Characterizing distribution rules for cost sharing games

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This thesis is dedicated to

my parents,

whose love and support have made everything possible.

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Abstract

Cost sharing games, traditionally studied in economics in the context of cooperative game theory, have garnered recent interest in the computer science community in the noncooperative setting, where individually strategic agents choose resources based on how the welfare (cost or revenue) generated at each resource (which depends on the set of agents that choose the resource) is distributed. The focus is on finding distribution rules that lead to stable allocations, which is formalized by the concept of *Nash equilibrium*.

There are many known distribution rules that guarantee the existence of a (pure Nash) equilibrium in the noncooperative setting, e.g., the Shapley value and its weighted variants (budget-balanced), and the marginal contribution (not budget-balanced). To better understand the limitations on the design of distribution rules, researchers have recently sought to provide characterizations of the class of distribution rules that guarantee equilibrium existence. Recent work has shown that the only *budget-balanced* distribution rules that guarantee equilibrium existence in all welfare sharing games are generalized weighted Shapley value (GWSV) distribution rules. Though this characterization seems general, its proof consists of exhibiting a specific 'worst-case' welfare function which requires that GWSV distribution rules be used. Thus, characterizing the space of distribution rules (not necessarily budget-balanced) for any specific local welfare functions remains an important open problem. Our work provides an exact characterization of this space for a general class of scalable and separable games, which includes a variety of applications such as facility location, routing, network formation, and coverage games as special cases.

Our main result states that all games conditioned on any fixed local welfare functions possess an equilibrium if and only if the distribution rules are equivalent to GWSV distribution rules on some 'ground' welfare functions. This shows that it is neither the existence of some worst-case welfare function, nor the restriction of budget-balance, which limits the design to GWSV distribution rules.

Our second result provides an alternative characterization of the set of distribution rules that guarantee the existence of an equilibrium. In particular, it states that all games conditioned on any fixed local welfare functions possess an equilibrium if and only if the distribution rules are equivalent to generalized weighted marginal contribution (GWMC) distribution rules on some 'ground' welfare functions. This result is actually a consequence of a connection between Shapley values and marginal contributions, namely that they can be viewed as equivalent given a transformation connecting their ground welfare functions.

These characterizations provide two alternatives for designing distribution rules, trading off budget-balance (easier to control with Shapley values) and computational tractability (marginal contributions are more tractable). Another important consequence of our result is that, in order to guarantee equilibrium existence in all games with any fixed welfare function, it is *necessary* to work within the class of potential games, since GWSV and GWMC distribution rules result in (weighted) potential games. Additionally, the proofs develop tools for analyzing cost sharing games which could be useful for related models, such as cost sharing mechanisms, and also expose a key relationship between Shapley values and marginal contributions, leading to a novel closed-form expression for the potential function.

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Chapter 1

Introduction

Fair division is an issue that is at the heart of social science. An important constituent of this broad concept is the problem of cost sharing—how should the cost incurred (or revenue generated) by a group of self-interested agents be shared among them? This central question has led to a large literature in economics over the last several decades.

While a few central ideas behind cost sharing can be traced back to the late nineteenth century in the context of trading between two agents (Edgeworth [23]), a formal and sustained study of cost sharing historically emerged in 1944 as the central theme of the theory of *cooperative games with transferable utility* (von Neumann and Morgenstern [86]), which provided valuable tools for a structured study on the subject. Moulin [58] traces the research advances in axiomatic cost sharing that was initiated in 1953 by the seminal work of Shapley [78], and credits this line of research with generating some of the deepest axiomatic results of modern microeconomic theory. Young [93] provides a discussion of equitable division of costs and benefits in cooperative enterprises, one of the earliest applications of a formal study of cost sharing. Young [92] further lists several diverse applications where the cost sharing problem emerges—involuntary settings such as allocation of infrastructure costs among different calls by a telephone company, allocating computing costs among different departments by a university, setting landing fees for aircraft of varying sizes by aviation authorities, determining road taxes for vehicles causing different levels of wear and tear to the roadways by highway departments, etc., as well as voluntary settings such as roommates sharing living costs, municipalities sharing setup and operating costs of a common water supply system, etc.

The common feature of all these diverse examples is that prices are not determined by market forces, but are set internally by mutual agreement or administrative decision. Practical cost sharing scenarios provide motivation for abstract solution concepts of cooperative game theory, and different frameworks emerged for modeling the common problem of cost sharing, depending on the context of the application (Young [92]).

A standard framework that we adopt in this thesis is that of *cost sharing games*, where a set of agents makes strategic choices of which resources to utilize. Each resource generates a welfare (cost or revenue)[1] depending on the set of agents that choose the resource, and the total welfare across all the resources is then shared among the agents. A wide variety of applications fit this broad description—the terms 'resource', 'utilize', 'welfare' are abstract, and could mean different things depending on the specific situation that is being modeled. For example, consider a setting where graduate students (agents) in a research group choose which projects (resources) they should work on. Not all students might be able to choose all projects—there will be constraints on what combinations of projects a student can work on, depending on factors such as research background and ability to multitask. These constraints determine the 'action sets' of the students. Once the students make their decisions, each project generates a certain reward (welfare) depending on the progress made by the students working on that project, and the total reward across all projects is distributed among the students according to some rule. Compatibility between students (whether two students can get along well with each other or not) can be taken into account while specifying the reward function of each project (since it affects progress).

This framework can also model resource allocation problems that arise in computer systems, such as network formation (Jackson [40]) and facility location (Goemans and Skutella [32]), as observed in recent years by researchers in computer science. This observation was one of many convergent factors that contributed to the genesis of the field of algorithmic game theory in the late 1990s. Jain and Mahdian [41] provide an introduction to cost sharing and cooperative game theory from this perspective. Moulin [60] surveys recent research in network cost sharing and lists several open problems in the area. In such cooperative cost sharing games, the problems studied typically involve a cost value $v(S)$ for each subset of agents S, which usually stems from the optimal solution to an underlying combinatorial optimization problem.[2] A canonical example for illustrating network cost sharing is the multicast network formation game (Granot and Huberman [36]), where a set of consumers (agents) N wishes to connect to a common source (a broadcaster) s by utilizing links (resources) of an underlying graph. Each link has a cost associated with its usage, and the total cost of all the links used needs to be split among the agents. In such a situation, any subset S of agents, if they cooperate, can form a coalition, and the best they can do is to choose the links of the minimum cost spanning tree for the set of vertices $S \cup \{s\}$, and incur its cost—denote it by $v(S)$.

A cooperative framework, in effect, models a 'binary choice' for the agents—opt out, or opt in and cooperate. An important solution concept in cooperative game theory, called the *core*, is defined as the set of all possible ways of distributing $v(N)$ to the agents in N in such a way that it is in their best interest to fully

[1] The term 'cost sharing games' has come to refer to both cost and revenue sharing games. Our convention is to interpret positive welfare as revenue and negative welfare as cost. Hence, the objective of the agents is always *welfare maximization*.

[2] Note that our focus in this thesis is on cost sharing *games* and not cost sharing *mechanisms* (Feigenbaum et al. [27]), which additionally involve soliciting agents' exogenous private valuations of attaining the end goal. We briefly discuss the applicability of our work to cost sharing mechanisms in Chapter 6.

cooperate (and hence form the grand coalition). In other words, a distribution rule $f^v : N \to \mathbb{R}$ is in the core, if $\sum_{i \in N} f^v(i) = v(N)$, and for every subset $S \subseteq N$, $\sum_{i \in S} f^v(i) \leq v(S)$. This solution concept carries with it a very strong notion of stability—for distribution rules in the core, the grand coalition is immune to arbitrary *coalitional deviations* by the agents. In a way, such a severe stability requirement is justified in the cooperative setting, where, to begin with, agents are assumed to be open to cooperating with each other, and cooperation is feasible in, if not facilitated by, the environment. In general, the core can be empty, though for multicast games it is not.

In contrast, in large distributed (and often unregulated) systems such as the Internet, agents' options are more complex as they have the opportunity to strategically choose the best action from multiple available options. Cooperation is rarely desired, and even then, rarely feasible in such settings; therefore, a *noncooperative cost sharing game* is a better model, where agents are concerned only with minimizing their individual share of the total cost across all the resources. Moreover, for the same reason, it is more appropriate for the distribution rule to be decentralized, i.e., composed of *local* resource-specific distribution rules that specify how the local cost generated at a resource is shared (only) among the agents who choose that resource. Accordingly, there has been an emerging focus recently on weaker notions of stability that suit the noncooperative model, such as the Nash equilibrium, which is an allocation that is immune to *unilateral deviations* by any single agent. This focus is driven by network cost sharing applications where, more recently, individually strategic behavior is a more suitable assumption. The first such models of noncooperative cost sharing appeared just a decade ago, in the context of facility location (Vetta et al. [84]) and network formation (Fabrikant et al. [24]).

Our previous example of multicast games also provides a useful illustration of the noncooperative cost sharing framework. Multicast games were first modeled as noncooperative games in Chekuri et al. [14], whose model also generalized facility location games. As explained above, the principal difference from the cooperative model is that here, the global cost share of an agent stems from local distribution rules which specify how the local cost (cost of each link) is split between the agents using that link. Accordingly, an agent's total cost share is simply the sum of her cost shares across all the links she uses. In addition, each agent can choose between potentially several link combinations that connect to the source. A pure Nash equilibrium corresponds to a choice of links by each agent such that each agent incurs the least possible cost given the links chosen by the other agents. Similarly to the fact that the core might be empty in the cooperative model, a pure Nash equilibrium may not exist in general, but for multicast games it does.

The primary objective for researchers is in identifying (possibly budget-balanced[3]) distribution rules that result in stable and/or fair outcomes. In the cooperative model, this could mean investigating global distribution rules that are in the *core* and hence guarantee a stable grand coalition. In the noncooperative model,

[3] A distribution rule is said to be *budget-balanced* if it exactly distributes the welfare to the agents. In other words, there is no surplus or deficit. Often, for reasons of practicality, or otherwise, some form of approximate (if not exact) budget-balance is desired.

it could mean investigating local distribution rules that guarantee the existence of a Nash equilibrium. An additional concern that arises in the noncooperative model is regarding the 'efficiency' of the outcome—for example, how close is the total welfare generated by the Nash equilibrium to the maximum total welfare that could be achieved, had there been a central authority who could control the agents' actions? (For example, in the noncooperative multicast game, a Nash equilibrium choice of links by the agents may not collectively result in the minimum spanning tree for $N \cup \{s\}$; so how inefficient could it be?) Hence, a secondary objective for researchers in the noncooperative model is to investigate for local distribution rules that guarantee the most efficient Nash equilibria. *It is important to note that the existence of such desirable distribution rules is not a priori guaranteed.* There are cooperative games that have empty cores, and there are noncooperative games that possess no Nash equilibria. Further, if the most efficient Nash equilibrium that could be guaranteed is not 'efficient enough' for the application at hand, then it could mean a bad modeling choice or a bad choice of a solution concept. *A principal contribution of this thesis lies in characterizing the space of local distribution rules for which interesting classes of scalable noncooperative games are guaranteed to possess Nash equilibria. We also initiate the study of efficiency analysis in these classes.*

Most of the previous literature in noncooperative cost sharing games (Anshelevich et al. [7], Corbo and Parkes [18], Fiat et al. [29], Chekuri et al. [14], Christodoulou et al. [16]) considered a fixed distribution rule that guarantees equilibrium existence, namely equal share (dubbed the 'fair cost allocation rule', equivalent to the Shapley value in these settings), and the focus was directed towards characterizing the efficiency of equilibria. Only recently is there an emerging focus on *designing* local distribution rules that guarantee equilibrium existence and studying the efficiency of the resulting equilibria. Perhaps, the most famous such distribution rule is the Shapley value (Shapley [78]), which is budget-balanced, guarantees the existence of a Nash equilibrium when used in a noncooperative setting, and for certain classes of cooperative games such as convex games, is always in the core. Generalizations of the Shapley value, e.g., weighted and generalized weighted Shapley values (Shapley [77]), exhibit many of the same properties.

With these goals in mind, researchers have sought to provide characterizations of the class of (local) distribution rules that guarantee equilibrium existence. The first step towards this goal was made by Chen et al. [15], who prove that the only *budget-balanced* distribution rules that guarantee equilibrium existence in all cost sharing games are generalized weighted Shapley value distribution rules. Following on Chen et al. [15], Marden and Wierman [54] provide the parallel characterization in the context of revenue sharing games. Though these characterizations seem general, they are actually just worst-case characterizations. In particular, the proofs in Chen et al. [15], Marden and Wierman [54] consist of exhibiting a specific 'worst-case' welfare function which requires that generalized weighted Shapley value distribution rules be used. *Thus, characterizing the space of distribution rules (not necessarily budget-balanced) for specific local welfare functions remains an important open problem, and is the focus of this thesis.* In practice, it is exactly this issue that is important: when designing a distribution rule, one *knows* the specific local welfare functions for the situation, wherein there may be distribution rules other than generalized weighted Shapley values that also guarantee the existence of an equilibrium.

Another recent example of work in this direction is by von Falkenhausen and Harks [85], who consider games where the action sets (strategy spaces) of the agents are either singletons or bases of a matroid defined on the ground set of resources. For such games, they focus on designing (possibly non-separable, non-scalable) distribution rules that result in efficient equilibria. They tackle the question of equilibrium existence with a novel characterization of the set of possible equilibria independent of the distribution rule, and then exhibit a family of distribution rules that result in any given equilibrium in this set. Our goal is fundamentally different from theirs, in that we seek to characterize distribution rules that guarantee equilibrium existence for a *class of games*, whereas they directly characterize the best/worst achievable equilibria of a *given game*.

An alternative approach towards noncooperative cost sharing is studied by Anshelevich et al. [8], Hoefer and Krysta [38]. They consider a fundamentally different model of a cost sharing game where agents not only choose resources, but also indicate their demands for the shares of the resulting welfare at these resources. Their model essentially defers the choice of the distribution rule to the agents. In such settings, they prove that an equilibrium may not exist in general.

A rather unconventional, though important, application of designing stable and efficient distribution rules for noncooperative cost sharing games has recently emerged in the context of designing application-independent utility functions for distributed agents in *game-theoretic control*. In this approach, distributed computer agents are modeled as players in a noncooperative cost sharing game—each agent is programmed to act individually rationally, using a distributed learning algorithm, to maximize a custom utility function that is derived from the global objective function of the control system. The resulting equilibrium then provides a stable operating point for the control system. This is a useful model when the resulting noncooperative game exhibits efficient equilibria, and admits efficient learning rules for attaining such equilibria. Thus, the two major modules of this approach are designing the agent utility functions (utility design) for an efficient equilibrium, and designing the distributed learning algorithm that steers the system towards such an equilibrium (learning design). Most literature on game-theoretic control provides application-specific co-designs of the utility functions and learning rules. Recently, Gopalakrishnan et al. [33] provide an 'architectural' view of game-theoretic control, proposing the notion of an added design constraint that facilitates the decoupling of utility design and learning design from one another, as well as from the specific application at hand. This work also opened the doors for applying cost sharing design techniques towards application-independent utility design for agents.

1.1 Contributions of the Thesis

The goal of this thesis is to study the design of stable, scalable, efficient distribution rules for a broad class of noncooperative cost sharing games. In so doing, importantly, we expose the fundamental scope and limitations of such a design. As surveyed above, our work has far reaching implications across various fields such as microeconomic theory, operations research, algorithmic game theory and mechanism design, and distributed control, where cost sharing has found important applications. We now outline the structure of this thesis, while highlighting our technical contributions.

1.1.1 Background and Preliminaries

In *Chapter 2*, we provide a historic overview of game theory, algorithmic game theory, and game-theoretic control, highlighting the connections to cost sharing. Then, the relevant mathematical preliminaries, including both set theoretic and game-theoretic concepts, are introduced. This chapter can be safely skipped by the expert reader, without significant loss in continuity.

1.1.2 Noncooperative Cost Sharing Games

While most of *Chapter 3* is concerned with the specifics of our model and a review of relevant existing literature on distribution rules, towards the end, we present two auxiliary contributions of this thesis which have fundamental implications beyond just our goals of characterizing distribution rules for noncooperative cost sharing games.

We begin the chapter with a formal description of our model for a noncooperative cost sharing game. We then discuss existing distribution rules for cost sharing games, such as the Shapley value and marginal contribution distribution rules, and their extensions. Thereafter, we introduce an important *basis representation framework for distribution rules* based on 'unanimity games', that greatly simplifies their specification, and provides a deeper understanding of the intuitive interpretation of these rules. We demonstrate the power of this basis framework by exposing a fundamental relationship between Shapley values and marginal contributions—these families of distribution rules are equivalent, given a transformation connecting the welfare functions they are defined on. To be precise, in Proposition 1, we show that *given any welfare function $v : 2^N \to \mathbb{R}$, there is another welfare function $v' : 2^N \to \mathbb{R}$ such that, for every player $i \in N$, her Shapley value according to v is identical to her marginal contribution according to v', and vice versa.* This connection is, to the best of our knowledge, not to be found in existing literature, and immediately leads to (previously unknown) closed-form expressions for the potential functions of cost sharing games that employ the Shapley value family of distribution rules. We present these closed-form expressions as part of a second contribution, Theorem 1, where we fill a gap in the seminal work of Hart and Mas-Colell [37] by presenting the details of how generalized weighted Shapley values (and, due to Proposition 1, equivalently, generalized weighted marginal contributions) result in 'generalized' weighted potential games.

1.1.3 Characterization Results

Chapter 4 details the core results of this thesis,[4] where we provide a complete characterization of the space of distribution rules (not necessarily budget-balanced) that guarantee the existence of a pure Nash equilibrium (which we will henceforth refer to as just an equilibrium) *for any specific local welfare functions*:

(a) Our main result (Theorem 2) states that all games conditioned on any fixed local welfare functions possess an equilibrium if and only if the distribution rules are equivalent to generalized weighted Shapley value distribution rules on some 'ground' welfare functions. This shows, perhaps surprisingly, that the results of Chen et al. [15], Marden and Wierman [54] hold much more generally. In particular, it is neither the existence of some worst-case welfare function, nor the restriction of budget-balance, which limits the design of distribution rules to generalized weighted Shapley values.

(b) Our second result (Theorem 3) provides an alternative characterization of the set of distribution rules that guarantee equilibrium existence. In particular, it states that all games conditioned on any fixed local welfare functions possess an equilibrium if and only if the distribution rules are equivalent to generalized weighted marginal contribution distribution rules on some 'ground' welfare functions. This result is actually a consequence of the connection between Shapley values and marginal contributions, namely that they can be viewed as equivalent given a transformation connecting their ground welfare functions (Proposition 1 in Chapter 3).

These characterizations provide two alternative approaches for the problem of designing distribution rules, with different design tradeoffs, e.g., between budget-balance and tractability. More specifically, a design through generalized weighted Shapley values provides direct control over how close to budget-balanced the distribution rule will be; however, computing these distribution rules often requires computing exponentially many marginal contributions (Matsui and Matsui [55], Conitzer and Sandholm [17]). On the other hand, a design through generalized weighted marginal contributions requires computing only one marginal contribution; however, it is more difficult to provide bounds on the degree of budget-balance.

Another important consequence of our characterizations is that *potential games are necessary to guarantee the existence of an equilibrium* in all games with fixed local welfare functions, since generalized weighted Shapley value and generalized weighted marginal contribution distribution rules result in ('weighted') potential games (Theorem 1, Hart and Mas-Colell [37], Ui [83]). This is particularly surprising, since the class of potential games is a relatively small subset of the class of games that possess an equilibrium (Sandholm [74]), and our characterizations imply that such a relaxation in game structure would offer no advantage in guaranteeing equilibria.

The final section of this chapter is devoted to an outline of the proof of Theorem 2, highlighting the key steps involved with illustrations. The complete proof is deferred to the following chapter.

[4] These results were incrementally published in Gopalakrishnan et al. [34, 35].

1.1.4 Proof of Theorem 2

Chapter 5 is the meat of this thesis, spanning more than 50 pages in detailing the proof of our main result, Theorem 2. The first phase of the proof consists of a quick reduction of the problem of characterizing *all* distribution rules to the problem of characterizing only *budget-balanced* rules. Though straightforward, this is a powerful reduction, because the rest of the proof uses the budget-balance property copiously. The second phase of the proof consists of a sequence of counterexamples that establish novel necessary conditions for budget-balanced distribution rules to guarantee the existence of an equilibrium. These necessary conditions are established by exhibiting one or more counterexample games with carefully chosen parameters, such that failure to satisfy a necessary condition would lead to these games having no equilibria. Every successive necessary condition eliminates a bunch of distribution rules, until finally, only generalized weighted Shapley values remain.

Within this analysis, new tools for studying distribution rules using their basis representation (see Section 3.4) are developed, including an inclusion-exclusion framework that is crucial for our proof. Specifically, they aid in constructing clever counterexamples that help establish necessary conditions. For example, our *inclusion-exclusion technique* is a generic procedure that enables 'singling out' the effect (on the incentives of an agent) of any desired unanimity game without affecting the incentives of other agents. The ability to elicit such an isolation for an arbitrary welfare function (which is a linear combination of unanimity games) is at the crux of several lemmas whose proofs involve induction on the size of the unanimity game, requiring a counterexample for every induction step. Due to their generic nature, such tools could also be useful for related models, such as cost sharing mechanisms.

Another example of a powerful proof technique is the basis framework for distribution rules, without which the current modular structure of the proof would not be possible—once we establish the necessary condition that every budget-balanced distribution rule that guarantees equilibrium existence must have a basis representation, it is the basis framework that enables the separation of the easier task of just focusing on the 2^{n-1} *basis distribution rules* (the distribution rules corresponding to unanimity games), which are *independent of the welfare function*, from the much more difficult task of establishing the 'consistency' of these basis distribution rules, which requires specifying counterexample games for distribution rules that are arbitrary linear combinations of these basis rules—we tackle this latter task by grouping all possible linear combinations into 3^{2^n-1} cases, and providing a counterexample 'template' for each case.

Not only does it help prove our main result in novel ways, but, as discussed in Chapter 3, the basis framework exposes a fundamental relationship between the Shapley value and marginal contribution distribution rules (Proposition 1), which in turn leads to a novel-closed form expression for the potential function of the resulting cost sharing games (Theorem 1). These auxiliary results have implications beyond just our model of noncooperative cost sharing games.

1.1.5 Conclusion and Future Work

Chapter 6 begins by taking a step back to place our core contributions in characterizing a class of stable distribution rules, in the context of broader research in designing distribution rules with 'good' static/dynamic properties for noncooperative cost sharing games. We discuss the limitations of our work, and then analyze the implications of our results on some of the other desirable properties such as efficiency, budget-balance, tractability and privacy, while suggesting future work in all of these areas. In the case of efficiency, we even take some first steps, with a short, but significant, proposition that shows that *when there are no other constraints, among generalized weighted Shapley rules (or equivalently, generalized marginal contribution rules), the efficiency of the resulting game is independent of the weights of the agents*; hence, one need only optimize over the possible ground welfare functions for efficiency. Finally, the chapter concludes by identifying a potential extension of our model in the realm of mechanism design, where agents have heterogeneous private valuations over different actions, and the designer must craft a mechanism that asks agents to provide their private information, based on which the distribution rules of the cost sharing game are designed. The mechanism must ensure that the agents cannot gain by lying about their values, and moreover, they make choices that maximize the effective 'social welfare', which includes the sum of the valuations of all the agents for their respective actions, in addition to the total welfare generated at the resources.

Chapter 2

Background and Preliminaries

In this chapter, we first provide, in Section 2.1, a historical introduction to game theory, followed by introductions to two major fields where the cost sharing framework has found applications—algorithmic game theory and game-theoretic control. These introductions, though self-contained, are by no means comprehensive—at all times, we focus on material that is either directly related to, or is a prerequisite for understanding our research on cost sharing games. We then introduce set theoretic preliminaries that will be required for an understanding of the rest of this thesis, in Section 2.2. Finally, in Sections 2.3 and 2.4, we establish the notation as well as formal definitions and discussions of relevant concepts in game theory and cost sharing games that are introduced in a semiformal, pedagogical setting in Section 2.1.

2.1 Background

Game theory is the study of decision making by strategic entities. It was initiated in 1944 by mathematician John von Neumann and economist Oskar Morgenstern [86]. Roger Myerson [62] defines game theory as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. The 'entities' and 'decision-makers' in these definitions are aptly termed 'players' in a 'game'. Specifying a game consists of specifying its 'rules'. Broadly, this involves specifying the players, what they can do, and how the outcome of their actions affects each other. Game theory consists of two major sub-theories—cooperative game theory (models of cooperation) and noncooperative game theory (models of conflict).

A game is *cooperative* if players are able to form binding commitments (e.g., legally enforceable contracts). A cooperative game involves specifying the set of players, and a *characteristic function* that specifies the value or worth of every possible *coalition* (group) of players, which is a measure of how well they can collectively perform, assuming that all the other players act in opposition to them. The focus is on studying which coalitions the players choose to form, according to their estimate of how the collective value will be divided among coalition members (e.g., as specified in some contract). The actual process of coalition formation, i.e., how the players communicate, negotiate, bargain, etc., is usually not addressed in the basic formulation of the game.

When players are unwilling to cooperate, or when cooperation fails, a *noncooperative* game can better model the problem. This is not to say that cooperation is ruled out, rather, any resulting 'cooperation' must be self-enforcing, as opposed to relying on a third party (such as a court of law) for enforcement. Simple noncooperative games are often specified in *normal form*[1], which involves specifying the set of players, the set of actions that each player can independently choose from, and a utility function for each player, which encodes the preferences of that player over all possible outcomes (i.e., all possible combinations of actions the players can collectively choose). The focus is on studying what actions the players will individually choose in order to maximize their own utility function, assuming, in the simplest of models, complete information, i.e., every player knows the specification of the entire game, and not just her own action set and utility function.

Specifying the game (either in the cooperative or noncooperative setting) is a relatively easy task in comparison to the challenge of predicting its outcome. The difficulty of this task stems from the fact that researchers have varying views on which outcome should be expected. A *solution concept* is a formal rule for predicting how a game will be played. These predictions constitute 'solutions' which specify what actions will be chosen by the players, and therefore, the outcome of the game. Cooperative and noncooperative games have different solution concepts, since they place different behavioral assumptions on the players:

(a) *Cooperative games.* In most cooperative games, the characteristic function is superadditive, that is, the value of the grand coalition (where all players cooperate) is at least as large as the total value of any disjoint set of coalitions that could be formed. Therefore, a reasonable assumption is that 'rational' players will agree to form the grand coalition.[2] The problem is then for the players to agree on how the resulting value of the grand coalition should be split among themselves. Several solution concepts have been proposed, that correspond to different notions of 'stability' and/or 'fairness'. Some solution concepts are *refinements* of others, meaning that the sets of outcomes satisfying them are contained within the set of outcomes of a less restrictive solution concept. Such refinements become necessary when a solution concept cannot predict a unique outcome. The two solution concepts of interest to us are the *core*, which consists of the set of outcomes that guarantee a stable grand coalition, and the *Shapley value*, which is the unique outcome that satisfies a certain set of fairness axioms. We discuss them in more detail in Section 2.3.

(b) *Noncooperative games.* In noncooperative games, a solution concept is any function that maps a game to a subset of its possible outcomes. Most solution concepts are *equilibrium concepts*, which select outcomes that satisfy varying notions of 'stability'. Perhaps the most famous of such concepts is the *Nash equilibrium*, which consists of the set of outcomes in which no player can strictly gain by unilaterally deviating from her current action. We discuss it in more detail in Section 2.4.

[1]The normal form representation is used to specify simple games where the timing/sequence of the players' actions is not modeled. In effect, the players could be thought of as making their moves simultaneously. In contrast, games where timing issues are important, e.g., when a player's next action could depend on the previous actions (history of the game), are specified in *extensive form*. For the purposes of our study in this thesis, normal form representations suffice, as we deal with distributed (and often large) environments where the players are typically unaware of the actions of other players.

[2]Even if the characteristic function is not superadditive and/or the grand coalition is not formed, these solution concepts can still be applied to the *subgames* defined by whatever coalitions actually form.

Different solution concepts have different characteristics, and no single solution concept is universally better than another. It is worth mentioning at this point that considerable efforts have been made to bridge the gap between cooperative and noncooperative game theory, and therefore, their solution concepts; for details on one such initiative called the Nash program, refer to Serrano [76]. Indeed, this thesis highlights the value of using cooperative solution concepts to incentivize players in a noncooperative game to choose some desired outcome, e.g., a Nash equilibrium. For a more inclusive treatment of solution concepts in both cooperative and noncooperative games, refer to a standard textbook on game theory, such as Osborne and Rubinstein [68].

Of independent interest is the question of how exactly players reach a desired outcome, and when applicable, how they could be aided in that process. For example, in cooperative games, how does a coalition actually form? How do the players communicate, bargain, etc.? In noncooperative games, how do players arrive at an equilibrium? What outcomes are observed when players play these games in the real world, and how do they arrive that these outcomes? The fields of experimental game theory, and behavioral economics at large, deal with such questions about game dynamics. There is also a large literature on 'learning in games' which studies distributed algorithms (sometimes intuitive and natural) that players could carry out to attain a desired outcome, often in the presence of limited information. A general discussion of this literature is beyond the scope of this thesis, but at times, we do cite some useful results from this body of work that support our results.

2.1.1 Algorithmic Game Theory

Primarily developed by economists during its early years, game theory has since found applications in a variety of areas such as political science, psychology, biology, logic, and computer science. Within computer science, the first connection to game theory was perhaps observed in 1977 by Andrew Yao [91], in the final paragraph of his seminal paper on (what is today known as) Yao's principle, an important proof technique for establishing lower bounds on the computational complexity of randomized algorithms, especially of online algorithms. It took several more years, however, before a compelling case was made, by Noam Nisan and Amir Ronen [66] in 1999, for incorporating strategic concerns and the resulting need for game-theoretic analysis while designing (distributed) algorithms. Since then, the study of game theory within the context of computer algorithms has quickly grown into a field of its own, known as algorithmic game theory or computational game theory.

There is a wide variety of problems that fall under the umbrella of algorithmic game theory. One area of research deals with the computational complexity of solving games (for a given solution concept). Another area revisits currently implemented algorithms using game-theoretic tools to examine their outcomes and the resulting efficiency loss when the inputs could be misreported by strategic participants. A third area attempts to take corrective action by designing algorithms while keeping in mind the strategic nature of the inputs, with the goal that the resulting game should have good game-theoretic properties such as the existence of an efficient equilibrium outcome, as well as good algorithmic properties like efficient running time. This last area is called algorithmic mechanism design. For a comprehensive introduction to algorithmic game theory, refer to Nisan et al. [67].

Our topic of study in this thesis, namely cost sharing, is most closely associated with mechanism design—they both share the underlying goal of designing a game with good static and dynamic properties. However, most often, mechanism design problems additionally involve soliciting some *private* information from the players, such as how much they value a scarce good or service. This information is then used to allocate these scare resources in an optimal way, e.g., in a single-item auction, 'social welfare' is maximized when the person who values the item the most gets it. Since the players cannot be trusted to reveal their true valuations, mechanism designers worry about designing a game in which players cannot benefit from lying about their private values. In this thesis, we only concern ourselves with *cost sharing games*, where private values are not part of the model. Towards the end, in Chapter 6, we do briefly discuss a parallel model of *cost sharing mechanisms* and how such a problem could be approached.

2.1.2 Game-Theoretic Control

Historically, game theory has been shaped largely by economists—indeed, today, game theory is regarded primarily as a branch of economics—and therefore, from their point of view, the strategic players were human beings, who come with fixed incentives (utility functions), perhaps unknown to begin with. Economists primarily concern themselves with learning these preferences, which would help model them better, leading to a better prediction of their collective behavior as a group or society, in response to a public policy or mechanism. However, as game theory found more applications outside of economics, the concept of a 'player' grew more abstract, and could broadly refer to any decision-making entity, not necessarily human. One such application is in the area of multiagent systems (Shoham and Leyton-Brown [80]), a branch of study that emerged from the artificial intelligence (AI) community in the late 1980s, when researchers realized the limitations of a single agent handling complex, large-scale tasks (Alonso [5]). The principal mathematical tool used by AI researchers until this point was statistical decision theory (also a fundamental tool used in econometrics), which was developed independently of game theory in the 1950s.[3] Briefly, decision theory deals with a single agent interacting with its environment. While this theory does, in principle, extend to

[3]Perhaps surprisingly, the discovery of the fundamental ideas of decision theory is rife with controversy. Blaise Pascal, in the 17th century, came up with the idea of expected value. However, the notion of what is today formalized as 'expected utility' was proposed in the 18th century by Daniel Bernoulli, who argued that the value of a resource (such as money) should depend on how much of it an individual currently possesses. This intuition of 'moral expectation' (as opposed to mathematical expectation) was endorsed in the early 19th century by Pierre-Simon Laplace. But this line of thought was either ignored or condemned for more than a hundred years by such famous mathematicians as William Feller, who, in his well-known book on probability theory (Feller [28]), simply rejected Bernoulli's resolution of the St. Petersburg paradox without even mentioning it, while assuring readers that he "tried in vain to solve it by the concept of moral expectation." Thankfully, the expected utility theory was rekindled with a more formal foundation in the second quarter of the 20th century in a series of papers by Abraham Wald [88], who extended the theory of statistical hypothesis testing (developed during the 1930s) of Jerzy Neyman and Egon Pearson [65]. This attempt was met with renewed skepticism, again from famous statisticians such as Maurice Kendall [45] and Ronald Fisher [30], even as von Neumann and Morgenstern [87] laid down their classic axioms of rationality under which the expected utility hypothesis holds. For more details and an extensive, albeit dramatic, account of the history of decision theory, refer to Edwin Jaynes [42].

multiple agents (by treating, for each agent, all other agents collectively as the 'environment'), it becomes exceedingly difficult to do so as the number of agents increases—it simply does not scale. Game theory, in contrast, provides a framework that is meant to model multiple interacting decision-makers. In fact, most of the situations modeled by classical decision theory can be thought of as one-player games.

Game theory saw increased adoption in the modeling and analysis of various problems in the field of multiagent systems since the late 1990s, so much so that today, the terms 'agent' and 'player' are used almost interchangeably. Modern texts on multiagent systems do not just deal with intelligent agents from the AI perspective, but treat the subject broadly, by including such diverse disciplines as distributed systems, economics (mostly microeconomic theory), operations research, analytic philosophy, and linguistics. Game-theoretic control is one such interdisciplinary area that emerged within the field of multiagent systems, in the context of a game-theoretic approach to distributed resource allocation.

Resource allocation is a fundamental problem that arises in nearly all computer systems. Traditional approaches to determine efficient allocations involved mostly centralized algorithms (Ageev and Sviridenko [3], Ahuja et al. [4], Feige and Vondrak [26]). However, in many modern applications, these centralized algorithms are not applicable and/or desirable, due to their poor reliability, efficiency and scalability. As a result, increasingly, resource allocation in a multiagent system is a problem that needs to be solved in a distributed, decentralized manner, e.g., power control and frequency selection problems in wireless networks (Altman and Altman [6], Campos-Náñez et al. [12], Falomari et al. [25], Kauffmann et al. [44]) and coverage problems in sensor networks (Cassandras and Li [13], Marden and Wierman [53]).

Game-theoretic control involves modeling the interactions of agents within a noncooperative cost sharing game where the agents are 'self-interested'; the resulting equilibria are viewed as stable operating points for the system. This is motivated by the fact that the underlying decision-making architecture in noncooperative cost sharing is identical to the desired decision-making architecture in distributed engineering systems, i.e., local decisions based on local information where the global behavior emerges from a compilation of these local decisions. This parallel makes it possible to utilize a broad set of game-theoretic tools in distributed control. A key distinction between game theory for economic systems and game theory for engineering systems is that decision-makers are inherited in economic systems while decision-makers are designed in engineering systems. There are wide-ranging advantages to the game-theoretic approach including robustness to failures and environmental disturbances, minimal communication requirements, and improved scalability.

Applying game-theoretic control requires specifying decision-makers, their choices and their objective/utility functions in such a way that the resulting game has good stability properties, such as the existence of an equilibrium, and then specifying the distributed learning rules to be followed in order to reach an equilibrium. The goal is to design the utility functions and the learning rules so that the emergent global behavior is desirable, that is, the equilibrium should be close to an optimal operating point, and the agents must be able to converge to it relatively quickly.

These two major modules of game-theoretic control are referred to as *utility design* and *learning design*, respectively. Most of the existing literature on game-theoretic control (e.g., most of the work cited above) provides application-specific co-designs of the utility functions and learning rules. Recently, Gopalakrishnan et al. [33] provided an architectural framework for game-theoretic control, proposing the notion of an added design constraint that facilitates decoupling utility design and learning design from one another. This allows for an independent development of a rich set of utility and learning designs, from which designs can be chosen 'off the shelf' according to the requirements of the resource allocation problem being considered. This work provides fresh motivation for unifying distribution rule design for cost sharing games with application-independent utility design for distributed agents. While the set of cost sharing design tools are developed from a broader, more abstract standpoint, the specific restrictions/constraints imposed by the application (e.g., economic versus engineering) dictate the appropriate design choice to be ultimately used/deployed.

2.2 Set Theoretic Concepts

In this section, we define a few concepts from set theory. Let N be a finite set. A partition of a set N is a set of nonempty subsets of N such that every element of N is in exactly one of these subsets.

Definition 1. *A collection $\mathcal{P}$ of subsets of N is a partition of N if and only if:*

(a) $\emptyset \notin \mathcal{P}$

(b) $\bigcup \mathcal{P} = N$

(c) $(\forall\, A, B \in \mathcal{P}) \quad A \cap B \neq \emptyset \Leftrightarrow A = B$

The partition $\{N\}$ is called the trivial partition of N.

2.2.1 Relations

Definition 2. *A (binary) relation R on a set N is a subset of $N \times N$.*

For any two elements $x, y \in N$, if $(x, y) \in R$, then we write xRy. Sometimes, a relation is better denoted directly by the operator that defines it. For example, the operator $\subseteq$ defines a relation on 2^N for a finite set N. For any two subsets $A, B \subseteq N$, we say $A \subseteq B$ if and only if $(A, B) \in \subseteq$. (Here, instead of a letter R, we have used the operator $\subseteq$ to denote the relation itself.) Some basic properties that a relation R on a set N could satisfy are:

(a) *Reflexive:* $\forall\, x \in N, xRx$.

(b) *Irreflexive (or strict):* $\forall\, x \in N, \neg xRx$.

(c) *Symmetric:* $\forall\, x, y \in N, xRy \Leftrightarrow yRx$.

(d) *Antisymmetric:* $\forall\, x \neq y \in N, xRy \Rightarrow \neg yRx$.

(e) *Transitive:* $\forall\, x, y, z \in N, xRy \wedge yRz \Rightarrow xRz$.

(f) *Total:* $\forall\, x, y \in N, xRy \vee yRx$.

