HYDROMAGNETIC STABILITY OF A STREAMING CYLINDRICAL PLASMA

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ABSTRACT

Dispersion relations for hydromagnetic stability were found for three related problems in which the effects of plasma motion were considered. The hydromagnetic differential equations and boundary conditions were linearized in an analysis which assumes small amplitude perturbations about an equilibrium configuration. This configuration consists of a dissipationless plasma flowing in an infinite cylinder with internal and external longitudinal and azimuthal magnetic field components.

Problem 1 is an extension of earlier work and includes electromagnetic radiation and compressibility effects. Problems 2 and 3 assume that the plasma is bound by a non-conducting compressible medium in addition to the magnetic fields. The equilibrium magnetic and velocity field vary as

$$\mathbb{B} = B(0,\ r/r_e,\ h), \qquad \mathbb{V} = \Lambda\ v_A(0,\ r/r_e,\ h)$$

where $v_A = B/(\mu_o \rho_p)^{1/2}$. In problem 2 incompressibility is assumed, while in 3 the assumption of compressibility is made where $v, v_A \ll$ sonic speed of the plasma. This allows a matrix-perturbation expansion about the incompressible solution. The effects of the moving boundary were included. It was found convenient to use the hydromagnetic pressure $\pi = p + B^2/2\mu_o$ as the basic dependent variable and to use the hydromagnetic equations in symmetric form. The equations were extended to a quasi-symmetrical form for treating the compressible medium.

An analytical-numerical study was made in which the dispersion relation for incompressible flow was treated as a function of a complex variable. In each of ten different physical situations the flow parameter, Λ , was varied over the range $0 \leq \Lambda \leq 1.5$ and the following conclusions were reached:

1. The oscillation frequencies are symmetrically distributed about the origin with $\Lambda = 0$. When $\Lambda > 0$ the mode frequencies are all shifted toward the negative and vary monotonically with Λ .
2. The growth rates are larger for large wave number disturbances.
3. The oscillation frequency for complex modes increases with increasing Λ .
4. Increasing the flow (Λ) removes sausage instabilities and enhances (the magnitude of) kink instabilities.
5. Adding a strong longitudinal magnetic field intensifies the sausage instabilities by increasing the magnitude of their growth rate and requiring a larger flow to remove them. Kink instabilities are removed.

1 INTRODUCTION

1.1 Previous Work in Hydromagnetic Stability

The need for a better understanding of phenomena associated with the interaction of ionized matter and electric and magnetic fields has stimulated a great activity in hydromagnetics in recent years. Of particular interest have been those problems involving the stability of plasma systems. The contributions to the literature (see Bibliography) by astrophysicists studying the behavior of sunspots, arches, prominences, magnetic variables, and spiral galactic arms, and by physicists studying the problems of the stability of a plasma fusion machine are evidence of this activity. Astrophysicists have made an effort to show that certain astrophysical phenomena are essentially a cosmic display of a hydromagnetic (hm) instability. On the other end of the magnitude scale, physicists have determined that the stability of a constricted gas discharge is an essential requirement for the successful operation of a fusion machine.

The partial differential equations (d.e.) of hydromagnetics present problems of formidable mathematical complexity primarily because of the non-linear coupling terms involving the velocity and magnetic fields. Most analyses of recent years have employed linearization techniques to overcome this handicap. For example, in current investigations of the dynamo problem, the velocity field is assumed known, and one neglects the velocity equations and reduces the problem to a linear one for characterizing the magnetic field. Similarly, in stability problems, one investigates the behavior of the system under small perturbations in the neighborhood of a stationary equilibrium. If one neglects the product of two or more perturbation quantities, one obtains a coupled set of linear partial differential equations with spatially variable coefficients. The latter procedure is used by the author in the problems investigated below.

Mathematically, it is convenient to treat the plasma as dissipationless (resistivity and conductivity both zero) and to neglect the heat flow. These assumptions are desirable, for they allow one to separate the basic instabilities in the first part of the motion (where linearity is valid) from the effects of natural decay. The first assumption reduces the order of the d.e., whereas the first and second together reduce the complexity of the energy equation of the fluid. The physical considerations upon which these assumptions rest are related to behavior of the temperature in the plasma, and are discussed in Section 2.1.

As in all physical problems, the geometry of the system determines the specific form of: the component d.e., the boundary conditions (b.c.), and the spatial dependence of the equilibrium variables. Cartesian geometry has been used to study the elementary solutions of the hm equations and their physical significance. These results are applicable to large scale cosmic phenomena where geometric effects are of second order. The spherical plasma is a natural model for the pulsating magnetic variables.

The stability of the cylindrical plasma has excited considerable activity because the results are applicable to such phenomena as spiral galactic arms, interstellar fields which confine cosmic ray particles, and the pinch effect in a fusion machine. Recently, the equilibrium of a toroidal plasma has been investigated because of its applicability to the "Stellerator" type fusion machine.

Two methods have been used to determine whether plasma systems are stable.

1. The "normal mode" technique (1) is the usual one for investigating stability and is the one employed in this investigation. It consists of solving the linearized hm equations for small perturbations about equilibrium. Boundary conditions are matched, and one obtains a dispersion relation (or secular equation) whose roots indicate whether the solution is unstable (grows with time) or stable (decays or retains the same amplitude). The dispersion relation contains transcendental functions which are characteristic of the geometry. This is to be compared with the polynomial functions obtained from the characteristic determinant of systems of linear constant coefficient d.e.

2. The energy principle technique (2,3), on the other hand, depends upon a variational formulation. It was used by Rayleigh (1877) in the calculation of frequencies of vibrating systems and was originally stated by Lundquist (4) for hm systems. Its advantage lies in the fact that if one seeks solely to determine stability and not rates of growth or oscillation frequencies, it is necessary only to discover whether there is any perturbation which decreases the potential energy from its equilibrium value. This makes practical the stability analysis of more complicated equilibrium geometries than does the normal mode analysis. The disadvantages of the energy technique are twofold. First, one must guess or have a prior knowledge of the functional dependence of the eigenfunctions.

Second, it is not applicable to cases where the equilibrium velocity field of the plasma is finite (the raison d'etre of this dissertation), for in such cases overstable* roots are present in the dispersion relation. This prevents one from writing a convenient variational principle.

A virial theorem derived by Chandrasekhar and Fermi (5) has been applied to absolute stability investigations of cylindrical and spherical geometries (6). It is an energy technique similar to the one above, but it can also be used to determine the frequency of the primary mode of oscillation.

1.2 Statement of the Problem

The aim of this dissertation is to study the hm stability of a cylindrical plasma in which the mass motion of the plasma plays as important a role as does the magnetic field. In the three problems described below the plasma is assumed to be non-dissipative (resistivity and viscosity vanish) and non-heat-conducting. The b.c. are matched across a "moving boundary." Figure 1-1 sketches the problems presented.

1. Electromagnetic radiation and compressibility effects are considered. The plasma contains internal and external, uniform (but different), longitudinal magnetic fields and an internal, uniform, longitudinal flow. The equilibrium state is maintained by the azimuthal and longitudinal surface currents.

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* Roots which lie in the complex plane - off the real and imaginary axis.

The latter produces the external azimuthal magnetic field.

$$\mathbb{V} = (0,0,v_z), \qquad \mathbb{B} = B(0,0,b_i); \qquad r < r_e$$

$$\mathbb{V} = 0 \qquad , \qquad \mathbb{B} = B(0, \frac{r_e}{r}, b_o); \qquad r > r_e$$

2. Electromagnetic radiation and compressibility effects are omitted. Hence, in equilibrium the plasma has a constant density ρ_p. The velocity field has azimuthal as well as longitudinal components. There is a magnetic field within and external to the plasma. In both cases there are azimuthal and longitudinal components, all of which differ in magnitude. The region surrounding the plasma contains a non-conducting compressible gas which is at rest and has a constant density ρ_o in the equilibrium state. If ρ_o vanishes, one has the case of a vacuum surrounding the plasma.

$$\mathbb{V}_e = \Lambda v_A(0, \frac{r}{r_e}, h), \qquad \mathbb{V}_{Ae} = v_A(0, \frac{r}{r_e}, h); \qquad r < r_e$$

$$\mathbb{V}_e = 0 \qquad , \qquad \mathbb{V}_{Ae} = v_A(0, b_\Theta \frac{r_e}{r}, hb_z); \quad r > r_e$$

where

$$v_A = (\mu_o \rho_p)^{-1/2} B$$

3. The equilibrium is the same as for the second problem, except that we allow the plasma to be slightly compressible. That is, we assume

$$\frac{|\mathbf{v}_e|^2}{(\text{sonic speed})^2} < 1, \qquad \frac{|\mathbf{v}_{Ae}|^2}{(\text{sonic speed})^2} < 1$$

This causes the plasma density to be spatially dependent, as shown in Chapter 4.

Section 3.1, which follows a discussion of the basic hm equations, will present an outline of the approximations made by previous investigators and will compare them to the ones made in this dissertation.

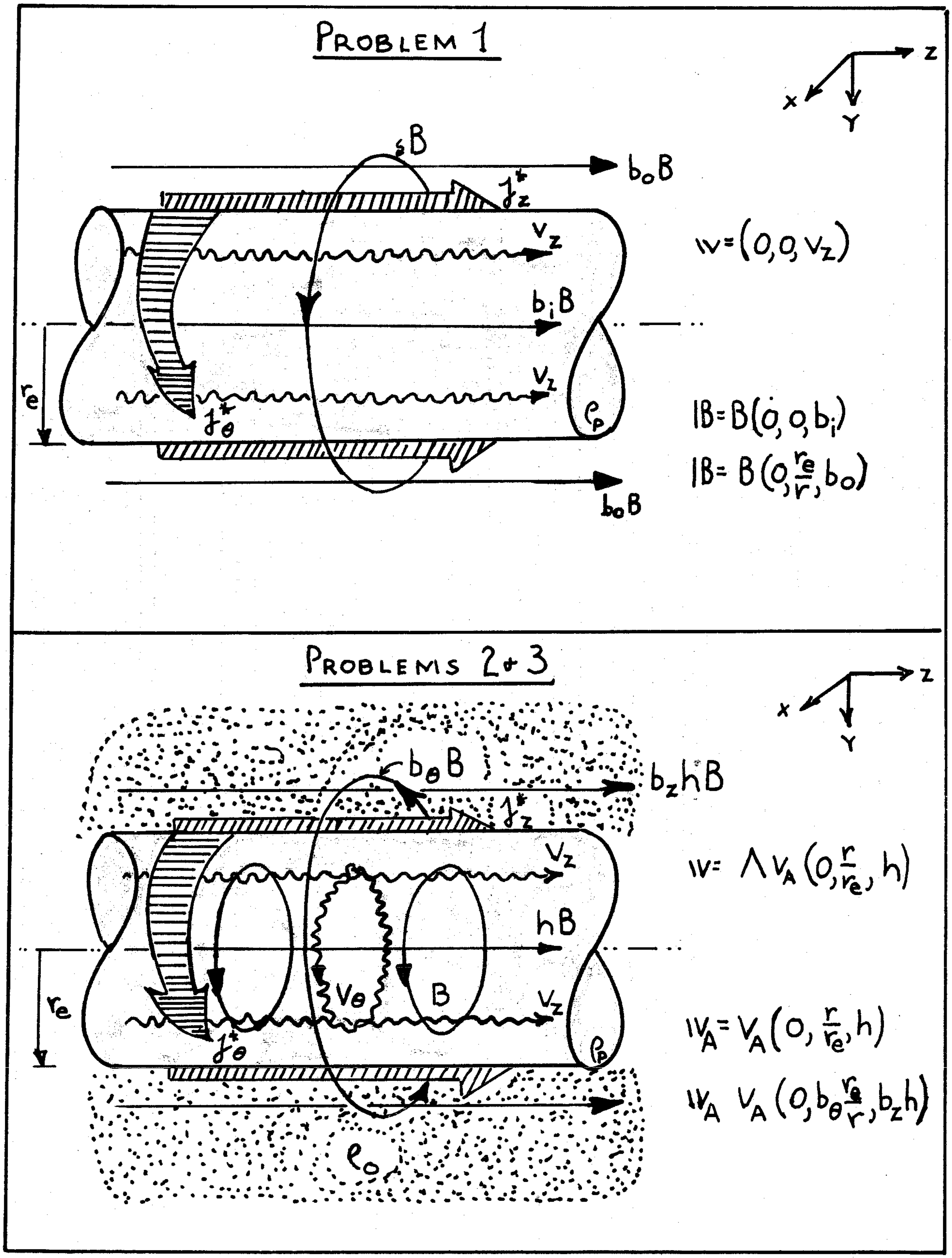

Fig. 1-1. The Equilibrium Configurations.

2 THE HYDROMAGNETIC DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

2.1 The Basic Set of Hydromagnetic Equations

The d.e. used to describe the dynamic behavior of an ionized gas are the hm equations. In the discussions which follow we will consider our gas as fully ionized - a plasma composed of two components, electrons and ions. To obtain a relativistically covariant form for the conservation equations (momentum and energy) one would take the four-dimensional divergence of the complete energy-momentum tensor. The complete tensor is the sum of the energy-momentum tensor of the electromagnetic field and the energy-momentum tensor of the plasma (fluid). Since we are interested in phenomena where the macroscopic fluid velocity is much smaller than the velocity of light, we will be satisfied with equations which include relativistic effects to the first order (problem 1).

Numerous investigators (7, 8 Chap. 18, 9, 10) have described procedures for obtaining such equations by starting with the Maxwell-Boltzmann collision equations. A separate collision equation describes the behavior of the distribution function for each componenet of the plasma. Moments of each component equation are formed with various powers of the velocity of each fluid component. Properly weighted linear combinations of the moment equations yield the macroscopic hm equations.

1. Conservation of mass (zeroth velocity moments summed)

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad (2.1)$$

2. Conservation of momentum or Navier-Stokes equation (first velocity moments summed)

$$\rho D_t \mathbb{v} = -\nabla \cdot \overleftrightarrow{\mathbb{P}} + \mathbb{j} \times \mathbb{B} + \sigma \mathbb{E} - \rho \nabla \Phi \qquad (2.2)$$

3. Conservation of energy (second velocity moments summed) (8, eqs. 18.2-18.6)

$$\frac{3}{2} \partial_t p + \frac{3}{2} \nabla \cdot (p \ \mathbb{v}) =$$

$$= -\nabla \cdot \mathbb{Q} + p(\nabla \Phi) \cdot \mathbb{v} + (\mathbb{j} - \sigma \mathbb{v}) \cdot (\mathbb{E} + \mathbb{v} \times \mathbb{B}) - (\overleftrightarrow{\mathbb{P}} \cdot \nabla) \cdot \mathbb{v} \qquad (2.3)$$

4. Generalized Ohm's Law (first velocity moments differenced) (7, eq. 2-12)

$$[\mathbb{E} + \mathbb{v} + \mathbb{B} - \overset{\textcircled{1}*}{\eta \mathbb{j}}] = \frac{m_e}{nq^2} \overset{\textcircled{2}}{\partial_t \mathbb{j}} - \frac{1}{nq} [\nabla p_e \overset{\textcircled{3}}{-} \mathbb{j} \times \mathbb{B}] \qquad (2.4)$$

where D_t = substantial derivative = $\partial_t + (\mathbb{v} \cdot \nabla)$ (2.5)

and n = number density of electrons = number density of ions

q = (MKS) charge on an electron (taken as positive)

m_e = electron mass; m_i = ion mass

ρ = mass density = $n(m_i + m_e)$

σ = charge density = $nq(Z-1)$

$\mathbb{v}$ = macroscopic fluid velocity

---

* The balloons are explained below.

$\mathbb{j}$ = macroscopic current

$\overleftrightarrow{\mathbb{P}} = p\,\overleftrightarrow{\mathbb{U}} + \overleftrightarrow{\mathbb{P}}'$ = total pressure tensor of the medium

p = 1/3 trace of $(\overleftrightarrow{\mathbb{P}}) = p_i + p_e = nkT$

$\mathbb{B}$, $\mathbb{E}$ = total magnetic and electric field

Φ = external potential energy per unit mass

$\mathbb{Q}$ = heat flow vector (associated with transport of kinetic energy)

η = resistivity of medium (assumed a scalar)

Equations 2.1, 2.2, and 2.3 are exact, and 2.1 through 2.4 are Galilean invariant. Unfortunately, the set of equations is incomplete. To get a complete set of equations which is also mathematically tractable, several simplifications will be introduced.

If one assumes a large collision rate between particles, the distribution function of the components is isotropic in velocity space. In this case $\overleftrightarrow{\mathbb{P}}$ is a diagonal tensor with similar elements, i.e., $\nabla \cdot \overleftrightarrow{\mathbb{P}} = \nabla p$. The off-diagonal terms of $\overleftrightarrow{\mathbb{P}}$ are related to the viscous shear forces and contribute to energy dissipation in the medium (the last term of 2.3). A better approximation to reality (9) would utilize a pressure tensor which is diagonal in a local rectangular coordinate system, one of whose axes points along $\mathbb{B}$. That is, for a magnetic field along $\mathbb{e}_z$

$$\overleftrightarrow{\mathbb{P}} = p_\perp\,(\mathbb{e}_x\,\mathbb{e}_x + \mathbb{e}_y\,\mathbb{e}_y) + p_{||}\,\mathbb{e}_z\,\mathbb{e}_z \qquad (2.6)$$

In this case the requirement for complete isotropy in pressure has been removed. Gravitational and other external forces are omitted ($\Phi = 0$) in the present study.

In 2.3 we assume:

1. Viscous and resistive dissipation are negligible ($\eta = 0$, $\overleftrightarrow{P}^1 = 0$).

2. Heat flow along and across the lines of force is negligible ($\nabla \cdot Q = 0$).

3. There are no external forces ($\Phi = 0$).

Hence, the energy equation reduces to

$$\frac{3}{2}\,\partial_t p + \frac{3}{2}\,\nabla\cdot(p\,w) + p\,\nabla\cdot w = \frac{3}{2}\,D_t p - \frac{5}{2}\frac{p}{\rho}\,D_t\rho = 0 \qquad (2.7)$$

or

$$D_t(p\rho^{-\gamma}) = 0, \qquad \gamma = 5/3 \qquad (2.8)$$

Such approximations are valid in a stability analysis where one is interested in initial motions. Furthermore, one can show (11) that dissipative effects can never remove a basic instability, but will only reduce the growth rate.

Ohm's Law simplifies* when one assumes:

1. A high temperature plasma (7, eq. 5-39). Physically, this result is derived from the assumption of a very small collision rate between electrons and ions. That is, $\eta \propto$ (collision rate) $\rightarrow 0$.

2. Plasma phenomena change slowly, so that if t_c is the characteristic time for plasma events, then

$$t_c \gg \frac{1}{\omega_p}, \qquad t_c \gg \frac{1}{\omega_L}$$

---

* The balloons above 2.4 refer to the following enumeration. Each term becomes zero when one makes the corresponding assumption.

where

$$\omega_p = \text{electron plasma frequency} = \left(\frac{nq^2}{m_e\,\epsilon_o}\right)^{1/2}$$

$$\omega_L = \text{ion Larmor frequency} = \frac{qB}{m_i} \qquad (2.9)$$

3. $\left(\frac{c_e^2}{vv_i}\right)\frac{r_L}{L} \ll 1$, where: (a) $c_e^2 = \mathcal{O}(c_s^2 + v_A^2)$; (b) v = the macroscopic fluid velocity; (c) v_i = the ion velocity in a Larmor circle; (d) r_L = the Larmor radius; and (e) L = a characteristic length over which the plasma changes. This assumption breaks down if $v = 0$ in the equilibrium state. In Ref. 2, paragraphs 3a and 3b, an examination is made which includes the last two terms of 2.4 in the case $\mathbb{v}_e = 0$, and it is found that the absolute stability is unaffected. Hence, condition 3 may be violated without changing the absolute stability of a system.

With these approximations in mind, the above equations reduce to the basic hm equations.

$$\partial_t \rho + \nabla \cdot (\rho \mathbb{v}) = 0 \qquad (2.10)$$

$$\rho D_t \mathbb{v} = -\nabla p + \mathbb{j} \times \mathbb{B} + \sigma \mathbb{E} \qquad (2.11)$$

$$D_t(p\rho^{-\gamma}) = 0 = \frac{1}{p} D_t p - \frac{\gamma}{\rho} D_t \rho = 0 \qquad (2.12)$$

$$\mathbb{E} + \mathbb{v} \times \mathbb{B} = 0 \qquad (2.13)$$

To obtain a complete set, one must add the exact, Lorentz-invariant Maxwell equations.

$$\nabla \cdot \mathbb{B} = 0 \tag{2.14}$$

$$\nabla \times \mathbb{B} = \mu_o \mathbb{j} + \frac{1}{c^2} \partial_t \mathbb{E} \tag{2.15}$$

$$\nabla \cdot \mathbb{E} = \frac{\sigma}{\epsilon_o} \tag{2.16}$$

$$\nabla \times \mathbb{E} = - \partial_t \mathbb{B} \tag{2.17}$$

Note that 2.14 is implied in 2.17. Equations 2.10 and 2.11, the curl of 2.13 (with the resistive term included), and 2.14-2.17 are written out as component equations in cylindrical coordinates in Appendix A2. There, $\sigma \mathbb{E}$ is omitted from the momentum equation. Since none of our problems contains equilibrium electric fields in the rest frame, the $\sigma \mathbb{E}$ will enter as a perturbation term of second order in our stability analysis and will be neglected. That is,

$$\sigma \mathbb{E} = \frac{(\nabla \cdot \mathbb{E})\ E}{\epsilon_o} \propto \frac{E^2}{L\epsilon_o} \tag{2.18}$$

An alternate discussion for omitting σ is presented in Section 3.1.

2.2 The Symmetric Form of the Hydromagnetic Equations

Where the velocity field within the plasma is of comparable magnitude to the magnetic field, a transformed set of equations is more convenient. If the medium is incompressible, ρ = constant, one can use the Elsasser-Biermann equations (12, 13) which are derived below.

By introducing the definition $\mathbb{v}_A = \frac{\mathbb{B}}{(\mu_o \rho_p)^{1/2}}$, one can re-write the curl of 2.13 as

$$-\partial_t \mathbb{V}_A + \nabla\times(\mathbb{V}\times\mathbb{V}_A) = -\partial_t \mathbb{V}_A + \nabla\cdot[\mathbb{V}_A \mathbb{V} - \mathbb{V} \mathbb{V}_A] = 0 \quad (2.19)$$

where A1.19 has been used. Equation 2.6 can be written in this form if we assume, in addition to incompressibility ($\rho = \rho_p$), that the displacement current and electric force terms are negligible. If we substitute 2.15 into 2.11 and divide by ρ_p, we obtain

$$\partial_t \mathbb{V} + (\mathbb{V}\cdot\nabla)\mathbb{V} = -\frac{1}{\rho_p}\nabla(p + \frac{B^2}{2\mu_o}) + (\mathbb{V}_A\cdot\nabla)\mathbb{V}_A \quad (2.20)$$

where we have used A1.13. Using A1.18 and the fact that $\nabla\cdot\mathbb{V} = \nabla\cdot\mathbb{V}_A = 0$, one can write this as

$$\partial_t \mathbb{V} + \nabla\cdot[\mathbb{V}\mathbb{V} - \mathbb{V}_A\mathbb{V}_A] = -\nabla\pi \quad (2.21)$$

where $\pi = \frac{1}{\rho_p}(p + \frac{B^2}{2\mu_o})$ = the normalized hm pressure. (2.22)

If one adds and subtracts 2.19 and 2.21, one obtains the symmetric hm equations.

$$\partial_t \mathbb{Q}_- + \nabla\cdot[\mathbb{Q}_+ \mathbb{Q}_-] = -\nabla\pi \quad (2.23)$$

$$\partial_t \mathbb{Q}_+ + \nabla\cdot[\mathbb{Q}_- \mathbb{Q}_+] = -\nabla\pi \quad (2.24)$$

where $\mathbb{Q}_\pm = \mathbb{V} \pm \mathbb{V}_A$ (not the heat flow vector) (2.25)

These equations are symmetric with respect to an interchange $\mathbb{Q}_+ \Longleftrightarrow \mathbb{Q}_-$. The mass and flux continuity equations take the form

$$\nabla\cdot(\mathbb{Q}_\pm) = 0 \quad (2.26)$$

2.3 The Quasi-Symmetric Form of the Hydromagnetic Equations

To deal with a compressible plasma, the author introduced new variables $\mathbb{R}_{\pm}$ to account for the density variation. Using 2.10 one can write the momentum equation, 2.11, as

$$\partial_t(\rho \mathbb{W}) + \nabla \cdot (\rho \mathbb{W}, \mathbb{W}) = -\rho_p \nabla \pi + \mathbb{W}_A \cdot \nabla \mathbb{W}_A \quad (2.27)$$

where we have again neglected the displacement current and charge density terms. By adding and subtracting 2.27 with 2.19, we obtain the quasi-symmetric equations of hm.

$$\partial_t \mathbb{R}_- + (1/2) \nabla \cdot \left\{ (\mathbb{R}_+ + \mathbb{Q}_+) \mathbb{Q}_- + (\mathbb{R}_+ - \mathbb{Q}_+) \mathbb{Q}_- \right\} = -\nabla \pi \quad (2.28)$$

$$\partial_t \mathbb{R}_+ + (1/2) \nabla \cdot \left\{ (\mathbb{R}_- + \mathbb{Q}_-) \mathbb{Q}_+ + (\mathbb{R}_- - \mathbb{Q}_-) \mathbb{Q}_+ \right\} = -\nabla \pi \quad (2.29)$$

where

$$\mathbb{Q}_{\pm} = \mathbb{W} \pm \mathbb{W}_A ; \qquad \mathbb{R}_{\pm} = \frac{\rho}{\rho_p} \mathbb{W} \pm \mathbb{W}_A \quad (2.30)$$

$$\pi = \frac{1}{\rho_p} \left(p + \frac{B^2}{2\mu_0}\right) \quad (2.31)$$

Equations 2.28 and 2.29 are symmetric with respect to the interchange of signs (-) <==> (+), but not with respect to the interchange $\mathbb{R}$ <==> $\mathbb{Q}$.

The mass and flux continuity equations (2.10 and 2.14) take the form

$$\nabla \cdot \mathbb{Q}_{\pm} = -\frac{1}{\rho} (D_t \rho) \quad (2.32)$$

$$\nabla \cdot \mathbb{R}_{\pm} = -\frac{1}{\rho_p} (\partial_t \rho) \quad (2.33)$$

p is related to π and $\mathbb{Q}$ through the relation

$$p = \rho_p \pi - (1/4) \left\{ \mathbb{Q}_+ - \mathbb{Q}_- \right\}^2 \tag{2.34}$$

$\mathbb{Q}$ and $\mathbb{R}$ are related to each other by

$$2\mathbb{R}_\pm = \mathbb{Q}_+[\frac{\rho}{\rho_p} \pm 1] + \mathbb{Q}_-[\frac{\rho}{\rho_p} \mp 1] \tag{2.35}$$

$$2\mathbb{Q}_\pm = \mathbb{R}_+[\frac{\rho}{\rho_p} \pm 1] + \mathbb{R}_-[\frac{\rho}{\rho_p} \mp 1] \tag{2.36}$$

2.4 The Boundary Conditions for Hydromagnetic Problems

The b.c. across surfaces which move with the fluid particles are obtained, as in electromagnetic problems, by integrating the d.e. of motion over vanishingly small "pill-boxes" which have areas in the adjacent media. The result of such an integration contains terms of the form

$$D_t \int \mathbb{E}\, d^3x = 0 \qquad \text{and} \qquad D_t \int [\rho \mathbb{V} + \epsilon_o \mathbb{E} \times \mathbb{B}]\, d^3x = 0 \tag{2.37}$$

which vanish in the limit of small volumes. Since each surface of the pill-box lies in a different medium, these b.c. are applicable only to situations where there is a sharp boundary between the plasma and the adjacent medium. Northrop (14, eqs. 9-12) and Kruskal and Tuck (15, eqs. 10-14) have presented the b.c., and they are summarized below with corrections made in certain of their signs.

1. Velocity continuity

$$\mathbb{V} \cdot \mathbb{n}|_p = \mathbb{V} \cdot \mathbb{n}|_o = u \tag{2.38}$$

2. Pressure continuity

$$\left\{ \left(p + \frac{B^2}{2\mu_o} + \frac{E^2}{2\mu_o c^2}\right) \mathbb{n} + \overleftrightarrow{\mathbb{P}}^{(1)} \cdot \mathbb{n} - \frac{\mathbb{B}(\mathbb{B} \cdot \mathbb{n})}{\mu_o} - \frac{\mathbb{E}(\mathbb{E} \cdot \mathbb{n})}{\mu_o c^2} - \frac{u\ E\ B}{\mu_o c^2} \right\} \Bigg|_p = \left\{ \begin{matrix} \text{similar} \\ \text{quantity} \\ \text{outside} \end{matrix} \right\} \Bigg|_o \qquad (2.39)$$

3. Magnetic flux continuity

$$\mathbb{n} \cdot \mathbb{B}\big|_p = \mathbb{n} \cdot \mathbb{B}\big|_o \qquad (2.40)$$

$$\left\{ \mathbb{n} \times \mathbb{B} + \frac{u}{c^2} \ \mathbb{E} \right\} \Bigg|_p - \left\{ \mathbb{n} \times \mathbb{B} + \frac{u}{c^2} \ \mathbb{E} \right\} \Bigg|_o = + \mu_o \ \mathbb{j}^* \qquad (2.41)$$

4. Electric flux continuity

$$\mathbb{n} \cdot \mathbb{E}\big|_p - \mathbb{n} \cdot \mathbb{E}\big|_o = + \frac{\sigma^*}{\epsilon_o} \qquad (2.42)$$

$$\left\{ \mathbb{n} \times \mathbb{E} - u \ \mathbb{B} \right\} \Big|_p = \left\{ \mathbb{n} \times \mathbb{E} - u \ \mathbb{B} \right\} \Big|_o \qquad (2.43)$$

where

a. $\overleftrightarrow{\mathbb{P}} = p \ \overleftrightarrow{\mathbb{U}} + \overleftrightarrow{\mathbb{P}}^{(1)}$

b. $\mathbb{n}$ is the unit vector, normal to the surface of discontinuity. It is <u>positive when pointing into the plasma</u>. Thus, if the plasma is the interior medium

$$d\$ = - \mathbb{n} dS \qquad (2.44)$$

c. the subscripts p and o stand for "plasma" and "outer"

d. $\mathbb{j}^*$ and σ^* are the result of integrating sheet currents and charge densities over small pill-boxes. That is,

$$j^* = \frac{\int j \, d^3x}{dS} \qquad \sigma^* = \frac{\int \sigma \, d^3x}{dS} \tag{2.45}$$

if j and σ have delta function distributions.

Not all these b.c. are independent. A judicious choice of the proper equations will often simplify the work in solving stability problems. One could also obtain a b.c. from the energy conservation equation, but this is redundant.

In deriving the above b.c. it was assumed that the surface of discontinuity between the two regions, $f(x,y,z,t) = 0$, moved with the fluid particles of each region. Hence, those particles which initially were present in the surface remained there, for they have the same velocity along the normal as the surface itself. Furthermore, no new particle can become a member of the surface. This defines a "stationary surface" or in the nomenclature of Hadamard (16), a "discontinuity of order zero." For such discontinuities one can specify arbitrarily the ratio of densities on each side. The "total" pressure is continuous across such surfaces. Its gradient, which is proportional to the acceleration of the fluid particles, will not be continuous.

Discontinuities of order "n" (shock discontinuities) can be obtained by assuming that the density and the acceleration and their first (n-1) time and space derivatives are continuous across the discontinuity. Such surfaces are not "stationary" with respect to the fluid but drift through it with a definite velocity, calculable from the physical assumptions which are made. For gas dynamics problems, this velocity is one of the sonic speeds in the medium.

The b.c. across such discontinuities are calculated from the basic d.e. of motion, using the pill-box idea. However, the details of the calculation are different, since the surface of discontinuity is located in one medium at time t and moves to the position of the next medium at $t + \Delta t$.

