Essays on Economic Networks

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To my parents Maria Sanchez and Jose Melo.

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Abstract

In this thesis we analyze three different problems raised in the context of economic networks. In Chapter 2 we study the problem of bargaining in networked markets, and we make two contributions. First, we characterize market equilibria in our model, and find that players' equilibrium payoffs coincide with their degree of centrality in the network, as measured by Bonacich's centrality measure. This characterization allows us to map in a simple way network structures into market equilibrium outcomes, so that payoffs dispersion in networked markets is driven by players' network positions. Second, we show that the market equilibrium for our model converges to the eigenvector centrality measure. We show that the economic condition to reach convergence is that the players' discount factor goes to one. In Chapter 3, we extend the traditional discrete choice theory to the case of markets with a network structure. Formally, we model the discrete choice process as the flow on a directed network, which is assigned in an optimal way through the solution of a dynamic programming problem. Combining these two elements we show that a demand system for hierarchical or sequential decision processes can be obtained as the outcome of utility maximization by a representative agent. Finally, in Chapter 4 we study the problem of price competition and free entry in networked markets subject to congestion effects. In our analysis, we consider a network with multiple origins and a common destination node, where each link is owned by a firm that sets prices in order to

maximize profits, whereas users want to minimize the total cost they face, which is given by the congestion cost plus the prices set by firms. In this environment, we introduce the notion of *Markovian Traffic Equilibrium* to establish the existence and uniqueness of a pure strategy price equilibrium, without assuming that the demand functions are concave nor imposing particular functional forms for the latency functions. Given this existence and uniqueness result, we apply our framework to study entry decisions and welfare, and establish that in congested markets with free entry, the number of firms exceeds the social optimum.

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Chapter 1

Introduction

This thesis belongs to the growing field of economic networks. In particular, we develop three essays in which we study the problem of bargaining, discrete choice representation, and pricing in the context of networked markets. Despite analyzing very different problems, the three essays share the common feature of making use of a network representation to describe the market of interest.

In Chapter 1 we present an analysis of bargaining in networked markets. We make two contributions. First, we characterize market equilibria in a bargaining model, and find that players' equilibrium payoffs coincide with their degree of centrality in the network, as measured by Bonacich's centrality measure. This characterization allows us to map, in a simple way, network structures into market equilibrium outcomes, so that payoffs dispersion in networked markets is driven by players' network positions. Second, we show that the market equilibrium for our model converges to the so called *eigenvector centrality measure*. We show that the economic condition for reaching convergence is that the players' discount factor goes to one. In particular, we show how the discount factor, the matching technology, and the network structure interact in a very particular way in order to see the eigenvector centrality as the limiting case of our market equilibrium.

We point out that the eigenvector approach is a way of finding the most central or relevant players in terms of the *global* structure of the network, and to pay less attention to patterns that are more *local*. Mathematically, the eigenvector centrality captures the relevance of players in the bargaining process, using the eigenvector associated to the largest eigenvalue of the adjacency matrix of a given network. Thus our result may be viewed as an economic justification of the eigenvector approach in the context of bargaining in networked markets.

As an application, we analyze the special case of seller-buyer networks, showing how our framework may be useful for analyzing price dispersion as a function of sellers and buyers' network positions.

From a technical viewpoint, we show how simple ideas from the theory of *linear complementarity problems* can be exploited in the context of networked markets and Nash bargaining. In particular, we show that both models, the exchange network and the seller-buyer networks, can be analyzed through the study of an associated quadratic optimization problem. This optimization problem encapsulates all the needed information to understand existence, uniqueness, and characterization of a market equilibrium.

Chapter 2 discusses how to extend the discrete choice theory approach to the case of networked markets. Formally, we provide an alternative way to model sequential decision processes, which is consistent with the random utility maximization hypothesis and the existence of a representative agent. Our result is stated in terms of a direct utility representation, and it does not depend on parametric assumptions. One of the main innovations in this Chapter, is the idea of viewing the decision process as the *flow* in a directed network, which is assigned in an optimal way through the solution of a dynamic programming problem. Combining these two elements we show that a demand system for hierarchical or sequential decision processes can be obtained as the outcome of utility maximization by a representative agent. We only require the mild condition that the distribution of the unobserved components must be absolutely continuous.

From an applied perspective we point out that our results can be useful for carrying out welfare analysis in networked markets, where the standard discrete choice theory may not apply. For example, our results can be applied to bundling decisions, merger analysis, or compatibility among goods in networked markets.

Finally, in Chapter 3 we study the problem of price competition and free entry in networked markets subject to congestion effects. In many environments, such as communication networks in which network flows are allocated, or transportation networks in which traffic is directed through the underlying road architecture, congestion plays an important role. In particular, we consider a network with multiple origins and a common destination node, where each link is owned by a firm that sets prices in order to maximize profits, whereas users want to minimize the total cost they face, which is given by the congestion cost plus the prices set by firms. In this environment, we introduce the notion of *Markovian traffic equilibrium* to establish the existence and uniqueness of a pure strategy price equilibrium, without assuming that the demand functions are concave nor imposing particular functional forms for the latency functions. We derive explicit conditions to guarantee existence and uniqueness of equilibria. Given this existence and uniqueness result, we apply our framework to study entry decisions and welfare, and establish that in congested markets with free entry, the number of firms exceeds the social optimum.

Chapter 2

Bargaining and centrality in networked markets

2.1 Introduction

Traditional models of economic exchange assume that all possible coalitions of agents can meet and trade. However, due to social relationships, institutional, legal, and physical barriers, it may be impossible for certain sets (pairs) of sellers and buyers to communicate or trade with one another directly. For example, financial markets, supply chains, international trade, and many other markets exhibit barriers limiting the kinds of coalitions that agents may form.

A simple way of capturing the barriers to the exchange process is via the network of connections/exchanges it allows for trading. Formally, in a given network the set of nodes represent players and the set of links represent the possibilities of exchange between any two players. Thus, the use of a network representation has the advantage of showing in a simple way how the different types of barriers to trade determine different market structures.

In this chapter we try to understand how the network topology affects players' payoffs. We focus on the following key question:

How does network topology determine players' payoffs (market outcomes)?

We address the question by studying a simple bargaining model in networked markets. Our model is based on random matching and the *Nash bargaining solution*, and has the advantage of being simple, and most importantly, providing an *explicit* connection between players' network positions and equilibrium payoffs. Concretely, we show that players' equilibrium payoffs are uniquely determined by their degree of centrality in the network, measured by Bonacich [1987]'s centrality measure.[1]

Our characterization shows how payoffs dispersion is driven by players' network positions, where players with a higher degree of centrality obtain higher payoffs than players with lower centrality. Thus players' positions determine their market power in the trading process.

In addition, we note that our result can be seen as economic justification of Bonacich's measure in the context of bargaining in networked markets. [2]

Our second contribution is the result that as long as the players' discount factor goes to one, the equilibrium payoff vector converges to the eigenvector centrality.

We point out that the eigenvector approach is a way to find the most central or relevant players in terms of the "*global*" structure of the network, and to pay less attention to patterns that are more "*local*". Mathematically, the eigenvector centrality captures the relevance of players in the bargaining process, using the eigenvector associated to the largest eigenvalue of the adjacency matrix of a given network.[3]

[1]For a survey of applications of Bonacich's measure in economics we refer the reader to Jackson [2008] and Goyal [2009].

[2]In fact, Bonacich [1987] proposes the measure as way to rationalize the outcome of lab experiments in the context of bargaining in networks, but without considering an economic model.

[3]From an applied perspective, the eigenvector approach has been applied in sociology and in the context of page rank analysis (see Langville and Meyer [2006]).

Thus our result may be viewed as an economic justification of the eigenvector approach in the context of bargaining in networked markets.

Formally, our convergence result relies on the specific relationship between, the discount factor, the matching technology, and the network structure, and intuitively it may be interpreted as follows: as long as players become more and more patient, then the market equilibrium is driven by the eigenvector centrality.

We remark that our result is not the first establishing the connection between Bonacich's measure and the eigenvector centrality. [4] However, the main contribution of our result is that it provides a simple and intuitive economic condition for understanding when Bonacich's measure can be interpreted as the eigenvector centrality

As a particular case of our model, we analyze the problem of seller-buyer networks. For this specific environment, we show that a convex minimization problem contains all the relevant information to study existence and uniqueness of a market equilibrium. The characterization of a market equilibrium as the solution of a minimization problem turns out to be useful in the context of networks with finitely many sellers and buyers, where standard procedures of convex optimization may be implemented. Furthermore, we exploit the bipartite structure of seller-buyer networks in order to provide an explicit characterization of equilibrium payoffs in terms of sellers and buyers' positions. This characterization rationalizes why buyers with the same valuation for a good may, in equilibrium, pay different prices.

For the case of the convergence of a market equilibrium to the eigenvector centrality, we show that for sellers and buyers two mutually exclusive conditions can be derived. The reason for this different condition is given by the bipartite structure of

[4]See for example Friedkin and Johnsen [1990], Friedkin [1991], Bonacich [1997], and Bonacich and Lloyd [2001]

seller-buyer networks.

It is worth remarking that our result of payoffs characterization as the convergence of a market equilibrium to the eigenvector centrality extend the findings in Corominas-Bosch [2004]. Concretely, our results fully characterize the sellers and buyers' payoffs in terms of networks positions without relying on a specific bargaining protocol, whereas Corominas-Bosch [2004]'s results rely on the kind of bargaining protocol implemented.

From a technical viewpoint, we show how simple ideas from the theory of *linear complementarity problems* can be exploited in the context of networked markets and Nash bargaining. In particular, we show that both models, the exchange network and the seller-buyer networks, can be analyzed through the study of an associated quadratic optimization problem. This optimization problem encapsulates all the needed information to understand the existence, uniqueness, and characterization of a market equilibrium.

2.1.1 Outline of the model

Our model may be described as follows. We consider an environment in which any pair of players can trade if and only they are connected in the network. The network represents the underlying social structure, where a link between two players represents the opportunity to create one unit of surplus. Given the possibility of jointly creating such surplus, both players (connected by a link) must agree on how to split it. We assume that players split the surplus according to the *symmetric Nash bargaining solution*.[5]

Borrowing ideas from the macroeconomics literature on search, we model the

[5] The symmetry assumption is made for simplicity, but it is not essential to our analysis. In fact, all of our results extend to the case of using an asymmetric Nash bargaining solution.

meeting process among players in dynamic and random way. In particular, at each point of time, a link connecting two agents is randomly drawn.[6]Then the chosen players must split the surplus (according to the Nash bargaining solution). An important feature is that the disagreement points are determined *endogenously.* This endogeneity captures the fact that if two chosen players do not reach an agreement, then they can wait until the next period with the expectation of being drawn again and so achieve a better payoff.[7] On the other hand, if two players reach an agreement, then they split the surplus and leave the market, and their positions in the network are occupied by two new players.[8] Players discount utilities using a common discount factor.

This random exchange process induces a dynamical system which shows how players' payoffs evolve over time. In order to study the equilibrium of the exchange process, we analyze its *steady state*, which we refer as a *market equilibrium.*

In this framework, it is easy to show the *existence* and *uniqueness* of a market equilibrium. However, the most important property of our analysis is the fact that a market equilibrium is exactly equivalent to the centrality measure proposed by Bonacich [1987]. For this characterization, we provide a condition such that the market equilibrium converges to the eigenvector centrality.

[6]It is worth remarking that the idea of drawing a link at random at each point of time was early proposed in Stolte and Emerson [1977] in the context of experiments on exchange networks in sociology. Thus our choice of the matching technology can be justified from an experimental perspective.

[7]The endogeneity of the disagreement points allows us to link the Nash' bargaining solution with the strategic approach. See Binmore et al. [1986] for a detailed discussion about this technical aspect.

[8]This assumption allows us to avoid strategic considerations that would be raised in the situation that players cannot be replaced. Formally the assumption allows us to avoid dealing with repeated games effects.

2.1.2 Related work

The model we have described in the previous section is related to two different branches of economic literature. Specifically, our model is related to macroeconomics models of search and matching[9], and models of bargaining in networked markets.[10] Instead of describing the extensive literature on these different approaches, we shall describe the paper by Manea [2011], which turns out to be the closest article to our work.

The paper by Manea [2011] develops a model of strategic bargaining in networked markets. In particular, Manea [2011] extends Gale [1987]'s strategic approach to network environments. However, there are two important differences between our paper and Manea [2011]'s approach. First, we analyze a networked market using the symmetric Nash bargaining solution, whereas Manea [2011] proposes a strategic model where the extensive form of the bargaining turns out to be critical. Formally, in Manea [2011] there are many possible sub game perfect Nash equilibria supporting a market outcome. Because we use the Nash bargaining solution we get rid of this multiplicity problem.

The second and most important difference, is that our approach allows us to give a simple and explicit expression for the equilibrium payoff vector in terms of Bonacich [1987]'s centrality measure. Furthermore, we are able to derive an economic condition to relate Bonacich's measure with the eigenvector centrality in the context

[9] The idea of combining random matching and Nash bargaining was first proposed in the macroecomomic literature on search. The first papers proposing this approach were Diamond and Maskin [1979] and Diamond [1982]. A survey of this literature can be found in Rogerson et al. [2005].

[10] Two early papers that analyze markets using a network representation are Kranton and Minehart [2001] and Calvo-Armengol [2001] Following Kranton and Minehart [2001] and Calvo-Armengol [2001]'s ideas, the following papers study networked markets: Corominas-Bosch [2004], Kleinberg and Tardos [2008], Nguyen [2011a,b], Chakraborty et al. [2011], Manea [2011], Elliott [2012]. All these papers pay attention to an array of different issues, such as strategic bargaining, imbalance outcomes, and inefficiencies.

of bargaining in networked markets. Neither of these results are in Manea [2011].

The rest of the chapter is organized as follows. Section 4.2 describes the basic model of exchange in networks. Section 4.3 analyzes the relationship between our market equilibrium and the eigenvector centrality. Section 4.4 studies the specific case of sellers-buyers networks. Section 2.5 concludes. Definitions and proofs are relegated to Appendix A.1 .

2.2 The model: exchange networks

Let $N = \{1, \ldots, n\}$, with $n \geq 3$, being the set of agents, and let i and j denote typical members of this set. Let $E \subset N \times N$ be the set of connections (relationships) among agents in N. Given the sets N and E, we define a networked market by the *undirected* graph $G = (N, E)$. We identify the network G with its adjacency matrix $\mathbf{G} = (\mathbf{g}_{ij})$, where $\mathbf{g}_{ij} = 1$ if there is a link between i and j, and $\mathbf{g}_{ij} = 0$ otherwise. We denote by $\mathcal{G}^N$ the set of all possible networks given the set of agents N. In particular, a link $ij \in G$ is viewed as the ability of agents i and j to generate a unit of surplus. We shall assume that any pair of agents in the network G, can be connected through a collection of links in E. Formally, we shall assume that any network $G \in \mathcal{G}$ is *strongly connected*.

For each player $i \in N$, we define the set of neighbors as $N_i^G = \{j \in N : ij \in G\}$, which describes the set of agents with which player i can meet and trade. We denote the cardinality of N_i^G as $\mathbf{d}_i = |N_i^G|$. We refer to $\mathbf{d}_i$ as agent i's *degree*, where the diagonal matrix $\mathbf{D}$ contains the degree of all players, i.e., its elements are given by the $\mathbf{d}_i$.

In order to analyze the exchange process, we introduce the following dynamic bargaining process. Each period of time $t = 0, 1, \ldots,$ a link $ij \in G$ is selected randomly with an *exogenous* probability π_{ij}. The probability π_{ij} is the probability that agents i and j are drawn to trade. We assume the π_{ij}s are uniform, with $\pi_{ij} = \pi = \frac{1}{E}$ for all $ij \in G$. We shall refer to the probability measure π as the *matching technology*.[11]

Once the link ij is drawn, players i and j split the generated surplus accordingly to the *Nash bargaining solution*. Players discount payoffs with a common discount factor $0 < \delta < 1$.

From the previous description, the dynamical process can be summarized as follows. At time t players i and j, connected through link ij, are chosen with probability π to bargain over one unit of surplus. If they do not reach an agreement, each stays in the market until $t+1$, waiting for a new opportunity to trade. In the case that both players reach an agreement at time t they split the surplus using the Nash solution. We point out that the disagreement points for players i and j correspond to they expected utility if they remains in the market until period $t+1$. In order to analyze a steady state situation, we assume that when a pair of players i and j reach an agreement, they leave the market, and they are replaced in the same positions by two new players.

In a steady state situation, let V_i be player i's *steady state expected payoff* of being in the market at time t. Then if at time t the pair of players i and j fail to reach an agreement, they remain in the market until time $t+1$, and from the point of view of time t, disagreement results in expected utilities δV_i and δV_j.

Formally, a steady state is represented by the following system of linear equations:

[11] We point out that this type of matching technology on networks was early proposed in the sociological literature under the label of *1-exchange rule*. For a discussion and applications of the 1-exchange rule on exchange networks we refer to Willer [1999].

$$V_i = \sum_{j \in N_i^G} \frac{\pi}{2}\left(1 - \delta V_j + \delta V_i\right) + (1 - \mathbf{d}_i \pi)\delta V_i, \quad \forall i. \tag{2.1}$$

The explanation for equation (2.1) is the following. When players i and j are drawn to trade, they split the surplus, and thanks to the Nash bargaining solution, player i gets $\frac{1}{2}(1 - \delta V_j + \delta V_i)$. Because the meeting between players i and j occurs with probability π, we find that the expected payoff for player i is $\frac{\pi}{2}(1 - \delta V_j + \delta V_i)$. Adding up over all $j \in N_i^G$, we find that player i's expected value of being drawn at time t is $\sum_{j \in N_i^G} \frac{\pi}{2}(1 - \delta V_j + \delta V_i)$. The second term is due to the fact that at time t with probability $(1 - \mathbf{d}_i \pi)$, player i has to wait till period $t+1$ for a new opportunity to trade. The expected value of this event is $(1 - \mathbf{d}_i \pi)\delta V_i$. Combining these two expected values we get expression (2.1).

Equation (2.1) can be written in a compact way using matrix notation. In particular, we get the following

$$(1 - \delta)[\mathbf{I} + \pi\kappa\mathbf{A}]V = \frac{\pi}{2}\mathbf{d}, \tag{2.2}$$

where $\mathbf{A} = \mathbf{D} + \mathbf{G}$, a n-square matrix, and $\mathbf{d} = (d_i)_{i \in N}$ the $n-$dimensional degree vector. The parameter κ is defined as $\kappa = \frac{\delta}{2(1-\delta)}$. Finally $V = (V_i)_{i \in N}$ is an $n-$dimensional vector, which captures players' payoffs.

Expression (2.2) may be rewritten as:

$$[\mathbf{I} + \pi\kappa\mathbf{W}]V \quad = \quad \hat{\mathbf{c}}, \tag{2.3}$$

where $\mathbf{W} \equiv [\mathbf{I} + \pi\kappa\mathbf{D}]^{-1}\mathbf{G}$ and $\hat{\mathbf{c}} = \frac{1}{(1-\delta)}[\mathbf{I} + \pi\kappa\mathbf{D}]^{-1}\mathbf{c} = \frac{1}{2(1-\delta)}\mathbf{W}\mathbf{1}$, with $\mathbf{1}$ a vector of ones. It is worth nothing that the matrix $\mathbf{W}$ can be interpreted as a weighted adjacency matrix.

We now are ready to give a definition of a *market equilibrium* for the networked market just described.

Definition 1 *A payoff vector V^* is a market equilibrium (ME) if it solves (2.3) with $V^* \geq 0$.*

In other words, Definition 1 establishes that a nonnegative solution of (2.3) is an ME. The intuition for $V^* \geq 0$ is that players can afford zero payoffs if they decide not to participate in the market. Thus the non negativity condition for a ME can be interpreted as an individual rationality condition.

Our first task, is to establish that such an ME exists. In order to analyze the existence and uniqueness of a ME, we borrow basic ideas from the theory of *linear complementarity problems* (Cottle et al. [2009]). The application of this technique is based on the the observation that a ME can be formulated as the following collection of linear inequalities:

$$
\begin{aligned}
V &\geq \mathbf{0} && (2.4)\\
(\mathbf{I} + \pi\kappa\mathbf{W})V - \hat{\mathbf{c}} &\geq \mathbf{0} && (2.5)\\
V^T([\mathbf{I} + \pi\kappa\mathbf{W}]V - \hat{\mathbf{c}}) &= \mathbf{0} && (2.6)
\end{aligned}
$$

Equations (2.4)-(2.6) describe an ME as a linear complementarity problem (LCP), where a vector V is a solution to a ME, if and only if V satisfies (2.4)-(2.6) *simultaneously*. For short, we refer to (2.4)-(2.6) as LCP$(\mathbf{W}, \hat{\mathbf{c}})$.

Proposition 1 below states the simple but important fact that the ME can be analyzed as the solution of LCP$(\mathbf{W}, \hat{\mathbf{c}})$.

Proposition 1 *V^* is an ME if and only if V^* solves the* LCP$(\boldsymbol{W}, \hat{\boldsymbol{c}})$.

The main advantage of studying the equilibrium problem through LCP is the powerful battery of results of existence and uniqueness of solutions. In fact, our next result exploits such connection.

Proposition 2 *There exists a unique ME which is given by the solution of the following optimization problem:*

$$\begin{aligned} \min_V \quad & \{\frac{1}{2} V^T \boldsymbol{M} V - \boldsymbol{c}^T V\} \\ s.t. \quad & V \geq 0, \end{aligned} \tag{2.7}$$

with $\boldsymbol{M} = (1-\delta)[\boldsymbol{I} + \pi\kappa\boldsymbol{D}][\boldsymbol{I} + \pi\kappa\boldsymbol{W}]$.