Any relation R has a corresponding *transitive closure*, R^+, which is a transitive relation obtained from R by iteratively adding pairs (x, z) to R whenever there exists y such that $(x, y) \in R$ and $(y, z) \in R$. If R is transitive to begin with, then it is its own transitive closure. Any reflexive relation $\preceq$ has a corresponding irreflexive or strict relation that is obtained by removing all pairs of the form (x, x) from $\preceq$. This relation is aptly denoted by $\prec$. (Sometimes, $\succeq$ and $\succ$ are used.) We now define a few important kinds of relations on a set N:

Definition 3. *A partial order on N is a binary relation $\preceq$ on N that is reflexive, antisymmetric, and transitive.*

A set N along with a partial order $\preceq$ on N, is called a *partially ordered set* or *poset*, and is denoted by the tuple $(N, \preceq)$. For example, $(2^N, \subseteq)$ is a poset. An element $x \in N$ is a *maximal* (respectively, *minimal*) element of the poset $(N, \preceq)$ if there is no element $y \in N \backslash \{x\}$ such that $x \preceq y$ (respectively, $y \preceq x$). The set of all maximal (respectively, minimal) elements of a poset $(N, \preceq)$ is denoted by $N^{\max}$ (respectively, $N^{\min}$).

Definition 4. *A total order on N is a binary relation $\preceq$ on N that is total, antisymmetric, and transitive.*

A set N along with a total order $\preceq$ on N, is called a *totally ordered set*, denoted by $(N, \preceq)$. For example, $(2^N, \subseteq)$ is not a totally ordered set, whereas $(\mathbb{Z}, \leq)$ is.

Definition 5. *An equivalence relation on N is a binary relation $\sim$ on N that is reflexive, symmetric, and transitive.*

The *equivalence class* of an element $x \in N$ under $\sim$, denoted $[x]$, is defined as $[x] = \{y \in X \mid x \sim y\}$. For example, $=$ is an equivalence relation on $\mathbb{Z}$, where the equivalence class of any integer is simply the singleton set containing that integer. As another example, 'is parallel to' is an equivalence relation on a set of lines L on a plane, where the equivalence class of any given line is the set of all lines in L that are parallel to it, including itself. *The set of all equivalence classes defined by an equivalence relation on a set N is a partition of N.*

2.2.2 Set Functions

Let N be a finite set. Then, any function $W : 2^N \to \mathbb{R}$ is called a set function. For example, the function that assigns to every subset of N, its cardinality, is a set function. A set function $W : 2^N \to \mathbb{R}$ is said to be:

(a) *additive*, if for every pair of disjoint sets $A, B \in 2^N$, $W(A \cup B) = W(A) + W(B)$.

(b) *superadditive*, if for every pair of disjoint sets $A, B \in 2^N$, $W(A \cup B) \geq W(A) + W(B)$.

(c) *supermodular* or *convex*, if for every pair of sets $A, B \in 2^N$, $W(A \cup B) + W(A \cap B) \geq W(A) + W(B)$.

(d) *subadditive*, if for every pair of disjoint sets $A, B \in 2^N$, $W(A \cup B) \leq W(A) + W(B)$.

(e) *submodular* or *concave*, if for every pair of sets $A, B \in 2^N$, $W(A \cup B) + W(A \cap B) \leq W(A) + W(B)$.

(f) *monotone*, if for every pair of sets $A, B \in 2^N$ with $A \subseteq B$, $W(A) \leq W(B)$.

Let $\mathcal{F}(N)$ denote the space of all set functions $W : 2^N \to \mathbb{R}$ with $W(\emptyset) = 0$. It is a vector space with the standard operations of addition and multiplication. Instead of working with set functions $W \in \mathcal{F}(N)$ directly, it is often easier to represent W as a linear combination of simple basis functions. A natural basis, first defined in Shapley [77], is the set of *inclusion functions*:

Definition 6. *An inclusion function of a nonempty set $T \in 2^N$, denoted by W^T, is given by:*

$$W^T(S) = \begin{cases} 1, & T \subseteq S \\ 0, & \textit{otherwise} \end{cases}$$

It is well known (Shapley [77]) that the set of all $2^n - 1$ inclusion functions constitutes a basis for $\mathcal{F}(N)$, i.e., given any function $W \in \mathcal{F}(N)$, there exists a unique support set $\mathcal{T}^W \subseteq 2^N$, and a unique sequence $Q^W = \{q_T^W\}_{T \in \mathcal{T}^W}$ of nonzero coefficients indexed by $\mathcal{T}^W$, such that:[4]

$$W = \sum_{T \in \mathcal{T}^W} q_T^W W^T$$

We sometimes denote the function W by the tuple $\left(\mathcal{T}^W, Q^W\right)$. Inclusion functions are of great importance in the context of cooperative game theory, where they are identified with *unanimity games*. We explore this in more detail in the next section.

2.3 Cooperative Games

Let $N = \{1, 2, \ldots, n\}$ denote a finite set of players. We begin by defining a cooperative game.[5]

Definition 7. *A cooperative game, or a game in coalitional form, denoted by the tuple (N, v), consists of a finite set of players N, and a characteristic function $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. A subgame of a cooperative game (N, v) induced by a subset $S \subseteq N$ is the cooperative game (S, v_S), where the characteristic function $v_S : 2^S \longrightarrow \mathbb{R}$ satisfies $v_S(T) = v(T)$ for all subsets $T \subseteq S$.*

[4] Notice that W and W^T are *functions*. Two functions are said to be equal to each other if they have the same domain, and agree on every element of their common domain.

[5] We only concern ourselves with *transferable utility*- or *TU*- games, where it is assumed that, for a coalition that is formed, the game does not specify each individual's payoff, but only the collective payoff of the coalition, which can then be freely shared among the players through some other mechanism such as bargaining.

Since specifying a function requires specifying its domain, we sometimes denote the game simply by v. The characteristic function describes how much collective payoff a set of players can gain by forming a coalition. The players then choose which coalitions to form, according to their beliefs about how the payoff from a coalition will be divided among coalition members, often agreed upon through a binding commitment/contract. Traditionally, in most applications, the characteristic function v is superadditive, and therefore, $v(N)$, the payoff of the grand coalition, is at least as much as the total payoff of any disjoint set of coalitions that could be formed. Therefore, a standard assumption in cooperative game theory is that 'rational' players will agree to form the grand coalition, and need only agree upon how $v(N)$ is shared among them. Thus, solution concepts for cooperative games are traditionally proposed only for such a distribution rule $f^v : N \to \mathbb{R}^n$.[6] Let Γ be the space of all cooperative games (N, v).

Definition 8. *A solution concept for cooperative games is a mapping that assigns/recommends, to each game* $(N, v) \in \Gamma$*, a distribution rule* $f^v : N \to \mathbb{R}^n$ *that specifies how the payoff of the grand coalition* $v(N)$ *could be split among the players in* N.

Different solution concepts have been proposed, based on different notions of fairness. Some of the important properties to expect from distribution rules f^v recommended by a solution concept, sometimes called *axioms of fairness*, are:

(a) *Efficiency.* The payoff from the grand coalition is completely divided among its players, i.e., for all $v \in \Gamma$, $\sum_{i \in N} f^v(i) = v(N)$.

(b) *Dummy.* A player i is called a *dummy* player or *null* player in game v, if her marginal contribution to any coalition that does not contain her is zero, i.e., $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$. This axiom states that, for all $v \in \Gamma$, if i is a dummy player in v, then $f^v(i) = 0$.

(c) *Symmetry.* Two players i, j are *symmetric* in game v, if swapping one for the other in any coalition containing only one of these two players does not change the coalition's payoff, i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. In other words, swapping their identities does not change the game v. This axiom states that, for all $v \in \Gamma$, if i, j are symmetric players in v, then $f^v(i) = f^v(j)$. This property is sometimes referred to as *anonymity*.

(d) *Additivity.* Let $(N, v), (N, w) \in \Gamma$ be two games. Then, the share of player i from the game $v + w$ is the sum of her shares from games v and w. That is, for all $v, w \in \Gamma$, $f^{v+w} = f^v + f^w$.

(e) *Individual Rationality.* The share of each player i is at least as much as the payoff she could have obtained on her own, i.e., for all $v \in \Gamma$, for all $i \in N$, $f^v(i) \geq v(\{i\})$.

[6] However, our forthcoming model of cost sharing games in Chapter 3 requires a broader application of this concept, because we allow for arbitrary characteristic functions (not just superadditive), and in addition, we allow for arbitrary external constrains on what coalitions are allowed to form. In such situations, it may be infeasible, if not impossible, for the grand coalition to form. But these solution concepts are still useful, because they are defined on the space of all games, and so the same solution concept can simply be applied to the subgame induced by the coalition that is actually formed. In other words, if coalition $S \subseteq N$ is formed in the cooperative game (N, v), then we can apply solution concepts to the subgame (S, v_S) to deduce the distribution rules f^{v_S} recommended by them.

In addition to these axioms,[7] it is also desirable that a solution concept recommends a nonempty (preferably singleton) set of distribution rules for all games.

An efficient distribution rule is called a *pre-imputation*, and an individually rational pre-imputation is called an *imputation*. Next, we define the core and the Shapley value, two important solution concepts proposed for cooperative games:

Definition 9. *The core of a game v, denoted by $C(v)$, is defined as the set of all imputations of v under which no coalition has a payoff that is greater than the sum of its members' shares, i.e.,*

$$C(v) = \left\{ f^v : N \to \mathbb{R} \mid \sum_{i \in N} f^v(i) = v(N) \text{ and } \sum_{i \in S} f^v(i) \geq v(S),\ \forall\, S \subseteq N \right\}$$

The concept of the core was first conceived by Francis Edgeworth [23], and later defined in game-theoretic terms in Donald Gillies [31]. By definition, the core guarantees that the grand coalition will be stable, in the sense that it is immune to coalitional deviations, i.e., no group of players, by breaking out of the grand coalition and forming a separate coalition, can collectively obtain a higher payoff. The core of a game could be empty, although for convex games, it is not.[8] Even if the core is nonempty, it could contain more than one imputation. Therefore, while this solution concept brings with it a strong sense of stability and is indeed a refinement of the set of all imputations, for some games it turns out to be too strong that it eliminates all imputations, and for some, too weak that it still retains many.

Definition 10. *The Shapley value of a game v is the distribution rule f^v_{SV}, given for all $i \in N$ by:*

$$f^v_{SV}(i) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \ (v(S \cup \{i\}) - v(S))$$

The Shapley value has the following intuitive interpretation—imagine that the grand coalition is being formed as players arrive into a room, one at a time. When a player i arrives into the room, and sees the set of players S who are already present in the room, she demands her marginal contribution $v(S \cup \{i\}) - v(S)$. The Shapley value assigns to each player i, her *average* or *expected* marginal contribution in joining the set of players who arrived before her, assuming that all $n!$ possible orders of arrival are equally likely. This elegant solution concept was proposed in his PhD dissertation by Lloyd Shapley [77], and it turns out to be the unique distribution rule that satisfies the first four of the five fairness axioms[9] (and is therefore a pre-imputation). For superadditive games, the Shapley value additionally satisfies the individual rationality axiom and is hence an imputation. Moreover, for convex games, the Shapley value is guaranteed to be in the core, and hence a meritorious refinement. We discuss the Shapley value much more extensively in Chapter 3.

[7] Technically, the name of the first axiom should be different. Traditionally, when superadditivity of the game was a common assumption, this axiom implies efficiency, but in general, it is not always efficient to form the grand coalition. As we shall see in Chapter 3, an analogous property in the context of noncooperative cost sharing is better termed as 'budget-balance'.

[8] A necessary and sufficient condition for a nonempty core is called *balancedness*, which was independently discovered by Olga Bondareva [11] and Lloyd Shapley [79]. It can be shown that convex games are balanced, and balanced games are superadditive.

[9] There have been many other axiomatic characterizations of the Shapley value since Shapley's original one, notable among which is one by Young [94], where he retains efficiency and symmetry, but replaces additivity and dummy with a *marginality* axiom, which states that if a player's vector of marginal contributions to all coalitions in two games are the same, then she should get the same payoff.

2.3.1 Unanimity Games

The set of inclusion functions (introduced in the previous section) forms a basis for the space of all cooperative games with a fixed set of players. The games with an inclusion function as their characteristic function are called *unanimity games*, for the following reason. Consider the inclusion function W^T for some nonempty $T \subseteq N$. Then, for the game (N, W^T), no coalition can receive a nonzero payoff unless all the players in T unanimously agree to participate in that coalition.

Unanimity games hold special significance, because most solution concepts for cooperative games (the ones that are additive) need only be defined for unanimity games, since every game can be expressed as a linear combination of unanimity games; therefore, these solution concepts would automatically extend to arbitrary games by linearity. In addition, it turns out that many solution concepts are vastly simpler to state for unanimity games. For example, the Shapley value of a unanimity game (N, W^T) distributes the payoff equally among all players in T:

$$f_{SV}^{W^T}(i) = \begin{cases} \frac{1}{|T|}, & i \in T \\ 0, & \text{otherwise} \end{cases}$$

Therefore, because it is an additive solution concept, the Shapley value of any game $(N, v) \in \Gamma$ can be written as:

$$f_{SV}^{v}(i) = \sum_{T \in \mathcal{T}^v} q_T^v f_{SV}^{W^T}(i),$$

where $v = \sum_{T \in \mathcal{T}^v} q_T^v W^T$. The proof of our core result in this thesis (Chapter 5) relies critically on a basis framework that is based on unanimity games.

2.4 Noncooperative Games

As before, we let $N = \{1, 2, \ldots, n\}$ denote a finite set of players. For each player $i \in N$, we let $\mathcal{A}_i$ denote a finite set of actions for her to choose from. We call the tuple $a = (a_1, a_2, \ldots, a_n)$ an outcome or (joint) action profile, where $a_i \in \mathcal{A}_i$. Then, the space of all possible action profiles is denoted by $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$. In order to emphasize a player i's particular action, we sometimes denote an action profile $a \in \mathcal{A}$ as (a_i, a_{-i}), where a_{-i} denotes the actions of all players but i in the action profile a. Each player $i \in N$ has a utility function $U_i : \mathcal{A} \to \mathbb{R}$ that assigns, to each outcome $a \in \mathcal{A}$, her payoff $U_i(a)$. Utility functions can be thought of as encoding players' preferences over outcomes. We are now ready to define a noncooperative game.

Definition 11. *A finite noncooperative game (in normal form), denoted by the tuple $(N, \{\mathcal{A}_i\}_{i \in N}, \{\mathcal{U}_i\}_{i \in N})$, consists of a finite set of players N, and for each player $i \in N$, a finite action set $\mathcal{A}_i$ and a utility function $U_i : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathbb{R}$.*

We assume the simplest of settings, where there is complete information, that is, each player knows the entire game, not just her action set and utility function. We also assume that all players choose their actions simultaneously, or, equivalently, they are not aware of the actions chosen by the other players while choosing their own action. The task is then to predict which of the outcomes in $\mathcal{A}$ would occur. Solution concepts for noncooperative games attempt to answer this question. Let Γ be the space of all noncooperative games.

Definition 12. *A solution concept for noncooperative games is a mapping that assigns/recommends, to each game $(N, \{\mathcal{A}_i\}, \{U_i\}) \in \Gamma$, a set of action profiles $a \in \mathcal{A}$ that indicates the possible outcomes of the game.*

Most solution concepts for noncooperative games are *equilibrium concepts*, i.e., they are based on some notion of stability. Perhaps, the most famous one is the Nash equilibrium:[10]

Definition 13. *A Nash equilibrium of a game $(N, \{\mathcal{A}_i\}, \{U_i\}) \in \Gamma$ is defined as any action profile $a^* \in \mathcal{A}$ such that*

$$(\forall\, i \in N) \quad U_i(a_i^*, a_{-i}^*) \;=\; \max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}^*).$$

In other words, a Nash equilibrium denotes a stable outcome of a game, where no player has an incentive to unilaterally deviate, as she cannot strictly improve her payoff. The above definition also motivates the common usage of the term *best response* in this context. An action profile is a Nash equilibrium if and only if every player's action is a best response to the other players' actions. A game might have no Nash equilibrium, or multiple Nash equilibria; both of these possibilities are frequent criticisms against the Nash equilibrium as a solution concept.[11] The basic idea behind this concept can be traced back to Antoine Cournot [19], who used a version of the Nash equilibrium concept in his theory of oligopoly. The modern game-theoretic concept of Nash equilibrium was first introduced for zero-sum games by von Neumann and Morgenstern [86], and subsequently significantly extended to general games by John Nash [64].

2.4.1 Potential Games

Of interest to us are classes of games for which a Nash equilibrium is guaranteed to exist. Indeed, this thesis investigates this principal question in the context of cost sharing games. Potential games is one such class.

Definition 14. *A finite game $(N, \{\mathcal{A}_i\}, \{U_i\}) \in \Gamma$ is an (exact) potential game if there exists a function $\Phi : \mathcal{A} \to \mathbb{R}$ such that, for all $i \in N$, all $a_{-i} \in \mathcal{A}_{-i}$, all $a_i', a_i'' \in \mathcal{A}_i$,*

$$U_i(a_i', a_{-i}) - U_i(a_i'', a_{-i}) = \Phi(a_i', a_{-i}) - \Phi(a_i'', a_{-i}).$$

[10] Note that in this thesis, we only consider *pure* Nash equilibrium, i.e., players make deterministic choices of which action to play. More general equilibrium concepts such as mixed Nash equilibrium and correlated equilibrium allow players to choose probabilistically.

[11] Other criticisms involve the *epistemic conditions* to be assumed in order for a Nash equilibrium to be realized. Some of these conditions such as common knowledge of rationality and common knowledge of the players' actions are considered to impose strong coordination requirements behind this solution concept. For more details, refer to Aumann and Brandenburger [9].

The function Φ is called the *potential function* of the game. The existence of such a potential function for a game is nontrivial, because of the fact that Φ is a single, player-independent function that encodes the differences in every player's utility function due to unilateral deviations. Finite potential games are guaranteed to possess at least one Nash equilibrium, because Φ must attain a maximum in $\mathcal{A}$, say a^*, and by definition of Φ, this means that at a^*, no player can unilaterally deviate and increase their utility, rendering it a Nash equilibrium. The set of local maxima of Φ (both interior and boundary) exactly determines the set of Nash equilibria.[12] In addition to these useful static properties, potential games also possess good dynamic properties—for example, simple best-response dynamics (where players keep taking turns to iteratively switch to their best response to the other players' actions) is guaranteed to converge to a Nash equilibrium. For all these reasons, potential games are a very special class of games indeed. The underlying idea of a potential was proposed in 1973 by Robert Rosenthal [70] for a special class of games called *congestion games*, and he showed that every congestion game had a potential function. Two decades later, in 1996, Dov Monderer and Lloyd Shapley generalized the concept for any noncooperative game, and showed that for every potential game, there is a congestion game with the same potential function [57].

The type of potential games defined above is sometimes explicitly called an *exact potential game*, since the difference in the potential function is exactly equal to the difference in the utility function, see (15). However, it can be seen that this definition can be relaxed further, while still maintaining the existence of a Nash equilibrium:

Definition 15. *A finite game $(N, \{\mathcal{A}_i\}, \{U_i\}) \in \Gamma$ is a weighted potential game if there exists a vector of strictly positive player weights $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$, and a function $\Phi : \mathcal{A} \to \mathbb{R}$ such that, for all $i \in N$, all $a_{-i} \in \mathcal{A}_{-i}$, all $a_i', a_i'' \in \mathcal{A}_i$,*

$$U_i(a_i', a_{-i}) - U_i(a_i'', a_{-i}) = \omega_i \left(\Phi(a_i', a_{-i}) - \Phi(a_i'', a_{-i}) \right).$$

Weighted potential games exhibit the same static properties of potential games. The slight difference in dynamic properties is that the convergence rates to Nash equilibria will depend on the weights.

A further generalization that is quite popular is the concept of *ordinal potential games*:

Definition 16. *A finite game $(N, \{\mathcal{A}_i\}, \{U_i\}) \in \Gamma$ is an ordinal potential game if there exists a totally ordered set $(\mathcal{R}, \succeq)$, and a function $\Phi : \mathcal{A} \to \mathcal{R}$ such that, for all $i \in N$, all $a_{-i} \in \mathcal{A}_{-i}$, all $a_i', a_i'' \in \mathcal{A}_i$,*

$$U_i(a_i', a_{-i}) > U_i(a_i'', a_{-i}) \Longleftrightarrow \Phi(a_i', a_{-i}) \succ \Phi(a_i'', a_{-i}).$$

It is easy to see that ordinal potential games also possess the same static properties like guaranteeing the existence of a Nash equilibrium. While best-response dynamics still converges to a Nash equilibrium, the convergence rate analysis is more complex.

[12] These results do not always hold for games that are not finite. For more details, refer to Sandholm [73].

There are many more generalizations in existing literature that we shall not discuss here, but there is one useful generalization called the *generalized weighted potential game*, which we contribute in this thesis, and will be relevant for our results in Chapter 4:

Definition 17. *A finite game $(N, \{\mathcal{A}_i\}, \{U_i\}) \in \Gamma$ is a generalized weighted potential game if there exists a vector of strictly positive player weights $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$, and a function $\boldsymbol{\Phi} : \mathcal{A} \to \mathbb{R}^m$ (where m is some positive integer), such that, for all $i \in N$, all $a_{-i} \in \mathcal{A}_{-i}$, all $a_i', a_i'' \in \mathcal{A}_i$,*

$$U_i(a_i', a_{-i}) - U_i(a_i'', a_{-i}) = \omega_i \left(\Phi_{k(i)}(a_i', a_{-i}) - \Phi_{k(i)}(a_i'', a_{-i}) \right),$$

where $k(i)$ denotes the index of the first nonzero term of $\boldsymbol{\Phi}\left(a_i', a_{-i}\right) - \boldsymbol{\Phi}\left(a_i'', a_{-i}\right)$.

It can be seen that all generalized weighted potential games are ordinal potential games, where the totally ordered set is simply $\mathbb{R}^m$ with lexicographic ordering. Note that weighted potential games are simply generalized weighted potential games with a one-dimensional potential function ($m = 1$).

Though the existence of a potential function is sufficient for a game to possess a Nash equilibrium, it is, in general, by no means necessary. Perhaps surprisingly, this thesis identifies a very general class of cost sharing games for which potential games are *necessary* to ensure the existence of a Nash equilibrium for all games in that class.

2.4.2 Inefficiency of Nash Equilibrium

Social situations that are modeled as games are often associated with an appropriate notion of *social welfare*, which is a function of the outcome of the game. Formally, we denote the social welfare function associated with a finite game $(N, \{\mathcal{A}_i\}, \{U_i\})$ by $\mathcal{W} : \mathcal{A} \to \mathbb{R}$. Classic examples of such functions include the *utilitarian welfare*, which is simply the sum of the utilities of the players, and the *egalitarian welfare*, which is the minimum of the utilities of the players. A benevolent planner, if she could control the outcome of the game, would choose an outcome $a^{OPT} \in \mathcal{A}$ that maximizes the social welfare. The solution concept we adopt predicts that the players will arrive at one of the Nash equilibrium outcomes, $\boldsymbol{a}^{NE} \subseteq \mathcal{A}$.[13] We discuss two popular metrics that have been proposed to quantify the 'inefficiency' that results from letting the players independently choose their actions, instead of mandating their actions in a way that would maximize the social welfare. Define $(N, \{\mathcal{A}_i\}, \{U_i\}, \mathcal{W})$ to be the (extended) game with welfare function $\mathcal{W}$.

Definition 18. *The price of anarchy, $PoA(G)$ (respectively, the price of stability, $PoS(G)$) of a game $G = (N, \{\mathcal{A}_i\}, \{U_i\}, \mathcal{W})$ is defined as the ratio of the maximum social welfare to the social welfare of the worst (respectively, best) Nash equilibrium:*

$$PoA(G) = \frac{\mathcal{W}(a^{OPT})}{\min\limits_{a \in \boldsymbol{a}^{NE}} \mathcal{W}(a)} \qquad\qquad PoS(G) = \frac{\mathcal{W}(a^{OPT})}{\max\limits_{a \in \boldsymbol{a}^{NE}} \mathcal{W}(a)}$$

[13] For the purposes of this discussion, we restrict ourselves to games with a nonempty set of Nash equilibria.

By definition, these measures are always at least 1. Often, we are interested in the worst-case price of anarchy or worst-case price of stability for a class of games $\mathcal{G}$:

$$PoA(\mathcal{G}) = \max_{G \in \mathcal{G}} PoA(G) \qquad\qquad PoS(\mathcal{G}) = \max_{G \in \mathcal{G}} PoS(G)$$

Obviously, games with 'low' price of anarchy/stability are desirable, in situations where there is some freedom in 'designing' the game (e.g., in the context of game-theoretic control, and, as we shall see in Chapter 3, cost sharing). The price of anarchy was first proposed by Elias Koutsoupias and Christos Papadimitriou [46] in 1999, where they termed it 'coordination ratio'. Papadimitriou is credited with coining the catchy phrase of 'price of anarchy', soon after the publication. The concept of price of stability was first discussed in Andreas Schulz and Nicolás Moses [75], but was first called so by Anshelevich et al. [7].

Chapter 3

Noncooperative Cost Sharing Games

We begin this chapter by presenting our formal model for noncooperative cost sharing, in Section 3.1. Then, in Section 3.2, we survey a few existing distribution rules for cost sharing that have been used in the literature. Section 3.3 briefly shows how existing, well-known families of noncooperative games can be thought of as special cases of our model. Then, in Section 3.4, we introduce a basis representation for distribution rules that derives naturally derives from the basis of inclusion functions introduced in Chapter 2. Not only is it critical to the proofs of our core results that appear in Chapter 5, but this powerful framework also helps unearth a fundamental connection between two seemingly unrelated families of distribution rules, which, to the best of our knowledge, has not been observed in the literature. This contribution, presented in Section 3.5, immediately leads to previously unknown closed-form expressions for the potential functions of cost sharing games with the Shapley value family of distribution rules. We present these closed-form expressions as part of a second contribution in our final section, namely Section 3.6, where we also extend an important piece of work by Hart and Mas-Colell [37] by showing how generalized weighted Shapley values (as well as generalized weighted marginal contributions) result in 'generalized' weighted potential games.

3.1 Model

We consider a simple, but general, model of a noncooperative welfare (cost or revenue) sharing game, where there is a set of self-interested agents/players $N = \{1, \dots, n\}$ ($n > 1$) that each select a collection of resources from a set $R = \{r_1, \dots, r_m\}$ ($m > 1$). That is, each agent $i \in N$ is capable of selecting potentially multiple resources in R; therefore, we say that agent i has an action set $\mathcal{A}_i \subseteq 2^R$. The resulting action profile, or (joint) allocation, is a tuple $a = (a_1, \dots, a_n) \in \mathcal{A}$ where the set of all possible allocations is denoted by $\mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$.

Each allocation generates a (social) welfare, $\mathcal{W}(a)$, which needs to be shared among the agents. In our model, we assume that $\mathcal{W}(a)$ is *(linearly) separable* across resources, i.e.,

$$\mathcal{W}(a) = \sum_{r \in R} W_r \left(\{a\}_r \right),$$

where $\{a\}_r = \{i \in N : r \in a_i\}$ is the set of agents that are allocated to resource r in a, and $W_r : 2^N \to \mathbb{R}$ is the local welfare function at resource r. This is a standard assumption (Anshelevich et al. [7], Chekuri et al. [14], Chen et al. [15], Marden and Wierman [53]), and is quite general. Note that we incorporate both revenue and cost sharing games, since we allow for the local welfare functions W_r to be either positive or negative.

The manner in which the welfare is shared among the agents determines the utility function $U_i : \mathcal{A} \to \mathbb{R}$ that agent i seeks to maximize. Because the welfare is assumed to be separable, it is natural that the utility functions should follow suit. Separability corresponds to welfare garnered from each resource being distributed among only the agents allocated to that resource, which is most often appropriate, e.g., in revenue and cost sharing. This results in

$$U_i(a) = \sum_{r \in a_i} f^r\left(i, \{a\}_r\right),$$

where $f^r : N \times 2^N \to \mathbb{R}$ is the local distribution rule at resource r, i.e., $f^r(i, S)$ is the portion of the local welfare W_r that is allocated to agent $i \in S$ when sharing with S. In addition, we assume that resources with identical local welfare functions have identical distribution rules, i.e., for any two resources $r, r' \in R$,

$$W_r = W_{r'} \;\Rightarrow\; f^r = f^{r'}.$$

In light of this assumption, for the rest of this thesis, we write f^{W_r} instead of f^r. For completeness, we define $f^{W_r}(i, S) := 0$ whenever $i \notin S$. A distribution rule f^{W_r} is said to be *budget-balanced* if, for any player set $S \subseteq N$, $\sum_{i \in S} f^{W_r}(i, S) = W_r(S)$.

We represent a welfare sharing game as $G = \left(N, R, \{\mathcal{A}_i\}_{i \in N}, \left\{f^{W_r}\right\}_{r \in R}, \{W_r\}_{r \in R}\right)$, and the design of f^{W_r} is our focus. *When there is only one local welfare function, i.e., when $W_r = W$ for all $r \in R$, we drop the subscripts and denote the local welfare function and its corresponding distribution rule by W and f^W, respectively.*

The primary goals when designing the distribution rules f^{W_r} are to guarantee (i) equilibrium existence, and (ii) equilibrium efficiency. Our focus in this work is entirely on (i) and we consider pure Nash equilibria; however it should be noted that other equilibrium concepts are also of interest (Adlakha et al. [2], Su and van der Schaar [81], Marden [48]). For simplicity, from now on, we simply write ‘equilibrium’ instead of ‘pure Nash equilibrium’.

3.2 Examples of Distribution Rules

Existing literature on cost sharing games predominantly focuses on the design and analysis of specific distribution rules. As such, there are a wide variety of distribution rules that are known to guarantee the existence of an equilibrium. Table 3.1 summarizes several well-known distribution rules (both budget-balanced and non-budget-balanced) from existing literature on cost sharing, and we discuss their salient features in the following.

Table 3.1: *Example distribution rules*

NAME	PARAMETER	FORMULA
Equal share	None	$f^W_{EQ}(i,S)=\frac{W(S)}{\lvert S\rvert}$
Proportional share	$\boldsymbol{\omega}=(\omega_1,\ldots,\omega_n)$ where $\omega_i>0$ for all $1\le i\le n$	$f^W_{PR}[\boldsymbol{\omega}](i,S)=\frac{\omega_i}{\sum_{j\in S}\omega_j}W(S)$
Shapley value	None	$f^W_{SV}(i,S)=\sum_{T\subseteq S\setminus\{i\}}\frac{(\lvert T\rvert)!(\lvert S\rvert-\lvert T\rvert-1)!}{\lvert S\rvert!}(W(T\cup\{i\})-W(T))$
Marginal contribution		$f^W_{MC}(i,S)=W(S)-W(S-\{i\})$
Weighted Shapley value	$\boldsymbol{\omega}=(\omega_1,\ldots,\omega_n)$ where $\omega_i>0$ for all $1\le i\le n$	$f^W_{WSV}[\boldsymbol{\omega}](i,S)=\sum_{T\subseteq S:i\in T}\frac{\omega_i}{\sum_{j\in T}\omega_j}\left(\sum_{R\subseteq T}(-1)^{\lvert T\rvert-\lvert R\rvert}W(R)\right)$
Weighted marginal contribution		$f^W_{WMC}[\boldsymbol{\omega}](i,S)=\omega_i\left(W(S)-W(S-\{i\})\right)$
Generalized weighted Shapley value	$\omega=(\boldsymbol{\lambda},\Sigma)$ $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ $\Sigma=(S_1,\ldots,S_m)$ where $\lambda_i>0$ for all $1\le i\le n$ and $S_i\cap S_j=\emptyset$ for $i\neq j$ and $\cup\Sigma=N$	$f^W_{GWSV}[\omega](i,S)=\sum_{T\subseteq S:i\in\overline{T}}\frac{\lambda_i}{\sum_{j\in\overline{T}}\lambda_j}\left(\sum_{R\subseteq T}(-1)^{\lvert T\rvert-\lvert R\rvert}W(R)\right)$ where $\overline{T}=T\cap S_k$ and $k=\min\{j\mid S_j\cap T\neq\emptyset\}$
Generalized weighted marginal contribution		$f^W_{GWMC}[\omega](i,S)=\lambda_i\big(W(\overline{S}_k)-W(\overline{S}_k-\{i\})\big)$ where $\overline{S}_k=S-\bigcup_{\ell=1}^{k-1}S_\ell$ and $i\in S_k$

3.2.1 Equal/Proportional Share Distribution Rules

Most prior work in network cost sharing (Anshelevich et al. [7], Corbo and Parkes [18], Fiat et al. [29], Chekuri et al. [14], Christodoulou et al. [16]) deals with the equal share distribution rule, f^W_{EQ}, defined in Table 3.1. Here, the welfare is divided equally among the players. The proportional share distribution rule, $f^W_{PR}[\boldsymbol{\omega}]$, is a generalization, parameterized (exogenously) by $\boldsymbol{\omega}\in\mathbb{R}^{\lvert N\rvert}_{++}$, a vector of strictly positive player-specific weights, and the welfare is divided among the players in proportion to their weights.

Both f^W_{EQ} and f^W_{PR} are budget-balanced distribution rules. However, for general welfare functions, they do not guarantee an equilibrium for all games.[1]

3.2.2 The Shapley Value Family of Distribution Rules

One of the oldest and most commonly studied distribution rules in the cost sharing literature is the Shapley value (Shapley [78]). Its extensions include the *weighted* Shapley value and the *generalized weighted* Shapley value, as defined in Table 3.1.

[1] When the local welfare functions $\{W_r\}$ are 'anonymous', i.e., when $W_r(S)$ is purely a function of $\lvert S\rvert$ for all $S\subseteq N$ and $r\in R$, $\left\{f^{W_r}_{EQ}\right\}$ guarantees an equilibrium for all games. This is a consequence of it being identical to the Shapley value distribution rule (Section 3.2.2) in this case. However, the analogous property for $f^W_{PR}[\boldsymbol{\omega}]$ does not hold.

The Shapley value family of distribution rules can be interpreted as follows. For any given subset of players S, imagine the players of S arriving one at a time to the resource, according to some order π. Each player i can be thought of as contributing $W\left(P_i^\pi \cup \{i\}\right) - W\left(P_i^\pi\right)$ to the welfare $W(S)$, where P_i^π denotes the set of players in S that arrived before i in π. This is the 'marginal contribution' of player i to the welfare, according to the order π. The Shapley value, $f_{SV}^W(i,S)$, is simply the *average* marginal contribution of player i to $W(S)$, under the assumption that all $|S|!$ orders are equally likely. The weighted Shapley value, $f_{WSV}^W[\boldsymbol{\omega}](i,S)$, is then a weighted average of the marginal contributions, according to a distribution with full support on all the $|S|!$ orders, determined by the parameter $\boldsymbol{\omega} \in \mathbb{R}_{++}^{|N|}$, a strictly positive vector of player weights. The (symmetric) Shapley value is recovered when all weights are equal.

The generalized weighted Shapley value, $f_{GWSV}^W[\omega]$, generalizes the weighted Shapley value to allow for the possibility of player weights being zero. It is parameterized by a *weight system* given by $\omega = (\boldsymbol{\lambda}, \Sigma)$, where $\boldsymbol{\lambda} \in \mathbb{R}_{++}^{|N|}$ is a vector of strictly positive player weights, and $\Sigma = (S_1, S_2, \ldots, S_m)$ is an ordered partition of the set of players N. Once again, players get a weighted average of their marginal contributions, but according to a distribution determined by $\boldsymbol{\lambda}$, with support only on orders that conform to Σ, i.e., for $1 \le k < \ell \le m$, players in S_ℓ arrive before players in S_k. Note that the weighted Shapley value is recovered when $|\Sigma| = 1$, i.e., when Σ is the trivial partition, (N).

The importance of the Shapley value family of distribution rules is that all distribution rules are budget-balanced, guarantee equilibrium existence in any game, and also guarantee that the resulting games are so-called 'potential games' (Hart and Mas-Colell [37], Ui [83]).[2] However, they have one key drawback—computing them is often[3] intractable (Matsui and Matsui [55], Conitzer and Sandholm [17]), since it requires computing the sum of exponentially many marginal contributions.[4]

3.2.3 The Marginal Contribution Family of Distribution Rules

Another classic and commonly studied distribution rule is f_{MC}^W, the marginal contribution distribution rule (Wolpert and Tumer [89]), where each player's share is simply his marginal contribution to the welfare, see Table 3.1. Clearly, f_{MC}^W is not always budget-balanced. However, an equilibrium is always guaranteed to exist, and the resulting game is an exact potential game, where the potential function is the same as the welfare function. Accordingly, the marginal contribution distribution rule always guarantees that the welfare maximizing allocation is an equilibrium, i.e., the 'price of stability' is one. Finally, unlike the Shapley value family of distribution rules, note that it is easy to compute, as only two calls to the welfare function are required.

[2] Shapley value distribution rules result in exact potential games, weighted Shapley value distribution rules result in weighted potential games, and generalized weighted Shapley value distribution rules result in a slight variation of weighted potential games (see Appendix 3.6 for details).

[3] The Shapley value has been shown to be efficiently computable in several applications (Deng and Papadimitriou [20], Mishra and Rangarajan [56], Aadithya et al. [1]), where specific welfare functions and special structures on the action sets enable simplifications of the general Shapley value formula.

[4] Technically, if the entire welfare function is taken as an input, then the input size is already $O(2^n)$, and Shapley values can be computed 'efficiently'. However, if access to the welfare function is by means of an oracle (Liben-Nowell et al. [47]), than the actual input size is still $O(n)$, and the hardness is exposed.

Note that, it is natural to consider weighted and generalized weighted marginal contribution distribution rules which parallel those for the Shapley value described above. These are defined formally in Table 3.1, and they inherit the equilibrium existence and potential game properties of f_{MC}^{W}, in an analogous manner to their Shapley value counterparts. These rules have, to the best of our knowledge, not been considered previously in the literature; however, they are crucial to the characterizations provided in this article.

3.3 Important Families of Cost/Revenue Sharing Games

Our model for welfare sharing games generalizes several existing families of games that have received significant attention in the literature. We illustrate a few examples below, in all of which the typical distribution rule studied is the equal share or Shapley value distribution rule:

(a) *Multicast and facility location games* (Chekuri et al. [14]) are a special case where N is the set of users, R is the set of links of the underlying graph, $\mathcal{A}_i$ consists of all feasible paths from user i to the source, and for all $r \in R$, $W_r = c_r W$ is the local welfare function, where c_r is the cost of the link r, and W is given by:

$$(\forall\, S \subseteq N) \quad W(S) = \begin{cases} -1, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases} \tag{3.1}$$

(b) *Congestion games* (Rosenthal [70]) are a special case where, for each $r \in R$, the local welfare function W_r is 'anonymous', i.e., $W_r(S)$ is purely a function of $|S|$, and is given by $|S|$ times the negative of the delay function at r, for all $S \subseteq N$.

