Descriptive set theory and the ergodic theory of countable groups

Thesis by

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Abstract

The primary focus of this thesis is on the interplay of descriptive set theory and the ergodic theory of group actions. This incorporates the study of turbulence and Borel reducibility on the one hand, and the theory of orbit equivalence and weak equivalence on the other. Chapter 2 is joint work with Clinton Conley and Alexander Kechris; we study measurable graph combinatorial invariants of group actions and employ the ultraproduct construction as a way of constructing various measure preserving actions with desirable properties. Chapter 3 is joint work with Lewis Bowen; we study the property MD of residually finite groups, and we prove a conjecture of Kechris by showing that under general hypotheses property MD is inherited by a group from one of its co-amenable subgroups. Chapter 4 is a study of weak equivalence. One of the main results answers a question of Abért and Elek by showing that within any free weak equivalence class the isomorphism relation does not admit classification by countable structures. The proof relies on affirming a conjecture of Ioana by showing that the product of a free action with a Bernoulli shift is weakly equivalent to the original action. Chapter 5 studies the relationship between mixing and freeness properties of measure preserving actions. Chapter 6 studies how approximation properties of ergodic actions and unitary representations are reflected group theoretically and also operator algebraically via a group's reduced C^*-algebra. Chapter 7 is an appendix which includes various results on mixing via filters and on Gaussian actions.

Contents

Chapter 1

Introduction

The questions addressed in this thesis lie at the interface of several fields including descriptive set theory, ergodic theory, representation theory, probability theory, and measurable group theory. A unified approach to studying these questions is facilitated by a global perspective which was initiated and greatly developed in [**Kec10**]. From this perspective, problems in ergodic theory may be seen as topological-dynamical and descriptive problems concerning continuous actions of the Polish group $\mathcal{A} = \mathcal{A}(X, \mu)$ of automorphisms of a standard (usually non-atomic) probability space (X, μ). Likewise, representation theory may be studied via continuous actions of the Polish group $\mathcal{U} = \mathcal{U}(\mathcal{H})$ of unitary operators on a separable (usually infinite-dimensional) Hilbert space $\mathcal{H}$.

More concretely, if Γ is a countable group then the set $A(\Gamma, X, \mu)$ of all measure preserving actions of Γ on (X, μ) naturally forms a Polish space on which $\mathcal{A}$ acts continuously by conjugation. What is significant here is that the natural ergodic theoretic notion of isomorphism ("conjugacy") of measure preserving actions of Γ is exactly the orbit equivalence relation generated by this action of the Polish group $\mathcal{A}$; analogous remarks hold for unitary representations of Γ and the Polish group $\mathcal{U}$. Descriptive set theorists have developed a general theory of *Borel reducibility*, which studies the set theoretic complexity of equivalence relations such as those arising from Polish group actions. Applications of this theory to actions of $\mathcal{A}$ and $\mathcal{U}$ have led to deep and surprising insights into the nature of conjugacy in ergodic theory and of unitary equivalence in representation theory. We begin with a brief introduction to the basic notions of this framework.

1. Borel reducibility and classification

If E and F are equivalence relations on standard Borel spaces X and Y, respectively, then E is called *Borel reducible* to F, denoted $E \leq_B F$, if there is a Borel map $\psi : X \to Y$ satisfying $xEy \Leftrightarrow \psi(x)F\psi(y)$ for all $x, y \in X$. Such a map ψ is called a *Borel reduction* from E to F. The substance of this notion lies in the requirement that this map be definable in some sense, and there are theoretical reasons for choosing Borel definability. The resulting richness of the ordering $\leq_B$ and its continuing success in comparing naturally occurring equivalence relations in mathematics may be taken as further justifications for this choice. A Borel reduction from E to F may be seen as providing an *explicitly definable* classification of elements of X up to E-equivalence using the F-classes as invariants.

An equivalence relation is said to be *classifiable by countable structures* if it is Borel reducible to the isomorphism relation on some standard Borel space of countable structures, for example, countable graphs, groups, or partial orders. More precisely, E admits classification by countable structures if there exists a countable language $\mathcal{L}$ and a Borel reduction from E to isomorphism on the standard Borel space $X_{\mathcal{L}}$ of all $\mathcal{L}$-structures with universe $\mathbb{N}$. A classical example of such a classification is the Halmos-von Neumann Theorem which completely classifies all ergodic measure preserving transformations with discrete spectrum, up to isomorphism, by their group of eigenvalues [**HvN42**]. Another example is Elliott's complete classification of unital AF-algebras by their pointed pre-ordered K_0-groups [**Ell76**], [**FTT11**]. On the other hand, Hjorth has isolated a dynamical property called *turbulence* that may hold of a Polish group action, and which is an obstruction to there being a classification by countable structures for the orbit equivalence relation of that action. In fact, turbulence is in a sense the only obstruction to the existence of such a classification [**Hjo02**].

2. Approximation and classification in the ergodic theory of countable groups

A *(probability-)measure preserving action* of a (discrete) countably infinite group Γ on (X, μ) is a homomorphism $a : \Gamma \to \mathcal{A}(X, \mu)$. The set of all measure preserving actions of Γ on (X, μ) naturally forms a Polish space $A(\Gamma, X, \mu)$ on which $\mathcal{A}$ acts continuously by coordinate-wise conjugation. The orbit $\mathcal{A} \cdot a$ of $a \in A(\Gamma, X, \mu)$ is called its *conjugacy class* and two actions a and b from $A(\Gamma, X, \mu)$ with the same conjugacy class are said to be *conjugate*. We say that b is *weakly contained* in a if it is in the closure of the conjugacy class of a, and we call a and b *weakly equivalent* if each weakly contains the other. If $a \in A(\Gamma, X, \mu)$ and $b \in A(\Gamma, Y, \nu)$ are actions with different underlying probabilities spaces then we say that b is weakly contained in a if it is a factor (i.e., quotient) of some $c \in A(\Gamma, X, \mu)$ that is weakly equivalent to a. The weak containment relation is reflexive and transitive, and weak equivalence is therefore an equivalence relation. Weak containment of measure preserving actions was introduced by Kechris in [**Kec10**] as an ergodic theoretic analogue of weak containment of unitary representations, and it has proven to be a remarkably robust notion that accurately captures an intuition that one measure preserving action asymptotically approximates or simulates another. Abért and Elek have recently defined a compact Polish topology on the set of weak equivalence classes in which many important invariants of weak equivalence become continuous functions [**AE11**], [**TD12c**]. A fundamental theorem regarding weak containment is due to Abért and Weiss and concerns the Bernoulli shift action of Γ which we now define.

Let Γ act on the set $[0, 1]^\Gamma$ of functions $f : \Gamma \to [0, 1]$ by shifting indices: $(\gamma \cdot f)(\delta) = f(\gamma^{-1}\delta)$. This action preserves the product measure ν^Γ where ν is Lebesgue measure, and we call this measure preserving action the *Bernoulli shift* of Γ and denoted it by s_Γ. The Bernoulli shift provides an ergodic theoretic counterpart to the left regular representation of Γ.

THEOREM 2.1 (Abért-Weiss [**AW11**]). *s_Γ is weakly contained in every free measure preserving action of Γ.*

Conversely, any measure preserving action weakly containing s_Γ must itself be free. Adrian Ioana conjectured that there is in fact an absorption principle at work which strengthens this.

CONJECTURE 2.2 (A. Ioana). *Let a be any free measure preserving action of a countably infinite group Γ. Then $s_\Gamma \times a$ is equivalent to a.*

Conjecture 2.2 strengthens Theorem 2.1 since the product action $s_\Gamma \times a$ is easily seen to weakly contain each of its factors. By combining ideas from [**AGV12**] with a close analysis of weak containment it is shown in Chapter 4 ([**TD12c**]) that an even more general absorption principle holds, of which Ioana's conjecture is a special case.

THEOREM 2.3 (T-D [**TD12c**]). *Conjecture 2.2 is true.*

Theorem 2.3 has interesting global consequences for the space $A(\Gamma, X, \mu)$, which are used in Chapter 4 to provide a strong negative answer to a question of Abért and Elek concerning the relationship between conjugacy and weak equivalence. Abért and Elek exhibited weak containment rigidity among E_0-ergodic profinite actions [**AE10**] and, prompted by the orbit equivalence superrigidity results of Popa, asked whether it is was possible to obtain full weak equivalence rigidity:

QUESTION 2.4 (Abért-Elek [**AE11**]). Does there exist a countably infinite group Γ with a free measure preserving action whose conjugacy class and weak equivalence class coincide?

Combining Theorem 2.3 with the work of Kerr, Li, and Pichot [**KLP10**] on turbulence in spaces of C^*-algebra representations, the following is shown:

THEOREM 2.5 (T-D [**TD12c**]). *Let a be any free measure preserving action of a countably infinite group Γ. Then the conjugacy relation on the weak equivalence class of a is not classifiable by countable structures.*

This implies that the weak equivalence class of a contains a continuum of conjugacy classes, and thus provides a negative answer to Question 2.4. But the conclusion is actually much stronger than this: there is no Borel way of assigning countable trees, groups, orderings, etc., as invariants to actions in the weak equivalence class of a that completely classifies these actions up to conjugacy.

3. Invariants of weak equivalence and measurable combinatorics

Theorem 2.5 shows that the degree to which countable invariants can provide meaningful distinctions, even within each weak equivalence class, is limited. Fortunately, the notion of weak equivalence turns out to be valuable in itself: many important properties of measure preserving actions have been shown to be invariants of weak equivalence. Furthermore, these invariants of weak equivalence usually turn out to exhibit interesting behavior under weak containment.

Many examples of this phenomenon arise in the study of measurable combinatorial invariants of measure preserving actions (another example is cost, discussed in §6 in this introduction). If Γ is a finitely generated group, then for any finite generating set S of $\Gamma \setminus \{e\}$ and action $a \in A(\Gamma, X, \mu)$ we consider the graph $G(S, a)$, with underlying vertex set X, and where x and y are connected by an edge if $s^a \cdot x = y$ or $s^a \cdot y = x$ for some $s \in S$. We let $E(S, a) \subseteq X \times X$ denote the set of edges of $G(S, a)$. Measurable combinatorial parameters are then associated to $G(S, a)$. For example:

(1) A subset $A \subseteq X$ of vertices is said to be *independent* in $G(S, a)$ if no two vertices in A are adjacent. The *independence number* of the graph $G(S, a)$, denoted $i_\mu(S, a)$, is then defined to be the supremum of the measures $\mu(A)$ as A ranges over measurable subsets of X which are independent in $G(S, a)$.
(2) The *measurable chromatic number* of $G(S, a)$, denoted $\chi_\mu(S, a)$ is the smallest natural number $k \in \mathbb{N}$ such that there exists a measurable function $c : X \to$

$\{0, 1, \ldots, k-1\}$ (called a *k-coloring*) assigning no two adjacent vertices the same value.[1]

(3) The *approximate chromatic number* of $G(S, a)$, denoted $\chi_\mu^{ap}(S, a)$ is the smallest natural number $k \in \mathbb{N}$ such that for every $\epsilon > 0$ there exists a Borel set $A \subseteq X$ with $\mu(A) > 1 - \epsilon$ along with a measurable coloring $c : A \to \{0, 1, \ldots, k-1\}$ of the induced subgraph $G(S, a) \upharpoonright A$.

