Relative Mirror Symmetry and Ramifications of a Formula for Gromov-Witten Invariants

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Michel van Garrel

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Abstract

For a toric Del Pezzo surface S, a new instance of mirror symmetry, said *relative*, is introduced and developed. On the A-model, this relative mirror symmetry conjecture concerns genus 0 relative Gromov-Witten of maximal tangency of S. These correspond, on the B-model, to relative periods of the mirror to S. Furthermore, two conjectures for BPS state counts are related. It is proven that the integrality of BPS state counts of K_S, the total space of the canonical bundle on S, implies the integrality for the relative BPS state counts of S. Finally, a prediction of homological mirror symmetry for the open complement is explored. The B-model prediction is calculated in all cases and matches the known A-model computation when $S = \mathbb{P}^2$.

Contents

Chapter 1

Introduction

The study of mirror symmetry in mathematics originated in theoretical physics and has, since the 1980s, led to an intensive interaction between the two fields. An essential ingredient of mirror symmetry is the curve counting theory of Gromov-Witten invariants. Denote by S a Del Pezzo surface and by D a smooth effective anti-canonical divisor on it. Furthermore, let $\beta \in \mathrm{H}_2(S, \mathbb{Z})$ be an effective curve class. This data is associated with the genus 0 Gromov-Witten invariants $I_\beta(K_S)$ of K_S, the total space of the canonical bundle on S. Local mirror symmetry for K_S asserts that these Gromov-Witten invariants are computed via periods on its mirror variety. Alternatively, one considers genus 0 relative Gromov-Witten invariants of S relative to D. Such invariants are virtual counts of genus 0 curves in S with a variety of tangency conditions along D. These are indexed by the weight partition of the cohomology of D. One such choice governs the genus 0 relative Gromov-Witten invariants of maximal tangency, denoted by $N_\beta(S, D)$. This requires that the curves meet D in exactly one point, thus assures that the relevant moduli space is zero-dimensional and hence removes the need for insertions. In chapter 4, we introduce a conjecture relating these relative invariants of maximal tangency to relative periods on the mirror to S. We call this new instance *relative mirror symmetry* and prove it when S is toric. A formulation of relative mirror symmetry for general genus 0 relative Gromov-Witten invariants is under development by the author. The proof we present for the invariants of maximal tangency relies on the following theorem, which was proven for $\mathbb{P}^2$ by Gathmann in [1], and then extended by Graber-Hassett to all Del Pezzo surfaces (unpublished):

Theorem 1. *(Gathmann, Graber-Hassett) With the above notation,*

$$N_\beta(S, D) = (-1)^{\beta \cdot D}\,(\beta \cdot D)\, I_\beta(K_S).$$

We explore further ramifications of this formula in chapters 2 and 5. In chapter 2 we consider BPS state counts, which are refinements of Gromov-Witten invariants. Whereas Gromov-Witten invariants are rational numbers in general, BPS state counts are expected to be integers. For Calabi-Yau three-folds, of which K_S are examples, this was conjectured by Gopakumar-Vafa in [2] and [3]. Relative BPS state counts for log Calabi-Yau surface pairs, of which (S, D) are examples, were introduced by Gross-Pandharipande-Siebert in [4]. The authors conjecture that these invariants are integers as well. We prove that the conjecture for K_S implies the conjecture for (S, D). In chapter 5, we are interested in the homological mirror symmetry conjecture for the open complement $S - D$. The conjecture states that the derived Fukaya category of $S - D$ ought to be equivalent to the bounded derived category of coherent sheaves of its mirror M_S. Taking Hochschild cohomology on both sides yields the expectation that the Hochschild cohomology of M_S is isomorphic to the symplectic cohomology of $S - D$. As a step towards verifying this prediction of homological mirror symmetry, we calculate the Hochschild cohomology of M_S as a module over the polynomial ring. For $\mathbb{P}^2 - D$, this matches up with the calculation of its symplectic cohomology by Nguyen-Pomerleano.

Chapter 2

Local and relative BPS state counts

This chapter are the results of joint work with Tony W. H. Wong and Gjergji Zaimi. Denote by S a Del Pezzo surface, by D a smooth effective anti-canonical divisor on it and by K_S the total space of the canonical bundle on S. Furthermore, let $\beta \in \mathrm{H}_2(S, \mathbb{Z})$ be an effective non-zero curve class. On one side are the genus 0 degree β relative Gromov-Witten invariants of maximal tangency $N_\beta(S, D)$ of (S, D). On the other side are the genus 0 degree β local Gromov-Witten invariants $I_\beta(K_S)$ of K_S. Theorem 1 relates these two sets of invariants via

$$N_\beta(S,D) = (-1)^{\beta\cdot D}\,(\beta \cdot D)\, I_\beta(K_S). \tag{2.1}$$

In general, Gromov-Witten invariants are rational numbers, since the relevant moduli spaces are Deligne-Mumford stacks. Since K_S is Calabi-Yau, generically genus 0 curves are embedded with normal bundle isomorphic to $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. For $d \geq 1$, a degree d cover of such a curve will contribute to the degree $d\beta$ invariant $I_{d\beta}(K_S)$. This contribution is quantified by the Aspinwall-Morrison formula to be $1/d^3$, proven by Manin in [5]. The *BPS state counts* $n(\beta)$ are the rational numbers defined via

$$I_\beta(K_S) = \sum_{k|\beta} \frac{n(\beta/k)}{k^3}. \tag{2.2}$$

If all embedded genus 0 curves were of the above form, the $n(\beta)$ would count actual genus 0 degree β curves in K_S. This is false in general. It is nevertheless conjectured by Gopakumar-

Vafa that $n(\beta) \in \mathbb{Z}$ for all Calabi-Yau 3-folds. This was proven in the case where K_S is toric by Peng in [6]. On the other side, in [4] Gross-Pandharipande-Siebert introduce *relative BPS state counts* for log Calabi-Yau surface pairs, of which (S, D) are examples. Assume that β is primitive and set $w = D \cdot \beta$. For $d \geq 1$, consider the relative GW invariant $N_{d\beta}(S, D)$. Adopting the same notation as in [4], we write

$$N_S[dw] = N_{d\beta}(S, D). \tag{2.3}$$

The authors consider the generating series

$$N_S = \sum_{d=1}^{\infty} N_S[dw]\, q^d. \tag{2.4}$$

Computing multiple cover contributions leads the authors to define the relative BPS numbers $n_S[dw] \in \mathbb{Q}$ via

$$N_S = \sum_{d=1}^{\infty} n_S[dw] \sum_{k=1}^{\infty} \frac{1}{k^2} \binom{k(dw-1)-1}{k-1} q^{dk}. \tag{2.5}$$

Analogously to the local case, Gross-Pandharipande-Siebert conjecture that the $n_S[dw]$ are integers for all $d \geq 1$. In this chapter, we prove the following theorem.

Theorem 2. *(Garrel-Wong-Zaimi) Let $\beta \in \mathrm{H}_2(S, \mathbb{Z})$ be an effective non-zero primitive curve class. For $d \geq 1$, consider the two sequences of rational numbers $N_{d\beta}(S, D)$ and $I_{d\beta}(K_S)$ and assume that they satisfy equation* (2.1) *for all $d\beta$. Define two sequences of rational numbers $n_S[dw]$ and $n(d)$ by means of the equations* (2.2), (2.3), (2.4) *and* (2.5)*. Then:*

$$n_S[dw] \in \mathbb{Z}, \forall d \geq 1 \iff dw \cdot n(d) \in \mathbb{Z}, \forall d \geq 1.$$

An immediate consequence then is:

Corollary 3. *The conjecture on the integrality of the local BPS invariants of K_S implies the conjecture on the integrality of the relative BPS invariants of (S, D).*

Moreover, the result in [6] on the integrality for the toric local case implies:

Corollary 4. *If S is toric, then its relative BPS numbers are integers.*

We prove theorem 2 in the next two sections.

2.1 A formula relating the invariants

Assuming that β is primitive, formula (2.2) applied to $d\beta$ gives

$$\sum_{d=1}^{\infty} I_{d\beta}(K_S)\, q^d = \sum_{d=1}^{\infty} \sum_{k|d} \frac{1}{k^3}\, n(\frac{d}{k}\beta)\, q^d.$$

Combining this with formula (2.1) and noting that $d\beta \cdot D = dw$ yields

$$\begin{aligned} N_S &= \sum_{d=1}^{\infty} N_S[dw]\, q^d \\ &= \sum_{d=1}^{\infty} (-1)^{dw}\, dw\, I_{d\beta}(K_S)\, q^d \\ &= \sum_{d=1}^{\infty} q^d\, (-1)^{dw}\, dw \sum_{k|d} \frac{1}{k^3} n_S(\frac{d}{k}\beta). \end{aligned}$$

Lemma 5. *The $n_S[dw]$ are related to the $n(d\beta)$ by the formula*

$$\begin{aligned} N_S &= \sum_{d=1}^{\infty} q^d \sum_{k|d} \frac{1}{k^2} \binom{k(\frac{d}{k}w-1)-1}{k-1} n_S[\frac{d}{k}w] \\ &= \sum_{d=1}^{\infty} q^d\, (-1)^{dw} dw \sum_{k|d} \frac{1}{k^3} n_S(\frac{d}{k}\beta). \end{aligned}$$

Proof. This follows from the change of variable $\tilde{d} = dk$:

$$\begin{aligned} N_S &= \sum_{d=1}^{\infty} n_S[dw] \sum_{k=1}^{\infty} \frac{1}{k^2} \binom{k(dw-1)-1}{k-1} q^{dk} \\ &= \sum_{\tilde{d}=1}^{\infty} q^{\tilde{d}} \sum_{k|\tilde{d}} \frac{1}{k^2} \binom{k(\frac{\tilde{d}}{k}w-1)-1}{k-1} n_S[\frac{\tilde{d}}{k}w]. \end{aligned}$$

$\square$

Fix a positive integer m. We write the formula of lemma (5), capped in degree $m+1$ and larger, in matrix form: Let row d encode the terms in q^d, and let $\frac{d}{k}$ parametrize the columns. Then, the above m equations turn into:

$$R\,\left[n_S[dw]\right]_d = A \cdot L \cdot A^{-1}\,\left[(-1)^{dw}\,dw\,n(d)\right]_d, \tag{2.6}$$

where

$$R_{ij} := \begin{cases} \frac{1}{(i/j)^2}\binom{i/j\,(jw-1)-1}{i/j-1} & \text{if } j|i, \\ 0 & \text{else;} \end{cases}$$

$$A_{ij} := (-1)^{iw}\,iw \cdot \delta_{ij};$$

$$L_{ij} := \begin{cases} \frac{1}{(i/j)^3} & \text{if } j|i, \\ 0 & \text{else.} \end{cases}$$

Note that multiplying with A^{-1} yields a matrix of determinant ± 1.

Notation. For a square-free integer n, let $\#_p(n)$ denote the number of primes in the prime factorization of n. Moreover, for integers k and m, write $k \in I(m)$ to mean that k divides m and that m/k is square-free.

Lemma 6. *Define the $m \times m$ matrix C as follows. If $t|s$, let*

$$C_{st} := \frac{(-1)^{sw}}{(s/t)^2} \sum_{k \in I(s/t)} (-1)^{\#_p(s/kt)}(-1)^{ktw}\binom{k(tw-1)-1}{k-1}, \tag{2.7}$$

If t does not divide s, set $C_{st} = 0$. Then the invariants $\{n_S[dw]\}$ and $\left\{(-1)^{dw}\,dw\,n(d)\right\}$, for $1 \le d \le m$, are related via

$$C \cdot \left[n_S[dw]\right]_d = \left[(-1)^{dw}\,dw\,n(d)\right]_d.$$

Moreover, C has determinant ± 1 and is lower triangular. It follows by Cramer's rule that

$$C \text{ integral} \iff C^{-1} \text{ integral.}$$

Proof. We start by writing $L = B \cdot \tilde{L} \cdot B^{-1}$, where

$$\tilde{L}_{ij} = \begin{cases} 1 & \text{if } j|i, \\ 0 & \text{else;} \end{cases}$$
$$B_{ij} = \frac{1}{i^3} \cdot \delta_{ij}.$$

By Möbius inversion, the inverse of $\tilde{L}$ is given by

$$\tilde{L}^{-1}_{ij} = \begin{cases} (-1)^{\#_p(i/j)} & \text{if } j|i \text{ and } i/j \text{ is square-free,} \\ 0 & \text{else.} \end{cases}$$

Moreover,

$$(AB)_{ij} = (-1)^{iw} \frac{w}{i^2} \cdot \delta_{ij},$$

and

$$\left((AB)^{-1}\right)_{ij} = (-1)^{iw} \frac{i^2}{w} \cdot \delta_{ij}.$$

It now follows from formula (2.6) that the matrix C is given by

$$C = AB \cdot \tilde{L}^{-1} \cdot (AB)^{-1} \cdot R.$$

We calculate that

$$\left(AB \cdot \tilde{L}^{-1}\right)_{sr} = \begin{cases} (-1)^{sw} \frac{w}{s^2} (-1)^{\#_p(s/r)} & \text{if } r|s \text{ and } s/r \text{ is square-free,} \\ 0 & \text{else;} \end{cases}$$

and that

$$\left((AB)^{-1}\cdot R\right)_{rt} = \begin{cases} (-1)^{rw}\, \frac{r^2}{w}\, \frac{1}{(r/t)^2} \binom{r/t\,(tw-1)-1}{r/t-1} & \text{if } t|r, \\ 0 & \text{else.} \end{cases}$$

If t does not divide s, then there is no integer r such that $t|r|s$, so that $C_{st} = 0$. If, however, $t|s$, then

$$\begin{aligned} C_{st} &= \sum (-1)^{sw}\, (-1)^{\#_p(s/r)}\, (-1)^{rw}\, \frac{1}{(s/t)^2} \binom{r/t\,(tw-1)-1}{r/t-1} \\ &= (-1)^{sw}\, \frac{1}{(s/t)^2} \sum (-1)^{\#_p(s/r)}\, (-1)^{rw} \binom{r/t\,(tw-1)-1}{r/t-1}, \end{aligned}$$

where the sum runs over all r such that $t|r|s$ and such that s/r is square-free. Set $k = r/t$, so that, for t dividing s,

$$C_{st} = (-1)^{sw}\, \frac{1}{(s/t)^2} \sum_{k\in I(s/t)} (-1)^{\#_p(s/kt)}\, (-1)^{ktw} \binom{k\,(tw-1)-1}{k-1},$$

finishing the proof. □

Lemma 6 reduces theorem 2 to proving that the coefficients of the matrix C are integers. We show this in lemmas 9 and 10.