2.5. The Normal-Vector Differential Equation

The vector normal to the surface of discontinuity, f,

$$\mathbb{n} = \frac{\nabla f}{|\nabla f|} \tag{2.46}$$

appears explicitly in the b.c. Thus, rather than describe the behavior of the surface by the d.e. for $f(x,y,z,t)$, we prefer to work directly with the d.e. for $\mathbb{n}$. This is derived in Appendix A3, and given in A3.20 as

$$\partial_t \mathbb{n} + u(\mathbb{n}\cdot\nabla)\mathbb{n} = \mathbb{n} \times [\mathbb{n} \times \nabla u] \tag{2.47}$$

where u = velocity of the surface normal to itself. For surfaces of discontinuity of order zero, then,

$$u = \mathbb{v}_p \cdot \mathbb{n} = \mathbb{v}_o \cdot \mathbb{n} \tag{2.48}$$

3 NORMAL MODE ANALYSIS

3.1. A Comparison of Approximations Introduced by Investigators

To reduce the mathematical complexity of stability studies, investigators have introduced assumptions which fall into three classes: (1) those which affect the d.e. of motion; (2) those which affect the b.c.; (3) those which affect the form which the equilibrium configuration assumes. These are outlined below so that comparisons can be made to the problems to be solved in this dissertation (as stated in Section 1.2).

1. Differential Equations

   a. Charge Density Term (σ). It is common to set $\sigma = 0$, because in a typical plasma the Debye shielding distance, $\lambda_D = (\epsilon_o kT_e/nq^2)^{1/2}$, is small compared to a characteristic length over which a plasma changes. For example, if $T = 10^{6\,o}K$ and $n = 10^{22}$ per m^3, then $\lambda_D = 6.9 \times 10^{-5}$ cm. This simplifies 2.3 and 2.4.

   b. Electromagnetic Radiation Terms. These terms arise from including the displacement current in Maxwell's induction equation. They are manifest in the dispersion relation as functions of the quantity $(\omega/c)^2$, where ω is the growth rate or oscillation frequency of the system. It is common to omit them for low-frequency disturbances, and this has been done in problems 2 and 3. These terms have been included in Refs. 1 and 15 and also in problem 1.

c. Incompressibility of the Plasma. The assumption that the material density $\rho = \rho_p$ = constant is valid when the macroscopic velocity of the plasma is much smaller than the sonic velocity, $(\gamma p/\rho)^{1/2}$, and the Alfvén velocity, $B/(\mu_o \rho)^{1/2}$, within the plasma. This assumption has been made by many investigators (5, 17, 18, 19) and is also made in problem 2. A partially compressible plasma is treated in problem 3.

d. Pressure Tensor. Most investigators, including the author, have assumed a diagonal tensor, all of whose terms are equal. Rosenbluth (20) uses a diagonal tensor with different components along and perpendicular to the magnetic field (his eq. 11).

e. Dissipation Terms. Terms due to viscosity (the off-diagonal elements of the pressure tensor) are present in 2.2; the resistivity, η, is present in 2.4; while both terms are contained in the energy equation, 2.3. In a high-temperature plasma the omission of η is justified, since it varies as $(T)^{-3/2}$. On the other hand, the viscosity varies as $(T)^{1/2}$ (21, p. 203), and one cannot justify its omission by a physical argument. However, for mathematical convenience we neglect both. Tayler (22,23) has considered the effects of "hydrodynamic viscosity" (i.e., he neglects the effect of the magnetic field on the coefficient of viscosity) in order to examine its effect on the infinite instability which arises in large wave number disturbances.

f. Heat Flow, (Q. For mathematical convenience we neglect the thermal conductivity of the plasma. This omission cannot be justified physically, because the thermal conductivity varies directly as the temperature $\rightarrow \frac{T}{\eta} \rightarrow T^{5/2}$ (21, p. 343).

2. Boundary Conditions

a. The Moving Boundary. Most investigators have matched hydrodynamic and electromagnetic b.c. across a moving boundary. Trehan (18, 19) introduced as his sole b.c. the assumption that the perturbation of the normalized hm pressure, π, was zero. This procedure is not valid, as discussed in Section 7.1.

3. Equilibrium Configurations

a. Magnetic Fields. The spatial distribution of the internal and external aximuthal (B_θ) and longitudinal (B_z) magnetic fields affects the stability. The following table summarizes some of the equilibrium magnetic field distributions used by various investigators, including the author.

b. Velocity Fields. The spatial distribution of the velocity field determines the pressure and density in a compressible plasma. The relations which exist between these fields and the magnetic fields are obtained from the hm equilibrium equations, as described in Chapter 4. All investigators but Trehan (18,23) have omitted the effect of a velocity field. He imposed a special requirement between the equilibrium velocity and the magnetic fields, of the form $\mathbb{W} = \mathbb{B}(\mu_o \rho_p)^{-1/2}$. This is equivalent to $\Lambda = 1$ in the statement of the problem given in Section 1.2.

TABLE 3-1

RADIAL DEPENDENCE OF MAGNETIC FIELD COMPONENTS

Investigator	Ref.	Internal B_θ	Internal B_z	External B_θ	External B_z
Tayler	23	0	0	1/r	0
		r	0	1/r	0
		r^{n+1}	0	1/r	0
Tayler	17	0	const.	1/r	const.
		f(r)	g(r)	1/r	const.
		r	const.	1/r	const.
Rosenbluth	20	0	const.	1/r	const.
Körper	24	0	const.	0	const.
Trehan	18	r	const.	not specified	not specified
Kruskal & Tuck	15	0	const.	1/r	const.
		r	const.	1/r	const.
Problem 1		0	const.	1/r	const.
Problems 2 & 3		r	const.	1/r	const.

c. Plasma Containment. Two schemes are avilable for balancing the plasma pressure: (1) the presence of a non-conducting compressible gas adjacent to the plasma; (2) a current sheet flowing on the surface of the plasma. The latter gives rise to a discontinuity in the magnetic field, and thus to the magnetic component of the stress tensor. Most investigators have used the second scheme. In problem 1, scheme 2 is used, while in problems 2 and 3 a combination of both is used.

3.2 Procedures for a Normal Mode Analysis

The equilibrium or stationary configuration is chosen in accordance with the physical situation being studied. An investigation is then made to determine the inter-relationships which exist among the equilibrium variables. Such a study is made in Chapter 4 for the problems given in Chapter 1. One then goes to the appropriate d.e. of motion for the medium and replaces each quantity with the sum of its equilibrium value, q_e, and a first order perturbation, $\tilde{q}$. The perturbation quantities are arbitrary functions of r and exponentially dependent on Θ, z, t. That is,

$$\begin{aligned} q = q_e + \tilde{q} &= q_e + \tilde{q}(r) \exp[i(m\Theta + kz + \omega t)] \\ &= q_e + \tilde{q}^o f(r) \end{aligned} \tag{3.1}$$

where $$\tilde{q}^o = \tilde{q}^o_o \exp[i(m\Theta + kz + \omega t)]$$

The subscript e will be omitted when we are obviously dealing with equilibrium quantities. The resulting equations are then linearized by neglecting products of two or more first order quantities.

It is the aim of the following analysis to determine the eigenvalues ω as a function of the equilibrium parameters, the azimuthal number m, and the wave number k of the perturbation. This is accomplished by solving the d.e. for the pertinent variables and relating these variables through the b.c. These substitutions result in the dispersion relation $f(\omega,...) = 0$, whose zeros are the required eigenvalues. This is demonstrated in Chapter 5.

In the plasma where the hm d.e. are applicable, the basic dependent variables are $\tilde{\pi}$ and $\tilde{Q}_r$ or $\tilde{v}_r$ (which are derived from $\tilde{\pi}$). The techniques for obtaining these variables are presented in Chapters 6, 7, and 8. If no current flows in the outer region, the electromagnetic and hydrodynamic (if there is a fluid present) effects are uncoupled and can be considered separately. Here, the most convenient dependent variables are the pressure $\tilde{p}_o$ and the z components of the electric and magnetic fields. In the next section we solve for these variables.

3.3. Perturbation Solutions in the Outer Region

Assume that the region surrounding the plasma contains a compressible, non-conducting, dissipationless gas. In equilibrium it is at rest and has a constant density ρ_o and pressure p_o. To describe the hydrodynamic effects we deal with the mass conservation equation (2.10), the momentum equation (2.11) with $\mathbf{j} = \sigma = 0$, and the adiabatic equation of state (2.12). If we substitute our equilibrium and perturbation quantities into these equations, they become

$$\partial_t \tilde{\rho} + \rho_o \nabla \cdot \tilde{\mathbf{v}} = 0 \tag{3.2}$$

$$\rho_o \, \partial_t \, \tilde{\mathbf{v}} = -\nabla \tilde{p} \tag{3.3}$$

$$\partial_t \, \tilde{p} = c_{sg}^2 \, \partial_t \tilde{\rho} \tag{3.4}$$

where

$$c_{sg}^2 = \frac{\gamma_o p_o}{\rho_o} \tag{3.5}$$

The acoustic wave equation is obtained by substituting 3.2 and 3.4 into the divergence of 3.3

$$\nabla^2 \tilde{p} - \frac{1}{c_{sg}^2} \, \partial_{tt}^2 \, \tilde{p} = \nabla^2 \tilde{p} + \frac{\omega^2}{c_{sg}^2} \tilde{p} = 0 \tag{3.6}$$

If one substitutes the exponential form of the perturbation quantity in 3.1 into 3.6, one obtains the modified Bessel d.e. Thus, in the outer region

$$\tilde{p} = \tilde{p}_o^o \, K_m(\zeta_g r) \tag{3.7}$$

where K_m is the modified Bessel function of the second kind and

$$\zeta_g^2 = k^2 - \frac{\omega^2}{c_{sg}^2} \qquad \text{and} \qquad \zeta_g = + \sqrt{k^2 - (\omega^2/c_{sg}^2)} \tag{3.8}$$

$\tilde{p}_o^o$ =(constant) exp[i(mΘ + kz + ωt)]

$\tilde{\mathbf{v}}_o$ is obtained from 3.3 as

$$\tilde{w}_o = \frac{\tilde{p}_o^o}{\rho_o \omega r} \left\{ [i \zeta_g r \, K_m'(\zeta_g r)], \; [-mK_m(\zeta_g r)], \; [-krK_m(\zeta_g r)] \right\} \tag{3.9}$$

Maxwell's equations describe the electromagnetic phenomena in the outer medium where the inductive capacities are the same as those in a vacuum, $\epsilon = \epsilon_o$ and $\mu = \mu_o$. These can be rearranged in the conventional manner to yield wave equations for $\tilde{\mathbb{E}}$ and $\tilde{\mathbb{B}}$.

$$\nabla^2 \tilde{\mathbb{E}} + \frac{\omega^2}{c^2} \tilde{\mathbb{E}} = 0, \qquad \nabla^2 \tilde{\mathbb{B}} + \frac{\omega^2}{c^2} \tilde{\mathbb{B}} = 0 \tag{3.10}$$

If one substitutes the perturbation exponent into the z components of the above equations, one obtains the modified Bessel equation. The solutions valid in the outer region are

$$\tilde{B}_z = \tilde{B}_z^o \, K_m(\zeta_o r) \tag{3.11}$$

$$\tilde{E}_z = \tilde{E}_z^o \, K_m(\zeta_o r) \tag{3.12}$$

where

$$\zeta_o^2 = k^2 - \frac{\omega^2}{c^2} \qquad \text{and} \qquad \zeta_o = + \sqrt{k^2 - (\omega^2/c^2)} \tag{3.13}$$

The remaining components of $\tilde{\mathbb{B}}$ and $\tilde{\mathbb{E}}$ can be obtained from 2.15 and 2.17 or their component representations, A2.12-A2.14 and A2.17-A2.19.

Let us consider now the special cases presented in problems 2 and 3. Here, electromagnetic radiation effects are neglected; that is, $\omega^2/c^2 \to 0$ and $\zeta_o \to |k|$. Hence, A2.12-A2.14 simplify because we neglect the displacement current and $\tilde{\mathbf{B}}$ is determined as

$$\tilde{\mathbf{B}}_o = \frac{\tilde{B}^o_o}{kr} \left\{ -i|kr|\, K'_m(kr),\ mK_m(kr),\ krK_m(kr) \right\} \qquad (3.14)$$

A similar expression can be written for $\tilde{\mathbf{E}}$.

4 THE EQUILIBRIUM STATE

As a point of departure for making hm stability investigations, one must have a complete knowledge of the equilibrium state. References 25, 26, and 27 are typical studies involving a cylindrical plasma at rest. In Chapter 1 three problems arose because different assumptions were made as to the behavior of the equilibrium variables. In this chapter we will derive these results after making certain fundamental and simplifying assumptions.

4.1 Assumptions in the Equilibrium State ($\partial_t = 0$)

1. The plasma is in motion.
2. All dependent variables are finite and independent of θ and z. $\partial_\theta = \partial_z = 0$.
3. The plasma density and kinetic pressure are everywhere > 0.
4. The plasma is compressible in problems 1 and 3. In problem 2 it is incompressible.
5. The plasma is electrically neutral $\sigma = 0$.
6. The plasma carries a current sheet $\mathbf{j}^* = (0, j^*_\theta, j^*_z)$.
7. The plasma is non-viscous. That is, the pressure tensor is diagonal. If one includes the viscosity (as a scalar), one finds that v_z has a parabolic distribution with radius. The analysis is identical to that given for the cylindrical Poiseuille flow problem in hydrodynamics.
8. The plasma is resistive. We assume that the resistivity is a scalar to see how it determines the magnetic field configuration. In the stability analysis we neglect it.

4.2 Magnetic and Velocity Field Components

1. Equations 2.10, 2.14 and 2.16, when subjected to assumptions 2 and 3 above, yield

$$E_r = B_r = v_r = 0 \qquad (4.1)$$

2. The r component of Ohm's Law, A2.21, gives us the important relation

$$v_\theta B_z = v_z B_\theta \qquad \text{or} \qquad v_\theta v_{Az} = v_z v_{A\theta} \qquad (4.2)$$

where

$$v_{Ai} = (\mu_0 \rho_p)^{-1/2} B_i \qquad (4.3)$$

That is, $\mathbb{V}$ is parallel to $\mathbb{B}$. This is a well known result for a perfectly conducting plasma and implies that matter streamlines coincide with magnetic field lines. Hence, if a plasma is streaming longitudinally in a combined B_θ, B_z field, it must have an azimuthal velocity component. (More about this below.)

3. The θ component of the curled Ohm's Law (with resistivity), A2.26, yields a second order linear d.e. in B_θ whose solution is

$$B_\theta = B_\theta \frac{r}{r_e} \qquad (4.4)$$

4. Similarly, the z component of the curled Ohm's Law (with resistivity), A2.27, yields a second order linear d.e. in B_z whose solution is

$$B_z = \text{a constant} \qquad (4.5)$$

If one had neglected the resistivity, one would not have the determined arbitrary functions of r for both B_Θ and B_z (4.4 and 4.5).

5. Since viscous effects have been omitted, v_z can be an arbitrary function of r. For convenience we take it to be uniform, v_z = constant. Thus

$$v_\Theta = v_\Theta \frac{r}{r_e} \tag{4.6}$$

as determined by 4.2. If we define

$$\Lambda = \frac{v_\Theta}{v_{A\Theta}} = \text{constant}, \qquad h = \frac{B_z}{B_\Theta} = \text{constant} \tag{4.7}$$

and use relation 4.2, we obtain the equilibrium distributions presented in Chapter 1.

A few notes are in order. A linearly dependent B_Θ means a uniform j_z. It can be shown that in order to have a j_Θ one must remove the requirement $\partial_\Theta = 0$.

The important requirement imposed by 4.2 led the author to study a similar problem from the particle point of view. At $t = 0$ a charged particle was moving with a velocity $\mathbb{W} = \mathbb{e}_z v_z$ at $r = r_e$ in a magnetic field $\mathbb{B} = (0, B_\Theta \frac{r}{r_e}, B_z)$. An analysis was performed which linearized the equations about an equilibrium configuration. If one assumes $(B_z/B_\Theta) \gg 1$, then the equilibrium equations impose the condition that the azimuthal velocity has a slow precession $\dot{\Theta}$, given by

$$\dot{\Theta} = \frac{v_z B_\Theta}{B_z r_e} \tag{4.8}$$

This is identical with 4.2. Furthermore, it should be noted that $\dot{\theta}$ is independent of both the mass and the charge of the particle. The same simple condition does not hold if $(B_z/B_\theta) \ll 1$.

For convenience we will write the θ components of the magnetic and velocity field as

$$B_\theta = B(r/r_e), \qquad v_\theta = v(r/r_e) \tag{4.9}$$

omitting the subscript θ.

4.3 Pressure and Density Distributions

To find the kinetic pressure distribution, one uses the r component of the momentum conservation equation, A2.6. If the proper assumptions are made, one obtains

$$-\rho \frac{v_\theta^2}{r} = -\partial_r \left[p + \frac{B_\theta^2 + B_z^2}{2\mu_o} \right] - \frac{B_\theta^2}{\mu_o r} \tag{4.10}$$

Using the assumptions given in 4.4-4.6, one obtains

$$\frac{1}{\rho_p} \partial_r p - \frac{\rho}{\rho_p} \left(\frac{v}{r_e}\right)^2 r = -2\left(\frac{v_A}{r_e}\right)^2 r \tag{4.11}$$

If we multiply and divide the first term by the normalization pressure p_p and define

$$\frac{\gamma p_p}{\rho_p} = c_{sp}^2; \quad \text{and} \quad M = \frac{v}{c_{sp}}, \qquad M_A = \frac{v_A}{c_{sp}} \tag{4.12}$$

and

$$x = \frac{r}{r_e} \tag{4.13}$$

we obtain

$$\partial_x\left(\frac{p}{p_p}\right) - \gamma\left(\frac{\rho}{\rho_p}\right) M^2 x = -2\gamma M_A^2 x \tag{4.14}$$

In the incompressible case (problem 2) $\rho = \rho_p$ and 4.14 is readily integrated to yield

$$\frac{p}{p_p} = 1 - \frac{\gamma x^2}{2}\left[2M_A^2 - M^2\right] \tag{4.15}$$

Thus the equilibrium normalized hm pressure is given by

$$\pi_p = \frac{P_m}{\rho_p} - \frac{c_s^2 x^2}{2}\left[M_A^2 - M^2\right] = \frac{P_m}{\rho_p} - \frac{v_A^2 x^2}{2}\left[1 - \Lambda^2\right]$$

(incompressible) (4.16)

where p_p = kinetic pressure at $x = 0$, and

$$P_m = p_p + \frac{h^2 B^2}{2\mu_o} = \text{constant} \tag{4.17}$$

If the medium is compressible (problem 3), ρ and p are related through the adiabatic equation of state

$$\frac{p}{p_p} = \left(\frac{\rho}{\rho_p}\right)^\gamma \tag{4.18}$$

Hence, equation 4.14 can be transformed to

$$y^{\gamma-1}\frac{dy}{dx} = \beta x\left[y - \alpha\right] \tag{4.19}$$

where

$$\beta = M^2, \qquad \alpha = \frac{2}{\Lambda^2} = 2\frac{M_A^2}{M^2}, \qquad y = \frac{\rho}{\rho_p} = \left(\frac{p}{p_p}\right)^{1/\gamma} \tag{4.20}$$

This equation can be integrated exactly, but unfortunately y will be implicitly related to x. That is,

$$\frac{\beta x^2}{2} = \int_{y_o}^{y} \frac{y^{\gamma-1}dy}{y-\alpha} = \alpha^{\gamma-1} \int_{y/\alpha}^{y_o/\alpha} t^{\gamma-1}(1-t)^{-1}\, dt$$

$$= \alpha^{\gamma-1} \left\{ B_{y_o/\alpha}[\gamma,0] - B_{y/\alpha}[\gamma,0] \right\} \qquad (4.21)$$

where $B_x[p,q]$ = incomplete beta function = $\int_o^x t^{p-1}(1-t)^{q-1}\, dt$ (4.22)

A more useful result is obtained by assuming that the solution of 4.19 can be expressed as a simple power series in x. The result of such a calculation yields the result

$$y = 1 - \beta\alpha_1 \frac{x^2}{2} - \frac{\beta^2\alpha_1}{8}(1+\alpha_1\gamma_1) - \frac{\beta^3\alpha_1}{48}[1+4\alpha_1\gamma_1+\alpha_1^2(3\gamma_1^2-\gamma_1\gamma_2)]x^6$$

$$+ \sigma\left\{\frac{\beta^4\alpha_1}{384} x^8\right\} \qquad (4.23)$$

where

$$\alpha_n = \alpha - n, \qquad \gamma_n = \gamma - n, \qquad x = \frac{r}{r_e} \le 1 \qquad (4.24)$$

If $\beta = M^2 < 1$, the series converges rapidly and one is justified in taking a two-term approximation to it. This is consistent with the approximations made in studying the dynamics of the compressible problem, Chapter 8. There we make a perturbation expansion in the small parameters M^2 and M_A^2.

A few notes follow.

α and β are not independent. This follows, since we require the density of the plasma to be > 0. If one considers 4.23 at the boundary, $x = 1$, one finds that α has an upper limit which depends on β. Thus

β	upper limit of α	$1+\frac{1.4}{\beta}$
1	2.17	2.40
$\frac{1}{2}$	3.76	3.80

It can also be shown that

$$0 \leq \alpha < 1 + \frac{1.40}{\beta} \tag{4.25}$$

The quantity on the right side of the inequality is given in the last column. The left-hand inequality follows from the fact that α is > 0 (the second equality in 4.20).

To get an estimate of the maximum error which one makes by neglecting the higher order terms in the series of 4.23, we evaluate the expression

$$\frac{|\text{3rd term + 4th term}|}{|\text{2nd term}|} = \frac{\beta}{4}\,[1+\alpha_1\gamma_1] + \frac{\beta^2}{24}\,[1+4\alpha_1\gamma_1+\alpha_1^2(3\gamma_1^2-\gamma_1\gamma_2)] \tag{4.26}$$

This is done at $x = 1$ for $\gamma = 3/2$ and results in the following table.

$\beta = M^2$	$\alpha = 2/\Lambda^2$	Eq. 4.26
1	2.17	.596
$\frac{1}{2}$	3.76	.447
$\frac{1}{4}$	5.60	.351
$\frac{1}{16}$	22.4	.290
0	∞	.257

For small Mach numbers, we see that the higher terms contribute a small amount and can be neglected.

If one substitutes $y = \rho/\rho_p$ into 4.18 and neglects all but the first two terms of the binomial expansion, one obtains

$$\frac{p}{p_p} = 1 - \gamma \frac{x^2}{2} M_A^2 [2 - \Lambda^2] \qquad (4.27)$$

which is identical with the incompressible pressure variation (4.15). We conclude that in the "partially compressible" approximation the hm pressure is given by

$$\pi_p = \frac{p_m}{\rho_p} - \frac{v_A^2 x^2}{2} [1 - \Lambda^2] \qquad \text{(compressible)} \qquad (4.28)$$

while the density variation is given by

$$\frac{\rho}{\rho_p} = 1 - \frac{x^2}{2} M_A^2 [2 - \Lambda^2] \qquad \text{(compressible)} \qquad (4.29)$$

4.4 Behavior of Equilibrium Quantities in the Outer Region

Using Stokes' theorem one can show that at $r = r_e$

$$B_{\theta o} = B_{\theta p} + \mu_o \, j_z^* \qquad \text{or} \qquad b_\theta = 1 + \frac{\mu_o \, j_z^*}{B} \tag{4.30}$$

$$B_{zo} = B_{zp} - \mu_o \, j_\theta^* \qquad \text{or} \qquad b_z = 1 - \frac{\mu_o \, j_\theta^*}{B} \tag{4.31}$$

where

$$b_\theta = \left.\frac{B_{\theta o}}{B_{\theta p}}\right|_{r_e} , \qquad b_z = \left.\frac{B_{zo}}{B_{zp}}\right|_{r_e} \tag{4.32}$$

The kinetic pressure in the outer gas at $r = r_e$ is obtained from the pressure continuity equation (2.39) as

$$p_o = p_p - \frac{\rho_p \, v_A^2}{2} \, [2 - \Lambda^2] + \frac{B^2}{2\mu_o} \, [(1-b_\theta^2) + h^2(1-b_z^2)] \tag{4.33}$$

or

$$p_o = p_p + \rho_p \, \frac{v_A^2}{2} \, [-(1-\Lambda^2) - b_\theta^2 + h^2(1-b_z^2)] = \text{constant} \tag{4.34}$$

We assume p_o is constant in the outer medium. Following an analysis similar to that given in Section 4.2 we show

$$B_{\theta o} = b_\theta \, B \, \frac{r_e}{r} \tag{4.35}$$

$$B_{zo} = b_z hB = \text{constant} \tag{4.36}$$

4.5 Necessary Conditions in the Equilibrium State

Two requirements must be satisfied in the equilibrium state: (1) the kinetic pressure must always be greater than zero, and (2) the hm pressure must be continuous across the boundary. Let us derive the necessary mathematical relations.

One can rewrite 4.15 (or its identical form, 4.27) as

$$p = p_p - \frac{x^2 B^2}{2\mu_o}(2 - \Lambda^2) \qquad (4.37)$$

Dividing through and normalizing, one writes

$$d = d_p - x^2(2 - \Lambda^2) \qquad (4.38)$$

where

$$d_i = \frac{p_i}{B^2/2\mu_o} \qquad (4.39)$$

Since $d > 0$, then at the boundary

$$d_p \geqslant 2 - \Lambda^2 \qquad (4.40)$$

Normalizing and rearranging the equation for the continuity of hm pressure (4.34), one writes

$$d_p = d_o + b_\theta^2 + h^2(b_z^2 - 1) + (1 - \Lambda^2) \qquad (4.41)$$

Thus the equilibrium parameters must satisfy two additional conditions: 4.40 and 4.41.

5 SATISFYING BOUNDARY CONDITIONS IN HYDROMAGNETIC PROBLEMS

In this chapter we show how b.c. are matched to obtain the dispersion relation as a function of the plasma variables, $\tilde{\pi}$, $\tilde{v}_{rp}$, and $\tilde{B}_{rp}$. The procedure will be applied to the physical circumstances described in problems 2 and 3, where we have omitted the effects of electromagnetic radiation. In Chapter 6 the analysis is performed for problem 1, where radiation effects are included.

The three b.c. which will most conveniently yield the dispersion relation are given in 2.38, 2.39, and 2.40. If one substitutes the equilibrium plus perturbation quantities into the r component of the pressure continuity equation (2.39) and neglects second order terms, one obtains

$$\mathcal{P} = \pi_{pe} - \pi_{oe} + \tilde{\pi}_p - \tilde{\pi}_o = 0 \qquad (5.1)$$

$\mathcal{P}$, the normalized hm pressure <u>difference</u>, is zero across the boundary surface, which we assume is a zeroth order discontinuity. $\mathcal{P}$ satisfies the same d.e. as does the boundary, namely,

$$D_t\mathcal{P} = 0 = i\omega \;+\; \mathbb{V}_{ep} \cdot \nabla\tilde{\mathcal{P}} + \tilde{v}_{rp}\, \partial_r \mathcal{P}_e = 0 \qquad (5.2)$$

Hence, rearranging, one obtains

$$i\bar{\omega}[\tilde{\pi}_p - \tilde{\pi}_o] + \tilde{v}_{rp}\, \partial_r[\pi_{pe} - \pi_{oe}] = 0 \qquad (5.3)$$

where

$$\bar{\omega} = \omega + \omega_p \tag{5.4}$$

$$\omega_p = \omega_\theta + \omega_z = \frac{m v_\theta}{r_e} + k v_z = \frac{v}{r_e}(m + Xh) \tag{5.5}$$

and

$$X = k r_e, \qquad m' = m + Xh \tag{5.6}$$

Before continuing with 5.3, let us derive several important relationships from the remaining b.c. With the velocity continuity equation (2.38) and the values of n given in A3.37, one can show that

$$\mathbb{V}_p \cdot \mathbb{n} = \mathbb{V}_o \cdot \mathbb{n} = -\frac{\omega}{\bar{\omega}} \tilde{v}_{rp} = -\tilde{v}_{ro} \tag{5.7}$$

Using the flux continuity condition (2.40) and the solutions for $\mathbb{n}$, one writes

$$\mathbb{B}_p \cdot \mathbb{n} = \mathbb{B}_o \cdot \mathbb{n} = -\tilde{B}_{rp} + \frac{\tilde{v}_{rp}}{\bar{\omega}}\left(m\frac{B_\theta}{r_e} + kB_z\right) = -\tilde{B}_{ro} + \frac{\tilde{v}_{rp}}{\bar{\omega}}\left(b_\theta m\frac{B_\theta}{r_e} + b_z k B_z\right) \tag{5.8}$$

or

$$\tilde{B}_{ro} = \tilde{B}_{rp} + \frac{\tilde{v}_{rp}}{\bar{\omega} r_e}(\delta B) \tag{5.9}$$

where

$$(\delta B) = B_\theta[m(b_\theta - 1) + Xh(b_z - 1)] \tag{5.10}$$

The hm pressure in the outer medium is

$$\pi_o = \pi_{oe} + \tilde{\pi}_o = \frac{p_{oe}}{\rho_p} + \frac{\tilde{p}_o}{\rho_p} + \frac{1}{2\mu_o\rho_p}\left(b_\theta B_\theta \frac{r_e}{r} + \tilde{B}_\theta\right)^2 + \frac{1}{2\mu_o\rho_p}\left(b_z h B_\theta + \tilde{B}_{zo}\right)^2 \tag{5.11}$$

Thus

$$\pi_{oe} = \frac{p_{oe}}{\rho_p} + \frac{v_A^2}{2}\left[\left(b_\theta \frac{r_e}{r}\right)^2 + (b_z h)^2\right] \qquad (5.12)$$

and

$$\tilde{\pi}_o = \frac{\tilde{p}_o}{\rho_p} + v_A\left[b_\theta\, \tilde{v}_{A\theta o} + b_z h \tilde{v}_{Azo}\right] \qquad (5.13)$$

We can relate $\tilde{p}_o$ to $\tilde{v}_{rp}$ through $\tilde{v}_{ro}$. Referring to 3.7 and 3.9, we can write

$$\tilde{p}_o = \tilde{p}_o^o K_m(\zeta_g r_e) = \tilde{v}_{ro} \left\{-i \rho_o \omega r_e \mathcal{K}_m^{-1}(\zeta_g r_e)\right\}$$

$$= \tilde{v}_{rp} \left\{-\frac{i\omega^2 r_e}{\bar{\omega}} \rho_o \mathcal{K}_m^{-1}(\zeta_g r_e)\right\} \qquad (5.14)$$

where

$$\mathcal{K}_m(z) = \frac{z\ K_m'(z)}{K_m(z)} \qquad (5.15)$$

Using the solutions in the outer region, 3.11 and 3.14, we can write $\tilde{v}_{Azo}$ and $\tilde{v}_{A\theta o}$ in terms of $\tilde{v}_{Aro}$. Thus

$$\tilde{v}_{A\theta o} = \frac{m}{X} \tilde{v}_{Azo}^o\ K_m(X) = \tilde{v}_{Aro} \left\{i m \mathcal{K}_m^{-1}(X)\right\} \qquad (5.16)$$

$$\tilde{v}_{Azo} = \tilde{v}_{Azo}^o\ K_m(X) = \tilde{v}_{Aro} \left\{i\ X\ \mathcal{K}_m^{-1}(X)\right\} \qquad (5.17)$$

Substituting 5.14, 5.16, and 5.17 into 5.13, we obtain

$$\tilde{\pi}_o\Big|_{r_e} = \tilde{v}_{rp} i \left\{ -\frac{\omega^2}{\bar{\omega}} r_e \frac{\rho_o}{\rho_p} \mathcal{K}_m^{-1}(\zeta_g r_e) + \frac{v_A(\delta v_A)}{\bar{\omega} r_e} \mathcal{K}_m^{-1}(X)[m b_\theta + X h b_z] \right\}$$

$$+ \tilde{v}_{Arp} i \left\{ v_A \mathcal{K}_m^{-1}(X)[m b_\theta + X h b_z] \right\} \qquad (5.18)$$

where

$$\delta v_A = (\mu_o \rho_p)^{-1/2} (\delta B) \quad \text{(see eq. 5.9)} \qquad (5.19)$$

Substituting π_{pe}, 4.28, and π_{oe}, 5.12, into $\partial_r(\pi_{pe} - \pi_{oe})$ yields

$$\partial_r[\pi_{pe} - \pi_{oe}] = \omega_A^2 r_e [\Lambda^2 - (1 - b_\theta^2)] \qquad (5.20)$$

where

$$\omega_A = v_A / r_e \qquad (5.21)$$

We now substitute 5.18 and 5.20 into 5.3 and divide through by $\tilde{\pi}_p$ and $(\omega_A v_A)/\bar{\omega}$. This yields the dispersion relation in terms of the plasma variables: $\tilde{\pi}_p$, $\tilde{v}_{rp}$, and $\tilde{v}_{Arp}$.

$$\underset{①}{\left(\frac{\tilde{\pi}_p}{\tilde{v}_{rp}}\right) \frac{i\bar{\omega}}{\omega_A^2 r_e}} + \left\{ \underset{②}{[\Lambda^2 - (1 - b_\theta^2)]} - \underset{③}{\frac{\rho_o}{\rho_p} u^2 \mathcal{K}_m^{-1}(\zeta_g r_e)} \right.$$

$$\left. + \underset{④}{(m b_\theta + X h b_z) \left[\frac{\delta v_A}{\omega_A r_e} + \frac{\bar{\omega}}{\omega_A} \frac{\tilde{v}_{Arp}}{\tilde{v}_{rp}} \right] \mathcal{K}_m^{-1}(X)} \right\} = 0 \qquad (5.22)$$

where

$$u = \frac{\omega}{\omega_A} \qquad (5.23)$$

It is instructive to point out the physical significance of each of the terms in the above expression:

1. $\left(\frac{\tilde{\pi}_p}{\tilde{v}_{rp}}\right) \frac{i\bar{\omega}}{\omega_A^2 r_e}$: perturbation of hm pressure of plasma.