(4) A *matching* of a graph G is a $M \subseteq E(G)$ of edges such that no two edges in M share a vertex. If M is a matching of $G(S, a)$ then we let X_M denote the set of matched vertices. The *matching number* of $G(S, a)$ is defined as $m_\mu(S, a) = \frac{1}{2} \sup_M \mu(X_M)$, where M ranges over all matchings of $G(S, a)$ which are measurable.

The parameters i_μ, χ_μ^{ap} and m_μ each respect weak containment: if a is weakly contained in b, then $i_\mu(S, a) \leq i_\mu(S, b)$, $\chi_\mu^{ap}(S, a) \geq \chi_\mu^{ap}(S, b)$ and $m_\mu(S, a) \leq m_\mu(S, b)$. In particular these parameters are invariants of weak equivalence.

Chapter 2 ([**CKTD11**]) is joint work with Clinton Conley and Alexander Kechris. We connect combinatorial properties of measure preserving actions to random graph-theoretic objects studied in probability theory. An *invariant random k-coloring* of an infinite countable graph G is a Borel probability measure on the compact space of k-colorings of G which is invariant under automorphisms of G. Using ultraproduct techniques we address a question raised by Aldous and Lyons [**AL07**] about the existence of invariant random colorings of Cayley graphs of groups.

THEOREM 3.1 (Conley-Kechris-T-D [**CKTD11**]). *Let Γ be a countably infinite group with finite generating set S. Let* Cay(Γ, S) *denote the Cayley graph of Γ with respect to S and let d denote the degree of* Cay(Γ, S) *(i.e., $d = |S \cup S^{-1}|$). Then* Cay(Γ, S) *admits and invariant random d-coloring.*

Aldous and Lyons had previously shown this to hold under the additional assumption that Γ is sofic.

[1] It is a non-trivial fact that this number is always finite.

4. Co-induction and weak containment

Chapter 3 is joint work with Lewis Bowen [**BTD11**]. A residually finite group Γ has property MD [**Kec12**] if the finite actions (i.e., actions coming from finite quotients of Γ) are dense in $A(\Gamma, X, \mu)$, and Γ has FD [**LS04**] if the finite representations are dense in the space $\mathrm{Rep}(\Gamma, \mathcal{H})$ of representations of Γ on $\mathcal{H}$. It is not difficult to show that MD implies FD, but the converse is unknown. It is known that free groups and residually finite amenable groups have MD [**Kec12**] and that MD is closed under taking subgroups [**Kec12**] and free products [**TD12c**]. The groups $SL_n(\mathbb{Z})$ for $n \geq 3$ are known to not have FD [**LS04**] and hence do not have MD.

In Chapter 3, Lewis Bowen and I answer affirmatively a question raised Kechris concerning the relationship between co-induced actions and weak containment. This leads to another closure property of MD which implies that surface groups have MD and - in light of the recent proof of the Virtual Fibration Conjecture by Agol [**AGM12**] - that fundamental groups of closed hyperbolic 3-manifolds have property MD.

5. Automatic freeness

The subject of non-free measure preserving actions has received significant attention recently, see, for example, [**AGV12, Bow12b, BGK12, CP12, Ele12, TD12c, TD12a, TD12b, Ver12, ABB$^+$11, AE11, Gri11, Ver11, BG04, SZ94**]. In [**SZ94**], Stuck and Zimmer proved a strong generalization of the Margulis Normal Subgroup Theorem for certain higher-rank semisimple Lie groups in terms of an automatic freeness property for many measure preserving actions of these groups. One consequence is that if Γ is an irreducible lattice in such a group then any non-atomic ergodic $a \in A(\Gamma, X, \mu)$ is *almost free*, i.e., there exists a finite normal subgroup N of Γ such that the stabilizer Γ_x of almost every $x \in X$ is equal to N. This is an example of automatic freeness at one extreme: by restricting considerably the group Γ, a minimal hypothesis on the action is needed to ensure that it is almost free. The main result of Chapter 5 ([**TD12a**]) is an automatic freeness result at the other extreme in which Γ is only assumed infinite, but a more serious ergodicity assumption is imposed on the action. A measure preserving action of Γ is called *totally ergodic* if each infinite subgroup of Γ acts ergodically and it is called *trivial* if the underlying measure is a point mass. The following is shown in Chapter 5:

THEOREM 5.1 (T-D [**TD12a**]). *All non-trivial totally ergodic actions of countably infinite groups are almost free. In particular, all non-trivial mixing actions and all non-trivial mildly mixing actions of countably infinite groups are almost free.*

This is new even for the case of mixing actions; Weiss had previously observed that actions of amenable groups with a much stronger mixing property called completely positive entropy are almost free. The total ergodicity assumption is close to optimal since there are examples due to Vershik [**Ver12**] of actions with mixing properties only slightly weaker than mild mixing, but which are *totally non-free*, which means that these examples are in some sense as far from free as possible. The most surprising aspect of Theorem 5.1 is that its proof ultimately relies on the Feit-Thompson odd order theorem from finite group theory! Indeed, the proof of Theorem 5.1 directly uses the group theoretic fact that every infinite locally finite group contains an infinite abelian subgroup, and all known proofs of this fact in turn rely on the Feit-Thompson theorem [**Kar63, HK64, Rob96**].

6. Expressions of non-amenability in ergodic theory and representation theory

Chapter 6 may be seen as an investigation into natural analogues of Theorem 5.1. These analogues turn out to have connections to well-known open questions about group C^*-algebras as well as to the theory of cost.

Amenable Invariant Random Subgroups The freeness properties of an action $a \in A(\Gamma, X, \mu)$ may be studied directly via that action's stabilizer distribution, obtained as the image of the measure μ under the stabilizer map $x \mapsto \Gamma_x$. This defines a Borel probability measure on the space of subgroups of Γ that is invariant under conjugation by elements of Γ. Any such probability measure is called an *invariant random subgroup* of Γ, so-named by Abért, Glasner, and Virag, who showed that every invariant random subgroup of Γ arises as the stabilizer distribution of some measure preserving action of Γ [**AGV12**]. Each normal subgroup of Γ is an invariant random subgroup when viewed as a Dirac distribution and many theorems originally concerning normal subgroups have been shown to generalize to invariant random subgroups, the Stuck-Zimmer Theorem being one prominent example. In what follows, an invariant random subgroup of Γ will be said to have a particular property if it has that property with probability 1.

OPEN QUESTION 6.1. Is every amenable invariant random subgroup of a countable group Γ contained in some amenable normal subgroup of Γ?

While this is open in general, Y. Glasner [**Gla12**] has obtained a positive answer for linear groups (see also the remark after (Diagram 0)). There is a useful way of restating Question 6.1 in terms of the amenable radical of a group. Day showed that every discrete group Γ contains a characteristic subgroup, called the *amenable radical* of Γ, denoted by AR_Γ, which is amenable and which contains all other amenable normal subgroups of Γ. Question 6.1 is then equivalent to the question of whether a countable group with trivial amenable radical has no non-trivial amenable invariant random subgroups.

Shift-minimality and C^*-simplicity If $\mathcal{C}$ is a class of groups then a measure preserving action of a group Γ is called $\mathcal{C}$-*ergodic* if each subgroup of Γ in $\mathcal{C}$ acts ergodically. An idea from the proof of Theorem 5.1 shows that if a non-trivial action of Γ is $\mathcal{NA}$-ergodic, where $\mathcal{NA}$ is the class of non-amenable groups, then the invariant random subgroup associated to this action is amenable. One may show that every measure preserving action weakly contained in the Bernoulli shift s_Γ is $\mathcal{NA}$-ergodic, and therefore any non-trivial action weakly contained in s_Γ gives rise to an amenable invariant random subgroup of Γ which will be non-trivial provided the original action was not free. Call a countable group Γ *shift-minimal* if every non-trivial action weakly contained in s_Γ is free.

OPEN QUESTION 6.2 (T-D). If the amenable radical of Γ is trivial then is Γ shift-minimal?

The Abért-Weiss characterization of free actions as those weakly containing s_Γ yields that Γ is shift-minimal if and only if every non-trivial action weakly contained in s_Γ is in fact weakly equivalent to s_Γ. It is well known that Γ is *C^*-simple*, i.e., the reduced C^*-algebra, $C^*_r(\Gamma)$, of Γ is simple, if and only if every non-zero unitary representation of Γ weakly contained in the left regular representation λ_Γ is actually weakly equivalent to λ_Γ [**dlH07**]. This is a tantalizing parallel, although there is no obvious implication between the two properties.

OPEN QUESTION 6.3 (T-D). Are all C^*-simple groups shift-minimal?

C^*-simplicity may be restated as a dynamical property of an action of the unitary group $\mathcal{U}(\mathcal{H})$, where $\mathcal{H} = \ell^2(\Gamma)$. The set $\mathrm{Irr}_\lambda(\Gamma, \mathcal{H})$ of all irreducible representations of Γ on $\mathcal{H}$ weakly contained in λ_Γ naturally forms a Polish space on which $\mathcal{U}(\mathcal{H})$ acts continuously by coordinate-wise conjugation. Then Γ is C^*-simple if and only if Γ is ICC and every unitary conjugacy class in $\mathrm{Irr}_\lambda(\Gamma, \mathcal{H})$ is dense.

Evidence suggests that C^*-simple groups should be shift-minimal. In Chapter 6 I show that shift-minimality of Γ follows from another property called the *unique trace property*, which means that $C^*_r(\Gamma)$ has a unique tracial state. In all known examples, the unique trace property and C^*-simplicity coincide, although it is open whether this is the case in general.

THEOREM 6.4 (T-D [**TD12b**]). *Groups with the unique trace property are shift-minimal. In fact, groups with the unique trace property have no non-trivial amenable invariant random subgroups.*

Powers [**Pow75**] demonstrated C^*-simplicity and the unique trace property for non-abelian free groups, and since then many large classes of groups have been shown to have both of these properties [**dLH85, BN88, B̀91, BCdLH94, AM07, dlH07, dlHP11**]. It is notable that in many cases, including the original argument of Powers, the proof given for a group's C^*-simplicity makes use of stronger hypotheses than the corresponding proof that the group has the unique trace property. The following diagram depicts the known implications among the five notions discussed. Any implication not addressed by the diagram is an open problem in general.

(Diagram 0)

$$\begin{tikzcd}[column sep=large]
C^*\text{-simple} \arrow[ddr, Rightarrow, bend right, "\textbf{[PS79]}"'] & \begin{array}{c}\text{No non-trivial}\\ \text{amenable IRS}\end{array} \arrow[d, Rightarrow, "\text{T-D}"'] & \text{Unique trace} \arrow[l, Rightarrow, "\text{T-D}"'] \arrow[dl, Rightarrow, bend left, "\text{T-D}"] \arrow[ddl, Rightarrow, bend left, "\textbf{[PS79]}"] \\
& \text{Shift-minimal} \arrow[d, Rightarrow, "\text{T-D}"'] & \\
& \begin{array}{c}\text{Trivial amenable}\\ \text{radical}\end{array} &
\end{tikzcd}$$

Theorem 6.4 and results of Poznansky [**Poz09**] imply these properties are all equivalent for linear groups.