(c) *Atomic routing games with unsplittable flows* (Roughgarden and Tardos [72]) are a special case where N is the set of source-destination pairs (s_i, t_i), each of which is associated with r_i units of flow, R is the set of edges of the underlying graph, and $\mathcal{A}_i$ consists of all feasible $s_i - t_i$ paths. If $c_r(x)$ denotes the latency function on edge r, then W_r is the negative of the cost of the total flow due to the players in S, i.e., $W_r(S) = -|S| c_r\left(\sum_{i \in S} r_i\right)$, for all $S \subseteq N$.

(d) *Network formation games* (Anshelevich et al. [7]) are a special subcase of the previous case, with a suitable encoding of the players. Suppose the set of players is $N = \{0, 1, \ldots, n-1\}$, and the cost of constructing each edge r is $C_r(S)$ when $S \subseteq N$ is the set of players who choose that edge. Then, one possibility is to set $r_i = 10^i$ so that $\sum_{i \in S} r_i$ can be decoded to obtain the set of players S. Therefore, c_r can be defined such that for all $S \subseteq N$, $c_r\left(\sum_{i \in S} r_i\right) = \frac{C_r(S)}{|S|}$.

Other notable specializations of our model that focus on the design of distribution rules are network coding (Marden and Effros [50]), graph coloring (Panagopoulou and Spirakis [69]), and coverage problems (Marden and Wierman [52], Marden and Wierman [53]). Designing distribution rules in our cost sharing model also has applications in distributed control (Gopalakrishnan et al. [33]).

3.4 A Basis for Distribution Rules

To gain a deeper understanding of the structural form of some of the distribution rules discussed in Section 3.2, it is useful to consider their 'basis' representations. Not only do these representations provide insight, they are crucial to the proofs in this paper. The basis framework we adopt was first introduced in Shapley [77] in the context of the Shapley value, and derives naturally from the set of 'inclusion functions', which we introduced in Chapter 2. First, recall Definition 6:

Definition. *An inclusion function of a nonempty set $T \in 2^N$, denoted by W^T, is given by:*

$$W^T(S) = \begin{cases} 1, & T \subseteq S \\ 0, & \textit{otherwise} \end{cases} \tag{3.2}$$

The set of all $2^n - 1$ inclusion functions forms a basis for the space of all welfare functions, i.e., given any welfare function W, there exists a unique support set $\mathcal{T}^W \subseteq 2^N$, and a unique sequence $Q^W = \{q_T^W\}_{T \in \mathcal{T}^W}$ of non-zero coefficients indexed by $\mathcal{T}^W$, such that:

$$W = \sum_{T \in \mathcal{T}^W} q_T^W W^T \tag{3.3}$$

Now, we can define a corresponding 'basis' for distribution rules, and then move on to introducing the basis representations of the specific distribution rules we introduced in Section 3.2.

To simplify notation in the following, we denote f^{W^T} by f^T, for each $T \in \mathcal{T}^W$. That is, $f^T : N \times 2^N \to \mathbb{R}$ is a *basis distribution rule* corresponding to the unanimity game W^T, where $f^T(i, S)$ is the portion of $W^T(S)$ allocated to agent $i \in S$ when sharing with S.

Given a set of basis distribution rules $\{f^T : T \subseteq N\}$, by linearity, the function f^W,

$$f^W := \sum_{T \in \mathcal{T}^W} q_T^W f^T, \tag{3.4}$$

defines a distribution rule corresponding to the welfare function W. Note that if each f^T is budget-balanced, meaning that for any player set $S \subseteq N$, $\sum_{i \in S} f^T(i, S) = W^T(S)$, then f^W is also budget-balanced. However, unlike the basis for welfare functions, some distribution rules do not have a basis representation of the form (3.4), e.g., equal and proportional share distribution rules (see Section 3.2.1). But, well-known distribution rules of interest to us, like the Shapley value family of distribution rules, were originally defined in this manner. *Further, our characterizations highlight that any distribution rule that guarantees equilibrium existence must have a basis representation.*

Table 3.2: *Definition of basis distribution rules*

NAME	PARAMETER	DEFINITION
Shapley value	None	$f^T_{SV}(i,S) = f^{W^T}_{EQ}(i,S) = \begin{cases} \frac{1}{\lvert T\rvert}, & i \in T \text{ and } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$ (3.5)
Marginal contribution		$f^T_{MC}(i,S) = \lvert T\rvert f^T_{SV}(i,S) = \begin{cases} 1, & i \in T \text{ and } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$ (3.6)
Weighted Shapley value	$\boldsymbol{\omega}=(\omega_1,\ldots,\omega_n)$ where $\omega_i>0$ for all $1\leq i\leq n$	$f^T_{WSV}[\boldsymbol{\omega}](i,S) = f^{W^T}_{PR}[\boldsymbol{\omega}](i,S) = \begin{cases} \frac{\omega_i}{\sum_{j\in T}\omega_j}, & i \in T \text{ and } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$ (3.7)
Weighted marginal contribution		$f^T_{WMC}[\boldsymbol{\omega}](i,S) = \left(\sum_{j\in T}\omega_j\right) f^T_{WSV}[\boldsymbol{\omega}](i,S) = \begin{cases} \omega_i, & i \in T \text{ and } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$
Generalized weighted Shapley value	$\omega=(\boldsymbol{\lambda},\Sigma)$ $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ $\Sigma=(S_1,\ldots,S_m)$ where $\lambda_i>0$ for all $1\leq i\leq n$ and $S_i\cap S_j=\emptyset$ for $i\neq j$ and $\cup\Sigma=N$	$f^T_{GWSV}[\omega](i,S)=\begin{cases} \frac{\lambda_i}{\sum_{j\in\overline{T}}\lambda_j}, & i \in \overline{T} \text{ and } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$ (3.8) where $\overline{T}=T\cap S_k$ and $k=\min\{j\mid S_j\cap T\neq\emptyset\}$
Generalized weighted marginal contribution		$f^T_{GWMC}[\omega](i,S)=\left(\sum_{j\in\overline{T}}\lambda_j\right)f^T_{GWSV}[\omega](i,S)=\begin{cases} \lambda_i, & i \in \overline{T} \text{ and } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$ (3.9) where $\overline{T}=T\cap S_k$ and $k=\min\{j\mid S_j\cap T\neq\emptyset\}$

Table 3.2 restates the distribution rules shown in Table 3.1 in terms of their basis representations, which, as can be seen, tend to be simpler and provide more intuition.

For example, the Shapley value distribution rule on a welfare function W is quite naturally defined through its basis—for each unanimity game W^T, the welfare is shared equally among the players, see (3.5). In other words, whenever there is welfare generated (when all the players in T are present), the resulting welfare is split equally among the contributing players (players in T). Similarly, the weighted Shapley value, for each unanimity game W^T, distributes the welfare among the players in proportion to their weights, see (3.7). Finally, the basis representation highlights that the generalized weighted Shapley value can be interpreted with Σ as representing a grouping of players into priority classes, and the welfare being distributed only among the contributing players of the highest priority, in proportion to their weights, see (3.8).

3.5 Connecting Shapley Values and Marginal Contributions

Interestingly, the marginal contribution distribution rule, though it was not originally defined this way, has a basis representation that highlights a core similarity to the Shapley value. In particular, though the definitions in Table 3.1 make f^W_{MC} and f^W_{SV} seem radically different; from Table 3.2, their basis distribution rules, f^T_{MC} and f^T_{SV}, are, in fact, quite intimately related, see (3.5) and (3.6). We formalize this in the following proposition, which forms the basis for extending our core characterization result in Chapter 4. To the best of our knowledge, this fundamental connection has not been observed in the literature.

Proposition 1. *For any two welfare functions $W' = (\mathcal{T}', Q')$ and $W'' = (\mathcal{T}'', Q'')$, and any weight system $\omega = (\boldsymbol{\lambda}, \Sigma)$,*

$$f^{W'}_{GWSV}[\omega] = f^{W''}_{GWMC}[\omega] \quad \Leftrightarrow \quad \mathcal{T}' = \mathcal{T}'' \quad \textit{and} \quad (\forall\, T \in \mathcal{T}')\ q'_T = \left(\sum_{j\in\overline{T}} \lambda_j\right) q''_T. \tag{3.10}$$

Proof. First, note that we only need to prove one direction, since from (3.8) and (3.9) in Table 3.2, it follows that,

$$q'_T = \left(\sum_{j\in\overline{T}} \lambda_j\right) q''_T \quad \Leftrightarrow \quad q'_T f^T_{GWSV}[\omega] = q''_T f^T_{GWMC}[\omega] \tag{3.11}$$

To prove the other direction, it suffices to show that,

$$f^{W'}_{GWSV}[\omega] = f^{W''}_{GWMC}[\omega] \quad \Rightarrow \quad (\forall\, T \subseteq N)\ q'_T = \left(\sum_{j\in\overline{T}} \lambda_j\right) q''_T \tag{3.12}$$

with the understanding that $q_T = 0$ whenever $T \notin \mathcal{T}$, for $\mathcal{T} = \mathcal{T}', \mathcal{T}''$. The proof is by induction on $|T|$. The base case, where $|T| = 1$ is immediate—suppose $T = \{i\}$ for some $i \in N$. From the definition in (3.8), $\overline{T} = T$ whenever $|T| = 1$. Then,

$$\begin{aligned} f^{W'}_{GWSV}[\omega] = f^{W''}_{GWMC}[\omega] \quad &\Rightarrow \quad f^{W'}_{GWSV}[\omega](i,T) = f^{W''}_{GWMC}[\omega](i,T) \\ &\Rightarrow \quad q'_T f^T_{GWSV}[\omega](i,T) = q''_T f^T_{GWMC}[\omega](i,T) \\ &\Rightarrow \quad q'_T = \lambda_i q''_T, \end{aligned}$$

as desired. Our induction hypothesis is that (3.12) holds for all subsets T such that $0 < |T| \le z$, for some $0 < z < |N|$. Assuming that this is true, we show that (3.12) holds for all subsets T of size $z + 1$:

$$\begin{aligned} f^{W'}_{GWSV}[\omega] = f^{W''}_{GWMC}[\omega] \quad &\Rightarrow \quad f^{W'}_{GWSV}[\omega](i,T) = f^{W''}_{GWMC}[\omega](i,T) \\ &\Rightarrow \quad \sum_{S\subseteq T} q'_S f^S_{GWSV}[\omega](i,S) = \sum_{S\subseteq T} q''_S f^S_{GWMC}[\omega](i,S) \\ &\Rightarrow \quad q'_T f^T_{GWSV}[\omega](i,T) + \sum_{S\subsetneq T} q'_S f^S_{GWSV}[\omega](i,S) \\ &\qquad\qquad = q''_T f^T_{GWMC}[\omega](i,T) + \sum_{S\subsetneq T} q''_S f^S_{GWMC}[\omega](i,S) \\ &\Rightarrow \quad q'_T = \left(\sum_{j\in\overline{T}} \lambda_j\right) q''_T, \qquad\qquad \big(\text{from (3.11)}\big) \end{aligned}$$

as desired. Note that we used the induction hypothesis in the last step for all strict subsets of T. This completes the proof. □

Informally, Proposition 1 says that generalized weighted Shapley values and generalized weighted marginal contributions are equivalent, except with respect to different welfare functions whose relationship is through their basis coefficients, as indicated in (3.10).

3.6 Generalized Weighted Potential Games

Hart and Mas-Colell [37] show that weighted Shapley values result in weighted potential games, by explicitly constructing a potential function (in a recursive form, almost identical to the one in (4.2)). They claim that their result extends to generalized weighted Shapley values, but do not provide a proof. In this section, we fill this gap by showing that generalized weighted Shapley values result in a slight variant of weighted potential games, called *generalized weighted potential games*. First, recall Definition 17 from Chapter 2:

Definition. *A finite game $(N, \{\mathcal{A}_i\}, \{U_i\}) \in \Gamma$ is a generalized weighted potential game if there exists a vector of strictly positive player weights $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$, and a function $\boldsymbol{\Phi} : \mathcal{A} \to \mathbb{R}^m$ (where m is some positive integer), such that, for all $i \in N$, all $a_{-i} \in \mathcal{A}_{-i}$, all $a_i', a_i'' \in \mathcal{A}_i$,*

$$U_i(a_i', a_{-i}) - U_i(a_i'', a_{-i}) = \omega_i \left(\Phi_{k(i)}(a_i', a_{-i}) - \Phi_{k(i)}(a_i'', a_{-i}) \right), \tag{3.13}$$

where $k(i)$ denotes the index of the first nonzero term of $\boldsymbol{\Phi}(a_i', a_{-i}) - \boldsymbol{\Phi}(a_i'', a_{-i})$.

This definition applies to any finite noncooperative game in normal form. However, recall from Section 3.1 that, in our model, the agent utility functions are separable, given by

$$U_i(a) = \sum_{r \in a_i} f^r(i, \{a\}_r) \tag{3.14}$$

Hence, in searching for a potential function for G, it is natural to seek a separable potential function $\boldsymbol{\Phi} : \mathcal{A} \to \mathbb{R}^m$ (where m is some positive integer), given by

$$\boldsymbol{\Phi}(a) = \sum_{r \in R} \boldsymbol{\phi}_r(\{a\}_r) \tag{3.15}$$

where $\boldsymbol{\phi}_r : 2^N \to \mathbb{R}^m$ is the 'local' potential function at resource r. Therefore, to show that $\boldsymbol{\Phi}$ is a potential function for G, it is sufficient to show that for every agent $i \in N$, there exists a positive weight $w_i > 0$ such that, for every resource $r \in R$, for every player subset $S \subseteq N$ containing i,

$$f^r(i, S) = w_i \left((\phi_r)_{k(i)}(S) - (\phi_r)_{k(i)}(S - \{i\}) \right), \tag{3.16}$$

where $k(i)$ denotes the index of the first nonzero term of $\boldsymbol{\phi}_r(S) - \boldsymbol{\phi}_r(S - \{i\})$. Verifying this is quite straightforward; use (3.14)-(3.16) to check that (3.13) is satisfied.

We now state our formal result. Recall that a weight system $\omega = (\boldsymbol{\lambda}, \Sigma)$ consists of a strictly positive vector of player weights $\boldsymbol{\lambda} \in \mathbb{R}^N_{++}$, and an ordered partition $\Sigma = (S_1, S_2, \ldots, S_m)$ of the set of players N.

Theorem 1. *For any welfare sharing game* $G = \left(N, R, \{\mathcal{A}_i\}_{i\in N}, \{f^r\}_{r\in R}, \{W_r\}_{r\in R}\right)$ *and any weight system* ω*, if for every* $r \in R$*, the distribution rule* $f^r = f_{GWSV}^{W_r'}[\omega] = f_{GWMC}^{W_r''}[\omega]$ *with* W_r', W_r'' *being any two ground welfare functions related according to (3.10), then,* G *is a generalized weighted potential game, with player weights* $\boldsymbol{\lambda}$*, and the local potential function at resource* r*,*

$$\boldsymbol{\phi}_r[\omega](S) = \left((\phi_r[\omega])_1(S), (\phi_r[\omega])_2(S), \ldots, (\phi_r[\omega])_m(S)\right),$$

where, for all $1 \le k \le m$*,* $(\phi_r[\omega])_k(S)$ *is given in terms of* W_r' *in the following recursive form:*

$$(\phi_r[\omega])_k(S) = \frac{1}{\sum_{i\in S}\lambda_i}\left(W_r'(\overline{S}_{m-k+1}) + \sum_{i\in S}\lambda_i\,(\phi_r[\omega])_k(S-\{i\})\right), \tag{3.17}$$

and in terms of W_r'' *in the following closed-form:*

$$(\phi_r[\omega])_k(S) = W_r''(\overline{S}_{m-k+1}), \tag{3.18}$$

where $\overline{S}_k = S - \cup_{\ell=1}^{k-1} S_\ell$*.*

Proof. First, we use the closed-form expression in (3.18) to show that $\boldsymbol{\phi}_r[\omega]$ satisfies (3.16). This involves proving, for any $1 \le k \le m$, for any subset S containing a player $i \in S_k$, the following two steps:

(a) $k(i) = m - k + 1$, i.e., $W_r''(\overline{S}_\ell) - W_r''\left(\left(\overline{S-\{i\}}\right)_\ell\right) = 0$ for all $k+1 \le \ell \le m$

(b) $f_{GWMC}^{W_r''}[\omega](i, S) = \lambda_i\left(W_r''(\overline{S}_k) - W_r''\left(\left(\overline{S-\{i\}}\right)_k\right)\right)$

Observe that $\left(\overline{S-\{i\}}\right)_\ell = \overline{S}_\ell - \{i\}$ for all $1 \le \ell \le m$. Part (a) is straightforward, since if $i \in S_k$, then by definition, $i \notin \overline{S}_\ell$ for all $k+1 \le \ell \le m$. We now focus on part (b), which is exactly the definition of $f_{GWMC}^{W_r''}[\omega]$ in Table 3.1. Hence, the following is simply an exercise in verifying the equivalence of the two definitions of $f_{GWMC}^{W_r''}[\omega]$ from Tables 3.1 and 3.2, using the basis representation discussed in Section 3.4. Evaluating the left hand side, we get,

$$\lambda_i\left(W_r''(\overline{S}_k) - W_r''\left(\left(\overline{S-\{i\}}\right)_k\right)\right) = \lambda_i\left(\sum_{T\in\mathcal{T}''(\overline{S}_k)} q_T'' - \sum_{T\in\mathcal{T}''(\overline{S}_k-\{i\})} q_T''\right)$$

where, for any player subset $S \subseteq N$, $\mathcal{T}''(S)$ denotes the set of all coalitions $T \in \mathcal{T}''$ that are contained in S ($T \subseteq S$). Notice that $\overline{S}_k$ does not contain any players in $S_1 \cup S_2 \cup \ldots S_{k-1}$. Therefore, $\mathcal{T}''(\overline{S}_k)$ consists of those coalitions contained in S that do not contain any player in $S_1 \cup S_2 \cup \ldots S_{k-1}$. Similarly, $\mathcal{T}''(\overline{S}_k - \{i\})$ consists of those coalitions contained in $S - \{i\}$ that do not contain any player in $S_1 \cup S_2 \cup \ldots S_{k-1}$. Therefore, the collection $\mathcal{T}''(\overline{S}_k) - \mathcal{T}''(\overline{S}_k - \{i\})$ consists precisely of those coalitions $T \in \mathcal{T}''(S)$ that do not contain any player in $S_1 \cup S_2 \cup \ldots S_{k-1}$, but contain player i. Since $i \in S_k$, this is the same as saying that the collection $\mathcal{T}''(\overline{S}_k) - \mathcal{T}''(\overline{S}_k - \{i\})$ contains precisely those coalitions $T \in \mathcal{T}''(S)$ for which $i \in \overline{T}$.

So, we get,

$$\begin{aligned}
\lambda_i \left(W_r''(\overline{S}_k) - W_r''\left(\left(\overline{S - \{i\}}\right)_k \right) \right) &= \lambda_i \left(\sum_{T \in \mathcal{T}''(\overline{S}_k)} q_T'' - \sum_{T \in \mathcal{T}''(\overline{S}_k - \{i\})} q_T'' \right) \\
&= \sum_{T \in \mathcal{T}''(S)\,:\, i \in \overline{T}} q_T'' \lambda_i \\
&= \sum_{T \in \mathcal{T}''} q_T'' f_{GWMC}^{T}[\omega](i, S) \qquad \big(\text{from (3.9)}\big) \\
&= f_{GWMC}^{W_r''}[\omega](i, S)
\end{aligned}$$

To complete the proof, observe that when W_r' and W_r'' are related according to (3.10), then, for all $1 \le k \le m$, and all $S \subseteq N$, the expression for $\left(\phi_r[\omega]\right)_k (S)$ (3.18) satisfies the recursion (3.17). □

Chapter 4

Designing Distribution Rules for Pure Nash Equilibria

Our goal is to characterize the space of distribution rules that guarantee the existence of an equilibrium in welfare sharing games. Towards this end, this paper builds on the recent works of Chen et al. [15], and Marden and Wierman [54], who take the first steps towards providing such a characterization. Proposition 2 combines the main contributions of these two papers into one statement. Let $\mathbb{W}$ denote a nonempty set of welfare functions. Let $f^{\mathbb{W}} = \{f^W\}_{W \in \mathbb{W}}$ denote the set of corresponding distribution rules. Let $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ denote the class of all welfare sharing games with player set N, local welfare functions $W_r \in \mathbb{W}$, and corresponding distribution rules $f^{W_r} \in f^{\mathbb{W}}$. We refer to $\mathbb{W}$ as the set of local welfare functions of the class $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$. Note that this class is quite general; in particular, it includes games with arbitrary resources and action sets. *When there is only one local welfare function, i.e., when $\mathbb{W} = \{W\}$, we denote this class simply by $\mathcal{G}(N, f^W, W)$. Note that $\mathcal{G}(N, f^W, W) \subseteq \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ for all $W \in \mathbb{W}$.*

Proposition 2 (Chen et al. [15], Marden and Wierman [54])**.** *There exists a local welfare function W for which all games in $\mathcal{G}(N, f^W, W)$ possess a pure Nash equilibrium for a budget-balanced f^W if and only if there exists a weight system ω for which f^W is the generalized weighted Shapley value distribution rule, $f^W_{GWSV}[\omega]$.*

Less formally, Proposition 2 states that if one wants to use a distribution rule that is budget-balanced and guarantees equilibrium existence for all possible welfare functions and action sets, then one is limited to the class of generalized weighted Shapley value distribution rules.[1] This result is shown by exhibiting a specific 'worst-case' local welfare function W^* (the one in (3.1)) for which this limitation holds. In reality, when designing a distribution rule, one *knows* the specific set of local welfare functions $\mathbb{W}$ for the situation, and Proposition 2 claims nothing in the case where it does not include W^*, where, in particular, there may be

[1] Chen et al. [15], and Marden and Wierman [54] use the term *ordered protocols* to refer to generalized weighted Shapley value distribution rules with $|\Sigma| = |N|$, i.e., where Σ defines a total order on the set of players N. They state their characterizations in terms of *concatenations* of positive ordered protocols, which are generalized weighted Shapley value distribution rules with an arbitrary Σ.

other budget-balanced distribution rules that guarantee equilibrium existence for all games. Recent work has shown that there are settings where this is the case (Marden and Wierman [53]), at least when the agents are not allowed to choose more than one resource. In addition, the marginal contribution family of distribution rules is a non-budget-balanced class of distribution rules that guarantee equilibrium existence in all games (no matter what the local welfare functions $\mathbb{W}$), and there could potentially be others as well.

In the rest of this chapter, we provide two equivalent characterizations of the space of distribution rules that guarantee equilibrium existence for all games with a fixed set of local welfare functions—one in terms of generalized weighted Shapley values and the other in terms of generalized weighted marginal contributions. We dedicate the entirety of the next chapter for the complete proof; however, we sketch an outline in Section 4.3, highlighting the proof technique and the key steps involved.

4.1 Characterization in Terms of Generalized Weighted Shapley Values

Our first characterization states that for *any fixed set of local welfare functions*, even if the distribution rules are *not budget-balanced*, the conclusion of Proposition 2 is still valid. That is, every distribution rule that guarantees the existence of an equilibrium in all games is equivalent to a generalized weighted Shapley value distribution rule:

Theorem 2. *Given any set of local welfare functions $\mathbb{W}$, all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ possess a pure Nash equilibrium if and only if there exists a weight system ω, and a mapping g_{SV} that maps each local welfare function $W \in \mathbb{W}$ to a corresponding ground welfare function $g_{SV}(W)$ such that its distribution rule $f^W \in f^{\mathbb{W}}$ is equivalent to the generalized weighted Shapley value distribution rule, $f^{W'}_{GWSV}[\omega]$, where $W' = g_{SV}(W)$ is the actual welfare that is distributed[2] by f^W, defined as,*

$$(\forall S \subseteq N) \quad W'(S) = \sum_{i \in S} f^W(i, S). \tag{4.1}$$

Refer to Chapter 5 for the complete proof, and Section 4.3 for an outline. While Proposition 2 states that there exists a local welfare function for which any budget-balanced distribution rule is required to be equivalent to a generalized weighted Shapley value (on that welfare function) in order to guarantee equilibrium existence, Theorem 2 states a much stronger result that, *for any set of local welfare functions*, the corresponding distribution rules must be equivalent to generalized weighted Shapley values on some ground welfare functions to guarantee equilibrium existence. This holds true even when the distribution rules are

[2] Note that $W' = W$ if and only if f^W is budget-balanced.

not budget-balanced. Proving Theorem 2 requires working with arbitrary local welfare functions, which is a clear distinction from the proof of Proposition 2, which exhibits a specific local welfare function, showing the result for that case.

From Theorem 2, it follows that designing distribution rules to ensure the existence of an equilibrium merely requires selecting a weight system $\omega = (\boldsymbol{\lambda}, \Sigma)$ and a ground welfare function W' for each local welfare function $W \in \mathbb{W}$ (this defines the mapping g_{SV}), and then applying the distribution rules $\left\{f^{W'}_{GWSV}[\omega]\right\}_{W\in\mathbb{W}}$. Budget-balance, if required, can be directly controlled through proper choice of W', since $\{W'\}$ are the actual welfares distributed. For example, if exact budget-balance is desired, then $W' = W$ for all $W \in \mathbb{W}$. Notions of approximate budget-balance (Roughgarden and Sundararajan [71]) can be similarly accommodated by keeping W' 'close' to W.

An important implication of Theorem 2 is that if one hopes to use a distribution rule that always guarantees equilibrium existence in games with any fixed set of local welfare functions, then one is limited to working within the class of 'potential games'. This is perhaps surprising since a priori, potential games are often thought to be a small, special class of games (Sandholm [74]).[3] More specifically, recall from Section 3.6 that generalized weighted Shapley value distribution rules result in a slight variation of weighted potential games (Hart and Mas-Colell [37], Ui [83]) whose potential function can be computed recursively as:

$$\boldsymbol{\Phi}[\omega](a) = \sum_{r\in R} \boldsymbol{\phi}_r[\omega](\{a\}_r),$$

where $\boldsymbol{\phi}_r[\omega] : 2^N \to \mathbb{R}^m$ is the local potential function at resource r (we denote the kth element of this vector by $(\phi_r[\omega])_k$), and for any $1 \le k \le m$ and any subset $S \subseteq N$,

$$\left(\phi_r[\omega]\right)_k(S) = \frac{1}{\sum_{i\in S}\lambda_i}\left(W'_r(\overline{S}_{m-k+1}) + \sum_{i\in S}\lambda_i\left(\phi_r[\omega]\right)_k(S-\{i\})\right), \tag{4.2}$$

where $W'_r = g_{SV}(W_r)$ and $\overline{S}_k = S - \cup_{\ell=1}^{k-1} S_\ell$.

Theorem 2 also has some negative implications. First, the limitation to generalized weighted Shapley value distribution rules means that one is forced to use distribution rules which may require computing exponentially many marginal contributions, as discussed in Section 3.2. Second, if one desires budget-balance, then there are efficiency limits for games in $\mathcal{G}(N, f^{\mathbb{W}}_{WSV}, \mathbb{W})$. In particular, there exists a submodular welfare function W such that, for any weight vector $\boldsymbol{\omega}$, there exists a game in $\mathcal{G}(N, f^{W}_{WSV}[\boldsymbol{\omega}], W)$ where the best equilibrium has welfare that is a multiplicative factor of two worse than the optimal welfare (Marden and Wierman [54]).

[3] In spite of this limitation, it is useful to point out that there are many well understood learning dynamics which guarantee equilibrium convergence in potential games (Blume [10], Marden et al. [49], Marden and Shamma [51]).

4.2 Characterization in Terms of Generalized Weighted Marginal Contributions

Our second characterization is in terms of the marginal contribution family of distribution rules. The key to obtaining this contribution is the connection between the marginal contribution and Shapley value distribution rules that we proved in Section 3.5. Recall Proposition 1:

Proposition. *For any two welfare functions $W' = (\mathcal{T}', Q')$ and $W'' = (\mathcal{T}'', Q'')$, and any weight system $\omega = (\boldsymbol{\lambda}, \Sigma)$,*

$$f_{GWSV}^{W'}[\omega] = f_{GWMC}^{W''}[\omega] \quad \Longleftrightarrow \quad \mathcal{T}' = \mathcal{T}'' \quad \textit{and} \quad (\forall\, T \in \mathcal{T}')\ q'_T = \left(\sum_{j \in \overline{T}} \lambda_j\right) q''_T. \tag{4.3}$$

This proposition immediately leads to the following equivalent statement of Theorem 2:

Theorem 3. *Given any set of local welfare functions $\mathbb{W}$, all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ possess a pure Nash equilibrium if and only if there exists a weight system ω, and a mapping g_{MC} that maps each local welfare function $W \in \mathbb{W}$ to a corresponding ground welfare function $g_{MC}(W)$ such that its distribution rule $f^W \in f^{\mathbb{W}}$ is equivalent to the generalized weighted marginal contribution distribution rule, $f_{GWMC}^{W''}[\omega]$, where $W'' = g_{MC}(W)$ is defined as,*

$$W'' = h(g_{SV}(W)), \tag{4.4}$$

where h denotes the mapping that maps W' to W'' according to (4.3).

Importantly, Theorem 3 provides an alternate way of designing distribution rules that guarantee equilibrium existence. The advantage of this alternate design is that marginal contributions are much easier to compute than the Shapley value, which requires computing exponentially many marginal contributions. However, it is much more difficult to control the budget-balance of marginal contribution distribution rules. Specifically, $\{W''\}$ are not the actual welfares distributed, and so there is no direct control over budget-balance as was the case for generalized weighted Shapley value distribution rules. Instead, it is necessary to start with desired welfares $\{W'\}$ to be distributed (equivalently, the desired mapping g_{SV}) and then perform a 'preprocessing' step of transforming them into the ground welfare functions $\{W''\}$ using (4.4), which requires exponentially many calls to each W'. However, this is truly a preprocessing step, and thus only needs to be performed once for a given $\{W'\}$.

Another simplification that Theorem 3 provides when compared to Theorem 2 is in terms of the potential function. In particular, in light of Proposition 1, the distribution rules $f_{GWSV}^{W'}[\omega]$ and $f_{GWMC}^{W''}[\omega]$, where $W'' = h(W')$, result in the same 'weighted' potential game with the same potential function $\boldsymbol{\Phi}[\omega]$. However, in terms of W'', there is a clear closed-form expression for the local potential function at resource r,

$\phi_r[\omega] \; : \; 2^N \rightarrow \mathbb{R}^m$. For any $1 \leq k \leq m$ and any subset $S \subseteq N$,

$$\left(\phi_r[\omega]\right)_k(S) = W''_r(\overline{S}_{m-k+1}),$$

where $W''_r = g_{MC}(W_r)$ and $\overline{S}_k = S - \cup_{\ell=1}^{k-1} S_\ell$. In other words, we have,

$$(\forall \; S \subseteq N) \quad \boldsymbol{\phi}_r[\omega](S) = \left(W''_r(\overline{S}_m), W''_r(\overline{S}_{m-1}), \ldots, W''_r(\overline{S}_1)\right).$$

This has been proved in Section 3.6. Having a simple closed-form potential function is helpful for many reasons. For example, it aids in the analysis of learning dynamics and in characterizing efficiency bounds through the well-known potential function method (Tardos and Wexler [82]).

4.3 Proof Sketch of Theorem 2

We now sketch an outline of the proof of Theorem 2 for the special case where there is just one local welfare function W, i.e., $\mathbb{W} = \{W\}$, highlighting the key stages. For an independent, self-contained account of the complete proof, refer to Chapter 5.

First, note that we only need to prove one direction since it is known that for any weight system ω and any two welfare functions W, W', all games in $\mathcal{G}(N, f_{GWSV}^{W'}[\omega], W)$ have an equilibrium (Hart and Mas-Colell [37]).[4] Thus, our focus is solely on proving that for distribution rules f^W that are not generalized weighted Shapley values on some ground welfare function, there exists $G \in \mathcal{G}(N, f^W, W)$ with no equilibrium.

The general proof technique is as follows. First, we present a quick reduction to characterizing only the budget-balanced distribution rules f^W that guarantees the existence of an equilibrium for all games in $\mathcal{G}(N, f^W, W)$. Then, we establish several necessary conditions for a budget-balanced distribution rule f^W that guarantees the existence of an equilibrium for all games in $\mathcal{G}(N, f^W, W)$, which effectively eliminate all but generalized weighted Shapley values on W, giving us our desired result. We establish these conditions by a series of counterexamples which amount to choosing a resource set R and the action sets $\{\mathcal{A}_i\}_{i \in N}$, for which failure to satisfy a necessary condition would lead to nonexistence of an equilibrium.

Throughout, we work with the basis representation of the welfare function W that was introduced in Section 3.4. Since we are dealing with only one welfare function W, we drop the superscripts from $\mathcal{T}^W$, Q^W, and q_T^W in order to simplify notation. It is useful to think of the sets in $\mathcal{T}$ as being 'coalitions' of players that contribute to the welfare function W (also referred to as "contributing coalitions"), and the corresponding coefficients in Q as being their respective contributions. Also, for simplicity, we normalize W by setting $W(\emptyset) = 0$ and therefore, $\emptyset \notin \mathcal{T}$. Before proceeding, we introduce some notation below:

[4]Notice that W has no role to play as far as equilibrium existence of games $G \in \mathcal{G}(N, f_{GWSV}^{W'}[\omega], W)$ is concerned, since it does not affect player utilities. This observation will prove crucial later.

(a) For any subset $S \subseteq N$, $\mathcal{T}(S)$ denotes the set of contributing coalitions in S:

$$\mathcal{T}(S) = \{T \in \mathcal{T} \mid T \subseteq S\}$$

(b) For any subset $S \subseteq N$, $N(S)$ denotes the set of contributing players in S:

$$N(S) = \bigcup \mathcal{T}(S)$$

(c) For any two players $i, j \in N$, $\mathcal{T}_{ij}$ denotes the set of all coalitions containing i and j:

$$\mathcal{T}_{ij} = \{T \in \mathcal{T} \mid \{i, j\} \subseteq T\}$$

(d) Let $\mathcal{B} \subseteq 2^N$ denote any collection of subsets of a set N. Then the relation $\subseteq$ induces a partial order on $\mathcal{B}$. $\mathcal{B}^{\min}$ denotes the set of minimal elements of the poset $(\mathcal{B}, \subseteq)$:

$$\mathcal{B}^{\min} = \{B \in \mathcal{B} \mid (\nexists B' \in \mathcal{B}) \ s.t. \ B' \subsetneq B\}$$

Example 1. *Let $N = \{i, j, k, \ell\}$ be the set of players. Table 4.1a defines a $W : 2^N \to \mathbb{R}$, as well as five different distribution rules for W. Table 4.1b shows the basis representation of W, and Table 4.1c illustrates the notation defined above for W. Throughout the proof sketch, we periodically revisit these distribution rules to illustrate the key ideas.*

The proof is divided into five sections—each section incrementally builds on the structure imposed on the distribution rule f by previous sections.

4.3.1 Reduction to Budget-Balanced Distribution Rules

First, we reduce the problem of characterizing *all* distribution rules f^W that guarantee equilibrium existence for all $G \in \mathcal{G}(N, f^W, W)$ to characterizing only *budget-balanced* distribution rules f^W that guarantee equilibrium existence for all $G \in \mathcal{G}(N, f^W, W)$:

Proposition 3. *For all welfare functions W, a distribution rule f^W guarantees the existence of an equilibrium for all games in $\mathcal{G}(N, f^W, W)$ if and only if it guarantees the existence of an equilibrium for all games in $\mathcal{G}(N, f^W, W')$, where, for all subsets $S \subseteq N$, $W'(S) := \sum_{i \in S} f^W(i, S)$.*

Proof. This proposition is actually a subtlety of our notation. Recall that for games $G \in \mathcal{G}(N, f^W, W)$, the utility function for an agent i is given by

$$U_i(a) = \sum_{r \in a_i} f^W(i, \{a\}_r).$$

Table 4.1: *Tables for Examples 1-6*

(a) *Definitions of welfare functions and distribution rules*

$S \subseteq N$	$\emptyset$	$\{i\}$	$\{j\}$	$\{k\}$	$\{\ell\}$	$\{i,j\}$	$\{i,k\}$	$\{i,\ell\}$	$\{j,k\}$	$\{j,\ell\}$	$\{k,\ell\}$	$\{i,j,k\}$	$\{i,j,\ell\}$	$\{i,k,\ell\}$	$\{j,k,\ell\}$	$\{i,j,k,\ell\}$
$W(S)$	0	5	3	0	3	6	2	8	0	6	3	0	7	5	3	1
$f_1^W(\cdot,S)$	–	(5)	(3)	(0)	(3)	$(3,3)$	$(1,1)$	$(4,4)$	$(0,0)$	$(3,3)$	$(\frac{3}{2},\frac{3}{2})$	$(0,0,0)$	$(\frac{7}{3},\frac{7}{3},\frac{7}{3})$	$(\frac{5}{3},\frac{5}{3},\frac{5}{3})$	$(1,1,1)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$
$f_2^W(\cdot,S)$	–	(5)	(3)	(0)	(3)	$(4,2)$	$(4,-2)$	$(5,3)$	$(2,-2)$	$(3,3)$	$(0,3)$	$(3,1,-4)$	$(3,1,3)$	$(4,-2,3)$	$(2,-2,3)$	$(2,0,-4,3)$
$f_3^W(\cdot,S)$	–	(5)	(3)	(0)	(3)	$(4,2)$	$(4,-2)$	$(5,3)$	$(2,-2)$	$(3,3)$	$(0,3)$	$(3,1,-4)$	$(\frac{10}{3},\frac{2}{3},3)$	$(4,-2,3)$	$(2,-2,3)$	$(\frac{7}{3},-\frac{1}{3},-4,3)$
$f_4^W(\cdot,S)$	–	(5)	(3)	(0)	(3)	$(4,2)$	$(4,-2)$	$(5,3)$	$(1,-1)$	$(3,3)$	$(0,3)$	$(3,0,-3)$	$(3,1,3)$	$(4,-2,3)$	$(1,-1,3)$	$(2,-1,-3,3)$
$f_5^W(\cdot,S)$	–	(5)	(3)	(0)	(3)	$(3,1)$	$(2,-3)$	$(5,3)$	$(0,-3)$	$(3,3)$	$(0,3)$	$(0,-2,-6)$	$(1,-1,1)$	$(2,-3,3)$	$(0,-3,3)$	$(-2,-4,-6,1)$
$W'(S)$	0	5	3	0	3	4	-1	8	-3	6	3	-8	1	2	0	-11

(b) *Basis representation of W*

Coalition $T \in \mathcal{T}$	Contribution $q_T \in Q$
$\{i\}$	5
$\{j\}$	3
$\{\ell\}$	3
$\{i,j\}$	-2
$\{i,k\}$	-3
$\{j,k\}$	-3
$\{i,j,\ell\}$	-2

(c) *Notation*

Symbol	Value
$\mathcal{T}(\{i,\ell\})$	$\{\{i\},\{\ell\}\}$
$\mathcal{T}(\{j,k\})$	$\{\{j\},\{j,k\}\}$
$N(\{i,k\})$	$\{i,k\}$
$\mathcal{T}_{ij}$	$\{\{i,j\},\{i,j,\ell\}\}$
$\mathcal{T}_{ij}^{\min}$	$\{\{i,j\}\}$

(d) *Basis distribution rules computed by recursion (5.14)*

Coalition $T \in \mathcal{T}$	$f_1^T(\cdot,T)$	$f_2^T(\cdot,T)$	$f_3^T(\cdot,T)$	$f_4^T(\cdot,T)$	$f_5^T(\cdot,T)$
$\{i\}$	(1)	(1)	(1)	(1)	(1)
$\{j\}$	(1)	(1)	(1)	(1)	(1)
$\{\ell\}$	(1)	(1)	(1)	(1)	(1)
$\{i,j\}$	$(1,0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(1,1)$
$\{i,k\}$	$(\frac{4}{3},-\frac{1}{3})$	$(\frac{1}{3},\frac{2}{3})$	$(\frac{1}{3},\frac{2}{3})$	$(\frac{1}{3},\frac{2}{3})$	$(1,1)$
$\{j,k\}$	$(1,0)$	$(\frac{1}{3},\frac{2}{3})$	$(\frac{1}{3},\frac{2}{3})$	$(\frac{2}{3},\frac{1}{3})$	$(1,1)$
$\{i,j,\ell\}$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{3},\frac{2}{3},0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(1,1,1)$

From this, it is clear that the welfare function W does not directly affect strategic behavior (it only does so through the distribution rule f^W). Therefore, in terms of strategic behavior and equilibrium existence, the classes $\mathcal{G}(N, f^W, W)$ and $\mathcal{G}(N, f^W, W')$ are identical, for any two welfare functions W, W'. Therefore, a distribution rule f^W guarantees equilibrium existence for all games in $\mathcal{G}(N, f^W, W)$ if and only if it guarantees equilibrium existence for all games in $\mathcal{G}(N, f^W, W')$. To complete the proof, simply pick W' to be the actual welfare distributed by f^W, as defined in (4.1). $\square$

Notice that $f^{\mathbb{W}}$ are budget-balanced distribution rules for the actual welfares they distribute, namely $g_{SV}(\mathbb{W})$ as defined in (4.1). Hence, it is sufficient to prove that for budget-balanced $f^{\mathbb{W}}$ that are not generalized weighted Shapley values, there exists a game in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ for which no equilibrium exists.

