $\mathbf{U}$ denotes the unipotent radical of $\mathbf{B}$. The Weyl group W of $(\mathbf{G}, \mathbf{T})$ is $\{1, w\} \cong \mathbb{Z}/2\mathbb{Z}$, where w is the reflection $w(k) = -k$, so the isomorphism $X^*(\mathbf{T}) \cong X_*(\mathbf{T})$ determined by the pairing is clearly compatible with the action of W.

The coroot dual to α is $\alpha^\vee = 1$, which is the cocharacter sending $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Let $X_*(\mathbf{T})_+ := \{k \in X_*(\mathbf{T}) : k \geq 0\}$ be the set of dominant cocharacters, i.e., those whose inner product with α, i.e. $k \cdot 2$, is nonnegative, and let $X_*(\mathbf{T})_- := \{k \in X_*(\mathbf{T}) : k \leq 0\}$ be the set of antidominant cocharacters, i.e., those whose inner product $k \cdot 2$ with α is nonpositive. Fix a uniformizer π of F, and note that $\mathbb{Z} \cong T(F)/T(\mathcal{O}_F)$ via $k \mapsto \begin{pmatrix} \pi^k & 0 \\ 0 & \pi^{-k} \end{pmatrix}$. Then the composite map $\lambda \mapsto T(F)/T(\mathcal{O}_F)$ given by

$$\left(t \mapsto \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix} \right) \mapsto \begin{pmatrix} \pi^k & 0 \\ 0 & \pi^{-k} \end{pmatrix}$$

is an isomorphism as well. Let $\alpha_0 = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}$; then in particular

$$X_*(\mathbf{T}) \cong \langle \alpha_0 \rangle \subset T(F)/T(\mathcal{O}_F),$$

where the brackets denote the cyclic subgroup, and the antidominant cocharacters correspond to the nonnegative powers of α_0.

1.1.2 Root data of GL_2 and PGL_2

Let $\mathbf{T_G}$ denote the maximal split torus in $\mathbf{GL_2}$. We associate a root datum $(X, \Phi, X^\vee, \Phi^\vee)$ to the pair $(\mathbf{GL_2}, \mathbf{T_G})$ as follows.

The group X of algebraic characters of $\mathbf{T_G}$ corresponds to the group of F-algebra homomorphisms $F[x, x^{-1}] \to F[y_1, y_1^{-1}, y_2, y_2^{-1}]$. Each such homomorphism is described by a pair of integers (m_1, m_2) such that $x \mapsto y_1^{m_1} y_2^{m_2}$, and distinct pairs of integers, including pairs containing the same integers in the opposite order, determine distinct F-algebra maps. Hence $X \cong \mathbb{Z}^2$. The group $X^\vee$ of algebraic cocharacters of $\mathbf{T_G}$ corresponds to the group of F-algebra homomorphisms $F[y_1, y_1^{-1}, y_2, y_2^{-1}] \to F[x, x^{-1}]$, and each such homomorphism is determined by the images of y_1 and y_2. These must be invertible in $F[x, x^{-1}]$ and so $y_1 \mapsto x^{m_1}$, $y_2 \mapsto x^{m_2}$. Again each distinct pair of integers (m_1, m_2) determines a different F-algebra map, so $X^\vee \cong \mathbb{Z}^2$ as well. X and $X^\vee$ are dual via the pairing $\langle x, y \rangle = (m_1, m_2) \cdot (k_1, k_2) = m_1 k_1 + m_2 k_2$.

We now describe the root system $\Phi \subset X$ and coroots $\Phi^\vee \subset X^\vee$ of $\mathbf{GL_2}$ with respect to $\mathbf{T_G}$. The Lie algebra of $\mathbf{GL_2}$ is the matrix algebra M_2, and the adjoint representation of $\mathbf{GL_2}$ is the conjugation action on M_2. Restricting Ad to $\mathbf{T_G}$, the action is

$$\mathrm{Ad}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e & \frac{t_1}{t_2} f \\ \frac{t_2}{t_1} g & h \end{pmatrix},$$

so an algebraic character α of T gives the adjoint action of T on a nontrivial subspace of $\mathfrak{g}$ if and only if α is one of the following: $\alpha = 1$, i.e., $\alpha = (0, 0)$ in $\mathbb{Z}^2$ (so $\mathfrak{g}_\alpha = T$), $\alpha = t_1/t_2$, i.e. $\alpha = (1, -1)$ in $\mathbb{Z}^2$ (so $\mathfrak{g}_\alpha$ is the set of matrices with zero entries except in the top right corner), or $\alpha = t_2/t_1$, i.e. $\alpha = (-1, 1)$ (so $\mathfrak{g}_\alpha$ is the set of matrices with zero entries except in the lower left corner). Thus, taking these characters and excluding the trivial character, $\Phi(\mathbf{GL_2}, \mathbf{T_G}) = \{(1, -1), (-1, 1)\}$.

Starting with $\alpha = (-1, 1)$, we have $\ker(\alpha) = \{$diagonal matrices (t_1, t_2) such that $t_2/t_1 = 1\}$, which is equal to the center $Z(\mathbf{GL_2})$. $Z(\mathbf{GL_2})$ is connected, so $T_\alpha = Z(\mathbf{GL_2})$ and $Z_\alpha = Z_{\mathbf{GL_2}}(Z) = \mathbf{GL_2}$. Then $\alpha^\vee$ is the cocharacter $\mathbb{G}_m \to T$ which, over F, sends $t \in \mathbb{G}_m$ to the diagonal matrix $(t, t^{-1}) \in SL_2(F)$ and then (via Φ_α) to the same diagonal matrix $(t, t^{-1}) \in GL_2(F)$. This composition corresponds to $(1, -1) \in \mathbb{Z}^2$ in the parametrization of $X^\vee$ used above. The coroot dual to α is defined to satisfy $\langle \alpha, \alpha^\vee \rangle = 2$, so we multiply $(1, -1)$ by -1 to get $\alpha^\vee = (-1, 1)$. As for the coroot corresponding to the root $(1, -1)$, the calculation is identical and gives the result $(1, -1) \in X^\vee$.

Let w be the Weyl element of $\mathbf{GL_2}$ considered as an element of $GL_2(F)$: $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let N_{T_G} be the normalizer of T_G in G. Then N_{T_G} acts on X by conjugation, and since T is commutative, the Weyl group $W_0 := N_{T_G}/T_G$ also acts on X by conjugation. Noting that N_{T_G} is the disjoint union of T_G with the set of antidiagonal elements of G, that is, $N_{T_G} = T_G \amalg wT_G$, we have $W_0 = \{1, w\}$. The nontrivial element of the Weyl group can be interpreted as the reflection through the hyperplane orthogonal to the opposite roots in Φ; it permutes the two roots.

Let Φ^+ be a choice of a positive root, say $\Phi^+ = (1, -1)$. This choice corresponds to the selection of $B = Lie(\mathbf{T_G}) \oplus \mathfrak{g}_{(1,-1)}$, which is the group of upper triangular matrices, as the preferred Borel subgroup.

Let $W = N_{T_G}(F)/T_G(\mathcal{O}_F)$. Using the decomposition

$$N_{T_G} = T_G \amalg wT_G,$$

we write $T_G(F)/T_G(\mathcal{O}_F) \amalg wT_G(F)/T_G(\mathcal{O}_F) = \Lambda \times W_0$, where $\Lambda = T_G(F)/T_G(\mathcal{O}_F)$. As was true for $SL_2(F)$, there is a canonical isomorphism

$$\Lambda \cong X^\vee : \Lambda \cong (F^*/\mathcal{O}_F^\times)^2 \cong \mathbb{Z}^2$$

via the map

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto (v_p(x), v_p(y)).$$

The nontrivial element $w \in W_0$ acts on Λ by

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto w \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} w = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$$

and on $X^v \cong \mathbb{Z}^2$ by permuting the two coordinates, so the isomorphism $\Lambda \cong \mathbb{Z}^2$ is compatible with the action of W_0.

To compute the root datum of $\mathbf{PGL_2}$, note that the characters of the torus of $\mathbf{PGL_2}$ are those characters of $\mathbf{T}_G$ which appear in the diagonal of $\mathbb{Z} \times \mathbb{Z}$, while the cocharacters of the torus of $\mathbf{PGL_2}$ are those of the form $(n, -n)$ in $\mathbb{Z} \times \mathbb{Z}$. Hence the root datum of $\mathbf{PGL_2}$ is the one obtained from the root datum of $\mathbf{SL_2}$ by switching X with $X^\vee$ and Φ with $\Phi^\vee$; that is, $\mathbf{SL_2}$ and $\mathbf{PGL_2}$ have *dual* root systems.

1.1.3 Decompositions in $SL_2(F)$ and $GL_2(F)$

We review the notation used by Barthel-Livné, Breuil, and Abdellatif to describe decompositions of $GL_2(F)$ and $SL_2(F)$. In $GL_2(F)$, define the following elements:

$$\alpha := \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}, \beta := \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}, w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Also put

$$z(\pi) := \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} \in Z(GL_2),$$

and note that $\beta = \alpha w$ and $\beta^2 = z(\pi)$.

In $SL_2(F)$, we define the analogous elements, some of which have appeared already:

$$\alpha_0 := \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}, \beta_0 := \begin{pmatrix} 0 & \pi \\ -\pi^{-1} & 0 \end{pmatrix}, s := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Every maximal compact subgroup of $GL_2(F)$ is conjugate to $K_G := GL_2(\mathcal{O}_F)$, while the maximal compact subgroups of $SL_2(F)$ lie in two distinct conjugacy classes: one which is represented by $K := SL_2(\mathcal{O}_F)$, and the other by $K' := \alpha K_0 \alpha^{-1}$. The existence of two non-conjugate maximal compacts has nontrivial consequences for the representation theory of $SL_2(F)$, but in this work we will only consider K.

The Cartan decomposition of $GL_2(F)$ is the disjoint union

$$GL_2(F) = \amalg_{n\in\mathbb{N}} K_G Z \alpha^n K_G. \tag{1.1}$$

Likewise, there is a Cartan decomposition of $SL_2(F)$ relative to each of its two representative maximal compacts:

$$SL_2(F) = \amalg_{n\in\mathbb{N}} K \alpha_0^n K = \amalg_{n\in\mathbb{N}} K \alpha_0^{-n}, \tag{1.2}$$

$$SL_2(F) = \amalg_{n\in\mathbb{N}} K'\alpha_0^n K' = \amalg_{n\in\mathbb{N}} K'\alpha_0^{-n} K' \tag{1.3}$$

Let I_G denote the standard Iwahori subgroup in $GL_2(F)$, which consists of those elements whose reduction mod π is upper triangular in $GL_2(k)$. It is compact and open in $GL_2(F)$. Denote $I_G \cap SL_2(F)$ by I; this is the standard Iwahori subgroup of $SL_2(F)$. From the Bruhat decomposition $G(k) = B(k) \amalg B(k)WB(k)$ of the groups $G = GL_2$ and SL_2 over the residue field, we obtain the following decompositions of $GL_2(F)$ and $SL_2(F)$ with respect to their Iwahori subgroups:

$$GL_2(F) = \amalg_{n\in\mathbb{Z}}\, I_G Z\alpha^n I_G \ \amalg_{n\in\mathbb{Z}}\ I_G Z\beta\alpha^n I_G \tag{1.4}$$

and

$$SL_2(F) = \amalg_{n\in\mathbb{Z}}\, I\alpha_0^n I \ \amalg_{n\in\mathbb{Z}}\ I\beta_0\alpha_0^n I. \tag{1.5}$$

Moreover, we have two decompositions of the Iwahori subgroup itself, both valid in $GL_2(F)$ and in $SL_2(F)$. Recall that U denotes the upper triangular unipotent subgroup, and let U' denote the lower triangular unipotent subgroup.

$$I_G = (U \cap I_G)(T_G \cap I_G)(U' \cap I_G) = (U' \cap I_G)(T \cap I)(U \cap I_G). \tag{1.6}$$

Intersecting with $SL_2(F)$, we get the analogous decompositions of I.

When H is any compact open subgroup of $G = GL_2(F)$, $PGL_2(F)$, or $SL_2(F)$ (or later, of the cover of such a group) and $x \in G$, define the *volume* of a double coset HxH to be the index

$$\mathrm{vol}(HxH) = [H : H \cap xHx^{-1}].$$

We record here a list of single-coset decompositions and volumes of certain I-double cosets in $SL_2(F)$ which will be needed in Chapter 4. Each calculation is an application of one of the two Iwahori decompositions (1.6) in $SL_2(F)$.

Lemma 1.1.1. *For $\ell > 0$,*

1. $I\alpha_0^{-\ell}I =$

$$I\begin{pmatrix}\pi^{\ell} & 0\\ 0 & \pi^{-\ell}\end{pmatrix}I = \coprod_{\substack{0\le v(y)\le 2\ell-1\\ or\ y=0}} I\begin{pmatrix}\pi^{\ell} & 0\\ 0 & \pi^{-\ell}\end{pmatrix}\begin{pmatrix}1 & 0\\ y\pi & 1\end{pmatrix},$$

and $vol(I\alpha_0^{-\ell}I) = q^{2\ell}$.

2. $I\alpha_0^{\ell}I =$

$$I\begin{pmatrix}\pi^{-\ell} & 0\\ 0 & \pi^{\ell}\end{pmatrix}I = \coprod_{\substack{0\le v(x)\le 2\ell-1\\ or\ x=0}} I\begin{pmatrix}\pi^{-\ell} & 0\\ 0 & \pi^{\ell}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$$

and $vol(I\alpha_0^{\ell}I) = q^{2\ell}$.

3. $Is\alpha_0^{-\ell}I =$

$$I\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\pi^{\ell} & 0\\ 0 & \pi^{-\ell}\end{pmatrix}I = \coprod_{\substack{0\le v(y)\le 2\ell-2\\ or\ y=0}} I\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\pi^{\ell} & 0\\ 0 & \pi^{-\ell}\end{pmatrix}\begin{pmatrix}1 & 0\\ y\pi & 1\end{pmatrix},$$

and $vol(Is\alpha_0^{-\ell}I) = q^{2\ell-1}$.

4. $Is\alpha_0^{\ell}I =$

$$I\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\pi^{-\ell} & 0\\ 0 & \pi^{\ell}\end{pmatrix}I = \coprod_{\substack{0\le v(x)\le 2\ell\\ or\ x=0}} I\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\pi^{-\ell} & 0\\ 0 & \pi^{\ell}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},$$

and $vol(Is\alpha_0^{\ell}I) = q^{2\ell+1}$.

5. $IsI =$

$$I\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}I = \coprod_{\substack{v(x)=0\\ or\ x=0}} I\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},$$

and $vol(IsI) = q$.

Proof. In the following calculations, x and y denote arbitrary elements of $\mathcal{O}_F$ and a denotes an arbitrary element of $\mathcal{O}_F^\times$.

1. We use the first Iwahori decomposition, and calculate:

 - $$\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x\pi^{2\ell} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix} \in I\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}.$$
 - $$\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix} \in I\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}$$
 - $$\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y\pi^{1-2\ell} & 1 \end{pmatrix}\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}.$$

 The element $\begin{pmatrix} 1 & 0 \\ y\pi^{1-2\ell} & 1 \end{pmatrix}$ is in $I \iff v(y) \geq 2\ell$, so $\mathrm{vol}(I\alpha_0^{-\ell}I) = q^{2\ell}$.

2. We use the second Iwahori decomposition, and calculate:

 - $$\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y\pi^{2\ell+1} & 1 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix} \in I\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}.$$
 - $$\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix} \in I\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}$$
 - $$\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x\pi^{-2\ell} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}$$

 The element $\begin{pmatrix} 1 & x\pi^{-2\ell} \\ 0 & 1 \end{pmatrix}$ is in $I \iff v(x) \geq 2\ell$, so $\mathrm{vol}(I\alpha_0^{\ell}I) = q^{2\ell}$.

3. We use the first Iwahori decomposition.

 - $$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x\pi^{2\ell} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}$$
 $$= \begin{pmatrix} 1 & 0 \\ -x\pi^{2\ell} & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix} \in I\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}$$

- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{\ell} & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{\ell} & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}$
$\in I\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{\ell} & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}$
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{\ell} & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y\pi^{1-2\ell} & 1 \end{pmatrix}\begin{pmatrix} \pi^{\ell} & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}$
$= \begin{pmatrix} 1 & -y\pi^{1-2\ell} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{\ell} & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}$,

and the element $\begin{pmatrix} 1 & -y\pi^{1-2\ell} \\ 0 & 1 \end{pmatrix}$ is in $I \iff v(y) \geq 2\ell - 1$, so $\mathrm{vol}(Is\alpha_0^{-\ell}I) = q^{2\ell-1}$.

4. We use the second Iwahori decomposition, and calculate:

- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y\pi^{2\ell+1} & 1 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}$
$= \begin{pmatrix} 1 & -y\pi^{2\ell+1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix} \in I\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}$.
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}$
$\in I\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}$
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x\pi^{-2\ell} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}$
$= \begin{pmatrix} 1 & 0 \\ -x\pi^{2\ell} & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{\ell} \end{pmatrix}$,

and the element $\begin{pmatrix} 1 & 0 \\ -x\pi^{2\ell} & 1 \end{pmatrix}$ is in $I \iff v(x) \geq 2\ell + 1$, so $\mathrm{vol}(Is\alpha_0^{\ell}I) = 2\ell + 1$.

5. We use the second Iwahori decomposition, and calculate

- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix} = \begin{pmatrix} 1 & -y\pi \\ 0 & 1 \end{pmatrix} \in I \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in I \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

and the element $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ is in $I \iff v(x) \geq 1$, so $\mathrm{vol}(IsI) = q$.

□

Finally, for future reference we note some results of Iwahori and Matsumoto on decompositions of products of I-double cosets:

Lemma 1.1.2 ([15] Prop. 2.8)**.** *1. If ℓ and m are either both ≥ 0 or both ≤ 0, we have $I\alpha_0^{\ell} I \alpha_0^m I = I\alpha_0^{\ell+m} I$.*

2. $IsIsI = I \amalg IsI$,

3. $Is\alpha_0^{-1} Is\alpha_0^{-1} I = I \amalg Is\alpha_0^{-1} I$.

1.2 Definition of the metaplectic cover of $SL_2(F)$

There are several ways to construct a nontrivial twofold cover of $SL_2(F)$. We will follow Deligne [10] and Savin [23] and define $\widetilde{SL}_2(F)$ to be the group generated by

$$\{e^+(a), \, e^-(a), \, h(a)\}_{a \in F^\times},$$

where

Definition 1.2.1. *1.* $e^+(a) := \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, 1 \right)$ *and* $e^-(a) := \left(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, 1 \right)$ *for* $a \in F$.

In particular, $\widetilde{SL}_2(F)$ *contains the following elements:*

$$w(a) := e^+(a)e^-(-a^{-1})e^+(a) = \left(\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}, 1 \right)$$

for all $a \in F^\times$.

The following two elements of $\widetilde{GL}_2(F)$ *will be used to define generators* $h(a)$ *of the torus of* $\widetilde{SL}_2(F)$*:*

- $d_1(a) := \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$ *for* $a \in F^\times$,
- $d_2(a) := w(1)d_1(a)w(1)^{-1} =$

$$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \cdot \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \cdot \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1 \right) = \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, (-1, a) \right)$$

  for $a \in F^\times$.

2. Then the torus $\widetilde{T} \subset \widetilde{SL}_2(F)$ *is defined to be the subgroup generated by the following elements:*

$$h(a) := d_1(a^{-1})d_2(a) = w(a^{-1})w(1)^{-1} = \left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, (-1, a) \right)$$

for $a \in F^\times$.

We can also view $\widetilde{SL}_2(F)$ as the group with underlying set $SL_2(F) \times \{\pm 1\}$ and multiplication

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 \sigma(g_1, g_2)),$$

where $\sigma(g_1, g_2)$ is a certain 2-cocycle on $SL_2(F)$ with values in $\{\pm 1\}$. Kubota [18] showed that one can define the multiplication in $\widetilde{SL}_2(F)$ with the following cocycle:

$$\sigma(g_1, g_2) = \left(\frac{X(g_1 g_2)}{X(g_1)}, \frac{X(g_1 g_2)}{X(g_2)} \right)_F, \tag{1.7}$$

where

$$X\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases},$$

and $(,)_F$ is the Hilbert symbol on F.

Remark 1.2.2. *In fact, the Kubota coycle σ given in (1.7) is simplified from the one written down by Kubota in [18]; see, e.g., [19] § 3 for (1.7).*

Thanks to our standing assumption that $p > 2$, the extension $\widetilde{SL}_2(F)$ of $SL_2(F)$ defined by the cocycle σ of (1.7) splits over the maximal compact subgroup K of $SL_2(F)$. The splitting is effected by the preferred section

$$g \mapsto (g, \theta(g)),$$

where we define

$$\theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} (c, d)_F & \text{if } cd \neq 0 \text{ and } c \text{ is not a unit}, \\ 1 & \text{otherwise.} \end{cases} \tag{1.8}$$

We note that the generators $e^+(a)$ and $e^-(a)$ defined in (1.2.1) are the lifts of $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and of $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, respectively, by $g \mapsto (g, \theta(g))$, and that d_1 is the lift of $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ by an extension of the preferred section to $GL_2(F)$.

The subgroup generated by $\{h(a) : a \in \mathcal{O}_F^\times\}$ is denoted by $\widetilde{T}(\mathcal{O}_F)$; the extension splits over $T(\mathcal{O}_F)$, since the Hilbert symbol on F is unramified. Let K^* denote the image of K in $\widetilde{SL}_2(F)$ under the preferred section, and let $\widetilde{K}$ denote the full preimage of K in the covering group. Then the image of $T(\mathcal{O}_F)$ in $\widetilde{SL}_2(F)$ under the section is $\widetilde{T} \cap K^*$.

The splitting of the extension over K is also compatible with the canonical splitting over the upper and lower unipotent subgroups U, U' of $SL_2(F)$; we define U^* to be the image of the upper triangular unipotent group in $\widetilde{SL}_2(F)$, and note that U^* is the subgroup of $\widetilde{SL}_2(F)$ generated by $\{e^+(a) : a \in F\}$. Likewise U^* is the subgroup of $\widetilde{SL}_2(F)$ generated by $\{e^-(a) : a \in F\}$. We denote the common lift of $U \cap K$ by $(U \cap K)^*$, and note that this is the same as $U^* \cap K^*$ (and likewise for U').

The extension does not split over the diagonal torus T of $SL_2(F)$. For a, $b \in F^\times$, we have

$$h(ab) = (1, (a,b)_F) \cdot h(a)h(b).$$

Note that $\widetilde{T}(F)$ contains the element $(1,-1)$, since F contains a unit u such that $(\pi, u)_F = -1$, and

$$h(u^{-1}\pi^{-1})h(u)h(\pi) = h(u^{-1}\pi^{-1})h(u\pi)(u,\pi)_F = h(1)(u^{-1}\pi^{-1}, u\pi)_F(u,\pi)_F$$

$$= (1, (u,\pi)_F^2(u,u)_F(\pi,\pi)_F(u,\pi)_F = (1,(u,\pi)_F) = (1,-1),$$

but that $(1,-1) \notin \widetilde{T} \cap K^*$.

Definition 1.2.3. *Define Λ to be the subgroup of $\widetilde{T}(F)$ generated by*

$$h(\pi) = \left(\begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}, (-1,\pi)_F \right),$$

and define $\tilde{\Lambda} := \Lambda \times (1,-1)$.

We set down some notation for the sign of $h(\pi)^n$, $n \in \mathbb{Z}$. Note that when the order of the residue field $q = |k|$ is congruent to $1 \pmod 4$, then $-1 \in F^\times$, so $(-1,\pi)_F = (\pi,\pi)_F = 1$.

Thus when $q \equiv 1 \pmod 4$, we have $\Lambda = \langle \alpha_0^n \rangle \times \{1\}$. Otherwise, the sign of $h(\pi)^n$ depends on the class of $n \pmod 4$:

$$h(\pi^m)h(\pi^n) = \begin{cases} h(\pi^{m+n}) & \text{if } 2|m \text{ or } 2|n, \\ (1, (-1,\pi)_F)h(\pi^{m+n}) & \text{otherwise.} \end{cases}$$

By induction, we get

$$h(\pi)^k = \begin{cases} h(\pi^k) & \text{if } k \equiv 0 \text{ or } 1 \pmod 4 \\ (1, (-1,\pi)_F)h(\pi^k) & \text{if } k \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

Hence

$$h(\pi)^n = \begin{cases} \left(\begin{pmatrix} \pi^{-n} & 0 \\ 0 & \pi^n \end{pmatrix}, 1 \right) & \text{if } n \equiv 0 \text{ or } 3 \pmod 4 \\ \left(\begin{pmatrix} \pi^{-n} & 0 \\ 0 & \pi^n \end{pmatrix}, (-1,\pi)_F \right) & \text{if } n \equiv 1 \text{ or } 2 \pmod 4. \end{cases}$$

Definition 1.2.4. *For convenience in future formulae, let $\phi(n)$ denote the sign of $h(\pi)^n$. Concretely,*

$$\phi(n) = \begin{cases} 1 & \textit{for all } n \textit{ if } q \equiv 1 \pmod 4, \\ -1 & \textit{if } q \equiv 3 \pmod 4 \textit{ and } n \equiv 1 \textit{ or } 2 \pmod 4 \\ 1 & \textit{if } q \equiv 3 \pmod 4 \textit{ and } n \equiv 3 \textit{ or } 4 \pmod 4. \end{cases}$$

The Cartan decomposition (1.2) of $SL_2(F)$ implies that we can choose representatives for $K^* \setminus \widetilde{SL}_2(F)/K^*$ which lie in $\tilde{\Lambda}$. We will show in Chapter 3 that in fact $\tilde{\Lambda}$ forms a set of representatives for the K^*-double cosets in $\widetilde{SL}_2(F)$.

Next, in preparation for lifting the decomposition (1.5) to the covering group, we describe the normalizer of $\widetilde{T} \cap K^*$ in $\widetilde{SL}_2(F)$.

Let W_0 denote the finite Weyl group of $\widetilde{SL}_2(F)$, i.e., the subgroup generated by $w(1)$, whose elements are

$$\{w(1)^0 = h(1) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right), w(1) = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right),$$

$$w(1)^2 = h(-1) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1\right), w(1)^3 = w(-1) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\}.$$

Lemma 1.2.5. *The normalizer of* $\widetilde{T} \cap K^*$ *in* $\widetilde{SL}_2(F)$ *is equal to* $\widetilde{T} \amalg \widetilde{T}w(1)$.

Proof. If $(g, \zeta) \in N_{\widetilde{SL}_2(F)}(\widetilde{T}(\mathcal{O}_F))$, then given $t \in \mathcal{O}_F^\times$,

$$(g, \zeta)\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, 1\right)(g, \zeta)^{-1} = \left(\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, 1\right)$$

for some $r \in \mathcal{O}_F^\times$. In particular, $g \in N_{SL_2(F)}(T(\mathcal{O}_F)) = T \amalg Ts$. Let $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in T$, and let $\eta \in \{\pm 1\}$ satisfy $(g, \zeta) = h(a^{-1}) \cdot (1, \eta)$. That is, η is defined by $\zeta = \eta \cdot (-1, a)_F$. We have $h(a^{-1})^{-1} = h(a) \cdot (1, (-1, a)_F)$, so

$$(g, \zeta)^{-1} = h(a) \cdot (1, \eta(-1, a)_F) = \left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \eta\right).$$

Then $(g,\zeta)\left(\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}, 1\right)(g,\zeta)^{-1} =\right.$

$$\begin{aligned}
&\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}, \zeta\right)\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}, 1\right)\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}, \zeta(-1,a)_F\right) \\
&= ((1,\eta)\, h(a^{-1})h(t^{-1}) \cdot (1,\zeta)h(a) \\
&= (1, \zeta\eta(a,t)_F)\, h(a^{-1}t^{-1})h(a) \\
&= (1, \zeta\eta(a,t)_F(at,a)_F)\, h(t^{-1}) \\
&= (1, (-1,a)_F(a,t)_F(a,a)_F(t,a)_F)\, h(t^{-1}) \\
&= \left(1, (-1,a)_F(a,t)_F^2(a,a)_F\right) h(t^{-1}) \\
&= \left(1, (-1,a)_F^2\right) h(t^{-1}) = h(t^{-1}).
\end{aligned}$$

So for any $a \in F^\times$ and $\eta \in \{\pm 1\}$, $h(a)\cdot(1,\eta)$ is in the normalizer of $\widetilde{T}(\mathcal{O}_F)$.

Next, suppose $g \in Ts$, so that $(g,\zeta) = (1,\eta)\cdot w(a) = (1,\eta)h(a)w(1)$ for some $\eta \in \{\pm 1\}$ and some $a \in F^\times$. Then $(g,\zeta)^{-1} = ((1,\eta)h(a)w(1))^{-1} = w(-1)h(a^{-1})\,(1,\eta(-1,a)_F)$, so

$$(g,\zeta)\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}, 1\right)(g,\zeta)^{-1} = (1,\eta)h(a)w(1)\cdot h(t^{-1})\cdot w(-1)h(a^{-1})\,(1,\eta(-1,a)_F)\,. \tag{1.9}$$

Calculating with the Kubota cocycle, we get $w(1)h(t^{-1})w(-1) = h(t)$, so (1.9) is equal to

$$\begin{aligned}
&\left(1, \eta^2(-1,a)_F\right) h(a)h(t))h(a^{-1}) \\
&= \left(1, (-1,a)_F(a,t^{-1})_F\right) h(at)h(a^{-1}) \\
&= \left(1, (-1,a)_F(a,t^{-1})_F(at,a^{-1})_F\right) h(t) \\
&= \left(1, (-1,a)_F^2(a,t)_F^2\right) h(t) \\
&= h(t) \in \widetilde{T}(\mathcal{O}_F).
\end{aligned}$$

Hence

$$N_{\widetilde{SL}_2(F)}(\widetilde{T}\cap K^*) = \widetilde{T} \amalg \widetilde{T}w(1).$$

□

We have shown that the normalizer of $\widetilde{T}\cap K^*$ in $\widetilde{SL}_2(F)$ is equal to the inverse image in $\widetilde{SL}_2(F)$ of the normalizer N of $T(\mathcal{O}_F)$ in $SL_2(F)$, and we denote it $N_{\widetilde{SL}_2(F)}(\widetilde{T}\cap K^*)$ by $\widetilde{N}$.

Since the Iwahori subgroup $I \subset SL_2(F)$ is contained in K, its image in $\widetilde{SL}_2(F)$ under the preferred section (1.8) is contained in K^*, and we denote it by I^*. The main significance of $\widetilde{N}$ to us will be that one can choose representatives of I^*-double cosets in $\widetilde{SL}_2(F)$ to lie in $\widetilde{N}$. In Chapter 4 we will show that in fact $\tilde{\Lambda} \ltimes W_0$ is a set of representatives for $I^* \setminus \widetilde{SL}_2(F)/I^*$.

Finally, we note that since the splitting of the extension over K is compatible with that over the unipotent subgroups of $SL_2(F)$, the two Iwahori decompositions of I^* lift to the covering group:

$$I^* = (U\cap I)^*(\widetilde{T}\cap I^*)(U'\cap I)^* = (U'\cap I)^*(\widetilde{T}\cap I^*)(U\cap I)^*. \tag{1.10}$$

These decompositions will be crucial in Chapter 4, when we will need to find single I^*-coset decompositions of products of I^*-double cosets lifting those given in of Lemma 1.1.1.

1.2.1 Norms and the Hilbert symbol on F

This section contains several results which will be needed in calculations with the cocycle. Namely, we will show that since $p > 2$, then the Hilbert symbol is *unramified*, i.e. trivial on $\mathcal{O}_F^\times \times \mathcal{O}_F^\times$; that given a uniformizer π of F, there is a unit $u \in \mathcal{O}_F^\times$ such that $(u,\pi)_F = -1$; and that the norms from the quadratic extension $F(\sqrt{\pi})$ have index 2 in $\mathcal{O}_F^\times$.

Given any degree-n extension of p-adic fields F'/F such that F contains n n-th roots of unity, let $N_{F'/F}$ denote the norm map $F'^\times \to F$. Recall that by local class field theory we have a symbol $(,)_F$ on $F^\times \times F^\times$ such that ([24], XIV Prop. 7):

1. $(aa',b)_F = (a,b)_F(a',b)_F$

2. $(a, bb')_F = (a, b)_F (a, b')_F$

3. $(a, b)_F = 1 \iff b \in N_{F'/F}(F')$.

4. $(a, -a)_F = (a, 1-a)_F = 1$, and in general if $a \in F^*$, $b \in F$, and $b^n - a \neq 0$, then $(a, b^n - a) = 1$.

5. $(a, b)_F (b, a)_F = 1$.

6. If $(a, b)_F = 1$ for all $b \in F^\times$, then $a \in (F^\times)^n$.

So in particular, the degree-2 Hilbert symbol on F is symmetric, multiplicative in each entry, and has the property that $(a^2, b)_F = 1$ for all a, $b \in F^\times$.

Recall our assumption that $p \neq 2$; hence the extension $F(\sqrt{a})/F$ is at most tamely ramified for all $a \in F^\times$. Then ([24] XIV § 3) if a, $b \in F^\times$, a formula for $(a, b)_F$ is given by the tame symbol

$$(a, b)_F = \left(\overline{(-1)^{v(a)v(b)} \frac{a^{v(b)}}{b^{v(a)}}} \right)^{\frac{q-1}{2}},$$

where the term inside the parentheses is considered to be in the residue field k of F. In particular, if b is a unit, then

$$(a, b)_F = \left(\overline{b^{v(a)}} \right)^{-\frac{q-1}{2}},$$

and if π is a uniformizer of F, then

$$(\pi, \pi)_F = (-1)^{\frac{q-1}{2}} = \begin{cases} -1 & \text{if } q - 1 \equiv 2 \pmod 4 \\ 1 & \text{if } q - 1 \equiv 0 \pmod 4. \end{cases}$$

Other important cases for our applications are: if a and b are both units, then

$$(a, b)_F = 1,$$

and if π is a uniformizer of $\mathcal{O}_F$ and b is a unit, then

$$(b,\pi)_F = \left(\overline{b}\right)^{\frac{q-1}{2}}.$$

So if $q \equiv 1 \pmod 4$, then -1 is a square in k, and so

$$(-1,\pi)_F = \left(\overline{-1}\right)^{\frac{q-1}{2}} = 1.$$

Using multiplicativity of the symbol, we get

$$(\pi,\pi)_F = (-1,\pi)_F = \begin{cases} -1 & \text{if } q \equiv 3 \pmod 4 \\ 1 & \text{if } q \equiv 1 \pmod 4. \end{cases}$$

Finally, it will be important to know that given a fixed uniformizer π of $\mathcal{O}_F$,

$$\{u \in \mathcal{O}_F^\times : (u,\pi)_F = 1\}$$

is an index-2 subgroup of $\mathcal{O}_F^\times$. This is because, by a quick argument in group cohomology, we have the following equality for any finite cyclic extension E/F of local fields:

$$[\mathcal{O}_F^\times : N_{E/F}(\mathcal{O}_E^\times)] = e(E/F),$$

where $e(E/F)$ is the ramification index of the extension. In particular, when $E = F(\sqrt{\pi})$ for π a uniformizer of F, then $e(E/F) = 2$ thanks to our standing assumption that $2 \neq p$. And of course if $u \in \mathcal{O}_F^\times$ is a norm from $F(\sqrt{\pi})$, then it is the norm of an element of $\mathcal{O}_E^\times$. So the norms are of index 2 in $\mathcal{O}_F^\times$.

1.2.2 Topology of the cover

It will be very important to know that the covering group is locally compact and totally disconnected.

As is any p-adic Lie group, $SL_2(F)$ is locally compact and totally disconnected (an "ℓ-group" in the terminology of Bernstein-Zelevinsky [4]), so has a neighborhood basis at 1 of open compact subgroups.

This structure transfers without trouble to the covering group. By construction, the cocycle $\sigma : G \times G \to \{\pm 1\}$ is continuous with respect to the usual topology on $SL_2(F)$, and by the results of the previous section, the extension splits over the open compact subgroup $K \subset G$. The subgroup K contains a neighborhood basis $\{K_n : n \in \mathbb{N}\}$ of open compacts at 1, and we define the set of images of the K_n under the splitting to be a neighborhood basis of $(1, 1)$ in $\widetilde{G}$. Let the topology on $\widetilde{G}$ be the one generated by this neighborhood basis. Then $\widetilde{G}$ is a Hausdorff ℓ-group, and the projection $p : \widetilde{G} \to G$ is continuous and open.

1.3 The tree of $SL_2(F)$

Many of the calculations in this work become easier when phrased in terms of the action of $SL_2(F)$ on its Bruhat-Tits building, which is a $q+1$-regular tree. Though the tree is a special case of a Bruhat-Tits building and many results about it can be generalized for unramified connected reductive groups, it is enough here to give the following concrete description. A reference for this section is [25].

A *lattice* in $F \oplus F$ is a finite-type $\mathcal{O}_F$-module whose F-span is equal to $F \oplus F$. Every such $\mathcal{O}_F$-module is free, so after fixing a basis for $\mathcal{O}_F \oplus \mathcal{O}_F \subset F \oplus F$ to correspond to the 2-by-2 identity matrix, any other lattice can be specified by a nonsingular 2-by-2 matrix with entries in F. A *homothety* of a lattice is a scaling of both basis vectors by the same factor, i.e., a transformation by a scalar 2-by-2 matrix. Homothety is an equivalence relation on the lattices in $F \oplus F$, so we can consider a set of vertices $\mathrm{Ver}(X)$ indexed by their homothety classes, i.e., by the elements of $PGL_2(F)$.

Thanks to the Cartan decomposition of $GL_2(F)$, any two elements $[L]$, $[L']$ of $PGL_2(F) \cong GL_2(F)/Z$ have representatives $L := a_1e_1 \oplus a_2e_2$, $L' := b_1e_1 \oplus b_2e_2$ such that $b_1 = \pi^n a_1$ and $b_2 = \pi^m a_2$ for some $n \leq m \in \mathbb{Z}$. The difference $m - n$ is independent of the choice of representatives, so we can define two vertices $[L]$, $[L'] \in \mathrm{Ver}(X)$ to be *adjacent*, i.e., connected by an edge, if and only if $m - n = 1$. Equivalently, two vertices are adjacent if they can be represented by lattices L and L' such that $L'/L \cong \mathcal{O}_F/\pi\mathcal{O}_F = k$. Let $\mathrm{Ed}(X)$ be the set of oriented edges generated by this principle. More generally, define the *distance* between any two vertices $[L]$, $[L']$ to be the integer $m - n$. One can show that the graph $(\mathrm{Ver}(X), \mathrm{Ed}(X))$ is a tree, each of whose vertices have degree $|\mathbb{P}^1(k)| = q + 1$.

