Equilibrium in Dynamic Economic Models

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Abstract

The main theme running through these three chapters is that economic agents are often forced to respond to events that are not a direct result of their actions or other agents actions. The optimal response to these shocks will necessarily depend on agents' understanding of how these shocks arise. The economic environment in the first two chapters is analogous to the classic chain store game. In this setting, the addition of unintended trembles by the agents creates an environment better suited to reputation building. The third chapter considers the competitive equilibrium price dynamics in an overlapping generations environment when there are supply and demand shocks.

The first chapter is a game theoretic investigation of a reputation building game. A sequential equilibrium model, called the "error prone agents" model, is developed. In this model, agents believe that all actions are potentially subjected to an error process. Inclusion of this belief into the equilibrium calculation provides for a richer class of reputation building possibilities than when perfect implementation is assumed.

In the second chapter, maximum likelihood estimation is employed to test

the consistency of this new model and other models with data from experiments run by other researchers that served as the basis for prominent papers in this field. The alternate models considered are essentially modifications to the standard sequential equilibrium. While some models perform quite well in that the nature of the modification seems to explain deviations from the sequential equilibrium quite well, the degree to which these modifications must be applied shows no consistency across different experimental designs.

The third chapter is a study of price dynamics in an overlapping generations model. It establishes the existence of a unique perfect-foresight competitive equilibrium price path in a pure exchange economy with a finite time horizon when there are arbitrarily many shocks to supply or demand. One main reason for the interest in this equilibrium is that overlapping generations environments are very fruitful for the study of price dynamics, especially in experimental settings. The perfect foresight assumption is an important place to start when examining these environments because it will produce the *ex post* socially efficient allocation of goods. This characteristic makes this a natural baseline to which other models of price dynamics could be compared.

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Chapter 1

Reputation Building with Error Prone Agents

1.1 Introduction

Kreps and Wilson's (1982a) sequential equilibrium (SE) has become the standard equilibrium concept for reputation building models. Imperfect information is the essential element of these models that allows the SE to make appealing predictions. This uncertainty allows agents to mimic other types of agents in order to mislead other agents, in other words, to strategically build reputations. Reputation building is in strong contrast to the equilibrium predictions under perfect information which are typically counterintuitive in these situations, such as the famous backward unraveling result in the chain store paradox. The models to which the SE has been applied generally assume that agents believe that all agents are able to perfectly implement their SE strategies. Under this assumption, certain actions will fully reveal the agent's true type, so that the continuation game becomes one of perfect information. So a single action can resolve the uncertainty that enabled the SE to make different predictions than the counterintuitive predictions of the perfect information case.

The current work introduces a new model of reputation building, in which the perfect implementation assumption is relaxed. In this model, no action is seen as a perfect signal, so the model will remain one of imperfect information for any observed history. Imperfect implementation will enter into the model in the form of a belief by agents that every action is subject to an error process. The error process is assumed to take the following form: after each move by a player, nature intervenes with some small probability. If nature intervenes, nature will simply randomize over the possible moves at that node. If nature does not intervene, the player's move is implemented. Other agents cannot tell whether nature intervened or not. The error process is assumed to be common knowledge. This model will be called the "error prone agents" model. The effect of this error process is similar to the introduction of imperfect monitoring in principal-agent models and repeated games,[1] in the sense that the imperfections make agents less willing to punish other agents when "undesirable" outcomes are observed because the link between observation and intended action is now uncertain. The difference between the two types of imperfections is that under imperfect implementation, agents also consider the possibility that they may accidentally take a suboptimal action in the future.

Consider an example in which a creditor is expecting a payment from a debtor in the mail, but does not receive it by the deadline. It may be that the debtor honestly thought she had made a payment but she actually forgot, or maybe the check got lost in the mail. Another possibility is that the debtor had no intention of making a payment. The first two possibilities are quite similar from the creditor's perspective, but the third case is very different. The error process introduced above is an attempt to include the first two explanations of the missing check into standard reputation building models, where before only the third case was considered. Notice that even though there are very different explanations behind the first two examples, the model developed here will treat them as if they were the same. The first example rests on the fallibility of the debtor, and the second on a noisy

[1] See Abreu, Pearce and Stacchetti (1986) and Kandori (1992).

environment. But in both cases, the debtor intended to make a payment and may be surprised to find out that the check was not received. It may be the case that the debtor knows she is forgetful or that the mail is unreliable, and thus would not be very surprised at all. The important point is that she intended to pay.

In the error prone agents model, agents believe that their own actions are also susceptible to the error process. This specification of the error process has the attractive characteristic that the perfect implementation model is obtained as the agents' beliefs about the potential for errors approaches zero. The resulting equilibrium provides for some interesting possibilities for reputation building. Consider the simple example above. Under perfect implementation, the creditor would likely contact a collection agency if the check is not received by the deadline. When imperfect implementation is considered, a debtor who had been dependable in the past may be granted a grace period, or sent a reminder, whereas a debtor with a bad credit history may not receive such leniency. The sequential equilibrium to the error prone agents model allows for this kind of distinction.

A sketch of the calculation of this equilibrium will be presented in the body of the chapter (a detailed equilibrium calculation appears in the appendix) for a game studied by Camerer and Weigelt (1988) (CW, henceforth). CW is primarily an experimental investigation of the predictive power of the reputation building SE with the standard assumption of perfect implementation. Their model is analyzed to facilitate the use of their data for model testing.

This chapter is organized as follows. Section 1.2 provides a brief introduction to the reputation literature. The development of the SE for the error prone agents model is sketched in Section 1.3. Section 1.4 will conclude this chapter with closing thoughts and ideas for extensions of this approach.

1.2 Reputation Building Literature

1.2.1 The Sequential Equilibrium

A classic reputation building problem is the chain store paradox. The chain store paradox involves a monopolist and a set of potential entrants. The monopolist sequentially plays a stage game against each of the potential entrants. The stage game starts with the potential entrant deciding whether to enter the market or not. If entry occurs then the monopolist has the choice of either acquiescing and sharing the market or fighting the entrant. The monopolist's preference ordering in the stage game is: no entry $\succ$ acquiescence $\succ$ fighting. Entrants' preferences are: acquiescence $\succ$ no entry $\succ$ fighting. The monopolist would like for no entry to occur, so she may fight early entrants as a signal to future entrants to stay out.

However, the following argument shows the weakness of the logic leading to such a strategy. In a finite sequence of the stage games, the entrant in the final stage game will enter because he knows that the monopolist would rather acquiesce than fight and that fighting can have no important effect on reputation. Notice that this does not depend on any of the previous moves in the game. Since entry is expected in the final stage game, the monopolist need not be concerned with the reputation effects of her action in the penultimate stage game. By induction, all entrants should enter and the monopolist should always acquiesce. This holds for any finite sequence of stage games. This unraveling result is intuitive for a small number of repetitions, but the backward induction argument starts to seem more tenuous as the number of entrants increases. Consider a forward induction argument under which monopolists would fight in early rounds in order to deter future entry. The early entrants will recognize that the monopolist may want to do this, and so they will not enter. This dichotomy between forward and backward induction is the reason these results are dubbed a 'paradox.'

Kreps and Wilson (1982b) suggest that if entrants have a small amount of uncertainty about the monopolist's payoffs, then there exists a game theoretic prediction in which reputations can form and some entrants will not enter in the finitely repeated game. For instance, entrants may think that there is some possibility that the monopolist's marginal cost curve is downward sloping so that he would prefer to fight rather than share the market. This type of monopolist would always fight entry. Monopolists who prefer not to fight would be willing to fight early entrants to deter future entrants. As the game approaches the final stage game, monopolists have less to gain by fighting, so at some point entrants will decide to test the monopolist. This is the basic intuition behind this SE. Kreps and Wilson (1988a) define a sequential equilibrium as a set of beliefs and a set of strategies for each player satisfying the following conditions: (i) a player has some probability assessment over the nodes in her information set conditioned on observed play whenever action is required; (ii) these probabilities are consistent with the equilibrium strategy; and (iii) play at each information set is optimal given future equilibrium behavior of others and the aforementioned probability distribution. The exact form of the SE for the chain store game will not be considered here as it is similar to the SE model of reputation building for the particular game to be studied throughout this chapter which shall be addressed shortly.

1.2.2 Imperfect Equilibrium

Another equilibrium concept, the imperfect equilibrium (Beja 1992), employs an alternate form of uncertainty, but is also able to obtain an equilibrium that also avoids the backward unraveling result. In this model, rather than asserting that some agents are inclined to play one particular strategy (i.e., always fight), it is instead assumed that agents have imperfect performance. Agents have in mind a particular strategy, but they sometimes fail to perform it correctly. The probability that a strategy accidentally gets played is inversely related to the potential loss the player would incur if it were adopted. The rationale for this is that players would ensure against costly mistakes.

It turns out that for the chain store scenario, this characteristic greatly diminishes the impact of the entrants having a dominant strategy. Beja discusses a two period repeated chain store game. His result is that the first entrant should choose to stay out and the second should enter, at which point the monopolist will acquiesce. The reason the first entrant stays out comes from a consideration of the effect monopolist's first round strategy has on the second entrant's expected payoff. The second entrant has four pure strategies to choose from, given that the first entrant actually enters. These can be denoted: $\{(e,e),(o,o),(o,e),$ and, $(e,o)\}$, where "e" stands for enter, "o" for out, and the pairs are ordered so that these are the responses to acquiescence and fighting, respectively, by the monopolist in the first round. It is clear that (e,e) is the dominant strategy in this equilibrium, as it is in standard complete information setups. Nothing the monopolist does in the first round can change that. However, since the probabilities that strategies are selected accidentally depends on their relative costs, the monopolist must consider his effect on the error rates. If the monopolist were simply to acquiesce in the first round, it would be very costly for the second entrant to adopt (o,e) as compared to (e,o), so the monopolist would normally face entry when the second entrant errs. By increasing the probability of fighting in the first round, the monopolist can increase the chance that the second entrant will accidentally not enter because she will now sometimes fail to enter under (e,o), and because (o,e) is selected more often due to its relative increase in expected payoff. This scenario was based on entry in the first round, but given that the monopolist will fight with high probability, the first entrant will opt to stay out. The details of this obviously depend on the payoffs and on the specification of how the performances depend on the relative payoffs.

The simplicity of this example shrouds the complexity that could arise when trying to analyze a multi-period repeated game with this technique. For instance, consider a three period chain store game. Entry is still dominant for the third entrant. To determine the monopolists's response to entry in the second period, we would need to consider the effect this would have on the performance of the third entrant. The third entrant, however, will have 64 pure strategies to consider, given entry in the second period.[2] It is clear that this analysis quickly becomes intractable for multi-period repeated games. The equilibrium developed later in this chapter bears some resemblance to the imperfect equilibrium, although it was developed independently. It differs in that it is developed in an imperfect information environment, and that it greatly narrows the state space by allowing players to use an updated "reputation" to summarize past moves in the game.

1.2.3 Camerer and Weigelt

This paper is primarily an experimental paper. It is included with these theoretical works as a way to introduce the model to be used for the original theoretical work that is to follow.

The model studied is a borrower-lender game in which a single borrower (entrepreneur or E-type) faces a sequence of eight lenders (bankers or B-types) who must decide whether to grant the borrower credit or not. If a borrower does procure a loan, she must then decide whether to repay the loan or to default. Of course, if no loan is granted, the borrower has no choice to make. The stage game consists of a banker decision followed by a borrower decision if necessary. The outcomes of all previous stage games are common knowledge. Bankers like to make loans, as long as they are repaid, but they would rather refuse the loan than to have it go into default. Assuming that the stage game is only played once, a typical profit-maximizing entrepreneur

[2]There are three possible outcomes in the first period, and two possible outcomes in the second, given entry. Thus, we must consider 2^6 or 64 possible pure strategies for entrant three.

should prefer to get a loan, and then default over repaying or not getting it at all. This is what CW call an X-type entrepreneur. These entrepreneurs prefer repaying a loan to being refused credit. If only these two types of agents are present, and everyone believes that to be true, then the finitely repeated game is susceptible to complete unraveling like the chain store game. In the final period, an X-type should default if given a loan, because she prefers defaulting to repaying and there are no more stage games to be effected by this decision. So the final banker should refuse to lend, to avoid the default. Then in the penultimate period, the entrepreneur knows she will not get a loan in the next period regardless of her actions in any previous periods, so she may as well default. Backward induction implies that no banker will lend to the X-type entrepreneur.

If bankers believe that there is some probability that the entrepreneur actually would rather repay than default, the sequential equilibrium model of reputation building predicts that some lending should occur early in the game. This new type of entrepreneur is called a Y-type. The preferences of each type of agent in this game are summarized below.

Banker (B-type): repayment $\succ$ no loan $\succ$ renege

X-type Entrepreneur: renege $\succ$ repayment $\succ$ no loan

Y-type Entrepreneur: repayment $\succ$ no loan $\succ$ renege

CW induced a given proportion of their entrepreneurs to have Y-type preferences in their experiments. By making this proportion common knowledge, they were able to investigate the SE's predictive power.

The sequential equilibrium to the game studied by CW under the perfect implementation assumption will not be considered in detail here because it is a special case of the SE to the more general model introduced in the next section. However, a good intuitive idea of the SE predictions for this model will help the reader to better understand the results for the more complicated model that follows. Even though the equilibrium developed later in this chapter is also an SE, and others do exist, unless otherwise specified, the term "the SE" will refer to the sequential equilibrium to this perfect implementation model throughout the rest of this chapter. This notation is adopted because this is the standard SE model considered in the reputation building literature.

Once a banker sees a default, he knows that the entrepreneur is an X-type, so he should not make a loan. (This is a strong implication of the perfect implementation assumption.) Early in the game, even an X-type entrepreneur will repay her loans to try to mislead future bankers into thinking she is a Y-type and continuing to lend her money. As the game's end approaches, default becomes more appealing to the X-type because there is not as much to gain by continuing to repay as there was in the beginning of the game. Bankers recognize this, so they need extra assurance that they are facing a Y-type to keep lending. If both X-types and Y-types are acting identically (always repaying), then bankers learn nothing from seeing a string of repayments. To have a repayment serve as support for the bankers believing the entrepreneur is a Y-type, an X-type must play mixed strategies that place positive probability on reneging. The later in the game, the more certain bankers must be that they are facing a Y-type for them to be willing to lend. Bankers will be indifferent following a repayment under this mixed strategy, and can thus play mixed strategies which will support the mixing by the X-types. As the game progresses, X-types must repay less frequently so that if they do repay, it makes bankers more likely to believe they are facing a Y-type entrepreneur and continue to give some loans. The SE prediction is that there will be repaid loans for the first few periods then a mixed strategy equilibrium in which no loans occur following a renege and repayment is followed by more mixing ("few" could be made more precise given information about the payoffs, number of periods and initial beliefs of bankers about what type of entrepreneur they are facing).

1.3 The Error Prone Agents Model

While this section is entirely theoretical in nature, the inspiration for it was a data concern of Camerer and Weigelt. For the first few periods in their borrower-lender game, the SE predicts pure strategies by all types of agents. Consider what a participant in one of the experiments would think about an observation that should never occur under the SE. If the participant knew the SE and assumed that everyone would play it, a deviation would completely dumbfound him. Of course, the SE does have contingency plans so that there are equilibrium moves even after an unexpected move. Is it reasonable to expect a person, who is working under the assumption that all players are following the SE, to continue to play the equilibrium after he has seen a deviation? He observed something that should have never happened according to his original beliefs. [3] It seems as though such a deviation must alter his belief structure in some manner. The SE to the error prone agents model has the characteristic that agents expect to see some out of equilibrium moves, and will respond optimally to these deviations by following the equilibrium.

One could model beliefs about imperfect implementation in many ways. Given the relative success the SE seemed to have in CW, a relatively simple modification of the traditional belief structure was chosen. People still believe that others intend to follow a common knowledge equilibrium, but they sometimes fail to do so. It is as if with some epsilon probability, nature can intervene and randomize over the possible actions. Agents realize that this may also happen to them. The main goal of this research is to see if the introduction of a simple belief structure which accounts for these errors can improve on both the qualitative and quantitative predictions of the SE under the perfect implementation assumption. Unfortunately, full construction of the SE for this new model is quite complex, so only a sketch of the development of this equilibrium is given in this section. Those interested in a more detailed account should consult the appendix.

[3] See Fudenberg and Tirole (1990) for a good discussion of the observation of zero probability events. This paper shows the relation between perfect Bayesian equilibria and sequential equilibria. They also offer a refinement of the perfect Bayesian equilibrium which is shown to be equivalent to the SE for finite games with finitely many types of agents.

1.3.1 Beliefs and Updating

Construction of the new model starts with a belief, ϵ, which is the frequency of intervention by nature. At any decision node in the borrower-lender game, there are only two options. It is arbitrarily assumed that nature picks between them with equal probability. So an agent assumes that with probability $1-\epsilon/2$ their intended move will successfully be enacted because nature will not intervene with probability $1-\epsilon$ and nature will intervene only to make the intended move with probability $\epsilon/2$. Consider the reputation building implication of a positive ϵ: If an entrepreneur is strongly believed to be a Y-type, a renege is likely to be interpreted as intervention by nature. This is mentioned just to fore-shadow one of the interesting characteristics of the new equilibrium, which is that it allows for reputations to be built that cannot be destroyed with one action. In the absence of beliefs about errors, $\epsilon = 0$, the SE under perfect implementation is obtained and one renege is a definite sign that the borrower is not a Y-type. So a positive ϵ allows for a richer type of reputation to develop.

Entrepreneurs, both X- and Y-types, will only have beliefs about the error rates. For ease of exposition, it is assumed that this is a common belief among all agents, including bankers. Bankers will also have beliefs about whether the entrepreneur in the current setting is an X-type or a Y-type. To be consistent with CW, P_t will be the belief of the banker in the t^{th} period that the current entrepreneur is a Y-type. Y-types most prefer to repay loans. The introduction of beliefs about errors will not alter this being a dominant strategy for them. That leaves only the bankers' and the X-types' strategies to be calculated.

Let L_t be the intended probability with which the banker in period t will grant a loan, and S_t be the intended probability with which an X-type will repay the loan. For $\epsilon > 0$, it will always be the case that bankers can update their beliefs about the entrepreneur using Bayes' rule. The probability that a Y-type will repay a loan in any period t is $1 - \epsilon/2$, made up of the $1 - \epsilon$ probability of intending to repay the loan, and the $\epsilon/2$ probability that nature intervenes and repays. Similarly, the probability that an X-type repays is $S_t(1 - \epsilon) + \epsilon/2$. A banker in period $t + 1$ would then update her belief P_t after observing a repayment according to the formula,

$$\begin{aligned} P_{t+1}^{PB} &= \frac{(1 - \epsilon/2)P_t}{(1 - \epsilon/2)P_t + [S_t(1 - \epsilon) + \epsilon/2](1 - P_t)} \\ &= \frac{(1 - \epsilon/2)P_t}{1 - \epsilon/2 - (1 - P_t)(1 - S_t)(1 - \epsilon)}. \end{aligned} \tag{1.1}$$

The superscript PB indicates that the entrepreneur paid the loan back in the previous period. Similarly, bankers can update their beliefs based on a renege by using

$$\begin{aligned} P_{t+1}^{RN} &= \frac{(\epsilon/2)P_t}{(\epsilon/2)P_t + [(1 - S_t)(1 - \epsilon) + \epsilon/2](1 - P_t)} \\ &= \frac{(\epsilon/2)P_t}{\epsilon/2 + (1 - P_t)(1 - S_t)(1 - \epsilon)}. \end{aligned} \tag{1.2}$$

Of course, if no loan was granted in period t, then $P_{t+1} = P_t$ because no new information about the entrepreneur was obtained.

The following theorem establishes that beliefs following repayment must be at least as great as those following a default, and gives the conditions under which this relation is strict.

Theorem 1.3.1 *For any* t, $P_{t+1}^{PB} \geq P_{t+1}^{RN}$. *Furthermore, for any* $\epsilon < 1$, $0 < P_t < 1$ *and* $S_t < 1$, $P_{t+1}^{PB} > P_{t+1}^{RN}$.

Proof: By way of contradiction, assume $P_{t+1}^{PB} < P_{t+1}^{RN}$. (1.1) and (1.2) imply the following.

$$\begin{aligned}
P_{t+1}^{PB} &< P_{t+1}^{RN} \\
\frac{(1-\epsilon/2)P_t}{1-\epsilon/2-(1-P_t)(1-S_t)(1-\epsilon)} &< \frac{(\epsilon/2)P_t}{\epsilon/2+(1-P_t)(1-S_t)(1-\epsilon)} \\
(1-P_t)(1-S_t)(1-\epsilon)(1-\epsilon/2)P_t &< (1-P_t)(1-S_t)(1-\epsilon)(\epsilon/2)P_t \\
(1-P_t)(1-S_t)(1-\epsilon)^2 P_t &< 0
\end{aligned}$$

Since all of the terms on the left of this last inequality are probabilities, it is clear that the inequality cannot be satisfied. To support the second claim of the theorem, simply use a weak inequality in the contradictory assumption. The conditions of the second claim clearly violate the weak inequality. ∎

Given a set of payoffs, the SE to the new model can be constructed. The parameters from CW's experiments in which 33 percent of the entrepreneurs were assigned to be Y-types will be used in this exposition. In these experiments, X-types received a payoff of 60 francs for repaying, 150 for reneging and 10 if no loan was granted. Payoffs were stated in francs, which were to be exchanged for U.S. currency at the end of the experiment. Bankers received 40 francs following repayment, -100 following a default, and 10 if they did not grant a loan.

In general, a banker who must decide whether to lend in period t, who shall be referred to as B_t, will have the following indifference condition:

$$\begin{aligned} 40P_t(1-\epsilon/2) - 100(\epsilon/2) + 40(1-P_t)[(1-\epsilon)S_t + \epsilon/2] & \\ -100(1-P_t)[(1-\epsilon)(1-S_t) + \epsilon/2] &= 10 \quad (1.3) \end{aligned}$$

which reduces to

$$(1-P_t)(1-S_t)(1-\epsilon) = 3/14 - \epsilon/2. \quad (1.4)$$

If the left side of (1.4) is less (greater) than the right, then the banker would (would not) be willing to lend.

It may appear from (1.3) that the banker is not considering the possibility that he will make an error himself. Theorem 1.3.2 shows that when comparing the expected payoffs to two actions, one can ignore the possibility of an error affecting that particular action.

Theorem 1.3.2 *The probability of an error on the current move is irrelevant to the choice between the two available pure strategies.*

Proof: Let EP_1 and EP_2 be the expected payoffs to actually carrying out the two possible moves at a node, and assume $EP_1 > EP_2$. The expected payoffs to intending to carry out these strategies would then be $EP_1(1-\epsilon/2)+EP_2(\epsilon/2)$ and $EP_2(1-\epsilon/2)+EP_1(\epsilon/2)$. Assume by way of contradiction that $EP_1(1-\epsilon/2)+EP_2(\epsilon/2) \leq EP_2(1-\epsilon/2)+EP_1(\epsilon/2)$. Collecting terms results in $EP_1(1-\epsilon) \leq EP_2(1-\epsilon)$. For $\epsilon < 1$ this contradicts the initial assumption that $EP_1 > EP_2$. ∎

1.3.2 Period Eight

The SE to the error prone agents model is developed through backward induction. Each entrepreneur faces a sequence of eight bankers. Clearly, if an X-type receives a loan in period eight, they will want to renege, so $S_8 = 0$. For $S_8 = 0$, (1.4) simplifies to the following indifference condition for B_8,

$$P_8 = \frac{11/14 - \epsilon/2}{1 - \epsilon}. \tag{1.5}$$

Higher beliefs will lead to an intended loan, and lower beliefs lead to an intended refusal. B_8's strategies when indifferent will depend on the strategy employed by X_7, so those mixed strategies will be discussed when X_7's strategies are developed.

1.3.3 Period Seven

B_8's indifference condition can be used to calculate mixed strategy equilibria for X-types to play in period seven. Two ways to satisfy (1.5) are: for P_8^{PB} to satisfy it; and for P_8^{RN} to satisfy it. (It is also possible for B_8 to be indifferent following a refusal, but that is a very special case which will only be addressed in the appendix.) That is to say, B_8 can be made indifferent following a repayment for certain values of P_7, and following a renege for others. Each of these possibilities requires a unique S_7. The S_7 that leads to indifference following a repayment will be denoted by S_7^{PB}, and following a renege by S_7^{RN}. S_7^{PB} is found by solving for the S_7 that forces P_8^{PB}, from (1.1), to satisfy (1.5). The closed form solution to this is not particularly enlightening, so it will only be presented in the appendix. This strategy basically corresponds to the mixed strategy SE under perfect implementation. Analogously, S_7^{RN} is obtained by making B_8 indifferent when she updates using P_8^{RN}. The X-type entrepreneur's strategies can best be understood by viewing a graph.