2.2 Integrality of C

The following lemma follows directly form the proof of lemma $A.1$ of [6].[1]

Lemma 7. *(Peng) Let a, b and α be positive integers and denote by p a prime number. If $p = 2$, assume furthermore that $\alpha \geq 2$. Then*

$$\binom{p^\alpha a - 1}{p^\alpha b - 1} \equiv \binom{p^{\alpha-1} a - 1}{p^{\alpha-1} b - 1} \mod \left(p^{2\alpha}\right).$$

[1] In the combinatorics literature, there are a significant number of results on the divisibility of binomial coefficients by prime powers, see e.g., [7].

In the case that $p = 2$ and $\alpha = 1$, we have the following lemma:

Lemma 8. *Let $k \geq 1$ be odd and let a be a positive integer. Then*

$$\binom{2ka-1}{2k-1} + (-1)^a \binom{ka-1}{k-1} \equiv 0 \mod (4)\,.$$

Proof. Note that

$$\begin{aligned}
\binom{2ka-1}{2k-1} &= \frac{2ka-1}{2k-1} \cdot \frac{2ka-2}{2k-2} \cdots \frac{2ka-2k+2}{2} \cdot \frac{2ka-2k+1}{1} \\
&= \frac{2ka-1}{2k-1} \cdot \frac{ka-1}{k-1} \cdots \frac{ka-k+1}{1} \cdot \frac{2ka-2k+1}{1} \\
&= \frac{(ka-1)(ka-2)\cdots(ka-k+1)}{(k-1)(k-2)\cdots 1} \cdot \frac{(2ka-1)(2ka-3)\cdots(2ka-2k+1)}{(2k-1)(2k-3)\cdots 1} \\
&= \binom{ka-1}{k-1} \cdot \frac{(2ka-1)(2ka-3)\cdots(2ka-2k+1)}{(2k-1)(2k-3)\cdots 1},
\end{aligned}$$

and hence

$$\begin{aligned}
&\binom{2ka-1}{2k-1} + (-1)^a \binom{ka-1}{k-1} \\
=&\binom{ka-1}{k-1} \left((-1)^a + \frac{(2ka-1)(2ka-3)\cdots(2ka-2k+1)}{(2k-1)(2k-3)\cdots 1} \right).
\end{aligned}$$

It thus suffices to show that

$$\frac{(2ka-1)(2ka-3)\cdots(2ka-2k+1)}{(2k-1)(2k-3)\cdots 1} \equiv (-1)^{a+1} \mod (4)\,. \tag{2.8}$$

Suppose first that a is even, so that the left-hand-side of (2.8) is congruent to

$$\begin{aligned}
&\frac{(-1)(-3)\cdots(-(2k-3))(-(2k-1))}{1 \cdot 3 \cdots (2k-3)(2k-1)} \\
\equiv&(-1)^k \equiv (-1)^{a+1} \mod (4),
\end{aligned}$$

where the last congruence follows form the fact that k is odd. Suppose now that a is odd.

Then the left-hand-side of the expression (2.8) is congruent to

$$\frac{2k(a-1)+(2k-1)}{2k-1}\cdot\frac{2k(a-1)+(2k-3))}{2k-3}\cdots\frac{2k(a-1)+1}{1}$$
$$\equiv 1\equiv(-1)^{a+1} \mod (4)\,.$$

$\square$

We return to the proof of theorem 2. If $s=t$, then $C_{st}=\pm 1$ and is therefore an integer. We assume henceforth that $t|s$, but $t\neq s$. Let p be a prime number and α a positive integer. For an integer n, we use the notation

$$p^{\alpha}||n,$$

to mean that $p^{\alpha}|n$, but $p^{\alpha+1}\nmid n$. In order to show that $C_{st}\in\mathbb{Z}$, we show, for every prime number p, that if

$$p^{\alpha}||\frac{s}{t},$$

then

$$p^{2\alpha}|\sum_{k\in I(s/t)}(-1)^{\#_p(s/kt)}\,(-1)^{ktw}\binom{k\,(tw-1)-1}{k-1}.$$

Fix a prime number p and a positive integer α such that

$$p^{\alpha}||\frac{s}{t}.$$

For $k\in I(s/t)$ to be square-free, it is necessary that $p^{\alpha-1}|k$. This splits into the two cases

$$k=p\cdot l,\ \text{or}\ k=l,$$

where

$$l\in I(s/pt).$$

Regrouping the terms of the expression of (2.7) accordingly yields

$$\sum_{l\in I(s/pt)}\sum_{k\in\{l,pl\}}(-1)^{\#_p(s/kt)}\,(-1)^{ktw}\binom{k\,(tw-1)-1}{k-1}.$$

Thus, it suffices to prove that for all $l \in I(s/pt)$,

$$f(l) := \sum_{k\in\{l,pl\}}(-1)^{\#_p(s/kt)}\,(-1)^{ktw}\binom{k\,(tw-1)-1}{k-1} \equiv 0 \mod \left(p^{2\alpha}\right),$$

which we proceed in showing. There are two cases: either the sign $(-1)^{ktw}$ in the above sum changes or not. The only case where the sign does not change is when $p = 2$, $\alpha = 1$, and both t and w are odd.

Lemma 9. *Assume that either $p \neq 2$ or, if $p = 2$, that $\alpha > 1$. Then*

$$f(l) \equiv 0 \mod \left(p^{2\alpha}\right).$$

Proof. In this situation,

$$\begin{aligned} f(l) &= \pm\left(\binom{pl(tw-1)-1}{k-1} - \binom{l(tw-1)-1}{k-1}\right) \\ &\equiv 0 \mod \left(p^{2\alpha}\right), \end{aligned}$$

by lemma 7. □

Lemma 10. *Assume that $p = 2$ and that $\alpha = 1$. Then*

$$f(l) \equiv 0 \mod (4).$$

Proof. In this case,

$$\begin{aligned} f(l) &= \pm\left(\binom{2l(tw-1)-1}{2l-1} + (-1)^{tw-1}\binom{l(tw-1)-1}{l-1}\right) \\ &\equiv 0 \mod (4)\,, \end{aligned}$$

follows from lemma 8. $\square$

Chapter 3

Mirror geometries to $\mathbb{P}^2$

Broadly speaking, mirror symmetry for a Calabi-Yau mirror pair $(X, \check{X})$ states that the association

$$\check{X} \xleftrightarrow{\text{mirror symmetry}} X\ ,$$

exchanges the complex and symplectic geometries of X and $\check{X}$. The complex side is called the *B-model*, while the symplectic side is referred to as the *A-model*. The classical formulation of mirror symmetry in addition produces a recipe for computing Gromov-Witten (GW) invariants of X in terms of the periods of $\check{X}$, schematically:

$$\text{Periods of } \check{X} \longleftrightarrow \text{GW invariants of } X.$$

The first part of this chapter consists in reviewing some of the mirror constructions and mirror symmetry statements relating to $\mathbb{P}^2$ and $K_{\mathbb{P}^2}$. This will serve to motivate the result on relative mirror symmetry of the next chapter, and the result on the consequence of homological mirror symmetry in the chapter after that. Additionally, in section 3.5 we explore how two families of affine elliptic curves, serving as mirrors for $K_{\mathbb{P}^2}$, are related. Throughout this chapter we emphasize an enumerative perspective. From this viewpoint, *local* mirror symmetry calculates the genus 0 *local* Gromov-Witten invariants of $\mathbb{P}^2$, which are the genus 0 Gromov-Witten invariants of $K_{\mathbb{P}^2}$. This terminology is justified by the fact that genus 0 stable maps into $K_{\mathbb{P}^2}$ factor through $\mathbb{P}^2$.

3.1 Mirror symmetry statements for $\mathbb{P}^2$ and $K_{\mathbb{P}^2}$

Denote by θ_z the logarithmic differential $z\frac{\partial}{\partial z}$. For holomorphic functions

$$f: \ \Delta^\times \to \mathbb{C},$$

consider the differential equation

$$\mathcal{L}\, f = 0, \tag{3.1}$$

where

$$\mathcal{L} = \theta_z^3 + 3z\theta_z(3\theta_z + 1)(3\theta_z + 2).$$

This is called the *A-hypergeometric differential equation* associated to $\mathbb{P}^2$. Chiang-Klemm-Yau-Zaslow in [8] show that (the Taylor coefficients of) solutions of this equation, via change of variable and analytic continuation, calculate the local GW invariants of $\mathbb{P}^2$. The solutions to the above equation are expressed as periods of various mirror geometries, which in turn are based on the mirror constructions developed by Batyrev in [9] and [10]. In [8], the authors consider the family of affine elliptic curves

$$B_\phi := \left\{ P_\phi(x,y) := xy - \phi(x^3 + y^3 + 1) = 0 \mid x, y \in \mathbb{C}^\times \right\}, \tag{3.2}$$

for $\phi \in \mathbb{C}^\times$ and periods of the 1-form

$$\mathrm{Res}_{P_\phi = 0}\,(\log P_\phi)\,\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{xy}.$$

Using the same family, Takahashi in [11] expresses mirror symmetry with relative periods. These are integrals of the relative cohomology class $\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{xy} \in \mathrm{H}^2(\mathbb{T}^2, B_\phi; \mathbb{Z})$ over relative homology classes in $\mathrm{H}_2(\mathbb{T}^2, B_\phi; \mathbb{Z})$. Based on [9], Stienstra in [12] proves that periods of the family

$$M^0_{\check{q}} := \left\{ 0 = 1 + t_1 + t_2 + \frac{\check{q}}{t_1 t_2} \,|\, (t_1, t_2) \in \mathbb{T}^2 \right\}, \tag{3.3}$$

for $\check{q} \in \mathbb{C}^\times$, correspond to solutions of (A.1). Here, the author considers integrals over the cohomology class $\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{xy} \in \mathrm{H}^2(\mathbb{T}^2, M^0_{\check{q}}; \mathbb{Z})$. We show how the two families are related in lemma 11 and proposition 13 below. Hori-Iqbal-Vafa in [13] consider the related family of open 3-folds

$$Z_{\check{q}} := \left\{ xy = F_{\check{q}}(t_1, t_2) := 1 + t_1 + t_2 + \frac{\check{q}}{t_1 t_2} \,|\, (x, y) \in \mathbb{C}^2,\, (t_1, t_2) \in \mathbb{T}^2 \right\}, \tag{3.4}$$

and integrals of the holomorphic 3-form

$$\mathrm{Res}\, \frac{1}{xy - F_{\check{q}}(t_1, t_2)} \frac{\mathrm{d}\, t_1 \,\mathrm{d}\, t_2}{t_1 t_2} \,\mathrm{d}\, x \,\mathrm{d}\, y \in \mathrm{H}^3(Z_{\check{q}}, \mathbb{Z}).$$

Hosono in [14] proves that these periods yield solutions to the differential equation (A.1). The mirror to $K_{\mathbb{P}^2}$ of this form was constructed by Gross-Siebert in [15], see also [16], by passing through tropical geometry. A construction avoiding tropical geometry and emphasizing semi-flat coordinates was elaborated by Chan-Lau-Leung in [17]. Geometrically, $M^0_{\check{q}}$ is the fixed locus of the $\mathbb{C}^\times$-action on $Z_{\check{q}}$ given by

$$\lambda \cdot (x, y, t_1, t_2) = (\lambda x, \lambda^{-1} y, t_1, t_2).$$

By instead taking the GIT-quotient of the same action, one gets a correspondence of the periods of each family. This was described by Gross in [18] and thoroughly developed by Konishi-Minabe in [19]. The latter authors describe the variation of mixed Hodge structures on $M^0_{\check{q}}$ and $Z_{\check{q}}$, yielding a natural language for the local B-model. Denote by ϕ a third root of $\check{q}$. After the coordinate change

$$t_i \mapsto \phi\, t_i,$$

the family $M^0_{\check{q}}$ is described as

$$\left\{ 0 = 1 + \phi \left(t_1 + t_2 + \frac{1}{t_1 t_2} \right) \;|\; t_1, t_2 \in \mathbb{C}^\times \right\}.$$