2. $\Lambda^2 - (1-b_\theta^2)$: convective effect of boundary; that is, the boundary moves into a region of a different equilibrium hm pressure.

3. $\frac{-\rho_o}{\rho_p} u^2 \mathcal{K}_m^{-1}(\zeta_g r_e)$: perturbation of kinetic pressure in outer region.

4. $(mb_\theta + X\, hb_z) \left[\frac{(\delta v_A)}{\omega_A r_e} + \frac{\bar{\omega}\tilde{v}_{Arp}}{\omega_A \tilde{v}_{rp}}\right] \mathcal{K}_m^{-1}(X)$: perturbation of magnetic (Maxwell stress) pressure in outer region.

Later it will be shown that $\tilde{v}_{Arp}$ and $\tilde{v}_{rp}$ can be expressed in terms of $\tilde{\pi}_p$ and $\partial_r \tilde{\pi}_p$. Thus, all arbitrary constants vanish and one is left with the dispersion relation.

6 STABILITY OF A LONGITUDINALLY STREAMING PLASMA BOUNDED BY A CURRENT SHEET--PROBLEM 1

The stability of a plasma contained by a current sheet was first studied by Kruskal and Schwarzschild in 1954 (1). Rosenbluth in 1956 (20), Tayler in 1957 (17), and Kruskal and Tuck in 1958 (15) included the effects of a uniform longitudinal magnetic field. The first two also included the presence of a perfectly conducting container.

In the present problem we consider an infinite cylindrical plasma which is in uniform motion along the z axis. The dispersion relation for this problem is readily obtained by making a coordinate transformation to a frame of reference which has a velocity = $e_z v_z$. This introduces an additional radial electric field in the outer region but simplifies the problem in the plasma to the form of the ones previously considered. However, the analysis has been included because the approach and the techniques are different.

Figure 1-1 shows a sketch of the details of the equilibrium configuration. The current sheet gives rise to a step discontinuity in the magnetic field at the cylindrical surface.

Using Stokes' Law, one shows immediately

$$j^*_\theta = \frac{B}{\mu_o}[b_i - b_o] , \qquad j^*_z = \frac{B}{\mu_o} \tag{6.1}$$

In equilibrium the hm pressure continuity equation (2.39) gives

$$p = \frac{B^2}{2\mu_o}\ [1 + b_o^2 - b_i^2] \tag{6.2}$$

6.1 Characteristic Equation of the Plasma

One introduces the above equilibrium assumptions into the plasma momentum equation (A2.6-A2.8) and disregards products of two or more perturbation quantities. The normalized hm pressure π is treated as the basic dependent variable. Rearranging, one obtains

$$\tilde{\mathbf{V}} = \begin{bmatrix} \tilde{v}_r \\ \\ \tilde{v}_\theta \\ \\ \tilde{v}_z \end{bmatrix} = \frac{1}{\bar{\omega}_z} \begin{bmatrix} i\partial_r\tilde{\pi} - \dfrac{\omega B_z^2}{\mu_o \rho_p c^2}\tilde{v}_r + \left(\dfrac{kB_z}{\mu_o \rho_p} + \dfrac{\omega B_z v_z}{\mu_o \rho_o c^2}\right)\tilde{B}_r \\ \\ -\dfrac{m\tilde{\pi}}{r} - \dfrac{\omega B_z^2}{\mu_o \rho_p c^2}\tilde{v}_\theta + \left(\dfrac{kB_z}{\mu_o \rho_p} + \dfrac{\omega B_z v_z}{\mu_o \rho_p c^2}\right)\tilde{B}_\theta \\ \\ -k\tilde{\pi} + \left(\dfrac{kB_z}{\mu_o \rho_p}\right)\tilde{B}_z \end{bmatrix} \tag{6.3}$$

where

$$\bar{\omega}_z = \omega + \omega_z = \omega + kv_z \tag{6.4}$$

$$B_z = b_i B = \text{plasma magnetic field} \tag{6.5}$$

Similarly, one introduces the equilibrium assumptions into the curled Ohm's Law, eqs. A2.25-A2.27, and onbtains

$$\tilde{\mathbf{B}} = \left\{\tilde{B}_r, \tilde{B}_\theta, \tilde{B}_z\right\} = \frac{b_i B}{\bar{\omega}_z}\left\{k\tilde{v}_r,\ k\tilde{v}_\theta,\ k\tilde{v}_z + \bar{\omega}_z \frac{\tilde{\rho}}{\rho_p}\right\} \tag{6.6}$$

To simplify the expression for $\tilde{B}_z$ we have used the flux and mass continuity equations appropriate to this problem, namely,

$$\frac{1}{r}\,\partial_r(r\tilde{B}_r) + \frac{im}{r}\,\tilde{B}_\theta + ik\tilde{B}_z = 0 \qquad (6.7)$$

$$\frac{1}{r}\,\partial_r(r\tilde{v}_r) + \frac{im}{r}\,\tilde{v}_\theta + ik\tilde{v}_z = -i\,\bar{\omega}_z\,\frac{\tilde{\rho}}{\rho_p} \qquad (6.8)$$

Substituting 6.6 into 6.3 and rearranging, one can solve for $\tilde{\mathbf{v}}$ in terms of $\tilde{\pi}$ and $\tilde{\rho}/\rho_p$, as

$$\left[\tilde{\mathbf{v}} = \frac{1}{\bar{\omega}_T}\left[\begin{array}{l} i\partial_r(\tilde{\pi}) \\ -\frac{m}{r}\,\tilde{\pi} \\ \frac{\bar{\omega}_T}{\bar{\omega}_1}\left\{-k\tilde{\pi} + \frac{\omega_{Az}^2}{k}\,\frac{\tilde{\rho}}{\rho_p}\right\} \end{array}\right.\right. \qquad (6.9)$$

where

$$\omega_{Az} = kb_i v_A = b_i X\omega_A$$

$$\bar{\omega}_T = \bar{\omega}_z - \frac{b_i^2 v_A^2 Y^2}{\bar{\omega}_z r_e^2} \qquad (6.10)$$

$$Y^2 = (\zeta_o r_e)^2 = [X^2 - (\frac{\omega r_e}{c})^2], \qquad Y = +\sqrt{X^2 - (\omega r_e/c)^2}$$

$$v_A = \frac{B}{(\mu_o\rho_p)^{1/2}}, \qquad \omega_A = \frac{v_A}{r_e}$$

The equation of state, 2.7, yields

$$\tilde{p} = \frac{\gamma\,p_e}{\rho_p}\,\tilde{\rho} = c_s^2\,\tilde{\rho} \qquad (6.11)$$

This follows because p_{pe} and p_{oe} are both constants. Thus, for the present case we relate $\tilde{\rho}/\rho_p$ to $\tilde{\pi}$ through the relation

$$\tilde{\pi} = \frac{\tilde{p}}{\rho_p} + \frac{b_i B\tilde{B}_z}{\mu_o \rho_p} = \frac{\tilde{\rho}}{\rho_p} c_s^2 + \frac{b_i B\tilde{B}_z}{\mu_o \rho_p} \qquad (6.12)$$

Using the definition of $\tilde{B}_z$ in 6.6, we can write

$$\tilde{\pi} = \frac{\tilde{\rho}}{\rho_p} \left\{ \frac{\bar{\omega}_1}{\bar{\omega}_z} c_s^2 + b_i^2 v_A^2 \right\} \qquad (6.13)$$

where

$$\bar{\omega}_1 = \bar{\omega}_z - \frac{\omega_{Az}^2}{\bar{\omega}_z} \qquad (6.14)$$

Now, by substituting the results of 6.9 and 6.13 into the mass conservation equation, 6.8, we obtain the modified Bessel equation

$$\frac{1}{r} \partial_r (r \partial_r \tilde{\pi}) - [\frac{m^2}{r^2} + \zeta_p^2] \tilde{\pi} = 0 \qquad (6.15)$$

where

$$U^2 = (\zeta_p r_e)^2 = \frac{[X^2 - b_i^2 M_A^2 \bar{u}_z^2]\,[\bar{u}_z^2 - b_i^2 Y^2]}{[\bar{u}_z^2 + b_i^2 (M_A^2 \bar{u}_z^2 - X^2)]} \qquad (6.16)$$

and

$$u = \frac{\omega}{\omega_A}, \qquad \bar{u}_z = \frac{\bar{\omega}_z}{\omega_A}, \qquad M_A = \frac{v_A}{c_s} \qquad (6.17)$$

Equation 6.10 contains other definitions. Hence,

$$\tilde{\pi} = \tilde{\pi}^o I_m(\zeta_p r) \qquad (6.18)$$

The solution of the second kind, $K_m(\zeta_p r)$, is disregarded in the region of the origin because we require a non-singular solution. The eigenfunctions $\tilde{\mathbb{v}}$ and $\tilde{\mathbb{B}}$ can easily be expressed in terms of 6.18 by using 6.9, 6.6, and 6.13.

6.2 Other Solutions

In the vacuum surrounding the plasma, Maxwell's equations apply. Following the procedure given in Section 3.3, we establish the solutions to the electric and magnetic z component equations as

$$\tilde{B}_{zo} = \tilde{B}^o_{zo} \, K_m(\zeta_o r), \qquad \tilde{E}_{zo} = \tilde{E}^o_{zo} \, K_m(\zeta_o r) \qquad (6.19)$$

The normal differential equation is easily solved as shown in Section A3.5 and yields

$$\mathbb{n} = (-1,0,0) + \frac{\tilde{v}_{rp}}{\bar{\omega}_z r_e} (0,m,X) \qquad (6.20)$$

Note that in this case $\bar{\omega} = \bar{\omega}_z$, since $v_{\theta e} = 0$. Thus,

$$\mathbb{v} \cdot \mathbb{n} = -\frac{\omega}{\bar{\omega}_z} \tilde{v}_{rp} \qquad (6.21)$$

6.3 Calculating the Dispersion Relation by Satisfying Boundary Conditions

Employing the methods described in Chapter 5, we follow the normalized hm pressure difference

$$\partial_t \mathcal{P} + (\mathbb{v} + \tilde{\mathbb{v}}) \cdot \nabla \mathcal{P} = 0 \qquad (6.22)$$

$$\mathcal{P} = [\pi_{pe} + \tilde{\pi}_p] - \frac{1}{2\mu_o\ p}[(B_\theta \frac{r_e}{r} + \tilde{B}_{\theta o})^2 + (b_o B_\theta + \tilde{B}_{zo})^2] \quad (6.23)$$

Recall,

$$w = kv_z\ \mathbb{e}_z = \omega_z\ \mathbb{e}_z$$

Substituting 6.23 into 6.22 and simplifying yields the dispersion relation in the form

$$i\bar{\omega}_z \left\{ \frac{\tilde{\pi}}{\tilde{v}_{rp}} - \frac{v_A}{(\mu_o \rho_p)^{1/2}} \left[\frac{\tilde{B}_{\theta o} + b_o \tilde{B}_{zo}}{\tilde{v}_{rp}} \right] \right\} + \frac{v_A^2}{r_e} = 0 \quad (6.24)$$

The ratio in brackets is obtained from Maxwell's equations in the vacuum. Substituting A2.12 into A2.18, rearranging, adding $b_o \tilde{B}_{zo}$, and finally dividing by $\tilde{v}_{rp}$ yields

$$\left[\frac{\tilde{B}_{\theta o} + b_o \tilde{B}_{zo}}{\tilde{v}_{rp}} \right] = \frac{1}{\zeta_o^2} \left\{ [b_o \zeta_o^2 + \frac{km}{r_e}] \frac{\tilde{B}_{zo}}{\tilde{v}_{rp}} + i \frac{\omega\, \partial_r(\tilde{E}_{zo})}{c^2\ \tilde{v}_{rp}} \right\} \quad (6.25)$$

The b.c. must be utilized to obtain relations for the terms $\tilde{B}_{zo}/\tilde{v}_r$ and $\partial_r(\tilde{E}_{zo})/\tilde{v}_r$ in terms of known constants. If one simplifies the $\mathbb{e}_\theta$ and $\mathbb{e}_z$ component of 2.43 one obtains

$$\tilde{E}_{zo} = \tilde{E}_{zp} - B \frac{\omega}{\bar{\omega}_z} \tilde{v}_{rp} \quad (6.26)$$

$$\tilde{E}_{\theta o} = \tilde{E}_{\theta p} - B \frac{\omega}{\bar{\omega}_z} (b_i - b_o) \tilde{v}_{rp} \quad (6.27)$$

$\tilde{E}_{zp}$ and $\tilde{E}_{\theta p}$ are determined from the θ and z components of the Ohm's Law equation in the plasma. Substituting these results into 6.26 and 6.27 yields

$$\tilde{E}_{zo} = - B \frac{\omega}{\bar{\omega}_z} \tilde{v}_{rp} \tag{6.28}$$

$$\tilde{E}_{\theta o} = b_o B \frac{\omega}{\bar{\omega}_z} \tilde{v}_{rp} \tag{6.29}$$

Equation 6.28 provides the necessary relation between $\tilde{E}_{zo}$ and $\tilde{v}_{rp}$ for use in 6.25. To obtain the relation between $\tilde{B}_{zo}$ and $\tilde{v}_{rp}$, substitute A2.13 into A2.17 and rearrange

$$\tilde{E}_{\theta o} = \frac{1}{\zeta_o^2} \left[\frac{mk}{r_e} \tilde{E}_{zo} - i\omega \partial_r (\tilde{B}_{zo}) \right] \tag{6.30}$$

This, together with 6.28 and 6.29 yields the ratio

$$\frac{\tilde{B}_{zo}}{\tilde{v}_{rp}} = \frac{i \, K_m(Y) B}{\zeta_o \, K_m'(Y) \quad z} \left[b_o \zeta_o^2 + \frac{km}{r_e} \right] \tag{6.31}$$

From the second equality of 6.19 we obtain

$$\frac{\partial_r (\tilde{E}_{zo})}{\tilde{v}_{rp}} = \frac{\tilde{E}^o_{zo}}{\tilde{v}_{rp}} \zeta_o \, K_m'(Y) \tag{6.32}$$

where ζ_o is as defined in 3.13 and Y as in 6.10. The ratio

$$\frac{\tilde{E}^o_{zo}}{\tilde{v}_{rp}} = - [K_m(Y)]^{-1} \, B \frac{\omega}{\bar{\omega}_z} \tag{6.33}$$

is determined from 6.19 and 6.28. Making the two substitutions (6.33 into 6.32 and then into 6.25, and 6.31 into 6.25) gives us the required ratio for our dispersion relation. When these results are substituted into 6.24 and the results simplified, the dispersion relation is obtained as

$$(\bar{u}_z^2 - b_i^2 Y^2) \mathcal{J}_m^{-1}(U)$$

$$= -1 - [\frac{mX}{Y} + b_o Y]^2 \mathcal{K}_m^{-1}(Y) + \frac{1}{Y^2} (\frac{\omega r_e}{c})^2 \mathcal{K}_m(Y) \quad (6.34)$$

where $\mathcal{K}_m(Y)$ is as defined in 5.15, and where

$$\mathcal{J}_m(z) = \frac{zI_m'(z)}{I_m(z)} \quad (6.35)$$

If one assumes that the normalized characteristic frequencies, ωr_e, are very much smaller than the velocity of light, that is,

$$(\frac{\omega r_e}{c})^2 \ll k^2 \quad \text{and} \quad (\frac{\omega r_e}{c})^2 \ll 1 \quad .$$

one can neglect these quantities. That is, $Y \rightarrow |X|$. Upon rearranging one obtains

$$\mathcal{J}_m(U) = - \frac{[\bar{u}_z^2 - (b_i X)^2]}{1 + [\frac{mX}{|X|} + b_o|X|]^2 \left\{\mathcal{K}_m(X)\right\}^{-1}} \quad (6.36)$$

where

$$U^2 = \frac{(\bar{u}_z^2 - b_i^2 X^2)(X^2 - b_i^2 M_A^2 \bar{u}_z^2)}{(\bar{u}_z^2 - b_i^2 X^2 + b_i^2 M_A^2 \bar{u}_z^2)} \quad (6.37)$$

6.4 Reductions to Previously Obtained Work

Tayler considered the plasma at rest ($\omega_z = 0$) and neglected electromagnetic radiation ($1/c^2$) $\rightarrow 0$. He also considered the effects of the presence of a perfect conductor at $r = R_o = \Lambda r_e$.* Thus,

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* His Λ is different from my Λ .

setting $\Lambda = \infty$, $\omega_z = 0$, and $(1/c^2) = 0$, we find that 6.34 reduces to his equation (3.12), page 1053. The transformation is not immediately obvious, because under the above conditions the coefficient of $[X\mathcal{l}_m^{-1}]$ in 6.36 becomes

$$b_i^2 X^2 - \frac{\gamma}{2} w_o^2(1 + b_o^2 - b_i^2); \qquad \text{where} \quad w_o = \frac{\omega r_e}{c_s} \tag{6.38}$$

whereas he gives the coefficient of $XU(I_m/I'_m)$ as

$$b_i^2 - \frac{\gamma w_o^2(1 + b_o^2 - b_i^2)}{2X_o^2 + w_o^2} \tag{6.39}$$

(taking into account the change in nomenclature). However, one can show that $(1/U^2)$ times eq. 6.38 reduces to 6.39, where U^2 takes on the reduced form

$$U^2 = \frac{(X^2 - w_o^2)\ [X^2 - (\frac{\omega r_e}{b_i v_A})^2\ ]}{[X^2 - w_o^2 - (\frac{\omega r_e}{b_i v_A})^2\ ]} \tag{6.40}$$

Kruskal and Tuck considered the problem where: (1) the plasma was at rest; (2) there were internal and external uniform longitudinal magnetic fields; (3) electromagnetic radiation was present. Making these substitutions, one easily obtains the result given in eq. 6.26, page 227, of their paper (of course taking into account the change in nomenclature).

7 THE STABILITY OF AN INCOMPRESSIBLE PLASMA WITH A TWO COMPONENT EQUILIBRIUM VELOCITY FIELD--PROBLEM 2

A plasma velocity field was first included in the equilibrium configuration by Trehan (18,19). His analysis introduced many simplifications, of which the most restricting was the requirement that Λ, the flow parameter, = 1. This was made at the very beginning of his analysis and greatly simplified the work for reasons which will soon become evident.

The problem being considered was described in Section 1.2 and illustrated in Figure 1-1. For convenience, we repeat the equilibrium conditions:

$$\mathbb{v} = \Lambda v_A(0, \frac{r}{r_e}, h), \qquad \mathbb{v}_A = v_A(0, \frac{r}{r_e}, h); \quad r \leq r_e \quad (7.1)$$

$$\mathbb{v} = 0 \quad , \qquad \mathbb{v}_A = v_A(0, b_\theta \frac{r_e}{r}, hb_z); \quad r > r_e \quad (7.2)$$

and

$$\pi_{pe} = \frac{P_m}{\rho_p} - \frac{v_A^2}{2}\left(\frac{r}{r_e}\right)^2 (1 - \Lambda^2) \qquad \text{(eq. 4.16)} \qquad (7.3)$$

Since the magnetic and velocity fields are of equal importance, we use the symmetric hm equations described in Section 2.2. If one introduces the equilibrium + perturbation quantities into these equations (2.23 and 2.24), cancels the equilibrium quantities, and linearizes the result to first order perturbation quantities, one obtains the set of six linear simultaneous equations

$$[a]\,[\tilde{Q} = [\tilde{\pi}$$

or

$$\begin{bmatrix} i\bar{\omega}_- & -\omega_{-\theta} & 0 & 0 & -\omega_{+\theta} & 0 \\ \omega_{-\theta} & i\bar{\omega}_- & 0 & \omega_{+\theta} & 0 & 0 \\ 0 & 0 & i\bar{\omega}_- & 0 & 0 & 0 \\ 0 & -\omega_{-\theta} & 0 & i\bar{\omega}_+ & -\omega_{+\theta} & 0 \\ \omega_{-\theta} & 0 & 0 & \omega_{+\theta} & i\bar{\omega}_+ & 0 \\ 0 & 0 & 0 & 0 & 0 & i\bar{\omega}_+ \end{bmatrix} \begin{bmatrix} \tilde{Q}_{+r} \\ \tilde{Q}_{+\theta} \\ \tilde{Q}_{+z} \\ \tilde{Q}_{-r} \\ \tilde{Q}_{-\theta} \\ \tilde{Q}_{-z} \end{bmatrix} = -i \begin{bmatrix} -i\partial_r\tilde{\pi} \\ \frac{m}{r}\tilde{\pi} \\ k\tilde{\pi} \\ -i\partial_r\tilde{\pi} \\ \frac{m}{r}\tilde{\pi} \\ k\tilde{\pi} \end{bmatrix} \tag{7.4}$$

where

$$\omega_{\pm\theta} = Q_{\pm\theta}/r_e = \omega_A(\Lambda \pm 1) \tag{7.5}$$

$$\omega_{\pm z} = kQ_{\pm z} = \omega_A Xh(\Lambda \pm 1) \tag{7.6}$$

$$\omega_{\pm p} = \omega_{\pm\theta} + \omega_{\pm z} = \omega_A m'(\Lambda \pm 1) \tag{7.7}$$

$$\omega_p = \Lambda\omega_A m' \qquad \text{(see 5.5 and 5.6)} \tag{7.8}$$

$$\bar{\omega}_{\pm} = \omega + \omega_{\pm p}\,, \qquad \bar{\omega} = \omega + \omega_p \tag{7.9}$$

and

$$\omega_A = v_A/r_e, \qquad \Lambda = v/v_A \tag{7.10}$$

$$X = kr_e\,, \qquad m' = m + Xh \tag{7.11}$$

If h were not a constant, additional terms would appear in elements (3,4) and (6,1). Note the peculiar symmetry possessed by these equations. The coefficients in (i,j) are obtained from those in (i-3, j-3) by changing - to + and + to -.

The inverse of matrix [a] is given by

$$[a]^{-1} = \frac{1}{D_a} \begin{vmatrix} A_{11} & A_{12} & 0 & A_{14} & A_{15} & 0 \\ -A_{12} & A_{11} & 0 & -A_{15} & A_{14} & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ \bar{A}_{14} & \bar{A}_{15} & 0 & \bar{A}_{11} & \bar{A}_{12} & 0 \\ -\bar{A}_{15} & \bar{A}_{14} & 0 & -\bar{A}_{12} & \bar{A}_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{A}_{33} \end{vmatrix} \qquad (7.12)$$

where

$$D_a = \bar{\omega}_- \alpha(\Omega^2 - 1) \qquad (7.13)$$

$$\alpha = \bar{\omega}_-^2 \bar{\omega}_+^3, \qquad \beta = \bar{\omega}_- / \bar{\omega}_+ \qquad (7.14)$$

$$\Omega = (x + y), \qquad x = \omega_{+\theta} / \bar{\omega}_+, \quad y = \omega_{-\theta} / \bar{\omega}_- \qquad (7.15)$$

and

$$A_{11} = -i\alpha(x^2 + xy - 1), \quad \bar{A}_{11} = -i\beta\alpha(y^2 + xy - 1), \quad A_{12} = \alpha y,$$

$$\bar{A}_{12} = \beta\,\alpha x \qquad (7.16)$$

$$A_{14} = i\alpha x\Omega, \; \bar{A}_{14} = i\beta\alpha y\Omega, \quad A_{15} = \alpha x, \quad \bar{A}_{15} = \beta\,\alpha y \qquad (7.17)$$

$$A_{33} = i\alpha(1 - \Omega^2), \quad \bar{A}_{33} = i\beta\alpha(1 - \Omega^2) \qquad (7.18)$$

The elements of the inverse matrix have certain interesting properties. First, A^{-}_{ij} is obtained from A_{ij} by changing $-$ to $+$ and $+$ to $-$. That is,

$$(x \rightarrow y), \quad (y \rightarrow x), \quad (\alpha \rightarrow \beta\,\alpha), \quad \Omega \text{ unchanged}$$

Second,

$$\beta\, A_{12} = A^{-}_{15} \qquad \text{and} \qquad \beta\, A_{15} = A^{-}_{12}$$

Thus $\tilde{Q}]$ can be obtained as $\tilde{Q}] = [a]^{-1}\,[\tilde{\pi}$, or

$$\tilde{Q}] = -\frac{1}{\bar{\omega}_{-}(1-\Omega^2)} \begin{bmatrix} -i & (\partial_r\tilde{\pi} & +(m/r)\,\Omega\,\tilde{\pi}) \\ + & (\Omega\partial_r\tilde{\pi} & +(m/r)\,\tilde{\pi}) \\ + & (1-\Omega^2)k\tilde{\pi} & \\ -\beta i & (\partial_r\tilde{\pi} & +(m/r)\Omega\tilde{\pi}) \\ \beta & (\Omega\partial_r\tilde{\pi} & +(m/r)\,\tilde{\pi}) \\ \beta & (1-\Omega^2)k\tilde{\pi} & \end{bmatrix} \tag{7.19}$$

Note that

$$\tilde{Q}_{-i} = \beta\,\tilde{Q}_{+i} \tag{7.20}$$

and hence

$$\tfrac{1}{2}(\tilde{Q}_{+i} + \tilde{Q}_{-i}) = \tilde{v}_i = \tfrac{1}{2}\,(1+\beta)\,\tilde{Q}_{+i} \tag{7.21}$$

$$\tfrac{1}{2}\,(Q_{+i} - Q_{-i}) = \tilde{v}_{Ai} = \tfrac{1}{2}\,(1-\beta)\,\tilde{Q}_{+i} \tag{7.22}$$

When $\Omega = \pm 1$, $D_a = 0$, and thus the inverse of $[a]$ is undetermined. This problem is considered in detail in Appendix A4. It will be shown that $\Omega = -1$ is always a solution of the dispersion relation, and this consideration is important.

If one substitutes 7.19 into 2.26 and rearranges, one obtains the modified Bessel equation for the normalized hm pressure.

$$\partial^2_{rr}\tilde{\pi} + \frac{1}{r}\partial_r\tilde{\pi} - [\frac{m^2}{r^2} + \zeta_p^2]\tilde{\pi} = 0 \tag{7.23}$$

where

$$\zeta_p^2 = k^2(1 - \Omega^2) \tag{7.24}$$

Thus

$$\tilde{\pi} = \tilde{\pi}^o\, I_m(\zeta_p r) \tag{7.25}$$

$$\tilde{v}_r = \frac{\frac{1}{2}(1+\beta)i}{\bar{\omega}_-(1-\Omega^2)}\,[\partial_r\tilde{\pi} + \frac{m}{r}\Omega\tilde{\pi}] \tag{7.26}$$

$$\tilde{v}_{Ar} = \frac{(1-\beta)}{(1+\beta)}\tilde{v}_r \tag{7.27}$$

Equations 7.26 and 7.27 follow from 7.21 and 7.22

Substituting these results (7.25-7.27) into the dispersion relation (5.22) and setting $r = r_e$ yields

$$\begin{aligned} -\frac{2\bar{\omega}}{\omega_A^2}\frac{\bar{\omega}-\bar{\omega}_+}{(\bar{\omega}_+ + \bar{\omega}_-)}(1-\Omega^2) \\ = \left\{\mathcal{J}_m(\zeta_p r_e) + m\Omega\right\}\left\{[\Lambda^2 - (1-b_\theta^2) - \frac{\rho_0}{\rho_p}u^2\mathcal{K}_m^{-1}(\zeta_g r_e) \right. \\ \left. + (mb_\theta + X\, hb_z)^2\mathcal{K}_m^{-1}(X)\right\} \end{aligned} \tag{7.28}$$

In deriving the last term in the second brace we made use of the relations

$$\frac{\delta v_A}{(\omega_A r_e)} = [mb_\theta + \chi\, hb_z - m']$$ (7.29)

$$\bar{\omega} = \omega + \omega_p = \tfrac{1}{2}(\bar{\omega}_+ + \bar{\omega}_-) = \frac{\bar{\omega}_+}{2}(1 + \beta)$$ (7.30)

$$\tfrac{1}{2}(\bar{\omega}_+ - \bar{\omega}_-) = \frac{\bar{\omega}_+}{2}(1 - \beta) = \omega_A m'$$ (7.31)

The basic structure of 7.28 is the same as that of 6.26, the dispersion relation for problem 1, except for the presence of the $m\,\Omega$, which is added to $\mathcal{J}_m$. This comes from the fact that $\tilde{v}_r$ in problem 2 contains this term (see 7.19, 7.21, and 7.26), while in problem 1, where $B_{\theta e}$ and $v_{\theta e}$ both vanish, this term is absent (see 6.9, r component). Hence, the presence of a longitudinal current (azimuthal flux) changes the basic form of the dispersion relation.

Now the transformation from (7.15) $u = \omega/\omega_A$ to Ω can be written in the form

$$\Omega = \frac{\Lambda + 1}{u + m'(\Lambda + 1)} + \frac{\Lambda - 1}{u + m'(\Lambda - 1)} = \frac{2[\Lambda u + m'(\Lambda^2 - 1)]}{\bar{u}_+ \bar{u}_-}$$ (7.32)

or

$$\frac{2}{m'\Omega} = \frac{u/m'}{\Lambda} + (1 + 1/\Lambda^2) - \frac{(\Lambda^2 - 1)}{\Lambda^2[u/m' + (\Lambda^2 - 1)]}$$ (7.33)

where

$$\bar{u}_\pm = u + m'(\Lambda \pm 1) \quad \text{and} \quad u = \frac{\omega}{\omega_A}$$ (7.34)

If one treats Ω as the independent variable, one obtains the inverse transformation of 7.32 as

$$u = -m' \left\{ \Lambda(1-y) \pm \sqrt{1 - 2y + \Lambda^2 y^2} \right\}$$ (7.35)

or

$$u = -1/\Omega \left\{ \Lambda(m'\Omega - 1) \pm \sqrt{(m'\Omega - 1)^2 + (\Lambda^2 - 1)} \right\} \tag{7.36}$$

where here

$$y = 1/m'\Omega$$

Note that u can be complex for y real if

$$1 - \sqrt{1 - \Lambda^2} < y < 1 + \sqrt{1 - \Lambda^2} \tag{7.37}$$

Hence, if $\Lambda \geq 1$, u cannot be complex for real y. The bounds on Ω are given in the stability diagram of Figure 7-1, where $m'\Omega$ is plotted against Λ.