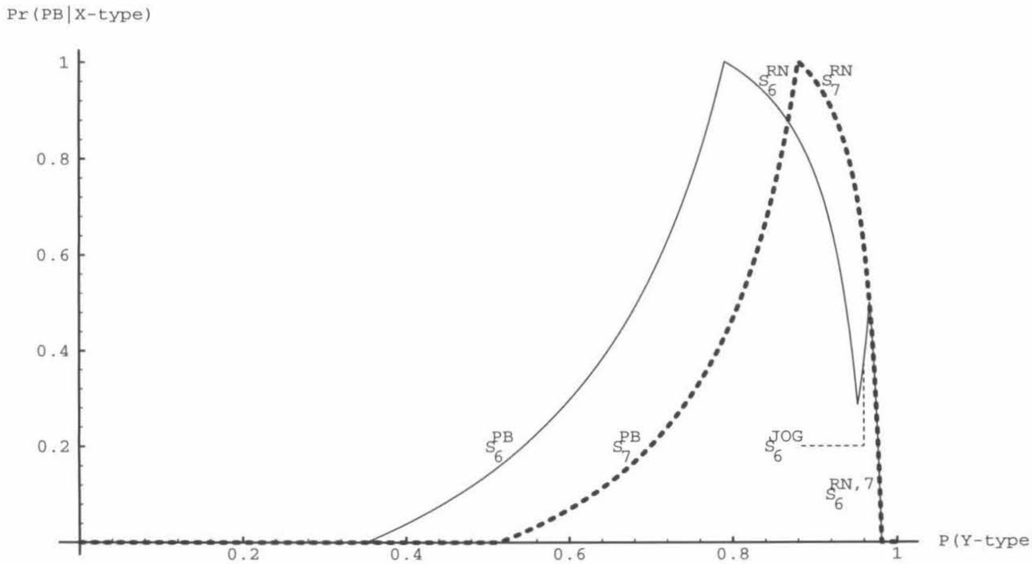

Figure 1.1: X-type strategies for periods six and seven: $\epsilon = .25$

Figure 1.1 graphs the X-type strategies for periods six and seven. The strategy for period seven is the thick, dotted curve that looks like a wave. The strategy for period six will be explained later. The Y-axis is the probability that the X-type repays a loan in period seven, S_7, and the X-axis is bankers belief that the entrepreneur is a Y-type, P_7. The strategies in the graph are calculated for $\epsilon = .25$. For very low beliefs, there is no reason to repay a loan at all, because B_8 will see the repayment as being more likely an accident than an intentional action and will have no intention of offering a loan. For somewhat higher P_7, S_7 starts to become positive. These are the levels of P_7 for which B_8 will be indifferent following a repayment given that a repayment was unlikely to have come from an X-type. The higher P_7 becomes, the less strong a signal a repayment needs to be to invoke indifference. This upward-sloping portion of the curve is S_7^{PB}. A renege for these values of P_7 would deter B_8 from lending. To make an X-type indifferent between defaulting and repaying under this equilibrium mixed strategy, B_8 must mix in response to a repayment to equate the expected payoff from repaying to that of reneging and being refused a loan in period eight. The closed form for B_8's strategy that satisfies this indifference is given in the appendix. These payoffs naturally must take into account the role of beliefs about errors on expectations. Eventually the curve reaches a point where $S_7 = 1$. For any P_7 above this point, a repayment will cause B_8 to want to lend. As shown in Theorem 1.3.3, unless a repayment leads to indifference, an X-type cannot have a pure strategy to repay.

Theorem 1.3.3 *For $\epsilon > 0$, $S_7 = 1$ is an equilibrium strategy only if P_8^{PB}, given that $S_7 = 1$, makes B_8 indifferent.*

Proof: Assume by way of contradiction that $S_7 = 1$ and B_8 will not be indifferent following a repayment. For $S_7 = 1$, (1.1) and (1.2) imply that $P_8^{PB} = P_8^{RN} = P_7$. Given that B_8 is not indifferent for this updated belief, B_8 must have a pure strategy to either lend or refuse to lend. This will be independent of X_7's move. Since X_7's action has no effect on the continuation game, X_7 should maximize short term payoffs by reneging. This contradicts the proposed equilibrium strategy of $S_7 = 1$. ∎

Under perfect implementation, S_7 levels off at one for all beliefs in this range. Theorem 1.3.3 eliminated that possibility. It turns out that the only equilibrium strategy for these values of P_7 is S_7^{RN}. In this range, a renege will cause indifference, and a repayment will imply an intended loan. An interesting characteristic of this P_7 which is the transition point from S_7^{PB} to S_7^{RN} is that it is the belief level at which B_8 is indifferent. If X_7 gets a loan and repays under S_7^{RN}, B_8's beliefs will be updated to this particular transition point. It is always the case that B_{t+1} will be indifferent if P_{t+1} is equal to the P_t at which $S_t^{PB} = S_t^{RN} = 1$. This gives the peaks of these "waves" a meaningful interpretation. B_8 will mix in response to a renege under S_7^{RN} so that the expected payoff is equal to the expected payoff from repaying and getting a loan in period eight. The higher P_7 gets, the less an X-type needs to mock a Y-type to have a renege result in indifference. Finally, for very high levels of P_7, an X-type can have a pure strategy to renege and still expect to get a loan following a renege.

1.3.4 Periods Six through One

In period six, the same strategies are employed with mixed strategies resulting from the indifference described by (1.4) under both P_7^{PB} and P_7^{RN}. However, there is an added level of complexity. For levels of P_6 just below the level at which B_7 would intend to lend following a renege, there is no mixed strategy available to B_7 that would support S_6^{RN} as an equilibrium. Two new strategies will be introduced for this set of beliefs. Where all the mixed strategies of X-types previously discussed were supported by mixed strategies played by the banker in the very next period, these new strategies depend on mixing by the banker two turns in the future. For reasons that are apparent from the graphical representation of these strategies, these mixed strategies will be called S_6^{JOG} and the $S_6^{RN,7}$.

Note from Figure 1.1, that as S_7^{RN} reaches zero, B_8 is indifferent following a renege. But at this point, the X-type has a pure strategy to renege. So there is sort of a spare degree of freedom in that B_8 can play a mixed strategy in response to a renege which only needs to make X_7 prefer to renege. This degree of freedom will be used in the construction of S_6^{JOG}. Under S_6^{JOG}, a repayment will result in P_7 being updated to the point where $S_7^{RN} = 0$. A renege will cause beliefs to be updated so that a loan is expected in period seven and the strategy S_7^{PB} is followed. These mixed strategies should properly be written as functions of beliefs, for example, $S_6^{JOG}(P_6)$. The second new strategy is called $S_6^{RN,7}$ because $S_6^{RN,7}(p) = S_7^{RN}(p)$. The ability to support $S_6^{RN,7}$ relies on the extra degree of freedom created by the situation when a renege causes beliefs to be updated such that $S_7^{PB} = S_7^{RN} = 1$. Again, a pure strategy leads to indifference by B_8, so that indifference can be used to equate expected payoffs for the entrepreneur in period six. So X_6's strategy is much like the wave pattern seen for X_7, but there is a little sawtooth for high levels of P_6 made up of S_6^{JOG} and $S_6^{RN,7}$.

Two new strategies corresponding to S_6^{JOG} and $S_6^{RN,7}$ are added each period. For instance, in period five, the X-type strategies from low values of

P_5 to high are ($S_5 = 0$, S_5^{PB}, S_5^{RN}, S_5^{JOG}, $S_5^{RN,6}$, $S_5^{JOG,6}$, $S_5^{RN,7}$, $S_5 = 0$). So in period one, there are 16 possible strategies for an X-type entrepreneur to follow depending on the initial beliefs of the bankers.

Figures 1.2 through 1.4 show the responsiveness of these strategies to ϵ, which varies from 0.00 to 0.10 in the figures. The wave pattern is less pronounced for small ϵ, as are the saw-teeth. For $\epsilon > .291$, X_7 will never want to repay because the chance that B_8 will accidentally refuse a loan is too high. So if agents believe errors are prevalent, no mixed strategy can be supported.

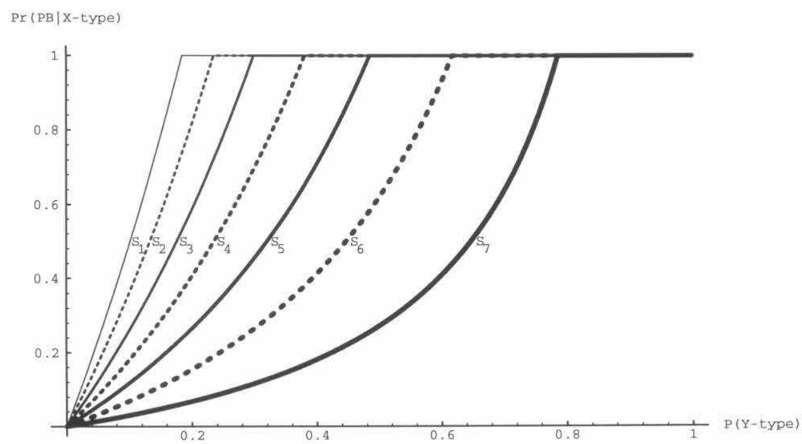

Figure 1.2: X-type strategies for periods one through seven: $\epsilon = 0.00$

1.4 Conclusion

This chapter introduced an equilibrium model of the borrower-lender game in which agents did not believe that all agents would be able to perfectly implement their equilibrium strategies. In this model, agents were able to

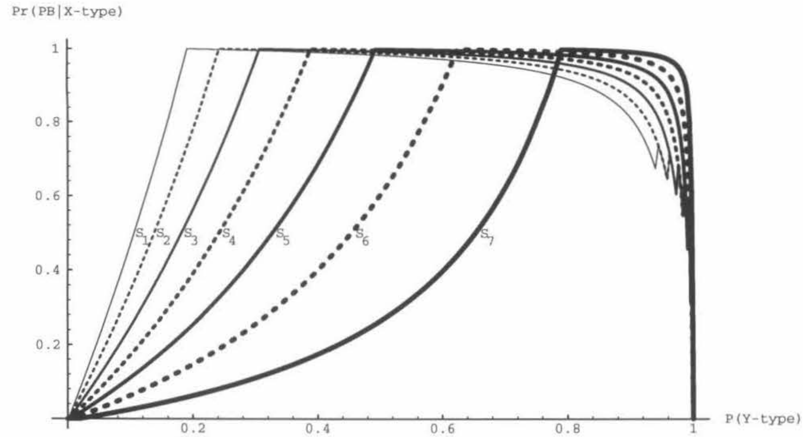

Figure 1.3: X-type strategies for periods one through seven: $\epsilon = 0.01$

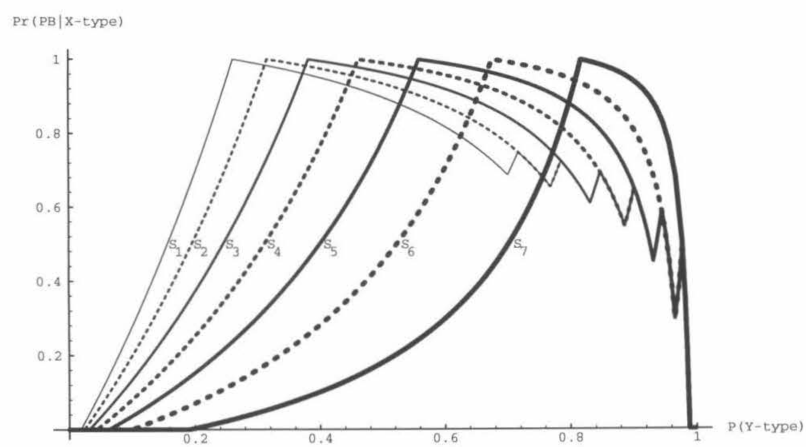

Figure 1.4: X-type strategies for periods one through seven: $\epsilon = 0.10$

build strong reputations that could not be destroyed with only one action. This feature allowed agents with good reputations to be treated with leniency when an undesirable outcome was achieved, while at the same time making sure that an agent with a bad reputation would be punished for such an observation.

A rather simple model of how agents fail to implement these strategies was introduced. This specification did reveal that relaxing the perfect implementation assumption can result in some interesting theoretical results and a much richer dynamical structure. Possible alternatives are to have the error rate be time dependent or to have it depend on the magnitude of the loss the agent would incur if an error did occur, as in the Beja paper. Another interesting line of research would be to investigate the implications of a relaxation of beliefs about perfect implementation in infinitely repeated game settings. Fudenberg, Levine, and Maskin (1990) have already studied the effects of imperfect observability on the folk theorems. The main difference between imperfect implementation and imperfect observability is that in the former, agents consider the fact that they may accidentally take suboptimal actions themselves. The impact of the differences between these two approaches is an open question worthy of further consideration.

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Chapter 2

Empirical Investigation of Reputation Building

2.1 Introduction

The previous chapter was a theoretical consideration of reputation building. This chapter is complementary. There is an expanding collection of literature which aims to empirically test the predictions of the sequential equilibrium. Rigorous statistical analysis of experimental data from these types of environments has proven problematic primarily because of the persistence of actions that should occur with zero probability according to the SE. The model introduced in the previous chapter lends itself to maximum likelihood estimation using the entire data set. This procedure is fairly general, and can thus be used to test the new model against various alternatives. It will also allow for a more efficient test of an empirical regularity found by Camerer and Weigelt.

This paper is organized as follows. Section 2.2 introduces a few of the important experimental papers on reputation building. Maximum likelihood estimation of the error prone agents model and various alternate models is presented in Section 2.3. Section 2.4 will conclude.

2.2 Reputation Building: Experimental Evidence

There is a vast literature regarding reputation building in experimental setting. Isaac and Smith (1985) first started looking for predatory pricing in experimental markets, although they did not do so in an incomplete information environment. The following papers all operationalize an incomplete information environment in most of their experiments. The degree to which reputation building behavior is observed varies.

2.2.1 Camerer and Weigelt - Homemade Priors

Camerer and Weigelt test the consistency of their data with the SE prediction by concentrating on the predicted reneging and lending probabilities (they considered lending following a string of repayments and lending following a refusal of a loan, both of which have distinct SE predictions). Initially they consider two experimental designs, differentiated primarily by the percentage of entrepreneurs who are assigned Y-type preferences. They start with six experiments, three with a 10 percent chance of the entrepreneur being a Y-type, and three with a 33 percent chance. Across experimental designs, they found that the observed frequency of reneging was lower than the SE prediction. Although, they also saw that the reneging frequencies increased almost monotonically in all of their designs, which is a qualitative prediction of the SE. They found that lending frequencies were often supportive of the SE, but not without exception. One particular exception was that bankers occasionally lent after observing a default. This analysis does help to give us an idea of the power of the SE prediction, but it does not lend itself to testing the SE against the predictions of other hypotheses.

One such hypothesis studied by CW is the homemade priors hypothesis. It simply suggests that bankers bring into the experiment a belief that some proportion of the entrepreneurs assigned X-type preferences will act as if they have Y-type preferences. This proportion is called the homemade prior. The inflated beliefs imply that X-types can have lower probabilities of reneging because they do not need to boost beliefs as much as without the homemade priors, and can thus afford to have a repayment be a weaker signal.

The addition of homemade priors will not qualitatively change the SE prediction; it will simply alter the timing of the switch from pure strategies to mixed strategies and the reneging probability in the first round of mixing. CW estimated the magnitude of the homemade priors by looking for the first period in which significant reneging takes place in their data. They then calculated the homemade prior necessary to make that period the first period of mixed strategies with the observed level of reneging. Notice that this estimation procedure only uses a small portion of the data.

This caveat aside, they find the surprising result that for both of their experimental designs, the homemade prior is estimated remarkably close to 17 percent. This is quite a striking result, so they ran another set of experiments in which none of the entrepreneurs were induced to be Y-types. Even though the bankers had no idea that any entrepreneurs had ever been assigned Y-type preferences, CW again found that a homemade prior of roughly 17 percent gave them the best fit. The consistency of the homemade prior throughout these three sets of experiments is supportive of the hypothesis, though not conclusive due to the small amount of data on which it is based. Because the homemade priors are calculated from only one period's data, it may be that data from the other periods do not support this hypothesis. A method to estimate homemade priors which uses all of the available data will be presented in Section 2.3.

2.2.2 Neral and Ochs

Neral and Ochs (1992) experimentally examine the same game as CW, but they use a comparative statics approach to test the predictions of the SE. They start by roughly replicating one of CW's experimental designs. They then decrease the payoff to X-types for reneging. The change in payoffs implies that under the SE, X-types' strategies should not change, but bankers should lend less often. They find that both of these predictions are contradicted by their data. A homemade prior cannot resolve this violation of SE predictions.

There are three main differences between the experiments here and those in CW. First, the stage game is only repeated six times in these experiments, as opposed to eight in CW. Second, for sequences involving Y-type entrepreneurs, the experimenters automatically repaid any loans. Assuming the subjects believed that the experimenters would not make mistakes when doing this, the foundation of the error prone agents model crumbles. Just as when agents believe strategies are perfectly implemented, the only way a default could be observed is if the entrepreneur was an X-type. Finally, the entrepreneurs in these experiments faced *one* banker for six consecutive periods, as opposed to the CW setup where they faced a sequence of eight different bankers. Although it is not an equilibrium strategy, this seems like a change that would be likely to make the banker try to get repaid loans through forward induction. We could easily think of having a new type of banker who simply lends until a default is observed. A normal type banker would gain from building such a reputation. These authors were also generous enough to provide their data for the analysis that follows. While it will provide a nice comparison to the CW data, it should be noted that the three differences in the experimental designs could have profound effects on subject behavior. Unfortunately, one can only speculate as to the effect of any particular variation.

2.2.3 Jung, Kagel, and Levin

Jung, Kagel, and Levin (1992) is an experimental study of the chain store game. In experiments with no experimenter-induced strong monopolists, they find no support for complete backward unraveling, as the theory would predict if agents truly believed all monopolists to be soft. In these experiments, and in experiments where there were two types of monopolists, there was consistent evidence of reputation building behavior. However, persistent deviations from the SE predictions were observed. Homemade prior formulations along the lines of CW would not explain the deviations.

2.2.4 McKelvey and Palfrey

McKelvey and Palfrey (1992) was an experimental study of yet another game with reputation building aspects, the centipede game. A full description of the game is not necessary for this discussion; it will suffice to note that in complete information models, the Nash prediction is for complete backward unraveling, just as in the previous games. The authors develop a sequential equilibrium to an incomplete information model of this game, where there is noisy play, as in the error prone agents model. The authors find support for this model. Furthermore, they found a significant bias in agents' beliefs which is much in the vein of the Camerer and Weigelt homemade priors hypothesis.

2.3 Maximum Likelihood Estimation

2.3.1 The Likelihood Function

The error prone agents model is suggestive of a maximum likelihood estimation procedure which will be employed to test the predictive powers of the standard SE, the error prone agents model, and variations on the homemade priors hypothesis. Some notation to describe this estimation procedure will now be discussed. Actions taken by the entrepreneur in a game are represented by $\mathbf{s} = (s_1, s_2, \ldots, s_N)$ where $s_t \in \{PB, RN\}$ for $t = 1, 2, \ldots, N$, where N is the number of stage games in the repeated game. The bankers' actions are denoted $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ where $b_t \in \{\text{lend}, \text{refuse}\}$ for $t = 1, 2, \ldots, N$. It is also important to keep track of the type of entrepreneur taking these actions, so let $T \in \{X, Y\}$ stand for the entrepreneurs type. An immediate problem with maximum likelihood estimation is that invariably the data includes numerous observations, $(\mathbf{s}, \mathbf{b}, T)$, which are assigned zero probability under the SE, with or without homemade priors. The likelihood function would then be zero for all the different experimental parameters and nothing would be learned. Omission of the offending observations would result in the loss of information about how well the models are followed when out-of-equilibrium behavior is observed. Instead, the outliers will be accounted for by estimating a parameter that is similar to the ϵ of the previous section.

This new parameter, e, is the probability that nature intervenes in the play of the game and randomizes over the available actions. Remember, ϵ is the *belief* about the error rate. If a given model summarizes the data well, one would expect to see a low estimate of e. This term is much like the residual included in standard regression analysis. As an example of how this fits into the likelihood function, say in period four that the entrepreneur's equilibrium strategy is to repay with probability S_4, and the banker's equilibrium strategy is to lend with probability L_4. The likelihood of a repaid loan in that period would be $[L_4(1-e)+e/2][S_4(1-e)+e/2]$.

The inclusion of e into the estimation procedure presents the opportunity to estimate the homemade priors from the entire data set. For a given set of experimental parameters, say the proportion of entrepreneurs induced with Y-type preferences is $\Pr(Y)$. The homemade prior, v, is the proportion of X-types believed to be acting as if they are Y-types. So in period one, the belief of the banker should be $P_1 = \Pr(Y) + v[1-\Pr(Y)]$. A maximum likelihood estimate of v for a given set of observations could be obtained by finding the $(\hat{v}, \hat{e})$ for which the estimated likelihood is maximized. Before such analysis is considered, two more parameters will be discussed.

The parameter ϵ, which is the common believed error rate, was introduced in the previous chapter. This parameter is only relevant when considering the error prone agents model. Although it is similar to e, the difference is very important. ϵ is the *believed* error rate, and e is the actual error rate. They need not be equal, although it would be desirable from a rational expectations point of view if they were.

The final parameter also lends itself to tests of rational expectations. The homemade prior was introduced as a belief that bankers brought into the experiment about the behavior of some X-type entrepreneurs. One might wonder why people would have these beliefs. Perhaps it is the case that some X-types really do act like Y-types. The final parameter considered, y, is the proportion of X-types acting as if they had Y-type preferences. Only when an agent is an X-type does y effect the likelihood of a given observation. How this enters into a likelihood calculation is best understood through an example. The likelihood of a repaid loan by an X-type in period four is $[L_4(1-e)+e/2]\{(1-y)[S_4(1-e)+e/2]+y(1-e/2)\}$. So y corresponds to υ as e corresponds to ϵ. The parameters assigned Roman letters reflect actual behavior, and those assigned Greek symbols represent beliefs.

The full model to be estimated has the following characteristics. Some proportion, e, of actions are subject to intervention by nature, in which case nature randomizes over the possible actions. Agents have a common belief, ϵ, about how often nature intervenes. Some proportion, y, of X-type entrepreneurs act as if they have Y-type preferences. Agents have a common belief that some fraction of all X-types, υ, will act as if they are Y-types.

2.3.2 Model Specification and Hypotheses

A number of models will now be formulated in terms of these parameters. The hypotheses are that each model best describes the data. The first model is the sequential equilibrium under the assumption that agents believe that strategies are perfectly implemented, more formally, $\epsilon = 0$. As was previously discussed, agents do not always precisely follow the SE, so e must be allowed to be positive to prevent the likelihood function from always being estimated as zero. Model specification SEE allows for $e > 0$ but a large e might indicate a weakness of the SE. In any case, a rejection of SEE would be a rejection of the perfect implementation sequential equilibrium because the precise perfect implementation SE is nested in the estimated model. The sequential equilibrium with errors model (SEE) is specified below.

$$\text{SEE: } e \in [0,1], \epsilon = y = \upsilon = 0.$$

Camerer and Weigelt's homemade priors hypothesis simply added homemade priors to the SE. The next model, the sequential equilibrium with errors and homemade priors (SEEHP), is an analogous extension of SEE. Likewise, it is a weaker statement than the CW specification.

$$\text{SEEHP: } (e, \upsilon) \in [0,1]^2, \epsilon = y = 0.$$

Consider one possible explanation for why people may have these homemade priors, which is that they are not homemade at all. Perhaps people have these beliefs about X-types because the behavior of the entrepreneurs suggests that some X-types are acting as if they were Y-types, so the priors are rational in a sense. The SEE with rational priors (SEERP) is defined as:

$$\text{SEERP: } (e, \upsilon, y) \in [0,1]^3, \epsilon = 0, y = \upsilon.$$

The three previous models did not allow for beliefs about the error rate; that is, they all required $\epsilon = 0$, but they did allow $e > 0$. If it is the case that erratic behavior is common, is it reasonable to assume people ignore this fact? There is a sort of internal inconsistency in each of the models above. The following models consider sequential equilibria where agents recognize that some observations may be errors, which has been introduced as the error prone agents model. The analysis here will be limited to beliefs about the error rate that are consistent with observed behavior. Agents with such beliefs will be called "rational error prone agents." The first model, the rational error prone agents model (REPA), allows rational beliefs about errors, but does not permit any X-types to act like Y-types or for anyone to believe that this may happen, that is, no homemade priors.

$$\text{REPA: } (e, \epsilon) \in [0,1]^2, e = \epsilon, y = \upsilon = 0.$$

The final hypothesis is that people are doubly rational in the sense that they have both homemade priors and beliefs about the error rate which are both consistent with observed behavior. This model is called the REPA with rational priors, or REPARP.

$$\text{REPARP: } (e, \epsilon, y, \upsilon) \in [0,1]^4, e = \epsilon, y = \upsilon.$$

2.3.3 Estimation Procedure

The likelihood function is estimated by performing a global grid search over the entire parameter space. Given a particular parameter vector $(e, \epsilon, y, \upsilon)$, the likelihood function, $\mathcal{L}(e, \epsilon, y, v)$, is estimated as follows. As shown in the appendix, intended equilibrium strategies are mappings from the subjective probability the banker assigns to the entrepreneur being a Y-type along with the observed history of play to probabilities of lending and repaying. Once these mappings are known, all that remains is to track bankers' beliefs through each observation. Let $L_t(P_t)$ be the mapping from beliefs to intended lending probabilities[1] and $S_t(P_t)$ be the mapping to intended repayment probabilities for X-types. Y-types always intend to repay, regardless of P_t. Given an objective prior, $\Pr(Y)$, which was announced by the experimenter, bankers' subjective prior is $P_1 = \Pr(Y) + v[1 - \Pr(Y)]$. This P_1 maps to a strategy triplet $(L_1(P_1), S_1(P_1), 1)$.