Cost and the first ℓ^2-Betti number The second half of Chapter 6 connects shift-minimality and cost. The *cost* of a measure preserving countable Borel equivalence relation is a $[0, \infty]$-valued orbit equivalence invariant introduced by Levitt [**Lev95**] and then developed considerably by Gaboriau [**Gab00**]. The cost of a measure preserving action of Γ is defined to be the cost of the equivalence relation generated by this action. The cost of a group Γ, denoted $C(\Gamma)$, is then defined as the infimum of the costs of its free measure preserving actions. When Γ is infinite, then $C(\Gamma) \geq 1$. Γ is said to have *fixed price* r, where $r \geq 0$, if every free action of Γ has cost r. For example, infinite amenable groups have fixed price 1, and Gaboriau has shown the free group of rank n has fixed price n. A major open question in the area is whether every countable group has fixed price. This is known to be the case for many groups, but is open in general. The following is shown in Chapter 6.

THEOREM 6.5 (T-D [**TD12b**]). *If a countable group Γ does not have fixed price 1 then $\Gamma/\mathrm{AR}_\Gamma$ is shift-minimal. In addition, if $C(\Gamma) > 1$ then every non-trivial invariant random subgroup of $\Gamma/\mathrm{AR}_\Gamma$ of infinite index has cost ∞, and in particular $\Gamma/\mathrm{AR}_\Gamma$ has no non-trivial amenable invariant random subgroups.*

Results of Gaboriau imply AR_Γ is finite in the above situation. Part of the proof of the first statement in Theorem 6.5 involves extending a result of Kechris [**Kec10**], that cost respects weak containment in finitely generated groups, to the setting of general countable groups; one consequence is a characterization of countable groups with fixed price 1, previously shown to hold in the finitely generated case by Abért and Weiss: a countable group has fixed price 1 if and only if its Bernoulli shift has cost 1. The second statement is an analogue of a theorem of Bergeron and Gaboriau [**BG04**] about the first ℓ^2-Betti number.

Theorems 6.4 and 6.5 along with Bergeron and Gaboriau's result provide evidence for the following conjecture:

Conjecture 1: If Γ is a countably infinite group with positive ℓ^2-Betti number, then $\Gamma/\mathrm{AR}_\Gamma$ has the unique trace property.

It is known that $C(\Gamma) \geq \beta_1^{(2)}(\Gamma) + 1$ for any countably infinite group Γ, where $\beta_1^{(2)}(\Gamma)$ is the first ℓ^2-Betti number of Γ. It is an open problem whether this is actually an equality. Regardless, the hypothesis $\beta_1^{(2)}(\Gamma) > 0$ is at least as strong as the hypothesis $C(\Gamma) > 1$ from Theorem 6.5. Peterson and Thom [**PT11**] have shown that if Γ is torsion-free and satisfies an additional technical hypothesis, then Conjecture 1 holds. What they actually show is that groups satisfying their hypotheses have many free subgroups, and then C^*-simplicity and the unique trace property are easily deduced using a Powers-like argument from [**BCdLH94**]. If the additional technical hypothesis is dropped then their methods still show that Γ has rather strong paradoxicality properties.

In light of the connections between cost and invariant random subgroups, a proof of Conjecture 1 would add an interesting dimension to the relationship between cost and the first ℓ^2-Betti number.

Chapter 2

Ultraproducts of measure preserving actions and graph combinatorics

Clinton T. Conley, Alexander S. Kechris, and Robin D. Tucker-Drob

1. Introduction

In this paper we apply the method of ultraproducts to the study of graph combinatorics associated with measure preserving actions of infinite, countable groups, continuing the work in Conley-Kechris [**CK13**].

We employ the ultraproduct construction as a flexible method to produce measure preserving actions a of a countable group Γ on a standard measure space (X, μ) (i.e., a standard Borel space with its σ-algebra of Borel sets and a Borel probability measure) starting from a sequence of such actions a_n on $(X_n, \mu_n), n \in \mathbb{N}$. One uses a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ to generate the ultraproduct action $\prod_n a_n/\mathcal{U}$ of (a_n) on a measure space $(X_\mathcal{U}, \mu_\mathcal{U})$, obtained as the ultraproduct of $((X_n, \mu_n))$ via the Loeb measure construction. The measure algebra of the space $(X_\mathcal{U}, \mu_\mathcal{U})$ is non-separable but by taking appropriate countably generated subalgebras of this measure algebra one generates factors a of the action $\prod_n a_n/\mathcal{U}$ which are now actions of Γ on a standard measure space (X, μ) and which have various desirable properties.

In §2, we discuss the construction of the ultrapower $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$ of a sequence of standard measure spaces $(X_n, \mu_n), n \in \mathbb{N}$, with respect to a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, via the Loeb measure construction. We follow largely the exposition in Elek-Szegedy [**ES07**], which dealt with the case of finite spaces X_n with μ_n the counting measure.

In §3, we define the ultraproduct action $\prod_n a_n/\mathcal{U}$ on $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$ associated with a sequence $a_n, n \in \mathbb{N}$, of measure preserving actions of a countable group Γ on (X_n, μ_n) and discuss its freeness properties. When $a_n = a$ for all n, we put $a_{\mathcal{U}} = \prod_n a_n/\mathcal{U}$.

In §4, we characterize the factors of the action $\prod_n a_n/\mathcal{U}$ associated with countably generated σ-subalgebras of the measure algebra of $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$.

For a measure space (X, μ) and a countable group Γ, we denote by $A(\Gamma, X, \mu)$ the space of measure preserving actions of Γ on (X, μ) (where, as usual, actions are identified if they agree a.e.). This space carries the *weak topology* generated by the maps $a \in A(\Gamma, X, \mu) \mapsto \gamma^a \cdot A$ $(\gamma \in \Gamma, A \in \mathrm{MALG}_\mu)$, from $A(\Gamma, X, \mu)$ into the measure algebra MALG_μ (with the usual metric $d_\mu(A, B) = \mu(A \Delta B)$), and where we put $\gamma^a \cdot x = a(\gamma, x)$. When (X, μ) is standard, $A(\Gamma, X, \mu)$ is a Polish space.

If $a \in A(\Gamma, X, \mu), a_n \in A(\Gamma, X_n, \mu_n), n \in \mathbb{N}$, and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, we say that a is *weakly $\mathcal{U}$-contained* in (a_n),in symbols

$$a \prec_{\mathcal{U}} (a_n)$$

if for every finite $F \subseteq \Gamma, A_1, \dots, A_N \in \mathrm{MALG}_\mu, \epsilon > 0$, for $\mathcal{U}$-almost all n:

$$\exists B_{1,n} \dots \exists B_{N,n} \in \mathrm{MALG}_{\mu_n} \forall \gamma \in F \forall i, j \leq N$$

$$|\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_{i,n} \cap B_{j,n})| < \epsilon,$$

(where a property $P(n)$ is said to hold for *$\mathcal{U}$-almost all* n if $\{n \colon P(n)\} \in \mathcal{U}$). In case $a_n = b$ for all n, then $a \prec_{\mathcal{U}} (a_n) \Leftrightarrow a \prec b$ (in the sense of weak containment of actions, see Kechris [**Kec10**]).

If $a, b_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, we write

$$\lim_{n\to\mathcal{U}} b_n = a$$

if for each open nbhd V of a in $A(\Gamma, X, \mu), b_n \in V$, for $\mathcal{U}$-almost all n. Finally $a \cong b$ denotes isomorphism (conjugacy) of actions.

We show the following (in 4.3):

THEOREM 1. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let (X, μ), (X_n, μ_n), $n \in \mathbb{N}$ be non-atomic, standard measure spaces and let $a \in A(\Gamma, X, \mu)$, $a_n \in A(\Gamma, X_n, \mu_n)$. Then the following are equivalent:

(1) $a \prec_{\mathcal{U}} (a_n)$,

(2) a is a factor of $\prod_n a_n/\mathcal{U}$,

(3) $a = \lim_{n\to\mathcal{U}} b_n$, for some sequence (b_n), with

$$b_n \in A(\Gamma, X, \mu), b_n \cong a_n, \forall n \in \mathbb{N},$$

In particular, for $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu), a \prec b$ is equivalent to "a is a factor of $b_{\mathcal{U}}$". Moreover one has the following curious compactness property of $A(\Gamma, X, \mu)$ as a consequence of Theorem 1: If $a_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, then there is $n_0 < n_1 < n_2 < \dots$ and $b_{n_i} \in A(\Gamma, X, \mu), b_{n_i} \cong a_{n_i}$, such that (b_{n_i}) converges in $A(\Gamma, X, \mu)$.

In §5, we apply the ultraproduct construction to the study of combinatorial parameters associated to group actions. Given an infinite group Γ with a finite set of generators S, not containing 1, and given a free action a of Γ on a standard space (X, μ), the (simple, undirected) graph $G(S, a)$ has vertex set X and edge set $E(S, a)$, where

$$(x, y) \in E(S, a) \Leftrightarrow x \neq y \;\&\; \exists s \in S(s^a \cdot x = y \text{ or } s^a \cdot y = x).$$

As in Conley-Kechris [**CK13**], we define the associated parameters $\chi_\mu(S, a)$ (the *measurable chromatic number*), $\chi_\mu^{ap}(S, a)$ (the *approximate chromatic number*) and $i_\mu(S, a)$ (the *independence number*), as follows:

• $\chi_\mu(S,a)$ is the smallest cardinality of a standard Borel space Y for which there is a ($\mu-$)measurable coloring $c\colon X \to Y$ (i.e., $xE(S,a)y \Rightarrow c(x) \neq c(y)$).

• $\chi_\mu^{ap}(S,a)$ is the smallest cardinality of a standard Borel space Y such that for each $\epsilon > 0$, there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \epsilon$ and a measurable coloring $c\colon A \to Y$ of the induced subgraph $G(S,a)|A = (A, E(S,A) \cap A^2)$.

• $i_\mu(S,a)$ is the supremum of the measures of Borel independent sets, where $A \subseteq X$ is *independent* if no two elements of A are adjacent.

Given a (simple, undirected) graph $G = (X,E)$, where X is the set of vertices and E the set of edges, a *matching* in G is a subset $M \subseteq E$ such that no two edges in M have a common vertex. We denote by X_M the set of matched vertices, i.e., the set of vertices belonging to an edge in M. If $X_M = X$ we say that M is a *perfect matching*.

For a free action a of Γ as before, we also define the parameter

$$m(S,a) = \text{ the } \textit{matching number,}$$

where $m(S,a)$ is 1/2 of the supremum of $\mu(X_M)$, with M a Borel (as a subset of X^2) matching in $G(S,a)$. If $m(S,a) = 1/2$ and the supremum is attained, we say that $G(S,a)$ admits an *a.e. perfect matching*.

The parameters $i_\mu(S,a), m(S,a)$ are monotone increasing with respect to weak containment, while $\chi_\mu^{ap}(S,a)$ is decreasing. Below we let $a \sim_w b$ denote *weak equivalence* of actions, where $a \sim_w b \Leftrightarrow a \prec b \;\&\; b \prec a$, and we let $a \sqsubseteq b$ denote that a is a factor of b. We now have (see 5.2)

THEOREM 2. Let Γ be an infinite, countable group and S a finite set of generators. Then for any free action a of Γ on a non-atomic, standard measure space (X,μ), there is a free action b of Γ on (X,μ) such that

(i) $a \sim_w b$ and $a \sqsubseteq b$,

(ii) $\chi_\mu^{ap}(S,a) = \chi_\mu^{ap}(S,b) = \chi_\mu(S,b)$,

(iii) $i_\mu(S,a) = i_\mu(S,b)$ and $i_\mu(S,b)$ is attained,

(iv) $m(S,a) = m(S,b)$ and $m(S,b)$ is attained.