We fix a standard apartment, or infinite path, in X, which will parametrize the diagonal part of the group action. Let v_0 denote the identity vertex, corresponding to the homothety class of the lattice $\mathcal{O}_F \oplus \mathcal{O}_F \subset F \oplus F$. Also recall the elements

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \in GL_2(F) \text{ and } \alpha_0 = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix} \in SL_2(F).$$

Let $v_n := \alpha^n \cdot v_0$; then $\{v_n : n \in \mathbb{Z}\}$ is the standard apartment in X. Its intermediate edges are numbered accordingly: let $e_{n,n\pm1}$ be the oriented edge originating at v_n and terminating at $v_{n\pm1}$.

Let K_G denote the maximal compact subgroup $GL_2(\mathcal{O}_F) \subset GL_2(F)$. Clearly $GL_2(F)$ acts on $\mathrm{Ver}(X)$ on the left; equally clearly this action is transitive and the stabilizer of v_0 is $K_G Z$, so $\mathrm{Ver}(X) \cong GL_2(F)/K_G Z$ as a left $GL_2(F)$-set. In fact, $\mathrm{Ver}(X)$ is a transitive and faithful $PGL_2(F)$-set. In particular, with the labelling conventions of Figure 1.1, which represents a portion of the tree of $\mathbb{Q}_3$, the action of α on v_0 is to move a vertex up once along the leftmost upwards edge emerging from it. The action of α^{-1} on v_0 is one downward move along the leftmost downward edge.

The action of $SL_2(F)$ on $\mathrm{Ver}(X)$ is *not* transitive: it has two orbits, corresponding to the two conjugacy classes of maximal compact subgroups in $SL_2(F)$. One of these classes is represented by $K = SL_2(\mathcal{O}_F)$, the stabilizer of v_0, while the other class is represented by the stabilizer of v_1, which is $K' := \alpha K \alpha^{-1}$. Since the difference of valuation between the two nonzero entries of any diagonal element of $SL_2(F)$ is *even*; the orbit of v_0 is the set of all vertices at even distance from v_0, while the orbit of v_1 is the set of all vertices at odd distance from v_0. The diagonal part of the action of $SL_2(F)$ on $\mathrm{Ver}(X)/K$ is parametrized

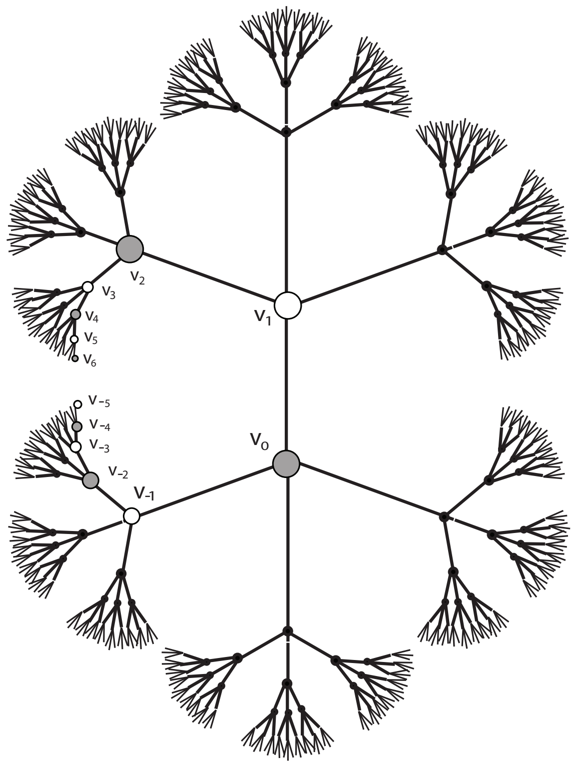

Figure 1.1

by powers of α_0, which is equivalent to α^2 modulo the center of $GL_2(F)$; the element α_0 sends v_0 to v_2. In Figure 1.1, the vertices of the standard apartment are shaded according to their parity: the gray dots are those which lie in the orbit of v_0 under the action of $SL_2(F)$.

Note that the action of $GL_2(F)$ is isometric, and that an oriented edge is fixed by $g \in GL_2(F)$ when g fixes both the origin and the vertex. A *generalized Iwahori subgroup* of $GL_2(F)$ is the pointwise stabilizer of any two adjacent vertices; the *standard Iwahori subgroup* is the one which fixes both v_0 and v_1:

$$I_G := GL_2(\mathcal{O}_F) \cap \alpha GL_2(\mathcal{O}_F)\alpha^{-1} = \Big\{ \begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} : a,\, b,\, c,\, d \in \mathcal{O}_F,\, ad - bc \neq 0 \Big\},$$

i.e. I_G is the pointwise stabilizer of $e_{0,1}$ and of $e_{1,0}$. (The coset

$$I_G \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} I_G$$

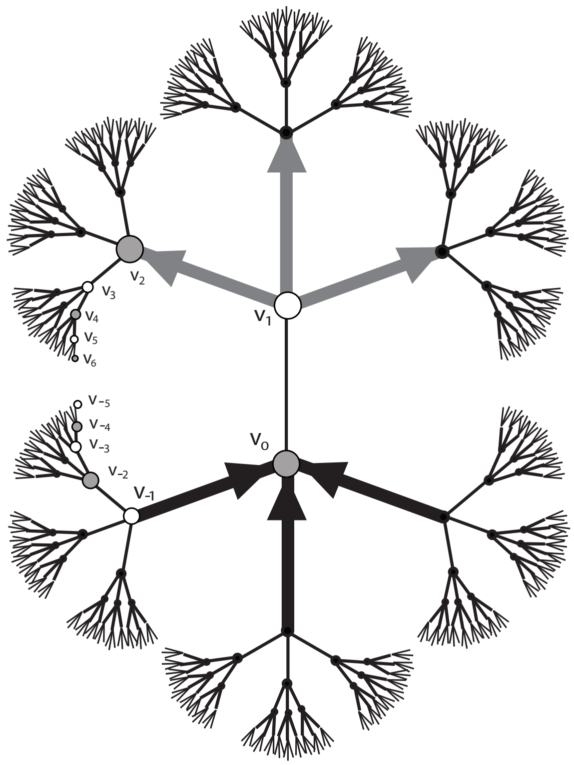

Figure 1.2

is the setwise stabilizer of $e_{0,1}$.) As the action of $GL_2(F)$ is transitive on edges as well, we have $\mathrm{Ed}(X) \cong GL_2(F)/I_G Z$ as a left G-set. Figure 1.2 shows a portion of the tree of $SL_2(\mathbb{Q}_3)$ with gray arrows representing the oriented edges in the I-orbit of $\alpha \cdot e_{0,1}$, and black arrows representing the I-orbit of $\alpha^{-1} \cdot e_{0,1}$. We define the direction of all of these arrows to be *upward*, and extend to put a compatible notion of direction on all oriented edges of the tree. Furthermore, we say that $e_{1,2}$ is the *leftmost* emerging from v_1, and extend this to a notion of "right" and "left" turns by labelling turns from the "upward" point of view. (This notion of direction is not particularly standard and certainly not canonical, but will be convenient and harmless to use here.)

The action of $SL_2(F)$ on $\mathrm{Ed}(X)$ again has two orbits: one consisting of the edges which originate at even distance from v_0 and the other of edges originating at odd distance from v_0. Hence the action of $SL_2(F)$ is transitive on non-oriented edges, but not on oriented edges.

One immediate and very useful consequence of the $GL_2(F)$-set isomorphisms $\mathrm{Ver}(X) \cong GL_2(F)/K_G Z$ and $\mathrm{Ed}(X) \cong GL_2(F)/I_G Z$ is that two elements of $GL_2(F)$ are in the same left $K_G Z$-coset (respectively, left $I_G Z$-coset) if and only if they send v_0 (resp., $e_{0,1}$) to the same vertex (resp., oriented edge) of X. Of course this holds for elements of $SL_2(F)$ as well.

Due to the Cartan decomposition (1.1), the K_G-orbit of the standard vertex v_n is the circle C_n of vertices at radius n from v_0. The K-orbit of v_{2n} is likewise all of C_{2n}. The actions of K_G and K on the tree should be thought of as analogous to the rotation action of the maximal compact $SO(2) \subset SL_2(\mathbb{R})$ on the symmetric space of $SL_2(\mathbb{R})$.

The n-th congruence subgroup

$$K_{G,n} := \{k \in K_G : k \equiv \mathbf{1} \pmod{\pi^n}\} \subset K_G$$

fixes all the vertices which are either on C_n or enclosed by it; if $m > n$, then the orbit of v_m under K_n is the subset of C_m consisting of all vertices from which a path to v_0 passes through v_n.

In particular, if $U_n := \begin{pmatrix} 1 & \pi^n \mathcal{O}_F \\ 0 & 1 \end{pmatrix} \in U$ with $n \geq 0$, then $u \in U_n$ fixes C_n and all vertices enclosed by it. On the other hand, if $n < 0$, then $u \in U_n$ fixes all vertices in the K_m-orbit of v_m (for all $m \geq n$), while for $m < n$ the U_n orbit of v_m is the set of all vertices at distance $m - n$ from v_n and for which the path to v_n does not pass through v_{n+1}. In Figure 1.3, the vertices which are in the U-orbit of v_0 (and which are on or enclosed by C_6) are marked with gray squares.

Working in $PGL_2(F)$, consider the intersection

$$K_G Z \alpha^n K_G \cap \alpha^m Z U,$$

where U is the full upper triangular unipotent subgroup. For future reference, we count the size of this intersection.

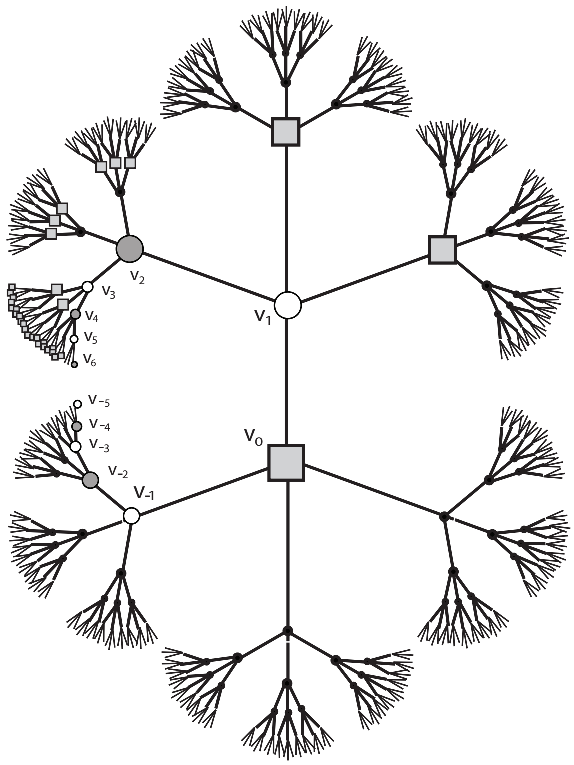

Figure 1.3

Lemma 1.3.1. *For $m > 0$, $(\alpha^m U \cdot v_0) \cap C_n \neq \emptyset$ only if $|n| = 2\ell + m$ for some $\ell \geq 0$, and*

$$|(\alpha^m U \cdot v_0) \cap C_{2\ell+m}| = \begin{cases} 1 & \text{if } \ell = 0 \\ q^{\ell-1}(q-1) & \text{if } \ell \geq 1. \end{cases}$$

Hence

$$|K_G Z \alpha^n K_G \cap \alpha^m U Z| = \begin{cases} 0 & \text{if } |n| \neq 2\ell + m \text{ for all } \ell > 0 \\ 1 & \text{if } |n| = m \\ q^{\ell-1}(q-1) & \text{if } |n| = 2\ell + m \text{ with } \ell \geq 1. \end{cases} \tag{1.11}$$

and in $SL_2(F)$, we have

$$|K\alpha_0^n K \cap \alpha_0^m U| = \begin{cases} 0 & \text{if } |n| < m \\ 1 & \text{if } |n| = m \\ q^{\ell-1}(q-1) & \text{if } |n| = \ell + m \text{ with } \ell \geq 1. \end{cases} \tag{1.12}$$

Proof. Given k_1, $k_2 \in K_G$, we have $k_2\alpha^n k_1 \cdot v_0 = k_2\alpha^n \cdot v_0 = k_2 \cdot v_n \in C_n$, and conversely for any $v \in C_n$ there exists $k_2 \in K_G$ such that $k_2 v_n = v$. By $GL_2(F)$-equivariance of the action, then, an element of $\alpha^m U$ is also in $K_G\alpha^n K_G$ if and only if it sends v_0 to a vertex in C_n.

Consider an element

$$u = \begin{pmatrix} 1 & a\pi^k \\ 0 & 1 \end{pmatrix} \in U$$

where $a \in \mathcal{O}_F^\times$ and $k \in \mathbb{Z}$, and more generally let U_k denote the subset of U whose upper-right entry has valuation exactly k. If $k \geq 0$, then $u \in K_G$, so $\alpha^m u \cdot v_0 = \alpha^m \cdot v_0 = v_m$. If $k < 0$, then $u \cdot v_0$ is one of the $q^{|k|} - 1$ vertices which sit at distance $|k|$ from $v_{|k|}$ and are neither equal to v_0 nor have a path back to v_0 which meets an edge of the standard apartment. As a ranges over $\mathcal{O}_F^\times$ and $k < 0$ stays fixed, the vertices $u \cdot v_0$ are distributed as follows: $q - 1$ of them are in C_2, and for each j $(1 \leq j \leq |k|)$, $q^{j-1}(q-1)$ of them are in C_{2j}.

Suppose we take $\ell < k$ and consider the orbit of v_0 under the subset U_ℓ of elements whose upper-right entry has valuation exactly ℓ. Then $U_\ell \cdot v_0 \cap U_k \cdot v_0 = U_k \cdot v_0$, i.e. for $j \leq |k|$, the intersection of $U_\ell \cdot v_0$ with C_{2j} is equal to that of $U_k \cdot v_0$ with C_{2j}. Hence for each $j \geq 1$ the intersection of the full U-orbit $U \cdot v_0$ with C_{2j} consists of $q^{j-1}(q-1)$ vertices, and the intersection of $U \cdot v_0$ with C_{2j-1} is empty.

Now apply α^m to an element of $U \cdot v_0$: if $u \cdot v_0 \in C_{2j}$, then $\alpha^m u \cdot v_0 \in C_{2j+m}$, and distinct vertices of C_{2j} are sent to distinct vertices in C_{2j+m} since $m > 0$. Thus we get $q^{j-1}(q-1)$ points in $\alpha^m U \cap C_{2j+m}$. This suffices to prove the first part of the lemma. Then (1.11) and (1.12) follow from the action of $PGL_2(F)$ and $SL_2(F)$, respectively, on the tree, noting that α_0 has the same action on v_0 as α^2. □

This calculation will be used in Chapter 4 to describe the image of the Satake transform.

Chapter 2

Genuine ordinary representations of $\widetilde{SL}_2(F)$

2.1 Summary

2.1.1 Abstract of the chapter

The main result of this chapter is a classification of the irreducible subquotients of the genuine mod p representations of $\widetilde{SL}_2(F)$ which are induced from the Borel subgroup $\widetilde{B}$. Such representations are called *ordinary* mod p representations of $\widetilde{SL}_2(F)$. Under the simplifying assumption that $q \equiv 1 \pmod 4$, we show that all genuine ordinary representations are induced from genuine characters of the torus $\widetilde{T}$, which are classified up to dependence on an additive character ψ of F; that they are all irreducible; and that inductions of distinct characters are inequivalent. We also compute the spaces of spherical, Iwahori-fixed, and pro-p-Iwahori-fixed vectors.

2.1.2 Main results

The chapter begins with a review of the smooth and compact induction functors for representations of a locally compact, totally disconnected group over an arbitary coefficient field. Among other background information, this preliminary section sets up notation for the Frobenius reciprocity maps which will be used frequently in this chapter and later ones.

Next it is shown that every character of $\widetilde{B}$ factors through a character of $\widetilde{T}$, and the genuine characters of $\widetilde{T}$ are classified. Moreover, as an easy but crucial point, it is checked that $\widetilde{SL}_2(F)$ has no nontrivial characters, and in particular has no genuine characters.

As a first step in the classification of the ordinary representations, we analyze the $\widetilde{B}$-module structure of representations induced from characters of $\widetilde{B}$:

Proposition 2.1.1. *Let F be a p-adic field with residue field of order $q \equiv 1 \pmod 4$. Let $\tilde{\chi}$ be the inflation to $\widetilde{B}$ of a smooth genuine character of $\widetilde{T}(F)$. As a $\widetilde{B}$-module, $I(\tilde{\chi})$ has the character $\tilde{\chi}$ as a quotient, and the kernel of this quotient map is an irreducible, smooth, genuine $\widetilde{B}$-module. The sequence does not split, so $I(\tilde{\chi})$ is an indecomposable $\widetilde{B}$-module of length 2.*

The $\widetilde{B}$-module structure together with Frobenius reciprocity implies the first part of the main result:

Theorem A (Theorem 2.3.5 (1), (2)). *1. The irreducible smooth, genuine, ordinary mod p representations of $\widetilde{SL}_2(F)$ are exactly those of the form $I(\tilde{\chi}) := Ind_{\widetilde{B}}^{\widetilde{SL}_2(F)}\tilde{\chi}$, where Ind is the smooth induction functor and $\tilde{\chi}$ is an arbitrary genuine character of $\widetilde{T}(F)$ (defined with respect to a fixed additive character of F).*

2. The dimension of $Hom_{\widetilde{SL}_2(F)}(I(\tilde{\chi}), I(\tilde{\chi}'))$ is 1 if $\tilde{\chi} = \tilde{\chi}'$ and is 0 otherwise, so $I(\tilde{\chi}) \cong I(\tilde{\chi}')$ if and only if $\tilde{\chi} = \tilde{\chi}'$.

Making use of various decompositions of $\widetilde{SL}_2(F)$ relative to the compact open subgroups K^*, I^*, and $I(1)^*$, we find the dimensions of invariant subspaces. Then:

Theorem B (Theorem 2.3.5 (3), (4)). *1. The $I(1)^*$-invariant space $I(\tilde{\chi})^{I(1)^*}$ is of dimension 2 over $\bar{\mathbb{F}}_p$ for all $\tilde{\chi}$.*

2. If the restriction of $\tilde{\chi}$ to $\widetilde{T} \cap K^*$ is not equal to $\tilde{1}$, then $I(\tilde{\chi})$ has no nontrivial I^*- or K^*-invariants. If $\tilde{\chi}|_{\widetilde{T}\cap K^*} = \tilde{1}$, i.e., when $\tilde{\chi}$ is unramified, then $I(\tilde{\chi})^{I^*} = I(\tilde{\chi})^{I(1)^*}$ (and so is 2-dimensional), and $I(\tilde{\chi})^{K^*}$ is 1-dimensional.

The chapter ends with a summary of Barthel and Livné's classification of the mod p unramified principal series representations of $PGL_2(F)$. In Chapter 3, we compute the genuine spherical mod p Hecke algebra of $\widetilde{G}$ in order to describe the relationship between those representations of $PGL_2(F)$ and the unramified genuine ordinary representations of $\widetilde{SL}_2(F)$.

2.2 Preliminaries: Smooth and compact induction

In this first section, let G denote a Hausdorff locally compact, totally disconnected (LCTD) topological group (for example, a p-adic reductive or metaplectic group), and let (π, V) be a smooth representation of G on a vector space V over a field E. The definitions and results in this section hold for any field E, though in practice they may be modified slightly (e.g. by normalization) depending on the characteristic of E. The exposition of this section mainly follows that of [26] § I.5.

Let H be a subgroup of G. Restriction of π from G to H makes $(\pi\big|_H, V)$ into a smooth H-representation, since $\mathrm{Stab}_H(v) = \mathrm{Stab}_G(v) \cap H$ is open in H for each v. The functor of induction from H to G should be defined so as to be adjoint to the functor of restriction. One can define a right adjoint functor, *smooth induction*, whenever H is a closed subgroup of G, and a subfunctor, *compact induction*, which is also a left adjoint when H is also an open subgroup of G; these two functors agree when $H \setminus G$ is compact.

Let (σ, W) denote a smooth representation of H on an E-vector space. The prototypical induction of (σ, W) to a representation of G has the underlying vector space

$$\{f : G \to W \,\big|\, f(hg) = \sigma(h)f(g) \text{ for all } h \in H,\ g \in G\},$$

together with the right-translation action of G. Depending on H, it is desirable to put one or two extra conditions on the functions in the underlying space.

1. **Smooth induction.** The space of functions above does not give a smooth representation of G in most cases, so the first desirable condition to put on the $\{f : G \to W\}$

is one which picks out the maximal smooth subrepresentation of the prototype. When H is a closed subgroup of G, the *smooth induction* of (σ, W) is the space

$$\begin{aligned}\mathrm{Ind}_H^G(\sigma) = &\{ f : G \to W \,\big|\, f(hg) = \sigma(h) f(g) \text{ for all } h \in H,\ g \in G, \text{ and } \exists \text{ an open} \\ &\quad \text{compact subgroup } K_0 \subset G \text{ such that } f(gk) = f(g) \text{ for all } g \in G,\ k \in K_0 \}\end{aligned}$$

with the right-translation action of G.

Proposition 2.2.1 (Smooth Frobenius reciprocity)**.** *([26] I.5.7.(i)) When G is a LCTD topological group and $H \subset G$ is a closed subgroup, then the smooth induction Ind_H^G is a right adjoint to the restriction map $\pi \mapsto \pi\big|_H$. In other words, whenever (π, V) is a smooth representation of G and (σ, W) is a smooth representation of H, there is a natural isomorphism*

$$Hom_G(\pi, Ind_H^G(\sigma)) \to Hom_H(\pi\big|_H, \sigma).$$

Proof. Let

$$\Phi : \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma)) \to \mathrm{Hom}_H(\pi\big|_H, \sigma)$$

be the map sending

$$\phi \mapsto \Phi(\phi) := (v \mapsto \phi(v)(1)) \,.$$

Then $\Phi(\phi)$ is an H-equivariant homomorphism $V \to W$:

$$\Phi(\phi)(h \cdot v) := \phi(h \cdot v)(1) = \phi(v)(h) = h \cdot (\phi(v)(1)) = h \cdot \Phi(\phi)(v)$$

since ϕ is G-equivariant. And Φ is injective, since if $\Phi(\phi) = 0$, then $\phi(v)(1) = 0$ for all v, which implies that $(g \cdot \phi)(v)(1) = \phi(v)(g) = 0$ for all v, g, so $\phi = 0$. Finally, Φ is surjective, since if $\varphi \in \mathrm{Hom}_H(\pi\big|_H, \sigma)$ sends $v \mapsto \varphi(v)$, then φ is the image under Φ of the map $\phi : V \to \mathrm{Ind}_H^G(\sigma)$ which is defined by

$$\phi(v)(1) = \varphi(v).$$

To see that this ϕ really is an element of $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma))$, let g, $g' \in G$ and note that

$$\phi(g \cdot v)(g') = \varphi(g'g \cdot v) = \phi(v)(g'g) = (g \cdot \phi(v))(g'),$$

so $v \mapsto \phi(v)$ is G-equivariant. Moreover, given any $v \in V$, smoothness of ϕ implies that $\mathrm{Stab}_G(v)$ is open in G; since G is locally profinite, we can choose a compact open subgroup $K_0 \subset \mathrm{Stab}_G(V)$. Then for all $h \in H$, $k \in K_0$, and $g \in G$, we have

$$(hgk \cdot \phi)(v)(1) = \varphi(hgk \cdot v) = \varphi(hg \cdot v) = h \cdot \varphi(g \cdot v) = h \cdot \phi(v)(g),$$

so $\phi(v) \in \mathrm{Ind}_H^G(\sigma)$.

□

2. **Induction with compact support.** When H is an open subgroup of an LCTD group G, then H is of course also closed (as it is the complement in G of the union of its nontrivial cosets, each of which is open), so the above definition of $\mathrm{Ind}_H^G(\sigma)$ still makes sense. But we also get a left adjoint to restriction by defining the *induction with compact support*, or *compact induction*, of (σ, W) to be the space

$$\mathrm{ind}_H^G(\sigma) = \{f \in \mathrm{Ind}_H^G(\sigma) \,\big|\, \mathrm{Supp}(f) \text{ is compact in } G\}$$

with the right-translation of G.

Proposition 2.2.2 (Compact Frobenius reciprocity)**.** *([26] I.5.7.(ii)) Keep the notation of the previous proposition, but now suppose that H is an open subgroup of the LCTD group G. Then the compact induction ind_H^G is a left adjoint to the restriction functor: that is, whenever (π, V) is a smooth representation of G and (σ, W) is a smooth representation of H, there is a natural isomorphism*

$$Hom_G(ind_H^G(\sigma), \pi) \to Hom_H(\sigma, \pi\big|_H).$$

Proof. Given $w \in W$, define the function

$$f_w := \left(g \mapsto \begin{cases} \sigma(g)w & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases} \right),$$

and let G act on $\{f_w : w \in W\}$ by

$$g \cdot f_w(g') := f_w(g'g) = \begin{cases} \sigma(g'g)w & \text{if } g' \in Hg^{-1} \\ 0 & \text{if } g' \notin Hg^{-1}. \end{cases}$$

Then the set $\{f_w : w \in W\}$ generates $\mathrm{ind}_H^G(\sigma)$ as a G-representation; concretely, if $f \in \mathrm{ind}_H^G(\sigma)$, then

$$f = \sum_{g \in H \backslash G} g^{-1} f_{f(g)}.$$

Define a map $\Phi_c : \mathrm{Hom}_G(\mathrm{ind}_H^G(\sigma), \pi) \to \mathrm{Hom}_H(\sigma, \pi\big|H)$ by setting

$$\phi \mapsto \Phi_c(\phi) := (w \mapsto \phi(f_w))$$

and extending by G-linearity. Then for all h, $h' \in H$,

$$\Phi_c(\phi)(h \cdot w)(h') = \phi(f_{h \cdot w})(h') = (h \cdot \phi)(f_w)(h') = (h \cdot \Phi_c(\phi))(h'),$$

so Φ_c does indeed have image in $\mathrm{Hom}_H(\sigma, \pi\big|_H)$.

The map Φ_c is injective, since

$$\Phi_c(\phi)(w) = 0 \text{ for all } w \in W \implies \phi(f_w) = 0 \text{ for all } w \in W \implies \phi = 0.$$

And it is surjective, since given $\varphi \in \mathrm{Hom}_H(\sigma, \pi\big|_H)$, the map

$$f_w \mapsto \varphi(w)$$

extends to a G-equivariant homomorphism $\mathrm{ind}_H^G(\sigma) \to \pi$. $\square$

In summary, the two isomorphisms of E-vector spaces above are given by the following formulae:

1. **Smooth Frobenius reciprocity.** When $H \subset G$ is closed, we have an isomorphism

$$\Phi : \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma)) \to \mathrm{Hom}_H(\pi\big|_H, \sigma)$$

defined by

$$\begin{aligned} (\phi : v \mapsto f) &\mapsto \big(\varphi : v \mapsto f(1)\big) \\ \big(\phi : v \mapsto (g \mapsto \varphi(g \cdot v))\,\big) &\leftarrow\!\shortmid (\varphi : v \mapsto w) \end{aligned}$$

2. **Compact Frobenius reciprocity.** When $H \subset G$ is open, we have an isomorphism

$$\Phi_c : \mathrm{Hom}_G(\mathrm{ind}_H^G(\sigma), \pi) \to \mathrm{Hom}_H(\sigma, \pi\big|_H)$$

defined by

$$\begin{aligned} (\phi : f \mapsto v) &\mapsto \big(\varphi : w \mapsto \phi(f_w)\big) \\ \Big(f \mapsto \sum_{g \in H\setminus G} \pi(g^{-1})\,\varphi(f(g))\Big) &\leftarrow\!\shortmid (\varphi : w \mapsto v)\,. \end{aligned}$$

When G is a p-adic reductive group, H is a parabolic subgroup, and (σ, W) is a smooth representation of H, then $\mathrm{Ind}_H^G(\sigma) = \mathrm{ind}_H^G(\sigma)$. This is a special case of the following:

Lemma 2.2.3. *If $H \subset G$ is an open subgroup, $H \setminus G$ is compact, and (σ, W) is a smooth representation of H, then the two functors are equal.*

Proof. For any closed H and smooth representation (σ, W) of H, consider the support of $f \in \mathrm{Ind}_H^G(\sigma)$ in $H \setminus G$. Since f is locally constant, its support is both closed and open in

$H \setminus G$. If H is open and $H \setminus G$ is itself compact, then the support of $f \in \mathrm{Ind}_H^G(\sigma)$ is compact in $H \setminus G$, hence is compact in G. $\square$

Note that the lemma applies equally well to a metaplectic cover $\widetilde{G}$ of a reductive group G and a subgroup $\widetilde{H} \subset \widetilde{G}$ which is the full preimage of a parabolic subgroup $H \subset G$.

In order to classify the principal series representations of $\widetilde{G} = \widetilde{SL}_2(F)$, it will be important to understand the $\widetilde{B}$-module decomposition of the smooth induction $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}(\sigma)$ after restriction to $\widetilde{B}$. The following statement of the Mackey decomposition theorem is true for an arbitrary LCTD group G and any coefficient field:

Lemma 2.2.4. *([26] §I.5.5) Let H, K be two closed subgroups of a LCTD group G such that the double cosets HgK, $g \in G$, are both open and closed. Let $g(H)$ denote the set $\{gHg^{-1} : g \in G\}$, and given a smooth right H-module σ, let $g(\sigma)$ denote the representation of $g(H)$ defined by*

$$g(\sigma)(ghg^{-1}) = \sigma(g).$$

Then

$$Ind_H^G(\sigma)\big|_K \cong \prod_{HgK} Ind_{K\cap g(H)}^K \left(g(\sigma)\big|_{K\cap g(H)} \right)$$

and

$$ind_H^G(\sigma)\big|_K \cong \bigoplus_{HgK} ind_K^{K\cap g(H)} \left(g(\sigma)\big|_{K\cap g(H)} \right).$$

The Iwasawa decomposition $SL_2(F) = BK$ lifts to give

$$\widetilde{SL}_2(F) = \widetilde{B}K^*,$$

in which both $\widetilde{B}$ and K^* are closed subgroups of $\widetilde{G}$. Since there is only one double class, the Mackey decomposition says that if σ is a smooth representation of $\widetilde{B}$, then there is a K^*-equivariant isomorphism

$$\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}(\sigma)\big|_{K^*} \cong \mathrm{Ind}_{K^*}^{\widetilde{G}}(\sigma)\big|_{K^*} \cong \mathrm{Ind}_{\widetilde{B}\cap K^*}^{K^*} \left(\sigma\big|_{\widetilde{B}\cap K^*}\right),$$

while if τ is a smooth representation of K^*, then there is a $\widetilde{B}$-equivariant isomorphism

$$\mathrm{ind}_{K^*}^{\widetilde{B}}(\tau)\big|_{\widetilde{B}} \cong \mathrm{ind}_{\widetilde{B}\cap K^*}^{\widetilde{B}}(\tau).$$

Furthermore, returning to the general case of an LCTD group G, if H is a closed subgroup of G and K is an open compact subgroup, then the Mackey decomposition implies that for a smooth representation (σ, W) of H,

$$(\mathrm{Ind}_H^G \sigma)^K \cong \prod_{HgK} W^{H\cap g(K)}$$

and

$$(\mathrm{ind}_H^G \sigma)^K \cong \bigoplus_{HgK} W^{H\cap g(K)}.$$

In particular, considering $\widetilde{G} = \widetilde{SL}_2(F)$, the closed subgroup $\widetilde{B}$, the open compact subgroup K^*, and a smooth representation (σ, W) of $\widetilde{B}$, then

$$(\mathrm{Ind}_{\widetilde{B}}^G \sigma)^{K^*} \cong W^{\widetilde{B}\cap K^*}.$$

Remark 2.2.5. *While the functor of invariants under a compact subgroup is exact on representations of p-adic groups in characteristic 0 or $\ell \neq p$, this is no longer necessarily true when the coefficient field has characteristic p (cf. [2]).*

2.3 Classification of the genuine ordinary representations of $\widetilde{SL}_2(F)$

2.3.1 A note on terminology

The goal of this section is to classify the irreducible subquotients of smooth genuine representations of $\widetilde{G} := \widetilde{SL}_2(F)$ which are induced from genuine representations of the Borel subgroup $\widetilde{B}$. As this kind of representation tends to be characterized in different ways depending on the characteristic of the coefficient field, we briefly explain the choice of terminology.

We will call a representation *ordinary* if it is a subquotient of a representation induced from $\widetilde{B}$. We do not require ordinary representations to be irreducible (though it turns out that all of the mod p ordinary representations of $\widetilde{SL}_2(F)$ are irreducible). The term appears in work of Barthel-Livné, e.g., [3], Breuil, and Herzig on mod p and p-adic representations, and refers to the fact that the 2-dimensional representation of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ associated to an *ordinary* elliptic curve defined over $\mathbb{Q}_p$ has image in the Borel subgroup.

In the study of mod p representations of p-adic GL_n ([2], [3], [6], [7], [13]), SL_2, and rank-1 unitary groups ([1]), there is a dichotomy between the ordinary representations and the *supersingular* ones, which can be read off from parametrization of mod p representations by eigenvalues of the generators of a generalized spherical Hecke algebra: The supersingular representations are those on which these generators act by 0. However, in this work we will only consider the spherical Hecke algebra of $\widetilde{G}$ with respect to the trivial representation of K^*, so we have to defer the question of this dichotomy to future work.

2.3.2 Abelianization of $\tilde{B}$ and of $\tilde{G}$

From now on, G stands for $SL_2(F)$ and $\widetilde{G}$ stands for $\widetilde{SL}_2(F)$.

The torus $\widetilde{T}(F)$ of $\widetilde{G}$ is commutative, though the extension does not split over it. The next lemma shows that $\widetilde{T}(F)$ is the abelianization of $\widetilde{B}$, while the abelianization of $\widetilde{G}$ is trivial. Hence a 1-dimensional representation of $\widetilde{B}$ is just the inflation of a representation of $\widetilde{T}$ by the trivial action on $\widetilde{B}/\widetilde{T} = U^*$, and every 1-dimensional representation of $\widetilde{G}$ is trivial.

Lemma 2.3.1. *$\widetilde{G}$ is equal to its commutator subgroup $[\widetilde{G}, \widetilde{G}]$, while $[\widetilde{B}, \widetilde{B}] = U^*$.*

Proof. It is well known that $[SL_2(E), SL_2(E)] = SL_2(E)$ whenever E is a field of cardinality > 3 (cf. [1] § 3.3.1). This implies that for each $g \in G$, there is some $\zeta \in \{\pm 1\}$ such that $(g, \zeta) \in [\widetilde{G}, \widetilde{G}]$. So to show that $[\widetilde{SL}_2(F), \widetilde{SL}_2(F)] = \widetilde{SL}_2(F)$ we only need to prove that $(1, -1)$ is generated by commutators.

Let

$$(g_1,\zeta_1)=\left(\begin{pmatrix} y & 0 \\ 0 & y^{-1}\end{pmatrix},\zeta_1\right),\ (g_2,\zeta_2)=\left(\begin{pmatrix} 1 & \frac{x}{y^2-1} \\ 0 & 1\end{pmatrix},\zeta_2\right),$$

$$(g_3,\zeta_3)=\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & y\end{pmatrix},\zeta_3\right),\ (g_4,\zeta_4)=\left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{x}{y^2-1}\end{pmatrix},\zeta_4\right),$$

where x is any element of $F^\times$, $y\in F^\times\setminus\{\pm1\}$, and the ζ_i are any elements of μ_2. Then

$$\sigma(g_1,g_1^{-1})=(y,y)_F \text{ so } (g_1,\zeta_1)^{-1}=(g_1^{-1},\zeta_1(y,y)_F),$$

$$\sigma(g_2,g_2^{-1})=1 \text{ so } (g_2,\zeta_2)^{-1}=(g_2^{-1},\zeta_2),$$

$$\sigma(g_3,g_3^{-1})=(y,y)_F \text{ so } (g_3,\zeta_3)^{-1}=(g_3^{-1},\zeta_3(y,y)_F),$$

$$\sigma(g_4,g_4^{-1})=\left(\frac{x}{y^2-1},\frac{-x}{y^2-1}\right)_F=1 \text{ so } (g_4,\zeta_4)^{-1}=(g_4^{-1},\zeta_4).$$

Commutators of these elements generate all nontrivial members of U^* and U'^*:

$$\begin{aligned}
[(g_1,\zeta_1),(g_2,\zeta_2)] &= \left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix},\zeta_1^2\zeta_2^2(y,y)_F\sigma(g_1,g_2)\sigma(g_1g_2,g_1^{-1})\sigma(g_1g_2g_1^{-1},g_2^{-1})\right)\\
&= \left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix},(y,y)_F(1,y)_F(y,y)_F(1,1)_F\right)\\
&= \left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix},1\right)
\end{aligned}$$

$$[(g_3,\zeta_3),(g_4,\zeta_4)] = \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \zeta_3^2\zeta_4^2(y,y)_F\sigma(g_3,g_4)\sigma(g_3g_4,g_3^{-1})\sigma(g_3g_4g_3^{-1},g_4^{-1}) \right)$$
$$= \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, (y,y)_F \left(\frac{x}{y^2-1}, y\right)_F \left(y, \frac{xy}{y^2-1}\right)_F \left(y^2-1, -(y^2-1)\right)_F \right)$$
$$= \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, 1 \right).$$

These elements generate all those of the form

$$\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, (-1,a)_F \right) \in \widetilde{T},$$

since

$$w(1) = \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, 1 \right) \in U^*U'^*U^*$$

and

$$\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, (-1,a)_F \right) = \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, 1 \right) w(1).$$

Now choose $u \in \mathcal{O}_F^\times$ such that $(u,\pi)_F = -1$. We have both

$$\left(\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, (-1,u)_F \right) \left(\begin{pmatrix} \pi & 0 \\ 0 & u\pi^{-1} \end{pmatrix}, (-1,\pi)_F \right) =$$

$$\left(\begin{pmatrix} u\pi & 0 \\ 0 & (u\pi)^{-1} \end{pmatrix}, (-1,u\pi)_F(u,\pi)_F \right) \in [\widetilde{G},\widetilde{G}]$$

and

$$\left(\begin{pmatrix} u\pi & 0 \\ 0 & (u\pi)^{-1} \end{pmatrix}, (-1, u\pi)\right)^{-1} = \left(\begin{pmatrix} (u\pi)^{-1} & 0 \\ 0 & u\pi \end{pmatrix}, (-1, u\pi)_F(u\pi, u\pi)_F\right) \in [\widetilde{G}, \widetilde{G}],$$

so

$$\left(\begin{pmatrix} u\pi & 0 \\ 0 & (u\pi)^{-1} \end{pmatrix}, (-1, u\pi)_F(u, \pi)_F\right)\left(\begin{pmatrix} (u\pi)^{-1} & 0 \\ 0 & u\pi \end{pmatrix}, (-1, u\pi)_F(u\pi, u\pi)_F\right)$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (u, \pi)_F\right) = (1, -1) \in [\widetilde{G}, \widetilde{G}].$$

To show that $[\widetilde{B}, \widetilde{B}] = U^*$, we need only recall that $\widetilde{B} = \widetilde{T}U^*$ and check that $\widetilde{T}$ normalizes U^*. Conjugating an element of U^* by an element of $\widetilde{T}$, we get

$$\begin{aligned}
&\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \zeta\right)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \zeta\right)^{-1} \\
&= \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \zeta\right)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \zeta(t, t)_F\right) \\
&= \left(\begin{pmatrix} t & tx \\ 0 & t^{-1} \end{pmatrix}, \zeta\right)\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \zeta(t, t)_F\right) \\
&= \left(\begin{pmatrix} 1 & t^2x \\ 0 & 1 \end{pmatrix}, \zeta^2(t, t)_F^2\right) \\
&= \left(\begin{pmatrix} 1 & t^2x \\ 0 & 1 \end{pmatrix}, 1\right) \in U^*.
\end{aligned}$$

Hence any commutator in $\widetilde{B}$ can be written as a product of elements in U^*.