Example 2. *Note that f_1^W through f_4^W are budget-balanced, whereas f_5^W, the marginal contribution rule f_{MC}^W, is not. Let W', shown in Table 4.1a, be the actual welfare distributed by f_5^W, as defined in (4.1). Then f_5^W is a budget-balanced distribution rule for W'. In fact, it is the Shapley value rule $f_{SV}^{W'}$.*

4.3.2 Three Necessary Conditions

The second step of the proof is to establish that for every subset $S \subseteq N$ of players, any budget-balanced distribution rule f^W must distribute the welfare $W(S)$ only among contributing players, and do so as if the noncontributing players were absent:

Proposition 4. *If f^W is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f^W, W)$, then,*

$$(\forall\, S \subseteq N) \quad (\forall\, i \in S) \quad f^W(i, S) = f^W(i, N(S))$$

Section 5.4.1 is devoted to the proof, which consists of incrementally establishing the following necessary conditions, for any subset $S \subseteq N$:

(a) If no contributing coalition is formed in S, then f^W does not allocate any utility to the players in S (Lemma 1).

(b) f^W distributes the welfare *only* among the contributing players in S (Lemma 2).

(c) f^W distributes the welfare among the contributing players in S as if all other players were absent (Lemma 3).

Example 3. *For the welfare function W, $S = N(S)$ for all subsets S, making Proposition 4 trivial, except for the two subsets $\{k\}$ and $\{k, \ell\}$ for which k is not a contributing player. Note that f_2^W through f_4^W allocate no welfare to k in these subsets, and ℓ gets the same whether k is present or not. But $f_1^W(k, \{k, \ell\}) \neq 0$ and $f_1^W(\ell, \{k, \ell\}) \neq f_1^W(\ell, \{\ell\})$. Therefore, f_1^W, which is the equal share distribution rule f_{EQ}^W, violates conditions (b)-(c), and hence, Proposition 4. So, it does not guarantee equilibrium existence in all games; see Counterexample 2 in the proof of Lemma 3.*

4.3.3 Decomposition of the Distribution Rule

The third step of the proof establishes that f^W must have a basis representation of the form (3.4), where the basis distribution rules are generalized weighted Shapley values:

Proposition 5. *If f is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f^W, W)$, then there exists a sequence of weight systems $\Omega = \{\omega^T\}_{T \in \mathcal{T}}$ such that*

$$f^W = \sum_{T \in \mathcal{T}} q_T f_{GWSV}^T[\omega^T]$$

Note that for now, the weight systems ω^T could be arbitrary, and need not be related in any way. We deal with how they should be 'consistent' in the next section.

In Section 5.4.2, we prove Proposition 5 by describing a procedure to compute the basis distribution rules, f^T, assuming they exist, and then showing the following properties of f^T:

(a) Each f^T is a budget-balanced distribution rule for W^T (Lemmas 4-5).

(b) f^W, and the basis distribution rules $\left\{f^T\right\}_{T\in\mathcal{T}}$, satisfy (3.4) (Lemma 7).

(c) Each f^T is nonnegative; so, $f^T = f^T_{GWSV}[\omega^T]$ for some ω^T (Lemma 8).

Example 4. *Table 4.1d shows the basis distribution rules computed by our recursive procedure in (5.14). Note that $f_j^T, 1 \le j \le 4$, are budget-balanced distribution rules for W^T (for each T, the shares sum up to 1). It can be verified that f_2^T, f_3^T, f_4^T are nonnegative and satisfy (3.4). Next, observe that from Table 4.1a, $f_1^W(i,\{i,\ell\}) = 4$, but from Table 4.1d, $\sum_{T\in\mathcal{T}} q_T f_1^T(i,\{i,\ell\}) = q_{\{i\}} f_1^{\{i\}}(i,\{i,\ell\}) = 5$, so f_1^W violates condition (b). Also, $f_1^{\{i,k\}}(k,\{i,k\}) < 0$, violating condition (c). So, f_1^W, the equal share distribution rule, does not have a basis representation, and hence does not guarantee equilibrium existence in all games; see Counterexamples 3-4 in the proofs of Lemmas 7 and 8.*

4.3.4 Consistency of Basis Distribution Rules

The fourth part of the proof establishes two important consistency properties that the basis distribution rules f^T must satisfy:

(a) *Global consistency:* If there is a pair of players i, j common to two coalitions T, T', then their local shares from these two coalitions must satisfy (Lemma 9):

$$f^T(i,T) f^{T'}(j,T') = f^{T'}(i,T') f^T(j,T)$$

(b) *Cyclic consistency:* If there is a sequence of $z \ge 3$ players, $(i_1, i_2, \ldots, i_z)$ such that for each of the z neighbor-pairs $\{(i_1,i_2),(i_2,i_3),\ldots,(i_z,i_1)\}$, $\exists\, T_1 \in \mathcal{T}^{\min}_{i_1 i_2}, T_2 \in \mathcal{T}^{\min}_{i_2 i_3}, \ldots, T_z \in \mathcal{T}^{\min}_{i_z i_1}$ and in each T_j, at least one of the neighbors i_j, i_{j+1} gets a nonzero share, then the shares of these z players must satisfy (Lemma 10):

$$f^{T_1}(i_1,T_1) f^{T_2}(i_2,T_2) \cdots f^{T_z}(i_z,T_z) = f^{T_1}(i_2,T_1) f^{T_2}(i_3,T_2) \cdots f^{T_z}(i_1,T_z)$$

Section 5.5.1 is devoted to the proofs. Since $f^T = f^T_{GWSV}[\omega^T]$ for some ω^T, the above translate into consistency conditions on the sequence of weight systems $\Omega = \left\{\omega^T\right\}_{T\in\mathcal{T}}$ (Corollaries 2 and 3, respectively). These conditions are generalizations of those used to prove Proposition 2 in Chen et al. [15], and Marden and Wierman [54]—the welfare function used, see (3.1), is such that $\mathcal{T} = 2^N \backslash \{\emptyset\}$, which is 'rich' enough to further simplify the above consistency conditions. In such cases, the distribution rule f^W is fully determined by 'pairwise shares' of the form $f^W(i,\{i,j\})$.

Example 5. *Among the three budget-balanced distribution rules that have a basis representation, namely f_2^W, f_3^W and f_4^W, only f_2^W satisfies both consistency conditions. f_3^W fails the global consistency test, since $f_3^{\{i,j\}}(i,\{i,j\})f_3^{\{i,j,\ell\}}(j,\{i,j,\ell\}) \neq f_3^{\{i,j,\ell\}}(i,\{i,j,\ell\})f_3^{\{i,j\}}(j,\{i,j\})$, and so, does not guarantee equilibrium existence in all games; see Counterexample 5(a) in the proof of Lemma 9. Similarly, f_4^W fails the cyclic consistency test, since*

$$f_4^{\{i,j\}}(i,\{i,j\})f_4^{\{j,k\}}(j,\{j,k\})f_4^{\{i,k\}}(k,\{i,k\}) \neq f_4^{\{i,j\}}(j,\{i,j\})f_4^{\{j,k\}}(k,\{j,k\})f_4^{\{i,k\}}(i,\{i,k\}),$$

and hence does not guarantee equilibrium existence in all games; see Counterexample 6 in the proof of Lemma 10.

4.3.5 Existence of a Universal Weight System

The last step of the proof is to show that there exists a universal weight system $\omega^* = (\boldsymbol{\lambda}^*, \Sigma^*)$ that is equivalent to all the weight systems in $\Omega = \{\omega^T\}_{T\in\mathcal{T}}$. That is, replacing ω^T with ω^* for any coalition T does not change the distribution rule $f_{GWSV}^T[\omega^T]$:

Proposition 6. *If $f^W = \sum_{T\in\mathcal{T}} q_T f_{GWSV}^T[\omega^T]$ is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f^W, W)$, then, there exists a weight system ω^* such that,*

$$(\forall\, T \in \mathcal{T}) \quad f_{GWSV}^T[\omega^T] = f_{GWSV}^T[\omega^*]$$

In Section 5.5.2, we prove this proposition by explicitly constructing ω^*, given a sequence of weight systems $\Omega = \{\omega^T\}_{T\in\mathcal{T}}$ that satisfies the consistency Corollaries 2 and 3.

Example 6. *The only budget-balanced distribution rule to have survived all the necessary conditions is f_2^W. Using the construction in Section 5.5.2, it can be shown that f_2^W is equivalent to the generalized weighted Shapley value distribution rule $f_{GWSV}^W[\omega^*]$, where the weight system $\omega^* = (\boldsymbol{\lambda}^*, \Sigma^*)$ is given by $\boldsymbol{\lambda}^* = (\frac{1}{2}, \frac{1}{2}, 1, a)$ where a is any strictly positive number, and $\Sigma^* = \{\{i,j,k\},\{\ell\}\}$.*

Chapter 5

Proof of Theorem 2

In this chapter, we present the complete proof of Theorem 2. It is our intent that this presentation be self-contained and independent of the partial outline presented in Section 4.3; therefore, it may contain some redundancies.

First, note that we only need to prove one direction since it is known that for any weight system ω and any mapping g_{SV}, all games in $\mathcal{G}(N, \{f_{GWSV}^{g_{SV}(W)}[\omega]\}_{W \in \mathbb{W}}, \mathbb{W})$ have an equilibrium (Hart and Mas-Colell [37]).[1] Thus, we present the bulk of the proof—the other direction—proving that for distribution rules $f^{\mathbb{W}}$ that are not generalized weighted Shapley values on some welfare function, there exists a game in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ for which no equilibrium exists.

The general technique of the proof is as follows. First, we present a quick reduction to characterizing only budget-balanced distribution rules $f^{\mathbb{W}}$ that guarantees equilibrium existence for all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$. Then, we establish several necessary conditions that these rules must satisfy. Effectively, for each $W \in \mathbb{W}$, these necessary conditions eliminate any budget-balanced distribution rule f^W that is not a generalized weighted Shapley value on W, and hence give us our desired result. We establish each of these conditions by a series of counterexamples which amount to choosing a resource set R, the local welfare functions $\{W_r\}_{r \in R}$, and the associated action sets $\{\mathcal{A}_i\}_{i \in N}$, for which failure to satisfy a necessary condition would lead to nonexistence of an equilibrium.

Most counterexamples involve multiple copies of the same resource. To simplify specifying such counterexamples, we introduce a scaling coefficient $v_r \in \mathbb{Z}_{++}$ for each resource $r \in R$, which denotes the number of copies of r, so that we have,

$$(\forall\, a \in \mathcal{A}) \qquad \mathcal{W}(a) = \sum_{r \in R} v_r W_r\left(\{a\}_r\right) \qquad \text{and} \qquad U_i(a) = \sum_{r \in a_i} v_r f^r\left(i, \{a\}_r\right).$$

Therefore, to exhibit a counterexample, in addition to choosing R, $\{W_r\}_{r \in R}$ and $\{\mathcal{A}_i\}_{i \in N}$, we also choose $\{v_r\}_{r \in R}$.

[1] In fact, notice that $\mathbb{W}$ has no role to play as far as equilibrium existence of games $\mathcal{G}(N, \{f_{GWSV}^{g_{SV}(W)}[\omega]\}_{W \in \mathbb{W}}, \mathbb{W})$ is concerned, since it does not *directly* affect player utilities. This observation will prove crucial later.

Table 5.1: *Summary of notation*

Symbol	Definition	Meaning
$\mathcal{T}^W(S)$	$\{T \in \mathcal{T}^W \mid T \subseteq S\}$	set of contributing coalitions in S
$N^W(S)$	$\bigcup \mathcal{T}^W(S)$	set of contributing players in S
$\mathcal{T}^W_{ij}$	$\{T \in \mathcal{T}^W \mid \{i,j\} \subseteq T\}$	set of coalitions containing both players i and j
$\mathcal{B}^{\max}$	$\{B \in \mathcal{B} \mid (\nexists B' \in \mathcal{B}) \ s.t. \ B \subsetneq B'\}$	set of maximal elements of the poset $(\mathcal{B}, \subseteq)$
$\mathcal{B}^{\min}$	$\{B \in \mathcal{B} \mid (\nexists B' \in \mathcal{B}) \ s.t. \ B' \subsetneq B\}$	set of minimal elements of the poset $(\mathcal{B}, \subseteq)$

Throughout, we work with the basis representation of the welfare function that was introduced in Section 3.4. For each $W \in \mathbb{W}$, it is useful to think of the sets in $\mathcal{T}^W$ as being 'coalitions' of players that contribute to the welfare function W (also referred to as contributing coalitions), and the corresponding coefficients in Q^W as being their respective contributions. Also, for simplicity, we normalize W by setting $W(\emptyset) = 0$ and therefore, $\emptyset \notin \mathcal{T}^W$. Before proceeding, we introduce some notation below, which is also summarized in Table 5.1 for easy reference.

5.1 Notation

For any subset $S \subseteq N$, let $\mathcal{T}^W(S)$ denote the set of contributing coalitions in S:

$$\mathcal{T}^W(S) = \{T \in \mathcal{T}^W \mid T \subseteq S\}$$

Using this notation, and the definition of inclusion functions from (3.2) in (3.3), we have an alternate way of writing W, namely,

$$W(S) = \sum_{T \in \mathcal{T}^W(S)} q^W_T \tag{5.1}$$

For any subset $S \subseteq N$, let $N^W(S)$ denote the set of contributing players in S:

$$N^W(S) = \bigcup \mathcal{T}^W(S)$$

Using this notation, and the alternate definition of W from (5.1), we have,

$$W(S) = W(N^W(S)) \tag{5.2}$$

For any two players $i, j \in N$, let $\mathcal{T}^W_{ij}$ denote the set of all coalitions containing i and j:

$$\mathcal{T}^W_{ij} = \{T \in \mathcal{T}^W \mid \{i,j\} \subseteq T\} \tag{5.3}$$

Let $\mathcal{B} \subseteq 2^N$ denote a collection of subsets of N. The relation $\subseteq$ induces a partial order on $\mathcal{B}$. Let $\mathcal{B}^{\max}$

Table 5.2: *Tables for Example 7*

(a) *Definition of W*

S	$W(S)$
$\emptyset$	0
$\{i\}$	1
$\{j\}$	2
$\{k\}$	3
$\{i,j\}$	3
$\{j,k\}$	3
$\{i,k\}$	3
$\{i,j,k\}$	4

(b) *Basis representation of W*

Coalition $T \in \mathcal{T}^W$	Contribution $q_T^W \in Q^W$
$\{i\}$	1
$\{j\}$	2
$\{k\}$	3
$\{j,k\}$	-2
$\{i,k\}$	-1
$\{i,j,k\}$	1

(c) *Illustration of notation*

Symbol	Value
$\mathcal{T}^W(\{i,j\})$	$\{\{i\},\{j\}\}$
$\mathcal{T}^W(\{j,k\})$	$\{\{j\},\{k\},\{j,k\}\}$
$N^W(\{i,k\})$	$\{i,k\}$
$\mathcal{T}_{ij}^W$	$\{\{i,j,k\}\}$
$\left(\mathcal{T}_{jk}^W\right)^{\max}$	$\{\{i,j,k\}\}$
$(\mathcal{T}_{ik}^W)^{\min}$	$\{\{i\},\{k\}\}$

and $\mathcal{B}^{\min}$ denote the set of maximal and minimal elements of the poset $(\mathcal{B}, \subseteq)$, respectively:

$$\mathcal{B}^{\max} = \{B \in \mathcal{B} \mid (\nexists B' \in \mathcal{B}) \ s.t. \ B \subsetneq B'\}$$

$$\mathcal{B}^{\min} = \{B \in \mathcal{B} \mid (\nexists B' \in \mathcal{B}) \ s.t. \ B' \subsetneq B\}$$

Example 7. *Let $N = \{i,j,k\}$ be the set of players, and $W : 2^N \to \mathbb{R}$ as defined in Table 5.2a. Table 5.2b shows the basis representation of W, and Table 5.2c illustrates our notation for this W.*

5.2 Proof Outline

The proof is divided into five sections—each section incrementally builds on the structure imposed on the distribution rule f by previous sections:

1. *Reduction to budget-balanced distribution rules.* We reduce the problem of characterizing all distribution rules $f^{\mathbb{W}}$ that guarantee equilibrium existence for all $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ to characterizing only *budget-balanced* distribution rules $f^{\mathbb{W}}$ that guarantee equilibrium existence for all $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$.

2. *Three necessary conditions.* We establish three necessary conditions that collectively describe, for any $W \in \mathbb{W}$, for any subset S of players, which players get shares of $W(S)$ and how these shares are affected by the presence of the other players.

3. *Decomposition of the distribution rule.* We use these conditions to show that for each $W \in \mathbb{W}$, f^W must be representable as a linear combination of *generalized weighted Shapley value distribution rules* (with possibly different weight systems) on the unanimity games corresponding to the coalitions in $\mathcal{T}^W$, with corresponding coefficients from Q^W.

4. *Consistency of basis distribution rules.* We establish two important consistency properties (one global and one cyclic) that these 'basis' distribution rules should satisfy, and restate these properties in terms of their corresponding weight systems.

5. *Existence of a universal weight system.* We use the two consistency conditions on the weight systems of the basis distribution rules to show that there exists a single universal weight system that can replace the weight systems of all the basis distribution rules without changing the resulting shares of any welfare function. This establishes, for each $W \in \mathbb{W}$, the equivalence of f^W to a generalized weighted Shapley value on W with this universal weight system.

5.3 Reduction to Budget-Balanced Distribution Rules

First, we reduce the problem of characterizing *all* distribution rules $f^{\mathbb{W}}$ that guarantee equilibrium existence for all $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ to characterizing only *budget-balanced* distribution rules $f^{\mathbb{W}}$ that guarantee equilibrium existence for all $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$:

Proposition 7. *Given any set of local welfare functions $\mathbb{W}$, their corresponding local distribution rules $f^{\mathbb{W}}$ guarantee the existence of an equilibrium for all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ if and only if they guarantee the existence of an equilibrium for all games in $\mathcal{G}(N, f^{\mathbb{W}}, g_{SV}(\mathbb{W}))$.*

Proof. This proposition is actually a subtlety of our notation. Recall that for games $G \in \mathcal{G}(N, f^W, W)$, the utility function for an agent i is given by

$$U_i(a) = \sum_{r \in a_i} f^W\left(i, \{a\}_r\right).$$

From this, it is clear that the welfare function W does not directly affect strategic behavior (it only does so through the distribution rule f^W). Therefore, in terms of strategic behavior and equilibrium existence, the classes $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ and $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W}')$ are identical, for any two sets of welfare functions $\mathbb{W}, \mathbb{W}'$. Therefore, a distribution rule $f^{\mathbb{W}}$ guarantees equilibrium existence for all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ if and only if it guarantees equilibrium existence for all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W}')$. To complete the proof, simply pick $\mathbb{W}' = g_{SV}(\mathbb{W})$, the actual welfares distributed by $f^{\mathbb{W}}$, as defined in (4.1). □

Notice that $f^{\mathbb{W}}$ are budget-balanced distribution rules for the actual welfares they distribute, namely $g_{SV}(\mathbb{W})$ as defined in (4.1). Hence, it is sufficient to prove that for budget-balanced $f^{\mathbb{W}}$ that are not generalized weighted Shapley values, there exists a game in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ for which no equilibrium exists.

5.4 Constraints on Individual Distribution Rules

In the next two sections, we establish common constraints that each budget-balanced distribution rule $f^W \in f^{\mathbb{W}}$ must satisfy, in order to guarantee equilibrium existence for all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ for any given set of local welfare functions $\mathbb{W}$. To do this, we deal with one welfare function at a time—for each $W \in \mathbb{W}$, we only focus on the corresponding distribution rule f^W guaranteeing equilibrium existence for all games in the class $\mathcal{G}(N, f^W, W)$. Note that this is justified by the fact that $\mathcal{G}(N, f^W, W) \subseteq \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ for all $W \in \mathbb{W}$, and so, if $f^{\mathbb{W}}$ guarantees equilibrium existence for all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$, then each $f^W \in f^{\mathbb{W}}$ must guarantee equilibrium existence for all games in $\mathcal{G}(N, f^W, W)$.

Since we are dealing with only one welfare function at a time, we drop the superscripts from f^W, $\mathcal{T}^W$, Q^W, q_T^W, etc. in order to simplify notation.

5.4.1 Three Necessary Conditions

Our goal in this section is to establish that, for every subset $S \subseteq N$ of players, any budget-balanced distribution rule f must distribute the welfare $W(S)$ only among contributing players, and do so as if the noncontributing players were absent:

Proposition 8. *If f is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, then,*

$$(\forall\, S \subseteq N) \quad (\forall\, i \in S) \quad f(i, S) = f(i, N(S))$$

We prove this proposition in incremental stages, by establishing the following necessary conditions, for any subset $S \subseteq N$:

(a) If no contributing coalition is formed in S, then f does not allocate any utility to the players in S. Formally, in Lemma 1, we show that if $\mathcal{T}(S) = \emptyset$, then for all players $i \in S$, $f(i, S) = 0$.

(b) f distributes the welfare *only* among the contributing players in S. Formally, in Lemma 2, we generalize Lemma 1 by showing that for all players $i \notin N(S)$, $f(i, S) = 0$.

(c) f distributes the welfare among the contributing players in S as if all other players were absent. Formally, in Lemma 3, we show that for all players $i \in N(S)$, $f(i, S) = f(i, N(S))$.

Lemma 1. *If f is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, then,*

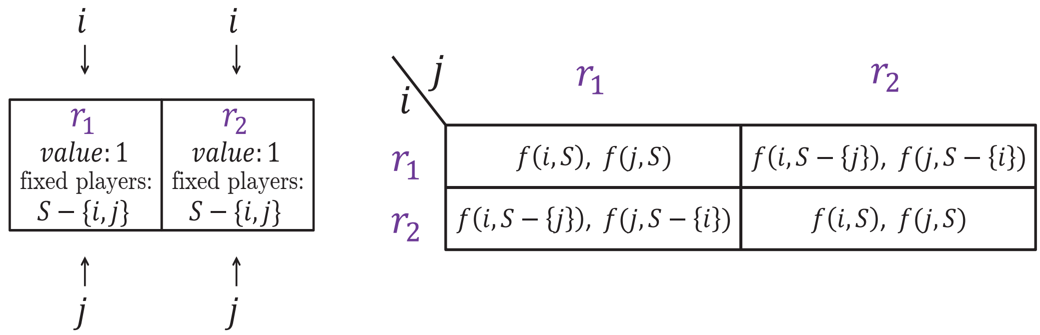
$$(\forall\, S \subseteq N \ s.t.\ \mathcal{T}(S) = \emptyset) \quad (\forall\, i \in S) \quad f(i, S) = 0 \tag{5.4}$$

Proof. The proof is by induction on $|S|$. The base case, where $|S| = 1$ is immediate, because from budget-balance, we have that for any player $i \in N$,

$$f(i, \{i\}) = \begin{cases} q_{\{i\}} & , \ \{i\} \in \mathcal{T} \\ 0 & , \ \text{otherwise} \end{cases}$$

Our induction hypothesis is that (5.4) holds for all subsets S of size z, for some $0 < z < |N|$. Assuming that this is true, we show that (5.4) holds for all subsets S of size $z + 1$. The proof is by contradiction, and proceeds as follows.

Assume to the contrary, that $f(i,S) \neq 0$ for some $i \in S$, for some $S \subseteq N$, where $\mathcal{T}(S) = \emptyset$ and $|S| = z+1$. Since f is budget-balanced, and $z+1 \geq 2$, it follows that there is some $j \in S - \{i\}$ with $f(j,S) \neq 0$, such that $f(i,S) \cdot f(j,S) < 0$, i.e., $f(i,S)$ and $f(j,S)$ have opposite signs. Without loss of generality, assume that $f(i,S) < 0$ and $f(j,S) > 0$.



(a) *The game* **(b)** *The payoff matrix*

Figure 5.1: *Counterexample 1*

Counterexample 1: Consider the game in Figure 5.1a, with resource set $R = \{r_1, r_2\}$ and local resource coefficients $v_{r_1} = v_{r_2} = 1$. Players i and j have the same action sets—they can each choose either r_1 or r_2. All other players in S have a fixed action—they choose both resources. Formally,

$$\mathcal{A}_k = \begin{cases} \{\{r_1\}, \{r_2\}\} & , k \in \{i,j\} \\ \{\{r_1, r_2\}\} & , k \in S - \{i,j\} \end{cases}$$

This is essentially a game between i and j, with the payoff matrix in Figure 5.1b.

Since $\mathcal{T}(S) = \emptyset$, it follows that $\mathcal{T}(S') = \emptyset$ for all $S' \subseteq S$. Therefore, by letting $S' = S - \{i\}$, we can apply the induction hypothesis to S' to obtain $f(j, S - \{i\}) = 0$. Similarly, by letting $S' = S - \{j\}$, we get $f(i, S - \{j\}) = 0$. We now use this to show that none of the four outcomes of Counterexample 1 is an equilibrium—this contradicts the fact that f guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. First, consider the outcome $(\{r_1\}, \{r_1\})$. Given that player j is in r_1, player i obtains a payoff of $f(i,S)$ in r_1, which, by our assumption in step (ii), is negative. By deviating to r_2, player i would obtain a payoff of $f(i, S - \{j\}) = 0$, which is strictly better for player i. Hence, $(\{r_1\}, \{r_1\})$ is not an equilibrium. By nearly identical arguments, it can be shown that the other three outcomes are also not equilibria. This completes the inductive argument. □

Lemma 2. *If f is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, then,*

$$(\forall\, S \subseteq N) \quad (\forall\, i \in S - N(S)) \quad f(i,S) = 0 \tag{5.5}$$

Proof. For $0 \leq p \leq |\mathcal{T}|$ and $0 \leq q \leq n$, let $\mathcal{P}_p^q$ denote the collection of all nonempty subsets S for which $|\mathcal{T}(S)| = p$ and $|S - N(S)| = q$, i.e., S has exactly p contributing coalitions and q noncontributing players in it. Then, $\mathbb{P} = \{\mathcal{P}_0^0, \mathcal{P}_0^1, \ldots, \mathcal{P}_0^n \ , \ \mathcal{P}_1^0, \mathcal{P}_1^1, \ldots, \mathcal{P}_1^n \ , \ \ldots \ , \ \mathcal{P}_{|\mathcal{T}|}^0, \mathcal{P}_{|\mathcal{T}|}^1, \ldots, \mathcal{P}_{|\mathcal{T}|}^n\}$ is an ordered partition of all nonempty subsets of N. Note that we have slightly abused the usage of the term 'partition', since it is possible that $\mathcal{P}_p^q = \emptyset$ for some p, q.

We prove the lemma by induction on $\mathbb{P} = \big\{\{\mathcal{P}_p^q\}\big\}$, i.e., the tuple (p, q). Our base cases are twofold:

(i) When $p = 0$, i.e., for any subset $S \in \bigcup_{q=0}^n \mathcal{P}_0^q$, $\mathcal{T}(S) = \emptyset$. So, (5.5) is true from Lemma 1.

(ii) When $q = 0$, i.e., for any subset $S \in \bigcup_{p=0}^n \mathcal{P}_p^0$, $S = N(S)$. So, (5.5) is vacuously true.

Our induction hypothesis is the following statement:

$$\text{(5.5) holds for all } S \in \bigcup_{p=0}^{z} \bigcup_{q=0}^{n} \mathcal{P}_p^q \bigcup_{q=0}^{y} \mathcal{P}_{z+1}^q, \text{ for some } 0 \leq z < |\mathcal{T}|, \text{ and for some } 0 \leq y < n.$$

Assuming that this is true, we prove that (5.5) holds for all $S \in \mathcal{P}_{z+1}^{y+1}$. In other words, assuming that for all subsets $S \in \bigcup\{\mathcal{P}_0^0, \mathcal{P}_0^1, \ldots, \mathcal{P}_0^n \ , \ \ldots \ , \ \mathcal{P}_z^0, \mathcal{P}_z^1, \ldots, \mathcal{P}_z^n \ , \ \ldots \ , \ \mathcal{P}_{z+1}^0, \mathcal{P}_{z+1}^1, \ldots, \mathcal{P}_{z+1}^y\}$, we have already proved the lemma, we focus on proving the lemma for $S \in \mathcal{P}_{z+1}^{y+1}$, the next collection in $\mathbb{P}$. The proof is by contradiction, and proceeds as follows.

Assume to the contrary, that $f(i, S) \neq 0$ for some $i \in S - N(S)$, for some $S \in \mathcal{P}_{z+1}^{y+1}$. Since $z + 1 \geq 1$ and $y + 1 \geq 1$, it must be that $|S| \geq 2$, i.e., S has at least two players. Also, because $i \notin N(S)$, it follows that $N(S) = N(S - \{i\})$, and so, from (5.2), we have, $W(S) = W(S - \{i\})$. Since f is budget-balanced, and $W(S) = W(S - \{i\})$, we can express $f(i, S)$ as,

$$f(i, S) = \sum_{k \in S - \{i\}} \left(f(k, S - \{i\}) - f(k, S)\right)$$

Because $f(i, S) \neq 0$, it is clear that at least one of the difference terms on the right hand side is nonzero and has the same sign as $f(i, S)$. That is, there is some $j \in S - \{i\}$ such that

$$f(i, S)\left(f(j, S - \{i\}) - f(j, S)\right) > 0 \tag{5.6}$$

Also, $f(i, S - \{j\}) = 0$. To see this, we consider the following two cases, where, for ease of expression, we let $S' = S - \{j\}$.

(i) If $j \in N(S)$, then $|\mathcal{T}(S')| < |\mathcal{T}(S)| = z + 1$, and so $S' \in \bigcup_{p=0}^z \bigcup_{q=0}^n \mathcal{P}_p^q$.

(ii) If $j \notin N(S)$, then $|\mathcal{T}(S')| = |\mathcal{T}(S)| = z + 1$, and $|S' - N(S')| < |S - N(S)| = y + 1$ and so $S' \in \bigcup_{q=0}^y \mathcal{P}_{z+1}^q$.

In either case, we can apply the induction hypothesis to $S - \{j\}$ to conclude that $f(i, S - \{j\}) = 0$, since $i \notin N(S - \{j\})$. Therefore, (5.6) can be rewritten as,

$$(f(i,S) - f(i,S-\{j\}))\,(f(j,S) - f(j,S-\{i\})) < 0$$

To complete the proof, let us first consider the case where $f(i,S) - f(i,S-\{j\}) < 0$ and $f(j,S) - f(j,S-\{i\}) > 0$. For this case, *Counterexample 1* illustrated in Figure 5.1, along with the arguments for nonexistence of equilibrium therein (the proof of Lemma 1), serves as a counterexample here too. The proof for when $f(i,S) - f(i,S-\{j\}) > 0$ and $f(j,S) - f(j,S-\{i\}) < 0$ is symmetric. □

Lemma 3. *If f is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, then,*

$$(\forall\, S \subseteq N) \quad (\forall\, i \in N(S)) \quad f(i,S) = f(i, N(S)) \tag{5.7}$$

Proof. Since this is a tautology when $N(S) = S$, let us assume that $N(S) \subsetneq S$. We consider two cases below.

Case 1: $|N(S)| = 1$. Without loss of generality, let $N(S) = \{i\}$. Since f is budget-balanced, and $W(S) = W(N(S))$, we can express $f(i, N(S))$ as,

$$f(i, N(S)) = \sum_{k \in S} f(k, S)$$

From Lemma 2, we know that $f(k,S) = 0$ for all $k \in S - N(S)$. Accordingly, $f(i,S) = f(i,N(S))$.

Case 2: $|N(S)| \neq 1$. For $0 \leq p \leq |\mathcal{T}|$, let $\mathcal{P}_p$ denote the collection of all nonempty subsets S such that $|N(S)| \neq 1$ and $N(S) \subsetneq S$, for which $|\mathcal{T}(S)| = p$, i.e., S has exactly p contributing coalitions in it. Then, $\mathbb{P} = \{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{|\mathcal{T}|}\}$ is an ordered partition of all nonempty subsets S such that $|N(S)| \neq 1$ and $N(S) \subsetneq S$. Note that we have slightly abused the usage of the term 'partition', since it is possible that $\mathcal{P}_p = \emptyset$ for some p.

We prove the lemma by induction on $\mathbb{P}$. The base case, where $S \in \mathcal{P}_0$, is vacuously true, since $N(S) = \emptyset$. Our induction hypothesis is that (5.7) holds for all subsets $S \in \bigcup_{p=0}^{z} \mathcal{P}_p$, for some $0 \leq z < |\mathcal{T}|$. Assuming that this is true, we show that (5.7) holds for all subsets $S \in \mathcal{P}_{z+1}$.

Before proceeding with the proof, we point out the following observation. Since f is budget-balanced, and $W(S) = W(N(S))$, we have,

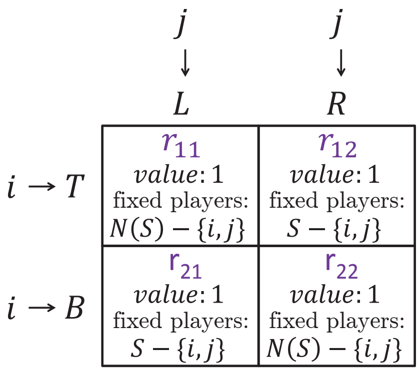
$$\sum_{k \in N(S)} f(k, N(S)) = \sum_{k \in S} f(k,S) = \sum_{k \in N(S)} f(k,S) \tag{5.8}$$

where the second equality comes from Lemma 2, which gives us $f(k,S) = 0$ for all $k \in S - N(S)$.

The proof by contradiction proceeds as follows. Assume to the contrary, that $f(k, S) \neq f(k, N(S))$ for some $k \in N(S)$, for some $S \in \mathcal{P}_{z+1}$. Since $z + 1 \geq 1$, and $|N(S)| \neq 1$, $|N(S)| \geq 2$. Then, from (5.8), we can pick $i, j \in N(S)$ such that,

$$f(i, S) < f(i, N(S)) \tag{5.9}$$

$$f(j, S) > f(j, N(S)) \tag{5.10}$$



(a) *The game*

$i \backslash j$	L	R
T	$f(i, N(S)) + f(i, S - \{j\})$, $f(j, N(S)) + f(j, S - \{i\})$	$f(i, N(S) - \{j\}) + f(i, S)$, $f(j, N(S) - \{i\}) + f(j, S)$
B	$f(i, N(S) - \{j\}) + f(i, S)$, $f(j, N(S) - \{i\}) + f(j, S)$	$f(i, N(S)) + f(i, S - \{j\})$, $f(j, N(S)) + f(j, S - \{i\})$

(b) *The payoff matrix*

Figure 5.2: *Counterexample 2*

Counterexample 2: Consider the game in Figure 5.2a, with resource set $R = \{r_{11}, r_{12}, r_{21}, r_{22}\}$ and local resource coefficients $v_{r_{11}} = v_{r_{12}} = v_{r_{21}} = v_{r_{22}} = 1$. Player i is the row player and player j is the column player. All other players in $N(S)$ have a fixed action—they choose all four resources. And all players in $S - N(S)$ also have a fixed action—they choose resources r_{12} and r_{21}. Formally,

$$\mathcal{A}_k = \begin{cases} \{T = \{r_{11}, r_{12}\}, B = \{r_{21}, r_{22}\}\} & , k = i \\ \{L = \{r_{11}, r_{21}\}, R = \{r_{12}, r_{22}\}\} & , k = j \\ \{\{r_{11}, r_{12}, r_{21}, r_{22}\}\} & , k \in N(S) - \{i, j\} \\ \{\{r_{12}, r_{21}\}\} & , k \in S - N(S) \end{cases}$$

This is essentially a game between players i and j, with the payoff matrix in Figure 5.2b. The set of joint action profiles can therefore be represented as $\mathcal{A} = \{TL, TR, BL, BR\}$.

Because $i \in N(S)$, $|\mathcal{T}(S - \{i\})| = |\mathcal{T}(N(S) - \{i\})| < |\mathcal{T}(S)|$. Also note that $N(S - \{i\}) = N(N(S) - \{i\})$. Now, consider two cases:

(i) If $j \notin N(S - \{i\})$, then $j \notin N(N(S) - \{i\})$, and so, from Lemma 2, $f(j, S - \{i\}) = f(j, N(S) - \{i\}) = 0$.

