ON RECONSTRUCTION THEOREMS IN NONCOMMUTATIVE RIEMANNIAN GEOMETRY

Thesis by

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Abstract

We present a novel account of the theory of commutative spectral triples and their two closest noncommutative generalisations, almost-commutative spectral triples and toric noncommutative manifolds, with a focus on reconstruction theorems, *viz*, abstract, functional-analytic characterisations of global-analytically defined classes of spectral triples. We begin by reinterpreting Connes's reconstruction theorem for commutative spectral triples as a complete noncommutative-geometric characterisation of Dirac-type operators on compact oriented Riemannian manifolds, and in the process clarify folklore concerning stability of properties of spectral triples under suitable perturbation of the Dirac operator. Next, we apply this reinterpretation of the commutative reconstruction theorem to obtain a reconstruction theorem for almost-commutative spectral triples. In particular, we propose a revised, manifestly global-analytic definition of almost-commutative spectral triple, and, as an application of this global-analytic perspective, obtain a general result relating the spectral action on the total space of a finite normal compact oriented Riemannian cover to that on the base space. Throughout, we discuss the relevant refinements of these definitions and results to the case of real commutative and almost-commutative spectral triples. Finally, we outline progess towards a reconstruction theorem for toric noncommutative manifolds.

Contents

Chapter 1

Introduction

I tell them that if they will occupy themselves with the study of mathematics, they will find in it the best remedy against the lusts of the flesh.

— T. Mann *apud* M. Reed and B. Simon, *Methods of Mathematical Physics* I

Vetustam fecit pellem meam et carnem meam; contriuit ossa mea.

— Lamentations 3:4 (Vulgate)

The starting point of operator-algebraic noncommutative geometry is Gel'fand's observation that the continuous complex-valued functions on a compact Hausdorff space X form a commutative unital C^*-algebra $C(X)$; *Gel'fand–Naĭmark duality* then establishes that the assignment $X \mapsto C(X)$ defines a contravariant equivalence of categories between the category of compact Hausdorff spaces and continuous maps and the category of commutative unital C^*-algebras and unital $*$-homomorphisms. This, together with related results such as the Serre–Swan theorem on vector bundles and Rieffel's theory of strong Morita equivalence and noncommutative quotients, has motivated the functorial identification in operator-algebraic noncommutative geometry of C^*-algebra theory as a theory of noncommutative topology.

Given such a theory of noncommutative topology, one can seek to refine it into a theory of noncommutative differential geometry, especially in light of possible applications to theoretical physics. With the Atiyah–Singer index theorem in mind, Connes proposed a candidate notion of noncommutative manifold in the form of a *spectral triple*, that is, a triple (A, H, D) where A is a unital pre-C^*-algebra, H is a Hilbert space admitting a faithful unital $*$-representation of A by bounded operators, and D is an essentially self-adjoint operator on H with compact resolvent, such that $[D, a]$ is bounded for all $a \in A$. The archetypal example of a spectral triple is $(C^\infty(X), L^2(X, \mathcal{S}), \not{D})$ for X a compact spin manifold, $\mathcal{S} \to X$ the spinor bundle, and $\not{D}$ the Dirac operator on $\mathcal{S}$.

To justify an identification of the theory of spectral triples as a theory of noncommutative differential geometry, one would need an analogue of Gel'fand–Naĭmark for suitable spectral triples. Indeed, at the level of objects, Connes [27] has defined a notion of *commutative* spectral triple, motivated by the spectral triple of a compact spin manifold, and proved the *reconstruction theorem*, which shows that a commutative unital Fréchet pre-C^*-algebra A is isomorphic to $C^\infty(X)$ for some compact oriented manifold X if and only if there exists a commutative spectral triple of the form (A, H, D) with the same algebra A. The primary goal of this thesis is to obtain analogous reconstruction theorems for the two classes of spectral triples closest to the commutative case, *almost-commutative* spectral triples and *toric noncommutative manifolds*.

We begin in Chapters 2 and 3 with a telegraphic review of necessary differential- and noncommutative-geometric background for the theory of spectral triples. In particular, Section 2.3, which has been adapted from [10, § 2.1], provides the differential-geometric motivation for the notion of *real* spectral triple from the perspective of Plymen's noncommutative-geometric characterisation of spin$^{\mathbb{C}}$ and spin manifolds.

Next, Chapter 4, which incorporates the material related to commutative spectral triples from [9, 10], mostly consists of an account of the theory of commutative spectral triples from an emphatically Riemannian perspective. In particular, we propose a weakening of the orientability condition in the definition of commutative spectral triple to accommodate Dirac-type operators in full generality, and provide the following re-interpretation of Connes's reconstruction theorem as precisely a complete noncommutative-geometric characterisation of Dirac-type operators on compact oriented Riemannian manifolds:

Corollary ([9, Cor. 2.19])**.** *Let (A, H, D) be a p-dimensional commutative spectral triple. Then there exist a compact oriented Riemannian p-manifold X and a Hermitian vector bundle $\mathcal{E} \to X$ such that $(A, H, D) \cong (C^\infty(X), L^2(X, \mathcal{E}), D)$, where D is identified with an essentially self-adjoint Dirac-type operator on $\mathcal{E}$.*

We conclude the chapter by proposing and discussing the following conservative noncommutative generalisation of the definition of commutative spectral triple, for later use in our discussion of toric noncommutative manifolds:

Definition. Let (A, H, D) be a spectral triple; we call (A, H, D) *two-sided* spectral triple if H admits a faithful unital $*$-representation of the opposite algebra A^o making H into an A-bimodule, and

$$AH^\infty \subset H^\infty, \quad H^\infty A \subset H^\infty.$$

We then call (A, H, D) a *p-dimensional Dirac-type spectral triple* for $p \in \mathbb{N}$ if it is two-sided and if the following conditions hold:

1. **Dimension**: The spectral triple (A, H, D) has metric dimension p.

2. **Order one**: For any a, $b \in A$, $[[D, a], b^o] = 0$.

3. **Finiteness**: The right A-module H^∞ is finitely generated projective.

4. **Strong regularity**: One has that $\mathrm{End}_{A^o}(H^\infty) \subset \cap_k \mathrm{Dom}(\mathrm{ad}\,|D|)^k$.

5. **Orientability**: There exists an antisymmetric Hochschild p-cycle $c \in Z_p(A, A)$ such that $\chi = \pi_D(c)$ is a self-adjoint unitary on $\mathcal{H}$, satisfying $a\chi = \chi a$ and $[D, a]\chi = (-1)^{p+1}\chi[D, a]$ for all $a \in A$.

6. **Absolute continuity**: The right A-module H^∞ admits a Hermitian structure $(\cdot, \cdot)_A$, satisfying $\langle \xi, \eta \rangle_A = \fint (\xi, \eta)_A \, (D^2 + 1)^{-p/2}$ for ξ, $\eta \in H^\infty$.

In Chapter 5, which incorporates the results of [9, Appendix A], we provide the necessary machinery that allows us to weaken the orientability hypothesis in Connes's reconstruction theorem. This consists of establishing the stability of various properties of spectral triples under suitable perturbation of the Dirac operator; whilst some of these questions have already been explicitly considered by Chakraborty–Mathai [14], the others have been folkloric results at best. In short, what we prove is the following:

Theorem ([9, Appendix A]). *Let (A, H, D) be a p-dimensional Dirac-type spectral triple (v. supra), and let M be a self-adjoint element of $\mathrm{End}_{A^o}(H^\infty)$ for $H^\infty := \cap_k \mathrm{Dom}\, D^k$, such that*

$$[(D + M)^2 - D^2, T] \in \mathcal{D}(\mathrm{End}_{A^o}(H^\infty), D^2)_{k+1}$$

for all $T \in \mathcal{D}(\mathrm{End}_{A^o}(H^\infty), D^2)_k$ (cf. Chapter 5). Then $D_M := D + M$ extends to an essentially self-adjoint operator on H with smooth core H^∞, making (A, H, D_M) into a p-dimensional Dirac-type spectral triple satisfying the following:

1. *For each $k \in \mathbb{N}$, $\mathrm{Dom}\, D_M^k = \mathrm{Dom}\, D^k$, and hence $\cap_k \mathrm{Dom}\, D_M^k = H^\infty$;*

2. *For each $k \in \mathbb{N}$,*

$$\mathrm{Dom}\,(\mathrm{ad}\,|D_M|)^k = \mathrm{Dom}\,(\mathrm{ad}\,|D|)^k \subset B(H),$$

and hence

$$\bigcap_k \mathrm{Dom}\,(\mathrm{ad}\,|D_M|)^k = \bigcap_k \mathrm{Dom}\,(\mathrm{ad}\,|D|)^k .$$

3. *For all ξ, $\eta \in H^\infty$,*

$$\fint (\xi, \eta)\,(D_M^2 + 1)^{-p/2} = \fint (\xi, \eta)\,(D^2 + 1)^{-p/2} = \langle \xi, \eta \rangle .$$

In Chapter 6, which incorporates the relevant material from [9, 10], we finally turn to almost-commutative spectral triples, which we present from the perspective first proposed in [9]. These were first proposed by Connes [23, 24] to provide semi-classical spacetimes for high energy physics models, morally constructed as the Cartesian products of classical (Euclidean) spacetimes with 0-dimensional noncommutative "internal" space. More precisely, an almost-commutative spectral triple is conventionally defined as the Cartesian product of the canonical spectral triple of a compact spin manifold, the classical spacetime, with a *finite* spectral triple, namely, a spectral triple with finite-dimensional Hilbert space (see [8, 46, 55] for the general theory, and [40–44] for classification results). The application of almost-commutative spectral triples to high energy physics is epitomised by the project of reformulating the classical field theory of the Standard Model in noncommutative-geometric terms, which culminated in 2006 with the near-simultaneous papers by Barrett [1] and by Chamsedinne–Connes–Marcolli [18] (see also [25, 30]).

Our first goal is to motivate and then translate into noncommutative-geometric terms the following modified definition of almost-commutative spectral triple:

Definition ([9, Def. 2.3])**.** A *concrete almost-commutative spectral triple* is a spectral triple of the form $(C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D)$, where X is a compact oriented Riemannian manifold, $\mathcal{A} \to X$ is an algebra bundle, $\mathcal{E} \to X$ is a Clifford $\mathcal{A}$-module, and D is a Dirac-type operator on $\mathcal{E}$.

This expanded definition still encompasses the original definition and still allows for the use of exactly the same heat-theoretic tools in computations related to applications to high energy physics. On the other hand, the new definition turns out to be stable under inner fluctuations of the metric, unlike the old one, and encompasses a number of global-analytically defined spectral triples already considered in the literature, for instance, by Zhang [70] and by Boeijink–Van Suijlekom [5]. Most importantly, because of its manifestly global-analytic nature, this definition lends itself immediately to the statement and proof of a suitable reconstruction theorem:

Theorem ([9, Thm. 2.17])**.** *Let (A, H, D) be a p-dimensional abstract almost-commutative spectral triple with base B, so that B is a central unital $*$-subalgebra of A such that (B, H, D) is a p-dimensional Dirac-type commutative spectral triple and A is a finitely generated projective unital B-module-$*$-subalgebra of $\operatorname{End}_B(\cap_k \operatorname{Dom} D^k)$ satisfying $[[D, b], a] = 0$ for $a \in A$, $b \in B$. Then (A, H, D) is unitarily equivalent to a concrete almost-commutative spectral triple, that is, there exist a compact oriented Riemannian p-manifold X, an algebra bundle $\mathcal{A} \to X$, a Clifford $\mathcal{A}$-module $\mathcal{E} \to X$, such that*

$$(A, H, D) \cong (C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D),$$

where D is identified with a symmetric Dirac-type operator on $\mathcal{E}$.

In Sections 6.4-5, consisting of the main content of [10], we consider the implications of these revised definitions of commutative and almost-commutative spectral triples for *real* almost-commutative spectral triples [10]. In particular, the correct abstract definition of real almost-commutative spectral triple turns out to have the following form:

Definition ([10, Def. 2.29])**.** A *real almost-commutative spectral triple* of KO-dimension $n \bmod 8$ and metric dimension p is a real spectral triple (A, H, D, J) of KO-dimension $n \bmod 8$, such that (A, H, D) is a p-dimensional (abstract) almost-commutative spectral triple with base

$$\tilde{A}_J := \{a \in A \mid Ja^* J^* = a\}.$$

The significance of this definition is that a real almost-commutative spectral triple automatically comes with a canonical commutative algebra encoding the base manifold; in the general case, by contrast, such a commutative algebra must be specified separately. This is a feature that had been observed in all the real almost-commutative spectral triples appearing in the literature on applications to high energy physics, but only on a case-by-case basis.

In Section 6.5, which incorporates the author's contribution to [11], we then apply this emphatically global-analytic perspective on almost-commutative spectral triples to provide a general explanation for the relationship between the spectral action on S^3 and the spectral action on the quotients of S^3 by the various finite subgroups of $SU(2)$ that had been observed by Teh in the course of explicit computations using explicit Dirac spectra and the Poisson summation formula [11,50,51,65]. By applying general heat kernel estimates, we are able to prove the following, together with a slight generalisation:

Theorem ([11, Thm. 3.6, cf. Thm. 3.10])**.** *Let $\widetilde{X} \to X$ be a finite normal Riemannian covering with $\widetilde{X}$ and X compact, connected and oriented, and let Γ be the deck group of the covering. Let $\widetilde{\mathcal{E}} \to \widetilde{X}$ be a Γ-equivariant Clifford module, and let $\widetilde{D}$ be a Γ-invariant symmetric Dirac-type operator on $\widetilde{\mathcal{E}}$. Let $\mathcal{E} := \widetilde{\mathcal{E}}/\Gamma \to \widetilde{X}/\Gamma =: X$, and let D be the pushforward of $\tilde{D}$ to D. Finally, let $f : \mathbb{R} \to \mathbb{C}$ be of the form $f(x) = \mathcal{L}[\phi](x^2)$ for $\phi \in \mathcal{S}(0, \infty)$. Then for $\Lambda > 0$,*

$$\operatorname{Tr}(f(D/\Lambda)) = \frac{1}{\#\Gamma} \operatorname{Tr}\left(f(\widetilde{D}/\Lambda)\right) + O(\Lambda^{-\infty}), \quad \textit{as } \Lambda \to +\infty.$$

In particular, this is likely one of the very first general qualitative results concerning the spectral action on almost-commutative spectral triples anywhere in the literature.

Finally, in Chapter 7, we describe progress towards a reconstruction theorem for toric noncommutative manifolds. In Sections 7.1-2 we generalise Yamashita's noncommutative generalisation [69] of Connes–Landi's original construction [29] of toric noncommutative manifolds, to accommodate $\mathbb{T}^N$-actions for $N \geq 2$. Then, in Section 7.3, we consider toric noncommutative manifolds specifically. First, we check that toric noncommutative manifolds satisfy the following proposed abstract definition:

Definition. Let (A, H, D) be a $\mathbb{T}^N$-equivariant p-dimensional Dirac-type spectral triple, and let $\theta \in \mathfrak{so}(N)$. We call (A, H, D) a *p-dimensional toric noncommutative manifold* with *deformation parameter θ* if

$$\forall a \in A, \quad a^o := \lambda_{-2\theta}(a).$$

Remark. The condition that the right A-action on H by given by $a^o := \lambda_{-2\theta}(a)$ is equivalent to requiring that the undeformed algebra $A_{-\theta}$ be commutative and that for any $a_{-\theta} \in A_{-\theta}$, $a^o := \lambda_\theta(a_\theta)^o = \lambda_{-\theta}(a_\theta)$, which is precisely what holds in the concrete case.

Then, we describe our progress to date towards a reconstruction theorem, which takes the following form:

Theorem. *Let (A, H, D) be a p-dimensional toric noncommutative manifold with deformation parameter $\theta \in \mathfrak{so}(N)$. Suppose, moreover, that the orientation cycle $c \in Z_p(A, A)$ corresponds to a $\mathbb{T}^N$-invariant antisymmetric cycle $c_\theta \in Z_p(A_{-\theta}, A_{-\theta})$ such that $\pi_D(c) = \pi_D(c_\theta)$. Then there exists a concrete $\mathbb{T}^N$-equivariant commutative spectral triple such that*

$$(A_{-\theta}, H, D) \cong (C^\infty(X), L^2(X, \mathcal{E}), D), \quad (A, H, D) \cong (C^\infty(X_\theta), L^2(X_\theta, \mathcal{E}), D),$$

i.e., (A, H, D) is unitarily equivalent to a concrete p-dimensional toric noncommutative manifold.

Whilst this is already a reconstruction theorem for toric noncommutative manifolds insofar as it provides a complete noncommutative-geometric characterisation thereof, a genuinely satisfactory reconstruction theorem would be obtained only after removal of the awkward and artificial additional orientability-related hypothesis.

Chapter 2

Differential-geometric preliminaries

J'ai de sérieuses raisons de croire que la planète d'où venait le petit prince est l'astéroïde B 612. Cet astéroïde n'a été aperçu qu'une fois au télescope, en 1909, par un astronome turc. Il avait fait alors une grande démonstration de sa découverte à un Congrès International d'Astronomie. Mais personne ne l'avait cru à cause de son costume...

— A. de Saint-Exupéry, *Le Petit Prince*

In this brief section, we recall some relevant differential-geometric background for the theory of spectral triples, particularly commutative spectral triples, almost-commutative spectral triples, and toric noncommutative manifolds.

2.1 Clifford modules and Dirac-type operators

Let us first recall a few definitions and facts from the theory of Dirac-type operators, mostly to establish notation and terminology; for a full account, see [2, §§ 3.1–3; 36, Chapter 5, §§ 9.1–3]. Throughout, we shall use the conventions of super-linear algebra [2, § 1.3]. Thus, if H is a $\mathbb{Z}_2$-graded vector bundle, we consider $\operatorname{End}(H)$ as $\mathbb{Z}_2$-graded as well, and we shall consider the $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded vector bundles and operators on them, which we denote by $\widehat{\otimes}$. If a vector bundle is not explicitly $\mathbb{Z}_2$-graded, as shall often be the case, we consider it as trivially $\mathbb{Z}_2$-graded.

Let us first recall the notion of Clifford module:

Definition 2.1.1. Let X be a Riemannian manifold with Riemannian metric g. A *Clifford module* over X is a Hermitian vector bundle $\mathcal{E} \to X$ together with a bundle morphism $c : T^*X \to \operatorname{End}(X)$, the *Clifford action*, satisfying the following:

1. for all ξ, $\eta \in \Omega^1(X)$, $c(\xi)c(\eta) + c(\eta)c(\xi) = -g^{-1}(\xi, \eta)$;

2. for all $\xi \in \Omega^1(X)$, $c(\xi)^* = -c(\xi)$.

If, moreover, $\mathcal{E}$ is $\mathbb{Z}_2$-graded, then we require $c(\xi)$ to be odd for all $\xi \in \Omega^1(X)$.

Example 2.1.2. If X is spin, then the spinor bundle $\mathcal{S} \to X$ is a Clifford module. More generally, $\wedge T^*X \to X$ is a Clifford module with Clifford action $c(\xi)\eta := \xi \wedge \eta - i_\xi \eta$ for $\eta \in \Omega(X)$, $\xi \in \Omega^1(X)$.

Let us now recall the central differential-geometic definition of this thesis:

Definition 2.1.3. Let $\mathcal{E}$ be a Clifford module bundle over a Riemannian manifold (X, g). A *Dirac-type operator* on E is a first-order differential operator on $\mathcal{E}$ such that

$$[D, f] = c(df), \quad f \in C^\infty(X),$$

where $c : \mathrm{Cl}(X) \to \mathrm{End}(E)$ denotes the Clifford action on E.

Example 2.1.4. If X is spin, then *the* Dirac operator is a Dirac-type operator. More generally, $d + d^*$ defined a Dirac-type operator on $\wedge T^*X$.

An immediate consequence of this definition is that if D is Dirac-type, then D^2 is a *generalised Laplacian*, that is, a second order differential operator such that in local coordinates,

$$D^2 = -g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + \text{lower order terms}.$$

The basic analytic properties of a Dirac-type operator are summarised as follows:

Proposition 2.1.5. *Let X be a compact oriented Riemannian manifold, let $\mathcal{E} \to X$ be a Clifford module, and let D be a symmetric Dirac-type operator on $\mathcal{E}$.*

1. *D is essentially self-adjoint with smooth core $C^\infty(X, \mathcal{E}) = \cap_k \mathrm{Dom}\, D^k$, and for each k, $\mathrm{Dom}\, D^k$ yields the k-th Sobolev space $H^k(X, \mathcal{E})$ of $\mathcal{E} \to X$.*

2. *D is elliptic as a partial differential operator.*

3. *D has* compact resolvent*, namely, $(D^2 + 1)^{-1/2}$ is a compact operator. In particular, D has pure point spectrum, and the eigenvalues of D have finite multiplicity and accumulate only at infinity.*

Much deeper is the following, the relevant version of Weyl's famous result:

Theorem 2.1.6 (Weyl's law for Dirac-type operators)**.** *Let X be a compact oriented p-Riemannian manifold, let $\mathcal{E} \to X$ be a Clifford module, and let D be a symmetic Dirac-type operator on $\mathcal{E}$. Let λ_k denote the k-th eigenvalue of the positive operator D^2 in increasing order, counted with multiplicity. Then*

$$\sum_k e^{-t\lambda_k} = (4\pi t)^{-p/2} \operatorname{rank}(\mathcal{E}) \operatorname{vol}(M) + O(t^{-p/2+1}), \quad t \to +\infty.$$

In particular, $\lambda_k = O(k^{-2/p})$ *as* $k \to +\infty$.

For more on Dirac-type operators, see the monographs by Roe [61] and by Berline, Getzler and Vergne [2], as well as the notes by Roepstorff and Vehns [62, 63]. Note that [2] and [62, 63] consider only odd Dirac-type operators on $\mathbb{Z}_2$-graded Clifford module bundles, a restriction that is unnecessary for our purposes.

2.2 Spin$^{\mathbb{C}}$ and spin manifolds

We now recall Plymen's noncommutative-geometric characterisation of spin and spin$^{\mathbb{C}}$ manifolds in differential geometric language. To state it, we shall need the following definition:

Definition 2.2.1. Let X be a compact manifold.

1. A *finite rank Azumaya bundle* is a bundle $\mathcal{C} \to X$ with fibre $M_n(\mathbb{C})$ and structure group $PU(n)$ for some n, where the projective unitary group $PU(n)$ acts on $M_n(\mathbb{C})$ by $([u], a) \mapsto uau^*$ for $[u] \in PU(n)$ and $a \in M_n(\mathbb{C})$.

2. Let $\mathcal{C} \to X$ be a finite rank Azumaya bundle. Then a Hermitian vector bundle $\mathcal{E} \to X$ is called an *irreducible* $\mathcal{C}$*-module* if $\mathrm{End}(\mathcal{E}) \cong \mathcal{C}$, where the isomorphism is given fibrewise by an adjoint-preserving algebra automorphism of $M_n(\mathbb{C})$.

Now, recall that if X is a compact oriented Riemannian manifold, we may define a finite rank Azumaya bundle $\mathbb{C}\mathrm{l}^{(+)}(X) \to X$ by

$$\mathbb{C}\mathrm{l}^{(+)}(X) := \begin{cases} \mathbb{C}\mathrm{l}(X), & \text{if } \dim X \text{ is even,} \\ \mathbb{C}\mathrm{l}^{+}(X), & \text{if } \dim X \text{ is odd,} \end{cases}$$

where $\mathbb{C}\mathrm{l}(X)$ is the Clifford bundle of X, formed from the cotangent bundle T^*X. The bundle $\mathbb{C}\mathrm{l}^{(+)}(X)$ admits a canonical $\mathbb{C}$-linear anti-involution τ defined by

$$\tau(\xi_1 \cdots \xi_m) := (-1)^m \xi_m \cdots \xi_1, \quad m \in \begin{cases} \mathbb{N}, & \text{if } \dim X \text{ is even,} \\ 2\mathbb{N}, & \text{if } \dim X \text{ is odd,} \end{cases} \quad \xi_i \in \Omega^1(X),$$

so that if $\mathcal{E}$ is a $\mathbb{C}\mathrm{l}^{(+)}(X)$-module with $\mathbb{C}\mathrm{l}^{(+)}(X)$-action denoted by

$$c : \mathbb{C}\mathrm{l}^{(+)}(X) \to \mathrm{End}(\mathcal{E}),$$

then the dual bundle $\mathcal{E}^\vee$ is also a $\mathbb{C}\mathrm{l}^{(+)}(X)$-module with $\mathbb{C}\mathrm{l}^{(+)}(X)$-action given by

$$c^\vee(\omega) := c(\tau(\omega))^T, \quad \omega \in C^\infty(X, \mathbb{C}\mathrm{l}^{(+)}(X)).$$

Plymen's characterisation then takes the following form:

Theorem 2.2.2 (Plymen [54, § 2]). *Let X be a compact oriented Riemannian manifold.*

1. *X is spin$^{\mathbb{C}}$ if and only if there exists an irreducible $\mathbb{C}\mathrm{l}^{(+)}(X)$-module, which is precisely a spin$^{\mathbb{C}}$ spinor bundle.*
2. *X is spin if and only if there exists an irreducible $\mathbb{C}\mathrm{l}^{(+)}(X)$-module $\mathcal{S}$ such that $\mathcal{S} \cong \mathcal{S}^{\vee}$ as irreducible $\mathbb{C}\mathrm{l}^{(+)}(X)$-modules, in which case the choice of isomorphism class of such an $\mathcal{S}$ corresponds to a choice of spin structure, for which $\mathcal{S}$ is the spinor bundle.*

Now, for $\varepsilon' = \pm 1$, define a $\mathbb{C}$-linear anti-involution $\tau_{\varepsilon'}$ on $\mathbb{C}\mathrm{l}(X)$ by

$$\tau_{\varepsilon'}|_{\Omega^1(X)} = -\varepsilon' \operatorname{Id}_{\Omega^1(X)};$$

by construction, $\tau_{\varepsilon'}$ defines an extension to $\mathbb{C}\mathrm{l}(X)$ of τ on $\mathbb{C}\mathrm{l}^{(+)}(X)$. Since for a Hermitian vector bundle $\mathcal{E}$, the dual bundle $\mathcal{E}^{\vee}$ is canonically isomorphic to the conjugate bundle $\overline{\mathcal{E}}$, it is traditional in the noncommutative-geometric literature to reformulate Plymen's characterisation of spin manifolds as follows:

Corollary 2.2.3 (cf. [36, Thms. 9.6, 9.20]). *Let X be a compact oriented Riemannian n-manifold. Then X is spin if and only if there exists an irreducible Clifford module $\mathcal{S}$ together with an antiunitary bundle endomorphism C on $\mathcal{S}$ satisfying*

1. *$C^2 = \varepsilon \operatorname{Id}_{\mathcal{S}}$,*
2. *$Cc(\omega^*)C^* = c(\tau_{\varepsilon'}(\omega))$ for all $\omega \in C^{\infty}(X, \mathbb{C}\mathrm{l}(X))$,*
3. *$C\chi = \varepsilon''\chi C$ for $\chi \in C^{\infty}(X, \mathbb{C}\mathrm{l}(X))$ the chirality element, when n is even,*

where $(\varepsilon, \varepsilon', \varepsilon'') := (\varepsilon(n), \varepsilon'(n), \varepsilon''(n)) \in \{\pm 1\}^3$ are determined by $n \bmod 8$ as follows (with $\varepsilon'' \equiv 1$ is suppressed for n odd):

n	0	1	2	3	4	5	6	7
$\varepsilon(n)$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
$\varepsilon'(n)$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
$\varepsilon''(n)$	$+$		$-$		$+$		$-$	

(2.2.1)

The above folkloric result is the origin of Connes's notion of real structures on spectral triples, to be discussed in § 4.3 and, in particular, the above table is the origin of the notion of the KO-dimension of a real spectral triple.

Remark 2.2.4. Condition (2) in the above result can be viewed as specifying the compatibility of C with the Clifford action on $\mathcal{S}$, for C, *a priori*, defines a $\mathbb{C}$-linear anti-involution $T \mapsto CT^*C^*$ on $\operatorname{End}(\mathcal{S})$.

Remark 2.2.5. Suppose that X is spin, and that $\mathcal{S}$ and C are as above. Then $\not{D}C = \varepsilon' C \not{D}$ for $\not{D}$ the Dirac operator on $\mathcal{S}$.

Finally, suppose that X is a compact spin n-manifold for n even, and that $\mathcal{S}$ and C are as given in the above corollary; in particular, we necessarily have that $\varepsilon' = 1$. Let $C_- = C\chi$. Then C_- is an antiunitary bundle automorphism on $\mathcal{S}$ satisfying

1. $C_-^2 = \varepsilon_- \operatorname{Id}_{\mathcal{S}}$,
2. $C_- c(\omega^*) C_-^* = c(\tau_{\varepsilon'}(\omega))$ for all $\omega \in C^\infty(X, \mathbb{C}\mathrm{l}(X))$,
3. $C_- \chi = \varepsilon''_- C_- \chi$, when n is even,

for $(\varepsilon_-, \varepsilon'_-, \varepsilon''_-) := (\varepsilon\varepsilon'', -1, \varepsilon'')$. Thus, as Dąbrowski–Dossena first observed, one could readily expand the above table to

n	0_+	0_-	1	2_+	2_-	3	4_+	4_-	5	6_+	6_-	7
$\varepsilon(n)$	$+$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$+$	$-$	$+$
$\varepsilon'(n)$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$
$\varepsilon''(n)$	$+$	$+$		$-$	$-$		$+$	$+$		$-$	$-$	

(2.2.2)

where for n even, n_+ and n_- denote the two (interchangeable!) possibilities, namely n_+ the "conventional" KO-dimension and n_- the new "exotic" KO-dimension. Since replacing C with $C\chi$ takes us reversibly between n_+ and n_- [32, § 2.3], the "exotic" KO-dimensions would seem to offer nothing more than additional notational flexibility. However, as Dąbrowski–Dossena show, we will need to consider both possibilities simultaneously in order to define consistently products of real spectral triples in Section 6.3.

Chapter 3

Noncommutative-geometric preliminaries

...Heureusement pour la réputation de l'astéroïde B 612, un dictateur turc imposa à son peuple, sous peine de mort, de s'habiller à l'Européenne. L'astronome refit sa démonstration en 1920, dans un habit très élégant. Et cette fois-ci tout le monde fut de son avis.

— A. de Saint-Exupéry, *Le Petit Prince*

We now review the noncommutative-geometric background for the description and discussion of our work. Gracia-Bondía–Várilly–Figueroa [36] is still the standard reference, though Khalkhali's recent introductory text [38] provides a more focussed and accessible account.

3.1 Noncommutative topology

Let us begin by recalling some basic definitions from the theory of C^*-algebras, so that we can state Gel'fand–Naĭmark duality, the starting point for functional-analytic noncommutative geometry.

The basic notion is that of a *C^*-algebra*, which, we shall soon see, can be interpreted as a noncommutative topological space.

Definition 3.1.1. A *C^*-algebra* if a $\mathbb{C}$-algebra A together with:

1. a norm $\|\cdot\|$ on A making $(A, \|\cdot\|)$ into a Banach space, such that for all a, $b \in A$, $\|ab\| \leq \|a\| \|b\|$;

2. a conjugate-linear map (*involution*) $* : A \to A$, such that $*^2 = \mathrm{Id}_A$ and, for all a, $b \in A$, $(ab)^* = b^* a^*$;

such that for all $a \in A$, $\|a^* a\| = \|a\|^2$.