For a given observation, $(\mathbf{s}, \mathbf{b}, T)$, first consider b_1. If b_1 =refuse, then $\log \mathcal{L}(e, \epsilon, y, v)$ is incremented by $\log\{[1 - L_1(P_1)](1 - e) + e/2\}$. Since no loan was granted, the entrepreneur could not move and bankers have no new information with which to update their beliefs, so the next step would be to consider b_2. If b_1 =lend, then $\log \mathcal{L}(e, \epsilon, y, v)$ is incremented by $\log[L_1(P_1)(1-e) + e/2]$. Since a loan was granted, the entrepreneur must act. If the entrepreneur is an X-type, $T = X$, then $\log \mathcal{L}(e, \epsilon, y, v)$ is incremented by $\log\langle(1 - y)\{[1 - S_1(P_1)](1 - e) + e/2\} + y(e/2)\rangle$ following a renege and by $\log\{(1-y)[S_1(P_1)(1-e)+e/2]+y(1-e/2)\}$ following a repayment. If $T = Y$,

[1]Unfortunately, due to the complexity of this problem, this entire mapping could not be solved for in closed form. Closed form solutions were used for all of the X-types' strategies and the bankers' strategies when only the S_t^{PB} and S_t^{RN} strategies were used by X-types. There was a strong suspicion that parameter values leading to strategies which would require estimation of the problematic portion of the likelihood function would not be the optimal parameter values. This suspicion was confirmed by using a very generous upper bound to the likelihood function in this problematic range.

reneging would add $\log(e/2)$ to the log-likelihood function, and repaying would add $\log(1-e/2)$. It is important to note that the entrepreneur's actual type is only used for calculating the likelihood of the entrepreneur's actions. This information does not enter the calculation of bankers' strategies. In fact, the next step in the estimation procedure is to update bankers' beliefs using either P_2^{PB} or P_2^{RN} as given by (1.1) and (1.2). Using the strategy triplet implied by the updated P_2, $\log \mathcal{L}(e, \epsilon, y, v)$ is incremented as above. This process is repeated for each of the periods, updating bankers' beliefs after each period. Then the next observation,$(\mathbf{s}, \mathbf{b}, T)$, is considered. This whole procedure is repeated for each observation in the data set. The resulting log-likelihood is then compared to previously calculated values to see if it is a maximum for any of the relevant restrictions on the parameters. Then the next point in the grid, representing a new set of parameters, is considered.

This grid search process is computer intensive, especially when searching over the entire four dimensional parameter space. Since future work in this field will likely require consideration of larger parameter spaces, it would be nice to have a procedure to quickly estimate these more complicated models. The problem considered here represents a very nice test case for alternate optimization algorithms, because it includes estimation of unconstrained models as well as models under simple constraints, and the results for a careful grid search have already been obtained. A publicly available genetic algorithm for optimization was employed for this purpose. The algorithm is called GENOCOP[2], which stands for GEnetic algorithm for Numerical Opti-

[2]Copyright ©Zbigniew Michalewicz. This program is available via anonymous ftp at unccsun.uncc.edu. See Michalewicz (1992) for a description of the algorithm.

mization for COnstrained Problems. This particular algorithm was selected because the C source code is publicly available, and it is written to easily handle linear constraints on the parameters. The results were very encouraging. Nearly five-hundred sets of parameters were estimated for the different models under the many subsets of the data considered. In every case, the optimal parameters found by the genetic algorithm were indistinguishable from those found by the grid search procedure. When only optimizing over one or two parameters, both methods were very fast. The advantage of the genetic algorithm is that the CPU time required to estimate the optimal parameters increases linearly in the dimension of the parameter space, as opposed to geometric increases in computing time for the grid search. So when even larger parameter spaces are considered, the genetic algorithm approach will be far quicker. The very positive results here suggest that researchers may want to use such algorithms to perform some preliminary analysis of complicated models, before large amounts of computer time are spent on a costly grid search. However, there is no guarantee that the genetic algorithm will converge to the global optimum, so a grid search should eventually be performed to verify the accuracy of the model estimates.

2.3.4 Data Sets

Data from the Camerer-Weigelt and Neral-Ochs papers are examined. The CW data is further classified by experimental design; the most recognizable difference between these is the proportion of entrepreneurs assigned to be Y-types, or $\Pr(Y)$. This can take on three values, $0.0, 0.1$, and 0.33; these designs will be referred to as CW00, CW10 and CW33. Within each of these designs are two or three experiments. Neral and Ochs have two experimental designs, which they call cells one and two. The payoffs in cell one are identical to those in CW's $\Pr(Y) = 0.33$ design. The payoffs in cell two are all the same except that the payoff to X-types for defaulting has been decreased. Cells one and two will be called NO1 and NO2.

CW focussed their analysis on the final two-thirds of their observations from each experiment. So for an experiment in which the eight period game is repeated 90 times, they would focus on the final 60 repetitions. They conjectured that participants would just be learning the nature of the experiment at first, and thus could not be expected to be in equilibrium. If this is the case, one would expect to see the models fit the early observations worse than they do the latter observations. The data from each experiment was partitioned into thirds to allow for testing of this conjecture.

2.3.5 Estimates for Partitioned Data

Tables 2.1 through 2.5 report the estimated error rates and log-likelihoods for each subset of the partitioned data sets under each of the six models, for a total of 360 model estimates. The conjecture was that in the early part of the experiments, subjects would be learning how to play the game and would consequently not be playing equilibrium strategies. One would not reject this conjecture from the results in these tables. Two indications of early learning would be higher error rates and lower log-likelihoods in the first part of the experiments than in the latter two thirds. Both of these conditions only

Table 2.1: MLE Results for Partitioned Data Sets: CW00

		Early		Middle		Late	
Experiment	Model	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.45	-439.0	.27	-358.6	.17	-290.2
	SEE	1.00	-543.4	1.00	-528.2	1.00	-524.7
	SEEHP	.48	-444.1	.34	-367.5	.19	-298.0
	SEERP	.46	-445.2	.32	-362.5	.18	-292.2
	REPA	1.00	-543.4	1.00	-528.2	1.00	-524.7
	REPARP	.23	-489.0	.21	-363.4	.17	-290.9
9	Unconstrained	.62	-242.5	.43	-206.8	.19	-129.6
	SEE	.96	-266.5	1.00	-259.2	1.00	-249.5
	SEEHP	.64	-244.2	.45	-207.6	.23	-133.8
	SEERP	.63	-245.0	.44	-207.0	.22	-136.1
	REPA	.96	-266.5	.99	-259.2	1.0	-249.5
	REPARP	.96	-266.5	.23	-216.1	.15	-133.7
10	Unconstrained	.25	-174.9	.12	-135.0	.08	-116.2
	SEE	1.00	-276.6	1.00	-268.9	.92	-273.8
	SEEHP	.26	-177.4	.15	-143.6	.08	-116.2
	SEERP	.25	-176.0	.13	-138.3	.08	-116.4
	REPA	1.00	-276.6	1.00	-268.9	.92	-273.8
	REPARP	.17	-188.6	.13	-139.3	.07	-118.2

Table 2.2: MLE Results for Partitioned Data Sets: CW10

		Early		Middle		Late	
Experiment	Model	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.25	-410.9	.10	-274.7	.11	-293.8
	SEE	.32	-516.0	.09	-355.5	.13	-353.9
	SEEHP	.28	-423.5	.10	-274.7	.14	-315.7
	SEERP	.27	-425.4	.10	-276.8	.14	-311.0
	REPA	.16	-543.0	.05	-363.6	.05	-350.8
	REPARP	.20	-433.7	.08	-289.3	.09	.297.1
6	Unconstrained	.14	-108.0	.03	-62.1	.04	-58.0
	SEE	.17	-159.1	.04	-113.4	.05	-118.6
	SEEHP	.15	-109.6	.05	-66.2	.05	-58.2
	SEERP	.15	-111.5	.04	-63.9	.04	-63.5
	REPA	.16	-168.8	.04	-113.4	.05	-118.6
	REPARP	.10	-118.6	.02	-64.7	.03	-67.1
7	Unconstrained	.38	-154.5	.16	-115.5	.20	-135.1
	SEE	.44	-180.4	.16	-130.9	.24	-153.7
	SEEHP	.39	-155.7	.16	-115.5	.27	-148.8
	SEERP	.39	-160.2	.16	-115.8	.25	-147.2
	REPA	.16	-199.5	.14	-141.3	.14	-160.1
	REPARP	.23	-169.4	.14	-119.6	.15	-137.5
8	Unconstrained	.20	-131.0	.07	-72.9	.04	-55.0
	SEE	.32	-169.1	.07	-105.9	.05	-70.3
	SEEHP	.30	-145.0	.07	-72.9	.06	-56.6
	SEERP	.26	-141.1	.07	-74.6	.06	-57.0
	REPA	.16	-173.9	.05	-107.3	.03	-69.6
	REPARP	.13	-134.1	.05	-76.2	.03	-55.5

Table 2.3: MLE Results for Partitioned Data Sets: CW33

		Early		Middle		Late	
Experiment	Model	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.27	-485.5	.14	-367.9	.12	-369.5
	SEE	.34	-561.4	.15	-379.1	.14	-397.4
	SEEHP	.30	-494.5	.18	-373.7	.12	-375.6
	SEERP	.30	-501.6	.14	-370.6	.12	-372.9
	REPA	.17	-605.0	.05	-407.8	.08	-422.4
	REPARP	.23	-505.3	.12	-375.7	.11	-385.0
3	Unconstrained	.25	-136.1	.10	-99.0	.10	-131.6
	SEE	.27	-155.2	.11	-105.1	.10	-136.1
	SEEHP	.25	-136.1	.10	-99.1	.10	-131.6
	SEERP	.25	-138.8	.10	-99.1	.10	-131.6
	REPA	.20	-149.6	.04	-112.9	.05	-144.2
	REPARP	.22	-140.8	.08	-101.7	.08	-136.4
4	Unconstrained	.25	-153.4	.19	-132.0	.13	-145.0
	SEE	.28	-181.5	.21	-134.0	.16	-157.0
	SEEHP	.26	-156.4	.21	-134.0	.14	-146.4
	SEERP	.26	-163.9	.20	-133.7	.13	-145.4
	REPA	.12	-192.0	.12	-146.6	.09	-157.6
	REPARP	.20	-166.2	.15	-135.0	.13	-153.8
5	Unconstrained	.32	-180.7	.12	-117.2	.13	-89.7
	SEE	.45	-219.3	.12	-137.1	.15	-102.9
	SEEHP	.39	-196.0	.13	-122.4	.15	-92.0
	SEERP	.37	-194.0	.12	-118.2	.13	-90.0
	REPA	.17	-247.6	.04	-146.8	.09	-110.4
	REPARP	.23	-192.1	.10	-124.9	.11	-94.2

Table 2.4: MLE Results for Partitioned Data Sets: NO1

		Early		Middle		Late	
Experiment	Model	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.06	-248.0	.04	-181.2	.01	-179.4
	SEE	.07	-295.5	.01	-203.6	.01	-189.0
	SEEHP	.09	-265.6	.04	-181.2	.01	-184.2
	SEERP	.08	-255.6	.04	-182.4	.01	-184.9
	REPA	.04	-302.0	.01	-205.3	.01	-190.0
	REPARP	.08	-254.8	.03	-183.7	.02	-184.4
1	Unconstrained	.06	-91.7	.03	-67.3	.01	-58.8
	SEE	.10	-103.5	.03	-70.0	.01	-60.5
	SEEHP	.11	-102.2	.03	-68.4	.01	-58.8
	SEERP	.11	-101.5	.03	-68.4	.01	-59.0
	REPA	.07	-100.6	.02	-71.3	.01	-60.9
	REPARP	.14	-96.5	.03	-68.4	.01	-59.1
2	Unconstrained	.07	-74.9	.03	-58.3	.02	-57.8
	SEE	.06	-99.5	.01	-66.9	.02	-62.6
	SEEHP	.08	-77.5	.03	-58.3	.02	-62.1
	SEERP	.07	-74.9	.03	-58.5	.02	-62.3
	REPA	.04	-102.1	.01	-67.6	.020	-62.6
	REPARP	.06	-75.9	.02	-59.0	.02	-61.9
3	Unconstrained	.02	-63.2	.00	-43.7	.03	-56.7
	SEE	.04	-91.0	.00	-64.4	.01	-65.8
	SEEHP	.05	-74.2	.00	-43.7	.03	-56.9
	SEERP	.02	-65.9	.00	-44.5	.03	-57.1
	REPA	.03	-92.0	.00	-64.4	.01	-66.0
	REPARP	.02	-65.7	.00	-44.5	.03	-57.1

Table 2.5: MLE Results for Partitioned Data Sets: NO2

		Early		Middle		Late	
Experiment	Model	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$	$\hat{e}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.10	-383.4	.04	-273.7	.01	-243.8
	SEE	.24	-675.8	.04	-481.4	.01	-391.8
	SEEHP	.16	-411.3	.05	-275.6	.01	-258.0
	SEERP	.13	-398.5	.04	-277.0	.01	-252.4
	REPA	.04	-714.9	.04	-481.4	.01	-392.5
	REPARP	.10	-386.7	.04	-280.2	.01	-252.2
4	Unconstrained	.11	-84.6	.00	-42.0	.00	-53.6
	SEE	.13	-139.6	.00	-108.3	.00	-90.5
	SEEHP	.12	-85.9	.00	-42.0	.04	-55.0
	SEERP	.12	-86.6	.00	-45.6	.04	-57.2
	REPA	.04	-142.1	.00	-108.3	.00	-90.5
	REPARP	.10	-89.2	.00	-45.6	.04	-59.4
5	Unconstrained	.17	-111.9	.01	-65.3	.01	-62.2
	SEE	.38	-185.9	.04	-111.4	.04	-109.4
	SEEHP	.25	-127.4	.07	-75.3	.07	-73.3
	SEERP	.20	-118.0	.06	-71.4	.01	-66.4
	REPA	.12	-199.3	.04	-111.2	.03	-109.7
	REPARP	.19	-112.1	.06	-71.9	.01	-66.3
6	Unconstrained	.05	-69.8	.09	-73.8	.00	-53.4
	SEE	.15	-160.8	.06	-131.5	.00	-87.5
	SEEHP	.09	-76.1	.09	-73.8	.00	-53.8
	SEERP	.08	-82.6	.09	-75.9	.00	-54.0
	REPA	.04	-166.4	.04	-132.0	.00	-87.5
	REPARP	.07	-77.6	.07	-79.2	.00	-54.0
7	Unconstrained	.11	-93.2	06	-70.3	.02	-66.8
	SEE	.28	-180.6	.07	-125.5	.02	-100.8
	SEEHP	.17	-109.1	.06	-70.3	.02	-67.5
	SEERP	.14	-104.3	.06	-72.0	.02	-67.4
	REPA	.04	-185.9	.04	-125.5	.02	-101.2
	REPARP	.09	-97.2	.05	-73.6	.02	-67.3

failed in 7 of the 120 triplets of model estimates shown in these tables. In comparison, at least one of these conditions failed roughly half of the time when comparing results from the middle data to those from late data. These results support CW's decision to concentrate on the final two-thirds of the observations, so the first third of the observations were omitted for all of the estimates in Tables 2.6–2.10.

2.3.6 Results for Model Comparison

Tables 2.6–2.10 show the maximum likelihood estimates of each of the parameters and the likelihood function value under each of the suggested models. One note of caution: the sample sizes given in these tables are the number of sequences used from each experiment. For the CW data, this represents the number of eight-period repeated games analyzed. However, for the NO data, each observation is a six-period repeated game. Thus, one would expect lower likelihoods from the CW data than from the NO data for comparable sample sizes.

One very striking result is the estimated error rate, $\hat{e}$, in the $\Pr(Y) = .00$ experiments. When no homemade priors were allowed, under SEE and REPA, the estimated error rate was equal to one, more than twice as large as it was under any of the other hypotheses. These models failed to describe the behavior in these experiments so badly that modeling each decision as a coin toss was an improvement over the models. This result emphasizes the importance of considering the estimated error rate. One could hardly say that either of these models were supported by this data, even if the log-

Table 2.6: Maximum Likelihood Estimates: CW00

Experiment	Model	$\hat{e}$	$\hat{\epsilon}$	$\hat{y}$	$\hat{v}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.23	.00	.38	.18	-657.2
	SEE	1.00	.00	.00	.00	-1052.9
	SEEHP	.25	.00	.00	.20	-672.7
	SEERP	.23	.00	.20	.20	-660.4
	REPA	1.00	1.00	.00	.00	-1052.9
	REPARP	.21	.21	.32	.32	-662.1
				n = 127		
9	Unconstrained	.31	.20	.23	.45	-358.0
	SEE	1.00	.00	.00	.00	-508.8
	SEEHP	.36	.00	.00	.55	-359.6
	SEERP	.34	.00	.40	.40	-365.0
	REPA	1.00	1.00	.00	.00	-508.8
	REPARP	.23	.23	.47	.47	-365.0
				n = 58		
10	Unconstrained	.14	.00	.34	.17	-265.6
	SEE	1.00	.00	.00	.00	-544.1
	SEEHP	.15	.00	.00	.19	-271.2
	SEERP	.14	.00	.18	.18	-266.6
	REPA	1.00	1.00	.00	.00	-544.1
	REPARP	.13	.13	.24	.24	-273.3
				n = 69		

Table 2.7: Maximum Likelihood Estimates: CW10

Experiment	Model	$\hat{e}$	$\hat{\epsilon}$	$\hat{y}$	$\hat{v}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.10	.00	.17	.10	-589.8
	SEE	.11	.00	.00	.00	-710.6
	SEEHP	.14	.00	.00	.19	-604.0
	SEERP	.14	.00	.19	.19	-602.4
	REPA	.05	.05	.00	.00	-714.9
	REPARP	.08	.08	.11	.11	-594.6
				n = 144		
6	Unconstrained	.04	.00	.10	.30	-125.8
	SEE	.04	.00	.00	.00	-232.1
	SEEHP	.05	.00	.00	.31	-128.4
	SEERP	.04	.00	.27	.27	-128.7
	REPA	.04	.04	.00	.00	-232.0
	REPARP	.02	.02	.23	.23	133.0
				n = 46		
7	Unconstrained	.18	.00	.00	.09	-252.1
	SEE	.20	.00	.00	.00	-285.5
	SEEHP	.21	.00	.00	.09	-266.3
	SEERP	.21	.00	.09	.09	-264.8
	REPA	.14	.14	.00	.00	-301.5
	REPARP	.14	.14	.10	.10	-260.0
				n = 53		
8	Unconstrained	.05	.03	.00	.10	-141.1
	SEE	.06	.00	.00	.00	-176.4
	SEEHP	.07	.00	.00	.10	-141.6
	SEERP	.07	.00	.10	.10	-142.5
	REPA	.04	.04	.00	.00	-177.7
	REPARP	.05	.05	.10	.10	-142.0
				n = 45		

Table 2.8: Maximum Likelihood Estimates: CW33

Experiment	Model	$\hat{e}$	$\hat{\epsilon}$	$\hat{y}$	$\hat{v}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.13	.00	.32	.07	-738.6
	SEE	.14	.00	.00	.00	-776.6
	SEEHP	.14	.00	.00	.07	-750.1
	SEERP	.13	.00	.07	.07	-744.6
	REPA	.05	.05	.00	.00	-839.6
	REPARP	.11	.11	.16	.16	-761.5
				n = 173		
3	Unconstrained	.10	.00	.09	.07	-230.6
	SEE	.10	.00	.00	.00	-241.2
	SEEHP	.10	.00	.00	.07	-230.7
	SEERP	.10	.00	.07	.07	-230.6
	REPA	.05	.05	.00	.00	-257.1
	REPARP	.08	.08	.13	.13	-238.1
				n = 62		
4	Unconstrained	.18	.00	.38	.05	-283.5
	SEE	.18	.00	.00	.00	-291.4
	SEEHP	.18	.00	.00	.07	-286.1
	SEERP	.17	.00	.07	.07	-283.9
	REPA	.12	.12	.00	.00	-301.5
	REPARP	.15	.15	.19	.19	-293.1
				n = 62		
5	Unconstrained	.13	.00	.23	.20	-211.8
	SEE	.13	.00	.00	.00	-240.2
	SEEHP	.14	.00	.00	.21	-214.6
	SEERP	.13	.00	.20	.20	-211.8
	REPA	.05	.05	.00	.00	-257.6
	REPARP	.11	.11	.18	.18	-219.4
				n=49		

Table 2.9: Maximum Likelihood Estimates: NO1

Experiment	Model	$\hat{e}$	$\hat{\epsilon}$	$\hat{y}$	$\hat{v}$	$\log \hat{\mathcal{L}}$
Pooled	Unconstrained	.01	.16	.00	.27	-367.7
	SEE	.01	.00	.00	.00	-394.4
	SEEHP	.01	.00	.00	.06	-381.1
	SEERP	.01	.00	.06	.06	-381.9
	REPA	.01	.01	.00	.00	-396.2
	REPARP	.01	.01	.07	.07	-380.5
				n = 180		
1	Unconstrained	.02	.07	.00	.13	-127.3
	SEE	.02	.00	.00	.00	-131.0
	SEEHP	.02	.00	.00	.06	-127.6
	SEERP	.02	.00	.05	.05	-127.9
	REPA	.01	.01	.00	.00	-132.3
	REPARP	.02	.02	.08	.08	-128.1
				n = 60		
2	Unconstrained	.01	.16	.00	.26	-122.2
	SEE	.01	.00	.00	.00	-129.6
	SEEHP	.02	.00	.00	.05	-126.6
	SEERP	.02	.00	.05	.05	-127.0
	REPA	.01	.01	.00	.00	-130.1
	REPARP	.02	.02	.07	.07	-126.7
				n = 60		
3	Unconstrained	.01	.11	.05	.20	-104.9
	SEE	.00	.00	.00	.00	-130.9
	SEEHP	.01	.00	.00	.11	-105.6
	SEERP	.01	.00	.09	.09	-105.9
	REPA	.00	.00	.00	.00	-131.1
	REPARP	.01	.01	.10	.10	-105.8
				n = 60		

Table 2.10: Maximum Likelihood Estimates: NO2

Experiment	Model	$\hat{e}$	$\hat{\epsilon}$	$\hat{y}$	$\hat{v}$	$\log\hat{\mathcal{L}}$
Pooled	Unconstrained	.05	.00	.17	.26	-536.2
	SEE	.03	.00	.00	.00	-875.6
	SEEHP	.05	.00	.00	.30	-541.9
	SEERP	.05	.00	.24	.24	-536.6
	REPA	.03	.03	.00	.00	-876.9
	REPARP	.05	.05	.28	.28	-544.8
				n = 240		
4	Unconstrained	.02	.00	.00	.27	-104.9
	SEE	.00	.00	.00	.00	-198.8
	SEEHP	.02	.00	.00	.27	-104.9
	SEERP	.02	.00	.23	.23	-107.5
	REPA	.00	.00	.00	.00	-198.8
	REPARP	.02	.02	.23	.23	-109.5
				n = 60		
5	Unconstrained	.01	.28	.42	.58	-128.6
	SEE	.04	.00	.00	.00	-220.8
	SEEHP	.07	.00	.00	.28	-148.7
	SEERP	.06	.00	.25	.25	-140.4
	REPA	.03	.03	.00	.00	-221.0
	REPARP	.06	.06	.27	.27	-141.2
				n = 60		
6	Unconstrained	.05	.05	.31	.27	-138.3
	SEE	.03	.00	.00	.00	-222.7
	SEEHP	.06	.00	.00	.33	-139.0
	SEERP	.05	.00	.28	.28	-139.2
	REPA	.03	.03	.00	.00	-223.3
	REPARP	.05	.05	.27	.27	-138.3
				n = 60		
7	Unconstrained	.07	.00	.00	.31	-143.9
	SEE	.05	.00	.00	.00	-227.4
	SEEHP	.07	.00	.00	.31	-143.9
	SEERP	.07	.00	.23	.23	-145.0
	REPA	.04	.04	.00	.00	-228.6
	REPARP	.06	.06	.29	.29	-148.4
				n = 60		

likelihood came out very high relative to other models. Fortunately, for the main hypotheses of interest, SEEHP, SEERP and REPARP, the error rates never approach that level. Remember that $e = .20$ implies that an observed action was not the intended action ten percent of the time.

Another interesting observation is how low the estimated error rates are for the NO data. To show how pronounced this is, compare the CW $\Pr(Y) = 0.33$ results to those from NO Cell 1, where the payoffs are the same. The *lowest* estimated error rate for any model using the CW data is five percent, which is more than twice as large as the *highest* estimated error rate for the NO data, which is two percent. Furthermore, for all of the NO data, the error rate never exceeds seven percent for any of the models. As was discussed previously, there are too many differences between the CW and NO procedures to permit us to infer that any result such as this is due to one particular change.

Also, comparing NO cell one results to results from cell two suggests that the theoretical models do not adequately predict the change in behavior caused by the change in payoffs. Notice that for all of the equilibrium models that allow for some kind of inflated prior (SEEHP, SEERP, and REPARP), no prior is estimated over 11 percent in cell one, but in cell two, the prior is never estimated under 23 percent. This indicates that the estimation procedure is using these inflated priors to compensate for some persistent deviation from the equilibrium predictions.