(ii) If $j \in N(S - \{i\})$, then $j \in N(N(S) - \{i\})$. If $N(S - \{i\}) = \{j\}$, then, applying our analysis in

Case 1 to $S - \{i\}$ and $N(S) - \{i\}$, we have,

$$\begin{aligned} f(j, S - \{i\}) &= f(j, N(S - \{i\})) \\ f(j, N(S) - \{i\}) &= f(j, N(N(S) - \{i\})) \end{aligned} \tag{5.11}$$

If $N(S - \{i\}) \neq \{j\}$, then we know that $|N(S - \{i\})| \geq 2$. Accordingly, we can apply our induction hypothesis to $S - \{i\}$ and $N(S) - \{i\}$ to obtain (5.11).

In either case, we have,

$$f(j, S - \{i\}) = f(j, N(S) - \{i\}) \tag{5.12}$$

By similar arguments, we obtain,

$$f(i, S - \{j\}) = f(i, N(S) - \{j\}) \tag{5.13}$$

We use the four properties in (5.9), (5.10), (5.12), and (5.13) to show that Counterexample 2 does not possess an equilibrium, thereby contradicting the fact that f guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. We show this for each outcome:

(i) TL is not an equilibrium, since player j has an incentive to deviate from L to R:

$$f(j, N(S) - \{i\}) + f(j, S) > f(j, S - \{i\}) + f(j, N(S))$$

This results from combining (5.10) and (5.12).

(ii) TR is not an equilibrium, since player i has an incentive to deviate from T to B:

$$f(i, S - \{j\}) + f(i, N(S)) > f(i, N(S) - \{j\}) + f(i, S)$$

This results from combining (5.9) and (5.13).

(iii) BR and BL are also not equilibria, because in these action profiles, players j and i, respectively, have incentives to deviate—the arguments are identical to cases (a) and (b) above, respectively.

This completes the inductive argument. □

5.4.2 Decomposition of the Distribution Rule

Our goal in this section is to use the necessary conditions above (Proposition 8) to establish that f must be representable as a linear combination of generalized weighted Shapley value distribution rules (see (3.8) in Table 3.2) on the unanimity games corresponding to the coalitions in $\mathcal{T}$, with corresponding coefficients from Q:

Proposition 9. *If f is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, then, there exists a sequence of weight systems $\Omega = \{\omega^T\}_{T \in \mathcal{T}}$ such that*

$$f = \sum_{T \in \mathcal{T}} q_T f^T_{GWSV}[\omega^T]$$

Note that for now, the weight systems ω^T could be arbitrary, and need not be related in any way. We deal with how they should be 'consistent' later, in Section 5.5.1.

Before proceeding, we define a useful abstract mathematical object. The $\min$-partition of a finite poset $(\mathcal{T}, \subseteq)$, denoted by $\mathbb{P}_{\min}(\mathcal{T}) = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\ell\}$, is an ordered partition of $\mathcal{T}$, constructed iteratively as specified in Algorithm 1.

Algorithm 1 Construction of $\mathbb{P}_{\min}(\mathcal{T})$

$\mathcal{P}_1 = \mathcal{T}^{\min}$
$z \leftarrow 2$
while $\mathcal{T} \neq \bigcup_{1 \leq p < z} \mathcal{P}_p$ **do**
    $\mathcal{P}_z = (\mathcal{T} - \bigcup_{1 \leq p < z} \mathcal{P}_p)^{\min}$
    $z \leftarrow z + 1$
end while

Example 8. *Let $(\mathcal{T}, \subseteq)$ be a poset, where $\mathcal{T} = \{\{i\}, \{j\}, \{j,k\}, \{k,\ell\}, \{j,\ell\}, \{i,j,k\}\}$. Then,*

$$\mathbb{P}_{\min}(\mathcal{T}) = \{\ \{\{i\}, \{j\}, \{k,\ell\}\}\ ,\ \{\{j,k\}, \{j,\ell\}\}\ ,\ \{\{i,j,k\}\}\ \}$$

Construction of basis distribution rules: Given a budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, we now show how to construct a sequence of basis distribution rules $\{f^T\}_{T \in \mathcal{T}}$ such that (3.4) is satisfied. Let $\mathbb{P}_{\min}(\mathcal{T}) = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\ell\}$ be the $\min$-partition of the poset $(\mathcal{T}, \subseteq)$, and let f be a distribution rule for W. Starting with $z = 1$, recursively define f^T for each $T \in \mathcal{P}_z$ as,

$$(\forall S \subseteq N)\ (\forall i \in N) \quad f^T(i, S) = \begin{cases} \frac{1}{q_T}\left(f(i, T) - \displaystyle\sum_{T' \in \mathcal{T}(T) - \{T\}} q_{T'} f^{T'}(i, S) \right) & , T \subseteq S \\ 0 & , \text{otherwise} \end{cases} \tag{5.14}$$

At the end of this procedure, we obtain the basis distribution rules $\{f^T\}_{T \in \mathcal{T}}$. Note that it is not obvious from this construction that these basis distribution rules satisfy (3.4), or that they are generalized weighted Shapley value distribution rules on their corresponding unanimity games. The rest of this section is devoted to showing these properties. But first, here is an example to demonstrate this recursive construction.

Example 9. *Consider the setting in Example 7, where $N = \{i, j, k\}$ is the set of players, and $W : 2^N \to \mathbb{R}$ is the welfare function defined in Table 5.2a. The basis representation of W is shown in Table 5.2b. The set of* *coalitions is therefore given by* $\mathcal{T} = \{\{i\},\{j\},\{k\},\{j,k\},\{i,k\},\{i,j,k\}\}$*. For the poset* $(\mathcal{T},\subseteq)$*, we have,*

$$\mathbb{P}_{\min}(\mathcal{T}) = \{\ \{\{i\},\{j\},\{k\}\}\ ,\ \{\{j,k\},\{i,k\}\}\ ,\ \{\{i,j,k\}\}\ \}$$

Consider the following two distribution rules for W*.*

(i) f_{SV}*, the Shapley value distribution rule (see Section 3.2.2).*

(ii) f_{EQ}*, the equal share distribution rule (see Section 3.2.1).*

Table 5.3a shows f_{SV} *and* f_{EQ} *for this welfare function. The basis distribution rules* f^T_{SV} *and* f^T_{EQ} *that result from applying our construction (5.14) above are shown in Table 5.3b. For simplicity, we show only* $f^T_{SV}(\cdot,T)$ *and* $f^T_{EQ}(\cdot,T)$*.*

Table 5.3: *Tables for Example 9*

(a) *Definition of* f_{SV} *and* f_{EQ}

S	$f_{SV}(\cdot,S)$	$f_{EQ}(\cdot,S)$
$\{i\}$	(1)	(1)
$\{j\}$	(2)	(2)
$\{k\}$	(3)	(3)
$\{i,j\}$	$(1,2)$	$(\frac{3}{2},\frac{3}{2})$
$\{j,k\}$	$(1,2)$	$(\frac{3}{2},\frac{3}{2})$
$\{i,k\}$	$(\frac{1}{2},\frac{5}{2})$	$(\frac{3}{2},\frac{3}{2})$
$\{i,j,k\}$	$(\frac{5}{6},\frac{4}{3},\frac{11}{6})$	$(\frac{4}{3},\frac{4}{3},\frac{4}{3})$

(b) *Basis distribution rules for* f_{SV} *and* f_{EQ}

Coalition $T\in\mathcal{T}$	$f^T_{SV}(\cdot,T)$	$f^T_{EQ}(\cdot,T)$
$\{i\}$	(1)	(1)
$\{j\}$	(1)	(1)
$\{k\}$	(1)	(1)
$\{j,k\}$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{4},\frac{3}{4})$
$\{i,k\}$	$(\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},\frac{3}{2})$
$\{i,j,k\}$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},\frac{4}{3})$

The proof of Proposition 9 consists of four lemmas, as outlined below:

(a) In Lemma 4, we show that each f^T, as constructed in (5.14) mimics f locally for its corresponding unanimity game W^T, i.e., f^T satisfies Proposition 8 for $W = W^T$.

(b) Using this property, in Lemma 5, we show that each f^T is a budget-balanced distribution rule for its corresponding unanimity game W^T.

(c) In Lemma 7, we show that f, and the basis distribution rules $\{f^T\}_{T\in\mathcal{T}}$, satisfy (3.4), i.e., $f = \sum_{T\in\mathcal{T}} q_T f^T$.

(d) Finally, in Lemma 8, we show that for each $T\in\mathcal{T}$, there exists a weight system ω^T such that $f^T = f^T_{GWSV}[\omega^T]$.

Lemma 4. *Each* f^T *as defined in (5.14) satisfies,*

$$(\forall S\subseteq N)\ (\forall i\in N)\quad f^T(i,S) = \begin{cases} f^T(i,T) & ,\ i\in T \text{ and } T\subseteq S \\ 0 & ,\ \text{otherwise} \end{cases} \tag{5.15}$$

Proof. The proof is by induction on $\mathbb{P}_{\min}(\mathcal{T})$. The base case, where $T \in \mathcal{P}_1$ is immediate, because from (5.14), for any $T \in \mathcal{P}_1$,

$$(\forall S \subseteq N)\ (\forall i \in N) \quad f^T(i,S) = \begin{cases} \frac{1}{q_T} f(i,T) & , T \subseteq S \\ 0 & , \text{otherwise} \end{cases}$$

Our induction hypothesis is that f^T satisfies (5.15) for all $T \in \bigcup_{p=1}^{z} \mathcal{P}_p$, for some $1 \leq z < \ell$. Assuming that this is true, we prove that f^T satisfies (5.15) for all $T \in \mathcal{P}_{z+1}$. To evaluate $f^T(i,S)$ for some $i \in S \subseteq N$, we consider the following three cases:

(i) $T \nsubseteq S$. In this case, from (5.14), $f^T(i,S) = 0$.

(ii) $i \notin T \subseteq S$. Here, we know that $f(i,T) = 0$ by definition. Also, for all $T' \in \mathcal{T}(T) - \{T\}$, we have, $T' \in \bigcup_{p=1}^{z} \mathcal{P}_p$ and $i \notin T'$; so, from the induction hypothesis, $f^{T'}(i,S) = 0$. Therefore, evaluating (5.14), we get $f^T(i,S) = 0$.

(iii) $i \in T \subseteq S$. In this case, we need to show that $f^T(i,S) = f^T(i,T)$. By (5.14), we have,

$$f^T(i,S) = \frac{1}{q_T}\left(f(i,T) - \sum_{T' \in \mathcal{T}(T) - \{T\}} q_{T'} f^{T'}(i,S)\right)$$
$$f^T(i,T) = \frac{1}{q_T}\left(f(i,T) - \sum_{T' \in \mathcal{T}(T) - \{T\}} q_{T'} f^{T'}(i,T)\right)$$

For each $T' \in \mathcal{T}(T) - \{T\}$, we know that $T' \in \bigcup_{p=1}^{z} \mathcal{P}_p$, and hence, from the induction hypothesis, we have $f^{T'}(i,S) = f^{T'}(i,T)$. Therefore, $f^T(i,T) = f^T(i,S)$, as desired.

Hence, f^T satisfies (5.15). $\square$

Lemma 5. *If f is a budget-balanced distribution rule for W, then each f^T as defined in (5.14) is a budget-balanced distribution rule for W^T, i.e.,*

$$(\forall T \in \mathcal{T})\ (\forall S \subseteq N) \quad \sum_{i \in S} f^T(i,S) = W^T(S)$$

Proof. Since f^T is of the form (5.15) from Lemma 4, to show (local) budget-balance, we need only show that

$$(\forall T \in \mathcal{T}) \quad \sum_{i \in T} f^T(i,T) = 1 \tag{5.16}$$

Once again, the proof is by induction on $\mathbb{P}_{\min}(\mathcal{T})$. The base case, where $T \in \mathcal{P}_1$ follows from the budget-balance of f. Our induction hypothesis is that f^T satisfies (5.16) for all $T \in \cup_{p=1}^{z} \mathcal{P}_p$, for some $1 \leq z < \ell$. Assuming that this is true, we prove that f^T satisfies (5.16) for all $T \in \mathcal{P}_{z+1}$. For any $T \in \mathcal{P}_{z+1}$, using

(5.14), we have,

$$\begin{aligned}\sum_{i\in T} f^T(i,T) &= \frac{1}{q_T}\left(\sum_{i\in T} f(i,T) - \sum_{i\in T}\sum_{T'\in\mathcal{T}(T)-\{T\}} q_{T'} f^{T'}(i,T')\right)\\ &= \frac{1}{q_T}\left(W(T) - \sum_{T'\in\mathcal{T}(T)-\{T\}}\sum_{i\in T'} q_{T'} f^{T'}(i,T')\right)\\ &= \frac{1}{q_T}\left(\sum_{T'\in\mathcal{T}(T)} q_{T'} - \sum_{T'\in\mathcal{T}(T)-\{T\}} q_{T'}\right) = \frac{1}{q_T}\left(q_T\right) = 1\end{aligned}$$

where we have used the budget-balance of f, followed by the induction hypothesis and (5.1). This completes the inductive argument and hence the proof. □

Example 10. *Consider the decomposition of f_{SV} and f_{EQ} illustrated in Example 9. Both f^T_{SV} and f^T_{EQ} are locally budget-balanced for all $T \in \mathcal{T}$, that is, they satisfy (5.16).*

Before continuing with the proof, in the next lemma, we present the *conditional inclusion-exclusion principle*, an important and useful property of the basis distribution rules $\{f^T\}_{T\in\mathcal{T}}$.

Lemma 6. *(Conditional inclusion-exclusion principle) For any $T \in \mathcal{T}$, there exist integers $\{n_T(T')\}_{T'\in\mathcal{T}}$ such that the basis distribution rules $\{f^T\}_{T\in\mathcal{T}}$ defined in (5.14) satisfy,*

$$(\forall i \in T) \quad q_T f^T(i,T) = \sum_{T'\in\mathcal{T}(T)} n_T(T') f(i,T') \tag{5.17}$$

Furthermore, if $\{f^T\}_{T\in\mathcal{T}}$ satisfies

$$(\forall S \subsetneq T)\ (\forall i \in S) \quad f(i,S) = \sum_{T'\in\mathcal{T}(S)} q_{T'} f^{T'}(i,T'), \tag{5.18}$$

then,

$$(\forall i \in T)\ (\forall j \in T-\{i\}) \quad 0 = q_T f^T(i,T-\{j\}) = \sum_{T'\in\mathcal{T}(T)} n_T(T') f(i,T'-\{j\}) \tag{5.19}$$

Proof. For any $T \in \mathcal{T}$, setting $S = T$ in (5.14), and using Lemma 4, we get,

$$(\forall i \in T) \quad q_T f^T(i,T) = f(i,T) - \sum_{T'\in\mathcal{T}(T)-\{T\}} q_{T'} f^{T'}(i,T') \tag{5.20}$$

It follows that by unravelling the recursion above, i.e., by repeatedly substituting for the terms $q_{T'} f^{T'}(i,T')$ that appear in the summation, we obtain (5.17), where $\{n_T(T')\}_{T'\in\mathcal{T}}$ are some integers.

Let $\mathcal{T}_i(T) = \{T' \in \mathcal{T}(T) : i \in T'\}$ denote the set of coalitions contained in T that contain i. Before proving (5.19), we make the following observation. From (5.17), we have,

$$\begin{aligned}
(\forall i \in T) \quad q_T f^T(i,T) &= \sum_{T' \in \mathcal{T}(T)} n_T(T') f(i,T') \\
&= \sum_{T' \in \mathcal{T}_i(T)} n_T(T') \sum_{T'' \in \mathcal{T}_i(T')} q_{T''} f^{T''}(i,T'') \quad \text{(from (5.14) and Lemma 4)} \\
&= \sum_{T' \in \mathcal{T}_i(T)} m_{i,T}(T') q_{T'} f^{T'}(i,T')
\end{aligned} \tag{5.21}$$

where $\{m_{i,T}(T')\}_{T' \in \mathcal{T}}$ are some integer coefficients. We now exploit the fact that (5.21) holds for all distribution rules f to show that the unique solution for the coefficients $m_{i,T}(T')$ is given by,

$$m_{i,T}(T') = \begin{cases} 1 & , T' = T \\ 0 & , \text{otherwise} \end{cases} \tag{5.22}$$

To see this, we first prove that, given $T \in \mathcal{T}$ and $i \in T$, $m_{i,T}(T'') = 0$ for all $T'' \in \mathcal{T}_i(T) - \{T\}$ by induction on $\mathbb{P}_{\min}(\mathcal{T}_i(T) - \{T\})$. To do this, we focus on the family of generalized weighted Shapley value distribution rules $\{f_{GWSV}[\omega^S]\}_{S \in \mathcal{T}(T)}$ with weight systems $\omega^S = (\boldsymbol{\lambda}, \Sigma^S)$, where $\boldsymbol{\lambda} = (1,1,\ldots,1)$ and $\Sigma^S = (N-S, S)$. By definition (see (3.8) in Table 3.2), for each $T' \in \mathcal{T}_i(T)$,

$$f^{T'}_{GWSV}[\omega^S](i,T') = \begin{cases} \frac{1}{|T'|} & , T' \subseteq S \\ 0 & , \text{otherwise} \end{cases} \tag{5.23}$$

(i) For the base case, when $T'' \in \mathcal{P}_1$, it follows from (5.23), with $S = T''$, that, for all $T' \in \mathcal{T}_i(T)$,

$$f^{T'}_{GWSV}[\omega^{T''}](i,T') = \begin{cases} \frac{1}{|T'|} & , T' = T'' \\ 0 & , \text{otherwise} \end{cases} \tag{5.24}$$

This results from the fact that since $T'' \in \mathcal{P}_1$, $T' \subseteq T''$ if and only if $T' = T''$. Now, we evaluate (5.21) for the distribution rule $f_{GWSV}[\omega^{T''}]$ to get,

$$q_T f^T_{GWSV}[\omega^{T''}](i,T) = \sum_{T' \in \mathcal{T}_i(T)} m_{i,T}(T') q_{T'} f^{T'}_{GWSV}[\omega^{T''}](i,T')$$

Using (5.24) to simplify the above equation, we get,

$$0 = m_{i,T}(T'') q_{T''} \frac{1}{|T''|}$$

since, for any $T' \neq T''$, $T' \cap (N - T') \neq \emptyset$. Therefore, $m_{i,T}(T'') = 0$.

(ii) Our induction hypothesis is that $m_{i,T}(T'') = 0$ for all $T'' \in \bigcup_{p=1}^{z} \mathcal{P}_p$, for some $1 \leq z < \ell$. Assuming that this is true, we prove that $m_{i,T}(T'') = 0$ for all $T'' \in \mathcal{P}_{z+1}$. If $T'' \in \mathcal{P}_{z+1}$, it follows from (5.23), with $S = T''$, that, for all $T' \in \{T\} \bigcup_{p=z+1}^{\ell} \mathcal{P}_p$, (5.24) holds, from a similar reasoning as above. Now, we evaluate (5.21) for the distribution rule $f_{GWSV}[\omega^{T''}](\cdot,\cdot)$ to get,

$$q_T f^T_{GWSV}[\omega^{T''}](i,T) = \sum_{T' \in \mathcal{T}_i(T)} m_{i,T}(T') q_{T'} f^{T'}_{GWSV}[\omega^{T''}](i,T')$$

By grouping together terms on the right hand side, we can rewrite this as,

$$q_T f^T_{GWSV}[\omega^{T''}](i,T) = m_{i,T}(T) q_T f^T_{GWSV}[\omega^{T''}](i,T) + \sum_{p=1}^{\ell} \sum_{T' \in \mathcal{P}_p} m_{i,T}(T') q_{T'} f^{T'}_{GWSV}[\omega^{T''}](i,T') \tag{5.25}$$

Using the induction hypothesis, we get that,

$$\sum_{p=1}^{z} \sum_{T' \in \mathcal{P}_p} m_{i,T}(T') q_{T'} f^{T'}_{GWSV}[\omega^{T''}](i,T') = 0 \tag{5.26}$$

Using (5.24), we get that,

$$\begin{aligned} \sum_{p=z+1}^{\ell} \sum_{T' \in \mathcal{P}_p} m_{i,T}(T') q_{T'} f^{T'}_{GWSV}[\omega^{T''}](i,T') &= m_{i,T}(T'') q_{T''} \frac{1}{|T''|} \\ f^T_{GWSV}[\omega^{T''}](i,T) &= 0 \end{aligned} \tag{5.27}$$

since, for all $T' \in \{T\} \bigcup_{p=z+1}^{\ell} \mathcal{P}_p$, if $T' \neq T''$, then $T' \cap (N - T') \neq \emptyset$. Therefore, using (5.26) and (5.27) in (5.25), we get $m_{i,T}(T'') = 0$.

This completes the inductive argument. From this, it is straightforward to see that $m_{i,T}(T) = 1$.

We now return to proving the remainder of the lemma, that is (5.19). The right hand side of (5.19) can be evaluated as,

$$\begin{aligned} \sum_{T' \in \mathcal{T}(T)} n_T(T') f(i, T' - \{j\}) &= \sum_{T' \in \mathcal{T}(T)} n_T(T') \sum_{T'' \in \mathcal{T}(T' - \{j\})} q_{T''} f^{T''}(i, T'') && \text{(from (5.18))} \\ &= \sum_{T' \in \mathcal{T}_i(T)} n_T(T') \sum_{T'' \in \mathcal{T}_i(T')} q_{T''} f^{T''}(i, T'' - \{j\}) && \text{(from Lemma 4)} \\ &= \sum_{T' \in \mathcal{T}_i(T)} m_{i,T}(T') q_{T'} f^{T'}(i, T' - \{j\}) && \text{(from (5.21))} \\ &= q_T f^T(i, T - \{j\}) = 0 && \text{(from (5.22))} \end{aligned}$$

This completes the proof. □

Lemma 7. *If f is a budget-balanced distribution rule that guarantees the equilibrium existence in all games $G \in \mathcal{G}(N, f, W)$, then the basis distribution rules $\{f^T\}_{T\in\mathcal{T}}$ defined in (5.14) satisfy*

$$(\forall S \subseteq N)\ (\forall i \in S) \quad f(i, S) = \sum_{T\in\mathcal{T}} q_T f^T(i, S) \tag{5.28}$$

Proof. From Lemma 4, (5.28) is equivalent to

$$(\forall S \subseteq N)\ (\forall i \in S) \quad f(i, S) = \sum_{T\in\mathcal{T}(S)} q_T f^T(i, T) \tag{5.29}$$

Let $S \subseteq N$. We consider three cases:

Case 1: $S \in \mathcal{T}$. The proof is immediate here, because, rearranging the terms in (5.14), we get,

$$f(i, S) = \sum_{T'\in\mathcal{T}(S)} q_{T'} f^{T'}(i, S) = \sum_{T'\in\mathcal{T}(S)} q_{T'} f^{T'}(i, T') \tag{5.30}$$

where the last equality follows from Lemma 4.

Case 2: $S \neq N(S)$. In this case, we can apply Proposition 8, i.e., $f(i, S) = f(i, N(S))$, to reduce it to the following case, replacing S with $N(S)$.

Case 3: $S = N(S)$. In other words, S is a union of one or more coalitions in $\mathcal{T}$. The remainder of the proof is devoted to this case.

For any subset $S \subseteq N$ such that $S = N(S)$, i.e., S is exactly a union of one or more coalitions in $\mathcal{T}$, we prove this lemma by induction on $|\mathcal{T}(S)|$. The base case, where $|\mathcal{T}(S)| = 1$ (and hence $S \in \mathcal{T}$) is true from (5.30). Our induction hypothesis is that (5.29) holds for all subsets $S \subseteq N$ such that $S = N(S)$, with $|\mathcal{T}(S)| \leq z$ for some $1 \leq z < |\mathcal{T}|$. Assuming that this is true, we prove that (5.29) holds for all subsets $S \subseteq N$ such that $S = N(S)$, with $|\mathcal{T}(S)| = z + 1$. If $S \in \mathcal{T}$, then the proof is immediate from (5.30), so let us assume $S \notin \mathcal{T}$. Before proceeding with the proof, we point out the following observation. Since f is budget-balanced, we have,

$$\begin{aligned}
\sum_{i\in S} f(i, S) = W(S) &= \sum_{T\in\mathcal{T}(S)} q_T && \text{(from (5.1))} \\
&= \sum_{T\in\mathcal{T}(S)} q_T \sum_{i\in T} f^T(i, T) && \text{(from Lemma 5)} \\
&= \sum_{i\in S} \sum_{T\in\mathcal{T}(S)} q_T f^T(i, T) && \text{(from Lemma 4)}
\end{aligned} \tag{5.31}$$

The proof is by contradiction, and proceeds as follows. Assume to the contrary, that for some $k \in S$, for some $S \subseteq N$ such that $S = N(S)$ and $|\mathcal{T}(S)| = z + 1$, $f(k, S) \neq \sum_{T\in\mathcal{T}(S)} q_T f^T(k, T)$. Since $z + 1 \geq 2$, and $S = N(S)$, it must be that $|S| \geq 2$, i.e., S has at least two players. Then, from (5.31), it follows that we can pick $i, j \in S$ such that,

$$f(i,S) > \sum_{T \in \mathcal{T}(S)} q_T f^T(i,T) \tag{5.32}$$

$$f(j,S) < \sum_{T \in \mathcal{T}(S)} q_T f^T(j,T) \tag{5.33}$$

Because $S = N(S)$, for any $S' \subsetneq S$, $|\mathcal{T}(S')| < |\mathcal{T}(S)|$. Hence, applying the induction hypothesis,

$$(\forall S' \subsetneq S)\ (\forall i \in S') \quad f(i,S') = \sum_{T \in \mathcal{T}(S')} q_T f^T(i,T) \tag{5.34}$$

Since every coalition $T \in \mathcal{T}(S)$ is a subset of S, (5.34) holds when S is replaced with any $T \in \mathcal{T}(S)$. Therefore, Lemma 6, the conditional inclusion-exclusion principle, can be applied to obtain, for any coalition $T \in \mathcal{T}(S)$,

$$(\forall i \in T) \quad q_T f^T(i,T) = \sum_{T' \in \mathcal{T}(T)} n_T(T') f(i,T')$$
$$(\forall i \in T)\ (\forall j \in T - \{i\}) \quad q_T f^T(i, T - \{j\}) = \sum_{T' \in \mathcal{T}(T)} n_T(T') f(i, T' - \{j\})$$

Summing up these equations over all $T \in \mathcal{T}(S)$, we get,

$$\begin{aligned} (\forall i \in S) \quad & \sum_{T \in \mathcal{T}(S)} q_T f^T(i,T) = \sum_{T \in \mathcal{T}(S)} \widetilde{n}_S(T) f(i,T) \\ (\forall i \in S)\ (\forall j \in S - \{i\}) \quad & \sum_{T \in \mathcal{T}(S)} q_T f^T(i, T - \{j\}) = \sum_{T \in \mathcal{T}(S)} \widetilde{n}_S(T) f(i, T - \{j\}) \end{aligned} \tag{5.35}$$

where the constants $\{\widetilde{n}_S(T)\}_{T \in \mathcal{S}}$ are given by,

$$\widetilde{n}_S(T) = \sum_{\substack{T' \in \mathcal{T}(S) \\ T \subseteq T'}} n_{T'}(T)$$

Counterexample 3. Our goal is to exploit inequalities (5.32) and (5.33) to build a counterexample that mimics *Counterexample 2* illustrated in Figure 5.2, leading to a similar best-response cycle involving just players i and j. Equation (5.35) suggests the following technique for achieving precisely this. Consider the game in Figure 5.3, which has the same underlying 2×2 box structure of *Counterexample 2*. The resources in the top half are added as follows:

(i) Add a resource r_1 to the top left box.

(ii) Add resources in $R_1^+ = \left\{ r_1^T : T \in \mathcal{T}(S) \text{ and } \widetilde{n}_S(T) > 0 \right\}$ to the top right box.

(iii) Add resources in $R_1^- = \left\{ r_1^T : T \in \mathcal{T}(S) \text{ and } \widetilde{n}_S(T) < 0 \right\}$ to the top left box.

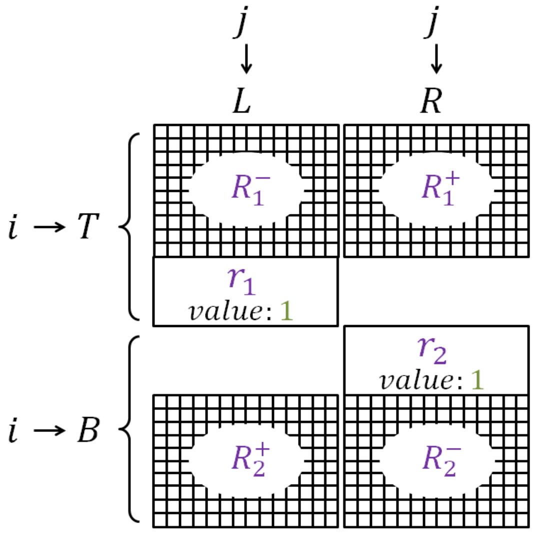


Figure 5.3: *Counterexample 3*

Then, the bottom half is symmetrically filled up as follows.

(i) Add a resource r_2 to the bottom right box.

(ii) Add resources in $R_2^+ = \left\{r_2^T : T \in \mathcal{T}(S) \text{ and } \widetilde{n}_S(T) > 0\right\}$ to the bottom left box.

(iii) Add resources in $R_2^- = \left\{r_2^T : T \in \mathcal{T}(S) \text{ and } \widetilde{n}_S(T) < 0\right\}$ to the bottom right box.

The resource set R is therefore given by,

$$R = \{r_1, r_2\} \cup R_1^+ \cup R_1^- \cup R_2^+ \cup R_2^-$$

The local resource coefficients are given by,

$$v_{r_1} = v_{r_2} = 1 \qquad \text{and} \qquad (\forall T \in \mathcal{T}(S)) \quad v_{r_1^T} = v_{r_2^T} = |\widetilde{n}_S(T)|$$

In resources r_1 and r_2, we fix players in $S - \{i, j\}$. For each $T \in \mathcal{T}(S)$, in resources r_1^T and r_2^T, we fix players in $T - \{i, j\}$. Effectively, all players other than i and j have a fixed action in their action set, determined by these fixtures. The action set of player i is given by $\mathcal{A}_i = \{T, B\}$, where,

$$T = \left\{r_1^T \in R_1^- : i \in T\right\} \cup \{r_1\} \cup \left\{r_1^T \in R_1^+ : i \in T\right\}$$

$$B = \left\{r_2^T \in R_2^+ : i \in T\right\} \cup \{r_2\} \cup \left\{r_2^T \in R_2^- : i \in T\right\}$$

The action set of player j is given by $\mathcal{A}_j = \{L, R\}$, where,

$$L = \left\{r_1^T \in R_1^- : j \in T\right\} \cup \{r_1\} \cup \left\{r_2^T \in R_2^+ : j \in T\right\}$$

$$R = \left\{r_1^T \in R_1^+ : j \in T\right\} \cup \{r_2\} \cup \left\{r_2^T \in R_2^- : j \in T\right\}$$

This is essentially a game between players i and j. The set of joint action profiles can therefore be represented as $\mathcal{A} = \{TL, TR, BL, BR\}$.

We use the four properties in (5.32), (5.33), (5.34), and (5.35) to show that *Counterexample 3* does not possess an equilibrium, thereby contradicting the fact that f guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. We show this for each outcome:

(i) TL is not an equilibrium, since player j has an incentive to deviate from L to R. To see this, consider the utilities of player j when choosing L and R,

$$U_j(T, L) = -\sum_{\substack{T \in \mathcal{T}(S) \\ \widetilde{n}_S(T) < 0}} \widetilde{n}_S(T) f(j, T) + f(j, S) + \sum_{\substack{T \in \mathcal{T}(S) \\ \widetilde{n}_S(T) > 0}} \widetilde{n}_S(T) f(j, T - \{i\})$$

$$U_j(T, R) = \sum_{\substack{T \in \mathcal{T}(S) \\ \widetilde{n}_S(T) > 0}} \widetilde{n}_S(T) f(j, T) + f(j, S - \{i\}) - \sum_{\substack{T \in \mathcal{T}(S) \\ \widetilde{n}_S(T) < 0}} \widetilde{n}_S(T) f(j, T - \{i\})$$

The difference in utilities for j between choosing R and L is therefore given by,

$$\begin{aligned}
&U_j(T, R) - U_j(T, L) \\
&\quad = \left(\sum_{T \in \mathcal{T}(S)} \widetilde{n}_S(T) f(j, T) - f(j, S)\right) \\
&\qquad\qquad + \left(f(j, S - \{i\}) - \sum_{T \in \mathcal{T}(S)} \widetilde{n}_S(T) f(j, T - \{i\})\right) \\
&\quad = \left(\sum_{T \in \mathcal{T}(S)} q_T f^T(j, T) - f(j, S)\right) + \left(f(j, S - \{i\}) - \sum_{T \in \mathcal{T}(S)} q_T f^T(j, T - \{i\})\right) \\
&\quad = \left(\sum_{T \in \mathcal{T}(S)} q_T f^T(j, T) - f(j, S)\right) + \left(f(j, S - \{i\}) - \sum_{T \in \mathcal{T}(S - \{i\})} q_T f^T(j, T)\right) \\
&\quad = \sum_{T \in \mathcal{T}(S)} q_T f^T(j, T) - f(j, S) > 0
\end{aligned}$$

This results from using (5.35) first, followed by (5.34), Lemma 4, and then (5.33).

(ii) TR is not an equilibrium, since player i has an incentive to deviate from T to B. The proof is along the same lines as the previous case. Using similar arguments, we get,

$$U_i(B, R) - U_i(T, R) = f(i, S) - \sum_{T \in \mathcal{T}(S)} q_T f^T(i, T)$$

which is positive, from (5.32).

(iii) BR and BL are also not equilibria, because in these action profiles, players j and i, respectively, have incentives to deviate—the arguments are identical to cases (a) and (b) above, respectively.

This completes the inductive argument. □

Example 11. *Consider the decomposition of f_{SV} and f_{EQ} into their respective basis distribution rules, as illustrated in Example 9. Let $S = \{i, j\}$.*

(i) $f_{SV}(i, S) = 1$, and $\sum_{T \in \mathcal{T}(S)} q_T f_{SV}^T(i, T) = q_{\{i\}} f_{SV}^{\{i\}}(i, \{i\}) = 1$. Thus, f_{SV} satisfies (5.28).

(ii) $f_{EQ}(i, S) = \frac{3}{2}$, and $\sum_{T \in \mathcal{T}(S)} q_T f_{EQ}^T(i, T) = q_{\{i\}} f_{EQ}^{\{i\}}(i, \{i\}) = 1$. Therefore, f_{EQ} does not satisfy (5.28), and hence does not guarantee the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$.

From Lemma 7, it follows that any budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$ satisfies (5.18) for all $T \in \mathcal{T}$. So, for such f, condition (5.18) can be stripped off of Lemma 6, leading to the (unconditional) *inclusion-exclusion principle*, a powerful tool that we use extensively in proving several subsequent lemmas. We formally state this in the following corollary:

Corollary 1. *(Inclusion-exclusion principle) If f is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, and $\left\{f^T\right\}_{T \in \mathcal{T}}$ are the basis distribution rules defined in (5.14), then, for every $T \in \mathcal{T}$, there exist integers $\{n_T(T')\}_{T' \in \mathcal{T}}$ such that the following equations hold:*

$$(\forall i \in T) \quad q_T f^T(i, T) = \sum_{T' \in \mathcal{T}(T)} n_T(T') f(i, T') \tag{5.36}$$

$$(\forall i \in T) \ (\forall j \in T - \{i\}) \quad 0 = q_T f^T(i, T - \{j\}) = \sum_{T' \in \mathcal{T}(T)} n_T(T') f(i, T' - \{j\}) \tag{5.37}$$

Example 12. *To illustrate the inclusion-exclusion principle, let the set of players be $N = \{i, j, k, \ell\}$, and let the set of contributing coalitions be $\mathcal{T} = \{\{i\}, \{i, j\}, \{i, k\}, \{i, \ell\}, \{i, j, k, \ell\}\}$. Then, for $T = \{i, j, k, \ell\}$, unraveling the recursion in (5.20) gives the following inclusion-exclusion formula for isolating $f^T(i, T)$, in terms of f:*

$$q_T f^T(i, T) = f(i, \{i, j, k, \ell\}) - f(i, \{i, j\}) - f(i, \{i, k\}) - f(i, \{i, \ell\}) + 2f(i, \{i\})$$

The corresponding coefficients are given by,

$$n_T(\{i, j, k, \ell\}) = 1 \qquad n_T(\{i, j\}) = n_T(\{i, k\}) = n_T(\{i, \ell\}) = -1 \qquad n_T(\{i\}) = 2$$

Lemma 8. *If f is a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, then, for each basis distribution rule f^T defined in (5.14), there exists a weight system ω^T, such that,*

$$f^T = f_{GWSV}^T[\omega^T] \tag{5.38}$$

Proof. First, we show that each basis distribution rule f^T is nonnegative. The proof is by contradiction, and proceeds as follows. Assume to the contrary, that $f^T(k,T) < 0$ for some $k \in T$. From (5.16), this is possible only if $|T| \geq 2$, and it follows that we can pick $i, j \in T$ such that $q_T f^T(i,T) < 0$ and $q_T f^T(j,T) > 0$.

Counterexample 4. Our goal is to exploit the inequalities $q_T f^T(i,T) < 0$ and $q_T f^T(j,T) > 0$ to build a counterexample that mimics *Counterexample 3* illustrated in Figure 5.3, leading to a similar best-response cycle involving just players i and j. The inclusion-exclusion principle (Corollary 1) suggests the following technique for achieving precisely this. Consider the game in Figure 5.4, which is nearly identical to *Counterexample 3*, except that resources r_1 and r_2 are absent. The resources in the top half are added as follows.

(i) Add resources in $R_1^+ = \left\{r_1^{T'} : T' \in \mathcal{T}(T) \text{ and } n_T(T') > 0\right\}$ to the top right box.

(ii) Add resources in $R_1^- = \left\{r_1^{T'} : T' \in \mathcal{T}(T) \text{ and } n_T(T') < 0\right\}$ to the top left box.

Then, the bottom half is symmetrically filled up as follows.

(i) Add resources in $R_2^+ = \left\{r_2^{T'} : T' \in \mathcal{T}(T) \text{ and } n_T(T') > 0\right\}$ to the bottom left box.

(ii) Add resources in $R_2^- = \left\{r_2^{T'} : T' \in \mathcal{T}(T) \text{ and } n_T(T') < 0\right\}$ to the bottom right box.

The resource set R is therefore given by,

$$R = R_1^+ \cup R_1^- \cup R_2^+ \cup R_1^-$$

The local resource coefficients are given by,

$$(\forall T' \in \mathcal{T}(T)) \quad v_{r_1^{T'}} = v_{r_2^{T'}} = |n_T(T')|$$

For each $T' \in \mathcal{T}(T)$, in resources $r_1^{T'}$ and $r_2^{T'}$, we fix players in $T' - \{i,j\}$. Effectively, all players other than i and j have a fixed action in their action set, determined by these fixtures.