A *Fréchet pre-C^*-algebra* is a a dense $*$-closed subalgebra A^∞ of a C^*-algebra A equipped with a family of submultiplicative seminorms making A^∞ into a nuclear Fréchet space such that the multiplication, involution, and the inclusion $A^\infty \hookrightarrow A$ are all continuous on A^∞ as a Fréchet space. One usually requires a Fréchet pre-C^*-algebra to be *closed under the holomorphic functional calculus*, a condition equivalent to requiring that $a \in A^\infty$ be invertible in A^∞ whenever it is invertible in A, with the same inverse.

Remark 3.1.2. In the absence of a norm, we shall call such an algebra simply a *$*$-algebra*.

Remark 3.1.3. In addition to complex $*$-algebras, one also encounters real $*$-algebras in the noncommutative-geometric literature, especially in applications to theoretical high energy physics, e.g., [18]. Everything we do can be made to accommodate real $*$-algebras, though for simplicity of exposition, we shall only consider complex $*$-algebras.

For our purposes, the following is the canonical example of a C^*-algebra:

Example 3.1.4. Let X be a compact Hausdorff space, and let $C(X)$ be the $\mathbb{C}$-algebra of continuous complex-valued functions on X. Then $C(X)$ is a commutative unital C^*-algebra for the supremum norm

$$\|f\| := \sup_{x\in X} |f(x)|\,, \quad f \in C(X),$$

the involution

$$f^* := \left(x \mapsto \overline{f(x)}\right), \quad f \in C(X),$$

and the unit

$$1_{C(X)} := (x \mapsto 1)\,.$$

Moreover, if X is a smooth manifold, then $C^\infty(X)$ is a Fréchet pre-C^*-algebra in $C(X)$, for instance, with submultiplicative seminorms given by the Sobolev norms on the Sobolev spaces $H^k(X) \supset C^\infty(X)$.

From a more general, noncommutative-geometric standpoint, the following is the quintessential example of a noncommutative C^*-algebra:

Example 3.1.5. Let H be a Hilbert space, and let $B(H)$ be the $\mathbb{C}$-algebra of continuous linear operators $H \to H$. Then $B(H)$ is a unital C^*-algebra for the operator norm

$$\|A\| := \sup_{0\neq\xi\in H} \frac{\|A\xi\|}{\|\xi\|}, \quad A \in B(H),$$

the involution $*$ given by taking the adjoint of an operator, and the unit $1_{B(H)} := \mathrm{Id}_H$.

Remark 3.1.6. By the (i.e., another) Gel'fand–Naĭmark theorem [34], any abstract C^*-algebra can be realised, up to isomorphism, as a C^*-subalgebra (*viz*, norm-closed, $*$-closed subalgebra) of $B(H)$

for some Hilbert space H.

The following, then, turns out to be the correct notion of morphism for C^*-algebras.

Definition 3.1.7. Let A and B be unital C^*-algebras. A unital $\mathbb{C}$-algebra homomorphism $\phi : A \to B$ is called an *$*$-homomorphism* if for any $a \in A$, $\phi(a^*) = \phi(a)^*$; if A and B are unital, we require that $\phi(1) = 1$.

Example 3.1.8. Let X and Y be compact Hausdorff spaces, and let $f : Y \to X$ be continuous. Then $C(f) : C(X) \to C(Y)$ defined by $C(f)(a) := a \circ f$ is a $*$-homomorphism.

Examples 3.1.4 and 3.1.8 are highly suggestive of a relation between the category of compact Hausdorff spaces and continuous maps and the category of commutative unital C^*-algebras and $*$-homomorphisms. The relation, which is as strong as could possibly be hoped for, is given by Gel'fand–Naĭmark duality:

Theorem 3.1.9 (Gel'fand–Naĭmark [34])**.** *Let* $\mathsf{CpctHaus}$ *denote the category of compact Hausdorff spaces and continuous maps, and let* $\mathsf{CommC^*Alg_1}$ *denote the category of commutative unital C^*-algebras and $*$-homomorphisms. Then the cofunctor C :* $\mathsf{CpctHaus} \to \mathsf{CommC^*Alg_1}$ *defined by*

$$X \mapsto C(X), \quad (f : Y \to Z) \mapsto (C(f) : C(Z) \to C(Y))$$

is a contravariant equivalence of categories.

From this, we see that the category of C^*-algebras can be cofunctorially identified with a category of noncommutative topological spaces. Operator-algebraic noncommutative geometry, in general, can be considered as an attempt to expand this *Gel'fand–Nauımark paradigm* to cover as much of topology and geometry as possible.

3.2 Noncommutative vector bundles

In this section, we recall how to extend the Gel'fand–Naĭmark paradigm to accomodate vector bundles. The key observation is that for $\mathcal{E} \to X$ a vector bundle over a compact Hausdorff space X, the vector space of $C(X, \mathcal{E})$ is a finitely-generated projective module over $C(X)$. One way to see this is to recall that $\mathcal{E} \to X$ is necessarily a direct summand of some globally trivial vector space $X \times V$, which implies that $C(X, \mathcal{E})$ is a direct summand, as a $C(X)$-module, of $C(X, X \times V) \cong C(X) \otimes_{\mathbb{C}} V$, which is manifestly free. Moreover, a bundle map $\phi : \mathcal{E} \to \mathcal{F}$ covering Id_X immediately defines a $C(X)$-linear map of $C(X)$-modules by

$$C(X, \mathcal{E}) \ni \xi \mapsto C(X, \phi)\xi := (x \mapsto \phi(\xi(x))) \,.$$

In a manner directly analogous to Gel'fand–Naĭmark, these observations motivate the following result of Swan, adapting to the topological context an algebro-geometric result of Serre:

Theorem 3.2.1 (Serre–Swan [64])**.** *Let X be a compact Hausdorff space, let $\mathsf{CVect}(X)$ be the category of continuous vector bundles over X and bundle maps covering Id_X, and let $\mathsf{FGPMod}(C(X))$ be the category of finitely generated projective $C(X)$-modules and $C(X)$-linear maps. Then the functor $C(X,\cdot) : \mathsf{CVect}(X) \to \mathsf{FGPMod}(C(X))$ defined by*

$$\mathcal{E} \mapsto C(X,\mathcal{E}), \quad (\phi : \mathcal{E} \to \mathcal{F}) \mapsto (C(X,\phi) : C(X,\mathcal{E}) \to C(X,\mathcal{F}))$$

is an equivalence of categories.

Suppose, moreover, that X is a smooth manifold. Let $\mathsf{Vect}(X)$ be the category of smooth vector bundles over X and smooth bundle maps covering Id_X, and let $\mathsf{FGPMod}(C^\infty(X))$ be the category of finitely generated projective $C^\infty(X)$-modules and $C^\infty(X)$-linear maps. Then the functor $C^\infty(X,\cdot) : \mathsf{Vect}(X) \to \mathsf{FGPMod}(C^\infty(X))$ defined by

$$\mathcal{E} \mapsto C^\infty(X,\mathcal{E}), \quad (\phi : \mathcal{E} \to \mathcal{F}) \mapsto (C^\infty(X,\phi) : C^\infty(X,\mathcal{E}) \to C^\infty(X,\mathcal{F}))$$

is an equivalence of categories.

Of particular importance are Hermitian vector bundles; the following definition will provide the correct noncommutative-geometric analogue thereof.

Definition 3.2.2. Let A be a $*$-algebra. Then a *pre-Hilbert A-module* is a finitely generated projective right A-module E together with a *Hermitian metric*, a sesquilinear map $(\cdot,\cdot) : E \times E \to A$ such that:

1. for all ξ, $\eta \in E$, $a \in A$, $(\xi, \eta a) = (\xi, \eta)\, a$,
2. for all ξ, $\eta \in E$, $(\xi, \eta) = (\eta, \xi)^*$,
3. for all $\xi \in E$, $(\eta, \eta) > 0$ is positive (i.e., takes the form $b^* b$ for some $b \in A$);

we use the mathematical physics convention that $(\cdot,\cdot)$, like our inner products, is linear in the second argument and conjugate-linear in the first.

Moreover, if A is a C^*-algebra and E is complete in the norm $E \ni \xi \mapsto \sqrt{\|(\xi,\xi)\|}$, then E is called a *Hilbert A-module*.

For instance, if $\mathcal{E} \to X$ is a Hermitian vector bundle, then the Hermitian metric on $C^\infty(X,\mathcal{E})$ is given by

$$\forall \xi, \eta \in C^\infty(X,\mathcal{E}), \quad (\xi,\eta) := (x \mapsto (\xi_x, \eta_x)_x)\,.$$

The appropriate refinement of Serre–Swan, therefore, is as follows:

Corollary 3.2.3. *Let X be a compact Hausdorff space, let $\mathsf{CHVect}(X)$ be the category of continuous Hermitian vector bundles over X and bundle maps covering Id_X, and let $\mathsf{Hilb}(C(X))$ be the category of Hilbert $C(X)$-modules and $C(X)$-linear maps. Then the functor $C(X,\cdot) : \mathsf{CHVect}(X) \to \mathsf{Hilb}(C(X))$ defined by*

$$\mathcal{E} \mapsto C(X,\mathcal{E}), \quad (\phi : \mathcal{E} \to \mathcal{F}) \mapsto (C(X,\phi) : C(X,\mathcal{E}) \to C(X,\mathcal{F}))$$

is an equivalence of categories.

Suppose, moreover, that X is a smooth manifold. Let $\mathsf{HVect}(X)$ be the category of smooth Hermitian vector bundles over X and smooth bundle maps covering Id_X, and let $\mathsf{PreHilb}(C^\infty(X))$ be the category of pre-Hilbert $C^\infty(X)$-modules and $C^\infty(X)$-linear maps. Then the functor $C^\infty(X,\cdot) : \mathsf{HVect}(X) \to \mathsf{PreHilb}(C^\infty(X))$ defined by

$$\mathcal{E} \mapsto C^\infty(X,\mathcal{E}), \quad (\phi : \mathcal{E} \to \mathcal{F}) \mapsto (C^\infty(X,\phi) : C^\infty(X,\mathcal{E}) \to C^\infty(X,\mathcal{F}))$$

is an equivalence of categories.

3.3 Noncommutative de Rham complexes

Now we recall what will turn out to be the relevant way to generalise the de Rham complex of a manifold in noncommutative geometry. The main definition is the following, adapted by Connes from the corresponding definition in commutative algebra:

Definition 3.3.1. Let A be an algebra. Then the *Hochschild complex* of A is the chain complex $(C_\bullet(A,A), b)$ defined as follows:

- For $n \geq 0$, $C_n(A,A) := A^{\otimes(n+1)}$.
- One has that $b_0 : C_0(A,A) \to 0$ is the zero map, whilst for $n > 0$, $b_n : C_n(A,A) \to C_{n-1}(A,A)$ is given by

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := a_0a_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{k=1}^{n-1} a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n \\ + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

The *Hochschild homology* $HH_\bullet(A)$ is therefore the homology of this chain complex; in particular, $Z_n(A,A) = \operatorname{Ker} b_n$ is the space of *Hochschild n-cycles.* If A is a Fréchet pre-C^*-algebra, we take the $A^{\otimes p}$ to be topological tensor products of nuclear Fréchet spaces.

Theorem 3.3.2 (Hochschild–Kostant–Rosenberg–Connes [20]). *Let X be a compact manifold. Then the maps $\mu_n : C^\infty(X)^{\otimes(n+1)} \to \Omega^n(X)$ defined by*

$$\mu_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := a_0 da_1 \wedge \cdots \wedge da_n, \quad a_k \in C^\infty(X),$$

define a morphism of complexes $\mu : (Z_\bullet(C^\infty(X), C^\infty(X)), b) \to (\Omega^\bullet(X), 0)$ that descends to an isomorphism $HH_\bullet(C^\infty(X)) \cong \Omega^\bullet(X)$.

Remark 3.3.3. In order to recover the de Rham *cohomology* of a manifold X from the Hochschild homology of $C^\infty(X)$, one needs to pass to a refinement of Hochschild homology called *periodic cyclic homology.*

Finally, for future reference, we shall record the following definition:

Definition 3.3.4. A p-cycle $c := \sum_{i=1}^n a_{i,0} \otimes a_{i,1} \otimes \cdots \otimes a_{i,p} \in Z_p(A, A)$ is *antisymmetric* if for all permutations $\sigma \in S_p$ on p elements, $\pi(c) := \sum_{i=1}^n a_{i,0} \otimes a_{i,\sigma(1)} \otimes \cdots \otimes a_{i,\sigma(p)}$ satisfies $\pi(c) = (-1)^\pi c$.

3.4 Noncommutative integration

Finally, we recall what will turn out to be the relevant way to generalise integration on a manifold. In short, we shall define a distinguished ideal of noncommutative "integrands," within the ideal of compact operators on a Hilbert space and containing all the trace-class operators, and then define a notion of "integration" of such "integrands" by defining a class of traces on this ideal that vanish on trace-class operators.

Before continuing, let us recall the following standard definition from functional analysis:

Definition 3.4.1. Let H be a Hilbert space, and let $0 < p < \infty$. Then the *Schatten p-class* $\mathcal{L}^p(H)$ is the ideal in $B(H)$ of all $T \in \mathcal{K}(H)$, the ideal of all compact operators on H, such that $\sum_{k=0}^\infty \sigma_k(T)^p < \infty$, where $\sigma_k(T)$ denotes the k-th singular value of T in decreasing order, counted with multiplicity. In particular, if $p \geq 1$, then $\mathcal{L}^p(X)$ is a Banach space with norm $\|T\| := \left(\sum_{k=0}^\infty \sigma_k(T)^p\right)^{1/p}$ for $T \in \mathcal{L}^p(H)$.

Example 3.4.2. One has that $\mathcal{L}^1(H)$ is the ideal of all trace-class operators with $\|T\|_1 := \operatorname{Tr}|T|$ for all $T \in \mathcal{L}^1(H)$, and that $\mathcal{L}^2(H)$ is the ideal of all Hilbert–Schmidt operators, with $\|T\|_2 = \operatorname{Tr} T^*T$ for all $T \in \mathcal{L}^2(H)$.

The Schatten ideals were originally defined as a sort of proto-noncommutative-geometric analogue of L^p spaces. Indeed, as befits such an analogue, the relevant analogue of Hölder's inequality applies:

Proposition 3.4.3 (Hölder's inequality). *Let α, r, $s > 0$ with $r^{-1} + s^{-1} = \alpha^{-1}$. Then for all $A \in \mathcal{L}^r(H)$ and $B \in \mathcal{L}^s(H)$, $AB \in \mathcal{L}^\alpha(H)$, and if α, r, $s > 1$, then $\|AB\|_\alpha \leq \|A\|_r \|A\|_s$. Moreover, if $A \in \mathcal{L}^1(H)$ and $B \in B(H)$, then $\|AB\| \leq \|A\|_1 \|B\|$.*

A key technical result related to Schatten ideals that we shall need later is the so-called BKS inequality, part of which is quoted below:

Theorem 3.4.4 (Birman–Koplienko–Solomjak [3, Thm. 1])**.** *Let A and $B \in B(H)$ be positive with $A - B$ compact. Then for any $0 < \alpha < 1$, and $p \geq 1$, if $|A-B|^{\alpha} \in \mathcal{L}^p(H)$ then $A^{\alpha} - B^{\alpha} \in \mathcal{L}^p(H)$.*

As it turns out, however, none of the Schatten ideals provide us with the ideal of noncommutative "integrands" that we want. The correct definition turns out to be as follows:

Definition 3.4.5. Let H be a Hilbert space. Then the *Dixmier ideal* $\mathcal{L}^{1+}(H)$ is the ideal in $B(H)$ of all $T \in \mathcal{K}(H)$ such that

$$\|T\|_{1+} := \sup_{k\geq 2} \frac{\sigma_k(T)}{\log k} < \infty;$$

equivalently, $T \in \mathcal{L}^{1+}(H)$ if

$$\sum_{k=0}^{n} \sigma_k(T) = O(\log n), \quad n \to +\infty.$$

In particular, $T \mapsto \|T\|_{1+}$ defines a norm making $\mathcal{L}^{1+}(H)$ into a Banach space.

Example 3.4.6. If $A \in \mathcal{K}(H)$ satisfies $\sigma_k(A) = O(k^{-1})$ as $k \to +\infty$, then $A \in \mathcal{L}^{1+}(H)$.

Remark 3.4.7. For all $1 < p < q$, one has inclusions $\mathcal{L}^1(H) \subset \mathcal{L}^{1+}(H) \subset \mathcal{L}^p(H) \subset \mathcal{L}^q(H)$.

We can now define the notion of noncommutative integration that we shall use:

Definition 3.4.8. Let ω be a *dilation-invariant state* on ℓ^∞, *viz*, a positive linear functional, such that

1. $\omega(a) = c$ if $\lim_{n\to\infty} a_n = c$,
2. $\omega \circ \sigma = \omega$, where $\sigma \in B(\ell^\infty)$ is the dilation $\sigma(a) := (a_1, a_1, a_2, a_2, \dots)$.

For $T \in \mathcal{L}^{1+}(H)$ positive, set

$$\mathrm{Tr}_\omega(T) := \omega\left(\left\{\frac{1}{\log(n)} \sum_{k=0}^{n} \lambda_k(T)\right\}_{n=2}^{\infty}\right).$$

Then $T \mapsto \mathrm{Tr}_\omega(T)$ extends to a unitarily invariant positive linear functional on $\mathcal{L}^{1+}(H)$, vanishing on $\mathcal{L}^1(H)$, called the *Dixmier trace* induced by ω.

In what follows, we shall fix a single dilation-invariant state ω on ℓ^∞, and we shall denote $\mathrm{Tr}_\omega(T)$ by $⨍ T$.

Remark 3.4.9. Let $T \in \mathcal{K}(H)$, and suppose that T is *measurable*, that is, $\lim_{k\to\infty} \frac{\sigma_k(T)}{\log n}$ exists. Then, not only is $T \in \mathcal{L}^{1+}(H)$, but necessarily $\mathrm{Tr}_\omega(T) = \lim_{k\to\infty} \frac{\sigma_k(T)}{\log k}$ for all dilation-invariant states ω on ℓ^∞. In the sequel, virtually every operator in $\mathcal{L}^{1+}(H)$ of interest will be measurable, at least *a posteriori*.

The justification for our identification of the theory of Dixmier traces as a theory of noncommutative integration is provided by the following result, a special case of a more general result, the so-called Connes trace formula:

Theorem 3.4.10 (Connes [21], cf. [37, Thm. 3.23])**.** *Let X be a compact oriented Riemannian p-manifold and let D be a symmetric Dirac-type operator on a Clifford module $\mathcal{E} \to X$. Then for any $B \in C^\infty(X, \operatorname{End}(\mathcal{E}))$, $B(D^2+1)^{-p/2} \in \mathcal{L}^{1+}(L^2(X,\mathcal{E}))$ is measurable, and*

$$⨍ B(D^2+1)^{-p/2} = \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2}+1)} \int_M \operatorname{tr}(B(x))d\operatorname{vol}(x),$$

where $\operatorname{tr}(B(x))$ denotes the trace of $B(x) \in \operatorname{End}_{\mathbb{C}}(\mathcal{E}_x)$.

Chapter 4

Commutative and Dirac-type spectral triples

O wear your tribulations like a rose.

— W. H. Auden, *Anthem for St. Cecilia's Day*

In this section, we introduce *spectral triples*, Connes's proposal for a notion of noncommutative smooth manifold, with an emphasis on commutative spectral triples. In particular, we recall Connes's *reconstruction theorem*, which we shall recognise as providing precisely a (not yet functorial) noncommutative-geometric characterisation of compact oriented Riemannian manifolds together with a Dirac-type operator. Then we shall introduce real structures on spectral triples, again with an emphasis on commutative spectral triples, and then finally we shall propose a close noncommutative generalisation of the definition of commutative spectral triple, which we shall later see is applicable to toric noncommutative manifolds.

Recommended references for the basic theory of spectral triples, especially in relation to Dirac-type operators, are the detailed lecture notes of Várilly [67, 68] and Landsman [47], and the excellent expository article of Carey–Phillips–Rennie [13].

4.1 Commutative spectral triples

Let us begin by introducing the basic definitions of the theory of spectral triples. The starting point is the following definition, which encapsulates the analytic behaviour of Dirac-type operators:

Definition 4.1.1. A *spectral triple* is a triple (A, H, D), where A is a unital $*$-algebra faithfully represented on a Hilbert space H, and D, the *Dirac operator*, is an essentially self-adjoint operator with compact resolvent on H, such that $[D, a] \in B(H)$ for all $a \in A$. Moreover:

- (A, H, D) is called *even* if there exists a self-adjoint unitary $\gamma \in B(H)$ such that $a\gamma = \gamma a$ for all $a \in A$ and $D\gamma = -\gamma D$;

- (A, H, D) is said to have *metric dimension* $p > 0$ if $\lambda_k((D^2+1)^{-1/2}) = O(k^{-1/p})$ as $k \to +\infty$, where $\lambda_k(T)$ denotes the k-th eigenvalue in decreasing order, counted with multiplicity, of a positive compact operator T;

- (A, H, D) is said to be *pre-regular* if for $H^\infty := \cap_k \operatorname{Dom} D^k$, $AH^\infty \subset H^\infty$;

- (A, H, D) is said to be *regular* if $A + [D, A] \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$.

Finally, two spectral triples (A_1, H_1, H_1) and (A_2, H_2, D_2) are said to be *unitarily equivalent* (written $(A_1, H_1, D_1) \cong (A_2, H_2, D_2)$) if there exists a unitary isomorphism $U : H_1 \to H_2$ such that $UD_1U^* = D_2$ and $a \mapsto UaU^*$ defines an isomorphism $A_1 \to A_2$ of C^*-algebras.

Remark 4.1.2. It follows from the definition of unitary equivalence that unitarily equivalent spectral triples share all qualitative behaviour.

An essential point in the sequel is that a spectral triple (A, H, D) gives rise, for each p, to a representation $\pi_D : Z_p(A, A) \to \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k \subset B(H)$ of Hochschild p-cycles on H by

$$\pi_D(a_0 \otimes a_1 \otimes \cdots a_p) := a_0[D, a_1] \cdots [D, a_p], \quad a_k \in A.$$

Now, in the case of a compact oriented Riemannian manifold X, a Clifford module $\mathcal{E} \to X$, and a symmetric Dirac-type operator D, $(C^\infty(X), L^2(X, \mathcal{E}), D)$ is a spectral triple, since D is essentially self-adjoint with compact resolvent, and since for all $a \in A$, $[D, a] = c(da)$ is a bundle endomorphism of $\mathcal{E}$, and hence, in particular, a bounded operator on $L^2(X, \mathcal{E})$. However, Connes observed almost immediately [24], the *concrete commutative spectral triple* $(C^\infty(X), L^2(X, \mathcal{E}), D)$ contains a great deal more structure than that of a spectral triple *simpliciter*. The following is our slight modification (cf. [9, Def. 2.7]) of Connes's original notion of *commutative* spectral triple [24, 27]:

Definition 4.1.3. Let (A, H, D) be a pre-regular spectral triple. We call (A, H, D) a *p-dimensional commutative spectral triple* for $p \in \mathbb{N}$ if A is commutative and the following conditions hold:

1. **Dimension**: The spectral triple (A, H, D) has metric dimension p.

2. **Order one**: For any a, $b \in A$, $[[D, a], b] = 0$.

3. **Finiteness**: The A-module H^∞ is finitely generated projective.

4. **Strong regularity**: One has that $\operatorname{End}_A(H^\infty) \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$.

5. **Orientability**: There exists an antisymmetric Hochschild p-cycle $c \in Z_p(A, A)$ such that $\chi = \pi_D(c)$ is a self-adjoint unitary on H satisfying $a\chi = \chi a$ and $[D, a]\chi = (-1)^{p+1}\chi[D, a]$ for all $a \in A$.

6. **Absolute continuity**: The A-module H^∞ admits a Hermitian structure $(\cdot,\cdot)$ satisfying $\langle\xi,\eta\rangle = \fint a\,(\xi,\eta)\,(D^2+1)^{-p/2}$ for $\xi,\,\eta\in H^\infty$.

If, in addition, $\chi D + D\chi = 0$ if p is even and $\chi = 1$ if p is odd, then we call (A,H,D) *strongly orientable.*

Remark 4.1.4. What we call "strong orientability" is the orientability condition originally proposed by Connes, which models commutative spectral triples specifically on the Dirac operator $\not{D}$ of a compact spin manifold; the orientability condition above was first proposed in the final arXiv version of [9], in order to accommodate correctly general Dirac-type operators on compact oriented Riemannian manifolds. Note that the orientability condition proposed in the published version of [9] is insufficiently general in the odd-dimensional case, for it only accommodates Clifford modules that are locally twisted spinor bundles.

That a concrete commutative spectral triple is indeed a commutative spectral triple in this abstract sense is then guaranteed by the following standard, indeed folkloric, result:

Proposition 4.1.5 (cf. Gracia-Bondía–Várilly–Figueroa [36, Thm. 11.1], Connes [27, Thm. 11.4])**.** *Let X be a compact oriented Riemannian p-manifold, let $\mathcal{E}\to X$ be a Hermitian vector bundle, and let D be a symmetric Dirac-type operator. Then $(C^\infty(X), L^2(X,\mathcal{E}), D)$ is a p-dimensional commutative spectral triple.*

Let us briefly sketch out the main points of the proof:

1. That $(C^\infty(X), L^2(X,\mathcal{E}), D)$ has metric dimension p is simply Weyl's law applied to the generalised Laplacian D^2.

2. The order one condition follows precisely since D is a first-order differential operator.

3. Finiteness follows since $\cap_k \operatorname{Dom} D^k = C^\infty(X,\mathcal{E})$, which in turn follows from the Sobolev theory applied to the elliptic operator D.

4. Strong regularity is a consequence of the Sobolev theory of the elliptic operator D on the compact manifold X; the point is that $\operatorname{Dom} D^k = \operatorname{Dom}|D|^k = H^k(X,\mathcal{E})$, the k-th Sobolev space of sections of $\mathcal{E}$, and that if $T\in C^\infty(X,\operatorname{End}(\mathcal{E}))$, then $T\in B(H^k(X,\mathcal{E}))$, and hence $T\in\operatorname{Dom}(\operatorname{ad}|D|)^k$.

5. Orientability follows from [27, proof of Thm. 11.4], where the Hochschild cycle c is constructed from the volume form on X, and thus acts as the chirality operator.

6. Absolute continuity follows from the Connes trace formula, Theorem 3.4.10, applied to pseudodifferential operators on the Hermitian vector bundle H of the form $a(D^2+1)^{-p/2}$, where $a\in C^\infty(X)$ (see [36, § 9.4; 61, Chapter 8] for details).

In general, then, any Dirac-type operator on a compact oriented Riemannian manifold gives rise to a commutative spectral triple. Let us single out two examples of particular significance:

Example 4.1.6. Let X be a compact spin manifold with fixed spinor structure, let $\mathcal{S} \to X$ be the spinor bundle, and let $\not{D}$ be the Dirac operator on $\mathcal{S}$. Then $(C^\infty(X), L^2(X,\mathcal{S}), \not{D})$ is the *canonical* commutative spectral triple of X.

Example 4.1.7 (cf. [27, Proof of Thm. 11.4])**.** Let X be a compact oriented Riemannian manifold. Then $\wedge T^*X \to X$ is a Clifford module for the Clifford action

$$c(\xi)\eta := \xi \wedge \eta - i_\xi \eta, \quad \xi \in \Omega^1(X),\ \eta \in \Omega(X),$$

and $d + d^*$ is a symmetric Dirac-type operator on $\wedge T^*X$; the resulting commutative spectral triple $(C^\infty(X), L^2(X, \wedge T^*X), D)$ is called the *Hodge–de Rham* spectral triple of X.

4.2 Connes's reconstruction theorem

Let us now see how spectral triples can be used to adapt the Gel'fand–Naĭmark paradigm to compact oriented Riemannian manifolds.

Indeed, already in 1996, Connes conjectured [24] that one could recover a commutative manifold from a commutative spectral triple, just as one can recover a topological space from a commutative C^*-algebra *via* Gel'fand–Naĭmark. Connes finally proved his conjecture, now called the *reconstruction theorem* for commutative spectral triples, in 2008 [27], following a substantial attempt by Rennie–Várilly in 2006 [59]:

Theorem 4.2.1 (Connes [27, Thm. 1.1])**.** *Let (A, H, D) be a strongly orientable p-dimensional commutative spectral triple. Then there exists a compact oriented manifold X such that $A \cong C^\infty(X)$.*

Once one has reconstructed the manifold itself, one can proceed to reconstruct the Hermitian vector bundle too, and realise the operator D as an elliptic first-order differential operator:

Theorem 4.2.2 (Connes [24], Gracia-Bondía–Várilly–Figueroa [36, Thm. 11.2])**.** *Let (A, H, D) be a strongly orientable p-dimensional commutative spectral triple with $A \cong C^\infty(X)$ for some compact orientable manifold X. Then there exists a Hermitian vector bundle $\mathcal{E} \to X$ such that $(A, H, D) \cong (C^\infty(X), L^2(X, \mathcal{E}), D)$, where D is identified with an essentially self-adjoint elliptic first-order differential operator on $\mathcal{E}$.*

In fact, this last result was stated and proved in the context of reconstructing spin manifolds with spin Dirac operators. The following result of Connes' gives a concise characterisation of spin$^\mathbb{C}$ manifolds and spin$^\mathbb{C}$ Dirac operators, possibly with torsion; we shall later recall Plymen's characterisation of spin manifolds amongst spin$^\mathbb{C}$ manifolds and the resulting theory of real spectral triples.

Corollary 4.2.3 (Connes [27, Thm. 1.2])**.** *If, moreover, A'' acts on H with multiplicity $2^{\lfloor p/2 \rfloor}$ (but without needing to assume strong regularity), then X is spin$^{\mathbb{C}}$, $\mathcal{E} \to X$ is a spinor bundle, and D is a Dirac-type operator.*

However, the conclusion of Theorem 4.2.2 can be strengthened by means of the following essential technical lemma, which appears merely as an off-hand observation within Connes's proof of the reconstruction theorem[1]:

Lemma 4.2.4 ([27, p. 28])**.** *Let (A, H, D) be a strongly orientable p-dimensional commutative spectral triple. Then for any $a \in A$, $[D, a]^2 \in A$.*

Remark 4.2.5. This lemma makes redundant the Dirac-type condition proposed by the author in [9, Def. 2.9], namely, that $[D, a]^2 \in A$ for all $a \in A$, as well as the irreducibility condition proposed by Gracia-Bondía–Várilly–Figueroa [36, Def. 11.2], insofar as they used it to reconstruct the Riemannian metric on the reconstructed manifold.