Setting

$$W_0(t_1,t_2) := t_1 + t_2 + \frac{1}{t_1 t_2},$$

we can rewrite $M^0_{\check{q}}$ as the family parametrized by the fibers

$$W_0(t_1,t_2) = -1/\phi,$$

of the superpotential

$$W_0 : \ \mathbb{T}^2 \longrightarrow \mathbb{C}. \tag{3.5}$$

A Landau-Ginzburg model consists of a complex manifold and a holomorphic function on it, called the superpotential. The present Landau-Ginzburg model, namely $(\mathbb{T}^2, W_0)$, is the mirror to $\mathbb{P}^2$ constructed by Givental in [20]. In terms of periods, mirror symmetry in this setting sets up a correspondence between oscillatory integrals on $(\mathbb{T}^2, W_0)$ and Gromov-Witten invariants of $\mathbb{P}^2$. Implementing the SYZ conjecture, the Gross-Siebert program relates both sides of this correspondence to tropical disk counts of tropical $\mathbb{P}^2$, see, e.g., [21]. Mirror symmetry for $\mathbb{P}^2$ also states that the quantum cohomology ring $\mathrm{Q\,H}^*(\mathbb{P}^2)$ of $\mathbb{P}^2$ is isomorphic to the Jacobian ring $\mathrm{Jac}(W_0)$ of W_0. In terms of the homological mirror symmetry (HMS) conjecture introduced by Kontsevich at the ICM in Zürich (cf. [22]), this is categorified as follows. Define M by compactifying the fibers of W_0, i.e.,

$$M = \left\{ t_1^2 t_2 + t_1 t_2^2 + t_3^3 - s t_1 t_2 t_3 \right\} \subset \mathbb{P}^2_{\mathbb{C}[s]},$$

and denote by W the extension of W_0 to M. The generic fiber of W is an elliptic curve (mirror to an elliptic curve in $\mathbb{P}^2$) and the three singular fibers are simple nodal curves of arithmetic genus 1. Homological mirror symmetry then predicts a set of correspondences

$$(M, W) \leftrightarrow \mathbb{P}^2,$$

$$\text{Complex geometry of } \ \mathrm{Crit}(W) \leftrightarrow \text{Symplectic geometry of } \mathbb{P}^2,$$

$$\text{Symplectic geometry of } \ \mathrm{Crit}(W) \leftrightarrow \text{Complex geometry of } \mathbb{P}^2,$$

where $\mathrm{Crit}(W)$ denotes the critical locus of W.[1] On one side, the symplectic geometry of $\mathrm{Crit}(W)$ is described by the derived category of Lagrangian vanishing-cycles $D\,\mathrm{Lag}_{vc}(W)$, while its complex geometry is encoded in its category of matrix factorizations $\mathrm{MF}(W)$. On the other side, the symplectic geometry of $\mathbb{P}^2$ is described by its Fukaya category $\mathcal{F}(\mathbb{P}^2)$, whereas its derived category of coherent sheaves $D\,\mathrm{Coh}(\mathbb{P}^2)$ encodes its complex geometry. Kontsevich's HMS conjecture states that these categories ought to be equivalent:[2]

$$\mathrm{MF}(W) \simeq \mathcal{F}(\mathbb{P}^2),$$
$$D\,\mathrm{Lag}_{vc}(W) \simeq D\,\mathrm{Coh}(\mathbb{P}^2).$$

The second correspondence was proven by Auroux-Katzarkov-Orlov in [23], building on work by Seidel of [24].[3] A proof of the first correspondence was announced by Abouzaid-Fukaya-Oh-Ohta-Ono. The data of Gromov-Witten invariants is encoded by these categories as well, albeit too abstractly to allow for calculations: $\mathrm{Q\,H}^*(\mathbb{P}^2)$ is the Hochschild cohomology of $\mathcal{F}(\mathbb{P}^2)$; and $\mathrm{Jac}(W) = \mathrm{Jac}(W_0)$ is the Hochschild cohomology of $\mathrm{MF}(W)$.

3.2 The SYZ conjecture

The discussion in the previous section elaborates on an *algebraic* viewpoint of mirror symmetry. That is, we introduced different varieties that function as mirrors, i.e., their periods calculate Gromov-Witten invariants of the A-model. These statements do not explain why mirror symmetry holds. Enters the Strominger-Yau-Zaslow (SYZ) conjecture, see [26], which provides a geometric explanation of mirror symmetry in terms of dualizing Lagrangian torus fibrations. This conjecture is not expected to hold in full generality, but adaptations of it have led to a geometric understanding of mirror symmetry. The mirror constructions for $\mathbb{P}^2$ and $K_{\mathbb{P}^2}$ are schematically summarized as in the diagram below. As usual, denote by D

[1] Concerning the second equivalence, if $\mathrm{Crit}(W)$ is not symplectic, the statement is adapted.

[2] Kontsevich originally formulated the conjecture for pairs of mirror Calabi-Yau manifolds. The statements here are an adaptation to Fano varieties.

[3] See also the paper [25] by Auroux-Katzarkov-Orlov.

a smooth effective anti-canonical divisor on $\mathbb{P}^2$, i.e., an elliptic curve, and denote by T the toric divisor on $\mathbb{P}^2$, namely the union of the coordinate axes.

$$\begin{tikzcd}
(M, W) \arrow[rr, leftrightarrow, "\text{Homological}", "\text{Mirror Symmetry}"'] & & (\mathbb{P}^2, D) \\
(\mathbb{T}^2, W) \arrow[u, "\text{Compactification of fibers}"] \arrow[rr, leftrightarrow, "\text{Gross-Siebert}", "\text{program}"'] & & (\mathbb{P}^2, T) \arrow[u, "\text{Smoothing of divisor}"'] \\
Z_{\check{q}} \arrow[u, "\mathbb{C}^\times\text{-fixed locus}"] \arrow[rr, leftrightarrow, "\text{Local}", "\text{Mirror Symmetry}"'] & & K_{\mathbb{P}^2}. \arrow[u, "0\text{-section and } \mathbb{C}^\times\text{-fixed locus}"']
\end{tikzcd}$$

The geometric construction of the family $Z_{\check{q}}$ was carried out by Gross-Siebert using tropical geometry in [16]. The Gross-Siebert program aims at explaining mirror symmetry in terms of the SYZ-conjecture by passing through tropical geometry, cf. [15], [21] for $\mathbb{P}^2$ and the book [27]. Chan-Lau-Leung in [17] give an alternative construction of the family $Z_{\check{q}}$ without passing through tropical geometry: The SYZ conjecture describes how to construct a mirror for an open subvariety V of $K_{\mathbb{P}^2}$. In order to get a mirror to the entire space, that mirror needs to be deformed by *quantum/instanton corrections*. These are encoded by open Gromov-Witten invariants that arise in the Lagrangian torus fibration constructed by Gross for V, see [18].

3.3 Batyrev's construction

In this section, we describe in more detail Batyrev's mirror constructions relating to $\mathbb{P}^2$, following the exposition of Konishi-Minabe in [19]. The families $M^0_{\check{q}}$ of A.3 and $Z_{\check{q}}$ of A.4 are affine open subsets of the families we introduce here. Start with

$$\mathbb{L}_{reg} = \left\{ (a_0, a_1, a_2, a_3) \in \mathbb{C}^4 \mid a_1 a_2 a_3 \neq 0,\ \frac{a_0^3}{a_1 a_2 a_3} + 27 \neq 0 \right\},$$

and consider the $\mathbb{T}^3$-action

$$(\lambda_0, \lambda_1, \lambda_2) \cdot (a_0, a_1, a_2, a_3) = \lambda_0 (a_0, \lambda_1 a_1, \lambda_2 a_2, \frac{1}{\lambda_1 \lambda_2} a_3),$$

as well as the character

$$\begin{aligned} \chi : \ \mathbb{T}^3 &\to \mathbb{C}^\times, \\ (\lambda_i) &\mapsto \lambda_0^3. \end{aligned}$$

Denote the associated GIT-quotient by $\mathcal{M}_\mathbb{C}$. All points of $\mathcal{M}_\mathbb{C}$ are stable and $\mathcal{M}_\mathbb{C}$ is identified to $\mathbb{P}^1 \backslash \{-1/27\}$ via

$$\begin{aligned} \mathcal{M}_\mathbb{C} &\to \mathbb{P}^1, \\ (a_i) &\mapsto [a_0^3, a_1 a_2 a_3]. \end{aligned}$$

For $a = (a_i) \in \mathcal{M}_\mathbb{C}$, define the Laurent polynomial

$$F_a(t_1, t_2) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}.$$

Denote by

$$M_a^0 \to \mathcal{M}_\mathbb{C},$$

the family of affine 3-folds whose fibers are given by

$$\left\{ F_a(t_1, t_2) + xy = 0 \,|\, (t_1, t_2) \in \mathbb{T}^2,\ (x, y) \in \mathbb{C}^2 \right\};$$

and by

$$Z_a \to \mathcal{M}_\mathbb{C},$$

the family of affine curves whose fibers are given by

$$\left\{F_a(t_1,t_2)=0\,|\,(t_1,t_2)\in\mathbb{T}^2\right\}.$$

Consider the affine open $\{a_0\neq 0\}\subset\mathcal{M}_{\mathbb{C}}$ with coordinate

$$\check{q}:=\frac{a_1a_2a_3}{a_0^3},$$

so that $\check{q}$ parametrizes $\mathbb{C}^\times\setminus\{-1/27\}$. This will yield the families $M^0_{\check{q}}$ and $Z_{\check{q}}$ with the singular fiber removed. Consider the change of variable

$$t_i\mapsto\frac{a_0}{a_i}t_i,$$

to get

$$F_a(t_1,t_2)=a_0\left(1+t_1+t_2+\frac{\check{q}}{t_1t_2}\right).$$

If $a_0\neq 0$, then

$$M^0_{\check{q}}\to\mathbb{C}^\times\setminus\{-1/27\},$$

is given by

$$\left\{0=1+t_1+t_2+\frac{\check{q}}{t_1t_2}\,|\,(t_1,t_2)\in\mathbb{T}^2\right\},$$

as in (A.3). On the other hand, by dividing, e.g., y by a_0, the family

$$Z_{\check{q}}\to\mathbb{C}^\times\setminus\{-1/27\},$$

is described as

$$\left\{xy=1+t_1+t_2+\frac{\check{q}}{t_1t_2}\,|\,(x,y)\in\mathbb{C}^2,\,(t_1,t_2)\in\mathbb{T}^2\right\},$$

which agrees with (A.4).

Remark. Two points are in order:

1. Described this way, the GIT-quotient $\mathcal{M}_{\mathbb{C}}$ may seem arbitrary. In fact, the authors

in [19] start by considering the respective families over $\mathbb{L}$ and then consider the GIT quotient induced by the action

$$F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2),\ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{T}^3.$$

The resulting families are the same as the ones described above.

2. Also, as detailed in [19], the construction readily generalizes to other toric Del Pezzo surfaces.

3.4 Local mirror symmetry for $\mathbb{P}^2$

We overview the mirror construction to $K_{\mathbb{P}^2}$ given by Chan-Lau-Leung in [17]. The authors consider as complex moduli of $Z_{\check{q}}$ the punctured unit open disk $\Delta^\times$ with complex parameter $\check{q}$. The Kähler moduli of $K_{\mathbb{P}^2}$ is isomorphic to $\Delta^\times$ and we denote by $q \in \Delta^\times$ the Kähler parameter. Denote by $c(q)$ a certain generating series of open Gromov-Witten invariants. The instanton-corrected[4] mirror to $K_{\mathbb{P}^2}$ is any one member of the family parametrized by q of non-compact Calabi-Yau varieties

$$\left\{ xy = 1 + t_1 + t_2 + \frac{q}{c(q)^3 t_1 t_2} \,|\, (x,y) \in \mathbb{C}^2,\ (t_1, t_2) \in \mathbb{T}^2 \right\}.$$

The complex parameter $q \in \Delta^\times$ parametrizes the symplectic structure of $K_{\mathbb{P}^2}$. The authors provide conjectural evidence that the map

$$q \mapsto \check{q} := \frac{q}{c(q)^3},$$

[4]The SYZ conjecture describes how to construct a mirror for an open subvariety V of $K_{\mathbb{P}^2}$. In order to get a mirror to the entire space, that mirror needs to be deformed by 'quantum/instanton corrections'. These are encoded by open Gromov-Witten invariants that arise in the Lagrangian fibration predicted by SYZ for V. (More precisely, a slightly different fibration needs to be considered.)

provides an isomorphism between the Kähler moduli of $K_{\mathbb{P}^2}$ and the complex moduli of $Z_{\check{q}}$, thus being inverse to the mirror map

$$\check{q} \mapsto \exp(-I_2(\check{q})),$$

where $I_2(\check{q})$ is defined in the next chapter to be the logarithmic solution to the differential equation (A.1).

3.5 Relating two families by coordinate change

We end this chapter by describing how the families of (A.2) and (A.3) are related. Recall that, for $\check{q}, \phi \in \mathbb{C}^\times$, these two families were given by

$$B_\phi = \left\{0 = xy - \phi(x^3 + y^3 + 1) \,|\, (x, y) \in \mathbb{T}^2\right\},$$

and

$$M^0_{\check{q}} = \left\{0 = 1 + t_1 + t_2 + \frac{\check{q}}{t_1 t_2} \,|\, (t_1, t_2) \in \mathbb{T}^2\right\}.$$

Consider the following embeddings into projective space:

- Denote by $N_\phi \subset \mathbb{P}^2 \times \mathbb{C}^\times$ the family given by

$$XYZ - \phi(X^3 + Y^3 + Z^3) = 0. \tag{3.6}$$

- Then B_ϕ is the affine variety $N_\phi - \{XYZ = 0\}$, so that

$$B_\phi \subset \mathbb{T}^2 \times \mathbb{C}^\times,$$

$$B_\phi : \; xy - \phi(x^3 + y^3 + 1) = 0,$$

  with affine coordinates $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$.