Three remarks about Proposition 2 are in order. First, we point out that Proposition 2 not only establishes the existence and uniqueness of an ME, but also provides an algorithm to compute the equilibrium payoffs. In particular, the convex minimization problem (2.14) can be used to analyze changes of the network topology, the surpluses, and of the matching technology. Second, Proposition 2 can be extended to deal with weighted networks, and for the case of heterogeneous discount factors δ. Finally, we stress that despite of its apparently similarity with the sort of problems analyzed in Bramoulle et al. [2011], their results do not apply to our problem.

Our second task is to establish the connection between network topologies and equilibrium payoffs. In order to provide a characterization of the equilibrium payoffs in terms of network topologies, we need to introduce the notion of centrality measure. Our next definition introduces Bonacich's centrality measure.

Definition 2 *(Bonacich [1987]) Let $G \in \mathcal{G}^N$, and let $\boldsymbol{H}$ be its (non negative) weighted adjacency matrix. Let $\mu \in \mathbb{R}$ be such that $\boldsymbol{K}(\mu, \boldsymbol{H}) = [\boldsymbol{I} - \mu\boldsymbol{H}]^{-1}$ is well defined and nonnegative. Let $\theta \in \mathbb{R}^n_+$. The vector of (weighted) Bonacich centralities of parameter μ for the matrix $\boldsymbol{H}$ is given by:*

$$\boldsymbol{b}(\mu, \boldsymbol{H}; \theta) = \boldsymbol{K}(\mu, \boldsymbol{H}) \cdot \boldsymbol{H} \cdot \theta.$$

In Definition 2, the parameter μ reflects the degree to which a player's payoff is a function of the payoffs of those to whom he is connected. If μ is positive, $\mathbf{b}(\mu, \mathbf{H}; \theta)$ is a conventional centrality measure in which each player's payoff is a positive function of the payoffs of the players with which it is in contact. When μ is negative, each player's payoff is reduced by the higher payoffs to those to which it is connected.

Intuitively, we can think of the magnitude of the μ as the degree to which distant links are taken into account. If $\mu = 0$, then $\mathbf{b}(\mu, \mathbf{H}; \theta)$ is proportional to the degree of player i, and the number of others that with it is connected, regardless of their centralities. As the magnitude of μ increases, the centralities of these other players are taken into account, i.e., as the magnitude of μ increases, $\mathbf{b}(\mu, \mathbf{H}; \theta)$ captures *local* and *global* network effects in the bargaining process.

We point out that Bonacich's measure was first applied in economics by Ballester et al. [2006] in the context of network games. In sociology, Bonacich's measure is widely used in empirical and experimental work.[12] To the best of our knowledge, in the context of bargaining in networked markets, no previous paper has characterized the equilibrium in terms of Bonacich's measure

Theorem 1 *The equilibrium payoff vector* V^* *satisfies:*

$$V^* = \boldsymbol{b}(-\pi\kappa,\, \boldsymbol{W}; \hat{\boldsymbol{c}}), \tag{2.8}$$

where $\hat{\boldsymbol{c}} = \frac{1}{2(1-\delta)}\boldsymbol{W} \cdot \boldsymbol{1}$.

In order to explain the economic intuition of Theorem 1, we may rewrite (2.8) as follows:

$$V^* = \hat{\mathbf{c}} - \pi\kappa\mathbf{W}V^*, \tag{2.9}$$

$$V^* = \underbrace{[\mathbf{I} - \pi\kappa\mathbf{W}]\hat{\mathbf{c}}}_{\text{Local Effect}} + \underbrace{(\pi\kappa)^2\mathbf{W}^2V^*}_{\text{Global Effect}}. \tag{2.10}$$

Equations (2.9) and (2.10) show how players' payoffs depend on the whole network structure. For example equation (2.9) shows how players' payoffs depend negatively on the amount of share of surplus that players must give to their direct neighbors. In addition, Equation (2.10) shows that the payoff characterization in Theorem 1 can be decomposed into local and global effects. The term $[\mathbf{I} - \pi\kappa\mathbf{W}]\hat{\mathbf{c}}$ shows the effect of direct links on players' payoffs. However, the interesting term is $(\pi\kappa)^2\mathbf{W}^2V^*$,

[12] See Jackson [2008] and the references therein.

which shows how the indirect links have a positive effect on players' payoffs. For example, if we consider player i, then $(\pi\kappa)^2\mathbf{W}^2V^*$ shows how the links of players in the set N_i^G (player i's neighbors) have a positive effect on his equilibrium payoff. The intuition for the global effect captured by $(\pi\kappa)^2\mathbf{W}^2V^*$, is that from player i's point of view, it is better if her neighbors in $j \in N_i^G$ are connected to players that demand a large share of the surplus. For example, we can think of the situation where the set of n buyers is connected to one common seller, i.e., the market is supplied by a monopolist. It follows that buyers do not have connections to other possible sellers, so the monopolist can exploit this fact in the bargaining process demanding a large share of the surplus. In this case, Theorem 1 predicts that the monopolist gets almost all the surplus.

In the previous analysis, the key element for deriving our intuitions is the negative term $-\pi\kappa$. Concretely, the fact that $-\pi\kappa < 0$ means that V^* is reduced when the connections of any player are themselves central (equation (2.9)), but increased by the centrality of those at distance two (equation (2.10)), whose centrality has reduced the centrality of those at distance one. Thus, a player can have bargaining power because his neighbors have no options (alternative players to trade) or because his neighbors are connected to players with a high degree of centrality.

In order to understand the utility of our caractherization, let us consider the network in Figure 2.1, which consists of seven players and six links. Because we are assuming that the matching technology is uniform, we get $\pi = \frac{1}{6}$. For the case of $\delta = 0$, we get that $\pi\kappa = 0$, and the equilibrium payoffs are $V_1^* = V_5^* = V_7^* = 0.08$ and $V_2^* = V_4^* = V_6^* = 0.17$, and $V_3^* = 0.25$. In this case players' payoffs are determined only for the local effect in payoffs (equation 2.10), and player 3 obtains the highest payoff. Thus for players highly impatience ($\delta \approx 0$), they only get benefits from their direct links, and the network topology does not play any role. However, for values of δ in the open interval (0,1), payoffs behave in a different way. In fact, for $\delta = 0.99$, we get that $\pi\kappa > 0$, which implies $V_5^* = V_7^* = 0.41$, $V_2^* = V_4^* = V_6^* = 0.55$, and $V_3^* = 0.44$. In other words, for $\delta = 0.99$ we get

$$V_2^* = V_4^* = V_6^* > V_3^* > V_5^* = V_7^*.$$

The reasons for this result are as follows. It is easy to see that players 5 and 7 are at disadvantage because they only can trade with players 4 and 6 respectively. This implies that players 5 and 7 have no possibility of demanding a large amount of the surplus, because players 4 and 6 have alternative trading partners. Similarly, players 4 and 6 can exploit this fact, getting a large amount of the available surplus. For the case of player 3, all of his neighbors (players 2, 4, and 6) have an alternative trading partner from which they can get a large amount of surplus. Thus player 3 in order to trade to with their neighbors, must be available to demand a small amount of surplus. In sum, when players are patient ($0 < \delta < 1$) they can take advantage of the whole network structure, incorporating in their payoffs not only the local effects but also the global effects.

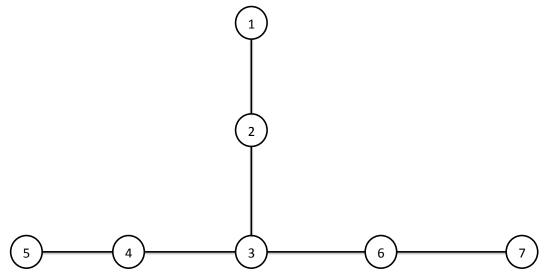

Figure 2.1: A network with seven players and six links

2.3 Market equilibrium and eigenvector centrality

The example in Figure 2.1 shows how players' equilibrium payoffs depend on the value of the discount factor δ. In particular, example 2.1 shows that when the discount factor δ goes to one, players' payoffs depend on the whole network structure.

In this section we show that a general conclusion can be obtained for the case when the discount factor δ goes to one. In order to estate our convergence result, we need to introduce some notation. We denote the maximum eigenvalue (in absolute value) of the matrix $\mathbf{A}$ by $\rho(\mathbf{A})$. Let $\mathbf{e}$ and $\mathbf{q}$ be the left and right eigenvectors associated to $\rho(\mathbf{A})$.

We remark that, thanks to Perron-Frobenius' Theorem (Ch. 8, Horn and Johnson [1990]), we get that $\rho(\mathbf{A}) > 0$, and the eigenvector $\mathbf{e}$ is strictly positive and uniquely determined (up to constant).

Finally, we introduce the notion of eigenvector centrality that is considered in our analysis.

Definition 3 *Let $G \in \mathcal{G}^N$, and let $\boldsymbol{H}$ be a non negative matrix induced by the network G. The eigenvector centrality $\boldsymbol{e}(\boldsymbol{H})$ of the matrix $\boldsymbol{H}$ is its right eigenvector associated to the spectral radius $\rho(\boldsymbol{H})$.*

The previous definition states that the eigenvector centrality is the right eigenvector associated to the largest eigenvalue of a matrix $\mathbf{H}$. In terms of our problem, the matrix $\mathbf{H}$ is given by the matrix $\mathbf{A}$, which is just the adjacency matrix plus the diagonal matrix $\mathbf{D}$.

Now we are ready to establish the main result of this section.

Theorem 2 *Let V^* be a market equilibrium, and let $W^* \equiv \sum_{i=1}^{n} V_i^*$. Then*

$$\lim_{\delta \uparrow \frac{1}{\pi\rho(\boldsymbol{A})/2+1}} \frac{V^*}{W^*} = \boldsymbol{e}(\boldsymbol{A}).$$

We note that Theorem 2 shows the explicit relationship among π, the discount factor δ and the network G. In particular, the condition $\delta \longrightarrow \frac{1}{\pi\rho(\mathbf{A})/2+1}$ provides an economic condition to establish the connection between our ME given by Bonacich's measure and the eigenvector centrality. Thus Theorem 2 has the economic interpretation that as long as the discount factor goes to one the ME is driven by the eigenvector centrality.

We remark that the result in Theorem 2 has been established in sociology by Friedkin and Johnsen [1990], Friedkin [1991], Bonacich [1997], and Bonacich and Lloyd [2001]. However, all of these results rely on parametric assumptions without an economic foundation, which makes difficult the interpretation of when Bonacich's measure can be interpreted as the eigenvector centrality. Thus our main contribution is that we show that a simple economic condition on the discount factor contains all the relevant information about the relationship between Bonacich's measure and the eigenvector approach. Furthermore, our result can be seen as an economic justification to the eigenvector centrality.

Proposition 3 below shows that for the case of regular networks a sharper condition may be obtained, which establishes that the number of players plays a role in the convergence. We recall that a network G is called regular of degree d if $\mathbf{d}_i = d$ for all $i \in N$.

Proposition 3 *Let G be a regular network of degree d. Then* $\lim_{\delta \uparrow \frac{1}{1/n*+1}} \frac{V^*}{W^*} = \boldsymbol{e}(\boldsymbol{A})$, *where* $n^* = \frac{n}{2}$.

The previous result shows how the number of players, in the case of regular networks, plays an important role. In particular, for the case of networks with finitely many players, we may expect $\frac{1}{n^*}$ to be close to zero, so that the condition $\delta \longrightarrow \frac{1}{1/n*+1}$ can be viewed as $\delta \longrightarrow 1$.

2.4 Seller-buyer networks

In this section we specialize the previous analysis to the case of seller-buyer networks. In particular, we assume that the set of agents can be partitioned into two disjoint sets, where the set of sellers and buyers are denoted as S and B, respectively. We assume that sellers own a unit of an indivisible good. The buyers value the good at 1, and each seller s charges a price p_s in order to sell the good. The price that seller

s receives when he sells the good to buyer b is given by

$$p_{sb} = \frac{1}{2}(1 - \delta V_b + \delta V_s) \quad \forall b \in N_s^G, \tag{2.11}$$

and buyer b gets a utility of

$$1 - p_{sb} = \frac{1}{2}(1 - \delta V_s + \delta V_b) \quad \forall s \in N_b^G. \tag{2.12}$$

Using expressions (2.11) and (2.12), the system of linear equations for the seller-buyer network can be written as:

$$\begin{aligned} V_s &= \sum_{b \in N_s^G} \pi p_{sb} + (1 - \sum_{b \in N_s^G} \mathbf{d}_s \pi) \delta V_s, \quad \forall s \in S, \\ V_b &= \sum_{s \in N_b^G} \pi (1 - p_{sb}) + (1 - \sum_{s \in N_b^G} \mathbf{d}_b \pi) \delta V_b, \quad \forall b \in B. \end{aligned} \tag{2.13}$$

In order to write the equilibrium problem using matrix notation we define the matrix $\mathbf{G}$ as follows:

$$\begin{pmatrix} \mathbf{0}_S & \mathbf{G}_S \\ \mathbf{G}_B & \mathbf{0}_B \end{pmatrix}$$

where $\mathbf{0}_S$ is a zero matrix with dimension $|S| \times |S|$, the matrix $\mathbf{G}_S$ has dimension $|S| \times |B|$, the matrix $\mathbf{0}_B$ has dimension $|B| \times |B|$, and $\mathbf{G}_B$ has dimension $|B| \times |S|$.

The matrices $\mathbf{G}_S$ and $\mathbf{G}_B$ show the links that sellers have with buyers and the links that buyers have with sellers, respectively. In addition, we define the following matrices $\mathbf{W}_l = [\mathbf{I}_s + \pi\kappa \mathbf{G}_l]^{-1} \mathbf{G}_l$ for $l = S, B$. Finally we rewrite the vectors V and

$\mathbf{c}$ as $V = [V_S', V_B']'$ and $\mathbf{c} = [\mathbf{c}_S', \mathbf{c}_B']'$ respectively.

Using previous notation we can specialize Proposition 2 to the case of seller-buyer networks.

Proposition 4 *There exists a unique ME which is given by the solution of the following optimization problem:*

$$\begin{array}{ll} \min_{V_S,V_B} & \left\{ \frac{1}{2} \sum_{l=S,B} [V_l' \boldsymbol{M}_l V_l - \boldsymbol{c}_l' V_l] + \frac{1}{2} \left[\pi\kappa V_S' \boldsymbol{G}_S V_B + V_B' \boldsymbol{G}_B V_S \right] \right\} \\ s.t. & V_S, V_B \geq 0, \end{array}$$

with $\boldsymbol{M}_l = (1-\delta)[\boldsymbol{I}_l + \pi\kappa \boldsymbol{D}_l]^{-1}[\boldsymbol{I}_l + \pi\kappa\, \boldsymbol{W}_l]$ *, and* $\boldsymbol{W}_l = [\boldsymbol{I}_l + \pi\kappa \boldsymbol{D}_l]^{-1} \boldsymbol{G}_l$ *for* $l = S, B$*.*

The previous proposition shows how the specific seller-buyer structure is captured by the minimization problem characterizing the ME. The main advantage of this characterization is that for the case of large networks, we can find the equilibrium solving the convex problem associated to it. Furthermore, comparative static exercises can be carried out using this characterization.

Combining the uniqueness of an ME with the fact seller s's price may be defined as $p_s = \sum_{b \in N_s^G} \pi p_{sb}$, we get the following straightforward corollary to Proposition 4.

Corollary 1 *Let* $p_s = \sum_{b \in N_s^G} \pi p_{sb}$ *be the expected price that seller* s *receives from selling his good. Then there exists a unique price vector* $p^* = (p_s^*)_{s \in S}$*.*

Our next task is to characterize sellers' and buyers' payoffs as a function of the network structure. Our next proposition shows how the specific structure of seller-buyer network, allows us to characterize an ME in terms of two vectors, which capture sellers and buyers' payoffs, respectively.

Proposition 5 *Let V_S^* and V_B^* be the sellers and buyers equilibrium payoff vectors respectively. Then*

$$
\begin{aligned}
V_S^* &= \boldsymbol{b}(\alpha, \boldsymbol{W}_{SB}; \theta_{SB}), \\
V_B^* &= \boldsymbol{b}(\alpha, \boldsymbol{W}_{BS}; \theta_{BS}),
\end{aligned}
$$

where $\alpha = (\pi\kappa)^2, \boldsymbol{W}_{SB} = \boldsymbol{W}_S \times \boldsymbol{W}_B$, $\boldsymbol{W}_{BS} = \boldsymbol{W}_B \times \boldsymbol{W}_S$, and $\theta_{SB} = \hat{\boldsymbol{c}}_S - \pi\kappa \boldsymbol{W}_S \hat{\boldsymbol{c}}_S$, $\theta_{BS} = \hat{\boldsymbol{c}}_B - \pi\kappa \boldsymbol{W}_S \hat{\boldsymbol{c}}_S$, with $\hat{\boldsymbol{c}}_l = [\boldsymbol{I}_l + \pi\kappa \boldsymbol{G}_l]^{-1} \boldsymbol{c}_l$ for $l = S, B$.

Proposition 5 characterizes sellers' (respectively buyers') payoffs in terms of their network positions. Concretely, Proposition 5 shows how an ME is determined by Bonacich's centrality measure. Thus, for the particular case of seller-buyer networks, Bonacich's measure can be viewed as a measure of market power, which is driven by sellers and buyers' positions.

Example 1 *Let G be the sellers-buyers network displayed in Figure 2.2. Let $\delta = 0.99$ be the value for the discount factor. From Proposition 5, we get the following equilibrium payoffs $V_{s_1} = 0.5$, $V_{s_2} = 0.48$, $V_{b_1} = 0.48$, and $V_{b_2} = 0.5$. The equilibrium payoffs show how seller s_1 and buyer b_2 obtain a higher (expected) share of the surplus. The difference in payoffs is driven by players' positions. In particular, seller s_1 and buyer b_2 can exploit buyer b_1 and s_2 respectively. In fact, they use their local monopoly power to extract a higher share of the surplus. In terms of network positions seller s_1 and buyer b_2 are identical, so that at equilibrium they get the same payoffs. A*

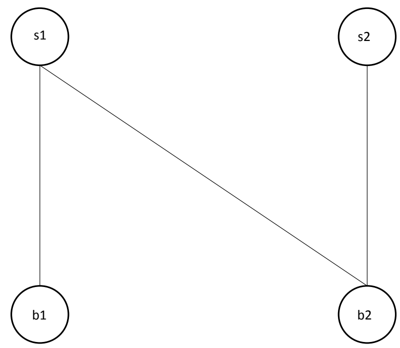

Figure 2.2: A network with two sellers and two buyers

similar reasoning applies to the case of seller s_2 *and buyer* b_1*, which in terms of network positions are identical.*

Example 2 *Let us consider the network displayed in Figure 2.3. The main feature of this network is that each seller (buyer) is connected to the two buyers (sellers) on the other side. Let us assume* $\delta = 0.99$*. Then the equilibrium payoffs are given* $V_{s_1} = V_{s_2} = V_{b_1} = V_{b_2} \approx 0.5$*. The equilibrium payoffs represent the fact that in terms of network positions, sellers and buyers do not have advantage. In terms of Proposition 5, the equilibrium payoffs show that sellers and buyers have the same centrality.*

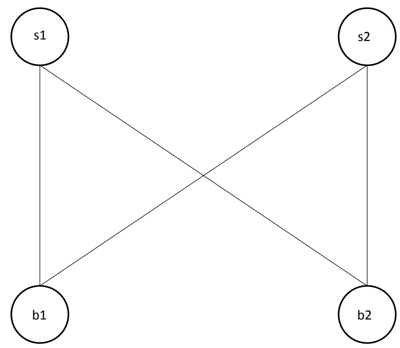

Figure 2.3: A network with two sellers and two buyers

It is worth emphasizing that Examples 2.2 and 2.3 show how our framework differs from the approach Corominas-Bosch [2004]. The main difference is the fact that our results do not rely on a specific protocol in the bargaining process, and, most importantly, our approach and results highlight the relevance of sellers and buyers' network positions.

2.4.1 Seller-buyer networks and eigenvector centrality

In this subsection we show how our convergence result in Theorem 2 may be specialized to the case of seller-buyer networks exploiting the bipartite structure of the network.

Before establishing our result, we need to introduce some notation. Let $\mathbf{p}_S$ and $\mathbf{q}_S$ be the right and left eigenvectors of $\mathbf{W}_S$ respectively. Similarly, let $\mathbf{p}_B$ and $\mathbf{q}_B$ be the right and left eigenvectors of $\mathbf{W}_B$, respectively. Let $\rho(\mathbf{W}_S)$ and $\rho(\mathbf{W}_B)$ be the largest eigenvalue of $\mathbf{W}_S$ and $\mathbf{W}_B$, respectively.

We now are ready to state our convergence result.

Proposition 6 *Let V_S^* be the sellers' equilibrium payoff vector, and let $W_S^* = \sum_{i \in S}^{n} V_i^*$. Then*

$$\lim_{\delta \uparrow \frac{1}{\pi\rho(\boldsymbol{W}_S)/2+1}} \frac{V_S^*}{W_S^*} = \boldsymbol{e}(\boldsymbol{W}_S).$$

Similarly, let V_B^* be buyers' equilibrium payoff vector with $W_B^* = \sum_{j \in B} V_j^*$. Then

$$\lim_{\delta \uparrow \frac{1}{\pi\rho(\boldsymbol{W}_B)/2+1}} \frac{V_B^*}{W_B^*} = \boldsymbol{e}(\boldsymbol{W}_B).$$

We note that the conditions for convergence of sellers and buyers' payoffs are mutually exclusive. The reason for this is because, in general, $\rho(\mathbf{W}_S) \neq \rho(\mathbf{W}_B)$. However, for the particular case of a complete seller-buyer network network with $|S| = |B|$, it holds that $\mathbf{W}_S = \mathbf{W}_B$, which implies that $\rho(\mathbf{W}_S) = \rho(\mathbf{W}_B)$. From this observation, the following corollary is a direct consequence of Proposition 6.