In §6, we study analogues of the classical Brooks' Theorem for finite graphs, which asserts that the chromatic number of a finite graph G with degree bounded by d is $\leq d$ unless $d = 2$ and G contains an odd cycle or $d \geq 3$ and G contains the complete subgraph with $d + 1$ vertices.

Let Γ, S be as in the preceding discussion, so that the graph $G(S, a)$ associated with a free action a of Γ on a standard space (X, μ) has degree $d = |S^{\pm 1}|$, where $S^{\pm 1} = S \cup S^{-1}$. It was shown in Conley-Kechris [**CK13**] that $\chi_{\mu}^{ap}(S, a) \leq d$, so one has an "approximate" version of Brooks' Theorem. Using this and the results of §5, we now have (see 6.11):

THEOREM 3. Let Γ be an infinite group and S a finite set of generators. Then for any free action a of Γ on a non-atomic, standard space (X, μ), there is a free action b on (X, μ) such that $a \sim_w b$ and $\chi_{\mu}(S, b) \leq d$ $(= |S^{\pm 1}|)$.

It is not the case that for *every* free action a of Γ we have $\chi_{\mu}(S, a) \leq d$, but the only counterexamples known are $\Gamma = \mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (with the usual sets of generators) and Conley-Kechris [**CK13**] show that these are the only counterexamples if Γ has finitely many ends.

The previous result can be used to answer a question in probability theory (see Aldons-Lyons [**AL07**]), namely whether for any Γ, S, there is an invariant, random d-coloring of the Cayley graph Cay(Γ, S) (an earlier result of Schramm (unpublished, 1997) shows that this is indeed the case with d replaced by $d + 1$). A *random d-coloring* is a probability measure on the Borel sets of the space of d-colorings of the Cayley graph Cay(Γ, S) and invariance refers to the canonical shift action of Γ on this space.

We now have (see 6.4):

THEOREM 4. Let Γ be an infinite group and S a finite set of generators with $d = |S^{\pm 1}|$. Then there is an invariant, random d-coloring. Moreover for any free action a of Γ on a non-atomic, standard space (X, μ), there is such a coloring weakly contained in a.

Let $G_{\Gamma, S}$ be the automorphism group of the Cayley graph $G(\Gamma, S)$ with the pointwise convergence topology. This is a Polish locally compact group containing Γ as a closed subgroup. One can consider invariant, random colorings under the canonical action of $G_{\Gamma,S}$ on the space of colorings, which we call $G_{\Gamma,S}$-*invariant, random colorings*. This appears as a stronger notion but we show in 6.6 that the existence of a $G_{\Gamma,S}$-invariant, random d-coloring is equivalent to the existence of an invariant, random d-coloring, so Theorem 4 works as well for $G_{\Gamma,S}$-invariant, random colorings.

One can also ask whether the last statement in Theorem 4 can be improved to "is a factor of" instead of "weakly contained in". This again fails for $\Gamma = \mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ and a the shift action of Γ on $[0,1]^\Gamma$, a case of primary interest, but holds for all other Γ that have finitely many ends. Moreover in the case of the shift action one has also $G_{\Gamma,S}$-invariance (see 6.7).

THEOREM 5. Let Γ be an infinite group and S a finite set of generators with $d = |S^{\pm 1}|$. If Γ has finitely many ends but is not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, then there is a $G_{\Gamma,S}$-invariant, random d-coloring which is a factor the shift action of $G_{\Gamma,S}$ on $[0,1]^\Gamma$.

In §7, we discuss various results about a.e. perfect matchings and invariant, random matchings. Lyons-Nazarov [**LN11**] showed that if Γ is a non-amenable group with a finite set of generators S and Cay(Γ, S) is bipartite (i.e., has no odd cycles), then there is a $G_{\Gamma,S}$-invariant, random perfect mateching of its Cayley graph, which is a factor of the shift action of $G_{\Gamma,S}$ on $[0,1]^\Gamma$. This also implies that $m(S, s_\Gamma) = \frac{1}{2}$, where s_Γ is the shift action of Γ on $[0,1]^\Gamma$, and in fact the graph associated with this action has an a.e. perfect matching. We do not know if $m(S,a) = \frac{1}{2}$ actually holds for *every* Γ, S and *every* free action a. We note in 7.4 that the only possible counterexamples are those Γ, S for which Γ is not amenable and S consists of elements of odd order. However we show in 7.7 the following:

THEOREM 6. Let $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ with the usual set of generators $S = \{s,t\}$, where $s^3 = t^3 = 1$. Then for any free action a of Γ on a non-atomic, standard measure space (X,μ), the associated graph $G(S,a)$ has an a.e. perfect matching.

In §8, we study independence numbers. In Conley-Kechris [**CK13**], the following was shown: Let Γ, S be as before. Then the set of independence numbers $i_\mu(S,a)$, as a varies over all free actions of Γ, is a closed interval. The question was raised about the structure of the set of all $i_\mu(S, a)$, where a varies over all free, *ergodic* actions of Γ. We show the following (in 8.1).

THEOREM 7. Let Γ be an infinite group with S a finite set of generators. If Γ has property (T), the set of $i_\mu(S, a)$ as a varies over all the free, ergodic actions of Γ is closed.

We do not know what happens if Γ does not have property (T).

In §9, we discuss the notion of sofic equivalence relations and sofic actions, recently introduced in Elek-Lippner [**EL10**]. We use ultraproducts and a result of Abért-Weiss [**AW11**] to give (in 9.6) an alternative proof of the theorem of Elek-Lippner [**EL10**] that the shift action of an infinite countable sofic group in sofic and discuss some classes of groups Γ for which *every* free action is sofic.

Elek-Lippner [**EL10**] raised the question of whether *every* free action of a sofic group is sofic.

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2. Preliminaries

We review here some standard terminology and notation that will be used throughout the paper.

(A) A *standard measure space* is a measure space (X, μ), where X is standard Borel space (i.e., a Polish space with its σ-algebra of Borel sets) and μ a probability measure on the σ-algebra $\boldsymbol{B}(X)$ of Borel sets. We do not assume in this paper that (X, μ) is non-atomic, since we do want to include in this definition also finite measure spaces. If (X, μ) is supposed to be non-atomic in a given context, this will be stated explicitly.

The *measure algebra* MALG_μ of a measure space (X, μ) is the Boolean σ-algebra of measurable sets modulo null sets equipped with the measure μ.

As a general convention in dealing with measure spaces, we will often neglect null sets, if there is no danger of confusion.

(B) If (X, μ) is a standard measure space and $E \subseteq X^2$ a countable Borel equivalence relation on X (i.e., one whose equivalence classes are countable), we say that E is *measure preserving* if for all Borel bijections $\varphi\colon A \to B$, where A, B are Borel subsets of X, such that $\varphi(x) E x$, μ-a.e.$(x \in A)$, we have that φ preserves the measure μ.

Such an equivalence relation is called *treeable* if there is a Borel acyclic graph on X whose connected components are the equivalence classes.

(C) If Γ is an infinite, countable group and S a finite set of generators, not containing 1, the *Cayley graph* Cay(Γ, S), is the (simple, undirected) graph with set of vertices Γ and in which $\gamma, \delta \in \Gamma$ are connected by an edge iff $\exists s \in S(\gamma s = \delta$ or $\delta s = \gamma)$.

Finally for such Γ, S the number of *ends* of Cay(Γ, S) is the supremum of the number of infinite components, when any finite set of vertices is removed. This number is independent of S and it is equal to $1, 2$ or ∞.

3. Ultraproducts of standard measure spaces

(A) Let $(X_n, \mu_n), n \in \mathbb{N}$, be a sequence of standard measure spaces and denote by $\boldsymbol{B}(X_n)$ the σ-algebra of Borel sets of X_n. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. For $P \subseteq \mathbb{N} \times X$ (X some set) we write

$$\mathcal{U}n P(n, x) \Leftrightarrow \{n\colon P(n, x)\} \in \mathcal{U}.$$

If $\mathcal{U}n P(n, x)$ we also say that for $\mathcal{U}$*-almost all* $n, P(n, x)$ holds. On $\prod_n X_n$ define the equivalence relation

$$(x_n) \sim_{\mathcal{U}} (y_n) \Leftrightarrow \mathcal{U}n(x_n = y_n),$$

let $[(x_n)]_{\mathcal{U}}$ be the $(\sim_{\mathcal{U}})$-equivalence class of (x_n) and put

$$X_{\mathcal{U}} = (\prod_n X_n)/\mathcal{U} = \{[(x_n)]_{\mathcal{U}}\colon (x_n) \in \prod_n X_n\}.$$

Given now $(A_n) \in \prod_n \boldsymbol{B}(X_n)$, we define $[(A_n)]_{\mathcal{U}} \subseteq X_{\mathcal{U}}$ by

$$[(x_n)]_{\mathcal{U}} \in [(A_n)]_{\mathcal{U}} \Leftrightarrow \mathcal{U}n(x_n \in A_n).$$

Note that

$$[(\sim A_n)]_{\mathcal{U}} = \sim [(A_n)]_{\mathcal{U}}$$

$$[(A_n \cup B_n)]_{\mathcal{U}} = [(A_n)]_{\mathcal{U}} \cup [(B_n)]_{\mathcal{U}}$$

$$[(A_n \cap B_n)]_{\mathcal{U}} = [(A_n)]_{\mathcal{U}} \cap [(B_n)]_{\mathcal{U}},$$

where $\sim$ denotes complementation. Put

$$\boldsymbol{B}^0_{\mathcal{U}} = \{[(A_n)]_{\mathcal{U}} \colon (A_n) \in \prod_n \boldsymbol{B}(X_n)\},$$

so that $\boldsymbol{B}^0_{\mathcal{U}}$ is a Boolean algebra of subsets of $X_{\mathcal{U}}$.

For $[(A_n)]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}$, put

$$\mu_{\mathcal{U}}([(A_n)]_{\mathcal{U}}) = \lim_{n \to \mathcal{U}} \mu_n(A_n),$$

where $\lim_{n \to \mathcal{U}} r_n$ denotes the ultrafilter limit of the sequence (r_n). It is easy to see that $\mu_{\mathcal{U}}$ is a finitely additive probability Borel measure on $\boldsymbol{B}^0_{\mathcal{U}}$. We will extend it to a (countably additive) probability measure on a σ-algebra containing $\boldsymbol{B}^0_{\mathcal{U}}$.

DEFINITION 3.1. A set $N \subseteq X_{\mathcal{U}}$ is *null* if $\forall \epsilon > 0 \exists A \in \boldsymbol{B}^0_{\mathcal{U}}$ ($N \subseteq A$ and $\mu_{\mathcal{U}}(A) < \epsilon$). Denote by $\boldsymbol{N}$ the collection of null sets.

PROPOSITION 3.2. The collection $\boldsymbol{N}$ is a σ-ideal of subsets of $X_{\mathcal{U}}$.

Proof. It is clear that $\boldsymbol{N}$ is closed under subsets. We will now show that it is closed under countable unions.