- On the other hand, denote by $M_{\check{q}} \subset \mathbb{P}^2 \times \mathbb{C}$ the family given by

$$T_1^2 T_2 + T_1 T_2^2 + T_3^3 - \check{q} T_1 T_2 T_3 = 0.$$

- Then $M^0_{\check{q}}$ is the affine variety $M_{\check{q}} - \{T_1 T_2 T_3 = 0\}$. Choosing affine coordinates $t_1 = \frac{T_1}{T_3}$, $t_2 = \frac{T_2}{T_3}$, we get

$$M^0_{\check{q}} \subset \mathbb{T}^2 \times \mathbb{C}^\times,$$

$$M^0_t : \; t_1^2 t_2 + t_1 t_2^2 + 1 - \check{q} t_1 t_2 = 0.$$

Comparing the Weierstrass forms of each families, we obtain the following result.

Lemma 11. *For $\phi \in \mathbb{C}^\times$, $\phi^3 \neq 1/27$; and for $\check{q} \in \mathbb{C}^\times$, $\check{q}^3 \neq 27$, the families*

$$N_{\phi^3} \text{ and } M_{\check{q}^3},$$

parametrized by ϕ^3 and $\check{q}^3$, have isomorphic fibers via the identification

$$\check{q}^3 = \frac{-1}{\phi^3 - 1/27}.$$

More precisely, the isomorphism of the fibers is given by the projective change of variable

$$\begin{aligned}
X &= 2\sqrt{3}\, i\, T_1 + \left(3 + \sqrt{3}\, i\right) T_2 - \left(1 + \sqrt{3}\, i\right) \check{q}\, T_3, \\
Y &= \bar{X} = -2\sqrt{3}\, i\, T_1 + \left(3 - \sqrt{3}\, i\right) T_2 + \left(-1 + \sqrt{3}\, i\right) \check{q}\, T_3, \\
Z &= 6\, \phi\, \check{q}\, T_3;
\end{aligned}$$

or, alternatively, by the projective change of variable

$$
\begin{aligned}
T_1 &= -3\left(1+\sqrt{3}\,i\right)\,\phi\,\check{q}\,X + 3\left(-1+\sqrt{3}\,i\right)\,\phi\,\check{q}\,Y + 2\,\check{q}\,Z,\\
T_2 &= 6\,\phi\,\check{q}\,(X+Y) + 2\,\check{q}\,Z,\\
T_3 &= 6\,Z.
\end{aligned}
$$

Proof. By direct verification. □

We remove the fibers above $\phi^3 = 1/27$ and $\check{q}^3 = 27$, so that $M_{\check{q}}$ and N_ϕ have isomorphic fibers. Moreover, lemma 11 gives an explicit isomorphism between the families. Consider now the family N_ϕ to be embedded into $\mathbb{P}^2 \times \mathbb{C}^\times$, via the map

$$
M_{\check{q}} \to N_\phi \subset \mathbb{P}^2 \times \mathbb{C}^\times.
$$

We proceed to removing the divisor

$$
\{XYZ = 0\}\,,
$$

from both families. On N_ϕ, we get the above family B_ϕ with affine coordinates

$$
\begin{aligned}
x &= \frac{X}{Z},\\
y &= \frac{Y}{Z}.
\end{aligned}
$$

Denote by $A_{\check{q}} \subset M_{\check{q}}$ the resulting family. Since Z is never zero on $A_{\check{q}}$, T_3 is never zero either and we we can use affine coordinates

$$
\begin{aligned}
t_1 &= \frac{T_1}{T_3},\\
t_2 &= \frac{T_2}{T_3}.
\end{aligned}
$$

Lemma 12. *For $\check{q} \in \mathbb{C}^\times$, $\check{q}^3 \neq 27$ and affine coordinates t_1 and t_2, the fiber $A_{\check{q}}$ is obtained* *by removing the union of two lines given by*

$$3(t_1+t_2)^2 - 3(t_1+\check{q})(t_2+\check{q}) + 4\check{q}^2.$$

This corresponds to the union of the coordinate axes $xy = 0$ on a fiber of B_ϕ such that $\check{q}^3 = \frac{-1}{\phi^3 - 1/27}$.

Denote by ζ a primitive third root of unity and consider the affine coordinates x, y on

$$B_\phi \subset \mathbb{T}^2 \times \ \mathbb{C}^\times \setminus \left\{ \frac{1}{27}\,\zeta^k \,|\, k = 0,1,2 \right\},$$

as well as t_1, t_2 on

$$M^0_{\check{q}} \subset \mathbb{T}^2 \times \ \mathbb{C}^\times \setminus \left\{ 27\,\zeta^k \,|\, k = 0,1,2 \right\}.$$

For Γ_ϕ a 2-cycle in $\mathbb{T}^2$ with its boundary supported in B_ϕ, Takahashi in [11] considers the relative periods

$$I(\phi) := \int_{\Gamma_\phi} \frac{\mathrm{d}\,x \wedge \mathrm{d}\,y}{xy},$$

and shows that $I(\phi^3)$ satisfies the A-hypergeometric differential equation (A.1) associated to $\mathbb{P}^2$. It follows that $I(\phi^3)$ calculates the local Gromov-Witten invariants of $\mathbb{P}^2$. Using lemma 11 to move the above integral to the family $M^0_{\check{q}}$, we get the following result:

Proposition 13. *Denote by $\Delta_{\check{q}}$ a 2-cycle in $\mathbb{T}^2$ with boundary supported in $M^0_{\check{q}}$. Consider the relative periods*

$$\Psi(\check{q}) := \int_{\Delta_{\check{q}}} \frac{3\sqrt{3}\,i\,\mathrm{d}\,t_1\,\mathrm{d}\,t_2}{3(t_1+t_2)^2 - 3(t_1+\check{q})(t_2+\check{q}) + 4\check{q}^2}.$$

Then the functions $\Psi(\frac{-1}{\phi^3-1/27})$, for $\phi \in \mathbb{C}^\times$, $\phi^3 \neq 0$, and for various 2-cycles $\Delta_{\check{q}}$, correspond to solutions of A.1.

Conclusion

Starting from the family $M_{\check{q}}$, we can either remove the divisor $\{T_1T_2T_3 = 0\}$ and consider relative periods of $\frac{\mathrm{d}\,t_1\,\mathrm{d}\,t_2}{t_1t_2}$ and get the result on relative periods from [12]. Or we can remove the divisor

$$\left\{\left(3(T_1+T_2)^2 - 3(T_1+\check{q})(T_2+\check{q}) + 4\check{q}^2\right)T_3 = 0\right\},$$

and consider the above relative periods to get the same result.

Chapter 4

Relative mirror symmetry

The goal of this chapter is to introduce relative mirror symmetry. We start by stating the theorem and then verify it separately for all toric Del Pezzo surfaces. In the appendix, we describe the Gromov-Witten invariant calculations and the solutions to the system of A-hypergeometric differential equation more explicitely in the cases of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up in one point. Recall the notation $\Delta^\times$ for the closed punctured unit disk.

Definition. For a Landau-Ginzburg model $(\mathbb{T}^2, W)$ and $x \in \mathbb{C}^\times$, let $\Gamma = \Gamma(x)$ be a coherent choice of relative 2-cycles of the fibers of W, i.e.,

$$\Gamma(x) \in \mathrm{H}_2(\mathbb{T}^2, W^{-1}(x); \mathbb{Z}).$$

Consider moreover the integrals

$$f_\Gamma^W(x) := \int_{\Gamma(x)} \omega_0,$$

where $\omega_0 = \left[\frac{\mathrm{d}\,t_1\,\mathrm{d}\,t_2}{t_1 t_2}\right] \in \mathrm{H}^2(\mathbb{T}^2, W^{-1}(x); \mathbb{Z})$. Then the *relative periods* associated to W and Γ are defined, for $x \in \Delta^\times$, by

$$I_\Gamma^W(x) := f_\Gamma^W(1/x).$$

Theorem 14. *Let S be a toric Del Pezzo surface and let D be a smooth effective anti-canonical divisor on it. Denote by $(\mathbb{T}^2, W^S)$ its mirror Landau-Ginzburg model and by r the dimension of the Kähler moduli of S. Then there is an $(r-1)$-dimensional family W_ϕ^S of deformations of the superpotential W^S, parametrized by $\phi \in (\Delta^\times)^{r-1}$, with the following* *property. For coherent choices of relative 2-cycles*

$$\Gamma_\phi = \Gamma_\phi(x) \in \mathrm{H}_2(\mathbb{T}^2, (W_\phi^S)^{-1}(x); \mathbb{Z}),$$

consider the relative periods

$$I_{\Gamma_\phi}^{W_\phi^S}(x),$$

where $(x, \phi) \in \Delta^\times \times (\Delta^\times)^{r-1}$. *Then these periods, via change of variable and analytic continuation, calculate the genus 0 relative Gromov-Witten invariants of maximal tangency of* (S, D).

Remark. Denote by K_S the total space of the canonical bundle on S. Theorem 1 yields a correspondence between the genus 0 Gromov-Witten invariants of K_S and the relative Gromov-Witten invariants of the conjecture. The solutions to the A-hypergeometric system of differential equations associated to S calculate the genus 0 Gromov-Witten invariants of K_S. Therefore, it is enough to show that, via change of variable and analytic continuation, the above relative periods yield the solutions to the A-hypergeometric system of differential equations.

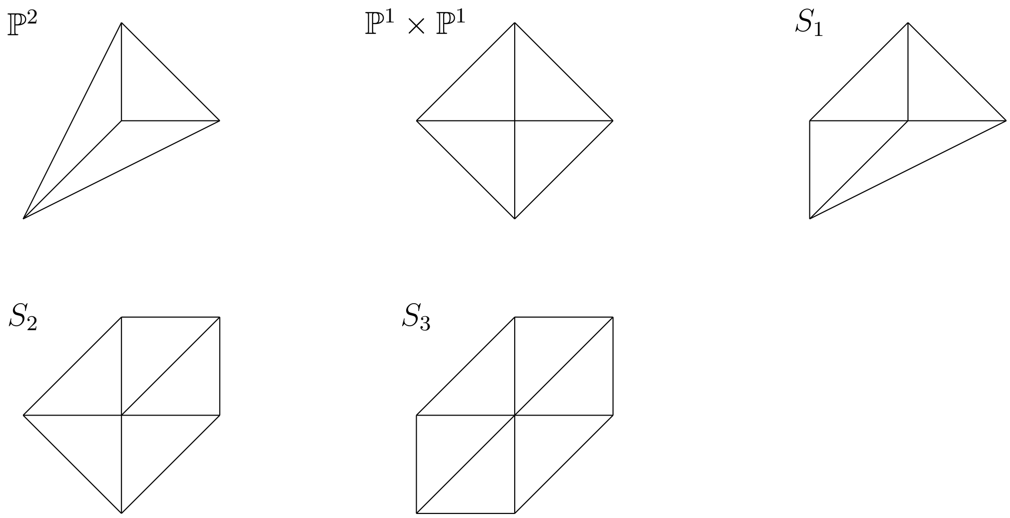
In the next section, we review some aspects of local mirror symmetry for toric Del Pezzo surfaces. In the sections thereafter, we prove theorem 14 separately for each toric Del Pezzo surface. Denote by S_i, for $i = 1, 2$ and 3 the toric Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in i general points.

4.1 Reflexive polytopes and local mirror symmetry

We start by reviewing the mirror constructions by Batyrev of [10], as well as parts of the local mirror symmetry calculations of Chiang-Klemm-Yau-Zaslow in [8]. For a toric Del Pezzo surface S, denote by Δ_S the 2-dimensional integral reflexive polytope such that

$$S = \mathbb{P}_{\Delta_S}.$$

Then these polytopes are as follows. Note that the number of 2-simplices equals the number of independent solutions to the associated system of A-hypergeometric differential equations.