Incorporating this into the proof of Theorem 4.2.2 then allows one to conclude that

$$df \mapsto -[D, f]^2, \quad f \in A \cong C^\infty(X)$$

defines a Riemannian metric on X, and hence that D is indeed a Dirac-type operator on $\mathcal{E} \to X$. In particular, D will be a Dirac-type operator inducing a Clifford action on $\mathcal{E}$ such that $D\chi + \chi D = 0$ if X is even-dimensional and $\chi = \mathrm{Id}_{\mathcal{E}}$ if X is odd-dimensional, where χ denotes the chirality operator on $\mathcal{E}$ as a Clifford module. This condition, however, fails for general Dirac-type operators:

Example 4.2.6. Let X be an even-dimensional compact spin manifold with spinor bundle $\mathcal{S} \to X$ and Dirac operator $\not{D}$, let $\mathcal{E}$ be a non-trivially $\mathbb{Z}_2$-graded Hermitian vector bundle over N, and let $\not{D}_{\mathcal{E}}$ be the twisted Dirac operator on $\mathcal{S} \widehat{\otimes} \mathcal{E}$ corresponding to any self-adjoint (super)connection on E. Then $(C^\infty(X), L^2(X, \mathcal{S} \widehat{\otimes} \mathcal{E}), \not{D}_{\mathcal{E}})$ is not strongly orientable, for any Hochschild p-cycle will act on $\mathcal{S} \widehat{\otimes} \mathcal{E}$ by a bundle endomorphism of the form

$$T \widehat{\otimes} 1 \in C^\infty(X, \mathrm{End}(\mathcal{S})) \widehat{\otimes} C^\infty(X, \mathrm{End}(\mathcal{E})) \cong C^\infty(X, \mathrm{End}(\mathcal{S} \widehat{\otimes} \mathcal{E}))$$

with T even, so that it cannot distinguish between $\mathcal{S} \widehat{\otimes} \mathcal{E}^+$ and $\mathcal{S} \widehat{\otimes} \mathcal{E}^-$, and thus cannot act as the $\mathbb{Z}_2$-grading on $S \widehat{\otimes} \mathcal{E}$.

In the case of an odd-dimensional compact oriented Riemannian manifold X, one can also readily construct Clifford modules $\mathcal{E} \to X$ such that the chirality operator defines a non-trivial $\mathbb{Z}_2$-grading on $\mathcal{E}$ [48, § II.5; 56, § 8].

[1]The author thanks Jan Jitse Venselaar for pointing out Connes's observation.

The lesson, then, is that Connes's reconstruction theorem as stated is tantalisingly close to a precise noncommutative-geometric characterisation of compact oriented Riemannian manifolds and Dirac-type operators, but just fails to be sufficiently general. Moreover, Example 4.2.6 will have the additional upshot that the reconstruction theorem, as stated, cannot be readily applied in the context of almost-commutative spectral triples. However, as we shall prove in the next chapter, one can indeed drop the strong orientability hypothesis:

Corollary 4.2.7 ([9, Cor. 2.19]). *Let (A, H, D) be a p-dimensional commutative spectral triple. Then there exist a compact oriented Riemannian p-manifold X and a Hermitian vector bundle $\mathcal{E} \to X$ such that $(A, H, D) \cong (C^\infty(X), L^2(X, \mathcal{E}), D)$, where D is identified with an essentially self-adjoint Dirac-type operator on $\mathcal{E}$.*

Thus, Connes's reconstruction theorem, with the strong orientability hypothesis dropped, really is a precise (if not obviously functorial) noncommutative-geometric characterisation of compact oriented Riemannian manifolds and symmetric Dirac-type operators, just as Gel'fand–Naĭmark duality really is a precise (and robustly functorial) noncommutative-geometric characterisation of compact Hausdorff spaces.

4.3 Real structures

Let us now consider the implications of the reconstruction theorem for *real* commutative spectral triples. Before continuing, let us recall the definition of real spectral triple, which generalises the example of $(C^\infty(X), L^2(X, \mathcal{S}), \not{D}, C)$ for X a compact spin manifold with spinor bundle $\mathcal{S} \to X$, Dirac operator $\not{D}$, and charge conjugation C.

Definition 4.3.1. A *real spectral triple of KO-dimension* $n \bmod 8$ is a spectral triple (A, H, D), even with $\mathbb{Z}_2$-grading γ if n is even, together with an antiunitary J on H satisfying:

1. $J^2 = \varepsilon \operatorname{Id}_H$,
2. $DJ = \varepsilon' JD$,
3. $J\gamma = \varepsilon'' \gamma J$ (if n is even),

for $(\varepsilon, \varepsilon', \varepsilon'') := (\varepsilon(n), \varepsilon'(n), \varepsilon''(n)) \in \{\pm 1\}^3$ depending on $n \bmod 8$ according to Table 2.2.1.

Before continuing on to examples, it is worth mentioning that a real spectral triple (A, H, D, J) defines, in particular, a canonical central unital $*$-subalgebra of A, a fact that will be key to our discussion of real commutative and almost-commutative spectral triples:

Lemma 4.3.2 (cf. [18, Prop. 3.1])**.** *Let (A, H, D, J) be a real spectral triple. Then*

$$\tilde{A}_J := \{a \in A \mid Ja^*J^* = a\}$$

defines a central unital $*$-subalgebra of A.

As mentioned above, by Section 2.2, a compact spin p-manifold X with spinor bundle $\mathcal{S}$, charge conjugation C, and Dirac operator $\not{D}$ gives rise to a real spectral triple $(C^\infty(X), L^2(X, \mathcal{S}), \not{D}, C)$ of KO-dimension $p \bmod 8$, the canonical, motivating example of a real spectral triple. Indeed, we have the following well-known consequence of the reconstruction theorem for commutative spectral triples, the original form of Theorem 4.2.2:

Corollary 4.3.3 (Connes [24], Gracia-Bondía–Várilly–Figueroa [36, Thm. 11.2])**.** *Let (A, H, D) be a strongly orientable p-dimensional commutative spectral triple, such that A'' acts on H with multiplicity $2^{\lfloor p/2 \rfloor}$, so that $(A, H, D) \cong (C^\infty(X), L^2(X, \mathcal{S}), D)$ for X a compact spin$^{\mathbb{C}}$ p-manifold and $\mathcal{S} \to X$ a spinor bundle, with D identified with an essentially self-adjoint Dirac-type operator on $\mathcal{S}$. If, in addition, there exists an antiunitary J making (A, H, D, J) a real spectral triple of KO-dimension $p \bmod 8$, with $JaJ^* = a^*$ for $a \in A \cong C^\infty(X)$, then X is spin, $\mathcal{S}$ is the spinor bundle on X, J is the charge conjugation on $\mathcal{S}$, and $D = \not{D} + M$ for $\not{D}$ the Dirac operator on X and M a suitable symmetric bundle endomorphism on $\mathcal{S}$.*

Now, just as in Section 2.2, in the case that n is even, we can go reversibly from the "conventional" KO-dimension n_+ to the "exotic" KO-dimension n_- by replacing J with $J\gamma$, so that we can expand Table 2.2.1 to Table 2.2.2 for free. By abuse of notation and terminology, then, we shall say that (A, H, D, γ, J) is *of KO-dimension* $n_+ \bmod 8$ if $(\varepsilon, \varepsilon', \varepsilon'')$ is given by n_+ in the above table, and that it is *of KO-dimension* $n_- \bmod 8$ if $(\varepsilon, \varepsilon', \varepsilon'')$ is given by n_- instead. Indeed, we shall find the following definition convenient:

Definition 4.3.4. Let (A, H, D, γ, J) be a real spectral triple of KO-dimension $n \bmod 8$ for n even, and let $\beta \in \{\pm 1\}$.

- If $\beta = 1$, then J_β is the element of $\{J, J\gamma\}$ such that (A, H, D, γ, J_1) has KO-dimension $n_+ \bmod 8$;
- If $\beta = -1$, then J_- is the element of $\{J, J\gamma\}$ such that $(A, H, D, \gamma, J_{-1})$ has KO-dimension $n_- \bmod 8$.

Thus, we are free to identify a real spectral triple (A, H, D, γ, J) of even KO-dimension $n \bmod 8$ simultaneously with the real spectral triple (A, H, D, γ, J_1) of KO-dimension $n_+ \bmod 8$ and the real spectral triple $(A, H, D, \gamma, J_{-1})$ of KO-dimension $n_- \bmod 8$.

Let us now consider the case of real *commutative* spectral triples. We have already seen the example of the canonical real spectral triple of a compact spin manifold with fixed spin structure. However, the canonical spectral triple of a compact oriented Riemannian manifold (*v. supra*) immediately gives rise to a canonical real spectral triple of KO-dimension $0 \bmod 8$, a seemingly trivial example that shall prove quite instructive indeed:

Example 4.3.5. Let X be a compact oriented Riemannian manifold. Since the operators $d + d^*$ and $(-1)^{|\cdot|}$ on $\wedge T^*_{\mathbb{C}}X$ are simply straightforward $\mathbb{C}$-linear extensions of operators on the real exterior bundle $\wedge T^*X$, we can realise the Hodge–de Rham spectral triple of X as a real triple $(C^\infty(X), L^2(X, \wedge T^*_{\mathbb{C}}X), d + d^*, (-1)^{|\cdot|}, K)$ of KO-dimension $0 \bmod 8$, where K is the complex conjugation operator on $\wedge T^*_{\mathbb{C}}X$ *qua* complexification of the real vector bundle $\wedge T^*X$.

In light of this last example, we already see that a generalisation of the "charge conjugation" operator of Cor. 2.2.3 can be usefully defined on more general Clifford modules:

Definition 4.3.6. Let X be a compact oriented Riemannian manifold, let $\mathcal{E} \to X$ be a Clifford module, which may or may not be $\mathbb{Z}_2$-graded. Let J be an antiunitary bundle automorphism on $\mathcal{E}$, and let $n \in \mathbb{Z}_8$. We call $(\mathcal{E}, J)$ a *real Clifford module of KO-dimension* $n \bmod 8$ if $\mathcal{E}$ is $\mathbb{Z}_2$-graded with $\mathbb{Z}_2$-grading γ when n is even, and C satisfies the following:

1. $J^2 = \varepsilon \operatorname{Id}_{\mathcal{E}}$,
2. $Jc(\omega^*)J^* = c(\tau_{\varepsilon'}(\omega))$ for all $\omega \in C^\infty(X, \mathbb{C}\mathrm{l}(X))$,
3. $J\gamma = \varepsilon''\gamma J$ if n is even,

where $(\varepsilon, \varepsilon', \varepsilon'') \in \{\pm 1\}^3$ is determined by $n \bmod 8$ according Table 2.2.1.

Remark 4.3.7. Just as in the spinor case, if n is even, then we can replace J with $J\gamma$ to go reversibly between the "conventional" KO-dimension n_+ and the "exotic" KO-dimension n_-.

In both examples, we have a real triple of the form $(C^\infty(X), L^2(X, \mathcal{E}), D, J)$, where X is a compact oriented Riemannian manifold, $(\mathcal{E}, J)$ is a real Clifford module, and D is a Dirac-type operator on $\mathcal{E}$ compatible with J in the following sense:

Definition 4.3.8. Let $(\mathcal{E}, J)$ be a real Clifford module of KO-dimension $n \bmod 8$ over a compact oriented Riemannian manifold X. Let D be a Dirac-type operator on $\mathcal{E}$. We shall call D *J-compatible* if $DJ = \varepsilon' JD$.

Thus, if $(\mathcal{E}, J)$ is a real Clifford module of KO-dimension $n \bmod 8$ over X, and D is a Dirac-type operator on $\mathcal{E}$, then $(C^\infty(X), L^2(X, \mathcal{E}), D, J)$ is a real spectral triple of KO-dimension $n \bmod 8$ if and only if D is J-compatible—let us call such a real spectral triple a *concrete real commutative spectral triple*. One can therefore ask if a Dirac-type operator on $\mathcal{E}$ is necessarily J-compatible. As it turns out, the answer is yes, up to perturbation by a symmetric bundle endomorphism:

Proposition 4.3.9. *Let X be a compact oriented Riemannian manifold, let $(\mathcal{E}, J)$ be a real Clifford module on X of KO-dimension $n \bmod 8$, and let D be a symmetric Dirac-type operator on $\mathcal{E}$. Then there exists a unique symmetric bundle endomorphism M on $\mathcal{E}$ such that $D - M$ is a J-compatible Dirac-type operator, and $MJ = -\varepsilon' JM$.*

Proof. For any $f \in C^\infty(X, \mathbb{R})$, $J[D, f]J^* = Jc(df)J^* = \varepsilon' c(df) = \varepsilon'[D, f]$, and hence $[D - \varepsilon' JDJ^*, f] = 0$. Thus, $M = \frac{1}{2}(D - \varepsilon' JDJ^*)$ is a symmetric bundle endomorphism, so that $D - M = \frac{1}{2}(D + \varepsilon' JDJ^*)$ is a symmetric Dirac-type operator; if n is even, so that $\mathcal{E}$ is $\mathbb{Z}_2$-graded, then $D - M$ is odd since D is, and since $J^2 = \varepsilon$ commutes with D and with M, $J(D - M) = \varepsilon'(D - M)J$ and $JM = -\varepsilon' MJ$, as required.

Finally, suppose that N is another symmetric bundle endomorphism on $\mathcal{E}$ such that $(D - N)J = \varepsilon' J(D - N)$ and $NJ = -\varepsilon' JN$. Then $(D - M) - (D - N) = N - M$ both commutes and anticommutes with J, and thus must vanish. □

Finally, let us show that a real commutative spectral triple, in the appropriate abstract sense, necessarily arises from a real Clifford module together with a compatible Dirac-type operator.

Consider a concrete real commutative spectral triple $(C^\infty(X), L^2(X, \mathcal{E}), D, J)$. On the one hand, $C^\infty(X)$ is already commutative, while on the other, by Lem. 4.3.2, $C^\infty(X)$ contains a canonical central unital $*$-subalgebra

$$\widetilde{C^\infty(X)}_J := \{a \in C^\infty(X) : Ja^*J^* = a\};$$

that J is an anti-linear bundle endomorphism on $\mathcal{E}$ is precisely equivalent to the fact that $\widetilde{C^\infty(X)}_J = C^\infty(X)$. This, then, motivates the following:

Definition 4.3.10. Let (A, H, D, J) be a real spectral triple. We call (A, H, D, J) a *real commutative spectral triple* if the following hold:

1. $A = \tilde{A}_J$, or equivalently, $JaJ^* = a^*$ for all $a \in A$;
2. (A, H, D) is a commutative spectral triple.

In particular, then, a concrete real commutative spectral triple is automatically a real commutative spectral triple in this abstract sense.

Remark 4.3.11. If one wants A to correspond to $C^\infty(X, \mathbb{R})$ instead of $C^\infty(X) = C^\infty(X, \mathbb{C})$, then one should take A to be a real (Fréchet) pre-C^*-algebra with trivial $*$-operation, in which case, condition (1) corresponds simply to the commutation of J with A.

The relevant refinement of the reconstruction theorem for commutative spectral triples is, thus, the claim that a real spectral triple is real commutative, if and only if it is unitarily equivalent to a concrete real commutative spectral triple:

Proposition 4.3.12. *Let (A, H, D, J) be a real commutative spectral triple of KO-dimension $n \bmod 8$ and metric dimension p. Then there exist a compact oriented Riemannian p-manifold X and a self-adjoint Clifford module $\mathcal{E} \to X$, such that $(A, H, D, J) \cong (C^\infty(X), L^2(X, \mathcal{E}), D, J)$, where D, viewed as an operator on $L^2(X, \mathcal{E})$, is an essentially self-adjoint Dirac-type operator on $\mathcal{E}$, and where J, viewed as an operator on $L^2(X, \mathcal{E})$, makes $(\mathcal{E}, J)$ a real Clifford module of KO-dimension $n \bmod 8$ such that D is J-compatible.*

Proof. Suppose that (A, H, D, J) is a real commutative spectral triple of KO-dimension $n \bmod 8$ and metric dimension p. In particular, (A, H, D) is a Dirac-type commutative spectral triple of metric dimension p, so that by Thm. 4.2.7, there exist a compact oriented Riemannian p-manifold X and a Hermitian vector bundle $\mathcal{E} \to X$, such that $(A, H, D) \cong (C^\infty(X), L^2(X, \mathcal{E}), D)$, where D, viewed as an operator on $L^2(X, \mathcal{E})$, defines an essentially self-adjoint Dirac-type operator on $\mathcal{E}$. In particular, then, D makes $\mathcal{E}$ into a Clifford module, so that it suffices to prove that J, viewed as an operator on $L^2(X, \mathcal{E})$, is an anitunitary bundle automorphism on $\mathcal{E}$, making $(\mathcal{E}, J)$ a real Clifford module of KO-dimension $n \bmod 8$.

First, since $JaJ^* = a^*$ for all $a \in C^\infty(X)$, J can be viewed as a unitary $C^\infty(X)$-linear morphism $C^\infty(X, \mathcal{E}) \to \overline{C^\infty(X, \mathcal{E})} = C^\infty(X, \overline{\mathcal{E}})$, where $\overline{C^\infty(X, \mathcal{E})}$ is the conjugate $C^\infty(X)$-module to $C^\infty(X, \mathcal{E})$, and $\overline{\mathcal{E}}$ is the conjugate bundle to $\mathcal{E}$. Hence, J defines a unitary bundle isomorphism $\mathcal{E} \cong \overline{\mathcal{E}}$, that is, an antiunitary bundle automorphism on $\mathcal{E}$. The rest then follows from the fact that $(C^\infty(X), L^2(X, \mathcal{E}), D, J)$ is a real spectral triple of KO-dimension $n \bmod 8$; in particular, since $DJ = \varepsilon' JD$, $Jc(df)J^* = J[D, f]J^* = \varepsilon'[D, f] = \varepsilon' c(df)$ for $f \in C^\infty(X, \mathbb{R})$, as required. ∎

Remark 4.3.13. One may ask which KO-dimensions are possible for real commutative spectral triples over a given compact oriented Riemannian manifold X. We shall soon see how to use the spectral triple of Ex. 4.3.5 to construct real commutative spectral triples over X of any KO-dimension.

4.4 Dirac-type spectral triples

We now propose a straightforward generalisation of the precise definition of commutative spectral triple to accommodate a noncommutative algebra:

Definition 4.4.1. Let (A, H, D) be a spectral triple; we call (A, H, D) *two-sided* spectral triple if H admits a faithful unital $*$-representation of the opposite algebra A^o making H into an A-bimodule, and

$$AH^\infty \subset H^\infty, \quad H^\infty A \subset H^\infty.$$

We then call (A, H, D) a *p-dimensional Dirac-type spectral triple* for $p \in \mathbb{N}$, if it is two-sided and the following conditions hold:

1. **Dimension**: The spectral triple (A, H, D) has metric dimension p.

2. **Order one**: For any a, $b \in A$, $[[D, a], b^o] = 0$.

3. **Finiteness**: The right A-module H^∞ is finitely generated projective.

4. **Strong regularity**: One has that $\operatorname{End}_{A^o}(H^\infty) \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$.

5. **Orientability**: There exists an antisymmetric Hochschild p-cycle $c \in Z_p(A, A)$, such that $\chi = \pi_D(c)$ is a self-adjoint unitary on H, satisfying $a\chi = \chi a$ and $[D, a]\chi = (-1)^{p+1}\chi[D, a]$ for all $a \in A$.

6. **Absolute continuity**: The right A-module H^∞ admits a Hermitian structure $(\cdot, \cdot)_A$, satisfying $\langle \xi, \eta \rangle_A = ⨍ (\xi, \eta)_A (D^2 + 1)^{-p/2}$ for ξ, $\eta \in H^\infty$.

Remark 4.4.2. This definition can be considered as a generalisation of the traditional notion of *noncommutative spin geometry* (see, e.g., [36, § 10.5]) to the case where no real structure is available. In particular, a noncommutative spin geometry with antisymmetric orientation cycle in $Z_p(A, A)$ and satisfying strong regularity and absolute continuity will be a Dirac-type spectral triple.

Of course, a Dirac-type spectral triple with commutative algebra is precisely a commutative spectral triple. Our discussion of toric noncommutative manifolds in Chapter 7, on the other hand, will offer a very wide range of genuinely noncommutative examples. In the meantime, however, let us establish the relation between this proposed definition and two closely related definitions recently proposed by Lord–Rennie–Várilly [49].

Now, for (A, H, D) a spectral triple, let $\mathcal{C}_D(A)$ denote the unital $*$-subalgebra of $B(H)$ generated by $A + [D, A]$, and recall that $H^\infty := \cap_k \operatorname{Dom} D^k$. Lord–Rennie–Várilly's first definition, modelled on the spectral triple of a spin$^\mathbb{C}$ Dirac operator on a spin$^\mathbb{C}$ manifold, takes the following form:

Definition 4.4.3 ([49, Def. 4.8])**.** A *p-dimensional noncommutative oriented spin*$^\mathbb{C}$ *manifold*, with $p \in \mathbb{N}$, is a spectral triple (A, H, D) together with a Hochschild p-cycle $c \in Z_p(A, A)$, satisfying the following conditions:

1. **Dimension**: The operator $(D^2 + 1)^{-p/2} \in B(H)$ lies in the Dixmier ideal, so that for any dilation-invariant state ω on ℓ^∞, $\psi_\omega : T \mapsto \operatorname{Tr}_\omega(T(D^2 + 1)^{-p/2})$ defines a positive linear functional on $\mathcal{C}_D(A)$.

2. **Regularity**: One has that $A + [D, A] \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$.

3. **Finiteness and absolute continuity**: One has that H^∞ is a finitely generated projective left $\mathcal{C}_D(A)$-module, and the functional ψ_ω on $\mathcal{C}_D(A)$ is faithful with $H = L^2(H^\infty, \psi_\omega)$.

4. **Orientability**: One has that $\chi = \pi_D(c)$ is a self-adjoint unitary on H satisfying $a\chi = \chi a$ for all $a \in A$ and $D\chi = (-1)^{p+1}\chi D$.

5. **Spin**$^{\mathbb{C}}$: The dense subspace H^∞ of H is a pre-Morita equivalence bimodule between $\mathcal{C}_D(A)$ and A, i.e., $\mathcal{C}_D(A) = \mathrm{End}_{A^o}(H^\infty)$.

If, in addition, there exists an orthogonal family of central projections $p_1, \ldots, p_n \in A$ with $\sum_k p_k = 1$, such that for any $a \in A$, $[D, a] = 0$ if and only if $a \in \operatorname{span}\{p_k\}$, then (A, H, D, c) is said to satisfy *connectivity.*

The relation between this definition and ours is established by the following result:

Proposition 4.4.4. *Let (A, H, D, c) be a p-dimensional noncommutative oriented spin$^{\mathbb{C}}$ manifold. Suppose that it has metric dimension p, that it satisfies connectivity, and that c is antisymmetric. Then (A, H, D) is Dirac-type.*

Before proceeding to the proof, we recall the following technical result from [49]:

Lemma 4.4.5 ([49, Prop. 4.10])**.** *Let (A, H, D, c) be a p-dimensional noncommutative oriented spin$^{\mathbb{C}}$ manifold satisfying connectivity. Then H^∞ is finitely generated projective as a right A-module and $\mathcal{C}_D(A)$ is finitely generated projective as a left or as a right A-module. Moreover,*

$$\forall \xi, \eta \in H^\infty, \quad \langle \eta, \xi \rangle = \psi_\omega((\xi, \eta)_A) = \mathrm{Tr}_\omega\left((\xi, \eta)_A (D^2 + 1)^{-p/2}\right)$$

Proof of Proposition 4.4.4. First, observe that by regularity, (A, H, D) is pre-regular, whilst by the spin$^{\mathbb{C}}$ condition, it is two-sided. Let us now check the conditions for a Dirac-type spectral triple one by one:

1. By hypothesis, (A, H, D) has metric dimension p.

2. By the spin$^{\mathbb{C}}$ condition, $A^0 \subset \mathcal{C}_D(A)'$, and hence $[[D, a], b^o] = 0$ for all a, $b \in A$.

3. By the spin$^{\mathbb{C}}$ condition, H^∞ is a pre-Morita equivalence bimodule between $\mathcal{C}_D(A)$ and A, and hence, in particular, H^∞ is a finitely generated projective right A-module.

4. By this last observation, $\mathrm{End}_{A^o}(H^\infty) = \mathcal{C}_D(A)$, so that strong regularity immediately follows from regularity.

5. Orientability immediately follows from orientability as a noncommutative oriented spin$^{\mathbb{C}}$ manifold and the hypothesis that c is antisymmetric.

6. By the spin$^{\mathbb{C}}$ condition, H^∞ is, in fact, a finitely generated projective right Hermitian A-module. Then, by the connectivity hypothesis, we can apply Lemma 4.4.5 to conclude that, indeed,
$$\forall \xi,\ \eta \in H^\infty, \quad \langle \eta, \xi \rangle = \mathrm{Tr}_\omega\left((\eta, \xi)_A (D^2 + 1)^{-p/2}\right).$$

Thus, (A, H, D) is Dirac-type, as was claimed. □

Let us now recall Lord–Rennie–Várilly's second definition, modelled on the Hodge–de Rham spectral triple of a compact oriented Riemannian manifold:

Definition 4.4.6 ([49, Def. 4.11])**.** A *p-dimensional noncommutative oriented Riemannian manifold*, with $p \in \mathbb{N}$, is a spectral triple (A, H, D), together with a Hochschild p-cycle $c \in Z_p(A, A)$ and a vector $\Phi \in H^\infty$, satisfying the following conditions:

1. **Dimension**: The operator $(D^2 + 1)^{-p/2} \in B(H)$ lies in the Dixmier ideal, so that for any dilation-invariant state ω on ℓ^∞, $\psi_\omega : T \mapsto \operatorname{Tr}_\omega(T(D^2 + 1)^{-p/2})$ defines a positive linear functional on $\mathcal{C}_D(A)$.

2. **Regularity**: One has that $A + [D, A] \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$.

3. **Finiteness and absolute continuity**: One has that H^∞ is a finitely generated projective left $\mathcal{C}_D(A)$-module, and the functional ψ_ω on $\mathcal{C}_D(A)$ is faithful with $H = L^2(H^\infty, \psi_\omega)$.

4. **Orientability**: One has that $\chi = \pi_D(c)$ is a self-adjoint unitary on H satisfying $a\chi = \chi a$ for all $a \in A$ and $D\chi = (-1)^{p+1}\chi D$.

5. **Riemannian**: The dense subspace H^∞ of H contains a cyclic and separating vector Φ for the action of $\mathcal{C} := \mathcal{C}_D(A)$ in the algebraic sense, so that $H^\infty = \mathcal{C}\Phi$ and $w = 0$ in $\mathcal{C}$ if and only if $w\Phi$ in H^∞; in particular, then, $\mathcal{H} = \mathcal{C}\Phi$ is a free left $\mathcal{C}$-module. Moreover, there exists a Hermitian metric ${}_{\mathcal{C}}(\cdot, \cdot)$ on H^∞ such that

$$\forall \eta,\ \xi \in \mathcal{C}, \quad \langle \eta, \xi \rangle = \psi_\omega\left({}_{\mathcal{C}}(\eta, \xi)\right) = \operatorname{Tr}_\omega\left({}_{\mathcal{C}}(\eta, \xi)\,(D^2 + 1)^{-p/2}\right),$$

   and ${}_{\mathcal{C}}(\Phi, \Phi)$ is a strictly positive central element of $\mathcal{C}$; without loss of generality, take $\|\Phi\| = 1$. Finally, one requires that H^∞ be finite projective as a left A-module, and that there exist a grading operator ϵ on H, making (A, H, D, ϵ) an even spectral triple.

Now, let (A, H, D, c, Φ) be a noncommutative oriented Riemannian manifold. By the remarks on [49, p. 1630], the Tomita–Takesaki theory implies the existence of an antiunitary operator J_Φ on H with $J_\Phi^2 = 1$, such that $\omega \mapsto J_\Phi \omega^* J_\Phi$ defines an anti-isomorphism $\mathcal{C}'' \to \mathcal{C}'$. Moreover, by [49, Lemma 4.18], it follows that $a \mapsto a^o := Ja^*J$ defines a right action of A on H, making (A, H, D) into a two-sided spectral triple. The connection between this definition and ours is established by the following result:

Proposition 4.4.7. *Let (A, H, D, c, Φ) be a p-dimensional noncommutative oriented Riemannian manifold. Suppose that it has metric dimension p, that c is antisymmetric, and that $J \operatorname{End}_A(H^\infty)J \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$. Then (A, H, D) is Dirac-type.*

Before proceeding to the proof, let us recall the following technical result from [49]:

Lemma 4.4.8 ([49, Cor. 4.20])**.** *Suppose that (A, H, D) satisfies conditions (1), (2), and (5) of the definition of noncommutative oriented Riemannian manifold. Then (A, H, D) satisfies condition (3), if and only if the following both hold:*

1. *$\mathcal{C}_D(A)$ is a finitely generated projective left A-module,*
2. *there exists an operator-valued weight $\Psi : \mathcal{C}_D(A) \to A$ such that $\psi_\omega = \psi_\omega \circ \Psi$ for all dilation-invariant states ω on ℓ^∞.*

Proof of Proposition 4.4.7. Let us check the conditions for a Dirac-type spectral triple one by one:

1. By hypothesis, (A, H, D) has metric dimension p.
2. By construction, $A^0 = JAJ \subset \mathcal{C}_D(A)'$, and hence $[[D, a], b^o] = 0$ for all a, $b \in A$.
3. By Lemma 4.4.8, H^∞ is a finitely generated projective left A-module, so that we have an isomorphism of left A-modules $\phi : H^\infty \cong A^n q$ for some $n \in \mathbb{N}$ a projection $q \in M_n(A)$. Then $\tilde{\phi} : H^\infty \to qA^n$ given by $\tilde{\phi}(\xi) := \phi(J\xi)^*$ is the desired isomorphism of right A-modules.
4. By hypothesis, $J \operatorname{End}_A(H^\infty)J \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$; since $\operatorname{End}_{A^o}(H^\infty) = J \operatorname{End}_A(H^\infty)J$, strong regularity therefore follows.
5. Orientability immediately follows from orientability as a noncommutative oriented Riemannian manifold and the hypothesis that c is antisymmetric.
6. By the proof of Lemma 4.4.8, there is an operator-valued weight $\Psi : \mathcal{C}_D(A) \to A$ such that

$$ {}_A(\xi, \eta) := \Psi\left({}_{\mathcal{C}}(\xi, \eta)\right), \quad \xi, \eta \in H^\infty $$

   defines a Hermitian metric on ${}_A H^\infty$, and such that $\psi_\omega = \psi_\omega \circ \Psi$ for all Dixmier limits ω. In this light, $\phi : H^\infty \cong A^n q$ is taken to be an isomorphism of Hermitian left A-modules, yielding $\tilde{\phi} : H^\infty \cong qA^n$ as an isomorphism of Hermitian right A-modules, where

$$ (\eta, \xi)_A := {}_A(J\xi, J\eta), \quad \xi, \eta \in H^\infty. $$

   It therefore follows that for $\psi_\omega : T \mapsto \operatorname{Tr}\left(T(D^2+1)^{-p/2}\right)$,

$$ \forall \xi, \eta \in H^\infty, \quad \psi_\omega\left((\eta, \xi)_A\right) = \psi_\omega\left({}_A(J\xi, J\eta)\right) = \psi_\omega\left({}_{\mathcal{C}}(J\xi, J\eta)\right) = \langle J\xi, J\eta\rangle = \langle \xi, \eta\rangle, $$

   as required.

Thus, (A, H, D) is indeed Dirac-type, as was claimed. □

Chapter 5

Stability results

Conclusit uias meas lapidibus quadris; semitas meas subuertit.

— Lamentations 3:9 (Vulgate)

In this chapter, we finally prove that the orientability hypothesis can be dropped from Connes's reconstruction theorem to yield the full generality of Corollary 4.2.7. This will depend on the non-trivial result that a Dirac-type spectral triple remains a Dirac-type spectral triple with the same qualitative features, under suitable perturbation of the Dirac operator. To state it, we shall need the following definition, adapted from Higson [37]:

Definition 5.0.9. Let Δ be a self-adjoint operator with compact resolvent on a separable Hilbert space H, and let C be a subset of $B(H)$ that leaves $H^\infty := \cap_k \operatorname{Dom} D^k$ invariant. Then the *algebra of differential operators* generated by C with respect to Δ is the smallest filtered algebra $\mathcal{D}(C, \Delta)$ of linear operators on H^∞ that contains C and is closed under the operator $T \mapsto [\Delta, T]$; this algebra is defined and filtered inductively as follows:

(a) $\mathcal{D}(C, \Delta)_0$ is the unital algebra generated by C itself.