Figures 2.1 and 2.2 show the estimated likelihood function under the constraints of the model SEEHP. The reported likelihoods are the inverse logs of the estimated log-likelihood per repetition of the stage game. This can be

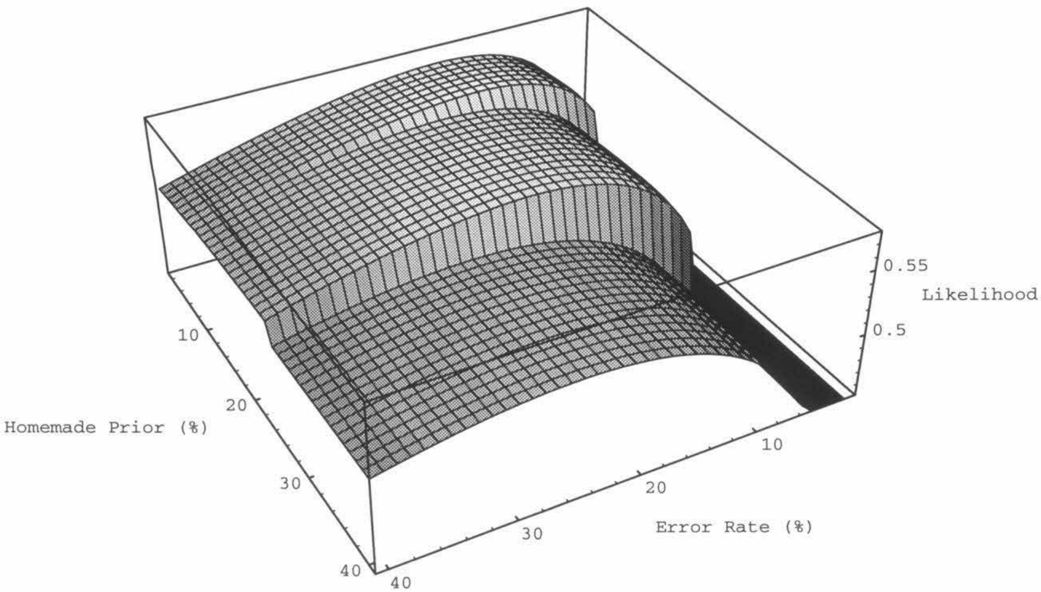

Figure 2.1: Estimated Likelihood Function for SEEHP: CW33 Pooled Data

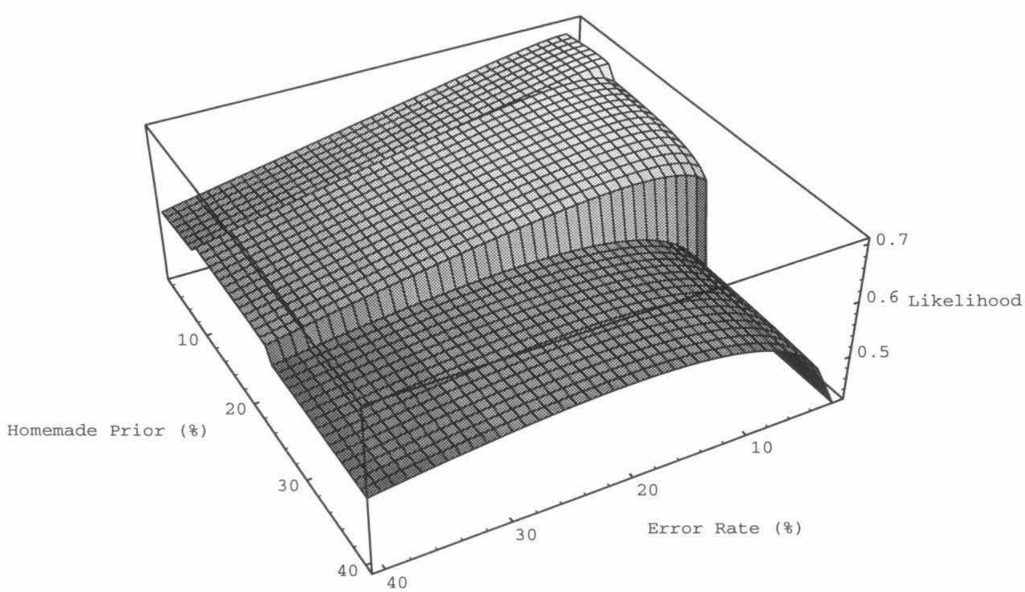

Figure 2.2: Estimated Likelihood Function for SEEHP: NO1 Pooled Data

interpreted as the the probability that the model under the given parameters would correctly predict the outcome of an arbitrary stage game. As the homemade prior, $\hat{v}$, increases, one sees ridges in the likelihood function. Recall that as the initial beliefs of bankers increase, the start of mixed strategy play is pushed farther back in the repeated game. Thus, these ridges correspond to the initial beliefs of the bankers reaching a level at which mixed strategy play is pushed farther back in the game.

In order to illustrate the implications of these parameters on predicted behavior, a simple example of equilibrium play under the REPARP model will be considered, because this model uses all four parameters. Notice in Table 2.8 that when using the pooled data, $\hat{\epsilon}$ is estimated at 0.11, which is close to the value of ϵ used to create Figure 1.4. Recall that these graphs were constructed using the payoff vectors for the $\Pr(Y) = .33$ experiments. Table 2.8 shows that for the REPARP model, the homemade prior, $\hat{v}$, is .16. This implies that the beliefs of the initial banker should be $P_1 = .33 + .16 \times .67 \approx .44$. Figure 1.4 shows that $S_1 = S_1^{RN}$ for this level of beliefs, and it is known that for beliefs higher than that at the apex of S_1, B_1 should always intend to lend. In the REPARP model, as in all of the other models, it is assumed that there truly is an error process. So while B_1 may fully intend to grant a loan, since $\hat{e} = 0.11$, a loan will actually only be granted with probability .95. Additionally, if an X-type is moving, there is a chance that this X-type may be acting like a Y-type. Thus, the actual probability of observing a repayment is $(1 - \hat{y})[(1 - \hat{e})S_t(P_t) + \hat{e}/2] + \hat{y}(1 - \hat{e}/2)$. Lending and repaying probabilities adjusted for $\hat{e}$ and $\hat{y}$ will be placed in parentheses.

- If B_1 does lend, X_1 should intend to repay about 93% (89%) of the time.
  - If X_1 does repay it will lead to $P_2 \approx .47$ and an intended loan with probability 1.00 (.95) from B_2 because .47 is past the apex of S_1.
    * If B_2 does grant a loan, X_2 should play S_2^{RN} under which she should repay almost 95% (91%) of the time.
    * If B_2 fails to lend, no updating occurs and $P_3 = .47$ which is past the apex of S_2, so B_3 should intend to lend with probability 1.00 (.95).
  - However if X_1 reneges, P_2 will drop to about .26 which corresponds to the apex of S_1, and B_2 will play a mixed strategy, lending about 29% (31%) of the time.
    * If B_2 does lend, X_2 should play S_2^{PB} and intend to repay with probability .75 (.76).
    * If B_2 does not lend, no future banker should intend to lend, or intend to lend with probability 0.00 (0.06), unless some other banker does lend and the entrepreneur repays under some S_t^{PB} mixed strategy.
- If B_1 does not lend, $P_2 = P_1 = .44$ which is past the apex of S_1, so B_2 should intend to lend with probability 1.00 (.95).
  - If B_2 does lend, X_2 should play S_2^{RN} under which she should intend to repay 95% (91%) of the time.

– If B_2 refuses to lend, $P_3 = P_2 = .44$ which is past the apex of S_2, so B_3 should intend to lend with probability 1.00 (.95).

2.3.7 Hypothesis Testing

Table 2.11 presents the likelihood ratio test statistics comparing each of the restricted models to the unconstrained estimates. Notice that the only model that ever fails to be rejected for pooled data is SEERP, which is rejected for three of the five pooled data sets. This evidence suggests that none of the models accurately describe the pooled data. This is not surprising given that SEERP, REPA and REPARP depend on agents' beliefs being consistent with observed behavior. One certainly should not expect beliefs to depend on behavior an agent has not observed, as is the case when pooled data is used.

Table 2.11 shows that for the 15 individual experiments, at a five percent significance level, both SEEHP and SEERP could not be rejected in 11 cases. REPARP is rejected in 11 cases at a five percent significance level, but fails to be rejected 7 times at a one percent significance level. SEE and REPA are rejected for all but one experiment each. This strongly suggests that the inclusion of some sort of inflated belief by the bankers does improve the fit of these models.

One may suspect that the flexibility created by having a number of free parameters in these econometric models is what is driving the positive results, rather than the reputation building structure.[3] A naive model with myopic

[3] I would like to thank Charles Plott for helpful comments regarding this issue.

Table 2.11: Likelihood Ratio Tests: Unconstrained vs Constrained

		-2 log-likelihood ratio				
Design	Experiment	SEE	SEEHP	SEERP	REPA	REPARP
CW00	Pooled	791.4	31.0	6.4†	791.4	9.8
	9	301.6	3.2†	14.0	301.6	14.0
	10	557.0	11.2	2.0†	557.0	15.4
CW10	Pooled	241.6	28.4	25.2	250.2	9.6
	6	212.6	5.2†	5.8†	212.4	14.4
	7	66.8	28.4	25.4	98.8	15.8
	8	70.6	1.0†	2.8†	73.2	1.8†
CW33	Pooled	76.0	23.0	12.0	202.0	44.8
	3	21.2	0.2†	0.0†	53.1	15.0
	4	15.8	5.2†	0.8†	36.0	19.2
	5	56.8	5.6†	0.0†	91.6	15.2
NO1	Pooled	53.4	26.8	28.4	57.0	25.6
	2	7.4†	0.6†	1.2†	10.0*	1.6†
	3	14.8	8.8*	9.6	15.8	9.0*
	5	52.0	1.4†	2.0†	52.4	1.8†
NO2	Pooled	678.8	11.4	0.8†	681.4	17.2
	6	187.8	0.0†	5.2†	187.8	9.2*
	7	184.4	40.2	23.6	184.8	25.2
	8	168.8	1.4†	1.8†	170.0	0.0†
	9	167.0	0.0†	2.2†	169.4	9.0*

*Not significant at 1%.
†Not significant at 5%.

agents is constructed in order to investigate this conjecture. Assume that all types of agents play mixed strategies. These mixed strategies are fixed at the beginning of the experiment and do not change regardless of observed behavior. One may ask, what is the best fit such a model could produce and how does this fit compare with the previous results? Maximum likelihood estimation is employed to find the triplet of mixed strategies that maximize the estimated likelihood function for each data set. This model has more free parameters than any of the restricted models considered above. The fit of this model is much worse than all of the previously studied models except for SEE and REPA, which achieve mixed results relative to the naive model. This suggests that the reputation building models are capturing a lot of the dynamics observed in the experiments.

2.4 Conclusion

The results presented here give some support to the findings of Camerer and Weigelt, that by inflating the initial belief that the entrepreneur prefers to repay, the descriptive power of the sequential equilibrium can be improved. I say "descriptive," rather than "predictive," because the rejection of these models using pooled data suggests that the optimal adjustment to the priors will depend on the subjects in the particular experiment.

The results are generally more in line with the findings of Neral and Ochs. There is evidence that some kind of reputation building is taking place, but the equilibrium models considered show no consistent bias when trying to explain the results of these experiments.

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Chapter 3

Price Dynamics in Overlapping Generations Environments[1]

[1]This paper is joint work with Charalambos Aliprantis. It was published in *Economic Theory* in July, 1993. I am the first author.

3.1 Introduction

This paper studies competitive equilibria price dynamics in overlapping generations environments. The environments considered consist of a single market in which agents trade two commodities, a good and a *numéraire*. We consider two types of agents who need only be differentiated by their endowments. One type is only endowed with positive supply of the *numéraire*. These agents shall be called buyers. The other agents start with only positive supply of the good. Naturally, these agents are the sellers. This market is considered over a sequence of discrete time intervals, or periods. In each period, a new generation is born. A new generation consists of a finite number of both types of agents whose endowments and utility functions will be summarized in generational aggregate supply and demand schedules. These agents are potentially active for a fixed but arbitrary number of periods, which shall be called a lifetime.

The current work is primarily intended to be an extension of the model presented in Aliprantis and Plott (AP) [1]. AP presented a unique perfect foresight competitive equilibrium to a finite horizon OLG environment in which there was a particular type of parametric shift (an "opposing shift") and agents lived for two periods. They also presented experimental evidence of convergence to the competitive price. One important contribution of AP is the introduction of a framework in which experimentalists could study price adjustment in a dynamic setting. The overlapping nature of the environment partitions the transition from the old market parameters (supply and demand) to the new into discrete time intervals. This is in contrast to the standard non-OLG experimental environments. This particular feature alone makes these settings very appealing for the study of price dynamics. The existence of a unique perfect foresight competitive equilibrium price path provides a natural baseline to which other models could be compared. Take a simple alternative, which is that the overlapping feature has no effect, and agents only trade within their generations. This will generally be *ex post* inefficient, but yet it is one of the many plausible sequences of temporary equilibria to this environment. Other sequences of temporary equilibria arise from various learning rules.[2] These will also be *ex post* inefficient. The unique, *ex post* efficient, competitive equilibrium is a natural baseline to which these models could be compared.

Naturally, the extent to which the theoretical predictions of various models differ will depend on the parameters of the given environment. From an econometric standpoint, the most powerful tests of these models will arise from environments in which the various models give very distinct predictions. A method to calculate the unique perfect foresight competitive equilibria for a more general class of OLG environments is presented here with this goal in mind. One possible application of these results is an investigation of the sort suggested above where the parameters of the environment are carefully selected to distinguish between the predictions of the rival models.

While this paper is an extension of the experimental literature, it is purely theoretical in nature and presents results that are of general interest. A number of results that characterize the competitive equilibrium price path for

[2] See Blume and Easley [2], Marcet and Sargent [3], and Marimon and Sunder [4].

environments with finite or countably infinite time horizons are presented. These results primarily show a degree of stability in the equilibrium prices. Under the assumptions of a finite time horizon and a single exogenous parametric shift, a unique perfect foresight competitive equilibrium is shown to exist. The environments here differ from those in AP in two ways. First, agents in the AP model had lifetimes of two periods. Here, the model is expanded to consider lifetimes of any finite number of periods. The main consequence of this in light of the previous discussion of price dynamics is that the number of periods spent in transition from the old market parameters to the new is proportional to the life span of the agents. Increasing the lifetime thus provides a better environment in which to study the era of transition from old market parameters to new. Additionally, AP required the shifts in supply and demand to be opposing, so that an increase in demand would be accompanied by a decrease in supply. This restriction on allowable shifts is dropped in the current paper.

The characteristics of this model which make it readily applicable to econometric investigation of price dynamics are the same characteristics that contrast this work with most of the overlapping generations literature. The most important difference between the model considered here and traditional OLG models is that our commodities are both non-perishable. This feature increases the possibilities for intergenerational trade, which is the driving force behind most of our results. Overlapping generations are not introduced in these environments to find contrasts with the standard Arrow–Debreu type results, but rather to facilitate the study of price dynamics in environments where standard results can be shown to apply.

The paper is organized as follows. Section 3.2 describes the environments and the equilibrium concept. Section 3.3 gives a quick intuitive explanation of the results that will follow in the next two sections. Section 3.4 presents some results that characterize the competitive equilibrium price path for OLG models with up to a countably infinite number of periods. Finally, Section 3.5 shows uniqueness and existence of the equilibrium price path in a finite time horizon model with a single parameter shift.

3.2 The OLG Model

The OLG model under consideration has two types of agents: buyers and sellers. The lifetime of each new generation of agents consists of n periods, where n is fixed *a priori*. Thus, a buyer or seller born in any period x will be alive for periods $x, x+1, \ldots, x+n-1$. Agents can be active in the market during any of the periods in their lifetime, but not in periods before their birth or after their death. Agents born in period x will be referred to as the **new agents** of generation x. Consequently, in any given period $x > n-1$ only the agents of generations $x-n+1, x-n+2, \ldots, x$ are alive.

The new buyers of generation x collectively have an aggregate demand function $D_x(p)$. Similarly, the new sellers in period x have an aggregate supply function $S_x(p)$. The *characteristics* of the new agents in period x are the pair of functions $(D_x(p), S_x(p))$.

All agents are assumed to be profit maximizers. Profit maximization is the driving force behind the trading transactions. The use of supply and demand functions and profit maximization stems from the genesis of this work being in the experimental economics literature, where this is the standard formulation of economic environments. Before introducing the notation to be used throughout this work, we shall briefly address the connection between this formulation and standard utility maximization. A full exploration of this topic is beyond the scope of this paper. However, consideration of a very simple economy does uncover some important details of this relation.

Consider a pure exchange economy, in which there are two commodities, x_1 and x_2. Suppose all agents' preferences over these commodities can be represented by a quasilinear utility function of the form, $U(x_1, x_2) = x_1 + u(x_2)$, where u is increasing and concave. Agents start with endowments (ω_1, ω_2), and prices are denoted (p_1, p_2). The agent's utility maximization problem can be written as

$$\max[x_1 + u(x_2)] \quad \text{subject to} \quad p_1\omega_1 + p_2\omega_2 \geq p_1x_1 + p_2x_2, x_1 \geq 0, x_2 \geq 0.$$

Without loss of generality, let $p_1 = 1$. The optimum may be reached at a corner, characterized by either $x_1 = 0$ or $x_2 = 0$, or in the interior, where $u'(x_2) = p_2$.

Consider a two agent economy in which both agents have a common utility function. The agents start with different endowments, $(\omega_{1,b}, \omega_{2,b})$ and $(\omega_{1,s}, \omega_{2,s})$. Figure 3.1 depicts how there could be gains to trade between these two agents. The marginal utility with respect to x_2 is downward sloping by the concavity assumption. The graph shows that the utility gain to b from acquiring an additional unit of x_2, which is $p(1)$, is greater than the utility s would lose from giving up a unit of x_2, $c(1)$. Both agents would be better off if

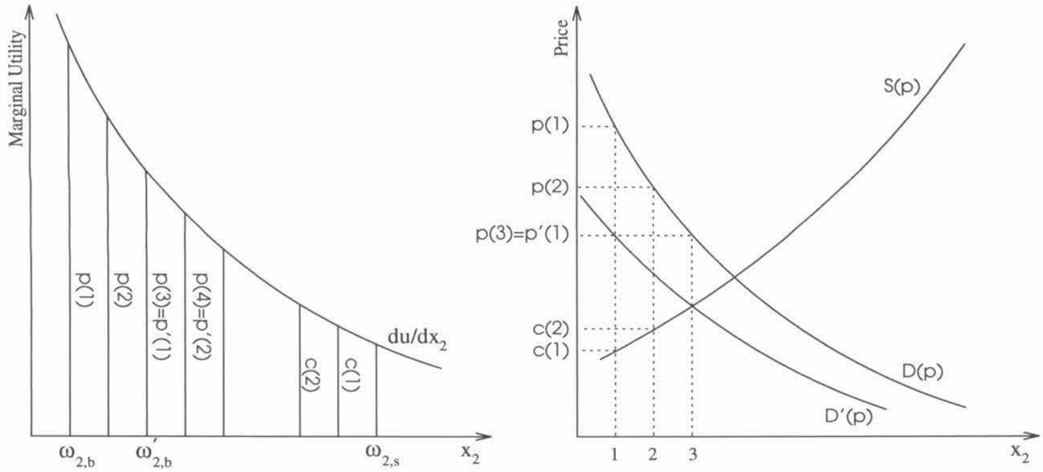

Figure 3.1: From Utility Maximization to Supply and Demand

b would trade s some quantity of x_1 between $c(1)$ and $p(1)$ for one unit of x_2, assuming b has adequate endowment of x_1. (Remember that $\partial U/\partial x_1 = 1$.) Naturally, equilibration would require a trade of q units of x_2 for $p_2 q$ units of x_1 where $p(q) = c(q) = p_2$. These changes in utility can equivalently be viewed as inverse demand and cost functions, which are then used to construct the supply and demand functions in the adjacent graph. These graphs also show that one potential cause of a shift in supply or demand is a change in endowments, as is the case when changing $\omega_{2,b}$ to $\omega'_{2,b}$ causes a shift from $D(p)$ to $D'(p)$. It is clear that a change in tastes, as represented by the utility function, may also shift supply or demand.

Henceforth, only the supply and demand formulation will be considered. Throughout this paper, both supply and demand will be expressed as functions. Supply is assumed to be strictly increasing in price and demand is taken to be strictly decreasing. Continuity is also always assumed. All results are easily extended to an alternate formulation using supply and demand correspondences and set notation. Such a formulation merely complicates the exposition without adding significantly to the economic content of the model. For this reason, the functional formulation is used throughout this paper. Alternate statements of all the major results could also be established for supply and demand that are weakly monotone, but this is omitted for the same reason. We shall employ the following notation.

1. The new buyers in period x may buy (collectively) b_x^x units in period x, b_{x+1}^x in period $x+1$, up through b_{x+n-1}^x in period $x+n-1$. In other words, the trading actions of the new buyers in period x are completely described by the vector

$$B_x = (b_x^x, b_{x+1}^x, \ldots, b_{x+n-1}^x).$$

   If $p_x, \ldots, p_{x+n-1}$ are the prevailing prices in periods $x, \ldots, x+n-1$, then by the profit maximization assumption, we must have

$$b_x^x + b_{x+1}^x + \cdots + b_{x+n-1}^x \leq D_x\left(\min\{p_x, p_{x+1}, \ldots, p_{x+n-1}\}\right).$$

   Of course, profit maximization and perfect foresight necessarily require this to hold with equality.

2. The situation of the new sellers in period x is analogous. They sell s_x^x units in period x, s_{x+1}^x in period $x+1$, through s_{x+n-1}^x in period $x+n-1$. That is, the actions of the new sellers born in period x are

given by the vector

$$C_x = (s_x^x, s_{x+1}^x, \ldots, s_{x+n-1}^x).$$

To maximize profit, their transactions must satisfy the inequality

$$s_x^x + s_{x+1}^x + \cdots + s_{x+n-1}^x \leq S_x\left(\max\{p_x, p_{x+1}, \ldots, p_{x+n-1}\}\right).$$

This will again hold with equality under profit maximization and perfect foresight.

We now come to the concept of a competitive equilibrium for the OLG model.

Definition 3.2.1 *A* **perfect foresight competitive equilibrium** *(or simply an* **equilibrium***) consists of two sequences,* $(p_1, p_2, \ldots)$ *and*

$$\left(\begin{bmatrix} b_1^1 \\ s_1^1 \end{bmatrix}, \begin{bmatrix} b_2^1 & b_2^2 \\ s_2^1 & s_2^2 \end{bmatrix}, \ldots, \begin{bmatrix} b_n^1 & \cdots & b_n^n \\ s_n^1 & \cdots & s_n^n \end{bmatrix}, \begin{bmatrix} b_{n+1}^2 & \cdots & b_{n+1}^{n+1} \\ s_{n+1}^2 & \cdots & s_{n+1}^{n+1} \end{bmatrix}, \ldots\right),$$

which satisfy the following conditions.

1. *The sequence* $(p_1, p_2, \ldots)$ *is called the* **price path** *and each price* p_t *represents the prevailing price at period* t.
2. *The sequence* $\left(\begin{bmatrix} b_1^1 \\ s_1^1 \end{bmatrix}, \ldots, \begin{bmatrix} b_t^{t-n+1} & \cdots & b_t^t \\ s_t^{t-n+1} & \cdots & s_t^t \end{bmatrix}, \ldots\right)$ *is the* **(trade) allocation sequence***. The matrix at each period represents all the transactions that take place in that period.*

3. *The traded units must satisfy:*

   (a) *the* **market clearance equations**,

$$b_1^1 = s_1^1, b_2^1 + b_2^2 = s_2^1 + s_2^2, \ldots, \sum_{i=j-n+1}^{j} b_j^i = \sum_{i=j-n+1}^{j} s_j^i \text{ for } j \geq n,$$

   (b) *the* **perfect foresight profit maximizing budget constraints**,

$$\begin{aligned} b_x^x + \cdots + b_{x+n-1}^x &= D_x\left(\min\{p_x, \ldots, p_{x+n-1}\}\right) \text{ and} \\ s_x^x + \cdots + s_{x+n-1}^x &= S_x\left(\max\{p_x, \ldots, p_{x+n-1}\}\right) \end{aligned}$$

   for each $x = 1, 2, \ldots$.

We shall only be concerned with environments where equilibrium prices support non-trivial trades. The following definition of **degenerate** prices will formalize this concept.

Definition 3.2.2 *The price path* $(p_1, p_2, \ldots)$ *will be called degenerate if for some period* t, *either* $D_t(\min\{p_t, \ldots, p_{t+n-1}\}) = 0$ *or* $S_t(\max\{p_t, \ldots, p_{t+n-1}\}) = 0$.

This could easily be avoided by assuming that both supply and demand functions are strictly positive.

3.3 Preview of Results

This section briefly describes some of the intuition behind the results of the next two sections. A graphical presentation of the results of Section 3.5 will exemplify what is to be shown in that section. Furthermore, the intuition behind these graphs will also carry through to the more abstract results of the next section. So, consider an OLG environment with a finite time horizon and one parameter shift. This shift takes the following form: all generations born before period k are endowed with $(D(\cdot), S(\cdot))$; and all generations born from periods k through $k+m$ are endowed with $(D^*(\cdot), S^*(\cdot))$. Let p_0 and p_0^* be the intragenerational equilibrium prices, i.e., $D(p_0) = S(p_0)$ and $D^*(p_0^*) = S^*(p_0^*)$. These are standard Arrow–Debreu equilibrium prices. Figures 3.2 through 3.5 show four possible shifts which would often lead to price paths similar to those presented adjacent to the supply and demand graphs.[3] Notice that there are at most two price changes corresponding to any of these shifts, and that these shifts occur at specific times. The time horizon can essentially be split into three relevant eras: one in which only agents with the initial characteristics, $(D(\cdot), S(\cdot))$, are alive; one in which some agents have the initial characteristics and some have $(D^*(\cdot), S^*(\cdot))$; and one in which only the latter characteristics are present. These stages shall be called the **initial**, **transitional**, and **final** eras, respectively.