Starting from Δ_S, Batyrev in [10] describes how to construct a family of affine elliptic curves whose mirror family is induced by the dual Δ_S^*. For the periods of the family induced by Δ_S, Chiang-Klemm-Yau-Zaslow in [8] derive associated systems of Picard-Fuchs equations. The authors moreover show how the solutions to these equations yield, via change of variable and analytic continuation, the genus 0 Gromov-Witten invariants of K_S. We proceed to describing how these families of affine elliptic curves are constructed. Denote by m the number of vertices of Δ_S. We label the vertices counterclockwise $v_1, \ldots, v_r$, starting with the vertex at $(1,0)$. Let moreover $v_0 = (0,0)$ and set

$$\overline{v}_i = (1, v_i),\ i = 0, \ldots, m.$$

Denote by r the dimension of the Kähler moduli of S. For $j = 1, \ldots, r$, consider an integral basis of linear relations $\left\{ l^j = (l^j_0, \ldots, l^j_m) \right\}$ among the $\overline{v}_i$. That is, such that

$$\sum_{i=0}^{r} l^j_i \overline{v}_i = 0.$$

In addition, the l^j are required to span the Mori cone of $\mathbb{P}_{\Delta_S}$. This condition uniquely determines the l^j for all S except for S_3. The authors obtain the following relations:

1. For $\mathbb{P}^2$:
$$l^1 = (-3, 1, 1, 1).$$

2. For $\mathbb{P}^1 \times \mathbb{P}^1$:
$$l^1 = (-2, 1, 0, 1, 0),\, l^2 = (-2, 0, 1, 0, 1).$$

3. For S_1:
$$l^1 = (-2, 1, 0, 1, 0),\, l^2 = (-1, 0, 1, -1, 1).$$

4. For S_2:
$$\begin{aligned} &l^1 = (-1, 1, -1, 1, 0, 0),\, l^2 = (-1, 0, 1, -1, 1, 0), \\ &l^3 = (-1, -1, 1, 0, 0, 1). \end{aligned}$$

5. For S_3 we use the following choice:
$$\begin{aligned} &l^1 = (-1, 1, -1, 1, 0, 0, 0), \quad l^2 = (-1, 0, 1, -1, 1, 0, 0), \\ &l^3 = (-1, -1, 2, -1, 0, 1, 0),\, l^4 = (-1, -1, 1, 0, 0, 0, 1). \end{aligned}$$

The authors furthermore describe how to obtain from this data the system of A-hypergeometric differential equations associated to S. We do not recall it, as it is not relevant here. Nevertheless:

Theorem 15. *(Chiang-Klemm-Yau-Zaslow in [8]) The solutions to the* A*-hypergeometric system of differential equations associated to S compute, via change of variable and analytic continuation, the genus 0 Gromov-Witten invariants of K_S.*

Following [19] and [10], let $\mathbb{L}(\Delta_S)$ be the space of Laurent polynomials with Newton polytope Δ_S. Write $v_i = (v_i^1, v_i^2)$. Then $F_a \in \mathbb{L}(\Delta_S)$ if

$$F_a(t_1, t_2) = \sum_{i=0}^{m} a_i t_1^{v_i^1} t_2^{v_i^2},$$

where $(t_1, t_2) \in \mathbb{T}^2$ and $a = (a_0, \ldots, a_r)$. In the case of $\mathbb{P}^2$ for instance, this yields

$$F_a(t_1, t_2) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}.$$

Moreover, F_a is defined to be Δ_S-*regular*, written $F_a \in \mathbb{L}_{reg}(\Delta_S)$, if in addition the following condition is satisfied. For any $0 < m \leq 2$ and any m-dimensional face $\Delta' \subset \Delta_S$, the equations

$$\begin{aligned} F_a^{\Delta'} &= 0, \\ \frac{\partial F_a^{\Delta'}}{\partial t_1} &= 0, \\ \frac{\partial F_a^{\Delta'}}{\partial t_2} &= 0, \end{aligned}$$

have no common solutions. Here,

$$F_a^{\Delta'} := \sum_{v_i \in \Delta'} a_i t_1^{v_i^1} t_2^{v_i^2}.$$

Denote by $\mathcal{Z}_S \to \mathbb{L}_{reg}(\Delta_S)$ the family of affine elliptic curves with fibers given by

$$\{F_a(t_1, t_2) = 0\} \subset \mathbb{T}^2.$$

Consider moreover the $\mathbb{T}^3$-action,

$$F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2),$$

where $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{T}^3$, as well as the GIT-quotient

$$\mathcal{Z}_S / \mathbb{T}^3 \to \mathcal{M}(\Delta_S) := \mathbb{L}_{reg}(\Delta_S) / \mathbb{T}^3.$$

Then each of l^j determines a $\mathbb{T}^3$-invariant complex structure coordinate

$$z_j := \prod_{i=0}^{r} a_i^{l_i^j},$$

for S. This yields:

1. For $\mathbb{P}^2$:
$$z = \frac{a_1 a_2 a_3}{a_0^3}.$$
2. For $\mathbb{P}^1 \times \mathbb{P}^1$:
$$z_1 = \frac{a_1 a_3}{a_0^2}, z_2 = \frac{a_2 a_4}{a_0^2}.$$
3. For S_1:
$$z_1 = \frac{a_1 a_3}{a_0^2}, z_2 = \frac{a_2 a_4}{a_0 a_3}.$$
4. For S_2:
$$z_1 = \frac{a_1 a_3}{a_0 a_2},\ z_2 = \frac{a_2 a_4}{a_0 a_3},\ z_3 = \frac{a_2 a_5}{a_0 a_1}.$$
5. For S_3:
$$z_1 = \frac{a_1 a_3}{a_0 a_2},\ z_2 = \frac{a_2 a_4}{a_0 a_3},\ z_3 = \frac{a_2^2 a_5}{a_0 a_1 a_3},\ z_4 = \frac{a_2 a_6}{a_0 a_1}.$$

Requiring that $z \in (\Delta^\times)^r$, for $z = (z_1, \ldots, z_r)$, determines a subset of $\mathcal{M}(\Delta_S)$. Denote by $M_z^S \to (\Delta^\times)^r$ the restriction to this subset. This yields the following list.

$$\begin{aligned}
M_z^{\mathbb{P}^2} &= \left\{0 = 1 + t_1 + t_2 + \frac{z}{t_1 t_2}\right\}, \\
M_z^{\mathbb{P}^1 \times \mathbb{P}^1} &= \left\{0 = 1 + t_1 + t_2 + \frac{z_1}{t_1} + \frac{z_2}{t_2}\right\}, \\
M_z^{S_1} &= \left\{0 = 1 + t_1 + z_1\left(t_2 + \frac{1}{t_1}\right) + \frac{z_2}{t_1 t_2}\right\}, \\
M_z^{S_2} &= \left\{0 = 1 + z_1\left(t_1 + t_1 t_2 + t_2\right) + \frac{z_2}{t_1} + \frac{z_3}{t_2}\right\}, \\
M_z^{S_3} &= \left\{0 = 1 + z_1\left(t_1 + t_1 t_2 + t_2\right) + \frac{z_2}{t_1} + \frac{z_3}{t_1 t_2} + \frac{z_4}{t_2}\right\}.
\end{aligned}$$

We will need the following result by Stienstra. The theorem is initially stated for the family $\mathcal{Z}$, but it translates readily to the family M_z^S. Note that the result applies more generally.

Theorem 16. *(Stienstra, in [12], see also [19]) Let S be a toric Del Pezzo surface S and consider the relative cohomology class $\omega_0 = \left[\frac{\mathrm{d}\,t_1\,\mathrm{d}\,t_2}{t_1 t_2}\right] \in \mathrm{H}^2(\mathbb{T}^2, M_z^S; \mathbb{Z})$. For coherent choices of relative 2-cycles $\Gamma_z \in \mathrm{H}_2(\mathbb{T}^2, M_z^S; \mathbb{Z})$, the period integrals*

$$\mathcal{P}_\Gamma^S(z) := \int_{\Gamma_z} \omega_0,$$

are in bijection with the solutions to the A*-hypergeometric system of differential equations associated to S.*

Combining this with theorem 1 readily yields:

Corollary 17. *The period integrals $\mathcal{P}_\Gamma^S$, via change of variable and analytic continuation, compute the genus 0 relative Gromov-Witten invariants of (S, D), where D is a smooth effective anti-canonical divisor on S.*

4.2 Proof of relative mirror symmetry

Denote by $(\mathbb{T}^2, W^S)$ the Landau-Ginzburg model mirror to the toric Del Pezzo surface S. Various non-zero values can be taken for the coefficients, so we set them all to 1. Then:

$$\begin{aligned}
W^{\mathbb{P}^2}(t_1,t_2) &= 1 + t_1 + t_2 + \frac{1}{t_1 t_2},\\
W^{\mathbb{P}^1\times\mathbb{P}^1}(t_1,t_2) &= 1 + t_1 + t_2 + \frac{1}{t_1} + \frac{1}{t_2},\\
W^{S_1}(t_1,t_2) &= 1 + t_1 + t_2 + \frac{1}{t_1} + \frac{1}{t_1 t_2},\\
W^{S_2}(t_1,t_2) &= 1 + t_1 + t_1 t_2 + t_2 + \frac{1}{t_1} + \frac{1}{t_2},\\
W^{S_3}(t_1,t_2) &= 1 + t_1 + t_1 t_2 + t_2 + \frac{1}{t_1} + \frac{1}{t_1 t_2} + \frac{1}{t_2}.
\end{aligned}$$

In light of corollary 17, it suffices to show that the relative periods of $(\mathbb{T}^2, W^S)$, via change of variable and analytic continuation, are in bijection with the periods $\mathcal{P}_\Gamma^S$. We prove theorem 14 for each surface separately.

Proof for $\mathbb{P}^2$

Denote by x a third root of the complex parameter z. The family

$$M_z^{\mathbb{P}^2} = \left\{0 = 1 + t_1 + t_2 + \frac{z}{t_1 t_2}\right\},$$

after the coordinate change

$$t_i \mapsto x\, t_i,$$

is described as the fibers

$$W^{\mathbb{P}^2}(t_1, t_2) = -1/x.$$

Note that this coordinate change does not change ω_0. Then

$$\mathcal{P}_\Gamma^{\mathbb{P}^2}(z) = I_\Gamma^{W^{\mathbb{P}^2}}(-x^3),$$

that is, the periods $I_\Gamma^{W^{\mathbb{P}^2}}(x)$, after the change of variables $z = -x^3$, are in bijection with the solutions to the A-hypergeometric differential equation associated to $\mathbb{P}^2$. This proves theorem 14 for $\mathbb{P}^2$.

Proof for $\mathbb{P}^1 \times \mathbb{P}^1$

Denote by $z = (z_1, z_2) \in (\Delta^\times)^2$ the complex parameter for $\mathbb{P}^1 \times \mathbb{P}^1$. For $i = 1, 2$ and for a choice of square-roots ϕ_i of z_i, consider the coordinate change

$$t_i \rightsquigarrow \phi_i t_i.$$

Then the family $M_z^{\mathbb{P}^1\times\mathbb{P}^1}$ is described as

$$\left\{0 = 1 + \phi_1\left(t_1 + \frac{1}{t_1} + \frac{\phi_2}{\phi_1}\left(t_2 + \frac{1}{t_2}\right)\right)\right\}.$$

Define, for $\psi \in \mathbb{C}^\times$,

$$W_\psi^{\mathbb{P}^1\times\mathbb{P}^1}(t_1,t_2) := t_1 + \frac{1}{t_1} + \psi\left(t_2 + \frac{1}{t_2}\right),$$

so that $M_z^{\mathbb{P}^1\times\mathbb{P}^1}$ is given by

$$\left\{0 = 1 + \phi_1 W_{\phi_2/\phi_1}^{\mathbb{P}^1\times\mathbb{P}^1}(t_1,t_2)\right\}.$$

For a coherent choice of relative 2-cycles $\Gamma_\psi(z) \in \mathrm{H}_2(\mathbb{T}^2, W_\psi^{-1}(z);\mathbb{Z})$, relative periods are

$$\begin{aligned} I_\Gamma^{W_\psi^{\mathbb{P}^1\times\mathbb{P}^1}}(-,\psi) : \Delta^\times &\to \mathbb{C}, \\ z &\mapsto \int_{\Gamma_\psi(1/z)} \omega_o. \end{aligned}$$

Then

$$\mathcal{P}_\Gamma^{\mathbb{P}^1\times\mathbb{P}^1}(z) = I_\Gamma^{W_\psi^{\mathbb{P}^1\times\mathbb{P}^1}}\left(-\phi_1^2, (\phi_2/\phi_1)^2\right).$$

Therefore the relative periods $I_\Gamma^{W^{\mathbb{P}^1\times\mathbb{P}^1}}$, via change of variable and analytic continuation, correspond to the periods $\mathcal{P}_\Gamma^{\mathbb{P}^1\times\mathbb{P}^1}$, proving theorem 14 for $\mathbb{P}^1\times\mathbb{P}^1$.

Proof for S_1

Denote by $z = (z_1,z_2) \in \Delta^\times \times \Delta^\times$ the complex parameter for S_1. By choosing a square root ϕ_1 of z_1 and by considering the change of coordinates

$$\begin{aligned} t_1 &\rightsquigarrow \phi_1 t_1, \\ t_2 &\rightsquigarrow \frac{1}{\phi_1} t_2, \end{aligned}$$

the family $M_z^{S_1}$ is described by

$$\left\{0 = 1 + \phi_1 \left(t_1 + t_2 + \frac{1}{t_1} + \frac{z_2}{\phi_1}\frac{1}{t_1 t_2}\right)\right\}.$$

Equivalently, it is given by the fibers of

$$W^{S_1}_{z_2/\phi_1}(t_1, t_2) = -\frac{1}{\phi_1},$$

where

$$W^{S_1}_{\psi}(t_1, t_2) := t_1 + t_2 + \frac{1}{t_1} + \frac{\psi}{t_1 t_2}.$$

Let $\Gamma_\psi(z) \in \mathrm{H}_2(\mathbb{T}^2, W_\psi^{-1}(z); \mathbb{Z})$ be a coherent choice of relative 2-cycles. Relative periods for S_1 are

$$\begin{aligned} I_\Gamma^{W_\psi^{S_1}}(-, \psi) : \Delta^\times &\to \mathbb{C}, \\ z &\mapsto \int_{\Gamma_\psi(1/z)} \omega_o. \end{aligned}$$

Note that

$$\mathcal{P}_\Gamma^{S_1}(z) = I_\Gamma^{W_\psi^{S_1}}\left(-\phi_1^2, z_2/\phi_1^2\right),$$

so that, via change of variable and analytic continuation, the relative periods $I_\Gamma^{W_\psi^{S_1}}$ correspond to the solutions to the system of A-hypergeometric differential equations associated to S_1, yielding theorem 14 for S_1.