(b) $\mathcal{D}(C, \Delta)_1 := \mathcal{D}(C, \Delta)_0 \, [\Delta, \mathcal{D}(C, \Delta)_0] \, \mathcal{D}(C, \Delta)_0$.

(c) $\mathcal{D}(C, \Delta)_k := \sum_{j=1}^{\infty} \mathcal{D}(C, \Delta)_j \mathcal{D}(C, \Delta)_{k-j} + \mathcal{D}(C, \Delta)_0 \, [\Delta, \mathcal{D}(C, \Delta)_{k-1}] \, \mathcal{D}(C, \Delta)_0$.

Moreover, we shall call $(\mathcal{D}(C, \Delta), \Delta)$ a *differential pair* if for every $X \in \mathcal{D}(C, \Delta)_k$, there exists some $\epsilon > 0$, such that for all $\xi \in H^\infty$, $\left\|\Delta^{k/2}\xi\right\| + \|\xi\| \geq \epsilon \, \|X\xi\|$.

Example 5.0.10. In the case of a concrete commutative spectral triple $(C^\infty(X), L^2(X, \mathcal{E}), D)$, one has that $\mathcal{D} := \mathcal{D}(C^\infty(X, \operatorname{End}(\mathcal{E}), D^2)$ is is indeed an algebra of differential operators on H; that $(\mathcal{D}, D^2)$ is a differential pair follows from the Gårding estimates of elliptic regularity theory [61, Chapter 5].

The key technical tool is, thus, the following theorem, collating the technical lemmas of [9, Appendix A], which in turn consist both of generalisations of folkloric results proved by Chakraborty–Mathai [14] and Iochum–Levy–Vassilevich [39], as well as folkloric results hitherto unproved in the literature.

Theorem 5.0.11. *Let (A, H, D) be a p-dimensional Dirac-type spectral triple, and let M be a self-adjoint element of $\mathrm{End}_{A^o}(H^\infty)$ for $H^\infty := \cap_k \mathrm{Dom}\, D^k$, such that*

$$[(D+M)^2 - D^2, T] \in \mathcal{D}(\mathrm{End}_{A^o}(H^\infty), D^2)_{k+1}$$

for all $T \in \mathcal{D}(\mathrm{End}_{A^o}(H^\infty), D^2)_k$. Then $D_M := D+M$ extends to an essentially self-adjoint operator on H with smooth core H^∞, making (A, H, D_M) into a p-dimensional Dirac-type spectral triple, satisfying the following:

1. *For each $k \in \mathbb{N}$, $\mathrm{Dom}\, D_M^k = \mathrm{Dom}\, D^k$, and hence $\cap_k \mathrm{Dom}\, D_M^k = H^\infty$;*

2. *For each $k \in \mathbb{N}$,*

$$\mathrm{Dom}\,(\mathrm{ad}\,|D_M|)^k = \mathrm{Dom}\,(\mathrm{ad}\,|D|)^k \subset B(H),$$

   and hence

$$\bigcap_k \mathrm{Dom}\,(\mathrm{ad}\,|D_M|)^k = \bigcap_k \mathrm{Dom}\,(\mathrm{ad}\,|D|)^k .$$

3. *For all $\xi,\ \eta \in H^\infty$,*

$$⨍ (\xi, \eta)\,(D_M^2 + 1)^{-p/2} = ⨍ (\xi, \eta)\,(D^2 + 1)^{-p/2} = \langle \xi, \eta \rangle .$$

Remark 5.0.12. If (A, H, D) is commutative, and $M \in \mathrm{End}_A(H^\infty)$ is self-adjoint and satisfies

$$\forall a \in A,\ M[D, a] = -[D, a]M,$$

then $D_M^2 - D^2 = MD + DM \in \mathrm{End}_A(H^\infty)$, and, hence, M satisfies the hypothesis of Theorem 5.0.11.

The essence of the proof of Corollary 4.2.7 is, then, contained in the following corollary of Theorem 5.0.11:

Corollary 5.0.13. *Let (A, H, D) be a p-dimensional commutative spectral triple with chirality operator χ and Hermitian metric $(\cdot, \cdot)$ on $H^\infty := \cap_k \mathrm{Dom}\, D^k$, satisfying*

$$\forall \xi, \eta \in H^\infty,\ \langle \xi, \eta \rangle = ⨍ (\xi, \eta)\,(D^2 + 1)^{-p/2}.$$

Define an operator D_0 on H^∞ by

$$D_0 = \frac{1}{2}\chi^p \left(D - (-1)^p \chi D \chi\right).$$

Then D_0 extends to an essentially self-adjoint operator on H, making (A, H, D_0) into a strongly orientable p-dimensional commutative spectral triple that satisfies the following:

1. *For each $k \in \mathbb{N}$, $\operatorname{Dom} D_0^k = \operatorname{Dom} D^k$, and hence $\cap_k \operatorname{Dom} D_0^k = H^\infty$;*
2. *For each $k \in \mathbb{N}$,*
$$\operatorname{Dom}\left(\operatorname{ad}|D_0|\right)^k = \operatorname{Dom}\left(\operatorname{ad}|D|\right)^k \subset B(H),$$
and hence
$$\bigcap_k \operatorname{Dom}\left(\operatorname{ad}|D_0|\right)^k = \bigcap_k \operatorname{Dom}\left(\operatorname{ad}|D|\right)^k.$$
3. *For all ξ, $\eta \in H^\infty$,*
$$⨍ (\xi, \eta)\,(D_0^2 + 1)^{-p/2} = ⨍ (\xi, \eta)\,(D^2 + 1)^{-p/2} = \langle \xi, \eta \rangle.$$

In particular, $M := D - \chi^p D_0$ extends to a self-adjoint element of $\operatorname{End}_B(H^\infty) \subset B(H)$ such that M restricts to an element of $B(\operatorname{Dom} D^k)$ for each $k \in \mathbb{N}$.

Proof. Let us begin, rather, by defining M on H^∞ by

$$M := \tfrac{1}{2}\left(D + (-1)^p \chi D \chi\right).$$

Recalling that $a\chi = \chi a$ and $[D, a]\chi = (-1)^{p+1}\chi[D, a]$ for all $a \in A$, we have that

$$\forall a \in A,\ [M, a] = \frac{1}{2}\left([D, a] + (-1)^p \chi [D, a] \chi\right) = 0,$$

so that $M \in \operatorname{End}_A(H^\infty)$. Since $\operatorname{End}_A(H^\infty) \subset B(H)$ by strong regularity, M is bounded, whilst by construction, M is symmetric on $H^\infty \subset H$ and hence self-adjoint as an element of $B(H)$. Moreover, since

$$D_{-M} := D - M = \tfrac{1}{2}\left(D - (-1)^p \chi D \chi\right),$$

it follows that

$$(D_{-M}^2 - D^2) = -M D_{-M} - D_{-M} M = \chi D^2 \chi - D^2.$$

Hence, by construction of $\mathcal{D}(\operatorname{End}_A(H^\infty), D^2)$, for any $T \in \mathcal{D}(\operatorname{End}_A(H^\infty), D^2)_k$,

$$[D_{-M}^2 - D^2, T] = [\chi D^2 \chi - D^2, T] = \chi [D^2, T] \chi - [D^2, T] \in \mathcal{D}(\operatorname{End}_A(H^\infty), D^2)_{k+1}.$$

Thus, we can apply Theorem 5.0.11 to conclude that (A, H, D_{-M}) is a p-dimensional Dirac-type spectral triple satisfying:

1. For each $k \in \mathbb{N}$, $\operatorname{Dom} D_{-M}^k = \operatorname{Dom} D^k$, and hence $\cap_k \operatorname{Dom} D_{-M}^k = H^\infty$;

2. For each $k \in \mathbb{N}$,
$$\operatorname{Dom}\left(\operatorname{ad}|D_{-M}|\right)^k = \operatorname{Dom}\left(\operatorname{ad}|D|\right)^k \subset B(H),$$
and hence
$$\bigcap_k \operatorname{Dom}\left(\operatorname{ad}|D_{-M}|\right)^k = \bigcap_k \operatorname{Dom}\left(\operatorname{ad}|D|\right)^k .$$

3. For all $\xi,\, \eta \in H^\infty$,
$$⨍ (\xi, \eta)\,(D_{-M}^2 + 1)^{-p/2} = ⨍ (\xi, \eta)\,(D^2 + 1)^{-p/2} = \langle \xi, \eta \rangle .$$

In particular, observe that by construction $D_{-M}\chi = (-1)^{p+1}\chi D_{-M}$.

Now, set
$$D_0 = \chi^p D_{-M} = \frac{1}{2}\chi^p\left(D - (-1)^p \chi D \chi\right).$$

Since $D_0^2 = D_{-M}^2$, it therefore follows that that (A, H, D_0) is again a p-dimensional commutative spectral triple, satisfying

1. For each $k \in \mathbb{N}$, $\operatorname{Dom} D_0^k = \operatorname{Dom} D^k$, and hence $\cap_k \operatorname{Dom} D_0^k = H^\infty$;

2. For each $k \in \mathbb{N}$,
$$\operatorname{Dom}\left(\operatorname{ad}|D_0|\right)^k = \operatorname{Dom}\left(\operatorname{ad}|D|\right)^k \subset B(H),$$
and hence
$$\cap_k \operatorname{Dom}\left(\operatorname{ad}|D_0|\right)^k = \cap_k \operatorname{Dom}\left(\operatorname{ad}|D|\right)^k .$$

3. For all $\xi,\, \eta \in H^\infty$,
$$⨍ (\xi, \eta)\,(D_0^2 + 1)^{-p/2} = ⨍ (\xi, \eta)\,(D^2 + 1)^{-p/2} = \langle \xi, \eta \rangle .$$

If p is even, then $D_0 = D_{-M}$, and hence $D\chi = -\chi D$, so that (A, H, D_0) is strongly orientable. If p is odd, then if
$$c = \sum_{k=1}^{N} a_{k,0} \otimes a_{k,1} \otimes \cdots \otimes a_{k,p}$$
is the orientation cycle of (A, H, D), then, since
$$\forall a \in A,\ [D_0, a] = [\chi^p(D - M), a] = \chi^p[D, a],$$

one has that

$$\pi_{D_0}(c) = \sum_{k=1}^{N} a_{k,0}[D_0, a_{k,1}] \cdots [D_0, a_{k,p}] = \chi^{p^2}\chi = 1,$$

as required. □

The rest of the chapter amounts to a step-by-step proof of Theorem 5.0.11.

5.1 Spectral triple; metric dimension

We begin by showing that perturbing the Dirac operator of a spectral triple by a bounded self-adjoint operator still results in a spectral triple.

Lemma 5.1.1 ([14, Lemma 2.1, Proposition 2.2])**.** *Let (A, H, D) be a spectral triple, let $M \in B(H)$ be self-adjoint, and set $D_M := D + M$. Then (A, H, D_M) is a spectral triple.*

Proof. First, by the Kato-Rellich theorem, D_M is self-adjoint on $\operatorname{Dom} D_M = \operatorname{Dom} D$ and essentially self-adjoint on any core of D. Next, since D has compact resolvent, for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$(D_M - \lambda)^{-1} = (D - \lambda)^{-1} - (D_M - \lambda)^{-1} M (D - \lambda)^{-1} \in \mathcal{K}(\mathcal{H}),$$

so that D_M too has compact resolvent. Finally, for any $a \in A$, since $[D, a] \in B(H)$ and since $M \in B(H)$,

$$[D_M, a] = [D, a] + [M, a] \in B(H).$$

Thus, (A, H, D_M) is indeed a spectral triple. □

Let us now consider stability of metric dimension.

Lemma 5.1.2. *Let (A, H, D) be a spectral triple, let $M \in B(H)$ be self-adjoint, and set $D_M := D + M$. If (A, H, D) has metric dimension $p > 0$, then so too does (A, H, D_M).*

To prove this, we shall need the following technical lemma from the literature:

Lemma 5.1.3 (Carey–Phillips [12, Lemma B.6])**.** *If D is an unbounded self-adjoint operator on a Hilbert space H, and $M \in B(H)$ is self-adjoint, then for $D_M := D + M$,*

$$\left(D_M^2 + 1\right)^{-1} \leq f(\|M\|) \left(D^2 + 1\right)^{-1},$$

where $f(x) := 1 + \frac{1}{2}x^2 + \frac{1}{2}x\sqrt{x^2 + 4}$.

Proof of Lemma 5.1.2. By Lemma 5.1.3 one has that

$$\frac{1}{f(\|M\|)}(D^2 + 1)^{-1} \leq (D_M^2 + 1)^{-1} \leq f(\|M\|)(D^2 + 1)^{-1},$$

where $f(x) := 1 + \frac{1}{2}x^2 + \frac{1}{2}x\sqrt{x^2+4}$, so that by [58, Lemma on p. 270], if $\lambda_n(C)$ denotes the n-th eigenvalue of the positive compact operator $C \in B(\mathcal{H})$, in decreasing order, then

$$\frac{1}{f(\|M\|)}\lambda_k((D^2+1)^{-1}) \leq \lambda_k((D_M^2+1)^{-1}) \leq f(\|M\|)\lambda_k((D^2+1)^{-1})$$

for all $n \in \mathbb{N}$. Since $\lambda_k((D^2+1)^{-1}) = O(k^{-2/p})$, it therefore follows that

$$\lambda_k((D_M^2+1)^{-1}) = O(k^{-2/p}).$$

Thus, $(A, \mathcal{H}, D_M)$ has metric dimension p, as was claimed. □

Now, in the context of Theorem 5.0.11, since M is bounded and self-adjoint, it immediately follows that (A, H, D_M) is a spectral triple of metric dimension p.

5.2 Finiteness

The following lemma will suffice to establish stability of finiteness in the proof of Theorem 5.0.11, and will also be necessary for our discussion below of [strong] regularity and absolute continuity; we shall follow the proof by Iochum–Levy–Vassilevich.

Lemma 5.2.1 (Iochum–Lévy–Vassilevich [39, Lemma 2.3])**.** *For $k \in \mathbb{N}$, let $H^k := \operatorname{Dom} D^k$ with the Sobolev inner product*

$$\langle \xi, \eta \rangle_k := \langle D^k\xi, D^k\eta \rangle + \langle \xi, \eta \rangle,$$

and similarly let $H_M^k := \operatorname{Dom} D_M^k$ with the Sobolev inner product

$$\langle \xi, \eta \rangle_{M,k} := \langle D_M^k\xi, D_M^k\eta \rangle + \langle \xi, \eta \rangle.$$

Suppose now that M restricts to an element of $B(H^k)$ for each $k \in \mathbb{N}$. Then $H^k = H_M^k$ for all $k \in \mathbb{N}$ with equivalent norms, and thus, in particular, $\cap_k \operatorname{Dom} D^k = \cap_k \operatorname{Dom} D_M^k$.

Proof. Let us first prove equality of vector spaces. We proceed by induction on k. First, by the Kato-Rellich theorem [57, Theorem X.12], D_M is self-adjoint on $\operatorname{Dom} D_M = \operatorname{Dom} D$ and essentially self-adjoint on any core of D, so that the claim holds for $k = 1$. Now, assume by induction that the claim holds for some $m \in \mathbb{N}$. Then, by the induction hypothesis and our restriction on M,

$$\begin{aligned}\operatorname{Dom} D_M^{m+1} &= \{\xi \in \operatorname{Dom} D_M^m \mid D_M\xi \in \operatorname{Dom} D_M^m\} \\ &= \{\xi \in \operatorname{Dom} D^m \mid (D+M)\xi \in \operatorname{Dom} D^m\} \\ &= \{\xi \in \operatorname{Dom} D^m \mid D\xi \in \operatorname{Dom} D^m\}\end{aligned}$$

$$= \operatorname{Dom} D^{m+1},$$

as required.

Let us now prove equivalence of the Sobolev norms. Before continuing, we will find it convenient to replace $\langle\cdot,\cdot\rangle_k$ and $\langle\cdot,\cdot\rangle_{M,k}$ with $(\cdot,\cdot)_k$ and $(\cdot,\cdot)_{M,k}$, respectively, where

$$(\xi,\eta)_k := \left\langle (D+i)^k\xi, (D+i)^k\eta \right\rangle + \langle \xi,\eta\rangle,$$
$$(\xi,\eta)_{M,k} := \left\langle (D_M+i)^k\xi, (D_M+i)^k\eta \right\rangle + \langle \xi,\eta\rangle.$$

Indeed, let us show, for instance, that $\langle\cdot,\cdot\rangle_k$ and $(\cdot,\cdot)_k$ define equivalent norms. On the one hand, for $\xi \in H^k$,

$$\left\|(D+i)^k\xi\right\| = \left\|\sum_{m=0}^{k} i^m D^{k-m}\xi\right\| \leq \sum_{m=0}^{k} \left\|D^{k-m}\xi\right\|,$$

so that by continuity of the inclusions $H^k \hookrightarrow H^{k-m}$ for the $\langle\cdot,\cdot\rangle_n$, there exists some $C > 0$, independent of ξ, such that

$$\left\|(D+i)^k\xi\right\|^2 + \|\xi\|^2 \leq C\left(\left\|D^k\xi\right\|^2 + \|\xi\|^2\right).$$

On the other hand since the $(\cdot,\cdot)_k$ is also simply the k-th Sobolev inner product for $\sqrt{D^2+1}$, the inclusions $H^k \hookrightarrow H^{k-m}$ are also continuous for the $(\cdot,\cdot)_n$, and hence, since

$$\left\|D^k\xi\right\| = \left\|((D+i)-i)^k\xi\right\| = \left\|\sum_{m=0}^{k}(-i)^m(D+i)^{k-m}\xi\right\| \leq \sum_{m=0}^{k}\left\|(D+i)^{k-m}\xi\right\|,$$

there exists some C', independent of ξ, such that

$$\left\|D^k\xi\right\|^2 + \|\xi\|^2 \leq C'\left(\left\|(D+i)^k\xi\right\|^2 + \|\xi\|^2\right).$$

Thus, $\langle\cdot,\cdot\rangle_k$ and $(\cdot,\cdot)_k$ do indeed define equivalent norms.

Now, fix $k \in \mathbb{N}$, and consider the linear map $B = (D_M - i)^k(D-i)^{-k}$ on $\mathcal{H}$; we claim that B is, in fact, bounded on $\mathcal{H}$. First, one has that on $\operatorname{Dom} D^k = \operatorname{Dom} D_M^k$,

$$(D_M - i)^k = ((D-i)+M)^k = (D-i)^k + \sum_{m=1}^{k} T_m,$$

where for each m, T_m is a product of k operators, each of which is either $(D-i)$ or M. By our assumption on M, then, each T_m therefore defines a continuous map $H_k \to H_1$, so that, since

$(D-i)^k : \mathcal{H}_k \to \mathcal{H}$ and $(D-i)^{-k} : \mathcal{H} \to \mathcal{H}_k$ are continuous,

$$B = (D_M - i)^k (D-i)^{-k} = \mathrm{Id}_{\mathcal{H}} + \sum_{m=1}^{k} T_m (D-i)^k$$

defines a bounded operator on $\mathcal{H}$. Since B is bijective, it therefore follows by the bounded inverse theorem that B has a bounded inverse. Thus, for $\xi \in \operatorname{Dom} D^k = \operatorname{Dom} D_M^k$, since $(D_M - i)^k = B(D-i)^k$ and $(D-i)^k = B^{-1}(D_M - i)^k$,

$$(\xi,\xi)_{M,k} \leq \max\left\{1, \|B\|^2\right\}(\xi,\xi)_k\,, \quad (\xi,\xi)_k \leq \max\left\{1, \left\|B^{-1}\right\|^2\right\}(\xi,\xi)_{M,k}\,,$$

which implies, by our earlier observation, that $\|\cdot\|_k$ and $\|\cdot\|_{M,k}$ are equivalent, as required. □

5.3 Regularity and strong regularity

We shall now use Higson's characterisation of regularity, first to allow us to apply Lemma 5.2.1 in context, and then to generalise a result on stability of regularity due to Chakraborty–Mathai [14].

We have already stated our adaptation of Higson's definition of abstract algebra of differential operators; the relevant stability result therefore is as follows:

Lemma 5.3.1 ([14, Proposition 4.2])**.** *Let (A, H, D) be a spectral triple, and let*

$$C \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k \subset B(H).$$

Let $M \in \mathcal{D}(C, D^2)$ be self-adjoint, set $D_M := D + M$, and suppose that $[D_M^2 - D^2, T] \in \mathcal{D}(C, D^2)_{k+1}$ for $T \in \mathcal{D}(C, D^2)_k$. Then $C \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D_M|)^k \subset B(H)$.

In the case that $C = A + [D, A]$, we get a sufficient condition for stability of regularity, and in the case that $C = \operatorname{End}_{A^o}(H^\infty)$, we get a sufficient condition for stability of strong regularity.

Remark 5.3.2. If M is an inner fluctuation of the metric, that is, if

$$M = \sum_{i=1}^{n} a_i [D, b_i]$$

for some a_i, $b_i \in \mathcal{A}$, then the condition that $[D_M^2 - D^2, T] \in \mathcal{D}_{k+1}$ for $T \in \mathcal{D}_k$ is automatically satisfied.

As was first observed by Chakraborty–Mathai, the essential tool is Higson's characterisation of regularity, stated here in the full generality actually provided for by Higson's proof:

Theorem 5.3.3 (Higson [37, Theorem 4.26])**.** *Let D be a self-adjoint operator with compact resolvent on a separable Hilbert space H, and let $H^\infty = \cap_k \operatorname{Dom} D^k$. Let C be a subalgebra of $B(H)$ that leaves H^∞ invariant. Then $C \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$ if and only if $(\mathcal{D}(C,\Delta), D^2)$ is a differential pair.*

The tool that will allow us to use Lemma 5.2.1 is the following lemma of Higson's:

Lemma 5.3.4 (Higson [37, Lemma 4.7])**.** *Let Δ be a self-adjoint operator with compact resolvent on a separable Hilbert space H. Let C be a subalgebra of $B(H)$ that leaves $H^\infty = \cap_k \operatorname{Dom} D^k$ invariant, and let $\mathcal{D} = \mathcal{D}(C,\Delta)$. If $(\mathcal{D}, D^2)$ is a differential pair, then for any $T \in \mathcal{D}_k$, T extends to a bounded operator $H^{k+m} \to H^m$ for all $m \in \mathbb{N} \cup \{0\}$.*

Finally, with Theorem 5.3.3 and Lemma 5.3.4 at our disposal, we can finally prove our stability result:

Proof of Lemma 5.3.1. For simplicity, let $\mathcal{D} := \mathcal{D}(C, D^2)$, and let $\mathcal{D}_M := \mathcal{D}(C, D_M^2)$. By our hypothesis on $D_M^2 - D^2$, there is a filtered inclusion $\mathcal{D}_M \subset \mathcal{D}$ of filtered algebras, so that by our hypothesis on C and Theorem 5.3.3, $(\mathcal{D}_M, D^2)$ is a differential pair, in the sense that for every $X \in (\mathcal{D}_M)_k$, there exists some $\epsilon > 0$, such that for all $\xi \in H^\infty$, $\left\| |D|^k \xi \right\| + \|\xi\| \geq \epsilon \|X\xi\|$. Thus, fix $X \in \mathcal{D}_M$ of order $\leq k$, so that there exists some $\epsilon > 0$, such that for all $\xi \in H^\infty := \cap_k \operatorname{Dom} D^k = \cap_k \operatorname{Dom} D_M^k$,

$$\left\| D^k \xi \right\| + \|\xi\| \geq \epsilon \|X\xi\| .$$

Then, since $\operatorname{Dom} D^k = \operatorname{Dom} D_M^k$ with equivalent Sobolev norms by Lemmas 5.3.4 and 5.2.1, it follows that that D^k is bounded as an operator from $\operatorname{Dom} D_M^k$ endowed with the Sobolev k-norm for D_M, to H, implying that $\left\| D^k \xi \right\| \leq \alpha \left\| D_M^k \xi \right\| + \beta \|\xi\|$ for some α, $\beta > 0$ independent of ξ, and hence that

$$\left\| D_M^k \xi \right\| + \|\xi\| \geq \epsilon' \|X\xi\|$$

for some $\epsilon' > 0$ independent of ξ. Thus, $(\mathcal{D}_M, D_M^2)$ is a differential pair, so that by Theorem 5.3.3, $C \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D_M|)^k$, as required. □

Now, let us apply these results to the proof of Theorem 5.0.11. First, by Theorem 5.3.3 and Lemma 5.3.4, M restricts to a bounded operator on $\operatorname{Dom} D^k$ with the relevant Sobolev norm, for each k. Hence, by Lemma 5.2.1, $\operatorname{Dom} D_M^k = \operatorname{Dom} D^k$ with equivalent Sobolev norms for each k, and thus, in particular, $\cap_k \operatorname{Dom} D_M^k = \cap_k \operatorname{Dom} D^k = H^\infty$, proving pre-regularity and finiteness for (A, H, D_M). Then, by Lemma 5.3.1 applied to $C = \operatorname{End}_{A^\circ}(H^\infty)$, we can conclude that (A, H, D_M) is strongly regular.

5.4 Absolute continuity

Finally, we consider stability of absolute continuity, which shall depend essentially on the BKS inequality, Theorem 3.4.4.

Lemma 5.4.1. *Let (A, H, D) be a spectral triple of metric dimension p, and let $M \in \cap_k B(H^k)$ be self-adjoint, where $H^k := \operatorname{Dom} D^k$ with the Sobolev inner product*

$$\langle \xi, \eta \rangle_k := \langle D^k \xi, D^k \eta \rangle + \langle \xi, \eta \rangle .$$

Let $D_M := D + M$. Then

$$\forall T \in B(H), \ \fint T(D_M^2 + 1)^{-p/2} = \fint T(D^2 + 1)^{-p/2}.$$

Proof. Let

$$n = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even,} \\ \frac{p+1}{2} & \text{if } p \text{ is odd,} \end{cases} \qquad \alpha = \frac{p}{2n} = \begin{cases} 1 & \text{if } p \text{ is even,} \\ \frac{p}{p+1} & \text{if } p \text{ is odd.} \end{cases}$$

First, we have that

$$\begin{aligned} &(D_M^2 + 1)^{-n} - (D^2 + 1)^{-n} \\ &= (D^2 + 1)^{-n} \left((D^2 + 1)^n - (D_M^2 + 1)^n \right) (D_M^2 + 1)^{-n} \\ &= (D^2 + 1)^{-n} \left(\sum_{i=0}^{n} \sum_{j=0}^{2i-1} \binom{n}{i} D^j M D_M^{2i-1-j} \right) (D_M^2 + 1)^{-n} \\ &= \sum_{i=0}^{n} \sum_{j=0}^{2i-1} \binom{n}{i} (D^2 + 1)^{-n+\frac{j}{2}} \left[D(D^2 + 1)^{-\frac{1}{2}} \right]^j M \left[D_M (D_M^2 + 1)^{-\frac{1}{2}} \right]^{2i-1-j} (D_M^2 + 1)^{-n+i-\frac{j+1}{2}}, \end{aligned}$$

which by Lemma 5.2.1 can be checked on the common core H^∞ of D and D_M.

Now, consider the term

$$(D^2 + 1)^{-n+\frac{j}{2}} \left[D(D^2 + 1)^{-\frac{1}{2}} \right]^j M \left[D_M (D_M^2 + 1)^{-\frac{1}{2}} \right]^{2i-1-j} (D_M^2 + 1)^{-n+i-\frac{j+1}{2}},$$

where $0 \leq i \leq n$ and $0 \leq j \leq 2i - 1$. Since (A, H, D) is of metric dimension p, so too is (A, H, D_M) by Lemma 5.1.2, so that

$$(D^2 + 1)^{-p/2}, \ (D_M^2 + 1)^{-p/2} \in \mathcal{L}^{1+}(\mathcal{H}),$$

and hence, for all $\epsilon > 0$,

$$(D^2 + 1)^{-1}, \ (D_M^2 + 1)^{-1} \in \mathcal{L}^{\alpha n+\epsilon}(\mathcal{H}).$$

Setting $\epsilon = \alpha(n - i + \frac{1}{2})$, we therefore find that

$$(D^2+1)^{-n+\frac{j}{2}} \in \mathcal{L}^r(\mathcal{H}),$$

$$\left[D(D^2+1)^{-\frac{1}{2}}\right]^j M \left[D_M(D_M^2+1)^{-\frac{1}{2}}\right]^{2i-1-j} (D_M^2+1)^{-n+i-\frac{j+1}{2}} \in \mathcal{L}^s(H),$$

for

$$r = \frac{\alpha n + \epsilon}{n - \frac{j}{2}}, \quad s = \frac{\alpha n + \epsilon}{n - i + \frac{j+1}{2}},$$

which satisfy $r^{-1} + s^{-1} = \alpha^{-1}$. Hence, by Hölder's inequality for Schatten ideals,

$$(D^2+1)^{-n+\frac{j}{2}} \left[D(D^2+1)^{-\frac{1}{2}}\right]^j M \left[D_M(D_M^2+1)^{-\frac{1}{2}}\right]^{2i-1-j} (D_M^2+1)^{-n+i-\frac{j+1}{2}} \in \mathcal{L}^\alpha(\mathcal{H});$$

since this is true for all i and j, it therefore follows that $(D_M^2+1)^{-n} - (D^2+1)^{-n} \in \mathcal{L}^\alpha(\mathcal{H})$. If p is even, then $\alpha = 1$ and $(D_M^2+1)^{-n} - (D^2+1)^{-n}$ is already trace-class; if p is odd, then since

$$(D_M^2+1)^{-p/2\alpha} - (D^2+1)^{-p/2\alpha} = (D_M^2+1)^{-n} - (D^2+1)^{-n} \in \mathcal{L}^\alpha(H)$$

for $0 < \alpha < 1$, we can apply the BKS inequality (Theorem 3.4.4) to

$$\left|(D_M^2+1)^{-p/2\alpha} - (D^2+1)^{-p/2\alpha}\right|^\alpha \in \mathcal{L}^1(H)$$

to conclude that $(D_M^2+1)^{-p/2} - (D^2+1)^{-p/2}$ is indeed trace-class. □

Let us now conclude the proof of Theorem 5.0.11. By Theorem 5.3.3 and Lemma 5.3.4, we have that $M \in \cap_k B(H^k)$, and we have likewise already seen that pre-regularity and finiteness are preserved. We can therefore apply Lemma 5.4.1 to conclude that (A, H, D) and (A, H, D_M) have the same noncommutative integration, and, hence, absolute continuity is preserved.

Chapter 6

Almost-commutative spectral triples

Nigdar ni tak bilo da ni nekak bilo,
pak ni vezda ne bu da nam nekak ne bu.
— M. Krleža, *Khevenhiller*

In the following, we shall motivate and propose both concrete (*viz*, global-analytic) and abstract (*viz*, noncommutative-geometric) definitions of almost-commutative spectral triples, and then state and prove a reconstruction theorem for almost-commutative spectral triples, thereby establishing the equivalence of the concrete and abstract definitions.