Corollary 2 *Let G be a complete network with $|S| = |B|$. Let $W_S^* = \sum_{i \in S} V_i^*$ and $W_B^* = \sum_{j \in B} V_j^*$. Then*

$$\lim_{\delta \uparrow \frac{1}{\pi\rho/2+1}} \frac{V_S^*}{W_S^*} = \lim_{\delta \uparrow \frac{1}{\pi\rho/2+1}} \frac{V_B^*}{W_B^*} = \boldsymbol{e}(\boldsymbol{W}),$$

with $\rho \equiv \rho(\boldsymbol{W}_S) = \rho(\boldsymbol{W}_B)$ and $\boldsymbol{W} = \boldsymbol{W}_S = \boldsymbol{W}_B$.

2.5 Conclusions

This chapter presents a simple model of bargaining in networked markets. We present two main results. First, we show that the equilibrium payoffs are given by Bonacich [1987]'s centrality measure. Our result may be viewed as an economic justification of this measure in the context of bargaining in networked markets.

Our second contribution is the proof that the market equilibrium converges to the eigenvector centrality as long as the discount factor goes to one. Our result highlights the specific connection between the network structure, the matching technology, and the discount factor, and provides an economic interpretation to the eigenvector centrality.

Chapter 3

A representative consumer theorem for discrete choice models in networked markets

3.1 Introduction

In this Chapter, we propose a way of modeling sequential discrete decision processes, which is consistent with the random utility hypothesis and the existence of a representative agent. In particular, our approach is based on a network representation for the consumers' decision process and dynamic programming.[1]Combining the aforementioned elements, we show that a demand system for hierarchical or sequential decision processes can be obtained as the outcome of the utility maximization by a representative agent. Our result differs from previous findings in two important aspects. First, our result is in terms of a *direct* utility representation, whilst most of

[1]The idea of analyzing discrete choice models using a network representation is also considered in Daly and Bierlaire [2006]. They derive their results under the assumption that the random variables are generalized extreme value variables. Despite using a network representation, our main result in section 2 differs from Daly and Bierlaire [2006]'s approach in two important aspects. First, our result does not require the assumption of generalized extreme value random variables. Second, and most importantly, we model in an explicit way the recursive choice process.

results available in discrete choice theory are based on an *indirect* utility approach.[2] Second, and most important, our result does not depend on parametric assumptions concerning the random components associated to the utilities of different choices. From a technical point of view, we need to assume that the random variables are independent within any bundle (path) of goods, but we do not rule out the possibility that the random components can be correlated among different bundles (paths).

Thus, given its generality, our approach and result can be useful in the study of demand systems with complex substitution patterns among the utilities associated with different choices.

An important feature of our result is that when we assume the specific double exponential distribution for the unobserved components, we show that the nested logit model can be seen as a particular case of our approach. In particular, we show that a sequential logit under specific parametric constraints coincides with the nested logit model(McFadden [1978a,b, 1981]).[3] This result generalizes previous findings in Borsch-Supan [1990], Konning and Ridder [1993], Herriges and Kling [1996], Verboven [1996], Konning and Ridder [2003], and Gil-Molto and Hole [2004]. All of these papers impose parametric constraints in order to be consistent with the random utility maximization. Our result shows that such constraints can be avoided using the assumption of sequential decision making.

Finally, from an applied perspective we point out that our results can be useful to carry out welfare analysis in networked markets, where the standard discrete choice theory may not apply. For example, our results can be applied to bundling decisions,

[2] For a survey of these results see Anderson et al. [1992].

[3] It is worth pointing out that the nested logit model does not need to be interpreted as a sequential decision process. In fact, its standard justification is based on preference correlation structure (see, e.g., Anderson et al. [1992]).

merger analysis, or compatibility among goods in networked markets.[4]

The chapter is organized as follows: section 3.2 presents the model, section 3.3 presents the main result of this chapter, and subsection 3.3.1 discusses the logit case. Finally, section 3 applies our demand framework to study price competition and merging in the context of networked markets. Appendix B.1 contains the proofs.

3.2 The model

Let $G = (N, A)$ be an *acyclic directed* graph with N being the set of nodes and A the set of links respectively. Without loss of generality, we assume that the graph G has a single origin-destination pair, where o and t stand for the origin node and destination node respectively.

We identify the set N as the set of decision nodes faced by consumers, and the set A is identified as the set of the available goods in this economy, i.e., the good a is represented by the link $a \in A$.[5] Thus, starting at the origin o, consumers can choose bundles of goods through the choice of links on A. The destination t is interpreted as the node that is reached once consumers have chosen their desired bundles of goods, and then they leave the market.

For each good $a \in A$, consumers' valuation is represented by $\theta_a \in \mathbb{R}_{++}$. Similarly, $p_a \in \mathbb{R}_+$ is the price associated to good a. Thus, the utility for good a may be written as $u_a = \theta_a - p_a$. We assume that there exists a continuum of users with unitary mass. According to this, let $d = (d_a)_{a \in A}$ a non-negative flow vector, where $d_a \geq 0$ denotes the demand for good a. Any flow vector d must satisfy the flow conservation

[4] For a survey of networked markets in economics see Economides [1996].

[5] The set A can also be called the set of choices.

constraints

$$\sum_{a \in A_i^-} d_a = \sum_{a \in A_i^+} d_a \quad \forall i \in N, \tag{3.1}$$

where A_i^- denotes the set of links ending at node i, and A_i^+ denotes the set of links starting at node i. The set of feasible flows is denoted by $\mathcal{D}$.

It is worth emphasizing that in this chapter we interpret each path in the graph as a bundle of goods.[6] This interpretation allows us to see the goods within a bundle as complements, and different paths can be viewed as substitute goods.

In order to introduce heterogeneity into the model, we assume that consumers are *randomly* drawn from a large population. According to this, the *random utility* $\widetilde{u}_a$ may be defined as

$$\widetilde{u}_a = u_a + \epsilon_a \quad \forall a \in A_i^+, i \in N,$$

with $\{\epsilon_a\}_{a \in A}$ being a collection of absolutely continuous random variables with $\mathbb{E}(\epsilon_a) = 0$ for all a. The random variables ϵ_a take into account the heterogeneity within the population. In particular, these random variables represent the variability of the valuation θ_a.

For all node $i \neq d$, we assume that the random variables $\epsilon_a \in A_i^+$ are independent with the random variables $\epsilon_b \in A_{j_a}^+$. In other words, we rule out the possibility that random variables within the same path can be correlated. Nonetheless, for all $i \neq d$ we allow for correlation among the random variables $\epsilon_a \in A_i^+$. Furthermore, we allow for correlation among the random variables in different paths.

[6]We point out that the standard discrete model can be viewed as particular case of our approach. In fact, we can define a network with the set of nodes N consisting of just two nodes, where one node is the source and the other one is the sink, and a collection of $|A|$ parallel links representing the goods available in the market.

In this networked market, consumers choose the optimal bundle of goods in a recursive way. Concretely, at each node consumers choose a good considering their utility plus the continuation value associated to their choices. Formally, at each node $i \neq d$ we define the random utility $\tilde{V}_a$ as

$$\tilde{V}_a \;=\; V_a + \epsilon_a \tag{3.2}$$

with $V_a = u_a + \varphi_{j_a}(V)$ and $\varphi_{j_a}(V) \equiv \mathbb{E}(\max_{b \in A^+_{j_a}} \{V_b + \epsilon_b\})$, where j_a denotes that node j_a has been reached using the link a.

Regarding equation (3.2) three remarks are important. First, thanks to the assumption of independence among the ϵ_a along the same path, the terms ϵ_a and $\varphi_{j_a}(\cdot)$ are independent. Second, equation (3.2) makes explicit the recursive nature of the consumers' choice process. In particular, consumers reaching node i observe the realization of the random variables $\tilde{V}_a$, and choose the link $a \in A^+_i$ with the highest utility, taking into account the current utility u_a plus the *continuation value* $\varphi_{j_a}(V)$.[7]The third observation is that (3.2) makes explicit the assumption that a consumer makes sequential choices. In other words, consumers maximize utility solving a dynamic programming problem.[8]

From previous discussion, it follows that the expected flow x_i entering node i splits among the goods $a \in A^+_i$ according to

[7]In the discrete choice literature the functions $\varphi_i(\cdot)$ are known as the *inclusive values* at node $i \neq d$ (See McFadden [1978a,b, 1981], Anderson et al. [1992]).

[8]We point out that the idea of modeling discrete choices through a sequential process was first proposed by Ben-Akiva and Lerman [1985] in order to justify the nested logit model. Another paper exploiting the idea of sequential discrete choice models to analyze price competition among multi-product firms is the work by Anderson and de Palma [2006].

$$d_a = x_i \mathbb{P}(\underbrace{V_a + \epsilon_a}_{\widetilde{V}_a} \geq \underbrace{V_b + \epsilon_b}_{\widetilde{V}_b},\ \forall b \in A_i^+). \tag{3.3}$$

This *recursive discrete choice model* generates the following stochastic conservation flow equations

$$x_i = \sum_{a \in A_i^-} d_a. \tag{3.4}$$

Using a well known result in discrete choice theory[9], equations (3.3)-(3.4) may be expressed in terms of the gradient of the function $\varphi_i(\cdot)$. In particular, the conservation flow equations (3.3) and (3.4) may be rewritten as

$$\begin{cases} d_a = x_i \frac{\partial \varphi_i(V)}{\partial V_a} \quad \forall a \in A_i^+, \\ x_i = \sum_{a \in A_i^-} d_a, \end{cases} \tag{3.5}$$

where $\frac{\partial \varphi_i(V)}{\partial V_a} = \mathbb{P}(V_a + \epsilon_a \geq V_b + \epsilon_b,\ \forall b \in A_i^+)$.

Following the previous description, it is easy to see that consumers' choice process can be expressed as a Markov chain. In particular, once a consumer reaches a specific node, say node i, then he must choose among the goods available in the set A_i^+.

The following definition formalizes the notion of Markovian assignment in a networked market.[10]

[9] For details see Ch. 2 in Anderson et al. [1992].

[10] We point that an equilibrium notion called Markovian traffic equilibrium has been introduced in Baillon and Cominetti [2008] and extended to oligopoly pricing problems in Melo [2011]. However, neither Baillon and Cominetti [2008] nor Melo [2011] analyze the problem that is studied in this chapter.

Definition 4 *Let $p \geq 0$ be a given price vector. A vector $d \in \mathbb{R}_+^{|A|}$ is a Markovian assignment if and only if the d_a's satisfy the flow distribution equation (3.5) with V solving $V_a = u_a + \varphi_{j_a}(V)$ for all $a \in A$.*

We stress that the previous setting defines consumers' utility in an indirect way. Assuming a specific distribution for the ϵ_a, we can solve $V_a = u_a + \varphi_{j_a}(V)$ and find the demand vector. The next section establishes the main result of this chapter: The Markovian assignment is equivalent to the demand system generated as the solution of a *direct utility function* by a representative consumer.

3.3 Main result

For the networked market described in section 3.2, we consider there exists a representative consumer endowed with income $Y \in \mathbb{R}_{++}$. There is a numeraire good which is indexed by 0, and its price p_0 is normalized to the unity. The budget constraint for the representative consumer is given by

$$B(p, Y) = \left\{ (d, d_0) \in \mathbb{R}_+^{|A|+1} : \sum_{a \in A} p_a d_a + d_0 \leq Y \right\}, \tag{3.6}$$

where $d = (d_a)_{a \in A}$ is the demand vector for the good at the network, and d_0 is the demand for good 0. We recall that the demand d must satisfy the flow constraint (3.4).

Theorem 3 *A representative consumer's utility function consistent with the Markovian assignment is given by*

$$U(d) = \begin{cases} \sum_{a\in A}\theta_a d_a + d_0 - \sum_{i\in N}\chi_i(d), & \textit{s.t. (3.4);} \\ -\infty, & \textit{otherwise,} \end{cases} \tag{3.7}$$

where $\chi_i(d) = \sup_V \left\{\sum_{a\in A_i^+}(V_a - \varphi_i(V))d_a\right\}$.

It is worth emphasizing four important features of Theorem 3.

First, we note that Theorem 3 does not require independence of the random variables ϵ_a. Thus this result can deal with complex correlation patterns among different alternatives.

Second, Theorem 3 is based on a direct utility function for the representative consumer. In particular, the direct utility function in expression (3.7) encapsulates two different components. The first component, given by the linear term $\sum_{a\in A}\theta_a d_a + d_0$, expresses the utility derived from the consumption of (d, d_0) in the absence of interaction among goods. Furthermore, the valuation parameters θ_a can be viewed as measuring the intrinsic contribution of good a to the total utility. The second effect is given by $-\sum_{i\in N}\chi_i(d)$, which expresses the *variety-seeking behavior* of the representative consumer.[11] The interpretation of variety-seeking behavior has been given in Anderson et al. [1988].

Third, we note that for the simple case where the number of goods is $|A|$, and there are just two nodes, an origin and destination, Theorem 3 provides a direct representation without imposing parametric assumptions on the collection of ϵ_a. For this simple case, Theorem 3 generalizes previous results in discrete choice theory.[12]

[11] The terms $-\sum_{i\in N}\chi_i(d)$ can be viewed as a generalized entropy. In section 3.3.1 this interpretation is clearer when we assume that the ϵ_a follow a double exponential.

[12] For a survey of the results of representative agents and demand systems in discrete choice models see Anderson et al. [1992, Ch. 3].

Finally, we note that Theorem 3 can be extended to the case of endogenous consumption. In particular, instead of considering the linear component $\sum_{a\in A}\theta_a d_a$, we can consider a strictly concave function $F(d;\theta)$ with continuous second derivatives. Using this function $F(d;\theta)$, the strict concavity of the optimization problem holds, so that we can apply the same reasoning given in the proof of Theorem 3.

3.3.1 The sequential logit case

In theoretical and applied work, the nested logit model is the leading case for modeling markets with a tree or network structure. In this section we show that the nested logit is a particular case of Theorem 3, which is obtained assuming that at each node i the ϵ_a are *i.i.d.* random variables following a double exponential distribution.

Proposition 7 *Assume that at each node i the random variables ϵ_a are i.i.d. following a double exponential distribution with location parameter $\beta_i \in \mathbb{R}_{++}$. Then, a representative consumer's utility function consistent with the Markovian assignment is given by*

$$U(d) = \begin{cases} \sum_{a\in A}\theta_a d_a + d_0 - \sum_{i\in N}\chi_i(d), & \textit{s.t. (3.1);} \\ -\infty, & \textit{otherwise,} \end{cases} \tag{3.8}$$

where $\chi_i(d) = \frac{1}{\beta_i}\left(\sum_{a\in A_i^+} d_a \log d_a - \sum_{a\in A_i^+} d_a \log\left(\sum_{a\in A_i^+} d_a\right)\right)$.

We stress that Proposition 7 generates a demand system based on a sequential logit model. However, after some simple algebra, it is possible to show that the choice probabilities in Proposition 7 can be written as a nested logit. In particular, we can find the explicit parametric constraints on the β_i such that Proposition 7 yields a demand system based on a nested logit model.[13]

3.4 Applications

In this section we apply our result to analyze price competition in the context of networked markets. Formally, we analyze the following two stage game; In the first stage firms set prices p in order to maximize the profits

$$\pi_a(p, d_a) = (p_a - c_a)d_a \quad \forall\ a \in A, \tag{3.9}$$

where d_a is the demand for good a, and c_a is firm a's marginal cost. In the second stage, and given the price vector p, consumers choose the optimal bundle (path). We solve the game using backward induction.

The equilibrium concept that we use is the subgame perfect Nash equilibrium, and for the first stage we introduce the following definition.

Definition 5 *A pair $(p^{OE}, d(p^{OE}))$ is a pure strategy Oligopoly Price Equilibrium (OE) if for all $a \in A$*

$$\pi_a(d_a(p_a^{OE}, p_{-a}^{OE})) \geq \pi_a(d_a(p_a, p_{-a}^{OE})) \quad \forall \ \ (p_a, p_{-a}^{OE}), \tag{3.10}$$

where $d(p_a, p_{-a}^{OE})$ is the Markovian assignment for the price vector (p_a, p_{-a}^{OE}).

In order to show the existence of an OE. Thanks to our sequential discrete choice

[13] See Appendix B.1 for details.

model, the pricing problem can be reduced to local oligopoly pricing problems. The following result exploits that feature of our demand model.

Proposition 8 *In the pricing game there exists at least one OE. Furthermore, if for all $a \in A$ the random variable ϵ_a follows a Gumbel distribution, then the OE is unique.*

Proposition 8 generalizes the findings in Chen and Nalebuff [2007], Casadesus-Masanell et al. [2007], and Chen and Nalebuff [2007].

A straightforward corollary of Proposition 8 is the following equilibrium characterization.

Corollary 3 *Let p^{OE} be an OE. Then*

$$p_a^{OE} = c_a + \mu_a(p^{OE}) \quad \forall a \in A,$$

where $\mu_a(p^{OE}) = \frac{F_a(\theta_a - p_a^{OE} + \varphi_{ja}(p^{OE}))}{f_a(\theta_a - p_a^{OE} + \varphi_{ja}(p^{OE}))}$.

Previous result shows how our demand model allows us to characterize the equilibrium prices taking into account the network topology through the terms $\varphi(\cdot)$.

3.4.1 Price competition in parallel serial link networks

In this section we analyze price competition in the case of parallel serial networks. From an economic point of view, a parallel serial link network represents the case where set of paths represent *disjoint bundles*, i.e., consumers cannot combine goods froms different paths.

In order to explain how our framework works, we embed the classical Cournot's complement model into a network (See Ch. 9, Cournot [1897]). Concretely, Cournot analyzes the situation where consumers in order to produce brass, they must combine copper and zinc. Each good is produced by a monopolist. In terms of a network, the market is shown by Figure 3.1.

The node s represents the situation where the consumers must decide if they want to buy zinc or leave the market. If a consumer decides to buy zinc, then he must buy cooper.

In terms of our two stage game, firms set p_z and p_c and then consumers choose the bundle zinc and cooper, or they decide to leave the market. The main result in Cournot's model is the following: cooper and zinc producers split the profits evenly, regardless of cost differences.

Exploiting our demand model, we can extend Cournot's insight to the case or parallel serial link networks. Formally, we get the following result.

Proposition 9 *Let G a parallel serial network. Let $\mathcal{R}$ be the set of paths (bundles). Then the equilibrium price vector p^{OE} is given by*

$$p_a^{OE} = c_a + (\bar{P}_r - \bar{C}_r) \quad \forall a \in r \quad \textit{and} \quad r \in \mathcal{R},$$

with $\bar{P}_r = \frac{1}{|r|}\sum_{b \in r} p_a$, and $\bar{C}_r = \frac{1}{|r|}\sum_{b \in r} c_b$.

It is easy to see that Cournot's result is a particular case of Proposition 9. Let us consider Figure 2, which displays a parallel serial link network with two paths. For this particular case, the set of paths is $\mathcal{R} = \{r_1, r_2\}$ where $r_1 = (a_1, a_2)$ and

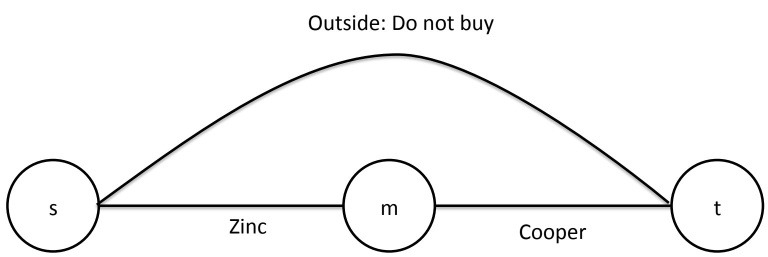

Figure 3.1: Cournot's complements model

$r_2 = (b_1, b_2)$, and the demand function for path r_1 is given by:

$$d_{r_1}(p) = d_{a_1}(p) = \mathbb{P}(\theta_{r_1} - p_{a_1} - p_{a_2} + \epsilon_{a_1} > \theta_{r_2} - p_{b_1} - p_{b_2} + \epsilon_{b_1}),$$

with $\theta_{r_1} = \theta_{a_1} + \theta_{a_2}$ and $\theta_{r_2} = \theta_{b_1} + \theta_{b_2}$.

Thus applying Proposition 9 it follows that an OE satisfies

$$p_{a_1} - c_{a_1} = p_{a_2} - c_{a_2}.$$

Similar argument applies to path r_2.

We can also exploit the network representation and our demand model to analyze the effect of adding serial links. Figure 3.2 shows the network resulting from adding a serial link, where a_2 connects m and m', whereas a_3 connects m' with t.