Proof for S_2

The family $M_z^{S_2}$, with complex parameter $z = (z_1, z_2, z_3)$, is given by

$$\left\{0 = 1 + z_1 \left(t_1 + t_1 t_2 + t_2\right) + \frac{z_2}{t_1} + \frac{z_3}{t_2}\right\},$$

which in turn is described as

$$\left\{0 = 1 + z_1\left((t_1 + t_1t_2 + t_2) + \frac{z_2}{z_1}\frac{1}{t_1} + \frac{z_3}{z_1}\frac{1}{t_2}\right)\right\}.$$

Setting

$$W^{S_2}_{(\psi_1,\psi_2)}(t_1,t_2) := t_1 + t_1t_2 + t_2 + \frac{\psi_1}{t_1} + \frac{\psi_2}{t_2},$$

it is given by the fibers of

$$W^{S_2}_{(z_2/z_1,z_3/z_1)}(t_1,t_2) = -\frac{1}{z_1}.$$

Hence

$$\mathcal{P}^{S_2}_{\Gamma}(z) = I_{\Gamma}^{W^{S_2}_{(\psi_1,\psi_2)}}\left(-z_1, z_2/z_1, z_3/z_1\right),$$

and the periods $I_{\Gamma}^{W^{S_2}_{(\psi_1,\psi_2)}}$ have the desired property.

Proof for S_3

Finally, denote by $z = (z_1, z_2, z_2, z_4)$ the complex parameter for the family $M^{S_3}_z$ given by

$$\left\{0 = 1 + z_1\left(t_1 + t_1t_2 + t_2\right) + \frac{z_2}{t_1} + \frac{z_3}{t_1t_2} + \frac{z_4}{t_2}\right\},$$

or, equivalently by

$$\left\{0 = 1 + z_1\left((t_1 + t_1t_2 + t_2) + \frac{z_2}{z_1}\frac{1}{t_1} + \frac{z_3}{z_1}\frac{1}{t_1t_2} + \frac{z_4}{z_1}\frac{1}{t_2}\right)\right\}.$$

Now,

$$W^{S_3}_{(\psi_1,\psi_2,\psi_3)}(t_1,t_2) := t_1 + t_1t_2 + t_2 + \frac{\psi_1}{t_1} + \frac{\psi_2}{t_1t_2} + \frac{\psi_3}{t_2},$$

yields the description of $M^{S_3}_z$ as the fibers of

$$W^{S_3}_{(z_2/z_1,z_3/z_1,z_4/z_1)}(t_1,t_2) = -\frac{1}{z_1}.$$

Therefore

$$\mathcal{P}_\Gamma^{S_2}(z) = I_\Gamma^{W^{S_2}_{(\psi_1,\psi_2,\psi_3,\psi_4)}}\left(-z_1, z_2/z_1, z_3/z_1, z_4/z_1\right),$$

which yields relative mirror symmetry for S_3, and finishes the proof of theorem 14.

Chapter 5

A prediction of homological mirror symmetry

In this chapter, we are interested in proving a prediction of the homological mirror symmetry conjecture for the open complement, yielding evidence for this conjecture in that setting. We perform the B-model side calculation for every Del Pezzo surface. The calculation of the A-model side in the case of $\mathbb{P}^2$, which matches our calculation, was done by Nguyen-Pomerleano and will be published in a forthcoming paper.

Set up

Denote by S a Del Pezzo surface and by D a smooth effective anti-canonical divisor on it, i.e., in the present setting, an elliptic curve. We additionally denote by S_k the Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in $0 \leq k \leq 8$ generic points. Note that every Del Pezzo surface but $\mathbb{P}^1 \times \mathbb{P}^1$ is obtained as such. If $k \leq 3$, then S_k is toric. Auroux-Katzarkov-Orlov in [25] construct the Landau-Ginzburg model

$$W_k : M_k \to \mathbb{A}^1$$

mirror to S_k. For S toric, we considered in the preceding chapter the Landau-Ginzburg model mirror to (S, T), where T is the toric divisor. In this chapter though, we consider the mirror Landau-Ginzburg model with respect of the smoothing of T to D. According to the construction of [25], $W_k : M_k \to \mathbb{A}^1$ is an elliptic fibration with $k+3$ nodal fibers. In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, it is an elliptic fibration with 4 nodal fibers. For toric S, these elliptic fibrations coincide with the fiber-wise compactification the superpotentials considered in the previous chapter. We will proceed with the following abbreviations.

Notation. Denote by $W : M \to \mathbb{A}^1$ the Landau-Ginzburg model mirror to the Del Pezzo surface S. Then M is an elliptic fibration with r nodal fibers, where $3 \leq r \leq 11$.

Homological mirror symmetry for S (at least one direction) states that the derived Fukaya category of S should be equivalent to the category of matrix factorizations of (M, W):

$$\mathrm{MF}(M, W) \simeq \mathcal{F}(S).$$

Taking Hochschild cohomology of both categories yields that the Jacobian ring of W is isomorphic to the quantum cohomology of S:

$$\mathrm{Jac}(W) \cong \mathrm{Q\,H}^*(S).$$

We need to understand how this statement is modified when the divisor D is removed on S. The mirror operation consists in removing the superpotential. Thus, conjecturally, M is the mirror to the open complement $S - D$. The homological mirror symmetry conjecture for $S - D$ states that the wrapped Fukaya category on $S - D$ ought to be equivalent to the bounded derived category of coherent sheaves on M:

$$\mathcal{D}^b \operatorname{Coh}(M) \simeq \mathcal{WF}(S - D).$$

Moreover, taking Hochschild cohomology on both sides yields the prediction that the Hochschild cohomology of M is isomorphic to the symplectic cohomology of $S - D$:

$$\mathrm{H\,H}^*(M) \cong \mathrm{S\,H}^*(S - D).$$

We aim in the present chapter at investigating this statement.

The Hochschild cohomology of M

Since M is Calabi-Yau, its Hochschild cohomology takes a particularly nice form. Denote by Ω_M the cotangent bundle on M. Denote by $\mathrm{H}^i(\mathcal{O}_M)$, respectively by $\mathrm{H}^i(\Omega_M)$ the sheaf cohomology groups $\mathrm{H}^i(M, \mathcal{O}_M)$, respectively $\mathrm{H}^i(M, \Omega_M)$. Denote moreover by $\mathrm{H}\,\mathrm{H}^i(M)$ the Hochschild cohomology groups of M. Then, $\mathrm{H}\,\mathrm{H}^i(M) = 0$ when $i < 0$ or $i > 3$, and

$$\begin{aligned}
\mathrm{H}\,\mathrm{H}^0(M) &= \mathrm{H}^0(\mathcal{O}_M),\\
\mathrm{H}\,\mathrm{H}^1(M) &= \mathrm{H}^1(\mathcal{O}_M) \oplus \mathrm{H}^0(\Omega_M),\\
\mathrm{H}\,\mathrm{H}^2(M) &= \mathrm{H}^0(\mathcal{O}_M) \oplus \mathrm{H}^1(\Omega_M),\\
\mathrm{H}\,\mathrm{H}^3(M) &= \mathrm{H}^1(\mathcal{O}_M).
\end{aligned}$$

Indeed, as vector spaces, we have the equality

$$\mathrm{H}\,\mathrm{H}_*(M) = \oplus_{p=0}^{2} \mathbf{R}\,\Gamma(M, \wedge^p \Omega_M).$$

This is the direct sum of the cohomology groups of the sheaves $\mathcal{O}_M$, Ω_M and $\wedge^2\Omega_M$. Moreover, M is Calabi-Yau and thus $\omega_M = \wedge^2\Omega_M = \mathcal{O}_M$. In degree i,

$$\mathrm{H}\,\mathrm{H}_i(M) = \oplus_{q-p=i}\, \mathrm{H}^q(M, \wedge^p \Omega_M).$$

For $q > 1$, the groups $\mathrm{H}^q(\mathcal{O}_M)$ and $\mathrm{H}^q(\mathcal{O}_M)$ are zero as M is a fibration over $\mathbb{A}^1$ of relative dimension 1. It follows that

$$\begin{aligned}
\mathrm{H}\,\mathrm{H}_{-2}(M) &= \mathrm{H}^0(\mathcal{O}_M),\\
\mathrm{H}\,\mathrm{H}_{-1}(M) &= \mathrm{H}^1(\mathcal{O}_M) \oplus \mathrm{H}^0(\Omega_M),\\
\mathrm{H}\,\mathrm{H}_0(M) &= \mathrm{H}^0(\mathcal{O}_M) \oplus \mathrm{H}^1(\Omega_M),\\
\mathrm{H}\,\mathrm{H}_1(M) &= \mathrm{H}^1(\mathcal{O}_M),\\
\mathrm{H}\,\mathrm{H}_i(M) &= 0 \text{ for } i > 1 \text{ or } i < -2.
\end{aligned}$$

Note that the ring structure of $\mathrm{H\,H}^*(M)$ is not apparent in this description. Moreover, since M is Calabi-Yau and of dimension 2, it follows that, after a translation of the degree by 2, $\mathrm{H\,H}_*$ is isomorphic to $\mathrm{H\,H}^*$. Hence the above description. In this chapter we compute the Hochschild cohomology groups $\mathrm{H\,H}^*(M)$, as modules over the polynomial ring.

Two short exact sequences and identities

Denote by $\Omega_{M/\mathbb{A}^1}$ be the sheaf of relative differentials, forming an exact sequence

$$W^*\Omega_{\mathbb{A}^1} \to \Omega_M \to \Omega_{M/\mathbb{A}^1} \to 0.$$

Now, $\Omega_{\mathbb{A}^1}$ is trivial, so that $W^*\Omega_{\mathbb{A}^1} \cong \mathcal{O}_M$. Moreover, since M is smooth, the above sequence is exact:

$$0 \to \mathcal{O}_M \to \Omega_M \to \Omega_{M/\mathbb{A}^1} \to 0. \tag{5.1}$$

Denote by $\omega_{M/\mathbb{A}^1}$ the relative dualizing sheaf, yielding a short exact sequence

$$0 \to \Omega_{M/\mathbb{A}^1} \to \omega_{M/\mathbb{A}^1} \to \prod_{i=1}^{r} \mathcal{O}_{p_i} \to 0, \tag{5.2}$$

where r is the number of nodal fibers and where the p_i are the ramification points of W. As mentioned above, $\omega_M \cong \mathcal{O}_M$. Moreover, $\omega_{M/\mathbb{A}^1} \cong \omega_M \otimes W^*(\omega_{\mathbb{A}^1}^\vee)$, so that $\omega_{M/\mathbb{A}^1}$ is trivial as well. Then, $R^2\pi_*(\mathcal{O}_M)$ is zero. Thus, by the theorem of cohomology and base change, for all $s \in \mathbb{A}^1$, the natural morphism

$$R^1W_*(\mathcal{O}_M) \otimes_{\mathbb{C}} \kappa(s) \to \mathrm{H}^1(M_s, \mathcal{O}_s),$$

is an isomorphism. As all the curves in the family are of arithmetic genus 1 (the family is flat), $\mathrm{H}^1(M_s, O_s) \cong \mathbb{C}$ and therefore $R^1W_*(O_M) \cong \mathrm{H}^1(M, O_M)^\sim \cong (\mathbb{C}[t])^\sim$. By the same reasoning, $\mathrm{H}^0(M, O_M)$ is free of rank 1 over $\mathbb{C}[t]$. Consequently, the non-zero cohomology groups of ω_M and $\omega_{M/\mathbb{A}^1}$ are free of rank 1 over $\mathbb{C}[t]$ as well.

Cohomology of $\Omega_{M/\mathbb{A}^1}$

The short exact sequence of (5.2) turns into a long exact sequence

$$0 \to \mathrm{H}^0(\Omega_{M/\mathbb{A}^1}) \to \mathrm{H}^0(\omega_{M/\mathbb{A}^1}) \cong \mathbb{C}[t] \to \mathbb{C}^r \to$$

$$\to \mathrm{H}^1(\Omega_{M/\mathbb{A}^1}) \to \mathrm{H}^1(\omega_{M/\mathbb{A}^1}) \cong \mathbb{C}[t] \to 0.$$

For principal ideal domains, submodules of free modules are free and thus $\mathrm{H}^0(\Omega_{M/\mathbb{A}^1}) \cong \mathbb{C}[t]$. We argue that the sequence

$$0 \to \mathrm{H}^0(\Omega_{M/\mathbb{A}^1}) \cong \mathbb{C}[t] \to \mathrm{H}^0(\omega_{M/\mathbb{A}^1}) \cong \mathbb{C}[t] \to \mathbb{C}^r \to 0$$

is exact: Elements of $\mathrm{H}^0(\omega_{M/\mathbb{A}^1})$ correspond to the global sections of $R^0W_*(\omega_{M/\mathbb{A}^1})$. As $\omega_{M/\mathbb{A}^1} \cong \mathcal{O}_M$, $R^0W_*(\omega_{M/\mathbb{A}^1}) \cong R^1W_*(O_M)$ is locally free of rank 1 as well. It follows that its global sections correspond to functions on $\mathbb{A}^1$. Moreover, the map

$$\mathrm{H}^0(\omega_{M/\mathbb{A}^1}) \to \mathrm{H}^0(\prod \mathcal{O}_{p_i}),$$

corresponds to evaluating a function at the $W(p_i)$'s. Since a polynomial can be chosen to take on any value on any number of chosen points, this map is surjective. Hence $\mathrm{H}^1(\Omega_{M/\mathbb{A}^1}) \cong \mathbb{C}[t]$.