Substituting 7.30 and 7.32 into the left side of 7.28 yields the dispersion relation in the final form:

$$\mathcal{I}_m(U) + m\Omega = - \frac{2U^2 [\Lambda u + m'(\Lambda^2 - 1)]}{\Omega X^2 D} \tag{7.38}$$

where $\mathcal{I}_m$ and $\mathcal{K}_m$ are defined (as previously) by

$$\mathcal{I}_m(U) = \frac{UI'_m(U)}{I_m(U)}, \qquad \mathcal{K}_m(X) = \frac{XK'_m(X)}{K_m(X)}$$

and where

$$U^2 = (\zeta_p r_e)^2 = X^2(1 - \Omega^2) \tag{7.39}$$

$$Y_g^2 = (\zeta_g r_e)^2 = X^2 - \left(\frac{\omega r_e}{c_{sg}}\right)^2 \tag{7.40}$$

$$D = [\Lambda^2 - (1 - b_\theta^2)] - \frac{\rho_o}{\rho_p} u^2 \mathcal{K}_m^{-1}(Y_g) + (mb_\theta + X\,hb_z)^2 \mathcal{K}_m^{-1}(X) \tag{7.41}$$

Other definitions are given in 7.10, 7.11, and 7.34.

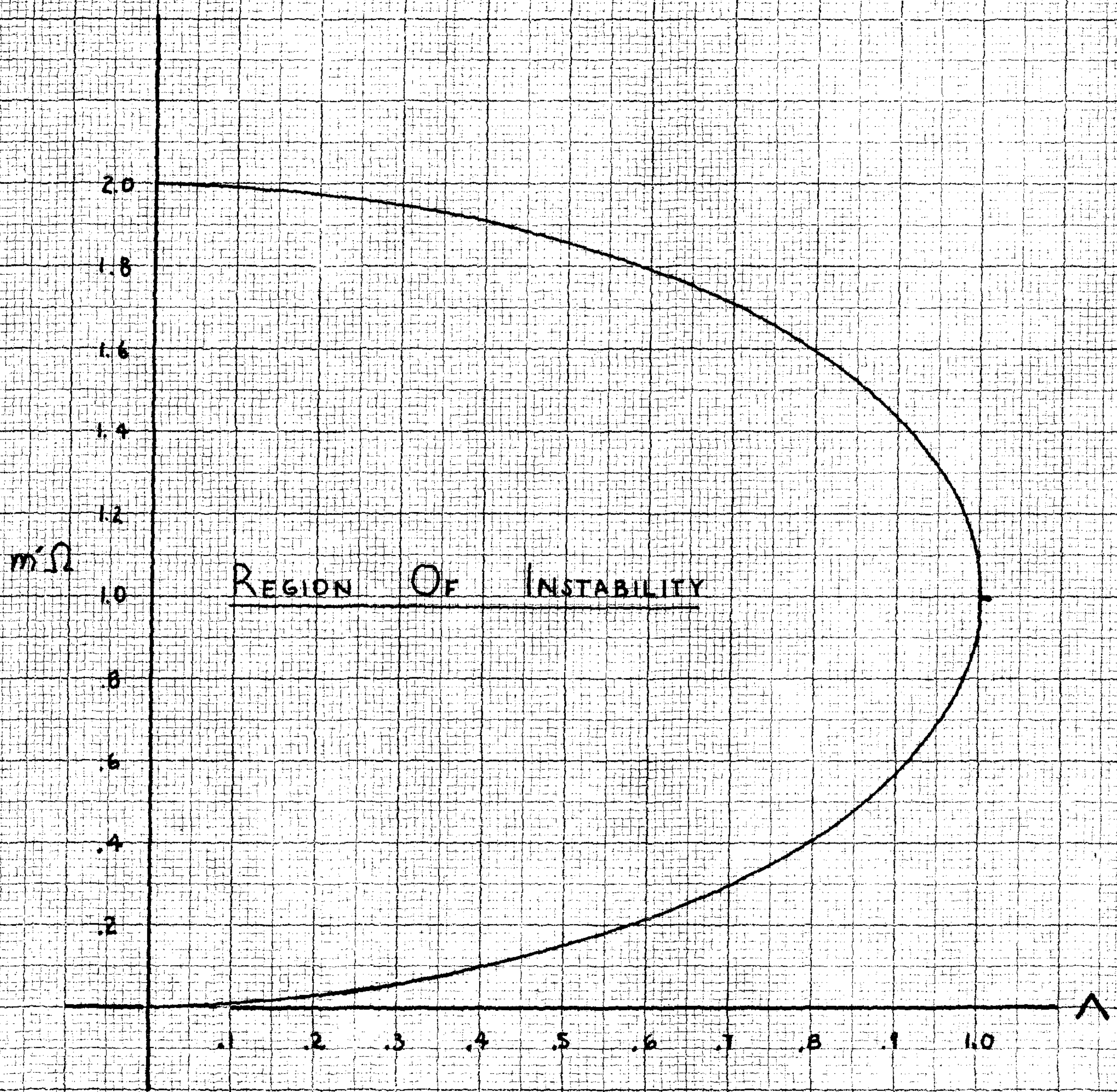

Fig. 7-1. The Stability Diagram For Incompressible Flow (Equation 7.37).

If $\Omega^2 = 1$, $U^2 \to 0$. As shown in Appendix A5,

$$\lim_{U \to 0} \mathcal{J}_m(U) = m + U^2/2(m+1) + \mathcal{O}(U^4) \qquad (7.42)$$

Hence, in the limit $\Omega^2 \to 1$, 7.38 takes on the form

$$m(1+\Omega) = U^2 \left[\frac{-2[\;\;]}{X^2 \Omega D} - \frac{1}{2(m+1)} \right] + \mathcal{O}(U^4) \qquad (7.43)$$

Equation 7.43 is satisfied for $m \geq 1$ by $\Omega = -1$ and for $m = 0$ by $\Omega = \pm 1$. These values of Ω cause the determinant of the system of equations, 7.4, to vanish, and the inversion procedure breaks down. A more careful examination of the basic equation is made in Appendix A4, in the limit $\Omega \to \pm 1$, and it is found that these values of Ω do indeed correspond to solutions of the dispersion relation with eigenfunctions which vary as powers of r. As shown in A4.10 and A.4.11, the eigenfunctions of the "basic modes" are of the form

$$\left.\begin{aligned} \tilde{\pi} &= \tilde{\pi}^o r^m \\ \tilde{Q}_{+r} &= \frac{i\tilde{\pi}^o}{2\bar{\omega}_-} r^{m-1} \left[m + \frac{(kr)^2}{1+m} \right] \end{aligned}\right\} \quad \begin{aligned} &\Omega = -1, \; m \geq 1 \\ &\Omega = \pm 1, \; m = 0 \end{aligned} \qquad (7.44)$$

For $\Omega = -1$, u takes on the form

$$u = -\left\{ \Lambda(m'+1) \pm \sqrt{(m'+1)^2 + (\Lambda^2 - 1)} \right\} \qquad (7.45)$$

7.1 Comparison with Previous Work.

Trehan (18) is the only investigator who has made a careful study of the stability of a cylindrical hm problem with flow. In his analysis he postulates, at the start, $\Lambda = 1$ and arrives at the d.e.

$$\partial_r^2 \tilde{\pi} + \frac{1}{r} \partial_r \tilde{\pi} + [k^2 \chi^2 - \frac{m^2}{r^2}] \tilde{\pi} = 0 \qquad \text{(his eq. 21)} \qquad (7.46)$$

where his

$$\chi^2 = \frac{4}{[u + 2m']^2} - 1 * \qquad \text{(his eq. 22)} \qquad (7.47)$$

These results are equivalent to 7.23 and 7.24 with $\Lambda = 1$, for in this case

$$\Omega = \frac{2}{u + 2m'} \qquad \text{and} \qquad \zeta_p^2 = 1 - \frac{4}{(u + 2m')^2} \qquad (7.48)$$

and thus his $\chi^2 = - \zeta_p^2$.

In Chapter 1 (in the discussion of b.c.) we mentioned that Trehan used as his dispersion relation the simple expression

$$\tilde{\pi}_p = 0 \qquad \text{(his eq. 26)} \qquad (7.49)$$

This was based on an incomplete derivation (his paragraph a, p. 448).

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* In making the transformations from his nomenclature to mine, it is convenient to have the following table:

His:	p	χ^2	k	ϖ	v_A	σ	p	K
Mine:	$\tilde{\pi}$	$-\zeta_p^2$	k	r	hv_A	ω	hr_e	m'/hr_e

If one examines 5.3, one notes that 7.49 is valid only if: (1) $\tilde{\pi}_o = 0$, <u>and</u> (2) $(\partial_r \pi_{pe} - \partial_r \pi_{oe}) = 0$. These approximations cannot be simultaneously justified.

The first assumes $\tilde{\pi}_o \ll \tilde{\pi}_p$, an assumption which is valid only when the gas pressure external to the plasma = 0 in the equilibrium state.

The second is valid only when

$$b_\theta^2 = 1 - \Lambda^2 \tag{7.50}$$

This imposes a requirement on b_z (see 4.41 with $d_o = 0$)

$$b_z^2 = \frac{d_p + 2(\Lambda^2 - 1)}{h^2} + 1 \tag{7.51}$$

Hence, with no gas external to the plasma, 7.49 is the dispersion relation only when 7.50 and 7.51 hold. The first condition says that $\Lambda \leq 1$. The second equation says that the external longitudinal magnetic field is larger than the internal longitudinal field if $d_p > 2(1-\Lambda^2)$. This is true for $\Lambda \approx 1$. From the works of others we know that a strong longitudinal magnetic field acts to stabilize a cylindrical plasma. This implicit condition is probably why Trehan did not observe any plasma instabilities.

8 A PERTURBATION PROCEDURE FOR STUDYING A PARTIALLY COMPRESSIBLE CYLINDRICAL PLASMA--PROBLEM 3

8.1 Introduction

In the analysis of the incompressible plasma treated in Chapter 7, one explicitly assumes that the density of the medium is a constant. This implies that the plasma velocity, v, and the Alfvén velocity, v_A, are negligible in comparison with the sonic speed, $c_s = (\gamma p_o/\rho_p)^{1/2}$. The assumption of small velocities would not be made in a compressible analysis. However, as will soon become evident, such a problem will yield results which are not readily interpretable. To overcome this handicap, a first order perturbation analysis is made in which the parameters of smallness are $(v/c_s)^2$ and $(v_A/c_s)^2$.

We begin with the same equilibrium velocity and magnetic fields given in problem 2. In Chapter 4 we expressed the equilibrium density variation, 4.23, as a power series in the term $M^2 = (v/c_s)^2$. Consistent with a first order perturbation analysis, we omit higher terms and obtain 4.29, which we write as

$$\frac{\rho_e}{\rho_p} = 1 + f \qquad (8.1)$$

where

$$f = -\frac{1}{2}\left(\frac{r}{r_e}\right)^2 M_A^2 \,(2 - \Lambda^2) \qquad (8.2)$$

If we define

$$f' = \partial_r f \qquad (8.3)$$

then

$$\frac{rf'}{2} = f \qquad (8.4)$$

In 4.28 we see that the hm pressure distribution to first order is the same as that for the incompressible problem, 4.16. Since the form of the dispersion relation derived in 5.22 depends only on the equilibrium distributions of the velocity field, magnetic field, and hm pressure, we conclude that 5.22 is applicable to a first order perturbation analysis. The following analyses will determine the term $(\tilde{\pi}_p/\tilde{v}_{rp})$ in 5.22 for the partially compressible plasma and will thus complete the dispersion relation.

To obtain the characteristic d.e. of the plasma one utilizes the quasi-symmetric hm equations (Section 2.3) and manipulates these so that the total solution is represented as a sum of the incompressible term plus a perturbation term. In many cases the calculation is straightforward, although usually exceedingly lengthy, and many of the intermediary steps will be omitted.

8.2 Formulating the Differential Equation of a Partially Compressible Plasma

According to the procedures given in Chapter 3, we substitute the equilibrium plus perturbation terms into 2.28, 2.29, and 2.35. We use 2.35 to eliminate $\mathbb{R}$ from 2.28 and 2.29, and are left with six equations in eight quantities: $(\tilde{Q}_+$, $(\tilde{Q}_-$, $\tilde{\rho}$, and $\tilde{\pi}$. Another relation among these variables is obtained from 2.34 and the adiabatic equation of state, 2.12, as

$$\frac{\tilde{\rho}}{\rho_p} = \frac{1}{c_s^2}\left[\tilde{\pi} - \frac{v_A}{2}\left\{\left(\frac{r}{r_e}\right)(\tilde{Q}_{+\theta} - \tilde{Q}_{-\theta}) + h(\tilde{Q}_{+z} - \tilde{Q}_{-z})\right\}\right] \qquad (8.5)$$

These seven equations can be written in matrix form as

$$(a_c) \left(\begin{matrix} \tilde{Q} \\ \tilde{\rho} \end{matrix} = (\tilde{r}_c \right. \tag{8.6}$$

where these matrices are as defined in Table 8-1. The definitions of the terms used are given in 7.5-7.11 and 8.1

We use the last equation of the set (8.5) to eliminate $\tilde{\rho}/\rho_p$ in each equation and obtain the set of six equations in six unknowns:

$$(a_T) (\tilde{Q} = (\tilde{r}_T \tag{8.7}$$

where

$$(a_T) = (a) + (\delta) \tag{8.8}$$

$$(\tilde{r}_T) = (\tilde{\pi} + (\delta\tilde{\pi} \tag{8.9}$$

(a), $(\tilde{Q}$, and $\tilde{\pi}$ are defined in 7.4 and are the same matrices one deals with in the incompressible analysis. (δ) and $(\delta\tilde{\pi}$ are both of order $(v/c_s)^2$ and are given in 8.10 and 8.11, respectively. This representation exhibits the statement made previously: that we are expanding the solution of the partially compressible problem in the neighborhood of the incompressible problem.

$$(\delta) = \begin{bmatrix} i\delta_{11} & \delta_{12} - \Lambda\delta_{21} & \delta_{13} & i\delta_{11} & -\delta_{12} - \Lambda\delta_{21} & -\delta_{13} \\ (\Lambda-1)\delta_{21} & i(\delta_{11}+\delta_{22}) & i\delta_{23} & (\Lambda-1)\delta_{21} & i(\delta_{11}-\delta_{22}) & -i\delta_{23} \\ -\delta_{31} & i\delta_{23} & i(\delta_{11}+\delta_{33}) & -\delta_{31} & -i\delta_{23} & i(\delta_{11}-\delta_{33}) \\ i\delta_{11} & \delta_{12} - \Lambda\delta_{21} & \delta_{13} & i\delta_{11} & -\delta_{12} - \Lambda\delta_{21} & -\delta_{13} \\ (\Lambda+1)\delta_{21} & i(\delta_{11}-\delta_{22}) & -i\delta_{23} & (\Lambda+1)\delta_{21} & i(\delta_{11}+\delta_{22}) & i\delta_{23} \\ \delta_{31} & -i\delta_{23} & i(\delta_{11}-\delta_{33}) & \delta_{31} & i\delta_{23} & i(\delta_{11}+\delta_{33}) \end{bmatrix}$$

(8.10)

$(a_c) =$

$$
\begin{bmatrix}
i(2\bar{\omega}_- + \bar{\omega} f) & -2\omega_A(\Lambda - 1 + \Lambda f) & 0 & i\bar{\omega} f & -2\Lambda\omega_A(2+f) & 0 & -2r\Lambda^2\omega_A^2 \\
2\omega_A(\Lambda - 1 + \Lambda f + r f'/2) & i(2\bar{\omega}_- + \bar{\omega} f) & 0 & 2\omega_A(\Lambda + 1 + \Lambda f - r f'/2) & i\bar{\omega}_- f & 0 & -i2\omega_A r(\bar{\omega} - \omega f) \\
-\omega_A r_e h f' & 0 & i(2\bar{\omega}_- + \bar{\omega} f) & -\omega_A r_e h f' & 0 & i\bar{\omega} f & -i2\omega_A r_e h(\bar{\omega} - f\omega) \\
i\bar{\omega} f & -2\Lambda\omega_A f & 0 & i(2\bar{\omega}_+ + \bar{\omega} f) & -2\omega_A(\Lambda + 1 + \Lambda f) & 0 & -2r\Lambda^2\omega_A^2 \\
2\omega_A(\Lambda - 1 + \Lambda f + r f'/2) & i\bar{\omega}_+ f & 0 & 2\omega_A(\Lambda + 1 + \Lambda f + r f'/2) & i(2\bar{\omega}_+ + \bar{\omega} f) & 0 & i2\omega_A r(\bar{\omega} - \omega f) \\
\omega_A r_e h f' & 0 & i\bar{\omega} f & \omega_A r_e h f' & 0 & i(2\bar{\omega}_+ + \bar{\omega} f) & i2\omega_A r_e h(\bar{\omega} - \omega f) \\
0 & r\omega_A/2c_s^2 & r_e h\omega_A/2c_s^2 & 0 & -r\omega_A/2c_s^2 & -r_e h\omega_A/2c_s^2 & 1.0
\end{bmatrix}
$$

$$\begin{pmatrix}\tilde{Q} \\ \tilde{\rho}\end{pmatrix} = \left[\tilde{Q}_{+r}, \tilde{Q}_{+\theta}, \tilde{Q}_{+z}, \tilde{Q}_{-r}, \tilde{Q}_{-\theta}, \tilde{Q}_{-z}, \rho/\rho_p\right]$$

$$(r_c = -2\left[\partial_r\tilde{\Pi}, \frac{im}{r}\tilde{\Pi}, ik\tilde{\Pi}, \partial_r\tilde{\Pi}, \frac{im}{r}\tilde{\Pi}, ik\tilde{\Pi}, -\frac{\tilde{\Pi}}{2c_s^2}\right]$$

Table 8-1. Definition of the Matrices in Equation 8.6: $(a_c)\begin{pmatrix}\tilde{Q}_i \\ \tilde{\rho}\end{pmatrix} = (r_c$

$$(\delta\tilde{\pi} = \frac{2\tilde{\pi}}{r\omega_A} [\delta_{12}, i\delta_{22}, i\delta_{23}, \delta_{12}, -i\delta_{22}, -i\delta_{23}] \qquad (8.11)$$

where

$$\delta_{11} = \frac{\bar{\omega} f}{2} = c_{11}\eta^2 \qquad \delta_{21} = \omega_A f = c_{21}\eta^2$$

$$\delta_{12} = \frac{(\Lambda M_A)^2}{2}\omega_A (r/r_e)^2 = c_{12}\eta^2 \qquad \delta_{22} = \frac{M_A^2}{2}(r/r_e)^2\bar{\omega} = c_{22}\eta^2$$

$$\delta_{13} = \frac{(\Lambda M_A)^2}{2} h\,\omega_A (r/r_e) = c_{13}\eta \qquad \delta_{23} = \frac{hM_A^2}{2}(r/r_e)\bar{\omega} = c_{23}\eta$$

(8.12)

and

$$\delta_{31} = \frac{h\,\omega_A r_e f'}{2} = c_{31}\eta \qquad \delta_{33} = \frac{h^2 M_A^2 \bar{\omega}}{2} = c_{33} \qquad (8.13)$$

where

$$\eta = r/r_e \qquad (8.14)$$

In Chapter 4 we used $x = r/r_e$. To avoid confusion with $x = \omega_{+\theta}/\bar{\omega}_+$, defined in 7.15, we will henceforth use r/r_e as defined in 8.14. The c_{ij} are constants. The δ matrix possesses certain local symmetrical properties but no overall symmetry.

We are now in a position to solve for $\tilde{Q})$ by a procedure consistent with the first order perturbation approximation. From 8.7 we can write

$$\tilde{Q}) = (a_T)^{-1} (\tilde{r}_T = (a + \delta)^{-1} (\tilde{\pi} + \delta\tilde{\pi}) \qquad (8.15)$$

We can express the matrix $(a + \delta)^{-1}$ as a power series in (δ), if it is considered small in comparison to (a) (as discussed below).

$$(a + \delta)^{-1} = (a)^{-1} - (a)^{-1}(\delta)\,(a)^{-1} + (a)^{-1}\,(\delta)\,(a)^{-1}(\delta)\,(a)^{-1} - \cdots \tag{8.16}$$

$$= (a^{-1})\,[1 - (\alpha) + (\alpha)^2 - (\alpha)^3 \ldots] \tag{8.17}$$

where

$$(\alpha) = (\delta)\,(a)^{-1} \tag{8.18}$$

The procedure can be justified, provided: (1) (a) has an inverse (that is, its determinant does not vanish), and (2) the eigenvalues of (α) are sufficiently small so that the series is a convergent one.

The first condition fails if $\bar{\omega}_- = \bar{\omega}_+ = 0$ or $\Omega^2 = 1$, as shown in 7.13. Thus, the basic modes corresponding to $\Omega = -1$ (and $\Omega = +1$ for $m = 0$) cannot be treated by this method, and one must return to the original d.e. for a more careful investigation. The procedure would probably take a form similar to that outlined in Appendix A4.

The precise meaning of the second condition is discussed by Bodewig (28) in paragraphs 3.12, 4, and 6.1. He shows that if the Norm of $(\alpha) = (\delta)\,(a)^{-1}$ is < 1 (that is, all eigenvalues of (α) lie in the unit circle), then

$$\lim_{m \to \infty} (\alpha)^m \longrightarrow 0$$

and the series converges. He defines the Norm as

$$N(\alpha) = \text{Norm}(\alpha) = \left\{\text{Trace}(\alpha)^+\,(\alpha)\right\}^{1/2} = \left\{\sum_i \sum_k |\alpha_{ik}|^2\right\}^{1/2} \tag{8.19}$$

and shows that

$$\sum_p \lambda_p^2 \le N^2(\alpha) \tag{8.20}$$

where λ_p are the eigenvalues of (α). Since each term of (δ) involves M^2 or M_A^2, we can make $(\delta)(a)^{-1}$ sufficiently small to satisfy condition 2.

Consistent with the first order analysis, we can drop all but the first two terms of 8.16 and write the solution as

$$\tilde{Q}) = (a)^{-1} (\tilde{\pi} + (a)^{-1} (\delta\tilde{\pi} - (a)^{-1}(\delta)(a)^{-1} (\tilde{\pi} \qquad (8.21)$$

or

$$\tilde{Q}) = \tilde{Q}_0) \qquad + \tilde{Q}_1) \qquad - \tilde{Q}_1') \qquad (8.22)$$

The first term, $\tilde{Q}_0)$, represents the solution of the incompressible problem. $\tilde{Q}_1)$ and $\tilde{Q}_1')$ represent the perturbation terms and are given in 8.23 and in Table 8-2.

$$\tilde{Q}_1) = (a)^{-1} (\delta\tilde{\pi} = \frac{2\tilde{\pi}/\omega_A}{\bar{\omega}_-(\Omega^2-1)r} \begin{bmatrix} i\,\delta_{12} + i(y-x)\delta_{22} \\ -\Omega\delta_{12} + (2x^2+2xy-1)\delta_{22} \\ -(1-\Omega^2)\delta_{23} \\ i\beta[\delta_{12} + (y-x)\delta_{22}] \\ -\beta[\Omega\delta_{12} + (2y^2+2xy-1)\delta_{22} \\ \beta(1-\Omega^2)\delta_{23} \end{bmatrix} \qquad (8.23)$$

The terms used in Table 8-2 have been defined in 7.13-7.15. In addition,

$$\gamma = \frac{1-\beta}{1+\beta} \qquad (8.24)$$

Substituting 7.19 ($\tilde{Q}_0$), 8.23 ($\tilde{Q}_1$), and Table 8-2 ($\tilde{Q}_1'$) into 8.22 we obtain, after rearranging, the matrix $\tilde{Q}$). The first three components of $\tilde{Q}$) are given below.

$$\tilde{Q}'_1 = (a)^{-1}(\delta)(a)^{-1}(\tilde{\pi}$$

$$\tilde{Q}'_{+r1} = -\frac{i\alpha^2(1+\beta)}{D_a^2}\Big(\partial_r\tilde{\pi}\{-\delta_{11}(1+\Omega^2)-\delta_{12}(\gamma\Omega)+\delta_{21}[2\Lambda\Omega+(x-y)]+\delta_{22}\gamma(x^2-y^2)\} + \frac{m}{r}\tilde{\pi}\{-\delta_{11}(2\Omega)-\delta_{12}\gamma$$

$$+\delta_{22}[\Lambda(1+\Omega^2)+(x^2-y^2)]+\delta_{21}\gamma(x-y)\} + \kappa\tilde{\pi}\{-\delta_{13}\gamma(1-\Omega^2)+\delta_{23}\gamma(x-y)(1-\Omega^2)\}\Big)$$

$$\tilde{Q}'_{+\theta 1} = -\frac{\alpha^2(1+\beta)}{D_a^2}\Big(\partial_r\tilde{\pi}\{\delta_{11}(2\Omega)+\delta_{12}(\gamma\Omega^2)-\delta_{21}[\Lambda(1+\Omega^2)+(2x^2+2xy-1)]-\delta_{22}(\gamma\Omega)(2x^2+2xy-1)\}$$

$$+\frac{m}{r}\tilde{\pi}\{\delta_{11}(1+\Omega^2)+\delta_{12}(\gamma\Omega)-\delta_{21}\Omega[2\Lambda+(2x^2+2xy-1)]-\delta_{22}\gamma(2x^2+2xy-1)\}$$

$$+\kappa\tilde{\pi}\{\delta_{13}\gamma\Omega(1-\Omega^2)-\delta_{23}\gamma(1-\Omega^2)(2x^2+2xy-1)\}\Big)$$

$$\tilde{Q}'_{+z1} = -\frac{\alpha^2(1+\beta)}{D_a^2}\Big(\partial_r\tilde{\pi}\{\delta_{31}(1-\Omega^2)+\delta_{23}\gamma\Omega(1-\Omega^2)\} + \frac{m}{r}\tilde{\pi}\{\delta_{31}\Omega(1-\Omega^2)-\delta_{23}\gamma(1-\Omega^2)\}$$

$$+\kappa\tilde{\pi}\{\delta_{11}(1-\Omega^2)^2+\delta_{33}\gamma(1-\Omega^2)^2\}\Big)$$

Table 8-2. Definition of the Elements of the Matrix $\tilde{Q}'_1$).

$$\tilde{Q}'_{-r1} = \beta \tilde{Q}'_{+r1}$$

$$\tilde{Q}'_{-\theta 1} = -\frac{\beta\alpha^2(1+\beta)}{D_a^2}\Big(\partial_r \tilde{\pi}\left\{\delta_{11}(2\Omega) + \delta_{12}(\gamma\Omega^2) + \delta_{21}\left[-\Lambda(1+\Omega^2) + (2y^2+2xy-1) + \delta_{22}\gamma\Omega(2y^2+2xy-1)\right\}\right.$$

$$+\frac{m}{r}\tilde{\pi}\left\{\delta_{11}(1+\Omega^2) + \delta_{12}(\gamma\Omega) + \delta_{21}\left[-2\Lambda\Omega + \Omega(2y^2+2xy-1)\right] + \delta_{22}\gamma(2y^2+2xy-1)\right\}$$

$$+\kappa\tilde{\pi}\left\{\delta_{13}\gamma\Omega(1-\Omega^2) + \delta_{23}\gamma(1-\Omega^2)(2y^2+2xy-1)\right\}\Big)$$

$$\tilde{Q}'_{-z1} = -\frac{\beta\alpha^2(1+\beta)}{D_a^2}\Big(\partial_r \tilde{\pi}\left\{-\delta_{31}(1-\Omega^2) - \delta_{23}\gamma\Omega(1-\Omega^2)\right\} + \frac{m}{r}\tilde{\pi}\left\{-\delta_{31}\Omega(1-\Omega^2) + \delta_{23}\gamma(1-\Omega^2)\right\}$$

$$+\kappa\tilde{\pi}\left\{\delta_{11}(1-\Omega^2)^2 - \delta_{33}\gamma(1-\Omega^2)^2\right\}\Big)$$

As previously defined:

$D_a = \bar{\omega}\alpha(\Omega^2-1)$ $\quad \alpha = \bar{\omega}_-^2\bar{\omega}_+^3$ $\quad \beta = \frac{\bar{\omega}_-}{\bar{\omega}_+}$ $\quad \gamma = \frac{1-\beta}{1+\beta}$

$\Omega = x+y$ $\quad x = \frac{\omega_{+\theta}}{\bar{\omega}_+}$ $\quad y = \frac{\omega_{-\theta}}{\bar{\omega}_-}$

Table 8-2 continued. Definition of the Elements of the Matrix $\tilde{Q}'_1$).

$$\tilde{Q}_{+r} = \frac{i\alpha}{D_a}\left\{-(\partial_r\tilde{\pi} + \frac{m}{r}\Omega\tilde{\pi}) + a_{ro}r^2\partial_r\tilde{\pi} + a_{r1}r\tilde{\pi}\right\} = \frac{\tilde{Q}_{-r}}{\beta} \qquad (8.25)$$

$$\tilde{Q}_{+\theta} = \frac{\alpha}{D_a}\left\{(\Omega\partial_r\tilde{\pi} + \frac{m}{r}\tilde{\pi}) + a_{\theta o}r^2\partial_r\tilde{\pi} + a_{\theta 1}r\tilde{\pi}\right\} \qquad (8.26)$$

$$\tilde{Q}_{+z} = \frac{\alpha}{D_a}\left\{(1-\Omega^2)k\tilde{\pi} + a_{zo}r\partial_r\tilde{\pi} + a_{z1}\tilde{\pi} + a_{z2}r^2\tilde{\pi}\right\} \qquad (8.27)$$

where a_{ri}, $a_{\theta i}$, and a_{zi} are all constants of order M^2 and are given in Table 8-3.

The characteristic d.e. of the plasma is obtained, as previously, by using the mass-flux continuity equation, 2.32. Making the proper substitutions and rearrangements, this is written as

$$\nabla\cdot(\tilde{Q}_+ = -\frac{\rho}{\rho_p}i\bar{\omega} - \frac{(\tilde{Q}_{+r} + \tilde{Q}_{-r})}{2}f' \qquad (8.28)$$

$$= \frac{-i\bar{\omega}}{c_s^2}\left\{\tilde{\pi} - \frac{v_A}{2}(1-\beta)(\eta\tilde{Q}_{+\theta 0} + h\tilde{Q}_{+z0})\right\} - \frac{(1+\beta)}{2}f'\tilde{Q}_{+r0} \qquad (8.29)$$

The $\tilde{Q}_1$) and $\tilde{Q}_1'$) terms do not appear on the right-hand side, since they would add second order terms in M^2. Substituting 7.19 into 8.29 and combining, we obtain

$$\nabla\cdot(\tilde{Q}_+ = \frac{\alpha}{D_a}[a_0 i\tilde{\pi} + a_1 ir\partial_r\tilde{\pi}] \qquad (8.30)$$

with a_0 and a_1 also defined in Table 8-3.