LEMMA 3.3. Let $A^i \in \boldsymbol{B}^0_{\mathcal{U}}, i \in \mathbb{N}$, and assume that $\lim_{m \to \infty} \mu_{\mathcal{U}}(\bigcup_{i=0}^m A^i) = t$. Then there is $A \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\mu_{\mathcal{U}}(A) = t$ and $\bigcup_i A^i \subseteq A$.

Granting this let $N^i \in \boldsymbol{N}, i \in \mathbb{N}, \epsilon > 0$ be given. Let $N^i \subseteq A^i \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\mu_{\mathcal{U}}(A^i) \leq \epsilon/2^i$. Then $\mu_{\mathcal{U}}(\bigcup_{i=0}^m A^i) \leq \epsilon$ and $\mu_{\mathcal{U}}(\bigcup_{i=0}^m A^i)$ increases with m. So

$$\lim_{m \to \mathcal{U}} \mu_{\mathcal{U}}(\bigcup_{i=0}^{m} A^i) = t \leq \epsilon$$

and by the lemma there is $A \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\mu_{\mathcal{U}}(A) \leq \epsilon$ and $\bigcup_i N^i \subseteq \bigcup_i A^i \subseteq A$. So $\bigcup_i N^i$ is null.

Proof of 2.3. Put $B^m = \bigcup_{i=0}^m A^i$, so that $\mu_{\mathcal{U}}(B^m) = t_m \to t$. Let $A^i = [(A^i_n)]_{\mathcal{U}}$, so that $B^m = [(B^m_n)]_{\mathcal{U}}$, with $B^m_n = \bigcup_{i=0}^m A^i_n$. Let

$$T_m = \left\{ n \geq m \colon |\mu_n(B^m_n) - t_m| \leq \frac{1}{2^m} \right\},$$

so that $\bigcap_m T_m = \varnothing$ and $T_m \in \mathcal{U}$, as $t_m = \mu_{\mathcal{U}}(B^m) = \lim_{n \to \mathcal{U}} \mu_n(B^m_n)$.

Let $m(n) =$ largest m such that $n \in \bigcap_{\ell \leq m} T_m$. Then $m(n) \to \infty$ as $n \to \mathcal{U}$, since for each M, $\{n \colon m(n) \geq M\} \supseteq \bigcap_{m=0}^M T_m \in \mathcal{U}$. Also $n \in T_{m(n)}$. So

$$|\mu_{m(n)}(B^{m(n)}_n) - t_{m(n)}| \leq \frac{1}{2^{m(n)}},$$

thus

$$\lim_{n \to \mathcal{U}} \mu_n(B^{m(n)}_n) = t.$$

Let $A = [(B^{m(n)}_n)]_{\mathcal{U}}$. Then $\mu_{\mathcal{U}}(A) = t$. Also for each i,

$$\{n \colon A^i_n \subseteq B^{m(n)}_n\} \supseteq \{n \colon m(n) \geq i\} \in \mathcal{U},$$

so $A^i = [(A^i_n)]_{\mathcal{U}} \subseteq [(B^{m(n)}_n)]_{\mathcal{U}} = A$, thus $\bigcup_i A^i \subseteq A$. $\dashv$

Put

$$\boldsymbol{B}_{\mathcal{U}} = \{A \subseteq X_{\mathcal{U}} \colon \exists A' \in \boldsymbol{B}^0_{\mathcal{U}} (A \Delta A' \in \boldsymbol{N})\},$$

and for $A \in \boldsymbol{B}_{\mathcal{U}}$ put

$$\mu_{\mathcal{U}}(A) = \mu_{\mathcal{U}}(A')$$

where $A' \in \boldsymbol{B}^0_{\mathcal{U}}$, $A \Delta A' \in \boldsymbol{N}$. This is clearly well-defined and agrees with $\mu_{\mathcal{U}}$ on $\boldsymbol{B}^0_{\mathcal{U}}$.

PROPOSITION 3.4. The class $\boldsymbol{B}_{\mathcal{U}}$ is a σ-algrebra of subsets of $X_{\mathcal{U}}$ containing $\boldsymbol{B}^0_{\mathcal{U}}$ and $\mu_{\mathcal{U}}$ is a probability measure on $\boldsymbol{B}_{\mathcal{U}}$.

Proof. It is easy to see that $\boldsymbol{B}_{\mathcal{U}}$ is a Boolean algebra containing $\boldsymbol{B}^0_{\mathcal{U}}$ and $\mu_{\mathcal{U}}$ is a finitely additive probability measure on $\boldsymbol{B}_{\mathcal{U}}$. It only remains to show that if $A_n \in \boldsymbol{B}_{\mathcal{U}}, n \in \mathbb{N}$, are pairwise disjoint, then $\bigcup_n A_n \in \boldsymbol{B}_{\mathcal{U}}$ and $\mu_{\mathcal{U}}(\bigcup_n A_n) = \sum_n \mu_{\mathcal{U}}(A_n)$.

For $A, A' \in B_{\mathcal{U}}$, let

$$A \equiv A' \Leftrightarrow A \Delta A' \in \boldsymbol{N}.$$

Let now $A'_n \in \boldsymbol{B}^0_{\mathcal{U}}$ be such that $A_n \equiv A'_n$. By disjointifying, we can assume that the A'_n are disjoint. Note also that $\bigcup_n A_n \equiv \bigcup_n A'_n$. It is thus enough to find $A' \in \boldsymbol{B}^0_{\mathcal{U}}$ with $A' \equiv \bigcup_n A'_n$ and $\mu_{\mathcal{U}}(A') = \sum_n \mu_{\mathcal{U}}(A'_n)$ $(= \sum_n \mu_{\mathcal{U}}(A_n))$.

By Lemma 2.3, there is $A' \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\bigcup_n A'_n \subseteq A'$ and $\mu_{\mathcal{U}}(A') = \sum_n \mu_{\mathcal{U}}(A'_n)$. Then for each N,

$$A' \setminus \bigcup_n A'_n \subseteq A' \setminus \bigcup_{n=0}^{N} A'_n \in \boldsymbol{B}^0_{\mathcal{U}}$$

and

$$\mu_{\mathcal{U}}(A' \setminus \bigcup_{n=0}^{N} A'_n) = \mu_{\mathcal{U}}(A') - \sum_{n=0}^{N} \mu_{\mathcal{U}}(A'_n) \to 0$$

as $N \to \infty$. So

$$A' \Delta \bigcup_n A'_n = A' \setminus \bigcup_n A'_n \in \boldsymbol{N}$$

i.e., $A' \equiv \bigcup_n A'_n$. $\dashv$

Finally, note that for $A \in \boldsymbol{B}_{\mathcal{U}}, \mu_{\mathcal{U}}(A) = 0 \Leftrightarrow A \in \boldsymbol{N}$.

(B) The following is straightforward.

PROPOSITION 3.5. The measure $\mu_{\mathcal{U}}$ is non-atomic if and only if $\forall \epsilon > 0\ \forall (A_n) \in \prod_n \boldsymbol{B}(X_n)\ \big((\mathcal{U}n(\mu_n(A_n) \geq \epsilon) \Rightarrow \exists \delta > 0\ \exists (B_n) \in \prod_n \boldsymbol{B}(X_n)\ [\mathcal{U}n(B_n \subseteq A_n\ \&\ \delta \leq \mu_n(B_n), \mu_n(A_n \setminus B_n))]\big)$.

For example, this condition is satisfied if each (X_n, μ_n) is non-atomic or if each X_n is finite, μ_n is normalized counting measure and $\lim_{n \to \mathcal{U}} \mathrm{card}(X_n) = \infty$.

Let $\mathrm{MALG}_{\mu_\mathcal{U}}$ be the measure algebra of $(X, \boldsymbol{B}_\mathcal{U}, \mu_\mathcal{U})$. If $\mu_\mathcal{U}$ is non-atomic, fix also a function $S_\mathcal{U}\colon \mathrm{MALG}_{\mu_\mathcal{U}} \to \mathrm{MALG}_{\mu_\mathcal{U}}$ such that $S_\mathcal{U}(A) \subseteq A$ and

$$\mu_\mathcal{U}(S_\mathcal{U}(A)) = \frac{1}{2}\mu_\mathcal{U}(A).$$

Let now $\boldsymbol{B}_0 \subseteq \mathrm{MALG}_{\mu_\mathcal{U}}$ be a countable subalgebra closed under $S_\mathcal{U}$. Let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0) \subseteq \mathrm{MALG}_{\mu_\mathcal{U}}$ be the σ-subalgebra of $\mathrm{MALG}_{\mu_\mathcal{U}}$ generated by $\boldsymbol{B}_0$. Since every element of $\boldsymbol{B}$ can be approximated (in the sense of the metric $d(A, B) = \mu_\mathcal{U}(A \Delta B)$) by elements of $\boldsymbol{B}_0$, it follows that $\boldsymbol{B}$ is countably generated and non-atomic. It follows (see, e.g., Kechris [**Kec95**, 17.44]) that the measure algebra $(\boldsymbol{B}, \mu_\mathcal{U}|\boldsymbol{B})$ is isomorphic to the measure algebra of (any) non-atomic, standard measure space, in particular MALG_ρ, where ρ is the usual product measure on the Borel sets of $2^\mathbb{N}$. Then we can find a Cantor scheme $(B_s)_{s\in 2^{<\mathbb{N}}}$, with $B_s \in \boldsymbol{B}_\mathcal{U}$, $B_\varnothing = X$, $B_{s\hat{}0} \cap B_{s\hat{}1} = \varnothing$, $B_s = B_{s\hat{}0} \cap B_{s\hat{}1}$, $\mu_\mathcal{U}(B_s) = 2^{-n}$, and (B_s) viewed now as members of $\mathrm{MALG}_{\mu_\mathcal{U}}$, belong to $\boldsymbol{B}$ and generate $\boldsymbol{B}$. Then define

$$\varphi\colon X_\mathcal{U} \to 2^\mathbb{N}$$

by

$$\varphi(x) = \alpha \Leftrightarrow x \in \bigcap_n B_{\alpha|n}.$$

Then $\varphi^{-1}(N_s) = B_s$, where $N_s = \{\alpha \in 2^\mathbb{N}\colon s \subseteq \alpha\}$ for $s \in 2^{<\mathbb{N}}$. Thus φ is $\boldsymbol{B}_\mathcal{U}$-measurable (i.e., the inverse image of a Borel set in $2^\mathbb{N}$ is in $\boldsymbol{B}_\mathcal{U}$) and $\varphi_*\mu_\mathcal{U} = \rho$, so that $(2^\mathbb{N}, \rho)$ is a factor of $(X_\mathcal{U}, \mu_\mathcal{U})$ and $A \mapsto \varphi^{-1}(A)$ is an isomorphism of the measure algebra MALG_ρ with $(\boldsymbol{B}, \mu_\mathcal{U}|\boldsymbol{B})$.