It will be shown that the four price paths shown in Figures 3.2 through 3.5 are the only possible equilibrium price paths when $p_0 < p_0^*$. Fixing the

[3]The uncertainty in this statement arises from the fact that the price path depends not only on the generational supply and demand, but also on the number of periods before and after the shift along with the length of agents' lifetimes.

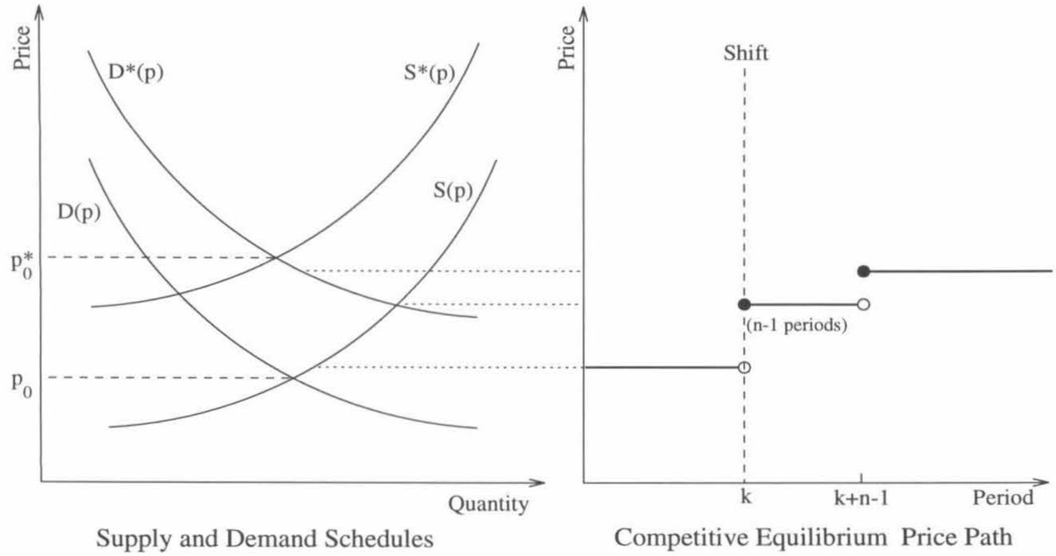

Figure 3.2: Supply and Demand Shifts Implying Three Distinct Prices

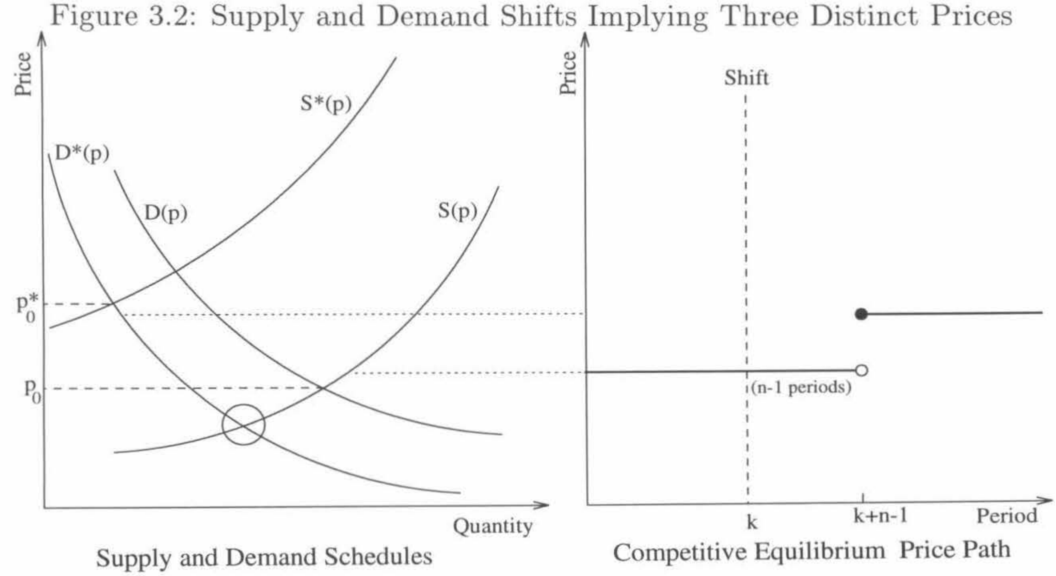

Figure 3.3: Supply and Demand Shifts Implying Two Distinct Prices: First Case

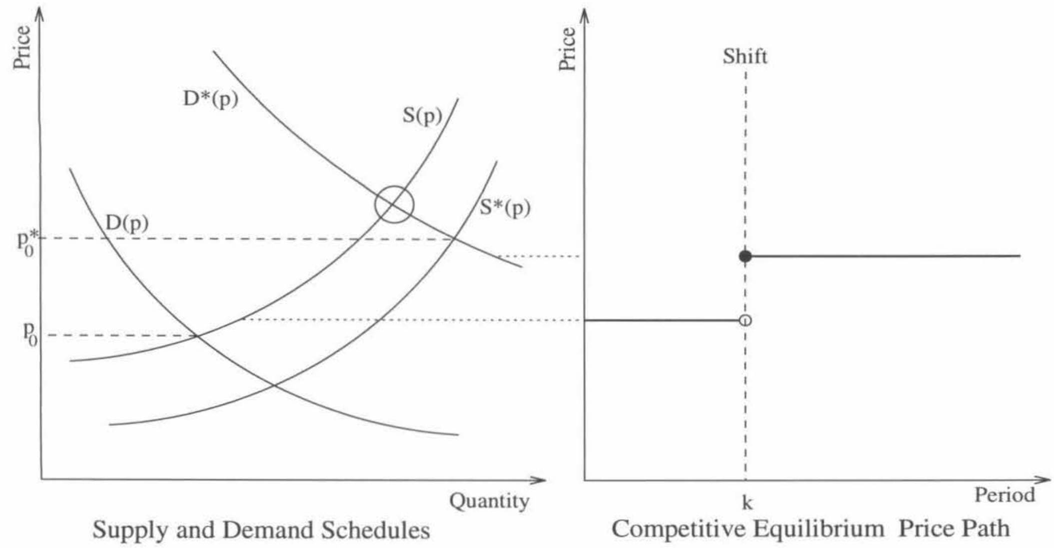

Figure 3.4: Supply and Demand Shifts Implying Two Distinct Prices: Second Case

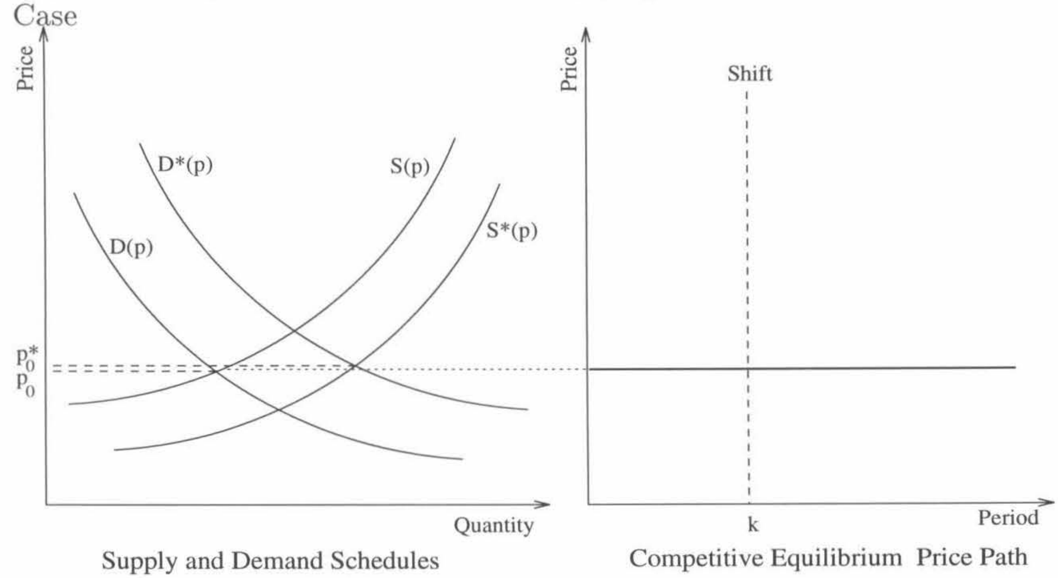

Figure 3.5: Supply and Demand Shifts Implying One Constant Price

number of periods before and after the shift and the length of agents' lifetimes, two properties of the supply and demand schedules help to determine which of these price paths is the equilibrium price path. If p_0 is sufficiently less than p_0^*, then the price path will depend on where $S(p)$ intersects $D^*(p)$. The determination of whether or not these prices are significantly different will depend on the time scales involved. Say this intersection occurs at $\bar{p}$. The intuition behind this price being of interest is that some sellers who were born with $S(p)$ will still be alive when new demanders are born with $D^*(p)$. These agents will interact if there are gains to be made from this interaction. So, for instance, take $p_0 < \bar{p} < p_0^*$ as in Figure 3.2. If the old sellers with $S(p)$ sell to the new buyers with $D^*(p)$, both agents are better off than if they had interacted with agents from their own generation. If $\bar{p} < p_0$, as in Figure 3.3, the new buyers would want to buy at this low price, but the old sellers have no reason to sell below the price they could have gotten from the buyers in their own generation. The new buyers are willing to pay more than the old intragenerational equilibrium price. It will be shown that market equilibration requires that the prevailing price of the good remain fixed at one level during the initial and transitional eras. Figure 3.4 shows a similar case in which one price will prevail over both the transitional and final eras. If p_0 is "close" to p_0^*, then one price will prevail for all periods as in Figure 3.5. This very casual discussion intentionally ignored some important details in order to present the intuition behind the results that follow.

3.4 The Structure of Equilibrium Price Paths

This section presents a series of theorems and lemmas that characterize some properties of an arbitrary competitive equilibrium price path. These results consider an environment with up to a countably infinite time horizon (none of the results depend on having a fixed final period) with arbitrarily many parameter shifts. The results establish that the competitive equilibrium price path is "well behaved" in a manner to be made more precise throughout our discussion here.

Our first result shows that no agent will ever live to see a reversal of a price change.

Theorem 3.4.1 ((Local Weak Monotonicity)) *For any non-degenerate equilibrium price path, $(p_1, p_2, \ldots)$, in an OLG environment, we have:*

1. *If $p_t < p_{t+1}$, then*

$$p_{t-n+1} \leq p_{t-n+2} \leq \cdots \leq p_t < p_{t+1} \leq p_{t+2} \leq \cdots \leq p_{t+n}.$$

2. *Similarly, if $p_t > p_{t+1}$, then*

$$p_{t-n+1} \geq p_{t-n+2} \geq \cdots \geq p_t > p_{t+1} \geq p_{t+2} \geq \cdots \geq p_{t+n}.$$ [4]

[4] If t and n are such that these inequalities characterize prices prior to the first period, the inequalities involving periods prior to period one can be truncated, and the meaningful inequalities remain intact.

Proof: We will establish only the first claim because the second follows from symmetry.

We are given $p_t < p_{t+1}$. Assume by way of contradiction that $p_{t+1} > p_{t+2}$, so that we have $p_t < p_{t+1}$ and $p_{t+1} > p_{t+2}$. All buyers who are alive in period $t+1$ are also alive in either period t or period $t+2$ when prices are lower, so no purchases will be made in period $t+1$. Market clearance in period $t+1$ requires

$$\sum_{i=t-n+2}^{t+1} b_{t+1}^{i} = \sum_{i=t-n+2}^{t+1} s_{t+1}^{i} = 0.$$

This leads to two useful observations.

1. $s_{t+1}^{t-n+2} = 0$ implies $p_j \geq p_{t+1}$ for some $t-n+2 \leq j \leq t-1$.

2. $s_{t+1}^{t+1} = 0$ implies $p_j \geq p_{t+1}$ for some $t+3 \leq j \leq t+n$.

We will focus on the second observation to build the set of inequalities working forward in time from period $t+1$. A similar argument based on the first observation results in the chain of inequalities working backward from period t.

Using the latter observation, we have $p_t < p_{t+1}$, $p_{t+1} > p_{t+2}$, and $p_j \geq p_{t+1}$. Consider the sellers in period $t+2$. Each one of them will also be alive in either period $t+1$ or period j, when prices will be higher than in period $t+2$. So, no seller will sell at p_{t+2}. Market clearance in period $t+2$ yields

$$\sum_{i=t-n+3}^{t+2} b_{t+2}^{i} \quad = \quad \sum_{i=t-n+3}^{t+2} s_{t+2}^{i} \quad = \quad 0.$$

As above, $b_{t+2}^{t+1} = 0$ implies that there exists $t+3 \leq u \leq t+n$ such that

$p_u \le p_{t+2}$. Since $p_u \le p_{t+2} < p_{t+1} \le p_j$, we see that $u \neq j$. So, we consider two cases: $j > u$ and $j < u$.

1. If $j > u$ then no seller wishes to sell in u because they are able to sell at a higher price in either $t+1$ or j, so $\sum_{i=u-n+1}^{u} s_u^i = 0$. Buyers born in period $t+1$ will never make a transaction at a price higher than p_u because they are profit maximizers, but no seller who is active during these buyers' lives will ever sell at p_u or lower. So, $D_{t+1}(\min\{p_{t+1}, \ldots, p_{t+n}\}) = \sum_{r=t+1}^{t+n} b_r^{t+1} = 0$, which shows that this is only consistent with a degenerate price path.

2. If $j < u$ then all buyers alive in j would prefer to buy in $t+2$ or u than in j, so $\sum_{i=j-n+1}^{j} b_j^i = 0$. Thus $S_{t+1}(\max\{p_{t+1}, \ldots, p_{t+n}\}) = \sum_{r=t+1}^{t+n} s_r^{t+1} = 0$. This is a degenerate price path again.

So, we have established the following: $p_t < p_{t+1}$ implies $p_{t+1} \le p_{t+2}$.

Now consider $p_{t+2} > p_{t+3}$. Thus, we have $p_t < p_{t+1} \le p_{t+2}$ and $p_{t+2} > p_{t+3}$. Again, no buyer will buy in $t+1$, as they prefer p_t and p_{t+3}. The same is true for period $t+2$. So, we have $s_{t+1}^{t+1} = s_{t+2}^{t+1} = 0$ which implies that there is some $j \in \{t+4, \ldots, t+n\}$ satisfying $p_j \ge p_{t+1}$. But then no sellers will want to sell in $t+3$, so there is a period u as above. Now apply the previous argument regarding j and u to get a contradiction. Thus, if $p_t < p_{t+1}$ then $p_{t+1} \le p_{t+2} \le p_{t+3}$.

Repeating this procedure up to period $t+n$ yields the stated result. If we would try to go past $t+n$, it would no longer be the case that all buyers in period $t+1$ could get a better price than p_{t+1} because those born in $t+1$ would die before $t+n+1$. ∎

The previous result showed a weak monotonicity property of the price path. Another interpretation of this is that if the environment were to produce cyclical prices, this theorem provides a lower bound to the periodicity.

The next few results address conditions that lead to local price stability. Before presenting these results, we present one more definition.

Definition 3.4.2 *The generations born in periods t through $t+x$ will be said to be* **homogeneous** *if $D_t(\cdot) = \cdots = D_{t+x}(\cdot)$ and $S_t(\cdot) = \cdots = S_{t+x}(\cdot)$.*

Theorem 3.4.3 *If the generations born in periods $1, \ldots, t-1$ are homogeneous, then $p_1 = \cdots = p_{t-1}$ for any non-degenerate equilibrium price path $(p_1, p_2, \ldots)$.*

Proof: Suppose $p_{j+1} > p_j$ for some $1 \leq j \leq t-2$. Then, with the appropriate truncations, Theorem 3.4.1 yields

$$p_{j-n-1} \leq \cdots \leq p_j < p_{j+1} \leq p_{j+2} \leq \cdots \leq p_{j+n}.$$

Clearance in period $j+1$ yields $b_{j+1}^{j-n+2} + \cdots + b_{j+1}^{j+1} = s_{j+1}^{j-n+2} + \cdots + s_{j+1}^{j+1}$, or $0 + \cdots + 0 + b_{j+1}^{j+1} = S(p_{j+1}) + s_{j+1}^{j-n+3} + \cdots + s_{j+1}^{j+1}$. Consequently,

$$D(p_{j+1}) \geq S(p_{j+1}). \tag{3.1}$$

Market clearance in period j yields $b_j^{j-n+1} + \cdots + b_j^j = s_j^{j-n+1} + \cdots + s_j^j$, or $b_j^{j-n+1} + \cdots + b_j^{j-1} + D(p_j) = S(p_j) + 0 + \cdots + 0$. Hence,

$$D(p_j) \leq S(p_j). \tag{3.2}$$

From (3.1) and (3.2), we see that $p_j \geq p_0 \geq p_{j+1}$, which contradicts $p_{j+1} > p_j$, and the proof is completed. ∎

The preceding theorem demonstrates that the price must stay fixed in all periods before a parameter shift. The next theorem will show that after a shift, prices will immediately adjust and then stay constant for either $n-1$ periods or until another shift occurs, whichever comes first. These prices come from interaction between agents who come from heterogeneous generations. These time intervals where there is such interaction will be called **transitional eras** because the market is in transition from one set of characteristics to another.

Theorem 3.4.4 *Let $(p_1, p_2, \ldots)$ be a non-degenerate equilibrium price path. For some $x \leq n-2$, assume the new generations in periods $t-n+1, \ldots, t-n+1+x$ are homogeneous as are the new generations in periods $t, \ldots, t+x$. If $p_{t-1} \leq p_t$ and $p_{t+x} \leq p_{t+x+1}$, then*

$$p_{t-1} \leq p_t = \cdots = p_{t+x} \leq p_{t+x+1}.$$

Similarly, $p_{t-1} \geq p_t$ and $p_{t+x} \geq p_{t+x+1}$ imply

$$p_{t-1} \geq p_{t+1} = \cdots = p_{t+x} \geq p_{t+x+1}.$$ [5]

Proof: Only the first claim will be verified as the second follows from an

[5] Also note that Theorem 3.4.1 ensures that the cases where the inequalities are in opposite directions need not be addressed because at least one of them would have to hold in equality and consequently fall into one of the cases already addressed here.

identical argument. Assume by way of contradiction that there is a $j \in \{t+1, \ldots, t+x\}$ satisfying $p_{j-1} < p_j$. Theorem 3.4.1 implies

$$p_{j-n} \leq \cdots \leq p_{j-1} < p_j \leq \cdots \leq p_{j+n-1}.$$

Notice that the only buyers who would possibly be active in period j are the buyers of generation j. Also notice that the sellers who will die in period j will certainly wait until that period to sell. Also, the only sellers who would potentially be active in period j-1 are those who are dying in that period. Any buyer alive in period j-1 will make their purchases in this period or a prior period. Market clearance in j-1 requires $b_{j-1}^{j-n} + \cdots + b_{j-1}^{j-1} = s_{j-1}^{j-n} + \cdots + s_{j-1}^{j-1}$, or

$$D_{j-1}(p_{j-1}) \leq b_{j-1}^{j-n} + \cdots + b_{j-1}^{j-2} + D_{j-1}(p_{j-1}) = s_{j-1}^{j-n} + 0 + \cdots + 0 \leq S_{j-n}(p_{j-1}).$$

Similarly, for period j, $b_j^{j-n+1} + \cdots + b_j^j = s_j^{j-n+1} + \cdots + s_j^j$. Therefore,

$$D_j(p_j) \geq 0 + \cdots + 0 + b_j^j = s_j^{j-n+1} + s_j^{j-n+2} + \cdots + s_j^j \geq S_{j-n+1}(p_j).$$

These two market clearing conditions and the homogeneity assumptions imply that $p_{j-1} \geq p_j$ which contradicts the assumption that $p_{j-1} < p_j$. ∎

The next two lemmas will set the groundwork for a price stability theorem for the time interval following a transitional era, assuming it is not another transitional era, in which case the previous theorem holds true.

Lemma 3.4.5 *Let $(p_1, p_2, \ldots)$ be a sequence of non-degenerate equilibrium* *prices in some OLG model such that* $p_t \neq p_{t+1}$ *for some* t.

1. *If the new generations in periods* $t\text{-}n+2, \ldots, t+2$ *are homogeneous, then* $p_{t+1} = p_{t+2}$.

2. *If the new generations in periods* $t\text{-}n+1, \ldots, t+1$ *are homogeneous, then* $p_{t-1} = p_t$.

Proof: (1) We prove only (1) and leave the identical argument of (2) for the reader. To this end, let

$$D = D_{t-n+2} = \cdots = D_{t+2} \quad \text{and} \quad S = S_{t-n+2} = \cdots = S_{t+2}.$$

Assume $p_t < p_{t+1}$. Then Theorem 3.4.1 implies

$$p_{t-n+1} \leq \cdots \leq p_t < p_{t+1} \leq \cdots \leq p_{t+n}.$$

Suppose by way of contradiction that $p_{t+2} > p_{t+1}$. Reapplying Theorem 3.4.1 gives

$$p_{t-n+1} \leq \cdots \leq p_t < p_{t+1} < p_{t+2} \leq \cdots \leq p_{t+n+1}.$$

So, the only buyers who will be active in period $t+1$ are the new buyers in that period, and the only sellers who would be active are those who are dying. Thus, market clearance in $t+1$ yields $D_{t+1}(p_{t+1}) = S_{t-n+2}(p_{t+1})$, or

$$D(p_{t+1}) = S(p_{t+1}). \tag{3.3}$$

Since $p_{t-n+1} \leq \cdots \leq p_{t+1} < p_{t+2}$, it must be the case that $\sum_{i=t-n+3}^{t+2} b_{t+2}^i =$

b_{t+2}^{t+2} and $\sum_{i=t-n+3}^{t+2} s_{t+2}^i = S_{t-n+3}(p_{t+2}) + \sum_{i=t-n+4}^{t+2} s_{t+2}^i$. In order for the market to clear in $t+2$,

$$D(p_{t+2}) \geq b_{t+2}^{t+2} \quad = \quad S_{t-n+3}(p_{t+2}) + \sum_{i=t-n+4}^{t+2} s_{t+2}^i \geq S(p_{t+2}). \qquad (3.4)$$

Now a glance at (3.3) and (3.4) yields $p_{t+1} = p_0 \geq p_{t+2}$, contrary to our assumption $p_{t+1} < p_{t+2}$. So, it must be the case that $p_{t+1} = p_{t+2}$. ∎

The previous lemma showed how changes in prices could imply that prices in surrounding periods would be stable. The next lemma gives conditions under which no price change from one period to the next will imply that prices stay fixed for at least one additional period.

Lemma 3.4.6 *Let $(p_1, p_2, \ldots)$ be a sequence of non-degenerate equilibrium prices in some OLG model. Assume that the new generations in $t-n+2, \ldots, t+2$ are homogeneous. If $p_t = p_{t+1}$, then $p_{t+1} = p_{t+2}$.*

Proof: Let $D = D_{t-n+2} = \cdots = D_{t+2}$ and $S = S_{t-n+2} = \cdots = S_{t+2}$. Also assume by way of contradiction, $p_t{=}p_{t+1} < p_{t+2}$. Theorem 3.4.1 implies

$$p_{t-n+2} \leq \cdots \leq p_t = p_{t+1} < p_{t+2} \leq p_{t+3} \leq \cdots \leq p_{t+n+1}.$$

Market clearance at $t+1$ yields

$$D(p_{t+2}) = b_{t+2}^{t+2} = \sum_{i=t-n+3}^{t+2} b_{t+2}^i = \sum_{i=t-n+3}^{t+2} s_{t+2}^i = S(p_{t+2}) + \sum_{i=t-n+4}^{t+2} s_{t+2}^i.$$

Therefore, $D(p_{t+2}) \geq S(p_{t+2})$, and so

$$p_{t+2} \leq p_0. \tag{3.5}$$

Similarly, market clearance at $t+1$ implies

$$D(p_{t+1}){=}D_{t+1}(p_{t+1}) \leq \sum_{i=t-n+2}^{t+1} b^i_{t+1} {=} \sum_{i=t-n+2}^{t+1} s^i_{t+1} \leq S_{t-n+2}(p_{t+1}){=}S(p_{t+1}).$$

Hence, $p_{t+1} \geq p_0$. A glance at (3.5) shows that $p_{t+1} \geq p_{t+2}$, which contradicts $p_{t+1} < p_{t+2}$. Thus, $p_{t+1} = p_{t+2}$ must hold true. ∎

The previous two lemmas are now employed to demonstrate price stability in eras with homogeneous agents which follow transitional eras.

Theorem 3.4.7 *If $(p_1, p_2, \ldots)$ is a sequence of non-degenerate equilibrium prices in some OLG model and the new generations in periods $t, t+1, \ldots, t+n+1$ all have the same characteristics, then $p_{t+n-1} = p_{t+n} = p_{t+n+1}$.*

Proof: If $p_{t+n} \neq p_{t+n+1}$, then by Lemma 3.4.5, $p_{t+n-1} = p_{t+n}$. Lemma 3.4.6 then implies $p_{t+n} = p_{t+n+1}$, a contradiction. Therefore, $p_{t+n} = p_{t+n+1}$. A similar argument shows the impossibility of $p_{t+n-1} \neq p_{t+n}$. ∎

The preceding results can be summarized as follows. Theorem 3.4.3 shows that prices in all periods prior to the first shift must be equal. Theorem 3.4.4 implies that after a shift, prices may make one jump and will then remain fixed until another shift or all of the generations with the prior characteristics die. Theorem 3.4.1 ensures that if another shift does occur during this transitional era, it cannot reverse the direction of the previous price change. Finally, Theorem 3.4.7 addresses the case where the transitional era passes without another shift occurring. In this case, prices will make one adjustment following the last period of the transitional era and then will remain fixed until another shift occurs. Clearly, the strongest result here comes from Theorem 3.4.1 which says that no agent will live to see a reversal of a price trend.