$\square$

2.3.3 Genuine characters of $\widetilde{T}$

A smooth character of $\widetilde{T}$ which is trivial on $\{(1, \pm 1)\}$ is just a smooth character $\chi : F^\times \to \bar{\mathbb{F}}_p^\times$:

$$\chi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \zeta\right) = \chi(a).$$

However, we interested in the characters of $\widetilde{T}$ which are nontrivial on μ_2. These, the *genuine* characters of $\widetilde{T}$, are described in terms of a certain basic character χ_ψ which depends on the choice of a nontrivial additive character ψ of F. Let γ be the *Weil index* of a quadratic form over F; in fact γ is a function on the Witt group $W(F)$ of equivalence classes of quadratic forms over F. As in [21], let γ define a map from the group of additive characters of F into the group of 8th roots of unity in $\mathbb{C}$, and for $a \in F^\times$ let

$$\gamma(a, \psi) := \frac{\gamma(\psi_a)}{\gamma(\psi)},$$

where ψ_a is the character $x \mapsto \psi(ax)$. The key properties of this map, proved by Rao in [21], are the following: for fixed $\psi \neq 1$ and arbitrary a, $b \in F^\times$, we have

1. $\gamma(ab^2, \psi) = \gamma(a, \psi)$,
2. $\gamma(ab, \psi) = (a, b)_F \gamma(a, \psi)\gamma(b, \psi)$,
3. $\gamma(a, \psi_b) = (a, b)_F \gamma(a, \psi)$,
4. $\gamma(a, \psi)^2 = (a, a)_F$,
5. $\gamma(a, \psi)^4 = 1$,

and, after fixing $\psi \in \hat{F}$, every genuine character of the group

$$F^\epsilon = \{(a, \zeta) : a \in F^\times,\ \zeta \in \mu_2,\ (a, \zeta) \cdot (b, \eta) = (ab, \zeta\eta(a, b)_F)\}$$

is of the form $\chi \cdot \gamma(-, \psi)$ for some character χ of $F^\times$.

Since F^ϵ above is isomorphic to $\widetilde{T}$ by construction, we can view $\gamma(-,\psi)$ as a character of $\widetilde{T} \subset \widetilde{G}$. Thus, after fixing ψ, every genuine smooth $\bar{\mathbb{F}}_p^\times$ character of $\widetilde{T}$ is equal to $\chi{\cdot}\gamma(-,\psi)$ for some smooth $\bar{\mathbb{F}}_p^\times$-valued character χ. In the following, denote this character by $\tilde{\chi}$, suppressing ψ from the notation but recalling that a choice has been made. At the end of the section we will comment on the effect of varying ψ.

The Hilbert symbol on F is unramified since we assume $p \neq 2$, so when χ is any unramified character of $F^\times$, then the restriction of $\tilde{\chi}$ to $(\widetilde{T}(F) \cap K^*) \times \{\pm 1\}$ has the property that

$$\tilde{\chi}(h(a), \zeta) = \zeta$$

for all $a \in \mathcal{O}_F^\times$. Conversely, we say that a genuine character of $\widetilde{G}$ is *unramified* if it is equal to $\tilde{\chi}$ for some unramified character χ of $F^\times$. In particular, with respect to a given choice of ψ, the representation $\tilde{1}$ is just $\gamma(-,\psi)$.

The unramified genuine characters of $\widetilde{T}$ are exactly the ones which are equal to $\tilde{1}$ on the subgroup $(\widetilde{T} \cap K^*) \times \{\pm 1\}$; any such smooth character is the product of $\tilde{1}$ with a smooth character of the lattice Λ, and so is determined by its value on $h(\pi)$.

2.3.4 Genuine ordinary representations: $\widetilde{B}$-module structure

All representations in the following are taken to be smooth, even if this is not specifically mentioned. The genuine ordinary representations of $\widetilde{G}$ are smooth inductions from $\widetilde{B}$ to $\widetilde{G}$ of genuine characters of $\widetilde{T}$. Let $\tilde{\chi}$ denote an arbitrary genuine character in the following, and let the name of a character of $\widetilde{T}$ ($\tilde{\theta}$, $\tilde{\chi}$, etc.) also denote its inflation to $\widetilde{B}$. Let $V_{\tilde{\chi}}$ denote the 1-dimensional $\bar{\mathbb{F}}_p$-vector space on which $\widetilde{B}$ acts by $\tilde{\chi}$.

Following the strategy of [8] and [1], we begin by studying the induced representations $I(\tilde{\chi}) := \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\tilde{\chi}$ as $\widetilde{B}$-modules. The result is the following:

Proposition 2.3.2. *Assume that $q \equiv 1 \pmod 4$. Let $\tilde{\chi}$ be the inflation to $\widetilde{B}$ of a smooth genuine character of $\widetilde{T}(F)$. As a $\widetilde{B}$-module, $I(\tilde{\chi})$ has the character $\tilde{\chi}$ as a quotient, and the kernel of this quotient map is an irreducible, smooth, genuine $\widetilde{B}$-module. The sequence does* *not split, so $I(\tilde{\chi})$ is an indecomposable $\widetilde{B}$-module of length 2.*

Remark 2.3.3. *The proof that ker$(I(\tilde{\chi}) \to \tilde{\chi})$ is irreducible is essentially by the method of [8] (§ 9.7) and [1] (Prop. 3.4.4). This is reasonable since it mainly relies on general properties of locally profinite groups, which $\widetilde{G}$ inherits from G; one just has to be careful that the necessary decompositions lift to the covering group. The assumption on q simplifies the calculations but is probably possible to remove by careful checking.*

Proof of Proposition 2.3.2. Let $\phi : I(\tilde{\chi}) \to \tilde{\chi}$ send $f \mapsto f((1,1))$; this is a surjective map of $\widetilde{B}$-modules, so to prove that $I(\tilde{\chi})$ is of length 2 as a $\widetilde{B}$-module, we need to prove that $\ker(\tilde{\chi})$ is an irreducible $\widetilde{B}$-module. The strategy of [1] Prop. 3.4.4 is to give a model of $\ker(\phi)$ on the space of smooth, compactly supported functions $U \to \bar{\mathbb{F}}_p$ and then to prove that this model is irreducible; we will do the same here, showing that U can be replaced by the canonical lift $U^* \subset \widetilde{G}$ of U.

Since the (refined) Bruhat decomposition in $SL_2(F)$ lifts to

$$\widetilde{G} = \widetilde{B} \amalg \widetilde{B}wU^*,$$

a function $f \in I(\tilde{\chi})$ is in $\ker(\phi)$ if and only if it is supported on $\widetilde{B}wU^*$. Following [1], we adapt a lemma of Bushnell-Henniart ([8] Lemma § 9.3) to show that the support of such an f is actually contained in $\widetilde{B}wU_0^*$ for some compact open subgroup $U_0 \subset U$.

Lemma 2.3.4. *A function $f \in I(\tilde{\chi})$ is in ker(ϕ) if and only if there is a compact open subgroup $U_0 \subset U$ such that the support of f is contained in $\widetilde{B}wU_0^*$.*

Proof. Since f is a smooth function on $\widetilde{G}$, there exists some compact open subgroup $H \subset \widetilde{G}$ such that $f(g) = f(gh)$ for all $g \in \widetilde{G}$, $h \in H$. In particular, if $f \in \ker(\phi)$, then

$$f(bh) = f(b) = 0$$

for all $b \in \widetilde{B}$. As a compact open subgroup of $\widetilde{G}$, H must be contained in one of the maximal compacts $K^* \times \mu_2$ or $K_0^* \times \mu_2$ of $\widetilde{G}$. Say that $H \subset K^* \times \mu_2$; the situation for the other conjugacy class of maximal compacts is identical. Then there exists some $m \geq 1$ such that either $K_m^* \subset H$ or $(1,-1)K_m^* \subset H$, where $K_m \subset K$ is the principal congruence subgroup of level m. By restriction of the Iwahori decomposition, we have $K_m = (K_m \cap U)(K_m \cap T)(K_m \cap U')$, so $\widetilde{B}(1, \pm 1)K_m^* = \widetilde{B}(K_m^* \cap U'^*)$. Hence there exists an $m \geq 1$ such that

$$f\left(b\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, 1\right)\right) = 0$$

for all $b \in \widetilde{B}$ and all $y \in \pi^m \mathcal{O}_F$. Similarly to the identity given in the proof of [8] 9.3 Lemma for $GL_2(F)$, we have for $y \neq 0$:

$$\begin{aligned}\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, 1\right) &= \left(\begin{pmatrix} 1 & y^{-1} \\ 0 & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}, 1\right)\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right)\left(\begin{pmatrix} 1 & y^{-1} \\ 0 & 1 \end{pmatrix}, 1\right) \\ &\in \widetilde{B}w\left(\begin{pmatrix} 1 & y^{-1} \\ 0 & 1 \end{pmatrix}, 1\right)\end{aligned}$$

Hence $f = 0$ on $\widetilde{B}w\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)$ when $v(x) \leq -m$. Let

$$U_0 = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : v(x) \geq -m+1\right\};$$

then U_0 is a compact open subgroup of U, so U_0^* is a compact open subgroup of U^*, and f is supported on $\widetilde{B}wU_0^*$. $\square$

Define a map

$$\Psi : f \mapsto (u \mapsto f(wu))$$

for $u \in U^*$, $f \in I(\tilde{\chi})$. Then $\Psi(f)$ is a smooth function $U^* \to V_{\tilde{\chi}} \cong \bar{\mathbb{F}}_p$. If $f \in \ker(\phi)$, then thanks to the lemma $\Psi(f)$ is compactly supported on U^*, so is in the set $C_c^\infty(U^*)$ of compactly supported smooth functions $U^* \to \bar{\mathbb{F}}_p$. Give $C_c^\infty(U^*)$ the following right action of

$\widetilde{B}$ (which factors through B):

$$\left[\left(\left(\begin{array}{cc} a & c \\ 0 & a^{-1} \end{array}\right), \zeta\right) \cdot \varphi\right]\left(\left(\begin{array}{cc} 1 & y \\ 0 & 1 \end{array}\right), 1\right) = \varphi\left[\left(\left(\begin{array}{cc} 1 & a^{-2}y + a^{-1}c \\ 0 & 1 \end{array}\right), 1\right)\right]$$

for $\varphi \in C_c^\infty(U^*)$. Note that if $b_1,\ b_2 \in \widetilde{B}$, then $(b_1 b_2) \cdot \varphi = b_2(b_1 \cdot \phi)$. Consider the representation

$$C_c^\infty(U^*) \otimes \tilde{\chi}^{-1};$$

since $\tilde{\chi}$ is a genuine character, this is a genuine representation of $\widetilde{B}$. In order to show that $\ker(\phi)$ is an irreducible $\widetilde{B}$-module, we'll show that Ψ gives a $\bar{\mathbb{F}}_p[\widetilde{B}]$-linear isomorphism of $\ker(\phi)$ with $C_c^\infty(U^*) \otimes \tilde{\chi}^{-1}$, and that the latter is irreducible.

To see that Ψ is an isomorphism of vector spaces, define the following map:

$$\Phi : C_c^\infty(U^*) \to \ker(\phi),\ \ \varphi \mapsto \left(g \mapsto \begin{cases} \tilde{\chi}(b)\varphi(u) & \text{if } g = bwu \in \widetilde{B}wU^* \\ 0 & \text{if } g \in \widetilde{B} \end{cases}\right).$$

By construction, $\Phi(\varphi) \in \ker(\phi)$, and Φ is the inverse of Ψ: if $f \in \ker(\phi)$, then

$$\Phi(\Psi(f)) = \Phi(u \mapsto f(wu)) = \left(g \mapsto \begin{cases} \tilde{\chi}(b)f(wu) & \text{if } g = bwu \in \widetilde{B}wU^* \\ 0 & \text{if } g \in \widetilde{B} \end{cases}\right) = f,$$

while if $\varphi \in C_c^\infty(U^*)$, then

$$\Psi(\Phi(\varphi)) = \Psi\left(g \mapsto \begin{cases} \tilde{\chi}(b)\varphi(u) & \text{if } g = bwu \in \widetilde{B}wU^* \\ 0 & \text{if } g \in \widetilde{B} \end{cases}\right) = (u \mapsto \varphi(u)) = \varphi.$$

Given $b = \left(\left(\begin{array}{cc} a & c \\ 0 & a^{-1} \end{array}\right), \zeta\right) \in \widetilde{B}$, $g \in \widetilde{G}$, and $f \in \ker(\phi)$, we have $(b \cdot f)(g) = f(gb)$; if

$g \in \widetilde{B}$, then $f(gb) = 0$, while if $g = b'w \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, 1 \right) \in \widetilde{B}wU^*$, then

$$gb = b' \left(\begin{pmatrix} 0 & -a^{-1} \\ a & a^{-1}y + c \end{pmatrix}, \zeta \right) = b' \left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \zeta \right) w \left(\begin{pmatrix} 1 & a^{-2}y + a^{-1}c \\ 0 & 1 \end{pmatrix}, 1 \right)$$

so

$$f(gb) = \tilde{\chi} \left[b' \left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \zeta \right) \right] \cdot f \left[w \left(\begin{pmatrix} 1 & a^{-2}y + a^{-1}c \\ 0 & 1 \end{pmatrix}, 1 \right) \right],$$

which, since $\left(\begin{pmatrix} a^{-1} & -c \\ 0 & a \end{pmatrix}, \zeta \right) = \left(\begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix}, \zeta \right)^{-1}$ when $q \equiv 1 \pmod 4$, is equal to

$$\tilde{\chi}(b') \tilde{\chi}^{-1}(b) \cdot f \left[w \left(\begin{pmatrix} 1 & a^{-2}y + a^{-1}c \\ 0 & 1 \end{pmatrix}, 1 \right) \right].$$

Then

$$\Psi(b \cdot f) \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, 1 \right) = \tilde{\chi}^{-1}(b) \cdot f \left[w \left(\begin{pmatrix} 1 & a^{-2}y + a^{-1}c \\ 0 & 1 \end{pmatrix}, 1 \right) \right],$$

while

$$(b \cdot (\Psi(f))) \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, 1 \right) = f \left[w \left(\begin{pmatrix} 1 & a^{-2}y + a^{-1}c \\ 0 & 1 \end{pmatrix}, 1 \right) \right].$$

Hence Ψ is a $\widetilde{B}$-isomorphism $\ker(\phi) \to C_c^\infty(U^*)$.

Now we prove that $C_c^\infty(U^*)$ is an irreducible representation of $\widetilde{B}$. Let $\varphi \neq 0 \in C_c^\infty(U^*)$, and let $U_0^* \subset U^*$ be a compact open subgroup which contains the support of φ. We want to show that φ generates all of $C_c^\infty(U^*)$ under the action of $\widetilde{B}$. (This argument is exactly as in [1] for $SL_2(F)$, since the extension splits over U and $C_c^\infty(U^*)$ factors through a representation of B.)

View φ as an element of $C_c^\infty(U_0^*)$ by restriction, and consider the representation of $\widetilde{B}$

on the cyclic subspace of $C_c^\infty(U_0^*)$ generated by φ. In particular, consider the restriction of this representation to $U_0^* \subset \widetilde{B}$. Since U_0^* is a pro-p group, $C_c^\infty(U_0^*)$ contains a nonzero U_0^*-invariant vector. Any U_0^*-invariant function $\psi \in C_c^\infty(U_0^*)$ must satisfy

$$\psi\left[\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)\right] = \psi((1,1))$$

for all $\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) \in U_0^*$, hence is a multiple of the characteristic function $\mathbf{1}_{U_0^*}$; thus $\mathbf{1}_{U_0^*} \in C_c^\infty(U_0^*)$. Furthermore, for $n \geq 0$, let

$$U_n^* := \left\{h(\pi)^n \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) h(\pi)^{-n} = \left(\begin{pmatrix} 1 & x\pi^{-2n} \\ 0 & 1 \end{pmatrix}, 1\right) : \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) \in U_0^*\right\}.$$

Then each U_n^* is a pro-p group contained in U^*, so by same argument as for U_0^*, the subspace of $C_c^\infty(U^*)$ generated (under the action of $\widetilde{B}$) by φ must contain $\mathbf{1}_{U_n^*}$ for each $n \geq 0$. The collection $\{U_n^*\}_{n\geq 0}$ is a neighborhood basis of open compacts for U^* at the identity, so the collection of characteristic functions

$$\left\{\mathbf{1}_{U_n^* \cdot u} : u \in U^*, n \geq 0\right\}$$

generates $C_c^\infty(U^*)$ as a vector space. And under the action of $\widetilde{B}$ on $\mathbf{1}_{U_n^*} \in \langle\varphi\rangle \subset C_c^\infty(U^*)$, we have

$$\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)^{-1} \cdot \mathbf{1}_{U_n}^* \left[\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, 1\right)\right]$$

$$= \mathbf{1}_{U_n^*}\left[\left(\begin{pmatrix} 1 & y-x \\ 0 & 1 \end{pmatrix}, 1\right)\right] = \mathbf{1}_{U_n^* u}\left[\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, 1\right)\right]$$

for $u = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)$ and $\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, 1\right)$ both in U^*. Thus $C_c^\infty(U^*)$ is a smooth irreducible representation of $\widetilde{B}$, and $C_c^\infty(U^*) \otimes \tilde{\chi}^{-1}$ is a smooth, irreducible, genuine representation of $\widetilde{B}$.

Suppose that $I(\tilde{\chi})$ is a decomposable $\widetilde{B}$-module, i.e. suppose there is a $\widetilde{B}$-equivariant splitting of the exact sequence

$$1 \longrightarrow \ker(\phi) \longrightarrow I(\tilde{\chi}) \xrightarrow{\phi} \tilde{\chi} \longrightarrow 1$$

Let $f \in I(\tilde{\chi})$ be the image of some nonzero vector $v \in V_{\tilde{\chi}}$ under this splitting. Then for all $b \in \widetilde{B}$, $g \in \widetilde{G}$, we have

$$f(gb) = \tilde{\chi}(b) f(g),$$

so the cyclic subspace of $I(\tilde{\chi})$ generated by f is stable by $\widetilde{B}$. We will now show that in fact it is stable under the action of $\widetilde{G}$.

Since $\tilde{\chi}$ is trivial on U^*, we have $u \cdot f = f$ for all $u \in U^*$. As shown in the proof of Lemma 2.3.4, f is fixed by

$$(U'^* \cap K_m^*) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ \pi^m x & 1 \end{pmatrix}, 1 \right) : x \in \mathcal{O}_F \right\}$$

for some $m \geq 1$. And if $u' = \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, 1 \right) \in U'^*$, then u' can be written as a product of elements of $\widetilde{T}$ and $(U'^* \cap K_m^*)$: let $k > 0$ be large enough so that $x\pi^{2k} \in \pi^m \mathcal{O}_F$, and check that

$$h(\pi)^k \left(\begin{pmatrix} 1 & 0 \\ x\pi^{2k} & 1 \end{pmatrix}, 1 \right) h(\pi)^{-k} = \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, (x, \pi^k)_F^2 \right) = u'.$$

So if $u' \in U'^*$,

$$(u' \cdot f) = (h(\pi)^k h(\pi)^{-k} f) = f,$$

i.e. f is fixed by U'^*. And since $w(1) \in U^*U'^*U^*$, namely

$$w(1) = \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, 1\right),$$

the refined Bruhat decomposition shows that the cyclic subspace of $I(\tilde{\chi})$ generated by f is in fact $\widetilde{G}$-stable.

Thus $\langle f \rangle$ is a one-dimensional genuine representation of $\widetilde{G}$. But since $[\widetilde{G}, \widetilde{G}] = \widetilde{G}$, every one-dimensional representation is trivial and so cannot be genuine. Hence the exact sequence does not split for any smooth genuine character $\tilde{\chi}$. $\square$

2.3.5 Classification of genuine ordinary mod p representations

Using the proposition of the previous section, we can classify the smooth, genuine, ordinary representations. We have shown that $I(\tilde{\chi})$ has length 2 as a $\widetilde{B}$-module and has a unique 1-dimensional quotient which is a genuine representation of $\widetilde{B}$. So if $I(\tilde{\chi})$ is a reducible $\widetilde{G}$-representation, then its length as a $\widetilde{G}$-module is exactly 2, and it has a unique 1-dimensional subquotient which restricts to a genuine representation of $\widetilde{B}$. But this is a contradiction: the only 1-dimensional representation of $\widetilde{G}$ is the trivial character, which cannot restrict to a genuine representation of $\widetilde{B}$. Hence $I(\tilde{\chi})$ is irreducible for each $\tilde{\chi}$.

The remaining question is when $I(\tilde{\chi}) \cong I(\tilde{\chi}')$. If these two principal series representations are isomorphic, then

$$\dim \operatorname{Hom}_{\widetilde{G}}(I(\tilde{\chi}), I(\tilde{\chi}')) \geq 1,$$

so by smooth Frobenius reciprocity,

$$\dim \operatorname{Hom}_{\widetilde{B}}(I(\tilde{\chi})\big|_{\widetilde{B}}, \tilde{\chi}') \geq 1.$$

But as a $\widetilde{B}$-module, $I(\tilde{\chi})$ contains has only one 1-dimensional subquotient, which is $\tilde{\chi}$ itself and occurs with multiplicity 1. So the space of $\widetilde{B}$-homomorphisms is 1-dimensional if $\tilde{\chi} = \tilde{\chi}'$ and 0-dimensional otherwise, showing that $I(\chi) \cong I(\chi')$ as $\widetilde{G}$-modules if and only if $\chi = \chi'$.

Next, we will describe the $I(1)^*$, I^*, and K^*-invariants of $I(\chi)$. The K^*-invariants will be useful in future exploration of the genuine spherical Hecke algebra and its modules, since the K^*-invariant subspace of a genuine representation gives a module over the genuine spherical Hecke algebra. Likewise, the I^*-invariants are of interest in applications of the result of Chapter 4, as the I^*-invariant subspace of a genuine representation of $\widetilde{G}$ which is generated by its I^*-invariants gives a module over the genuine Iwahori Hecke algebra. It will turn out that $I(\tilde{\chi})$ has nontrivial I^*- and K^*-invariants if and only if $\tilde{\chi}$ is unramified.

Since $(1,-1) \in \widetilde{B}$, the decomposition $G = BI(1) \amalg B\beta_0 I(1)$ lifts to $\widetilde{G}$:

$$\widetilde{G} = \widetilde{B}I(1)^* \amalg \widetilde{B}\left(\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}, 1\right) I(1)^*,$$

and so an $I(1)^*$-invariant vector in $I(\tilde{\chi})$ is determined (as a function $\widetilde{G} \to \bar{\mathbb{F}}_p$) by its values on $(1,1)$ and $\left(\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}, 1\right)$. These values are independent for a given function, so the $I(1)^*$-invariants of $I(\tilde{\chi})$ are 2-dimensional for each $\tilde{\chi}$. Let $\{f_1, f_2\}$ be the basis for $I(1)^*$ determined by the coset representatives $(1,1)$ and $\left(\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}, 1\right)$ respectively. Thanks to this decomposition into double cosets as $\widetilde{B}\backslash \widetilde{SL}_2(F)/I(1)^*$, the $\widetilde{SL}_2(F)$-translates of the $I(1)^*$-invariants generate the whole representation.

Now we can use this to find the I^*-invariants. The decomposition $I = T(\mathcal{O}_F)I(1)$ in G lifts to $\widetilde{G}$ since it lives in K; hence

$$I^* = (\widetilde{T} \cap K^*)I(1)^*,$$

so if $i \in I^*$, then $i = h(a) \cdot i_0$ for some $a \in \mathcal{O}_F^\times$ and some $i_0 \in I(1)^*$. If a vector $f \in I(\chi)$ is to be I^*-invariant, then it is certainly $I(1)^*$-invariant, so it is $\widetilde{G}$-generated by the functions f_1 and f_2. An element $i = h(a) \cdot i_0 \in I^*$ sends f_1 to $h(a) \cdot f_1 = \tilde{\chi}(h(a)) \cdot f_1$, while sending f_2 to $h(a) \cdot f_2 = \tilde{\chi}(a^{-1}) \cdot f_2$. Since this action preserves the decomposition of $I(\tilde{\chi})^{I(1)^*}$, a function f is I^*-invariant if and only if its f_1-component and f_2-component are each I^*-invariant, if and only if $\tilde{\chi}(h(a)) = \tilde{\chi}(h(a^{-1})) = 1$ for all $a \in \mathcal{O}_F^\times$. Hence $I(\tilde{\chi})^{I^*} = I(\chi)^{I(1)^*}$ if $\tilde{\chi}$ is an unramified genuine character, and $I(\tilde{\chi})$ has no I^*-invariants otherwise.

Note also that (as in [19] Lemma 6.3), thanks to the decomposition $\widetilde{G} = U^*\widetilde{T}K^*$, the map $I(\tilde{\chi}) \to \tilde{\chi}$ defined by $f \mapsto f((1,1))$ gives an isomorphism from $I(\tilde{\chi})^{K^*}$ to $\tilde{\chi}^{\widetilde{T}\cap K^*}$. Since $\tilde{\chi}$ is a 1-dimensional representation of $\widetilde{T}$, its $\widetilde{T} \cap K^*$-invariant space is 1-dimensional if $\tilde{\chi}$ is unramified and 0-dimensional otherwise.

These results are summarized in the following theorem:

Theorem 2.3.5. *Let F be a p-adic field with residue field of order $q \equiv 1 \pmod 4$.*

1. *The smooth, genuine, ordinary mod p representations of $\widetilde{SL_2}(F)$ are all those of the form $I(\tilde{\chi}) := Ind_{\widetilde{B}}^{\widetilde{SL}_2(F)}\tilde{\chi}$, where Ind is the smooth induction functor and $\tilde{\chi}$ is an arbitrary genuine character of $\widetilde{T}(F)$ (defined with respect to a fixed additive character of F).*

2. *The dimension of $Hom_{\widetilde{SL}_2(F)}(I(\tilde{\chi}), I(\tilde{\chi'}))$ is 1 if $\tilde{\chi} = \tilde{\chi}'$ and is 0 otherwise, so $I(\tilde{\chi}) \cong I(\tilde{\chi'})$ only if $\tilde{\chi} = \tilde{\chi}'$.*

3. *The $I(1)^*$-invariant space $I(\tilde{\chi})^{I(1)^*}$ is of dimension 2 over $\bar{\mathbb{F}}_p$.*

4. *If $\tilde{\chi}$ is not unramified, then $I(\tilde{\chi})$ has no nontrivial I^*- or K^*-invariants. If $\tilde{\chi}$ is unramified, then $I(\tilde{\chi})^{I^*} = I(\tilde{\chi})^{I(1)^*}$, so is 2-dimensional, and $I(\tilde{\chi})^{K^*}$ is 1-dimensional.*

2.4 Unramified principal series of $PGL_2(F)$

The unramified principal series representations of $PGL_2(F)$ are just those of $GL_2(F)$ with trivial central character, so can be extracted from Barthel and Livné's classification in [2]. The (smooth) unramified principal series representations of $GL_2(F)$ with trivial central character are exactly those of form $B(\chi) := \mathrm{ind}_B^G(\chi \otimes \chi^{-1})$ where $\chi : F^\times \to \bar{\mathbb{F}}_p^\times$ is a smooth character, and $B(\chi)$ is irreducible if and only if $\chi = \chi^{-1}$. Hence the irreducible unramified principal series representations of $PGL_2(F)$ are indexed by the unramified $\bar{\mathbb{F}}_p^\times$-characters of

$F^\times$ such that $\chi(\pi)^2 \neq 1$. The remaining unramified characters are the trivial character $\mathbf{1}$ and the sign character sgn (which sends $x \in F^\times$ to $(-1)^{\mathrm{val}(x)}$), and their induced representations $B(\mathbf{1})$ and $B(\mathrm{sgn})$ are isomorphic. Decomposing $B(\mathbf{1})$, we get two more irreducible unramified representations of $PGL_2(F)$, namely the trivial representation $\mathbf{1}$ and the infinite-dimensional Steinberg representation $St = \mathrm{Ind}(\mathbf{1} \otimes \mathbf{1})/\mathbf{1}$.

Barthel and Livné also study the I- and K-invariants of the unramified principal series in [3]: if $\chi^2 \neq 1$, then $\dim_{\bar{\mathbb{F}}_p} B(\chi)^I = 2$ and $\dim_{\bar{\mathbb{F}}_p} B(\chi)^K = 1$. The Steinberg representation has a 1-dimensional I-invariant subspace and has no K-invariants ([3] Prop. 32), while $\mathbf{1}$ is of course equal to its I- and K-invariants.

Chapter 3

The genuine spherical Hecke algebra

3.1 Summary

3.1.1 Abstract of the chapter

In this chapter, we show that the genuine spherical mod p Hecke algebra of $\widetilde{G}$ is a polynomial algebra in one variable over $\bar{\mathbb{F}}_p$, and we find an explicit generator. Next, we recall that the spherical mod p Hecke algebra of $PGL_2(F)$ is also a polynomial algebra in one variable over $\bar{\mathbb{F}}_p$, so the two algebras are abstractly isomorphic. Then we demonstrate that a certain explicit isomorphism between the mod p spherical Hecke algebras gives a bijection between the unramified genuine ordinary representations of $\widetilde{G}$ (except for the one induced from the sign character) and the unramified principal series representations of $PGL_2(F)$, and that this is a natural correspondence in the sense that corresponding representations have isomorphic Hecke module structures on their spherical vectors. This bijection agrees with the one defined by theta correspondence for unramified principal series representations in characteristic 0, including its dependence on the choice of an additive character of F.

3.1.2 Main results

The first section introduces some notation and then reviews the Satake isomorphism in various settings, since we will define a version of it to compute the genuine mod p spherical Hecke algebra $\mathcal{H}_p^{\epsilon}(\widetilde{G}, K^*)$ of $\widetilde{G}$. In the second section, we show that the mod p Satake isomorphism for reductive groups can be adapted to prove that $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ is a polynomial algebra in one variable, isomorphic to a certain subalgebra of the genuine spherical Hecke algebra of the torus $\widetilde{T}$.

Theorem (3.4.1)**.** *Define a map*

$$\mathcal{S} : \mathbb{H}_p^\epsilon(\widetilde{G}, K^*) \to \mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$$

by

$$f \mapsto \left(t \mapsto \sum_{u \in U^*/(U^* \cap K^*)} f(tu) \right).$$

Then $\mathcal{S}$ is injective and gives an algebra isomorphism

$$\mathcal{S} : \mathbb{H}_p^\epsilon(\widetilde{G}, K^*) \to \mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*),$$

where $\mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$ is the antidominant submonoid of $\mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$.

We also note that $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ is isomorphic to the group algebra $\bar{\mathbb{F}}_p[X_*^-(T)]$ of the antidominant coweights of $SL_2(F)$. Thanks to a result of Barthel-Livné, this is enough to show that $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ is abstractly isomorphic to the spherical Hecke algebra of $PGL_2(F)$ with respect to the trivial representation of K_G (Lemma 3.4.6).

We calculate the spherical Hecke module structure of the K^*- and K_G- invariants of the unramified ordinary representations of, respectively, $\widetilde{SL}_2(F)$ and $PGL_2(F)$ (Lemma 3.4.4) and use the results to define a bijection of principal series representations induced from unramified characters $\chi \neq \mathrm{sgn}$. This bijection is shown to be compatible with the most obvious choice of concrete isomorphism between the spherical Hecke algebras. The results of the second half of the chapter are summarized in the following:

Theorem C (Theorem 3.4.7)**.** *1. The $\bar{\mathbb{F}}_p$-linear map $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*) \to \mathcal{H}_p(PGL_2(F), K_G)$ defined by $t \mapsto t_1$ is an algebra isomorphism. Furthermore, it induces a bijection (which depends on the additive character ψ) of irreducible unramified ordinary representations associated to characters χ of $F^\times$ such that $\chi^2 \neq sgn$.*

2. *When $\tilde{\chi} = \chi \cdot \gamma_\psi$ is defined with respect to a fixed choice of ψ (as in §2.3.3) and χ is a smooth unramified character of $F^\times$ such that $\chi^2 \neq 1$, the irreducible unramified ordinary representation $I(\tilde{\chi})$ of $\widetilde{G}$ corresponds under the bijection to the irreducible ordinary representation $I(\chi \otimes \chi^{-1})$ of $PGL_2(F)$.*

3. *The dependence of the bijection on ψ is as follows. For $a \in F^\times/(F^\times)^2$, let χ_a denote the quadratic character of $F^\times$ given by the Hilbert symbol $(-, a)_F$. If $I(\chi \otimes \chi^{-1})$ corresponds to $I(\tilde{\chi})$ when the bijection is defined with respect to a nontrivial additive character ψ, then $I(\chi \otimes \chi^{-1})$ corresponds to $I(\chi_a \cdot \tilde{\chi})$ when the bijection is defined with respect to the character $\psi_a : x \mapsto \psi(ax)$.*

3.1.3 Review of related results in characteristic 0

For context, we mention that the main theorem of [22] implies the existence of an isomorphism between the center of the genuine $\mathbb{C}$-valued Iwahori Hecke algebra $\mathbb{H}^\epsilon_{\mathbb{C}}(\widetilde{G}, I^*)$ and the center of the $\mathbb{C}$-valued Iwahori Hecke algebra of $PGL_2(F)$. Note that Savin's results are more general, but we specialize them to the pair $(\widetilde{G},\ PGL_2(F))$ in this discussion. We give a full statement and further discussion of Savin's isomorphism of Iwahori Hecke algebras in §4.1.3.

The center of $\mathbb{H}^\epsilon_{\mathbb{C}}(\widetilde{G}, I^*)$ is the isomorphic image of the genuine $\mathbb{C}$-valued spherical Hecke algebra under the composition of a Satake isomorphism with a Bernstein isomorphism, and likewise the center of the Iwahori Hecke algebra $\mathbb{H}_{\mathbb{C}}(PGL_2(F), I_G)$ is the isomorphic image of the spherical $\mathbb{C}$-valued Hecke algebra of $PGL_2(F)$. Hence Savin's explicit isomorphism of Iwahori Hecke algebras $\mathbb{H}^\epsilon_{\mathbb{C}}(\widetilde{G}, I^*) \cong \mathbb{H}_{\mathbb{C}}$ also gives an isomorphism $\mathbb{H}^\epsilon_{\mathbb{C}}(\widetilde{G}, K^*) \cong \mathbb{H}_{\mathbb{C}}(PGL_2(F), K_G)$.

This induced map of spherical Hecke algebras, viewed via the Satake isomorphism as a map of Weyl orbits in the group algebras of the respective cocharacter lattice, sends the Weyl orbit of the dominant coroot of $SL_2(F)$ to the Weyl orbit of the dominant coroot of $PGL_2(F)$. As noted in [23] (p. 20), one gets a bijection between subquotients of unramified principal series representations (over $\mathbb{C}$) associated to a given unramified $\mathbb{C}$-valued character

χ of $F^\times$. Over $\mathbb{C}$, the unramified principal series representation $I(\tilde{\chi})$ is isomorphic to $I(\chi^w)$ for w in the Weyl group of $\widetilde{SL}_2(F)$, so the W-invariance of the image of the Satake map is crucial for the existence of the bijection.

Over $\bar{\mathbb{F}}_p$, we have seen (cf. Theorem A) that, for fixed choice of an additive character ψ, the ordinary representations associated to distinct $\bar{\mathbb{F}}_p$ characters χ, χ' of $F^\times$ are irreducible and nonisomorphic. Away from those χ such that $\chi^2 = 1$, we have the same result ([2], Thm.25) for the mod p principal series representations of $PGL_2(F)$. Hence Weyl invariance no longer plays a role in the mod p setting, and in fact the image of our Satake map is not W-invariant.

In the next sections, we give a more detailed review of Satake isomorphisms in different settings.

3.2 Preliminaries

3.2.1 Notation and definitions

Let $G = SL_2(F)$, $\widetilde{G} = \widetilde{SL}_2(F)$, and $K = SL_2(\mathcal{O}_F)$. Let K^* be the image of K in $\widetilde{G}$ under the preferred section θ defined in (ref to ch.1), and recall that $K \cong K^*$ since the extension splits over K.

The rest of this section sets down notation for the spherical mod p Hecke algebras of $\widetilde{G}$ and of $\widetilde{T}$.

Definition 3.1.

1. Define $\mathbb{H}_p(\widetilde{G}, K^*)$ to be the algebra of K^*-biinvariant, smooth, compactly supported $\bar{\mathbb{F}}_p$-valued functions on $\widetilde{G}$. The product of two functions f_1, f_2 in $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ is their convolution $f_1 * f_2$, defined by

$$(f_1 * f_2)(g') := \sum_{g \in \widetilde{G}/K^*} f_1(g'g) \cdot f_2(g^{-1}).$$

2. Let $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ denote the subset of $\mathbb{H}_p(\widetilde{G}, K^*)$ consisting of genuine functions, i.e., define

$$\mathbb{H}_p^\epsilon(\widetilde{G}, K^*) := \Big\{ f \in \mathbb{H}_p(\widetilde{G}, K^*) : f(g(1,\zeta)) = \zeta f(g) \text{ for all } g \in \widetilde{G},\, \zeta \in \{\pm 1\} \Big\}.$$

The convolution of two genuine functions in $\mathbb{H}_p(\widetilde{G}, K^*)$ is again a genuine function, so $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ is an algebra under the convolution product.

Since K^* is a compact open subgroup of $\widetilde{G}$, the Frobenius reciprocity map (Prop. 2.2.1) is an $\bar{\mathbb{F}}_p$-algebra isomorphism

$$\mathbb{H}_p(\widetilde{G}, K^*) \to \mathrm{End}_{\widetilde{G}}(\mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*}),$$

where $\mathbf{1}_{K^*}$ is the trivial representation of K^* and $\mathrm{ind}_{K^*}^{\widetilde{G}}$ is the functor of compact induction $Rep(K^*) \to Rep(\widetilde{G})$.