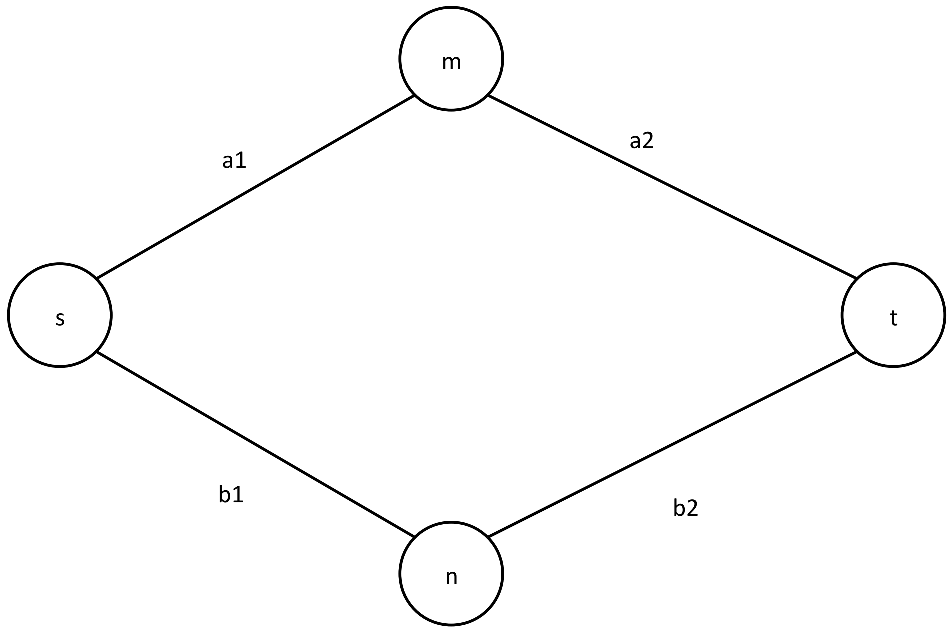

Figure 3.2: A parallel serial link network

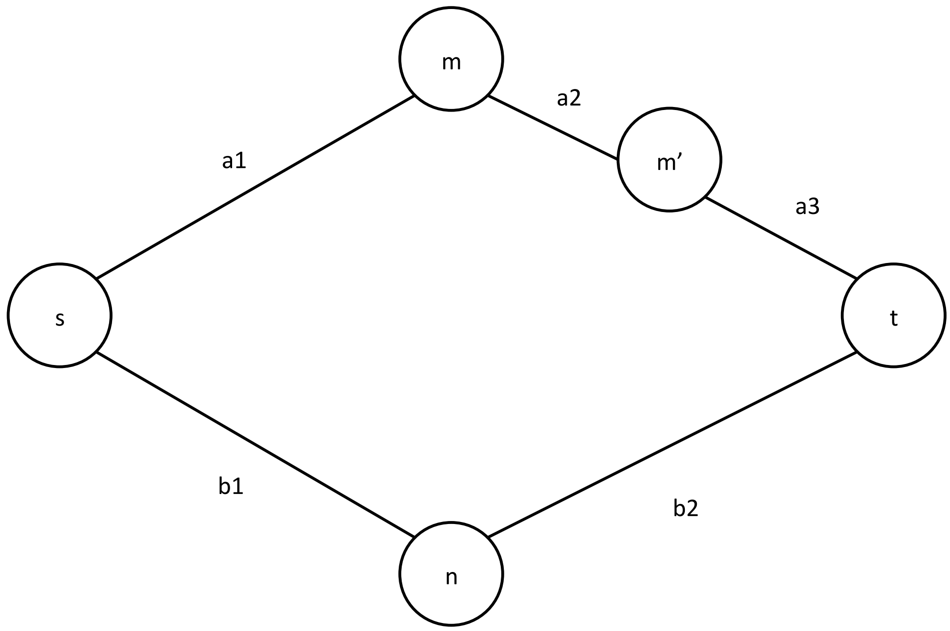

Figure 3.3: The effect of adding a serial link

For this new network the set of paths is $\mathcal{R} = \{r_1, r_2\}$ with $r_1 = (a_1, a_2, a_3)$ and $r_2 = (b_1, b_2)$. and the demand function for path r_1 is given by:

$$d_{a_1}(p) = \mathbb{P}(\theta_{r_1} - p_{a_1} - p_{a_2} - p_{a_3} + \epsilon_{a_1} > \theta_{r_2} - p_{b_1} - p_{b_2} + \epsilon_{b_1}),$$

with $\theta_{r_1} = \theta_{a_1} + \theta_{a_2} + \theta_{a_3}$ and $\theta_{r_2} = \theta_{b_1} + \theta_{b_2}$.

Then applying Proposition 9 it follows that an OE satisfies

$$p_{a_1} - c_{a_1} = p_{a_2} - c_{a_2} = p_{a_3} - c_{a_3}.$$

The main effect of adding the link a_3 is captured by θ_{r_1}. In particular, for $\theta_{a_3} > 0$ the new link a_3 affect the demand $d_{a_1}(p)$.

3.4.2 Merging analysis

In this section we show how our demand model can be useful to analyze merging decisions between two firms producing complementary goods. In terms of a networked market, we analyze the effect on equilibrium prices when two firms within the same path (bundle) decide to merge.

Formally, we are analyze the market with the structure displayed in Figure 3.4. In particular, we can think of firm a as the producer the main good, whereas firm b_1 and b_2 are the complement ors to this main good. This simple directed network captures the situation when consumers first need to buy the main good, and then they can enjoy the complements (b_1 or b_2).

An example of a real world situation that is captured by Figure 3.4 is the case where the main producer is $Microsoft$, and b_1 and b_2 are two complementors to the main good produced by *Microsoft*. Concretely, b_1 and b_2 can be the complements for *Microsoft* Windows. Our goal is to analyze the effect of merging firm a with firm b_1 on the equilibrium prices.

From Proposition 3, we know the equilibrium prices for firm a and b_1 takes the form:

$$p_a^{OE} = c_a + \mu_a(p^{OE}) \quad \text{and} \quad p_{b_1}^{OE} = c_{b_1} + \mu_{b_1}(p^{OE}),$$

where $\mu_a(p^{OE}) = \frac{F_a(\theta_a - p_a^{OE} + \varphi(p^{OE}))}{f_a(\theta_a - p_a^{OE} + \varphi(p^{OE}))}$, $\mu_{b_1}(p^{OE}) = \frac{F_{b_1}(\theta_{b_1} - \theta_{b_2} - p_{b_1}^{OE} + p_{b_2}^{OE})}{f_{b_1}(\theta_{b_1} - p_{b_1}^{OE} + p_{b_2}^{OE})}$, and

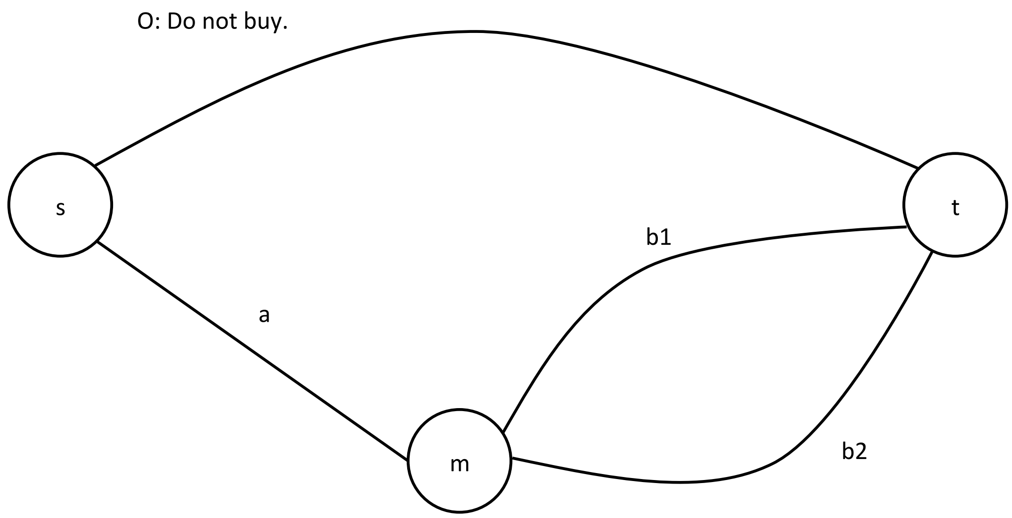

Figure 3.4: A network of complements and substitutes

$\varphi(p^{OE}) = \mathbb{E}(\max\{\theta_{b_1} - p_{b_1}^{OE} + \varepsilon_{b_1}, \theta_{b_2} - p_{b_2}^{OE} + \varepsilon_{b_2}\})$.

Now let us suppose that firm a and b_1 merge, i.e., they maximize their joint profit coordinating their pricing decisions. The equilibrium prices after merging are given by:

$$p_{a_1}^{OE} = c_a + \mu_a(p^{OE}) \quad \text{and} \quad p_{b_1}^{OE} = c_{b_1}.$$

The important property of the equilibrium prices after merging is the optimal price for firm b_1 is to set price equal to marginal cost. The intuition for this result is that once firm a and b_1 coordinate their pricing decisions, they internalize the effect on being in the same bundle (path).

Despite its simplicity, the prices after merging captures real world situations. A typical example is given the situation of Windows and some of its complements, e.g. Windows media player, which is given for free. The feature that the Windows media player is given for free is captured by our result assuming $c_{b_1} = 0$.

Chapter 4

Price competition, free entry, and welfare in congested markets

4.1 Introduction

In many environments, such as communication networks in which network flows are allocated, or transportation networks in which traffic is directed through the underlying road architecture, congestion plays an important role in terms of efficiency. In fact, over the last decade the phenomenon of congestion in traffic networks has received attention in a number of in different disciplines: economics, computer science, and operations research.

The main question is how to achieve a socially optimal outcome, which is intimately linked to the assessment of congestion effects. This feeds into the identification of socially optimal regulatory actions in such markets. Indeed, a social planner may use a sort of economic mechanisms in order to induce users' behavior toward the socially optimal outcome. In fact, since the seminal work of Pigou [1920], it is well known that an efficient outcome in a network subject to congestion can be reached through the *centralized implementation* of a toll scheme based on the principle of *marginal cost pricing.* Under this mechanism users pay for the negative externality that they impose on everybody else.

Concretely, under a Pigouvian tax scheme users face two sources of cost: one due to the congestion cost and the second due to the toll. Nonetheless, Pigou's solution is hard to implement in practice, because it requires that the social planner charges the tolls in a centralized way, which from a practical and computational perspective is a very complex task. Thus, the natural alternative is to consider a market based solution, where every route (or link) of the network is owned by independent firms who compete setting prices in order to maximize profits.[1]

Despite the relevance of and the increasing interest in implementing decentralized pricing mechanisms to reduce and control congestion in networks, little is known about the theoretical properties of such solutions. Indeed, little is known about the existence and uniqueness of equilibrium prices for general classes of network topologies.

In addition to the problem of the existence and uniqueness of a price equilibrium, a second problem that is raised in congested markets is the analysis of free entry and welfare. In particular, every firm can be viewed as a link, so the number of firms that enter the market will determine the network topology. Thus the socially optimal topology can be identified with the optimal number of firms in the market. Similarly to the study of existence and uniqueness of a price equilibrium, little is known about the free entry problem in general networks.

We point out that the analysis of existence, uniqueness and free entry in a network with multiple sources and a common sink is not only relevant from a theoretical perspective, but also from an applied point point of view. A real world situation

[1]For an early discussion of price mechanisms in congested networks, we refer the reader to Luski [1976], Levhari and Luski [1978], Reitman [1991], and MacKie-Mason and Varian [1995].

where our model may be useful, is the case of the airline industry. In particular, we can think of in a situation where consumers from different locations (sources) want to travel to a common destination, let say d. Then consumers can choose among airlines (and combinations of them) to reach d, where the total cost faced by consumers is given by the price charged by airlines plus the waiting time (congestion). An understanding of the conditions when a unique price equilibrium exists and how free entry affects social welfare turns out to provide critical information for policy makers regulating this industry.

A second example where a general network analysis is useful, is the case of *combinatorial markets*. In particular, in a large combinatorial market, such as the Internet, the firms own links and set prices in order to maximize profits whereas consumers purchase bundles of products in order to maximize their utility. Concretely, we can consider the situation where each consumer is interested in buying bandwidth along a path from its source (origin) to its destination, and obtains a value per unit of flow that it can send along this path. In this environment, the analysis of the existence and uniqueness of a price equilibrium provides us information concerning how prices can be used to organize the market. For instance the understanding of the existence of price equilibria may be useful in the analysis of the efficiency of these markets.[2]

4.1.1 Our contribution

In this chapter we develop and study a general oligopoly model in a network subject to congestion effects. Our contribution is threefold. First, we introduce an alternative notion of equilibrium in traffic networks, which we denote as *Markovian traffic equilibrium*. Our equilibrium concept is based on the idea that users choose their

[2] For an analysis of the efficiency of combinatorial markets we refer to Chawla and Roughgarden [2008].

optimal paths in a recursive way. The idea that users can find their optimal paths in a recursive way turns out to be different to the standard notion of Wardrop equilibria. We are not aware of previous papers studying price competition in congested markets using the notion of Markovian traffic equilibrium.

Second, we show the *existence* and *uniqueness* of a pure strategy price equilibrium. Our result is very general: we do not assume that demand functions are concave nor impose particular functional forms for the latency functions (congestion costs) as is commonly assumed in the extant literature. We derive *explicit conditions* to guarantee existence and uniqueness of equilibria. We stress that our existence and uniqueness result does not rely on a specific network topology. In fact, our result applies to any directed network with multiple origins and a common destination node.

Our third contribution is the study of entry decisions and welfare in congested markets. We show that the number of firms that enter the network exceeds the social optimum. In terms of network design the excess entry result means that the observed topology will not be the socially optimal. Because we obtain this result for a general network, we think of that our excess entry result may be useful in studying problems of optimal design of networks.

Formally, we study a network with multiple origins and a common destination node, where every link is owned by a firm that sets prices in order to maximize profits. In this environment, users face two sources of cost: the congestion cost plus the price set by the firms. The congestion in every link is captured by a latency function, which is strictly increasing in the number of users utilizing it. In order to solve the users' problem, we adapt the Markovian traffic model proposed by Baillon and Cominetti [2008], to the study of price competition in congested networks. This

Markovian model is based on *random utility models* and *dynamic programming*. The use of random utility models allows for heterogeneity in users' behavior, i.e., instead of assuming homogenous users, we model the utility of choosing a certain route as a random variable. In addition, and considering the stochastic structure of users' utilities, we assume that users solve a dynamic programming problem in order to construct the optimal path in a recursive way. Thus, at each node users consider the utility derived from the available links plus the continuation values associated to each link.

Furthermore, the introduction of random utility models has the advantage of generating a *demand system*, which shows how prices and congestion externalities induce users' choices.[3]

Combining the previous elements, we solve a complete information two stage game, which can be described as follows: In the first stage, firms owning the links maximize profits setting competitive prices *a là* Bertrand. In the second stage, given firms' prices, users choose routes in order to maximize their utility, namely the cheapest route. We solve this game using backward induction, looking for a pure strategy sub-game perfect Nash equilibrium, which we call the *Oligopoly price equilibrium*.

Related Work: The pricing game that we study in this chapter is not new at all. In fact, this class of games is studied in Cachon and Harker [2002], Engel et al. [2004], Hayrapetyan et al. [2005], Acemoglu and Ozdaglar [2007], Baake and Mitusch [2007], Allon and Federgruen [2007], Allon and Federgruen [2008], Chawla and Roughgar-

[3]We stress that our approach differs with the one known as aggregation in oligopoly markets proposed by Caplin and Nalebuff [1991]. The main technical difference is due to our demand system being defined in terms of a fixed point equation, which reflects the existence of congestion externalities in users' choices, while the results in Caplin and Nalebuff [1991] do not apply to the case of demand systems with externalities (positive or negative).

den [2008], and Weintraub et al. [2010]. In order to establish the existence of an oligopoly price equilibrium, these papers assume the following: First, in order to describe users' behavior these papers make use of the concept of *Wardrop equilibria*, which establishes that utilities (costs) on all the routes actually used are equal, and greater (less) than those which would be experienced by a single user on any unused route. Second, these papers impose assumptions on the demand generated by users' behavior or assumptions on the class of latency functions. In particular, Cachon and Harker [2002], Engel et al. [2004], Hayrapetyan et al. [2005], and Weintraub et al. [2010] assume that the demand functions are concave (or log-concave) functions of the price charged by firms. Thanks to the concavity assumption, the previous papers show the existence of an oligopoly price equilibrium. On the other hand, the papers of Acemoglu and Ozdaglar [2007], Baake and Mitusch [2007], and Chawla and Roughgarden [2008] show the existence of a pure strategy equilibrium assuming that the latency functions are affine.

Moreover, all of the papers mentioned above, consider a simple network consisting of a single origin-destination pair, with a collection of parallel links. This specific network topology rules out some interesting examples from an applied point of view,[4] thus limiting the application of the available existence results.

The recent papers of Allon and Federgruen [2007, 2008], use random utility models to study price in competition in the context of queuing games, where the latency functions represent the waiting time that users must wait to be served. These papers establish the existence and uniqueness of an oligopoly equilibrium. However, the results in Allon and Federgruen [2007, 2008] differ from ours in two important aspects. First, Allon and Federgruen [2007, 2008] consider a network consisting of a single origin-destination pair with parallel links, thus ruling out important cases from an

[4]For instance, this specific topology rules out the case of hub-spoke networks.

applied perspective. Second, Allon and Federgruen [2007, 2008] do not study the entry and welfare problem.

Regarding our result on free entry and welfare, similar findings in the context of traditional oligopoly theory can be found in Mankiw and Whinston [1986] and Anderson et al. [1995]. These papers do not, however, deal with network structures on congestion; features that are crucial components of our result. For the case of congested networks, a similar result to ours can be found in the recent paper of Weintraub et al. [2010] for the particular case of a network with a single pair source-sink and assuming parallel links. Summarizing, our results can be viewed as a generalization of previous findings of the free entry and welfare problem.

The rest of the chapter is organized as follows: Section 4.2 presents the model. Section 4.3 studies the free entry and welfare problem. Section 4.4 shows the existence and uniqueness of a oligopoly price equilibrium for a general class of latency functions. Finally, Section 4.5 concludes. Proofs and technical lemmas are presented in Appendix C.1.

4.2 The Model

Let $G = (N, A)$ be an acyclic directed graph representing a traffic network, with N being the set of nodes and A the set of *links* respectively. Let $d \in N$ be the destination node (or sink). For each node $i \neq d$, $g_i \geq 0$ denotes the numbers of users starting at that node. We interpret g_i as a continuum of users. For all $i \neq d$, we denote $\mathcal{R}_i$ as the set of available routes connecting node i with the destination node d. Every link a is represented by a convex and strictly increasing continuous latency function $l_a : \mathbb{R} \mapsto (0, \infty)$, which we assume to be twice continuously differentiable.

A flow vector is a nonnegative vector $v = (v_a)_{a \in A}$, where $v_a \geq 0$ denotes the mass of users using link a. Any flow vector v must satisfy the flow conservation constraint:

$$g_i + \sum_{a \in A_i^-} v_a = \sum_{a \in A_i^+} v_a, \quad \forall i \neq d, \tag{4.1}$$

where A_i^- denotes the set of links ending at node i, and A_i^+ denotes the set of links starting at node i. The set of feasible flows is denoted by V.

We introduce firms into the network through the assumption that each link a is operated by a different firm that sets prices in order to maximize profits. In particular, firm a's profits are given by:

$$\pi_a(p, v_a) \quad = \quad p_a v_a \quad \forall a \in A. \tag{4.2}$$

Profit maximization generates a nonnegative price vector p, $p = (p_a)_{a \in A}$.

In addition, and without loss of generality, we set the parameter $R > 0$ to be the users' reservation utility at each link a. Thus, given a flow v and a price vector p, the users' utility is given by:

$$u_a = R - p_a - l_a(v_a), \ \forall a \in A.$$

In this environment, firms and users strategically interact in the following way: at every node $i \neq d$ the firms owning the set of links starting in node i set prices in order to maximize profits. Then, considering the price vector generated by firms' behavior, users choose routes in order to maximize their utility. The solution concept for this game is a sub-game perfect Nash equilibrium, which we shall refer to the Oligopoly price equilibria.

We look for an Oligopoly price equilibrium using backward induction, i.e., given a price vector p, we solve the users' problem. Given the optimal solution for users, we solve the firms' problem.

It is worth noting that the previous framework is deterministic, so the notion of Wardrop equilibria turns out to be suited for solving the users' problem. Thus the firms' maximization profit considers the demand generated by this solution concept. This way of analysis has been traditional in the context of pricing in congested markets, and examples of its use are the papers of Acemoglu and Ozdaglar [2007], Engel et al. [2004], Hayrapetyan et al. [2005], Weintraub et al. [2010], and Anselmi et al. [Forthcoming].

In this chapter we propose an alternative model to study pricing in congested networks. In particular, we consider heterogenous consumers, where one of the main features of our approach is that the users' optimal solution is based on the combination of random utility models and dynamic programming. The next section describes in detail these ideas.

4.2.1 Markovian traffic equilibrium

In this section we introduce our equilibrium concept for the users' problem, which is based on two important features. First, to solve the users' problem we introduce the idea of random utility, which takes into account the heterogeneity of users' preferences. Second, we use techniques borrowed from dynamic programming to find in a sequential way the optimal path for users. We now proceed to explain in detail our approach.

We introduce heterogeneity in the model assuming that users are *randomly* drawn from a large population having variable perceptions of the utility of each link a.