The action set of player i is given by $\mathcal{A}_i = \{T, B\}$, where,

$$T = \left\{r_1^{T'} \in R_1^- : i \in T'\right\} \cup \left\{r_1^{T'} \in R_1^+ : i \in T'\right\}$$

$$B = \left\{r_2^{T'} \in R_2^+ : i \in T'\right\} \cup \left\{r_2^{T'} \in R_2^- : i \in T'\right\}$$

The action set of player j is given by $\mathcal{A}_j = \{L, R\}$, where,

$$L = \left\{r_1^{T'} \in R_1^- : j \in T'\right\} \cup \left\{r_2^{T'} \in R_2^+ : j \in T'\right\}$$

$$R = \left\{r_2^{T'} \in R_1^+ : j \in T'\right\} \cup \left\{r_1^{T'} \in R_2^- : j \in T'\right\}$$

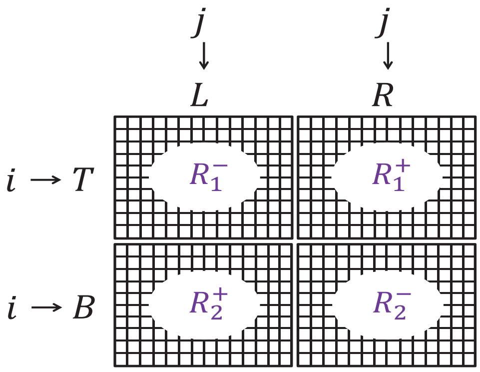


Figure 5.4: *Counterexample 4*

This is essentially a game between players i and j. The set of joint action profiles can therefore be represented as $\mathcal{A} = \{TL, TR, BL, BR\}$.

We use the inclusion-exclusion principle (Corollary 1) to show that Counterexample 4 does not possess an equilibrium, thereby contradicting the fact that f guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. We show this for each outcome:

(i) TL is not an equilibrium, because player j has an incentive to deviate from L to R. To see this, consider the utilities of player j when choosing L and R,

$$U_j(T, L) = -\sum_{\substack{T' \in \mathcal{T}(T) \\ n_T(T') < 0}} n_T(T') f(j, T') + \sum_{\substack{T' \in \mathcal{T}(T) \\ n_T(T') > 0}} n_T(T') f(j, T' - \{i\})$$

$$U_j(T, R) = \sum_{\substack{T' \in \mathcal{T}(T) \\ n_T(T') > 0}} n_T(T') f(j, T') - \sum_{\substack{T' \in \mathcal{T}(T) \\ n_T(T') < 0}} n_T(T') f(j, T' - \{i\})$$

The difference in utilities for j between choosing R and L is therefore given by,

$$\begin{aligned} U_j(T, R) - U_j(T, L) &= \sum_{T' \in \mathcal{T}(T)} n_T(T') f(j, T') - \sum_{T' \in \mathcal{T}(T)} n_T(T') f(j, T' - \{i\}) \\ &= \sum_{T' \in \mathcal{T}(T)} n_T(T') f(j, T') \qquad \text{(from (5.37))} \\ &= q_T f^T(j, T) > 0 \qquad \text{(from (5.36))} \end{aligned}$$

(ii) TR is not an equilibrium, because player i has an incentive to deviate from T to B. The proof resembles the previous case. By using similar arguments, we get,

$$U_i(B, R) - U_i(T, R) = q_T f^T(i, T) < 0$$

(iii) BR and BL are also not equilibria, because in these action profiles, players j and i, respectively, have incentives to deviate—the arguments are identical to cases (a) and (b) above, respectively.

This completes the inductive argument. Now, since each f^T is nonnegative, budget-balanced, and satisfies Lemma 4, it is completely specified by $|T|$ nonnegative values, $\{f^T(i,T) : i \in T\}$, that sum to 1. Let $\Sigma^T = \left(S_1^T, S_2^T\right)$ be an ordered partition of T, where $S_1^T = \left\{i : f^T(i,T) > 0\right\}$, and $S_2^T = T - S_1^T$. Define a weight vector $\boldsymbol{\lambda}^T$ as follows:

$$\lambda_i^T = \begin{cases} f^T(i,T) & , \ i \in S_1^T \\ \text{arbitrary positive value} & , \ i \in S_2^T \end{cases}$$

Then, it follows that f^T satisfies (5.38) with weight system $\omega^T = \left(\boldsymbol{\lambda}^T, \Sigma^T\right)$ constructed above. (See (3.8) in Table 3.2 to recall the definition of the generalized weighted Shapley value distribution rule.) This completes the proof. $\square$

Example 13. *Consider the decomposition of f_{SV} and f_{EQ} into their respective basis distribution rules, from Example 9. Clearly, f_{SV}^T is nonnegative for all $T \in \mathcal{T}$, whereas f_{EQ}^T is not.*

5.5 Consistency of Basis Distribution Rules

It follows from Proposition 9, that each budget-balanced distribution rule $f^W \in f^{\mathbb{W}}$ that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ is completely specified by a sequence of weight systems $\Omega^W = \left\{\omega^{W,T}\right\}_{T \in \mathcal{T}^W}$. But, these weight systems could be 'inconsistent' across different coalitions and across different welfare functions. Thus, our next steps focus on proving that all the weight systems $\omega^{T,W}$ are consistent—in other words, there exists a universal weight system ω^* that is equivalent to all the $\omega^{W,T}$ (replacing $\omega^{W,T}$ with ω^* for any coalition $T \in \mathcal{T}_W$ for any $W \in \mathbb{W}$ does not affect the distribution rule $f^{W,T} = f^T_{GWSV}[\omega^{W,T}]$).

To address the consistency of $\omega^{W,T}$ across different coalitions T under the same welfare function W, it is sufficient to work with one welfare function at a time, just like in the previous module. However, to address the consistency across different welfare functions, it is necessary to work with more than one welfare function at a time—for every subset of welfare functions $\mathbb{V} \subseteq \mathbb{W}$, we only focus on the corresponding distribution rules $f^{\mathbb{V}}$ that guarantee equilibrium existence for all games in the class $\mathcal{G}(N, f^{\mathbb{V}}, \mathbb{V})$. The justification is similar—$\mathcal{G}(N, f^{\mathbb{V}}, \mathbb{V}) \subseteq \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ for all $\mathbb{V} \subseteq \mathbb{W}$, and so, if $f^{\mathbb{W}}$ guarantees equilibrium existence for all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$, then for every subset $\mathbb{V} \subseteq \mathbb{W}$, $f^{\mathbb{V}} \subseteq f^{\mathbb{W}}$ must guarantee equilibrium existence for all games in $\mathcal{G}(N, f^{\mathbb{V}}, \mathbb{V})$.

In what follows, we work with $k > 1$ welfare functions at a time, say $W_1, W_2, \ldots, W_k$ (not necessarily distinct), to address consistency across welfare functions, and then use the special case of $W_1 = W_2 = \ldots = W_k = W$ to address consistency across coalitions under the same welfare function W. In order to simplify notation, we drop W from the superscripts. That is, for $1 \leq j \leq k$, we write f^j instead of f^{W_j}, $\mathcal{T}^j$ instead of $\mathcal{T}^{W_j}$, q_T^j instead of $q_T^{W_j}$, $n_T^j(T')$ instead of $n_T^{W_j}(T')$, etc.

5.5.1 Two Consistency Conditions

Our goal in this section is to establish the following two important consistency properties that the basis distribution rules $f^{W,T}$ must satisfy, in order for the budget-balanced distribution rules $f^W = \sum_{T\in\mathcal{T}^W} q_T^W f^{W,T}$ to guarantee the existence of an equilibrium in all games $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$. Recall that $\mathcal{T}_{ij}^W$, defined in (5.3), refers to the set of coalitions in $\mathcal{T}^W$ containing both players i and j. In addition, let $\mathcal{T}_{ij}^W(S) = \{T \in \mathcal{T}^W(S) \mid \{i,j\} \subseteq T\}$ denote the set of coalitions in $\mathcal{T}^W(S)$ containing both players i and j.

(a) *Global consistency:* If there is a pair of players common to two coalitions (under the same or different welfare functions), then their shares from these two coalitions (given by the corresponding $f^{W,T}$ values) must be 'consistent', as formalized in Lemma 9. Here, we deal with at most two welfare functions at a time.

(b) *Cyclic consistency:* If there is a sequence of $k \geq 3$ players, $(i_1, i_2, \ldots, i_k)$ such that for each of the k neighbor-pairs $\{(i_1,i_2), (i_2,i_3), \ldots, (i_k,i_1)\}$, $\exists\, T_1 \in \left(\mathcal{T}_{i_1 i_2}^1\right)^{\min}, T_2 \in \left(\mathcal{T}_{i_2 i_3}^2\right)^{\min}, \ldots, T_k \in \left(\mathcal{T}_{i_k i_1}^k\right)^{\min}$ and in each T_j, at least one of the neighbors i_j, i_{j+1} gets a nonzero share (given by the corresponding f^{j,T_j} value), then the shares of these k players from these k coalitions must satisfy a 'cyclic consistency' condition, as formalized in Lemma 10. Here, we deal with an arbitrary number of welfare functions at a time.

Lemma 9. *Given any two local welfare functions W_1, W_2, if $f^1 = \sum\limits_{T\in\mathcal{T}^1} q_T^1 f^{1,T}$ and $f^2 = \sum\limits_{T\in\mathcal{T}^2} q_T^2 f^{2,T}$ are corresponding budget-balanced distribution rules that guarantee equilibrium existence in all games $G \in \mathcal{G}(N, \{f^1, f^2\}, \{W_1, W_2\})$, then, for any two players $i, j \in N$, any two coalitions $T' \in \mathcal{T}_{ij}^1$ and $T \in \mathcal{T}_{ij}^2$,*

$$f^{1,T'}(i,T')f^{2,T}(j,T) = f^{2,T}(i,T)f^{1,T'}(j,T') \tag{5.39}$$

Proof. Note that it is sufficient to show (5.39) for only those coalitions in $\mathcal{T}_{ij}^1$ and $\mathcal{T}_{ij}^2$ in which at least one among i and j get a nonzero share. Formally, define the collections $\mathcal{T}_{ij}^{1+}$ and $\mathcal{T}_{ij}^{2+}$ as,

$$\begin{aligned}
\mathcal{T}_{ij}^{1+} &= \left\{T \in \mathcal{T}_{ij}^1 \;:\; f^{1,T}(i,T) > 0 \text{ or } f^{1,T}(j,T) > 0\right\} \\
\mathcal{T}_{ij}^{2+} &= \left\{T \in \mathcal{T}_{ij}^2 \;:\; f^{2,T}(i,T) > 0 \text{ or } f^{2,T}(j,T) > 0\right\}
\end{aligned} \tag{5.40}$$

Let S be a minimal element (coalition) in the poset $\left(\mathcal{T}_{ij}^{1+}, \subseteq\right)$, and without loss of generality, assume $f^{1,S}(i,S) > 0$. Then, we need only show[2] that for any coalition $T \in \mathcal{T}_{ij}^{2+}$,

$$f^{2,T}(i,T)f^{1,S}(j,S) = f^{1,S}(i,S)f^{2,T}(j,T) \tag{5.42}$$

[2]It can be shown that (5.42) implies (5.39): For the special case when $W_2 = W_1$, (5.42) implies that for all $T' \in \mathcal{T}_{ij}^{1+}$,

$$f^{1,T'}(i,T')f^{1,S}(j,S) = f^{1,S}(i,S)f^{1,T'}(j,T') \tag{5.41}$$

Let $T' \in \mathcal{T}_{ij}^1$ and $T \in \mathcal{T}_{ij}^2$. By assumption, $f^{1,S}(i,S) \neq 0$. If $f^{1,S}(j,S) = 0$, then (5.42) and (5.41) imply that $f^{2,T}(j,T) = 0$ and $f^{1,T'}(j,T') = 0$, in which case both sides of (5.39) are zero. If $f^{1,S}(j,S) \neq 0$, then none of the four terms in equations (5.42) and (5.41) are zero, and therefore, by eliminating $f^{1,S}(i,S)$ and $f^{1,S}(j,S)$ between them, we get (5.39).

The proof is by contradiction. Assume to the contrary, that for some $T \in \mathcal{T}_{ij}^{2+}$, $f^{2,T}(i,T)f^{1,S}(j,S) \neq f^{1,S}(i,S)f^{2,T}(j,T)$. We consider the following two cases:

Case 1: $q_S^1 q_T^2 > 0$.

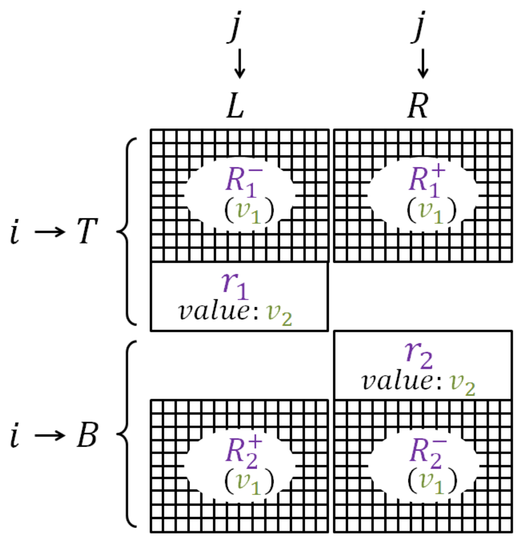


Figure 5.5: *Counterexample 5(a)*

Counterexample 5(a). Our goal is to build a counterexample that mimics *Counterexample 4* illustrated in Figure 5.4, leading to a similar best-response cycle involving just players i and j. As before, we use the inclusion-exclusion principle (Corollary 1) to isolate just $f^{2,T}$, by appropriately adding resources and setting action sets. Consider the game in Figure 5.5, which is identical to *Counterexample 4*, except for the following changes:

(i) There are two additional resources, r_1 and r_2, so the resource set is now

$$R = \{r_1, r_2\} \cup R_1^+ \cup R_1^- \cup R_2^+ \cup R_1^-$$

(ii) The welfare function at r_1 and r_2 is W_1. At all other resources, the welfare function is W_2.

(iii) The local resource coefficients are given by,

$$v_{r_1} = v_{r_2} = v_2 \qquad \text{and} \qquad (\forall T' \in \mathcal{T}^2(T)) \quad v_{r_1^{T'}} = v_{r_2^{T'}} = v_1 |n_T^2(T')|$$

where $v_1 > 0$ and $v_2 > 0$. We will discuss the specific choice of v_1, v_2 later.

(iv) In resources r_1 and r_2, we fix players in $S - \{i, j\}$.

(v) The actions T, B, L, R are modified to accommodate the two new resources:

$$
\begin{aligned}
T &= \left\{ r_1^{T'} \in R_1^- : i \in T' \right\} \cup \{r_1\} \cup \left\{ r_1^{T'} \in R_1^+ : i \in T' \right\} \\
B &= \left\{ r_2^{T'} \in R_2^+ : i \in T' \right\} \cup \{r_2\} \cup \left\{ r_2^{T'} \in R_2^- : i \in T' \right\} \\
L &= \left\{ r_1^{T'} \in R_1^- : j \in T' \right\} \cup \{r_1\} \cup \left\{ r_2^{T'} \in R_2^+ : j \in T' \right\} \\
R &= \left\{ r_1^{T'} \in R_1^+ : j \in T' \right\} \cup \{r_2\} \cup \left\{ r_2^{T'} \in R_2^- : j \in T' \right\}
\end{aligned}
$$

To complete the specification of *Counterexample 5(a)*, we need to specify the values of $v_1 > 0$ and $v_2 > 0$. We now show that if $f^{2,T}(i,T) f^{1,S}(j,S) \neq f^{1,S}(i,S) f^{2,T}(j,T)$, then these values can be picked carefully in such a way that *Counterexample 5(a)* does not possess an equilibrium, thereby contradicting the fact that f^1 and f^2 guarantee equilibrium existence in all games $G \in \mathcal{G}(N, \{f^1, f^2\}, \{W_1, W_2\})$. Consider each of the four outcomes:

(i) In action profiles TL and BR, player j has an incentive to deviate if

$$U_j(T,R) - U_j(T,L) = U_j(B,L) - U_j(B,R) > 0.$$

This happens if,

$$v_1 \left(\sum_{T' \in \mathcal{T}^2(T)} n_T^2(T') f^2(j,T') - \sum_{T' \in \mathcal{T}^2(T)} n_T^2(T') f^2(j, T' - \{i\}) \right) - v_2 \left(f^1(j,S) - f^1(j, S - \{i\}) \right) > 0$$

Using the inclusion-exclusion principle (Corollary 1) to simplify the terms in the first bracket, and the basis representation of f^1 to simplify the difference in the second bracket, this condition is equivalent to,

$$v_1 \left(q_T^2 f^{2,T}(j,T) \right) - v_2 \left(\sum_{T' \in \mathcal{T}_{ij}^1(S)} q_{T'}^1 f^{1,T'}(j,T') \right) > 0$$

Since S is minimal in $\mathcal{T}_{ij}^{1+}$, this reduces to,

$$v_1 q_T^2 f^{2,T}(j,T) > v_2 q_S^1 f^{1,S}(j,S) \tag{5.43}$$

(ii) Similarly, in action profiles TR and BL, player i has an incentive to deviate if

$$U_i(B,R) - U_i(T,R) = U_i(T,L) - U_i(B,L) > 0.$$

This happens if,

$$-v_1 \left(\sum_{T' \in \mathcal{T}^2(T)} n_T^2(T') f^2(i,T') - \sum_{T' \in \mathcal{T}^2(T)} n_T^2(T') f^2(i, T' - \{j\}) \right) + v_2 \left(f^1(i,S) - f^1(i, S - \{j\}) \right) > 0$$

By similar arguments as above, this condition reduces to,

$$v_1 q_T^2 f^{2,T}(i,T) < v_2 q_S^1 f^{1,S}(i,S) \tag{5.44}$$

Without loss of generality, assume $q_T^2 > 0$ and $q_S^1 > 0$ (For the symmetric case when $q_T^2 < 0$ and $q_S^1 < 0$, the same arguments apply, but the deviations in the best-response cycle are reversed). By assumption, $f^{1,S}(i,S) > 0$. Now we consider two cases for $f^{1,S}(j,S)$:

(i) $f^{1,S}(j,S) = 0$. In this case, $f^{2,T}(j,T) > 0$ (for otherwise, (5.42) would be satisfied). It follows then, that (5.43) and (5.44) always have a solution in strictly positive integers v_1 and v_2.

(ii) $f^{1,S}(j,S) > 0$. In this case, suppose $f^{2,T}(i,T)f^{1,S}(j,S) < f^{1,S}(i,S)f^{2,T}(j,T)$ (For the other case when $f^{2,T}(i,T)f^{1,S}(j,S) > f^{1,S}(i,S)f^{2,T}(j,T)$, the same arguments apply, but the deviations in the best-response cycle are reversed). Combining (5.43) and (5.44), we get,

$$\frac{q_T^2}{q_S^1}\frac{f^{2,T}(i,T)}{f^{1,S}(i,S)} < \frac{v_2}{v_1} < \frac{q_T^2}{q_S^1}\frac{f^{2,T}(j,T)}{f^{1,S}(j,S)}$$

This inequality has a solution in strictly positive integers v_1 and v_2, if and only if,

$$\begin{aligned} & \frac{q_T^2}{q_S^1}\frac{f^{2,T}(i,T)}{f^{1,S}(i,S)} < \frac{q_T^2}{q_S^1}\frac{f^{2,T}(j,T)}{f^{1,S}(j,S)} \\ \iff \quad & f^{2,T}(i,T)f^{1,S}(j,S) < f^{1,S}(i,S)f^{2,T}(j,T) \end{aligned}$$

which is true by assumption.

Case 2: $q_S^1 q_T^2 < 0$.

Counterexample 5(b). Our goal remains the same—to build a counterexample in which a best-response cycle involving just players i and j exists. This counterexample breaks from symmetry, and we use the inclusion-exclusion principle (Corollary 1) thrice here, to isolate two more basis distribution rules, in addition to $f^{2,T}$. We now present the formal details.

Consider the game in Figure 5.6, where we have various boxes with labels on them indicating which resource or set of resources is present. Let T_i and T_j be some coalitions that contain i and j, respectively. We will discuss the specific choice of T_i, T_j later. As before, we use the resource sets $\left(R_1^+, R_1^-\right)$ (with W_2 the welfare function at all these resources) for isolating $f^{2,T}$. In addition, we use resource sets $\left(R_3^+, R_3^-\right)$ (with W_x as the welfare function at all these resources) and $\left(R_4^+, R_4^-\right)$ (with W_y as the welfare function at all these resources) to isolate two more basis distribution rules, f^{x,T_j} and f^{y,T_i}, respectively, where the choice of $x, y \in \{1,2\}$ will be discussed later. In addition to these six sets, we also have a single resource r_2 whose

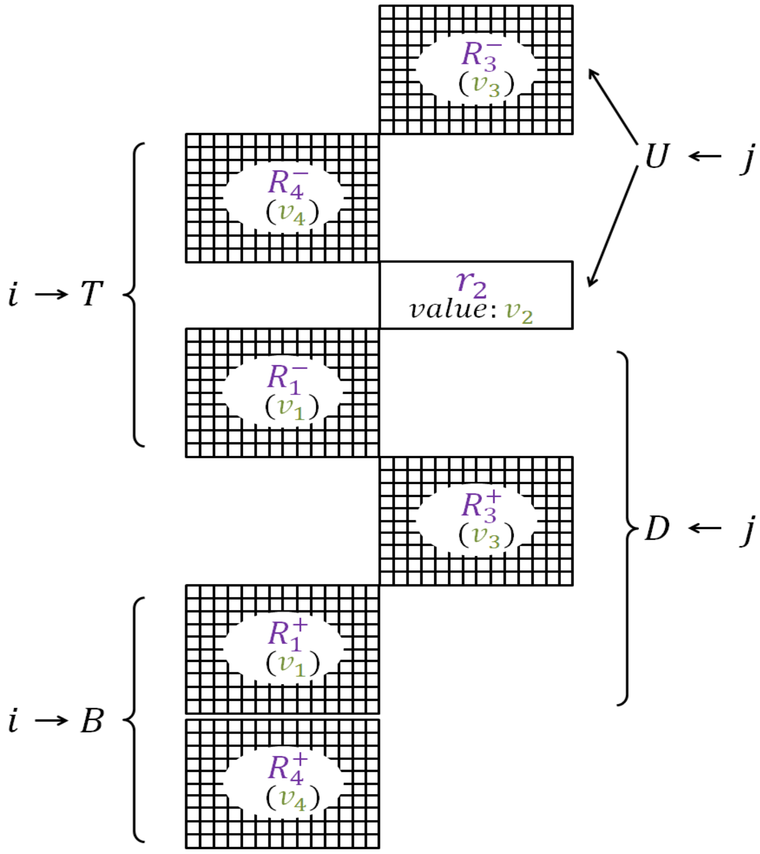


Figure 5.6: *Counterexample 5(b)*

welfare function is W_1. Formally,

$$R_1^+ = \left\{ r_1^{T'} : T' \in \mathcal{T}^2(T) \text{ and } n_T^2(T') > 0 \right\}$$
$$R_1^- = \left\{ r_1^{T'} : T' \in \mathcal{T}^2(T) \text{ and } n_T^2(T') < 0 \right\}$$
$$R_3^+ = \left\{ r_3^{T'} : T' \in \mathcal{T}^x(T_j) \text{ and } n_{T_j}^x(T') > 0 \right\} \qquad R_4^+ = \left\{ r_4^{T'} : T' \in \mathcal{T}^y(T_i) \text{ and } n_{T_i}^y(T') > 0 \right\}$$
$$R_3^- = \left\{ r_3^{T'} : T' \in \mathcal{T}^x(T_j) \text{ and } n_{T_j}^x(T') < 0 \right\} \qquad R_4^- = \left\{ r_4^{T'} : T' \in \mathcal{T}^y(T_i) \text{ and } n_{T_i}^y(T') < 0 \right\}$$

The resource set R is therefore given by,

$$R = \{r_2\} \cup R_1^+ \cup R_1^- \cup R_3^+ \cup R_3^- \cup R_4^+ \cup R_4^-$$

The local resource coefficients are given by,

$$v_{r_2} = v_2$$
$$(\forall T' \in \mathcal{T}^2(T)) \quad v_{r_1^{T'}} = v_1 |n_T^2(T')|$$
$$(\forall T' \in \mathcal{T}^x(T_j)) \quad v_{r_3^{T'}} = v_3 |n_{T_j}^x(T')| \qquad \text{and} \qquad (\forall T' \in \mathcal{T}^y(T_i)) \quad v_{r_4^{T'}} = v_4 |n_{T_i}^y(T')|$$

where $v_1, v_2, v_3, v_4 > 0$. We will discuss the specific choice of v_1, v_2, v_3, v_4 later. In resource r_2, we fix players in $S - \{i, j\}$. For each $T' \in \mathcal{T}^2(T)$, in resource $r_1^{T'}$, we fix players in $T' - \{i, j\}$. For each $T' \in \mathcal{T}^x(T_j)$, in resource $r_3^{T'}$, we fix players in $T' - \{j\}$. For each $T' \in \mathcal{T}^y(T_i)$, in resource $r_4^{T'}$, we fix players in $T' - \{i\}$. Effectively, all players other than i and j have a fixed action in their action set, determined by these fixtures. In addition, these fixtures might also specify mandatory sets of resources R_i and R_j that players i and j must always be present in. The action sets of players i and j are given by $\mathcal{A}_i = \{T, B\}$ and $\mathcal{A}_j = \{U, D\}$, where,

$$
\begin{aligned}
T &= \left\{r_4^{T'} \in R_4^- : i \in T'\right\} \cup \{r_2\} \cup \left\{r_1^{T'} \in R_1^- : i \in T'\right\} \cup R_i \\
B &= \left\{r_1^{T'} \in R_1^+ : i \in T'\right\} \cup \left\{r_4^{T'} \in R_4^+ : i \in T'\right\} \cup R_i \\
U &= \left\{r_1^{T'} \in R_3^- : j \in T'\right\} \cup \{r_2\} \cup R_j \\
D &= \left\{r_1^{T'} \in R_1^- : j \in T'\right\} \cup \left\{r_1^{T'} \in R_3^+ : j \in T'\right\} \cup \left\{r_1^{T'} \in R_1^+ : j \in T'\right\} \cup R_j
\end{aligned}
$$

This is essentially a game between players i and j. The set of joint action profiles can therefore be represented as $\mathcal{A} = \{TU, TD, BU, BD\}$.

To complete the specification of *Counterexample 5(b)*, we need to specify the values of $v_1, v_2, v_3, v_4 > 0$, $x, y \in \{1, 2\}$, and T_i, T_j. We now show that if $f^{2,T}(i, T)f^{1,S}(j, S) \neq f^{1,S}(i, S)f^{2,T}(j, T)$, then these values can be picked carefully in such a way that *Counterexample 5(b)* does not possess an equilibrium, thereby contradicting the fact that f^1 and f^2 guarantee equilibrium existence in all games $G \in \mathcal{G}(N, \{f^1, f^2\}, \{W_1, W_2\})$. Consider each of the four outcomes:

(i) In action profile TU, player i has an incentive to deviate if $U_i(B, U) - U_i(T, U) > 0$. This happens if,

$$
v_4 \left(\sum_{T' \in \mathcal{T}^y(T_i)} n_{T_i}^y(T') f^y(i, T') \right) + v_1 \left(\sum_{T' \in \mathcal{T}^2(T)} n_T^2(T') f^2(i, T' - \{j\}) \right) - v_2 f^1(i, S) > 0
$$

Note that $U_i(B, U)$ and $U_i(T, U)$ include utilities to player i from resources in R_i, but while taking the difference, this cancels out, since i is fixed in these resources, and between these two action profiles, all other players also have a fixed action. Now, using the inclusion-exclusion principle (Corollary 1) to simplify the terms in the first two brackets, we get,

$$
v_4 q_{T_i}^y f^{y,T_i}(i, T_i) > v_2 f^1(i, S) \tag{5.45}
$$

(ii) In action profile BD, player i has an incentive to deviate if $U_i(T, D) - U_i(B, D) > 0$. This happens if:

$$
-v_4 \left(\sum_{T' \in \mathcal{T}^y(T_i)} n_{T_i}^y(T') f^y(i, T') \right) - v_1 \left(\sum_{T' \in \mathcal{T}^2(T)} n_T^2(T') f^2(i, T') \right) + v_2 f^1(i, S - \{j\}) > 0
$$

As before, the utility to player i from resources in R_i cancels out. Using the inclusion-exclusion principle to simplify the terms, we get,

$$v_2 f^1(i, S - \{j\}) - v_1 q_T^2 f^{2,T}(i,T) > v_4 q_{T_i}^y f^{y,T_i}(i, T_i) \tag{5.46}$$

(iii) In action profile TD, player j has an incentive to deviate if $U_j(T,U) - U_j(T,D) > 0$. This happens if:

$$-v_3 \left(\sum_{T' \in \mathcal{T}^x(T_j)} n_{T_j}^x(T') f^x(j,T') \right) + v_2 f^1(j,S) - v_1 \left(\sum_{\substack{T' \in \mathcal{T}^2(T) \\ n_T^2(T')>0}} n_T^2(T') f^2(j, T' - \{i\}) - \sum_{\substack{T' \in \mathcal{T}^2(T) \\ n_T^2(T')<0}} n_T^2(T') f^2(j,T') \right) > 0$$

The utility to player j from resources in R_j cancels out. Using the inclusion-exclusion principle to simplify the first term, we get,

$$v_2 f^1(j,S) - v_1 \left(\sum_{\substack{T' \in \mathcal{T}^2(T) \\ n_T^2(T')>0}} n_T^2(T') f^2(j, T' - \{i\}) - \sum_{\substack{T' \in \mathcal{T}^2(T) \\ n_T^2(T')<0}} n_T^2(T') f^2(j,T') \right) > v_3 q_{T_j}^x f^{x,T_j}(j, T_j) \tag{5.47}$$

(iv) In action profile BU, player j has an incentive to deviate if $U_j(B,D) - U_j(B,U) > 0$. This happens if:

$$v_3 \left(\sum_{T' \in \mathcal{T}^x(T_j)} n_{T_j}^x(T') f^x(j,T') \right) - v_2 f^1(j, S - \{i\}) + v_1 \left(\sum_{\substack{T' \in \mathcal{T}^2(T) \\ n_T^2(T')>0}} n_T^2(T') f^2(j,T') - \sum_{\substack{T' \in \mathcal{T}^2(T) \\ n_T^2(T')<0}} n_T^2(T') f^2(j, T' - \{i\}) \right) > 0$$

As before, the utility to player j from resources in R_j cancels out. Using the inclusion-exclusion principle to simplify the first term, we get,

$$v_3 q_{T_j}^x f^{x,T_j}(j, T_j) > v_2 f^1(j, S - \{i\}) - v_1 \left(\sum_{\substack{T' \in \mathcal{T}^2(T) \\ n_T^2(T')>0}} n_T^2(T') f^2(j,T') - \sum_{\substack{T' \in \mathcal{T}^2(T) \\ n_T^2(T')<0}} n_T^2(T') f^2(j, T' - \{i\}) \right) \tag{5.48}$$

Combining inequalities (5.45) and (5.46), we get,

$$v_1 q_T^2 f^{2,T}(i,T) + v_2 \left(f^1(i,S) - f^1(i, S - \{j\}) \right) < 0$$

Using the basis representation of f^1 to simplify the difference in the bracket, this condition is equivalent to,

$$v_1 q_T^2 f^{2,T}(i,T) + v_2 \left(\sum_{T' \in \mathcal{T}_{ij}^1(S)} q_{T'}^1 f^{1,T'}(i,T') \right) < 0$$

Since S is minimal in $\mathcal{T}_{ij}^{1+}$, this reduces to,

$$v_1 q_T^2 f^{2,T}(i,T) + v_2 q_S^1 f^{1,S}(i,S) < 0 \tag{5.49}$$

Combining inequalities (5.47) and (5.48), we get,

$$v_1 \left(\sum_{T' \in \mathcal{T}^2(T)} n_T^2(T') f^2(j,T') - \sum_{T' \in \mathcal{T}^2(T)} n_T^2(T') f^2(j,T' - \{i\}) \right) + v_2 \left(f^1(j,S) - f^1(j,S - \{i\}) \right) > 0$$

Using the inclusion-exclusion principle to simplify the terms in the first bracket, and the basis representation of f^1 to simplify the difference in the second bracket, this condition is equivalent to,

$$v_1 q_T^2 f^{2,T}(j,T) + v_2 \left(\sum_{T' \in \mathcal{T}_{ij}^1(S)} q_{T'}^1 f^{1,T'}(j,T') \right) > 0$$

Since S is minimal in $\mathcal{T}_{ij}^{1+}$, this reduces to,

$$v_1 q_T^2 f^{2,T}(j,T) + v_2 q_S^1 f^{1,S}(j,S) > 0 \tag{5.50}$$

Without loss of generality, assume $q_T^2 > 0$ and $q_S^1 < 0$ (For the symmetric case when $q_T^2 < 0$ and $q_S^1 > 0$, the same arguments apply, but the deviations in the best-response cycle are reversed). By assumption, $f^{1,S}(i,S) > 0$. Now we consider two cases for $f^{1,S}(j,S)$:

(i) $f^{1,S}(j,S) = 0$. In this case, $f^{2,T}(j,T) > 0$ (for otherwise, (5.42) would be satisfied). It follows then, that (5.49) and (5.50) always have a solution in strictly positive integers v_1 and v_2.

(ii) $f^{1,S}(j,S) > 0$. In this case, suppose $f^{2,T}(i,T) f^{1,S}(j,S) < f^{1,S}(i,S) f^{2,T}(j,T)$ (For the other case when $f^{2,T}(i,T) f^{1,S}(j,S) > f^{1,S}(i,S) f^{2,T}(j,T)$, the same arguments apply, but the deviations in the best-response cycle are reversed). Combining (5.49) and (5.50), we get,

$$-\frac{q_T^2}{q_S^1} \frac{f^{2,T}(i,T)}{f^{1,S}(i,S)} < \frac{v_2}{v_1} < -\frac{q_T^2}{q_S^1} \frac{f^{2,T}(j,T)}{f^{1,S}(j,S)}$$

Therefore, this inequality has a solution in strictly positive integers v_1 and v_2, if and only if,

$$\frac{q_T^2}{q_S^1}\frac{f^{2,T}(i,T)}{f^{1,S}(i,S)} > \frac{q_T^2}{q_S^1}\frac{f^{2,T}(j,T)}{f^{1,S}(j,S)}$$
$$\iff \quad f^{2,T}(i,T)f^{1,S}(j,S) < f^{1,S}(i,S)f^{2,T}(j,T)$$

which is true by assumption.

Finally, we need to show that given these carefully chosen values for v_1 and v_2, it is possible to find $v_3 > 0$, $v_4 > 0$, $x, y \in \{1,2\}$, T_i and T_j such that the inequalities (5.45)-(5.48) are satisfied. These four inequalities can be consolidated as,

$$LHS_j < v_3 q_{T_j}^x f^{x,T_j}(j,T_j) < RHS_j$$
$$LHS_i < v_4 q_{T_i}^y f^{y,T_i}(i,T_i) < RHS_i$$

We describe the procedure to find $v_3 > 0$, $x \in \{1,2\}$, and T_j here. Finding $v_4 > 0$, $y \in \{1,2\}$, and T_i is analogous. Specifically, we consider the case where $RHS_j > 0$ (we discuss the other case later). Consider the two quantities $f^{1,S}(j,S)$ and $f^{2,T}(j,T)$. They are not both zero (for otherwise, (5.42) would be satisfied). We consider two subcases:

(i) If $f^{2,T}(j,T) > 0$, choose $x = 2$, $T_j = T$. Then, it is possible to find $v_3 > 0$ such that $LHS_j < v_3 q_T^2 f^{2,T}(j,T) < RHS_j$ because by assumption, $q_T^2 > 0$.

(ii) If $f^{2,T}(j,T) = 0$, then $f^{1,S}(j,S) > 0$. Here, we slightly alter *Counterexample 5(b)* by modifying player j's action set $\mathcal{A}_j = \{U, D\}$ as follows—we simply switch the resources in R_3^- and R_3^+ between action U and action D. Formally,

$$U' = \{r_2\} \cup \left\{r_1^{T'} \in R_3^+ : j \in T'\right\} \cup R_j$$
$$D' = \left\{r_1^{T'} \in R_3^- : j \in T'\right\} \cup \left\{r_1^{T'} \in R_1^- : j \in T'\right\} \cup \left\{r_1^{T'} \in R_1^+ : j \in T'\right\} \cup R_j$$

This alteration does not affect any of the arguments above, except that we now need to find $v_3 > 0$, $x \in \{1,2\}$, and T_j such that $LHS_j < -v_3 q_{T_j} f^{T_j}(j,T_j) < RHS_j$. (In particular, (5.49) and (5.50) remain unchanged.) Choose x=1, $T_j = S$. Then, it is possible to find $v_3 > 0$ such that $LHS_j < -v_3 q_S^1 f^{1,S}(j,S) < RHS_j$ because by assumption, $q_S^1 < 0$.

The case when $RHS_j < 0$ is symmetric—we just interchange S and T, and the choice of x, throughout in the two subcases. That is, if $f^{1,S}(j,S) > 0$, we choose $x = 1$, $T_j = S$. Otherwise, we choose $x = 2$, $T_j = T$, and alter *Counterexample 5(b)* the same way as above. This completes the proof.[3] ☐

[3] Note that v_1 and v_2 can be scaled by an arbitrary positive constant if necessary, in order to ensure that an integral solution to v_3, v_4 exists.

From Lemma 8, we inferred that each $f^{W,T}$ is a generalized weighted Shapley value for the corresponding unanimity game W^T, with weight system $\omega^{W,T} = \left(\boldsymbol{\lambda}^{W,T}, \Sigma^{W,T}\right)$ that we constructed using $f^{W,T}$. As a result, the conditions on $\left\{\{f^{W,T}\}_{T\in\mathcal{T}^W}\right\}_{W\in\mathbb{W}}$ imposed by Lemma 9 translate to equivalent conditions on the weight systems $\left\{\{\omega^{W,T}\}_{T\in\mathcal{T}^W}\right\}_{W\in\mathbb{W}}$. The following corollary restates Lemma 9 in terms of the weight systems.