3. Denote the endomorphism algebra $\mathrm{End}_{\widetilde{G}}(\mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*})$ by $\mathcal{H}_p(\widetilde{G}, K^*)$, and let $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ denote the subalgebra of $\mathcal{H}_p(\widetilde{G}, K^*)$ which is the image under Frobenius reciprocity of the genuine subalgebra $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$. In particular, Frobenius reciprocity is an isomorphism of $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ with $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$.

The isomorphic algebras $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ and $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ will both be called the *genuine mod p spherical Hecke algebra* of $\widetilde{G}$, and we will work in one or the other as context requires.

We make the analogous definitions for the torus $\widetilde{T}$ of $\widetilde{G}$ and its compact open subgroup $\widetilde{T} \cap K^*$.

Definition 3.2.

1. Let $\mathbb{H}_p(\widetilde{T}, \widetilde{T} \cap K^*)$ denote the algebra of compactly supported functions $f : \widetilde{T} \to \bar{\mathbb{F}}_p$ which are invariant under multiplication by $(\widetilde{T} \cap K^*)$; note that since $\widetilde{T}$ is abelian, left- and right-$(\widetilde{T} \cap K^*)$-invariance are equivalent. The algebra product in $\mathbb{H}_p(\widetilde{T}, \widetilde{T} \cap K^*)$

is the convolution

$$(f_1 * f_2)(t') = \sum_{t \in \widetilde{T}/(\widetilde{T} \cap K^*)} f_1(t't) f_2(t^{-1}).$$

2. Let $\mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$ denote the subalgebra of genuine functions in $\mathbb{H}_p(\widetilde{T}, \widetilde{T} \cap K^*)$.

3. Since $\widetilde{T} \cap K^*$ is a compact open subgroup of $\widetilde{T}$, Frobenius reciprocity gives an isomorphism of $\mathbb{H}_p(\widetilde{T}, \widetilde{T} \cap K^*)$ with $\mathrm{End}_{\widetilde{T}}(\mathrm{ind}_{\widetilde{T} \cap K^*}^{\widetilde{T}} \mathbf{1}_{\widetilde{T} \cap K^*})$, where $\mathrm{ind}_{\widetilde{T} \cap K^*}^{\widetilde{T}}$ is the functor of compact induction $Rep(\widetilde{T} \cap K^*) \to Rep(\widetilde{T})$.

   Denote $\mathrm{End}_{\widetilde{T}}(\mathrm{ind}_{\widetilde{T} \cap K^*}^{\widetilde{T}} \mathbf{1}_{\widetilde{T} \cap K^*})$ by $\mathcal{H}_p(\widetilde{T}, \widetilde{T} \cap K^*)$ and let $\mathcal{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$ denote the injective image under Frobenius reciprocity of $\mathbb{H}^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$ in $\mathcal{H}_p(\widetilde{T}, \widetilde{T} \cap K^*.)$

   The *genuine mod p spherical Hecke algebra* of $\widetilde{T}$ is defined to be either one of $\mathbb{H}^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$ or $\mathcal{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$, depending on the context. We will usually omit the modifier "mod p," but unless otherwise specified, the coefficient field should be assumed to be $\bar{\mathbb{F}}_p$.

Finally, we define the antidominant submonoid of $\mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$. In preparation, recall from §1.1.1 that we have a canonical (up to choice of a uniformizer π of F) isomorphism between the cocharacter group $X_*(T)$ and $T(F)/T(\mathcal{O}_F)$, where T is the diagonal torus of $SL_2(F)$, and this isomorphism sends a cocharacter $\lambda \in X_*(T)$ to the class of $\lambda(\pi)$ in $T/T(\mathcal{O}_F)$. The antidominant coroot $(-1, 1)$ is sent to the class modulo $T(\mathcal{O}_F)$ of

$$\alpha_0 := \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix},$$

while the dominant coroot $(1, -1)$ is sent to α_0^{-1}. Moreover we identified $T/T(\mathcal{O}_F)$ with Λ, the one-parameter subgroup of $\widetilde{T}/(\widetilde{T} \cap K^*)$ generated by the class modulo $\widetilde{T} \cap K^*$ of

$$h(\pi) := \left(\begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}, (-1, \pi)_F \right),$$

by identifying the class of α_0 modulo $T(\mathcal{O}_F)$ with the class of $h(\pi)$ modulo $\widetilde{T} \cap K^*$. Accordingly, we will say that a class of $\widetilde{T}$ modulo $\widetilde{T} \cap K^*$ is *antidominant* if it is represented by $h(\pi)^k$ for some $k \geq 0$, and *dominant* if it is represented by $h(\pi)^k$ for some $k \leq 0$.

Definition 3.3. Define $\mathbb{H}_p^-(\widetilde{T}, \widetilde{T} \cap K^*)$ to be the submonoid of $\mathbb{H}_p(\widetilde{T}, \widetilde{T} \cap K^*)$ consisting of functions which are supported only on antidominant classes of $\widetilde{T}/(\widetilde{T} \cap K^*)$.

The aim of this chapter is to describe the structure of the genuine spherical Hecke algebra $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ and then to use it to relate certain unramified representations of $\widetilde{G}$ to those of $SL_2(F)$. To explain how this has been done in similar settings, we review three known versions of the Satake map: for split reductive groups and then for their metaplectic covers when the coefficient field is $\mathbb{C}$, and for unramified reductive groups when the coefficient field is $\bar{\mathbb{F}}_p$.

3.2.2 The classical Satake isomorphism for split groups

In this expository section, we break from the convention of the main text and take all representations over $\mathbb{C}$. All representations are still assumed to be smooth. The main reference for this section is [9].

Let $\mathbf{G}$ be a reductive algebraic group which splits over a nonarchimedean local field F, and let $G = \mathbf{G}(F)$. Such a G has a *hyperspecial* maximal compact K, meaning that $K \cong \mathcal{G}(\mathcal{O}_F)$ for a smooth integral model model $\mathcal{G}$ of G.

More generally, one can take $\mathbf{G}$ to be *unramified* over F, meaning that it has a minimal parabolic subgroup B defined over F and a maximal compact subgroup K such that $G(F) = B(F)K(F)$ (i.e., $\mathbf{G}$ has an Iwasawa decomposition over F). Note that while not every unramified reductive group has a hyperspecial maximal compact (since not every such group has a smooth model over $\mathcal{O}_F$), it is true that if G is defined over a global field, then it has a smooth integral model at almost every place.

Let (σ, V) be an irreducible representation of K. The *spherical Hecke algebra of G with* *respect to* σ is the $\mathbb{C}$-algebra of functions

$$\mathbb{H}(G, K, \sigma) = \{ f : G \to \mathrm{End}_{\mathbb{C}} V \ : \ f(k_1 g k_2) = \sigma(k_1) \circ f(g) \circ \sigma(k_{2)} \, \forall \, k_i \in K, \, g \in G \}$$

which are compactly supported mod K on the left, on which multiplication is given by convolution and G acts by translation on the right. Although it is easy to give a $\mathbb{C}$-vector space basis for $\mathbb{H}(G, K, \sigma)$ using a Cartan decomposition of G, its algebra structure (e.g., its generators, relations, and properties such as commutativity and semisimplicity) is generally less clear. The Satake isomorphism transfers these questions to the group algebra of the cocharacter group of a maximal torus in G, or equivalently to the spherical Hecke algebra of a maximal torus, where the structure is much easier to see.

Let $\Psi = (X_*(T), \Phi, X^*(T), \Phi^\vee)$ denote the root datum attached to the pair (G, T): here $X_*(T)$ is the character group of T, $X^*(T)$ is the cocharacter group, and Φ and $\Phi^\vee$ are the sets of roots and coroots respectively. Also let W denote the Weyl group of (G, T). Then the dual group $G^\vee$ (which comes with a maximal torus $T^\vee \subset G^\vee$) to G is the connected reductive group such that the root system of $(G^\vee, T^\vee)$ is $\Psi^\vee = (X^*(T), \Phi^\vee, X_*(T), \Phi)$. (The justification for calling this the "dual group" in this situation is essentially the second statement of the Satake isomorphism below, which will relate characters of the Hecke algebra of G to representations of $G^\vee$.) In particular, $T^\vee$ is the maximal torus in $G^\vee$ whose character group is $X^*(T)$ and whose cocharacter group is $X_*(T)$, and W again acts on both of these groups. Here is the simplest statement of the Satake isomorphism theorem:

Theorem 3.2.1 (Satake isomorphism for split groups). *Let G be a split reductive group over a local field F, and let T be a split maximal torus of G and $K = G(\mathcal{O}_F)$. Then there is a canonical isomorphism between the spherical Hecke algebra $\mathcal{H}(G, K)$ and $\mathbb{C}[T^\vee]^W$.*

The W-action on $\mathbb{C}[T^\vee]$ is worth a quick note. The group algebra $\mathbb{C}[T^\vee]$ is canonically isomorphic to the ring of regular functions on the character group of $T^\vee$: given an element $f \in \mathbb{C}[T^\vee]$, f acts on $\mathbb{C}^\times$ by $\chi(f)\big|_{V_\chi}$ where $\chi \in \mathrm{Hom}(T^\vee, \mathbb{C}^\times)$ is any character. Since

$\mathrm{End}(V_\chi) \cong \mathbb{C}$, we can define a map $\mathbb{C}[T^\vee] \to \mathrm{Reg}(\mathrm{Hom}(T^\vee, \mathbb{C}^\times))$ by:

$$f \mapsto (\chi \mapsto \pi_\chi(f) \in \mathbb{C})\,,$$

where $\mathrm{Reg}(\mathrm{Hom}(T^\vee, \mathbb{C}^\times))$ is the ring of regular functions on the space of algebraic homomorphisms. Hence the W-action on $\mathbb{C}[T^\vee]$ is given by the usual action of W on $X_*(T)$.

With a bit more work, we can reformulate the isomorphism to replace $\mathbb{C}[T^\vee]^W$ with the group of unramified complex characters of T, as follows. Identify $X_*(T)$ with $\mathrm{Hom}_{F-\text{alg.gp.}}(\mathbb{G}_m, T)$ so that if $\phi \in X_*(T)$, then the corresponding element $\phi' \in \mathrm{Hom}_{F-\text{alg.gp.}}(\mathbb{G}_m, T)$ is the map such that for all $t \in F$, $\lambda \in X^*(T)$,

$$\lambda(\phi'(t)) = t^{\langle \phi, \lambda \rangle}.$$

(From now on, we will just call ϕ' by ϕ, and consider $X_*(T)$ to be the cocharacter group of T as an algebraic group, though we will also use the fact that it is the $\mathbb{Z}$-dual of $X^*(T)$.) From that point of view, we have an isomorphism $X_*(S) \cong T(F)/T(\mathcal{O}_F)$ which identifies the element $\phi \in X_*(T) = \mathrm{Hom}_\mathbb{Z}(X^*(T), \mathbb{Z})$ with the class modulo $T(\mathcal{O}_F)$ of elements $t \in T(F)$ such that

$$\langle \phi, \lambda \rangle := \phi(\lambda) = v_F(\lambda(t)).$$

The algebraic characters $\lambda \in X^*(T)$ are polynomial functions on T, and $\{\lambda \in X^*(T)\}$ is a basis for the F-algebra of polynomial functions on T. T is exactly the spectrum of the algebra of polynomials defined on it, so $T = \mathrm{Spec}(F[X^*(T)])$.

We now have a convenient way of describing the points of T under an extension K of k: if K is a commutative k-algebra, then the K-points of T are

$$T(K) = \mathrm{Hom}_{k-\text{alg.gp.}}(K, T) = \mathrm{Hom}_{k-\text{algebra.}}(k[X^*(T)], K^\times) \cong \mathrm{Hom}_\mathbb{Z}(X^*(T), K^\times).$$

Since $X_*(T)$ is the $\mathbb{Z}$-dual of $X^*(T)$, $\mathrm{Hom}_\mathbb{Z}(X^*(T), K^\times) \cong X_*(T) \otimes_\mathbb{Z} K^\times$. In particular, the

k-points of T are $X_*(T) \otimes_{\mathbb{Z}} k^\times$, which indeed is isomorphic to $(k^\times)^{\mathrm{rk}(T)} \cong T$.

Let $\Lambda(T) = T(F)/T(\mathcal{O}_F)$. Then $\Lambda(T) = X_*(T)$, and the group of unramified characters of T is $\mathrm{Hom}(X_*(T), \mathbb{C}^\times)$. Now let $S = \mathrm{Spec}(\mathbb{C}[X_*(T)])$, a complex torus. We have

$$X^*(S) \cong X_*(T)$$

and

$$S(\mathbb{C}) \cong \mathrm{Hom}(X^*(S), \mathbb{C}^\times) \cong \mathrm{Hom}(X_*(T), \mathbb{C}^\times) \cong \{\text{unram. chars. of } T\}.$$

So given a split torus T, we have the following recipe for a complex algebraic group S which parametrizes unramified characters of T (in the sense that its group of $\mathbb{C}$-points is isomorphic to the group of unramified characters): Let S be the complex torus whose character group is the same as the cocharacter group of S, and vice versa. The definition of the dual group to a split reductive group grows from this idea. In particular, the group of unramified complex characters of $SL_2(F)$ is isomorphic to the W-invariants of the complex group algebra of the torus (over $\mathbb{C}$) of PGL_2, and vice versa.

The Satake transform. One can again reformulate the theorem in a way which is less convenient to state but whichgives an explicit isomorphism. Let G be a split connected reductive group with a hyperspecial maximal compact K, Borel subgroup B such that $G = BK$ and $T \subset B$ for T a maximal split torus of G. Let $\mathcal{H}(T, T(\mathcal{O}_F))$ denote the algebra of $T(\mathcal{O}_F)$-biinvariant, compactly supported smooth functions on T. Let $\mathrm{d}u$ be the Haar measure on the unipotent radical U of B, normalized so that $\int_{U\cap K} \mathrm{d}u = 1$. Finally, let $\delta : T \to F^\times$ be the determinant of the action of $T(F)$ on $\mathrm{Lie}(U)$. In the split case, this is the modular character of the standard Borel containing the chosen split maximal torus. Then:

Theorem 3.2.2 (Satake isomorphism, [9] Thm. 4.1)**.** *The map*

$$\mathcal{H}(G, K) \to \mathcal{H}(T, T(\mathcal{O}_F))$$

defined by

$$f \mapsto \left(u \mapsto \delta(u)^{1/2} \int_U f(tu)\, du \right)$$

is an injective homomorphism of algebras, and its image is exactly $\mathcal{H}(T, T(\mathcal{O}_F))^W$.

In fact, as in [12], one can replace $\mathbb{C}$ with $\mathbb{Z}$ in the definition of the spherical Hecke algebras of G and T. Then the transform goes through as written, except that since $\delta(\mu(\pi))^{1/2} \in \mathbb{Z}[q^{\pm 1/2}]$, the image of $\mathcal{S}$ is in $\mathbb{Z}[q^{\pm 1/2}]$, and we need to adjoin $q^{\pm 1/2}$ before getting an isomorphism

$$\mathcal{S} : \mathcal{H}(G, K) \otimes \mathbb{Z}[q^{\pm 1/2}] \to (\mathcal{H}(T, {}^o T) \otimes \mathbb{Z}[q^{\pm 1/2}])^W \cong R(\hat{G}) \otimes \mathbb{Z}[q^{\pm 1/2}],$$

where $R(\hat{G})$ is the representation ring of $\hat{G}$. (This is because the irreducible representations of $\hat{G}$ are parametrized by their highest-weight vectors, which are W-orbits of characters of $T^\vee$.)

This formula cannot be reduced mod p as it stands, since it contains the inverse of q. However, this problem can be avoided by omitting the modular character δ, if one forfeits W-invariance of the image. As it turns out, this is unproblematic, as explained in the introduction to [14], and reduction mod p gives the statement of Herzig's mod p Satake isomorphism in the case where V is the trivial representation of K. (However, there are some extra difficulties when considering nontrivial representations of K.)

Application in characteristic 0: unramified principal series representations. The Satake transform gives an isomorphism between the spherical Hecke algebra of G and the group of unramified characters of T via $\mathcal{H}(T, T(\mathcal{O}_F))^W$, or $\mathbb{C}[T^\vee]^W$. Now we tie this to the larger representation theory of G and its dual group $\hat{G}$. Given an unramified character χ of T, inflate χ to a character of B and then induce to G to form the principal series representation $I(\chi)$.

Proposition 3.2.3 ([5]). *1.* $I(\chi) \cong I(\chi')$ *if and only if* $\chi' = \chi^w$ *for some* $w \in W$.

2. $I(\chi)^K$ *is 1-dimensional if* χ *is unramified, and 0 if* χ *is ramified.*

Let $(I(\chi), V_\chi)$ be a principal series representation of G, and identify it with the W-orbit of χ. If and only if χ is unramified, the K-invariant subspace is 1-dimensional, so V_χ^K is a 1-dimensional $\mathcal{H}(G, K)$-module, i.e. the W-orbit of a character of $\mathcal{H}(G, K)$. Precomposing with the Satake isomorphism, we can view this as a character of $\mathbb{C}[T^\vee]^W$. And

$$\mathrm{Hom}(\mathbb{C}[T^\vee], \mathbb{C}^\times) \cong \mathrm{Hom}(X^*(T^\vee), \mathbb{C}^\times) \cong T^\vee(\mathbb{C}),$$

so a character of $\mathbb{C}[T^\vee]^W$ matches up uniquely with a $\mathbb{C}$-point of $T^\vee/W$.

Note that this matching works for any unramified representation of G, so more generally let $s(\pi)$ be the semisimple conjugacy class of $\hat{G}$, or *Satake parameter*, associated to an unramified π.

Proposition 3.2.4 ([12], Prop. 6.4)**.** *The map $\pi \to s(\pi)$ gives a bijection between the set of isomorphism classes of unramified irreducible representations of G and the set of semisimple conjugacy classes in G.*

In particular, if the spherical Hecke algebras of two such groups are isomorphic, then transfer of characters from the spherical Hecke algebra of one group to the other induces a bijection of unramified principal series representations.

3.2.3 The Satake isomorphism for the metaplectic group in characteristic 0

This section is again expository, and again takes place in characteristic 0. The Satake transform in characteristic 0 can be defined similarly for a metaplectic group, and gives an isomorphism of the (appropriately defined) spherical Hecke algebra with the Weyl-invariants of the group algebra of the (again, appropriately defined) coweight lattice. The main point is to choose these two definitions carefully. The following results are due to Kazhdan and Patterson ([16], [17]) in the case of GL_n, and were studied by McNamara [19] for metaplectic covers of split reductive groups.

Let G be a split reductive group over F, and suppose that G has a hyperspecial maximal compact subgroup $K = \mathcal{G}(\mathcal{O}_F)$ where $\mathcal{G}$ is a smooth group scheme over $\mathcal{O}_F$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ and $\mathbf{T} \subset \mathbf{B}$ a maximal split torus; let $B = \mathbf{B}(F)$ and $T = \mathbf{T}(F)$. Let $\widetilde{G}$ be the central extension of G by μ_n and assume that $2n \big| (q-1)$. The extension splits (non-canonically) over K; let K^* denote some choice of lifting of K to $\widetilde{G}$ which is compatible with the canonical lifting of the unipotent radical $U \subset B$ to $U^* \subset \widetilde{G}$.

Define the antigenuine spherical Hecke algebra $\mathcal{H}^\epsilon(\widetilde{G}, K^*)$ to be the algebra of K^*-bi-invariant smooth, compactly supported, antigenuine functions $f : \widetilde{G} \to \mathbb{C}$. The algebra product is convolution with respect to a Haar measure on $\widetilde{G}$ which is normalized so that $\mathrm{vol}(K \times \mu_n) = 1$. Then ([19], Thm. 9.2) the support of $\mathcal{H}(\widetilde{G}, K^*)$ is $\mu_n K^* H K^*$, where H is the centralizer in $\widetilde{T}$ of $\widetilde{T} \cap K^*$. Note that H is a maximal abelian subgroup of $\widetilde{T}$, as proved in [19] Lem. 5.3. When $n = 2$ and $G = SL_2$, we have $H = \widetilde{T}$ and $\mu_2 \subset \widetilde{T}$, so the support of $\mathcal{H}^\epsilon(\widetilde{G}, K)$ is $K^* \widetilde{T} K^*$.

Let Y be the group of cocharacters of $\mathbf{T}$, viewed as a subgroup of T via $\lambda \mapsto \lambda(\pi)$. Recall that, thanks to the commutator formula in $\widetilde{T}$ when G is a split group, the extension splits trivially over Y when $2n \big| (q-1)$. Let $s : Y \subset \widetilde{G}$ denote such a splitting. Define

$$\Lambda = \{\lambda \in Y : \; s(\lambda(\pi)) \in H\},$$

and let $\mathbb{C}[\Lambda]$ denote the group algebra of Λ. Denote the modular quasicharacter of $\widetilde{B}$ by δ.

Theorem 3.2.5 ([19], Thm. 10.4)**.** *Define the Satake map $\mathcal{S} : \mathcal{H}^\epsilon(\widetilde{G}, K) \to \mathbb{C}[\Lambda]$ as follows:*

$$(\mathcal{S}f)(\lambda) = \delta^{1/2}(\lambda(\pi)) \int_{U^*} f(\lambda(\pi)u)\, du,$$

where du is a Haar measure on $\widetilde{G}$ such that vol$(K \times \mu_n) = 1$.

Then $\mathcal{S}$ is an injective homomorphism, and gives an algebra isomorphism

$$\mathcal{H}^\epsilon(\widetilde{G}, K) \to \mathbb{C}[\Lambda]^W.$$

To define a Satake transform on the mod p spherical Hecke algebra of $\widetilde{SL}_2(F)$, we will need to combine these modifications to the reductive case with some modifications to the characteristic-0 case. We now move back to characteristic p to explain what has been done for reductive groups in that setting.

3.2.4 The Satake isomorphism for unramified reductive groups in characteristic p

Herzig [14] has defined a Satake transform in characteristic p and shown that it is an isomorphism of every spherical Hecke algebra of an unramified connected reductive group with the group algebra of its antidominant coweights. Herzig considers spherical Hecke algebras with respect to all irreducible representations of the maximal compact K; if one is only interested in the spherical Hecke algebra with respect to the trivial representation of K, as is the case for us in the rest of this chapter, then the result can be deduced from a renormalized version of the integral Satake isomorphism mentioned in §3.2.2. However, we state Herzig's result in full.

The setting is as follows. G is an unramified connected reductive group over F with Iwasawa decomposition $G(F) = BK$, and T is a maximal torus such that $B = T \ltimes U$. Let (π, V) be an irreducible representation of K. The spherical Hecke algebra of G with respect to V has the following two equivalent definitions:

$$\mathcal{H}_G(V) = \mathrm{End}_{\overline{\mathbb{F}}_p G}\left(\mathrm{ind}_K^{G(F)} V\right),$$

where $\mathrm{ind}_K^{G(F)} V_{\vec{r}}$ is the compact induction, i.e., the space

$$\mathcal{I}(K, G, V) = \{f : G \to V_{\vec{r}} : \; f(kg) = \pi(k) f(g) \, \forall k \in K, \, g \in G\}$$

where f is locally constant and compactly supported mod K on the left, and with the right-translation action of G;

$$\mathbb{H}_G(V) = \{f : G \to \mathrm{End}_{\bar{\mathbb{F}}_p} V : \ f(k_1 g k_2) = \pi(k_1) f(g) \pi(k_2) \text{ for all } k_1,\, k_2 \in K,\, g \in G\}$$

where f is compactly supported and the $\bar{\mathbb{F}}_p$-algebra structure is given by convolution. The two algebras $\mathcal{H}_G(V)$ and $\mathbb{H}_G(V)$ are isomorphic by compact Frobenius reciprocity.

Let k denote the residue field of F, and let $U(k)$ denote the image of $U(\mathcal{O}_F)$ in $G(k)$. When V is an irreducible representation of K, then the $U(k)$-invariant subspace of $V_{\vec{r}}$ is a one-dimensional representation of $T(k)$, and the Hecke algebra of $T(k)$ with respect to this representation is defined to be

$$\mathbb{H}_T(V^{U(k)}) = \{f : T(F) \to \mathrm{End}_{\bar{\mathbb{F}}_p}(V^{U(k)}) \cong \bar{\mathbb{F}}_p : f(k_1 g k_2) = f(g)\, \forall k_i \in T(\mathcal{O}_F),\, g \in T(F)\}.$$

Given an irreducible representation (π, V) of $G(k)$, let $\mathbb{H}_{T^-}(V^{U(k)})$ be the subalgebra of $\mathbb{H}_T(V^{U(k)})$ in which all functions are supported on T^-.

Theorem 3.2.6 ([14] Thm. 1.2)**.** *Suppose that V is an irreducible representation of $G(k)$ over $\bar{\mathbb{F}}_p$. Then the map*

$$\mathcal{S} : \mathbb{H}_G(V) \to \mathbb{H}_T(V^U)$$

given by

$$f \mapsto \left(t \mapsto \sum_{u \in U(F)/U(\mathcal{O}_F)} f(tu)\big|_{V_{\vec{r}}^{U(k)}} \right)$$

is an injective $\bar{\mathbb{F}}_p$-algebra homomorphism with image $\mathbb{H}_{T^-}(V^{U(k)})$.

Note that the image of the transform is isomorphic, via evaluation of coweights on a uniformizer of $\mathcal{O}_F$, to the group algebra $\bar{\mathbb{F}}_p[X_*^-(S)]$, where S the maximal F-split torus in G which is normalized by T.

3.3 $\bar{\mathbb{F}}_p$-vector space structure of $\mathcal{H}_p^{\epsilon}(\widetilde{G}, K^*)$

We now begin to describe the structure of the genuine mod p spherical Hecke algebra $\mathcal{H}_p^{\epsilon}(\widetilde{G}, K^*)$ defined in §3.2.1.

3.3.1 Support of the genuine spherical Hecke algebra

The goal of this section is to prove the following proposition:

Proposition 3.3.1. *Every element of $\widetilde{SL}_2(F)$ is contained in the support of a function belonging to the genuine spherical Hecke algebra $\mathbb{H}_p^{\epsilon}(\widetilde{SL}_2(F), K^*)$.*

Remark 3.3.2. *The statement of Proposition 3.3.1 is well-known for the $\mathbb{C}$-valued spherical Hecke algebra of $\widetilde{G}$; for example, see [19], Thm. 9.2 for a more general statement which reduces to Proposition 3.3.1 in the case $G = SL_2$, $n = 2$. The proof in the context of $\mathbb{C}$-valued functions remains valid for $\bar{\mathbb{F}}_p$-valued functions, but for completeness we give an elementary argument for $\widetilde{G} = \widetilde{SL}_2(F)$.*

We first reduce the proof of Proposition 3.3.1 to showing that $K^*h(\pi)^nK^* \neq K^*h(\pi)^n(1,-1)K^*$ for all $n \geq 0$.

Lemma 3.3.3. *For each $n \in \mathbb{Z}$, there is a genuine function supported on*

$$K^*h(\pi)^nK^* \bigcup K^*h(\pi)^n(1,-1)K^*$$

if and only if $K^*h(\pi)^nK^* \neq K^*h(\pi)^n(1,-1)K^*$. In particular, if $K^*h(\pi)^nK^* \neq K^*h(\pi)^n(1,-1)K^*$, then the function

$$\mathbf{1}_{K^*h(\pi)^nK^*} - \mathbf{1}_{K^*h(\pi)^n(1,-1)K^*}$$

is in $\mathbb{H}_p^{\epsilon}(\widetilde{G}, K^*)$.

Proof of Lemma 3.3.3. If $K^*h(\pi)^kK^* = K^*h(\pi)^k(1,\zeta)K^*$ for some $n \in \mathbb{Z}$, then there exist k_1, $k_2 \in K^*$ for which $h(\pi)^n(1,-1) = k_1h(\pi)^nk_2$. Then a genuine function f must satisfy

$$-f(h(\pi)^n) = f(h(\pi)^n(1,-1)) = f(k_1h(\pi)^nk_2) = f(h(\pi)^n),$$

so $f(h(\pi)^n) = 0$. Hence $f(g) = 0$ for all $g \in K^* h(\pi)^n K^* \bigcup K^* h(\pi)^n (1, -1) K^*$ if the union is not disjoint.

Conversely, if $K^* h(\pi)^n K^* \cap K^* h(\pi)^n (1, -1) K^* = \emptyset$, then the function

$$\mathbf{1}_{K^* h(\pi)^n K^*} - \mathbf{1}_{K^* h(\pi)^n (1,-1) K^*}$$

is in $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ and is supported on $K^* h(\pi)^n K^* \cap K^* h(\pi)^n (1, -1) K^*$.

$\square$

Lifting the Cartan decompositions (1.2) to $\widetilde{G}$, we can write $\widetilde{G}$ as a union over K^*-double cosets in the following ways:

$$\widetilde{G} = \coprod_{n \geq 0} \left(\bigcup_{\zeta \in \{\pm 1\}} K^* h(\pi)^n (1, \zeta) K^* \right) = \coprod_{n \leq 0} \left(\bigcup_{\zeta \in \{\pm 1\}} K^* h(\pi)^n (1, \zeta) K^* \right) \tag{3.1}$$

Hence every element of $\widetilde{G}$ is contained in $K^* h(\pi)^n K^* \cup K^* h(\pi)^n (1, -1) K^*$ for some $n \geq 0$. So if $K^* h(\pi)^n K^* \neq K^* h(\pi)^n (1, -1) K^*$ for all $n \geq 0$, then Lemma 3.3.3 shows that every element of $\widetilde{G}$ is contained in the support of a genuine function.

Proof of Proposition 3.3.1. By the discussion following the proof of Lemma 3.3.3, it is enough to show that $K^* h(\pi)^n K^* \neq K^* h(\pi) n K^*$ for all $n \geq 0$. As it is no harder to show this for arbitrary $n \in \mathbb{Z}$, we will prove:

Claim. For each $n \in \mathbb{Z}$, $K^* h(\pi)^n K^* \neq K^* h(\pi)^n (1, -1) K^*$.

Proof of Claim. Since $(1, -1) \notin K^*$, the claim is clear for $n = 0$. Suppose that $K^* h(\pi)^n K^* = K^* h(\pi)^n (1, -1) K^*$ for some $n \in \mathbb{Z}$, $n \neq 0$. Then $h(\pi)^n (1, -1) \in K^* h(\pi)^n K^*$, so there exist k_1, $k_2 \in K$ such that

$$(k_1, \theta(k_1)) h(\pi)^n (k_2, \theta(k_2))^{-1} = h(\pi)^n (1, -1);$$

or equivalently, such that

$$(k_1,\theta(k_1))h(\pi)^n = h(\pi)^n(1,-1)(k_2,\theta(k_2)). \tag{3.2}$$

Recall from (ch. 1. ref) that $\phi(n) \in \{\pm 1\}$ is defined by

$$h(\pi)^n = (\alpha_0^n, \phi(n)),$$

and write $k_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $k_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. In terms of these matrix entries, the equality (3.2) can be rewritten as

$$\left(\begin{pmatrix} a\pi^{-n} & b\pi^{n} \\ c\pi^{-n} & d\pi^{n} \end{pmatrix}, \theta(k_1)\cdot\phi(n)\cdot\sigma(k_1,\alpha_0^n)\right) = \left(\begin{pmatrix} e\pi^{-n} & f\pi^{-n} \\ g\pi^{n} & h\pi^{n} \end{pmatrix}, \theta(k_2)\cdot(-\phi(n))\cdot\sigma(\alpha_0^n,k_2)\right), \tag{3.3}$$

where θ is the map $G \to \{\pm 1\}$ defined in (1.8). Equality of the $SL_2(F)$-parts implies that $a = e$, $b\pi^{2n} = f$, $c = g\pi^{2n}$, and $d = h$. Applying the formula for θ, we have

$$\theta(k_1) = \begin{cases} (g\pi^{2n}, d)_F = (g,d)_F & \text{if } 0 < |g\pi^{2n}|_F < 1, \\ 1 & \text{otherwise.} \end{cases} \tag{3.4}$$

Since $n \neq 0$ and $g \in \mathcal{O}_F$, $g\pi^{2n}$ is never a unit, so the first case of (3.4) occurs if and only if $g = 0$. Applying θ to k_2, we get

$$\theta(k_2) = \begin{cases} (g,d)_F & \text{if } 0 < |g|_F < 1, \\ 1 & \text{otherwise .} \end{cases} \tag{3.5}$$

It is now clear that $\theta(k_1) = \theta(k_2)$ whenever $g \notin \mathcal{O}_F^\times$.

In fact, $\theta(k_1) = \theta(k_2)$ when $g \in \mathcal{O}_F^\times$ as well; we prove this now. Suppose that $g \in \mathcal{O}_F^\times$. As c is in $\mathcal{O}_F$, $g = c\pi^{-2n} \notin \mathcal{O}_F^\times$ whenever $n < 0$, so we may assume that $n > 0$. Then

$v(c) = v(g\pi^{2n}) = 2n > 0$, and considering $\det(k_1) = ad - bc = ad - bg\pi^{2n} = 1$, we have

$$0 = v(ad - bg\pi^{2n})$$

if and only if $v(ad) = 0$. Since a, d are both in $\mathcal{O}_F$, $v(ad) = 0$ if and only if a, d are both in $\mathcal{O}_F^\times$. Then, since the Hilbert symbol on F is unramified and $g, d \in \mathcal{O}_F^\times$, we have

$$\theta(k_1) = (g, d)_F = 1,$$

and $\theta(k_2) = 1$ by definition. Thus $\theta(k_1) = \theta(k_2)$ for all values of $g \in \mathcal{O}_F$.

Next we show that the values of the cocycle σ agree on the two sides of (3.3). On the right-hand side, we have

$$\sigma(\alpha_0^n, k_2) = \begin{cases} (\pi^n, g)_F = (\pi^n, c\pi^{-2n})_F = (\pi^n, c)_F & \text{if } c \neq 0, \\ (\pi^n, h)_F = (\pi^n, d)_F & \text{if } c = 0, \end{cases} \tag{3.6}$$

which is exactly the value of $\sigma(k_1, \alpha_0^n)$ on the left-hand side.

If the two sides of the equation (3.3) have equal projections to G, then their projections to $\{\pm 1\}$ are, respectively,

$$\theta(k_1) \cdot \phi(n) \cdot \sigma(k_1, \alpha_0^n) \tag{3.7}$$

and

$$\theta(k_2) \cdot (-\phi(n)) \cdot \sigma(\alpha_0^n, k_2). \tag{3.8}$$

Since $\theta(k_1) = \theta(k_2)$ and $\sigma(k_1, \alpha_0^n) = \sigma(\alpha_0^n, k_2)$, we have

$$\theta(k_2) \cdot (-\phi(n)) \cdot \sigma(\alpha_0^n, k_2) = -\theta(k_1) \cdot \phi(n) \cdot \sigma(k_1, \alpha_0^n),$$

and so (3.7) $\neq$ (3.8). Hence there do not exist k_1, k_2 which satisfy the equation (3.3), implying that $h(\pi)^n(1, -1) \notin K^* h(\pi)^n K^*$. We conclude that $K^* h(\pi)^n (1, -1) K^*$ and $K^* h(\pi)^n K^*$ are disjoint.

The statement of Proposition 3.3.1 now follows from Lemma 3.3.3 and the Cartan decomposition (3.1).

□

3.3.2 Vector space bases for $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ and $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$

In this section, we use Proposition 3.3.1 and the Cartan decomposition of $\widetilde{G}$ to give an $\bar{\mathbb{F}}_p$-vector space basis for $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$, and then apply the Frobenius reciprocity map to get a $\bar{\mathbb{F}}_p$-vector space basis for $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$. The result is:

Lemma 3.3.4. *1. For $n \geq 0$, let*

$$t_n = \frac{1}{2}\left(\mathbf{1}_{K^*h(\pi)^nK^*} - \mathbf{1}_{K^*h(\pi)^n(1,-1)K^*}\right).$$

Then $\{t_n\}_{n\geq 0}$ is an $\bar{\mathbb{F}}_p$-vector space basis for $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$.

2. For $n \geq 0$, let T_n be the element of $End_{\widetilde{G}}(ind_{K^*}^{\widetilde{G}}\mathbf{1}_{K^*})$ which is determined by

$$T_n(\mathbf{1}_{K^*}) = t_n.$$

Then $\{T_n\}_{n\geq 0}$ is a $\bar{\mathbb{F}}_p$-vector space basis for $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$.

Proof. 1. While proving Proposition 3.3.1, we showed in particular that $K^*h(\pi)^nK^* \neq K^*h(\pi)^n(1,-1)K^*$ for all $n \geq 0$. Then by the last statement of Lemma 3.3.3, we have

$$\mathbf{1}_{K^*h(\pi)^nK^*} - \mathbf{1}_{K^*h(\pi)^n(1,-1)K^*} \in \mathbb{H}_p^\epsilon(\widetilde{G}, K^*),$$

so also $t_n \in \mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$. Since the proof of Proposition 3.3.1 shows that all of the unions are disjoint in the Cartan decomposition (3.1), we have

$$\widetilde{G} = \coprod_{\substack{n\geq 0 \\ \zeta\in\{\pm 1\}}} K^*h(\pi)^n(1,\zeta)K^*.$$

The disjointness of the union implies that the set $\{t_n\}_{n\geq 0}$ is linearly independent over $\bar{\mathbb{F}}_p$, and its exhaustion of $\widetilde{G}$ implies that every $g \in \widetilde{G}$ is contained in

$$K^* h(\pi)^n K^* \amalg K^* h(\pi)^n (1,-1) K^*$$

for some $n \geq 0$. By K^*-biinvariance, an arbitrary function $f \in \mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ is determined by its values on the set of representatives $\{h(\pi)^n,\ h(\pi)^n(1,-1)\}_{n\geq 0}$ of $K^* \setminus \widetilde{G}/K^*$, and since f is genuine it is in fact determined by its values on $\{h(\pi)^n\}_{n\geq 0}$. Hence f can be written as a linear combination

$$f = \sum_{n\geq 0} a_n \frac{1}{2} \left(\mathbf{1}_{K^* h(\pi)^n K^*} - \mathbf{1}_{K^* h(\pi)^n (1,-1) K^*}\right),$$

where $a_n = 0$ for almost all n since f is compactly supported. Thus the linearly independent set $\{t_n\}_{n\geq 0}$ also generates $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ as an $\bar{\mathbb{F}}_p$-vector space.

2. We can obtain a basis for $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ as a vector space over $\bar{\mathbb{F}}_p$ by applying compact Frobenius reciprocity to the basis $\{t_n\}_{n\geq 0}$ for $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$. Denote the image of t_n in $\mathcal{H}_p^\epsilon(\widetilde{SL}_2(F), K^*)$ by T_n. Using the explicit description of Frobenius reciprocity from (2.2.2), we see that $T_n \in \mathrm{End}_{\widetilde{G}} \mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*}$ sends $f \in \mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*}$ to

$$\left(g' \mapsto \sum_{g \in \mathcal{S}'} t_n\left(g' \cdot g^{-1}\right) f(g)\right) \in \mathrm{ind}_{K^*}^{\widetilde{SL}_2} \mathbf{1}_{K^*}, \tag{3.9}$$

where $\mathcal{S}'$ is any set of left coset representatives for K^* in $\widetilde{G}$.