Cohomology of Ω_M

Recall the short exact sequence of (5.1):

$$0 \to \mathcal{O}_M \to \Omega_M \to \Omega_{M/\mathbb{A}^1} \to 0.$$

Since Ω_M is locally free, but not $\Omega_{M/\mathbb{A}^1}$, it follows that Ω_M is not the trivial extension. Hence the boundary map

$$\mathbb{C}[t] \cong \mathrm{H}^0(\Omega_{M/\mathbb{A}^1}) \to \mathrm{H}^1(\mathcal{O}_M) \cong \mathbb{C}[t],$$

is non-zero. Since it is a map of $\mathbb{C}[t]$-modules, it is therefore injective. It follows that (5.1) induces the long exact sequence

$$0 \to \mathrm{H}^0(\mathcal{O}_M) \cong \mathbb{C}[t] \to \mathrm{H}^0(\Omega_M) \xrightarrow{0} \mathrm{H}^0(\Omega_{M/\mathbb{A}^1}) \cong \mathbb{C}[t] \to$$
$$\to \mathrm{H}^1(\mathcal{O}_M) \cong \mathbb{C}[t] \to \mathrm{H}^1(\Omega_M) \to \mathrm{H}^1(\Omega_{M/\mathbb{A}^1}) \cong \mathbb{C}[t] \to 0,$$

so that

$$\mathrm{H}^0(\Omega_M) \cong \mathbb{C}[t].$$

Now, by Serre duality in families,

$$\mathrm{H}^1(\Omega_M) \cong \mathrm{H}^0(\Omega_M^\vee \otimes \omega_{M/\mathbb{A}^1})^\vee = \mathrm{H}^0(\Omega_M^\vee)^\vee.$$

Finally, as $\Omega_M^\vee$ is locally free, $\mathrm{H}^0(\Omega_M^\vee)$ has no torsion. Thus the map

$$\mathrm{H}^1(\mathcal{O}_M) \to \mathrm{H}^1(\Omega_M),$$

is trivial,

$$\mathrm{H}^1(\Omega_M) \to \mathrm{H}^1(\Omega_{M/\mathbb{A}^1}),$$

is an isomorphism and

$$\mathrm{H}^1(\Omega_M) \cong \mathbb{C}[t].$$

Conclusion

We end by assembling the above cohomology groups together.

Theorem 18. *Let S be a Del Pezzo surface and denote by M its mirror, as constructed in [25]. As $\mathbb{C}[t]$-modules, the Hochschild cohomology groups $\mathrm{H\,H}^0(M)$ and $\mathrm{H\,H}^3(M)$ are free of rank 1, $\mathrm{H\,H}^1(M)$ and $\mathrm{H\,H}^2(M)$ are free of rank 2, and $\mathrm{H\,H}^i(M) = 0$ for $i < 0$ or $i \geq 4$.*

Appendix A

Local mirror symmetry for $\mathbb{P}^2$

In [1], Gathmann describes how mirror symmetry calculates the genus 0 relative GW invariants of maximal tangency of $\mathbb{P}^2$. This result, under a somewhat different form, is stated in corollary 20. We start by recalling some notions of the preceding chapters. Denote by θ_z the logarithmic differential $z\frac{\partial}{\partial z}$. For holomorphic functions

$$f:\ \Delta^\times \to \mathbb{C},$$

the A-hypergeometric differential equation associated to $\mathbb{P}^2$ is

$$\mathcal{L}\, f = 0, \tag{A.1}$$

where

$$\mathcal{L} = \theta_z^3 + 3z\theta_z(3\theta_z + 1)(3\theta_z + 2).$$

We start by recalling the following two families of affine elliptic curves. For $\phi \in \mathbb{C}^\times$, in [8] and [11] is considered the family

$$B_\phi := \left\{ 0 = xy - \phi(x^3 + y^3 + 1) \,|\, x, y \in \mathbb{C}^\times \right\}. \tag{A.2}$$

For $a \in \mathcal{M}_{\mathbb{C}} = \mathbb{P}^1 \backslash \{-1/27\}$, the family of [12] reads as

$$M_a^0 := \left\{ 0 = 1 + t_1 + t_2 + \frac{a}{t_1 t_2} \,|\, (t_1, t_2) \in \mathbb{T}^2 \right\}. \tag{A.3}$$

Finally, recall the family of open 3-folds

$$Z_a := \left\{ xy = F_a(t_1, t_2) := 1 + t_1 + t_2 + \frac{a}{t_1 t_2} \,|\, (x, y) \in \mathbb{C}^2,\ (t_1, t_2) \in \mathbb{T}^2 \right\}, \tag{A.4}$$

which was introduced in [13]. Following Konishi-Minabe in [19], we proceed to describe how the periods of the families Z_a and M_a^0 are related. The periods of Z_a are given by integrals of the relative cohomology class

$$\omega_0 = \frac{\mathrm{d}\, x\, \mathrm{d}\, y}{xy} \in \mathrm{H}^2(\mathbb{T}^2, M_a^0; \mathbb{Z}),$$

over relative 2-cycles. The periods of M_a^0 are given by integrating the 3-form

$$\omega_a = \mathrm{Res}\, \frac{1}{xy - F_a(t_1, t_2)} \frac{\mathrm{d}\, t_1\, \mathrm{d}\, t_2}{t_1 t_2}\, \mathrm{d}\, x\, \mathrm{d}\, y \in \mathrm{H}^3(Z_a, \mathbb{Z}),$$

over 3-cycles. Konishi-Minabe in [19] describe an isomorphism of mixed Hodge structures

$$\mathrm{H}^2(\mathbb{T}^2, M_a^0; \mathbb{Z}) \cong \mathrm{H}^3(Z_a, \mathbb{Z}),$$

that sends ω_0 to ω_a. This isomorphism respects the Gauss-Manin connection, which explains why periods of ω_0 over relative 2-cycles correspond to periods of ω_a over 3-cycles. On the other hand, Gross in [18] describes an isomorphism

$$\mathrm{H}_2(\mathbb{T}^2, M_{\check{q}}^0; \mathbb{Z}) \cong \mathrm{H}_3(Z_{\check{q}}, \mathbb{Z}),$$

where $\check{q} \in \Delta^\times$. Put together, this yields a correspondence

$$\text{periods of } M_{\check{q}}^0 \longleftrightarrow \text{periods of } Z_{\check{q}}.$$

We proceed by describing a basis of solutions to the equation (A.1), following Takahashi in [11]. See also [8] and [28]. The following functions are a constant solution, a logarithmic solution and a doubly logarithmic solution satisfying (A.1) as holomorphic functions (with possibly one or two branch cuts) $\Delta^\times \to \mathbb{C}$:

$$\begin{aligned} I_1(z) &= 1, \\ I_2(z) &= \log z + I_2^{(0)}(z), \\ I_3(z) &= \partial_\rho^2 \omega(z;\rho)|_{\rho=0} = (\log z)^2 + \cdots, \end{aligned}$$

where

$$\begin{aligned} I_2(z) &:= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{(3k)!}{(k!)^3} z^k, \\ \omega(z;\rho) &:= \sum_{k=0}^{\infty} \frac{(3\rho)_{3k}}{(1+\rho)_k^3} (-1)^k z^{k+\rho}, \\ (\alpha)_k &:= \alpha \cdot (\alpha+1) \cdots (\alpha+k-1). \end{aligned}$$

Local mirror symmetry asserts that the solutions spanned by these functions are given as both periods of $Z_{\check{q}}$ and relative periods of $M^0_{\check{q}}$. Translating these to Gromov-Witten invariants is achieved via the change of coordinate

$$q := -\exp(I_2(z)).$$

Indeed:

Theorem 19. *(Chiang-Klemm-Yau-Zaslow, in [8]) Denote by K_d the genus 0 degree d Gromov-Witten invariants of $K_{\mathbb{P}^2}$. Written in the coordinate q and via analytic continuation,*

$$I_3(q) = \frac{(\log(-q))^2}{2} - \sum_{d=1}^{\infty} 3d\, K_d\, q^d.$$

Recall the notation $N_d(\mathbb{P}^2, D)$ for the genus 0 degree d relative Gromov-Witten invariants of maximal tangency of $(\mathbb{P}^2, D)$, where D is an elliptic curve. Recall also from chapter 2 the notation $n_{\mathbb{P}^2}[3d]$ for the associated relative BPS numbers (here $w = 3$). The two following results follow from theorem 1. The first was described by Gathmann in [1], albeit from a different perspective.

Corollary 20. *With the same notation as above,*

$$I_3(q) = \frac{(\log(-q))^2}{2} - \sum_{d=1}^{\infty} (-1)^d N_d(\mathbb{P}^2, D)\, q^d$$

calculates the genus 0 relative Gromov-Witten invariants of maximal tangency of $(\mathbb{P}^2, D)$.

In terms of BPS state counts, this yields:

Corollary 21.

$$I_3(q) = \frac{(\log(-q))^2}{2} - \sum_{d=1}^{\infty} (-1)^d\, n_S[dw] \sum_{k=1}^{\infty} \frac{1}{k^2} \binom{k(dw-1)-1}{k-1} q^{dk}.$$

Appendix B

Local mirror symmetry for $\mathbb{P}^1 \times \mathbb{P}^1$ and S_1

Following the exposition of [8] and [19], we describe a basis of solutions to the A-hypergeometric differential equation associated to $\mathbb{P}^1 \times \mathbb{P}^1$ and S_1, which is the blow up of $\mathbb{P}^2$ in one point. In both of these cases, the complex moduli is of dimension 2, isomorphic to $\Delta^\times \times \Delta^\times$. Denote by $z = (z_1, z_2)$ the complex parameter. For $i = 1, 2$, denote by θ_i the partial differential operator $z_i \frac{\partial}{\partial z_i}$. Then the A-hypergeometric differential system associated to $\mathbb{P}^1 \times \mathbb{P}^1$ is of order 2. For functions $f : \Delta^\times \times \Delta^\times \to \mathbb{C}$, it reads

$$\mathcal{L}_1 f = 0,$$
$$\mathcal{L}_2 f = 0,$$

where

$$\mathcal{L}_1 = \theta_1^2 - 2z_1(\theta_1 + \theta_2)(2\theta_1 + 2\theta_2 + 1),$$
$$\mathcal{L}_2 = \theta_2^2 - 2z_2(\theta_1 + \theta_2)(2\theta_1 + 2\theta_2 + 1).$$

A basis of solutions consists of

$$
\begin{aligned}
I_1(z_1, z_2) &= 1, \\
I_2(z_1, z_2) &= \log z_1 + H^{\mathbb{P}^1\times\mathbb{P}^1}(z_1, z_2), \\
I_3(z_1, z_2) &= \log z_2 + H^{\mathbb{P}^1\times\mathbb{P}^1}(z_1, z_2), \\
I_4(z_1, z_2) &= \partial_{\rho_1}\partial_{\rho_2}\omega^{\mathbb{P}^1\times\mathbb{P}^1}(z, \rho)|_{\rho_1=\rho_2=0} = \log z_1 \log z_2 + \cdots ,
\end{aligned}
$$

where

$$
\begin{aligned}
H^{\mathbb{P}^1\times\mathbb{P}^1}(z_1, z_2) &= \sum_{\substack{n_1,n_2\geq 0\\(n_1,n_2)\neq(0,0)}} \frac{1}{n_1+n_2}\frac{(2n_1+2n_2)!}{(n_1!)^2(n_2!)^2} z_1^{n_1} z_2^{n_2}, \\
\omega^{\mathbb{P}^1\times\mathbb{P}^1}(z, \rho) &= \sum_{n_1,n_2\geq 0} \frac{(2\rho_1+2\rho_2)_{2n_1+2n_2}}{(\rho_1+1)_{n_1}^2(\rho_2+1)_{n_2}^2} z_1^{n_1+\rho_1} z_2^{n_2+\rho_2}.
\end{aligned}
$$

The case of S_1 proceeds analogously. Denote again by $z = (z_1, z_2) \in \Delta^\times \times \Delta^\times$ the complex parameter. The A-hypergeometric system of differential equations associated to S_1 is

$$
\begin{aligned}
\mathcal{L}_1 f &= 0, \\
\mathcal{L}_2 f &= 0,
\end{aligned}
$$

for functions $f : \Delta^\times \times \Delta^\times \to \mathbb{C}$, where

$$
\begin{aligned}
\mathcal{L}_1 &= \theta_1(\theta_1 - \theta_2) - z_1(2\theta_1 + \theta_2)(2\theta_1 + \theta_2 + 1), \\
\mathcal{L}_2 &= \theta_2^2 + z_2(2\theta_1 + \theta_2)(\theta_1 - \theta_2).
\end{aligned}
$$