Corollary 2. *Given any set of local welfare functions $\mathbb{W}$, if $f^{\mathbb{W}}$ are budget-balanced distribution rules that guarantee equilibrium existence in all games $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$, where, for each $W \in \mathbb{W}$, $f^W := \sum_{T\in\mathcal{T}^W} q_T^W f_{GWSV}^T[\omega^{W,T}]$, where $\omega^{W,T} = \left(\boldsymbol{\lambda}^{W,T}, \Sigma^{W,T} = \left(S_1^{W,T}, S_2^{W,T}\right)\right)$, then, for any two players $i, j \in N$,*

$$(a)\ (\exists\, W,T) \quad i \in S_1^{W,T},\ j \in S_2^{W,T} \ \Rightarrow\ (\forall\, W',T') \quad j \in S_2^{W',T'}$$

$$(b)\ (\exists\, W,T) \quad i \in S_1^{W,T},\ j \in S_1^{W,T} \ \Rightarrow\ (\forall\, W',T') \quad \begin{cases} (i)\ i \in S_1^{W',T'} \Leftrightarrow j \in S_1^{W',T'} \\ (ii)\ \{i,j\} \subseteq S_1^{W',T'} \Rightarrow \frac{\lambda_i^{W,T}}{\lambda_i^{W',T'}} = \frac{\lambda_j^{W,T}}{\lambda_j^{W',T'}} \end{cases}$$

Explanation. In essence, this corollary states consistency constraints that the various weight systems that define the distribution rules must satisfy, for every pair of players. It is obtained by applying Lemma 9 for all pairs of welfare functions $\mathbb{W} \times \mathbb{W}$. Suppose $W, W' \in \mathbb{W}$, and let $T \in \mathcal{T}_{ij}^W$ and $T' \in \mathcal{T}_{ij}^{W'}$. Then, from Lemma 9, we have,

$$f^{W,T}(i,T) f^{W',T'}(j,T') = f^{W',T'}(i,T') f^{W,T}(j,T) \tag{5.51}$$

The different parts of the corollary then follow directly from applying the definition of the generalized weighted Shapley value (see (3.8) in Table 3.2) and simplifying the above equation for the corresponding cases.

Lemma 10. *For any $k \geq 3$, given any k welfare functions $W_1, W_2, \ldots, W_k$, if $\left\{f^j := \sum_{T\in\mathcal{T}^j} q_T^j f^{j,T}\right\}_{j=1}^k$ are corresponding budget-balanced distribution rules that guarantee equilibrium existence in all games $G \in \mathcal{G}(N, \{f^1, f^2, \ldots, f^k\}, \{W_1, W_2, \ldots, W_k\})$, and $i_1, i_2, \ldots, i_k \in N$ are any k players such that $\exists\ T_1 \in \left(\mathcal{T}_{i_1 i_2}^{1+}\right)^{\min}, T_2 \in \left(\mathcal{T}_{i_2 i_3}^{2+}\right)^{\min}, \ldots, T_k \in \left(\mathcal{T}_{i_k i_1}^{k+}\right)^{\min}$, then,*

$$f^{1,T_1}(i_1,T_1) f^{2,T_2}(i_2,T_2) \cdots f^{k,T_k}(i_k,T_k) = f^{1,T_1}(i_2,T_1) f^{2,T_2}(i_3,T_2) \cdots f^{k,T_k}(i_1,T_k) \tag{5.52}$$

Proof. Recall that $\mathcal{T}_{xy}^{j+}$ denotes the collection of those coalitions from $\mathcal{T}_{xy}^j$ in which at least one of x, y obtains a nonzero share (according to f^j). Refer to (5.40) for the formal definition.

Index arithmetic: In the rest of this proof, the index set is $\{1, 2, \ldots, k\}$, and when we add an integer ℓ to an index j, $j + \ell$ denotes the index that is ℓ positions away from index j (cycling around if necessary). For example, suppose $k = 3$. Then, for index $j = 2$, $j + 2 = 1$ and $j - 2 = 3$.

For simplicity, denote $q^j_{T_j}$ by q_j, $f^{j,T_j}(i_j, T_j)$ by a_j, and $f^{j,T_j}(i_{j+1}, T_j)$ by b_j. Note that $a_j \geq 0$ and $b_j \geq 0$. Now, (5.52) can be written as:

$$\prod_{j=1}^{k} a_j = \prod_{j=1}^{k} b_j \tag{5.53}$$

Our proof technique mirrors those in previous sections. Assuming (5.53) is not satisfied, we present a game where no equilibrium exists. Without loss of generality, let

$$\prod_{j=1}^{k} a_j < \prod_{j=1}^{k} b_j \tag{5.54}$$

We present a family of counterexamples, each corresponding to a specific sign profile of the coefficients q_j. (Recall that for a given choice of the local welfare functions, q_j are fixed.) The proof is in three stages:

(i) We present the details of the counterexample game (*Counterexample 6*).

(ii) We present four *validity conditions* on the action profiles of *Counterexample 6*, and define those action profiles that satisfy at least one of them as *valid action profiles*, observing that such action profiles are never equilibria.

(iii) We show that every action profile in *Counterexample 6* is valid.

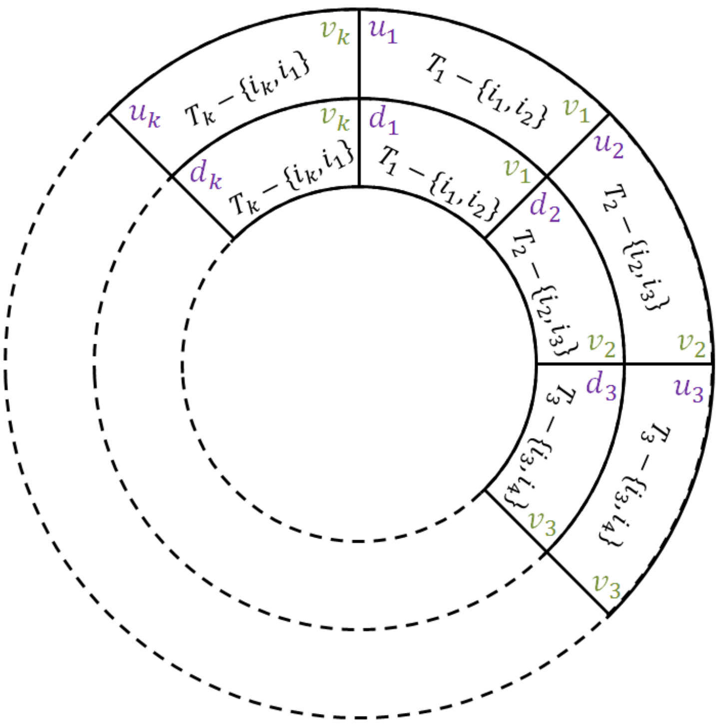


Figure 5.7: *Counterexample 6*

Counterexample 6. Consider the game in Figure 5.7, that involves only players $i_1, i_2, \ldots, i_k$. There are

$2k$ resources, arranged in two circular rows of k resources each. For each column j, both resources u_j and d_j share the same resource-specific coefficient $v_j > 0$ and same local welfare function W_j, and in both these resources, we fix players in $T_j - \{i_j, i_{j+1}\}$. Effectively, all players other than $i_1, i_2, \ldots, i_k$ have a fixed action in their action set, determined by these fixtures. In addition, these fixtures might also specify mandatory sets of resources $R_1, R_2, \ldots, R_k$ that the players $i_1, i_2, \ldots, i_k$ must always be present in, respectively. However, for simplicity, we will not explicitly represent this, since such actions do not affect strategic behavior (utilities from these resources cancel out as far as unilateral deviations are concerned, just like they did in the proof of Lemma 9). Next, we need to specify the action sets of the players $i_1, i_2, \ldots, i_k$, which will depend intimately on properties regarding the sequence of signs of the coefficients q_j. We begin by relabeling the indices according to a cyclic transformation that is without loss of generality, to ensure that the following properties will be satisfied after the transformation:

(i) The first coefficient, q_1, is negative, unless all coefficients are positive.

(ii) The last coefficient, q_k, is positive, unless all coefficients are negative.

(iii) The penultimate coefficient, q_{k-1}, is positive, unless no two adjacent coefficients are both positive.

In essence, we cut down on the different sequences of signs of the coefficients that we need to consider. Formally, we define two special index sets, J and J^*, as follows:

$$J = \{j \mid 1 \le j \le k \text{ and } q_{j-1} > 0 \text{ and } q_j < 0\} \qquad J^* = \{j \mid j \in J \text{ and } q_{j-2} > 0\}$$

Now, we define a special index k^* as follows. If $J^* \neq \emptyset$, pick any $k^* \in J^*$. Otherwise, if $J \neq \emptyset$, then pick any $k^* \in J$. If $J = J^* = \emptyset$, set $k^* = 1$. Now, we perform a cyclic transformation of the indices that resets $k^* = 1$, by rotating Figure 5.7 counter-clockwise by $k^* - 1$ columns. In other words, index j becomes index $j - k^* + 1$. In the rest of the proof, we assume that Figure 5.7 represents the counter-example after this transformation.

Next, we observe that given any profile of the signs of the coefficients q_j, the k columns of resources in Figure 5.7 can be grouped into several segments, each of which can be classified as one the following three kinds:

(i) $\mathbb{P}_\ell$, *a maximal plus segment of length* ℓ*:* This segment consists of $\ell > 1$ contiguous columns $i, i+1, \ldots, i+\ell-1$ such that $q_j > 0$ for $i \le j \le i+\ell-1$. In addition, we require maximality, i.e., if $\ell \neq k$, then $q_{i-1} < 0$ and $q_{i+\ell} < 0$.

(ii) $\mathbb{M}_\ell$, *a maximal minus segment of length* ℓ*:* This segment consists of $\ell > 1$ contiguous columns $i, i+1, \ldots, i+\ell-1$ such that $q_j < 0$ for $i \le j \le i+\ell-1$. In addition, we require maximality, i.e., if $\ell \neq k$, then $q_{i-1} > 0$ and $q_{i+\ell} > 0$.

(iii) $\mathbb{Z}_\ell$, *a maximal alternating minus-plus segment of length* ℓ*:* This segment consists of $\ell > 1$ contiguous columns (ℓ being even) $i, i+1, \ldots, i+\ell-1$ such that $q_j < 0$ and $q_{j+1} > 0$ for $j \in \{i, i+2, i+4, \ldots, i+\ell-2\}$. In addition, we require maximality, i.e., if $\ell \neq k$, then $q_{i-1} > 0 \Rightarrow q_{i-2} > 0$ and $q_{i+\ell} < 0 \Rightarrow q_{i+\ell+1} < 0$. Note that this kind of segment may share its first / last column with a preceding minus / succeeding plus segment.

Example 14. *Consider the sign profile* $(-,-,+,-,+,+,-)$*. Our special index sets are given by* $J = \{4,7\}$*,* $J^* = \{7\}$*. Hence,* $k^* = 7$*, and so, without loss of generality, we transform this sign profile to* $(-,-,-,+,-,+,+)$*. Now, the first three columns constitute an* $\mathbb{M}_3$ *segment, and the last two columns constitute a* $\mathbb{P}_2$ *segment. In between, we have a* $(+,-)$ *segment that does not fit any of our three definitions above. But, if the immediate neighbors on either side are taken into consideration, we have* $(-,+,-,+)$*, which is a* $\mathbb{Z}_4$ *segment. So, the (unique) decomposition of this sign profile is given by* $\mathbb{M}_3\mathbb{Z}_4\mathbb{P}_2$*, where columns* 3 *and* 6 *are shared between two neighboring segments.*

Technically, the above definition of a $\mathbb{Z}_\ell$ segment allows for a spurious $\mathbb{Z}_2$ segment to be sandwiched between an $\mathbb{M}$ and an adjacent $\mathbb{P}$ segment. For example, the sign profile $(-,-,+,+)$ could be decomposed as either $\mathbb{M}_2\mathbb{P}_2$ or $\mathbb{M}_2\mathbb{Z}_2\mathbb{P}_2$. We exclude this possibility by requiring that every $\mathbb{Z}_\ell$ segment have at least one column that is not shared with a neighboring segment. Note that this requirement also guarantees that the above decomposition is always unique.

We now specify the action sets for the players $i_1, i_2, \ldots, i_k$. For any player i_j, his actions involve only the resources in adjacent columns $j-1$ and j, and specifically, in only one of two ways:

(i) *Straight players* have the following action set:

$$\mathcal{A}_{i_j} = \{(u_{j-1}, u_j), (d_{j-1}, d_j)\} \tag{5.55}$$

(ii) *Diagonal players* have the following action set:

$$\mathcal{A}_{i_j} = \{(u_{j-1}, d_j), (d_{j-1}, u_j)\} \tag{5.56}$$

Whether player i_j is straight or diagonal is determined as follows. We consider two cases:

Case 1: $j = 1$. Player i_1 is straight if and only if one of the following conditions is satisfied:

(i) $q_1 > 0$

(ii) column 1 is at the beginning of an $\mathbb{M}$ segment and column k is at the end of a $\mathbb{P}$ segment

Case 2: $j \neq 1$. Player i_j is straight if and only if one of the following conditions is satisfied:

(i) column j is at the end of a $\mathbb{P}_{2\ell+1}$, $\mathbb{M}_\ell$ or $\mathbb{Z}_\ell$ segment

(ii) $q_j < 0$, and column j is in a $\mathbb{Z}_\ell$ segment

(iii) $q_j > 0$, $q_{j-1} < 0$, and column j is at the beginning of a $\mathbb{P}_\ell$ segment

(iv) $q_j < 0$, $q_{j-1} > 0$, $q_{j-2} < 0$, and column j is at the beginning of an $\mathbb{M}$ segment

A player is diagonal if and only if he is not straight.

Example 15. *Consider the $\mathbb{M}_3\mathbb{Z}_4\mathbb{P}_2$ sign profile of Example 14, namely, $(-,-,-,+,-,+,+)$. There are seven players, $i_1,\ldots,i_7$. Their action sets are represented pictorially (in blue) in Figure 5.8. Formally, players i_1, i_3, i_5, i_6 are straight, whereas players i_2, i_4, i_7 are diagonal.*

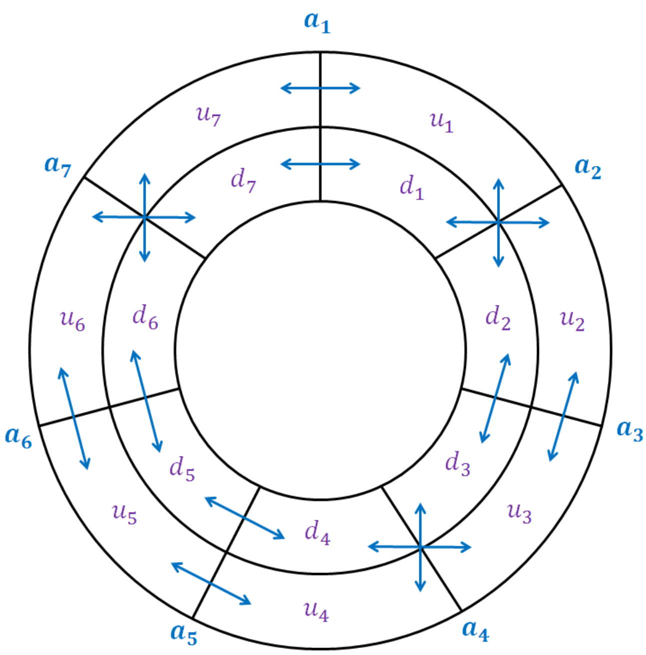


Figure 5.8: *Action sets of the players in Example 15*

Note. Each blue arrow connecting two resources in columns $j-1$ and j denotes the action of player j choosing those two resources.

Counterexample 6 is essentially a game between the players $i_1, i_2, \ldots, i_k$. To complete its specification, we need to specify the resource-specific coefficients v_j. For this, we pick any $v_j > 0$ satisfying the following set of inequalities:

$$\begin{aligned} v_1|q_1|b_1 &> v_2|q_2|a_2 \\ v_2|q_2|b_2 &> v_3|q_3|a_3 \\ &\vdots \\ v_k|q_k|b_k &> v_1|q_1|a_1 \end{aligned} \tag{5.57}$$

We argue that if (5.54) is satisfied, then this set of inequalities has a solution in strictly positive integers v_j. To see this, first observe that if (5.54) is satisfied, then $b_j \neq 0$ for all $1 \leq j \leq k$. If $a_j = 0$ for some j, say for $j = 1$, then (5.57) can be solved recursively as follows.

First, pick any integer $v_k > 0$; this satisfies the last inequality. Then, for $k-1 \geq j \geq 1$, pick any integer $v_j > 0$ that satisfies the jth inequality, i.e.,

$$v_j > v_{j+1} \frac{|q_{j+1}|}{|q_j|} \frac{a_{j+1}}{b_j}$$

Next, consider the case where $a_j \neq 0$ for all $1 \leq j \leq k$. For simplicity, denote $v_i|q_i|$ by v'_i.

For $i \neq 1$, multiplying the first $i-1$ inequalities results in an upper bound for the ratio $\frac{v'_i}{v'_1}$, and multiplying the last $k-i+1$ inequalities results in a lower bound for the ratio $\frac{v'_i}{v'_1}$. These bounds are given by:

$$(\forall 2 \leq i \leq k) \qquad \frac{\displaystyle\prod_{j=i+1}^{k+1} a_j}{\displaystyle\prod_{j=i}^{k} b_j} < \frac{v'_i}{v'_1} < \frac{\displaystyle\prod_{j=1}^{i-1} b_j}{\displaystyle\prod_{j=2}^{i} a_j} \tag{5.58}$$

It can be seen that (5.58) is feasible if and only if (5.54) is satisfied. Algorithm 2 describes a procedure for obtaining $v'_i > 0$ that solves (5.57).

Algorithm 2 Solving (5.57)

$v'_1 \leftarrow 1$
Pick v'_2 satisfying (5.58) for $i = 2$
$i \leftarrow 3$
while $i \leq k$ **do**
  Pick v'_i satisfying the inequality given by,

$$\frac{\prod_{j=i}^{k+1} a_j}{\prod_{j=i}^{k} b_j} < a_i v'_i < b_{i-1} v'_{i-1} \tag{5.59}$$

  $i \leftarrow i+1$
end while

Note that during the ith iteration, (5.59) is feasible for v'_i, because, by using the inequalities in (5.58) for i and $i-1$, it follows that,

$$\frac{\displaystyle\prod_{j=i}^{k+1} a_j}{\displaystyle\prod_{j=i}^{k} b_j} < a_i v'_i \ , \ b_{i-1} v'_{i-1} < \frac{\displaystyle\prod_{j=1}^{i-1} b_j}{\displaystyle\prod_{j=2}^{i-1} a_j}$$

and once again, this is feasible if and only if (5.54) is satisfied.

Now, we verify that the $v'_i > 0$ obtained through this procedure satisfy (5.57):

(i) From (5.59), it is clear that the second through $(k-1)$th inequalities are satisfied.

(ii) The first inequality is satisfied, because $v'_1 = 1$ and we picked v_2 satisfying (5.58) for $i = 2$, from which we get $v'_2 < \frac{b_1}{a_2}$.

(iii) The last inequality is satisfied, since $v'_1 = 1$, and the v_k that we picked satisfying (5.59) for $i = k$ also satisfies (5.58) for $i = k$, from which we get $v_k > \frac{a_1}{b_k}$.

From these v'_j, we obtain v_j by dividing out $|q_j|$. Note that it is always possible to choose v'_j such that all v_j are rational. If this is done, then these v_j can all be scaled by a single positive constant to make them integers, while still satisfying (5.57).

We observe that our definition of the players' action sets in (5.55) and (5.56) ensures that in any action profile, every column j (consisting of resources u_j and d_j) must have both players i_j and i_{j+1}. (There are four possible ways in which this can happen.) We call an action profile $a = (a_1, \ldots, a_k)$ a *valid action profile* if there exists some $1 \leq j \leq k$ (called a *valid index*) such that one of four *validity conditions* is true. Each validity condition involves a configuration consisting of two adjacent columns. We now present the four validity conditions: (The valid configurations that are referenced in these conditions are illustrated in Figure 5.9.)

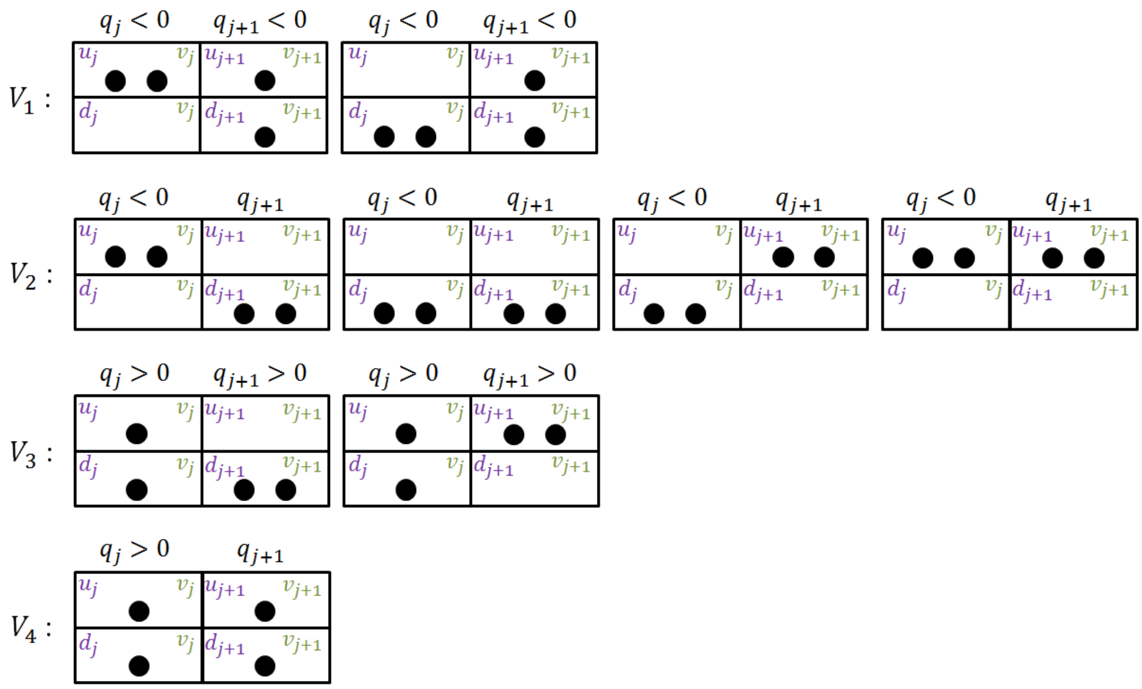


Figure 5.9: *Possible (anonymous) configurations for valid action profiles*

Note. Each black circle denotes a player. Recall that each column j must contain two players, i_j and i_{j+1}. Hence, in every configuration, there are two black circles per column. In every configuration, at the top of each column, the sign of its coefficient is indicated. If there is no sign indicated, then it could be either positive or negative.

(i) $q_j < 0$, $q_{j+1} < 0$, $a_j \cap a_{j+1} \neq \emptyset$. Visually, this corresponds to V_1 or V_2.

(ii) $q_j > 0$, $q_{j+1} > 0$, $a_j \cap a_{j+1} = \emptyset$. Visually, this corresponds to V_3 or V_4.

(iii) $q_j < 0$, $q_{j+1} > 0$, $a_j \cap a_{j+1} \neq \emptyset$, $a_{j+1} \cap a_{j+2} \neq \emptyset$. Visually, this corresponds to V_2.

(iv) $q_j > 0$, $q_{j+1} < 0$, $a_j \cap a_{j+1} = \emptyset$, $a_{j+1} \cap a_{j+2} = \emptyset$. Visually, this corresponds to V_4.

Figure 5.10 is more detailed, where we enumerate all possible ways the black circles in Figure 5.9 can correspond to players of their respective columns. For each set of valid configurations, we also show the utility to player i_{j+1} for the action that he is shown taking (a_{j+1}), as well as for the action he could otherwise have chosen ($\overline{a_{j+1}}$).

We now show that a valid action profile cannot be an equilibrium. From (5.57), we get,

$$(\forall\, 1 \leq j \leq k) \qquad v_j|q_j|f^{T_j}\left((i_{j+1}, T_j\right) > v_{j+1}|q_{j+1}|f^{T_{j+1}}\left(i_{j+1}, T_{j+1}\right) \tag{5.60}$$

Using (5.60), it can be seen that in any valid action profile with a valid index j, i.e., action profiles containing any of the configurations of V_1 through V_4, player i_{j+1} always has an incentive to deviate.

For example, consider any of the four configurations of V_1, where $q_j < 0$, $q_{j+1} < 0$. The difference in the utilities to player i_{j+1} between deviating to $\overline{a_{j+1}}$ and staying in a_{j+1} is given by,

$$\begin{aligned}
\Delta_{j+1} &= U_{j+1}(\overline{a_{j+1}}) - U_{j+1}(a_{j+1}) \\
&= -v_j\left(f(i_{j+1}, T_j) - f(i_{j+1}, T_j - \{i_j\})\right) + v_{j+1}\left(f(i_{j+1}, T_{j+1}) - f(i_{j+1}, T_{j+1} - \{i_{j+2}\})\right)
\end{aligned}$$

Using the basis representation of f, this can be simplified as,

$$\Delta_{j+1} = -v_j \sum_{T \in \mathcal{T}_{i_j i_{j+1}}(T_j)} q_T f^T(i_{j+1}, T) + v_{j+1} \sum_{T \in \mathcal{T}_{i_{j+1} i_{j+2}}(T_{j+1})} q_T f^T(i_{j+1}, T)$$

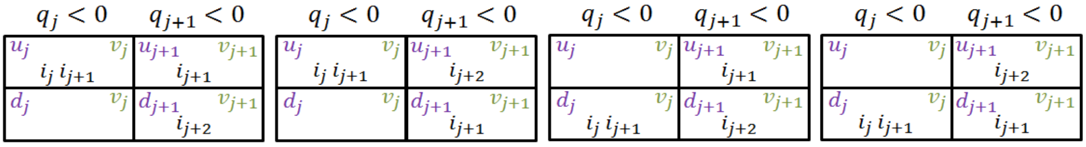
But T_j and T_{j+1} are minimal in $\mathcal{T}^+_{i_j i_{j+1}}$ and $\mathcal{T}^+_{i_{j+1} i_{j+2}}$, respectively. Therefore, we get,

$$\begin{aligned}
\Delta_{j+1} &= -v_j q_j f^{T_j}(i_{j+1}, T_j) + v_{j+1} q_{j+1} f^{T_{j+1}}(i_{j+1}, T_{j+1}) \\
&= v_j|q_j|f^{T_j}(i_{j+1}, T_j) - v_{j+1}|q_{j+1}|f^{T_{j+1}}(i_{j+1}, T_{j+1})
\end{aligned}$$

which is strictly positive, from (5.60). Hence, in configuration V_1, player i_{j+1} has an incentive to deviate. Similar arguments can be constructed for configurations of V_2, V_3, and V_4.

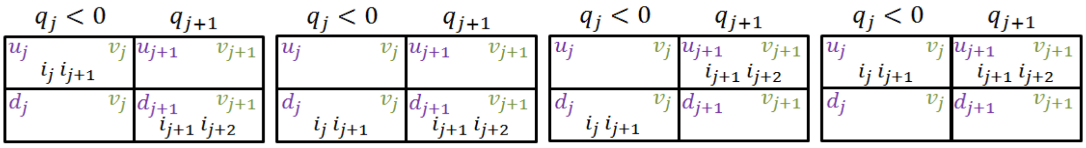
The final step is to show that no invalid action profile exists in *Counterexample 6*. We do this by showing that any attempt to construct an invalid action profile by choosing actions from the players' action sets must fail. Before presenting the formal details, we return to Example 15 to highlight the intuition behind our approach.

V_1 :



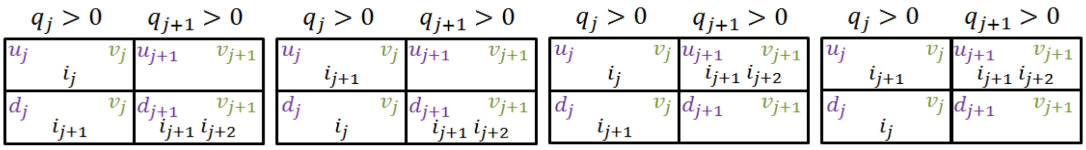
$$U_{j+1}(a_{j+1}) = v_j\, q_j\, f(i_{j+1}, T_j) + v_{j+1}\, q_{j+1}\, f(i_{j+1}, T_{j+1} - \{i_{j+2}\})$$
$$U_{j+1}(\overline{a_{j+1}}) = v_j\, q_j\, f(i_{j+1}, T_j - \{i_j\}) + v_{j+1}\, q_{j+1}\, f(i_{j+1}, T_{j+1})$$

V_2 :



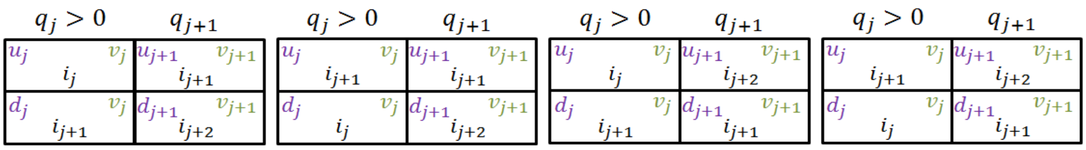
$$U_{j+1}(a_{j+1}) = v_j\, q_j\, f(i_{j+1}, T_j) + v_{j+1}\, q_{j+1}\, f(i_{j+1}, T_{j+1})$$
$$U_{j+1}(\overline{a_{j+1}}) = v_j\, q_j\, f(i_{j+1}, T_j - \{i_j\}) + v_{j+1}\, q_{j+1}\, f(i_{j+1}, T_{j+1} - \{i_{j+2}\})$$

V_3 :



$$U_{j+1}(a_{j+1}) = v_j\, q_j\, f(i_{j+1}, T_j - \{i_j\}) + v_{j+1}\, q_{j+1}\, f(i_{j+1}, T_{j+1})$$
$$U_{j+1}(\overline{a_{j+1}}) = v_j\, q_j\, f(i_{j+1}, T_j) + v_{j+1}\, q_{j+1}\, f(i_{j+1}, T_{j+1} - \{i_{j+2}\})$$

V_4 :



$$U_{j+1}(a_{j+1}) = v_j\, q_j\, f(i_{j+1}, T_j - \{i_j\}) + v_{j+1}\, q_{j+1}\, f(i_{j+1}, T_{j+1} - \{i_{j+2}\})$$
$$U_{j+1}(\overline{a_{j+1}}) = v_j\, q_j\, f(i_{j+1}, T_j) + v_{j+1}\, q_{j+1}\, f(i_{j+1}, T_{j+1})$$

Figure 5.10: *Possible configurations for valid action profiles*

Note. The action set of player i_{j+1} is given by $\mathcal{A}_{i_{j+1}} = \{a_{j+1}, \overline{a_{j+1}}\}$. The specific resources that these two actions include will depend on whether the player is straight or diagonal, which is not important here. We do not explicitly highlight the actions of all players other than players i_j, i_{j+1}, i_{j+2}. In each configuration, action a_{j+1} corresponds to the action that player i_{j+1} is shown taking. $U_{j+1}(a_{j+1})$ denotes the utility to player i_{j+1} in the configurations shown, and it only depends on whether it is V_1, V_2, V_3, or V_4. In every configuration, at the top of each column, the sign of its coefficient is indicated where relevant. If no sign is indicated, it could be either positive or negative.

Example 16. *In Example 15, we specified the action sets of the seven players involved in an $\mathbb{M}_3\mathbb{Z}_4\mathbb{P}_2$ sign profile, namely, $(-,-,-,+,-,+,+)$. Here, we show that for this sign profile, every admissible action profile is valid, i.e., it satisfies one of the four validity properties. We do this by showing that any attempt to construct an invalid action profile must fail:*

(i) First, consider the $\mathbb{M}_3$ segment. Recall that player i_2 is diagonal and player i_3 is straight. It can be seen that in any invalid action profile, there are only four possible ways in which this segment can be filled up—configurations M_1, M_2, or their symmetric counterparts, M_1', M_2', as illustrated in Figure 5.11a. To see this, take M_1, for example:

- *If i_2 switches, the first two columns form a valid configuration of V_2.*
- *If i_3 switches, the second and third columns form a valid configuration of V_2.*
- *If both i_2 and i_3 switch, the first two columns form a valid configuration of V_1.*

(ii) Next, consider the $\mathbb{Z}_4$ segment. Recall that player i_4 is diagonal, and players i_5 and i_6 are straight. It can be seen that in any invalid action profile, there are only ten possible ways in which this segment can be filled up—configurations Z_1 through Z_4, or their symmetric counterparts, Z_1' through Z_4', as illustrated in Figure 5.11b. Here, if any configuration other than these ten occurs, there will be adjacent columns what would form valid configurations of either V_2 or V_4.

(iii) Finally, consider the $\mathbb{P}_2$ segment. Recall that player i_7 is diagonal. It can be seen that in any invalid action profile, there are only four possible ways in which this segment can be filled up—configurations P_1, P_2, or their symmetric counterparts, P_1', P_2', as illustrated in Figure 5.11c. Here, if any configuration other than these four occurs, the two columns would form valid configurations of either V_3 or V_4.

It follows that any invalid action profile must be constructed by picking one configuration from each of Figures 5.11a-5.11c and 'stringing' them together. Note that in doing so, two columns shaded with the same color must be identical to be strung together, since they correspond to overlapping columns. Therefore, it can be seen that there are only four ways of gluing together such configurations: $M_1Z_4P_1$, $M_1Z_4P_2$, and their symmetric counterparts, $M_1'Z_4'P_1'$, $M_1'Z_4'P_2'$. None of these four action profiles are invalid:

(i) $M_1Z_4P_2$ and $M_1'Z_4'P_2'$ are illegal action profiles, since player i_1 is straight, whereas in these two action profiles, he chooses a diagonal action.

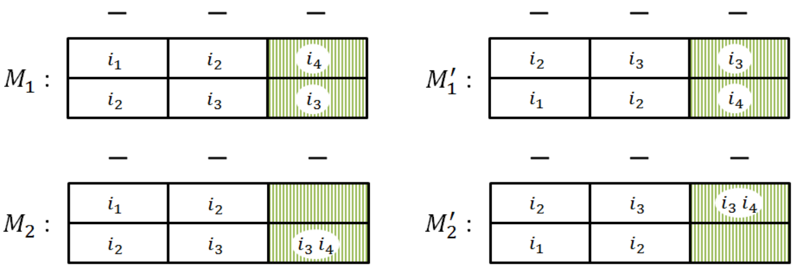
(ii) $M_1Z_4P_1$ and $M_1'Z_4'P_1'$ are legal action profiles, but are valid, because the last column and the first column (when wrapped around) form a valid configuration of V_4.

Hence, all action profiles are valid.

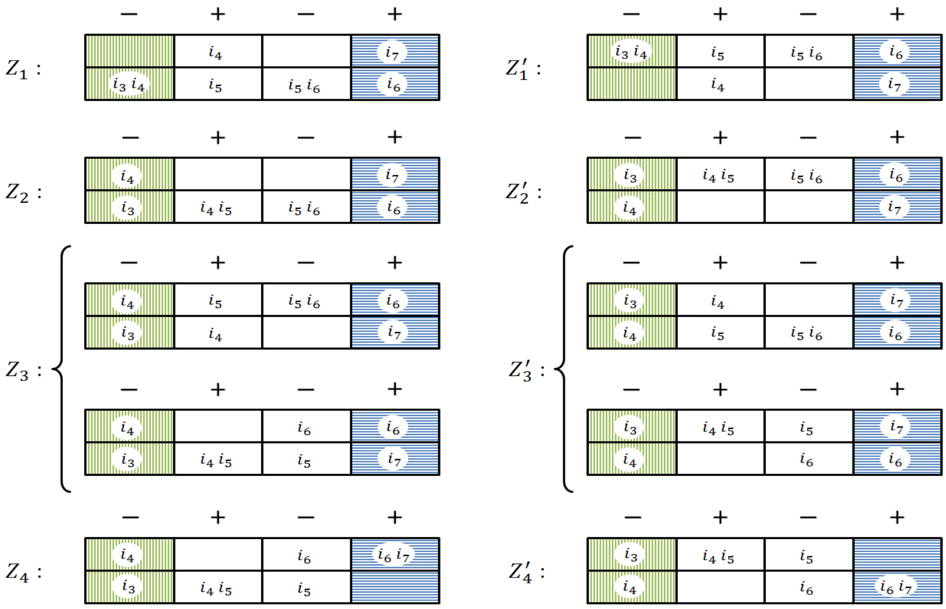
We now build on the general intuition that was demonstrated in the example above to provide a complete proof. First, we state some necessary conditions that any invalid action profile must satisfy within a $\mathbb{P}_\ell$, $\mathbb{M}_\ell$, and $\mathbb{Z}_\ell$ segment. The proofs are by induction, and involve arguing that in order to avoid any of the valid configurations V_1 through V_4, while still respecting how the action sets are defined, such segments must satisfy these necessary conditions. The visual configurations that are referenced in these conditions are illustrated in Figure 5.12.

(i) Let columns $j, j+1, \ldots, j+\ell-1$ form an $\mathbb{M}_\ell$ segment. Then, in any invalid action profile, $a_j \cap a_{j+1} = \emptyset$. In addition,

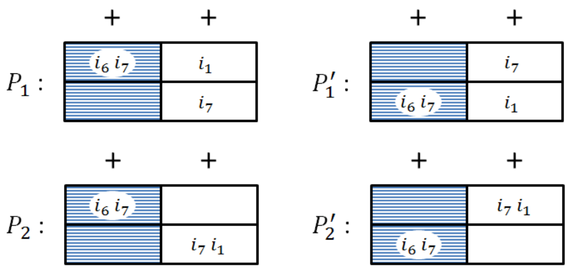
- $u_j \in a_j \Rightarrow d_{j+\ell-1} \in a_{j+\ell-1}$



(a) *Invalid configurations for an $\mathbb{M}_3$ segment.*
Note. The vertical green shade indicates that the third column of this $\mathbb{M}_3$ segment must match the first column of the succeeding $\mathbb{Z}_4$ segment, since they overlap.



(b) *Invalid configurations for a $\mathbb{Z}_4$ segment.*
Note. We have clubbed together two configurations as Z_3 (and similarly, Z_3'), since they have identical boundaries (first and last columns)—only boundary compatibility matters when gluing together different segments. The vertical green shade indicates that the first column of this $\mathbb{Z}_4$ segment must match the third column of the preceding $\mathbb{M}_3$ segment, since they overlap. The horizontal blue shade indicates that the fourth column of this $\mathbb{Z}_4$ segment must match the first column of the succeeding $\mathbb{P}_2$ segment, since they overlap.



(c) *Invalid configurations for a $\mathbb{P}_2$ segment.*
Note. The horizontal blue shade indicates that the first column of this $\mathbb{P}_2$ segment must match the fourth column of the preceding $\mathbb{Z}_4$ segment, since they overlap.

Figure 5.11: *Possible configurations for $\mathbb{M}_3$, $\mathbb{Z}_4$, $\mathbb{P}_2$ segments within an invalid action profile for Example 16*

- $d_j \in a_j \Rightarrow u_{j+\ell-1} \in a_{j+\ell-1}$

Visually, any $\mathbb{M}_\ell$ segment of an invalid action profile must match configurations M_1, M_2 or their symmetric counterparts M'_1, M'_2. If not, there will be adjacent columns that would form valid configurations of either V_1 or V_2.