6.1 Concrete definitions

In order to motivate our new definitions, let us recall the conventional definition of almost-commutative spectral triple, or, for convenience, *Cartesian almost-commutative spectral triples*. In order to do so, however, we must recall the following definition:

Definition 6.1.1. Let $X_1 = (A_1, H_1, D_1)$ and $X_2 = (A_2, H_2, D_2)$ be spectral triples. Then the *product* $X_1 \times X_2$ of X_1 and X_2 is the spectral triple defined as follows:

1. If X_1 and X_2 are both even with $\mathbb{Z}_2$-gradings γ_1 and γ_2 respectively, then

$$X_1 \times X_2 := (A_1 \otimes A_2, H_1 \otimes H_2, D_1 \otimes 1 + \gamma_1 \otimes D_2, \gamma_1 \otimes \gamma_2).$$

2. If X_1 is even with $\mathbb{Z}_2$-grading γ_1 and X_2 is odd, then

$$X_1 \times X_2 := (A_1 \otimes A_2, H_1 \otimes H_2, D_1 \otimes 1 + \gamma_1 \otimes D_2).$$

3. If X_1 is odd and X_2 is even with $\mathbb{Z}_2$-grading γ_2, then

$$X_1 \times X_2 := (A_1 \otimes A_2, H_1 \otimes H_2, D_1 \otimes \gamma_2 + 1 \otimes D_2).$$

4. If X_1 and X_2 are odd, then

$$X_1 \times X_2 := (A_1 \otimes A_2, H_1 \otimes H_2 \otimes \mathbb{C}^2, D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_2, 1 \otimes 1 \otimes \sigma_3),$$

where the σ_k are the Pauli sigma matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark 6.1.2. In the case where both X_1 and X_2 are even, one could alternatively construct the Dirac operator of $X_1 \times X_2$ as $D_1 \otimes \gamma_2 + 1 \otimes D_2$; the resulting spectral triple is, then, unitarily equivalent to $X_1 \times X_2$ as constructed above.

That the product of spectral triples is indeed a spectral triple does require verification—see [32, 53] for details. In particular, Otgonbayar proves that the product of regular spectral triples is again regular [53, Prop. 3.1.32].

The conventional definition of almost-commutative spectral triple then reads as follows:

Definition 6.1.3. A *Cartesian almost-commutative spectral triple* is a spectral triple of the form $X \times F$, where X is a compact spin manifold with fixed spin structure, identified, by abuse of notation, with its canonical commutative spectral triple $(C^\infty(X), L^2(X, \mathcal{S}), \not{D})$, and F is a finite spectral triple.

Remark 6.1.4. Note that this definition is not stable under inner fluctuations of the metric, for if $M = \sum_{i=1}^n a_i[D, b_i]$ for non-constant a_i, $b_i \in C^\infty(X) \otimes \mathcal{A}_F$, then M is generally not of the form $1 \widehat{\otimes} T$ for some constant $T \in B(H_F)$.

Now, for simplicity, let X be an even-dimensional compact spin manifold with fixed spinor bundle $\mathcal{S} \to X$ and corresponding Dirac operator $\not{D}$, and let $F = (A_F, H_F, D_F)$ be an even finite spectral triple. Then, by the above definition,

$$X \times F := (C^\infty(X) \widehat{\otimes} A_F, L^2(X, \mathcal{S}) \widehat{\otimes} H_F, \not{D} \widehat{\otimes} 1 + 1 \widehat{\otimes} D_F).$$

Let us now make some observations:

1. $L^2(X, \mathcal{S}) \widehat{\otimes} H_F = L^2(X, \mathcal{E})$, where $\mathcal{E} := \mathcal{S} \otimes (X \times H_F)$ is a twisted spinor bundle, and hence, in particular, a Clifford module.

2. $\mathcal{A}$ is a locally trivial bundle of finite-dimensional C^*-algebras, and that the inclusion $\mathcal{A} \hookrightarrow \mathrm{End}(\mathcal{E})$ is a morphism of local trivial bundles of finite-dimensional C^*-algebras, and sections of $\mathcal{A}$ act on $\mathcal{E}$ as even bundle endomorphisms that commute with the Clifford action on $\mathcal{E}$.

3. $\not{D}\widehat{\otimes}1$ is the twisted Dirac operator on the twisted spinor bundle $\mathcal{E} = \mathcal{S} \otimes (X \times H_F)$ arising from the trivial connection on $X \times H_F$ and $1\widehat{\otimes}D_F$ is a symmetric bundle endomorphism, so that $D := \not{D}\widehat{\otimes}1 + 1\widehat{\otimes}D_F$ is a symmetric Dirac-type operator on the Clifford module $\mathcal{E}$.

It is on the basis of these observations, then, that we shall propose our manifestly global-analytic definition of almost-commutative spectral triple.

First, let us characterise the datum $\mathcal{A} := X \times A_F$:

Definition 6.1.5. Let X be a compact manifold. We define an *algebra bundle* to be a locally trivial bundle of finite-dimensional C^*-algebras. We also define a *representation* of an algebra bundle $\mathcal{A} \to X$ on a Hermitian vector bundle $\mathcal{E} \to X$ to be an injective morphism $\mathcal{A} \to \mathrm{End}(\mathcal{E})$ of locally-trivial bundles of algebra bundles, in which case we call $\mathcal{E} \to X$ an *$\mathcal{A}$-module.*

Next, let us characterise the datum $\mathcal{E} := \mathcal{S} \otimes (X \times H_F)$ and its relation to $\mathcal{A}$:

Definition 6.1.6. Let X be a compact oriented Riemannian manifold, and let $\mathcal{A} \to X$ be a bundle of algebras. We define a *Clifford $\mathcal{A}$-module* to be a Clifford module $\mathcal{E} \to X$ together with a faithful $*$-representation of $\mathcal{A}$, commuting with the Clifford action; if $\mathcal{E}$ is $\mathbb{Z}_2$-graded, we require sections of $\mathcal{A}$ to act as even operators on $\mathcal{E}$.

With these definitions in place, let us finally give our proposed definition:

Definition 6.1.7. A *concrete almost-commutative spectral triple* is a spectral triple of the form $(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D)$, where X is a compact oriented Riemannian manifold, $\mathcal{A} \to X$ is a bundle of algebras, $\mathcal{E} \to X$ is a Clifford $\mathcal{A}$-module, and D is a Dirac-type operator on $\mathcal{E}$.

Remark 6.1.8. This definition is stable under inner fluctuation of the metric, for a perturbation of a symmetric Dirac-type operator by a symmetric bundle endomorphism is a symmetric Dirac-type operator, inducing the same Clifford action.

Of course, one should check that our definition does indeed give rise to a spectral triple. Suppose that X is a compact oriented Riemannian manifold, $\mathcal{A}$ is an algebra bundle, $\mathcal{E}$ is a Clifford $\mathcal{A}$-bundle, and D is a symmetric Dirac-type operator on H. Then by standard analytic results about Dirac-type operators [37, Theorem 3.23], together with the fact that sections of A act as even bundle endomorphisms supercommuting with the Clifford action H, so that $[D, a]$ is a bundle endomorphism for all $a \in C^\infty(X, A)$, $(C^\infty(M, A), L^2(M, H), D)$ is indeed a spectral triple of metric dimension $\dim X$.

Now, since the square of a Dirac-type operator is a generalised Laplacian, this definition manifestly lends itself to perturbative computation of the spectral action [15] via heat kernel methods [35] (see [33] for a comprehensive account for product almost-commutative spectral triples). Another feature of this definition is that it encompasses non-trivial "fibrations" in the following sense:

Lemma 6.1.9. *Let X be a compact spin manifold with spinor bundle $\mathcal{S} \to X$ and Dirac operator $\not{D}$, and let $F = (A_F, H_F, D_F)$ be a finite spectral triple. Let G be a compact Lie group, and let ρ be an action of G on F, namely, a unitary representation of G on H_F, such that for each $g \in G$, $\rho(g)A_F\rho(g)^* \subset A_F$, and $\rho(g)D_F\rho(g)^* = D_F$; if F is even, we moreover require the action of G to commute with the $\mathbb{Z}_2$-grading. Let $\mathcal{P}$ be a principal G-bundle over X, and let $\nabla^{\mathcal{P}}$ be a connection on $\mathcal{P}$. Define $\mathcal{E}$ and $\mathcal{A}$ by*

$$\mathcal{E} := S\widehat{\otimes}(\mathcal{P} \times_\rho H_F), \quad \mathcal{A} := \mathcal{P} \times_\rho A_F,$$

and let $D = \not{D}_{\mathcal{P}\times_\rho \mathcal{H}_F} + 1\widehat{\otimes}D_F$, where $\not{D}_{\mathcal{P}\times_\rho H_F}$ is the twisted Dirac operator on the twisted spinor bundle $\mathcal{E}$ corresponding to the connection on $\mathcal{P} \times_\rho H_F$ induced by $\nabla^{\mathcal{P}}$. Then

$$X \times_{(\mathcal{P},\nabla^{\mathcal{P}})} F := (C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D)$$

is an almost-commutative spectral triple.

Proof. It follows immediately that $(C^\infty(X), L^2(X, \mathcal{E}), D)$ is at least an almost-commutative spectral triple. However, since $C^\infty(X, \mathcal{A}) = C^\infty(X, \mathcal{P} \times_\rho A_F)$ acts on $\mathcal{E}$ by

$$a\,(\sigma \otimes \xi) := \sigma \otimes a\xi, \quad a \in C^\infty(X, \mathcal{A}),\ \sigma \in C^\infty(X, \mathcal{S}),\ \xi \in C^\infty(X, \mathcal{P} \times_\rho H_F),$$

sections of A act as even bundle endomorphisms supercommuting with the Clifford action on H, so that $(C^\infty(X, A), L^2(X, H), D)$ is also an almost-commutative spectral triple. □

We can view $X \times_{(\mathcal{P},\nabla^{\mathcal{P}})} F$ as the product of X and F *twisted* by $(\mathcal{P}, \nabla_{\mathcal{P}})$; a concrete example of this construction has already been studied in detail by Boeijink–Van Suijlekom [5] in connection with the Yang-Mills theory. It is also worth noting that the data $(\mathcal{P} \times_\rho H_F, \nabla^{\mathcal{P}\times_\rho H_F}, D_F)$ can be viewed as defining a non-trivial morphism $X \times_{(\mathcal{P},\nabla^{\mathcal{P}})} F \to X$ in the category of spectral triples proposed by Mesland [52].

6.2 A reconstruction theorem

Now, we shall give an abstract definition of almost-commutative spectral triple, which shall depend upon an abstract definition of commutative spectral triple, identical to that proposed by Connes [24, 27], except for a weakening of the orientability condition.

Now, let $(C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D)$ be a concrete almost-commutative spectral triple. We may just as well consider it as being composed of two pieces:

1. a concrete commutative spectral triple $(C^\infty(X), L^2(X,\mathcal{E}), D)$.

2. an algebra bundle $\mathcal{A}$, together with a monomorphism $\mathcal{A} \to \mathrm{End}(\mathcal{E})$ of algebra bundles over X, such that sections of $\mathcal{A}$ acts as even operators that commute with the Clifford action on $\mathcal{E}$.

In order to obtain an abstract definition of almost-commutative spectral triple, it therefore suffices to translate these two components into the language of noncommutative geometry. We already know how to translate the first component via Corollary 4.2.7, our refinement of Connes's reconstruction theorem: $(C^\infty(X), L^2(X,\mathcal{E}), D)$ is simply a commutative spectral triple of metric dimension $\dim X$. So, it remains only to characterise the second component.

Now, what we would like to use is some suitable refinement of Serre–Swan for algebra bundles. Such a refinement does indeed exist, if for a slightly weaker notion of algebra bundle:

Definition 6.2.1 (cf. [5, Def. 3.1])**.** A *weak algebra bundle* is a complex vector bundle $\mathcal{A} \to X$, together with the following data:

- a morphism of complex vector bundles $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ over Id_X, such that $\mu \circ (\mathrm{Id}_\mathcal{A} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{Id}_\mathcal{A})$,

- a section $1_\mathcal{A} \in C^\infty(X,\mathcal{A})$, such that $\mu(\bullet \otimes 1_\mathcal{A}) = \mathrm{Id}_\mathcal{A} = \mu(1_\mathcal{A} \otimes \bullet)$,

- a fibre-wise conjugate-linear morphism of real vector bundles $J : A \to A$ over Id_X, such that $J^2 = \mathrm{Id}_\mathcal{A}$ and
$$\forall a_1, a_2 \in C^\infty(X,\mathcal{A}), \quad J \circ \mu(a_1 \otimes a_2) = \mu(Ja_2 \otimes Ja_1).$$

in other words, a weak algebra bundle over X is a unital $*$-algebra in the category of complex vector bundles over X.

Moreover, the category $\mathsf{WAlgB}(X)$ is the category whose objects are weak algebra bundles and whose morphisms are vector bundle morphisms $\phi : \mathcal{A} \to \mathcal{A}'$ over Id_X such that

$$\phi \circ \mu_\mathcal{A} = \mu_{\mathcal{A}'} \circ (\phi \otimes \phi), \quad \phi(1_\mathcal{A}) = 1_{\mathcal{A}'}, \quad \phi \circ J_\mathcal{A} = J_{\mathcal{A}'} \circ \phi.$$

It follows immediately that an algebra bundle is necessarily a weak algebra bundle. However, a weak algebra bundle need not be an algebra bundle, for there is no *a priori* reason why $(\mathcal{A}_x, \mu_x, J_x)$ and $(\mathcal{A}_y, \mu_y, J_y)$ should be isomorphic as $*$-algebras for $x \neq y$. However, since we are concerned specifically with algebra bundles $\mathcal{A}$ admitting Clifford $\mathcal{A}$-modules, the following lemma guarantees that the distinction between algebra bundles and weak algebra bundles is irrelevant for our purposes:

Lemma 6.2.2. *Let $\mathcal{A} \to X$ be a weak algebra bundle, and suppose that there exists an injective morphism $\iota : \mathcal{A} \to \mathcal{B}$ of weak algebra bundles for some algebra bundle $\mathcal{B} \to X$. Then $\mathcal{A}$ is an algebra bundle.*

Proof. By the hypothesis, we may treat $\mathcal{A}$ as a weak algebra sub-bundle of $\mathcal{B}$. However, since $\mathcal{B}$ is an algebra bundle, i.e., a locally trivial bundle of finite-dimensional C^*-algebras, $\mathcal{A}$ is precisely a vector sub-bundle of $\mathcal{B}$, such that for each $x \in X$, the unital $*$-algebra structure of $\mathcal{A}_x$ is precisely that of $\mathcal{A}_x$ as a vector subspace of the finite-dimensional C^*-algebra $\mathcal{B}_x$ closed under the multiplication and $*$-operation of $\mathcal{B}_x$ and containing the unit of $\mathcal{B}_x$. Since the transition functions of $\mathcal{A}$ are simply the restrictions of the transition functions of $\mathcal{B}$, which are fibrewise $*$-automorphisms, it follows that the weak algebra structure of $\mathcal{A}$ is precisely the structure of a locally trivial bundle of finite-dimensional C^*-algebras inherited from $\mathcal{B}$, as required. □

We have already seen, via the Serre–Swan theorem, that vector bundles over X correspond to finitely generated projective modules over $C^\infty(X)$. Let us now see what weak algebra bundles correspond to:

Definition 6.2.3. Let B be a commutative $*$-algebra. Then a *B-module $*$-algebra* is a finitely generated projective B-module A together with the following data:

- an A-module morphism $m : A \otimes_B A \to A$ such that $m \circ (\mathrm{Id}_A \otimes m) = m \circ (m \otimes \mathrm{Id}_A)$,
- an element $1_A \in A$ such that $m \circ (\bullet \otimes 1_A) = \mathrm{Id}_A = m \circ (1_A \otimes \bullet)$,
- a map $* : A \to A$ such that $*^2 = \mathrm{Id}_A$ and

$$\forall a_1, a_2 \in A,\ b \in B, \quad (ba_1 + a_2)^* = b^* a_1^* + a_2^*, \quad m(a_1 \otimes a_2)^* = m(a_2^* \otimes a_1^*);$$

in other words, a B-module algebra is a unital $*$-algebra in the category of finitely generated projective B-modules. Moreover, the category $\mathsf{ModAlg}(B)$ is the category whose objects are B-module algebras and whose morphisms are morphisms of B-modules $\phi : A \to A'$, such that

$$\phi \circ m_A = m_{A'} \circ \phi, \quad \phi(1_A) = 1_{A'}, \quad \phi \circ *_A = *_{A'} \circ \phi.$$

If $\mathcal{A} \to X$ is a weak algebra bundle with multiplication μ, unit $1_{\mathcal{A}}$, and involution J, then $A = C^\infty(X, \mathcal{A})$ has the structure of a $C^\infty(X)$-module $*$-algebra with multiplication m, unit $1_A := 1_{\mathcal{A}}$, and involution $*$, where

$$\forall a, b \in A\ m(a \otimes b) := \mu(a \otimes b), \quad a^* := J(a).$$

That, conversely, any $C^\infty(X)$-module $*$-algebra is isomorphic to $C^\infty(X, \mathcal{A})$ for some weak algebra bundle $\mathcal{A} \to X$ is, then, guaranteed by the following refinement of the Serre–Swan theorem for weak algebra bundles:

Theorem 6.2.4 (Boeijink–Van Suijlekom [5, Thm. 3.8])**.** *Let X be a compact manifold. Then the map $\mathcal{A} \mapsto C^\infty(X, \mathcal{A})$ defines an equivalence of categories $\mathsf{WAlgB}(X) \to \mathsf{ModAlg}(C^\infty(X))$.*

As a result, a concrete almost-commutative spectral triple $(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D)$ can be broken down into two constituents:

1. a commutative spectral triple $(C^\infty(X), L^2(X, \mathcal{E}), D)$,
2. a $C^\infty(X)$-module subalgebra $C^\infty(X, \mathcal{A})$ of $\mathrm{End}_{C^\infty(X)}(C^\infty(X, \mathcal{E})) = C^\infty(X, \mathrm{End}(\mathcal{E}))$ whose elements are even (if the spectral triple is even) and commute with the Clifford action on $\mathcal{E}$, i.e., $[[D, b], a] = 0$ for all $b \in C^\infty(X)$ and $a \in C^\infty(X, \mathcal{A})$.

Thus, the basic structure of $(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D)$ is encapsulated by the following proposed definition:

Definition 6.2.5 ([9, Def. 2.16])**.** Let (A, H, D) be a spectral triple, let B be a central unital $*$-subalgebra of A, and let $p \in \mathbb{N}$. We call (A, H, D) a p-dimensional *almost-commutative spectral triple* with *base* B if the following conditions hold:

1. (B, H, D) is a p-dimensional commutative spectral triple;
2. A is a B-module subalgebra of $\mathrm{End}_B(H^\infty)$.
3. $[[D, b], a] = 0$ for all $a \in A$, $b \in B$.

Our proposed reconstruction theorem for almost-commutative spectral triples is therefore as follows:

Theorem 6.2.6 ([9, Thm. 2.17])**.** *Let (A, H, D) be a p-dimensional almost-commutative spectral triple with base B. Then there exist a compact oriented Riemannian p-manifold X, a bundle of algebras $\mathcal{A} \to X$, and a Clifford $\mathcal{A}$-module $\mathcal{E} \to X$, such that $B \cong C^\infty(X)$ and $(A, H, D) \cong (C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D)$, where D is identified with an essentially self-adjoint Dirac-type operator on $\mathcal{E}$.*

Proof. First, by Corollary 4.2.7 applied to (B, H, D), there exist a compact oriented Riemannian p-manifold X and a self-adjoint Clifford module bundle $\mathcal{E} \to X$, such that $B = C^\infty(X)$, $H = L^2(X, \mathcal{E})$, and D is an essentially self-adjoint Dirac-type operator on $\mathcal{E}$.

Next, condition (2), together with Theorem 6.2.4, implies that $\mathcal{A} = C^\infty(X, \mathcal{A})$ for $\mathcal{A}$ a weak algebra bundle over X.

Finally, condition (3) for almost-commutative spectral triples, together with the fact that D is a Dirac-type operator on the Clifford module bundle $\mathcal{E}$, implies that $\mathcal{A}$ can be identified as a weak algebra sub-bundle of $\operatorname{End}(\mathcal{E})$, whose sections act as even operators (if our spectral triple is even), commuting with the Clifford action on $\mathcal{E}$ induced by D. Hence, by Lemma 6.2.2, $\mathcal{A}$ is in fact an algebra bundle and, hence, $\mathcal{E}$ is a Clifford $\mathcal{A}$-module. $\square$

6.3 Real structures

At last, let us consider real structures on almost-commutative spectral triples, both concrete and abstract. To see what a real structure looks like on a concrete almost-commutative spectral triple, it suffices to consider the traditional Cartesian product construction. Let us therefore recall the construction of a product of real spectral triples as formulated by Dąbrowski–Dossena, after Vanhecke [66]:

Theorem 6.3.1 (Dąbrowski–Dossena [32, §4])**.** *For $i = 1, 2$, let $X_i = (A_i, H_i, D_i, J_i)$ be a real spectral triple of KO-dimension $n_i \bmod 8$. Then $X_1 \times X_2$ can made into a real spectral triple of KO-dimension $n_1 + n_2 \bmod 8$ with real structure J defined as follows:*

1. *If n_1 and n_2 are both even, then $J_\pm := (J_1)_{\pm\varepsilon''(n_1)} \otimes (J_2)_\pm$;*
2. *If n_1 is even and n_2 is odd, then $J := (J_1)_{\varepsilon'(n_1+n_2)} \otimes J_2$;*
3. *If n_1 is odd and n_2 is even, then $J := J_1 \otimes (J_2)_{\varepsilon'(n_1+n_2)}$;*
4. *If n_1 and n_2 are both odd, then $J_\pm := J_1 \otimes J_2 \otimes M_\pm K$, for K the complex conjugation on $\mathbb{C}^2$ and (M_+, M_-) chosen as follows, with rows indexed by n_1 and columns indexed by n_2:*

	1	3	5	7
1	$(i\sigma_2, \sigma_1)$	(σ_3, σ_0)	$(i\sigma_2, \sigma_1)$	(σ_3, σ_0)
3	(σ_0, σ_3)	$(\sigma_1, i\sigma_2)$	(σ_0, σ_3)	$(\sigma_1, i\sigma_2)$
5	$(i\sigma_2, \sigma_1)$	(σ_3, σ_0)	$(i\sigma_2, \sigma_1)$	(σ_3, σ_0)
7	(σ_0, σ_3)	$(\sigma_1, i\sigma_2)$	(σ_0, σ_3)	$(\sigma_1, i\sigma_2)$

(6.3.1)

Now, let $X := (C^\infty(X), L^2(X, \mathcal{S}), \not{D}, C)$ be the canonical spectral triple of a compact spin n_1-manifold X with fixed spin structure, and let $F = (A_F, H_F, D_F, J_F)$ be a finite real spectral triple of KO-dimension $n_2 \bmod 8$. Ignoring the real structures, one has that $X \times F$ takes the form $(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D)$, where, in particular, $\mathcal{A} := X \times A_F$ is an algebra bundle, $\mathcal{E}$, formed from $\mathcal{S}$ and H_F, is a Clifford $\mathcal{A}$-module with Clifford action defined by $c(df) := [D, f]$ for $f \in C^\infty(X)$, and $\mathcal{A}$-module structure induced by the representation of A_F on H_F, and D, formed from $\not{D}$ and D_F, is a Dirac-type operator on the Clifford module $\mathcal{E}$. Now, since X and F are real spectral triples, by the above theorem, $X \times F$ is a real spectral triple with real structure J. Taking this into account, we see that $\mathcal{E}$ is a Clifford $\mathcal{A} \otimes \mathcal{A}^o$-module with $\mathcal{A} \otimes \mathcal{A}^o$-module structure defined by

$$(a \otimes b^o)\xi := aJb^*J^*\xi, \quad a, b \in C^\infty(X, \mathcal{A}),\ \xi \in C^\infty(X, \mathcal{E}),$$

that D therefore satisfies the addition constraint that

$$[[D, a], b^o] = 0, \quad a, b \in C^\infty(X, \mathcal{A}),$$

and, hence, that J is an antiunitary bundle endomorphism on $\mathcal{E}$, satisfying

$$Jc(\omega^* \otimes a^* \otimes (b^*)^o)J^* = c(\tau_{\varepsilon'}(\omega) \otimes b \otimes a^o), \quad \omega \in C^\infty(X, \mathbb{C}\mathrm{l}(X)),\ a, b \in C^\infty(X, \mathcal{A}).$$

Thus, the additional structure provided by the real structure J is encoded in the following definition:

Definition 6.3.2. Let X be a compact oriented Riemannian manifold, let $\mathcal{A} \to X$ be an algebra bundle, and let $\mathcal{E} \to X$ be a Clifford $\mathcal{A} \otimes \mathcal{A}^o$-module, which may or may not be $\mathbb{Z}_2$-graded. Let J be an antiunitary bundle automorphism on $\mathcal{E}$, and let $n \in \mathbb{Z}_8$. We call $(\mathcal{E}, J)$ a *real Clifford $\mathcal{A}$-bimodule of KO-dimension* n mod 8 if $\mathcal{E}$ is $\mathbb{Z}_2$-graded with $\mathbb{Z}_2$-grading γ when n is even, and J satisfies the following:

1. $J^2 = \varepsilon \,\mathrm{Id}_{\mathcal{E}}$,

2. for all $\omega \in C^\infty(X, \mathbb{C}\mathrm{l}(X))$ and a, $b \in C^\infty(X, \mathcal{A})$,

$$J(c(\omega^* \otimes a^* \otimes (b^*)^o)J^* = c(\tau_{\varepsilon'}(\omega) \otimes b \otimes a^o),$$

3. $J\gamma = \varepsilon''\gamma J$ if n is even,

where $(\varepsilon, \varepsilon', \varepsilon'') \in \{\pm 1\}^3$ is determined by n mod 8 according Table 2.2.1.

Remark 6.3.3. Once more, just as before, if n is even, then we can replace J with $J\gamma$ to go reversibly between the "conventional" KO-dimension n_+ and the "exotic" KO-dimension n_-.

Remark 6.3.4. Condition (2) in the above definition can be viewed as specifying the compatibility of J with the Clifford $\mathcal{A}$-bimodule structure on $\mathcal{E}$, for J, *a priori*, defines a $\mathbb{C}$-linear anti-involution $T \mapsto JT^*J^*$ on $\mathrm{End}(\mathcal{E})$.

Compatibility of the Dirac-type operator D with the real Clifford $\mathcal{A}$-bimodule $(\mathcal{E}, J)$ is then encoded in the following definition:

Definition 6.3.5. Let X be a compact oriented Riemannian manifold, let $\mathcal{A} \to X$ be a bundle of algebras, and let $(\mathcal{E}, J)$ be a real Clifford $\mathcal{A}$-bimodule of KO-dimension $n \bmod 8$. Let D be a Dirac-type operator on $\mathcal{E}$. We say that D is $(\mathcal{A}, J)$-*compatible* if it is J-compatible, and if

$$[[D, a], b^o] = 0, \quad a, b \in C^\infty(X, \mathcal{A}).$$

Thus, if $(\mathcal{E}, J)$ is a real Clifford $\mathcal{A}$-bimodule of KO-dimension $n \bmod 8$ over X, and D is a Dirac-type operator on $\mathcal{E}$, then $(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D, J)$ defines a real spectral triple of KO-dimension $n \bmod 8$ if and only if D is $(\mathcal{A}, J)$-compatible. Indeed, one can therefore give the following definition, generalising the example of the product of a spin manifold with a finite real spectral triple:

Definition 6.3.6. A *concrete real almost-commutative spectral triple* is a real spectral triple of the form $(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D, J)$, where X is a compact oriented Riemannian manifold, $\mathcal{A} \to X$ is an algebra bundle, $(\mathcal{E}, J)$ is a real Clifford $\mathcal{A}$-bimodule, and D is a $(\mathcal{A}, J)$-compatible Dirac-type operator on $\mathcal{E}$.

In fact, in light of this more general definition, we can take any concrete real commutative spectral triple $V := (C^\infty(X), L^2(X, \mathcal{V}), D_\mathcal{V}, J_\mathcal{V})$ of KO-dimension $n_1 \bmod 8$ and any finite real spectral triple F of KO-dimension $n_2 \bmod 8$ to form the concrete real almost-commutative spectral triple $V \times F$ of KO-dimension $n_1 + n_2 \bmod 8$. Applying this to Example 4.3.5, we immediately obtain the following

Proposition 6.3.7. *Let X be a compact oriented Riemannian manifold. Then for any $n \in \mathbb{Z}_8$ there exists a concrete real almost-commutative spectral triple of KO-dimension $n \bmod 8$, namely, $(C^\infty(X), L^2(X, \wedge T^*_\mathbb{C} X), d + d^*, (-1)^{|\cdot|}, K) \times F$ for any finite real spectral triple F of KO-dimension $n \bmod 8$, and, hence, a concrete real commutative spectral triple of KO-dimension $n \bmod 8$.*

Given a real Clifford $\mathcal{A}$-bimodule $(\mathcal{E}, J)$, one can again ask, just as in the commutative case, if a Dirac-type operator on $\mathcal{E}$ compatible with its Clifford action and the bimodule structure is necessarily $(\mathcal{A}, J)$-compatible. Once more, the answer is yes, up to perturbation by a symmetric bundle endomorphism:

Proposition 6.3.8. *Let X be a compact oriented Riemannian manifold, let $(\mathcal{E}, J)$ be a real Clifford module on X of KO-dimension $n \bmod 8$, and let D be a Dirac-type operator on $\mathcal{E}$, satisfying*

$$[[D, a], b^o] = 0, \quad a, b \in C^\infty(X, \mathcal{A}).$$

Then there exists a unique symmetric bundle endomorphism M on $\mathcal{E}$, such that $D - M$ is an $(\mathcal{A}, J)$-compatible Dirac-type operator, and $MJ = -\varepsilon' JM$.

Now, let us derive the corresponding abstract definition of real almost-commutative spectral triple, so that we can get the appropriate refinement of the reconstruction theorem for almost-commutative spectral triples.

Let $(A, H, D, J) := (C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D, J)$ be a concrete real almost-commutative spectral triple. On the one hand, (A, H, D) is, in particular, an almost-commutative spectral triple with base (viz, distinguished central unital $*$-subalgebra of A) $B := C^\infty(X)$. On the other hand, by Lemma 4.3.2, A contains a canonical central unital $*$-subalgebra $\tilde{A}_J$, which, moreover, contains B precisely because J is, in particular, an antilinear bundle endomorphism of $\mathcal{E}$. It has already been observed in specific examples (e.g., the noncommutative-geometric Standard Model [18, Lemma 3.2])that B and $\tilde{A}_J$ are, in fact, equal—as it turns out, this is a completely general phenomenon.