The plasma d.e. is obtained by substituting the results of 8.25-8.27 into 8.30 and rearranging.

$$a_{r0} = \Delta\left\{(2-\Lambda)^2 \frac{\bar{u}}{2}(1+\Omega^2) - \Lambda^2 \gamma \Omega - (2-\Lambda^2)(2\Lambda\Omega + x - y) + \gamma \bar{u}\,(x^2 - y^2)\right\}$$

$$a_{r1} = (M_A/r_e)^2 \left\{\Lambda^2 + \bar{u}(y-x)\right\} + \Delta\left\{m(2-\Lambda^2)\bar{u}\Omega - m\Lambda^2\gamma - m(2-\Lambda^2)\left[\Lambda(1+\Omega^2) + x^2 - y^2\right]\right.$$

$$\left. + m\bar{u}\gamma(x-y) + X h \gamma (1-\Omega^2)\left[-\Lambda^2 + \bar{u}(x-y)\right]\right\}$$

$$a_{\theta 0} = \Delta\left\{+(2-\Lambda^2)\left[-\bar{u}\Omega + \Lambda(1+\Omega^2) + 2x^2 + 2xy - 1\right] + \Lambda^2 \gamma \Omega^2 - \bar{u}\gamma\Omega\,(2x^2 + 2xy - 1)\right\}$$

$$a_{\theta 1} = (M_A/r_e)^2 \left\{-\Lambda^2\Omega + \bar{u}(2x^2 + 2xy - 1)\right\} + \Delta\left\{m(2-\Lambda^2)\left[-\frac{\bar{u}}{2}(1+\Omega^2) + \Omega(2\Lambda + 2x^2 + 2xy - 1)\right] + m\Lambda^2\gamma\Omega\right.$$

$$\left. - m\bar{u}\gamma(2x^2 - 2xy - 1) + X h (1-\Omega^2)\left[\Lambda^2\gamma\Omega - \bar{u}(2x^2 + 2xy - 1)\right]\right\}$$

Table 8-3. The Perturbation Constants of the Differential Equation of a Partially Compressible Fluid.

$$a_{z0} = \Delta\,(\Omega^2-1)\,hr_e\,(2-\Lambda^2-\bar{u}\,\gamma\,\Omega)$$

$$a_{z1} = (M_A/r_e)^2\,hr_e\,\bar{u}\,(\Omega^2-1) + \Delta\,(\Omega^2-1)\,hr_e\left\{m(2-\Lambda^2)\,\Omega + m\,\bar{u}\,\gamma - X\,h\,\bar{u}\,\gamma\,(1-\Omega^2)\right\}$$

$$a_{z2} = -\left(\tfrac{1}{2}\right)(2-\Lambda^2)\,\Delta\,(1-\Omega^2)^2\,K\,\bar{u}$$

$$a_0 = \left(\tfrac{1}{2}\right)(M_A/r_e)^2\,\bar{u}\left\{-(2\,\bar{u}_-)(\Omega^2-1) + (1-\beta)\left[m + X\,h\,(1-\Omega^2)\right] - \frac{m}{\bar{u}}(2-\Lambda^2)(1+\beta)\,\Omega\right\}$$

$$a_1 = \left(\tfrac{1}{2}\right)(M_A/r_e)^2\,\bar{u}\left\{(1-\beta)\,\Omega + \frac{(1+\beta)}{\bar{u}}(2-\Lambda^2)\right\}$$

where $$\Delta = \left(\tfrac{1}{2}\right)(M_A/r_e)^2\,\frac{(1+\beta)}{\bar{u}_-\,(\Omega^2-1)}$$

Table 8-3 continued. The Perturbation Constants of the Differential Equation of a Partially Compressible Fluid.

$$(1-a_{ro}r^2)\,\partial_r^2\tilde{\pi} + [\frac{1}{r} - r(-a_1 + 3a_{ro} + ma_{\theta o} + ka_{zo} + a_{r1})]\,\partial_r\tilde{\pi}$$

$$- [\frac{m^2}{r^2} + \zeta_p^2 + \zeta_{p1}^2 + r^2(ka_{z2})]\tilde{\pi} = 0 \qquad (8.31)$$

where ζ_p^2 is as defined in 7.24 and

$$\zeta_{p1}^2 = -a_o + 2a_{r1} + ma_{\theta 1} + ka_{z1} \qquad (8.32)$$

If we let

$$\tau^2 = (\zeta_{pT}^2 r^2) \qquad \text{and} \qquad \zeta_{pT}^2 = \zeta_p^2 + \zeta_{p1}^2 \qquad (8.33)$$

then 8.31 takes the form

$$(1+\lambda_1\tau^2)\,\partial_\tau^2\tilde{\pi} + (\frac{1}{\tau} + \lambda_2\tau)\,\partial_\tau\tilde{\pi} - [\frac{m^2}{\tau^2} + 1 + \lambda_3\tau^2]\,\tilde{\pi} \qquad (8.34)$$

where

$$\lambda_1 = -a_{ro}(\zeta_p^2 + \zeta_{p1}^2)^{-1} \qquad (8.35)$$

$$\lambda_2 = (a_1 - 3a_{ro} - ma_{\theta o} - ka_{zo} - a_{r1})(\zeta_p^2 + \zeta_{p1}^2)^{-1} \qquad (8.36)$$

$$\lambda_3 = ka_{z2}(\zeta_p^2 + \zeta_{p1}^2)^{-2} \qquad (8.37)$$

Thus we see that the plasma d.e. is no longer the Bessel equation. The coefficients are modified and contain powers of r times λ_i, where all λ_i are of order M^2. If $\lambda_i = 0$ (that is, the incompressible case), then 8.34 reduces to 7.23.

8.3 Solution of the Characteristic Differential Equation of a Partially Compressible Plasma

One recalls that the plasma manifests itself in the dispersion relation for the incompressible problem, 7.38, by the presence of a Bessel function ratio, I'_m/I_m. A similar result applies to the compressible problem, as will become evident in Section 8.4. Thus, the procedure suggested is to transform the plasma d.e. to Ricatti's form* by introducing the variable

$$y = \frac{\partial_\tau \tilde{\pi}}{\tilde{\pi}} = y_0 + y_1 = \frac{I'_m(\zeta_{pT} r)}{I_m(\zeta_{pT} r)} + y_1 \qquad (8.38)$$

The y given in 8.38 is not to be confused with $y = \omega_{-0}/\bar{\omega}_-$, defined in 7.15. If $M^2 \to 0$, then $y \to y_0$ (with $\zeta_{pT} \to \zeta_p$). Thus y_1 represents the perturbation term and is at least of order M^2. If we divide 8.34 by $\tilde{\pi}$ and note that

$$\frac{\tilde{\pi}''}{\tilde{\pi}} = y' + y^2 \qquad (8.39)$$

then 8.34 takes the form

$$(1+\lambda_1\tau^2)(y'+y^2) + \frac{1}{\tau}(1+\lambda_2\tau^2)y = \frac{m^2}{\tau^2} + 1 + \lambda_3\tau^2 \qquad (8.40)$$

where the prime represents differentiation with respect to τ. Substituting $y = y_0 + y_1$ and neglecting second order terms of the form

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* The properties of this non-linear first order d.e. are discussed at great length by Watson (29, paragraphs 4.1 and 4.2).

y_1^2 and $\lambda_i y_1$, we obtain the linearized Ricatti equation

$$y_1' + y_1\left(\frac{1}{\tau} + 2y_0\right) = -m^2\lambda_1 + \tau y_0(\lambda_1 - \lambda_2) - \tau^2(\lambda_1 - \lambda_3) \qquad (8.41)$$

This is a first order linear d.e. and has the particular solution

$$y_1 = \frac{1}{\tau I_m^2(\tau)} \int^{\tau} \xi I_m^2(\xi)\left[-m^2\lambda_1 + \xi(\lambda_1 - \lambda_2)\frac{I_m'}{I_m} - \xi^2(\lambda_1 - \lambda_3)\right] d\xi \qquad (8.42)$$

We disregard the homogeneous solution, since if $\lambda_i = 0$, then $y_1 = 0$. We now have three integrals ($J_{1,m}$, $J_{2,m}$, and $J_{3,m}$) to consider before we can finally express our first order solution in closed form. $J_{1,m}$ and $J_{2,m}$ are readily evaluated from tabulated results. $J_{3,m+1}$ is derived in Section A5.4 (eq. A5.25). The results are presented below.

$$J_{1,m} = \int^{\tau} \xi I_m^2 \, d\xi = \frac{\tau^2}{2}\left\{ I_m^2(\tau) - I_{m-1}(\tau)\, I_{m+1}(\tau)\right\} \; * \qquad (8.43)$$

$$J_{2,m} = \int^{\tau} \xi^2 I_m I_m' \, d\xi = \frac{\tau^2 I_m^2}{2} - J_{1,m} = \left[\frac{\tau^2}{2} I_{m-1}(\tau)\, I_{m+1}(\tau)\right] ** \qquad (8.44)$$

and

$$J_{3,m+1} = J_{3,m-1} - 4m\tau^2\left[\frac{m}{2} I_{m+1} I_{m-1} + \sum_{p=1}^{m-1} (-1)^{p+1} I_{m-p+1} I_{m-p-1} + \frac{(-1)^{m+1}}{2} I_1^2\right]$$

$$+ 2m^2 \tau^2 I_m^2 \qquad (8.45)$$

---

* Given by Jahnke and Emde (30, p. 146).

** Integrate $J_{2,m} = (\frac{1}{2}) \int^{\tau} \xi^2 \, dI_m^2$ by parts.

To use this recursion formula one must have $J_{3,0}$ and $J_{3,1}$.

$$J_{3,0} = \frac{\tau^4}{2}\left\{\frac{I_0^2(\tau)}{2} - \frac{I_1^2(\tau)}{3} - \frac{I_2^2(\tau)}{6}\right\} \tag{8.46}$$

$$J_{3,1} = \frac{\tau^4}{6}\left\{I_1^2(\tau) - I_2^2(\tau)\right\} \tag{8.47}$$

The method for deriving 8.46 and 8.47 is also given in Section A5.4.

Hence, we can express our total solution as

$$y = \frac{\tilde{\pi}'}{\tilde{\pi}} = \frac{I_m'(\tau)}{I_m(\tau)} - \frac{\lambda_1 m^2 \tau}{2} + \frac{\tau}{2}\,[\,\lambda_1(m^2+1) - \lambda_2]\,[\frac{I_{m-1}I_{m+1}}{I_m I_m}]$$

$$- \frac{(\lambda_1 - \lambda_3)}{\tau I_m^2}\, J_{3,m} \tag{8.48}$$

where all I_m are functions of the independent variable τ (8.33).

8.4 Formulation of the Dispersion Relation for a Partially Compressible Plasma

The discussion given in the introduction (Section 8.1) justified the use of the generalized dispersion relation derived in 5.22. To complete this expression we must calculate the terms $(\tilde{v}_{rp}/\tilde{\pi}_p)$ and $(\tilde{v}_{Arp}/\tilde{v}_{rp})$ at the boundary $r = r_e$. From the definition, 2.25, and the derivation, 8.25, we can write

$$\tilde{v}_{rp} = \frac{1}{2}(\tilde{Q}_{+rp} + \tilde{Q}_{-rp}) = (\tfrac{1}{2})(1 + \beta)\tilde{Q}_{+r} \tag{8.49}$$

$$\tilde{v}_{Arp} = \frac{1}{2}(\tilde{Q}_{+rp} - \tilde{Q}_{-rp}) = (\tfrac{1}{2})(1 - \beta)\tilde{Q}_{+r} \tag{8.50}$$

Thus

$$\frac{\tilde{v}_{rp}}{\tilde{\pi}_p} = \frac{-i(1+\beta)a}{2r_e D_a}\left\{\frac{r_e \partial_r \tilde{\pi}}{\tilde{\pi}}(1 - a_{ro}r_e^2) + m\Omega - a_{r1}r_e^2\right\} \tag{8.51}$$

$$\frac{\tilde{v}_{Arp}}{\tilde{v}_{rp}} = \frac{(1-\beta)}{(1+\beta)} = \gamma \qquad \text{(equation 8.24)} \tag{8.52}$$

Equation 8.52 is identical with the result of $(\tilde{v}_{Arp}/\tilde{v}_{rp})$ for the incompressible problem, 7.27. To obtain the complete dispersion relation, one replaces the left-hand side of 7.38 (the incompressible dispersion relation) by the braced quantity in 8.51 and obtains

$$\frac{r_e \partial_r \tilde{\pi}}{\tilde{\pi}} (1 - a_{ro} r_e^2) + m\Omega - a_{r1} r_e^2 = \frac{-2(1-\Omega^2)}{\Omega D} [\Lambda u + m'(\Lambda^2 - 1)] \qquad (8.53)$$

Note that

$$\frac{r_e \partial_r \tilde{\pi}}{\tilde{\pi}} = \frac{\mathcal{T} \partial_{\mathcal{T}} \tilde{\pi}}{\tilde{\pi}} = \text{equation } 8.48$$

If we neglect products of first order quantities and substitute 8.48 into 8.53, we can write the final form of the dispersion relation as

$$\mathcal{J}_m(\mathcal{T}) + m\Omega - \Big\{ r_e^2 [a_{ro} \mathcal{J}_m(\mathcal{T}) + a_{r1}] + \frac{\mathcal{T}^2}{2} \lambda_1 m^2$$
$$+ \frac{\mathcal{T}^2}{2} [\lambda_2 - \lambda_1 (m^2+1)] [\frac{I_{m-1} I_{m+1}}{I_m I_m}] + \frac{(\lambda_1 - \lambda_3)}{I_m^2(\mathcal{T})} J_{3,m} \Big\} \qquad (8.54)$$
$$= \frac{-2(1-\Omega^2)}{\Omega D} [\Lambda u + m'(\Lambda^2 - 1)]$$

The left-hand side corresponds to the plasma, and thus the first order terms a_{ri} and λ_i are present in the brace. The right-hand side corresponds to the external medium and boundary phenomena and so is the same as in 7.38.

Note that $\Omega = -1$ is no longer a root of the dispersion relation.

9 AN ANALYTICAL-NUMERICAL STUDY OF THE PROPERTIES OF THE DISPERSION RELATION FOR INCOMPRESSIBLE FLOW

9.1 Introduction

The problems of hm are frequently of such complexity that many simplifying assumptions must be made to render them mathematically tractable. In many cases, the solution obtained after simplifying assumptions have been made is still too **unwieldy** for numerical analysis. Chandrasekhar remarked (26, p. 232), "... by not considering all the equations of a problem [in hm is inferred], we may miss discovering certain essential novelties which result as a consequence of imposing on a system conformity with two different sets of laws: such as conformity with the laws of electrodynamics and hydrodynamics." The author would like to apply this statement to the necessity for considering the complete b.c. in an eigenvalue problem. Just as with the equations of motion, one may miss many phenomena by oversimplifying the b.c. in hm problems.

The dispersion relation which was studied by means of a Model 205 Datatron digital computer was obtained from 7.38 by setting $\rho_o = 0$. Thus, there is no gas external to the plasma, and it is bound in its equilibrium state by the combined action of an azimuthal and longitudinal magnetic field. A brief examination of 7.38 indicates that one must be prepared to deal with functions of a complex variable, and the computer program must be written for complex numbers.

For example, suppose that when $\Lambda = 0$ (no flow) there are unstable (non-oscillatory) modes which correspond to real values of Ω. This is plausible, because other investigations have shown instabilities when the plasma is at rest. Non-oscillatory, unstable modes correspond to imaginary values of u, as seen in 3.1. When Λ is small but finite, u appears explicitly in 7.38. The purely divergent unstable mode, which satisfied 7.38 with $\Lambda = 0$, must become a complex number when $\Lambda \neq 0$ in order for 7.38 to be satisfied in the same neighborhood of variables and parameters. The computer techniques which were developed to seek the zeros of a function of a complex variable are outlined in Appendix A6.

One last note is in order on complex eigenvalues. In the literature on hm stability there are proofs which establish the reality of the eigenvalues. These are based on demonstrations of the self-adjoint property of the differential operator and often neglect the b.c. integrals. However, the eigenvalue problem at hand is of a class defined by the equations

$$\text{Eq.:} \qquad L\psi(z) = \lambda(u)\, r(z)\, \psi(z)$$

$$\text{B.C.:} \qquad \psi(0) = \text{finite}, \quad [A(u)\psi'(z) = B(u)\psi(z)] \qquad (9.1)$$

Such problems are discussed by Morse and Feshbach (31, paragraph 6.3, pp. 719-728). If one generalizes their arguments given on pp. 727 and 728, one can show that

$$\int_0^b \psi_m(z)\, \psi_n(z)\, r(z)\, dz \neq 0 \qquad (9.2)$$

because

$$[B(u_m)/A(u_m) - B(u_n)/A(u_n)] \neq 0$$

In the above, ψ_m and ψ_n are different eigenfunctions corresponding to the eigenvalues u_m and u_n. To prove the reality of eigenvalues, 9.2 must be satisfied. Because the b.c. are functions of the eigenvalues, one cannot prove a priori that the eigenvalues are real.

9.2 Some Properties of the u, Ω Transformation

The growth rate or oscillation frequency u is related to the basic plasma variable Ω by the transformation introduced in Chapter 7 (7.32, 7.33, 7.35, and 7.36). It can also be written in the form

$$u^2 + u\,[2\Lambda m'(1-y)] + 2m'^2(\Lambda^2-1)(1/2 - y) = 0 \qquad (9.3)$$

where

$$y = 1/(m'\Omega) \qquad (9.4)$$

Thus, as $m'\Omega \to \infty$,

$$\lim_{m'\Omega \to \infty} u \to -m'(\Lambda \pm 1) \qquad (9.5)$$

It will be shown that the dispersion relation has an infinite number of zeros which correspond to large Ω. These are transformed into two regions on the u axis.

In Fig. 9-1 we sketch some of the u, Ω transformations and come to the following conclusions:

a. For $|\Lambda| \gtrless 1$ the transformation from Ω to u is double-valued, while for $|\Lambda| = 1$ the transformation is single-valued.

b. For $\Lambda = 0$ each value of Ω corresponds to values of u which are symmetrically distributed about the origin. For $|\Lambda| > 0$ (and $\Lambda \neq 1$) the symmetry (or degeneracy) is removed. For $\Lambda > 0$ the values of u are displaced toward the negative region, while for $\Lambda < 0$ the shift is toward the positive region.

c. As $|\Omega| \to \infty$, the corresponding values of u are clustered around the singularities of the transformation whose values are given in 9.5.

d. As Λ is increased from 0 to 1, the portion of the curve between the singularities has a minimum of smaller magnitude (as shown in Fig. 9-1b). The minimum occurs at

$$u_{min} = m'(1 - \Lambda^2)^{1/2} \left\{ \frac{(1-\Lambda)^{1/2} - (1+\Lambda)^{1/2}}{(1-\Lambda)^{1/2} + (1+\Lambda)^{1/2}} \right\} \qquad (9.6)$$

Those zeros of the dispersion relation produced by positive values of Ω which fall below this minimum correspond to instabilities; that is, u may be a complex number. As shown on the figure, the size of the region which can correspond to instabilities is proportional to $1/m'$. Therefore, it is largest for: $m = 0$; a small wave number, $X = kr_e$; and a small longitudinal magnetic field, h. For $|\Lambda| > 1$ this region vanishes and no unstable modes arise which correspond to real values of Ω. The foregoing is a more careful restatement of the result made evident in the stability diagram given in Fig. 7-1.

9.3 The Form of the Dispersion Relation Investigated by the Computer

The dispersion relation given in 7.38 can be written as

$$F = m'g + \Lambda u g_1 = 0 \qquad (9.7)$$

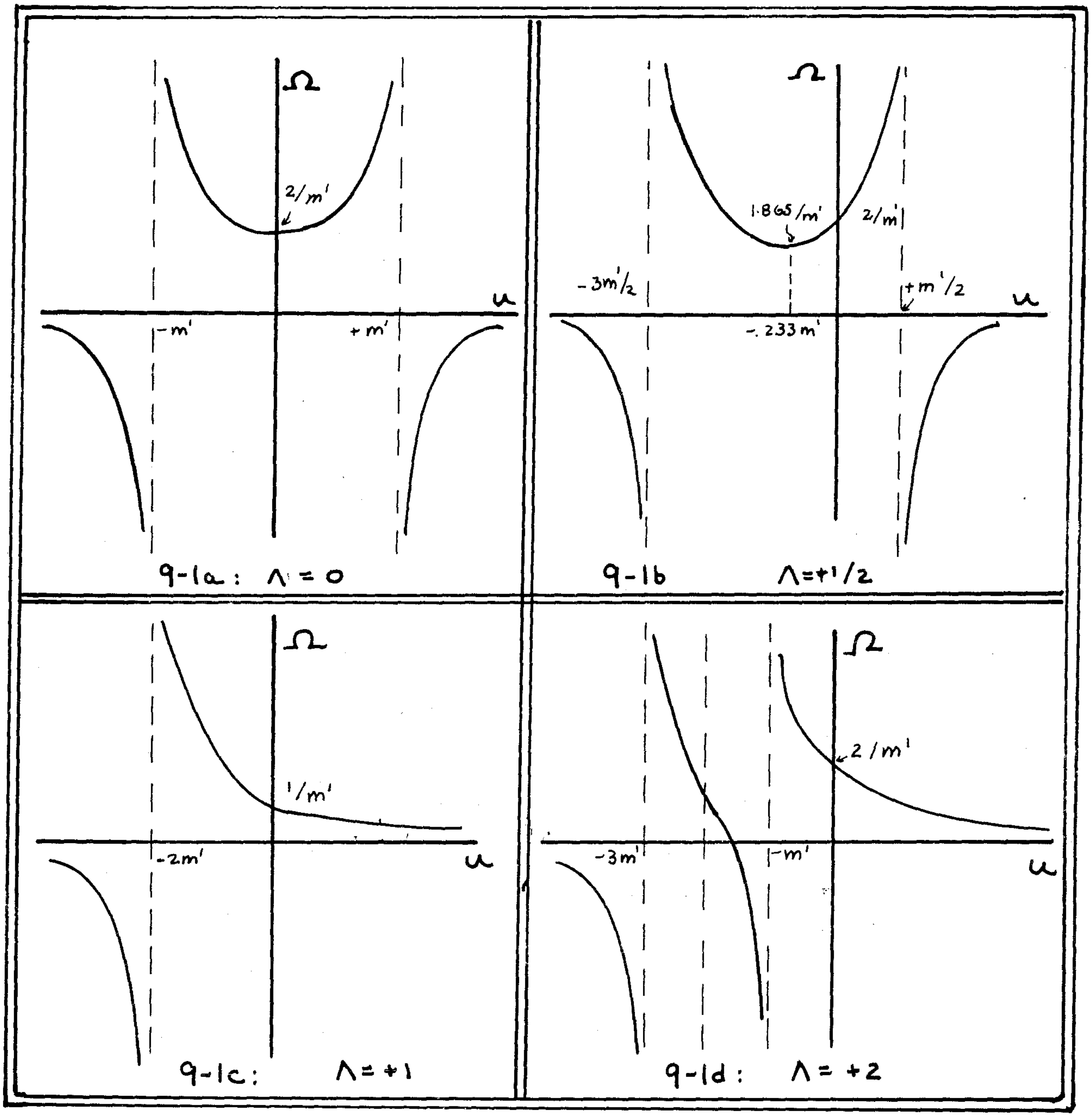

Fig. 9-1. Properties of the u,Ω Transformation for Real Values of u.

where

$$g = Dg_2 + (\Lambda^2 - 1)g_1 \tag{9.8}$$

$$g_1 = m'(1-\Omega^2) \tag{9.9}$$

$$g_2 = (1/2)\,[\Omega\mathcal{J}_m(U) + m\,\Omega^2] \tag{9.10}$$

$$D = \Lambda^2 - 1 + b_\theta^2 + (mb_\theta + X\,hb_z)^2\mathcal{K}_m^{-1}(X) \tag{9.11}$$

where we have set $\rho_o = 0$ in D of 7.41. For $\Lambda = 0$, Ω is taken as the independent variable of the function, $f = g$, whose zeros were sought. The u, Ω transformation calculated two values of u corresponding to each value of Ω which satisfied 9.7.

To avoid dealing simultaneously with the complex u and complex Ω plane when $\Lambda \neq 0$, we rearrange 9.7 as

$$u = \frac{-m'g}{\Lambda g_1} \tag{9.12}$$

and substitute in 9.3, obtaining

$$f = g^2 - g[2\Lambda^2 g_1(1-y)] + 2(\Lambda g_1)^2\,(\Lambda^2-1)(1/2-y) = 0 \tag{9.13}$$

or

$$f = \left\{g - \Lambda g_1[\Lambda(1-y) - \sqrt{1 - 2y + \Lambda^2 y^2}\,]\right\} \times$$

$$\left\{g - \Lambda g_1[\Lambda(1-y) + \sqrt{1 - 2y + \Lambda^2 y^2}\,]\right\} = 0 \tag{9.14}$$

The f of 9.14 is a function in one complex variable, Ω, and was used to study the properties of the $\Lambda \neq 0$ cases. Each value of $\Omega = \Omega_{root}$ which satisfied 9.14 yields one value of u as determined by 9.12, thereby removing the ambiguity which arises in determining u

from the u,Ω transformation. The exceptions to this are $\Omega = -1$ and $\Omega = +1$, $m = 0$, for which cases 9.7 is identically satisfied. The calculation of u in 9.12 was designated by U in the computer print-out (as shown in Fig. A6-2a), in order to distinguish it from the two values of u calculated by the u,Ω transformation, 7.36. A genuine root was distinguished by the fact that U (9.12) agreed with one of the values of u calculated by the u,Ω transformation (Fig. A6-2a).

The functions $\mathcal{J}_m$, and therefore f, have singularities where $I_m(X \sqrt{1-\Omega^2}) = 0$. This occurs when $\Omega > 1$ and

$$X \sqrt{\Omega^2 - 1} = j_{m,p} \qquad (9.15)$$

where $j_{m,p}$ is the p^{th} zero of $J_m(z)$. Thus, equation 9.14 has an infinite number of singularities along the real Ω axis. For large Ω, these singularities are separated by π/X.

When $\Lambda^2 = 1$, that is, where the Alfvén wave velocity equals the fluid velocity, 9.13 becomes

$$f = g \left\{ g - 2g_1(1-y) \right\} = 0 \qquad (9.16)$$

This reduces to zero if

$$g = 0 \qquad \text{or} \qquad g = 2g_1(1-y) \qquad (9.17)$$

* * * * * * * *

The physical situations investigated by the computer were established by fixing two parameters

$$b_z = 1.0 \qquad \text{and} \qquad d_p = \frac{p(r=0)}{B^2/2\mu_0} = 2.0 \qquad (9.18)$$

The first establishes equal longitudinal magnetic fields inside and outside, whereas the second establishes the ratio of the energy density due to thermal motion to the energy density of the magnetic field.* If one now chooses h and Λ and sets $d_o = 0$, b_θ can be evaluated from 4.41.

9.4 Discussion of Results

In the computer study, eq. 9.14 was investigated in the neighborhood of the origin of the Ω plane, $|\Omega| < 4.0$, Imag $\Omega > 0$. For large Ω one can develop asymptotic approximations.** The results are presented graphically in Figures 9-2 through 9-11, where we plot the eigenvalues $u = \omega/\omega_A$ versus the flow parameter Λ. The following table summarizes the parameters which describe the physical situations studied.

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* $d_p = 2.0$ is a reasonable number to expect in an "infinite cylindrical" fusion machine. Spitzer (32, paragraph 3.2) states that, "In an infinite cylinder, values of β [we call it d_p] as great as unity might be envisaged." In particular, if one has a plasma density of 10^{15} particles/cm^3, $T = 5.0$ kilovolt $= 5.8 \times 10^{7}$ °K, $B = 10^4$ gauss, then $d_p = 2.05$.

** For large X, Ω,

$$D \gtrsim - |X| h^2 b_z^2$$

$$F \gtrsim |X| \tan(X\Omega - m\pi/2 - \pi/4) - m + 2(1 \pm \Lambda)/h b_z^2 = 0$$

Table 9-1

A SURVEY OF THE PHYSICAL SITUATIONS STUDIED

Fig.	Group	m	h	X	u asymptotic	
9-2	1	0	0.1	0.1	$-.01(\Lambda \pm 1)$	
9-3	1	0	0.1	1.0	$-.1\ (\Lambda \pm 1)$	
9-4	1	0	0.1	3.0	$-.3\ (\Lambda \pm 1)$	sausage deformation
9-5	2	0	1.0	0.10	$-.1\ (\Lambda \pm 1)$	
9-6	2	0	1.0	1.0	$-1.0\ (\Lambda \pm 1)$	
9-7	3	1.0	0.1	0.1	$-1.01(\Lambda \pm 1)$	
9-8	3	1.0	0.1	1.0	$-1.1\ (\Lambda \pm 1)$	
9-9	3	1.0	0.1	3.0	$-1.3\ (\Lambda \pm 1)$	kink deformation
9-10	4	1.0	1.0	0.1	$-1.1\ (\Lambda \pm 1)$	
9-11	4	1.0	1.0	1.0	$-2.0\ (\Lambda \pm 1)$	

$b_z = 1.0$, $d_p = 2.0$, $b_\theta = (1 + \Lambda^2)^{1/2}$

$X = kr_e$ = wave number of deformation = 0.1 (small), 1.0 (medium), 3.0 (large)

h = longitudinal magnetic field = 0.1 (small), 1.0 (large)

u (asymptotic) denotes the position of the singularities of the u, Ω transformation.

We will discuss the results in groups (Column 2). Care should be exercised in comparing the ordinates of different figures, as the graphs were made to optimize the figures. Not all the modes discovered were plotted, as this would crowd certain graphs. The unstable modes were designated by dashed lines, the oscillation frequency (Re u) by circles, and the growth rate (Im u) by triangles. The latter was always plotted as a positive number. The complex conjugate curve (its mirror image) is understood. All other oscillatory modes were designated by circles connected by solid lines. The diamond form on the Λ axis at $\Lambda = 1$ designates a point which is not present on the curve. The zero value of the oscillation frequency is therefore excluded. This follows because the u, Ω transformation is single-valued at $\Lambda = 1$.

Group 1 (Sausage Deformation - Weak Magnetic Field):

a. The magnitude of the growth rate of the "basic " instability ($\Omega = +1$) increases with increasing wave number.

b. The presence of flow removes the basic instability; the smaller the wave number, the smaller the flow required. This is also seen by examining the stability diagram (Fig. 7-1).

c. In each case there exist other overstable modes with growth rates of smaller magnitude. For small wave numbers ($X = 0.1, 1.0$) these arise with $\Lambda > 0$, then build up to a maximum and vanish with increasing Λ. With $X = 3$ the mode is present at $\Lambda = 0$ as a pure divergence. The magnitude of the growth rate decreases monotonically with increasing Λ. These subsidiary modes are still present after basic modes have been removed and in general show the characteristic of increasing magnitude with increasing wave number.

d. In all the subsidiary modes the magnitude of the oscillation frequency increases with increasing Λ.

Group 2 (Sausage Deformation - Strong Magnetic Field):

a. The basic instabilities show the same behavior as in Group 1, except that the corresponding magnitudes of the growth rates are larger. A greater flow rate Λ is required to remove the basic instabilities than is required for Group 1's instabilities.

c. The subsidiary modes have been removed by the strong field.

Group 3 (Kink Deformation - Weak Magnetic Field):

a. The magnitude of the growth rate varies directly as the wave number.

b. The oscillation frequencies associated with the modes increase monotonically with increasing Λ for all wave numbers.

Group 4 (Kink Deformation - Strong Magnetic Field):

a. All instabilities have been removed and the oscillation frequencies, which were symmetrically distributed at $\Lambda = 0$, are shifted downward as Λ increases.

****

In general, one concludes that:

1. The oscillation frequencies are symmetrically distributed about the origin with $\Lambda = 0$. When $\Lambda > 0$ the mode frequencies are all shifted toward the negative and vary monotonically with Λ.
2. The growth rates are larger for large wave number disturbances.
3. The oscillation frequency for complex modes increases with increasing Λ.
4. Increasing the flow (Λ) removes sausage (m = 0) instabilities and enhances (the magnitude of) kink instabilities.