Given any such set S', the set of characteristic functions $\{\mathbf{1}_{K^* g}\}_{g \in \mathcal{S}'}$ forms a basis for $\mathrm{ind}_{K^*}^{\widetilde{SL}_2} \mathbf{1}_{K^*}$ as a vector space. Thus the characteristic function $\mathbf{1}_{K^*}$ generates $\mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*}$ under the right-translation action of $\widetilde{G}$. Since $T_n \in \mathrm{End}_{\widetilde{G}}(\mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*})$ and in particular is $\widetilde{G}$-equivariant, T_n is determined by its value on $\mathbf{1}_{K^*}$. Applying the formula (3.9) with $f = \mathbf{1}_{K^*}$,

$$T_n(\mathbf{1}_{K^*}) = \left(g' \mapsto \sum_{g \in \mathcal{S}'} t_n\left(g' \cdot g^{-1}\right) \mathbf{1}_{K^*}(g)\right)$$

$$= (g' \mapsto t_n (g')) = t_n.$$

Hence a vector space basis for $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ is given by $\{T_n\}_{n\geq 0}$ where T_n is determined by $T_n(\mathbf{1}_{K^*}) = t_n$.

□

One could directly compute relations between the vector space generators t_n (respectively, T_n) to find a presentation for $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ (respectively, for $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$) as an algebra. However, it will be easier and more illuminating to describe of the genuine spherical Hecke algebra of $\widetilde{G}$ in terms of that of the torus $\widetilde{T}$. In the next section, we will explicitly describe $\mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$ and its antidominant submonoid $\mathbb{H}_p^{\epsilon,+}(\widetilde{T}, \widetilde{T} \cap K^*)$, and in §3.4 we prove that $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ is isomorphic to $\mathbb{H}_p^{\epsilon,-}(\widetilde{G}, K^*)$.

3.3.3 The genuine spherical Hecke algebra of $\widetilde{T}$

In this section we check that that the genuine spherical Hecke algebra of the torus $\widetilde{T}$ is a polynomial algebra in one variable, and we give a concrete description algebra generator:

Lemma 3.3.5. *The function*

$$\tau_1 := \frac{1}{2}\left(\mathbf{1}_{K^*h(\pi)K^*} - \mathbf{1}_{K^*h(\pi)(1,-1)K^*}\right)$$

belongs to $\mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$*, and in fact we have an isomorphism of* $\bar{\mathbb{F}}_p$*-algebras*

$$\mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*) \cong \bar{\mathbb{F}}_p[\tau_1^{\pm 1}].$$

We have a similarly concrete presentation for the antidominant submonoid $\mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$:

Lemma 3.3.6. *Let* τ_1 *be the function defined in Lemma 3.3.5. Then* τ_1 *also belongs to* $\mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$*, and we have an isomorphism of* $\bar{\mathbb{F}}_p$*-algebras*

$$\mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*) \cong \bar{\mathbb{F}}_p[\tau_1].$$

Proof of Lemma 3.3.5. We have the decomposition

$$\widetilde{T} = \coprod_{\substack{n\in\mathbb{Z}\\ \zeta\in\{\pm 1\}}} (\widetilde{T}\cap K^*)h(\pi)^n(1,\zeta),$$

so a vector space basis for $\mathbb{H}_p^{\epsilon}(\widetilde{T}, \widetilde{T}\cap K^*)$ is given by

$$\left\{\tau_n := \frac{1}{2}\left(\mathbf{1}_{(\widetilde{T}\cap K^*)h(\pi)^n} - \mathbf{1}_{(\widetilde{T}\cap K^*)h(\pi)^n(1,-1)}\right)\right\}_{n\in\mathbb{Z}}.$$

For $a \in \mathcal{O}_F^\times$, $n \in \mathbb{Z}$, we have the formula (cf. §1.2)

$$\begin{aligned} h(a)h(\pi)^n &= \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} \pi^{-n} & 0 \\ 0 & \pi^n \end{pmatrix}, \phi(n)\right) \\ &= \left(\begin{pmatrix} a\pi^{-n} & 0 \\ 0 & a^{-1}\pi^n \end{pmatrix}, (a,\pi^n)_F\phi(n)\right). \end{aligned}$$

Hence for $k \in \mathbb{Z}$, $a \in \mathcal{O}_F^\times$, $\zeta \in \{\pm 1\}$,

$$\tau_n\left(\begin{pmatrix} a\pi^{-k} & 0 \\ 0 & a^{-1}\pi^k \end{pmatrix}, \zeta\right) = \begin{cases} \frac{1}{2} & \text{if } k = n \text{ and } \zeta = \phi(n)(a,\pi^n)_F \\ -\frac{1}{2} & \text{if } k = n \text{ and } \zeta = -\phi(n)(a,\pi^n)_F \\ 0 & \text{if } k \neq n. \end{cases} \tag{3.10}$$

For each pair n, $m \geq 0$, the convolution $\tau_n * \tau_m$ is genuine and $\widetilde{T}\cap K^*$-invariant, so is determined by its values on $\{h(\pi)^k\}_{k\in\mathbb{Z}}$. To find relations among the elements of the vector space basis $\{\tau_n\}_{n\in\mathbb{Z}}$, we compute the value of a convolution $\tau_n * \tau_m$ on an arbitrary $\widetilde{T}\cap K^*$-

coset representative $h(\pi)^k$.

$$(\tau_n * \tau_m)(h(\pi)^k) = \sum_{t \in \widetilde{T}/(\widetilde{T}\cap K^*)} \tau_n(h(\pi)^k t)\, \tau_m(t^{-1})$$
$$= \sum_{j\in\mathbb{Z},\, \zeta\in\{\pm 1\}} \tau_n(h(\pi)^k h(\pi)^j (1,\zeta))\, \tau_m((h(\pi)^{-j}(1,\zeta)).$$

The summand $\tau_n(h(\pi)^k h(\pi)^j(1,\zeta))\,\tau_m((h(\pi)^{-j}(1,\zeta))$ is nonzero only if $k+j = n$ and $m = -j$, so $(\tau_n * \tau_m)(h(\pi)^k) = 0$ unless $k = n + m$.

When $k = n+m$, we are left with the summands indexed by $j = n$, $\zeta \in \{\pm 1\}$:

$$\begin{aligned}
(\tau_n * \tau_m)(h(\pi)^{n+m}) &= \sum_{\zeta\in\{\pm 1\}} \tau_n(h(\pi)^n(1,\zeta))\, \tau_m((h(\pi)^m(1,\zeta)) \\
&= \frac{1}{4}\left(\tau_n(h(\pi)^n)\tau_m(h(\pi)^m) + \tau_n(h(\pi)^n(1,-1))\tau_m(h(\pi)^m(1,-1))\right) \\
&= \frac{1}{4}\left(1^2 + (-1)^2\right) \\
&= \frac{1}{2}.
\end{aligned}$$

So, since $\tau_n * \tau_m$ is genuine and $\widetilde{T} \cap K^*$-invariant, we have for all $t \in \widetilde{T}$,

$$\begin{aligned}
\tau_n * \tau_m(t) &= \begin{cases} \frac{1}{2}\zeta & \text{if } t \in K^* h(\pi)^{n+m}(1,\zeta)K^* \\ 0 & \text{otherwise} \end{cases} \\
&= \tau_{n+m}(t).
\end{aligned}$$

In particular,

$$\tau_n * \tau_0 = \tau_0 * \tau_n = \tau_n$$

for all $n \in \mathbb{Z}$, so $\tau_0 = \frac{1}{2}\left(\mathbf{1}_{K^*} - \mathbf{1}_{K^*(1,-1)}\right)$ is the identity element of the algebra $\mathbb{H}_p^{\epsilon}(\widetilde{T}, \widetilde{T}\cap K^*)$. And

$$\tau_n * \tau_{-n} = \tau_0,$$

so

$$\tau_n^{-1} = \tau_{-n}$$

for each $n \in \mathbb{Z}$. Hence for each $n \in \mathbb{Z}$ we have

$$\tau_n = \tau_1^n, \tag{3.11}$$

and so

$$\mathbb{H}_p^{\epsilon}(\widetilde{T}, \widetilde{T} \cap K^*) \cong \bar{\mathbb{F}}_p[\tau_1, \tau_{-1}] = \bar{\mathbb{F}}_p[\tau_1^{\pm 1}], \tag{3.12}$$

as desired. $\square$

Proof of Lemma 3.3.6. The class of $h(\pi)^n$ modulo $\widetilde{T} \cap K^*$ is identified with the cocharacter $(-n, n)$, which is antidominant if and only if $n \geq 0$. Hence

$$\tau_1 = \frac{1}{2}\left(\mathbf{1}_{K^* h(\pi) K^*} - \mathbf{1}_{K^* h(\pi)(1,-1) K^*}\right)$$

is supported on an antidominant class in $\widetilde{T}/(\widetilde{T} \cap K^*)$. By (3.11), as n runs over the nonnegative integers, the powers

$$\tau_1^n = \tau_n$$

run over the basis elements of $\mathbb{H}_p^{\epsilon}(\widetilde{T}, \widetilde{T} \cap K^*)$ which are supported on antidominant classes. Hence the isomorphism (3.12) given in Lemma 3.3.5 restricts to an isomorphism of $\bar{\mathbb{F}}_p$-algebras

$$\mathbb{H}_p^{\epsilon}(\widetilde{T}, \widetilde{T} \cap K^*) \cong \bar{\mathbb{F}}_p[\tau_1].$$

$\square$

Remark 3.3.7. *The map $\tau_n \mapsto (\alpha^\vee)^n$ is an isomorphism*

$$\mathbb{H}_p^{\epsilon}(\widetilde{T}, \widetilde{T} \cap K^*) \cong \bar{\mathbb{F}}_p[X_*(T)],$$

where T is the diagonal torus of $SL_2(F)$. Restricting the map to $\{\tau_n\}_{n \geq 0}$, we also get an *isomorphism of* $\mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$ *with the submonoid of* $\mathbb{F}[X_*(T)]$ *generated by antidominant cocharacters:*

$$\mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*) \cong \mathbb{F}[X_*^-(T)].$$

3.4 Isomorphisms of spherical mod p Hecke algebras

In this section, we define a Satake transform $\mathcal{S} : \mathbb{H}_p^\epsilon(\widetilde{G}, K^*) \to \mathbb{H}^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$ and prove that it is an isomorphism of $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ with the antidominant submonoid $\mathbb{H}^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$ of $\mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$.

3.4.1 The mod p Satake isomorphism for $\widetilde{SL}_2(F)$

Before stating the theorem, we recall from §1.2 that the extension defining $\widetilde{G}$ is canonically split over the unipotent subgroup U, and that a preferred section $g \mapsto (g, \theta(t))$ was chosen so that the extension splits over the maximal compact subgroup K of G. Furthermore, these two splittings are compatible on the intersection $(U \cap K)$. We let U^* (respectively, K^*) denote the image of U (resp., K) in $\widetilde{G}$ under the canonical (resp., preferred) section, and define $(U \cap K)^*$ to be the image of $U \cap K$ in $\widetilde{G}$ under either one of the two sections. In particular, $(U \cap K)^* = U^* \cap K^*$.

Theorem 3.4.1. *Define a map*

$$\mathcal{S} : \mathbb{H}_p^\epsilon(\widetilde{G}, K^*) \to \mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$$

by

$$f \mapsto \left(t \mapsto \sum_{u \in U^*/(U^* \cap K^*)} f(tu) \right).$$

Then $\mathcal{S}$ *is injective and gives an algebra isomorphism*

$$\mathcal{S} : \mathbb{H}_p^\epsilon(\widetilde{G}, K^*) \to \mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*),$$

Remark 3.4.2. *The proof closely follows that of the main theorem of [14], which in the case of the spherical Hecke algebra with respect to the trivial representation of K^* (the only case we consider here) also follows the classical proof in most ways. The vanishing outside the antidominant range is a particularity of the mod p situation.*

When extending Herzig's argument from the case of a reductive group to that of the covering group $\widetilde{G} = \widetilde{SL}_2(F)$, the main additional point is to check that no extraneous signs are introduced by the Satake transform or by convolution products in the genuine spherical Hecke algebras.

Proof of Theorem 3.4.1. The proof is in four steps.

1. We first verify that $\mathcal{S}$ defines a $\bar{\mathbb{F}}_p$-linear map of $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ into $\mathbb{H}_p^\epsilon(\widetilde{T}, \widetilde{T} \cap K^*)$.

   Let $f \in \mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$. By Lemma 3.3.4,

$$f = \sum_{n \geq 0} a_n t_n$$

   where t_n is the vector space basis element defined in Lemma 3.3.4 and $a_n \in \bar{\mathbb{F}}_p$ with $a_n = 0$ for almost all n. Furthermore, since K^* is a compact open subgroup of $\widetilde{G}$, each K^*-double coset $K^* h(\pi)^n (1, \zeta) K^*$ in $\widetilde{G}$ is a finite union of left K^*-cosets. Thus f is supported on a finite number of representatives of $\widetilde{G}/K^*$. By the Iwasawa decomposition

$$\widetilde{G} = \widetilde{B} K^*,$$

   we can choose representatives of $\widetilde{G}/K^*$ to lie in $\widetilde{B}/(\widetilde{B} \cap K^*)$, so $f(b) = 0$ for all but finitely many representatives b of $\widetilde{B}/(\widetilde{B} \cap K^*)$. We also have the decomposition

$$\widetilde{B} = \widetilde{T} U^*,$$

   so $f(tu) = 0$ for all but finitely many representatives t of $\widetilde{T}/(\widetilde{T} \cap K^*)$ when u runs over $U^*/(U \cap K)^*$. Furthermore, when $t \in \widetilde{T}$ is fixed, we have $f(tu) = 0$ for all but finitely many representatives u of $U^*/(U \cap K)^*$. Hence the support of $\mathcal{S}(f)$ is finite in

$\widetilde{T}/\widetilde{T}(\mathcal{O}_F)$, and if a representative $t \in \widetilde{T}/(\widetilde{T} \cap K)^*$ is in the support of $\mathcal{S}$, then $\mathcal{S}(f)(t)$ is given by a finite sum of values $f(tu) \in \bar{\mathbb{F}}_p$. Thus $\mathcal{S}(f) \in \mathbb{H}_p(\widetilde{T}, \widetilde{T} \cap K^*)$, and $\mathcal{S}$ defines a map of $\bar{\mathbb{F}}_p$-vector spaces.

Since f is a genuine function on $\widetilde{G}$, we have

$$\mathcal{S}(f)(t(1,\zeta)) = \sum_{u \in U^*/(U \cap K)^*} f(t(1,\zeta)u) = \sum_{u \in U^*/(U \cap K)^*} \zeta f(tu) = \zeta \mathcal{S}(f)(t),$$

so $\mathcal{S}(f)$ is in the genuine subalgebra $\mathbb{H}_p^{\epsilon}(\widetilde{T}, \widetilde{T} \cap K^*)$ of $\mathbb{H}_p(\widetilde{T}, \widetilde{T} \cap K^*)$.

2. Next we check that $\mathcal{S}$ is a homomorphism of algebras.

   Let $f_1,\ f_2 \in \mathbb{H}_p^{\epsilon}(\widetilde{G}, K^*)$ and $t \in \widetilde{T}$. Then $[\mathcal{S}(f_1) * \mathcal{S}(f_2)](t) =$

$$\begin{aligned} &= \sum_{t' \in \widetilde{T}/(\widetilde{T} \cap K^*)} \left[\left(\sum_{u' \in U^*/(U^* \cap K^*)} f_1(tt'u') \right) \left(\sum_{u \in U^*/(U^* \cap K^*)} f_2(t'^{-1}u) \right) \right] \\ &= \sum_{t' \in \widetilde{T}/(\widetilde{T} \cap K^*)} \sum_{u' \in U^*/(U^* \cap K^*)} \sum_{u \in U^*/(U^* \cap K^*)} f_1(tt'u') f_2(t'^{-1}u). \end{aligned}$$

   On the other hand,

$$\mathcal{S}(f_1 * f_2)(t) = \sum_{u \in U^*/(U \cap K)^*} \sum_{\tilde{G}/K^*} f_1(tug) f_2(g^{-1}).$$

   Since $\widetilde{G} = \widetilde{B}K$ we can choose representatives for $\widetilde{G}/K^*$ in

$$\widetilde{B}/(\widetilde{B} \cap K^*) = (\widetilde{T}/(\widetilde{T} \cap K^*))(U^*/(U^* \cap K^*)).$$

Thus $\mathcal{S}(f_1 * f_2)(t)v =$

$$\begin{aligned}
&= \sum_{u \in U^*/(U^* \cap K^*)} \sum_{b \in \widetilde{B}/(\widetilde{B} \cap K^*)} f_1(tub) f(b^{-1}) \\
&= \sum_{u \in U^*/(U^* \cap K^*)} \sum_{t' \in \widetilde{T}/(\widetilde{T} \cap K^*)} \sum_{u' \in U^*/(U^* \cap K^*)} f_1(tut'u') f_2(u'^{-1} t'^{-1})
\end{aligned}$$

As $\widetilde{T}$ normalizes U^*, we have $t'^{-1}ut'u' \in U^*$, so can substitute $u' \mapsto t'^{-1}ut'u'$. Then $u'^{-1}t'^{-1} = [(t'^{-1}ut')u']^{-1}t'^{-1}u \mapsto u'^{-1}t'^{-1}u$, and we can replace the above sum with the following:

$$= \sum_{t' \in \widetilde{T}/(\widetilde{T} \cap K^*)} \sum_{u' \in U^*/(U^* \cap K^*)} \sum_{u \in U^*/(U^* \cap K^*)} f_1(tt'u') f_2(u'^{-1} t'^{-1} u).$$

And finally, substitute $t'u'^{-1}t'^{-1}u \mapsto u$:

$$= \sum_{t' \in \widetilde{T}/(\widetilde{T} \cap K^*)} \sum_{u' \in U^*/(U^* \cap K^*)} \sum_{u \in U^*/(U^* \cap K^*)} f_1(tt'u') f_2(t'^{-1} u),$$

which is equal to $[\mathcal{S}(f_1) * \mathcal{S}(f_2)](t)$, so $\mathcal{S}$ is a homomorphism.

3. The next step is to compute the transforms of the $\bar{\mathbb{F}}_p$-basis elements $\{t_n\}_{n \geq 0}$ of $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$.

   Let $n \geq 0$, and let t_n be the vector space basis element of $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ defined in Lemma 3.3.4. The transform $\mathcal{S}(t_n)$ is a genuine $\widetilde{T} \cap K^*$-invariant function on $\widetilde{T}$, so it is determined by its values on $\{h(\pi)^m\}_{m \in \mathbb{Z}}$. We calculate

$$\begin{aligned}
\mathcal{S}(t_n)(h(\pi)^m) &= \sum_{u \in U^*/(U \cap K)^*} t_n(h(\pi)^m u) \\
&= \frac{1}{2} \sum_{u \in U^*/(U \cap K)^*} \left(\mathbf{1}_{K^* h(\pi)^n K^*}(h(\pi)^m u) - \mathbf{1}_{K^* h(\pi)^n (1,-1) K^*}(h(\pi)^m u) \right).
\end{aligned}$$

Now we need to know how the value of

$$\mathbf{1}_{K^*h(\pi)^nK^*} - \mathbf{1}_{K^*h(\pi)^n(1,-1)K^*}$$

on $h(\pi)^m u$ varies with u. In fact, the value is either always nonnegative or always nonpositive as u runs over U^*, and we can calculate it:

Lemma 3.4.3. *Let $n \geq 0$, $m \in \mathbb{Z}$. If $u \in U^*$ and*

$$h(\pi)^m u \in K^*h(\pi)^nK^* \amalg K^*h(\pi)^n(1,-1)K^*,$$

then

$$\left(\mathbf{1}_{K^*h(\pi)^nK^*}(h(\pi)^m u) - \mathbf{1}_{K^*h(\pi)^n(1,-1)K^*}(h(\pi)^m u)\right) = \phi(n)\phi(m).$$

Proof of Lemma 3.4.3. Recall from §1.2 that the projection of u to $\{\pm 1\}$ is equal to 1 for all $u \in U^*$, while the projection of u to G is equal to $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for some $x \in F$. So, given $u \in U^*$, we can write the product $h(\pi)^m u$ as

$$\begin{aligned}
&\left(\begin{pmatrix} \pi^{-m} & 0 \\ 0 & \pi^m \end{pmatrix}, \phi(m)\right)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) \\
=& \left(\begin{pmatrix} \pi^{-m} & x\pi^{-m} \\ 0 & \pi^m \end{pmatrix}, \phi(m)\,\sigma\left(\begin{pmatrix} \pi^{-m} & 0 \\ 0 & \pi^m \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\right) \\
=& \left(\begin{pmatrix} \pi^{-m} & x\pi^{m} \\ 0 & \pi^m \end{pmatrix}, \phi(m)(1,\pi^m)_F\right) \\
=& \left(\begin{pmatrix} \pi^{-m} & x\pi^{-m} \\ 0 & \pi^m \end{pmatrix}, \phi(m)\right).
\end{aligned}$$

Hence the projection of $h(\pi)^m u$ to $\{\pm 1\}$ is equal to $\phi(m)$ for all $u \in U^*$. Suppose that for some particular pair $u \in U^*$, $\zeta \in \{\pm 1\}$ we have

$$h(\pi)^m u \in K^* h(\pi)^n (1, \zeta) K^*.$$

Then $\mathbf{1}_{K^* h(\pi)^n K^*}(h(\pi)^m u) - \mathbf{1}_{K^* h(\pi)^n (1,-1) K^*}(h(\pi)^m u) = \phi(m)\phi(n)$.

If u' is another element of U^* such that $h(\pi)^m u' \in K^* h(\pi)^n (1, \zeta) K^* \amalg K^* h(\pi)^n (1, -\zeta) K^*$, then $u^{-1} u' \in (U \cap K)^*$, so the K^*- double cosets represented by $h(\pi)^m u$ and $h(\pi)^m u'$ are in fact equal. This proves the lemma.

$\square$

A summand of $\mathcal{S}t_n(h(\pi)^m(1, \zeta)) =$

$$\sum_{u \in U^*/(U^* \cap K^*)} \frac{1}{2} \left(\mathbf{1}_{K^* h(\pi)^n K^*}(h(\pi)^m u) - \mathbf{1}_{K^* h(\pi)^n (1,-1) K^*}(h(\pi)^m u) \right)$$

is nonzero if and only if the projection of $h(\pi)^m u$ to $SL_2(F)$ is contained in $\alpha_0^m U \cap K \alpha_0^n K$, so by Lemma 3.4.3, the value of the sum over $U^*/(U^* \cap K^*)$ is equal to

$$\frac{1}{2} \phi(n)\phi(m) |\alpha_0^m U \cap K \alpha_0^n K|. \tag{3.13}$$

When $m > 0$, then by (1.12) of Lemma 1.3.1, we have

$$|\alpha_0^m U \cap K \alpha_0^n K| = \begin{cases} 0 & \text{if } n < m \\ 1 & \text{if } n = m \\ q^{\ell-1}(q-1) & \text{if } n = \ell + m \text{ with } \ell \geq 1. \end{cases}$$

So $\mathcal{S}t_n(h(\pi)^n) \equiv 0 \pmod p$ unless $m = n$ or $m = n-1$. We calculate $\phi(n)^2 = 1$ and $\phi(n)\phi(n-1) = (-1)^{n\frac{q-1}{2}}$. Thus

$$\mathcal{S}t_n(h(\pi)^n) = \frac{1}{2}\phi(n)^2 \equiv \frac{1}{2} \pmod p \tag{3.14}$$

while

$$\mathcal{S}t_n(h(\pi)^{n-1}) = \frac{1}{2}(q-1)\phi(n)\phi(n-1)\frac{1}{2}(-1)^{n\frac{q-1}{2}} \equiv \frac{1}{2}(-1)^{1+n\frac{q-1}{2}} \pmod p. \quad (3.15)$$

This completes the calculation for $m \geq 0$.

Now we show that $\mathcal{S}t_n(h(\pi)^m)$ vanishes (mod p) for all $m < 0$. Note that

$$\left(\begin{pmatrix} \pi^{-m} & 0 \\ 0 & \pi^m \end{pmatrix}, \phi(m)\right)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) =$$
$$\left(\begin{pmatrix} 1 & x\pi^{-2m} \\ 0 & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} \pi^{-m} & 0 \\ 0 & \pi^m \end{pmatrix}, \phi(m)\right),$$

and when $m < 0$, the U^*-factor on the left is in $U \cap K^*$ whenever $v(x) \geq 2m$. Then, since t_n is $U \cap K^*$-invariant, the transform $\mathcal{S}t_n(h(\pi)^m)$ is equal to $[U(\pi\mathcal{O}_F) : U(\pi^{-2m}\mathcal{O}_F)] = q^{-2m-1}$ times a sum over $U^*/(U \cap K)^*$:

$$\begin{aligned}
\sum_{u \in U^*/(U\cap K)^*} t_n\left(h(\pi)^m u\right) &= \sum_{u \in U^*/h(\pi)^m(U\cap K^*)h(\pi)} t_n\left(uh(\pi)^m\right) \\
&= [U(\pi\mathcal{O}_F) : U(\pi^{-2m})] \sum_{u' \in U^*/(U\cap K)^*} t_n(u'h(\pi)^m) \\
&= q^{-2m-1} \sum_{u' \in U^*/(U\cap K)^*} t_n(u'h(\pi)^m) \\
&\equiv 0 \pmod p \text{ for all } m < 0.
\end{aligned}$$

Combining the vanishing on $\{h(\pi)^m\}_{m<0}$ with the values (3.14) and (3.15) gives the following formula:

$$\mathcal{S}t_n = \begin{cases} \tau_0 & \text{if } n = 0 \\ \tau_n + (-1)^{1+n\frac{q-1}{2}}\,\tau_{n-1} & \text{if } n > 0. \end{cases} \quad (3.16)$$

In particular,

$$\mathcal{S}t_1 = \begin{cases} \tau_1 - \tau_0 & \text{if } q \equiv 1 \pmod 4 \\ \tau_1 + \tau_0 & \text{if } q \equiv 3 \pmod 4). \end{cases}$$

4. Finally, we use the results of step (3) to show that the image of $\mathcal{S}$ is equal to $\mathbb{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$, and that it is an injective map.

   Every $f \in \mathbb{H}_p^{\epsilon}(\widetilde{G}, K^*)$ is a linear combination of the basis elements $\{t_n\}_{n\geq 0}$, and we showed in step (2) that $\mathcal{S}$ is a homomorphism, so the vanishing of $\mathcal{S}(t_n)$ on $h(\pi)^m$ for $m < 0$ is enough to show that the image of $\mathcal{S}$ is contained in $\mathbb{H}^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$. And the algebra generator $\tau_1 \in \mathbb{H}^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$ is in the image of $\mathcal{S}$, in particular equal to

$$\begin{cases} \mathcal{S}(t_1 + t_0) = \tau_1 - \tau_0 + \tau_0 & \text{if } q \equiv 1 \pmod 4, \\ \mathcal{S}(t_1 - t_0) = \tau_1 + \tau_0 - \tau_0 & \text{if } q \equiv 3 \pmod 4, \end{cases}$$

   so $\mathcal{S}$ is onto $\mathbb{H}^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$.

   Suppose $\mathcal{S}(f) = 0$ for some $f = \sum c_n t_n \in \mathbb{H}_p^{\epsilon}(\widetilde{G}, K^*)$, where $n \geq 0$ and $c_n = 0$ for almost all n. Then

$$0 = \sum c_n(\mathcal{S}t_n) = \begin{cases} c_0\tau_0 + \sum_{n\geq 1} c_n(\tau_n - \tau_{n-1}) & \text{if } q \equiv 1 \pmod 4 \\ c_0\tau_0 + \sum_{n\geq 1} c_n(\tau_n + (-1)^{\psi(n)}\tau_{n-1}) & \text{if } q \equiv 3 \pmod 4). \end{cases}$$

   As $\{\tau_n\}_{n\geq 0}$ is linearly independent over $\bar{\mathbb{F}}_p$, so are

$$\{\tau_n - \tau_{n-1}, \tau_0\}_{n\geq 1} \text{ and } \{\tau_n + (-1)^{n+1}\tau_{n-1}, \tau_0\}_{n\geq 1},$$

   so we have $c_n = 0$ for all n. Hence $f = 0$, showing that $\mathcal{S}$ is injective.

   This concludes the proof of Theorem 3.4.1.

$\square$

As a consequence, we find the following generator for $\mathbb{H}_p^\epsilon(\widetilde{G}, K^*)$ as a polynomial algebra in one variable over $\bar{\mathbb{F}}_p$:

$$\begin{cases} t_1 + t_0 & \text{if } q \equiv 1 \pmod 4 \\ t_1 - t_0 & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

3.4.2 Action of $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ on spherical vectors

We work out the action of $\mathcal{H}^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*) \cong \mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ on a spherical vector in an unramified genuine ordinary representation. The result is the following:

Lemma 3.4.4. *Fix a nontrivial additive character ψ of F, and let χ be a smooth unramified character of $F^\times$. Let $\tilde{\chi}$ be the genuine character of $\widetilde{T}$ defined with respect to ψ in §2.3.3, and let $I(\tilde{\chi})$ be the unramified ordinary representation induced from $\tilde{\chi}$. Then the K^*-fixed subspace $I(\tilde{\chi})^{K^*}$ is isomorphic to $\tilde{\chi}^{-1} = (\chi \cdot \gamma_\psi)^{-1}$ as a right $\mathcal{H}^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$ module.*

Before giving the proof of Lemma 3.4.4, we briefly explain how the genuine spherical Hecke algebra acts on the spherical vectors of a general smooth representation of $\widetilde{G}$. Let (π, V) be a smooth representation of $\widetilde{G}$ such that $V^{K^*} \neq 0$. Then V^{K^*} is a $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$-module:

$$V^{K^*} \cong \mathrm{Hom}_{K^*}(\mathbf{1}_{K^*}, \pi) \cong \mathrm{Hom}_{\widetilde{G}}(\mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*}, \pi)$$

where the second isomorphism is Frobenius reciprocity for compact induction. The image of $v \in V^{K^*}$ in $\mathrm{Hom}_{\widetilde{G}}(\mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*}, \pi)$ is the map

$$\Phi_v : \mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*} \to \pi$$

which sends $\mathbf{1}_{K^*} \mapsto v$, and hence (by $\widetilde{G}$-equivariance) sends

$$\mathbf{1}_{K^* g^{-1}} \mapsto \pi(g) \cdot v.$$

The effect of an element $T \in \mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ is to precompose Φ_v with the image of T in

$\mathcal{H}_p^\epsilon(\widetilde{G}, K^*) \cong \mathrm{End}_{\widetilde{G}}(\mathrm{ind}_{K^*}^{\widetilde{G}} \mathbf{1}_{K^*})$:

$$(\Phi_v \cdot T)(\mathbf{1}_{K^*}) = \Phi_v(T(\mathbf{1}_{K^*})).$$

Hence V^{K^*} is a right $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$-module.

Proof of Lemma 3.4.4. Let χ be an unramified character of $F^\times$, and let $I(\tilde{\chi})$ be the induced ordinary representation. Then, as shown in Theorem B, $I(\tilde{\chi})^{K^*}$ is 1-dimensional. Since $\widetilde{G} = U^*\widetilde{T}K^*$ and $I(\chi)$ is trivial on U^*, we have an isomorphism of $\bar{\mathbb{F}}_p[\widetilde{T}]$-modules

$$I(\tilde{\chi})^{K^*} \cong I(\chi)^{\widetilde{T}\cap K^*}$$

given by $f \mapsto f((1,1))$, where f is any nontrivial function in $I(\tilde{\chi})^{K^*}$.

Fix $v = f((1,1)) \neq 0 \in I(\tilde{\chi})^{K^*}$. Then

$$(v \cdot \tau_1)(\mathbf{1}_{\widetilde{T}\cap K^*}) = (v \cdot \tau_1)(\mathbf{1}_{\widetilde{T}\cap K^*}) \tag{3.17}$$

$$\begin{aligned}
&= \frac{1}{2}\Phi_v\left(\mathbf{1}_{\widetilde{T}\cap K^* h(\pi)} - \mathbf{1}_{\widetilde{T}\cap K^* h(\pi)(1,-1)}\right) \\
&= \frac{1}{2}\left(\tilde{\chi}(h(\pi)^{-1}) \cdot v - \tilde{\chi}(h(\pi)^{-1}(1,-1)) \cdot v\right) \\
&= \frac{1}{2}\left(\tilde{\chi}(h(\pi)^{-1}) \cdot v + \tilde{\chi}(h(\pi)^{-1}) \cdot v\right) = \tilde{\chi}^{-1}(h(\pi)) \cdot v \\
&= \chi(\pi) \cdot \gamma_\psi(\pi) v,
\end{aligned}$$

where ψ is the additive character of F with respect to which the genuine characters of $\widetilde{T}$ are defined. So for fixed ψ, $I(\tilde{\chi})^{\widetilde{T}\cap K^*} \cong (\chi \cdot \gamma_\psi)^{-1}$ as a right $\mathcal{H}_p^{\epsilon,-}(\widetilde{T}, \widetilde{T} \cap K^*)$-module. $\square$

3.4.3 Comparison with the spherical Hecke algebra of $PGL_2(F)$

The spherical Hecke algebra of $PGL_2(F)$ with respect to the trivial representation was shown by Barthel and Livné to be a polynomial algebra in one variable. Recall that $\alpha =$

$\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \in GL_2(F)$.

Proposition 3.4.5 ([3], Prop. 4)**.** *Let t be the element of*

$$\mathcal{H}_p(GL_2(F), K_G Z) \cong End_{GL_2(F)}(ind_{K_G Z}^{GL_2(F)} \mathbf{1})$$

defined by

$$t : \mathbf{1}_{K_G Z} \mapsto \mathbf{1}_{K_G Z \alpha K_G}.$$

Then $\mathcal{H}(GL_2(F), K_G Z) \cong \bar{\mathbb{F}}_p[t]$.

Hence we immediately have:

Proposition 3.4.6. *The genuine spherical Hecke algebra $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*)$ of $\widetilde{G}$ is abstractly isomorphic to the spherical Hecke algebra of $PGL_2(F)$.*

In fact, we can choose an isomorphism which induces a bijection of unramified ordinary representations on each side, except for $I(\tilde{\text{sgn}})$ on the $\widetilde{G}$ side, and which is compatible with the bijection between unramified ordinary representations and characters of the spherical Hecke algebras:

Theorem 3.4.7. *The $\bar{\mathbb{F}}_p$-linear map $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*) \to \mathcal{H}_p(PGL_2(F), K_G)$ defined by $t \mapsto t_1$ is an algebra isomorphism. Furthermore, it induces a bijection (which depends on the additive character ψ) of irreducible unramified ordinary representations on each side, except for $I(\tilde{sgn})$. This bijection is compatible with the correspondence of unramified ordinary representations to characters of the spherical Hecke algebra.*

Remark 3.4.8. *Of course, the bijection can be completed by matching $I(\tilde{sgn})$ on the $\widetilde{G}$ side with the Steinberg representation St on the $PGL_2(F)$ side, and it may be that this can be made natural in another way (e.g., from a matching of Iwahori Hecke module structure). However, as St has no K_G-fixed vectors, we cannot bring it into the correspondence via the spherical Hecke algebra.*

Proof. By the mod p Satake isomorphism for the reductive group $PGL_2(F)$, we have

$$\mathcal{H}_p(PGL_2, K_G) \cong \mathcal{H}_p^-(T_G, T_G(\mathcal{O}_F)) \cong \bar{\mathbb{F}}_p[X_{G,*}^-(T_G)],$$

where $X_{G,*}(T_G)$ is the cocharacter lattice of $(PGL_2(F), X_{G,*})$ and $X_{G,*}^-(T_G)$ is the antidominant submonoid generated by α. The preimage of α in $\mathcal{H}_p(PGL_2, K_G)$ is the element $1+t$, so $1+t$ is a generator of $\mathcal{H}_p(PGL_2, K_G)$ as an algebra. Hence the $\bar{\mathbb{F}}_p$-linear map $\mathcal{H}_p^\epsilon(\widetilde{G}, K^*) \to \mathcal{H}_p(PGL_2(F), K_G)$ defined by $t \mapsto t_1$ is an algebra isomorphism.

Next we construct a bijection between irreducible unramified principal series representations of $\widetilde{G}$ and of $PGL_2(F)$ which are associated to characters χ of $F^\times$ such that $\chi^2 \neq 1$. The bijection is defined via the action of $\mathcal{H}_p(PGL_2(F), K_G)$ on the $\bar{\mathbb{F}}_p$-span of a spherical vector in an unramified principal series representation $I(\chi \otimes \chi^{-1})$ of $PGL_2(F)$, which we now calculate.

Let $\Phi_v \in \mathrm{Hom}_{T_G}(\mathrm{ind}_{T_G(\mathcal{O}_F)}^{T_G} \mathbf{1}_{T_G(\mathcal{O}_F)}, I(\chi \otimes \chi^{-1}))$ be the T_G-equivariant map defined by $\mathbf{1}_{T_G(\mathcal{O}_F)} \mapsto v$. Then

$$(v \cdot t)\left(\mathbf{1}_{T_G(\mathcal{O}_F)}\right) = \Phi_v\left(\mathbf{1}_{T_G(\mathcal{O}_F)\alpha}\right) \tag{3.18}$$

$$\begin{aligned} &= \Phi_v\left(\mathbf{1}_{T_G(\mathcal{O}_F)\alpha}\right) \\ &= \Phi_v\left(\alpha^{-1} \cdot \mathbf{1}_{T_G(\mathcal{O}_F)}\right) \\ &= I(\chi \otimes \chi^{-1})(\alpha^{-1})v \\ &= \chi(\pi)v \end{aligned}$$

so $I(\chi \otimes \chi^{-1})^{K_G} \cong \chi^{-1}$ as a $\mathcal{H}(PGL_2, K_G)$-module.

As $I(\tilde{\chi})$ and $I(\chi \otimes \chi^{-1})$ are both irreducible for $\chi^2 \neq 1$ and $I(\tilde{\chi}) \ncong I(\tilde{\chi'})$, $I(\chi \otimes \chi^{-1}) \ncong I(\chi' \otimes \chi'^{-1})$ if $\chi^2 \neq 1$,$\chi'^2 \neq 1$, and $\chi \neq \chi'$, the map $\mathrm{Rep}_{\widetilde{G}} \to \mathrm{Rep}_{PGL_2(F)}$ which identifies $I(\tilde{\chi})$ with $I(\chi \otimes \chi^{-1})$ is a bijection of unramified principal series representations associated to characters χ of F such that $\chi^2 \neq 1$. Moreover, as the calculations (3.17), (3.18) show, this bijection is compatible with the isomorphism $t \leftrightarrow \tau_1$ of the spherical Hecke algebras (for a fixed choice of ψ).