4. Ultraproducts of measure preserving actions

(A) Let $(X_n, \mu_n), \mathcal{U}$ be as in §2. Let Γ be a countable group and let $\{\alpha_n\}$ be a sequence of Borel actions $\alpha_n\colon \Gamma \times X_n \to X_n$, such that α_n preserves $\mu_n, \forall n \in \mathbb{N}$. We can define then the action $\alpha_\mathcal{U}\colon \Gamma \times X_\mathcal{U} \to X_\mathcal{U}$ by

$$\gamma^{\alpha_\mathcal{U}} \cdot [(x_n)]_\mathcal{U} = [(\gamma^{\alpha_n} \cdot x_n)]_\mathcal{U},$$

where we let $\gamma^{\alpha_\mathcal{U}} \cdot x = \alpha_\mathcal{U}(\gamma, x)$ and similarly for each α_n.

PROPOSITION 4.1. The action $\alpha_\mathcal{U}$ preserves $\boldsymbol{B}^0_\mathcal{U}$, $\boldsymbol{B}_\mathcal{U}$ and the measure $\mu_\mathcal{U}$.

Proof. First let $A = [(A_n)]_\mathcal{U} \in \boldsymbol{B}^0_\mathcal{U}$. We verify that $\gamma^{\alpha_\mathcal{U}} \cdot A = [(\gamma^{\alpha_n} \cdot A_n)]_\mathcal{U}$, from which it follows that the action preserves $\boldsymbol{B}^0_\mathcal{U}$. Indeed

$$
\begin{aligned}
[(x_n)]_\mathcal{U} \in \gamma^{\alpha_\mathcal{U}} \cdot [(A_n)]_\mathcal{U} &\Leftrightarrow (\gamma^{-1})^{\alpha_\mathcal{U}} \cdot [(x_n)]_\mathcal{U} \in [(A_n)] \\
&\Leftrightarrow \mathcal{U}n((\gamma^{-1})^{\alpha_n} \cdot x_n \in A_n) \\
&\Leftrightarrow \mathcal{U}n(x_n \in \gamma^{\alpha_n} \cdot A_n) \\
&\Leftrightarrow [(x_n)]_\mathcal{U} \in [(\gamma^{\alpha_n} \cdot A_n)]_\mathcal{U}.
\end{aligned}
$$

Also

$$
\begin{aligned}
\mu_\mathcal{U}(\gamma^{\alpha_\mathcal{U}} \cdot A) &= \lim_{n\to\mathcal{U}} \mu_n(\gamma^{\alpha_n} \cdot A_n) \\
&= \lim_{n\to\mathcal{U}} \mu_n(A_n) = \mu_\mathcal{U}(A),
\end{aligned}
$$

so the action preserves $\mu_\mathcal{U}|\boldsymbol{B}^0_\mathcal{U}$.

Next let $A \in \boldsymbol{N}$ and for each $\epsilon > 0$ let $A \subseteq A_\epsilon \in \boldsymbol{B}^0_\mathcal{U}$ with $\mu_\mathcal{U}(A_\epsilon) < \epsilon$. Then $\gamma^{\alpha_\mathcal{U}} \cdot A \subseteq \gamma^{\alpha_\mathcal{U}} \cdot A_\epsilon$ and $\mu_\mathcal{U}(\gamma^{\alpha_\mathcal{U}} \cdot A_\epsilon) < \epsilon$, so $\gamma^{\alpha_\mathcal{U}} \cdot A \in \boldsymbol{N}$, i.e., $\boldsymbol{N}$ is invariant under the action.

Finally, let $A \in \boldsymbol{B}_\mathcal{U}$ and let $A' \in \boldsymbol{B}^0_\mathcal{U}$ be such that $A \Delta A' \in \boldsymbol{N}$, so that $\gamma^{\alpha_\mathcal{U}}(A) \Delta \gamma^{\alpha_\mathcal{U}}(A') \in \boldsymbol{N}$, thus $\gamma^{\alpha_\mathcal{U}}(A) \in \boldsymbol{B}_\mathcal{U}$ and $\mu_\mathcal{U}(\gamma^{\alpha_\mathcal{U}} \cdot A) = \mu_\mathcal{U}(\gamma^{\alpha_\mathcal{U}} \cdot A') = \mu_\mathcal{U}(A') = \mu_\mathcal{U}(A)$. $\dashv$

If (X, μ) is a probability space and $\alpha, \beta \colon \Gamma \times X \to X$ are measure preserving actions of Γ, we say the α, β are equivalent if $\forall \gamma \in \Gamma(\gamma^\alpha = \gamma^\beta, \mu\text{-a.e.})$. We let $A(\Gamma, X, \mu)$ be the space of equivalence classes and we call the elements of $A(\Gamma, X, \mu)$ also measure preserving actions. Note that if for each n, α_n, α'_n as above are equivalent, then it is easy to check that $\alpha_\mathcal{U}, \alpha'_\mathcal{U}$ are also equivalent, thus if $a_n \in A(\Gamma, X_n, \mu_n), n \in \mathbb{N}$, is a sequence of measure preserving actions and we pick α_n a representative of a_n, then we can define unambiguously the ultraproduct action

$$\prod_n a_n/\mathcal{U}$$

with representative $\alpha_{\mathcal{U}}$. This is a measure preserving action of Γ on $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$, i.e., $\prod_n a_n/\mathcal{U} \in A(\Gamma, X_{\mathcal{U}}, \mu_{\mathcal{U}})$. When $a_n = a$ for all n, we put

$$a_{\mathcal{U}} = \prod_n a/\mathcal{U}.$$

(B) Recall that if $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu)$, we say that b is a *factor* of a, in symbols

$$b \sqsubseteq a,$$

if there is a measurable map $\varphi\colon X \to Y$ such that $\varphi_*\mu = \nu$ and $\varphi(\gamma^a \cdot x) = \gamma^b \cdot \varphi(x)$, μ-a.e.(x). We denote by MALG_μ the measure algebra of (X, μ). Clearly Γ acts on MALG_μ by automorphisms of the measure algebra. If (Y, ν) is a non-atomic, standard measure space, the map $A \in \mathrm{MALG}_\nu \mapsto \varphi^{-1}(A) \in \mathrm{MALG}_\mu$ is an isomorphism of MALG_ν with a countably generated, non-atomic, σ-subalgebra $\boldsymbol{B}$ of MALG_μ, which is Γ-invariant, and this isomorphism preserves the Γ-actions. Conversely, we can see as in §1,**(B)** that every countably generated, non-atomic, σ-subalgebra $\boldsymbol{B}$ of MALG_μ, which is Γ-invariant, gives rise to a factor of a as follows: First fix an isomorphism π between the measure algebra $(\boldsymbol{B}, \mu|\boldsymbol{B})$ and the measure algebra of (Y, ν), where $Y = 2^{\mathbb{N}}$ and $\nu = \rho$ is the usual product measure. Use this to define the Cantor scheme $(B_s)_{s \in 2^{<\mathbb{N}}}$ for $\boldsymbol{B}$ as in §1, **(B)** and define $\varphi\colon X \to Y$ as before. Now the isomorphism π gives an action of Γ on the measure algebra of (Y, ν), which by definition preserves the Γ-actions on $(\boldsymbol{B}, \mu|\boldsymbol{B})$ and MALG_ν. The Γ-action on MALG_ν is induced by a (unique) action $b \in A(\Gamma, Y, \nu)$ (see, e.g., Kechris [**Kec95**, 17.46]) and then it is easy to check that φ witnesses that $b \sqsubseteq a$ (notice that for each $s \in 2^{<\mathbb{N}}, \gamma \in \Gamma, \varphi(\gamma^a \cdot x) \in N_s \Leftrightarrow \gamma^b \cdot \varphi(x) \in N_s$, μ-a.e.(x)).

In particular, the factors $b \in A(\Gamma, Y, \nu)$ of $a = \prod_n a_n/\mathcal{U}$ where (Y, ν) is a non-atomic, standard measure space, correspond exactly to the countably generated, non-atomic, Γ-invariant (for a) σ-subalgebras of $\mathrm{MALG}_{\mu_\mathcal{U}}$. For non-atomic $\mu_\mathcal{U}$, we can construct such subalgebras as follows: Start with a countable Boolean subalgebra $\boldsymbol{B}_0 \in \mathrm{MALG}_{\mu_\mathcal{U}}$, which is closed under the Γ-action and the function $S_\mathcal{U}$ of §2, **(B)**. Then let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ be the σ-subalgebra of $\mathrm{MALG}_{\mu_\mathcal{U}}$ generated by $\boldsymbol{B}_0$. This has all the required properties.

(C) We will next see how to insure, in the notation of the preceding paragraph, that the corresponding to $\boldsymbol{B}$ factor is a free action. Recall that $a \in A(\Gamma, X, \mu)$ is *free* if $\forall \gamma \in \Gamma \setminus \{1\}(\gamma^a \cdot x \neq x$, μ-a.e. $(x))$.

PROPOSITION 4.2. The action $a = \prod_n a_n/\mathcal{U}$ is free iff for each $\gamma \in \Gamma \setminus \{1\}$,

$$\lim_{n \to \mathcal{U}} \mu_n(\{x \colon \gamma^{a_n} \cdot x \neq x\}) = 1.$$

Proof. Note that, modulo null sets,

$$\{x \in X_\mathcal{U} \colon \gamma^a \cdot x \neq x\} = [(A_n)]_\mathcal{U},$$

where $A_n = \{x \in X_n \colon \gamma^{a_n} \cdot x \neq x\}$. ⊣

In particular, if all a_n are free, so is $\prod_n a_n/\mathcal{U}$.

PROPOSITION 4.3. Suppose the action $a = \prod_n a_n$ is free. Then for each $A \in \mathrm{MALG}_{\mu_\mathcal{U}}, A \neq \varnothing$ and $\gamma \in \Gamma \setminus \{1\}$, there is $B \in \mathrm{MALG}_{\mu_\mathcal{U}}$ with $B \subseteq A$, $\mu_\mathcal{U}(B) \geq \frac{1}{16}\mu_\mathcal{U}(A)$ and $\gamma^a \cdot B \cap B = \varnothing$.

Proof. It is clearly enough to show that if $\gamma \neq 1$, $A \in \boldsymbol{B}^0_\mathcal{U}, \mu_\mathcal{U}(A) > 0$, then there is $B \in \boldsymbol{B}^0_\mathcal{U}, B \subseteq A$, with $\mu_\mathcal{U}(B) \geq \frac{1}{16}\mu_\mathcal{U}(A)$ and $\gamma^a \cdot B \cap B = \varnothing$.

Let $A = [(A_n)]_\mathcal{U}$ and $\mu_\mathcal{U}(A) = \epsilon > 0$. Then there is $U \subseteq \mathbb{N}, U \in \mathcal{U}$ with $n \in U \Rightarrow (\mu_n(A_n) > \frac{\epsilon}{2}$ and $\mu(\{x \in X_n \colon \gamma^{a_n} \cdot x \neq x\}) > 1 - \frac{\epsilon}{4})$. We can assume that each X_n is Polish and γ^{a_n} is represented (a.e.) by a homeomorphism γ^{α_n} of X_n. Let

$$C_n = \{x \in A_n \colon \gamma^{\alpha_n} \cdot x \neq x\},$$

so that $\mu_n(C_n) > \frac{\epsilon}{4}$. Fix also a countable basis $(V_i^n)_{i\in\mathbb{N}}$ for X_n.