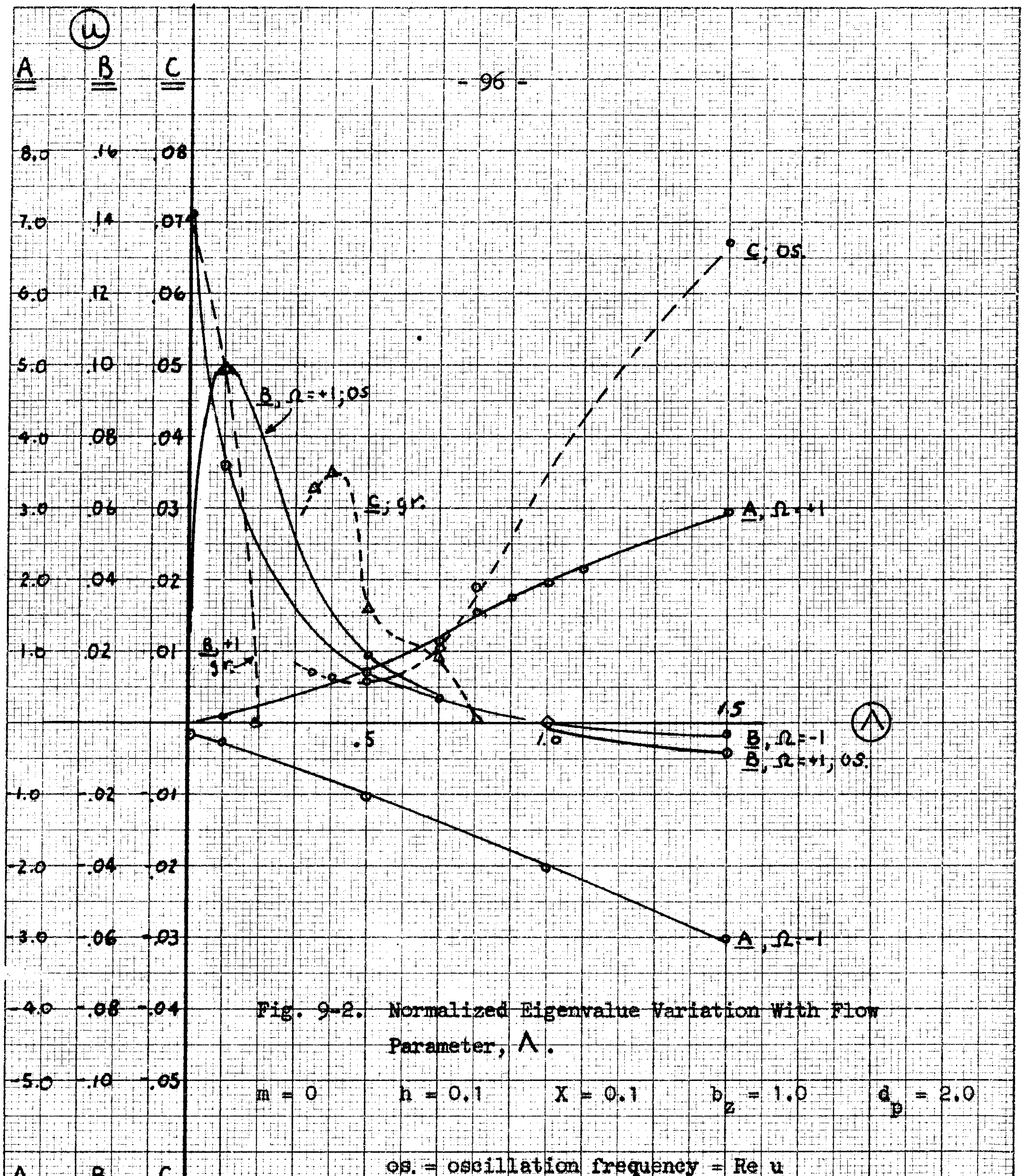


Fig. 9-2. Normalized Eigenvalue Variation With Flow Parameter, Λ.

m = 0 h = 0.1 X = 0.1 b_z = 1.0 d_p = 2.0

os. = oscillation frequency = Re u

gr. = growth rate = |Im u|

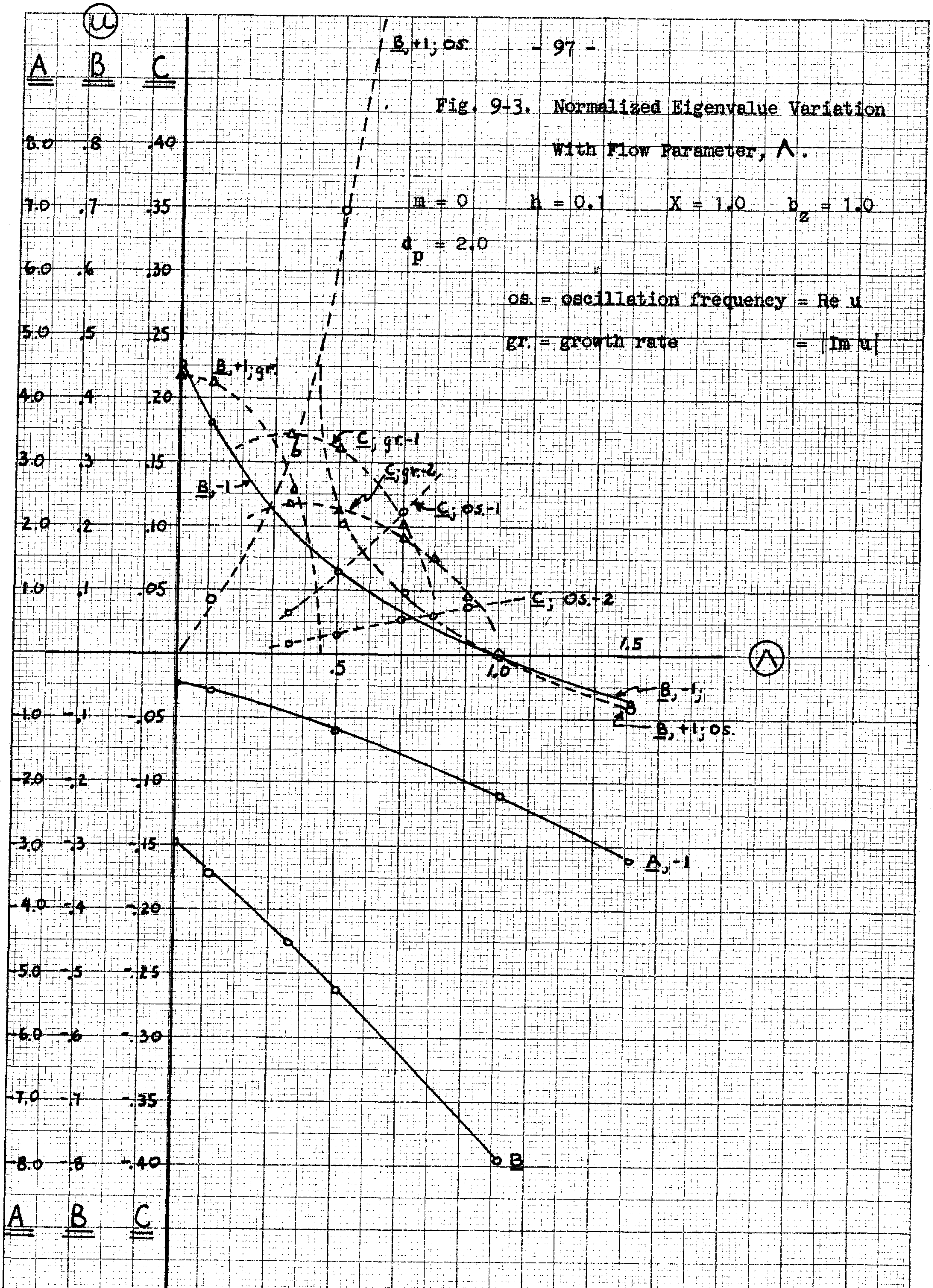


Fig. 9-3. Normalized Eigenvalue Variation With Flow Parameter, Λ.

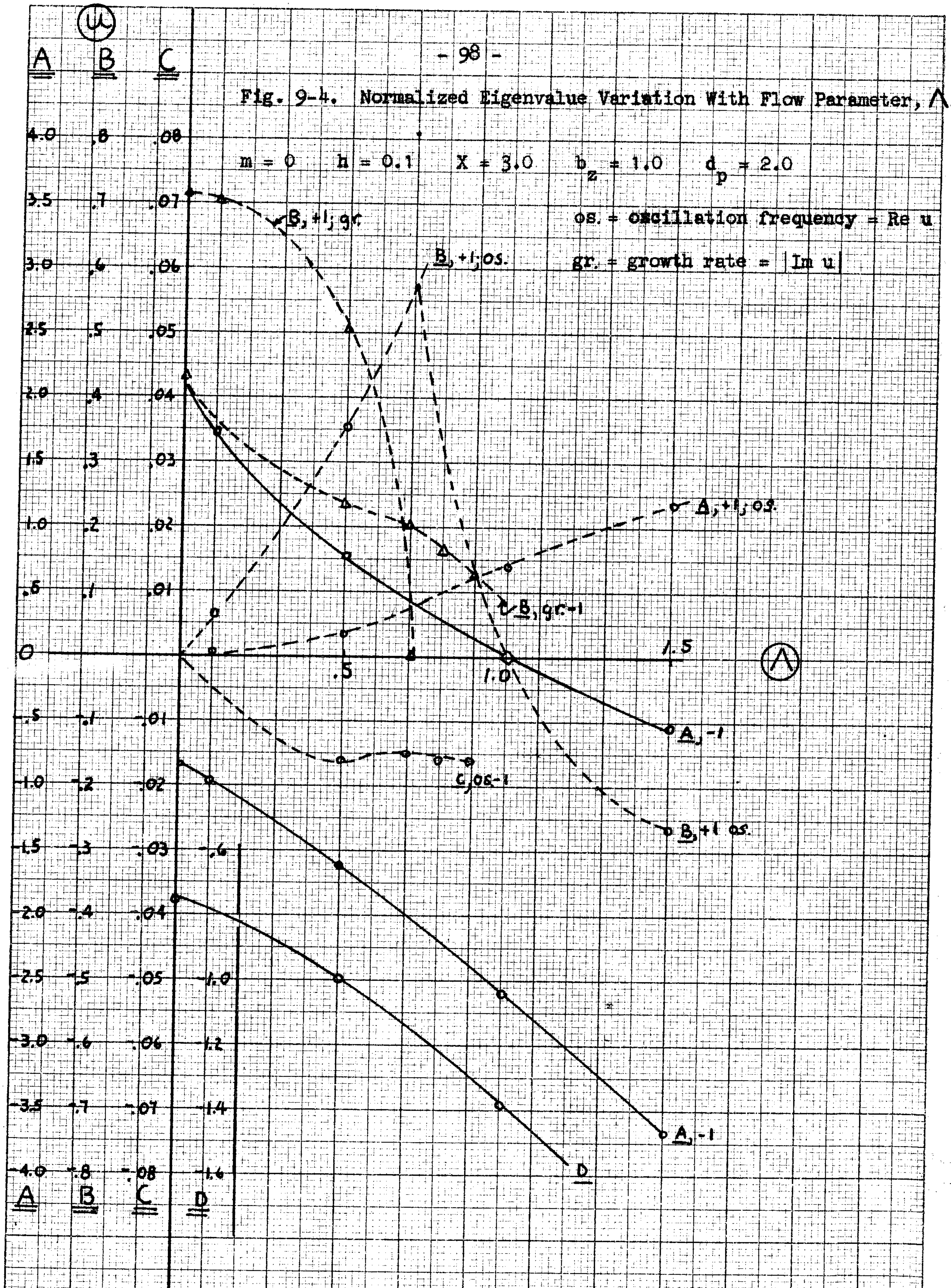
m = 0 h = 0.1 X = 1.0 $b_z = 1.0$

$d_p = 2.0$

os. = oscillation frequency = Re u

gr. = growth rate = |Im u|

Fig. 9-4. Normalized Eigenvalue Variation With Flow Parameter, Λ.



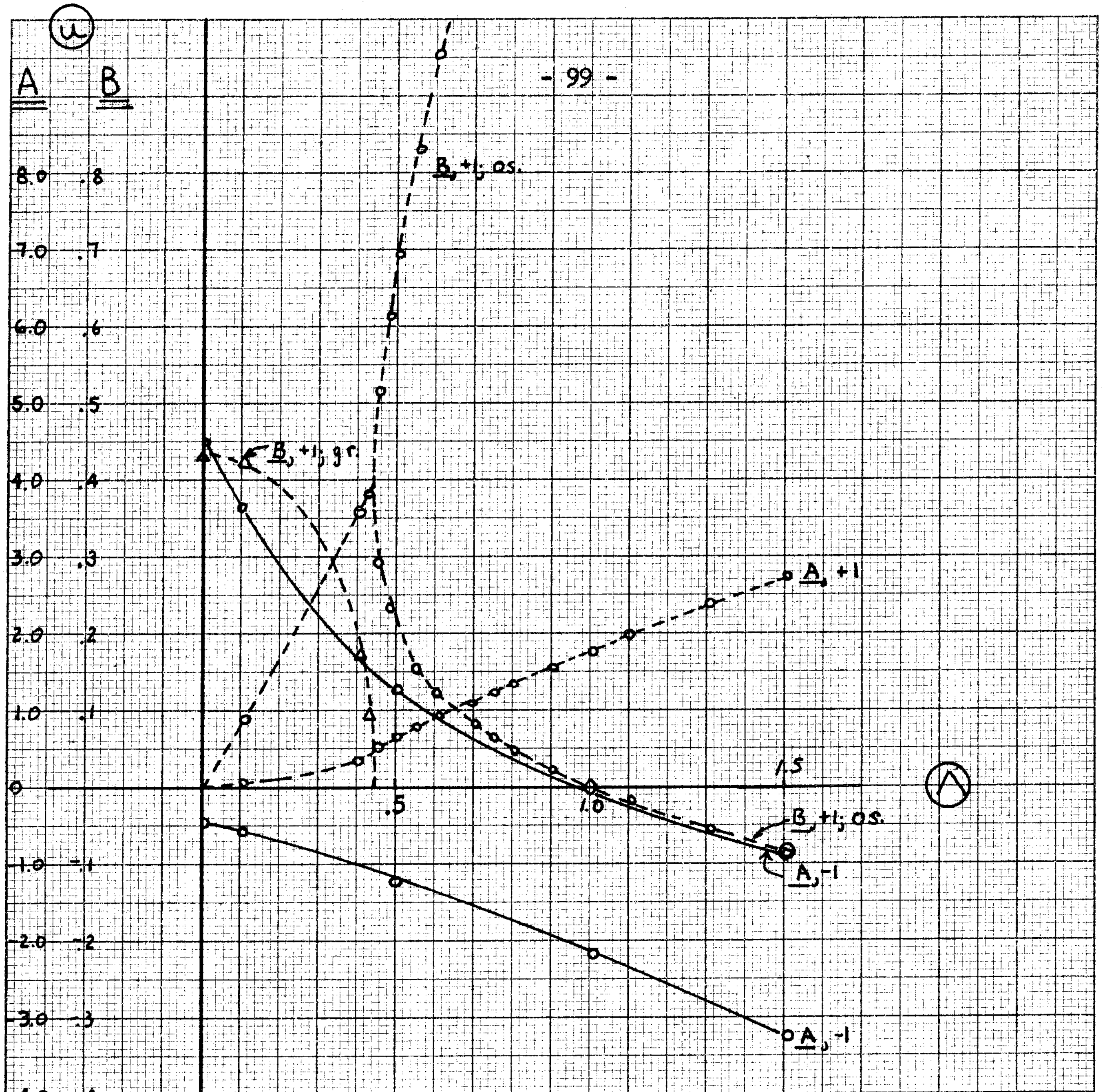


Fig. 9-5. Normalized Eigenvalue Variation With Flow Parameter, Λ.

$m = 0 \quad h = 1.0 \quad X = 0.1 \quad b_z = 1.0 \quad d_p = 2.0$

A B

os. = oscillation frequency = Re u

gr. = growth rate = |Im u|

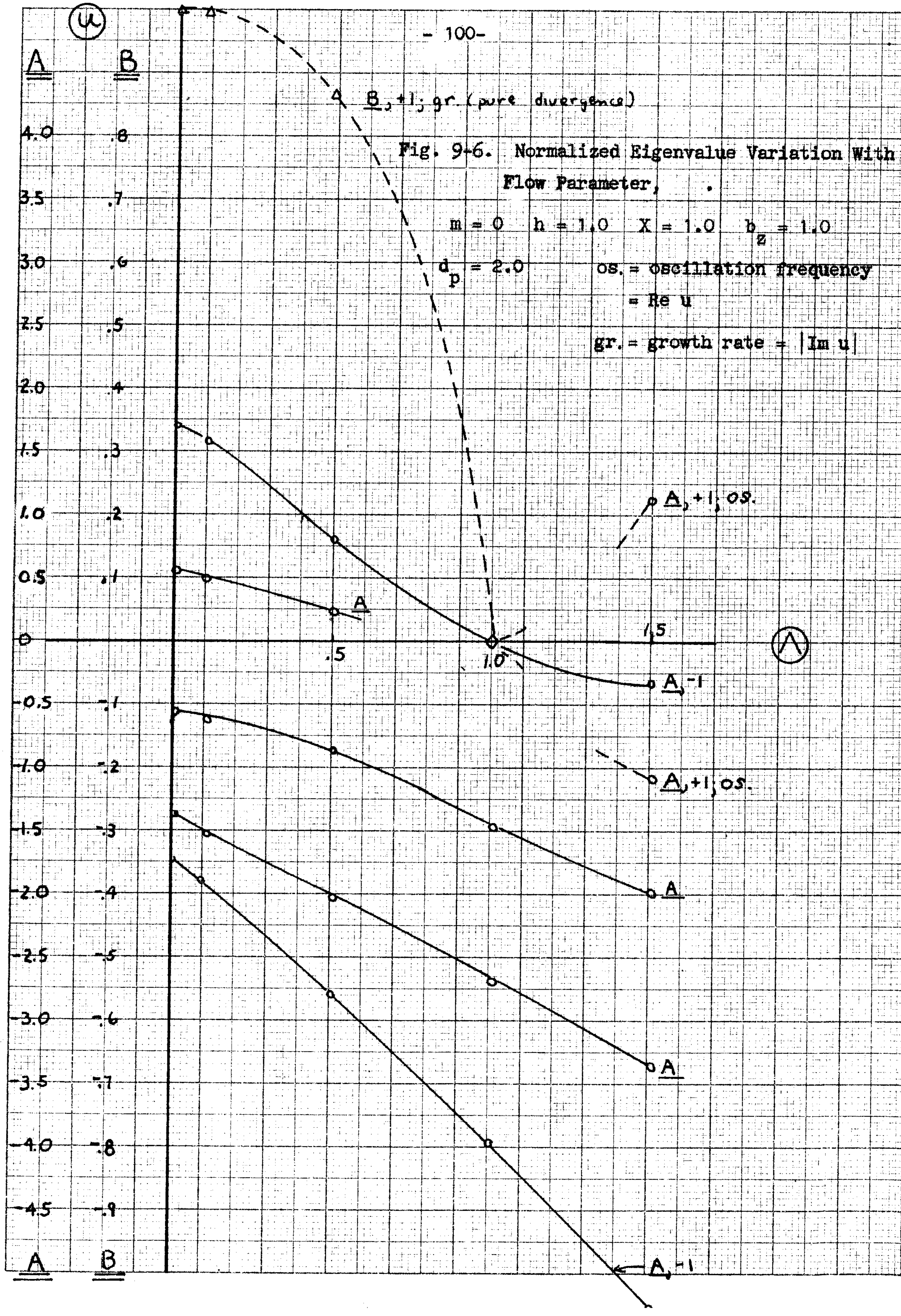


Fig. 9-6. Normalized Eigenvalue Variation With Flow Parameter, .

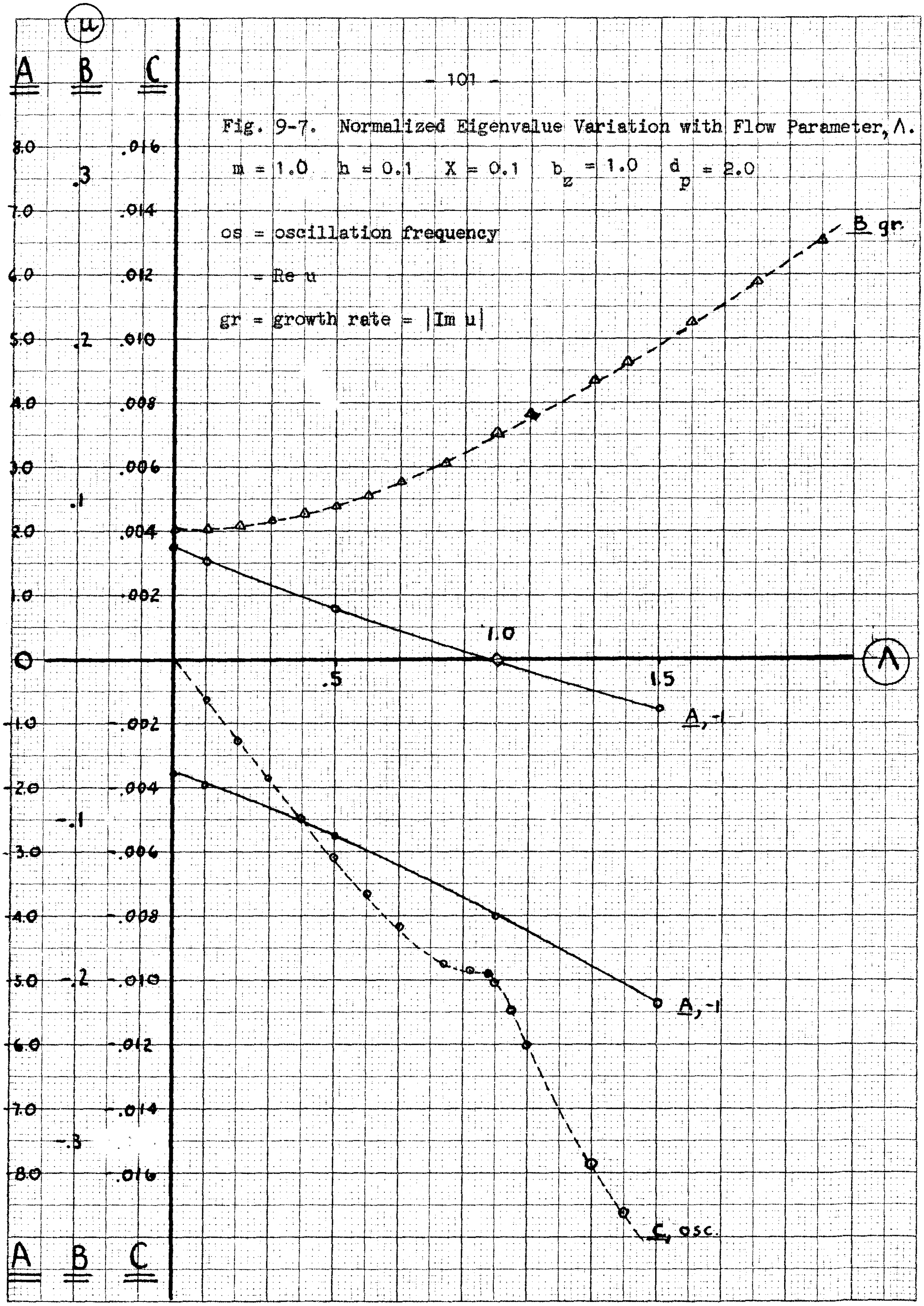


Fig. 9-7. Normalized Eigenvalue Variation with Flow Parameter, Λ.

$m = 1.0 \quad h = 0.1 \quad X = 0.1 \quad b_z = 1.0 \quad d_p = 2.0$

os = oscillation frequency

= Re u

gr = growth rate = |Im u|

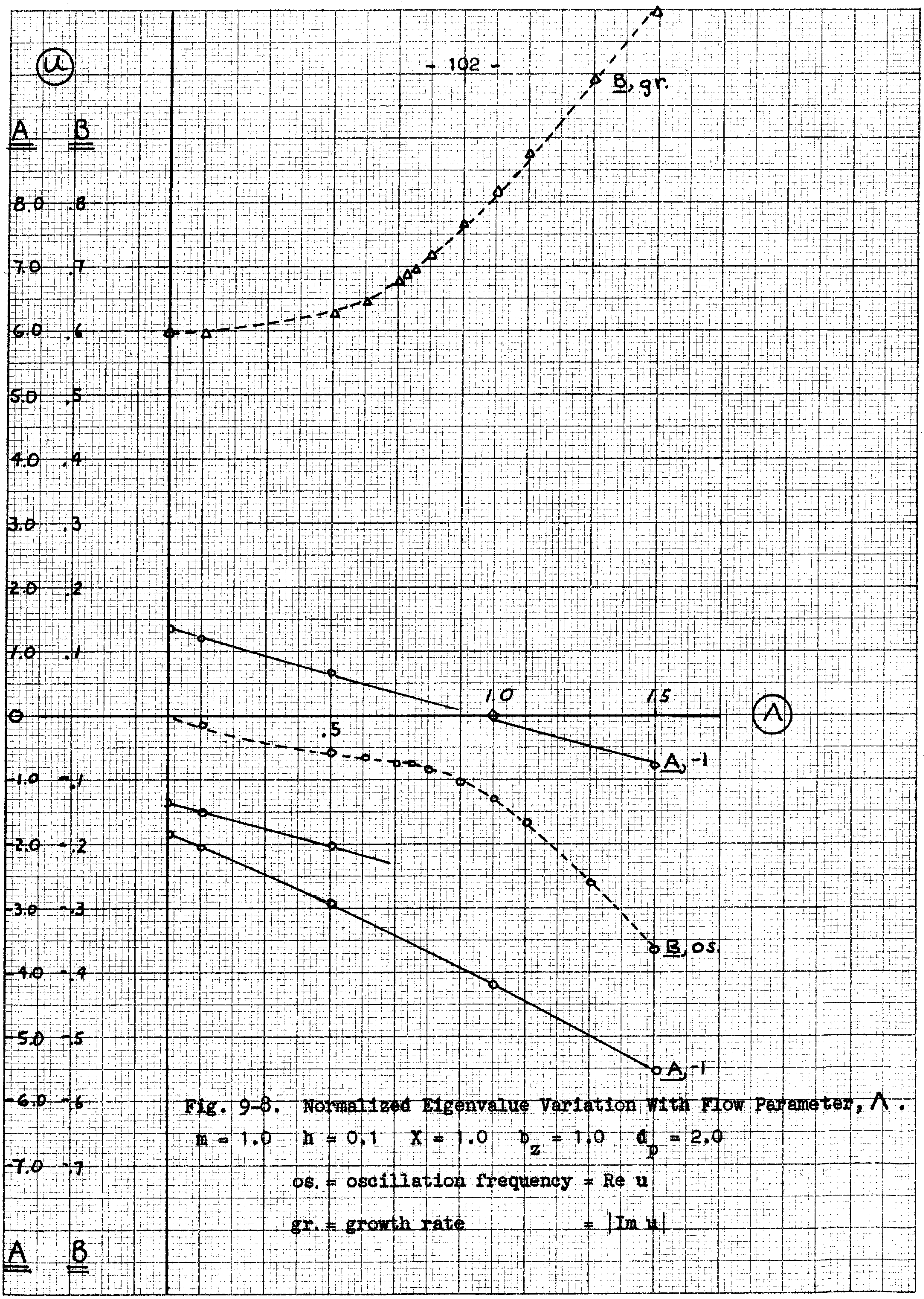


Fig. 9-8. Normalized Eigenvalue Variation With Flow Parameter, Λ.

$m = 1.0 \quad h = 0.1 \quad X = 1.0 \quad b_z = 1.0 \quad d_p = 2.0$

os. = oscillation frequency = Re u

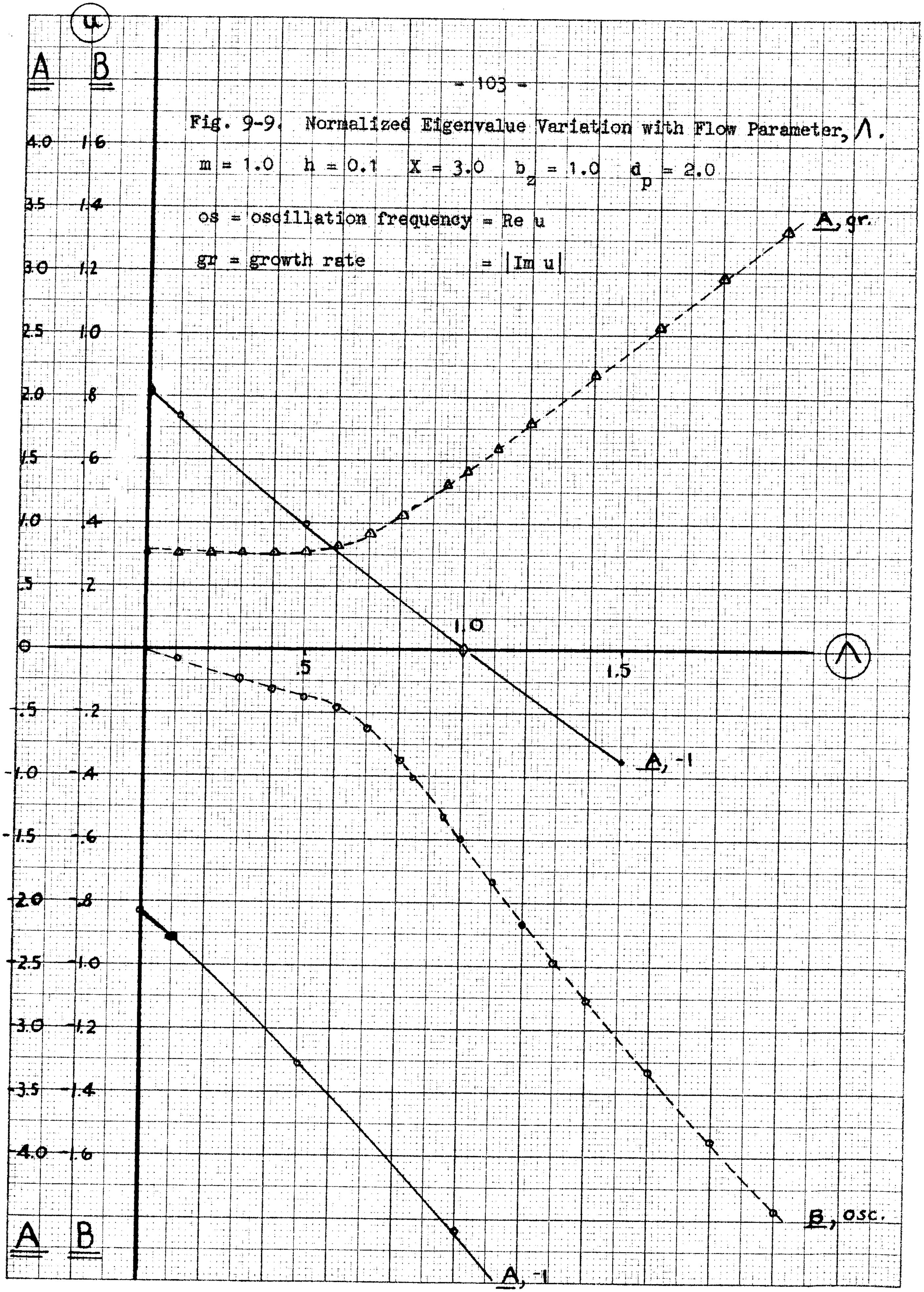
gr. = growth rate = |Im u|

Fig. 9-9. Normalized Eigenvalue Variation with Flow Parameter, Λ.