3.5 Finite Time Horizon and a Single Shift

Next, consider a simple OLG model in which there is a finite time horizon and a single parametric shift. For instance, let us say that the shift occurs in period k, meaning that generations born starting in period k have different characteristics than those of the previous generations. Also assume that period $k+m$ is the final period. Agents will still live for n periods. Recall that none of the results in the previous section were dependent on an infinite number of periods. In examining the price path, we can distinguish three eras.

Theorem 3.4.3 addressed the first era, consisting of the first $k-1$ periods, which will be called the **initial era** because all agents who are active in these periods have the initial characteristics $(D(\cdot), S(\cdot))$. The second era is called **transitional** because some of the active agents have the initial characteristics and some have $(D^*(\cdot), S^*(\cdot))$; this era will last for periods k through $k+n-2$. Theorem 3.4.4 shows that prices must remain constant over this era. In the **final era**, periods $k+n-1$ through $k+m$, all agents have the latter characteristics. Theorem 3.4.7 can be applied to this **final era** to show that $p_{k+n-1} = \cdots = p_{k+m}$. We will pick k and m so that there is the potential to have different prices in each of the three eras. This will require that $k \geq n$ and $m \geq n-1$. As these inequalities approach being binding, the parameter shifts must become increasingly severe to support three distinct price eras.

Theorems 3.4.1, 3.4.3, 3.4.4 and 3.4.7 show that the equilibrium price path can only take one of two following two forms:

1) $p_1 = \cdots = p_{k-1} \leq p_k = \cdots = p_{k+n-2} \leq p_{k+n-1} = \cdots = p_{k+m}$, or

2) $p_1 = \cdots = p_{k-1} \geq p_k = \cdots = p_{k+n-2} \geq p_{k+n-1} = \cdots = p_{k+m}$.

We must now specify what kind of shifts we will consider. Let p_0 be the price at which the market would clear if agents with the initial characteristics could only trade with one another, i.e., $D(p_0) = S(p_0)$. Similarly, let p_0^* be the intragenerational equilibrium price for agents with the final characteristics, or $D^*(p_0^*) = S^*(p_0^*)$. All shifts must satisfy one of the three following conditions: $p_0 > p_0^*$, $p_0 = p_0^*$, or $p_0 < p_0^*$. It is not difficult to show that if $p_0 = p_0^*$, then the price path will be flat at this price. We will only explicitly consider shifts such that $p_0 < p_0^*$, because the results are easily modified to apply for $p_0 > p_0^*$ by utilizing the symmetry.

The next lemma shows that equilibrium prices must be weakly increasing when $p_0 < p_0^*$.

Lemma 3.5.1 *For $p_0 < p_0^*$, where $D(p_0) = S(p_0)$ and $D^*(p_0^*) = S^*(p_0^*)$, a non-degenerate equilibrium price path, $(p_1, \ldots, p_{k+m})$, must be weakly in-*

creasing in the following manner:

$$p_1 = \cdots = p_{k-1} \leq p_k = \cdots = p_{k+n-2} \leq p_{k+n-1} = \cdots = p_{k+m}.$$

Proof: Suppose $p_{k+n-2} > p_{k+n-1}$. Then Theorem 3.4.1 asserts

$$p_{k-1} \geq \cdots \geq p_{k+n-2} > p_{k+n-1} \geq \cdots \geq p_{k+2n-2}.$$

But we know that prices are constant within the three eras. So,

$$p_1 = \cdots = p_{k-1} \geq p_k = \cdots = p_{k+n-2} > p_{k+n-1} = \cdots = p_{k+m}.$$

Let $\rho_1 = p_1 = \cdots = p_{k-1}$, $\rho_2 = p_k = \cdots = p_{k+n-2}$, and $\rho_3 = p_{k+n-1} = \cdots = p_{k+m}$ and note that $\rho_1 \geq \rho_2 > \rho_3$.

Aggregating the market clearing conditions for periods 1 through $k+n-2$ yields

$$(k-n)D(\rho_1) + (n-1)D(\rho_2) = (k-1)S(\rho_1) + (n-1)S^*(\rho_2).$$

From $\rho_2 \leq \rho_1$, we obtain

$$(k-1)D(\rho_2) \geq (k-1)S(\rho_2) + (n-1)S^*(\rho_2) \geq (k-1)S(\rho_2).$$

Hence, $D(\rho_2) \geq S(\rho_2)$ from which it follows that

$$\rho_2 \leq p_0. \tag{3.6}$$

Similarly, aggregating the market clearing conditions for periods $k+n-1$ through $k+m$ yields $(m+1)D^*(\rho_3) = (m-n+2)S^*(\rho_3)$. This implies that $D^*(\rho_3) < S^*(\rho_3)$, which leads to $\rho_3 \geq p_0^*$. This, along with (3.6) and the premise that $p_0 < p_0^*$, yields $\rho_2 \leq p_0 < p_0^* \leq \rho_3$, contrary to $\rho_2 > \rho_3$.

A similar argument shows that $p_{k-1} > p_k$ cannot happen. ∎

Note that this implies that the equilibrium price path $(p_1, \ldots, p_{k+m})$ can be fully specified by the triplet (p_1, p_k, p_{k+m}). So far, we have examined the levels of equilibrium prices in each of the three eras relative to the equilibrium prices in other eras. We will now concentrate on locating these equilibrium prices relative to the characteristics of the OLG model.

Lemma 3.5.2 *Let $(p_1, p_2, \ldots)$ be a sequence of non-degenerate equilibrium prices. For $p_0 < p_0^*$, equilibrium prices in the initial era are bounded from below by p_0, i.e.,*

$$p_1 = \cdots = p_{k-1} \geq p_0.$$

Similarly,

$$p_{k+n-1} = \cdots = p_{k+m} \leq p_0^*.$$

Proof: We shall only establish the first claim as the second follows from an identical argument. We already know that

$$p_1 = \cdots = p_{k-1} \leq p_k = \cdots = p_{k+n-2} \leq p_{k+n-1} = \cdots = p_{k+m}.$$

We now distinguish two cases: $p_1 < p_k$ and $p_1 = p_k$.

Consider first $p_1 < p_k$. In this case, the aggregate market clearance in the initial era (periods 1 through $k-1$) yields $(k-1)D(p_1) = (k-n)S(p_1)$. So $D(p_1) \leq S(p_1)$, from which it follows that $p_1 \geq p_0$.

Now consider $p_1 = p_k$. In this case, we distinguish two subcases: $p_1{=}p_k{=}p_{k+m}$ and $p_1{=}p_k \leq p_{k+m}$.

If $p_1 = p_k = p_{k+m}$, then aggregating the market clearance conditions through all periods, we get

$$(k-1)D(p_1) + (m+1)D^*(p_1) = (k-1)S(p_1) + (m+1)S^*(p_1).$$

Rewriting this as $(k-1)[D(p_1) - S(p_1)] = (m+1)[S^*(p_1) - D^*(p_1)]$ shows that $D(p_1) - S(p_1)$ and $S^*(p_1) - D^*(p_1)$ have the same sign, which happens only when $p_0 \leq p_1 \leq p_0^*$.

Finally, let us consider the case $p_1 = p_k < p_{k+m}$. Here, market clearing aggregated over the first $k+n-2$ periods (the initial and transitional eras) implies

$$(k-1)D(p_1) + (n-1)D^*(p_1) = (k-1)S(p_1).$$

So, $(k-1)D(p_1) \leq (k-1)S(p_1)$, or $D(p_1) \leq S(p_1)$, from which it follows that $p_1 \geq p_0$ holds true in this case too. ∎

In order to determine the nature of any given shift, we only need to calculate three critical prices. These prices will help us to classify all shifts into one of four possible types. Intuitively, think of these critical prices as the prices that would clear the market in each of the eras if the shift causes the price to increase greatly. The shift is so extreme that all sellers who can possibly supply in the era after they are born will postpone activity until the next era, and buyers will make all of their purchases in the era of their birth.

These three critical prices are π_1, π_2 and π_3 and will by definition satisfy the following equations:

$$(k-1)D(\pi_1) = (k-n)S(\pi_1) \tag{3.7}$$

$$D^*(\pi_2) = S(\pi_2) \tag{3.8}$$

$$(m-n+2)D^*(\pi_3) = (m+1)S^*(\pi_3) \tag{3.9}$$

We can now establish an upper bound for p_1.

Lemma 3.5.3 *Assume that $p_0 < p_0^*$ and that $(p_1, p_2, \ldots)$ is a non-degenerate equilibrium path of prices. Then:*

1. *Prices in the initial era are bounded from above by π_1, i.e.,*

$$p_1 = \cdots = p_{k-1} \leq \pi_1.$$

2. *Prices in the final era are bounded from below by π_3, i.e.,*

$$p_{k+n-1} = \cdots = p_{k+m} \geq \pi_3.$$

Proof: We shall establish the first case only. The market clearing aggregated over the initial era (periods 1 through $k-1$) can have a demand at most equal to $(k-1)D(p_1)$, while supply is at least equal to $(k-n)S(p_1)$. Therefore, we have

$$(k-1)D(p_1) \geq (k-n)S(p_1).$$

If $p_1 > \pi_1$ holds true, then strict monotonicity implies

$$(k-1)D(\pi_1) > (k-1)D(p_1) \geq (k-n)S(p_1) > (k-n)S(\pi_1),$$

contrary to $(k-1)D(\pi_1) = (k-n)S(\pi_1)$. So, $p_1 \leq \pi_1$ must be true. ∎

We can also use these critical prices to distinguish between types of shifts. For a non-degenerate equilibrium price path, $(p_1, p_2, \ldots)$, we have the following cases:

1. $\pi_1 < \pi_2 < \pi_3$. Here, prices satisfy

$$p_1 = \pi_1, \; p_k = \pi_2, \text{ and } p_{k+m} = \pi_3.$$

2. $\pi_2 \leq \pi_1 < \pi_3$. In this case

$$p_1 = p_k, \;\; p_{k+m} = \pi_3, \text{ and} (k-1)D(p_1) + (n-1)D^*(p_1) = (k-1)S(p_1).$$

3. $\pi_1 < \pi_3 \leq \pi_2$. This leads to

$$p_1 = \pi_1, \; p_k = p_{k+m}, \text{ and } (m+1)D^*(p_k) = (n-1)S(p_k) + (m+1)S^*(p_k).$$

4. $\pi_3 \leq \pi_1$. In which case

$$p_1 = p_k = p_{k+m} \text{ and}$$

$$(k-1)D(p_1) + (m+1)D^*(p_1) = (k-1)S(p_1) + (m+1)S^*(p_1).$$

These four cases correspond directly to Figures 3.2-3.5. In order to show existence of these equilibrium paths, we must show that some set of transactions that are supported by these prices will clear the market in all periods.

Before doing so, we will make some simplifying assumptions and introduce some notation. The simplifying assumptions are adopted only to ease exposition; they do not affect the existence of the equilibrium transactions. First, assume that $k-1$ is divisible by $n-1$, i.e., $\alpha(n-1) = k-1$ for some integer α. Also, let $m+1$ be divisible by $n-1$, so that $\beta(n-1) = m+1$ for some integer β. These assumptions require that the number of periods in both the initial and final eras are proportional to the number of periods in the transitional era. This greatly simplifies the expression of intergenerational transactions because it allows us to partition the set of all generations into groups of size $n-1$. This is useful because we will be able to specify transactions for each of these groups, rather than by generation. We will need some notation to help us classify each generation into the corresponding group. First, for each generation i, we let $\nu(i) = \max\{\lambda \in \{0, 1, 2, \ldots, k+m\} : (n-1)\lambda < i\}$. Also let $\mathcal{D} = D(p_1)$, $\mathcal{S} = S(p_1)$, $\widehat{\mathcal{D}^*} = D^*(p_k)$, $\hat{\mathcal{S}} = S(p_k)$, $\mathcal{D}^* = D^*(p_{k+m})$, and $\mathcal{S}^* = S^*(p_{k+m})$.

Using the transactions given in Table 3.1, we will now show by a series of propositions that these are the unique competitive equilibrium price paths under each of the conditions on the π's.

Proposition 3.5.4 *The transactions given in Table 3.1 are consistent with profit maximization.*

Proof: Since prices have been shown to be weakly monotonically increas-

Table 3.1: Examples of Equilibrium Transactions

i	Buyers	Sellers*
$1,\ldots,k\text{-}n$	$b^i_i = \mathcal{D}$ $b^i_{i+1} = \cdots = b^i_{i+n-2} = 0$ $b^i_{i+n-1} = 0$	$s^i_i{=}\mathcal{D}\text{-}\nu(i)(\mathcal{S}\text{-}\mathcal{D})$ $s^i_{i+1} = \cdots = s^i_{i+n-2} = 0$ $s^i_{i+n-1} = [\nu(i)+1](\mathcal{S}-\mathcal{D})$
$k\text{-}n{+}1,\ldots,k\text{-}1$	$b^i_i = \mathcal{D}$ $b^i_{i+1} = \cdots = b^i_{i+n-2} = 0$ $b^i_{i+n-1} = 0$	$s^i_i = \mathcal{D} - \nu(i)(\mathcal{S}-\mathcal{D})$ $s^i_{i+1} = \cdots = s^i_{i+n-2} = 0$ $s^i_{i+n-1} = \hat{\mathcal{S}} - \mathcal{D} + \nu(i)(\mathcal{S}-\mathcal{D})$
$k,\ldots,k{+}n\text{-}2$	$b^i_i = \widehat{\mathcal{D}^*}$ $b^i_{i+1} = \cdots = b^i_{i+n-2} = 0$ $b^i_{i+n-1} = 0$	$s^i_i{=}\mathcal{S}^*\text{-}\nu(k{+}m{+}1\text{-}i)(\mathcal{D}^*\text{-}\mathcal{S}^*)$ $s^i_{i+1} = \cdots = s^i_{i+n-2} = 0$ $s^i_{i+n-1}{=}\nu(k{+}m{+}1\text{-}i)(\mathcal{D}^*\text{-}\mathcal{S}^*)$
$k{+}n\text{-}1,\ldots,k{+}m$	$b^i_i = \mathcal{D}^*$ $b^i_{i+1} = \cdots = b^i_{i+n-2} = 0$ $b^i_{i+n-1} = 0$	$s^i_i{=}\mathcal{S}^*\text{-}\nu(k{+}m{+}1\text{-}i)(\mathcal{D}^*\text{-}\mathcal{S}^*)$ $s^i_{i+1} = \cdots = s^i_{i+n-2} = 0$ $s^i_{i+n-1}{=}\nu(k{+}m{+}1\text{-}i)(\mathcal{D}^*\text{-}\mathcal{S}^*)$

*As above, $\nu(i) = \max\{\lambda \in \{0,1,2,\ldots,k+m\} : (n-1)\lambda < i\}$.

ing, profit maximizing behavior simply requires that buyers' total purchases equal their demand at the prevailing price at the time of their *birth.* So, for generations 1 through $k-1$, $\sum_{j=i}^{i+n-1} b_j^i = \mathcal{D}$ satisfies the profit maximizing assumption. Table 3.1 clearly shows that the suggested transactions satisfy this condition. For generations k through $k+n-2$, profit maximization requires $\sum_{j=i}^{i+n-1} b_j^i = \widehat{\mathcal{D}^*}$, which is also easily verified. Finally, for generations $k+n-1$ through $k+m$, we need $\sum_{j=i}^{i+n-1} b_j^i = \mathcal{D}^*$, which is trivial to see from Table 3.1. Similarly, sellers total sales must equal their supply at the prevailing price at the time of their *death.* This is also easily verified, and is thus omitted. ∎

Proposition 3.5.5 *The transactions specified in Table 3.1 are all non-negative.*

Proof: It is clear that no buyer is ever required to buy a negative quantity, so we can simply concentrate on sellers.

The initial transaction from Table 3.1 for sellers born in periods 1 through $k-1$ is $s_i^i = \mathcal{D} - \nu(i)(\mathcal{S} - \mathcal{D})$. Lemma 3.5.2 establishes that $p_1 \geq p_0$, which implies that $\mathcal{S} \geq \mathcal{D}$, thus this s_i^i is decreasing in $\nu(i)$. Since $\nu(i)$ is nondecreasing in i, we need only verify non-negativity for the greatest i, or $k-1$. Since $\nu(k-1) = \alpha - 1$, we need only verify that $s_{k-1}^{k-1} = \mathcal{D} - (\alpha - 1)(\mathcal{S} - \mathcal{D}) = \alpha\mathcal{D} - (\alpha - 1)\mathcal{S} \geq 0$. Lemma 3.5.3 showed that $p_1 \leq \pi_1$, which implies $(k-1)\mathcal{D} \geq (k-n)\mathcal{S}$. Dividing both sides of this inequality by $n-1$ produces $\alpha\mathcal{D} \geq (\alpha-1)\mathcal{S}$, which establishes $s_{k-1}^{k-1} \geq 0$. The only other nontrivial periods for these sellers is their final period, $i+n-1$.

We must distinguish between those who will die in the initial era and those who will die in the transitional era. For those who die in the initial era, $s^i_{i+n-1} = [\nu(i) + 1](\mathcal{S} - \mathcal{D}) \geq 0$, because $\mathcal{S} \geq \mathcal{D}$. For those who die in the transitional era, $s^i_{i+n-1} = \hat{\mathcal{S}} - D + \nu(i)(\mathcal{S} - \mathcal{D})$. But $p_k \geq p_1$ by Lemma 3.5.1, so that $\hat{\mathcal{S}} \geq \mathcal{S}$, which clearly implies non-negativity.

For sellers born in periods k through $k + m$, Table 3.1 lists the initial transaction as $s^i_i = \mathcal{S}^* - \nu(k + m + 1 - i)(\mathcal{D}^* - \mathcal{S}^*)$. From Lemma 3.5.2 we know that $p_{k+m} \leq p^*_0$, and thus $\mathcal{D}^* \geq \mathcal{S}^*$. So, s^i_i is *increasing* in $\nu(k+m+1-i)$; it is consequently *decreasing* in i. We need only verify non-negativity for the lowest i, which is $i = k$. It is easy to show that $\nu(k + m + 1 - k) = \beta - 1$, so $s^k_k = \beta\mathcal{S}^* - (\beta - 1)\mathcal{D}^*$. From Lemma 3.5.3 we get $p_{k+m} \geq \pi_3$, which implies $(m-n)\mathcal{D}^* \leq (m+1)\mathcal{S}^*$. Dividing both sides by $n-1$ yields $(\beta-1)\mathcal{D}^* \leq \beta\mathcal{S}^*$, which clearly shows that $s^k_k \geq 0$. Finally, note that the inequality $s^i_{i+n-1} = \nu(k + m + 1 - i)(\mathcal{D}^* - \mathcal{S}^*) \geq 0$ follows from $\mathcal{D}^* \geq \mathcal{S}^*$. ∎

Before addressing the issue of market clearance, it will first be shown that Table 3.1 does not specify any market activity after the final period.

Proposition 3.5.6 *No transaction specified in Table 3.1 for the periods after period $k + m$ are positive under any of the four proposed equilibrium price paths.*

Proof: This condition is clearly met for the buyers. The only sellers who are potentially active in period $k + m + 1$ and beyond are those born in period $k+m-n+2$ through $k+m$. The only non-trivial transactions specified after the final period are these sellers' activity in their final period, where they sell

$s^i_{i+n-1} = \nu(k+m+1-i)(\mathcal{D}^* - \mathcal{S}^*)$. It is easily verified that $\nu(k+m+1-i) = 0$ for $k + m - n + 2 \leq i \leq k + m$. ∎

And now we come to the main theorem of this section. It shows that under each of the four conditions on the π's, the corresponding prices are the only prices that will clear the markets.

Theorem 3.5.7 *For each of the following relations between π_1, π_2 and π_3, the specified price path $(p_1, p_2, \ldots)$ is the unique competitive equilibrium price path.*

1. $\pi_1 < \pi_2 < \pi_3 \iff p_1 = \pi_1,\ p_k = \pi_2,$ *and* $p_{k+m} = \pi_3$.

2. $\pi_2 \leq \pi_1 < \pi_3 \iff p_1 = p_k,\ \ p_{k+m} = \pi_3,$ *and*

$$(k-1)D(p_1) + (n-1)D^*(p_1) = (k-1)S(p_1).$$

3. $\pi_1 < \pi_3 \leq \pi_2 \iff p_1 = \pi_1,\ p_k = p_{k+m},$ *and*

$$(m+1)D^*(p_k) = (n-1)S(p_k) + (m+1)S^*(p_k).$$

4. $\pi_3 \leq \pi_1 \iff p_1 = p_k = p_{k+m},$ *and*

$$(k-1)D(p_1) + (m+1)D^*(p_1) = (k-1)S(p_1) + (m+1)S^*(p_1).$$

Proof: Showing that the transactions in Table 3.1 clear the market in the **initial** and **final era** will not depend on the specific characteristics of each of these price paths, so this will be established first. Then each of the four price paths will be considered in turn. It will be shown that the market clears in the **transitional era** along these price paths for the corresponding types of shifts and that all other prices fail to do so.

We will start with the initial era, periods 1 through $k-1$. Take some arbitrary period, i where $1 \leq i \leq k-1$. Market clearance in period i then requires $b_i^{i-n+1} + \cdots + b_i^i = s_i^{i-n+1} + \cdots + s_i^i$, or

$$0 + \cdots + 0 + \mathcal{D} = [\nu(i-n+1)+1](\mathcal{S}-\mathcal{D}) + 0 + \cdots + 0 + \mathcal{D} - \nu(i)(\mathcal{S}-\mathcal{D}),$$

which is equivalent to $\mathcal{D} = \mathcal{D}$ because $\nu(i-n+1)+1 = \nu(i)$.

Next, consider the final era, periods $k+n-1$ through $k+m$. Market clearance in some arbitrary period i in this era requires $b_i^{i-n+1} + \cdots + b_i^i = s_i^{i-n+1} + \cdots + s_i^i$, or $\mathcal{D}^* = \nu(k+m-i+n)(\mathcal{D}^* - \mathcal{S}^*) + 0 + \cdots + 0 + \mathcal{S}^* - \nu(k+m+1-i)\mathcal{D}^* - \mathcal{S}^*$, which, given that $\nu(k+m-i+n) = \nu(k+m+1-i)+1$, is equivalent to $\mathcal{D}^* = \mathcal{D}^* - \mathcal{S}^* + \mathcal{S}^*$. As in the initial era, market clearance in the final era follows trivially from the proposed equilibrium transactions.

Market clearance in the transitional era will depend on specific qualities of the price paths. We will first establish a general statement of market clearance in this era and then show that it is met under each price path. Again, take some arbitrary $i \in \{k, \ldots, k+n-2\}$. Notice that $\nu(i-n+1) = \alpha - 1$ and $\nu(k+m+1-i) = \beta - 1$ for each i by the definitions of ν, α, and β. Market clearance in period i yields $b_i^{i-n+1} + \cdots + b_i^i = s_i^{i-n+1} + \cdots + s_i^i$ is equivalent to $\widehat{\mathcal{D}^*} = \hat{\mathcal{S}} - \mathcal{D} + \nu(i-n+1)(\mathcal{S}-\mathcal{D}) + \mathcal{S}^* - \nu(k+m+1-i)(\mathcal{D}^* - \mathcal{S}^*)$. Rewriting, we have $\widehat{\mathcal{D}^*} = \hat{\mathcal{S}} - \mathcal{D} + (\alpha-1)(\mathcal{S}-\mathcal{D}) + \mathcal{S}^* - (\beta-1)(\mathcal{D}^* - \mathcal{S}^*)$, or

$$\widehat{\mathcal{D}^*} = \hat{\mathcal{S}} - \alpha\mathcal{D} + (\alpha - 1)\mathcal{S} + \beta\mathcal{S}^* - (\beta - 1)\mathcal{D}^*. \tag{3.10}$$

We now consider each of the four price paths in turn.

First consider price path 1, where $p_1 = \pi_1$, $p_k = \pi_2$, and $p_{k+m} = \pi_3$. Under $p_1 = \pi_1$, $\alpha\mathcal{D} = (\alpha-1)\mathcal{S}$. Similarly, $p_{k+m} = \pi_3$ implies $(\beta-1)\mathcal{D}^* = \beta\mathcal{S}^*$. So (3.10) becomes $\widehat{\mathcal{D}^*} = \hat{\mathcal{S}}$, which is true by the definition of π_2.