If $x \in C_n$, let V_n^x be a basic open set such that $\gamma^{\alpha_n} \cdot V_n^x \cap V_n^x = \varnothing$ (this exists by the continuity of γ^{α_n} and the fact that $\gamma^{\alpha_n} \cdot x \neq x$). It follows that there is $x_0 \in C_n$ with $\mu_n(C_n \cap V_n^{x_0}) > 0$ and $\gamma^{\alpha_n} \cdot (C_n \cap V_n^{x_0}) \cap (C_n \cap V_n^{x_0}) = \varnothing$. Thus there is $C \subseteq C_n$ with $\mu_n(C) > 0$ and $\gamma^{a_n} \cdot C \cap C = \varnothing$. By Zorn's Lemma or transfinite induction there is an element B_n of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ which is maximal, under inclusion, among all $D \in \mathrm{MALG}_{\mu_{\mathcal{U}}}$ satifying: $D \subseteq C_n$ (viewing C_n as an element of the measure algebra), $\mu_n(D) > 0$, $\gamma^{a_n} \cdot D \cap D = \varnothing$. We claim that $\mu_n(B_n) \geq \frac{\epsilon}{16}$. Indeed let

$$E_n = C_n \setminus (B_n \cup \gamma^{a_n} \cdot B_n \cup (\gamma^{-1})^{a_n} \cdot B_n).$$

If $\mu_n(B_n) < \frac{\epsilon}{16}$, then $E_n \neq \varnothing$, so as before we can find $F_n \subseteq E_n$ with $\mu_n(F_n) > 0$ and $\gamma^{a_n} \cdot F_n \cap F_n = \varnothing$. Then $\gamma^{a_n} \cdot (B_n \cup F_n) \cap (B_n \cup F_n) = \varnothing$, contradicting to maximality of B_n. So $\mu_n(B_n) \geq \frac{\epsilon}{16}$. Let now $B = [(B_n)]_{\mathcal{U}}$. $\dashv$

So if the action $a = \prod_n a_n/\mathcal{U}$ is free, let

$$T_{\mathcal{U}} \colon \Gamma \times \mathrm{MALG}_{\mu_{\mathcal{U}}} \to \mathrm{MALG}_{\mu_{\mathcal{U}}}$$

be a function such that for each $\gamma \neq 1, A \in \mathrm{MALG}_{\mu_{\mathcal{U}}} \setminus \{\varnothing\}, T_{\mathcal{U}}(\gamma, A) \subseteq A, \mu(T_{\mathcal{U}}(\gamma, A)) \geq \frac{1}{16}\mu(A)$ and $\gamma^a \cdot T_{\mathcal{U}}(\gamma, A) \cap T_{\mathcal{U}}(\gamma, A) = \varnothing$. Now, if in the earlier construction of countably generated, non-atomic, Γ-invariant σ-subalgebras of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$, we start with a countable Boolean subalgebra $\boldsymbol{B}_0$ closed under the Γ-action, the function $S_{\mathcal{U}}$ of §2, **(B)** and $T_{\mathcal{U}}$ (i.e., $\forall\gamma(A \in \boldsymbol{B}_0 \Rightarrow T_{\mathcal{U}}(\gamma, A) \in B_0)$), then the factor b corresponding to $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ is a free action.

5. Characterizing factors of ultraproducts

In sections §4–8 all measure spaces will be non-atomic and standard. Also Γ is an arbitrary countable infinite group.

(A) For such a measure space (X, μ), $\mathrm{Aut}(X, \mu)$ is the Polish group of measure preserving automorphisms of (X, μ) equipped with the *weak topology* generated by the maps

$T \mapsto T(A), A \in \mathrm{MALG}_\mu$, from Aut$(X,\mu)$ into MALG$_\mu$ (equipped with the usual metric $d_\mu(A,B) = \mu(A\Delta B))$. We can identify $A(\Gamma,X,\mu)$ with the space of homomorphisms of Γ into Aut(X,μ), so that it becomes a closed subspace of Aut$(X,\mu)^\Gamma$ with the product topology, thus also a Polish space.

DEFINITION 5.1. Let $a \in A(\Gamma,X,\mu), a_n \in A(\Gamma,X_n,\mu_n), n \in \mathbb{N}$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. We say that a is *weakly $\mathcal{U}$-contained* in (a_n), in symbols

$$a \prec_{\mathcal{U}} (a_n)$$

if for every finite $F \subseteq \Gamma, A_1,\dots,A_N \in \mathrm{MALG}_\mu, \epsilon > 0$, for $\mathcal{U}$-almost all n:

$$\exists B_{1,n}\dots B_{N,n} \in \mathrm{MALG}_{\mu_n} \forall\gamma \in \Gamma \forall i,j \le N$$

$$|\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_{i,n} \cap B_{j,n})| < \epsilon.$$

Note that if $a_n = b$ for all n, then $a \prec_{\mathcal{U}} (a_n) \Leftrightarrow a \prec b$ in the sense of weak containment of actions, see Kechris [**Kec10**].

One can also trivially see that $a \prec_{\mathcal{U}} (a_n)$ is equivalent to the statement:

For every finite $F \subseteq \Gamma, A_1,\dots A_n \in \mathrm{MALG}_\mu, \epsilon > 0$, there are $[(B_{1,n}]_{\mathcal{U}},\dots,$ $[(B_{N,n})]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}(X_{\mathcal{U}})$ such that for $\mathcal{U}$-almost all n:

$$\forall\gamma \in F \forall i,j \le N |\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_{i,n}) \cap B_{j,n}| < \epsilon).$$

DEFINITION 5.2. For $a, b_n \in A(\Gamma,X,\mu)$, we write

$$\lim_{n\to\mathcal{U}} b_n = a$$

if for each open nbhd V of a in $A(\Gamma,X,\mu)$, $\mathcal{U}n(b_n \in V)$.

Since the sets of the form

$$V = \{b\colon \forall\gamma \in F \forall i,j \le N |\mu(\gamma^a \cdot A_i \cap A_j) - \mu(\gamma^b \cdot A_i \cap A_j)| < \epsilon\},$$

for $A_1, \dots, A_n$ a Borel partition of $X, \epsilon > 0, F \subseteq \Gamma$ finite containing 1, form a nbhd basis of $a, \lim_{n\to\mathcal{U}} b_n = a$ iff $\mathcal{U}n(b_n \in V)$, for any V of the above form.

Below $\cong$ denotes isomorphism of actions.

THEOREM 5.3. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $a \in A(\Gamma, X, \mu)$, and let $a_n \in A(\Gamma, X_n, \mu_n), n \in \mathbb{N}$. Then the following are equivalent

(1) $a \prec_{\mathcal{U}} (a_n)$,

(2) $a \sqsubseteq \prod_n a_n/\mathcal{U}$,

(3) $a = \lim_{n\to\mathcal{U}} b_n$, for some sequence (b_n), $b_n \in A(\Gamma, X, \mu)$ with $b_n \cong a_n, n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2): Let $1 \in F_0 \subseteq F_1 \subseteq \dots$ be a sequence of finite subsets of Γ with $\Gamma = \bigcup_m F_m$. We can assume that $X = 2^{\mathbb{N}}, \mu = \rho$ (the usual product measure on $2^{\mathbb{N}}$). Let $N_s = \{\alpha \in 2^{\mathbb{N}} \colon s \subseteq \alpha\}$, for $s \in 2^{<\mathbb{N}}$.

By (1), we can find for each $m, s \in 2^{\leq m}$, $[(B_n^{s,m})] \in \boldsymbol{B}^0_{\mathcal{U}}$ such that $U_m \in \mathcal{U}$, where

$$U_m = \{n \geq m \colon \forall \gamma \in F_m \forall s, t \in 2^{\leq m}$$

$$|\mu(\gamma^a \cdot N_s \cap N_t| - \mu_n(\gamma^{a_n} \cdot B_n^{s,m} \cap B_n^{t,m})| < \epsilon_m\},$$

where $\epsilon_m \to 0$. Since $\bigcap_m U_m = \varnothing$, let $m(n) =$ largest m such that $n \in \bigcap_{i\leq m} U_i$. Then $n \in U_{n(m)}$ and $\lim_{n\to\mathcal{U}} m(n) = \infty$. Put

$$B_s = [(B_n^{s,m(n)})]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}.$$

Since for all $n, n \in U_{m(n)}$, it follows (taking $\gamma = 1, s = t$ in the definition of U_m) that for all n with $m(n) > \mathrm{length}(s)$,

$$(*) \qquad |\mu(N_s) - \mu_n(B_n^{s,m(n)})| < \epsilon_{m(n)}.$$

So for any $\epsilon > 0$, if $M > \mathrm{length}(s)$ and $\epsilon_M < \epsilon$, then $\mathcal{U}n(m(n) > M)$, so $(*)$ holds with ϵ replacing $\epsilon_{m(n)}$ for $\mathcal{U}$-almost all n, thus

$$\mu_{\mathcal{U}}(B_s) = \lim_{n\to\mathcal{U}} \mu_n(B_n^{s,m(n)}) = \mu(N_s).$$

In general, we have that

$$\forall \gamma \in F_{m(n)} \forall s, t \in 2^{\leq m(n)}$$

$$|\mu(\gamma^a \cdot N_s \cap N_t) - \mu_n(\gamma^{a_n} \cdot B_n^{s,m(n)} \cap B_n^{t,m(n)})| < \epsilon_{m(n)}.$$

So if $\gamma \in F, s, t \in 2^{<\mathbb{N}}, \epsilon > 0$, and if M is large enough so that $M > \max\{\text{length}(s), \text{length}(t)\}, \gamma \in F_M, \epsilon_M < \epsilon$, then on $\{n \colon m(n) \geq M\} \in \mathcal{U}$ we have

$$|\mu(\gamma^a \cdot N_s \cap N_t) - \mu_n(\gamma^{a_n} \cdot B_n^{s,m(n)} \cap B_n^{t,m(n)})| < \epsilon,$$

so

(**) $$\mu_{\mathcal{U}}(\gamma^{\prod_n a_n/\mathcal{U}} \cdot B_s \cap B_t) = \mu(\gamma^a \cdot N_s \cap N_t).$$

Viewing each B_s as an element of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$, we have $B_\varnothing = X_{\mathcal{U}}$, $B_{s\hat{\ }0} \cap B_{s\hat{\ }1} = \varnothing$, $B_s = B_{s\hat{\ }0} \cup B_{s\hat{\ }1}$ (for the last take $\gamma = 1, t = s\hat{\ }i$ in $(**)$) and $\mu_{\mathcal{U}}(B_s) = 2^{-n}$, if $s \in 2^n$. Then the map $\pi(N_s) = B_s$ gives a measure preserving isomorphism of the Boolean subalgebra $\boldsymbol{A}_0$ of MALG_μ generated by (N_s) and the Boolean algebra $\boldsymbol{B}_0$ in $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ generated by (B_s). Let $\boldsymbol{B}$ be the σ-subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ generated by (B_s). Since π is an isometry of $\boldsymbol{A}_0$ with $\boldsymbol{B}_0$ (with the metrics they inherit from the measure algebra), and $\boldsymbol{A}_0$ is dense in MALG_μ, $\boldsymbol{B}_0$ is dense in $\boldsymbol{B}$, it follows that π extends uniquely to an isometry, also denoted by π, from MALG_μ onto $\boldsymbol{B}$. Since $\pi(\varnothing) = \varnothing$, π is actually an isomorphism of the measure algebra MALG_μ with the measure algebra $\boldsymbol{B}$ (see Kechris [**Kec10**, pp. 1-2]), it is thus enough to show that $\boldsymbol{B}$ is Γ-invariant (for $\prod_n a_n/\mathcal{U}$) and that π preserves the Γ-action.