According to this, the *random utility* $\widetilde{u}_a$ may be defined as

$$\widetilde{u}_a = u_a + \epsilon_a \quad \forall a \in A,$$

with $\{\epsilon_a\}_{a\in A}$ being a collection of absolutely continuous random variables with $\mathbb{E}(\epsilon_a) = 0$ for all a. At least two justifications for introducing $\{\epsilon_a\}_{a\in A}$ can be given. The first explanation comes from the fact that at each link a, the random variable ϵ_a takes into account the variability of users' reservation utility. This means that at each link a we can model the reservation utility as a random variable defined as $R_a = R + \epsilon_a$, with $\mathbb{E}(R_a) = R$. A similar justification can be given if we model the congestion costs as random variables. Concretely, for any given flow vector v, at each link a we can consider the random cost defined as $\widetilde{l}_a(v_a) = l_a(v_a) + \epsilon_a$, where $\mathbb{E}(\widetilde{l}_a(v_a)) = l_a(v_a)$. [5] For all $i \neq d$, let $\mathcal{R}_i$ denote the set of routes connecting node i with destination d. Thus, for a route $r \in \mathcal{R}_i$, we define its utility as $\tilde{u}_r = \sum_{a\in r} \tilde{u}_a$, and therefore the optimal utility $\tilde{\tau}_i = \max_{r\in\mathcal{R}_i} \tilde{u}_r$ as well as the utility $\widetilde{z}_a = \widetilde{u}_a + \widetilde{\tau}_{j_a}$ can be rewritten as $\tilde{\tau}_i = \tau_i + \theta_i$ and $\tilde{z}_a = z_a + \epsilon_a$, where j_a denotes that node j has been reached using the link a, and $\mathbb{E}(\theta_i) = \mathbb{E}(\epsilon_a) = 0$. Each user traveling towards the final node, and reaching the node i, observes the realization of the variables $\tilde{z}_a$ and then chooses the link $a \in A_i^+$ with the highest utility. This process is repeated in each subsequent node giving rise a *recursive discrete choice model*, where the expected flow x_i entering node $i \neq d$ splits among the arcs $a \in A_i^+$ according to

$$v_a = x_i \mathbb{P}(\widetilde{z}_a \geq \widetilde{z}_b,\ \forall b \in A_i^+). \tag{4.3}$$

[5] We can also consider the case where the utilities of every link are deterministic and the variability within the population is captured by the distribution of tastes in regard each link. Both justifications yield the same mathematical structure in terms of expected demand. For a detailed discussion see Anderson et al. [1992].

Furthermore, the recursive discrete choice model generates the following conservation flow equations

$$x_i = g_i + \sum_{a \in A_i^-} v_a. \tag{4.4}$$

Using a well known result in discrete choice theory (see Anderson et al. [1992]), equations (4.3)-(4.4) may be expressed in terms of the gradient of the function $\varphi_i(\cdot)$ defined as $\varphi_i(z) \equiv \mathbb{E}(\max_{a \in A_i^+}\{z_a + \epsilon_a\})$.[6] In particular, the conservation flow equations (4.3) and (4.4) may be rewritten as

$$\begin{cases} v_a = x_i \frac{\partial \varphi_i}{\partial z_a}(z) \quad \forall a \in A_i^+, \\ x_i = g_i + \sum_{a \in A_i^-} v_a, \end{cases} \tag{4.5}$$

where $\frac{\partial \varphi_i}{\partial z_a}(z) = \mathbb{P}(\widetilde{z}_a \geq \widetilde{z}_b,\ \forall b \in A_i^+)$.

Given the recursive structure of the problem, we may write the corresponding Bellman's equation in the form $\widetilde{\tau}_i = \max_{a \in A_i^+} \widetilde{z}_a$ using $\widetilde{z}_a = \widetilde{u}_a + \widetilde{\tau}_{j_a}$. Thus, taking expectation we get

$$z_a = u_a + \varphi_{j_a}(z) \tag{4.6}$$

or in terms of the variables τ_i

$$\tau_i = \varphi_i\left((u_a + \tau_{j_a})_{a \in A}\right) \tag{4.7}$$

In order to simplify our analysis, we assume the following two conditions for the random variables $\{\epsilon_a\}_{a \in A}$.

[6] The functions $\varphi_i(\cdot)$ are known as the social surplus in the literature of discrete choice models. In particular, this definition is introduced in McFadden [1981].

Assumption 1 *For all $i \neq d$ and for all $r \in \mathcal{R}_i$, the collection of random variables $\{\epsilon_a\}_{a \in r}$ are independent.*

Assumption 2 *For each node $i \neq d$, the collection of random variables $\{\epsilon_a\}_{a \in A_i^+}$ are i.i.d following a Gumbel (double exponential) distribution with localization parameter $0 < \beta_i < \infty$.*[7]

We stress that assumption 1 rules out the possibility that the ϵ_a in the same path may exhibit dependence. In terms of our model, assumption 1 implies that realizations of ϵ_a do not affect $\varphi_{j_a}(\cdot)$. However, assumption 1 does not impose independence among different paths.

Regarding assumption 2, we note that is made for expositional simplicity, but all our results hold for a general collection of random variables $\{\epsilon_a\}_{a \in A_i^+}$ with the technical requirement that the density of each ϵ_a belongs to $\mathcal{C}^2$. In particular, we can allow for very complex patterns of correlation among the ϵ_a at each node $i \neq d$.

Assumption 2 allows us to write the functions $\varphi_i(z)$ in a closed form (see Anderson et al. [1992]):

$$\varphi_i(z) = \frac{1}{\beta_i} \log \left(\sum_{b \in A_i^+} e^{\beta_i z_b} \right), \quad \forall i \neq d.$$

Using this log-sum formula, it follows that $\frac{\partial \varphi_i}{\partial z_a}(z) = \frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}}$, i.e., $\frac{\partial \varphi_i}{\partial z_a}(z)$ is the logit-choice rule.

From equations (4.6) and (4.7), it follows that for every price vector p, users recursively find the routes with the highest utility through the solution of a dynamic

[7]The Gumbel (or double exponential) distribution for ϵ_a is given by $\mathbb{P}(\epsilon_a \leq x) = exp(-exp(-\{\beta_i x + \gamma\}))$ where γ is Euler's constant and $0 < \beta_i < \infty$.

programming problem. This means that instead of choosing routes, users recursively choose links considering the utilities and the continuation values at every node. The method of solving recursively the users' problem turns out to be completely different from the standard notion of Wardrop equilibrium.[8] We shall call this solution concept *Markovian traffic equilibrium*. Its formal definition is:

Definition 6 *Let $p \geq 0$ be a given price vector. A vector $v \in \mathbb{R}_+^{|A|}$ is a Markovian traffic equilibrium (MTE) iff the v_a's satisfy the flow distribution equation (4.5), with z solving (4.6).*

Definition 6 formalizes the idea that for a given price vector p, users solve the associated dynamic programming problem such that the flow vector v is distributed in an optimal way. That is, the flow v is distributed such that users' utility is maximized. The notion of MTE has been introduced in Baillon and Cominetti [2008], and it generalizes the concept of *stochastic user equilibrium* considered by Daganzo and Sheffi [1977] and Fisk [1980].[9] In this Chapter we use the concept of MTE because it allows us to introduce heterogeneity within users through the stochastic terms ϵ_a. More importantly, the notion of MTE allows us to study price competition among firms exploiting the recursive structure in users' decisions. In particular, at every node $i \neq d$, and thanks to the Markovian structure, we can study price competition just considering the firms owning the links at every node. In other words, we exploit the recursive structure of the users' problem to decompose the problem of price competition for the whole network into a collection of local oligopoly pricing problems at each node $i \neq d$. In addition, the Markovian structure

[8] Recall that a Wardrop equilibrium establishes that utilities (costs) on all the routes actually used are equal, and greater (less) than those which would be experienced by a single user on any unused route.

[9] For a discussion of different equilibrium concepts used in traffic networks see the recent survey by Correa and Stier-Moses [2010].

makes it possible to study general network topologies in a simple fashion. Example 3 below shows how MTE works for a small network.

Example 3 *This example shows how the notion of MTE works. Consider the following network:*

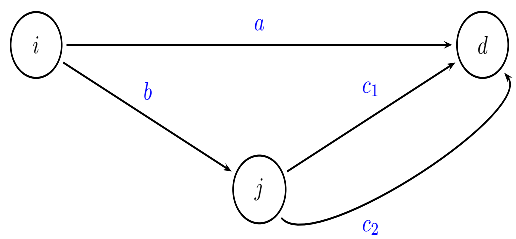

Figure 4.1: Finding an MTE.

The set of nodes is $N = \{i, j, d\}$*, where* i *and* d *represent the origin and destination node respectively. The set of links is represented by* $A = \{a, b, c_1, c_2\}$*. For each link* k *the users' cost is given by* $p_k + l_k(v_k)$ *with* $k = a, b, c_1, c_2$*, where* v_k *is the flow of users choosing link* k*. For a fixed price vector* p*, MTE requires that users solve a dynamic programming problem. According to this, at node* i *the users compare the cost of links* a *and* b *taking into account the associated continuation values. Thus,* *the users will choose link b if and only if* $u_b + \tau_{j_b} + \epsilon_b \geq u_a + \epsilon_a$. *Conditional on the choice of link b, the users reach node j. Then, they must choose between* c_1 *and* c_2*, considering the total costs and the associated continuation values. Finally, and noting that for this case the associated continuation values to* c_1 *and* c_2 *are zero, the users will choose link* c_1 *if and only if* $u_{c_1} + \epsilon_{c_1} \geq u_{c_1} + \epsilon_{c_2}$. *The same logic applies to how other paths are chosen. The key point is that the optimal path is constructed in a recursive fashion.*

4.2.2 Existence and uniqueness of an MTE

Now we are ready to characterize the MTE as the *unique* solution of a concave optimization program.

Proposition 10 *Given any price vector* $p \geq 0$*, the MTE is the unique optimal solution* v^* *of*

$$\max_{v \in V} \left\{ \sum_{a \in A} (R - p_a) v_a - \sum_{a \in A} \int_0^{v_a} l_a(s) ds - \chi(v) \right\}, \qquad (P)$$

where $\chi(v) = \sum_{i \neq d} \frac{1}{\beta_i} \left[\sum_{a \in A_i^+} v_a \ln v_a - \left(\sum_{a \in A_i^+} v_a \ln \left(\sum_{a \in A_i^+} v_a \right) \right) \right]$.

We stress three important points regarding Proposition 10. First, we point out that the result of Proposition 10 is a slight variation of Theorem 2 in Baillon and Cominetti [2008]. We have adapted their result to incorporate the price vector p. Second, the result in Proposition 10 is the stochastic version of the classical characterization for Wardrop equilibria introduced by Beckmann et al. [1956] (See also Ch. 18 in

Nisan et al. [2007]). In fact, in the deterministic case with $\varphi_i(z) = \min\{z_a : a \in A_i^+\}$ we get $\chi(v) \equiv 0$, so that the characterization in Proposition 10 coincides with the one given by Beckmann et al. [1956].[10] Intuitively, the variational problem in Proposition 10 can be viewed as a perturbed version of the deterministic problem, where the perturbation is given by $\chi(v)$, which takes into account the heterogeneity of users' utilities.[11].

Our final remark is that Proposition 10 establishes that an MTE gives us the optimal flow v^* in terms of an implicit equation. To see this, we note that at each node $i \neq d$, v^* can be rewritten as

$$v_a^* = x_i \frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}} = x_i \frac{e^{\beta_i (R - p_a - l_a(v_a^*) + \tau_{j_a})}}{\sum_{b \in A_i^+} e^{\beta_i (R - p_b - l_a(v_b^*) + \tau_{j_b})}} \quad \forall a \in A_i^+. \tag{4.8}$$

Expression (4.8) makes clear the fact that the optimal solution v^* is in terms of an implicit equation.[12]

The uniqueness of an MTE allows us to define v^* as $v^* \equiv D(p)$, where $D(p) = (D_a(p))_{a \in A}$. We called $D(p)$ the *demand function* for the traffic problem. By the maximum theorem, $D(p)$ is a continuous function of p. Thus, the following corollary is straightforward.

[10] We point out that Beckmann et al. [1956]'s characterization of Wardrop equilibria provides uniqueness of an optimal flow over links, but their decomposition of the optimal flow over paths is not unique. The result in Proposition 10 establishes the uniqueness of an optimal flow over links and paths.

[11] We note that $\chi(v) = 0$ means that $\beta_i \longrightarrow \infty$ for all i. This follows because under Assumption 2, for all $i \neq d$, $a \in A_i^+$ we get $\mathbb{V}(\epsilon_a) = \frac{\pi}{\beta_i \sqrt{6}}$. Thus the homogoneous case corresponds to a situation where the random variables ϵ_a are degenerate with mean and variance equal to zero. In particular, as long as $\beta_i \longrightarrow \infty$ for all $i \neq d$, the MTE coincides with the notion of Wardrop equilibrium.

[12] The derivation of (4.8) is as follows. Given the flow x_i at node i, the probability of choosing link $a \in A_i^+$ is given by $\mathbb{P}(u_a + \varphi_{j_a}(z) + \epsilon_a > u_b + \varphi_{j_b}(z) + \epsilon_b, \forall b \in A_i^+)$. Using Assumption 2 combined with the expression for u_a, u_b, $\varphi(\cdot)$ we get (4.8).

Corollary 4 *Let $D(p)$ be an MTE. Then the profit function (4.2) is a continuous function of p. Furthermore, (4.2) can be written as*

$$\pi_a(p) = p_a \left(x_i \frac{e^{\beta_i(R-p_a-l_a(D_a(p))+\tau_{ja})}}{\sum_{b\in A_i^+} e^{\beta_i(R-p_b-l_b(D_b(p))+\tau_{ja})}} \right) \quad \forall a \in A_i^+, i \neq d, \tag{4.9}$$

where x_i satisfies (4.5).

The previous corollary is important because it illustrates two important features of the profit functions represented by $\pi_a(\cdot)$'s. First, it explicitly shows how the congestion levels affects the shape of the profit functions. In fact, we shall see that this feature play a central role in establishing the existence and uniqueness of an OE.

The second feature is that a firm setting prices will worry either about its link being excluded from the optimal path, or, when retained in the optimal path, about the reduction on the overall demand for the path. The prices p_a and continuation values τ capture these effects in our model. We shall refer to these effects as the *path effect* and the *demand effect*, respectively.

4.3 Oligopoly pricing: existence and uniqueness of a symmetric price equilibrium

In this section we begin the study of a price equilibrium by considering a symmetric model. The general definition of an oligopoly price equilibrium is the following.

Definition 7 *A pair $(p^{OE}, D(p^{OE}))$ is a pure strategy Oligopoly price equilibrium (OE) if for all $a \in A$*

$$p_a^{OE} \in \arg\max_{p_a \in [0,R]} \left\{ \pi_a(D_a(p_a, p_{-a}^{OE})) \right\} \quad \forall \;\; p_{-a}^{OE}, \tag{4.10}$$

where $D(p_a, p_{-a}^{OE})$ *is the MTE for the price vector* (p_a, p_{-a}^{OE}).

Definition 7 is the standard notion of a sub-game perfect Nash equilibrium applied to the pricing game under study and it does not impose any restriction on the network topology.

We now specialize Definition 7 to the case of a symmetric model. In particular, in the symmetric model we assume that the congestion at each link a is captured by the same latency function, namely $l(\cdot)$. We assume $\beta_i = \beta$ and $g_i = g$ for all $i \neq d$. Furthermore, we assume that any pair of nodes is connected by at least two links, where the number of available links is denoted by n_i for any node $i \neq d$. Combining Definition 7 with the symmetry in the model, we can define the notion of a symmetric OE as follows.

Definition 8 *We say that a pure strategy OE given by* $(p^{OE}, D(p^{OE}))$ *is symmetric if and only if for all* $i \neq d$

$$p_a^{OE} = p_{n_i}^{OE} \quad \forall a \in A_i^+,$$

with $n_i = |A_i^+|$.

Definition 8 just says that at each node $i \neq d$, firms set the same price, which depends on the number of firms on that node. In fact, Proposition 14 and Corollary 5 in Appendix A show that for a symmetric OE, the prices and profits are decreasing in the number of firms. These results use that $l(\cdot)$ is a convex function.

Our first result is the existence and uniqueness of a symmetric OE. Formally we get:

Theorem 4 *There exists a unique symmetric OE.*

The proof of Theorem 4 is based on checking the technical conditions on Assumption 3 in Section 4.4 below. In particular, Assumption 3 is a condition on the latency functions which guarantees that the profit functions are concave, so that the Kakutani fixed point theorem can be invoked. Interestingly, for the symmetric case such technical conditions are automatically satisfied, and no further assumptions on the class of latency functions are required to guarantee the existence and uniqueness of a symmetric OE.[13]

4.3.1 Welfare analysis and entry decisions

Provided the existence and uniqueness of a symmetric OE, we ask the following question: Under free entry, *will the number of firms be socially optimal*? We interpret entry decisions as new links in the network. Considering a fixed set of nodes, whenever a firm enters the market, the network topology changes. In this environment, a social planner will look for the optimal number of links connecting the nodes, i.e., he will look for the socially optimal design of the network. Our main result in this section establishes that under free entry, and given a fixed cost, the number of firms that enter the market is larger than the socially optimal, i.e., there is excess entry in this setting. The excess entry result is due to a new firm entering the market reduces the demand and prices of the existing firms in the network. This phenomenon is known as " *The business-stealing effect*" (Mankiw and Whinston

[13]Intuitively, Theorem 4 establishes that the demand system induced by the MTE is strictly concave, which implies that the firms' best response map is convex. See section 4.4 for the details of the derivation of the strict concavity of the firms' profit functions in the general case.

[1986]). Intuitively, the business-stealing by a marginal entrant drives a wedge between the entrant's evaluation of the desirability of his entry and the social planner's, generating the discrepancy between n^E and n^S.

We introduce the presence of a social welfare measure, which is given by the sum of *firms' surplus* and *users' surplus.*

Definition 9 *Let* $(p^{OE}, D(p^{OE}))$ *be a pure strategy OE. We define the aggregate welfare as*

$$\mathbb{W}(p^{OE}, D(p^{OE})) \equiv \sum_{i \neq d} \mathbb{W}_i(p^{OE}, D(p^{OE})), \tag{4.11}$$

where $\mathbb{W}_i(p^{OE}, D(p^{OE})) \equiv \sum_{a \in A_i^+} \pi_a(D_a(p^{OE})) + x_i \mathbb{E}\left(\max_{a \in A_i^+} \{z_a + \epsilon_a\}\right)$ *is the welfare at node* $i \neq d$ *with* x_i *satisfying the flow constraint (4.5) and* $z_a = u_a + \varphi_{j_a}(z)$*.*

In Definition 9, the term $\sum_{a \in A_i^+} \pi_a(D_a(p^{OE}))$ represents firms' surplus while $x_i \mathbb{E}\left(\max_{a \in A_i^+} \{z_a + \epsilon_a\}\right)$ represents users' surplus. Thanks to Assumption 2, formula (4.11) can be written in a closed form, where

$$\mathbb{W}_i(p^{OE}, D(p^{OE})) = \sum_{a \in A_i^+} \pi_a(D_a(p^{OE})) + \frac{x_i}{\beta} \log\left(\sum_{a \in A_i^+} e^{\beta z_a}\right) \quad \forall i \neq d. \tag{4.12}$$

We note that Definition 9 explicitly uses the Markovian structure of the model. In fact, the aggregate welfare is just the sum of welfare at each node $i \neq d$, which follows from the recursive structure on users' decisions.

In order to analyze entry decisions, we consider a fixed entry cost, which is denominated as *sunk cost* and is denoted by F.[14] Thus, given a price vector p and the sunk cost F, the profit functions may be written as:

$$\pi_a(D_a(p)) = p_a D_a(p) - F, \quad \forall a \in A. \tag{4.13}$$

Using this simple framework, we are able to answer whether the market will provide the optimal number of firms or not. In particular, we compare the solution obtained by a social planner with the solution obtained by the market. The social planner maximizes the social welfare choosing the optimal number of firms. Formally, the planner solves the following optimization problem:

$$\max_n \left\{ \mathbb{W}(p_n^{OE}, D(p_n^{OE})) \right\} = \max_n \left\{ \sum_{i \neq d} \left(\sum_{a \in A_i^+} \pi_a(D_a(p_n^{OE})) + \frac{x_i}{\beta} \log \left(\sum_{a \in A_i^+} e^{\beta z_a} \right) \right) \right\}, \tag{4.14}$$

where $(p_n^{OE}, D(p_n^{OE}))$ denotes a symmetric equilibrium. Due to symmetry, denote $\mathbb{W}(p_n^{OE}, D(p_n^{OE}))$ as $\mathbb{W}(n)$, so that expression (4.14) may be rewritten as:

$$\max_n \{\mathbb{W}(n)\} = \max_n \left\{ \sum_{i \neq d} \left(\frac{x_i}{\beta} \log(n_i) - x_i l(x_i/n_i) - n_i F \right) \right\}, \tag{4.15}$$

[14]We can interpret the term F as the cost that a firm must pay to participate in the market.

where the last expression is obtained due to the symmetry of the problem.

We note that expression (4.15) is a strictly concave function in $n = (n_i)_{i \neq d}$, so the first order conditions are necessary and sufficient for finding the socially optimal number of firms $n^S = (n_i^S)_{i \neq d}$ for the whole network. On the other hand, the equilibrium condition for firms entering at each node $i \neq d$, is given by the zero profit condition:

$$\forall a \in A_i^+ \quad \pi_a(p_{n_i}^{OE}) = 0. \tag{4.16}$$

Thus solving equations (4.16) we get the equilibrium number of firms n_i^E, where for the whole network the number of firms is given by $n^E = (n_i^E)_{i \neq d}$. We remark that thanks to the convexity of the latency functions, the system of equations (4.16) has a unique solution.

Previous setting allows us to formalize our initial question in the following way: What is the relationship between n^E and n^S ?

The following theorem gives an answer to this question.

