Appendix A

Expressions for $f_r(T)$ and $Q_i(T)$ from Mattis-Bardeen theory

The Mattis-Bardeen theory of the anomalous skin effect in superconductors [10] may be used to derive the behavior of the resonance as the superconductor's temperature is varied.

A.1 Temperature dependence of f_r

Given conductivity $\sigma_s = \sigma_1 - i\sigma_2$, the superconducting resistivity is

$$\rho_s \equiv \frac{1}{\sigma_s} = \frac{\sigma_1 + i\sigma_2}{\sigma_1^2 + \sigma_2^2}.$$

The superconducting resistance is thus

$$R_s = \frac{\sigma_1}{\sigma_1^2 + \sigma_2^2} R_N$$

where R_N is the resistance in the normal state. Likewise, the superconducting inductance is given by

$$\omega L_s = \frac{\sigma_2}{\sigma_1^2 + \sigma_2^2} R_N$$

where ω is the angular frequency. The resonant frequency of an LC circuit is

$$2\pi f_r = \omega_r = \frac{1}{\sqrt{LC}}.$$

Let f_0 be the resonant frequency of the circuit at 0K. Then

$$x \equiv \frac{f_r - f_0}{f_0} = \frac{f_r}{f_0} - 1 = \sqrt{\frac{(L_r C)^{-1}}{(L_0 C)^{-1}}} - 1 = \sqrt{\frac{L_0}{L_r}} - 1$$

$$= \sqrt{\frac{\sigma_{2_0}}{\omega_0(\sigma_{1_0}^2 + \sigma_{2_0}^2)} \frac{\omega_r(\sigma_{1_r}^2 + \sigma_{2_r}^2)}{\sigma_{2_r}}} - 1.$$

Because we are dealing with frequency changes on the order of a few thousandths of f_0 (i.e. a few MHz), $\omega_r/\omega_0 \approx 1$. Also, for $T \ll T_c$, $\sigma_2 \gg \sigma_1$, so $\sigma_1^2 + \sigma_2^2 \approx \sigma_2^2$. Using these approximations, we have:

$$x \approx \sqrt{\frac{\sigma_{2_r}}{\sigma_{2_0}}} - 1.$$

σ_2 is related to the temperature by

$$\frac{\sigma_2(\omega)}{\sigma_n} = \frac{1}{\hbar\omega} \int_{\Delta}^{\Delta+\hbar\omega} dE \frac{E^2 + \Delta^2 - \hbar\omega E}{\sqrt{E^2 - \Delta^2}\sqrt{\Delta^2 - (E - \hbar\omega)^2}} [1 - 2f(E)], \tag{A.1}$$

where σ_n is the normal state conductivity, $\Delta \approx 3.5 k_B T_c$ is half the Cooper pair binding energy, ω is the angular resonant frequency and $f(E)$ is the distribution function for quasiparticles, given by $f(E) = 1/(e^{E/kT} + 1)$ in thermal equilibrium.

A.2 Temperature dependence of Q_i

The internal quality factor Q_i is the ratio of ω times the kinetic inductance L_k to the resistance R_s in the circuit. The kinetic inductance fraction $\alpha \equiv L_k/L_s$. Using the equations for R_s and ωL_s in section A.1, we can express Q_i as a function of σ_1 and σ_2:

$$Q_i = \frac{\omega L_k}{R_s} = \frac{1}{\alpha} \frac{\omega L_s}{R_s} = \frac{1}{\alpha} \frac{\sigma_2}{\sigma_1}.$$

σ_2 is given by Equation (A.1), while

$$\frac{\sigma_1(\omega)}{\sigma_n} = \frac{2}{\hbar\omega} \int_{\Delta}^{\infty} dE \frac{E^2 + \Delta^2 + \hbar\omega E}{\sqrt{E^2 - \Delta^2}\sqrt{(E + \hbar\omega)^2 - \Delta^2}} [f(E) - f(E + \hbar\omega)]$$

with the quantities σ_n, Δ, ω and $f(E)$ defined in section A.1.