Proposition 6.3.9. *Let $(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D, J)$ be a concrete real almost-commutative spectral triple. Then $\widetilde{C^\infty(X, \mathcal{A})}_J = C^\infty(X)1_{\mathcal{A}}$.*

This result is an immediate corollary of the following algebraic observation, applied pointwise:

Lemma 6.3.10. *Let A_F be a finite C^*-algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then*

$$\{a \in A_F \mid a \otimes 1 = 1 \otimes a \in A_F \otimes_{\mathbb{K}} A_F\} = \mathbb{K}1_{A_F}.$$

Proof. By Wedderburn's theorem for finite-dimensional C^*-algebras, write

$$A_F = \oplus_{k=1}^N M_k(\mathbb{K}_k),$$

where $\mathbb{K}_k \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ if $\mathbb{K} = \mathbb{R}$, and $\mathbb{K}_k = \mathbb{C}$ if $\mathbb{K} = \mathbb{C}$. By construction, then,

$$\{a \in A_F \mid a \otimes 1 = 1 \otimes a \in A_F \otimes_{\mathbb{K}} A_F\} \subset Z(A_F) \cong \bigoplus_{k=1}^N \mathbb{K}_k',$$

where $\mathbb{K}_k' := \mathbb{C}$ if $\mathbb{K}_k = \mathbb{C}$, and $\mathbb{K}_k' := \mathbb{R}$ otherwise, so that

$$Z(A_F) \otimes_{\mathbb{K}} Z(A_F) = \bigoplus_{k,l=1}^N \mathbb{K}_k' \otimes_{\mathbb{K}} \mathbb{K}_l'.$$

Now, let $a \in Z(A_F)$, which we identify with $(\lambda_k)_{k=1}^N \in \oplus_{k=1}^N \mathbb{K}_k'$. Then

$$a \otimes 1 - 1 \otimes a = (\lambda_k \otimes 1 - 1 \otimes \lambda_l)_{k,l=1}^N,$$

so that $a \otimes 1 = 1 \otimes a$, if and only if $\lambda_k \otimes 1 = 1 \otimes \lambda_k$ for all $1 \leq k, l \leq N$. We have two cases. First, suppose that $\mathbb{K} = \mathbb{C}$. It therefore follows that $a \otimes 1 = 1 \otimes a$, if and only if $\lambda_k = \lambda_l$ for all k, l, if and only if $a \in \mathbb{C}1_{A_F}$. Now, suppose that $\mathbb{K} = \mathbb{R}$. Then, similarly, $a \otimes 1 = 1 \otimes a$, if and only if $\lambda_k = \lambda_l \in \mathbb{R}$ for all k, l, if and only if $a \in \mathbb{R}1_{A_F}$. ∎

Our observations motivate the following definition:

Definition 6.3.11. Let (A, H, D, J) be a real spectral triple. We call (A, H, D, J) a *real almost-commutative spectral triple* if (A, H, D) is an almost-commutative spectral triple with base $\tilde{A}_J$.

We have just seen that every concrete real almost-commutative spectral triple is a real almost-commutative spectral triple; the reconstruction theorem for almost-commutative spectral triples readily implies the converse.

Theorem 6.3.12. *Let (A, H, D, J) be a real almost-commutative spectral triple of KO-dimension $n \bmod 8$ and metric dimension p. Then there exist a compact oriented Riemannian p-manifold X, a bundle of algebras $\mathcal{A} \to X$, and a Clifford $\mathcal{A}$-bimodule $\mathcal{E} \to X$, such that $\tilde{A}_J \cong C^\infty(X)$ and*

$$(A, H, D, J) \cong (C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D, J),$$

where, viewing D and J as operators on $\mathcal{E}$, $(\mathcal{E}, J)$ is a real Clifford $\mathcal{A}$-bimodule of KO-dimension $n \bmod 8$, and D is a $(\mathcal{A}, J)$-compatible essentially self-adjoint Dirac-type operator on $\mathcal{E}$.

Proof. First, by Theorem 6.2.6, there exist a compact oriented Riemannian p-manifold X, an algebra bundle $\mathcal{A} \to X$, and a Clifford $\mathcal{A}$-module $\mathcal{E} \to X$, such that $\tilde{A}_J \cong C^\infty(X)$ and $(A, H, D) \cong (C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D)$, where D, viewed as an operator on $\mathcal{E}$, is an essentially self-adjoint Dirac-type operator. Viewing J as an operator on $L^2(X, \mathcal{E})$, in light of Proposition 4.3.12, we therefore have that $(C^\infty(X), L^2(X, \mathcal{E}), D, J)$ is a real Dirac-type commutative spectral triple of KO-dimension $n \bmod 8$, and hence that $(\mathcal{E}, J)$ is a real Clifford module of KO-dimension $n \bmod 8$. Finally, the conditions for a real spectral triple imply that $(\mathcal{E}, J)$ is a real Clifford $\mathcal{A}$-bimodule, with $\mathcal{A} \otimes \mathcal{A}^o$-module structure given by

$$(a \otimes b^o)\xi := aJb^*J^*\xi, \quad a, b \in C^\infty(X, \mathcal{A}),\ \xi \in C^\infty(X, \mathcal{E}),$$

and that D is $(\mathcal{A}, J)$-compatible, as required. □

6.4 Twistings

We have already seen how to generalise the conventional definition of real almost-commutative spectral triple into a form suited to a reconstruction theorem. For physical applications, however, it is useful to consider a more conservative generalisation, where we take the product of a concrete real commutative spectral triple not with a single finite real spectral triple, but with a family of such spectral triples, equipped with suitable connection:

Definition 6.4.1. Let X be a compact oriented Riemannian manifold. A *real family of KO-dimension $n \bmod 8$* over X is a quintuple of the form $(\mathcal{A}, \mathcal{F}, \nabla^{\mathcal{F}}, D_{\mathcal{F}}, J_{\mathcal{F}})$, where:

1. $\mathcal{A} \to X$ is an algebra bundle;

2. $\mathcal{F} \to X$ is an $\mathcal{A} \otimes \mathcal{A}^o$-module endowed, if n is even, with a $\mathbb{Z}_2$-grading $\gamma_{\mathcal{F}}$, commuting with all sections of $\mathcal{A} \otimes \mathcal{A}^o$;

3. $\nabla^{\mathcal{F}}$ is a self-adjoint connection on $\mathcal{F}$, odd if n is even, such that the induced connection on $\mathrm{End}(\mathcal{F})$ restricts to connections on $\mathcal{A}$ and on $\mathcal{A}^o$;

4. $D_{\mathcal{F}}$ is a symmetric bundle endomorphism on $\mathcal{F}$, odd if n is even, satisfying

$$[[D_{\mathcal{F}}, a], b^o] = 0, \quad a, b \in C^\infty(X, \mathcal{A}),$$

and anticommuting with $\gamma_{\mathcal{F}}$ if n is even;

5. $J_{\mathcal{F}}$ is an antiunitary bundle endomorphism on $\mathcal{F}$ such that

   (a) $J_{\mathcal{F}}^2 = \varepsilon \, \mathrm{Id}_{\mathcal{F}}$,

   (b) $D_{\mathcal{F}} J_{\mathcal{F}} = \varepsilon' J_{\mathcal{F}} D_{\mathcal{F}}$, $\nabla^{\mathcal{F}} \circ J_{\mathcal{F}} = J_F \circ \nabla^{\mathcal{F}}$, and $J_{\mathcal{F}} a^* J_{\mathcal{F}}^* = a^o$ for all $a \in C^\infty(X, \mathcal{A})$,

   (c) $\gamma_{\mathcal{F}} J_{\mathcal{F}} = \varepsilon'' J_{\mathcal{F}} \gamma_{\mathcal{F}}$, if n is even,

   where $(\varepsilon, \varepsilon', \varepsilon'') \in \{\pm 1\}^3$ depend on n mod 8 according to Table 2.2.1 (or, equivalently, according to Table 2.2.2).

Remark 6.4.2. For each $x \in X$, $(\mathcal{A}_x, \mathcal{F}_x, (D_{\mathcal{F}})_x, (J_{\mathcal{F}})_x)$ is a finite real spectral triple of KO-dimension n mod 8, and the real family can be viewed as a family $(\mathcal{A}, \mathcal{F}, D_{\mathcal{F}}, J_{\mathcal{F}})$ of finite real spectral triples over X together with Bismut superconnection $D_{\mathcal{F}} + \nabla^{\mathcal{F}}$.

If $F = (A_F, H_F, D_F, J_F)$ is a single finite real spectral triple of KO-dimension n mod 8, and X is a compact oriented Riemannian manifold, then for $\mathcal{A} := X \times A_F$, we can define a real family of KO-dimension n mod 8 by

$$(\mathcal{A}, \mathcal{F}, \nabla^{\mathcal{F}}, D_{\mathcal{F}}, J_{\mathcal{F}}) := (X \times A_F, X \times H_F, d, \mathrm{Id}_X \times D_F, \mathrm{Id}_X \times J_F).$$

More generally, let G be a compact Lie group acting on F, in the sense that there exists a unitary representation $U : G \to U(H_F)$ such that for all $g \in G$,

$$U(g) A_F U(g)^* \subset A_F, \quad [U(g), D_F] = 0, \quad [U(g), J_F] = 0,$$

with each $U(g)$ even if n is even, and let $\mathcal{P} \to X$ be a principal G-bundle with connection $\nabla^{\mathcal{P}}$. Then we can define a real family of KO-dimension n mod 8 by

$$(\mathcal{A}, \mathcal{F}, \nabla^{\mathcal{F}}, D_{\mathcal{F}}, J_{\mathcal{F}}) := (\mathcal{P} \times_G A_F, \mathcal{P} \times_G H_F, \nabla^{\mathcal{P} \times_G H_F}, \mathrm{Id}_{\mathcal{P}} \times D_F, \mathrm{Id}_{\mathcal{P}} \times J_F),$$

with $\mathbb{Z}_2$-grading, if n is even, given by $\gamma_{\mathcal{F}} := \mathrm{Id}_{\mathcal{P}} \times \gamma_F$ for γ_F the $\mathbb{Z}_2$-grading of F.

In light of Definition 6.1.1 and Theorem 6.3.1, one therefore defines the twisting of a concrete real commutative spectral triple by a real family as follows:

Definition 6.4.3. Let $(C^\infty(X), L^2(X,\mathcal{V}), D_{\mathcal{V}}, J_{\mathcal{V}})$ be a concrete real commutative spectral triple of KO-dimension $m \bmod 8$ and metric dimension p, with $\mathbb{Z}_2$-grading $\gamma_{\mathcal{V}}$ if m is even, and let $(\mathcal{A}, \mathcal{F}, \nabla^{\mathcal{F}}, D_{\mathcal{F}}, J_{\mathcal{F}})$ be a real family of KO-dimension $n \bmod 8$, with $\mathbb{Z}_2$-grading $\gamma_{\mathcal{F}}$ is n is even. Then the *twisting* of $(C^\infty(X), L^2(X,\mathcal{V}), D_{\mathcal{V}}, J_{\mathcal{V}})$ by $(\mathcal{A}, \mathcal{F}, \nabla^{\mathcal{F}}, D_{\mathcal{F}}, J_{\mathcal{F}})$ is the concrete real almost-commutative spectral triple

$$(\mathcal{A}, \mathcal{F}, \nabla^{\mathcal{F}}, D_{\mathcal{F}}, J_{\mathcal{F}}) \times (C^\infty(X), L^2(X,\mathcal{V}), D_{\mathcal{V}}, J_{\mathcal{V}}) := (C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D, J),$$

of KO-dimension $m+n \bmod 8$ and metric dimension p, where $\mathcal{E}$, D and J are defined as follows:

1. If m and n are both even, then

$$\mathcal{E} := \mathcal{V} \otimes \mathcal{F}, \quad D := D_{\mathcal{V}} \otimes_{\nabla^{\mathcal{F}}} 1 + \gamma_{\mathcal{V}} \otimes D_{\mathcal{F}}, \quad J := (J_{\mathcal{V}})_{\varepsilon''(m)} \otimes J_{\mathcal{F}},$$

   with $\mathbb{Z}_2$-grading $\gamma := \gamma_{\mathcal{V}} \otimes \gamma_{\mathcal{F}}$.

2. If m is even and n is odd, then

$$\mathcal{E} := \mathcal{V} \otimes \mathcal{F}, \quad D := D_{\mathcal{V}} \otimes_{\nabla^{\mathcal{F}}} 1 + \gamma_{\mathcal{V}} \otimes D_{\mathcal{F}}, \quad J := (J_{\mathcal{V}})_{\varepsilon'(m+n)} \otimes J_{\mathcal{F}}.$$

3. If m is odd and n is even, then

$$\mathcal{E} := \mathcal{V} \otimes \mathcal{F}, \quad D := D_{\mathcal{V}} \otimes_{\nabla^{\mathcal{F}}} \gamma_{\mathcal{F}} + 1 \otimes D_{\mathcal{F}}, \quad J := J_{\mathcal{V}} \otimes (J_{\mathcal{F}})_{\varepsilon'(m+n)}.$$

4. If m and n are both odd, then

$$\mathcal{E} := \mathcal{V} \otimes \mathcal{F} \otimes \mathbb{C}^2, \quad D := D_{\mathcal{V}} \otimes_{\nabla^{\mathcal{F}}} 1 \otimes \sigma_1 + 1 \otimes D_{\mathcal{F}} \otimes \sigma_2, \quad J := J_{\mathcal{V}} \otimes J_{\mathcal{F}} \otimes MK,$$

   with $\mathbb{Z}_2$-grading $\gamma := 1 \otimes 1 \otimes \sigma_3$, where K is the complex conjugation on $\mathbb{C}^2$, and $M := M_+$ is given by Table 6.3.1.

In the expressions above, if $T = 1$ or $\gamma_{\mathcal{F}}$, then $D_{\mathcal{V}} \otimes_{\nabla^{\mathcal{F}}} T$ is defined locally by

$$(D_{\mathcal{V}} \otimes_{\nabla^{\mathcal{F}}} T)(\eta \otimes \xi) := (D_{\mathcal{V}}\eta) \otimes \xi + \sum_i (c(e^i)\eta) \otimes \nabla^{\mathcal{F}}_{e_i}(T\xi), \quad \eta \in C^\infty(X,\mathcal{V}),\ \xi \in C^\infty(X,\mathcal{F}),$$

where $\{e_i\}$ is a local vielbein on TX.

The essential point in checking that this definition makes sense is checking that one does, indeed, get a concrete almost-commutative spectral triple.

Remark 6.4.4. Let $(C^\infty(X), L^2(X,\mathcal{V}), D_\mathcal{V}, J_\mathcal{V})$ be a concrete real commutative spectral triple and let $(\mathcal{A}, \mathcal{F}, \nabla^\mathcal{F}, D_\mathcal{F}, J_\mathcal{F})$ be a real family; for simplicity, suppose that both are of even KO-dimension. Then $(C(X,\mathcal{F}), D_\mathcal{F}, \nabla^\mathcal{F})$ can be viewed as an unbounded $(C(X,\mathcal{A}), C(X))$-bimodule in the sense of Mesland, and the twisting

$$(\mathcal{A}, \mathcal{F}, \nabla^\mathcal{F}, D_\mathcal{F}, J_\mathcal{F}) \times (C^\infty(X), L^2(X,\mathcal{V}), D_\mathcal{V}, J_\mathcal{V}) =: (C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D, J)$$

can be viewed as an unbounded Kasparov product, that is,

$$(L^2(X,\mathcal{E}), D) \cong (C(X,\mathcal{F}), D_\mathcal{F}, \nabla^\mathcal{F}) \times (L^2(X,\mathcal{V}), D_\mathcal{V}),$$

and, hence, $(\mathcal{F}, \nabla^\mathcal{F}, D_\mathcal{F})$ defines a morphism

$$(C(X,\mathcal{F}), D_\mathcal{F}, \nabla^\mathcal{F}) : (C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D) \to (C^\infty(X), L^2(X,\mathcal{V}), D_\mathcal{V})$$

in Mesland's category of spectral triples.

Finally, let us record the consequences of this construction for the structure of inner fluctuations of the metric:

Proposition 6.4.5. *Let $(C^\infty(X), L^2(X,\mathcal{V}), D_\mathcal{V}, J_\mathcal{V})$ be a concrete real commutative spectral triple of KO-dimension $m \bmod 8$ and let $(\mathcal{A}\mathcal{F}, \nabla^\mathcal{F}, D_\mathcal{F}, J_\mathcal{F})$ be a real family of KO-dimension $n \bmod 8$. Let*

$$(C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D, J) := (\mathcal{A}, \mathcal{F}, \nabla^\mathcal{F}, D_\mathcal{F}, J_\mathcal{F}) \times (C^\infty(X), L^2(X,\mathcal{V}), D_\mathcal{V}, J_\mathcal{V}),$$

which is a concrete real almost-commutative spectral triple of KO-dimension $m+n \bmod 8$. Let $\mathbb{A} \in C^\infty(X, \operatorname{End}(\mathcal{E}))$ be an inner fluctuation of the metric on the twisting, i.e., symmetric and of the form $\mathbb{A} = \sum_i a_i[D, b_i]$ for a_i, $b_i \in C^\infty(X,\mathcal{A})$, and let

$$\omega_\mathbb{A} := \sum_i \left(a_i \wedge (\nabla^{\operatorname{End}\mathcal{F}} b_i) - (\nabla^{\operatorname{End}\mathcal{F}} b_i^o) \wedge a_i^o\right) \in \Omega^1(X, \mathcal{A} \otimes \mathcal{A}^o),$$
$$\Phi_\mathbb{A} := \sum_i (a_i[D_\mathcal{F}, b_i] - [D_\mathcal{F}, b_i^o]a_i^o) \in C^\infty(X, \operatorname{End}(\mathcal{F})).$$

Then $(\mathcal{A}, \mathcal{F}, \nabla^\mathcal{F} + \omega_\mathbb{A}, D_\mathcal{F} + \Phi_\mathbb{A}, J_\mathcal{F})$ is a real family of KO-dimension $n \bmod 8$, such that

$$(C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D + \mathbb{A} + \varepsilon' J\mathbb{A}J^*, J) = (\mathcal{A}, \mathcal{F}, \nabla^\mathcal{F} + \omega_\mathbb{A}, D_\mathcal{F} + \Phi_\mathbb{A}, J_\mathcal{F}) \times (C^\infty(X), L^2(X,\mathcal{V}), D_\mathcal{V}, J_\mathcal{V}).$$

Proof. For simplicity, let us work in the case where m and n are even; the other cases will follow *mutatis mutandis*. Let $\mathbb{A} = \sum_i a_i[D, b_i]$ for $a_i,\ b_i \in C^\infty(X, \mathcal{A})$, and suppose that $\mathbb{A}$ is self-adjoint, so that

$$\sum_i a_i[D, b_i] = \mathbb{A} = \mathbb{A}^* = -\sum_i [D, b_i^*]a_i^*.$$

Now, let us work on a coordinate patch $U \subset X$ with local vielbein $\{e_k\}$, so that

$$D = D_{\mathcal{V}} \otimes 1 + \sum_k c(e^k) \otimes \nabla^{\mathcal{F}}_{e_k} + \chi_{\mathcal{F}} \otimes D_{\mathcal{F}}.$$

On the one hand, then,

$$\begin{aligned}
\mathbb{A} &= \sum_i a_i \left[D_{\mathcal{V}} \otimes 1 + \sum_k c(e^k) \otimes \nabla^{\mathcal{F}}_{e_k} + \chi_{\mathcal{F}} \otimes D_{\mathcal{F}}, b_i \right] \\
&= \sum_i (1 \otimes a_i) \left(\sum_k c(e^k) \otimes \nabla^{\operatorname{End}\mathcal{F}}_{e_k} b_i + \chi_{\mathcal{F}} \otimes [D_{\mathcal{F}}, b_i] \right) \\
&= c\left(\sum_i a_i \wedge \nabla^{\operatorname{End}\mathcal{F}} b_i \right) + \chi_{\mathcal{F}} \otimes \left(\sum_i a_i [D_{\mathcal{F}}, b_i] \right),
\end{aligned}$$

whilst on the other hand,

$$\begin{aligned}
\varepsilon' J \mathbb{A} J^* &= \varepsilon' J \left(-\sum_i [D, b_i^*] a_i^* \right) J^* \\
&= -\sum_i [D, J b_i^* J^*] J a_i^* J^* \\
&= -\sum_i [D, b_i^o] a_i^o \\
&= -\sum_i \left[D_{\mathcal{V}} \otimes 1 + \sum_k c(e^k) \otimes \nabla^{\operatorname{End}\mathcal{F}}_{e_k} + \chi_{\mathcal{F}} \otimes D_{\mathcal{F}}, b_i^o \right] a_i^o \\
&= -c\left(\sum_i \nabla^{\operatorname{End}\mathcal{F}} b_i^o \wedge a_i^o \right) + \chi_{\mathcal{F}} \otimes \left(\sum_i [D_{\mathcal{F}}, b_i^o] a_i^o \right).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{A} + \varepsilon' J \mathbb{A} J^* =& c\left(\sum_i a_i \wedge \nabla^{\operatorname{End}\mathcal{F}} b_i \right) + \chi_{\mathcal{F}} \otimes \left(\sum_i a_i [D_{\mathcal{F}}, b_i] \right) \\
& - c\left(\sum_i \nabla^{\operatorname{End}\mathcal{F}} b_i^o \wedge a_i^o \right) + \chi_{\mathcal{F}} \otimes \left(\sum_i [D_{\mathcal{F}}, b_i^o] a_i^o \right) \\
=& c\left(\sum_i \left(a_i \wedge \nabla^{\operatorname{End}\mathcal{F}} b_i - \nabla^{\operatorname{End}\mathcal{F}} b_i^o \wedge a_i^o \right) \right) + \chi_{\mathcal{F}} \otimes \left(\sum_i \left(a_i [D_{\mathcal{F}}, b_i] - [D_{\mathcal{F}}, b_i^o] a_i^o \right) \right) \\
=& c(\omega_{\mathbb{A}}) + \chi_{\mathcal{F}} \otimes \Phi_{\mathbb{A}},
\end{aligned}$$

as required. □

Thus, for a real almost-commutative spectral triple formed by the twisting of a concrete real commutative spectral triple by a real family, inner fluctuations of the metric are are effected at the level of the real family, so that the concrete real commutative spectral triple may be viewed strictly as background data. For a general, KK-theoretic discussion of this kind of phenomenon, see [7].

6.5 Applications to the spectral action

Let us illustrate the potential of our manifestly global-analytic approach to almost-commutative spectral triples with an application to noncommutative-geometric mathematical physics, and in particular, to certain computations by Teh of the spectral action on quotients of S^3 by finite subgroups of $S^3 \cong \mathrm{SU}(2)$. This work was the author's contribution to [11].

In what follows, let $\mathcal{L}$ denote the Laplace transform, and let $\mathcal{S}(0,\infty) = \{\phi \in \mathcal{S}(\mathbb{R}) \mid \phi(x) = 0,\ x \leq 0\}$. Let us begin by recalling the definition of spectral action, together with its basic features.

Proposition 6.5.1. *Let (A, H, D) be a spectral triple of metric dimension p, and let $f : \mathbb{R} \to \mathbb{C}$ be even. If $|f(x)| = O(|x|^{-\alpha})$ as $x \to \infty$, for some $\alpha > p$, then for any $\Lambda > 0$, $f(D/\Lambda) = f(|D|/\Lambda)$ is trace-class. If, in addition, $f(x) = \mathcal{L}[\phi](x^2)$ for some measurable $\phi : \mathbb{R}_+ \to \mathbb{C}$, then*

$$\mathrm{Tr}\left(f(D/\Lambda)\right) = \int_0^\infty \mathrm{Tr}\left(e^{-sD^2/\Lambda^2}\right)\phi(s)ds. \tag{6.5.1}$$

The quantity $\mathrm{Tr}\, f\left(D/\Lambda\right))$ is then called the *spectral action* on (A, H, D) with *cutoff function* f and *energy scale* $\Lambda > 0$.

Proof. Fix $\Lambda > 0$. Let μ_k denote the k-th eigenvalue of D^2 in increasing order, counted with multiplicity; since (A, H, D) has metric dimension p, $(\mu_k + 1)^{-p/2} = O(k^{-1})$ as $k \to \infty$, and, hence, for $k > \dim\ker D$, $\mu_k^{-1} = O(k^{-2/p})$ as $k \to \infty$. By our hypothesis on f, then, for $k > \dim\ker D$,

$$|f(\mu_k^{1/2}/\Lambda)| = O(k^{-\alpha/p}),\ k \to \infty;$$

since $2\alpha/p > 1$, this implies that $\sum_{k=1}^{\infty} f(\mu_k^{1/2}/\Lambda)$ is absolutely convergent, as required.

Now, suppose, in addition, that $f(x) = \mathcal{L}[\phi](x^2)$ for some measurable $\phi : [0,\infty) \to \mathbb{C}$. Then

$$\begin{aligned}\mathrm{Tr}\left(f(D/\Lambda)\right) &= \sum_{k=1}^{\infty} \mathcal{L}[\phi](\mu_k/\Lambda^2)\\ &= \sum_{k=1}^{\infty}\int_0^\infty e^{-s\mu_k/\Lambda^2}\phi(s)ds\\ &= \int_0^\infty \left[\sum_{k=1}^{\infty} e^{-s\mu_k/\Lambda^2}\right]\phi(s)ds\end{aligned}$$

$$= \int_0^\infty \mathrm{Tr}\left(e^{-sD^2/\Lambda^2}\right)\phi(s)ds,$$

as was claimed. □

The above result raises the question of when a function $\phi : \mathbb{R}_+ \to \mathbb{C}$ defines a function $f(x) = \mathcal{L}[\phi](x^2)$, such that $f(D/\Lambda)$ is trace-class; a sufficient condition is given by the following lemma.

Lemma 6.5.2. *If $\phi \in \mathcal{S}(0,\infty)$, then $\mathcal{L}[\phi](s) = O(s^{-k})$ as $s \to +\infty$, for all $k \in \mathbb{N}$.*

Proof. Since $\phi \in \mathcal{S}(0,\infty)$, $\phi^{(k)}$ is a bounded function with $\phi^{(k)}(0) = 0$ for all $k \in \mathbb{N}$, and hence $s^n\mathcal{L}[\phi](s) = \mathcal{L}[\phi^{(n)}](s)$ too is bounded, as required. □

Now, suppose that $(C^\infty(X,\mathcal{A}), L^2(X,\mathcal{E}), D)$ is a concrete almost-commutative spectral triple. Since we can write

$$\mathrm{Tr}(e^{-tD^2}) = \int_X \mathrm{tr}\left(K(t,x,x)\right) d\,\mathrm{vol}(x), \quad t > 0,$$

for $K(t,x,y)$ the heat kernel of D^2, it follows that for f of the form $f(x) = \mathcal{L}[\phi](x^2)$ for $\phi \in \mathcal{S}(0,\infty)$,

$$\mathrm{Tr}\left(f(D/\Lambda)\right) = \int_0^\infty \left[\int_M \mathrm{tr}\left(K(s/\Lambda^2,x,x)\right) d\,\mathrm{vol}(x)\right]\phi(s)ds, \quad \Lambda > 0. \tag{6.5.2}$$

Now, conventionally, the spectral action on almost-commutative spectral triples has been computed only asymptotically, using the Seeley–De Witt asymptotic expansion of the heat trace of a generalised Laplacian. However, Chamseddine–Connes were able to make the following non-asymptotic computation by means of an explicit Dirac spectrum and the Poisson summation formula:

Theorem 6.5.3 (Chameseddine–Connes [17, §2.2])**.** *Let S^3 denote the round sphere with trivial spin structure, and let $D\!\!\!/$ be the corresponding Dirac operator. Let $f : \mathbb{R} \to \mathbb{C}$ be of the form $f(x) = \mathcal{L}[\phi](x^2)$ for $\phi \in \mathcal{S}(0,\infty)$. Then*

$$\mathrm{Tr}\, f(D\!\!\!/ /\Lambda) = \Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda\widehat{f}(0) + O(\Lambda^{-\infty}),$$

where $\widehat{f}^{(2)}$ denotes the Fourier transform of $u \mapsto u^2 f(u)$.

Teh's aim, then, was to apply the methods of Chameseddine–Connes to the spectral action on quotients of S^3 by finite subgroups of SU(2). He carried this out, case by case, over the course of several papers [50, 51, 65], using explicit Dirac spectral and the Poisson summation formula, culminating in the following result:

Theorem 6.5.4 (Teh [65])**.** *Let S^3 denote the round sphere with trivial spin structure, let $\Gamma <$ $\mathrm{SU}(2) \cong S^3$ be finite, therefore acting on S^3 by orientation-preserving isometries, and let $D\!\!\!/^{\Gamma}$ denote* *the Dirac operator of S^3/Γ acting on the spinor bundle $\mathcal{S} \to S^3/\Gamma$. Let $f : \mathbb{R} \to \mathbb{C}$ be of the form $f(x) = \mathcal{L}[\phi](x^2)$ for $\phi \in \mathcal{S}(0,\infty)$. Then*

$$\operatorname{Tr} f(\not{D}^{\Gamma}/\Lambda) = \frac{1}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}),$$

where $\widehat{f}^{(2)}$ denotes the Fourier transform of $u \mapsto u^2 f(u)$.

More recently, Teh generalised these computations to more general almost-commutative spectral triples over a quotient S^3/Γ, allowing for twisting (in the sense of Lemma 6.1.9) by a unitary representation of Γ. Again, use was made of explicit Dirac spectra, which had been computed for precisely the relevant twisted Dirac operators on finite quotients of S^3 by Cisneros-Molina [19].

Theorem 6.5.5 (Teh [11, Thm. 1.1])**.** *Let S^3 denote the round sphere with trivial spin structure, let $\Gamma < \mathrm{SU}(2) \cong S^3$ be finite, therefore acting on S^3 by orientation-preserving isometries, and let $\mathcal{S} \to S^3/\Gamma$ be the resulting spinor bundle of S^3/Γ. Let $\alpha : \Gamma \to U(N)$ be a unitary representation, let $\mathcal{V}_\alpha = S^3 \times_\alpha \mathbb{C}^N$ be the resulting flat bundle, and let $\not{D}^{\Gamma}_{\alpha}$ denote the twisted Dirac operator on the twisted spinor bundle $\mathcal{S} \otimes \mathcal{V}_\alpha \to S^3/\Gamma$ corresponding to the canonical flat connection on $\mathcal{V}$. Then*

$$\operatorname{Tr} f(\not{D}^{\Gamma}_{\alpha}/\Lambda) = \frac{N}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}),$$

where $\widehat{f}^{(2)}$ denotes the Fourier transform of $u^2 f(u)$.

Let us now treat these considerations in a fully general, manifestly global-analytic light. Let $\widetilde{X} \to X$ be a finite normal Riemannian covering with $\widetilde{X}$ and X compact, connected and oriented, and let Γ be the deck group of the covering. Let $\widetilde{\mathcal{E}} \to \widetilde{X}$ be a Γ-equivariant Clifford module, and let $\widetilde{D}$ be a Γ-invariant symmetric Dirac-type operator on $\widetilde{\mathcal{E}}$. We can therefore form the quotient Clifford module bundle $\mathcal{E} := \widetilde{\mathcal{E}}/\Gamma \to \widetilde{X}/\Gamma =: X$, with $\widetilde{D}$ descending to a symmetric Dirac-type operator D on $\mathcal{E}$; under the identification $L^2(X,\mathcal{E}) \cong L^2(\widetilde{X},\widetilde{\mathcal{E}})^{\Gamma}$, we can identify D with the restriction of $\widetilde{D}$ to $C^\infty(\widetilde{X},\widetilde{\mathcal{E}})^{\Gamma}$, where the unitary action $U : \Gamma \to U(L^2(\widetilde{X},\widetilde{\mathcal{E}}))$ is given by $U(\gamma)\xi(\widetilde{x}) := \xi(\widetilde{x}\gamma^{-1})\gamma$.

Our first goal is to prove the following result, relating the spectral action of D to the spectral action of $\widetilde{D}$ in the high energy limit; in particular, it immediately implies Theorem 6.5.4 as a corollary of Theorem 6.5.3.