Theorem 5 *In the symmetric congestion pricing game $n^E > n^S$.*

The proof of Theorem 5 relies on finding n^S and n^E solving (4.15) and (4.16) respectively. Once n^E and n^S have been found, we proceed to compare them concluding that $n^E > n^S$.

It is worth noting two underlying aspects in Theorem 5. First, as we said before, our result is based in the idea of the business stealing effect. In fact, in Appendix A we show that prices and profits are decreasing in the number of firms (Proposition 14 and Corollary 5 respectively). Thus an entering firm does not internalize such an effect, while the social planner's solution considers this externality. Second, the proof of Theorem 5 also relies on the assumption of convexity of the latency function $l(\cdot)$. If the latency functions are not convex, then the result in Theorem 5 no longer holds.

Theorem 5 generalizes the results in Anderson et al. [1995] and Mankiw and Whinston [1986] to the case of a congestion pricing game with a general network topology. Similarly, Theorem 5 generalizes the result in Weintraub et al. [2010] to the case of a general network.

4.4 Existence and uniqueness of an OE: The general case

The goal of this section is to establish the existence and uniqueness of an OE for a general class of latency functions. In our study, we shall restrict attention to an OE such that at any node $i \neq d$, the users' utilities satisfy[15]

$$R - p_a - l_a(D_a(p)) + \tau_{j_a} = R - p_b - l_b(D_b(p)) + \tau_{j_b} \quad \text{for all}\, a \neq b \in A_i^+.$$

This condition makes explicit the fact that any firm a setting prices takes into account the path effect and the demand effect of its price setting behavior. Moreover, restricting our attention to this class of equilibrium has the advantages of its simplicity and comparability with previous results in the literature.[16]

[15] Recall that the index j_a denotes that node j has been reached using link a.

[16] As we said before, most of the existent results on pricing in congested networks make use of

4.4.0.1 Existence

In order to study the existence of an OE, we exploit the Markovian structure of the users' decisions combined with the assumption that every link is owned by a different firm. In fact, due to the Markovian structure, we can decompose the pricing problem for the whole network into a collection of pricing problems at each node $i \neq d$. Thus the firms competing at node $i \neq d$ set their prices taking as given the flow of users starting at node i, and the prices set by firms in different nodes.

Using this structure, we study the problem of existence through the application of Kakutani's fixed point Theorem. In order to apply Kakutani's result, we need to check that the best response map is non empty, upper semi-continuous, and convex valued. For the pricing game under analysis, the fact that the best response is not empty and upper semi-continuous follows a straightforward application of the maximum theorem. However, the best response map is not convex valued, which makes the application of Kakutani's Theorem problematic. In this Chapter we provide a specific condition in order to guarantee the convexity of the best response. The condition depends on the latency functions and it is automatically satisfied in the symmetric case we analyzed in Section 4.2.

Formally, given any price vector $p \geq 0$, we define firm a's best response map $B_{ia}(p_{-a})$ as follows: For all $i \neq d$, $a \in A_i^+$,

$$B_{ia}(p_{-a}) = \arg \max_{p_a \in [0,R]} \{\pi_a(D_a(p_a, p_{-a}))\}.$$

To study the convexity of $B(\cdot)$, we analyze the concavity of the profit functions π_a. Recall that for each firm a the profit function is given by $\pi_a(D_a(p)) = p_a D_a(p)$. In

the concept of Wardrop equilibrium.

order to establish the concavity of the π_a we note that for all p^{OE}, profit maximization implies that the following optimality condition must hold

$$\forall a \in A \quad \frac{\partial \pi_a(D_a(p^{OE}))}{\partial p_a} = D_a(p^{OE}) + p_a^{OE} \frac{\partial D_a(p^{OE})}{\partial p_a} = 0. \tag{4.17}$$

Using (4.17), we get

$$\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2} = 2\frac{\partial D_a(p^{OE})}{\partial p_a} + p_a^{OE} \frac{\partial^2 D_a(p^{OE})}{\partial p_a^2}. \tag{4.18}$$

The profit function is concave if and only if expression (4.18) is non-positive . In particular, for the case of uncongested markets, and under the assumption that the distribution of the ϵ_a is double exponential, expression (4.18) is negative, and the concavity of the profit function holds. Thus, for the case of uncongested markets, the existence of an OE follows directly. However, we know from (4.8) that the presence of congestion effects implies that the demand function is defined in implicit terms, so when computing $\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2}$ we must take into account this feature. From (4.18) it follows that the concavity depends on $\frac{\partial D_a(p^{OE})}{\partial p_a}$ and $\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2}$. In Appendix C.1 we show that $\frac{\partial D_a(p^{OE})}{\partial p_a} < 0$ and $\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2}$ can be written as

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} = -\frac{1}{D_a(p^{OE})} \left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^2 \left[K_{ia}(p^{OE}) - 2\right],$$

where $K_{ia}(p^{OE})$ is the term that takes into account the effect of the latency functions $l_a(\cdot)$ into the sign (and shape) of $\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2}$. Using the previous expression combined with (4.17), we can rewrite (4.18) as[17]

$$\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2} = \frac{\partial D_a(p^{OE})}{\partial p_a} K_{ia}(p^{OE}). \tag{4.19}$$

Equation (4.19) shows that concavity of the profit function relies on understanding the term $K_{ia}(p^{OE})$. In particular, the profit function will be concave if $K_{ia}(p^{OE}) > 0$.

The derivation of the $K_{ia}(p^{OE})$ is quite involved, and we refer the reader to Appendix C.1 for details. We define the $K_{ia}(p^{OE})$ in the following way; for all $i \neq d, a \in A_i^+$

$$K_{ia}(p^{OE}) \quad = \quad 2 + \frac{\beta_i D_a(p^{OE})}{J_{ia}} \left(\Omega_{ia}(p^{OE}) + (1 - 2\mathbb{P}_a) \left[\frac{\partial D_a(p^{OE})}{\partial p_a} \right]^{-1} \right), \tag{4.20}$$

where $\mathbb{P}_a = \frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}}$, and $J_{ia} = 1 + \beta_i D_a(p^{OE}) \left((1 - \mathbb{P}_a) l_a'(D_a(p^{OE})) + \sum_{b \neq a} \frac{\mathbb{P}_b l_b'(D_b(p^{OE}))}{(n_i - 1)} \right)$.

The term $\Omega_{ia}(p^{OE})$, can be decomposed as

$$\Omega_{ia}(p^{OE}) = \underbrace{\text{Firm } a\text{'s congestion}}_{\mathcal{C}_a(p^{OE})} + \underbrace{\text{Competitors' congestion}}_{\mathcal{C}_{-a}(p^{OE})},$$

[17] The derivation of (4.19) is the following. From the first order condition, we know that $D_a(p^{OE}) = -p_a^{OE} \frac{\partial D_a(p^{OE})}{\partial p_a}$. Using this fact we note that $p_a^{OE} \frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} = 2 \frac{\partial D_a(p^{OE})}{\partial p_a}$. Then replacing the last expression into $\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2}$ the expression follows at once.

with $\mathcal{C}_a(p^{OE}) = (1-2\mathbb{P}_a)l'_a(D_a(p^{OE})) + D_a(p^{OE})(1-\mathbb{P}_a)l''_a(D_a(p^{OE}))$, and $\mathcal{C}_{-a}(p^{OE}) = \sum_{b\neq a} l'_b(D_b(p^{OE}))\left(\frac{(n_i-1)\mathbb{P}_b-\mathbb{P}_a}{(n_i-1)^2}\right) - \sum_{b\neq a}\frac{D_a(p^{OE})\mathbb{P}_b l''_b(D_b(p^{OE}))}{(n_i-1)^2}$.

The component $\mathcal{C}_a(p^{OE})$ shows the effect of firm a's latency function l_a, while $\mathcal{C}_{-a}(p^{OE})$ can be viewed as the average effect of firm a's competitors' latency functions l_b, with $b \neq a$. Thanks to this decomposition, the term $\Omega_{ia}(p^{OE})$ captures all relevant information needed to determine the concavity of the profit function.[18] First, we note that $\Omega_{ia}(p^{OE})$ depends on the l'_a and l''_a. By assumption, we know that the latency functions l_a's are strictly increasing and convex, so it follows that $\mathcal{C}_a(p^{OE})$ is strictly positive. For the case of $\mathcal{C}_{-a}(p^{OE})$, a more careful analysis must be carried out.

In fact, in Appendix A we analyze how $\mathcal{C}_{-a}(p^{OE})$ determines the sign of $K_{ia}(p^{OE})$ through its effect on $\Omega_{ia}(p^{OE})$. The main message from that analysis is that for highly congested networks, the sign of $K_{ia}(p^{OE})$ can be negative, which implies that the condition of strict concavity of the profit function can be violated.

4.4.0.2 Uniqueness

Similar to the study of existence, an explicit condition can be derived to analyze the uniqueness of an OE. In particular, we study the uniqueness based on the *dominant diagonal property* (see Vives [2001, Ch. 2]), which establishes that the equilibrium is unique if the following condition holds:

[18]Throughout the analysis, and without loss of generality, we shall assume that at each node $i \neq d$ the equilibrium probabilities satisfy $\sum_{b\neq a}\mathbb{P}_b \geq \mathbb{P}_a$ for all $a \in A_i^+$.

$$\forall i \neq d,\, \forall\, a, b, \in A_i^+; \quad -\sum_{b\neq a} \frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a \partial p_b} \left[\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2}\right]^{-1} < 1.$$

The previous condition shows that the uniqueness depends on the positivity of $\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a \partial p_b}$, which can be written as

$$\frac{\partial^2 \pi(D_a(p^{OE}))}{\partial p_a \partial p_b} = \frac{\partial D_a(p^{OE})}{\partial p_b} + p_a^{OE} \frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b}. \tag{4.21}$$

In Appendix C.1, Lemma 3 shows that $\frac{\partial D_a(p^{OE})}{\partial p_a^{OE}} > 0$, and Lemma 5 shows that $\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b}$ can be written as

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b} = -\frac{1}{D_a(p^{OE})} \left[\frac{\partial D_a(p^{OE})}{\partial p_a} \frac{\partial D_a(p^{OE})}{\partial p_b}\right] \left[\bar{K}_{iab}(p^{OE}) - 1\right],$$

where $\bar{K}_{iab}(p^{OE})$ is the term that takes into account the effect of the $l_a'(\cdot)$ and $l_a''(\cdot)$ in the sign (and shape) of $\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b}$. Using the previous expression, we rewrite (4.21) as

$$\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} = \frac{\partial D_a(p^{OE})}{\partial p_b} \bar{K}_{iab}(p^{OE}). \tag{4.22}$$

Thus expression (4.22) is positive if $\bar{K}_{iab}(p^{OE})$ is strictly positive. For uncongested markets, and under Assumption 2, expression (4.22) is strictly positive.

Similar to the case of $K_{ia}(p^{OE})$, the derivation of $\bar{K}_{iab}(p^{OE})$ is involved and we refer the reader to Appendix C.1. These terms are defined as follows; for all $i \neq d$, $a, b \in A_i^+$

$$\bar{K}_{iab}(p^{OE}) \quad = \quad 1 + \frac{\beta_i D_a(p^{OE})}{J_{ia}} \left(\Omega_{ia}(p^{OE}) - \left[\frac{\partial D_a(p^{OE})}{\partial p_b} \right]^{-1} \left[\mathbb{P}_b - \frac{\mathbb{P}_a}{(n_i - 1)} \right] \right) \qquad (4.23)$$

with $\Omega_{ia}, \mathbb{P}_a, \mathbb{P}_b$,and J_{ia} defined as before. Furthermore, we note that points 1) and 2) made for the $K_{ia}(p^{OE})$ also apply to the case of the $\bar{K}_{iab}(p^{OE})$.

From the previous analysis, it is clear that the existence and uniqueness of an OE follows if $K_{ia}(p^{OE}) > 0$ and $\bar{K}_{iab}(p^{OE}) > 0$. Formally

Assumption 3 *For any node $i \neq d$, and for all $a, b \in A_i^+$, let $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$ be given by expressions (4.20) and (4.23) respectively. The latency functions are such that: $K_{ia}(p^{OE}) > 0$ and $\bar{K}_{iab}(p^{OE}) > 0$ for all p^{OE}.*

We point out that Assumption 3 is not vacuous. The following examples illustrate how Assumption 3 can apply to two classes of latency functions.

Example 4 (Linear class) *Our first example is the class of linear latency functions. This class is given by the functions $l_a(D_a(p)) = \delta_a D_a(p)$, with $\delta_a > 0$ for all $a \in A$. A straightforward computing shows that for this class of functions $K_{ia}(p^{OE})$ is strictly positive. The reason for this is because $l_a''(D_a(p)) = 0$ for all $a \in A$. This implies that the negative term in $\mathcal{C}_{-a}$ does not play any role. Thus for the case of linear latency functions, the profit function is concave in its own price. Furthermore,* *it is easy to see that for the linear case $\bar{K}_{iab} > 0$. Thus, for the case of linear latency functions, the assumption 3 is satisfied.* $\square$

Example 5 (Load balancing class) *Let us consider the class of latency functions given by $l_a(D_a(p)) = (\mu_a - D_a(p))^{-1}$ with $\mu_a > D_a(p)$ for all $p \geq 0$ and $a \in A$. The parameter $\mu_a > 0$ represents the capacity of each link a. As we said before, this class of functions is the leading case in the context of queueing games (see Hassin and Haviv [2003]). This class is strictly increasing and strictly convex, where $l'_a(D_a(p)) = l^2_a(D_a(p))$ and $l''_a(D_a(p)) = 2l^3_a(D_a(p))$. From this latter property, it follows that $l''_a(D_a(p)) \longrightarrow \infty$ as $D_a(p) \longrightarrow \mu_a$, which implies that $\mathcal{C}_{-a}(p^{OE}) + (1 - 2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1}$ can be arbitrarily large and negative if some of firm a's competitors are operating very close to their link capacities. This behavior can make $K_{ia}(p^{OE})$ negative, and as a consequence, the concavity of the profit function will fail. The intuition for this observation is that for highly congested networks operating close to the capacity of their links, an OE may not exist.[19] Using the converse argument, it can be established that if at each node $i \neq d$ the μ_a are such that the strict inequality $\sum_{a \in A_i^+} \mu_a > \sum_{a \in A_i^+} D_a(p)$ holds, then the conditions of Assumption 3 will apply, and the profit functions will be concave. A similar reasoning can be applied to analyze the sign of $\bar{K}_{iab}(p^{OE})$.* $\square$

From an applied point of view, we note that the conditions in Assumption 3 provide information for the design of large scale simulation exercises having a unique OE.

Finally, the main result can be formally written as:

[19] A similar observation in the context of load balancing games can be found in Anselmi et al. [Forthcoming]. However, they do not provide conditions to study the existence of an OE. In particular, expression (4.20) formalizes their intuition.

Theorem 6 *Suppose that Assumptions 2 and 3 hold. Then, there exists a unique OE.*

Three aspects are worth emphasizing with regards to Theorem 6. First, we note that Theorem 6 is a generalization of the results available in the literature of oligopoly pricing in congested markets. Indeed, practically all environments considered in the literature satisfy Assumption 3. Furthermore, for the case of networked markets without congestion effects, Theorem 6 also applies. In fact, Assumption 3 trivially holds. Thus Theorem 6 can also be viewed as an extension of the results of existence and uniqueness of an OE in the standard oligopoly theory. Second, Theorem 6 establishes the *existence* and *uniqueness* of an OE for a general class of network topologies. Finally, we stress that Theorem 6 considers certain heterogeneity among the random variables because of the different β_i at each node, i.e. the result applies to the case where at every node the population heterogeneity is different. Indeed, Theorem 6 can be stated for a general collection of absolutely continuous random variables $\{\epsilon_a\}_{a\in A}$, at the cost of a more cumbersome notation and no additional insights.

In addition to the existence and uniqueness of an OE, we provide an explicit characterization of the equilibrium price vector p^{OE}.

Proposition 11 *Let $(p^{OE}, D(p^{OE}))$ be a pure strategy OE. Then, for all $i \neq d$, and $a \in A_i^+$:*

$$p_a^{OE} = \frac{1}{\beta_i(1-\mathbb{P}_a)} + D_a(p^{OE})\left[(1-\mathbb{P}_a)l'_a(D_a(p^{OE})) + \frac{\sum_{b\neq a}\mathbb{P}_b l'_b((D_b(p^{OE}))}{n_i - 1}\right],$$

where $n_i = |A_i^+|$, and $\mathbb{P}_a = \frac{e^{\beta_i z_a}}{\sum_{b\in A_i^+} e^{\beta_i z_b}}$ for all $a \in A_i^+$.

Proposition 11 establishes that the equilibrium price vector p^{OE} can be expressed as a function of two components. The first component is due to the fact that p^{OE} depends on the dispersion parameters β_i. In particular, at each node $i \neq d$, the equilibrium prices include the terms $\frac{1}{\beta_i(1-\mathbb{P}_a)}$. This contrasts with the expression that we would get if the notion of Wardrop equilibrium were considered. The reason for this discrepancy is because our approach allows for heterogeneity within users, whereas Wardrop equilibrium works for a homogenous population of users. However, as $\beta_i \longrightarrow \infty$ for all $i \neq d$, the term $\frac{1}{\beta_i(1-\mathbb{P}_a)}$ goes to zero, which implies that equilibrium prices resemble the ones obtained when Wardrop equilibrium is considered as the solution concept.[20]

The second component is $D_a(p^{OE})\left[(1-\mathbb{P}_a)l'_a(D_a(p^{OE})) + \frac{\sum_{b\neq a}\mathbb{P}_b l'_b(D_b(p^{OE}))}{n_i-1}\right]$. From the previous expression, it is easy to see that $D_a(p^{OE})(1-\mathbb{P}_a)l'_a(D_a(p^{OE}))$ is the Pigouvian pricing, which must be charged by firms such that users internalize the congestion externality. Regarding the term $D_a(p^{OE})\frac{\sum_{b\neq a}\mathbb{P}_b l'_b(D_b(p^{OE}))}{n_i-1}$, it has the interpretation of *an extra markup* due to oligopolistic competition among firms at each node $i \neq d$.

Summarizing, the equilibrium price vector p^{OE} can be viewed as the sum of three factors: *heterogeneity, pigouvian pricing, and extra markup.*

Finally, we note that Proposition 11 shows that equilibrium prices solve a fixed point equation, where for each firm a the equilibrium price p_a^{OE} depends on two factors: the prices of its competitors and the continuation values τ_{j_a} associated to its link. These two elements make explicit that a firm setting prices considers the path and demand effects. The previous effects combined with the heterogeneity in users

[20] Recall that at each node $i \neq d$, the variance of the random variable $\{\epsilon_a\}_{a\in A_i^+}$ is given by $\mathbb{V}(\epsilon_a) = \frac{\pi}{\beta_i\sqrt{6}}$. Then, $\beta_i \longrightarrow \infty$ implies that the variance goes to zero, meaning that utility within the population is homogenous. This latter interpretation allows us to compare our result in Proposition 11 with the prices that would be obtained using Wardrop equilibria as the equilibrium concept for solving users' problem.

(through the β_i) turn out to be new elements in the study of price competition in congested networks.

4.5 Conclusion and final remarks

In this Chapter we have studied the problems of free entry and welfare, and the existence and uniqueness of an OE in congested markets for a general class of networks.

In particular, we provided conditions under which an OE exists and is unique in a general class of environments, encompassing many settings studied in the literature. These results allow to us to inspect the welfare properties of congested networks under free entry. To the best of our knowledge, our work is the first to establish the result of excess entry for the case of congestion pricing games in a general network. The closest result to ours is the recent paper by Weintraub et al. [2010], who consider a simple network having a single origin-destination pair with a collection of parallel links. Consequently, we think that our result may provide insights regarding the design of optimal networks subject to congestion effects.

Finally,the introduction of random utility models to the study of pricing in congested networks opens the possibility of carrying out two interesting exercises. The first exercise is related to the evaluation of changes in users' welfare, using the demand function generated at every node. Concretely, and given the result in Theorem 6, we can evaluate the impact in users's welfare of different pricing policies. From an econometric viewpoint, the second exercise is related to the estimation of a congestion pricing game, mimicking the empirical I.O. literature.

Appendix A

Appendix to Chapter 1

A.1 Definitions and proofs

A.1.1 Definitions

Definition 10 (P-matrices) *A matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is said to be a $\boldsymbol{P}$-matrix if all its principal minors are positive. The class of such matrices is denoted $\boldsymbol{P}$.*

Definition 11 *A matrix $\boldsymbol{M} \in \mathbb{R}^{n}$ is row diagonally dominant if*

$$|m_{ii}| > \sum_{j\neq i} |m_{ij}|, \quad i = 1, \ldots, n$$

and strictly row diagonally dominant if strict inequality holds for all i.

Theorem 7 (Cottle et al. [2009]) *A matrix $M \in \mathbb{R}^{n\times n}$ is a $\boldsymbol{P}$-matrix if and only if the $LCP(q, \boldsymbol{\Sigma})$ has a unique solution for all vectors $q \in \mathbb{R}^{n}$.*

Proof. See Cottle et al. [2009]. ☐

Proof of Proposition 1:

($\Longrightarrow$) If V^* is a ME, then it satisfies $[\mathbf{I} + \pi\kappa\mathbf{W}]V^* = \hat{\mathbf{c}}$ with $V^* \geq 0$. Moreover, $V^{*\prime}([\mathbf{I} + \pi\kappa\mathbf{W}]V^* - \hat{\mathbf{c}}) = 0$. Thus V^* is a solution of the LCP($\mathbf{W}, \hat{\mathbf{c}}$).