$m = 1.0 \quad h = 0.1 \quad X = 3.0 \quad b_z = 1.0 \quad d_p = 2.0$

os = oscillation frequency = Re u

gr = growth rate = |Im u|



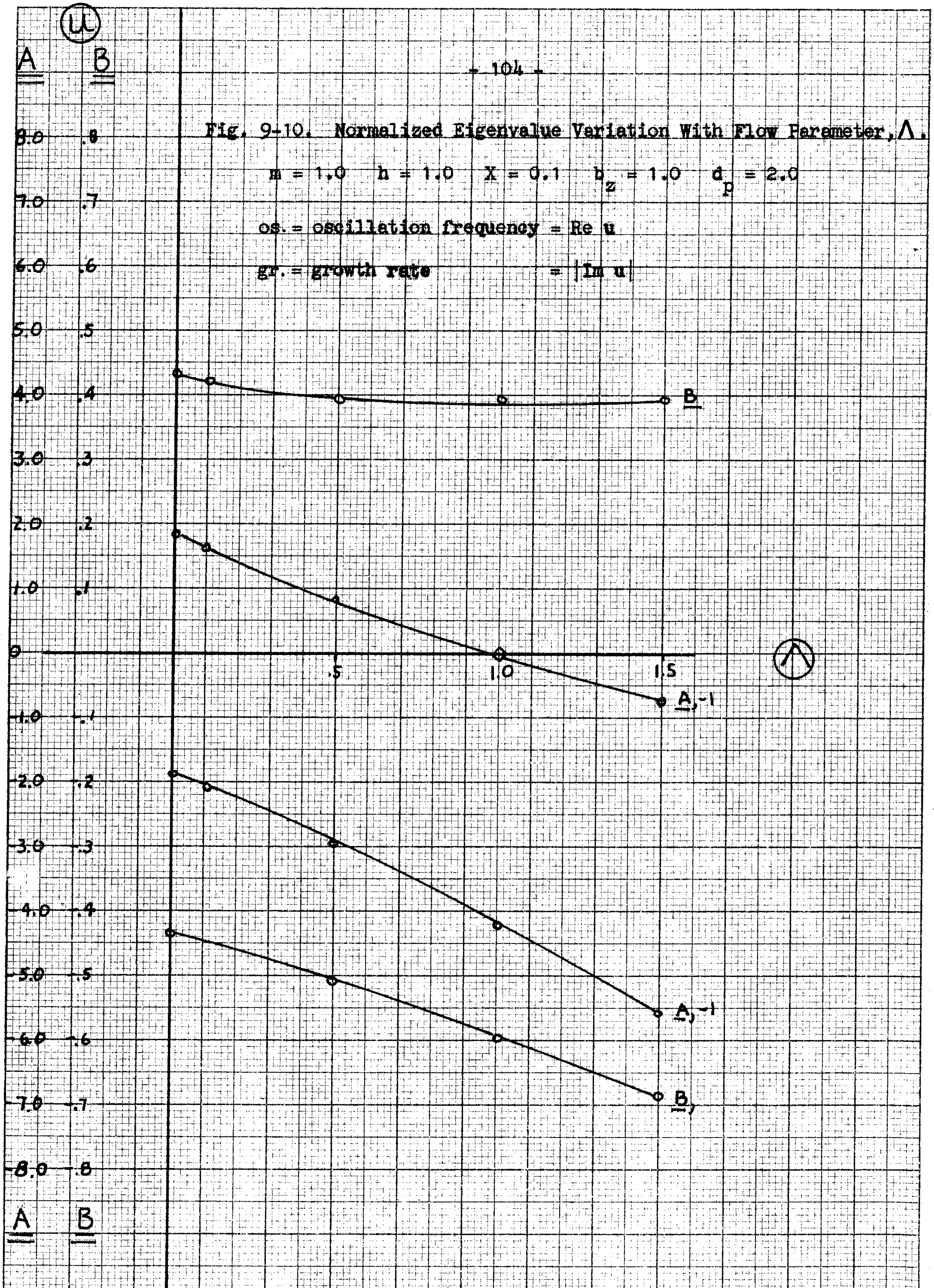


Fig. 9-10. Normalized Eigenvalue Variation With Flow Parameter, Λ.

m = 1.0 h = 1.0 X = 0.1 b_z = 1.0 d_p = 2.0

os. = oscillation frequency = Re u

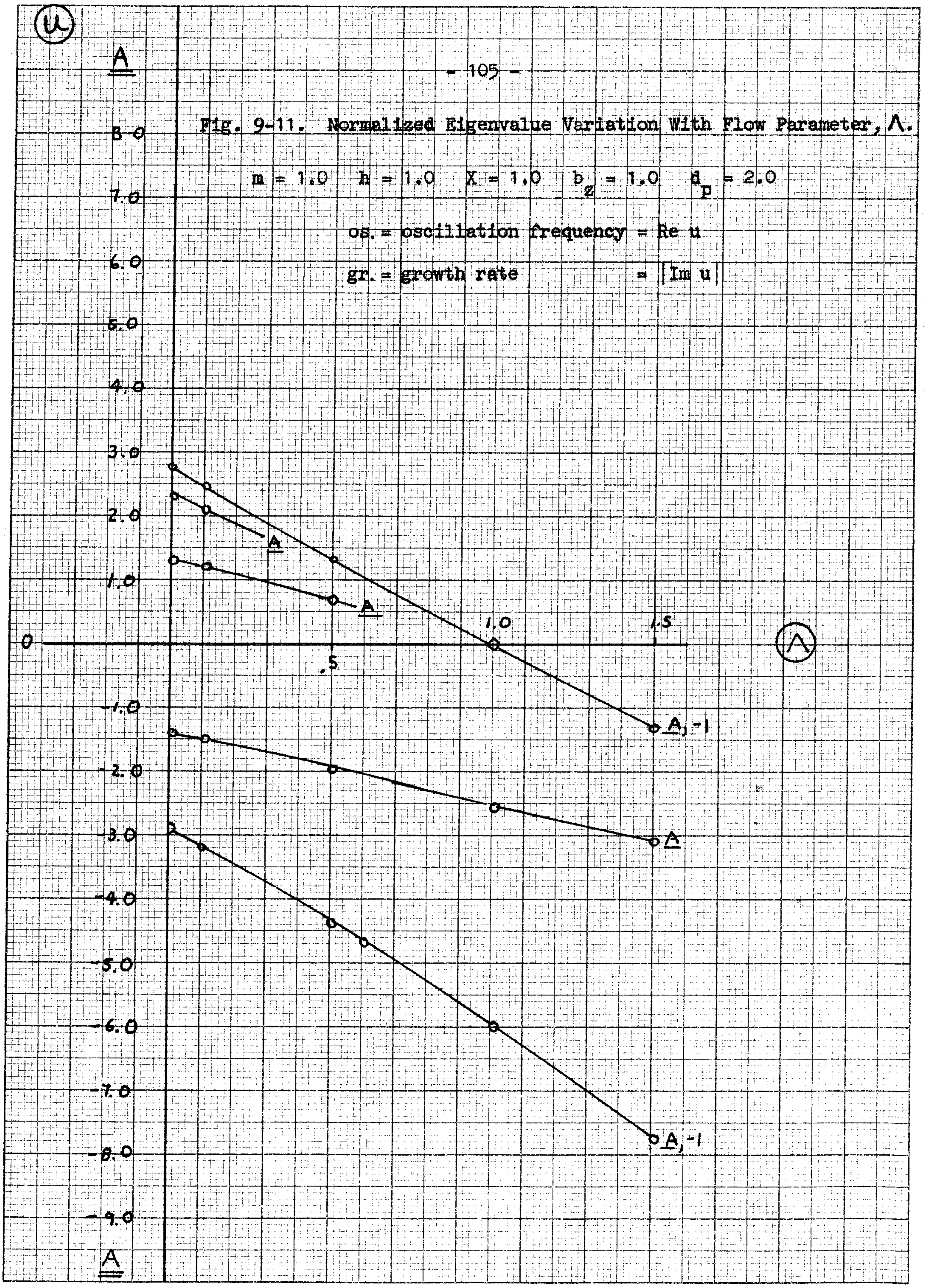
gr. = growth rate = |Im u|

Fig. 9-11. Normalized Eigenvalue Variation With Flow Parameter, Λ.

$m = 1.0 \quad h = 1.0 \quad X = 1.0 \quad b_z = 1.0 \quad d_p = 2.0$

os. = oscillation frequency = Re u

gr. = growth rate = |Im u|



A1 COMMON AND UNCOMMON VECTOR AND DYADIC IDENTITIES

$$(\mathbf{A}+\mathbf{B})\times(\mathbf{A}-\mathbf{B}) = -2\,(\mathbf{A}\times\mathbf{B}) \tag{1}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C} = -\mathbf{B}\times(\mathbf{C}\times\mathbf{A}) - \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) \tag{2}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \tag{3}$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \tag{4}$$

$$\nabla\cdot(\phi\mathbf{A}) = \nabla\phi\cdot\mathbf{A} + \phi\,\nabla\cdot\mathbf{A} \tag{5}$$

$$\nabla\times(\phi\mathbf{A}) = \nabla\phi\times\mathbf{A} + \phi\,\nabla\times\mathbf{A} \tag{6}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \tag{7}$$

$$\mathbf{A}\cdot\left[(\mathbf{B}\cdot\nabla)\mathbf{C}\right] = \mathbf{B}\cdot\left[(\mathbf{A}\cdot\nabla)\mathbf{C}\right] \tag{8}$$

$$\mathbf{A}\times(\nabla\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{9}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\nabla\mathbf{A})\cdot\mathbf{B} + (\nabla\mathbf{B})\cdot\mathbf{A} = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \tag{10}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{11}$$

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \frac{1}{2}\left\{\nabla(\mathbf{A}\cdot\mathbf{B}) - \nabla\times(\mathbf{A}\times\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla\times\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + \mathbf{A}(\nabla\cdot\mathbf{B})\right\} \tag{12}$$

$$(\mathbf{A}\cdot\nabla)\mathbf{A} = (\nabla\times\mathbf{A})\times\mathbf{A} + \frac{1}{2}(\nabla\mathbf{A}^2) \tag{13}$$

$$(\mathbf{A}\cdot\nabla)(\phi\mathbf{B}) = \mathbf{B}(\mathbf{A}\cdot\nabla\phi) + \phi(\mathbf{A}\cdot\nabla)\mathbf{B} \tag{14}$$

$$\nabla\times\left[\nabla\times\mathbf{A}\right] = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{15}$$

$$(\mathbf{A}\cdot\nabla)(\mathbf{B}\cdot\mathbf{C}) = \mathbf{B}\cdot\left[(\mathbf{A}\cdot\nabla)\mathbf{C}\right] + \mathbf{C}\cdot\left[(\mathbf{A}\cdot\nabla)\mathbf{B}\right] \tag{16}$$

$$(\mathbf{A}\cdot\nabla)(\mathbf{B}\times\mathbf{C}) = \left[(\mathbf{A}\cdot\nabla)\mathbf{B}\right]\times\mathbf{C} + \mathbf{B}\times\left[(\mathbf{A}\cdot\nabla)\mathbf{C}\right] \tag{17}$$

$$\nabla\cdot(\mathbf{A}\mathbf{B}) = \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{18}$$

$$\nabla\cdot(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}) = \nabla\times(\mathbf{B}\times\mathbf{A}) \tag{19}$$

$$\nabla\cdot(\phi^2\mathbf{A}\mathbf{B}) = \nabla\cdot(\phi\mathbf{A}\,\phi\mathbf{B}) = \nabla\cdot(\mathbf{A}\,\phi^2\mathbf{B}) \tag{20}$$

$$\nabla\cdot(\overleftrightarrow{\mathbf{A}}\cdot\mathbf{B}) = \left[\nabla\cdot\overleftrightarrow{\mathbf{A}} + \overleftrightarrow{\mathbf{A}}'\cdot\nabla\right]\cdot\mathbf{B} \tag{21}$$

$$\nabla\cdot(\mathbf{B}\cdot\overleftrightarrow{\mathbf{A}}) = \left[\nabla\cdot\overleftrightarrow{\mathbf{A}}' + \overleftrightarrow{\mathbf{A}}\cdot\nabla\right]\cdot\mathbf{B} \tag{22}$$

(21, 22): $\overleftrightarrow{\mathbf{A}}'$ = transpose of dyadic $\overleftrightarrow{\mathbf{A}}$

$$\int_V \nabla \phi \, d^3x = \int_S \phi \, d\mathbb{S} \tag{23}$$

$$\int_V \nabla \cdot \mathbb{A} \, d^3x = \int_S \mathbb{A} \cdot d\mathbb{S} \tag{24}$$

$$\int_V \nabla \times \mathbb{A} \, d^3x = -\int_S \mathbb{A} \times d\mathbb{S} \tag{25}$$

$$\int_V \mathbb{B} \cdot (\nabla \times \mathbb{A}) \, d^3x = \int_V \mathbb{A} \cdot (\nabla \times \mathbb{B}) \, d^3x + \int_S (\mathbb{A} \times \mathbb{B}) \cdot d\mathbb{S} \tag{26}$$

$$\int_S d\mathbb{S} \times \nabla \phi = \int_L \phi \, d\mathcal{L} \tag{27}$$

$$\int_S (\nabla \times \mathbb{A}) \cdot d\mathbb{S} = \int_L \mathbb{A} \cdot d\mathcal{L} \tag{28}$$

$d\mathbb{S}$ is the outward pointing normal vector.

A2 THE COMPONENT EQUATIONS IN CYLINDRICAL COORDINATES

A2.1 VECTOR IDENTITIES

$$(\mathbb{A}\cdot\nabla\mathbb{B}) = \mathbb{e}_r\left\{A_r\partial_r B_r + (\tfrac{1}{r})A_\theta\left[\partial_\theta B_r - B_\theta\right] + A_z\partial_z B_r\right\}$$
$$\mathbb{e}_\theta\left\{A_r\partial_r B_\theta + (\tfrac{1}{r})A_\theta\left[\partial_\theta B_\theta + B_r\right] + A_z\partial_z B_\theta\right\} \qquad (A2.1)$$
$$\mathbb{e}_z\left\{A_r\partial_r B_z + (\tfrac{1}{r})A_\theta\partial_\theta B_z + A_z\partial_z B_z\right\}$$

$$(\nabla\times\mathbb{v})\times\mathbb{v} = \mathbb{e}_r\left\{-(\tfrac{1}{2})\partial_r(v_\theta^2 + v_z^2) - (\tfrac{1}{r})v_\theta^2 + (\tfrac{1}{r})v_\theta\partial_\theta v_r + v_z\partial_z v_r\right\}$$
$$\mathbb{e}_\theta\left\{v_r\partial_r v_\theta + (\tfrac{1}{r})v_r v_\theta - (\tfrac{1}{2r})\partial_\theta(v_r^2 + v_z^2) + v_z\partial_z v_\theta\right\} \qquad (A2.2)$$
$$\mathbb{e}_z\left\{v_r\partial_r v_z + (\tfrac{1}{r})v_\theta\partial_\theta v_z - (\tfrac{1}{2})\partial_z(v_r^2 + v_\theta^2)\right\}$$

A2.2 THE MASS CONTINUITY EQUATION

$$\nabla\cdot(\rho\mathbb{v}) + \partial_t\rho = \rho\nabla\cdot(\mathbb{v}) + (\mathbb{v}\cdot\nabla)\rho + \partial_t\rho = 0 \qquad (A2.3)$$

$$\partial_t\rho + \frac{\rho}{r}\left[\partial_r(rv_r) + \partial_\theta v_\theta + r\partial_z v_z\right] + v_r\partial_r\rho + (\tfrac{1}{r})v_\theta\partial_\theta\rho + v_z\partial_z\rho = 0 \qquad (A2.4)$$

A2.3 THE DISSIPATIONLESS MOMENTUM CONSERVATION EQUATIONS

$$\rho\partial_t\mathbb{v} + \rho\left[(\mathbb{v}\cdot\nabla)\mathbb{v}\right] = -\nabla p + \mathbb{j}\times\mathbb{B} = -\rho_p\nabla\pi + (\tfrac{1}{\mu_0})\left[\mathbb{B}\cdot\nabla\mathbb{B}\right] - \frac{1}{\mu_0 c^2}(\partial_t\mathbb{E})\times\mathbb{B} \qquad (A2.5)$$

$$\langle\mathbb{e}_r\rangle : \rho\partial_t v_r + \rho\left[(\mathbb{v}\cdot\nabla)\mathbb{v}\right]_r = -\rho_p\partial_r\pi + (\tfrac{1}{\mu_0})\left[(\mathbb{B}\cdot\nabla)\mathbb{B}\right]_r$$
$$-(\tfrac{1}{\mu_0})(\tfrac{1}{c^2})\left\{B^2\partial_t v_r - (\mathbb{B}\cdot\mathbb{v})\partial_t B_r - (\mathbb{B}\cdot\partial_t\mathbb{v})B_r + (\tfrac{1}{2})v_r\partial_t B^2\right\} \qquad (A2.6)$$

$$\langle\mathbb{e}_\theta\rangle : \rho\partial_t v_\theta + \rho\left[(\mathbb{v}\cdot\nabla)\mathbb{v}\right]_\theta = -\rho_p(\tfrac{1}{r})\partial_\theta\pi + (\tfrac{1}{\mu_0})\left[\mathbb{B}\cdot\nabla\mathbb{B}\right]_\theta$$
$$-(\tfrac{1}{\mu_0 c^2})\left\{B^2\partial_t v_\theta - (\mathbb{B}\cdot\mathbb{v})\partial_t B_\theta - (\mathbb{B}\cdot\partial_t\mathbb{v})B_\theta + (\tfrac{1}{2})v_\theta\partial_t B^2\right\} \qquad (A2.7)$$

$$\langle e_z \rangle : \rho \partial_t v_z + \left[\rho(\mathbf{v}\cdot\nabla)\mathbf{v}\right]_z = -\rho_p \partial_z \pi + \left(\frac{1}{\mu_0}\right)\left[\mathbf{B}\cdot\nabla\mathbf{B}\right]_z$$
$$-\left(\frac{1}{\mu_0 c^2}\right)\left\{B^2 \partial_t v_z - (\mathbf{B}\cdot\mathbf{v})\partial_t B_z - (\mathbf{B}\cdot\partial_t \mathbf{v})B_z + \left(\tfrac{1}{2}\right)v_z \partial_t B^2\right\} \qquad \text{(A2.8)}$$

where $B^2 = \mathbf{B}\cdot\mathbf{B}$; $\mathbf{B}\cdot\partial_t\mathbf{v} = B_r \partial_t v_r + B_\theta \partial_t v_\theta + B_z \partial_t v_z$ (A2.9)

A2.4 MAXWELL'S EQUATIONS

$$\nabla\cdot\mathbf{B} = 0 \;;\quad \frac{1}{r}\left[\partial_r(rB_r) + \partial_\theta B_\theta + r\,\partial_z B_z\right] = 0 \qquad \text{(A2.10)}$$

$$\nabla\times\mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\partial_t \mathbf{E} \qquad \text{(A2.11)}$$

$$\langle e_r \rangle : \frac{1}{r}\left[\partial_\theta B_z - \partial_z(rB_\theta)\right] = \mu_0 j_r + \frac{1}{c^2}\partial_t E_r \qquad \text{(A2.12)}$$

$$\langle e_\theta \rangle : \partial_z B_r - \partial_r B_z = \mu_0 j_\theta + \frac{1}{c^2}\partial_t E_\theta \qquad \text{(A2.13)}$$

$$\langle e_z \rangle : \frac{1}{r}\left[\partial_r(rB_\theta) - \partial_\theta B_r\right] = \mu_0 j_z + \frac{1}{c^2}\partial_t E_z \qquad \text{(A2.14)}$$

$$\nabla\cdot\mathbf{E} = \frac{\sigma}{\epsilon_0} \;;\quad \frac{1}{r}\left[\partial_r(rE_r) + \partial_\theta E_\theta + r\,\partial_z E_z\right] = \frac{\sigma}{\epsilon_0} \qquad \text{(A2.15)}$$

$$\nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0 \qquad \text{(A2.16)}$$

$$\langle e_r \rangle : \frac{1}{r}\left[\partial_\theta E_z - \partial_z(rE_\theta)\right] + \partial_t B_r = 0 \qquad \text{(A2.17)}$$

$$\langle e_\theta \rangle : \partial_z E_r - \partial_r E_z + \partial_t B_\theta = 0 \qquad \text{(A2.18)}$$

$$\langle e_z \rangle : \frac{1}{r}\left[\partial_r(rE_\theta) - \partial_\theta E_r\right] + \partial_t B_z = 0 \qquad \text{(A2.19)}$$

A2.5 OHM'S LAW (ASSUMING $\frac{1}{c^2}\partial_t \mathbf{E} = 0$)

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\,\mathbf{j} = \lambda\,\nabla\times\mathbf{B} \qquad \text{(A2.20)}$$

$$\langle e_r \rangle : E_r + v_\theta B_z - v_z B_\theta = \eta j_r = \lambda\left[\frac{1}{r}\partial_\theta B_z - \partial_z B_\theta\right] \qquad \text{(A2.21)}$$

$$\langle e_\theta \rangle : E_\theta + v_z B_r - v_r B_z = \eta j_\theta = \lambda\left[\partial_z B_r - \partial_r B_z\right] \qquad \text{(A2.22)}$$

$$\langle e_z \rangle : E_z + v_r B_\theta - v_\theta B_r = \eta j_z = \frac{\lambda}{r}\left[\partial_r(rB_\theta) - \partial_\theta B_r\right] \qquad \text{(A2.23)}$$

A2.6 OHM'S LAW CURLED

$$-\partial_t \mathbb{B} + \nabla \times (\mathbb{v} \times \mathbb{B}) = \eta \nabla \times \mathbb{j} = -\lambda \nabla^2 \mathbb{B} - \frac{\lambda}{c^2} \partial^2_{tt} \mathbb{B} \tag{A2.24}$$

$$\langle \mathbb{e}_r \rangle : -\partial_t B_r + \frac{1}{r}\left[\partial_\theta (v_r B_\theta - v_\theta B_r)\right] - \partial_z (v_z B_r - v_r B_z) = -\lambda \left[\nabla^2 B_r - \frac{1}{r^2} B_r - \frac{2}{r^2} \partial_\theta B_\theta - \frac{1}{c^2} \partial^2_{tt} B_r\right] \tag{A2.25}$$

$$\langle \mathbb{e}_\theta \rangle : -\partial_t B_\theta + \partial_z (v_\theta B_z - v_z B_\theta) - \partial_r (v_r B_\theta - v_\theta B_r) = -\lambda \left[\nabla^2 B_\theta - \frac{B_\theta}{r^2} + \frac{2}{r^2} \partial_\theta B_r - \frac{1}{c^2} \partial^2_{tt} B_\theta\right] \tag{A2.26}$$

$$\langle \mathbb{e}_z \rangle : -\partial_t B_z + \frac{1}{r}\left[\partial_r (r v_z B_r - r v_r B_z) - \partial_\theta (v_\theta B_z - v_z B_\theta)\right] = -\lambda \left[\nabla^2 B_z - \frac{1}{c^2} \partial^2_{tt} B_z\right] \tag{A2.27}$$

where $\lambda = \eta / \mu_0$.

A3 THE DIFFERENTIAL EQUATION OF THE NORMAL VECTOR

The spatial and temporal variations of surfaces can be described in two ways: first, through the implicit relation

$$f(x,y,z,t) = 0 \tag{A3.1}$$

(implying $z = \phi(x,y,t)$; second, through the direction of the normal vector, $\mathbb{n}$,

$$\mathbb{n} = \frac{\nabla f}{|\nabla f|} = \frac{\nabla f}{a} \tag{A3.2}$$

and the velocity of each point on the surface along the normal

$$u = (\mathbb{w} \cdot \mathbb{n}) \tag{A3.3}$$

The latter procedure is more useful, since the normal vector appears explicitly in our b.c. The d.e. of motion of the normal vector is obtained from the d.e. of the surface, namely,

$$D_t f = \partial_t f + (\mathbb{w} \cdot \nabla) f = 0 \tag{A3.4}$$

Two procedures are used for deriving the normal-vector d.e. Although apparently different, they will be shown to be equivalent.

A3.1 Procedure 1

If we take the gradient of A3.4, we obtain

$$\partial_t (a\,\mathbb{n}) + \nabla [\,\mathbb{w} \cdot a\,\mathbb{n}] = 0 \tag{A3.5}$$

Using identity A1.10 and the fact that

$$[\nabla(a\,\mathbb{n})]\cdot \mathbb{v} = \mathbb{v} \times [\nabla \times (a\,\mathbb{n})] + (\mathbb{v}\cdot\nabla)a\,\mathbb{n} = (\mathbb{v}\cdot\nabla)a\,\mathbb{n} \qquad \text{(A3.6)}$$

we can write

$$\nabla[\mathbb{v}\cdot a\,\mathbb{n}] = [\nabla\mathbb{v}]\cdot a\,\mathbb{n} + (\mathbb{v}\cdot\nabla)(a\,\mathbb{n}) \qquad \text{(A3.7)}$$

By substituting A3.7 into A3.5 and dividing through by a, we obtain

$$\partial_t \mathbb{n} + (\mathbb{v}\cdot\nabla)\mathbb{n} + \frac{\mathbb{n}}{a} D_t a + [\nabla\mathbb{v}]\cdot\mathbb{n} = 0 \qquad \text{(A3.8)}$$

Now we separate each vector term in A3.8 into a component along $\mathbb{n}$ and one orthogonal to $\mathbb{n}$. Thus:

a. $\partial_t \mathbb{n}$ is orthogonal to $\mathbb{n}$ since $\partial_t(\mathbb{n}\cdot\mathbb{n}) = \mathbb{n}\cdot\partial_t\mathbb{n} = 0$

b. $(\mathbb{v}\cdot\nabla)\mathbb{n} - \mathbb{n}\left\{\mathbb{n}\cdot[(\mathbb{v}\cdot\nabla)\mathbb{n}]\right\} = -\mathbb{n} \times \left\{\mathbb{n} \times [(\mathbb{v}\cdot\nabla)\mathbb{n}]\right\}$ (A3.9)

c. $(\nabla\mathbb{v})\cdot\mathbb{n} - \mathbb{n}\left\{\mathbb{n}\cdot[(\nabla\mathbb{v})\cdot\mathbb{n}]\right\} = -\mathbb{n} \times \left\{\mathbb{n} \times [(\nabla\mathbb{v})\cdot\mathbb{n}]\right\}$

Combining all the components orthogonal to $\mathbb{n}$ yields

$$\partial_t \mathbb{n} - \mathbb{n} \times \left\{\mathbb{n} \times [(\mathbb{v}\cdot\nabla)\mathbb{n}]\right\} = \mathbb{n} \times \left\{\mathbb{n} \times [(\nabla\mathbb{v})\cdot\mathbb{n}]\right\} \qquad \text{(A3.10)}$$

Another form is

$$D_t \mathbb{n} = \mathbb{n} \times \left\{\mathbb{n} \times [(\nabla\mathbb{v})\cdot\mathbb{n}]\right\} + \mathbb{n}\left\{\mathbb{v}\cdot[(\mathbb{n}\cdot\nabla)\mathbb{n}]\right\} \qquad \text{(A3.11)}$$

The last term follows from A3.9b and the fact that

$$\mathbb{n}\cdot[\mathbb{v}\cdot\overleftrightarrow{A}] = \mathbb{v}\cdot[\mathbb{n}\cdot\overleftrightarrow{A}]$$

A3.2 Procedure 2

Rewrite A3.4 as

$$\partial_t f + au = 0 \qquad \text{(A3.12)}$$

If we divide by a, rearrange, and take the gradient, we obtain

$$-\nabla u = \frac{1}{a}\,\partial_t(\nabla f) + (\partial_t f)\,\nabla(\frac{1}{a}) \qquad (A3.13)$$

Using A3.2 and A3.12 we can show

$$\frac{1}{a}\partial_t(\nabla f) = \partial_t \mathbb{n} + \frac{\mathbb{n}}{a}\,\partial_t a = \partial_t \mathbb{n} - \mathbb{n}\left\{\frac{u}{a}\,(\mathbb{n}\cdot\nabla a) + (\mathbb{n}\cdot\nabla)u\right\} \qquad (A3.14)$$

since

$$\frac{1}{a}\,\partial_t a = \frac{1}{a}\,\partial_t[\nabla f\cdot\nabla f]^{1/2} = \frac{\mathbb{n}}{a}\cdot\nabla(\partial_t f) = -\frac{\mathbb{n}}{a}\cdot\nabla(au)$$

and

$$(\partial_t f)\,\nabla(\frac{1}{a}) = \frac{u}{a}\nabla a \qquad (A3.15)$$

If one substitutes A3.14 and A3.15 into A3.13 and makes use of the identity

$$\mathbb{n}\times[\mathbb{n}\times \mathbb{A}] = \mathbb{n}[\mathbb{n}\cdot\mathbb{A}] - \mathbb{A} \qquad (A3.16)$$

two times, one obtains

$$\partial_t \mathbb{n} - \frac{u}{a}\,\mathbb{n}\times[\mathbb{n}\times\nabla a] = \mathbb{n}\times[\mathbb{n}\times\nabla u] \qquad (A3.17)$$

Since

$$\mathbb{n}\times\nabla a = a\nabla\times\mathbb{n} \qquad (A3.18)$$

we can show that

$$(\frac{1}{a})\left\{\mathbb{n}\times[\mathbb{n}\times\nabla a]\right\} = -(\mathbb{n}\cdot\nabla)\mathbb{n} \qquad (A3.19)$$

and thus A3.17 becomes

$$\partial_t \mathbb{n} + u(\mathbb{n}\cdot\nabla)\mathbb{n} = \mathbb{n}\times[\mathbb{n}\times\nabla u] \qquad (A3.20)$$

A3.3 The Equivalence of Procedures 1 and 2

We will demonstrate the equivalence by deriving A3.11 from A3.20. First, examine ∇u.

$$\nabla u = \nabla(\mathbf{w}\cdot\mathbf{n}) = (\nabla\mathbf{w})\cdot\mathbf{n} + (\nabla\mathbf{n})\cdot\mathbf{w}$$
$$= (\nabla\mathbf{w})\cdot\mathbf{n} + (\mathbf{w}\cdot\nabla)\mathbf{n} + \mathbf{w}\times(\nabla\times\mathbf{n}) \qquad \text{(A3.21)}$$

Thus

$$\mathbf{n}\times[\mathbf{n}\times\nabla u] = \mathbf{n}\times\left\{\mathbf{n}\times[(\nabla\mathbf{w})\cdot\mathbf{n}]\right\} + \mathbf{n}\left\{\mathbf{n}\cdot[(\mathbf{w}\cdot\nabla)\mathbf{n} + \mathbf{w}\times(\nabla\times\mathbf{n})]\right\}$$
$$- \left\{(\mathbf{w}\cdot\nabla)\mathbf{n} + \mathbf{w}\times(\nabla\times\mathbf{n})\right\} \qquad \text{(A3.22)}$$

Since

$$\mathbf{n}\cdot[(\mathbf{w}\cdot\nabla)\mathbf{n}] = \mathbf{w}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{n}] \qquad \text{(A3.23)}$$

$$\mathbf{n}\cdot[\mathbf{w}\times\nabla\times\mathbf{n}] = \mathbf{w}\cdot[(\nabla\times\mathbf{n})\times\mathbf{n}] = \mathbf{w}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{n}] \qquad \text{(A3.24)}$$

we can rewrite A3.20 using the results derived above, namely,

$$\partial_t\mathbf{n} + (\mathbf{w}\cdot\nabla)\mathbf{n} = \mathbf{n}\times\left\{\mathbf{n}\times[(\nabla\mathbf{w})\cdot\mathbf{n}]\right\} + 2\mathbf{n}\left\{\mathbf{w}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{n}]\right\}$$
$$- \left\{\mathbf{w}\times(\nabla\times\mathbf{n}) + u(\mathbf{n}\cdot\nabla)\mathbf{n}\right\} \qquad \text{(A3.25)}$$

The last term can be rewritten as

$$-\left\{\mathbf{w} - u\mathbf{n}\right\}\times[\nabla\times\mathbf{n}] = -[\nabla\times\mathbf{n}]\times\left\{\mathbf{n}\times[\mathbf{n}\times\mathbf{w}]\right\}$$
$$= -\mathbf{n}[\nabla\times\mathbf{n}]\cdot[\mathbf{n}\times\mathbf{w}] = \mathbf{n}\cdot(\nabla\times\mathbf{n})[\mathbf{n}\times\mathbf{w}]$$
$$= -\mathbf{n}\left\{\mathbf{w}\cdot[(\nabla\times\mathbf{n})\times\mathbf{n}]\right\}$$
$$= -\mathbf{n}\left\{\mathbf{w}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{n}]\right\} \qquad \text{(A3.26)}$$

Substituting A3.26 into A3.25 and combining yields

$$D_t \mathbb{n} = \mathbb{n} \times \left\{ \mathbb{n} \times [(\nabla \mathbb{v}) \cdot \mathbb{n}] \right\} + \mathbb{n} \left\{ \mathbb{v} \, [(\mathbb{n} \cdot \nabla) \mathbb{n}] \right\} \tag{A3.27}$$

which is the result given in A3.11.

A3.4 Comments on the Differential Equation As It Appears in the Literature

Kruskal and Schwarzschild (1) give the d.e. (their eq. 9) as

$$D_t \mathbb{n} = \mathbb{n} \times [\mathbb{n} \times \nabla u] \tag{A3.28}$$

Tayler (23, eq. 2.24) gives the same equation, apparently taking the equation from Ref. 1 without checking it. Northrop (14, eq. 9) and Kruskal and Tuck (15, eq. 9b) give the d.e. as

$$D_t \mathbb{n} = \mathbb{n} \times [\mathbb{n} \times (\nabla \mathbb{v}) \cdot \mathbb{n}] \tag{A3.29}$$

The latter state that "equation (9b) which is given here is corrected from its previous erroneous form [Ref. 1]."