Appendix B

Resonator ring-down time derivation

The total quality factor Q_r is given by:

$$Q_r = \frac{\omega_0 \epsilon}{P}$$

where $\omega_0 = 2\pi f_0$ is the angular resonant frequency, $\epsilon = \frac{1}{2}LI^2$ is the energy stored in the resonance and $P = \frac{1}{2}I^2R$ is the power dissipated. Plugging in for ϵ and P yields:

$$Q_r = \frac{\omega_0 \frac{1}{2}LI^2}{\frac{1}{2}I^2R} = \frac{\omega_0 L}{R}$$

The attenuation can be obtained by solving the equation of motion for an RLC circuit and is equal to $\frac{R}{2L}$. Then we have:

$$\text{attenuation} = \frac{R}{2L} = \frac{\omega_0}{2}\frac{R}{\omega_0 L} = \frac{\omega_0}{2}\frac{1}{Q_r} = \frac{2\pi f_0}{2Q_r} = \frac{\pi f0}{Q_r}$$

The resonator ring-down time τ_{res} is defined as the inverse of attenuation, so, in terms of quantities easily found from fitting resonances,

$$\boxed{\tau_{res} = \frac{Q_r}{\pi f_0}}. \tag{B.1}$$

τ_{res} can also be expressed in terms of the bandwidth Δf by substituting $Q_r = \frac{f_0}{\Delta f}$:

$$\tau_{res} = \frac{1}{\pi \Delta f}.$$

Physically, the resonator ring-down time is the timescale on which the resonator loses energy during oscillation. By design, the bolometer time constant (sec 4.2) is much greater than the resonator ring-down time, so that quantity dominates in our measurements.

Appendix C

HFSS modeling calculations

An accurate NEP under loading can only be determined if the optical power reaching the detector is well known. This requires knowledge of both the absorption of the filters in front of the detector and the spectral dependence of the absorption of the detector itself. The former is provided by specification sheets for the filters, but the latter must be measured or modeled. In order to confidently report the amount of radiation absorbed by the detector, we have modeled the absorption of a single pixel. Details of the methodology are reported here.

C.1 Description of circuit

See section 1.3.1.

C.2 Analytic solution to simplest case

The simplest approximation treats the meandering inductor as a uniform 80 Ω sheet resistance on one surface of the silicon substrate in free space (see Fig. C.1). The Si substrate has a thickness of $l = 500$ μm. Standing waves can occur in the silicon. Maximum transmission occurs when the

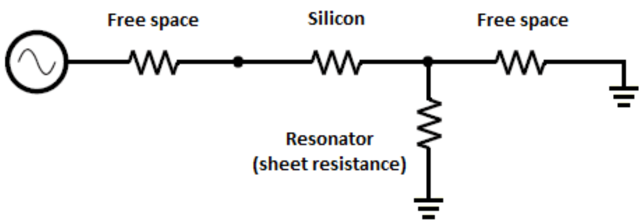

Figure C.1: In the simplest approximation, the light sensitive part of the resonator, the meandering inductor, is treated as a sheet resistance on the boundary between the silicon substrate and free space.

substrate thickness is an odd integer multiple of a quarter of a wavelength. Minimum transmission occurs when the substrate thickness is an even integer multiple of a quarter of a wavelength. To find the frequencies of minimum and maximum transmission, we evaluate:

$$\frac{n\lambda}{4} = \frac{n}{4}\frac{c}{\nu\sqrt{\epsilon}}$$

where $\lambda/4 = l = 500\mu$m and $\epsilon_{Si} = 11.9$. Solving yields

$$n\nu = n * 4.35 * 10^{10}\text{Hz}.$$

In the case that n is even, we are dealing with half-wavelength multiples, so the magnitude of the voltage is the same at either end of the silicon section. This means we can eliminate it from the circuit. The simplified circuit is free space in series with free space and a 50 Ω sheet resistance to ground in parallel. The equivalent impedance for the parallel section is

$$\frac{1}{\frac{1}{50\Omega} + \frac{1}{377\Omega}} = 44.1\ \Omega.$$

And the reflection coefficient for the transition between free space and this load is

$$\Gamma_{\frac{1}{2}\lambda} = \frac{377 - 44.1}{377 + 44.1} = 0.79.$$

In the case that n is odd, the equivalent impedance of the Si section plus load (free space in parallel with 50 Ω sheet resistance) is Z_{Si}^2/Z_{load}, so that the reflection coefficient is

$$\Gamma_{\frac{1}{4}\lambda} = \frac{377 - (377/\sqrt{11.9})^2/44.1}{377 + (377/\sqrt{11.9})^2/44.1} = 0.16.$$

|S11| should oscillate between these two extremes.