Finally, we describe the dependence on ψ in the bijection. Let $a \in F^\times/(F^\times)^2$, and define ψ_a to be the character $x \mapsto \psi(ax)$ of F. Then by Property (3) of the Weil index given in §2.3.3, we have

$$\gamma(x, \psi_a) = (x, a)_F \gamma(x, \psi).$$

Let χ_a denote the quadratic character $(-, a)_F$ of $F^\times$. Then if $I(\chi \otimes \chi^{-1})$ corresponds to $I(\tilde{\chi}) = I(\chi \cdot \gamma(-, \psi))$ under the bijection defined in this section with respect to ψ, then $I(\chi \otimes \chi^{-1})$ corresponds to $\chi_a \cdot I(\tilde{\chi})$ under the bijection defined with respect to ψ_a.

$\square$

Remark 3.4.9. *On unramified ordinary representations not associated to sgn, the bijection between unramified principal series representations agrees with that defined by theta correspondence in characteristic 0, including the its dependence on an additive character ψ of F.*

3.4.4 Comments on $\chi^2 = 1$

Recall from §2.4 that $I(1 \otimes 1) \cong I(\mathrm{sgn} \otimes \mathrm{sgn})$ as representations of $PGL_2(F)$, and that this representation is reducible with the trivial representation $\mathbf{1}$ as a subrepresentation and the Steinberg representation St as a quotient. On the other hand, $I(\tilde{1})$ and $I(\widetilde{\mathrm{sgn}})$ are distinct and irreducible representations of $\widetilde{SL}_2(F)$. The K^*-invariants of $I(\tilde{1})$ and of $I(\mathrm{s\tilde{g}n})$ and the K_G-invariants of 1 are all 1-dimensional, and the calculations of their spherical Hecke module structure go through as in (3.4.4) and in (3.18), respectively, for $\chi^2 \neq 1$. Thus the bijection for $\chi^2 \neq 1$ extends naturally to $\chi = 1$, identifying $I(\tilde{1})$ with the trivial representation of $PGL_2(F)$. However, since $(St)^{K_G} = 0$, we cannot expect to identify St with an unramified principal series representation of $\widetilde{G}$ via the action of the spherical Hecke algebras.

On the other hand, $(St)^{I_G}$ is 1-dimensional, so it is a nontrivial right module for the Iwahori Hecke algebra $\mathcal{H}(PGL_2, I_G)$. This is a reason for the comparison of $\mathcal{H}(PGL_2, I_G)$ with the genuine Iwahori Hecke algebra $\mathcal{H}^\epsilon(\widetilde{SL}_2(F), I^*)$ in the next chapter.

Chapter 4

The genuine Iwahori Hecke algebra

4.1 Summary

4.1.1 Abstract of the chapter

The goal of this chapter is to give a presentation for the genuine mod p Iwahori Hecke algebra of $\widetilde{G}$, and then to show that this algebra is not isomorphic to the mod p Iwahori Hecke algebra of $PGL_2(F)$ computed by Barthel and Livné.

This is in contrast to the situation in characteristic 0, where Savin ([22], [23]) has shown that if $\widetilde{H}$ is the n-fold metaplectic cover (with $p \not| n$) of a simply laced Chevalley group H, then the genuine Iwahori Hecke algebra of $\widetilde{H}$ is isomorphic to the Iwahori Hecke algebra of a dual group to H. When $H = SL_2(F)$, the dual group to H is $PGL_2(F)$, and Savin's isomorphism induces an equivalence of categories between genuine representations of $\widetilde{H}$ which are generated by their I^*-fixed vectors, and representations of the dual group which are generated by their Iwahori-fixed vectors.

The motivation for comparing $\mathcal{H}_p^{\epsilon}(\widetilde{G}, I^*)$ to $\mathcal{H}_p(PGL_2(F), I_G)$ was to eventually define a correspondence between mod p representations of $\widetilde{G}$ and of $PGL_2(F)$ in cases which are not addressed by the spherical Hecke algebra isomorphism of Chapter 3 (alternatively, to explain why no natural one should exist in some cases). Though the two algebras are not isomorphic, we relate some elements of $\mathcal{H}_p(PGL_2(F), I_G)$ to elements of $\mathcal{H}_p(PGL_2(F), I_G)$ using the tree of $SL_2(F)$.

4.1.2 Main results

There is quite a lot of notation to set up, most of which is done in the first section. We define a graph Δ in analogy with the tree of $SL_2(F)$, explain how its edges identify with I^*-double cosets in $\widetilde{G}$, and define a $\bar{\mathbb{F}}_p$-vector space basis for $\widetilde{G}$ in bijection with the edges of a "positive half" of Δ.

Next, we review some results of Savin and Iwahori-Matsumoto for use in calculating relations between the vector space basis operators and compute some I^*-double coset identities in $\widetilde{G}$. The results are used to show the main result: for certain operators $x := T^1_{0,-1}$ and $y := T^0_{2,1} \in \mathcal{H}^\epsilon(\widetilde{G}, I^*)$, we have

Proposition 4.1.1. *The following is a complete list of positive powers of* $x := T^1_{0,-1}$ *and* $y := T^0_{2,1}$*: for* $k \geq 1$,

1. $x^k = (-1)^{k-1} T^1_{0,-1} = (-1)^{k-1} x$,
2. $y^k = \begin{cases} T^0_{2,1} & \textit{if } k = 1 \\ 0 & \textit{if } k \geq 2, \end{cases}$
3. $(xy)^k = T^{-k}_{2k,2k+1}$,
4. $(yx)^k = (-1)^{k\frac{q-1}{2}} T^k_{-2k,-2k+1}$,
5. $y(xy)^k = T^{-k}_{2k+2,2k+1}$
6. $x(yx)^k = (-1)^{k\frac{q-1}{2}} T^{k+1}_{-2k,-2k-1}$.

Moreover we can show that these products are linearly independent and span $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ as a $\bar{\mathbb{F}}_p$-algebra. Thus $x = T^1_{0,-1}$ and $y = T^0_{2,1}$ generate $\mathcal{H}^\epsilon(\widetilde{G}, I^*)$, and we show that the algebra has the following presentation as a noncommutative polynomial algebra:

Theorem D (Theorem 4.3.7)**.**

$$\mathcal{H}^\epsilon(\widetilde{G}, I^*) = \bar{\mathbb{F}}_p\langle x,\, y\rangle / (x^2 + x, y^2).$$

We compare this algebra to a known presentation for $\mathcal{H}_p(PGL_2(F), I_G)$:

Proposition 4.1.2 ([3], Prop. 7)**.** *The mod p Iwahori Hecke algebra of $PGL_2(F)$ has the noncommutative presentation*

$$\mathcal{H}(PGL_2, I_G) \cong \bar{\mathbb{F}}_p\langle a, b\rangle/\{a^2 - 1,\, bab + b\}.$$

Comparing the number of $\bar{\mathbb{F}}_p$-characters on each side, we get

Theorem E (Corollary 4.3.8)**.** *$\mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$ is not isomorphic to $\mathcal{H}_p(PGL_2(F), I_G)$. Indeed, their abelianizations are not isomorphic.*

However, using the graph Δ we can identify $x \in \mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$ with $ba \in \mathcal{H}_p(PGL_2(F), I_G)$ and y with ab. The subalgebra of $\mathcal{H}_p(PGL_2(F), I_G)$ generated by ba and ab is the image of the embedding of the Iwahori Hecke algebra $\mathcal{H}_p(G, I)$ of $SL_2(F)$ in $\mathcal{H}_p(PGL_2(F), I_G)$. Hence we interpret the identification of x with ba and y with ab as an identification of $\mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$ with the quotient of $\mathcal{H}_p(G, I)$ by the square of one its two generators, namely the one which maps to ab in when $\mathcal{H}_p(G, I)$ embeds in $\mathcal{H}_p(PGL_2(F), I_G)$.

4.1.3 Savin's isomorphism in characteristic 0

As already mentioned, the motivation for the work of this chapter is a theorem of Savin which recasts the correspondence between certain $\mathbb{C}$-representations of a metaplectic group with certain $\mathbb{C}$-representations of a reductive group as an isomorphism between their Iwahori Hecke algebras. The main result is:

Theorem 4.1.3 (([23], Thm. 7.8)**.** *Let n be an integer and p a prime not dividing n. Let Z_n be the n-torsion subgroup in the center of a simply laced Chevalley group G over a p-adic field F which contains n n-th roots of unity, and let $\widetilde{G}$ be the central extension of G by the n-th rooths of unity. Then the genuine Iwahori Hecke algebra of $\widetilde{G}$ is isomorphic to the Iwahori Hecke algebra of $G_n(F)$, where G_n is the algebraic group isomorphic to G/Z_n.*

By a theorem of Borel, in characteristic 0 the functor of Iwahori-invariant vectors is an equivalence of categories

$$Rep_I(G) \to Mod(H(G,I)),$$

where $Rep_I(G)$ is the category of smooth representations of a reductive group G which are generated by their Iwahori-fixed vectors, and $Mod(H(G,I))$ is the category of right modules over the Iwahori Hecke algebra. Savin notes that the same result holds for a metaplectic group $\widetilde{G}$ when the adjective "genuine" is applied both to the representations and to the Hecke algebra. Hence the isomorphism of Theorem 4.1.3 induces an equivalence of categories between $Rep_I(G_n)$ and $Rep^{\epsilon}_{I^*}(\widetilde{G})$.

Remark 4.1.4. *To our knowledge, there is no analogue of Borel's theorem for mod p representations of metaplectic groups; even if the genuine mod p Iwahori Hecke algebra of $\widetilde{G}$ had been found to be isomorphic to the mod p Iwahori Hecke algebra of G, we would not have been able conclude equivalence of the categories of mod p representations. The relationship between the mod p representations of a metaplectic group and the modules over its Hecke algebras is an interesting point which we hope to explore in future work.*

4.2 A presentation for $\mathcal{H}^{\epsilon}_p(\widetilde{SL}_2(F), I^*)$

Let $R = \mathbb{Z}$ or $R = \bar{\mathbb{F}}_p$. Define $\mathbb{H}_R(\widetilde{G}, I^*)$ to be the algebra of functions

$$\Big\{ f : \widetilde{G} \to R : f(i_1 g i_2) = f(g) \text{ for all } g \in \widetilde{G}, i_j \in I^* \text{ and } f \text{ is compactly supported } \Big\},$$

where the product on $\mathbb{H}_R(\widetilde{G}, I^*)$ is given by convolution. Let $\mathbb{H}^{\epsilon}_R(\widetilde{G}, I^*) \subset \mathbb{H}^{\epsilon}_R(\widetilde{G}, I^*)$ be the subalgebra of *genuine* functions, i.e., of those f such that $f(g(1,\zeta)) = \zeta f(g)$ for all $g \in \widetilde{G}$ and $\zeta \in \{\pm 1\}$. The algebra $\mathbb{H}_R(\widetilde{G}, I^*)$ is isomorphic to the full Iwahori Hecke algebra $\mathcal{H}_R(\widetilde{G}, I^*) := \mathrm{End}_{R[\widetilde{G}]}(\mathrm{ind}^{\widetilde{G}}_{I^*})$ by compact Frobenius reciprocity; explicitly, a function

$\psi \in \mathbb{H}(\widetilde{G}, I^*)$ maps to the endomorphism of $\mathrm{ind}_{I^*}^{\widetilde{G}}(\mathbf{1}_{I^*})$ which sends $f \in \mathrm{ind}_{I^*}^{\widetilde{G}} \mathbf{1}_{I^*}$ to

$$\left(g' \mapsto \sum_{g \in S'} \psi(g' g^{-1}) f(g) \right) \in \mathrm{ind}_{I^*}^{\widetilde{G}} \mathbf{1}_{I^*},$$

where S' is any set of left coset representatives for I^* in $\widetilde{G}$. We define the *genuine* Iwahori Hecke algebra to the image of $\mathbb{H}_R^{\epsilon}(\widetilde{G}, I^*)$ in $\mathcal{H}_p(\widetilde{G}, I^*)$ under Frobenius reciprocity.

4.2.1 A vector space basis for $\mathcal{H}_R^{\epsilon}(\widetilde{SL}_2(F), I^*)$, $R = \bar{\mathbb{F}}_p$ or $R = \mathbb{Z}$

Recall that the finite Weyl group W_0 of $\widetilde{G}$ is generated by

$$w(1) = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1 \right),$$

and that W_0 is the homomorphic image of the finite Weyl group of $G = SL_2(F)$ under the section which splits the cover over K. Also recall that $\Lambda = T(F)/T(\mathcal{O}_F)$ is isomorphic to the cocharacter group of G by evaluation on the uniformizer π; the analogue for the covering group $\widetilde{G}$ is $\Lambda^* = \widetilde{T}(F)/(\widetilde{T} \cap K^*)$, which we will identify with the subgroup of $\widetilde{T}$ generated by $(1, -1)$ and $h(\pi) = \left(\begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}, (-1, \pi) \right)$. and let $\widetilde{\Lambda} = \Lambda^* \times \{(1, \pm 1)\}$. Then $\widetilde{T} = \widetilde{\Lambda} \times (\widetilde{T} \cap K^*)$.

The *affine Weyl group* of $\widetilde{G}$ is the semidirect product $\widetilde{\Lambda} \ltimes W_0$, which we denote by W. Note that W is contained in the normalizer of $\widetilde{T} \cap K^*$ in $\widetilde{G}$, which was calculated to be $\widetilde{T} \amalg \widetilde{T} w(1)$ in § 2.2. The projection of W to G is equal to the semidirect product of Λ with the finite Weyl group of G, which is a system of representatives for $I \setminus G/I$. Hence

$$\widetilde{G} = I^* W I^*,$$

so we can choose representatives in W for the I^*-double cosets in $\widetilde{G}$.

In order to relate our presentation for $\mathcal{H}_p^{\epsilon}(\widetilde{G}, I^*)$ to the Barthel-Livné presentation for

$\mathcal{H}(PGL_2, I_G)$, we will choose representatives for $I^* \setminus \widetilde{G}/I^*$ whose $SL_2(F)$-parts agree (after a dilation by a factor of 2 on the diagonal part) with their representatives modulo the center of $GL_2(F)$. Recall the following elements of $GL_2(F)$:

- $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$,
- $\beta = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$,
- $z(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ for $x \in F^\times$,
- $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

Barthel and Livné take the following set of representatives for $I_G Z_G \setminus GL_2(F)/I_G$ ([3], Lemma 5):

$$\{\alpha^n, \beta\alpha^n\}_{n\in\mathbb{Z}}.$$

Then they choose the vector space basis

$$\{\mathbf{1}_{I_G Z_G \alpha^{-n} I_G}, \mathbf{1}_{I_G Z_G \beta\alpha^{-n}}\}_{n\in\mathbb{Z}}$$

for the convolution algebra $\mathbb{H}_p(GL_2(F), I_G Z_G)$, and define $T_{n,n+1}$ (respectively, $T_{n+1,n}$) to be the image in $\mathcal{H}(GL_2(F), I_G Z_G)$ of $\mathbf{1}_{I_G Z_G \alpha^{-n} I_G}$ (respectively, $\mathbf{1}_{I_G Z_G \beta\alpha^{-n} I_G}$) under Frobenius reciprocity. Then $\{T_{n,n\pm 1}\}_{n\in\mathbb{Z}}\}$ is a vector space basis for $\mathcal{H}(GL_2(F), I_G Z_G)$. After computing relations between these basis elements, Barthel and Livné give the following presentation for $\mathcal{H}_p(GL_2(F), I_G Z_G) = \mathcal{H}_p(PGL_2(F), I_G)$:

Proposition 4.2.1 ([3], Prop. 9)**.** *$\mathcal{H}_p(GL_2(F), I_G Z_G)$ is (non-commutatively) presented by*

$$\mathcal{H}_p(GL_2(F), I_G Z_G) \cong \bar{\mathbb{F}}_p[T_{1,0}, T_{1,2}]/(T_{1,0}^2 - 1, T_{1,2}T_{1,0}, T_{1,2} + T_{1,2}).$$

.

Hence $\mathcal{H}_p(PGL_2(F), I_G)$ is generated as an algebra by the operators $T_{1,0}$ and $T_{1,2}$, which correspond, respectively, to the characteristic functions $\mathbf{1}_{I_G Z_G \beta I_G}$ and $\mathbf{1}_{I_G Z_G \alpha^{-1} I_G}$. The operators $T_{1,0}$ and $T_{1,2}$ can be interpreted as $PGL_2(F)$- equivariant correspondences on $\mathrm{Ed}^o(X)$, where X is the tree of $SL_2(F)$: $T_{1,0}$ sends the unit edge $e_{0,1}$ to $e_{1,0}$, while $T_{1,2}$ sends $e_{0,1}$ to the I_G-orbit of $e_{1,2}$. (See § 1.3 for a picture of the tree and more details of the I_G-action on its edges.)

Recall that we have defined the following elements of $G = SL_2(F)$:

- $\alpha_0 = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}$,
- $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $\beta_0 = \alpha_0 s = \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}$.

Then

$$\alpha_0 = z(\pi^{-1})\alpha^2,\ \beta_0 = \alpha_0 s = \gamma^{-1} z(\pi^{-1})\alpha\beta,\ \alpha_0^k s = \gamma^{-1} z(\pi^{-k})\alpha^{2k-1}\beta,$$

and

$$\alpha_0^{-1} = z(\pi)\alpha^{-2},\ (\alpha_0^k s)^{-1} = \gamma z(\pi^{k-1})\beta\alpha^{-(2k-1)}.$$

Under the projection $\widetilde{G} \to G$ we have

$$h(\pi)^{-k}(1, \pm 1) \mapsto \alpha_0^{-k},\ w(1)(1, \pm 1) \mapsto s,\ \text{and } (h(\pi)^k w(1))^{-1}(1, \pm 1) = w(-1)h(\pi)^{-k}(1, \pm 1) \mapsto (\alpha_0^k s)^{-1}.$$

Define

$$S^\zeta = \{h(\pi)^{-k}(1, \zeta), w(-1)h(\pi)^{-k}(1, \zeta)\}_{k \in \mathbb{Z}, \zeta \in \{\pm 1\}},$$

and let $S = S^+ \amalg S^-$. We will check that S is a set of representatives for $I^* \setminus \widetilde{G}/I^*$, and then explain how to associate the double cosets $\{I^*gI^* : g \in S\}$ to sums of edges on a disjoint union of copies of the tree X.

Since $\Lambda \ltimes W_0$ is a set of representatives for $I \setminus G/I$ and $S = (\Lambda \ltimes W_0) \times \{(1, \pm 1)\}$ as a set, we only need to check that $I^*gI^* \neq I^*g(1,-1)I^*$ for all $g \in S$. If this is not the case for some g, then

$$g(1,-1) \in I^*gI^* \subset K^*gK^*.$$

But $K^*h(\pi)^kK^* \neq K^*h(\pi)^k(1,-1)K^*$ for all $k \in \mathbb{Z}$, so the inclusion is impossible both for $g = h(\pi)^{-k}$ and (since $w(-1)^{-1} = w(1) \in K^*$) for $w(-1)h(\pi)^{-k}$. Hence the set $\{\mathbf{1}_{I^*gI^*} : g \in S\}$ is an $\bar{\mathbb{F}}_p$-vector space basis for $\mathbb{H}_p(\widetilde{G}, I^*)$, and the set of genuine functions

$$\{\mathbf{1}_{I^*h(\pi)^{-k}I^*} - \mathbf{1}_{I^*h(\pi)^{-k}I^*},\, \mathbf{1}_{I^*w(-1)h(\pi)^{-k}I^*} - \mathbf{1}_{I^*w(-1)h(\pi)^{-k}(1,-1)I^*}\}$$

forms a vector space basis for $\mathbb{H}^{\epsilon}(\widetilde{G}, I^*)$.

Next, we associate these basis elements to sums of edges of a graph. Let Δ denote the disjoint union

$$\Delta = \amalg_{\substack{k,\zeta\in\\ \mathbb{Z}\times\pm}} X_{k,\eta},$$

where each $X_{k,\eta} \cong X$ is preserved by the usual action of $PGL_2(F)$. Fix a unit vertex $v_0^{k,\eta}$ and standard apartment $v_n^{k,\eta} = \alpha^n v_0^{k,\eta}$ in each $X_{k,\eta}$, and give Δ an action of $GL_2(F)$ by letting the central element $z(x) \in Z_G$ send the unit vertex $v_0^{k,\eta} \in X_{k,\eta}$ to the unit vertex $v^{k-1,\eta} \in X_{k-1,\eta}$ and defining its effect on the rest of $X_{k,\eta}$ by $PGL_2(F)$-equivariance. The action of $T_G(\mathcal{O}_F)$ is trivial. Then the action of $GL_2(F)$ on the oriented edges of Δ has two orbits: the set of oriented edges of $\Delta^+ = \amalg_{k\in\mathbb{Z}} X_{k,+}$, and of $\Delta^- = \amalg_{k\in\mathbb{Z}} X_{k,-}$. Let $e_{0,1}^{0,+}$ be the unit edge of $X_{0,+}$. The orbit of $e_{0,1}^{0,+}$ under $SL_2(F)$ is the set of edges $\{e_{2k,2k\pm 1}^{k,+}\}_{k\in\mathbb{Z}}$. Finally, let $(1,-1)v_n^{k,\eta} = v_n^{k,-\eta}$ for all k, $n \in \mathbb{Z}$.

In Barthel and Livné's notation, the (characteristic function of the) left coset $I_G Z_G \alpha^{-n}$ is identified with the edge $\alpha^n e_{0,1} = e_{n,n+1}$ of the standard apartment of X, while the left coset $I_G Z_G \beta\alpha^{-n} I_G$ is identified with $\alpha^n\beta^{-1}e_{0,1} = \alpha^n\beta e_{0,1} = e_{n+1,n}$. These left cosets are identified, respectively, with the edges $e_{n,n+1}^{0,+}$ and $e_{n+1}^{0,+}$ of $X_{0,+} \subset \Delta$. Identify the left cosets

$\{I^*g : g \in S'\}$ with edges of Δ as follows:

- $h(\pi)^k(1,\zeta) = (\alpha_0^k, \zeta\phi(k))$, so $h(\pi)^k(1,\zeta)e_{0,1}^{0,+} = (z(\pi)^k\alpha^{-2k}, \zeta\phi(k))e_{0,1}^{0,+} \leftrightarrow e_{2k,2k+1}^{-k,\zeta}$, which is identified with $I^*h(\pi)^{-k}(1,\zeta)$.

- $h(\pi)^k w(1)(1,\zeta) = (\alpha_0^k s, \zeta\phi(k))$, so $h(\pi)^k w(1)(1,\zeta)e_{0,1}^{0,+} = (z(\pi)^{k-1}\beta\alpha^{-(2k-1)}, \zeta\phi(k))e_{0,1}^{0,+} \leftrightarrow e_{2k,2k-1}^{-(k-1),\zeta}$, which is identified with $I^*w(-1)h(\pi)^{-k}(1,\zeta)$.

Let S' be a set of representatives for the left I^*-cosets in $\widetilde{G}$ such that $S \subset S'$, and for $g \in S'$ define $S_g = \{g' \in S : I^*g' \subset I^*gI^*\}$ so that

$$I^*gI^* = \coprod_{g' \in S_g} I^*g'.$$

Identify the double coset I^*gI^* with the sum of those edges associated to I^*g' for $g' \in S_g$. Then as k ranges over $\mathbb{Z}$, each non-oriented edge of Δ is identified with exactly one left I^*-coset of G'. If I^*g is identified with $e_{n,m}^{k,\eta}$, then let $\phi_{n,m}^{k,\eta}$ denote the sum of edges identified with I^*gI^*.

Finally, if $w \in S^+$, let $T_{n,m}^k$ denote the image of $t_w := \frac{1}{2}\left(\mathbf{1}_{I^*wI^*} - \mathbf{1}_{I^*w(1,-1)I^*}\right)$ in $\mathcal{H}_R^\epsilon(\widetilde{G}, I^*)$ under Frobenius reciprocity. Then the set of all such $T_{n,m}^k$ forms a vector space basis for $\mathcal{H}_R^\epsilon(\widetilde{G}, I^*)$. The labeling conventions for generators of $\mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$ are summarized in the following definition:

Definition 4.2.2. *For $k \in \mathbb{Z}$,*

1. *The function $t_{h(\pi)^{-k}} \in \mathbb{H}_p^\epsilon(\widetilde{G}, I^*)$ corresponds by Frobenius reciprocity to the operator $T_{2k,2k+1}^{-k} \in \mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$.*

2. *The function $t_{w(-1)h(\pi)^{-k}} \in \mathbb{H}_p^\epsilon(\widetilde{G}, I^*)$ corresponds by Frobenius reciprocity to the operator $T_{2k,2k-1}^{-(k-1)} \in \mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$.*

Remark 4.2.3. *Note that the labeling convention defines a bijection, via*

$$t_w \leftrightarrow T_{n,m}^k,$$

between S^+ *and the set of triples*

$$\{(-k, 2k, 2k+1),\, (-k, 2k, 2k-1)\}_{k\in\mathbb{Z}}.$$

Since $\{t_w : w \in S^+\}$ *forms a* $\bar{\mathbb{F}}_p$*-vector space basis for* $\mathbb{H}_p^\epsilon(\widetilde{G}, I^*)$*, the set*

$$\{T^{-k}_{2k,2k+1},\, T^{-k}_{2k,2k-1}\}_{k\in\mathbb{Z}}$$

forms a $\bar{\mathbb{F}}_p$*-vector space for* $\mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$*.*

Remark 4.2.4. *Of course, the labeling system for the basic operators* $T^k_{n,m}$ *is redundant: the corresponding basis element of* $\mathbb{H}_p^\epsilon(\widetilde{G}, I^*)$ *can be recovered from any two of* k*,* n*,* m*. We originally included the superscript as an error check when calculating in* $GL_2(F)$*, and retain it here as a way of distinguishing our operators from Barthel-Livné's while also emphasizing their relationship. We hope that the superscript will not be confused with an exponent, and will always use parentheses when writing powers of the* $T^k_{n,m}$*.*

4.2.2 Effect of basic operators on the unit edge of Δ

Any two $\widetilde{G}$-equivariant endomorphisms of $\mathrm{ind}_{I^*}^{\widetilde{G}} \mathbf{1}_{I^*}$ are equal if and only if they agree on the characteristic function $\mathbf{1}_{I^*}$; equivalently, under the identification of the previous section, if they agree on the unit edge $e^{0,+}_{0,1}$. In preparation for the calculation of relations between the basic operators $T^k_{n,m}$, we note the effect of $T^k_{n,m}$ on $e^{0,+}_{0,1}$ and on $\mathbf{1}_{I^*}$.

Lemma 4.2.5. *If* $T^k_{n,m}$ *is the image of* t_w *under Frobenius reciprocity, then*

$$T^k_{n,m}(\mathbf{1}_{I^*}) = t_w.$$

Equivalently, under our identification between I^**-double cosets and edges of* Δ*, we may write*

$$T^k_{n,m}(e^{0,+}_{0,1}) = \frac{1}{2}\left(\phi^{k,+}_{n,m} - \phi^{k,-}_{n,m}\right).$$

Proof. By Frobenius reciprocity,

$$\begin{aligned} T^k_{n,m}(\mathbf{1}_{I^*}) &= \left(g' \mapsto \sum_{g \in S} t_w(g'g^{-1}) \cdot \mathbf{1}_{I^*}(g) \right) \\ &= \left(g' \mapsto t_w(g') = \begin{cases} \frac{1}{2}\zeta & \text{if } g' \in I^* w(1,\zeta) I^* \\ 0 & \text{if not} \end{cases} \right) \\ &= t_w. \end{aligned}$$

Recall that the unit edge $e^{0,+}_{0,1}$ is identified with the coset I^* and with its characteristic function $\mathbf{1}_{I^*}$, while the function t_w is identified with the linear combination $\frac{1}{2}\left(\phi^{k,+}_{n,m} - \phi^{k,-}_{n,m}\right)$ of edges of Δ. Hence we also have

$$T^k_{n,m}(e^{0,+}_{0,1}) = \frac{1}{2}\left(\phi^{k,+}_{n,m} - \phi^{k,-}_{n,m}\right).$$

□

We will move freely between the two points of view depending on the context.

4.2.3 Products in $\mathbb{H}^{\epsilon}_R(\widetilde{G}, I^*)$ and in $\mathcal{H}^{\epsilon}_R(\widetilde{G}, I^*)$

The product in $\mathcal{H}^{\epsilon}_R(\widetilde{G}, I^*)$ is easy to describe: an element $T \in \mathcal{H}^{\epsilon}_R(\widetilde{G}, I^*)$ is an $R[\widetilde{G}]$-equivariant endomorphism of $\operatorname{ind}^{\widetilde{G}}_{I^*}(\mathbf{1}_{I^*})$. Hence if $T, T' \in \mathcal{H}^{\epsilon}_R(\widetilde{G}, I^*)$, the product $T \cdot T'$ is the composite endomorphism $T \circ T'$ of $\operatorname{ind}^{\widetilde{G}}_{I^*}(\mathbf{1}_{I^*})$.

Since it will sometimes be more convenient to compute products in the convolution algebra $\mathbb{H}^{\epsilon}_R(\widetilde{G}, I^*)$, we give an explicit formula for the convolution of two basic genuine functions t_w, $t_{w'}$. Recall that for $w \in S^+$, we have defined

$$t_w = \frac{1}{2}\left(\mathbf{1}_{I^* w I^*} - \mathbf{1}_{I^* w(1,-1) I^*}\right).$$

Lemma 4.2.6. *Let w, $w' \in S^+$. Then*

$$t_{w'} \cdot t_w = \sum_{w'' \in S^+} (c^{w''}_{w,w'} - c^{w''}_{w(1,-1),w'}) \cdot t_{w''},$$

where

$$c^{w''}_{w(1,\zeta),w'} = |I^* \setminus I^* w^{-1}(1,\zeta) I^* w'' \cap I^* w' I^*|.$$

Proof. The standard convolution product on $\mathbb{H}_R(\widetilde{G}, I^*)$ is given, on characteristic functions of I^*-double cosets with representatives in W, by

$$\mathbf{1}_{I^*w'I^*} \cdot \mathbf{1}_{I^*wI^*} = \sum_{w'' \in W} c^{w''}_{w,w'} \mathbf{1}_{I^*w''I^*},$$

where $c^{w''}_{w,w'} := |I^* \setminus I^* w^{-1} I^* w'' \cap I^* w' I^*|$. Note that the index $c^{w''}_{w,w'}$ is nonzero in $\mathbb{Z}$ if and only if $I^* w'' I^* \subset I^* w I^* w' I^*$. Since $I^* w(1,-1) I^* w'(1,-1) I^* = I^* w I^* w' I^*$, we have

$$c^{w''}_{w(1,-1),w'(1,-1)} = c^{w''}_{w,w'},$$

and since $I^* w I^* w(1,-1) I^* = I^* w(1,-1) I^* w' I^*$,

$$c^{w''}_{w(1,-1),w'} = c^{w''}_{w,w'(1,-1)}.$$

Then $\mathbf{1}_{I^*w'I^*} \cdot \mathbf{1}_{I^*wI^*} = \mathbf{1}_{I^*w'(1,-1)I^*} \cdot \mathbf{1}_{I^*w(1,-1)I^*}$ and $\mathbf{1}_{I^*w'I^*} \cdot \mathbf{1}_{I^*w(1,-1)I^*} = \mathbf{1}_{I^*w'(1,-1)I^*} \cdot \mathbf{1}_{I^*wI^*}$, so

$$\begin{aligned} t_{w'} \cdot t_w &= \frac{1}{4}(\mathbf{1}_{I^*w'I^*} \cdot \mathbf{1}_{I^*wI^*} - \mathbf{1}_{I^*w'I^*} \cdot \mathbf{1}_{I^*w(1,-1)I^*} \\ &\quad - \mathbf{1}_{I^*w'(1,-1)I^*} \cdot \mathbf{1}_{I^*wI^*} + \mathbf{1}_{I^*w'(1,-1)I^*} \cdot \mathbf{1}_{I^*w(1,-1)I^*}) \\ &= \frac{1}{2}\left(\mathbf{1}_{I^*w'I^*} \cdot \mathbf{1}_{I^*wI^*} - \mathbf{1}_{I^*w'I^*} \cdot \mathbf{1}_{I^*w(1,-1)I^*}\right). \end{aligned}$$

By definition of the convolution product in $\mathbb{H}_R(\widetilde{G}, I^*)$, this is equal to

$$\frac{1}{2}\left(\sum_{w''\in W} c_{w,w'}^{w''}\, \mathbf{1}_{I^*w''I^*} - \sum_{w''\in W} c_{w(1,-1),w'}^{w''}\, \mathbf{1}_{I^*w''I^*}\right)$$
$$= \sum_{w''\in S^+} \frac{1}{2}(c_{w,w'}^{w''} - c_{w(1,-1),w'}^{w''})\left(\mathbf{1}_{I^*w''I^*} - \mathbf{1}_{I^*w''(1,-1)I^*}\right),$$

which by definition of $t_{w''}$ is equal to

$$\sum_{w''\in S^+} (c_{w,w'}^{w''} - c_{w(1,-1),w'}^{w''}) \cdot t_{w''}.$$

$\square$

4.2.4 Double-coset decompositions in $\widetilde{SL}_2(F)$

Recall the Iwahori decompositions of $I \subset G$ from (1.6):

$$I = (U \cap I)T(\mathcal{O}_F)(U' \cap I) = (U' \cap I)T(\mathcal{O}_F)(U \cap I),$$

where U is upper triangular unipotent subgroup of G and U' is the lower triangular unipotent subgroup. The extension defining $\widetilde{G}$ is split over I since $I \subset K$, so the Iwahori decompositions lift to $\widetilde{G}$:

$$I^* = (U \cap I)^*(\widetilde{T} \cap K^*)(U' \cap I)^* \quad \text{(first Iwahori decomposition in } \widetilde{G});$$

$$I^* = (U' \cap I)^*(\widetilde{T} \cap K^*)(U \cap I)^* \quad \text{(second Iwahori decomposition in } \widetilde{G})$$

Note that the preferred section θ is trivial on U' as well as on U, so both $(U \cap I)^*$ and $(U' \cap I)^*$ are contained in $G \times \{1\}$.

Lemma 4.2.7. *The following commutation relations hold in $\widetilde{G}$.*

1. *For $k > 0$, $(U' \cap I)^* h(\pi)^{-k} \subsetneq h(\pi)^{-k}(U' \cap I)^*$;*

2. *for* $k<0$, $h(\pi)^{-k}(U'\cap I)^* \subsetneq (U'\cap I)^* h(\pi)^{-k}$;

3. *for* $k>0$, $h(\pi)^{-k}(U\cap I)^* \subsetneq (U\cap I)^* h(\pi)^{-k}$;

4. *for* $k<0$, $(U\cap I)^* h(\pi)^{-k} \subsetneq h(\pi)^{-k}(U\cap I)^*$;

5. $(U'\cap I)^* w(-1) \subsetneq w(-1)(U\cap I)^*$;

6. $w(-1)(U'\cap I)^* \subsetneq (U\cap I)^* w(-1)$;

7. *for* $k>0$, $(U\cap I)^* w(-1)h(\pi)^{-k} \subsetneq w(-1)h(\pi)^{-k}(U'\cap I)^*$;

8. *for* $k>0$, $w(-1)h(\pi)^{-k}(U\cap I)^* \subsetneq (U'\cap I)^* w(-1)h(\pi)^{-k}$;

9. *for* $k\in\mathbb{Z}$, $w(-1)h(\pi)^{-k}w(-1)h(\pi)^{-k} = (-1,(\pi^k,\pi^k)_F)$;

10. *for* $k\in\mathbb{Z}$ *and* $a\in\mathcal{O}_F^\times$, $h(\pi)^{-k}h(a) = h(a)h(\pi)^{-k}$;

11. *for* $k\in\mathbb{Z}$, $w(-1)h(\pi)^k w(-1) = h(\pi)^k(-1,(-1)^{k(\frac{q-1}{2})})$.

Proof.

(1) and (2) follow from the calculation

$$\begin{aligned}
&\left(\begin{pmatrix}\pi^k & 0\\ 0 & \pi^{-k}\end{pmatrix},\phi(-k)\right)\cdot\left(\begin{pmatrix}1 & 0\\ y\pi & 1\end{pmatrix},1\right)\\
&=\left(\begin{pmatrix}\pi^k & 0\\ y\pi^{1-k} & \pi^{-k}\end{pmatrix},\phi(-k)\cdot\left(y\pi,\pi^k\right)_F\right)\\
&=\left(\begin{pmatrix}1 & 0\\ y\pi^{1-2k} & 1\end{pmatrix},1\right)\cdot\left(\begin{pmatrix}\pi^k & 0\\ 0 & \pi^{-k}\end{pmatrix},\phi(-k)\right).
\end{aligned}$$

(3) and (4) follow from the calculation

$$
\begin{aligned}
&\left(\begin{pmatrix} \pi^k & 0 \\ 0 & \pi^{-k} \end{pmatrix}, \phi(-k)\right) \cdot \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) \\
&= \left(\begin{pmatrix} \pi^k & x\pi^k \\ 0 & \pi^{-k} \end{pmatrix}, \phi(-k)\right) \\
&= \left(\begin{pmatrix} 1 & x\pi^{2k} \\ 0 & 1 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} \pi^k & 0 \\ 0 & \pi^{-k} \end{pmatrix}, \phi(-k)\right)
\end{aligned}
$$

(5) follows from

$$
w(-1)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix}, 1\right) = \left(\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}, 1\right) w(-1).
$$

(6) follows from

$$
w(-1)\left(\begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} y\pi & 1 \\ -1 & 0 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} 1 & -y\pi \\ 0 & 1 \end{pmatrix}, 1\right) w(-1).
$$

(7) follows from

$$
\begin{aligned}
&\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) w(-1)h(\pi)^{-k} \\
&= \left(\begin{pmatrix} -x\pi^k & \pi^{-k} \\ -\pi^k & 0 \end{pmatrix}, \phi(-k)(-1, \pi^k)_F\right) \\
&= \left(\begin{pmatrix} 0 & \pi^{-k} \\ -\pi^k & 0 \end{pmatrix}, \phi(-k)(-1, \pi^k)_F\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ -x\pi^{2k} & 1 \end{pmatrix}, 1\right) \\
&= w(-1)h(\pi)^{-k}\left(\begin{pmatrix} 1 & 0 \\ -x\pi^{2k} & 1 \end{pmatrix}, 1\right).
\end{aligned}
$$

(8) follows from

$$\begin{aligned}
&w(-1)h(\pi)^{-k}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) \\
&= \left(\begin{pmatrix} 0 & \pi^{-k} \\ -\pi^k & -x\pi^k \end{pmatrix}, \phi(-k)(-1,\pi^k)_F\right) \\
&= \left(\begin{pmatrix} 1 & 0 \\ -x\pi^{2k} & 1 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} 0 & \pi^{-k} \\ -\pi^k & 0 \end{pmatrix}, \phi(-k)(-1,\pi^k)_F\right) \\
&= \left(\begin{pmatrix} 1 & 0 \\ -x\pi^{2k} & 1 \end{pmatrix}, 1\right) w(-1)h(\pi)^{-k}.
\end{aligned}$$

(9) follows from the calculation

$$\begin{aligned}
w(-1)h(\pi)^{-k}w(-1)h(\pi)^{-k} &= \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \phi(-k)^2(-1,\pi^k)_F^2(\pi^k,\pi^k)_F\right) \\
&= \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, (\pi^k,\pi^k)_F\right).
\end{aligned}$$

(10) is true since $\widetilde{T}$ is abelian.