In order for another set of prices to also clear the market, it must be the case that these prices change the intergenerational nature of the transactions. For instance, if p_k were to decrease, but not enough to make the sellers of generation $k - 1$ indifferent between selling in period $k - 1$ and period k, i.e., $p_1 < p_k$, then the market could not possibly clear in the transitional era, as demand would increase and supply would decrease. This argument shows that price changes must be accompanied by a corresponding change in the generational distribution of transactions across eras. In order for this to be the case, it must be that the relation between the prices in these eras also changes. As long as $p_1 < p_k < p_{k+m}$, the generational distribution of prices will not change, and the proposed price path is the only price path that can clear the market. Consider a new set of prices where $p'_1 = p'_k < p'_{k+m}$. The market will certainly not clear for $p'_1 \leq p_1$ or $p'_1 \geq p_k$. By Lemma 3.5.3, the price in the initial era cannot increase because along the current price path, $p_1 = \pi_1$, which is the upper bound for prices in the initial era. Thus, the only way for prices to change would be for p'_k to decrease to p_1. This will cause increased supply and decreased demand in the transition era and no change in the initial era. Markets could not clear under these conditions. The same holds true for an increase in p_k.

Next, consider price path 2, where $p_1 = p_k$, $(k-1)\mathcal{D} + (n-1)\widehat{\mathcal{D}^*} = (k-1)\mathcal{S}$, and $p_{k+m} = \pi_3$. The second statement is an aggregation of the market clearing conditions for the first $k+n-2$ periods. The terms involving $\mathcal{S}^*$ and $\mathcal{D}^*$ will again cancel each other in the supply side of (3.10) due to $(\beta-1)\mathcal{D}^* = \beta\mathcal{S}^*$. Dividing the aggregate market clearing condition by $n-1$ results in $\alpha\mathcal{D} + \widehat{\mathcal{D}^*} = \alpha\mathcal{S}$. Setting $p_1 = p_k$, we get $\hat{\mathcal{S}} = \mathcal{S}$, or $\alpha\mathcal{D} + \mathcal{D}^* = \hat{\mathcal{S}} + (\alpha-1)\mathcal{S}$, which is identical to (3.10).

The only plausible alternative price path is a (p'_1, p'_k, p'_{k+m}), where $p'_1 \leq p_1$, $p'_k \geq p_k$, $p'_1 < p'_k$ and $p'_{k+m} = p_{k+m}$. Any other price path would either trivially violate market clearance, the known bounds to prices or weak monotonicity. But $p'_1 < p'_k$ implies that the generational distribution of transactions must be as they were under price path 1 in order for markets to clear, and thus $p'_1 = \pi_1$, $p'_k = \pi_2$ and $p'_{k+m} = \pi_3$. But we are considering an environment in which $\pi_1 \geq \pi_2$, so $p'_1 < p'_k$ is a contradiction.

Under price path 3, $p_1{=}\pi_1$, $p_k{=}p_{k+m}$, and $(m{+}1)\mathcal{D}^*{=}(n{-}1)\hat{\mathcal{S}}{+}(m{+}1)\mathcal{S}^*$. The terms involving $\mathcal{S}$ and $\mathcal{D}$ will again cancel each other in the supply side of (3.10) due to $\alpha\mathcal{D} = (\alpha-1)\mathcal{S}$ just as they did under price path 1. Dividing the aggregate market clearing condition by $n-1$ results in $\beta\mathcal{D}^* = \hat{\mathcal{S}} + \beta\mathcal{S}^*$. From $p_k = p_{k+m}$ we see that $\widehat{\mathcal{D}^*} = \mathcal{D}^*$, and so (3.10) yields $\mathcal{D}^* = \hat{\mathcal{S}} + \beta\mathcal{S}^* - (\beta-1)\mathcal{D}^*$ which is equivalent to the restatement of the condition that $p_k = p_{k+m}$ given in the previous sentence.

Similar to the previous price path, the only plausible alternative price path is (p'_1, p'_k, p'_{k+m}) where $p'_1 = p_1$, $p'_k \leq p_k$, $p'_{k+m} \geq p_{k+m}$ and $p'_k < p'_{k+m}$. The argument showing that market clearance under this condition will violate the condition that $\pi_2 \geq \pi_3$ is identical to the argument of the previous section.

Under the final price path, $p_1 = p_k = p_{k+m}$. Aggregating all $k+m$ market clearing conditions requires that $(k-1)\mathcal{D}+(m+1)\mathcal{D}^* = (k-1)\mathcal{S}+(m+1)\mathcal{S}^*$. Dividing this by $n-1$ results in $\alpha\mathcal{D} + \beta\mathcal{D}^* = \alpha\mathcal{S} + \beta\mathcal{S}^*$. Constant prices imply that $\widehat{\mathcal{D}^*} = \mathcal{D}^*$ and $\hat{\mathcal{S}} = \mathcal{S}$, so (3.10) can be rewritten as $\mathcal{D}^* = \alpha(\mathcal{S} - \mathcal{D}) + \beta(\mathcal{S}^* - \mathcal{D}^*) + \mathcal{D}^*$ which is clearly equivalent to the restatement of the aggregate market clearing condition.

Any alternative set of prices must include either a decrease in initial prices or an increase in final prices. Either of these cases would result in a price path satisfying $p'_1 \leq p'_k \leq p'_{k+m}$ and one of the following inequalities: either $p'_1 < p'_k$ or $p'_k < p'_{k+m}$. Consider $p'_1 < p'_k$; this will lead to the same market clearing condition for the initial era as held under price path 1. So market clearance requires $p'_1 = \pi_1$. Notice that at this price level, initial prices are at their upper bound, so it must be the case that the inequality was created by an increase in p_k. Note that this also implies that $p'_{k+m} > p_{k+m}$ because $p_k = p_{k+m}$ and prices are weakly increasing. This price change has two effects: it at least weakly increases the proportion of sellers to buyers in the transitional and final eras because $p'_1 < p'_k$, and it increases prices in these eras. These two effects are inconsistent with market clearance. ∎

While this section concentrated on a single shift in the parameters of the economy, it is clear that this is just a special case of a class of environments in which any finite number of shifts could occur. Think of adding another t periods onto the $k+m$ periods already considered. The generations born in periods $k+m+1$ through $k+m+t$ are born with characteristics denoted

$(D'(\cdot), S'(\cdot))$. If $m < n$ then this next shift occurs before a steady state is reached after the first shift. If agents born before period k are still alive in period $k + m + 1$, there will be a transitional era that lasts until either agents born in $k - 1$ die or another shift occurs. In the former case, another transitional era would start in period $k + n - 1$ and would last until the agents born in period $k + m$ die. Assuming no more shifts have occurred to that point, a stable period would begin in which only agents characterized by $(D'(\cdot), S'(\cdot))$ are active. Naturally, the price changes in these eras would be constrained by the weak monotonicity implied by Theorem 3.4.1.

3.6 Concluding Remarks

This paper presented some local monotonicity and stability results for a wide class of overlapping generations environments in which market parameters may vary greatly. A special case with a finite time horizon and a single parameter shift was shown to produce a unique perfect foresight competitive equilibrium. All of these results were developed for agents with arbitrarily long finite lifetimes.

One of the main goals of this paper was to develop a set of unique perfect foresight competitive equilibria for a class of finite horizon overlapping generations models that would be broad enough to allow alternate models to make diverse predictions. A simple example will show that these results would help to differentiate between two plausible models. The first is the perfect foresight model developed here. The second model assumes that people do not have perfect foresight and that they trade as if they were only trading with agents of their generation. So the equilibrium price path will simply be the intragenerational prices. When n is small relative to k and m, the stable prices before and after the shift are relatively close to the intragenerational prices and the transitional period is short. This makes it hard to econometrically distinguish between the two models. However, as n increases, these price paths start to diverge, and econometric tests become more plausible.

Natural extensions of this work include the types of experimental investigations of the perfect foresight assumption and various expectation formation models. On a theoretical level, many possibilities exist. The relaxation of perfect foresight could take many forms. Also, we could consider models of imperfect competition, such as monopoly and duopoly in these environments. These may be of particular interest because experimental evidence in oligopoly situations tends to show very interesting price dynamics. Perhaps some insight into these dynamics could be gained by examining these situations in an overlapping generations context. Additionally, we could consider a model where the lengths of agents' lives are stochastic rather than deterministic, as was assumed here.

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Appendix A

Derivation of the Error Prone Agents Equilibrium

This appendix will present a partial calculation of the sequential equilibrium in the error prone agents model. Most of the strategies were found by using a set of *Mathematica*[1] programs. The equilibrium for the final three periods is presented here. Given the length of this appendix, it should be clear why all eight periods were not included. This appendix is intended to show how one constructs such equilibria. It does not pretend to be a complete development of the equilibrium.

As in the body of the paper, the payoff vectors used by CW in their $\Pr(Y) = .33$ experiments will be considered. As stated in Section 1.3, the payoffs to an X-type for repaying, reneging and not getting a loan are 60, 150 and 10 respectively. The payoffs to bankers in the same order are 40, -100 and 10. The bankers' update their beliefs about the type of entrepreneur they are facing according to (1.1) and (1.2). The indifference condition of a banker in period t is given in (1.4). Y-types still have a dominant strategy to repay. Development of the equilibrium proceeds through backward induction.

A.1 Period 8

It is clear that in the last period, X-types will not worry about reputation effects of their actions. They strictly prefer to renege in the absence of reputation effects, so $S_8 = 0$. Given that knowledge, the bankers' decision criterion can be simplified to a simple condition on P_7. Given that $S_8 = 0$, (1.4) leads to the following rule.

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For $P_8 > \frac{11/14-\epsilon/2}{1-\epsilon}$ B_8 wants to lend.

For $P_8 = \frac{11/14-\epsilon/2}{1-\epsilon}$ B_8 is indifferent.

For $P_8 < \frac{11/14-\epsilon/2}{1-\epsilon}$ B_8 does not want to lend.

Notice that for certain values of ϵ, no viable P_8 will give B_8 the inclination to lend. Consider the condition that will make this indifference threshold greater than one.

$$\begin{aligned} \frac{11/14-\epsilon/2}{1-\epsilon} &> 1 \\ 11/14-\epsilon/2 &> 1-\epsilon \\ \epsilon &> 3/7 \end{aligned} \tag{A.1}$$

This is mentioned now because this limit on ϵ will reappear and it is useful to know from where it came. The intuition behind it is that as ϵ increases, P_8 plays a diminishing role in B_8's expected payoff from granting a loan. Given the disutility B_8 receives from a default, as the probability of observing a default approaches one-half with a high ϵ, B_8 strictly prefers to refuse to lend.

A.2 Period 7

For X-types to have mixed strategies in period seven, it must be the case that B_8 is indifferent in order to balance the expected payoffs from either of the two actions available to X_7 (this is shorthand for an X-type taking action in period seven). Since B_8's indifference corresponds to a unique value for P_8, strategies based on each of the three updating rules can induce this indifference.

Start by considering the case where P_8^{PB} satisfies B_8's indifference condition. Setting P_8^{PB} from (1.1) equal to the level of P_8 at which B_8 is indifferent, and solving for S_7 will result in a mixed strategy that will lead B_8 to be indifferent following a repayment in period seven. This strategy will be called S_7^{PB}. Performing the suggested algebra results in:

$$S_7^{PB} = 1 - \frac{(1-\epsilon/2)[11/14 - \epsilon/2 - (1-\epsilon)P_7]}{(1-P_7)(11/14-\epsilon/2)(1-\epsilon)}. \tag{A.2}$$

Two more conditions must be met to fully specify this equilibrium path. First, the set of values of P_7 for which $0 \leq S_7^{PB} \leq 1$ must be specified. This can easily be shown to be the case for P_7 satisfying

$$\frac{(\epsilon/2)(11/14-\epsilon/2)}{(1-\epsilon)(3/14)} \leq P_7 \leq \frac{11/14-\epsilon/2}{1-\epsilon}. \tag{A.3}$$

The upper bound to this range is, by no coincidence, the belief at which B_8 would be indifferent. The intuition for this will be covered later. The second condition to check is that there exists a strategy for B_8 that will make X_7 indifferent between repaying and reneging. As shown in Theorem 1.3.2, the possibility of an error in period seven by the entrepreneur need not be considered. Let L_8^{PB} be the mixed strategy B_8 will play if he is indifferent following a repayment. Before this can be calculated, it is necessary to find what the reaction of B_8 would be following a renege by X_7.

For all valid S_7^{PB} except when $S_7^{PB} = 1$, Theorem 1.3.1 implies that B_8 will not intend to lend following a renege because P_8^{PB} makes B_8 indifferent and $P_8^{RN} < P_8^{PB}$. The case where $S_7^{PB} = 1$ will play an important role later, so it is not considered here. The expected payoff to reneging under S_7^{PB} is $150 + (1 - \epsilon/2)10 + (\epsilon/2)[(1 - \epsilon/2)150 + (\epsilon/2)60]$. The expected payoff to repaying is $60 + [(1-\epsilon)L_8^{PB} + \epsilon/2][(1-\epsilon/2)150 + (\epsilon/2)60] + [(1-\epsilon)(1-L_8^{PB}) + \epsilon/2]10$. L_8^{PB} must equate these two expected payoffs.

$$L_8^{PB} = \frac{18}{28 - 37\epsilon + 9\epsilon^2} \tag{A.4}$$

Note that for $\epsilon \geq .291$, B_8 would have to lend with a probability greater than one under L_8^{PB}. This is due to the fact that for such a high error rate, any gains that X_7 may get from repaying are outweighed by the expected loss from an error occurring in the next round. For $\epsilon \geq .291$, the expected payoff to reneging is increased due to the greater possibility of accidentally receiving a loan to the extent that it dominates repaying. Clearly, for this range of beliefs, $S_7 = 0$. So B_7 will have the same indifference condition B_8 had. But then X_6 will be facing a situation identical to X_7's, so X_6 will also always intend to renege. This propagates all the way to the beginning of the game. So for $\epsilon \geq .291$, X-types should always renege, and bankers will always base their lending decision on the criterion given for B_8. The rest of the appendix considers $\epsilon < .291$.

Next consider the case where B_8 is indifferent following a renege. This requires that P_8^{RN} satisfies B_8's indifference condition. X_7's strategy that meets this condition will be denoted by S_7^{RN}. Setting P_8^{RN} from (1.2) equal to B_8's indifference threshold and solving for S_7 produces

$$S_7^{RN} = 1 + \frac{(\epsilon/2)[11/14 - \epsilon/2 - (1-\epsilon)P_7]}{(1-P_7)(11/14 - \epsilon/2)(1-\epsilon)}. \tag{A.5}$$

This is a valid probability for P_7 satisfying

$$\frac{(11/14 - \epsilon/2)(1-\epsilon/2)}{(1-\epsilon)11/14} \geq P_7 \geq \frac{11/14 - \epsilon/2}{1-\epsilon}. \tag{A.6}$$

Notice that the lower bound here is the upper bound on P_7 for S_7^{PB} to be valid. At this critical level of beliefs, $S_7^{PB} = S_7^{RN} = 1$.

To find the strategy by B_8 that will support S_7^{RN}, it must first be determined what X_7's expected payoff is if she repays the loan. Theorem 1.3.1 shows that if a renege makes B_8 indifferent, then a repayment should imply that B_8 intends to lend. The payoff to repayment is $60+(1-\epsilon/2)[(1-\epsilon/2)150+(\epsilon/2)60]+(\epsilon/2)10$. In response to a renege, B_8 will intend to lend with probability L_8^{RN}. The expected payoff to X_7 from reneging is $150+[(1-\epsilon)L_8^{RN}+\epsilon/2][(1-\epsilon/2)150+(\epsilon/2)60]+[(1-\epsilon)(1-L_8^{RN})+\epsilon/2]10$. Setting these expected payoffs equal and solving for L_8^{RN} results in:

$$L_8^{RN} = \frac{10 - 37\epsilon + 9\epsilon^2}{28 - 37\epsilon + 9\epsilon^2} \tag{A.7}$$

For $\epsilon \geq .291$, B_8 would have to lend with negative probability in response to a renege to equate the two expected payoffs. This is the same value of ϵ at which L_8^{PB} became greater than one. The common factor is that in both cases, an increasing ϵ implies that X_7 would prefer to take the relatively certain payoff from the renege in period seven over repaying and hoping an error is not made in the future.

From (1.2), it is clear that P_8^{RN} is increasing in P_7. Consider the upper bound of the support of S_7^{RN}. At that point, $S_7 = 0$, and a renege leads B_8 to be indifferent. Since P_8^{RN} is increasing in P_7, it must be the case that for P_7 greater than that upper bound, B_8 would intend to lend following a renege. That implies that the equilibrium for these very high beliefs is $S_7 = 0$.

Similarly, from the first expression of P_{t+1}^{PB} in (1.1), P_8^{PB} is clearly increasing in P_7 and decreasing in S_7. So consider the lower bound of P_7 for which S_7^{PB} is valid. At that point, $S_7^{PB} = 0$ and B_8 is only indifferent following a repayment. For $S_7 = 0$, P_7 lower than this lower bound will imply that B_8 will not even want to lend following a repayment. Since P_8^{PB} is decreasing in S_7, there is no strategy that would make B_8 inclined to make a loan following a repayment. Theorem 1.3.1 implies that no loan will be intended following a renege either, so for these low beliefs, $S_7 = 0$. Table A.1 summarizes the possible equilibrium strategies for period seven.

Table A.1: X_7 and B_8's Equilibrium Strategies

P_7 lower	P_7 upper	S_7	L_8 after a renege	L_8 after a repayment
0	$\frac{(\epsilon/2)(11/14-\epsilon/2)}{(1-\epsilon)(3/14)}$	$S_7 = 0$	$L_8 = 0$	$L_8 = 0$
$\frac{(\epsilon/2)(11/14-\epsilon/2)}{(1-\epsilon)(3/14)}$	$\frac{11/14-\epsilon/2}{1-\epsilon}$	S_7^{PB}	$L_8 = 0$	L_8^{PB}
$\frac{11/14-\epsilon/2}{1-\epsilon}$	$\frac{(11/14-\epsilon/2)(1-\epsilon/2)}{(1-\epsilon)11/14}$	S_7^{RN}	L_8^{RN}	$L_8 = 1$
$\frac{(11/14-\epsilon/2)(1-\epsilon/2)}{(1-\epsilon)11/14}$	1	$S_7 = 0$	$L_8 = 1$	$L_8 = 1$

Now turn to the lending decision of B_7. This decision will naturally depend on the equilibrium strategy of X_7, so the lending decision needs to be considered under each of the three strategies: $S_7 = 0$, S_7^{PB} and S_7^{RN}. For

$S_7 = 0$, B_7's lending criterion will be the same as B_8's, except that B_7 will be concerned with P_7 instead of P_8. This is the case because bankers are one-shot players, so as long as the X-types equilibrium strategy is to always renege, the lending criterion will always be the same as B_8's. Table A.1 shows that the indifference level of P_7 is the point where equilibrium switches from S_7^{PB} to S_7^{RN}. This necessarily implies that this level of P_7 is above the upper bound of the first range where $S_7 = 0$ is the equilibrium and is below the lower bound of the second range where $S_7 = 0$. So when $S_7 = 0$ because of low P_7, B_7 should not intend to lend. For the high levels of P_7 for which the equilibrium also is $S_7 = 0$, B_7 should intend to lend.

For the path that includes S_7^{PB}, finding B_7's strategy is somewhat more difficult. The general form for banker indifference given in (1.4) with $S_7 = S_7^{PB}$ is

$$(1 - S_7^{PB})(1 - P_7)(1 - \epsilon) = 3/14 - \epsilon/2,$$

which can be shown to reduce to

$$\frac{(11/14)(11/14 - \epsilon/2)}{(1 - \epsilon/2)(1 - \epsilon)} = P_7. \tag{A.8}$$

Some simple algebra shows that for $\epsilon < 3/7$, the indifference threshold given by (A.8) is in the interior of the support of S_7^{PB}. This means that for certain values of P_7 for which S_7^{PB} is an equilibrium strategy for an X-type, B_7 *will not* intend to lend, for certain values B_7 *will* intend to lend and for a particular P_7 given on the left side of (A.8), B_7 will be indifferent. If the left side is less (greater) than the right side, then B_7 intends to (not) lend.

To calculate B_7's strategy along the path where S_7^{RN} is X_7's equilibrium strategy, one must evaluate (1.4) using S_7^{RN}. As above,

$$(1 - S_7^{RN})(1 - P_7)(1 - \epsilon) = 3/14 - \epsilon/2$$

can be shown to reduce to

$$P_7 = \frac{(3/14)(11/14 - \epsilon/2)}{(\epsilon/2)(1 - \epsilon)}. \tag{A.9}$$

If the left side is less (greater) than the right side, then B_7 intends to (not) lend. Note that for S_7^{RN} there is an upper bound on P_7 for which B_7 will be willing to lend. This is because S_7^{RN} is decreasing in P_7. Once again, the constraint $\epsilon < 3/7$ shows up. This time, for $\epsilon < 3/7$, the right side of (A.9) is greater than the largest value of P_7 for which $0 \leq S_7^{RN} \leq 1$. So B_7 will intend to lend for all P_7 in the support of S_7^{RN}.

What happens at the transition points between the equilibrium strategies has been left unstated. For instance, it has not yet been specified how B_8 would respond to a renege when P_7 takes the value where $S_7^{PB} = S_7^{RN} = 1$. This omission is intentional. These transition points will be used to construct the strategies for period six. For now, just note that the strategies put forth are only valid away from the transition points.

Transition points aside, that completes the mapping from beliefs to strategies for period seven. One important thing to keep in mind is that the only equilibrium in which B_7 is indifferent is when X_7 plays S_7^{PB} and P_7 satisfies (A.8). Table A.2 summarizes this mapping. In addition, it introduces

Table A.2: X_7 and B_7's Equilibrium Strategies

P_7 lower	P_7 upper	S_7	L_7	$E\pi_7[S_7, L_7]$
0	$\frac{(\epsilon/2)(11/14-\epsilon/2)}{(1-\epsilon)(3/14)}$	$S_7 = 0$	$L_7 = 0$	$E\pi_7[0,0]$
$\frac{(\epsilon/2)(11/14-\epsilon/2)}{(1-\epsilon)(3/14)}$	$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon/2)(1-\epsilon)}$	S_7^{PB}	$L_7 = 0$	$E\pi_7[S_7^{PB}, 0]$
$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon/2)(1-\epsilon)}$	$\frac{11/14-\epsilon/2}{1-\epsilon}$	S_7^{PB}	$L_7 = 1$	$E\pi_7[S_7^{PB}, 1]$
$\frac{11/14-\epsilon/2}{1-\epsilon}$	$\frac{(11/14-\epsilon/2)(1-\epsilon/2)}{(1-\epsilon)11/14}$	S_7^{RN}	$L_7 = 1$	$E\pi_7[S_7^{RN}, 1]$
$\frac{(11/14-\epsilon/2)(1-\epsilon/2)}{(1-\epsilon)11/14}$	1	$S_7 = 0$	$L_7 = 1$	$E\pi_7[0,1]$

some shorthand to be used in the development of period six strategies. Let $E\pi_7[S_7, L_7]$ be the a priori expected payoff to X_7 from having a P_7 such that her strategy is S_7 and B_7's strategy is L_7.

$$\begin{aligned}
E\pi_7[0,0] &= (1-\epsilon/2)\{10 + (1-\epsilon/2)10 + (\epsilon/2)[(1-\epsilon/2)150 + (\epsilon/2)60]\} \\
&\quad +(\epsilon/2)\{(1-\epsilon/2)150 + (\epsilon/2)60 + (1-\epsilon/2)10 \\
&\quad +(\epsilon/2)[(1-\epsilon/2)150 + (\epsilon/2)60]\} \\
&= 20 + 140\epsilon - 45\epsilon^2 \\
E\pi_7[S_7^{PB}, 0] &= (1-\epsilon/2)\{10 + (1-\epsilon/2)10 + (\epsilon/2)[(1-\epsilon/2)150 + (\epsilon/2)60]\} \\
&\quad +(\epsilon/2)\{150 + (1-\epsilon/2)10 + (\epsilon/2)[(1-\epsilon/2)150 + (\epsilon/2)60]\} \\
&= 20 + 140\epsilon - (45/2)\epsilon^2 \\
E\pi_7[S_7^{PB}, 1] &= (\epsilon/2)\{10 + (1-\epsilon/2)10 + (\epsilon/2)[(1-\epsilon/2)150 + (\epsilon/2)60]\} \\
&\quad +(1-\epsilon/2)\{150 + (1-\epsilon/2)10 + (\epsilon/2)[(1-\epsilon/2)150 + (\epsilon/2)60]\} \\
&= 160 - (45/2)\epsilon^2
\end{aligned}$$

$$
\begin{aligned}
E\pi_7[S_7^{RN},1] &= (\epsilon/2)\{10+(1-\epsilon/2)[(1-\epsilon/2)150+(\epsilon/2)60]+(\epsilon/2)10\} \\
&\quad +(1-\epsilon/2)\{60+(1-\epsilon/2)[(1-\epsilon/2)150+(\epsilon/2)60]+(\epsilon/2)10\} \\
&= 210-140\epsilon+(45/2)\epsilon^2 \\
E\pi_7[0,1] &= (\epsilon/2)\{10+(1-\epsilon/2)[(1-\epsilon/2)150+(\epsilon/2)60]+(\epsilon/2)10\} \\
&\quad +(1-\epsilon/2)\{150+(1-\epsilon/2)[(1-\epsilon/2)150+(\epsilon/2)60]+(\epsilon/2)10\} \\
&= 300-185\epsilon+(45/2)\epsilon^2
\end{aligned}
$$

Each of these expected payoffs corresponds to a range of P_7. They are currently listed as if sorted from low values of P_7 to high values. The expected payoffs have the desirable characteristic that they are non-decreasing in P_7.