Theorem 6.5.6. *Let $f : \mathbb{R} \to \mathbb{C}$ be of the form $f(x) = \mathcal{L}[\phi](x^2)$ for $\phi \in \mathcal{S}(0,\infty)$. Then for $\Lambda > 0$,*

$$\operatorname{Tr}\left(f(D/\Lambda)\right) = \frac{1}{\#\Gamma}\operatorname{Tr}\left(f(\widetilde{D}/\Lambda)\right) + O(\Lambda^{-\infty}), \quad \text{as } \Lambda \to +\infty. \tag{6.5.3}$$

Remark 6.5.7. Theorem 6.5.6 continues to hold even when inner fluctuations of the metric are introduced, since for $A \in C^\infty(X, \mathrm{End}(\mathcal{V}))$ symmetric, $D + A$ on $\mathcal{E}$ lifts to $\widetilde{D} + \widetilde{A}$ on $\widetilde{\mathcal{E}}$, where $\widetilde{A}$ is the lift of A to $\widetilde{\mathcal{E}}$.

To prove this result, we shall need a couple of lemmas. First, we have the following well-known general fact:

Lemma 6.5.8. *Let Γ be a finite group acting unitarily on a Hilbert space H, and let A be a Γ-invariant self-adjoint trace-class operator on H. Let H^Γ denote the subspace of H consisting of Γ-invariant vectors. Then the restriction $A \mid H^\Gamma$ of A to H^Γ is also trace-class, and*

$$\operatorname{Tr}\left(A \mid H^\Gamma\right) = \frac{1}{\#G} \sum_{g \in G} \operatorname{Tr}(gA).$$

Proof. This immediately follows from the observation that $\frac{1}{\#G}\sum_{g\in G} g$ is the orthogonal projection onto H^G. □

Now, we can compute the heat kernel trace of D using the heat kernel for $\widetilde{D}$:

Lemma 6.5.9. *For $t > 0$,*

$$\operatorname{Tr}\left(e^{-tD^2}\right) = \frac{1}{\#\Gamma}\operatorname{Tr}\left(e^{-t\widetilde{D}^2}\right) + \frac{1}{\#\Gamma}\sum_{\gamma\in\Gamma\setminus\{e\}} \int_{\widetilde{X}} \operatorname{tr}\left(\rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t, \widetilde{x}\gamma^{-1}, \widetilde{x})\right) d\operatorname{vol}(\widetilde{x}), \qquad (6.5.4)$$

where $\widetilde{K}(t,\widetilde{x},\widetilde{y})$ denotes the heat kernel of $\widetilde{D}^2$, and ρ denotes the right action of Γ on the total space of $\widetilde{\mathcal{E}}$.

Proof. Let $\gamma \in \Gamma$. Then for any $\xi \in C^\infty(\widetilde{X}, \widetilde{\mathcal{E}})$,

$$\begin{aligned}\left(U(\gamma)e^{-t\widetilde{D}^2}\right)\xi(\widetilde{x}) &= U(\gamma)\left(\int_{\widetilde{X}} \widetilde{K}(t,\widetilde{x},\widetilde{y})\xi(\widetilde{y})d\operatorname{vol}(\widetilde{y})\right)\\ &= \rho(\gamma)(\widetilde{x}\gamma^{-1})\left(\int_{\widetilde{X}} (\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{y})\xi(\widetilde{y})d\operatorname{vol}(\widetilde{y})\right)\\ &= \int_{\widetilde{X}} \rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{y})\xi(\widetilde{y})d\operatorname{vol}(\widetilde{y})\end{aligned}$$

so that the operator $U(\gamma)e^{-t\widetilde{D}^2}$ has the integral kernel

$$(t,\widetilde{x},\widetilde{y}) \mapsto \rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{y}).$$

Since $L^2(X,\mathcal{E}) \cong L^2(\widetilde{X},\widetilde{\mathcal{E}})^\Gamma$, we can therefore apply Lemma 6.5.8 to obtain the desired result. □

Finally, we can proceed with our proof:

Proof of Theorem 6.5.6. By our earlier observation and Lemma 6.5.9, it suffices to show that for $\gamma \in G \setminus \{e\}$,

$$\int_0^\infty \left[\int_{\widetilde{X}} \operatorname{tr}\left(\rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(s/\Lambda^2, \widetilde{x}\gamma^{-1}, \widetilde{x})\right) d\operatorname{vol}(\widetilde{x})\right] \phi(s)ds = O(\Lambda^{-\infty}),$$

as $\Lambda \to \infty$.

Now, since $\widetilde{X}$ is compact, and since the finite group Γ acts freely and properly,

$$\inf_{(\widetilde{x},\gamma)\in\widetilde{X}\times\Gamma} d(\widetilde{x}\gamma^{-1},\widetilde{x}) = \min_{(\widetilde{x},\gamma)\in\widetilde{X}\times\Gamma} d(\widetilde{x}\gamma^{-1},\widetilde{x}) > 0.$$

Hence, by [45, Prop. 3.24], there exist constants $C > 0$, $c > 0$ such that

$$\sup_{\widetilde{x}\in\widetilde{X}} \|\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\|_2 \leq Ce^{-c/t}, \quad t > 0,$$

for $\|\cdot\|_2$ the fibre-wise Hilbert-Schmidt norm; this implies, in turn, that for every $n \in \mathbb{N}$, there exists a constant $C_n > 0$, such that

$$\sup_{\widetilde{x}\in\widetilde{X}} \|\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\|_2 \leq C_n t^n, \quad t > 0.$$

Hence, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\left|\int_0^\infty \left[\int_{\widetilde{X}} \operatorname{tr}\left(\rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(s/\Lambda^2,\widetilde{x}\gamma^{-1},\widetilde{x})\right) d\operatorname{vol}(\widetilde{x})\right]\phi(s)ds\right| \\
&\leq \int_0^\infty \operatorname{vol}(X)\left(\sup_{\widetilde{x}\in\widetilde{X}}\|\rho(\gamma)(\widetilde{x})\|_2\right)\left(\sup_{\widetilde{x}\in\widetilde{X}}\|\widetilde{K}(s/\Lambda^2,\widetilde{x}\gamma^{-1},\widetilde{x})\|_2\right)|\phi(s)|ds \\
&\leq \operatorname{vol}(X)\cdot\left(\sup_{\widetilde{x}\in\widetilde{X}}\|\rho(\gamma)(\widetilde{x})\|_2\right)\cdot C_n\int_0^\infty (s/\Lambda^2)^n|\phi(s)|ds \\
&= \left(\operatorname{vol}(X)\cdot\left(\sup_{\widetilde{x}\in\widetilde{X}}\|\rho(\gamma)(\widetilde{x})\|_2\right)\cdot C_n\cdot\int_0^\infty s^n|\phi(s)|ds\right)\Lambda^{-2n},
\end{aligned}$$

yielding the desired result. □

Now, let $\alpha : \Gamma \to \mathrm{GL}_N(\mathbb{C})$ be a representation of Γ; by endowing $\mathbb{C}^N$ with a Γ-invariant inner product, we take $\alpha : \Gamma \to U(N)$. Since $\widetilde{X} \to X$ is a principal Γ-bundle, we form the associated Hermitian vector bundle $\mathcal{F} := \widetilde{X} \times_\alpha \mathbb{C}^X \to M$; since Γ is finite, we endow $\mathcal{F}$ with the trivial flat connection d. We can therefore form the Clifford module $\mathcal{E}\otimes\mathcal{F} \to X$, which admits the symmetric Dirac-type operator D_α obtained from D by twisting by d, that is,

$$D_\alpha = D\otimes 1 + c(1\otimes d),$$

where c denotes the Clifford action on $\mathcal{V}\otimes\mathcal{F}$.

We now obtain the following generalisation of Theorem 6.5.6, which explains the factor of $N/\#\Gamma$ appearing in Theorem 6.5.5 above:

Theorem 6.5.10. *Let $f : \mathbb{R} \to \mathbb{C}$ be of the form $f(x) = \mathcal{L}[\phi](x^2)$ for $\phi \in \mathcal{S}(0,\infty)$. Then for $\Lambda > 0$,*

$$\operatorname{Tr}\left(f(D_\alpha/\Lambda)\right) = \frac{N}{\#\Gamma} \operatorname{Tr}\left(f(\widetilde{D}/\Lambda)\right) + O(\Lambda^{-\infty}), \quad \text{as } \Lambda \to +\infty. \tag{6.5.5}$$

Remark 6.5.11. This result is again compatible with inner fluctuations of the metric, insofar as if $A \in C^\infty(M, \operatorname{End}(\mathcal{E}))$ is symmetric, then $D_\alpha + A \otimes 1$ on $\mathcal{E} \otimes \mathcal{F}$ is induced from $\widetilde{D} + \widetilde{A}$ on $\widetilde{\mathcal{E}}$, where $\widetilde{A}$ is A viewed as a Γ-invariant element of $C^\infty(\widetilde{X}, \operatorname{End}(\widetilde{\mathcal{E}}))$.

Proof of Theorem 6.5.10. On the one hand, consider the trivial bundle $\widetilde{\mathcal{F}} := \widetilde{X} \times \mathbb{C}^N$ over $\widetilde{X}$, together with the trivial flat connection d. Then for the action $(\widetilde{x}, v)\gamma := (\widetilde{x}\gamma, \alpha(\gamma^{-1})v)$, $\widetilde{\mathcal{F}}$ is a Γ-equivariant Hermitian vector bundle, and d is a Γ-equivariant Hermitian connection on $\widetilde{\mathcal{F}}$. Then, by taking the tensor product of Γ-actions, we can endow $\widetilde{\mathcal{E}} \otimes \widetilde{\mathcal{F}}$ with the structure of a Γ-equivariant Clifford module, admitting the Γ-invariant symmetric Dirac-type operator $\widetilde{D_\alpha} = \widetilde{D} \otimes 1 + c(1 \otimes d)$. As a vector bundle, however, we may simply identify $\widetilde{\mathcal{E}} \otimes \widetilde{\mathcal{F}}$ with $\widetilde{\mathcal{F}}^{\oplus N}$, in which case we may identify $\widetilde{D_\alpha}$ with $\widetilde{D} \otimes 1_N$.

On the other hand, by construction, the bundle $\mathcal{F}$ defined above is the quotient of $\widetilde{\mathcal{F}}$ by the action of Γ. Hence, under the action of Γ, the quotient of $\widetilde{\mathcal{V}} \otimes \widetilde{\mathcal{F}}$ is the Clifford module $\mathcal{E} \otimes \mathcal{F}$, with $\widetilde{D_\alpha}$ descending to the operator $D \otimes 1 + c(1 \otimes d) = D_\alpha$.

Finally, by Theorem 6.5.6 and our observations above,

$$\begin{aligned}
\operatorname{Tr}\left(f(D_\alpha/\Lambda)\right) &= \frac{1}{\#\Gamma} \operatorname{Tr}\left(f(\widetilde{D_\alpha}/\Lambda)\right) + O(\Lambda^{-\infty}) \\
&= \frac{1}{\#\Gamma} \operatorname{Tr}\left(f(\widetilde{D}/\Lambda) \otimes 1_N\right) + O(\Lambda^{-\infty}) \\
&= \frac{N}{\#\Gamma} \operatorname{Tr}\left(f(\widetilde{D}/\Lambda)\right) + O(\Lambda^{-\infty}), \qquad \text{as } \Lambda \to +\infty,
\end{aligned}$$

as was claimed. □

One can apply these results to give a quick second proof of Theorem 6.5.5.

Second proof of Theorem 6.5.5. Recall that $\Gamma \subset \mathrm{SU}(2)$ is a finite group acting by isometries on S^3, identified with $SU(2)$ endowed with the round metric, and that $\alpha : \Gamma \to U(N)$ is a representation. Since S^3 is parallelizable and Γ acts by isometries, the spinor bundle $\mathbb{C}^2 \to \mathcal{S}_{S^3} \to S^3$ and the Dirac operator $\not{D}_{S^3}$ are trivially Γ-equivariant. Then, by construction, the Dirac-type operator D^Γ_α on $\mathcal{S}_{S^3} \otimes \mathcal{V}_\alpha$ is precisely the induced operator D_α corresponding to $\widetilde{D} = \not{D}_{S^3}$, so that by Theorem 6.5.10,

$$\operatorname{Tr}\left(f(D_\alpha/\Lambda)\right) = \frac{N}{\#\Gamma} \operatorname{Tr}\left(f(\not{D}_{S^3}/\Lambda)\right) + O(\Lambda^{-\infty}), \quad \text{as } \Lambda \to +\infty.$$

However, by Theorem 6.5.3, one has that

$$\mathrm{Tr}\left(f(\not{D}_{S^3}/\Lambda)\right) = \Lambda^3 \widehat{f^{(2)}}(0) - \frac{1}{4}\Lambda \widehat{f}(0) + O(\Lambda^{-\infty}),$$

where $\widehat{f^{(2)}}$ denotes the Fourier transform of $u^2 f(u)$. Hence,

$$\mathrm{Tr}\left(f(D_\alpha/\Lambda)\right) = \frac{N}{\#\Gamma}\left(\Lambda^3 \widehat{f^{(2)}}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}),$$

as required. □

Chapter 7

Toric noncommutative manifolds

I don't care how God-damn smart
these guys are: I'm bored.

It's been raining like hell all day long
and there's nothing to do.

— R. Brautigan, *At the California Institute of Technology*

Finally, we describe progress towards a reconstruction theorem for the other noncommutative class of spectral triples closest to the commutative case, *toric noncommutative manifolds*. These are spectral triples formed from torus-equivariant commutative spectral triples via a deformation quantisation (in the sense of Rieffel [60]) of the algebra along the torus action, that is, by a *Connes–Landi deformation* or *isospectral deformation*.

Our account is based on Yamashita's noncommutative generalisation [69] of Connes–Landi's original construction [29], though we generalise further to allow $\mathbb{T}^N$-actions for $N \geq 2$; another, equivalent approach to toric noncommutative manifolds is outlined in the preprint by Brain–Landi–Van Suijlekom [6].

7.1 Strict deformation quantisation

Let us begin with the basic construction of the deformed product on suitable bounded operators on a Hilbert space with a strongly continuous unitary action of $\mathbb{T}^N$.

In what follows, let H be a Hilbert space together with a strongly continuous unitary action $U_\cdot : \mathbb{T}^N \to B(H)$ of $\mathbb{T}^N$ on H. Then for each $r \in \mathbb{Z}^N$,

$$H_r := \left\{\xi \in H \mid \forall t \in \mathbb{T}^N,\ U_t\xi = e^{2\pi i r\cdot t}\xi\right\}$$

is a closed subspace of H, and

$$H = \bigoplus_{r \in \mathbb{Z}^N} H_r;$$

for each $r \in \mathbb{Z}^N$, let P_r then be the orthogonal projection onto H_r. Moreover, for each $r \in \mathbb{Z}^N$,

$$B(H)_r := \left\{T \in B(H) \mid \forall t \in \mathbb{T}^N,\ U_t T U_t^* = e^{2\pi i r \cdot t}\right\}$$

is a closed subspace of H, and for all $r,\ s \in \mathbb{Z}^N$,

$$(B(H)_r)^* \subset B(H)_{-r}, \quad B(H)_r B(H)_s \subset B(H)_{r+s};$$

one can also readily check that for $r,\ s \in \mathbb{Z}^N$, if $T \in B(H)_r$ then $TH_s \subset H_{r+s}$, so that $TP_s = P_{r+s}TP_s$. Finally, note that

$$B^\infty(H) := \{T \in B(H) \mid\ t \mapsto U_t T U_t^* \text{ is smooth in the weak topology on } B(H)\ \}$$

is a unital $*$-subalgebra of $B(H)$; for each $1 \leq k \leq N$, let ∂_k denote the derivation $\partial_k : B^\infty(H) \to B^\infty(H)$ defined for $T \in B^\infty(H)$ by

$$\forall \xi,\ \eta \in H, \quad \langle \partial_k T\xi, \eta\rangle := \lim_{s \to 0} \frac{\langle U_{se_k} T U_{se_k}^* \xi, \eta\rangle - \langle \xi, \eta\rangle}{s}.$$

Now, in what follows, let $\theta \in \mathfrak{so}(N, \mathbb{R})$ be a fixed skew-symmetic real $N \times N$ matrix; let $\chi_\theta : \mathbb{Z}^N \times \mathbb{Z}^N \to U(1)$ be the bicharacter defined as follows:

$$\forall r, s \in \mathbb{Z}^N, \quad \chi(r, s) := e^{\pi i r \cdot \theta s}.$$

Let us first define a key technical tool for the construction of the deformed product.

Lemma 7.1.1. *Let $r \in \mathbb{Z}^N$. Then $V_r^\theta := \sum_{s \in \mathbb{Z}^N} \chi_\theta(r, s) P_s$ is strongly convergent in $B(H)$ with $\left\|V_r^\theta\right\| = 1$ and $U_t V_r^\theta U_t^* = V_r^\theta$ for all $t \in \mathbb{T}^N$, and is unitary with $(V_r^\theta)^* = V_r^{-\theta} = V_{-r}^\theta$. Moreover, for $r,\ s \in \mathbb{Z}^N$, $V_r^\theta V_s^\theta = V_{r+s}^\theta$, whilst for $T \in B(H)_s$, $V_r^\theta T (V_r^\theta)^* = \chi_\theta(r, s) T$.*

Proof. Let $\xi \in H$. Since $H = \oplus_{s \in \mathbb{Z}^N} H_s$, we have that

$$\sum_{s \in \mathbb{Z}^N} \|\chi_\theta(r, s) P_s \xi\|^2 = \sum_{s \in \mathbb{Z}^N} \|P_s \xi\|^2 = \|\xi\|^2,$$

so that, since $\{\chi_\theta(r, s) P_s \xi\}_{s \in \mathbb{Z}^N}$ is orthogonal, $V_r^\theta \xi := \sum_{s \in \mathbb{Z}^N} \chi_\theta(r, s) P_s \xi$ converges in H, and, hence, $V_r^\theta = \sum_{s \in \mathbb{Z}^N} \chi_\theta(r, s) P_s$ strongly converges in $B(H)$ with $\left\|V_r^\theta\right\| = 1$.

Next, for ξ, $\eta \in H$,

$$\left\langle V_r^\theta \xi, \eta \right\rangle = \sum_{s \in \mathbb{Z}^N} \left\langle \chi_\theta(r,s) P_s \xi, \eta \right\rangle = \sum_{s \in \mathbb{Z}^n} \left\langle \xi, \chi_\theta(s,r) P_s \eta \right\rangle,$$

so that $(V_r^\theta)^* = V_r^{-\theta} V_{-r}^\theta$, since $\chi_\theta(s,r) = \chi_\theta(-r,s) = \chi_{-\theta}(r,s)$. But now, for any $\xi \in H$,

$$V_r^{-\theta} V_r^\theta \xi = \sum_{s \in \mathbb{Z}^N} V_r^{-\theta}(\chi_\theta(r,s) P_s \xi) = \sum_{s \in \mathbb{Z}^N} \chi_{-\theta}(r,s)\chi_\theta(r,s) P_s \xi = \sum_{s \in \mathbb{Z}^N} P_s \xi = \xi,$$

so that $V_r^{-\theta} V_r^\theta = 1$; replacing θ by $-\theta$, the same calculation yields $V_r^\theta V_r^{-\theta}$, so that, indeed, $(V_r^\theta)^* = V_r^{-\theta} = (V_r^\theta)^{-1}$, as was claimed.

Next, let $t \in \mathbb{T}^N$. Then for any $\xi \in H$,

$$\begin{aligned} U_t V_r^\theta U_t^* \xi &= U_t V_r \sum_{s \in \mathbb{Z}^N} e^{-2\pi i t \cdot s} P_s \xi \\ &= U_t \sum_{s \in \mathbb{Z}^N} \chi_\theta(r,s) e^{-2\pi i t \cdot s} P_s \xi \\ &= \sum_{s \in \mathbb{Z}^N} e^{2\pi i t \cdot s} \chi_\theta(r,s) e^{-2\pi i t \cdot s} P_s \xi \\ &= V_r^\theta \xi, \end{aligned}$$

as required.

Now, let r, $s \in \mathbb{Z}^N$. Then for any $\xi \in H$,

$$V_r^\theta V_s^\theta \xi = \sum_{t \in \mathbb{Z}^N} V_r^\theta \chi_\theta(s,t) P_t \xi = \sum_{t \in \mathbb{Z}^N} \chi_\theta(r,t)\chi_\theta(s,t) P_t \xi = \sum_{t \in \mathbb{Z}^N} \chi_\theta(r+s,t) P_t \xi = V_{r+s}^\theta \xi,$$

as was claimed.

Finally, let r, $s \in \mathbb{Z}^N$, and let $T \in B(H)_s$. Then for all $\xi \in H$,

$$\begin{aligned} V_r^\theta T V_r^{-\theta} \xi &= \sum_{t \in \mathbb{Z}^N} V_r^\theta \chi_{-\theta}(r,t) T P_t \xi \\ &= \sum_{t \in \mathbb{Z}^N} V_r^\theta \chi_\theta(-r,t) P_{t+s} T P_t \xi \\ &= \sum_{s \in \mathbb{Z}^N} \chi_\theta(r,t+s)\chi_\theta(r,-s)\chi_\theta(r,r-s) P_{r-s} T P_s \xi \\ &= \chi_\theta(r,s) \sum_{s \in \mathbb{Z}^N} P_{t+s} T P_t \\ &= \chi_\theta(r,s) T \xi, \end{aligned}$$

so that $V_r^\theta T (V_r^\theta)^* = \chi_\theta(r,s) T$, as was claimed. □

Now, let us define the deformed product (and accompanying deformed action) on homogeneous elements of $B(H)$.

Lemma 7.1.2. *For $T \in B(H)_r$, let*

$$\lambda_\theta(T) := TV_r^\theta, \quad \rho_\theta(T) := TV_r^{-\theta} = \lambda_\theta(T).$$

Then the assignments $T \mapsto \lambda_\theta(T)$ and $T \mapsto \rho_\theta(T)$ define linear contractions $\lambda_\theta, \rho_\theta : B(H)_r \to B(H)_r$, such that for any $T \in B(H)_r$, $\lambda_\theta(T)^* = \lambda_\theta(T^*) \in B(H)_{-r}$ and $\rho_\theta(T)^* = \rho_\theta(T^*) \in B(H)_{-r}$.

Moreover, if $T \in B(H)_r$ and $T' \in B(H)_s$, then

$$\lambda_\theta(T)\lambda_\theta(T') = \lambda_\theta(\chi_\theta(r,s)TT').$$

Proof. That λ_θ and ρ_θ are linear contractions on $B(H)_r$ follows immediately from the fact that $\|V_r^\theta\| = \|V_r^{-\theta}\| = 1$, and that $\lambda_\theta^{-1} = \lambda_{-\theta}$ and $\rho_\theta^{-1} = \rho_{-\theta}$ follows immediately from the fact that $(V_r^\theta)^{-1} = V_r^{-\theta}$.

Now, let $T \in B(H)_r$, and recall that $T^* \in B(H)_{-r}$. Then by Lemma 7.1.1, $V_r^\theta T (V_r^\theta)^* = \chi_\theta(r,r)T = T$, and hence

$$\lambda_\theta(T)^* = (TV_r^\theta)^* = (V_r^\theta T)^* = T^*(V_r^\theta)^* = T^*V_{-r}^\theta = \lambda_\theta(T^*);$$

replacing θ with $-\theta$ then yields $\rho_\theta(T)^* = \rho_\theta(T^*)$.

Finally, let $T \in B(H)_r$ and $T' \in B(H)_s$. Then by Lemma 7.1.1,

$$\lambda_\theta(T)\lambda_\theta(T') = TV_r^\theta T'V_s^\theta = TV_r^\theta T'(V_r^\theta)^*V_r^\theta V_s^\theta = \chi_\theta(r,s)TT'V_{r+s}^\theta = \lambda_\theta(\chi_\theta(r,s)TT'),$$

as was claimed. □

In order to extend our deformed product to more general bounded operators on $B(H)$, we must first show how to canonically decompose a bounded operator into homogeneous components.

Lemma 7.1.3. *Let $r \in \mathbb{Z}^N$. If $T \in B(H)$, then*

$$T_r := \sum_{s \in \mathbb{Z}^N} P_{r+s}TP_s$$

strongly converges in $B(H)$ to an element of $B(H)_r$, and the assignment $T \mapsto T_r$ defines a linear contraction $B(H) \to B(H)_r$, such that $(T_r)^* = (T^*)_{-r}$ for all $T \in B(H)$. Moreover, if $T \in B(H)_{r'}$, then $T_r = \delta_{rr'}T$.

Proof. First, let $T \in B(H)$. Let $\xi \in H$. Then

$$\sum_{s\in\mathbb{Z}^N} \|P_{r+s}TP_s\xi\|^2 \leq \sum_{s\in\mathbb{Z}^N} \|TP_s\xi\|^2 \leq \|T\|^2 \sum_{s\in\mathbb{Z}^N} \|P_s\xi\|^2 = \|T\|^2 \|\xi\|^2 ,$$

so that since $\{P_{r+s}TP_s\xi\}_{s\in\mathbb{Z}^N}$ is orthogonal, $T_r\xi := \sum_{s\in\mathbb{Z}^N} P_{r+s}TP_s\xi$ converges in H, and, hence, $T_r := \sum_{s\in\mathbb{Z}^N} P_{r+s}TP_s$ converges strongly in $B(H)$ with $\|T_r\| \leq \|T\|$. Moreover, if $t \in \mathbb{T}^N$, then for any $\xi \in H$,

$$\begin{aligned} U_tT_rU_t^*\xi &= U_t \sum_{s\in\mathbb{Z}^N} P_{r+s}TP_sU_t^*\xi \\ &= \sum_{s\in\mathbb{Z}^N} (U_tP_{r+s})T(U_{-t}P_s)\xi \\ &= \sum_{s\in\mathbb{Z}^N} e^{2\pi it\cdot(r+s)}e^{2\pi i(-t)\cdot s}P_{r+s}TP_s\xi \\ &= e^{2\pi ir\cdot t} \sum_{s\in\mathbb{Z}^N} P_{r+s}TP_s\xi \\ &= e^{2\pi ir\cdot t}T_r\xi, \end{aligned}$$

so that $T_r \in B(H)_r$. Once can readily check that $T \mapsto T_r$ is linear in T, so that $T \mapsto T_r$ does indeed define an linear contraction $B(H) \to B(H)_r$.

Now, for any ξ, $\eta \in H$,

$$\begin{aligned} \langle T_r^*\xi, \eta\rangle &= \langle \xi, T_r\eta\rangle \\ &= \sum_{s\in\mathbb{Z}^N} \langle \xi, P_{r+s}TP_s\eta\rangle \\ &= \sum_{s\in\mathbb{Z}^N} \langle P_sT^*P_{r+s}\xi, \eta\rangle \\ &= \sum_{s'\in\mathbb{Z}^N} \langle P_{-r+s'}T^*P_{s'}\xi, \eta\rangle \\ &= \langle (T^*)_{-r}\xi, \eta\rangle , \end{aligned}$$

so that $(T_r)^* = (T^*)_{-r}$.

Finally, suppose that $T \in B(H)_{r'}$. Then

$$T_r = \sum_{s\in\mathbb{Z}^N} P_{r+t}TP_t = \sum_{s\in\mathbb{Z}^N} P_{r+s}P_{r'+s}TP_s = \sum_{s\in\mathbb{Z}^N} \delta_{rr'}P_{r'+t}TP_t = \delta_{rr'}T,$$

as was claimed. ☐

Let us now characterise those bounded operators on $B(H)$ that admit a well-behaved decomposition into homogeneous components. We first shall need the following definition:

Definition 7.1.4. We define a *rapidly decreasing sequence* in $B^\infty(H)$ to be a sequence $\{T_\sigma\}_{\sigma\in\mathbb{Z}^N}$ in $B^\infty(H)$ with $T_\sigma \in B(H)_\sigma$ for each $\sigma \in \mathbb{Z}^N$, such that for all $k \in \mathbb{N}$,

$$\sup_{\sigma\in\mathbb{Z}^N} (1+\sigma\cdot\sigma)^{k/2} \|T_\sigma\| < \infty.$$

Now we can characterise the domain of our deformed product:

Lemma 7.1.5. *If $T \in B^\infty(H)$, then $\{T_r\}_{r\in\mathbb{Z}^N}$ is rapidly decreasing. Conversely, if $\{T_r\}_{r\in\mathbb{Z}^N} \subset B^\infty(H)$ is rapidly decreasing, then $\sum_{\sigma\in\mathbb{Z}^N} T_\sigma$ converges in norm to an element $T \in B^\infty(H)$. In particular, then, if $T \in B^\infty(H)$, then $T = \sum_{\sigma\in\mathbb{Z}^N} T_\sigma$ and $\partial_m T = \sum_{\sigma\in\mathbb{Z}^N} 2\pi i\sigma_k T_\sigma$ for each $1 \le m \le N$, with convergence in norm.*

To prove this, we shall need the following ancillary lemma:

Lemma 7.1.6 ([69, Proof of Lemma 1])**.** *Let $T \in B(H)$, and suppose that*

$$\partial_k T := \text{w-lim}_{s\to 0} \frac{1}{s}\left(U_{se_k} T U_{se_k}^* - T\right)$$

exists. Then for any $r \in \mathbb{Z}^N$,

$$(\partial_k T)_r = 2\pi i r_k T_r, \quad 1 \le k \le N.$$

Proof. First, if $\xi \in H_s$, $\eta \in H_t$, then

$$\begin{aligned}
\langle \xi, \partial_k T\eta\rangle &= \lim_{h\to 0} \frac{1}{h}\left(\langle \xi, U_{he_k} T U_{he_k}^* \eta\rangle - \langle \xi, T\eta\rangle\right) \\
&= \lim_{h\to 0} \frac{1}{h}(e^{2\pi i h e_k\cdot(s-t)} - 1)\langle \xi, T\eta\rangle \\
&= 2\pi i(s-t)_k \langle \xi, T\eta\rangle,
\end{aligned}$$

and, hence,

$$\begin{aligned}
\langle \xi, (\partial_k T)_r \eta\rangle &= \sum_{\sigma\in\mathbb{Z}^N} \langle \xi, P_{r+\sigma}\partial_k T P_\sigma \eta\rangle \\
&= \sum_{\sigma\in\mathbb{Z}^N} 2\pi i(s-t)_k \langle P_{r+\sigma}\xi, T P_\sigma\eta\rangle \\
&= \sum_{\sigma\in\mathbb{Z}^N} \delta_{r+\sigma,s}\delta_{\sigma,t} 2\pi i(s-t)_k \langle \xi, T\eta\rangle \\
&= \delta_{s,r+t} 2\pi i r_k \langle \xi, T\eta\rangle \\
&= 2\pi i r_k \langle \xi, T_r\eta\rangle.
\end{aligned}$$

Since the algebraic direct sum of the H_σ is dense in $H = \oplus_{\sigma\in\mathbb{Z}^N} H_\sigma$, it therefore follows that $(\partial_k T)_r = 2\pi i r_k T_r$, as was claimed. □

Proof of Lemma 7.1.5. First, let $T \in B^\infty(H)$. Then by Lemma 7.1.6, for any $r \in \mathbb{Z}^N$, $(\Delta T)_r = -4\pi^2 r\cdot r T_r$, and, hence,

$$(1+r\cdot r)^k T_r = \left(\left(1 - \tfrac{1}{4\pi^2}\Delta\right)^k T\right)_r.$$

Thus, for any $k \in \mathbb{N}\cup\{0\}$,

$$(1+r\cdot r)^{k/2}\|T_r\| \le (1+r\cdot r)^k \|T_r\| = \left\|\left(\left(1-\tfrac{1}{4\pi^2}\Delta\right)^k T\right)_r\right\| \le \left\|\left(1-\tfrac{1}{4\pi^2}\Delta\right)^k T\right\|,$$

so that $\{T_r\}$ is indeed rapidly decreasing.