None of these agrees with the results derived previously. The essence of the disagreement comes from the fact that they assume that $\mathbb{n} \cdot D_t \mathbb{n} = 0$, which implies that

$$\mathbb{n} \cdot [(\mathbb{v} \cdot \nabla) \mathbb{n}] = \mathbb{v} \cdot [(\nabla \times \mathbb{n}) \times \mathbb{n}] = 0 \tag{A3.30}$$

This term is of second **order**, $\mathcal{O}(2)$, in their stability investigations, since $\mathbb{v}_e = 0$, and thus their final results are unaffected. In the case where

$$\mathbb{v}_e = v_\theta \mathbb{e}_\theta + v_z \mathbb{e}_z$$

one can show that

$$(\mathbb{w}\cdot\nabla)\mathbb{n} = -\mathbb{e}_r[\frac{1}{r} v_\theta \tilde{n}_\theta] + \mathbb{e}_\theta[\frac{1}{r} v_\theta(\partial_\theta \tilde{n}_\theta - 1) + v_z \partial_z \tilde{n}_\theta]$$
$$+ \mathbb{e}_z[\frac{1}{r} v_\theta \partial_\theta \tilde{n}_z \qquad + v_z \partial_z \tilde{n}_z] + \mathcal{O}(2) \qquad (A3.31)$$

and thus

$$\mathbb{n} \cdot [(\mathbb{w}\cdot\nabla)\mathbb{n}] = \mathcal{O}(2)$$

We conclude that their equation will yield the correct results in the present first order stability analysis.

A3.5 First Order Solution of the Normal Vector Differential Equation

Consider the problem where the equilibrium quantities are

$$\mathbb{v}_e = (0, v_\theta, v_z) \qquad \text{and} \qquad \mathbb{n} = (-1, 0, 0) \qquad (A3.32)$$

We have a spiralling circular cylinder of fluid. If the total velocity field is written as

$$\mathbb{v} = (\tilde{v}_r, v_\theta + \tilde{v}_\theta, v_z + \tilde{v}_z) \qquad (A3.33)$$

then one can postulate a solution of the normal vector d.e. to have the form

$$\mathbb{n} = (-1 + \tilde{n}_r, \tilde{n}_\theta, \tilde{n}_z) \qquad (A3.34)$$

The tilde designates first order quantities as described in Chapter 3 and eq. 3.1.

Thus

$$\tilde{u} = \mathbb{w}\cdot\mathbb{n} = -\tilde{v}_r + v_\theta \tilde{n}_\theta + v_z \tilde{n}_z \qquad (A3.35)$$

Substituting these results into A3.20, one obtains

$$\partial_t \mathbb{n} = \left\{0, -\frac{1}{r}\partial_\theta \tilde{u}, -\partial_z \tilde{u}\right\} \qquad (A3.36)$$

since $\tilde{u}(\mathbb{n}\cdot\nabla)\mathbb{n}$ is a second order quantity. If one replaces the partial derivatives in A3.36

$$\partial_\theta \longrightarrow im, \qquad \partial_z \longrightarrow ik, \qquad \partial_t \longrightarrow i\omega$$

and substitutes A3.35 into A3.36, one obtains three linear simultaneous equations which can be solved for the components of $\tilde{\mathbb{n}}$ as

$$\tilde{\mathbb{n}} = \frac{\tilde{v}_r}{\bar{\omega} r}[0,m,kr] \qquad (A3.37)$$

where

$$\bar{\omega} = \omega + \omega_p$$
$$\omega_p = \frac{1}{r} m v_\theta + k v_z \qquad (A3.38)$$

$\tilde{v}_r = \tilde{v}_{rp}$ is the first order radial velocity component in the plasma. Note that in problems 2 and 3 the azimuthal equilibrium velocity is proportional to r, while the longitudinal velocity is uniform. Hence ω_p is a constant.

A4 $\Omega^2 = 1$ IN THE INCOMPRESSIBLE PROBLEM

If $\Omega^2 = 1$, the inverse matrix of [a] in the incompressible problem is not defined and one must return to the original equations, 7.4, to see what can be learned. If one applies 7.20 to the first three equations of 7.4, one obtains three equations in three unknowns

$$\begin{bmatrix} i\bar{\omega}_- & -(\omega_{-\theta}+\beta\omega_{+\theta}) & 0 \\ \omega_{-\theta}+\beta\omega_{+\theta} & i\bar{\omega}_- & 0 \\ 0 & 0 & i\bar{\omega}_- \end{bmatrix} \begin{bmatrix} \tilde{Q}_{+r} \\ \tilde{Q}_{+\theta} \\ \tilde{Q}_{+z} \end{bmatrix} = \begin{bmatrix} -\partial_r\tilde{\pi} \\ -\frac{im\tilde{\pi}}{r} \\ -ik\tilde{\pi} \end{bmatrix} \qquad (A4.1)$$

Thus

$$\tilde{Q}_{+z} = -\frac{k\tilde{\pi}}{\bar{\omega}_-} \qquad (A4.2)$$

and the remaining pair of equations can be written as

$$\begin{bmatrix} i & -\Omega \\ \Omega & i \end{bmatrix} \begin{bmatrix} \tilde{Q}_{+r} \\ \tilde{Q}_{+\theta} \end{bmatrix} = -\frac{1}{\bar{\omega}_-} \begin{bmatrix} \partial_r\tilde{\pi} \\ \frac{im\tilde{\pi}}{r} \end{bmatrix} \qquad \begin{matrix} (A4.3) \\ (A4.4) \end{matrix}$$

If $\Omega^2 = 1$, one must proceed more carefully. If A4.3 is substituted into $\nabla\cdot(Q_A = 0$ and rearranged, one obtains

$$\partial_r\tilde{Q}_{+r} + 1/r\,(1-m/\Omega)\,\tilde{Q}_{+r} = \frac{i}{\bar{\omega}_-}\left[\frac{-m}{r\Omega}\,\partial_r\tilde{\pi} + k^2\tilde{\pi}\right] \qquad (A4.5)$$

Similarly, for A4.4 one obtains

$$\partial_r\tilde{Q}_{+r} + 1/r\,(1-m\Omega)\,\tilde{Q}_{+r} = \frac{i}{\bar{\omega}_-}\left[\frac{m^2}{r^2} + k^2\right]\tilde{\pi} \qquad (A4.6)$$

Subtracting A4.5 from A4.6 yields

$$[1/\Omega - \Omega]\, \tilde{Q}_{+r} = \frac{i}{\bar{\omega}_-}\, [1/\Omega\ \partial_r \tilde{\pi} + (m/r)\tilde{\pi}] \tag{A4.7}$$

Substituting A4.7 into A4.6 yields the same modified Bessel equation for the plasma as 7.23.

$$\partial^2_{rr}\tilde{\pi} + \frac{1}{r}\,\partial_r\tilde{\pi} - [\frac{m^2}{r^2} + \zeta_p^2]\tilde{\pi} = 0 \tag{A4.8}$$

This is to be expected, since the value of the determinant cancelled out of the procedure used in deriving 7.23 from 7.12. If we set $\Omega = \pm 1$ in A4.7, we obtain the simple result

$$\partial_r\tilde{\pi} + \frac{m}{r}\Omega\tilde{\pi} = 0 \tag{A4.9}$$

from which

$$\tilde{\pi} = \tilde{\pi}^o (r)^{-m\Omega} \tag{A4.10}$$

Substituting A4.10 into A4.6 yields a first order linear d.e. in $\tilde{Q}_{+r}$, which is easily solved as

$$\tilde{Q}_{+r} = i\tilde{\pi}^o\ \frac{r^{m-1}}{2\bar{\omega}_-}\, [m + \frac{k^2 r^2}{(1+m)}] \tag{A4.11}$$

If $m > 0$ and $\Omega = -1$, $\tilde{\pi}$ is finite at the origin and a satisfactory solution to the problem. If $\Omega = +1$, $\tilde{\pi}$ is singular at $r = 0$, and this solution is disregarded.

If $m = 0$, both $\Omega = \pm 1$ can be solutions, since in both these cases $\tilde{\pi}=\tilde{\pi}^o$ is independent of r. Equation A4.11 yields

$$\tilde{Q}_{+r} = (\frac{i\tilde{\pi}^o k^2}{2\bar{\omega}_-})\, r \quad ; \qquad (m = 0,\ \Omega = \pm 1) \tag{A4.12}$$

Observe that although $\Omega = \pm 1$ yield the same perturbation in hm pressure, $\tilde{\pi}$, the slope of $\tilde{Q}_{+r}$ is different in all four* cases because $\bar{\omega}_{-}$ is different. Hence, each mode, u, corresponds to a different overall spatial behavior.

If $m < 0$, $= +1$ is a root and $= -1$ is disregarded.

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* Both $\Omega = -1$ and $\Omega = +1$ correspond to two roots, u, as shown in 7.35.

A5 THE BESSEL FUNCTION AND RELATED FUNCTIONS

A5.1 Introduction

This appendix provides a central location for properties of those Bessel Function ratios useful in dealing with the dispersion relations of problems with cylindrical geometry. The works of Watson (Ref. 29: 4.1, 4.2, 9.6, 15.41)*, Jahnke and Emde (30), and Dwight (33), and The Bateman Manuscript, Higher Transcendental Functions, II (Ref. 34; 7.2, 7.13) contain many useful formulas. Recently, Onoe** (35) published a book listing the properties of the Bessel Function ratios and including tables for $zJ_{m-1}(z)/J_m(z)$ (with $1 \le m \le 16$ and $0 \le |z| \le 20$; spacing, $|\Delta z| = .01$). Both real and imaginary values of z are tabulated.

A5.2 The Bessel Function Ratios

Definitions

$$\mathcal{I}_m(z) = \mathcal{J}_m(iz) = zI'_m(z)/I_m(z) = -m+zI_{m-1}(z)/I_m(z) \qquad (A5.1)$$

$$\mathcal{J}_m(z) = \mathcal{I}_m(iz) = zJ'_m(z)/J_m(z) = -m+zJ_{m-1}(z)/J_m(z) \qquad (A5.2)$$

$$\mathcal{K}_m(z) = zK'_m(z)/K_m(z) = -m-zK_{m-1}(z)/K_m(z) \qquad (A5.3)$$

$$\mathcal{K}_m(iz) = \mathcal{H}_m^{(2)}(z) = zH_m^{(2)\prime}(z)/H_m^{(2)}(z) \qquad (A5.4)$$

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* The numbers following the reference number refer to the paragraphs of particular use for dealing with the Bessel Function of a complex variable.

** His notation is different from the author's. His eqs. 2.16-2.19 are to be compared with A5.1-A5.4.

These functions are the result of multiplying z by the logarithmic derivative of the ordinary Bessel Function, e.g., $\mathscr{I}_m(z) = z\, d_z(\log I_m)$.

Initial Series

$$\mathscr{I}_m(z) = m + \sum_{p=1}^{\infty} a_{2p}\, z^{2p} \qquad \text{(A5.5)}$$

where

$$a_2 = \frac{1}{2(m+1)}\ ; \quad a_{2p} = \frac{1}{2(m+p)}\left(a_2 a_{2p-2} + a_4 a_{2p-4} + \cdots\ a_{2p-4} a_2\right) \qquad \text{(A5.6)}$$

Asymptotic Series

$$\mathscr{I}_m(z) \simeq z - (1/2) + \sum_{p=1}^{\infty} b_p (z)^{-p} \qquad \text{(A5.7)}$$

where

$$b_1 = b_2 = \frac{(4m^2-1)}{8};\ b_{p+1} = \frac{1}{2}\left[(p+1)b_p - \sum_{k=1}^{k=p-1} b_{p-k} b_k\right] \qquad \text{(A5.8)}$$

$$\mathscr{J}_m(z) = \frac{-z\left[P_m^{(1)}(z)\tan u + Q_m^{(1)}(z)\right]}{\left[P_m(z) + Q_m(z)\tan u\right]} \qquad \text{(A5.9)}$$

where $u = z - m\pi/2 - \pi/4$ and P_m, Q_m, $P_m^{(1)}$, and $Q_m^{(1)}$ are defined in (33; 808.31, 808.32, 808.41, 808.42). The leading terms of this series are

$$\mathscr{J}_m(z) \simeq \frac{-z\left[\tan u + (4m^2+3)/8z\right]}{\left[1 + (4m^2-1)/8z\right]} \simeq -z\tan u \qquad \text{(A5.10)}$$

Infinite Product Expansion

$$\mathcal{J}_m(z) = -m + 2m \prod_{p=1}^{\infty} \left\{1 - (\frac{z}{j_{m-1,p}})^2\right\} \bigg/ \left\{1-(\frac{z}{j_{m,p}})^2\right\} \qquad \text{(A5.11)}$$

A5.3 The Lommel Polynomials

As shown in the discussion in problem 3, functions of the form $I_{m+r}(z)/I_m(z)$ may be encountered. These are best treated with the Lommel Polynomials (29, paragraph 9.6). One can write

$$J_{m+r}(z)/J_m(z) = R_{r,m}(z) - [J_{m-1}(z)/J_m(z)]R_{r-1,m+1}(z) \qquad \text{(A5.12)}$$

where

$$R_{r,m}(z) = \sum_{p=0}^{\leq r/2} (-1)^p \left[\frac{(r-p)!}{p!(r-2p)!}\right] \left[\frac{\Gamma(m+r-p)}{\Gamma(m+p)}\right] (\frac{z}{2})^{-r+2p} \qquad \text{(A5.13)}$$

Thus

$$I_{m+r}(z)/I_m(z) = [i^{-r} R_{r,m}(iz)] + [i^{-(r-1)}R_{r-1,m+2}(iz)][I_{m-1}(z)/I_m(z)]$$

(A5.14)

The bracketed quantities are real.

A5.4 Evaluation of $J_{3,m+1} = \int^z x^3 I^2_{m+1}(x)dx$

We start with the **Bessel Function** recursion relation (33, eq. 803.3):

$$xI_{m+1} = xI_{m-1} - 2mI_m \qquad \text{(A5.15)}$$

If we square both sides, multiply by x, and integrate to z, we obtain

$$J_{3,m+1} = J_{3,m-1} - 4m\int^{z} x^2\, I_m\, I_{m-1}\, dx + 4m^2\int^{z} x\, I_m^2\, dx \qquad (A5.16)$$

By using the recursion relation (33, eq. 803.4)

$$I_m = 2I'_{m-1} - I_{m-2} \qquad (A5.17)$$

we can write the first integral as

$$\int^{z} x^2\, I_m\, I_{m-1}\, dx = 2\int^{z} x^2\, I_{m-1}\, I'_{m-1}\, dx - \int^{z} x^2\, I_{m-1}\, I_{m-2}\, dx \qquad (A5.18)$$

$$= \int^{z} x^2\, d(I_{m-1}^2) - \int^{z} x^2\, I_{m-1}\, I_{m-2}\, dx \qquad (A5.19)$$

If we continue this process $m-1$ times, we can write

$$\int^{z} x^2\, I_m\, I_{m-1}\, dx = \sum_{p=1}^{m-1} (-1)^{p+1}\int^{z} x^2\, d(I_{m-p})^2 + \frac{(-1)^{m+1}}{2}\int^{z} x^2 dI_o^2 \qquad (A5.20)$$

since

$$\int^{z} x^2\, I_1 I_0\, dx = (1/2)\int^{z} x^2\, d(I_o^2)$$

Integrating $\int^{z} x^2\, d(I_{m-p}^2)$ by parts we obtain

$$\int^{z} x^2\, d(I_{m-p}^2) = z^2\, I_{m-p}^2 - 2\int^{z} xI_{m-p}^2\, dx \qquad (A5.21)$$

The last integral is evaluated using Ref. 30, p. 146, namely,

$$\int^{z} x\, I_m^2\, dx = (z^2/2)\,(I_m^2 - I_{m-1}\, I_{m+1}) \qquad (A5.22)$$

Thus, A5.21 reduces to

$$\int^{z} x^2 \, d(I_{m-p}^2) = z^2 \, I_{m-p-1} \, I_{m-p+1} \tag{A5.23}$$

and A5.20 becomes

$$\int^{z} x^2 \, I_m \, I_{m-1} \, dx = \sum_{p=1}^{m-1} (-1)^{p+1} \, z^2 \, I_{m-p+1} \, I_{m-p-1} + \frac{(-1)^{m+1}}{2} \, I_1^2 \tag{A5.24}$$

If we substitute A5.24 into A5.16 and use A5.22 again, we obtain the recursion formula

$$J_{3,m+1} = J_{3,m-1} - 4mz^2 \left[\frac{m}{2} \, I_{m+1} \, I_{m-1} + \sum_{p=1}^{m-1} (-1)^{p+1} \, I_{m-p+1} \, I_{m-p-1} + \frac{(-1)^{m+1}}{2} \, I_1^2 \right] + 2m^2 z^2 I_m^2 \tag{A5.25}$$

For this recursion relation to be useful we must know $J_{3,0}$ and $J_{3,1}$. These are obtained from the integral representation of I_m^2 (29, paragraph 13.72, eq. 2).

$$I_m^2(x) = 2/\pi \int_0^{\pi/2} I_{2m}(2x \cos \theta) \, d\theta = 2/\pi \int_0^{\pi/2} I_{2m}(2x \sin \theta) \, d\theta \tag{A5.26}$$

Multiplying A5.26 by x^3 and integrating to z yields

$$J_{3,m} = 2/\pi \int_0^{\pi/2} d\theta \int^{z} x^3 \, I_{2m}(2x \sin \theta) \, dx \tag{A5.27}$$

Consider the inner integral for the case $m = 0$ and let

$$\alpha = 2 \sin \theta \qquad w = 2x \sin \theta = \alpha x \tag{A5.28}$$

Thus $\int^{z} x^3 I_o(\alpha x)\, dx = \frac{1}{\alpha^4}\int^{\alpha z} dw\, w^2[wI_o(w)] = \frac{1}{\alpha^4}[(\alpha z)^3 I_1 - 2(\alpha z)^2 I_2]$ (A5.29)

We have integrated by parts and used Ref. 33, eq. 803.6. Simplifying

$$\int^{z} x^3 I_o(\alpha x)\, dx = \frac{z^3}{2 \sin \theta} I_1 - \frac{2z^2 I_2}{4 \sin^2 \theta} = z^4 \left[\frac{I_o}{4} - \frac{I_2}{6} - \frac{I_4}{12}\right] \quad (A5.30)$$

The argument of the Bessel Functions is $(2z \sin \theta)$. The last relation is obtained by repeated use of the recurrence formula, A5.15. Substituting A5.30 into A5.27 (with $m = 0$) yields

$$J_{3,0} = \left(\frac{2z^4}{\pi}\right)\int_o^{\pi/2} d\theta[(\tfrac{1}{4})\, I_o(2z \sin \theta) - (\tfrac{1}{6})\, I_2(2z \sin \theta) - (\tfrac{1}{12})\, I_4(2z \sin \theta)] \quad (A5.31)$$

or

$$J_{3,0} = z^4[(\tfrac{1}{4})\, I_o^2(z) - (\tfrac{1}{6})\, I_1^2(z) - (\tfrac{1}{12})\, I_2^2(z)] \quad (A5.32)$$

A5.32 is obtained by comparing each term of A5.31 with A5.26. A similar manipulation yields

$$J_{3,1} = (z^4/6)[I_1^2(z) - I_2^2(z)] \quad (A5.33)$$

A6 AN OUTLINE OF THE COMPUTATION PROGRAM

The argument presented in Chapter 9 demonstrated that the dispersion relations for flow problems (e.g., 7.38) would have complex modes. Thus, these relations must be treated as functions of a complex variable. At the time the program was begun (December 1958) there were no subroutines available at the Caltech Computing Center for the Model 205 Datatron which could help the author. This necessitated the design of fundamental computer operations for complex numbers. The total program (including the dispersion relation being analyzed) was written for the "floating point" system of numbers and occupied 78 percent of the Datatron's memory of 4,000 words.

A6.1 Subroutines for Functions of a Complex Variable

In designing these subroutines the complex number is treated as an ordered pair of numbers upon which the elementary operations are performed. If $z = x + iy$ and $f = u + iv$, then the subroutines were constructed as follows:

a. Multiplication (M): $zf = (xu - yv) + i(yu + xv)$ (A6.1)

b. Divide (D): $z/f = [(xu + yv) + i(yu - xv)](u^2+v^2)^{-1}$ (A6.2)

c. Square Root (SR): $(z)^{1/2} = \pm (\alpha + i\beta)$ (A6.3)

where $\sqrt{2}\,\binom{\alpha}{\beta} = (\pm x + \sqrt{x^2 + y^2})^{1/2}$ (A6.4)

d. Bessel Function Ratio (BFR)*:

---

* See Section A5.1 and reference 35.

$$\frac{z^n I_{m-n}(z)}{I_m(z)} = c_{mn} \frac{\sum_{p=o}^{\infty} T_{m-n,p}(z)}{\sum_{p=o}^{\infty} T_{m,p}(z)} \tag{A6.5}$$

where

$$c_{mn} = \frac{2^n n!}{(m-n)!} \quad , \qquad T_{\beta,p}(z) = \frac{(z/2)^{2p}}{p!\ (\beta+p)!} \tag{A6.6}$$

e. Complex Iterator or Zero Searcher (CI):

$$z_{n+1} = \frac{-z_n f_{n-1} + z_{n-1} f_n}{f_n - f_{n-1}} = z_n - \frac{(z_n - z_{n-1})}{1-\alpha_n} \tag{A6.7}$$

where

$$\alpha_n = f_{n-1}/f_n \tag{A6.8}$$

Equation A6.7 is essentially Newton's method applied to the function $f(z)$ (whose zeros are sought), with derivatives replaced by differences. The subscripts refer to successive values obtained from the iteration procedure. To derive this relation, one expands f in a power series in a region, z_o, where a zero, z_r, is expected.

$$f(z) = f(z_o) + f'(z_o)\,\Delta z + \frac{f''(z_o)}{2}(\Delta z)^2 + \ldots \tag{A6.9}$$

This expansion is valid, since we are dealing with analytic functions of a complex variable. If derivatives higher than the second are neglected, one obtains Δz, an estimate of the quantity $z_r - z_o$, by setting the left side equal to zero and solving the resulting quadratic equation.

$$\Delta z = -\frac{f}{f'}\left[1 + \frac{ff''}{f'f'}\right] \tag{A6.10}$$

Equation A6.7 is derived from A6.10 by setting: (1) $f'' = 0$; (2) $\Delta z = z_{n+1} - z_n$; (3) $f = f_n$; and (4) $f' = \Delta f_{n-1}/\Delta z_{n-1} = (f_n - f_{n-1})/(z_n - z_{n-1})$. This procedure can be shown to converge if $\left|\frac{f_n - f_{n-1}}{f_n}\right| = |1-\alpha_n| > 1$ in the vicinity of a root. Thus the procedure converges if f is diminished by a factor > 2 in each step of the iteration.

The (CI) program operated as follows: (1) one stores z_o and z_1, two estimates of z_r; (2) f_o and f_1 are calculated and the iteration procedure begins with the calculation of z_2, using A6.7; (3) the machine is instructed to print out that a root has been located if either of the following requirements is met:

$$|f_n|^2 \leq 10^{-16} \qquad \text{or} \qquad \Delta z_n = z_{n+1} - z_n \leq 10^{-8}$$

The iteration procedure proved to be adequate except in those rare cases where: (1) $f(z)$ had a minimum or a maximum of very small magnitude; and (2) there was a multiple root or two nearby roots. In such cases the machine was instructed to print out the successive iterations, and one could examine them and draw the proper conclusions.

A6.2 Testing the Subroutines

Each subroutine was tested before the entire group was assembled into a unified program. The tests were constructed for examples which could easily be calculated by hand and for examples which tested critical numerical values. Operations A6.1, A6.2, and A6.3 give results accurate to one part in the eighth significant figure (there are 8 significant figures in a floating point number). Since A6.5 utilized an initial series expansion, one expects its accuracy to fall off far from the origin. Within a circle of radius $|z| < 8$ the results were accurate to one part in the sixth significant figure. Near the real axis, $z = x + i\epsilon$ (ϵ small), the accuracy was greatly improved, and at $z = x = 20$ the results were accurate to five parts in the eighth significant figure. In other words, the lines of "iso-accuracy" were ellipses with major axes along the real axis.

A6.3 The Composite Program

The dispersion relation, 7.38, was rearranged into the best form for computation (see Section 9.3) and utilized subroutines A6.1, A6.2, and A6.5. The function f was studied with the aid of three additional "system" programs.

a. Grid or Complex-Plane-Scanner (G): This program prints out the real and imaginary values of f along a line parallel to the real axis. The total range covered and the spacing between points along the horizontal and vertical axes are adjustable. Figure A6-1 exhibits a typical grid printed out with floating point numbers* of three digits and a sign. Thus, each value is given to one significant figure.

---

* The first two digits determine the exponent n. That is, n = (first two digits) - 50. The third and following digits give the number of significant figures up to a maximum of eight. For example, the number -yyxxxx is $-(0.xxxx) \times 10^n$, n = yy-50. If -yyx = -492, it corresponds to -0.02.

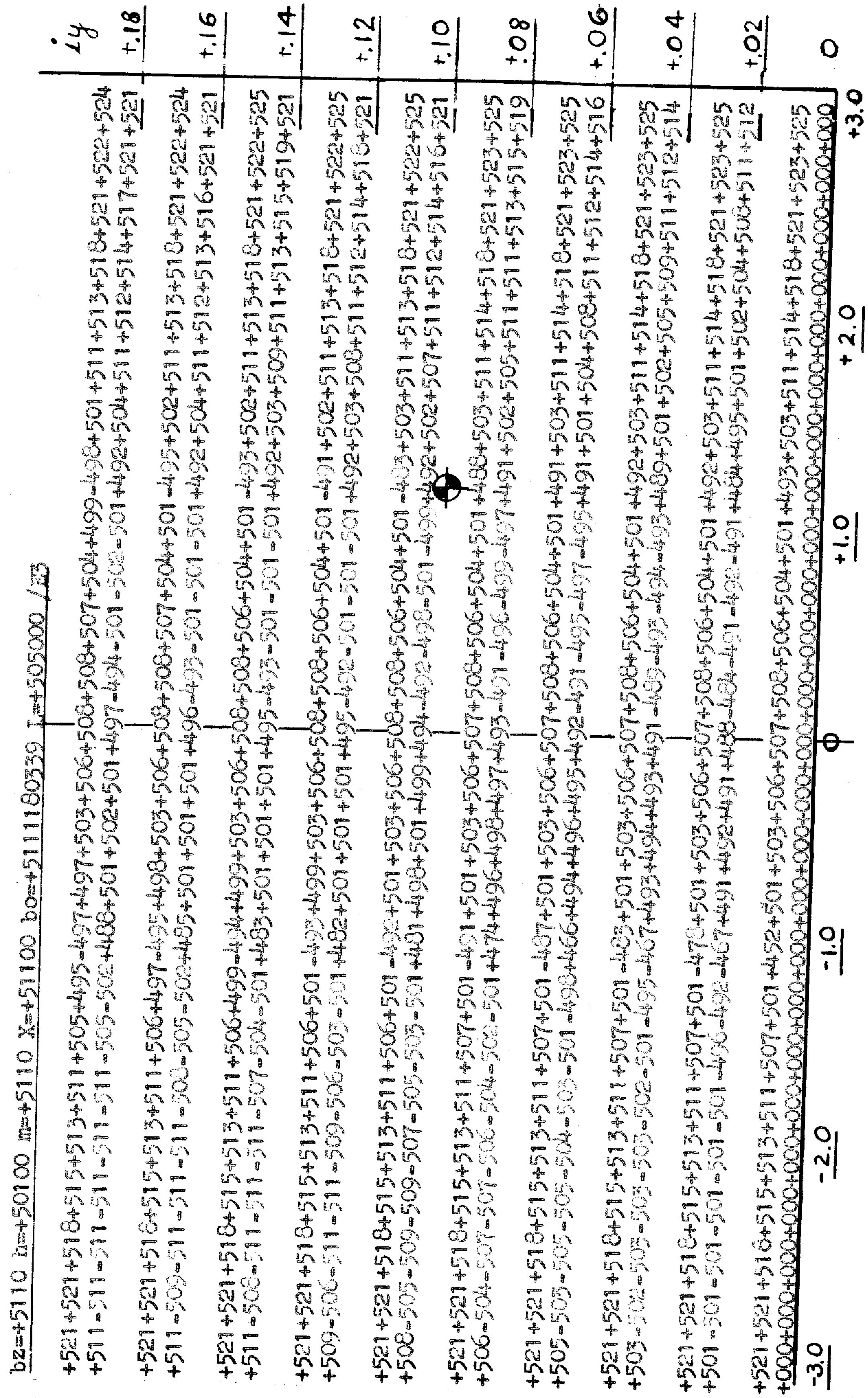

Fig. A6-1. A Grid Print-Out For a Function of a Complex Variable.

For each point in the plane the real part of f is printed over the imaginary part of f. Negative numbers are printed in red and positive in black (as seen in the original copy of this thesis) for quick scanning of zero locations. The heading gives b_z(bz), h, m, X, b_θ(bo), and Λ(L), symbols which are defined in the thesis.

b. Root-Locator (RL): After a line is calculated, the RL program examines successive pairs of adjacent values of f starting at the left. When both the real part and the imaginary part of f change sign, the machine searches that region for a zero by switching to the Complex Iterator and informing it of the region where the double sign change was found. The Grid and Root-Locator are frequently used together (GRL).

c. Special-Root-Search (SRS): The location of the singularities of f is determined by the values of m and X as shown in Section 9.3 Since f is changing rapidly in the neighborhood of such points, one expects that a zero (or zeros) may be present there. The SRS is designed to inform the Complex Iterator to search a pre-specified number of regions of the complex plane, depending upon the values of m and X which are being studied.

A6.4 Typical Print-Outs

When the Complex Iterator locates a root, the machine prints out a format as shown in Fig. A6-2a. The first line indicates that it is a root (in red) and the number of iterations ("its") required to get there. The next three lines give the last three iterations before print-out in the format

```
bz=+5110 h=+50100 m=+5110 X=+51100 bo=+5111180339 L=+505000 /E3
***root its.=+09
+51121823(+49922289).  +43220000(-42100000).
+51121823(+49922289).  +43430000(-44126000).
+51121823(+49922289).  +43120000(-43940000).
 u=-50226613(-50684489).  u=-49572079(+50622698).  U=-49572080(+50622698).
```

Fig.A6-2a. Format of the Computer Print-Out When A Root Is Located.

```
+51130000(+50100000).  +49119616(+49579392).
+51140000(+50110000).  +49949171(+50164239).
+51126261(+49766246).  +49133943(+49236899).
+51123966(+49707814).  +49120214(+49107395).
+51121375(+49844916).  +48456471(-48198233).
+51121729(+49937278).  -47817959(-47643585).
+51121825(+49921397).  +46507510(+46155150).
+51121824(+49922284).  +44258000(+44468000).
+51121823(+49922289).  +43120000(-43940000).
+51121823(+49922289).  +43430000(-44126000).
+51121823(+49922289).  +43220000(-42100000).
+00000000(+00000000).  +00000000(+00000000).
```

Fig.A6-2b. The Successive Iterates Corresponding to the Above Print-Out and the Grid of Figure A6-1.

Fig.A6-2. Format

Re z(Im z). Re f(Im f).

The fourth line gives the two values of u as determined from the u, Ω transformation, 7.36. The subroutine for the u, Ω transformation utilizes A6.1, A6.2, and A6.3. U is described in Section 9.3.

Figure A6-2b shows a sequence of nine iterations starting from $z_0 = 1.3 + i0.1$ and $z_1 = 1.4 + 0.11i$ and converging to $z_{10} = 1.21823 + i.0922289$. This case corresponds to the root printed out in Figure A6-2a and to the grid given in Fig. A6-1. In fact, between $x = 1.0$ and $x = 1.25$ one sees that both the real and imaginary values of f change sign at $y = 0.10 \rightarrow y = 0.18$, indicating the presence of a root in this region. Note also that f has a minimum on the real axis at $x = 1.25$, another indication of a root in the complex plane.

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