(11) follows from the calculation $w(-1)h(\pi)^k w(-1) =$

$$\begin{aligned}
&\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\left(\begin{pmatrix} \pi^{-k} & 0 \\ 0 & \pi^k \end{pmatrix}, \phi(k)\right)\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right) \\
&= \left(\begin{pmatrix} 0 & \pi^k \\ -\pi^{-k} & 0 \end{pmatrix}, \phi(k)(-1,\pi^k)_F\right)\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right) \\
&= \left(\begin{pmatrix} -\pi^k & 0 \\ 0 & -\pi^{-k} \end{pmatrix}, \phi(k)(-1,\pi^k)_F\right) \\
&= \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} \pi^k & 0 \\ 0 & \pi^{-k} \end{pmatrix}, \phi(k)\right) \\
&= \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \phi(k)\phi(-k)\right) h(\pi)^{-k} \\
&= (-1, (-1)^{k(\frac{q-1}{2})})h(\pi)^{-k}.
\end{aligned}$$

□

Next, we find I^*-double coset decompositions in $\widetilde{G}$ of the following products, using the commutation relations of Lemma 4.2.7 and Iwahori-Matsumoto's $I-$double coset decompositions in G from (1.1.2):

Lemma 4.2.8. *1. If k and j are both ≥ 0, or if k and j are both ≤ 0, then*

$$I^*h(\pi)^{-k}I^*h(\pi)^{-j}I^* = I^*h(\pi)^{-(k+j)}I^*.$$

2. $I^*w(-1)I^*w(-1)h(\pi)^{-1}I^* = I^*h(\pi)^{-1}I^*$.

3. $I^*w(-1)h(\pi)^{-1}I^*w(-1)I^* = I^*h(\pi)(1,(-1)^{\frac{q-1}{2}})I^*$.

4. For $k > 0$, $I^*w(-1)h(\pi)^{-1}I^*h(\pi)^{-k}I^* = I^*w(-1)h(\pi)^{-(k+1)}I^*$.

5. For $k < 0$, $I^*w(-1)I^*h(\pi)^{-k}I^* = I^*w(-1)h(\pi)^{-k}I^*$.

6. $I^*w(-1)I^*w(-1)I^* = I^* \amalg I^*w(-1)I^*$.

7. $I^*w(-1)h(\pi)^{-1}I^*w(-1)h(\pi)^{-1}I^* =$

$$I^*(1,(-1)^{\frac{q-1}{2}}) \amalg I^*w(-1)h(\pi)^{-1}I^* \amalg I^*w(-1)h(\pi)^{-1}(1,-1)I^*.$$

Proof. 1. First suppose k, j are both ≥ 0. By the first Iwahori decomposition in $\widetilde{G}$,

$$I^*h(\pi)^{-k}I^*h(\pi)^{-j}I^* = I^*h(\pi)^{-k}(U \cap I)^*(\widetilde{T} \cap I^*)(U' \cap I)^*h(\pi)^{-j}I^*.$$

By (3) and (10) of Lemma 4.2.7,

$$I^*h(\pi)^{-k}(U \cap I)^*(\widetilde{T} \cap I^*)(U' \cap I)^*h(\pi)^{-j}I^* \subset I^*h(\pi)^{-k}(U' \cap I)^*h(\pi)^{-j}I^*,$$

and by (1) of Lemma 4.2.7,

$$I^*h(\pi)^{-k}(U' \cap I)^*h(\pi)^{-j}I^* \subset I^*h(\pi)^{-k}h(\pi)^{-j}I^* = I^*h(\pi)^{-(k+j)}I^*.$$

Conversely, $I^*h(\pi)^{-(k+j)}I^* \subset I^*h(\pi)^{-k}I^*h(\pi)^{-j}I^*$.

When k, j are both ≤ 0, we use the second Iwahori decomposition in $\widetilde{G}$:

$$I^*h(\pi)^{-k}I^*h(\pi)^{-j}I^* = I^*h(\pi)^{-k}(U' \cap I)^*(\widetilde{T} \cap I^*)(U \cap I)^*h(\pi)^{-j}I^*.$$

By (2) and (10) of Lemma 4.2.7,

$$I^*h(\pi)^{-k}(U' \cap I)^*(\widetilde{T} \cap I^*)(U \cap I)^*h(\pi)^{-j}I^* \subset I^*h(\pi)^{-k}(U \cap I)^*h(\pi)^{-j}I^*,$$

and by (4) of Lemma 4.2.7,

$$I^*h(\pi)^{-k}(U \cap I)^*h(\pi)^{-j}I^* \subset I^*h(\pi)^{-k}h(\pi)^{-j}I^* = I^*h(\pi)^{-(k+j)}I^*.$$

Again we have the reverse inclusion $I^*h(\pi)^{-(k+j)}I^* \subset I^*h(\pi)^{-k}I^*h(\pi)^{-j}I^*$.

2. By the second Iwahori decomposition in $\widetilde{G}$,

$$I^*w(-1)I^*w(-1)h(\pi)^{-1}I^* = I^*w(-1)(U' \cap I)^*(\widetilde{T} \cap I^*)(U \cap I)^*w(-1)h(\pi)^{-1}I^*.$$

By (6) and (10) of Lemma 4.2.7,

$$I^*w(-1)(U' \cap I)^*(\widetilde{T} \cap I^*)(U \cap I)^*w(-1)h(\pi)^{-1}I^* \subset I^*w(-1)(U \cap I)^*w(-1)h(\pi)^{-1}I^*,$$

and by (7) of Lemma 4.2.7,

$$I^*w(-1)(U \cap I)^*w(-1)h(\pi)^{-1}I^* \subset I^*w(-1)^2h(\pi)^{-1}I^* = I^*h(\pi)^{-1}I^*.$$

Conversely,

$$I^*h(\pi)^{-1}I^* = I^*w(-1)^2h(\pi)^{-1}I^* \subset I^*w(-1)I^*w(-1)h(\pi)^{-1}I^*.$$

3. By the first Iwahori decomposition in $\widetilde{G}$,

$$I^*w(-1)h(\pi)^{-1}I^*w(-1)I^* = I^*w(-1)h(\pi)^{-1}(U \cap I)^*(\widetilde{T} \cap I^*)(U' \cap I)^*w(-1)I^*.$$

By (8) and (10) of Lemma 4.2.7,

$$I^*w(-1)h(\pi)^{-1}(U \cap I)^*(\widetilde{T} \cap I^*)(U' \cap I)^*w(-1)I^* \subset I^*w(-1)h(\pi)^{-1}(U' \cap I)^*w(-1)I^*,$$

and by (5) of Lemma 4.2.7,

$$I^*w(-1)h(\pi)^{-1}(U' \cap I)^*w(-1)I^* \subset I^*w(-1)h(\pi)^{-1}w(-1)I^*.$$

The reverse inclusion $I^*w(-1)h(\pi)^{-1}w(-1)I^* \subset I^*w(-1)h(\pi)^{-1}I^*w(-1)I^*$ is clear, so

$$I^*w(-1)h(\pi)^{-1}w(-1)I^* = I^*w(-1)h(\pi)^{-1}I^*w(-1)I^*,$$

and by (11) of Lemma 4.2.7,

$$I^*w(-1)h(\pi)^{-1}w(-1)I^* = I^*h(\pi)(1,(-1)^{\frac{q-1}{2}})I^*.$$

4. By the first Iwahori decomposition in $\widetilde{G}$,

$$I^*w(-1)h(\pi)^{-1}I^*h(\pi)^{-k}I^* = I^*w(-1)h(\pi)^{-1}(U \cap I)^*(\widetilde{T} \cap I^*)(U' \cap I)^*h(\pi)^{-k}I^*.$$

By (8) and (10) of Lemma 4.2.7,

$$I^*w(-1)h(\pi)^{-1}(U \cap I)^*(\widetilde{T} \cap I^*)(U' \cap I)^*h(\pi)^{-k}I^* \subset I^*w(-1)h(\pi)^{-1}(U' \cap I)^*h(\pi)^{-k}I^*,$$

and by (1) of Lemma 4.2.7,

$$I^*w(-1)h(\pi)^{-1}(U' \cap I)^*h(\pi)^{-k}I^* \subset I^*w(-1)h(\pi)^{-1}h(\pi)^{-k}I^* = I^*w(-1)h(\pi)^{-(k+1)}I^*.$$

Conversely, $I^*w(-1)h(\pi)^{-(k+1)}I^* \subset I^*w(-1)h(\pi)^{-1}I^*h(\pi)^{-k}I^*$.

5. By the second Iwahori decomposition in $\widetilde{G}$,

$$I^*w(-1)I^*h(\pi)^{-k}I^* = I^*w(-1)(U' \cap I)^*(\widetilde{T} \cap I^*)(U \cap I)^*h(\pi)^{-k}I^*.$$

By (6) of Lemma 4.2.7,

$$I^*w(-1)(U' \cap I)^*(\widetilde{T} \cap I^*)(U \cap I)^*h(\pi)^{-k}I^* \subset I^*w(-1)(\widetilde{T} \cap I^*)(U \cap I)^*h(\pi)^{-k}I^*,$$

and by (4) and (10) of Lemma 4.2.7,

$$I^*w(-1)(\widetilde{T}\cap I^*)(U\cap I)^*h(\pi)^{-k}I^* \subset I^*w(-1)h(\pi)^{-k}I^*.$$

Conversely, $I^*w(-1)h(\pi)^{-k}I^* \subset I^*w(-1)I^*h(\pi)^{-k}I^*$.

6. Since the extension defining $\widetilde{G}$ is split over K and we have both $I^* \subset K^*$ and $w(-1) \in K^*$, the product $I^*w(-1)I^*w(-1)I^*$ is equal to $\theta(Pr(I^*w(-1)I^*w(-1)I^*))$ where θ is our preferred section $G \to G^*$.

   We have $Pr(I^*w(-1)I^*w(-1)I^*) = IsIsI$. By (1.1.2),

$$IsIsI = I \amalg IsI,$$

   so $I^*w(-1)I^*w(-1)I^* = \theta(I) \amalg \theta(IsI) = I^* \amalg I^*(-1,1)w(-1)I^*. = I^* \amalg I^*w(-1)I^*$.

7. The projection of the product $I^*w(-1)h(\pi)^{-1}I^*w(-1)h(\pi)^*I^*$ to G is

$$Pr(I^*w(-1)h(\pi)^{-1}I^*w(-1)h(\pi)^{-1}I^*) = Is\alpha_0^{-1}Is\alpha_0^{-1}I.$$

   By (1.1.2),

$$Is\alpha_0^{-1}Is\alpha_0^{-1}I = I \amalg Is\alpha_0^{-1}I.$$

   If $w' \in S$ satisfies $I^*w'I^* \subset I^*w(-1)h(\pi)^{-1}I^*w(-1)h(\pi)^{-1}I^*$, then $Pr(w') = 1$ or $Pr(w') = -s\alpha_0^{-1}$, and hence $w' = (1, \pm 1)$ or $w' = w(-1)h(\pi)^{-1}(1, \pm 1)$. So to prove the statement, it suffices to show that $I^*w'I^* \subset I^*w(-1)h(\pi)^{-1}I^*w(-1)h(\pi)^{-1}I^*$ when $w' = (1, (-1)^{\frac{q-1}{2}})$, when $w' = w(-1)h(\pi)^{-1}$, and when $w' = w(-1)h(\pi)^{-1}(1, -1)$, but not when $w = (1, (-1)^{1+\frac{q-1}{2}})$.

We have

$$
\begin{aligned}
h(\pi)w(1) &= \left(\begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}, \phi(1)\right)\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}, \phi(1)\right) \\
&= \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \phi(1)\phi(-1)(-1,-\pi)_F\right)\left(\begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix}, \phi(-1)\right) \\
&= (-1, \phi(1)\phi(-1)) \cdot w(-1)h(\pi)^{-1} \\
&= (-1, (-1)^{\frac{q-1}{2}}) \cdot w(-1)h(\pi)^{-1}.
\end{aligned}
$$

So, working over $\mathbb{Z}$, we have $c^{(1,\zeta)}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} =$

$$
\begin{aligned}
&2 \cdot |I^* \setminus I^*h(\pi)w(1)I^*(1,\zeta) \cap I^*w(-1)h(\pi)^{-1}I^*| \\
&= 2 \cdot |I^* \setminus I^*w(-1)h(\pi)^{-1}(1, \zeta(-1)^{\frac{q-1}{2}})I^* \cap I^*w(-1)h(\pi)^{-1}I^*|.
\end{aligned}
$$

When $\zeta(-1)^{\frac{q-1}{2}} = 1$, the intersection $I^*w(-1)h(\pi)^{-1}(1, \zeta(-1)^{\frac{q-1}{2}})I^* \cap I^*w(-1)h(\pi)^{-1}I^*$ is equal to $I^*w(-1)h(\pi)^{-1}I^*$, and when $\zeta(-1)^{\frac{q-1}{2}} = -1$, the intersection is empty. Hence

$$
c^{(1,\zeta)}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} = \begin{cases} 2 \cdot \mathrm{vol}_{\widetilde{G}}(I^*w(-1)h(\pi)^{-1}I^*) & \text{if } \zeta = (-1)^{\frac{q-1}{2}} \\ 0 & \text{if } \zeta = (-1)^{1+\frac{q-1}{2}}. \end{cases} \tag{4.1}
$$

In particular, $I^*(1,\zeta)I^* \subset I^*w(-1)h(\pi)^{-1}I^*w(-1)h(\pi)^{-1}I^*$ when $\zeta = (-1)^{\frac{q-1}{2}}$ but not when $\zeta = (-1)^{1+\frac{q-1}{2}}$.

It is shown in the proof of Lemma 4.3.5 that $I^*w(-1)h(\pi)^{-1}I^*$ and $I^*w(-1)h(\pi)^{-1}(1,-1)I^*$ are contained in $I^*w(-1)h(\pi)^{-1}I^*w(-1)h(\pi)^{-1}I^*$.

□

4.3 A presentation for $\mathcal{H}^\epsilon(\widetilde{SL}_2(F), I^*)$

In this section we compute relations between elements of the vector space basis given in §4.2. Before doing so, we collect some useful facts about convolution products in Iwahori Hecke algebras.

In [15], Iwahori and Matsumoto prove the following results for Chevalley groups:

Proposition 4.3.1. *Let G be a Chevalley group over a p-adic field F with residue field of order q and uniformizer π, T_G a torus of G, R the root system of G with respect to T_G, and I an Iwahori subgroup of G. Let λ denote the length function on the extended Weyl group of G with respect to T_G. For $x \in G$, let vol$(IxI) = [I : I \cap x^{-1}Ix]$. Then*

1. *([15], p. 44) Then the map $\mathcal{H}_{\mathbb{Z}}(G, I) \to \mathbb{Z}$ defined by*

$$\sum_w d_w \cdot IwI \to \sum_w d_w \cdot vol(IwI)$$

(where w runs over the extended Weyl group of G and $d_w \in \mathbb{Z}$ such that $d_w = 0$ for almost all w), is a surjective ring homomorphism.

2. *([15], Prop. 3.2) For w in the extended Weyl group of G,*

$$vol(w) = q^{\lambda(w)}.$$

Let $Pr : \widetilde{G} \to G$ denote the projection $(g, \zeta) \mapsto g$. The following lemma is an adaptation to our situation of an observation by Savin in the proof of [23], Prop. 6.1.

Lemma 4.3.2. *Normalize the volumes of the I^*-double cosets in $\widetilde{G}$ by setting $vol_{\widetilde{G}}(I^*) = \frac{1}{2}$, so that $vol_{\widetilde{G}}(Pr^{-1}(I)) = 1$, and normalize the volumes of I-double cosets in G so that $vol_G(I) = 1$. If w is an element of $W = \widetilde{\Lambda} \ltimes W_0$, then $vol_{\widetilde{G}}(I^*wI^*) = \frac{1}{2} vol_G(I\,Pr(w)\,I)$.*

Proof. Since $\mathrm{vol}_{\widetilde{G}}(Pr^{-1}(I)) = 1 = \mathrm{vol}_G(I)$, we have

$$\mathrm{vol}_{\widetilde{G}}(Pr^{-1}(I\,Pr(w)\,I)) = \cdot \mathrm{vol}_G(I\,Pr(w)\,I).$$

The inverse image $Pr^{-1}(I\,Pr(w)\,I)$ is equal to the union

$$I^*wI^* \cup I^*w(1,-1)I^*,$$

which was shown in §4.2 to be disjoint for each $w \in W$. Since $(1,-1)$ is central in $\widetilde{G}$, we have

$$\mathrm{vol}_{\widetilde{G}}(I^*wI^*) = \mathrm{vol}_{\widetilde{G}}(I^*w(1,-1)I^*),$$

so

$$\mathrm{vol}_{\widetilde{G}}(Pr^{-1}(I\,Pr(w)\,I)) = 2 \cdot \mathrm{vol}_{\widetilde{G}}(I^*wI^*).$$

Thus

$$\mathrm{vol}_{\widetilde{G}}(I^*wI^*) = \frac{1}{2}\mathrm{vol}_G(I\,Pr(w)\,I).$$

□

For an element $w \in W$, define the *Weyl length* $\lambda(w)$ of w to be $\lambda(Pr(w))$. Then

$$q^{\lambda(w)} = \mathrm{vol}_G(I\,Pr(w)\,I) = 2 \cdot \mathrm{vol}_{\widetilde{G}}(I^*wI^*),$$

where the first equality is by Proposition 4.3.1 (2) and the second is by Lemma 4.3.2.

Lemma 4.3.3. *The map $\mathcal{H}^{\epsilon}_{\mathbb{Z}}(\widetilde{G}, I^*) \to \mathbb{Z}$ defined by*

$$\sum_{w \in S^+} d_w\, t_w \mapsto \sum_{w \in S^+} 2 \cdot d_w\, vol_{\widetilde{G}}(I^*wI^*)$$

(where $d_w \in \mathbb{Z}$, $d_w = 0$ for almost all $w \in S^+$) is a ring homomorphism.

Recall that the volumes of the double cosets IwI were calculated in Lemma 1.1.1 for certain elements w of the affine Weyl group of $SL_2(F)$. For convenience, the results are listed again here: For $k > 0$,

1. $\mathrm{vol}(I\alpha_0^{-k}I) = q^{2k}$.

2. $\mathrm{vol}(I\alpha_0^k I) = q^{2k}$

3. $\mathrm{vol}(I(-s)\alpha_0^{-k} I) = q^{2k-1}$

4. $\mathrm{vol}(I(-s)\alpha_0^k I) = q^{2k+1}$

5. $\mathrm{vol}(IsI) = q$.

We now state the key proposition of this section.

Proposition 4.3.4. *The following is a complete list of positive powers of* $x := T^1_{0,-1}$ *and* $y := T^0_{2,1}$. *For* $k \geq 1$,

1. $x^k = (-1)^{k-1} T^1_{0,-1} = (-1)^{k-1} x$,

2. $y^k = \begin{cases} T^0_{2,1} & \textit{if } k = 1 \\ 0 & \textit{if } k \geq 2, \end{cases}$

3. $(xy)^k = T^{-k}_{2k,2k+1}$,

4. $(yx)^k = (-1)^{k\frac{q-1}{2}} T^k_{-2k,-2k+1}$,

5. $y(xy)^k = T^{-k}_{2k+2,2k+1}$

6. $x(yx)^k = (-1)^{k\frac{q-1}{2}} T^{k+1}_{-2k,-2k-1}$.

Proof. 1. $T^1_{0,-1}$ is the image of $t_{w(-1)}$ under Frobenius reciprocity, so we calculate the convolution product

$$t_{w(-1)} \cdot t_{w(-1)} = \sum_{w'' \in S^+} \left(c^{w''}_{w(-1),w(-1)} - c^{w''}_{w(-1)(1,-1),w(-1)} \right) \cdot t_{w''},$$

working with $\mathbb{Z}$-coefficients until the last step of the calculation. The coefficient

$$c^{w''}_{w(-1)(1,\zeta),w(-1)} := |I^* \setminus I^* w(1)(1,\zeta) I^* w'' \cap I^* w(-1) I^*|$$

is nonzero in $\mathbb{Z}$ if and only if $I^*w''I^* \subset I^*w(-1)(1,\zeta)I^*w(-1)I^*$. By Lemma 4.2.8 (6) we have

$$I^*w(-1)I^*w(-1)I^* = I^* \amalg I^*w(-1)I^*. \tag{4.2}$$

Multiplying both sides of (4.2) by $(1,-1)$ preserves disjointness of the union, so

$$I^*w(-1)(1,-1)I^*w(-1)I^* = I^*(1,-1) \amalg I^*w(-1)(1,-1)I^*.$$

Hence $c^{w''}_{w(-1),w(-1)} \neq 0$ only if $w'' = (1,1)$ or $w'' = w(-1)$, and $c^{w''}_{w(-1)(1,-1),w(-1)} = 0$ for all $w'' \in S^+$.

Since both $w(-1) \in K^*$ and $I^* \subset K^*$, the index

$$c^{(1,1)}_{w(-1),w(-1)} = |I^* \setminus I^*w(1)I^*|$$

is equal to the index $c^1_{s,s}$ of $I = Pr(I^*)$ in $IsIsI = Pr(I^*w(-1)I^*w(-1)I^*)$, and

$$c^{w(-1)}_{w(-1),w(-1)} = |I^* \setminus I^*w(1)I^*w(-1) \cap I^*w(-1)I^*|$$

is equal to the index $c^s_{s,s}$ of $IsI = Pr(I^*w(-1)I^*)$ in $IsIsI = Pr(I^*w(-1)I^*w(-1)I^*)$.

The coefficient $c^1_{s,s}$ is easily calculated:

$$c^1_{s,s} = |I \setminus IsI| = \mathrm{vol}_G(IsI) = q,$$

with the last equality from Lemma 1.1.1.

We calculate $c^s_{s,s}$ by applying vol_G to both sides of the equation

$$\mathbf{1}_{IsI} \cdot \mathbf{1}_{IsI} = c^1_{s,s}\mathbf{1}_I + c^s_{s,s}\mathbf{1}_{IsI}.$$

Recall from Proposition 4.3.1 (1) that the map defined by $\mathbf{1}_{IwI} \mapsto \mathrm{vol}_G(IwI)$ is a ring homomorphism $\mathcal{H}_{\mathbb{Z}}(G,I) \to \mathbb{Z}$. By Lemma 1.1.1, $\mathrm{vol}_G(IsI) = q$, so

$$\mathrm{vol}_G(IsIsI) = \mathrm{vol}_G(IsI)^2 = q^2,$$

while

$$c^1_{s,s}\mathrm{vol}_G(I) + c^s_{s,s}\mathrm{vol}_G(IsI) = c^1_{s,s} + c^s_{s,s}q = q(1 + c^s_{s,s}).$$

Hence $q^2 = q(1 + c^s_{s,s})$, which implies $c^s_{s,s} = q - 1$.

Thus, over $\mathbb{Z}$, we have

$$c^{(1,1)}_{w(-1),w(-1)} = c^1_{s,s} = q$$

and

$$c^{w(-1)}_{w(-1),w(-1)} = c^s_{s,s} = q - 1$$

so

$$t_{w(-1)} \cdot t_{w(-1)} = qt_{(1,1)} + (q-1)t_{w(-1)} \equiv -t_{w(-1)} \pmod p.$$

Hence $T^1_{0,-1} \circ T^1_{0,-1}(e^{0,+}_{0,1}) = -T^1_{0,-1}(e^{0,+}_{0,1})$ over $\bar{\mathbb{F}}_p$, so

$$T^1_{0,-1} \cdot T^1_{0,-1} = -T^1_{0,-1}$$

in $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$. Letting $x = T^1_{0,-1}$, we have $x^k = (-1)^{k-1}x$ for $k > 0$.

2. $T^0_{2,1}$ is the image of $t_{w(-1)h(\pi)^{-1}}$ under Frobenius reciprocity, so we calculate

$$t_{w(-1)h(\pi)^{-1}} \cdot t_{w(-1)h(\pi)^{-1}} =$$

$$\sum_{w'' \in S^+} \left(c^{w''}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} - c^{w''}_{w(-1)h(\pi)^{-1}(1,-1),w(-1)h(\pi)^{-1}} \right) \cdot t_{w''}.$$

We have shown that $I^*w(-1)h(\pi)^{-1}I^*w(-1)h(\pi)^{-1}I^*$ is contained in

$$I^*(1,(-1)^{\frac{q-1}{2}}) \amalg I^*w(-1)h(\pi)^{-1} \amalg I^*w(-1)h(\pi)^{-1}(1,-1),$$

so $c^{w''}_{w(-1)h(\pi)^{-1}(1,\zeta),w(-1)h(\pi)^{-1}} \neq 0$ for at least one $\zeta \in \{\pm 1\}$ only if $w'' = (1,1)$ or $w'' = w(-1)h(\pi)^{-1}$. The calculations for $w'' = (1,1)$ are easy to complete: we showed in (4.1) that

$$c^{(1,\zeta)}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} = \begin{cases} 2 \cdot \mathrm{vol}_{\widetilde{G}}(I^* w(-1)h(\pi)^{-1} I^*) & \text{if } \zeta = (-1)^{\frac{q-1}{2}} \\ 0 & \text{if } \zeta = (-1)^{1+\frac{q-1}{2}}. \end{cases}$$

Note that $c^{(1,\zeta)}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} = c^{(1,1)}_{w(-1)h(\pi)^{-1}(1,\zeta),w(-1)h(\pi)^{-1}}$, so it only remains to compute the volume in the case $\zeta = (-1)^{\frac{q-1}{2}}$. By Proposition 4.3.2,

$$2 \cdot \mathrm{vol}_{\widetilde{G}}(I^* w(-1)h(\pi)^{-1} I^*) = \mathrm{vol}_G(I\, Pr(w(-1)h(\pi)^{-1})\, I) = \mathrm{vol}_G(I s \alpha_0^{-1} I).$$

By Part (3) of Lemma 1.1.1 with $\ell = 1$,

$$\mathrm{vol}_G(I s \alpha_0^{-1} I) = q.$$

Hence the coefficients for $w'' = (1,1)$, $\zeta \in \{\pm 1\}$ are

$$c^{(1,1)}_{w(-1)h(\pi)^{-1}(1,\zeta),w(-1)h(\pi)^{-1}} = \begin{cases} q & \text{if } \zeta = (-1)^{\frac{q-1}{2}} \\ 0 & \text{if } \zeta = (-1)^{1+\frac{q-1}{2}}. \end{cases}$$

Now we turn to the calculation of the coefficients $c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi^{-1})(1,\zeta),w(-1)h(\pi)^{-1}}$, $\zeta \in \{\pm 1\}$. Since the calculation is rather involved, we state it as a lemma and warn that several sub-lemmata are contained within.

Lemma 4.3.5.

$$c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi^{-1}),w(-1)h(\pi)^{-1}} = c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi^{-1})(1,-1)),w(-1)h(\pi)^{-1}} = \frac{q-1}{2}.$$

Proof. By definition, $c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1}(1,\zeta),w(-1)h(\pi)^{-1}} =$

$$\big|I^* \setminus I^*h(\pi)w(1)I^* \cap I^*w(-1)h(\pi)^{-1}I^*h(\pi)w(1)(1,\zeta)\big|. \tag{4.3}$$

We first identify those $g \in G$ such that

$$(g,\delta)^{-1} \in I^*h(\pi)w(1)I^* \cap I^*w(-1)h(\pi)^{-1}I^*h(\pi)w(1)(1,\zeta)$$

for some δ, $\zeta \in \{\pm 1\}$. We have the projections

$$Pr(I^*h(\pi)w(1)I^*) = I\alpha_0 sI,$$

$$Pr(I^*w(-1)h(\pi)^{-1}I^*h(\pi)w(1)(1,\zeta)) = I(-s)\alpha_0^{-1}I\alpha_0 s,$$

so if $(g,\delta) \in I^*w(-1)h(\pi)^{-1}I^*h(\pi)w(1)(1,\zeta) \cap I^*h(\pi)w(1)I^*$ for some δ, $\zeta \in \{\pm 1\}$,

$$g^{-1} \in I\alpha_0 sI \cap I(-s)\alpha_0^{-1}I\alpha_0 s.$$

Conversely, if $g^{-1} \in I\alpha_0 sI \cap I(-s)\alpha_0^{-1}I\alpha_0 s$, then

$$(g,\delta)^{-1} \in I^*w(-1)h(\pi)^{-1}I^*h(\pi)w(1)(1,\zeta) \cap I^*h(\pi)w(1)I^*$$

for some δ, $\zeta \in \{\pm 1\}$.

We pause the proof of Lemma 4.3.5 to prove an auxiliary statement.

Lemma 4.3.6. *Those $g \in G$ which satisfy*

$$g^{-1} \in I\alpha_0 sI \cap I(-s)\alpha_0^{-1}I\alpha_0 s \tag{4.4}$$

are exactly those which, modulo I on the left, are of the form

$$g = \begin{pmatrix} a & u\pi^{-1} \\ b\pi^2 & d \end{pmatrix} \tag{4.5}$$

with a, u, and $d \in \mathcal{O}_F^\times$ and $b \in \mathcal{O}_F$.

Proof. In the proof of [19] Thm. 12.3 (3'), McNamara asserts:

Claim 1. *Those $h \in SL_2(F)$ such that*

$$h^{-1} \in I\alpha_0 I \cap Is\alpha_0^{-1} I\alpha_0$$

are exactly those which, modulo I on the left, are of the form

$$h = \begin{pmatrix} b\pi^2 & d \\ a & u\pi^{-1} \end{pmatrix} \tag{4.6}$$

with a, d, and $u \in \mathcal{O}_F^\times$ and $b \in \mathcal{O}_F$.

For completeness, we prove Claim 1 here, together with an additional statement:

Claim 2. *If $h^{-1} \in SL_2(F)$ is in the intersection $I\alpha_0 I \cap Is\alpha_0^{-1} I\alpha_0$, then there is a representative $\begin{pmatrix} j & k \\ \ell & m \end{pmatrix} \in I$ such that $v(k) = 1$ and*

$$h^{-1} \in I\alpha_0 \begin{pmatrix} j & k \\ \ell & m \end{pmatrix}.$$

Proof of Claims 1 and 2. If $h^{-1} \in Is\alpha_0^{-1} I\alpha_0$, then modulo I on the left,

$$h^{-1} = \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix} = \begin{pmatrix} -c\pi^{-2} & -d \\ a & b\pi^2 \end{pmatrix} \tag{4.7}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$. If also $h^{-1} \in I\alpha_0 I$, then (again modulo I on the left)

$$h^{-1} = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix} \begin{pmatrix} j & k \\ \ell & m \end{pmatrix} = \begin{pmatrix} j\pi^{-1} & k\pi^{-1} \\ \ell\pi & m\pi \end{pmatrix}$$

where $\begin{pmatrix} j & k \\ \ell & m \end{pmatrix} \in I$. By definition of I, we must have $v(a) = v(d) = v(j) = v(m) = 0$, $v(b) \geq 0$, $v(k) \geq 0$, $v(c) \geq 1$, and $v(\ell) \geq 1$. In particular, $v(j\pi^{-1}) = -1$ and $v(\ell\pi) \geq 2$.

Let $\begin{pmatrix} t & x \\ y & z \end{pmatrix}$ be an arbitrary element of I. Then $v(t) = v(y) = 0$, $v(x) \geq 0$, $v(z) \geq 1$. Consider the valuation of the upper-left entry of the product

$$\begin{pmatrix} t & x \\ y & z \end{pmatrix} \begin{pmatrix} j\pi^{-1} & k\pi^{-1} \\ \ell\pi & m\pi \end{pmatrix} = \begin{pmatrix} tj\pi^{-1} + x\ell\pi & tk\pi^{-1} + xm\pi \\ yj\pi^{-1} + z\ell\pi & yk\pi^{-1} + zm\pi \end{pmatrix}.$$

Since $v(x\ell\pi) \geq 2$ while $v(tj\pi^{-1}) = -1$, we have $v(tj\pi^{-1} + x\ell\pi) = -1$. Hence the valuation of the upper-left entry of h^{-1} is invariant under left multiplication by I. So $v(-c\pi^{-2}) = v(j\pi^{-1}) = -1$, i.e., $c = u\pi$ for some $u \in \mathcal{O}_F^\times$. We conclude that every $h \in G$ such that $h^{-1} \in I\alpha_0 I \cap Is\alpha_0^{-1}I\alpha_0$ is of the form (4.6), proving one direction of Claim 1.

On the other hand, suppose that h is of form (4.6), i.e., suppose that $h = \begin{pmatrix} b\pi^2 & d \\ -a & -u\pi^{-1} \end{pmatrix} \in G$ with $a, u, d \in \mathcal{O}_F^\times$ and $b \in \mathcal{O}_F$. Then $\det(h) = ad - ub\pi = 1$, so $i := \begin{pmatrix} a & b \\ u\pi & d \end{pmatrix} \in G$, and the assumptions on $v(a)$, $v(b)$, $v(u)$ and $v(d)$ imply that $i \in I$. The inverse of h is $h^{-1} = \begin{pmatrix} -u\pi^{-1} & -d \\ a & b\pi^2 \end{pmatrix}$, and the second equality of (4.7) (taking $c = u\pi$) implies that $h^{-1} = s\alpha_0^{-1}i\alpha_0 \in Is\alpha_0^{-1}I\alpha_0$. To finish the proof of Claim 1, we have to show that h^{-1} is equivalent under multiplication by I on the left to an element of $I\alpha_0 I$. Let $i' := \begin{pmatrix} j & k \\ \ell & m \end{pmatrix}$ denote an arbitrary element of I, i.e., assume $v(j) = v(m) = 0$, $v(k) \geq 0$, $v(\ell) \geq 1$, and $jm - k\ell = 1$, and consider the product

$$\alpha_0 i' h = \begin{pmatrix} j\pi^{-1} & k\pi^{-1} \\ \ell\pi & m\pi \end{pmatrix} \begin{pmatrix} b\pi^2 & d \\ -a & -u\pi^{-1} \end{pmatrix} = \begin{pmatrix} bj\pi - ak\pi^{-1} & dj\pi^{-1} - ku\pi^{-2} \\ b\ell\pi^3 - am\pi & d\ell\pi - mu \end{pmatrix}. \tag{4.8}$$

If (4.8) is in I for some j, k, ℓ, m such that $i' \in I$, then $h^{-1} \in I\alpha_0 I$ as desired.

We now show that such j, k, ℓ, m exist, and furthermore that we must have $v(k) = 1$.

- *(Upper-left entry of (4.8).)* Since $v(b) \geq 0$, we have $v(bj\pi) = v(b) + 1 \geq 1$ when $v(j) = 0$, and since $v(a) = 0$, we have $v(ak\pi^{-1}) = v(k) - 1 \geq -1$. Thus

$$v(jb\pi - ak\pi^{-1}) \geq \min(v(b) + 1, v(k) - 1)$$

  when $v(j) = 0$. Since $v(b) + 1 \geq 1$, then under the condition that $v(j) = 0$, we have $v(jb\pi - ak\pi^{-1}) = 0$ if and only if $v(k) = 1$.

- *(Upper-right entry of (4.8).)* Recall that $v(d) = v(u) = 0$ and, due to the conclusion from the upper-left entry of (4.8), we assume that $v(k) = 1$. Write $k = k_0\pi$ where $k_0 \in \mathcal{O}_F^\times$. Then

$$dj\pi^{-1} - uk\pi^{-2} = (dj - uk_0)\pi^{-1},$$

  so $v(dj\pi^{-1} - uk\pi^{-2}) = v(dj - uk_0) - 1$. When $v(j) = 0$, we have

$$v(dj - uk_0) \geq 0,$$

  and can get the desired bound

$$v(dj - uk_0) \geq 1$$

by requiring that $j = (\gamma\pi + uk_0)/d$ for some $\gamma \in \mathcal{O}_F$.

- *(Lower-left entry of (4.8).)* Since $v(b) \geq 0$, whenever $v(\ell) \geq 1$ we have $v(b\ell\pi^3) \geq 4$, and since $v(a) = 0$, whenever $v(m) = 0$ we have $v(am\pi) = 1$. Hence $v(b\ell\pi^3 - am\pi) = 1$ whenever $v(\ell) \geq 1$ and $v(m) = 0$.

- *(Lower-right entry of (4.8).)* Since $v(d) = v(u) = 0$, whenever $v(\ell) \geq 1$ we have $v(d\ell\pi) \geq 2$, and whenever $v(m) = 0$ we have $v(mu) = 0$. Hence $v(d\ell\pi - mu) = 0$ whenever $v(\ell) \geq 1$ and $v(m) = 0$.

- *(Determinant.)* Finally, we show that the conditions we have imposed so far are compatible with the condition that $jm - k\ell = 1$. Suppose that $v(j) = v(m) = 0$, $v(\ell) \geq 0$, $k = k_0\pi$ for some $k_0 \in \mathcal{O}_F^\times$, and $j = (\gamma\pi + uk_0)/d$ for some $\gamma \in \mathcal{O}_F$. Then
$$jm - k\ell = \left(\frac{m\gamma}{d} - \ell k_0\right)\pi + \frac{muk_0}{d},$$
where $muk_0/d \in \mathcal{O}_F^\times$ and the coefficient of π is in $\mathcal{O}_F$. Hence $jm - k\ell = 1$ if and only if both
$$\frac{muk_0}{d} = 1 \text{ and } \frac{m\gamma}{d} = \ell k_0. \tag{4.9}$$
It is easy to satisfy the conditions of (4.9): for example, fix ℓ and m and take $k_0 = d/mu$; then $\gamma = d^2\ell/m^2u \in \mathcal{O}_F$.

We have shown that if $i' = \begin{pmatrix} j & k \\ \ell & m \end{pmatrix}$ is any element of I such that $v(k) = 1$, then
$$i'' := \alpha_0 i' h \in I.$$
Then $h^{-1} = i''^{-1}\alpha_0 i' \in I\alpha_0 I$, and moreover
$$h^{-1} \in I\alpha_0 \begin{pmatrix} j & k \\ \ell & m \end{pmatrix}$$

for an element $\begin{pmatrix} j & k \\ \ell & m \end{pmatrix} \in I$ such that $v(k) = 1$.

We conclude that if h is of the form (4.6), then $h^{-1} \in I\alpha_0 I \cap Is\alpha^{-1}I\alpha_0$, proving Claim 1. Now if $h^{-1} \in I\alpha_0 I \cap Is\alpha_0^{-1}I\alpha_0$, then h is of form (4.6), so the half of the proof of Claim 1 proceeding from that assumption shows that

$$h^{-1} \in I\alpha_0 \begin{pmatrix} j & k \\ \ell & m \end{pmatrix}$$

for some $\begin{pmatrix} j & k \\ \ell & m \end{pmatrix}$ such that $v(k) = 1$. This proves Claim 2.