It may seem odd that $E\pi_7[0,0] \neq E\pi_7[S_7^{PB},0]$, but there is a good reason. In calculating $E\pi_7[0,0]$ if B_7 accidentally lends, then X_7 should intend to renege. However, it is possible that X_7 will unintentionally repay the loan, which will not convince B_8 to lend. On the other hand, $E\pi_7[S_7^{PB},0]$ factors in the effect that B_8 will mix in response to a repayment. This is why $E\pi_7[0,0] \leq E\pi_7[S_7^{PB},0]$, although the difference is small because it is caused by the low probability event that an errant loan is followed by an errant repayment.

A.3 Period 6

Once again there are strategies such that if X_6 repays or reneges, B_7 will be indifferent. These strategies correspond directly to S_7^{PB} and S_7^{RN} and thus are called S_6^{PB} and S_6^{RN}. These strategies come from the conditions that

P_7^{PB} and P_7^{RN} lead to B_7 being indifferent as specified in (A.8).

Solving for the S_6 that allows P_7^{PB} to satisfy (A.8) results in S_6^{PB}.

$$S_6^{PB} = 1 - \frac{(1-\epsilon/2)[(11/14)(11/14-\epsilon/2) - (1-\epsilon/2)(1-\epsilon)P_6]}{(11/14)(1-P_6)(11/14-\epsilon/2)(1-\epsilon)} \qquad \text{(A.10)}$$

It is easy to show that $0 \leq S_6^{PB} \leq 1$ for P_6 satisfying

$$\frac{(\epsilon/2)(11/14)(11/14-\epsilon/2)}{(1-\epsilon)[(1-\epsilon/2)^2 - (11/14)(11/14-\epsilon/2)]} \leq P_6 \leq \frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)(1-\epsilon/2)}.$$

To determine the strategy to be played by B_7 to support this, consider the payoffs X_6 will expect from either of her two action choices. If X_6 reneges under S_6^{PB}, B_7 will not want to lend according to Theorem 1.3.1. P_7^{RN} can take on values such that if B_7 does accidentally lend, X_7 should either play $S_7 = 0$ or S_7^{PB}. To determine for what values of P_6 this will be the case, consider P_7^{RN} knowing that S_6^{PB} was the equilibrium strategy in period six. Given that S_6 is known, P_7^{RN} is only a function of P_6 and ϵ. The P_6 such that P_7^{RN} is the lower bound of the support of S_7^{PB} can be shown to be

$$P_6 = \frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)[(1-\epsilon/2)^2 + (11/14)(11/14-\epsilon/2)]}. \qquad \text{(A.11)}$$

For $\epsilon < 3/7$, this P_6 is in the interval of P_6 for which $S_6^{PB} \in [0,1]$. So a renege from S_6^{PB} can lead to either $E\pi_7[0,0]$ or $E\pi_7[S_7^{PB},0]$. For any P_6 for which S_6^{PB} is valid, a repayment will lead to S_7^{PB}.

Let $L_7^{PB,0}$ be the probability that B_7 will grant a loan following a renege, given that X_6 is playing S_6^{PB} and would play S_7=0 if a loan were granted following a renege. This is found by setting the expected payoff to reneging, $150 + E\pi_7[0,0]$, equal to the expected payoff to repaying, $60+L_7^{PB,0}E\pi_7[S_7^{PB},1]+(1-L_7^{PB,0})E\pi_7[S_7^{PB},0]$. The $L_7^{PB,0}$ that satisfies this condition is

$$L_7^{PB,0} = \frac{9(1-\epsilon/2)^2}{14(1-\epsilon)}. \tag{A.12}$$

Similarly, let $L_7^{PB,PB}$ be the probability that B_7 will lend following a renege, given that X_6 is following S_6^{PB} and would play S_7^{PB} if a loan were granted following a renege. The expected payoff to reneging is now $150 + E\pi_7[S_7^{PB},0]$ and the payoff to repaying is $60 + L_7^{PB,PB}E\pi_7[S_7^{PB},1] + (1 - L_7^{PB,PB})E\pi_7[S_7^{PB},0]$. Equating these two expected payoffs results in

$$L_7^{PB,PB} = \frac{9}{14(1-\epsilon)}. \tag{A.13}$$

To solve for S_6^{RN}, find the S_6 that will make P_7^{RN} satisfy (A.8). The unique S_6 that satisfies this condition is

$$S_6^{RN} = 1 + \frac{(\epsilon/2)[(11/14)(11/14-\epsilon/2)-(1-\epsilon)(1-\epsilon/2)P_6]}{(11/14)(1-P_6)(11/14-\epsilon/2)(1-\epsilon)}. \tag{A.14}$$

This is valid for P_6 satisfying

$$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)(1-\epsilon/2)} \leq P_6 \leq \frac{(11/14)(11/14-\epsilon/2)(1-\epsilon/2)}{(1-\epsilon)[(\epsilon/2)(1-\epsilon/2)+(11/14)(11/14-\epsilon/2)]}.$$

Once again, the lower bound here is the upper bound on beliefs for which S_6^{PB} is valid. Both strategies have X_6 strictly intending to repay at this transition point. It will turn out that for a certain range of beliefs toward the high end of this range, no mixed strategy by B_7 will make X_6 indifferent between reneging and repaying.

When solving for B_7's response to a repayment under S_6^{PB}, it first had to be specified what would happen in case of a renege. Now that a strategy that will make B_7 indifferent following a renege is under consideration, it must first be determined what happens following a repayment. Remembering that bankers cannot update beliefs if both X- and Y-types are strictly repaying, a quick glance at Figure 1.1 shows that a repayment can lead to S_7^{PB}. If no updating happens at the apex of S_6, then a repayment would certainly imply S_7^{PB}, and P_7^{PB} is continuous in S_6, so S_7^{PB} is a possibility. It is also clear that S_7^{RN} could also be played following a renege. Solving for the P_6 such that P_7^{PB} is equal to the transition point from S_7^{PB} to S_7^{RN} results in the following

$$P_6 = \frac{(11/14)(11/14 - \epsilon/2)}{(1-\epsilon)[11/14 + (3/28)\epsilon - (7/28)\epsilon^2]}. \tag{A.15}$$

Consistent with the intuition from Figure 1.1, it can be shown that for $\epsilon < 3/7$, this is in the range of P_6 for which $S_6^{RN} \in [0,1]$.

First consider the case where a repayment under S_6^{RN} would lead to S_7^{PB}. Theorem 1.3.1 implies that since B_7 is indifferent following a renege, he must prefer to lend following a repayment for $S_6^{RN} < 1$. Once again, discussion of equilibrium at the transition point will be postponed. Repayment carries the expected payoff of $60 + E\pi[S_7^{PB}, 1]$. Let $L_7^{RN,PB}$ be the probability with which B_7 intends to lend following a renege by X_6, who is playing S_6^{RN} and who will play S_7^{PB} if a loan is granted after repayment. The expected payoff to reneging is $150 + L_7^{RN,PB} E\pi_7[S_7^{PB}, 1] + (1 - L_7^{RN,PB}) E\pi_7[S_7^{PB}, 0]$. Making

X_6 indifferent requires the following $L_7^{RN,PB}$.

$$L_7^{RN,PB} = \frac{5/14 - \epsilon}{1 - \epsilon} \tag{A.16}$$

If repayment leads to S_7^{RN}, then the expected payoff to repaying is $60 + E\pi_7[S_7^{RN}, 1]$ and from reneging is $150 + L_7^{RN,RN} E\pi_7[S_7^{PB}, 1] + (1 - L_7^{RN,RN}) E\pi_7[S_7^{PB}, 0]$. Solving for $L_7^{RN,RN}$ results in

$$L_7^{RN,RN} = \frac{20 - 56\epsilon + 9\epsilon^2}{28(1 - \epsilon)} \tag{A.17}$$

One further possibility is that repayment will push beliefs so high that X_7 can strictly renege and still expect B_8 to lend. It will quickly be shown that no $L_7 \in [0, 1]$ can support S_7^{RN} for such a case. This is the case for

$$P_6 > \frac{11(11 - 7\epsilon)}{121 - 23\epsilon - 147\epsilon^2 + 49\epsilon^3}.$$

This P_6 is in the support of S_6^{RN}. Once again, the expected payoff to reneging is $150 + L_7 E\pi_7[S_7^{PB}, 1] + (1 - L_7) E\pi_7[S_7^{PB}, 0]$, while X_6 would expect a payoff of $60 + E\pi_7[0, 1]$ from repayment. Solving for L_7 results in

$$L_7 = \frac{38 - 65\epsilon + 9\epsilon^2}{28(1 - \epsilon)}, \tag{A.18}$$

which is greater than one for $\epsilon < .291$, which is the range of ϵ over which L_8^{PB} was valid. So this implies that there must be some other equilibrium strategy in this range of beliefs. Since B_7 is indifferent in this range only when X_6 reneges under S_6^{RN} and P_7^{RN} is increasing in S_6, a pure strategy to renege would lead one to not expect a loan in period seven. However, if the equilibrium were $S_6 = 0$, since P_7^{PB} is decreasing in S_6, a repayment would still push beliefs so high that X_7 would be able to renege and still get a loan in period eight. The payoff to repaying is far greater for the range of ϵ being considered, so X_6 would deviate from $S_6 = 0$ for these beliefs. Theorem 1.3.3 shows that $S_6 = 1$ can only be an equilibrium for $\epsilon > 0$ if B_7 is indifferent following a renege or repayment (remember, for $S_t = 1$ $P_{t+1}^{PB} = P_{t+1}^{PB} = P_t$). But the only strategy that makes B_7 indifferent following repayment is S_6^{PB} which is greater than one for this set of beliefs. So no pure strategy can be an equilibrium in this range, nor can any mixed strategy that depends on the indifference of B_7 be supported.

It was mentioned earlier that the transition points between strategies would play an important role. That role will now be described. Consider first the P_7 for which $S_7^{RN} = 0$. X_7 has a pure strategy to renege, but reneging will lead to B_8 being indifferent. So B_8 can play any mixed strategy as long as X_7 still prefers reneging to repaying. Consider a strategy, S_6^{JOG}, such that P_7^{PB} implies $S_7^{RN} = 0$. Solving for this strategy gives

$$S_6^{JOG} = \frac{(\epsilon/2)[(1-\epsilon)P_6 - 11/14 + \epsilon/2]}{(1-\epsilon)(11/14 - \epsilon/2)(1-P_6)}. \tag{A.19}$$

By construction, the probability of repaying under this strategy equals the probability of repaying under S_6^{RN} at the point where S_6^{RN} fails. A certain amount of path dependence must must be introduced to support this. If X_6 reneges while following this strategy, it can be shown that P_7^{RN} will be such that B_7 intends to lend and X_7 plays S_7^{PB}. The expected payoff to reneging is $150 + E\pi_7[S_7^{PB}, 1]$. When X_6 repays under this strategy, B_7 would clearly intend to lend, and $S_7 = 0$. Following a renege by X_7, B_8 would be indifferent and could mix such that X_6 is indifferent and X_7 weakly prefers to renege. Let $L_8^{JOG,6}$ be B_8's mixed strategy that supports S_6^{JOG}. Then X_6's expected payoff to repaying is

$$\begin{aligned} E\pi_6[PB|S_6^{JOG}] &= 60{+}(1{-}\epsilon/2)\langle(1{-}\epsilon/2)\{150 + [(1{-}\epsilon)L_8^{JOG} + \epsilon/2] \\ &\quad [(1{-}\epsilon/2)150 + (\epsilon/2)60] + [(1{-}\epsilon)(1 - L_8^{JOG,6}) + \epsilon/2]10\} \\ &\quad +(\epsilon/2)60 + (1{-}\epsilon/2)[(1{-}\epsilon/2)150 + (\epsilon/2)60] + (\epsilon/2)10\rangle \\ &\quad +(\epsilon/2)\{10 + (1{-}\epsilon/2)[(1{-}\epsilon/2)150 + (\epsilon/2)60] + (\epsilon/2)10\} \end{aligned}$$

Solving for $L_8^{JOG,6}$ produces

$$L_8^{JOG,6} = \frac{72 - 76\epsilon + 158\epsilon^2 - 73\epsilon^3 + 9\epsilon^4}{4(1-\epsilon/2)^2(1-\epsilon)(28-9\epsilon)}. \tag{A.20}$$

For all $\epsilon \in [0, 1]$, this will satisfy the condition that X_7 will prefer to renege after repaying in period six.

P_7^{RN} is increasing in both P_6 and S_6, and S_6^{JOG} is increasing in P_6, so it may be the case that a renege under S_6^{JOG} would lead to S_7^{RN}. This is true for

$$P_6 > \frac{11/14 - \epsilon/2}{(1-\epsilon)(11/14 + \epsilon/2)}. \tag{A.21}$$

So the expected payoff to reneging would be $150 + E\pi_7[S_7^{RN}, 1]$. The expected payoff to repaying would be as given above. This would require B_8 to mix with probability

$$L_8 = \frac{112 - 188\epsilon + 194\epsilon^2 - 73\epsilon^3 + 9\epsilon^4}{4(1-\epsilon/2)(28 - 37\epsilon + 9\epsilon^2)}. \tag{A.22}$$

This can easily be shown to be greater than one for all $\epsilon \in [0,1]$, so S_6^{JOG} fails at the P_6 given in (A.21).

The value of P_6 given by (A.21) has an interesting characteristic. If X-types' mixed strategies are written as functions of beliefs, then at this particular belief $S_6^{JOG}[P_6] = S_7^{RN}[P_6]$. This can be seen graphically in Figure 1.1. The point at which S_6^{JOG} intersects S_7^{RN} is exactly the belief at which a renege would lead to S_7^{RN}. This turns out to be less than coincidental. If S_7^{RN} is used as a mapping from beliefs in period *six* to probabilities of repaying in period *six*, P_7^{RN} would be such that $S_7 = S_7^{PB} = S_7^{RN} = 1$ and B_7 would intend to lend. Once again, X_6's mixed strategy in period six cannot be supported by B_7, but it can be supported by B_8. If X_6 reneges and then gets a loan, B_8 will be indifferent regardless of what X_7 does. B_8 must mix such that X_6 is indifferent and X_7 weakly prefers to repay.

The functional form for this strategy which shall be called $S_6^{RN,7}$ is already known; just replace P_7 with P_6 in S_7^{RN} to get

$$S_6^{RN,7} = 1 + \frac{(\epsilon/2)[11/14 - \epsilon/2 - (1-\epsilon)P_6]}{(1-P_6)(11/14 - \epsilon/2)(1-\epsilon)}. \tag{A.23}$$

Since B_8 would be indifferent following a renege, a repayment or even a refused loan, it would be possible to have him mix in response to any of these actions. Notation becomes somewhat tricky here, because of the path dependence of B_8's strategy. Let $L_8^{PB}[S_6^{RN,7}]$ be B_8's probability of lending given that X_6 reneged under $S_6^{RN,7}$ and repaid in period seven. $L_8^{RN}[S_6^{RN,7}]$ and $L_8^{NL}[S_6^{RN,7}]$ where NL stands for "no loan" are defined similarly. There is some latitude in picking these strategies because they only need to satisfy one equation and one weak inequality. Two somewhat arbitrary rules will be employed to help in the selection of these strategies. To take care of the case where B_8 is indifferent after B_7 did not lend, B_8 will just respond to the entrepreneur's previous move. In this case, B_8 is indifferent only after a renege by X_6, so following no loan in period seven, B_8 would lend with the same probability as if X_7 had reneged. The second rule says that if X_6 reneges and B_7 lends, B_8 will mix such that X_7 will be indifferent between reneging and repaying. This need not be the case; it is only required that X_7 weakly prefers repaying, which this rule will satisfy. This particular selection also preserves the weak monotonicity of X-types' expected payoffs in beliefs, for both periods six and seven.

The easiest equation to satisfy with these strategies is

$$L_8^{RN}[S_6^{RN,7}] = L_8^{NL}[S_6^{RN,7}]$$

which corresponds to my first assumption about the strategies. The second equation is the indifference between repaying and reneging in period seven given a renege in period six. Let $E\pi_7[PB|S_6^{RN,7}]$ be the expected payoff from successfully repaying in period seven along this path, and define

$E\pi_7[RN|S_6^{RN,7}]$ similarly.

$$E\pi_7[PB|S_6^{RN,7}] = 60+\{(1-\epsilon)L_8^{PB}[S_6^{RN,7}] + \epsilon/2\}[(1-\epsilon/2)150 + (\epsilon/2)60] + \{(1-\epsilon)(1 - L_8^{PB}[S_6^{RN,7}]) + \epsilon/2\}10$$

$$E\pi_7[RN|S_6^{RN,7}] = 150 + \{(1-\epsilon)L_8^{RN}[S_6^{RN,7}] + \epsilon/2\}[(1-\epsilon/2)150 + (\epsilon/2)60] + \{(1-\epsilon)(1 - L_8^{RN}[S_6^{RN,7}]) + \epsilon/2\}10$$

Finally, X_6 must be made indifferent between repaying and reneging. Let $E\pi_6[PB|S_6^{RN,7}]$ and $E\pi_6[RN|S_6^{RN,7}]$ be the expected payoffs along this path in period six.

$$\begin{aligned} E\pi_6[PB|S_6^{RN,7}] &= 60 + E\pi_7[0,1] \\ E\pi_6[RN|S_6^{RN,7}] &= 150 + (1-\epsilon/2)E\pi_7[PB|S_6^{RN,7}] \\ &\quad +(\epsilon/2)[(1-\epsilon)L_8^{NL}[S_6^{RN,7}][(1-\epsilon/2)150 + (\epsilon/2)60] \\ &\quad [(1-\epsilon)(1 - L_8^{NL}[S_6^{RN,7}])10] \end{aligned}$$

This is a set of three linear equations in three unknowns. The unique strategies satisfying these conditions are given below.

$$\begin{aligned} L_8^{PB}[S_6^{RN,7}] &= 1 \\ L_8^{RN}[S_6^{RN,7}] &= \frac{10 - 37\epsilon + 9\epsilon^2}{(1-\epsilon)(28 - 9\epsilon)} \\ L_8^{NL}[S_6^{RN,7}] &= L_8^{RN}[S_6^{RN,7}] \end{aligned}$$

For P_6 higher than those for which $S_6^{RN,7}$ is positive, X_6 should strictly renege. For any beliefs in this range, a repayment should imply an expected payoff for periods seven and eight of $E\pi_7[0, 1]$. When P_6 is toward the lower end of this range, a renege will imply S_7^{RN}. It is clear to see that this has a higher expectation. A repayment in period six gives X_6 a certain 60 and two payoffs of 150 that are subject to error. A renege makes one of the 150's certain, and subjects a 60 and a 150 to error. Reneging in period six clearly yields a better payoff. For some very high values of P_6, it is the case that P_7^{RN} will imply an expected payoff for the final two periods of $E\pi_7[0, 1]$. This clearly is better than repayment in period six.

Similarly, for very low values of P_6, X_6 should also intend to renege. For some P_6 in this range below the support of S_6^{PB}, a repayment would imply an expected payoff in the last two periods of $E\pi_7[S_7^{PB}, 0]$, but it is better to take the sure renege in period six than to hope for an accidental loan in period seven.

Much the same as in period seven, it can be shown that the only strategy for X_6 under which B_6 is indifferent is S_6^{PB}. The belief that causes that indifference will be calculated here, but the lack of indifference for all of the other strategies will not be shown.

Consider once again the general indifference condition of bankers given in (1.4):

$$(1 - S_6^{PB})(1 - P_6)(1 - \epsilon) = 3/14 - \epsilon/2.$$

Substituting S_6^{PB} for S_6 and simplifying gives the following expression for P_6

at which B_6 is indifferent:

$$p_6 = \frac{(11/14)^2(11/14 - \epsilon/2)}{(1-\epsilon/2)^2(1-\epsilon)}.$$

Table A.3 summarizes the mapping of P_6 to X-type strategies and expected payoffs, as well as B_6's strategies.

The expected payoffs for period six in terms of period seven's expected payoffs are given below. $E\pi_6[0,0] = (1-\epsilon/2)\{10 + E\pi_7[0,0]\} + (\epsilon/2)\{150 + E\pi_7[0,0]\}$

$E\pi_6[(S_6^{PB}, S_7 = 0), 0] = (1-\epsilon/2)\{10 + E\pi_7[0,0]\} + (\epsilon/2)\{150 + E\pi_7[0,0]\}$

$E\pi_6[(S_6^{PB}, S_7^{PB}), 0] = (1-\epsilon/2)\{10 + E\pi_7[0,0]\} + (\epsilon/2)\{150 + E\pi_7[S_7^{PB}, 0]\}$

$E\pi_6[(S_6^{RN}, S_7^{PB}), 1] = (1-\epsilon/2)\{60 + E\pi_7[S_7^{PB}, 1]\} + (\epsilon/2)\{10 + E\pi_7[S_7^{PB}, 1]\}$

$E\pi_6[(S_6^{RN}, S_7^{RN}), 1] = (1-\epsilon/2)\{60 + E\pi_7[S_7^{RN}, 1]\} + (\epsilon/2)\{10 + E\pi_7[S_7^{RN}, 1]\}$

$E\pi_6[S_6^{JOG}, 1] = (1-\epsilon/2)\{150 + E\pi_7[S_7^{PB}, 1]\} + (\epsilon/2)\{10 + E\pi_7[S_7^{RN}, 1]\}$

$E\pi_6[S_6^{RN,7}, 1] = (1-\epsilon/2)\{60 + E\pi_7[0,1]\} + (\epsilon/2)\{10 + E\pi_7[S_7^{RN}, 1]\}$

$E\pi_6[(S_6 = 0, S_7^{RN}), 1] = (1-\epsilon/2)\{150 + E\pi_7[S_7^{RN}, 1]\} + (\epsilon/2)\{10 + E\pi_7[0,1]\}$

$E\pi_6[(S_6 = 0, S_7 = 0), 1] = (1-\epsilon/2)\{150 + E\pi_7[0,1]\} + (\epsilon/2)\{10 + E\pi_7[0,1]\}$

It is clear to see why this appendix is limited to a partial derivation of the equilibrium. Even for this simple error process, solving for the closed form solution of the equilibrium quickly becomes quite complicated, both notationally and computationally. The most interesting characteristics of this equilibrium derivation are flushed out in this appendix. What is omitted is merely a matter of computation.

Table A.3: X_6 and B_6's Equilibrium Strategies

P_6 lower	P_6 upper	S_6	L_6	$E\pi_6[S_6, L_6]$
0	$\frac{(\epsilon/2)(11/14)(11/14-\epsilon/2)}{(1-\epsilon)[(1-\epsilon/2)^2-(11/14)(11/14-\epsilon/2)]}$	$S_6 = 0$	$L_6 = 0$	$E\pi_6[0, 0]$
$\frac{(\epsilon/2)(11/14)(11/14-\epsilon/2)}{(1-\epsilon)[(1-\epsilon/2)^2-(11/14)(11/14-\epsilon/2)]}$	$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)[(1-\epsilon/2)^2+(11/14)(11/14-\epsilon/2)]}$	S_6^{PB}	$L_6 = 0$	$E\pi_7[(S_6^{PB}, S_7 = 0), 0]$
$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)[(1-\epsilon/2)^2+(11/14)(11/14-\epsilon/2)]}$	$\frac{(11/14)^2(11/14-\epsilon/2)}{(1-\epsilon/2)^2(1-\epsilon)}$	S_6^{PB}	$L_6 = 0$	$E\pi_7[(S_6^{PB}, S_7^{PB}), 0]$
$\frac{(11/14)^2(11/14-\epsilon/2)}{(1-\epsilon/2)^2(1-\epsilon)}$	$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)(1-\epsilon/2)}$	S_6^{PB}	$L_6 = 1$	$E\pi_6[S_7^{PB}, 1]$
$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)(1-\epsilon/2)}$	$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)[(11/14+(3/28)\epsilon)-(7/28)\epsilon^2]}$	S_6^{RN}	$L_6 = 1$	$E\pi_6[(S_6^{RN}, S_7^{PB}), 1]$
$\frac{(11/14)(11/14-\epsilon/2)}{(1-\epsilon)[(11/14+(3/28)\epsilon)-(7/28)\epsilon^2]}$	$\frac{11(11-7\epsilon)}{121-23\epsilon-147\epsilon^2+49\epsilon^3}$	S_7^{RN}	$L_6 = 1$	$E\pi_6[[(S_6^{RN}, S_7^{RN}), 1]$
$\frac{11(11-7\epsilon)}{121-23\epsilon-147\epsilon^2+49\epsilon^3}$	$\frac{11/14-\epsilon/2}{(1-\epsilon)(11/14+\epsilon/2)}$	S_6^{JOG}	$L_6 = 1$	$E\pi_6[S_6^{JOG}, 1]$
$\frac{11/14-\epsilon/2}{(1-\epsilon)(11/14+\epsilon/2)}$	$\frac{(11/14-\epsilon/2)(1-\epsilon/2)}{(11/14)(1-\epsilon)}$	$S_6^{RN,7}$	$L_6 = 1$	$E\pi_6[S_6^{RN,7}, 1]$
$\frac{(11/14-\epsilon/2)(1-\epsilon/2)}{(11/14)(1-\epsilon)}$	$\frac{2(1-\epsilon/2)^2(11-7\epsilon)}{(1-\epsilon)(22-14\epsilon+7\epsilon^2)}$	$S_6 = 0$	$L_6 = 1$	$E\pi_6[(S_6 = 0, S_7^{RN}), 1]$
$\frac{2(1-\epsilon/2)^2(11-7\epsilon)}{(1-\epsilon)(22-14\epsilon+7\epsilon^2)}$	1	$S_6 = 0$	$L_6 = 1$	$E\pi_6[(S_6 = 0, S_7 = 0), 1]$