($\Longleftarrow$) Let V^* be a solution to the LCP$(\mathbf{W}, \hat{\mathbf{c}})$. Noting that $-\hat{\mathbf{c}} \leq 0$, and using the fact that $V^* \geq 0$, it follows that $V^{*'}([\mathbf{I}+\pi\kappa\mathbf{W}]V^* - \hat{\mathbf{c}}) = 0$ implies $[\mathbf{I}+\pi\kappa\mathbf{W}]V^* = \hat{\mathbf{c}}$. Then we conclude that V^* is a ME. $\square$

Proof of Proposition 2: It is easy to see that the matrix $\mathbf{M} = [\mathbf{I} + \pi\kappa\mathbf{D}][\mathbf{I} + \pi\kappa\mathbf{W}]$ satisfies Definition 11, so that $\mathbf{M}$ is dominant diagonal. This implies that $\mathbf{M}$ is invertible and positive definitive, as long as the condition $0 < \delta < 1$ holds. This fact combined with Theorem 7 implies that the $LCP(\mathbf{W}, \hat{\mathbf{c}})$ has a unique solution. Then the conclusion follows from Proposition 1. $\square$

Proof of Theorem 1 : Let V^* be an equilibrium. Then we know that it satisfies $[\mathbf{I} + \pi\kappa\mathbf{W}]V^* = \hat{\mathbf{c}}$. Noting that $[\mathbf{I} + \pi\kappa\mathbf{W}]$ satisfies Definition 11, we can solve for V^*, so that we get

$$V^* = [\mathbf{I} + \pi\kappa\mathbf{W}]^{-1}\hat{\mathbf{c}}.$$

This expression is equivalent to $[\mathbf{I} - (-\pi\kappa\mathbf{W})]^{-1}\hat{\mathbf{c}}$, and using Definition 2, we conclude that $V^* = \mathbf{b}(-\pi\kappa, \mathbf{W}; \hat{\mathbf{c}})$. $\square$

A.1.2 Proof of Theorem 2

Let $\mathbf{T} = \frac{\mathbf{A}}{\rho(\mathbf{A})}$, and let $\mathbf{P}$ be an n-square strictly positive matrix. In particular, $\mathbf{P}$ is defined as $\mathbf{P} = \frac{\mathbf{e}\mathbf{q}^t}{\mathbf{q}^t\mathbf{e}}$, where $\mathbf{e}$ and $\mathbf{q}$ are the right and left eigenvectors of $\mathbf{A}$ respectively.

The following Lemma establishes the connection between $\mathbf{T}$ and $\mathbf{P}$.

Lemma 1 $\lim_{k \longrightarrow \infty} \boldsymbol{T}^k = \boldsymbol{P}$.

Proof. Thanks to the assumption of G being a strongly connected network it follows that $\mathbf{T}$ is primitive, i.e., $\lim_{k\longrightarrow\infty}\mathbf{T}^k$ exists. Furthermore, this limit corresponds to the matrix $\mathbf{P}$ (Ch. 8, Horn and Johnson [1990]). $\square$

Lemma 2 $\lim_{a\longrightarrow 1}(1+a)[\boldsymbol{I}+a\boldsymbol{T}]^{-1}=\boldsymbol{P}$.

Proof. From Lemma 1 we know that for a given $\frac{\epsilon}{2}$ there exists K such that for all $k>K$ we get $\|\mathbf{T}^k-\mathbf{P}\|_\infty<\frac{\epsilon}{2}$. Let us choose $a<1$ such that $|\sum_{k=0}^{K}(1+a)(-a)^k|<\frac{\epsilon}{2\theta}$, where $\theta=\max_{k<K}\|\mathbf{T}^k-\mathbf{P}\|$. Using this fact we get:

$$\begin{aligned}\|(1+a)[\mathbf{I}+a\mathbf{T}]^{-1}-\mathbf{P}\| &= \|\sum_{k=0}^{\infty}(1+a)(-a)^k\mathbf{T}^k-\mathbf{P}\| \\ &\leq \underbrace{\|\sum_{k=0}^{K-1}(1+a)(-a)^k[\mathbf{T}^k-\mathbf{P}]\|}_{\mathbf{b1}}+\underbrace{\|\sum_{k=K}^{\infty}(1+a)(-a)^k[\mathbf{T}^k-\mathbf{P}]\|}_{\mathbf{b2}}.\end{aligned}$$

For $\mathbf{b2}$ we obtain:

$$\mathbf{b2} \leq |\sum_{k=K+1}^{\infty}(1+a)(-a)^k|\frac{\epsilon}{2}<\frac{\epsilon}{2}.$$

For the term $\mathbf{b1}$ we get:

$$\begin{aligned}\mathbf{b1} &\leq |\sum_{k=0}^{K}(1+a)(-a)^k|\theta, \\ &< \frac{\epsilon}{2\theta}\theta=\frac{\epsilon}{2}.\end{aligned}$$

Thus we obtain

$$\|\ (1+a)[\mathbf{I}+a\mathbf{T}]^{-1}-\mathbf{P}\|<\epsilon.$$

$\square$

Proof of Theorem 2: Thanks to previous lemma, we know that for ϵ_1, there exists ϵ such that

$$\| (1+a)[\mathbf{I}+a\mathbf{T}]^{-1} - \mathbf{P}\| < \epsilon,$$

which implies that

$$\left| \frac{\mathbf{b}_i(-a,\mathbf{T};\mathbf{1})}{\sum_{j=1}^{n} \mathbf{b}_j(-a,\mathbf{T};\mathbf{1})} - \mathbf{e}_i(\mathbf{A}) \right| < \epsilon_1 \quad \text{for all } i \in N.$$

Thus it follows that

$$\left\| \frac{\mathbf{b}(-a,\mathbf{T};\mathbf{1})}{\sum_{j=1}^{n} \mathbf{b}_j(-a,\mathbf{T};\mathbf{1})} - \mathbf{e}(\mathbf{A}) \right\| < \epsilon_1.$$

Then the conclusion follows from the fact that the eigenvectors are invariant to scale combined with the identification $\frac{a}{\rho(\mathbf{A})} = \pi\kappa$. $\square$

Proof of Proposition 3: For the regular case, we note that (2.2) can be written as

$$(1-\delta)[\mathbf{I}+\pi\kappa d\mathbf{A}]V^* = \mathbf{c},$$

where d is the degree. The important fact is that $d\mathbf{A}$ is given by $d\mathbf{A} = d[\mathbf{I}+\mathbf{G}']$, where $\mathbf{I}$ is the identity matrix, and G' is the original adjacency matrix for the network G weighted by $\frac{1}{d}$. Using the fact that $\rho(\mathbf{A}) = \rho(\mathbf{I}+\mathbf{G}') = 1+\rho(\mathbf{G}')$, and using the fact that the rows of $\mathbf{G}'$ add to one, we get $\rho(\mathbf{A}) = 2$. Furthermore, for the case of a regular network the number of edges E is given by $E = \frac{nd}{2}$. Thus π is equal to

$\pi = \frac{1}{nd/2}$. The previous observations imply that the condition $\pi\kappa d < \frac{1}{\rho(\mathbf{A})}$ may be written as $\delta < \frac{1}{n^*+1}$ where $n^* = \frac{n}{2}$. Thus the conclusion follows at once. $\square$

A.1.2.1 Proofs for the case of sellers-buyers networks

The proofs of this subsection follows from the previous arguments.

Appendix B

Appendix to Chapter 2

B.1 Proofs

Proof of Theorem 3: Noting that the utility function $U(d)$ is strictly concave, the first order conditions are necessary and sufficient for finding a maximum. Using this fact, we can write the Lagrangian for the consumer's optimization problem as

$$\mathcal{L} \;=\; \sum_{a\in A}\theta_a d_a + d_0 - \sum_{i\in N}\chi_i(d) + \lambda[Y - \sum_{a\in A} p_a d_a - d_0] + \sum_{i\in N}\mu_i[\sum_{a\in A_i^-} d_a - \sum_{a\in A_i^+} d_a] + \sum_{a\in A}\lambda_a d_a.$$

The multipliers μ_i and $\lambda \in \mathbb{R}$ correspond to constraints (4.1), (3.6), and $d_a \geq 0$ respectively. For a stationary point we get $\lambda = 1$, $u_a^* = \theta_a - p_a$ and $\zeta \in \partial(-\chi(v))$ with $\zeta_a = u_a^* + \mu_{j_a} - \mu_{i_a}$ and $\chi(d) = \sum_{i\in N}\chi_i(d)$. For the multipliers λ_a we simply set $\lambda_a = 0$ for all $a \in A$. Taking $\mu_i = \varphi_i((u_a^* + \varphi_{j_a}(V))_{a\in A})$, and combining (4.1) and (4.5) we get

$$d_a = \frac{\partial \varphi_{i_a}}{\partial d_a}(V)\sum_{a\in A_i^+} d_a,$$

which shows that V is an optimal solution for $-\chi(d)$. Therefore, setting $g_a =$

$\varphi_{i_a}(V) - V_a$ we get $g \in \partial(-\chi(d))$. Combining $\varphi_{i_a}(V)$ with $V_a = u_a^* + \varphi_{j_a}(V^*)$, it follows that $\zeta = g \in \partial(-\chi(d))$ as required. $\square$

Proof of Proposition 7 : We can write the Lagrangian for the consumer's optimization problem as

$$\begin{aligned} \mathcal{L} &= \sum_{a\in A} \theta_a d_a + d_0 - \sum_{i\in N} \frac{1}{\beta_i} \left(\sum_{a\in A_i^+} d_a \log d_a - \sum_{a\in A_i^+} d_a \log \left(\sum_{a\in A_i^+} d_a \right) \right) + \\ & \lambda[Y - \sum_{a\in A} p_a d_a - d_0] + \sum_{i\in N} \mu_i [\sum_{a\in A_i^-} d_a - \sum_{a\in A_i^+} d_a] + \sum_{a\in A} \lambda_a d_a. \end{aligned}$$

As in Theorem 3, the multipliers μ_i and $\lambda \in \mathbb{R}$ correspond to constraints (4.1), (3.6), and $d_a \geq 0$ respectively. Taking the first order condition and setting $\lambda_a = 0$, we get

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial d_0} &= 1 - \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial d_a} &= \theta_a - \lambda p_a - \frac{1}{\beta_i} \left[\log d_a - \log \left(\sum_{a\in A_i^+} d_a \right) \right] + \mu_{j_a} - \mu_{i_a} = 0 \qquad \forall i \in N. \end{aligned}$$

Combining (4.1) and (4.5), and after some simple algebra we find that

$$d_a = x_i \frac{e^{\beta_i(\theta_a - p_a + \varphi_{j_a})}}{\sum_{b\in A_i^+} e^{\beta_i(\theta_b - p_b + \varphi_{j_b})}} \quad \forall a \in A_i^+,$$

which is equivalent to

$$d_a = x_i \frac{e^{\beta_i V_a}}{\sum_{b \in A_i^+} e^{\beta_i V_b}} \quad \forall a \in A_i^+,$$

and the conclusion follows at once. □

B.2 Appendix: Sequential MNL and its relationship with the nested logit model

In this appendix we show how Proposition 7 is related to the nested logit model. To make as clear as possible our comparison, we describe a model in terms of a tree decision process. In particular, we follow Train [2009]'s description for a tree process.

Formally, let A be the set of goods, and $|A|$ the number of available choices for consumers. The set A can be partitioned into K nonoverlapping subsets denoted B_1,...,B_K, which we call nests. The utility associated to good j in nest k is given by $\tilde{u}_{jk} = w_k + u_{jk} + \epsilon_{jk}$, where u_{jk} is the deterministic utility for good j in nest k, w_k is a constant term within nest k, but it varies across nests, and the ϵ_{jk} is the random component for good j in nest k. The nested logit is obtained by assuming that the joint distribution of the random variables ϵ_{ik} is given by

$$F(\epsilon) = e^{\left(-\sum_{k=1}^{K}\left(\sum_{j \in B_k} e^{-\epsilon_{jk}/\lambda_k}\right)^{\lambda_k}\right)}, \tag{B.1}$$

where $0 < \lambda_k \leq 1$ for all k.

The expression (B.1) is a generalized extreme value distribution. It is a generalization of the double exponential distribution that gives rise to the logit model. The parameter λ_k is a measure of the degree of independence among goods in nest k.

Let $\mathbb{P}_{jk}$ be the probability that consumer chooses good j from nest k. Using (B.1) we get

$$\mathbb{P}_{jk} \quad = \quad \frac{e^{u_{jk}/\lambda_k}}{\sum_{i \in B_k} e^{u_{ik}/\lambda_k}} \frac{e^{w_k + \lambda_k I_k}}{\sum_{l=1}^{K} e^{w_l + \lambda_l I_l}} \tag{B.2}$$

where $I_l = \log(\sum_{i \in B_l} e^{u_{il}/\lambda_k})$ for $l = 1, ..., K$. In expression (B.2), as $\lambda_k \longrightarrow 0$ the alternatives in nest B_k are highly correlated, whereas $\lambda_k \longrightarrow 1$ means that the alternatives in B_k are independent. As we said in section 3.3.1, the random utility hypothesis holds for $0 < \lambda_k \leq 1$. This condition is formalized in McFadden [1978a,b] and McFadden [1981].

This parametric constraint for the λ_k can be violated in applied work, which is interpreted as violations of the random utility maximization hypothesis. In fact, evidence of this violation can be found in Kling and Herriges [1995].

From Proposition 7, we know that the probability of choosing good j from nest k is given by

$$\bar{\mathbb{P}}_{jk} \quad = \quad \frac{e^{\beta_k u_{jk}}}{\sum_{i \in B_k} e^{\beta_k u_{ik}}} \frac{e^{\beta \tau_k}}{\sum_{l=1}^{K} e^{\beta \tau_l}} \tag{B.3}$$

where $\tau_l = \mathbb{E}(\max_{j \in B_l}\{w_l + u_{jl} + \epsilon_{jl}\})$ for $l = 1, ..., K$. Moreover, for this case we know that $\tau_l = \mathbb{E}(\max_{j \in B_l}\{w_l + u_{jl} + \epsilon_{jl}\}) = \frac{1}{\beta_l} \log(\sum_{j \in B_l} e^{\beta_l (w_l + u_{il})})$. Then it follows that $\tau_l = w_l + \frac{1}{\beta_l} I_l$, and redefining $\beta_l = \frac{1}{\lambda_l}$ we get

$$\bar{\mathbb{P}}_{jk} = \frac{e^{u_{jk}/\lambda_k}}{\sum_{i \in B_k} e^{u_{ik}/\lambda_k}} \frac{e^{\beta w_k + \beta \lambda_k I_k}}{\sum_{l=1}^{K} e^{\beta w_l + \beta \lambda_l I_l}}. \tag{B.4}$$

Then, it follows that equations (B.2) and (B.4) coincides if and only if $\beta = 1$. This means, that as a particular case, we obtain the nested logit model from Proposition 7.

Appendix C

Appendix to Chapter 3

C.1 Proofs

We begin this Appendix with the proof of the existence and uniqueness of an MTE.

Proof of Proposition 10: For a given $p \geq 0$, let v^* be a MTE. Since (P) is a strictly concave program with respect to v, it suffices to check that $v^* \in V$ is a stationary point of the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \sum_{a \in A} (R - p_a) v_a - \sum_{a \in A} \int_0^{v_a} l_a(s) ds - \chi(v) + \sum_{i \neq d} \mu_i \left[g_i + \sum_{a \in A_i^-} v_a - \sum_{a \in A_i^+} v_a \right] \\ &- \sum_{a \in A} \lambda_a v_a. \end{aligned}$$

The multipliers $\mu_i \in \mathbb{R}$, and $\lambda_a \geq 0$ correspond to (4.1) and $v_a \geq 0$ respectively, and stationary amounts to $R - p_a - l_a(v_a^*) = u_a^*$, and $\zeta = \nabla \chi(v)$ where $\zeta = \mu_{i_a} + u_a^* - \mu_{j_a} - \lambda_a$. For the multipliers λ_a, we simply set $\lambda_a = 0$. To check the last condition take $\mu_i = \tau_i(u^*)$. Combining the (4.1) and (4.5) we get

$$v_a \;=\; \frac{\partial \varphi_i}{\partial z_a}(z) \sum_{a \in A_i^+} v_a = \left(\frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}} \right) \sum_{a \in A_i^+} v_a$$

which shows that z is a optimal solution for $\chi(v)$ and therefore setting $g_a = \varphi_{i_a}(z) - z_a$ we get $g = \nabla \chi(v)$. Since $\varphi_{i_a}(z) = \tau_i$ and $z_a = u_a^* + \tau_{j_a}$ we deduce that $g = \zeta = \nabla \chi(v)$ as required. $\square$

C.1.1 Quasi concavity of the profit function

In this section we derive the conditions for the quasi concavity of the profit function $\pi_a(\cdot)$ for all $a \in A$. In particular, we find expressions for $\frac{\partial^2 \pi(p_a, p_{-a}^{OE})}{\partial p_a^2}$ and $\frac{\partial^2 \pi(p_a, p_{-a}^{OE})}{\partial p_a \partial p_b}$, for $a \neq b$. Our first result is the characterization of $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}$ and $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_b}$ for $b \neq a$.

Lemma 3 *Let $(p^{OE}, D(p^{OE}))$ be a pure strategy OE. Then, $\forall i \neq d, a, b \in A_i^+$, with $a \neq b$ we get:*

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} \;=\; \frac{-\beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a)}{J_{ia}}, \tag{C.1}$$

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_b} \;=\; \frac{\beta_i D_a(p_a, p_{-a}^{OE})\mathbb{P}_b}{J_{ia}}, \tag{C.2}$$

where

$$J_{ia} \equiv 1 + \beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a) \left[l_a'(D_a(p_a, p_{-a}^{OE})) + \frac{\sum_{b \neq a} q_b l_b'(D_b(p_a, p_{-a}^{OE}))}{(n_i - 1)} \right],$$

with $n_i = |A_i^+|$, $\mathbb{P}_a = \frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}}$, *and* $q_b = \frac{\mathbb{P}_b}{1-\mathbb{P}_a}$.

Proof. Let us fix a node $i \neq d$. Considering $D_a(p_a, p_{-a}^{OE})$ and taking partial derivative with respect to p_a we get

$$\begin{aligned}\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} &= -\beta_i D_a(p_a, p_{-a}^{OE})(1-\mathbb{P}_a)\left[1 + l_a'(D_a(p_a, p_{-a}^{OE}))\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}\right] + \\ &\quad \beta_i D_a(p_a, p_{-a}^{OE}) \sum_{b \neq a} \mathbb{P}_b l_b'(D_b(p_a, p_{-a}^{OE}))\frac{\partial D_b(p_a, p_{-a}^{OE})}{\partial p_a}.\end{aligned}$$

On the other hand, for the entering flow x_i, we know that it must satisfy $\sum_{b \in A_i^+} D_b(p_a, p_{-a}^{OE}) = x_i$. Using the fact that p^{OE} is a pure strategy OE it follows that

$$\begin{aligned}\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} &= -\beta_i D_a(p_a, p_{-a}^{OE})(1-\mathbb{P}_a)\left[1 + l_a'(D_a(p_a, p_{-a}^{OE}))\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}\right] - \\ &\quad \frac{\beta_i D_a(p_a, p_{-a}^{OE})}{n_i - 1}\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} \sum_{b \neq a} \mathbb{P}_b l_b'(D_b(p_a, p_{-a}^{OE})).\end{aligned}$$

Solving for $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}$ we get:

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} = \frac{-\beta_i D_a(p_a, p_{-a}^{OE})(1-\mathbb{P}_a)}{1 + \beta_i D_a(p_a, p_{-a}^{OE})(1-\mathbb{P}_a)\left[l_a'(D_a(p_a, p_{-a}^{OE})) + \sum_{b \neq a} \frac{\mathbb{P}_b}{1-\mathbb{P}_a}\frac{l_b'(D_b(p_a, p_{-a}^{OE}))}{n_i - 1}\right]}$$

Finally, using the definition for q_b (with $b \neq a$) and J_{ia}, we find:

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} = \frac{-\beta_i D_a(p_a, p_{-a}^{OE})(1-\mathbb{P}_a)}{J_{ia}}.$$

As the previous analysis holds for any node $i \neq d$, the conclusion follows. For the case of $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_b}$, the same logic yields

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_b} = \frac{\beta_i D_a(p_a, p_{-a}^{OE})\mathbb{P}_b}{J_{ia}}, \quad \forall b \neq a \in A.$$

□

When there is no congestion at the network we get $l_a'(x_a) = 0$ for all x_a and for all $a \in A$, so that we find $\frac{\partial D_a(p_a, p_a^{OE})}{\partial p_a} = -\beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a)$, and $\frac{\partial D_a(p_a, p_a^{OE})}{\partial p_b} = \beta_i D_a(p_a, p_{-a}^{OE})\mathbb{P}_b$ for $b \neq a$. Thus, Lemma 3 can be viewed as a generalization of the demand behavior to the case of oligopoly competition in congested markets.

We now introduce the terms $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$ as follows: For all $i \neq d$, and $a \in A_i^+$, define

$$K_{ia}(p^{OE}) \equiv 2 + \frac{\beta_i D_a}{J_{ia}}\Omega_{ia} + \frac{\beta_i D_a}{J_{ia}}\left[\frac{\partial D_a}{\partial p_a}\right]^{-1}(1 - 2\mathbb{P}_a),$$

where $\Omega_{ia} = \left[(1 - 2\mathbb{P}_a)l_a' + \sum_{b\neq a} l_b'\left(\frac{(n_i-1)\mathbb{P}_b - \mathbb{P}_a}{(n_i-1)^2}\right) + D_a(1 - \mathbb{P}_a)l_a'' - \sum_{b\neq a}\frac{D_a\mathbb{P}_b l_b''}{(n_i-1)^2}\right]$, $\mathbb{P}_a = \frac{e^{\beta_i z_a}}{\sum_{b\in A_i^+} e^{\beta_i z_b}}$, $J_{ia} \equiv 1 + \beta_i\left(D_a(1 - \mathbb{P}_a)l_a' + \sum_{b\neq a}\frac{D_a\mathbb{P}_b l_b'}{(n_i-1)}\right)$, $D_a \equiv D_a(p^{OE})$, $l_a' \equiv l_a'(p^{OE})$, and $l_a'' \equiv l_a''(p^{OE})$ for all $a \in A$.