A basis of solutions is given by

$$
\begin{aligned}
I_1(z_1, z_2) &= 1, \\
I_2(z_1, z_2) &= \log z_1 + 2H^{S_1}(z_1, z_2), \\
I_3(z_1, z_2) &= \log z_2 + H^{S_1}(z_1, z_2), \\
I_4(z_1, z_2) &= (\frac{1}{2}\partial_{\rho_1}^2 + \partial_{\rho_1}\partial_{\rho_2})\omega^{S_1}(z, \rho)|_{\rho_1=\rho_2=0},
\end{aligned}
$$

for

$$
\begin{aligned}
H^{S_1}(z_1, z_2) &= \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \geq n_2}} \frac{(2n_1 + n_2 - 1)!}{n_1!(n_1 - n_2)!(n_2!)^2}(-1)^{n_2} z_1^{n_1} z_2^{n_2}, \\
\omega^{S_1}(z_1, z_2) &= \sum_{n_1, n_2 \geq 0} \frac{(2\rho_1 + \rho_2)_{2n_1+n_2}}{(\rho_1 + 1)_{n_1}(\rho_2 + 1)_{n_2}^2} \frac{\Gamma(1 + \rho_1 - \rho_2)}{\Gamma(1 + \rho_1 - \rho_2 + n_1 - n_2)} z_1^{n_1+\rho_1} z_2^{n_2+\rho_2}.
\end{aligned}
$$

Appendix C

Some Gromov-Witten invariants

In this appendix, we provide some Gromov-Witten invariants of Del Pezzo surfaces. We consider local and relative invariants, as well as local and relative BPS state counts. We are not concerned with calculations via geometric tools or mirror symmetry. Rather, emphasis is put on how these numbers are related to each other. Denote by S_1 the (degree 8) Hirzebruch surface given by blowing up $\mathbb{P}^2$ in one point and by S_2 the (degree 7) Del Pezzo surface given by blowing up $\mathbb{P}^2$ in two general points. Alternatively, S_2 is obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ in one point. Denote by $K_{\mathbb{P}^2}$, $\mathcal{K}_{\mathbb{P}^1\times\mathbb{P}^1}$, K_{S_1} and K_{S_2} the total spaces of the responding canonical bundles. Most of the computations below are performed with the open-source software Sage. Hu proves the following formula in [29], which we use as a means of checking some of our calculations. Let S be a Del Pezzo surface, and assume that its blowup at one point, $p : \tilde{S} \to S$, is a Del Pezzo surface as well. Denote by $\beta \in \mathrm{H}_2(S, \mathbb{Z})$ an effective curve class. Denote moreover by

$$p_!(\beta) := PD(p^*(PD(\beta)),$$

its push-forward. Here PD stands for Poincaré dual. Then Hu proves that the Gromov Witten invariant of $\mathcal{K}_S$ of class β equals the Gromov-Witten invariant of $\mathcal{K}_{\tilde{S}}$ of class $p_!(\beta)$. In examples below, pulling back the class of a line in S yields the class of a line in $\tilde{S}$ (away from the exceptional divisor) and the class of the exceptional divisor. We proceed to introducing some notation:

- $\mathrm{H}_2(\mathbb{P}^2) = \mathrm{H}_2(K_{\mathbb{P}^2}) \cong \mathbb{Z}$. For either groups, we denote by $d \geq 0$ an effective curve class of degree d.

- $\mathrm{H}_2(S_1) = \mathrm{H}_2(K_{S_1}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by the classes pulled back from $\mathbb{P}^2$ and by multiples of the exceptional divisor. We use as a basis B, the class of a line away from the exceptional divisor, and F, the fiber class, which is the class of the exceptional divisor. Moreover, we denote by $(d_B, d_F) \geq (0,0)$ the effective curve class $d_B \cdot B + d_F \cdot F$ of degree $d_B + d_F$. In particular, the pullback of a line in $\mathbb{P}^2$ is $(1,1)$.

- Denote by l_2 and l_3 the classes of two lines generating $\mathrm{H}_2(\mathbb{P}^1 \times \mathbb{P}^1)$. In this setting, we use the notation (d_2, d_3) to indicate the class $d_2 \cdot l_2 + d_3 \cdot l_3$.

- Consider S_2 as the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in one point, denote by L_2 and L_3 the pullbacks of l_2 and l_3. Denote by E the class of the exceptional divisor. In accordance with [8], we use the notation (d_1, d_2, d_3) to mean

  $$d_1 \cdot E + d_2 \cdot L_2 + d_3 \cdot L_3.$$

  In particular, the pullback of $a \cdot l_1 + b \cdot l_2$ is $(a+b, a, b)$.

- For whichever surface of the above surfaces, D denotes its anti-canonical divisor.

Let S, respectively K_S, be one of the above varieties and let $\beta \in \mathrm{H}_2(S, \mathbb{Z})$ be an effective curve class. Recall the notation $I_\beta(K_S)$ for the genus 0 degree β Gromov-Witten invariants of K_S, as well as the notation $N_\beta(S, D)$ for the genus 0 degree β relative Gromov-Witten invariants of maximal tangency. These invariants are related by theorem 1 as

$$N_\beta(S, D) = (-1)^{\beta \cdot D} (\beta \cdot D) I_\beta(K_S).$$

For the Calabi-Yau threefold K_S, instanton numbers or BPS state counts n_γ are defined via the equation

$$K_\beta = I_\beta(K_S) = \sum_{a|\beta} \frac{n_{\beta/a}}{a^3}.$$

Relative invariants of $\mathbb{P}^2$

In [30], p. 43, the authors compute the GW invariants for $K_{\mathbb{P}^2}$. They get:

d	K_d
1	3
2	$-\frac{45}{8}$
3	$\frac{244}{9}$
4	$-\frac{12333}{64}$
5	$\frac{211878}{125}$
6	$-\frac{102365}{6}$
7	$\frac{64639725}{343}$
8	$-\frac{1140830253}{512}$
9	$\frac{6742982701}{243}$
10	$-\frac{36001193817}{100}$

Now, $N_d(\mathbb{P}^2, D) = (-1)^{3d} \cdot 3d \cdot I_d(K_{\mathbb{P}^2})$, so that:

d	$N_d(\mathbb{P}^2, D)$
1	-9
2	$-\frac{135}{4}$
3	-244
4	$-\frac{36999}{16}$
5	$-\frac{635634}{25}$
6	-307095
7	$-\frac{193919175}{49}$
8	$-\frac{3422490759}{64}$
9	$-\frac{6742982701}{9}$
10	$-\frac{108003581451}{10}$

The instanton numbers of $K_{\mathbb{P}^2}$ are calculated in [8] to be:

d	n_d
1	3
2	-6
3	27
4	-192
5	1695
6	-17064
7	188454
8	-2228160
9	27748899
10	360012150

Applying the formulae discussed in chapter 2 yields the following relative BPS state counts:

d	$n[3d]$
1	-9
2	-27
3	-234
4	-2232
5	-25380
6	-305829
7	-3957219
8	-53462160
9	-749211021
10	-10800167040

Relative invariants of S_1

In [8], the authors calculate the instanton numbers of $\mathcal{K}_{S_1}$ to be:

$n_{(d_B,d_F)}$	d_F	0	1	2	3	4	5	6
d_B								
0			-2	0	0	0	0	0
1		1	3	5	7	9	11	13
2		0	0	-6	-32	-110	-288	-644
3		0	0	0	27	286	1651	6885
4		0	0	0	0	-192	-3038	-25216
5		0	0	0	0	0	1695	35870
6		0	0	0	0	0	0	-17064

The result by Hu explains why the above diagonal coincides with the instanton numbers of $K_{\mathbb{P}^2}$. We proceed to calculate the Gromov-Witten invariants of K_{S_1}. As an illustration, we perform a few computations by hand. For instance, $K_{(0,1)} = n_{(0,1)} = -2$, since no class divides $(0,1)$ other than itself. The same holds for the line corresponding to $(1,i)$. But not so for $K_{(2,2)}$. Indeed

$$K_{(2,2)} = \frac{n_{(2,2)}}{1} + \frac{n_{(1,1)}}{2^3} = -6 + \frac{3}{8} = -\frac{45}{8}.$$

For the same reason as above, $K_{(2,3)} = n_{(2,3)} = -32$. But not so for $(2,4)$:

$$K_{(2,4)} = \frac{n_{(2,4)}}{1} + \frac{n_{(1,2)}}{2^3} = -110 + \frac{5}{8} = -\frac{875}{8}.$$

Doing these calculation with the help of Sage, we get that:

$K_{(d_B,d_F)}$	d_F	0	1	2	3	4	5	6
d_B								
0			-2	0	0	0	0	0
1		1	3	5	7	9	11	13
2		0	0	$-\frac{45}{8}$	-32	$-\frac{875}{8}$	-288	$-\frac{5145}{8}$
3		0	0	0	$\frac{244}{9}$	286	1651	$\frac{185900}{27}$
4		0	0	0	0	$-\frac{12333}{64}$	-3038	-25220
5		0	0	0	0	0	$\frac{211878}{125}$	35870
6		0	0	0	0	0	0	$-\frac{102365}{6}$

Since

$$(d_B, d_F) \cdot D = 3 \cdot (d_B + d_F),$$

using theorem 1 yields:

$N_{(d_B,d_F)}$	d_F	0	1	2	3	4	5	6
d_B								
0			6	0	0	0	0	0
1		-3	18	-45	84	-135	198	-273
2		0	0	$-\frac{135}{2}$	480	$-\frac{7875}{4}$	6048	-15435
3		0	0	0	488	-6006	39624	-185900
4		0	0	0	0	$-\frac{36999}{8}$	82026	-756600
5		0	0	0	0	0	$\frac{1271268}{25}$	-1183710
6		0	0	0	0	0	0	-614190

Relative invariants of $\mathbb{P}^1 \times \mathbb{P}^1$

As previously mentioned, the authors of [8] calculate the relevant instanton numbers. A very similar calculation as above yields the following relative Gromov-Witten invariants for $(\mathbb{P}^1 \times \mathbb{P}^1, D)$:

	d_3	0	1	2	3	4	5	6
d_2								
0			6	0	0	0	0	0
1		6	-24	54	-96	150	-216	294
2		0	54	-390	1650	$-\frac{10395}{2}$	13524	-30744
3		0	-96	1650	$-\frac{40832}{3}$	74676	-313728	1089132
4		0	150	$-\frac{10395}{2}$	74676	-654435	4139748	$-\frac{41424825}{2}$
5		0	-216	13524	-313728	4139748	$-\frac{939030024}{25}$	259947930
6		0	294	-30744	1089132	$-\frac{41424825}{2}$	259947930	$-\frac{7236946916}{3}$

Relative invariants of S_2

In [8], the authors compute the following instanton invariants:

$d_1 = 0$	d_3	0	1
d_2			
0			1
1		1	0

$d_1 = 1$	d_3	0	1
d_2			
0		1	-2
1		-2	3

$d_1 = 2$	d_3	0	1	2
d_2				
0				
1			-4	5
2			5	-6

$d_1 = 3$	d_3	0	1	2	3
d_2					
0					
1				-6	7
2			-6	35	-32
3			7	-32	27

$d_1 = 4$	d_3	0	1	2	3	4
d_2						
0						
1					-8	9
2				-32	135	-110
3			-8	135	-400	286
4			9	-110	286	-192

$d_1 = 5$	d_3	0	1	2	3	4	5
d_2							
0							
1						-10	11
2					-110	385	-288
3				-110	1100	-2592	1651
4			-10	385	-2592	5187	-3038
5			11	-288	1651	-3038	1695

The symmetry is due to the fact that we pulled back our classes from $\mathbb{P}^1 \times \mathbb{P}^1$. Via similar calculations as above, we get for the local GW invariants:

$d_1 = 0$	d_3	0	1
d_2			
0			1
1		1	0

$d_1 = 1$	d_3	0	1
d_2			
0		1	-2
1		-2	3

$d_1 = 2$	d_3	0	1	2
d_2				
0				
1			-4	5
2			5	$-\frac{45}{8}$

$d_1 = 3$	d_3	0	1	2	3
d_2					
0					
1				-6	7
2			-6	35	-32
3			7	-32	$\frac{244}{9}$

$d_1 = 4$	d_3	0	1	2	3	4
d_2						
0						
1					-8	9
2				$-\frac{65}{2}$	135	$-\frac{875}{8}$
3			-8	135	-400	286
4			9	$-\frac{875}{8}$	286	$-\frac{12333}{64}$

$d_1 = 5$	d_3	0	1	2	3	4	5
d_2							
0							
1						-10	11
2					-110	385	-288
3				-110	1100	-2592	1651
4			-10	385	-2592	5187	-3038
5			11	-288	1651	-3038	$\frac{211878}{125}$

Applying the appropriate formula, and noting that the degree of (d_1, d_2, d_3) is $d_1+d_2+d_3$, we get the following relative invariants of (S_2, D):

$d_1 = 0$	d_3	0	1
d_2			
0			-3
1		-3	0

$d_1 = 1$	d_3	0	1
d_2			
0		-3	-12
1		-12	-27

$d_1 = 2$	d_3	0	1	2
d_2				
0				
1			-48	-75
2			-75	-108

$d_1 = 3$	d_3	0	1	2	3
d_2					
0					
1				-108	-147
2			-108	-735	-768
3			-147	-768	-732

$d_1 = 4$	d_3	0	1	2	3	4
d_2						
0						
1					-192	-243
2				-780	-3645	$-\frac{13125}{4}$
3			-192	-3645	-12000	-9438
4			-243	$-\frac{13125}{4}$	-9438	$-\frac{110997}{16}$

$d_1 = 5$	d_3	0	1	2	3	4	5
d_2							
0							
1						-300	-363
2					-3300	-12705	-10368
3				-3300	-36300	-93312	-64389
4			-300	-12705	-93312	-202293	-127596
5			-363	-10368	-64389	-127596	$-\frac{1906902}{25}$

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