We now turn to the proof of Lemma 4.3.6. Suppose that $g \in G$ is of the form (4.5). Then

$$sg = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & u\pi^{-1} \\ b\pi^2 & d \end{pmatrix} = \begin{pmatrix} -b\pi^2 & -d \\ a & u\pi^{-1} \end{pmatrix},$$

which is of the form (4.6) in Claim 1. Hence, by Claim 1,

$$(sg)^{-1} \in I\alpha_0 I \cap Is\alpha_0^{-1}I\alpha_0,$$

so by Claim 2, there is moreover an element $\begin{pmatrix} j & k \\ \ell & m \end{pmatrix} \in I$ such that $v(k) = 1$ and

$$(sg)^{-1} \in I\alpha_0 \begin{pmatrix} j & k \\ \ell & m \end{pmatrix}.$$

Then

$$g^{-1} \in I\alpha_0 \begin{pmatrix} j & k \\ \ell & m \end{pmatrix} s = I\alpha_0 s(-s) \begin{pmatrix} j & k \\ \ell & m \end{pmatrix} s = I\alpha_0 s \begin{pmatrix} j & \ell \\ k & m \end{pmatrix},$$

and since $v(k) = 1$, we have $\begin{pmatrix} j & \ell \\ k & m \end{pmatrix} \in I$. Hence

$$g^{-1} \in I\alpha_0 sI.$$

And since $(sg)^{-1} \in Is\alpha_0^{-1}I\alpha_0$ as well, we have

$$g^{-1} \in Is\alpha_0^{-1}I\alpha_0 s = I(-s)\alpha_0^{-1}I\alpha_0 s,$$

so

$$g^{-1} \in I\alpha_0 sI \cap I(-s)\alpha_0^{-1}I\alpha_0,$$

which proves one direction of Lemma 4.3.6.

Conversely, suppose that $g \in G$ satisfies $g^{-1} \in I\alpha_0 sI \cap I(-s)\alpha_0^{-1}I\alpha_0 s$. Then, since $\alpha_0 s = s\alpha_0^{-1}$, taking $\ell = 1$ in Part (3) of Lemma (cite ch. 1 decomps) we get the left coset decomposition

$$I\alpha_0 sI = \coprod_{y \in \mathcal{O}_F^\times \cup \{0\}} I\alpha_0 s \begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix}.$$

Hence

$$g^{-1} \in I\alpha_0 s \begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix}$$

for some $y \in \mathcal{O}_F^\times \cup \{0\}$. Then

$$(sg)^{-1} \in I\alpha_0 s \begin{pmatrix} 1 & 0 \\ y\pi & 1 \end{pmatrix} s = I\alpha_0 \begin{pmatrix} -1 & -y\pi \\ 0 & -1 \end{pmatrix} \subset I\alpha_0 I.$$

And $g^{-1} \in I(-s)\alpha_0^{-1}I\alpha_0 s$ implies that

$$(sg)^{-1} \in I(-s)\alpha_0^{-1}I\alpha_0,$$

so we have

$$(sg)^{-1} \in I\alpha_0 I \cap I(-s)\alpha_0^{-1} I\alpha_0.$$

Then by Claim 1, up to multiplication by I on the left we have

$$sg = \begin{pmatrix} b\pi^2 & d \\ a & u\pi^{-1} \end{pmatrix}$$

for some a, d, $u \in \mathcal{O}_F^\times$ and $b \in \mathcal{O}_F$. Then, again up to multiplication by I on the left,

$$g = -s \begin{pmatrix} b\pi^2 & d \\ a & u\pi^{-1} \end{pmatrix} = \begin{pmatrix} a & u\pi^{-1} \\ -b\pi^2 & -d \end{pmatrix},$$

which is of the desired form (4.5). This completes the proof of Lemma 4.3.6. $\square$

We now return to the proof of Lemma 4.3.5. By Lemma 4.3.6, the $g \in G$ such that $g^{-1} \in I\alpha_0 sI \cap I(-s)\alpha_0^{-1} I\alpha_0 s$ are exactly those of the form

$$g = \begin{pmatrix} a & u\pi^{-1} \\ b\pi^2 & d \end{pmatrix}$$

for some a, d, $u \in \mathcal{O}_F^\times$ and $b \in \mathcal{O}_F$.

Next, fixing $g = \begin{pmatrix} a & u\pi^{-1} \\ b\pi^2 & d \end{pmatrix}$ with a, d, $u \in \mathcal{O}_F^\times$ and $b \in \mathcal{O}_F$, we identify δ and $\zeta \in \{\pm 1\}$ such that

$$(g, \delta)^{-1} \in I^* w(-1) h(\pi)^{-1} I^* h(\pi) w(1)(1, \zeta) \cap I^* h(\pi) w(1) I^*.$$

Define the following elements of I^*:

$$i_1 = \left(\begin{pmatrix} u^{-1} & 0 \\ -d\pi & u \end{pmatrix}, (u,\pi)_F \right),$$
$$i_2 = \left(\begin{pmatrix} 1 & 0 \\ -au^{-1}\pi & 1 \end{pmatrix}, 1 \right),$$
$$i_3 = \left(\begin{pmatrix} d & -b \\ -u\pi & a \end{pmatrix}, (a,\pi)_F \right).$$

Then (using the fact that $\det(g) = ad - ub\pi = 1$, as well as the fact that the Hilbert symbol on F is unramified) we calculate $i_1(g,\delta)i_2 =$

$$\left(\begin{pmatrix} u^{-1} & 0 \\ -d\pi & u \end{pmatrix}, (u,\pi)_F \right) \left(\begin{pmatrix} a & u\pi^{-1} \\ b\pi^2 & d \end{pmatrix}, \delta \right) \left(\begin{pmatrix} 1 & 0 \\ -au^{-1}\pi & 1 \end{pmatrix}, 1 \right)$$
$$= \begin{cases} \left(\begin{pmatrix} u^{-1}a & \pi^{-1} \\ -\pi & 0 \end{pmatrix}, \delta(u,\pi)_F(d,b\pi)_F \right) \left(\begin{pmatrix} 1 & 0 \\ -au^{-1}\pi & 1 \end{pmatrix}, 1 \right) & \text{if } b \neq 0 \\ \left(\begin{pmatrix} u^{-1}a & \pi^{-1} \\ -\pi & 0 \end{pmatrix}, \delta(u,\pi)_F(d,\pi)_F \right) \left(\begin{pmatrix} 1 & 0 \\ -au^{-1}\pi & 1 \end{pmatrix}, 1 \right) & \text{if } b = 0 \end{cases}$$
$$= \begin{cases} \left(\begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix}, \delta(u,\pi)_F(d,b\pi)_F \right) & \text{if } b \neq 0 \\ \left(\begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix}, \delta(u,\pi)_F(d,\pi)_F \right) & \text{if } b = 0 \end{cases}$$
$$= \begin{cases} w(-1)h(\pi)^{-1}(1, \delta(-ud,\pi)_F(d,b)_F) & \text{if } b \neq 0, \\ w(-1)h(\pi)^{-1}(1, \delta(-ud,\pi)_F) & \text{if } b = 0. \end{cases}$$

Thus

$$(g,\delta) \in \begin{cases} I^* w(-1)h(\pi)^{-1}(1, \delta(-ud,\pi)_F(d,b)_F)I^* & \text{if } b \neq 0 \\ I^* w(-1)h(\pi)^{-1}(1, \delta(-ud,\pi)_F)I^* & \text{if } b = 0. \end{cases} \tag{4.10}$$

We have

$$(g,\delta)^{-1} = \begin{cases} (g^{-1}, \delta(b\pi^2, -b\pi^2)_F) = (g^{-1},\delta) & \text{if } b \neq 0 \\ (g^{-1}, \delta(d,a)_F) = \delta(g^{-1},1) & \text{if } b = 0, \end{cases}$$

so $(g,\delta)^{-1}w(-1)h(\pi)^{-1}i_3 =$

$$\left(\begin{pmatrix} d & -u\pi^{-1} \\ -b\pi^2 & a \end{pmatrix}, \delta\right)\left(\begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix}, (-1,\pi)_F\right)\left(\begin{pmatrix} d & -b \\ -u\pi & a \end{pmatrix}, (a,\pi)_F\right)$$

$$= \begin{cases} \left(\begin{pmatrix} u & d\pi^{-1} \\ -a\pi & -b\pi \end{pmatrix}, \delta(a,b\pi)_F\right)\left(\begin{pmatrix} d & -b \\ -u\pi & a \end{pmatrix}, (a,\pi)_F\right) & \text{if } b \neq 0, \\ \left(\begin{pmatrix} u & d\pi^{-1} \\ -a\pi & -b\pi \end{pmatrix}, \delta(a,\pi)_F\right)\left(\begin{pmatrix} d & -b \\ -u\pi & a \end{pmatrix}, (a,\pi)_F\right) & \text{if } b = 0 \end{cases}$$

$$= \begin{cases} \left(\begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix}, \delta(a,b)_F\right) & \text{if } b \neq 0 \\ \left(\begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix}, \delta\right) & \text{if } b = 0 \end{cases}$$

$$= \begin{cases} w(-1)h(\pi)^{-1}(1, \delta(-1,\pi)_F(a,b)_F) & \text{if } b \neq 0 \\ w(-1)h(\pi)^{-1}(1, \delta(-1,\pi)_F) & \text{if } b = 0. \end{cases}$$

Thus

$$(g,\delta)^{-1}w(-1)h(\pi)^{-1} \in \begin{cases} I^* w(-1)h(\pi)^{-1}(1, \delta(-1,\pi)_F(a,b)_F)I^* & \text{if } b \neq 0 \\ I^* w(-1)h(\pi)^{-1}(1, \delta(-1,\pi)_F I^* & \text{if } b = 0. \end{cases} \tag{4.11}$$

Hence, by (4.10) and (4.11), we have $t_{w(-1)h(\pi)^{-1}(1,-1)}(g,\delta)\, t_{w(-1)h(\pi)^{-1}}((g,\delta)^{-1}w(-1)h(\pi)^{-1}) =$

$$\begin{cases} \delta(-ud,\pi)_F(d,b)_F\cdot\delta(-1,\pi)_F(a,b)_F = (ud,\pi)_F(ad,b)_F & \text{if } b\neq 0, \\ \delta(-ud,\pi)_F\cdot\delta(-1,\pi)_F = (ud,\pi)_F & \text{if } b=0. \end{cases} \tag{4.12}$$

The value of $c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1}(1,\zeta),w(-1)h(\pi)^{-1}}$ is equal to the volume of the union of left I^*-cosets represented by those $(g,\delta)\in\widetilde{G}$ such that $g=\begin{pmatrix} a & u\pi^{-1} \\ b\pi^2 & d\end{pmatrix}$ with a, d, $u\in\mathcal{O}_F^\times$ and $b\in\mathcal{O}_F$ and such that the value of (4.12) is equal to ζ. We will now show that

$$c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} = c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1}(1,-1),w(-1)h(\pi)^{-1}},$$

i.e., that (4.12) takes the two values ± 1 on equal volumes in $\widetilde{G}$.

Fix $b\in\mathcal{O}_F$, and note that if $b\neq 0$, then for any a, $d\in\mathcal{O}_F^\times$ we have

$$(ad,b)_F = (ad,\pi^{v(b)})_F = \begin{cases} 1 & \text{if } 2\big| b \\ (ad,\pi)_F & \text{if } 2\not\big| b. \end{cases}$$

Hence the value of (4.12) depends on a, u, d, and b as follows:

$$(4.12) = \begin{cases} (ud,\pi)_F & \text{if } b\neq 0 \text{ and } 2\big| b \\ (ua,\pi)_F & \text{if } b\neq 0 \text{ and } 2\not\big| b \\ (ud,\pi)_F & \text{if } b=0. \end{cases}$$

Note that, when $b\in\mathcal{O}_F$ is fixed, we have the freedom to choose any two of a, d, $u\in\mathcal{O}_F^\times$ to define

$$g=\begin{pmatrix} a & u\pi^{-1} \\ b\pi^2 & d\end{pmatrix}.$$

Hence as a, u, and d run over $\mathcal{O}_F^\times$ such that this g is in G for the fixed choice of b, the first argument of the Hilbert symbol in (4.12) (corresponding to the valuation of b) runs uniformly over $\mathcal{O}_F^\times$. Then, since

$$\{x \in \mathcal{O}_F^\times : (x, \pi)_F = 1\}$$

has index 2 in $\mathcal{O}_F^\times$, the value of (4.12) is equal to 1 on half of the total volume of the union of left I^*-cosets represented by (g, δ) with $g = \begin{pmatrix} a & u\pi^{-1} \\ b\pi^2 & d \end{pmatrix} \in G$ (with fixed $b \in \mathcal{O}_F$ and with a, d, $u \in \mathcal{O}_F^\times$), and is equal to -1 on the other half. Since this holds for each $b \in \mathcal{O}_F$, we conclude that

$$c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} = c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1}(1,-1),w(-1)h(\pi)^{-1}}.$$

Since the two coefficients are equal, for each $\zeta \in \{\pm 1\}$ we have

$$|I^* \setminus I^*h(\pi)w(1)I^* \cap I^*w(-1)h(\pi)^{-1}I^*h(\pi)w(1)(1,\zeta)| = \frac{1}{2}|I \setminus I\alpha_0 sI \cap Is\alpha_0^{-1}I\alpha_0 s|,$$

so to finish the calculation it suffices to show that

$$|I \setminus I\alpha_0 sI \cap Is\alpha_0^{-1}I\alpha_0 s| = q - 1.$$

Consider the convolution product

$$\mathbf{1}_{Is\alpha_0^{-1}I} \cdot \mathbf{1}_{Is\alpha_0^{-1}I} = c^1_{s\alpha_0^{-1},s\alpha_0^{-1}}\mathbf{1}_I + c^{s\alpha_0^{-1}}_{s\alpha_0^{-1},s\alpha_0^{-1}}\mathbf{1}_{Is\alpha_0^{-1}I} \tag{4.13}$$

in $\mathcal{H}_\mathbb{Z}(G, I)$, and note that

$$c^1_{s\alpha_0^{-1},s\alpha_0^{-1}} = |I \setminus I\alpha_0 sI \cap Is\alpha_0^{-1}I| = |I \setminus Is\alpha_0^{-1}I| = \mathrm{vol}_G(Is\alpha_0^{-1}I),$$

$$c^{s\alpha_0^{-1}}_{s\alpha_0^{-1},s\alpha_0^{-1}} = |I \setminus I\alpha_0 sI \cap Is\alpha_0^{-1}I\alpha_0 s|,$$

Applying the ring homomorphism $\mathrm{vol}_G : \mathcal{H}_{\mathbb{Z}}(G, I) \to \mathbb{Z}$ to both sides of (4.13), we have

$$\left(\mathrm{vol}_G(Is\alpha_0^{-1}I)\right)^2 = c^1_{s\alpha_0^{-1},s\alpha_0^{-1}}\mathrm{vol}_G(I) + c^{s\alpha_0^{-1}}_{s\alpha_0^{-1},s\alpha_0^{-1}}\,\mathrm{vol}_G(Is\alpha_0^{-1}I). \tag{4.14}$$

By Part (3) of Lemma (cite ch. 1 volume calcs) with $\ell = 1$, we have $\mathrm{vol}_G(Is\alpha_0^{-1}I) = q$ while $\mathrm{vol}_G(I) = 1$, so the equation (4.14) becomes

$$q^2 = q(1 + c^{s\alpha_0^{-1}}_{s\alpha_0^{-1},s\alpha_0^{-1}}).$$

Thus

$$c^{s\alpha_0^{-1}}_{s\alpha_0^{-1},s\alpha_0^{-1}} = q - 1,$$

so

$$c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1}(1,\zeta),w(-1)h(\pi)^{-1}} = \frac{q-1}{2}$$

for each $\zeta \in \{\pm 1\}$. This concludes the lemma. $\square$

We now return to the calculation of the convolution product $t_{w(-1)h(\pi)^{-1}} \cdot t_{w(-1)h(\pi)^{-1}}$:

$$\begin{aligned}
t_{w(-1)h(\pi)^{-1}}\cdot t_{w(-1)h(\pi)^{-1}} =& \\
&\left(c^{(1,1)}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} - c^{(1,1)}_{w(-1)h(\pi)^{-1}(1,-1),w(-1)h(\pi)^{-1}}\right) t_{(1,1)} \\
&+ \left(c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1},w(-1)h(\pi)^{-1}} - c^{w(-1)h(\pi)^{-1}}_{w(-1)h(\pi)^{-1}(1,-1),w(-1)h(\pi)^{-1}}\right) t_{w(-1)h(\pi)^{-1}} \\
&= \left((-1)^{\frac{q-1}{2}} q\right) t_{(1,1)} + \left(\frac{q-1}{2} - \frac{q-1}{2}\right) t_{w(-1)h(\pi)^{-1}}. \\
&= \left((-1)^{\frac{q-1}{2}} q\right) t_{(1,1)} \equiv 0 \pmod{p}.
\end{aligned}$$

Thus the product $t_{w(-1)h(\pi)^{-1}} \cdot t_{w(-1)h(\pi)^{-1}}$ is equal to 0 in $\mathbb{H}^{\epsilon}_p(\widetilde{G}, I^*)$; equivalently, $T^0_{2,1} \circ T^0_{2,1}(e^{0,+}_{0,1}) = 0$ over $\bar{\mathbb{F}}_p$. Hence $T^0_{2,1} \cdot T^0_{2,1} = 0$ in $\mathcal{H}^{\epsilon}_p(\widetilde{G}, I^*)$.

3. We first show that $xy = T^1_{0,-1} \circ T^0_{2,1} = T^{-1}_{2,3}$. The corresponding convolution product is

$$t_{w(-1)} \cdot t_{w(-1)h(\pi)^{-1}} = \sum_{w'' \in S^+} \left(c^{w''}_{w(-1),w(-1)h(\pi)^{-1}} - c^{w''}_{w(-1)(1,-1),w(-1)h(\pi)^{-1}} \right) \cdot t_{w''}.$$

By Lemma 4.2.8 (2),

$$I^* w(-1) I^* w(-1) h(\pi)^{-1} I^* = I^* h(\pi)^{-1} I^*,$$

so for $w'' \in S^+$, we have

$$c^{w''}_{w(-1)(1,\zeta),w(-1)h(\pi)^{-1}} = \begin{cases} 1 & \text{if } \zeta = 1 \text{ and } w'' = h(\pi)^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$t_{w(-1)} \cdot t_{w(-1)h(\pi)^{-1}} = t_{h(\pi)^{-1}},$$

and $t_{h(\pi)^{-1}}$ corresponds by Frobenius reciprocity to $T^{-1}_{2,3}$.

Now we show that $(xy)^k = (T^{-1}_{2,3})^k = T^{-k}_{2k,2k+1}$ for all $k \geq 1$. The base case $k = 1$ is done; suppose that for some $k \geq 1$ we have $(xy)^k = T^{-k}_{2k,2k+1}$, and consider $T^{-k}_{2k,2k+1} \circ T^{-1}_{2,3}$. The corresponding product in $\mathbb{H}^{\epsilon}_p(\widetilde{G}, I^*)$ is

$$t_{h(\pi)^{-k}} \cdot t_{h(\pi)^{-1}} = \sum_{w'' \in S^+} \left(c^{w''}_{h(\pi)^{-k},h(\pi)^{-1}} - c^{w''}_{h(\pi)^{-k}(1,-1),h(\pi)^{-1}} \right) \cdot t_{w''}.$$

By Lemma 4.2.8 (1),

$$I^* h(\pi)^{-k} I^* h(\pi)^{-1} I^* = I^* h(\pi)^{-(k+1)} I^*,$$

so for $w'' \in S^+$,

$$c^{w''}_{h(\pi)^{-k}(1,\zeta),h(\pi)^{-1}} = \begin{cases} 1 & \text{if } \zeta = 1 \text{ and } w'' = h(\pi)^{-(k+1)} \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$t_{h(\pi)^{-k}} \cdot t_{h(\pi)^{-1}} = t_{h(\pi)^{-(k+1)}},$$

which implies that

$$T^{-k}_{2k,2k+1} \circ T^{-1}_{2,3} = T^{-(k+1)}_{2(k+1),2(k+1)+1}$$

in $\mathcal{H}^{\epsilon}_p(\widetilde{G}, I^*)$. So by induction,

$$(xy)^k = (T^{-1}_{2,3})^k = T^{-k}_{2k,2k+1}$$

for all $k \geq 1$.

4. We will first show that $yx = T^0_{2,1} \circ T^1_{0,-1} = (-1)^{\frac{q-1}{2}} T^1_{-2,-1}$. Calculating in $\mathbb{H}^{\epsilon}_p(\widetilde{G}, I^*)$,

$$t_{w(-1)h(\pi)^{-1}} \cdot t_{w(-1)} = \sum_{w'' \in S^+} \left(c^{w''}_{w(-1)h(\pi)^{-1},w(-1)} - c^{w''}_{w(-1)h(\pi)^{-1}(1,-1),w(-1)} \right) t_{w''}.$$

By Lemma 4.2.8 (3),

$$I^* w(-1) h(\pi)^{-1} w(-1) I^* = I^* h(\pi)(1, (-1)^{\frac{q-1}{2}}) I^*.$$

Hence for $w'' \in S^+$,

$$c^{w''}_{w(-1)h(\pi)^{-1}(1,\zeta),w(-1)} = \begin{cases} 1 & \text{if } \zeta = (-1)^{\frac{q-1}{2}} \text{ and } w'' = h(\pi) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$t_{w(-1)h(\pi)^{-1}} \cdot t_{w(-1)} = (-1)^{\frac{q-1}{2}} t_{h(\pi)}, \tag{4.15}$$

which implies, via Frobenius reciprocity, that in $\mathcal{H}_p^{\epsilon}(\widetilde{G}, I^*)$ we have

$$T^0_{2,1} \circ T^1_{0,-1} = (-1)^{\frac{q-1}{2}} T^1_{-2,-1}.$$

Next we show that $T^k_{-2k,-2k+1} \circ T^1_{-2,-1} = T^{k+1}_{-2(k+1),-2(k+1)+1}$ for all $k \geq 1$. The relevant convolution product is

$$t_{h(\pi)^k} \cdot t_{h(\pi)} = \sum_{w'' \in S^+} \left(c^{w''}_{w(-1),w(-1)h(\pi)^{-1}} - c^{w''}_{w(-1)(1,-1),w(-1)h(\pi)^{-1}} \right) \cdot t_{w''}.$$

By Lemma 4.2.8 (1), $I^*h(\pi)^k I^* h(\pi) I^* = I^* h(\pi)^{k+1} I^*$, so for $w'' \in S^+$,

$$c^{w''}_{h(\pi)^k(1,\zeta),h(\pi)} = \begin{cases} 1 & \text{if } \zeta = 1 \text{ and } w'' = h(\pi)^{k+1} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$t_{h(\pi)^k} \cdot t_{h(\pi)} = t_{h(\pi)^{k+1}},$$

which implies that

$$T^k_{-2k,-2k+1} \circ T^1_{-2,-1} = T^{k+1}_{-2(k+1),-2(k+1)+1}$$

for all $k \geq 1$. By induction on k,

$$(T^1_{-2,-1})^k = T^k_{-2,-1} \tag{4.16}$$

for all $k \geq 1$.

Now, by (4.15) and then (4.16),

$$(yx)^k = (-1)^{k\frac{q-1}{2}} (T^1_{-2,-1})^k = (-1)^{k\frac{q-1}{2}} T^k_{-2k,-2k+1}.$$

5. By Prop. 4.3.4 (3) above, $y(xy)^k = T^0_{2,1} \circ T^{-k}_{2k,2k+1}$ for all $k \geq 1$. We calculate the

corresponding product in $\mathbb{H}_p^\epsilon(\widetilde{G}, I^*)$:

$$t_{w(-1)h(\pi)^{-1}} \cdot t_{h(\pi)^{-k}} = \sum_{w'' \in S^+} \left(c^{w''}_{w(-1)h(\pi)^{-1},h(\pi)^{-k}} - c^{w''}_{w(-1)h(\pi)^{-1}(1,-1),h(\pi)^{-k}} \right) \cdot t_{w''}.$$

By Lemma 4.2.8 (4),

$$I^* w(-1)h(\pi)^{-1} I^* w(-k) I^* = I^* w(-1)h(\pi)^{-(k+1)},$$

so

$$c^{w''}_{w(-1)h(\pi)^{-1}(1,\zeta),h(\pi)^{-k}} = \begin{cases} 1 & \text{if } \zeta = 1 \text{ and } w'' = h(\pi)^{-(k+1)} \\ 0 & \text{otherwise,} \end{cases}$$

which gives the result

$$t_{w(-1)h(\pi)^{-1}} \cdot t_{h(\pi)^{-k}} = t_{w(-1)h(\pi)^{-(k+1)}}. \tag{4.17}$$

Applying Frobenius reciprocity to both sides of (4.17), we get

$$T^0_{2,1} \circ T^{-k}_{2k,2k+1} = T^{-k}_{2k+2,2k+1},$$

so $y(xy)^k = T^{-k}_{2k+2,2k+1}$ as desired.

6. By Prop. 4.3.4 (4) above,

$$x(yx)^k = T^1_{0,-1} \circ (-1)^{k\frac{q-1}{2}} T^k_{-2k,-2k+1} = (-1)^{k\frac{q-1}{2}} \left(T^1_{0,-1} \circ T^k_{-2k,-2k+1} \right).$$

We compute the convolution product in $\mathbb{H}_p^\epsilon(\widetilde{G}, I^*)$ which corresponds to $T^1_{0,-1} \circ T^k_{-2k,-2k+1}$:

$$t_{w(-1)} \circ t_{h(\pi)^k} = \sum_{w'' \in S^+} \left(c^{w''}_{w(-1),h(\pi)^k} - c^{w''}_{w(-1)(1,-1),h(\pi)^k} \right) \cdot t_{w''}.$$

Since $k > 0$, Lemma 4.2.8 (5) gives

$$I^* w(-1) I^* h(\pi)^k I^* = I^* w(-1) h(\pi)^k I^*,$$

so for $w'' \in S^+$,

$$c^{w''}_{w(-1)(1,\zeta),h(\pi)^k} = \begin{cases} 1 & \text{if } \zeta = 1 \text{ and } w'' = w(-1)h(\pi)^k \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$t_{w(-1)} \circ t_{h(\pi)^k} = t_{w(-1)h(\pi)^k},$$

which implies by Frobenius reciprocity that

$$T^1_{0,-1} \circ T^k_{-2k,-2k+1} = T^{k+1}_{-2k,-2k-1}.$$

Hence

$$x(yx)^k = (-1)^{k\frac{q-1}{2}} \left(T^1_{0,-1} \circ T^k_{-2k,-2k+1} \right) = (-1)^{k\frac{q-1}{2}} T^{k+1}_{-2k,-2k-1}.$$

□

Theorem 4.3.7. *The genuine mod p Iwahori Hecke algebra $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ is generated by $x := T^1_{0,-1}$ and $y := T^0_{2,1}$, and the algebra has the following presentation as a noncommutative polynomial algebra:*

$$\mathcal{H}^\epsilon_p(\widetilde{G}, I^*) \cong \bar{\mathbb{F}}_p\langle x, y\rangle / (x^2 + x, y^2). \tag{4.18}$$

Proof. Consider the $\bar{\mathbb{F}}_p$-linear homomorphism

$$\bar{\mathbb{F}}_p\langle x, y\rangle \to \mathcal{H}^\epsilon_p(\widetilde{G}, I^*) \tag{4.19}$$

defined by

$$x \mapsto T^1_{0,-1},\ y \mapsto T^0_{2,1}.$$

Since $(T^1_{0,-1})^2 = -T^1_{0,-1}$ by Proposition 4.3.4 (1) and $(T^0_{2,1})^2 = 0$ by Proposition 4.3.4 (2), the map (4.19) factors through the relations $x^2 + x$ and y^2. We now show that the induced map

$$\Omega : \bar{\mathbb{F}}_p\langle x, y\rangle/(x^2 + x, y^2) \to \mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$$

is an $\bar{\mathbb{F}}_p$-algebra isomorphism.

An $\bar{\mathbb{F}}_p$-vector space basis for $\bar{\mathbb{F}}_p\langle x, y\rangle/(x^2 + x, y^2)$ is given by the set

$$A = \{1, x, y, (xy)^k, (yx)^k, x(yx)^k, y(xy)^k\}_{k\geq 1}.$$

On the other hand, a $\bar{\mathbb{F}}_p$-vector space basis for $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ is given (cf. Remark 4.2.3) by the set

$$B = \{T^{-k}_{2k,2k+1}, T^{-k}_{2k,2k-1}\}_{k\in\mathbb{Z}}.$$

Proposition 4.3.4 demonstrates that the following elements of $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ are collinear with distinct basis elements $T^\ell_{m,n} \in B$:

$$\left\{1, T^1_{0,-1}, T^0_{2,1}, (T^1_{0,-1}\circ T^0_{2,1})^k, (T^0_{2,1}\circ T^1_{0,-1})^k, T^0_{2,1}\circ(T^1_{0,-1}\circ T^0_{2,1})^k, T^1_{0,-1}\circ(T^0_{2,1}\circ T^1_{0,-1})^k\right\}_{k\geq 1} \quad (4.20)$$

and that each basis element $T^\ell_{m,n} \in B$ is collinear with exactly one element of (4.20). So $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ is generated as an $\bar{\mathbb{F}}_p$-algebra by $T^1_{0,-1}$ and $T^0_{2,1}$, and the map $x \mapsto T^1_{0,-1}$, $y \mapsto T^0_{2,1}$ is a bijection between the vector space basis A of $\bar{\mathbb{F}}_p\langle x, y\rangle/(x^2 + x, y^2)$ and the vector space basis B of $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$. Hence Ω is a bijective map of $\bar{\mathbb{F}}_p$-vector spaces. Since Ω was also an $\bar{\mathbb{F}}_p$-algebra homomorphism by construction, we conclude that Ω is an $\bar{\mathbb{F}}_p$-algebra isomorphism. Thus $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ has the presentation

$$\bar{\mathbb{F}}_p\langle x, y\rangle/(x^2 + x, y^2)$$

as a noncommutative polynomial algebra over $\bar{\mathbb{F}}_p$. □

Corollary 4.3.8. *The genuine mod p Iwahori Hecke algebra* $\mathcal{H}^\epsilon_p(\widetilde{SL}_2(F), I^*)$ *is not isomor-*

phic to the mod p Iwahori Hecke algebra $\mathcal{H}_p(PGL_2(F), I_G)$.

Proof. If the two algebras are isomorphic, then they must have equal numbers of $\bar{\mathbb{F}}_p$-characters. However, we will show that $\mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$ has only two distinct $\bar{\mathbb{F}}_p$-characters while $\mathcal{H}_p(PGL_2(F), I_G)$ has four.

We have the following presentations for the two algebras:

$$\mathcal{H}_p^\epsilon(\widetilde{SL}_2(F), I^*) \cong \bar{\mathbb{F}}_p\langle x, y\rangle/(x^2+x, y^2) \qquad \text{(Theorem 4.3.7)}$$

$$\mathcal{H}_p(PGL_2(F), I_G) \cong \bar{\mathbb{F}}_p\langle a, b\rangle/(a^2-1, bab+b) \qquad \text{([3] Prop. 7, cf. Proposition 4.2.1)}$$

Consider an $\bar{\mathbb{F}}_p$-character of $\mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$, by which we mean an $\bar{\mathbb{F}}_p$-linear homomorphism $\chi : \bar{\mathbb{F}}_p\langle x, y\rangle/(x^2+x, y^2) \to \bar{\mathbb{F}}_p$. Such a map is determined by its values on x and y, which must satisfy

$$\chi(y)^2 = \chi(y^2) = \chi(0) = 0 \tag{4.21}$$

and

$$\chi(x)^2 = \chi(x^2) = \chi(-x) = -\chi(x). \tag{4.22}$$

From (4.21) we deduce $\chi(y) = 0$, and from (4.22) we deduce that $\chi(x)$ is a root of the polynomial $z^2 + z = z(z+1)$, hence $\chi(x) = 0$ or $\chi(x) = -1$. So $\mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$ has two $\bar{\mathbb{F}}_p$-characters,

$$\chi_{0,0} : x \mapsto 0,\ y \mapsto 0, \text{ and}$$
$$\chi_{-1,0} : x \mapsto -1,\ y \mapsto 0.$$

We will call $\chi_{0,0}$ the *trivial character* and $\chi_{-1,0}$ the *sign character* of $\mathcal{H}_p^\epsilon(\widetilde{G}, I^*)$.

On the other hand, consider an $\bar{\mathbb{F}}_p$-linear homomorphism $\chi : \bar{\mathbb{F}}_p\langle a, b\rangle/(a^2-1, bab+b) \to \bar{\mathbb{F}}_p$. Such a map must satisfy

$$\chi(a)^2 - 1 = \chi(a^2-1) = 0, \tag{4.23}$$

$$\chi(b)^2\chi(a)+\chi(b)=\chi(b)\chi(a)\chi(b)+\chi(b)=\chi(bab+b)=0. \tag{4.24}$$

By (4.23), we must have $\chi(a)=1$ or $\chi(a)=-1$. If $\chi(a)=1$, (4.24) implies that

$$\chi(b)^2+\chi(b)=0,$$

so $\chi(b)$ is a root of $z(z+1)$ and hence $\chi(b)=0$ or $\chi(b)=-1$. If $\chi(a)=-1$, then (4.24) implies that

$$-\chi(b)^2+\chi(b)=0,$$

so $\chi(b)$ is a root of $z(1-z)$ and hence $\chi(b)=0$ or $\chi(b)=1$. Conversely, each of the four possibilities we have listed does define a $\bar{\mathbb{F}}_p$-character of $\mathcal{H}_p(PGL_2(F),I_G)$. We label them as follows:

$$\begin{aligned}
\chi_{1,0}&: a\mapsto 1,\ \ b\mapsto 0,\\
\chi_{1,-1}&: a\mapsto 1,\ \ b\mapsto -1,\\
\chi_{-1,0}&: a\mapsto -1,\ \ b\mapsto 0,\ \text{and}\\
\chi_{-1,-1}&: a\mapsto -1,\ \ b\mapsto -1.
\end{aligned}$$

Since $\mathcal{H}_p^\epsilon(\widetilde{G},I^*)$ has only two distinct $\bar{\mathbb{F}}_p$-characters while $\mathcal{H}_p(PGL_2(F),I_G)$ has four, we conclude that the two algebras are not isomorphic. □

4.4 Comparison of $\mathcal{H}^\epsilon(\widetilde{SL}_2(F),I^*)$ with other Iwahori Hecke algebras

Though the two algebras are not isomorphic, we can describe some relationships between $\mathcal{H}_p^\epsilon(\widetilde{G},I^*)$ and $\mathcal{H}_p(PGL_2,I_G)$.

Calculating in $\mathcal{H}(PGL_2(F),I_G)$, we have $ab=T_{1,0}\circ T_{1,2}=T_{2,1}$ and $ba=T_{1,2}\circ T_{1,0}=T_{0,-1}$. Hence there is a natural identification, in terms of correspondences on the tree of SL_2, of the generators x and y of $\mathcal{H}^\epsilon(\widetilde{G},I^*)$ with these elements of $\mathcal{H}_p(PGL_2(F),I_G)$. Identify $x=T^1_{0,-1}$

with $ba = T_{1,0}$ and $y = T^0_{2,1}$ with $ab = T_{2,1}$; if we try to extend this to a map of $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ into $\mathcal{H}_p(PGL_2(F), I_G)$, the result is well-defined on the one-parameter subalgebras of $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ generated by x, by xy, and by yx, but not on the one-parameter subalgebra generated by y, since $y^2 = 0$ but $(ab)^2 = -ab \neq 0$. So $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ is the quotient of the subalgebra of $\mathcal{H}_p(PGL_2(F), I_G)$ generated by $\langle ab, ba\rangle$ by the relation $(ab)^2 = 0$, and in particular it is a subquotient of $\mathcal{H}_p(PGL_2(F), I_G)$.

Note that the subalgebra of $\mathcal{H}_p(PGL_2(F), I_G)$ generated by $\langle ab, ba\rangle$ is just the Iwahori Hecke algebra of $G = SL_2(F)$ in terms of Barthel-Livné generators: it is the algebra of operators corresponding to edges of the tree which originate at vertices lying at even distances from the base vertex, with composition relations calculated as in $\mathcal{H}(GL_2(F))$. Hence it is natural to identify $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ with the quotient of $\mathcal{H}_p(G, I)$ by the square of one of its generators, in particular the one sent to $ab = T_{1,0}$ when $\mathcal{H}_p(G, I_G)$ embeds in $\mathcal{H}_p(PGL_2(F), I_G)$.

4.5 Further questions

We emphasize the fact that Theorem 4.3.4 and Corollary 4.3.8 give a picture quite different from the one in characteristic 0, where the two algebras are isomorphic. This section lists some questions for future work; their answers should help explain the impact of Corollary 4.3.8 on the mod p representation theory of $\widetilde{G}$.

The first question concerns the relationship between $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ and $\mathcal{H}^\epsilon_p(\widetilde{G}, K^*)$. Ollivier [20] has shown the compatibility of the Satake and Bernstein maps for split reductive groups mod p. In that situation the Bernstein map gives an explicit isomorphism of the group algebra of the dominant cocharacters with the center of the Iwahori Hecke algebra, so the spherical Hecke algebra embeds as the center of the Iwahori Hecke algebra. It will be interesting to know whether $\mathcal{H}^\epsilon_p(\widetilde{G}, K^*)$ embeds in $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$, and if so whether its image is central.

Related to the spherical Hecke algebra, we would like to know whether the partial bijection of unramified principal series representations can be completed in a natural way by identifying the representation $I(\widetilde{\mathrm{sgn}})$ of $\widetilde{G}$ with the Steinberg representation St via the actions of $\mathcal{H}^\epsilon_p(\widetilde{G}, I^*)$ and of $\mathcal{H}_p(PGL_2(F), I_G)$, respectively, on their Iwahori-fixed vectors.

Likewise, we can ask whether the partial bijection we have defined using the spherical Hecke algebra can also be defined in terms of actions of the Iwahori Hecke algebras on Iwahori-fixed vectors.

More generally, it is not known whether the functor of I^*-invariants gives an equivalence of categories, a bijection, or neither between the category of smooth irreducible genuine mod p representations of $\widetilde{G}$ generated by their I^*-fixed vectors and the category of simple modules over the genuine Iwahori Hecke algebra of $\widetilde{G}$.

The previous question becomes particularly interesting when I^* is replaced by its pro-p subgroup $I(1)^*$. Since every mod p representation of a p-adic group has a vector fixed by its pro-p-Iwahori subgroup, one expects modules over the pro-p Iwahori Hecke algebra to give the most complete information about the mod p representation theory. As a starting point, one can ask for a presentation of the genuine pro-p Iwahori Hecke algebra of $\widetilde{G}$ and whether the genuine Iwahori Hecke algebra embeds in it.

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