Similarly, we define $\bar{K}_{iab}(p^{OE})$ as:

$$\bar{K}_{iab}(p^{OE}) = 1 + \frac{\beta_i D_a}{J_{ia}}\Omega_{ia} - \frac{\beta_i D_a}{J_{ia}}\left[\frac{\partial D_a}{\partial p_b}\right]^{-1}\left(\frac{(n_i-1)\mathbb{P}_b - \mathbb{P}_a}{(n_i-1)}\right),$$

with $\Omega_{ia}, \mathbb{P}_a, J_{ia}, D_a, l'_a$, and l''_a defined as before.

As we pointed out in the main text, the terms $K_{ia}(p^{OE})$ and $\overline{K}_{iab}(p^{OE})$ can be viewed as technical conditions on the class of latency functions.

The term $K_{ia}(p^{OE})$ is derived as follows. From Lemma 3 we know that $\frac{\partial D_a(p_a, p_a^{OE})}{\partial p_a}$ satisfies the following equation:

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} J_{ia} = -\beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a).$$

Derivating this expression with respect to p_a we find

$$\frac{\partial^2 D_a(p_a, p_{-a}^{OE})}{\partial p_a^2} J_{ia} + \frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} \frac{\partial J_{ia}}{\partial p_a} = -\beta_i \frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}(1 - \mathbb{P}_a) - \beta_i D_a(p_a, p_{-a}^{OE}) \frac{\partial(1 - \mathbb{P}_a)}{\partial p_a}.$$

Using implicit differentiation on $\frac{\partial J_{ia}}{\partial p_a}$, the previous equation can be written in terms of $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}$ and $\frac{\partial^2 D_a(p_a, p_{-a}^{OE})}{\partial p_a^2}$. Thus solving for $\frac{\partial^2 D_a(p_a, p_{-a}^{OE})}{\partial p_a^2}$ we can identify $K_{ia}(p^{OE})$ (see Lemma 4 below). A similar reasoning allows us to identify $\bar{K}_{iab}(p^{OE})$ (see Lemma 5 below).

C.1.2 Analysis of $\mathcal{C}_{-a}(p^{OE})$

In this section we discuss the terms $\mathcal{C}_{-a}(p^{OE})$. In particular, we show how the sign of the $\mathcal{C}_{-a}(p^{OE})$ depends on the latency functions. We consider two cases.

<u>CASE 1</u> $\mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1} \geq 0$: Noting that $\mathcal{C}_a(p^{OE}) > 0$, it follows that

$$\Omega_{ia}(p^{OE}) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1} > 0.$$

The previous condition implies that $K_{ia}(p^{OE}) > 0$, and from (4.19) we conclude that the profit function is concave.

<u>CASE 2</u> $\mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1} < 0$: Using this condition and noting that

$$\Omega_{ia}(p^{OE}) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1} = \mathcal{C}_a(p^{OE}) + \mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1},$$

we get the following: If $\mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1}$ dominates $\mathcal{C}_a(p^{OE})$, then we obtain $K_{ia}(p^{OE}) < 0$ and the concavity of the profit function will fail.[1] This implies that the existence of an OE cannot be established. Conversely, if $\mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1}$ is dominated by $\mathcal{C}_a(p^{OE})$, it follows that $K_{ia}(p^{OE})$ is strictly positive, and by the same argument used in CASE 1, we conclude the existence of an OE.

The previous analysis show us that the problem of establishing the concavity of the profit function, occurs when $\mathcal{C}_{-a}(p^{OE})$ and $(1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1}$ dominate the term $\mathcal{C}_a(p^{OE})$. In other words, the complicated case is when the latency functions are such that the $K_{ia}(p^{OE})$ are strictly negative implying that the concavity of the profit functions does not hold.

[1]We employ the term *dominate* in an absolute value sense.

C.1.3 Analysis of the existence and uniqueness of an OE

After this discussion we are ready to establish the technical Lemmas in order to show the existence and uniqueness of an OE.

Lemma 4 *Suppose that Assumption 3 holds. Then, for all $i \neq d$, $a \in A_i^+$*

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} = -\frac{1}{D_a}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^2 [K_{ia}(p^{OE}) - 2] < 0.$$

Proof. From Lemma 3 we know that

$$\frac{\partial D_a}{\partial p_a} = \frac{-\beta_i D_a(1-\mathbb{P}_a)}{J_{ia}},$$

where $D_a \equiv D_a(p_a, p_{-a}^{OE})$ for all $a \in A_i^+$. Thus, we can rewrite the previous expression as:

$$\frac{\partial D_a}{\partial p_a} J_{ia} \quad = \quad -\beta_i D_a(1-\mathbb{P}_a).$$

Recalling the definition of J_{ia} and taking the derivative with respect to p_a we get:

$$\frac{\partial^2 D_a}{\partial p_a^2} J_{ia} + \frac{\partial D_a}{\partial p_a}\frac{\partial J_{ia}}{\partial p_a} \quad = \quad -\frac{1}{D_a}\frac{\partial D_a}{\partial p_a}\beta_i(1-2\mathbb{P}_a).$$

Computing the derivative $\frac{\partial J_{ia}}{\partial p_a}$, evaluating at p^{OE}, and solving for $\frac{\partial^2 D_a}{\partial p_a^2}$, we get

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} = -\frac{1}{D_a}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^2 [K_{ia}(p^{OE}) - 2],$$

with $0 < D_a < x_i$ and thanks to Assumption 3, $K_{ia}(p^{OE}) - 2 > 0$ for all a. Thus, we conclude that $\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} < 0$. □

Now we establish a key result to guarantee the existence of an OE. Concretely, we utilize Lemma 2 to show the quasi-concavity of $\pi_a(\cdot, p_{-a}^{OE})$.

Proposition 12 *Suppose that Assumption 3 holds. Then, for all firm a, $\pi_a(p_a, p_{-a}^{OE})$ is strictly quasi-concave in its own price p_a.*

Proof. Taking the first order condition we get that an OE satisfies :

$$\frac{\partial \pi_a(p^{OE})}{\partial p_a} = D_a(p^{OE}) + p_a^{OE}\frac{\partial D_a(p^{OE})}{\partial p_a} = 0.$$

Now, taking the second order condition evaluated at p^{OE} we find that

$$\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} = 2\frac{\partial D_a(p^{OE})}{\partial p_a} + p_a^{OE}\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} < 0,$$

where last inequality follows from Lemmas 3 and 4. Thus, we conclude that for all $a \in A$, the profit function $\pi_a(\cdot, p_{-a}^{OE})$ is strictly quasi-concave in its own price p_a. □

The following Proposition establishes the properties of the best response map. In particular, we establish that under assumption 3 the best response map is convex valued.

Proposition 13 *Suppose that Assumption 3 holds, and let* $(p^{OE}, D(p^{OE}))$ *be a pure strategy OE. Then, at each node* $i \neq d$ *the best response map* $B_{ia}(p^{OE}_{-a})$ *is non empty, upper semi-continuous and convex valued for all* $a \in A^+_i$.

Proof. Fix a node $i \neq d$. Using Corollary 4.9 we know that for every firm $a \in A^+_i$, the profit function is continuous and $S_a = [0, R_a]$ is a compact set, then there exists at least one maximizer, which implies that $B_{ia}(p^{OE}_{-a})$ is non empty. By the maximum theorem, $B_{ia}(p^{OE}_{-a})$ is upper semi-continuous. The fact that $B_{ia}(p^{OE}_{-a})$ is a convex set follows from Proposition 12. □

To establish the uniqueness of an OE we use the dominant diagonal property (*cf.* Vives [2001, Ch. 2]). In order to apply such a property we need to establish two technical results, which are given in Lemmas 5 and 6.

Lemma 5 *For all* $i \neq d$, $a \neq b \in A^+_i$*:*

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b} = -\frac{1}{D_a}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\frac{\partial D_a(p^{OE})}{\partial p_b}\right][\bar{K}_{iab} - 1] > 0.$$

Proof. From Lemma 3 we know that

$$\frac{\partial D_a}{\partial p_b} = \frac{\beta_i D_a \mathbb{P}_b}{J_{ia}},$$

where $D_a \equiv D_a(p_a, p^{OE}_{-a})$ for all $a \in A^+_i$. Thus, we can rewrite the previous expression as:

$$\frac{\partial D_a}{\partial p_b} J_{ia} \quad = \quad \beta_i D_a \mathbb{P}_b.$$

Recalling the definition of J_{ia} and taking derivative with respect to p_a we get:

$$\frac{\partial^2 D_a}{\partial p_a \partial p_b} J_{ia} + \frac{\partial D_a}{\partial p_a}\frac{\partial J_{ia}}{\partial p_a} \;\; = \;\; \frac{\partial D_a}{\partial p_a}\beta_i \left[\mathbb{P}_b - \frac{\mathbb{P}_a}{n_i - 1}\right].$$

Computing the derivative $\frac{\partial J_{ia}}{\partial p_a}$, evaluating at p^{OE}, and solving for $\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b}$, we get

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b} \;\; = \;\; -\frac{1}{D_a}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\frac{\partial D_a(p^{OE})}{\partial p_b}\right][\bar{K}_{ib}(p^{OE}) - 1],$$

where $0 < D_a < x_i$ and by Assumption 3, $\bar{K}_{ib}(p^{OE}) - 1 > 0$ for all b. Thus, we conclude that

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b} > 0, \quad \forall a, b \in A.$$

□

Lemma 6 *For all $i \neq d$, $a, b \in A_i^+$ with $a \neq b$*

$$\sum_{b \neq a} \frac{\mathbb{P}_b}{1 - \mathbb{P}_a}\frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})} < 1 \quad \forall p^{OE}.$$

Proof. Note that for $a \neq b$, $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$ can be written as:

$$\begin{aligned} K_{ia}(p^{OE}) &= 1 + \frac{D_a}{J_{ia}}\overline{\Omega}_{ia} + \frac{\mathbb{P}_a}{1 - \mathbb{P}_a}, \\ \bar{K}_{iab}(p^{OE}) &= \frac{D_a}{J_{ia}}\overline{\Omega}_{ia} + \frac{\mathbb{P}_a}{\mathbb{P}_b(n_i - 1)}, \end{aligned}$$

where $\bar{\Omega}_{ia}$ is defined as:

$$\overline{\Omega}_{ia} \equiv \beta_i \left[(1-2\mathbb{P}_a) l'_a + \sum_{b\neq a} l'_b \left(\frac{(n_i-1)\mathbb{P}_b - \mathbb{P}_a}{(n_i-1)^2} \right) + D_a(1-\mathbb{P}_a) l''_a - \sum_{b\neq a} \frac{D_a \mathbb{P}_b l''_b}{(n_i-1)^2} \right].$$

Using this fact we get:

$$\begin{aligned}
\sum_{b\neq a} \frac{\mathbb{P}_b}{1-\mathbb{P}_a} \frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})} &= \sum_{b\neq a} \frac{D_a}{(1-\mathbb{P}_a)K_{ia}(p^{OE})} \mathbb{P}_b \left(\frac{\overline{\Omega}_{ia}}{J_{ia}} + \frac{1}{D_b(n_i-1)} \right), \\
&= \frac{D_a}{(1-\mathbb{P}_a)K_{ia}(p^{OE})} \left(\sum_{b\neq a} \mathbb{P}_b \frac{\overline{\Omega}_{ia}}{J_{ia}} + \frac{1}{x_i} \right), \\
&= \frac{D_a}{(1-\mathbb{P}_a)K_{ia}(p^{OE})} \left((1-\mathbb{P}_a) \frac{\overline{\Omega}_{ia}}{J_{ia}} + \frac{1}{x_i} \right),
\end{aligned}$$

where the last equality follows because of $\sum_{b\neq a} \mathbb{P}_b = 1 - \mathbb{P}_a$. On the other hand, for $K_{ia}(p^{OE})$ we get:

$$K_{ia}(p^{OE}) = \frac{D_a}{1-\mathbb{P}_a} \left(\frac{1}{D_a} + (1-\mathbb{P}_a) \frac{\overline{\Omega}_{ia}}{J_{ia}} \right).$$

Combining the expressions for $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$, we find

$$\sum_{b\neq a} \frac{\mathbb{P}_b}{1-\mathbb{P}_a} \frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})} = \frac{(1-\mathbb{P}_a)\frac{\overline{\Omega}_{ia}}{J_{ia}} + \frac{1}{x_i}}{(1-\mathbb{P}_a)\frac{\overline{\Omega}_{ia}}{J_{ia}} + \frac{1}{D_a}}.$$

Using the fact $0 < D_a < x_i$, we conclude that $\sum_{b\neq a} \frac{\mathbb{P}_b}{1-\mathbb{P}_a} \frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})} < 1$. $\square$

Now we are ready to proof Theorem 6.

Proof of Theorem 6:

<u>EXISTENCE</u>: First, thanks to Proposition 13, the correspondence $B(p^{OE})$ is non empty, upper semi-continuous and convex valued. Then, by Kakutani's fixed point theorem, it follows that there exists a price vector p^{OE} such that $p^{OE} = B(p^{OE})$. Second, we show that for p^{OE} there exists an MTE given by $D(p^{OE})$, such that the condition (4.10) is satisfied. In particular, we show that for any node $i \neq d$ and given p_{-a}^{OE}, the firm $a \in A_i^+$ does not have a profitable deviation. In fact, noting that $D_a(p^{OE})$ can be written as $D_a(p^{OE}) = x_i - \sum_{b \neq a} D_b(p^{OE})$ for all $b \in A_i^+$, and thanks to Proposition 10, it follows that the flow is uniquely determined, which means that firm a does not have an incentive to deviate from p_a^{OE}. As this argument is valid at any node $i \neq d$, we conclude that $(p^{OE}, D(p^{OE}))$ is an OE.

<u>UNIQUENESS</u>: As we said before, to establish the uniqueness we apply the dominant diagonal property. Concretely, at every node $i \neq d$ and for $a, b \in A_i^+$, we compute the term:

$$-\sum_{b \neq a} \frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} \left(\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} \right)^{-1}, \quad \forall a, b \in A_i^+.$$

Using Lemmas 4 and 5 we get:

$$\begin{aligned}
\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} &= \frac{\partial D_a(p^{OE})}{\partial p_b} \bar{K}_{iab}, \quad \text{for all } b \neq a \in A_i^+, \\
\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} &= \frac{\partial D_a(p^{OE})}{\partial p_a} K_{ia}, \quad \text{for all } a \in A_i^+.
\end{aligned}$$

Thus, we find that:

$$-\sum_{b \neq a} \frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} \left(\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} \right)^{-1} = \sum_{b \neq a} \frac{\mathbb{P}_b}{1 - \mathbb{P}_a} \frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})}.$$

Then, thanks to Lemma 6, it follows that

$$-\sum_{b\neq a}\frac{\partial^2\pi_a(p^{OE})}{\partial p_a\partial p_b}\left(\frac{\partial^2\pi_a(p^{OE})}{\partial p_a^2}\right)^{-1}<1\quad\forall a,b\in A_i^+,$$

and we conclude that the equilibrium is unique. □

Proof of Proposition 11:

Let p_{-a}^{OE} be an OE for all firms $b\neq a$. Then the best response for firm a is characterized by $\frac{\partial\pi(p_a,p_{-a}^{OE})}{\partial p_a}=0$. Thus, it follows that p_a^{OE} being a best response to p_{-a}^{OE} must satisfy

$$D_a(p^{OE})+p_a^{OE}\frac{\partial D_a(p^{OE})}{\partial p_a}\quad=\quad 0.$$

Then, using the expression for $\frac{\partial D_a(p^{OE})}{\partial p_a}$ given in Lemma 3, we find

$$p_a^{OE}=\frac{1}{\beta_i(1-\mathbb{P}_a)}+D_a(p^{OE})\left[l_a'(D_a(p^{OE}))+\frac{\sum_{b\neq a}q_b l_b'(D_b(p^{OE}))}{n_i-1}\right].$$

□

C.1.4 Symmetric case

Proposition 14 *Let $(p_n^{OE},D(p_n^{OE}))$ be a symmetric price equilibrium. Then, the following holds*

$$p_{n_i+1}^{OE}<p_{n_i}^{OE}\quad\forall i\neq d.$$

Proof. From a symmetric pure strategy OE condition it follows that

$$\begin{aligned}
p^{OE}_{n_i+1} &= \frac{n_i+1}{\beta n_i} + \frac{x_i}{n_i} l'(x_i/(n_i+1)), \quad \forall i \neq d \\
p^{OE}_{n_i} &= \frac{n_i}{\beta(n_i-1)} + \frac{x_i}{n_i-1} l'(x_i/n_i).
\end{aligned}$$

Computing $p^{OE}_{n_i+1} - p^{OE}_{n_i}$ we get:

$$p^{OE}_{n_i+1} - p^{OE}_{n_i} = -\frac{1}{\beta n_i(n_i-1)} + \frac{x_i}{n_i(n_i-1)}[n_i(l'(x_i/(n_i+1)) - l'(x_i/n_i)) - l'(x_i/(n_i+1))].$$

Thus, thanks to the convexity of $l(\cdot)$, the term $(l'(x_i/(n_i+1)) - l'(x_i/n_i))$ is negative. Combining this fact with $l'(\cdot) > 0$, it follows that $p^{OE}_{n_i+1} - p^{OE}_{n_i} < 0$, or equivalently $p^{OE}_{n_i+1} < p^{OE}_{n_i}$. □

Corollary 5 *Let $(p^{OE}_n, D(p^{OE}_n))$ be a symmetric equilibrium. Then, the following holds*

$$\pi_a(D_a(p^{OE}_{n+1})) < \pi_a(D_a(p^{OE}_n)) \quad \forall a \in A.$$

Proof. For all firms $a \in A$ consider the symmetric equilibriums $p^{OE}_{n_i+1}$ and $p^{OE}_{n_i}$ with the associated profits $\pi_a(D_a(p^{OE}_{n+1}))$ and $\pi_a(D_a(p^{OE}_n))$. Computing $\pi_a(D_a(p^{OE}_{n+1})) - \pi_a(D_a(p^{OE}_n))$ we get:

$$\begin{aligned}
\pi_a(D_a(p^{OE}_{n+1})) - \pi_a(D_a(p^{OE}_n)) &= \frac{x_i}{n_i(n_i+1)}[n_i(p^{OE}_{n_i+1} - p^{OE}_{n_i}) - p^{OE}_{n_i}] \\
&< 0,
\end{aligned}$$

where the last inequality follows from Proposition 14. Thus we conclude that profits are decreasing in n. □

Proof of Theorem 4:

<u>EXISTENCE</u>: Noting that for a symmetric OE we have that for all $i \neq d$, $a \in A$:

$$K_{ia}(p_n^{OE}) \;=\; \frac{n_i}{n_i-1} + \frac{\beta D_a}{J_{ia}}\left(\frac{n_i-2}{n_i(n_i-1)}\right)[(l'(D_a)(2n_i-1)+l''(D_a))] > 0,$$

with $D_a = \frac{x_i}{n_i}$. Thus, we find that Assumption 3 is satisfied and the existence of a symmetric OE follows from Theorem 6.

<u>UNIQUENESS</u>: In order to show the uniqueness, note that for all $i \neq d$, $b \in A$ it holds that:

$$\bar{K}_{iab}(p_n^{OE}) \;=\; \frac{1}{n_i-1} + \frac{\beta D_a}{J_{ia}}\left(\frac{n_i-2}{n_i-1}\right)[l'(D_a)+D_a l''(D_a)] > 0,$$

with $D_a = \frac{x_i}{n_i}$. In particular, we see that $K_{ia}(p^{OE}) > \bar{K}_{iab}(p_n^{OE})$, which implies that Lemma 6 applies, so we conclude that the symmetric equilibrium is unique. □

Proof of Theorem 5: First, as we noted the function $\mathbb{W}(p_n^{OE})$ is strictly concave in n. Thus, taking the first order conditions and solving for n we get:

$$\frac{\partial \mathbb{W}(p_n^{OE})}{\partial n_i} = \frac{x}{\beta n_i} + \left[\frac{x_i}{n_i}\right]^2 l'(x_i/n_i) - F = 0, \quad \forall i \neq d.$$

Thus, the optimal number of firms at each node is given by

$$\forall i \neq d \qquad \frac{x_i}{\beta n_i^S} + \left[\frac{x_i}{n_i^S}\right]^2 l'(x_i/n_i^S) = F \qquad (S).$$

Moreover, thanks to the convexity of $l(\cdot)$ the left hand side in (S) is a decreasing function of n_i^s, which implies that there exists a unique optimal solution n_i^S . Now considering the zero profit condition we find that $\pi_a(D_a(p_n^{OE})) = 0$ yields the following equation

$$\frac{x}{\beta(n_i^E - 1)} + \frac{x_i^2}{n_i^E(n_i^E - 1)} l'(x_i/n_i^E) = F \qquad (E).$$

Once again, the convexity of the left hand side in (E) implies that n^E is uniquely determined. Finally, from (S) and (E) it follows that $n^E > n^S$. $\square$

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