Now, let $\{T_r\}_{r\in\mathbb{Z}^N} \subset B^\infty(H)$ be rapidly decreasing. In particular, then,

$$\sum_{r\in\mathbb{Z}^N}\|T_r\| \le \sum_{r\in\mathbb{Z}^N}(1+r\cdot r)^{-(N+1)/2}\sup_{\sigma\in\mathbb{Z}^N}(1+\sigma\cdot\sigma)^{-(N+1)/2}\|T_\sigma\| < \infty,$$

so that $\sum_{r\in\mathbb{Z}^N} T_r$ converges in norm to some $T \in B(H)$. Moreover, for each $m \in \mathbb{N}$ and each $\{k_1,\dots,k_m\} \in \{1,\dots,N\}^m$,

$$\begin{aligned}\sum_{r\in\mathbb{Z}^N}\|(2\pi i r)_{k_1}\cdots(2\pi i r)_{k_m}T_r\| &\le \sum_{r\in\mathbb{Z}^N}(2\pi)^m(1+\sigma\cdot\sigma)^{m/2}\|T_r\|\\ &\le \sum_{r\in\mathbb{Z}^N}(2\pi)^m(1+r\cdot r)^{-(N+1)/2}\sup_{\sigma\in\mathbb{Z}^N}(1+\sigma\cdot\sigma)^{(N+m+1)/2}\|T_\sigma\|\\ &< \infty,\end{aligned}$$

so that

$$T^{k_1,\dots,k_m} := \sum_{r\in\mathbb{Z}^N}(2\pi i r)_{k_1}\cdots(2\pi i r)_{k_m}T_r$$

converges in norm to an element of $B(H)$; we shall prove by induction on m that $T^{k_1,\dots,k_m} = \partial_{k_1}\cdots\partial_{k_m}T$. Let $\{k_0,\dots,k_m\} \in \{1,\cdots,N\}^{m+1}$, and suppose by induction that $T^{k_1,\dots,k_m} = \partial_{k_1}\cdots\partial_{k_m}T$. Then for any $\xi \in H_s$, $\eta \in H_t$, by the proof of Lemma 7.1.6, on the one hand,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\left\langle U_{\varepsilon e_{k_0}}\partial_{k_1}\cdots\partial_{l_m}TU^*_{\varepsilon e_{k_0}}\xi,\eta\right\rangle - \langle\partial_{k_1}\cdots\partial_{k_m}T\xi,\eta\rangle\right) = -2\pi i(s-t)_{k_0}\langle\partial_{k_1}\cdots\partial_{k_m}\xi,\eta\rangle,$$

whilst on the other,

$$\begin{aligned}\left\langle T^{k_0,\dots,k_m}\xi,\eta\right\rangle &= \sum_{r\in\mathbb{Z}^N}(-2\pi i r)_{k_0}\cdots(-2\pi i r)_{k_m}\langle T_r\xi,\eta\rangle\\ &= \sum_{r\in\mathbb{Z}^N}(-2\pi i r)_{k_0}\cdots(-2\pi i r)_{k_m}\langle P_{r+s}T_rP_s\xi,\eta\rangle\end{aligned}$$

$$= -2\pi i(s-t)_{k_0} \left\langle (\partial_{k_1} \cdots \partial_{k_m} T)_{s-t}\xi, \eta \right\rangle;$$

since

$$\left\langle T^{k_0,\ldots,k_m}\xi, \eta \right\rangle = \sum_{r \in \mathbb{Z}^N} \left\langle (T^{k_0,\ldots,k_m})_r \xi, \eta \right\rangle = \sum_{r \in \mathbb{Z}^N} \left\langle P_{r+s} T^{k_0,\ldots,k_m}\xi, \eta \right\rangle = \left\langle (T^{k_0,\ldots,k_m})_{s-t}\xi, \eta \right\rangle,$$

it therefore follows that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\left\langle U_{\varepsilon e_{k_0}} \partial_{k_1} \cdots \partial_{l_m} T U^*_{\varepsilon e_{k_0}} \xi, \eta \right\rangle - \langle \partial_{k_1} \cdots \partial_{k_m} T\xi, \eta \rangle \right) = \left\langle T^{k_0,\ldots,k_m}\xi, \eta \right\rangle.$$

Since the algebraic direct sum of the H_σ is dense in H, this implies that

$$\text{w-lim}_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(U_{\varepsilon e_{k_0}} \partial_{k_1} \cdots \partial_{k_m} T U^*_{\varepsilon e_{k_0}} - \partial_{k_1} \cdots \partial_{k_m} T \right) = T^{k_0,\ldots,k_m},$$

as required.

Finally, let $T \in B^\infty(H)$; all that remains to be shown is that $T = \sum_{r \in \mathbb{Z}^N} T_r$ in $B(H)$. Since $\{T_r\}$ is rapidly decreasing, we certainly know that $\sum_{r \in \mathbb{Z}^N} T_r$ converges in norm to some $T' \in B^\infty(H)$. But then, if $\xi \in H_s$, $\eta \in H_t$ for s, $t \in \mathbb{Z}^N$, then

$$\begin{aligned} \langle T'\xi, \eta \rangle &= \sum_{r \in \mathbb{Z}^N} \langle T_r \xi, \eta \rangle \\ &= \sum_{r \in \mathbb{Z}^N} \sum_{\sigma \in \mathbb{Z}^N} \langle P_{r+\sigma} T P_\sigma \xi, \eta \rangle \\ &= \sum_{r \in \mathbb{Z}^N} \sum_{\sigma \in \mathbb{Z}^N} \delta_{\sigma,s} \delta_{r+\sigma,t} \langle T\xi, \eta \rangle \\ &= \langle T\xi, \eta \rangle; \end{aligned}$$

since the algebraic direct sum of the H_σ is dense in H, it follows that $T = T'$, as was claimed. □

We now construct the deformed action of $B^\infty(H)$ on H that will induce the deformed product on $B^\infty(H)$.

Lemma 7.1.7 ([69, Lemma 1])**.** *If $T \in B^\infty(H)$, then*

$$\lambda_\theta(T) := \sum_{r \in \mathbb{Z}^N} \lambda_\theta(T_r), \quad \rho_\theta(T) := \sum_{r \in \mathbb{Z}^N} \rho_\theta(T_r)$$

converge in the operator norm in $B(H)$, and the assignments $T \mapsto \lambda_\theta(T)$, $T \mapsto \rho_\theta(T)$ define invertible $\mathbb{T}^N$-equivariant linear contractions $\lambda_\theta, \rho_\theta : B^\infty(H) \to B^\infty(H)$, such that for any $T \in B^\infty(H)$, $\lambda_\theta(T)^* = \lambda_\theta(T^*)$ and $\rho_\theta(T)^* = \rho_\theta(T^*)$, with inverses $\lambda_\theta^{-1} = \lambda_{-\theta}$ and $\rho_\theta^{-1} = \rho_{-\theta}$. Moreover, for any

S, $T \in B^\infty(H)$ with $[S,T]=0$, $[\lambda_\theta(S), \rho_\theta(T)] = 0$.

Proof. By Lemma 7.1.5, $T = \sum_{r \in \mathbb{Z}^N} T_r$ is absolutely convergent in $B(H)$, so that

$$\sum_{r \in \mathbb{Z}^N} \|\lambda_\theta(T_r)\| \leq \sum_{r \in \mathbb{Z}^N} \|T_r\| < \infty,$$

and, hence, $\lambda_\theta(T) := \sum_{r \in \mathbb{Z}^N} \lambda_\theta(T_r)$ converges absolutely in $B(H)$; by replacing θ with $-\theta$, we get the same claim for $\rho_\theta(T)$. It then follows that the properties of $\lambda_\theta : B^\infty(H) \to B(H)$ (and, hence, also of $\rho_\theta : B^\infty(H) \to B(H)$) follow from Lemma 7.1.3.

Next, since all the P_r and V_s^θ commute with the action of $\mathbb{T}^N$ on H, it follows that if $T \in B^\infty(H)$, then for all $t \in \mathbb{T}^N$, $U_t \lambda_\theta(T) U_t^* = \lambda_\theta(U_t T U_t^*)$, so that $\lambda_\theta(B^\infty(H)) \subset \lambda_\theta(B^\infty(H))$, and $\lambda_\theta : B^\infty(H) \to B^\infty(H)$ is $\mathbb{T}^N$ equivariant; by replacing θ with $-\theta$, we prove the same for ρ_θ. It then follows from Lemma 7.1.3 and a direct computation that $\lambda_\theta^{-1} = \lambda_{-\theta}$ and $\rho_\theta^{-1} = \rho_{-\theta}$ in this more general context.

Finally, suppose that S, $T \in B^\infty(H)$ commute. First, since the various P_σ commute amongst themselves, it follows that S_r and T_s commute for all r, $s \in \mathbb{Z}^N$. But then, for each r, $s \in \mathbb{Z}^N$,

$$\begin{aligned}
\lambda_\theta(S_r)\rho_\theta(T_s) - \rho_\theta(T_s)\lambda_\theta(S_r) &= S_r V_r^\theta T_s V_s^{-\theta} - T_s V_s^{-\theta} S_r V_r^\theta \\
&= \chi_\theta(r,s) S_r T_s V_{r-s}^\theta - \chi_{-\theta}(s,r) T_s S_r V_{r-s}^\theta \\
&= \chi_\theta(r,s)[S_r, T_s] = 0.
\end{aligned}$$

It then follows from the construction of $\lambda_\theta(S)$ and $\rho_\theta(T)$ that they commute, as was claimed. □

Remark 7.1.8. If $T \in B(H)_0$, i.e., if $U_t T U_t^* T = T$ for all $t \in \mathbb{T}^N$, then $\lambda_\theta(T) = \rho_\theta(T) = T$.

Putting everything together, we see that we have constructed our deformed product on $B^\infty(H)$ as follows:

Theorem 7.1.9 (Rieffel, cf. [60]). *For S, $T \in B^\infty(H)$, let $S \star_\theta T := \lambda_\theta^{-1}(\lambda_\theta(S)\lambda_\theta(T))$. Then $(B^\infty(H), \star_\theta)$ is a unital $*$-algebra for the involution $*$ inherited from $B(H)$, and*

$$\forall S,\ T \in B^\infty(H), \quad \lambda_\theta(S \star_\theta T) = \lambda_\theta(S)\lambda_\theta(T).$$

Moreover, if $S \in B(H)_0$ or $T \in B(H)_0$, then $S \star_\theta T = ST$.

Remark 7.1.10. In light of Lemmas 7.1.5 and 7.1.7, the algebraic properties of $(B^\infty(H), \star_\theta)$ can be checked on homogeneous elements of $B^\infty(H)$.

7.2 Connes–Landi deformations

Let us now proceed to the construction of Connes–Landi deformations of torus-equivariant spectral triples. Just what we mean by a torus-equivariant spectral triple is given by the following definition:

Definition 7.2.1. We say that a regular spectral triple (A, H, D) is $\mathbb{T}^N$*-equivariant* if there exist a smooth isometric action $\sigma : \mathbb{T}^N \to A$ of $\mathbb{T}^N$ on A and a strongly continuous action $U_\bullet : \mathbb{T}^N \to U(H)$ of $\mathbb{T}^N$ on H satisfying the following:

1. For all $a \in A$, for all $t \in \mathbb{T}^N$, $U_t a U_t^* = \sigma_t(a)$.

2. For all $t \in \mathbb{T}^N$, $U_t D U_t^* = D$.

3. If (A, H, D) is even with $\mathbb{Z}_2$-grading γ, then for all $t \in \mathbb{T}^N$, $U_t \gamma U_t^* = \gamma$.

4. If (A, H, D) is two-sided, then for all $a \in A$, for all $t \in \mathbb{T}^N$, $U_t a^o U_t^* = \sigma_t(a)^o$.

If (A, H, D) is regular and $\mathbb{T}^N$-equivarant, then we take A to be a Fréchet pre-C^*-algebra with seminorms, for $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{N}^N$,

$$\nu_{k,\alpha}(a) := \left\|(\operatorname{ad}|D|)^k(\partial_\alpha a)\right\| + \left\|(\operatorname{ad}|D|)^k([D, \partial_\alpha a])\right\|, \quad a \in A.$$

Remark 7.2.2. If (A, H, D) is a $\mathbb{T}^N$-equivariant regular spectral triple, then for all $t \in \mathbb{T}^N$, $U_t H^\infty \subset H^\infty$, since by $\mathbb{T}^N$-invariance of D, $[|D|, U_t] = 0$, and, hence, $U_t \in \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$, which guarantees that $U_t H^\infty \subset H^\infty$, as was claimed.

Given this, we define the Connes–Landi deformation of a torus-equivariant spectral triple as follows:

Definition 7.2.3 (Yamashita [69], after Connes–Landi [29])**.** Let (A, H, D) be a $\mathbb{T}^N$-equivariant regular spectral triple, so that $A \subset B^\infty(H)$, let $\theta \in \mathfrak{so}(N)$, and let $A_\theta := (A, \star_\theta)$. Then (A_θ, H, D) is the *Connes–Landi deformation* of (A, H, D) with *deformation parameter* θ.

Remark 7.2.4. Since $\lambda_{\theta'} \circ \lambda_\theta = \lambda_{\theta+\theta'}$, it will follow that $((A_\theta)_{\theta'}, H, D) = (A_{\theta+\theta'}, H, D)$; in particular, $((A_\theta)_{-\theta}, H, D) = (A_0, H, D) = (A, H, D)$.

That Connes–Landi deformations are indeed well-defined are guaranteed by the following result:

Theorem 7.2.5 (Yamashita [69, Prop. 5])**.** *Let (A, H, D) be a $\mathbb{T}^N$-equivariant regular spectral triple, and let $\theta \in \mathfrak{so}(N)$. Then (A_θ, H, D) is a $\mathbb{T}^N$-equivariant regular spectral triple, even whenever (A, H, D) is, and of metric dimension p whenever (A, H, D) is.*

To prove this, we shall need two ancillary lemmas. The first is a refinement of Lemma 7.1.5:

Lemma 7.2.6. *Let (A, H, D) be a $\mathbb{T}^N$-equivariant regular spectral triple. If $a \in A$, then $a_r \in A_r \subset A$ and $a = \sum_{r\in\mathbb{Z}^N} a_r$ converges absolutely in A. Conversely, if $\{a_r\}_{r\in\mathbb{Z}^N} \subset A$ is rapidly decreasing in $B^\infty(H)$, then $a := \sum_{r\in\mathbb{Z}^N} a_r$ converges in A.*

Proof. First, observe that for any $a \in A$, $r \in \mathbb{Z}^N$, we can write

$$a_r = \int_{\mathbb{T}^N} e^{2\pi i r\cdot t} \sigma_t(a) d^N t \in A,$$

where the integral converges in A *qua* Fréchet space, since the $\mathbb{T}^N$-action on A is smooth and isometric.

Now, suppose that $\{a_r\}_{r\in\mathbb{Z}^N} \subset A$ is rapidly decreasing in $B^\infty(H)$. Then $a := \sum_{r\in\mathbb{Z}^N} a_r$ converges absolutely in $B^\infty(H) \subset B(H)$, and, hence, by $\mathbb{T}^N$-invariance of D, for any $c \in A$, $(\operatorname{ad}|D|)^k(\partial_\alpha a)$, $(\operatorname{ad}|D|)^k([D, \partial_\alpha a]) \in B^\infty(H)$ with

$$\forall r \in \mathbb{Z}^N, \quad (\operatorname{ad}|D|)^k(\partial_\alpha a_r) = (\operatorname{ad}|D|)^k(\partial_\alpha a)_r, \quad (\operatorname{ad}|D|)^k([D, \partial_\alpha a_r]) = (\operatorname{ad}|D|)^k([D, \partial_\alpha a])_r;$$

this implies, then, that

$$\begin{aligned}\sum_{r\in\mathbb{Z}^N} \nu_{k,\alpha}(a) &= \sum_{r\in\mathbb{Z}^N} \left(\left\|(\operatorname{ad}|D|)^k(\partial_\alpha a_r)\right\| + \left\|(\operatorname{ad}|D|)^k([D, \partial_\alpha a_r])\right\|\right)\\ &= \sum_{r\in\mathbb{Z}^N} \left\|(\operatorname{ad}|D|)^k(\partial_\alpha a)_r\right\| + \sum_{r\in\mathbb{Z}^N} \left\|(\operatorname{ad}|D|)^k([D, \partial_\alpha a])_r\right\|\\ &< \infty,\end{aligned}$$

as required.

Finally, if $a \in A$ and $a' = \sum_{r\in\mathbb{Z}^N} a_r$, then, in particular, $a' = \sum_{r\in\mathbb{Z}^N} a_r$ in $B(H)$, and, hence, by Lemma 7.1.5, $a = a'$, as required. ☐

The second lemma is a refinement of Proposition 7.1.9:

Lemma 7.2.7 (Yamashita [69, Lemma 2])**.** *Let (A, H, D) be a $\mathbb{T}^N$-equivariant regular spectral triple. Then for any $a, b \in A$, $a \star_\theta b \in A$, and, hence, $A_\theta := (A, \star_\theta)$ is a unital Fréchet pre-C^*-algebra for the same seminorms as A, with the same $\mathbb{T}^N$ action $\sigma : \mathbb{T}^N \to \operatorname{Aut}(A_\theta)$ and continuous $\mathbb{T}^N$-equivariant $*$-representation on H given by $\lambda_\theta|_{A_\theta} : A_\theta \to B(H)$.*

Proof. In light of Proposition 7.1.9, it suffices to show that for any $a, b \in A$,

$$a \star_\theta b = \sum_{r,s\in\mathbb{Z}^N} \chi_\theta(r, s) a_r b_s$$

converges absolutely in the Fréchet algebra A. However, by Lemma 7.2.6, it follows that

$$ab = \sum_{r,s \in \mathbb{Z}^N} a_r b_s$$

with absolute convergence in A, so that for any k and α,

$$\sum_{r,s \in \mathbb{Z}^N} \nu_{k,\alpha}(\chi_\theta(r,s) a_r b_s) = \sum_{r,s \in \mathbb{Z}^N} \nu_{k,\alpha}(a_r b_s) < \infty,$$

as required. □

Proof. First, since D is $\mathbb{T}^N$-invariant, for all $a \in A$, $[D, a] \in B^\infty(H)$, and, hence, $[D, \lambda_\theta(a)] = \lambda_\theta([D, a])$; since H and D are unchanged, this shows that (A_θ, H, D) is a spectral triple, indeed with metric dimension p if (A, H, D) has metric dimension p.

Next, again, since D is $\mathbb{T}^N$-invariant, for all $T \in B^\infty(H) \cap \left(\cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k\right)$, $(\operatorname{ad}|D|)^k(T) \in B^\infty(H)$, and, hence, $(\operatorname{ad}|D|)^k(\lambda_\theta(T)) = \lambda_\theta((\operatorname{ad}|D|)^k(T))$. Applying this to $T \in A + [D, A]$, we therefore find that (A, H, D) is regular; since A_θ is endowed with the same $\mathbb{T}^N$-action as A, and since $\lambda_\theta : A_\theta \to B(H)$ is $\mathbb{T}^N$-equivariant, it therefore follows that (A, H, D) is $\mathbb{T}^N$-equivariant.

Finally, if (A, H, D) is even with grading γ, since $\gamma \in B(H)_0$, it follows that for all $a \in A_\theta$, $[\lambda_\theta(a), \gamma] = \lambda_\theta([a, \gamma]) = 0$, as required. □

7.3 Toric noncommutative manifolds

At last, let we are in a position to discuss toric noncommutative manifolds and a potential reconstruction theorem for such spectral triples.

Let X be a compact oriented Riemannian manifold with an orientation-preserving isometric $\mathbb{T}^N$ action $c : \mathbb{T}^N \to \operatorname{Iso}^+(M)$, let $\mathcal{E} \to X$ be a $\mathbb{T}^N$-equivariant Clifford module with $\mathbb{T}^N$ action $t \mapsto (c_t, U_t)$, and let D be a $\mathbb{T}^N$-invariant symmetric Dirac-type operator on $\mathcal{E}$. Then $(C^\infty(X), L^2(X, \mathcal{E}), D)$, in particular, is a $\mathbb{T}^N$-equivariant regular spectral triple with $\mathbb{T}^N$-action on $C^\infty(X)$ given by

$$\forall \in C^\infty(X), t \in \mathbb{T}^N, \quad \sigma_t(f) := f \circ c_{-t},$$

and compatible $\mathbb{T}^N$ action on $L^2(X, \mathcal{E})$, given by $t \mapsto U_t$, with U_t viewed as an element of $B(L^2(X, \mathcal{E}))$. Given all this, we therefore call $(C^\infty(X), L^2(X, \mathcal{E}), D)$ a *concrete $\mathbb{T}^N$-equivariant commutative spectral triple*. We can therefore define a (concrete) toric noncommutative manifold, the type of Connes–Landi deformation originally considered by Connes–Landi themselves, as follows:

Definition 7.3.1. A *concrete toric noncommutative manifold* is a $\mathbb{T}^N$-equivariant spectral triple of the form

$$(C^\infty(X_\theta), L^2(X_\theta, \mathcal{E}), D) := (C^\infty(X)_\theta, L^2(X, \mathcal{E}), D),$$

where $(C^\infty(X), L^2(X, \mathcal{E}), D)$ is a concrete $\mathbb{T}^N$-equivariant commutative spectral triple.

Now, in order to obtain an abstract definition of toric noncommutative manifold, we shall need the following refinement of the definition of Dirac-type spectral triple:

Definition 7.3.2. We say that a p-dimensional Dirac-type spectral triple (A, H, D) is $\mathbb{T}^N$*-equivariant* if it is $\mathbb{T}^N$-equivariant as a regular spectral triple, and if the following also hold:

1. For all $a \in A$ and $t \in \mathbb{T}^N$, $U_t a^o U_t^* = \sigma_t(a)^o$.
2. The orientation cycle $c \in Z_p(A, A)$ is $\mathbb{T}^N$-invariant.
3. For any $\xi, \eta \in H^\infty$ and $t \in \mathbb{T}^N$, $(U_t\xi, U_t\eta) = (\xi, \eta)$.

Our proposed abstract definition of toric noncommutative manifold is therefore motivated by the following result:

Proposition 7.3.3 (Connes–Landi [29], Connes–Dubois-Violette [28])**.** *Let X be a compact oriented Riemannian p-manifold with orientation-preserving isometric $\mathbb{T}^N$-action, let $\mathcal{E} \to X$ be a $\mathbb{T}^N$-equivariant Clifford module, and let D be a $\mathbb{T}^N$-invariant symmetric Dirac-type operator on $\mathcal{E}$; let $\theta \in \mathfrak{so}(N)$. Then $(C^\infty(X_\theta), L^2(X_\theta, \mathcal{E}), D)$ is a $\mathbb{T}^N$-equivariant p-dimensional Dirac-type spectral triple with right $C^\infty(X_\theta)$-action defined by ρ_θ.*

Proof. First, we have that $(C^\infty(X_\theta), L^2(X_\theta, \mathcal{E}), D)$ is regular and of metric dimension p by Theorem 7.2.5. Indeed, it is two-sided for the right action $\lambda_\theta(a)^o := \rho_\theta(a)$; on the one hand, by $\mathbb{T}^N$-invariance of D, $(\operatorname{ad}|D|)^k(\lambda_\theta(a))^o = \rho_\theta((\operatorname{ad}|D|)^k(a))$, so that $\lambda_\theta(a)^o H^\infty \subset H^\infty$ for all $a \in C^\infty(X_\theta)$, and on the other hand, since $C^\infty(X)$ is commutative, $[\lambda_\theta(a), \rho_\theta(b)] = 0$ by Lemma 7.1.7. Moreover, by $\mathbb{T}^N$-equivariance of ρ_θ, it follows that

$$\forall a \in C^\infty(X_\theta),\ t \in \mathbb{T}^N, \quad U_t\lambda_\theta(a)^o U_t^* = U_t\rho_\theta(a)U^* = \rho_\theta(\sigma_t(a)) = \lambda_\theta(\sigma_t(a))^o.$$

Let us now check the conditions for a $\mathbb{T}^N$-equivariant p-dimensional Dirac-type spectral triple one by one:

1. As we have already seen, $(C^\infty(X_\theta), L^2(X_\theta, \mathcal{E}), D)$ has metric dimension p.
2. Since $[[D, a], b] = 0$ for all a, $b \in C^\infty(X)$, we have that

   $$[[D, \lambda_\theta(a)], \lambda_\theta(b)^o] = [[D, \lambda_\theta(a)], \rho_\theta(b)] = [\lambda_\theta([D, a]), \rho_\theta(b)] = 0$$

   by $\mathbb{T}^N$-invariance of D and Lemma 7.1.7.

3. Recall that $\cap_k \operatorname{Dom} D^k = C^\infty(X, \mathcal{E})$. Since $\mathcal{E} \to X$ is $\mathbb{T}^N$-equivariant, it follows [6, Lemma 3.3] that there exists a finite-dimensional $\mathbb{T}^N$-module V, such that $\mathcal{E}$ is $\mathbb{T}^N$-equivariantly a direct summand of $X \times V$, and, hence, we have a $\mathbb{T}^N$-equivariant identification $C^\infty(X,\mathcal{E})_{C^\infty(X)} \cong pC^\infty(X)^N$ for some $\mathbb{T}^N$-invariant orthogonal projection $p \in M_N(C^\infty(X))$. By $\mathbb{T}^N$-equivariance of this identification and $\mathbb{T}^N$-invariance of p, it therefore follows that we have a $\mathbb{T}^N$-equivariant identification
$$C^\infty(X_\theta, \mathcal{E})_{C^\infty(X_\theta)} := C^\infty(X, \mathcal{E})_{C^\infty(X_\theta)} \cong pC^\infty(X_\theta)^N,$$
immediately implying finiteness for the deformed spectral triple.

4. By invertibility of λ_θ and Lemma 7.1.7, we can readily check that $\operatorname{End}_{C^\infty(X_\theta)^o}(C^\infty(X_\theta,\mathcal{E})) = \lambda_\theta\left(\operatorname{End}_{C^\infty(X)}(C^\infty(X,\mathcal{E}))\right)$, so that $\operatorname{End}_{C^\infty(X_\theta)^o}(C^\infty(X_\theta,\mathcal{E})) \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k$, since
$$\operatorname{End}_{C^\infty(X)}(C^\infty(X,\mathcal{E})) \subset \cap_k \operatorname{Dom}(\operatorname{ad}|D|)^k.$$
Thus, strong regularity holds.

5. The orientation cycle $c \in Z_p(C^\infty(X), C^\infty(X))$ of $(C^\infty(X), L^2(X,\mathcal{E}), D)$ is given by the $\mathbb{T}^N$-invariant volume form on X. Hence, by the proof of [28, Theorem 13.9], it follows that the corresponding cycle $c_\theta \in Z_p(C^\infty(X_\theta), C^\infty(X_\theta))$ yields the desired antisymmetric $\mathbb{T}^N$-invariant orientation cycle for the deformed spectral triple.

6. Going back to the proof of finiteness, one can observe that $\mathcal{E}$ is a $\mathbb{T}^N$-equivariantly a direct summand of $X \times V$ as a Hermitian vector bundle. Thus $C^\infty(X,\mathcal{E})_{C^\infty(X)} \cong pC^\infty(X)^N$ as $\mathbb{T}^N$-equivariant right pre-Hilbert $C^\infty(X)$-modules, and, hence, $C^\infty(X_\theta,\mathcal{E})_{C^\infty(X_\theta)} \cong pC^\infty(X_\theta)^N$ as $\mathbb{T}^N$-equivariant right pre-Hilbert $C^\infty(X_\theta)$-modules; in particular, $\mathbb{T}^N$-invariance of the Hermitian metric on $C^\infty(X_\theta,\mathcal{E})$, obtained from the Hermitian metric on $C^\infty(X,\mathcal{E})$, implies that it takes its values in $C^\infty(X_\theta)_0 = C^\infty(X)_0$, so that absolute continuity is indeed preserved

Thus, $(C^\infty(X_\theta), L^2(X_\theta,\mathcal{E}), D)$ is indeed a $\mathbb{T}^N$-equivariant p-dimensional Dirac-type spectral triple. □

Example 7.3.4. If $(C^\infty(\mathbb{T}^N), L^2(\mathbb{T}^N,\mathcal{S}), \not{D})$ is the canonical spectral triple of $\mathbb{T}^N$ with a given spin structure, then $(C^\infty(\mathbb{T}^N_\theta), L^2(\mathbb{T}^N_\theta,\mathcal{S}), D)$ is the corresponding *noncommutative N-torus.*

In light of Proposition 7.3.3, we define an abstract toric noncommutative manifold as follows:

Definition 7.3.5. Let (A,H,D) be a $\mathbb{T}^N$-equivariant p-dimensional Dirac-type spectral triple, and let $\theta \in \mathfrak{so}(N)$. We call (A,H,D) a *p-dimensional toric noncommutative manifold* with *deformation parameter* θ if
$$\forall a \in A, \quad a^0 := \lambda_{-2\theta}(a).$$

Our progress towards a reconstruction theorem, then, is summarised in the following theorem:

Theorem 7.3.6. *Let (A, H, D) be a p-dimensional toric noncommutative manifold with deformation parameter $\theta \in \mathfrak{so}(N)$, and suppose, moreover, that the orientation cycle $c \in Z_p(A, A)$ corresponds to a $\mathbb{T}^N$-invariant antisymmetric cycle $c_\theta \in Z_p(A_{-\theta}, A_{-\theta})$, such that $\pi_D(c) = \pi_D(c_\theta)$. Then there exists a concrete $\mathbb{T}^N$-equivariant commutative spectral triple, such that*

$$(A_{-\theta}, H, D) \cong (C^\infty(X), L^2(X, \mathcal{E}), D), \quad (A, H, D) \cong (C^\infty(X_\theta), L^2(X_\theta, \mathcal{E}), D),$$

i.e., (A, H, D) is unitarily equivalent to a concrete p-dimensional toric noncommutative manifold.

Proof. In light of Proposition 7.3.3, it suffices to show that $(A_{-\theta}, H, D)$ is a p-dimensional commutative spectral triple, for then by Corollary 4.2.7, we can reconstruct X, $\mathcal{E}$, and D, and from there reconstruct the $\mathbb{T}^N$-action on X and on $\mathcal{E}$ from $t \mapsto \sigma_t$ and $t \mapsto U_t$, respectively.

Now, let a, $b \in A$. Then, in particular, $\rho_{-\theta}(b^o) = \lambda_\theta(b^o) = \lambda_\theta(\lambda_{-2\theta}(b)) = \lambda_{-\theta}(b)$, so that by Lemma 7.1.7,

$$[\lambda_{-\theta}(a), \lambda_{-\theta}(b)] = [\lambda_{-\theta}(a), \rho_{-\theta}(b^o)] = 0,$$

and, similarly,

$$[[D, \lambda_{-\theta}(a)], \lambda_{-\theta}(b)] = [\lambda_{-\theta}([D, a]), \rho_{-\theta}(b^o)] = 0.$$

The rest then follows from the proof of Proposition 7.3.3, *mutatis mutandis*, together with the explicit orientability-related hypothesis; in patricular, to prove finiteness and absolute continuity, one needs the noncommutative generalisation [4, Prop. 11.2.3] of [6, Lemma 3.3]. □

In fact, in light of the proof of Proposition 7.3.3, one could take Theorem 7.3.6 as a working reconstruction theorem for abstract noncommutative toric manifolds. However, the additional orientability-related hypothesis is decidedly awkward, indeed unnatural, and its removal is an immediate research priority. However, the author expects that this may well be somewhat tricky, given the delicacy of working with the Hochschild homology of deformation quantisations, including even the noncommutative 2-torus.

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