The proof is by induction on ℓ, the length of the segment. For the base case, when $\ell = 2$, the arguments are similar to those for the $\mathbb{M}_3$ segment in Example 16. Our induction hypothesis is that every invalid $\mathbb{M}_\ell$ segment must match one of the four configurations M_1, M_2, M'_1, M'_2, for some $\ell > 2$. Assuming this is true, now consider an invalid $\mathbb{M}_{\ell+1}$ segment. Keep in mind that from the definition of action sets, player $i_{j+\ell}$ is required to be straight, since he is at the end of this segment. Let $\widetilde{\mathbb{M}}_\ell$ denote the subsegment formed by its first ℓ columns. From the induction hypothesis, $\widetilde{\mathbb{M}}_\ell$ must match one of the four configurations M_1, M_2, M'_1, M'_2:

- $\widetilde{\mathbb{M}}_\ell$ cannot match M_2 or M'_2, because in either case, in $\mathbb{M}_{\ell+1}$, columns ℓ and $\ell+1$ together will form valid configurations of either V_1 or V_2 (depending on how column $\ell+1$ is occupied by the players $i_{j+\ell}$ and $i_{j+\ell+1}$).
- If $\widetilde{\mathbb{M}}_\ell$ matches M_1 or M'_1, then $\mathbb{M}_{\ell+1}$ will match one of the four configurations M_1, M_2, M'_1, M'_2 (depending on how column $\ell+1$ is occupied by the players $i_{j+\ell}$ and $i_{j+\ell+1}$).

(ii) Let columns $j, j+1, \ldots, j+\ell-1$ form a $\mathbb{P}_\ell$ segment. Then, in any invalid action profile, $a_j \cap a_{j+1} \neq \emptyset$. In addition,

- $u_j \in a_j \Rightarrow d_{j+\ell-1} \in a_{j+\ell-1}$
- $d_j \in a_j \Rightarrow u_{j+\ell-1} \in a_{j+\ell-1}$

Visually, any $\mathbb{P}_\ell$ segment of an invalid action profile must match configurations P_1, P_2 or their symmetric counterparts P'_1, P'_2. If not, there will be adjacent columns that would form valid configurations of either V_3 or V_4. The proof is by a similar inductive argument as the $\mathbb{M}_\ell$ case above, except that it is more complicated—we need to consider segments of odd and even lengths separately, because whether player $i_{j+\ell-1}$ is straight or diagonal in a $\mathbb{P}_\ell$ segment depends on whether ℓ is even or odd. We omit the proof for brevity.

(iii) Let columns $j, j+1, \ldots, j+\ell-1$ form a $\mathbb{Z}_\ell$ segment. Then, in any invalid action profile, one of the following three statements must hold:

- $a_j \cap a_{j+1} \neq \emptyset$ and $a_{j+\ell-1} \cap a_{j+\ell} = \emptyset$. In addition,
  - $u_j \in a_j \Rightarrow (u_{j+\ell-1} \in a_{j+\ell-1}$ AND $d_{j+\ell-1} \in a_{j+\ell})$
  - $d_j \in a_j \Rightarrow (d_{j+\ell-1} \in a_{j+\ell-1}$ AND $u_{j+\ell-1} \in a_{j+\ell})$

  Visually, this corresponds to configuration Z_1 or its symmetric counterpart Z'_1.

- $a_j \cap a_{j+1} = \emptyset$ and $a_{j+\ell-1} \cap a_{j+\ell} = \emptyset$. Visually, this corresponds to configurations Z_2, Z_3 or their symmetric counterparts Z'_2, Z'_3.
- $a_j \cap a_{j+1} = \emptyset$ and $a_{j+\ell-1} \cap a_{j+\ell} \neq \emptyset$. In addition,
  - $u_j \in a_j \Rightarrow d_{j+\ell-1} \in a_{j+\ell-1} \cap a_{j+\ell}$
  - $d_j \in a_j \Rightarrow u_{j+\ell-1} \in a_{j+\ell-1} \cap a_{j+\ell}$

  Visually, this corresponds to configuration Z_4 or its symmetric counterpart Z'_4.

Note that if none of these conditions are satisfied, then there will be adjacent columns that would form valid configurations of either V_2 or V_4. Once again, the proof is by a similar inductive argument, and is omitted for brevity.

It follows that any invalid action profile must somehow be constructed by 'stringing' together different-length instantiations of these sixteen configurations. Figure 5.12 illustrates how this is done. There are four possible configurations for an $\mathbb{M}_\ell$ segment, four possible configurations for a $\mathbb{P}_\ell$ segment, and eight possible configurations for a $\mathbb{Z}_\ell$ segment. Note that, by definition, these are maximal segments, so when stringing together two configurations, they cannot be of the same type of segment. We discuss all possible ways of putting together an invalid action profile below:

(i) First, observe that an invalid action profile cannot be constructed using exactly one of these sixteen configurations ($j = 1, \ell = k$ in this case), because, when wrapped around, at the boundary that is formed by the last column and the first column, either of the following two scenarios occur:

- The boundary player, i_1, ends up making an illegal choice, rendering the whole action profile illegal.
- The boundary configuration (the configuration formed by the last column and the first column) ends up being a valid configuration.

(ii) Now, we need at least two configurations to be strung together to create an invalid action profile. We investigate possible ways of stringing together two configurations. Observe that configurations Z_1, Z_2, Z_3 and their symmetric counterparts Z'_1, Z'_2, Z'_3 cannot be hooked to any configurations of a $\mathbb{P}_\ell$ segment (since the overlapping columns do not match), and cannot be glued to any configurations of an $\mathbb{M}_\ell$ segment (since this results in a valid boundary configuration of V_4). So, these six configurations cannot be used to construct an invalid action profile, and can be eliminated.

(iii) Among the remaining ten configurations, observe that P_1 and its symmetric counterpart P'_1 cannot be glued to any configurations of an $\mathbb{M}_\ell$ or any remaining configurations of a $\mathbb{Z}_\ell$ segment (since this results in a valid boundary configuration of V_4). Also, M_2 and its symmetric counterpart M'_2 cannot be hooked to any remaining configurations of a $\mathbb{Z}_\ell$ segment (since the overlapping columns do not match), and cannot be glued to any configurations of a $\mathbb{P}_\ell$ segment (since this results in a valid boundary

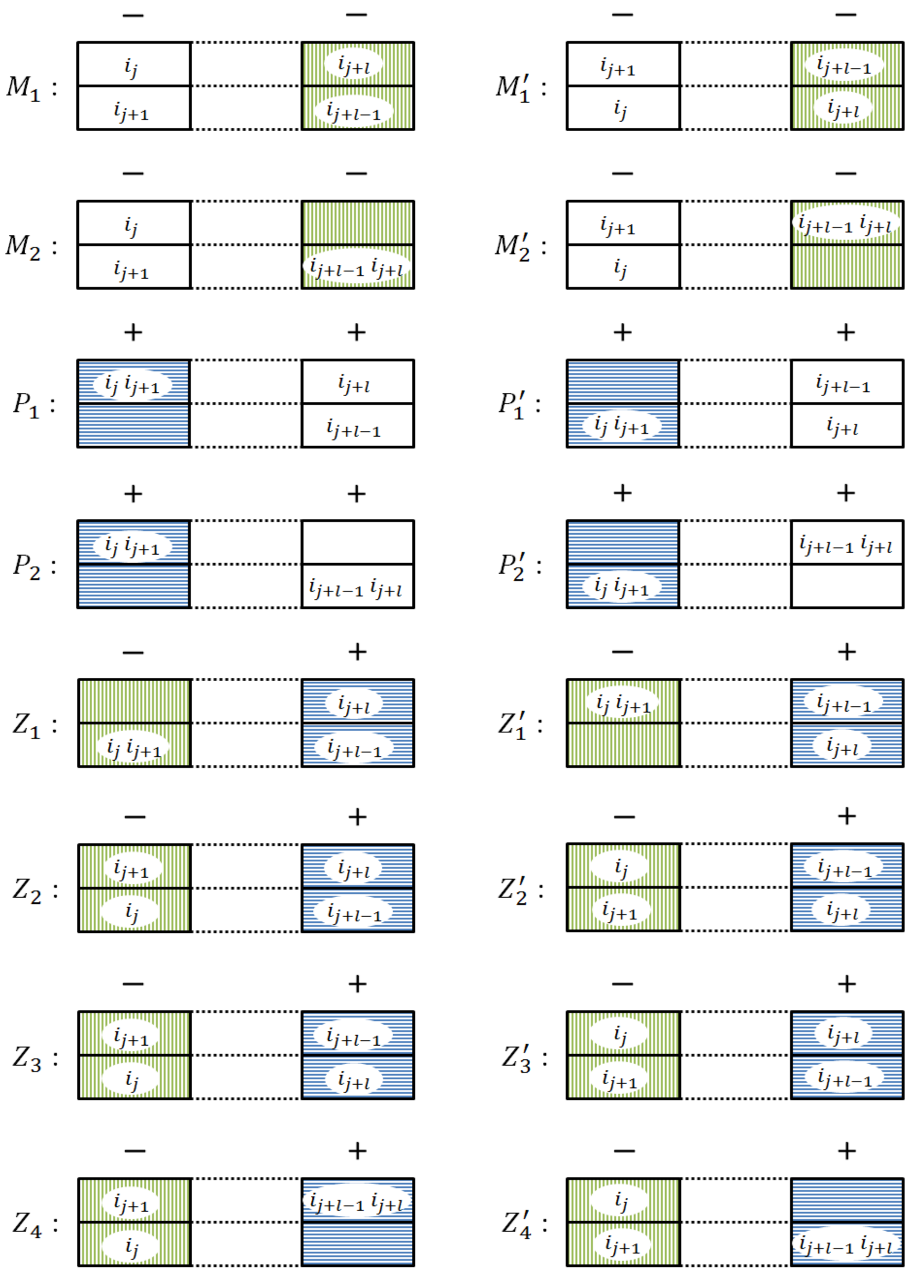


Figure 5.12: *Possible configurations for $\mathbb{M}_\ell$, $\mathbb{P}_\ell$, and $\mathbb{Z}_\ell$ segments (spanning columns j through $j+\ell-1$) within an invalid action profile*

Note. An action profile can be constructed by performing one of two operations repeatedly: (1) Two segments can be "hooked" together to form a larger segment if the last column of the first segment and the first column of the second segment are identical in both occupancy and shading scheme. In this case, these two columns overlap to become one column. For example, M_1 and Z_2 can be hooked together. (2) Two segments can potentially be "glued" together to form a larger segment if the last column of the first segment and the first column of the second segment do not have matching shading schemes. In this case, these two columns would not overlap, but sit next to each other. Such a gluing is permitted only if the resulting action of the boundary player $i_{j+\ell}$ is legal (permissible according to the definition of the action sets). For example, Z_2 and M_1 can be glued together, since it results in the boundary player $i_{j+\ell}$ being straight, which respects the definition of a straight player.

configuration of V_2). So, these four configurations cannot be used to construct an invalid action profile, and can be eliminated.

(iv) We are left with M_1, P_2, Z_4, which can all be hooked or glued with each other, and their symmetric counterparts, M_1', P_2', Z_4', which can also all be hooked or glued with each other. Also, none of M_1, P_2, Z_4 can be hooked or glued with any of M_1', P_2', Z_4' and vice versa (since either overlapping columns (if any) do not match, or the boundary player $i_{j+\ell}$ ends up making an illegal choice).

(v) Therefore, an invalid action profile must be constructed by using only M_1, P_2, Z_4 or only M_1', P_2', Z_4'. Also, such invalid action profiles must begin with a minus sign and end with a plus sign (due to our cyclic transformation at the beginning). There are now only three cases to be considered:

- Begin with M_1 (respectively, M_1') and end with Z_4 (respectively, Z_4')
- Begin with M_1 (respectively, M_1') and end with P_2 (respectively, P_2')
- Begin with Z_4 (respectively, Z_4') and end with P_2 (respectively, P_2')

But, in each of these cases, when wrapped around, the boundary player i_1 ends up making an illegal choice, rendering the whole action profile illegal.

Hence, there exist no invalid action profiles. This concludes the proof. □

As before, we now present a useful inference from Lemma 10 in terms of the weight systems:

Corollary 3. *Given any set of local welfare functions $\mathbb{W}$, let $f^{\mathbb{W}}$ be budget-balanced distribution rules that guarantee equilibrium existence in all games $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$, where, for each $W \in \mathbb{W}$, $f^W := \sum_{T \in \mathcal{T}^W} q_T^W \, f_{GWSV}^T[\omega^{W,T}]$, where $\omega^{W,T} = \left(\boldsymbol{\lambda}^{W,T}, \Sigma^{W,T} = \left(S_1^{W,T}, S_2^{W,T}\right)\right)$. Let $i_1, i_2, \ldots, i_k \in N$ be any k players ($k \geq 3$) such that $\exists\, T_1 \in \left(\mathcal{T}_{i_1 i_2}^{1+}\right)^{\min}, T_2 \in \left(\mathcal{T}_{i_2 i_3}^{2+}\right)^{\min}, \ldots, T_k \in \left(\mathcal{T}_{i_k i_1}^{k+}\right)^{\min}$, for any k welfare functions $W_1, W_2, \ldots, W_k \in \mathbb{W}$. Then,*

(a) $(\forall\, 1 \leq j \leq k)\ \{i_j, i_{j+1}\} \subseteq S_1^{j,T_j}$

(b) $\displaystyle\prod_{j=1}^{k} \frac{\lambda_{i_j}^{j,T_j}}{\lambda_{i_{j+1}}^{j,T_j}} = 1$

5.5.2 Existence of a Universal Weight System

In order to establish the global consistency of the sequence of weight systems $\Omega = \left\{\{\omega^{W,T}\}_{T \in \mathcal{T}^W}\right\}_{W \in \mathbb{W}}$, we need to show that there exists a universal weight system $\omega^* = (\boldsymbol{\lambda}^*, \Sigma^*)$ that is equivalent to all the weight systems in Ω, i.e., replacing $\omega^{W,T}$ with ω^* for any coalition $T \in \mathcal{T}_W$ for any $W \in \mathbb{W}$ does not affect the distribution rule $f^{W,T} = f_{GWSV}^T[\omega^{W,T}]$. We show this by explicitly constructing Σ^* and $\boldsymbol{\lambda}^*$. Before doing so, we use Ω to define two useful relations $\succeq_\Omega$ and $=_\Omega$ on the set N, as follows. For any two elements $i, j \in N$,

$$\begin{aligned} i \succeq_\Omega j &\iff \Big((\exists\, W \in \mathbb{W})\ (\exists\, T \in \mathcal{T}_{ij}^{W+})\ s.t.\ i \in S_1^{W,T} \Big) \text{ OR } \Big(i = j \Big) \\ i =_\Omega j &\iff (i \succeq_\Omega j) \text{ AND } (j \succeq_\Omega i) \end{aligned} \tag{5.61}$$

Using Corollary 2(a) and 2(b)(i), we can write down an equivalent set of definitions for these relations:

$$\begin{aligned} i \succeq_\Omega j &\iff \Big((\forall\, W \in \mathbb{W})\ (\forall\, T \in \mathcal{T}_{ij}^{W+})\ i \in S_1^{W,T} \Big) \text{ OR } \Big(i = j \Big) \\ i =_\Omega j &\iff \Big((\forall\, W \in \mathbb{W})\ (\forall\, T \in \mathcal{T}_{ij}^{W+})\ \{i, j\} \subseteq S_1^{W,T} \Big) \text{ OR } \Big(i = j \Big) \end{aligned} \tag{5.62}$$

We denote the transitive closures of these relations by $\succeq_\Omega^+$ and $=_\Omega^+$, respectively.

Lemma 11. *Given any set of local welfare functions $\mathbb{W}$, if $f^{\mathbb{W}}$ are budget-balanced distribution rules that guarantee equilibrium existence in all games $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$, described completely by the sequence of weight systems Ω, then, $\succeq_\Omega^+$ constitutes a partial order on N.*

Proof. By definition, $\succeq_\Omega^+$ is both *reflexive* and *transitive*. To prove that it is a partial order on N, we need only show *antisymmetry*, i.e., we need to show that for any $i, j \in N$, if $i \succeq_\Omega^+ j$ and $j \succeq_\Omega^+ i$, then $i =_\Omega^+ j$. This is equivalent to showing that if there is a cycle in $\succeq_\Omega$, i.e., if there exists a sequence of k distinct players $i_1, \ldots, i_k \in N$ such that $i_1 \succeq_\Omega i_2 \succeq_\Omega \ldots \succeq_\Omega i_k \succeq_\Omega i_1$, then it must be that $i_1 =_\Omega i_2 =_\Omega \ldots =_\Omega i_k =_\Omega i_1$. The case where $k = 1$ is trivial. For $k = 2$, the proof is vacuous by definition of $=_\Omega$. For $k \geq 3$, suppose there is a cycle in $\succeq_\Omega$. Then, using the definition of $\succeq_\Omega$ from (5.62), $\forall\, 1 \leq j \leq k\ \forall\, W \in \mathbb{W}\ \forall\, T_j \in \mathcal{T}_{i_j i_{j+1}}^{W+}$, $i_j \in S_1^{W,T_j}$. Then, from Corollary 3(a), $\forall\, 1 \leq j \leq k\ \exists W \in \mathbb{W}\ \exists\, T_j \in \mathcal{T}_{i_j i_{j+1}}^{W+}\ s.t.\ \{i_j, i_{j+1}\} \subseteq S_1^{W,T_j}$. The conclusion then follows by using the definitions from (5.61). ☐

We now present the construction of a universal weight system, $\omega^* = (\boldsymbol{\lambda}^*, \Sigma^*)$.

- **Construction of Σ^*.** From Lemma 11, the relation $\succeq_\Omega^+$ constitutes a partial order on N. And the corresponding relation $=_\Omega^+$ is an equivalence relation. We let $\Sigma^* = (S_1^*, S_2^*, \ldots, S_k^*)$ be an ordered partition of N into its equivalence classes according to $=_\Omega^+$, ordered in any manner that does not violate $\succeq_\Omega^+$, i.e., for any $1 \leq j < \ell \leq k$, any $i_j \in S_j^*$ and $i_\ell \in S_\ell^*$, $i_\ell \not\succeq_\Omega^+ i_j$.
- **Construction of $\boldsymbol{\lambda}^*$.** We construct $\boldsymbol{\lambda}^* = (\lambda_i^*)_{i \in N}$ in a piecewise fashion as follows. For each equivalence class $S_r^* \in \Sigma^*$, consider the following two cases:
  (1) $|S_r^*| = 1$. In this case, for $i \in S_r^*$, set λ_i^* to an arbitrary strictly positive number.
  (2) $|S_r^*| = k > 1$. Let $S_r^* = \{i_1, i_2, \ldots, i_k\}$. S_r^* is an equivalence class determined by the relation $=_\Omega^+$, the transitive closure of $=_\Omega$. So, by definition, it must be that for some permutation of its elements, without loss of generality the identity permutation, $i_1 =_\Omega i_2 =_\Omega \ldots =_\Omega i_k$. Using the

definition of $=_\Omega$ from (5.62), this means,

$$(\forall\, 1 \le j < k) \quad (\forall\, W \in \mathbb{W})\, (\forall\, T_j \in \mathcal{T}^{W+}_{i_j i_{j+1}}) \quad \{i_j, i_{j+1}\} \subseteq S_1^{W,T_j}$$

For each $1 \le j < k$, pick any[4] $W_j \in \mathbb{W}$ and $T_j \in \left(\mathcal{T}^{j+}_{i_j i_{j+1}}\right)^{\min}$. To begin with, set $\left(\lambda^*_{i_1}, \lambda^*_{i_2}\right) = \left(\lambda^{1,T_1}_{i_1}, \lambda^{1,T_1}_{i_2}\right)$. If $k = 2$, we are done. Otherwise, for $3 \le j \le k$, recursively set:

$$\lambda^*_{i_j} = \frac{\lambda^{j-1,T_{j-1}}_{i_j}}{\lambda^{j-1,T_{j-1}}_{i_{j-1}}} \lambda^*_{i_{j-1}} \tag{5.63}$$

Example 17. *Let $N = \{i, j, k, \ell, m, n\}$ be the set of players, and let there be just one local welfare function W, $\mathcal{T}^W = \left\{T^W_1 = \{i, j\}, T^W_2 = \{j, k, \ell\}, T^W_3 = \{m, n\}, T^W_4 = \{i, m, n\}\right\}$. Also, let f be a distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, described by the sequence of weight systems $\Omega = \left\{\omega^{W,T_i} = (\boldsymbol{\lambda}^{W,T_i}, \Sigma^{W,T_i})\right\}_{i=1}^{4}$, where,*

$$\boldsymbol{\lambda}^{W,T_1} = \boldsymbol{\lambda}^{W,T_3} = (1, 2) \qquad \boldsymbol{\lambda}^{W,T_2} = \boldsymbol{\lambda}^{W,T_4} = (1, 2, 3)$$
$$\Sigma^{W,T_1} = \{\{i, j\}, \{\}\} \quad \Sigma^{W,T_2} = \{\{j, k\}, \{\ell\}\} \quad \Sigma^{W,T_3} = \{\{m, n\}, \{\}\} \quad \Sigma^{W,T_4} = \{\{i\}, \{m, n\}\}$$

Using the definitions in (5.61) or (5.62), it can be seen that the players are related as follows:

$$i \;=^+_\Omega\; j \;=^+_\Omega\; k \;\succeq^+_\Omega\; \ell \quad \textit{and} \quad i \;\succeq^+_\Omega\; m \;=^+_\Omega\; n$$

Using the construction shown above, it can be seen that for Σ^*, both $\{\{i, j, k\}, \{m, n\}, \{\ell\}\}$, as well as $\{\{i, j, k\}, \{\ell\}, \{m, n\}\}$ are admissible orderings of the three equivalence classes of $=^+_\Omega$ (they do not violate $\succeq^+_\Omega$). As for the weights, we get $\boldsymbol{\lambda}^* = (1, 2, 4, a, 1, 2)$, where a can be any strictly positive number. Any strictly positive scaling of $\boldsymbol{\lambda}^*$ would also be admissible.

Before proceeding to show that ω^* as constructed above is equivalent to all the weight systems in Ω, we prove an important property of $\boldsymbol{\lambda}^*$ in a quick lemma:

Lemma 12. *With $\boldsymbol{\lambda}^*$ as derived above, for any $S^*_r \in \Sigma^*$ with $|S^*_r| > 1$, for any two players $i, j \in S^*_r$, for any $W \in \mathbb{W}$, for any coalition $T \in \mathcal{T}^{W+}_{ij}$ with $\{i, j\} \subseteq S_1^{W,T}$,*

$$\frac{\lambda^{W,T}_i}{\lambda^{W,T}_j} = \frac{\lambda^*_i}{\lambda^*_j} \tag{5.64}$$

Proof. Let $|S^*_r| = k > 1$. Equivalently, we show that for all $m \in \{1, 2, \ldots, k-1\}$, for all $\ell \in \{1, 2, \ldots, k - m\}$, (5.64) holds for players $i = i_\ell$ and $j = i_{\ell+m}$. The base case, where $m = 1$ follows by construction (5.63), and by using Corollary 2(b)(ii). For $m \ge 2$, suppose there exists a welfare function $W \in \mathbb{W}$ and

[4] At least one such (W_j, T_j) pair is guaranteed by the definitions in (5.61).

a coalition $T \in \mathcal{T}_{ij}^{W+}$, with $\{i, j\} \subseteq S_1^{W,T}$. Recall the welfare functions and coalitions (W_j, T_j), $1 \leq j < k$, that were picked for constructing $\boldsymbol{\lambda}^*$. From Corollary 2(b), it is sufficient to prove this lemma for $T \in \left(\mathcal{T}_{ij}^{W+}\right)^{\min}$. Now, using the definitions in (5.61), $j =_\Omega i$. Therefore, it follows that the players $i = i_\ell, i_{\ell+1}, \ldots, i_{\ell+m} = j$ form a cycle in $=_\Omega$, i.e., $i_\ell =_\Omega i_{\ell+1} =_\Omega \ldots =_\Omega i_{\ell+m} =_\Omega i_\ell$. This means that, $i_\ell \in S_1^{\ell,T_\ell}, i_{\ell+1} \in S_1^{\ell+1,T_{\ell+1}}, \ldots, i_{\ell+m-1} \in S_1^{\ell+m-1,T_{\ell+m-1}}, i_{\ell+m} \in S_1^{W,T}$. Applying Corollary 3(b) and (5.63), we have:

$$\left(\prod_{j=\ell}^{\ell+m-1} \frac{\lambda_{i_j}^{j,T_j}}{\lambda_{i_{j+1}}^{j,T_j}}\right) \frac{\lambda_{i_{\ell+m}}^{W,T}}{\lambda_{i_\ell}^{W,T}} = 1 \quad \Rightarrow \quad \frac{\lambda_i^{W,T}}{\lambda_j^{W,T}} = \prod_{j=\ell}^{\ell+m-1} \frac{\lambda_{i_j}^{j,T_j}}{\lambda_{i_{j+1}}^{j,T_j}} \quad \Rightarrow \quad \frac{\lambda_i^{W,T}}{\lambda_j^{W,T}} = \prod_{j=\ell}^{\ell+m-1} \frac{\lambda_{i_j}^*}{\lambda_{i_{j+1}}^*} \quad \Rightarrow \quad \frac{\lambda_i^{W,T}}{\lambda_j^{W,T}} = \frac{\lambda_i^*}{\lambda_j^*}$$

This concludes the proof. $\square$

Now we present the final lemma that establishes the global consistency of Ω.

Lemma 13. *Given any set of local welfare functions $\mathbb{W}$, if $f^{\mathbb{W}}$ are budget-balanced distribution rules that guarantee equilibrium existence in all games $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$, where, for each $W \in \mathbb{W}$, $f^W := \sum_{T \in \mathcal{T}^W} q_T^W f_{GWSV}^{W,T}[\omega^{W,T}]$, then, there exists a weight system ω^*, such that,*

$$(\forall\, W \in \mathbb{W})\ (\forall T \in \mathcal{T}^W) \quad f_{GWSV}^{W,T}[\omega^{W,T}] = f_{GWSV}^{T}[\omega^*] \tag{5.65}$$

Proof. We prove that ω^* as constructed above satisfies (5.65). Consider any welfare function $W \in \mathbb{W}$ and any coalition $T \in \mathcal{T}^W$. Let $k = \min\{r | S_r^* \cap T \neq \emptyset\}$. Then, we need only show the following:

(i) $S_1^{W,T} \subseteq S_k^*$

(ii) $S_2^{W,T} \cap S_k^* = \emptyset$

(iii) $\left(\forall i, j \in S_1^{W,T}\right)\ \frac{\lambda_i^{W,T}}{\lambda_j^{W,T}} = \frac{\lambda_i^*}{\lambda_j^*}$

Of these, the first two are immediate from the construction of Σ^*, and the third follows from Lemma 12. This completes the proof. $\square$

Chapter 6

Conclusion

In this thesis, we set out to study the design of *good/desirable* distribution rules for noncooperative cost sharing games. The exact set of properties that a designer considers 'good' or 'desirable' depends on the specific application at hand. Some of the important properties that a distribution rule may be expected to satisfy are stability, efficiency, budget-balance, tractability, locality[1] and (quite recently) privacy.[2] We chose to investigate a very specific property (stability in the sense of existence of a pure Nash equilibrium) in a very general model (accommodates several well-known classes of cost sharing games as special cases), and exposed a fundamental limitation in distribution rule design from the stability point of view—*a distribution rule guarantees equilibrium existence in all games with fixed local welfare functions if and only if it is equivalent to a generalized weighted Shapley value (or, equivalently, a generalized weighted marginal contribution) distribution rule.*

In this concluding chapter, we begin with a brief discussion of the limitations of the statement of this result, and related future work on further expanding its scope. Then, we also discuss the limitations imposed by this result when other factors such as efficiency, budget-balance, tractability, and privacy are desired, and the resulting tradeoffs that arise. Finally, we take a step back, and investigate possible extensions this work could have in the closely related area of mechanism design.

6.1 Limitations of Our Result

It is important to highlight that our characterizations in Theorems 2 and 3 crucially depend on the fact that an equilibrium must be guaranteed in *all games*, i.e., for all possibilities of resources, action sets, and choice of local welfare functions from $\mathbb{W}$. (This is the same for the characterizations given in the previous work of Chen et al. [15], Marden and Wierman [54].) If this requirement is relaxed it may be possible to find situations where distribution rules that are not equivalent to generalized weighted Shapley values can guarantee

[1] Our model, by default, assumes a local distribution rule, that is, the welfare at a resource is distributed only among agents who choose that resource.

[2] This is by no means an exhaustive list, even among static properties of the game. In applications such as distributed control, dynamic properties are important too, since distributed learning algorithms are based on convergent dynamics. Distribution rules that guarantee potential games are particularly attractive for both their static and dynamic properties.

equilibrium existence. For example, Marden and Wierman [53] give such a rule for a coverage game where players can select only one resource at a time. A challenging open problem is to determine the structure on the action sets that is necessary for the characterizations in Theorems 2 and 3 to hold.

It is also of interest to investigate equilibria under different notions of stability. Relaxing the solution concept beyond pure Nash equilibrium, for finite games, renders existence trivial, due to the theorem of Nash [63, 64]. Moreover, one may obtain more 'efficient' and 'dynamics-friendly' equilibria in doing so. However, there is also an increased chance of multiple equilibria, whence refinement is a bigger issue, and the variance of equilibrium efficiency is also higher.

6.2 Implications for Efficiency/Budget-Balance Tradeoff

Our entire focus has been on characterizing distribution rules that guarantee equilibrium existence. However, guaranteeing efficient equilibria is also an important goal for distribution rules. Existing efficiency results provide a lower bound of 2 for both the price of anarchy and price of stability for *submodular* welfare functions, if the distribution rule is restricted to be budget-balanced. On the other hand, the marginal contribution family of distribution rules guarantees a price of stability of 1 for any welfare function, but is not budget-balanced. The characterizations in Theorems 2 and 3 provide important new tools to optimize the efficiency of distribution rules for general cost sharing and revenue sharing games through proper choice of the weight system and ground welfare functions. An important open problem in this direction is to understand the resulting tradeoffs between budget-balance and efficiency.

In this regard, we make the following contribution to initiate this line of study. The following short proposition claims that when optimizing for efficiency (price of anarchy, price of stability, etc.) within the class of generalized weighted Shapley values, the weights do not matter, if there are no other constraints such as budget-balance. This means, on top of our characterization results reducing the search space to just the set of all generalized weighted Shapley values, the following proposition (potentially) knocks off several more dimensions. For simplicity, we state the result when there is just one local welfare function, but it can be extended to when there are multiple local welfare functions.

Proposition 10. *Let* $G_1 \in \mathcal{G}\left(N, f_{GWMC}^{W''}[(\boldsymbol{\lambda}, \Sigma)], W\right)$ *and* $G_2 \in \mathcal{G}\left(N, f_{GWMC}^{W''}[(\mathbf{1}, \Sigma)], W\right)$*. Then,* $\boldsymbol{a}^{NE}(G_1) = \boldsymbol{a}^{NE}(G_2)$*.*

Proof. It is well known that in any noncooperative game $G_1 = (N, \{\mathcal{A}_i\}, \{U_i\})$, scaling the utility function of a player by a positive constant does not affect the game. To be precise, for each player i, we can scale U_i by some $\chi_i > 0$, and the scaled game $G_2 = (N, \{\mathcal{A}_i\}, \{\chi_i U_i\})$ has exactly the same equilibria as the original game. That is, $\boldsymbol{a}^{NE}(G_1) = \boldsymbol{a}^{NE}(G_2)$.

In our model, the utility function U_i is determined by the distribution rule f as,

$$U_i(a) = \sum_{r \in a_i} v_r f\left(i, \{a\}_r\right),$$

Therefore, the set of scaled utility functions, $\{\chi_i U_i\}$, is equivalent to the following "scaling" of the distribution rule f:

$$(\forall\, i \in N) \quad (\forall S \subseteq N) \quad f(i,S) \longmapsto \chi_i f(i,S)$$

With this in mind, and recalling the definition of the generalized weighted marginal contribution distribution rule, $f^{W''}_{GWMC}[(\boldsymbol{\lambda}, \Sigma)]$,

$$f^{W''}_{GWMC}[(\boldsymbol{\lambda}, \Sigma)] = \lambda_i \left(W''(\overline{S}_k) - W''(\overline{S}_k - \{i\})\right) = \lambda_i f^{W''}_{GWMC}[(\mathbf{1}, \Sigma)],$$

it is clear that, by choosing the scaling constants $\chi_i = \frac{1}{\lambda_i}$, G_1 and G_2 have the same set of Nash equilibria. □

6.3 Implications for Tractability/Budget-Balance Tradeoff

Another tradeoff that arises is between budget-balance and tractability of computing Shapley values.[3] Given an arbitrary welfare function, are there tractable approximations that tradeoff with budget-balance, without losing the potential function property, i.e., the approximation must also be a Shapley value?[4] The answer to one extreme is well known to begin with—if exact budget-balance is desired, then the Shapley value must be computed exactly on the true welfare function, which is intractable in general. The two equivalent characterizations of this thesis now provide an answer for the other extreme—if budget-balance is completely relaxed, then one can select any ground welfare function and apply a marginal contribution rule to it (and it will be a Shapley value on some other, arbitrarily non-budget-balanced function).

In between, the question is wide open. One possibility that is provided by our equivalence result is to start with a welfare function satisfying the budget-constraints and then perform the required transformation (which is still of exponential complexity) in order to distribute the same shares using a marginal contribution rule. A small advantage is that this transformation need only be done once, as a 'preprocessing' step. Can we do better? Perhaps not, for an arbitrary welfare function. But, when the welfare function is known to have some structure, for example, submodular welfare functions, are there other tractable ways of approaching this problem? For example, it is tractable to compute the Shapley value for anonymous welfare functions.

[3] A different budget-balance/tractability tradeoff arises when there is interest in the tractability of *finding equilibria*. This tradeoff is quite direct—equilibria in finite potential games correspond to local maxima of the potential function, and hence one would like to choose potential functions that are easy to maximize. From Theorem 1, it is clear that Shapley values are nothing but marginal contributions on their potential functions; therefore, the choice of potential function affects budget-balance.

[4] There are known approximation algorithms for computing Shapley values (e.g., Liben-Nowell et al. [47]), but the approximations themselves are not guaranteed to be Shapley values, which might result only in a 'near-potential game' and approximate equilibria, as opposed to a weighted potential game and exact equilibria. It is also of interest to investigate such a stability vs. tractability tradeoff.

Therefore, an upper bound on the maximum distance between a submodular welfare function and its 'closest' anonymous welfare function (that can be found in polynomial time) would be a good starting point, since this would be also be an upper bound on the how much should necessarily be given up on budget-balance so that it becomes easy to compute the Shapley value. Since budget-balance has an effect on efficiency, this tradeoff affects efficiency too, and results in an induced tractability/efficiency tradeoff.

6.4 Implications for Privacy

Privacy is increasingly becoming a major concern of the modern public. In our cost sharing model, we may be required to design distribution rules that lead to a 'privacy preserving' outcome. Consider, for example, a health insurance firm, which recovers its collective healthcare costs from its clients by through premiums. Suppose it is legal for the firm to charge different clients different premiums based on their medical history. More likely than not, a client will desire that not 'much' information about her general health condition is revealed to other clients through their premiums. Now, to provide perfect privacy, the firm could charge everyone the same premium, regardless of their medical history. However, it will probably be more profitable for the firm (especially in the presence of competing firms) to set premiums based on the medical history of each client.

Several notions of 'privacy preserving' outcomes have been proposed—one of the more popular concepts in a statistical context is called *differential privacy* (Dwork [22]). Roughly, it means that when sensitive data corresponding to any one person is changed or removed, the result of the computation does not change 'much'. Most differential privacy algorithms require adding calculated noise to the output, thus trading off accuracy for privacy. In light of our characterization results, Shapley values and marginal contributions are deterministic computations and are not differentially private. This seems to indicate a deeper fundamental tradeoff between the achievable notion of stability and differential privacy. It would be of interest to quantify such tradeoffs in our noncooperative cost sharing model. Privacy issues are all the more relevant in the context of cost sharing *mechanisms* (discussed next), where the private information of a person could be much more sensitive than just the outcome of a resource allocation problem—how this plays into the incentive compatibility constraints is also an interesting question to study.

6.5 Cost Sharing Mechanisms

Finally, it is important to remember that our focus has been on cost sharing *games*; however it is natural to ask if similar characterizations can be obtained for cost sharing *mechanisms* (Moulin and Shenker [61], Dobzinski et al. [21], Immorlica and Pountourakis [39], Johari and Tsitsiklis [43], Yang and Hajek [90], Moulin [59]). More specifically, the model considered in this paper extends immediately to situations where players have independent heterogeneous valuations over actions, by adding more welfare functions to $\mathbb{W}$. To see this, consider a welfare sharing game $G \in \mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$. Let the action set of player i be $\mathcal{A}_i = \{a_1, \ldots, a_\ell\}$, and suppose he values action a_j at u_j, for $1 \leq j \leq \ell$. Then, we modify G to G' by adding ℓ more resources $r_1, \ldots, r_\ell$ to R, setting $W_{r_j}(S) = \begin{cases} u_j, & i \in S \\ 0, & i \notin S \end{cases}$ and $f^{W_{r_j}} = f_{SV}^{W_{r_j}}$ for $1 \leq j \leq \ell$, and augmenting each action in $\mathcal{A}_i$ with its corresponding resource, so that $a_j \rightarrow a_j \cup \{r_j\}$. Then, $G' \in \mathcal{G}(N, f^{\mathbb{W}'}, \mathbb{W}')$, where $\mathbb{W}' = \mathbb{W} \cup_{j=1}^{\ell} \{W_{r_j}\}$ and $f^{\mathbb{W}'} = f^{\mathbb{W}} \cup_{j=1}^{\ell} \{f^{W_{r_j}}\}$. Notice that all games in $\mathcal{G}(N, f^{\mathbb{W}}, \mathbb{W})$ have an equilibrium if and only if all games in $\mathcal{G}(N, f^{\mathbb{W}'}, \mathbb{W}')$ have an equilibrium.

The above transformation was simply to show mathematically that our model can accommodate agents' heterogeneous valuations over their action sets, and hence that our results extend to this case. However, in reality, we do not need to obtain these valuations and set up hypothetical resources, because the private valuations (whatever they may be, unknown to the designer) are implicitly added to the agents' utility functions (and hence taken into consideration) while they decide which action to take. The case for mechanism design arises only when the social welfare function that the designer seeks to optimize includes, for example, the sum of the valuations of the selected actions of the agents, in addition to the sum of the welfare generated at each resource. Here, there may be a need for the distribution rules to (possibly) depend on the valuations in order for an efficient outcome to emerge. However, player valuations are private, which adds a challenging wrinkle to this situation. Thus, extending our characterizations to the setting of cost sharing mechanisms is a difficult, but important, open problem.

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