Let $b = \prod_n a_n/\mathcal{U}$. It is enough to show that $\pi(\gamma^a \cdot N_s) = \gamma^b \cdot B_s$ (since (B_s) generates $\boldsymbol{B}$).

Fix $\gamma \in \Gamma, \epsilon > 0, s \in 2^{<\mathbb{N}}$. There is $A \in \boldsymbol{A}_0$ with $\mu(\gamma^a \cdot N_s \Delta A) < \epsilon/2$. Now $A = \bigsqcup_{i=1}^{m_1} N_{t_i}, \sim A = \bigsqcup_{j=1}^{m_2} N_{t'_j}$ and $\sim N_s = \bigsqcup_{k=1}^{m_3} N_{s_k}$ (disjoint unions), so

$$\begin{aligned}\gamma^a \cdot N_s \Delta A &= (\gamma^a \cdot N_s \cap (\sim A)) \sqcup (\gamma^a \cdot (\sim N_s) \cap A) \\ &= (\bigsqcup_{j=1}^{m_2} \gamma^a \cdot N_s \cap N_{t'_j}) \sqcup (\bigsqcup_{k=1}^{m_3} \bigsqcup_{i=1}^{m_1} (\gamma^a \cdot N_{s_k} \cap N_{t_i})).\end{aligned}$$

If $B = \pi(A) \in \boldsymbol{B}_0$, then we also have

$$\begin{aligned}\gamma^b \cdot B_s \Delta B =& (\bigsqcup_{j=1}^{m_2} \gamma^b \cdot B_s \cap B_{t'_j}) \sqcup \\ & (\bigsqcup_{k=1}^{m_3} \bigsqcup_{i=1}^{m_1} (\gamma^b \cdot B_{s_k} \cap B_{t_i})),\end{aligned}$$

so by $(**)$

$$\mu_{\mathcal{U}}(\gamma^b \cdot B_s \Delta B) = \mu(\gamma^a \cdot N_s \Delta A) < \epsilon/2.$$

Since π preserves measure, we also have $\mu_{\mathcal{U}}(\pi(\gamma^a \cdot N_s) \Delta B) < \epsilon/2$, thus

$$\mu_{\mathcal{U}}(\gamma^b \cdot B_s \Delta \pi(\gamma^a \cdot N_s)) < \epsilon.$$

Therefore $\gamma^b \cdot B_s = \pi(\gamma^a \cdot N_s)$.

(2) $\Rightarrow$ (1): Suppose that $a \sqsubseteq b = \prod_n a_n/\mathcal{U}$. Let $\pi \colon \mathrm{MALG}_\mu \to \mathrm{MALG}_{\mu_{\mathcal{U}}}$ be a measure preserving embedding preserving the Γ-actions (so that the image $\pi(\mathrm{MALG}_\mu)$ is a Γ-invariant σ-subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$). Fix $F \subseteq \Gamma$ finite, $A_1, \ldots, A_n \in \mathrm{MALG}_\mu$ and $\epsilon > 0$. Let $B^1, \ldots, B^N \in \boldsymbol{B}^0_{\mathcal{U}}$ represent $\pi(A_1), \ldots, \pi(A_N)$. Let $B^i = [(B^i_n)]_{\mathcal{U}}$. Then for $\gamma \in F, j, k \leq N$,

$$\begin{aligned}\mu(\gamma^a \cdot A_j \cap A_k) &= \mu_{\mathcal{U}}(\gamma^b \cdot B^j \cap B^k) \\ &= \lim_{n \to \mathcal{U}} \mu_n(\gamma^{a_n} \cdot B^j_n \cap B^k_n),\end{aligned}$$

so for $\mathcal{U}$-almost all n,

$$|\mu(\gamma^a \cdot A_j \cap A_k) - \mu_n(\gamma^{a_n} \cdot B_n^j \cap B_n^k)| < \epsilon,$$

and thus for $\mathcal{U}$-almost all n, this holds for all $\gamma \in F, j, k \leq N$. Thus $a \prec_{\mathcal{U}} (a_n)$.

(3) $\Rightarrow$ (1): Fix such b_n, and let $A_1, \dots, A_N \in \mathrm{MALG}_\mu$, $F \subseteq \Gamma$ finite, $\epsilon > 0$. Then there is $U \in \mathcal{U}$ such that for $n \in U$ we have

$$\forall \gamma \in F \forall i, j \leq N(|\mu(\gamma^a \cdot A_i \cap A_j) - \mu(\gamma^{b_n} \cdot A_i \cap A_j)| < \epsilon).$$

Let $\varphi_n \colon (X, \mu) \to (X_n, \mu_n)$ be an isomorphism that sends b_n to a_n and put $\varphi_n(A_i) = B_n^i$. Then $\varphi_n(\gamma^{b_n} \cdot A_i \cap A_j) = \gamma^{a_n} \cdot B_n^i \cap B_n^j$, so for $n \in U$:

$$\forall \gamma \in F \forall i, j \leq N(|\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_n^i \cap B_n^j)| < \epsilon),$$

thus $a \prec_{\mathcal{U}} (a_n)$.

(1) $\Rightarrow$ (3): Suppose $a \prec_{\mathcal{U}} (a_n)$. Let

$$V = \{c \in A(\Gamma, X, \mu) \colon \forall \gamma \in F \forall i, j \leq N(|\mu(\gamma^a \cdot A_i \cap A_j) - \mu(\gamma^c \cdot A_i \cap A_j)| < \epsilon),$$

where $A_1, \dots, A_n \in \mathrm{MALG}_\mu$ is a Borel partition of X, $\epsilon > 0$ and $F \subseteq \Gamma$ is finite with $1 \in F$, be a basic nbhd of a.

Claim. It suffices to show that for any such V we can find $U \in \mathcal{U}$ such that for $n \in U$ there is $b_n \in V$ with $b_n \cong a_n$.

Assume this for the moment and complete the proof of (1) $\Rightarrow$ (3) by verifying that indeed for any such V we can find a corresponding U as in the claim.

Since $a \prec_{\mathcal{U}} (a_n)$, for any $\delta > 0$, we can find $[(B_{1,n})]_{\mathcal{U}}, \dots, [(B_{N,n})]_{\mathcal{U}} \in \boldsymbol{B}_{\mathcal{U}}^0$ and $U_\delta \in \mathcal{U}$ such that for $n \in U_\delta$ we have

$$\forall \gamma \in F \forall i, j \leq N(|\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_{i,n} \cap B_{j,n})| < \delta).$$

Taking $\delta < \epsilon/20N^3$ and $U = U_\delta$, the proof of Proposition 10.1 in Kechris [**Kec10**] shows that for $n \in U$ there is $b_n \cong a_n$ with $b_n \in V$.

Proof of the claim. Let $V_0 \supseteq V_1 \supseteq V_2 \subseteq \dots$ be a nbhd basis for a consisting of sets of the above form, and assume that for each m there is $U_m \in \mathcal{U}$ such that for $n \in U_m$, there is $b_{n,m} \in V_m$ with $b_{n,m} \cong a_n$. We can also assume that $\bigcap_m U_m = \varnothing$. Let $m(n) =$ largest m such that $n \in \bigcap_{i \leq m} U_i$. We have $a_n \cong b_{n,m(n)} \in V_{m(n)}$, and for any nbhd V of a as above, if M is so large that $V_M \subseteq V$, then $b_{n,m(n)} \in V_{m(n)} \subseteq V_M \subseteq V$, for $n \in \{n \colon m(n) \geq M\} \in \mathcal{U}$. So $a = \lim_{n \to \mathcal{U}} b_{n,m(n)}$. ⊣

COROLLARY 5.4. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the actions $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu)$. Then the following are equivalent:

(1) $a \prec b$,

(2) $a \sqsubseteq b_{\mathcal{U}}$.

Theorem 4.3 also has the following curious consequence, a compactness property of the space $A(\Gamma, X, \mu)$.

COROLLARY 5.5. Let $a_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, be a sequence of actions. Then there is a subsequence $n_0 < n_1 < n_2 < \dots$ and $b_{n_i} \in A(\Gamma, X, \mu), b_{n_i} \cong a_{n_i}$, such that (b_{n_i}) converges in $A(\Gamma, X, \mu)$.

Proof. Let $a \in A(\Gamma, X, \mu)$ be such that $a \sqsubseteq \prod_n a_n/\mathcal{U}$ (such exists by §3, **(B)**). Then by 4.3, we can find $b_n \cong a_n$, with $\lim_{n \to \mathcal{U}} b_n = a$. This of course implies that there is $n_0 < n_1 < \dots$ with $\lim_{i \to \infty} b_{n_i} = a$. ⊣

Benjy Weiss pointed out that for *free* actions a stronger version of 4.5 follows from his work with Abért, see Abért-Weiss [**AW11**]. In this paper it is shown that if s_Γ is the shift action of an infinite group Γ on $[0,1]^\Gamma$, then $s_\Gamma \prec a$ for *any* free action a of Γ. From this it follows that given free $a_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, there is $b_n \cong a_n$ with $\lim_{n \to \infty} b_n = s_\Gamma$.

Another form of compactness for $A(\Gamma, X, \mu)$ that is an immediate consequence of 4.5 is the following:

Any cover of $A(\Gamma, X, \mu)$ by open, *invariant under* $\cong$ sets, has a finite subcover.

(B) Consider now $a \in A(\Gamma, X, \mu)$ and the action $a_{\mathcal{U}}$ on $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$. Clearly $\mu_{\mathcal{U}}$ is non-atomic as μ is non-atomic. Fix also a countable Boolean subalgebra $\boldsymbol{A}_0$ of MALG_μ which generates MALG_μ and is closed under the action a. The map

$$\pi(A) = [(A)]_{\mathcal{U}}$$

(where (A) is the constant sequence $(A_n), A_n = A, \forall n \in \mathbb{N}$) embeds $\boldsymbol{A}_0$ into a Boolean subalgebra $\boldsymbol{C}_0$ of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$, invariant under $a_{\mathcal{U}}$, preserving the measure and the Γ-actions (a on $\boldsymbol{A}_0$ and $a_{\mathcal{U}}$ on $\boldsymbol{C}_0$).

Let $\boldsymbol{B}_0 \supseteq \boldsymbol{C}_0$ be any countable Boolean subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ closed under the action $a_{\mathcal{U}}$ and the function $S_{\mathcal{U}}$ of §2, **(B)** and let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ be the σ-algebra generated by $\boldsymbol{B}_0$. Let b be the factor of $a_{\mathcal{U}}$ corresponding to $\boldsymbol{B}$, so that $b \sqsubseteq a_{\mathcal{U}}$ and thus $b \prec a$ by 4.4. We also claim that $a \sqsubseteq b$ and thus $a \sim_w b$, where

$$a \sim_w b \Leftrightarrow a \prec b \;\&\; b \prec a.$$

Indeed, let $\boldsymbol{D}_0 = \sigma(\boldsymbol{C}_0)$ be the σ-subalgebra of $\boldsymbol{B}$ generated by $\boldsymbol{C}_0$. Then $\boldsymbol{D}_0$ is also closed under the action $a_{\mathcal{U}}$. The map π is an isometry of $\boldsymbol{A}_0$ with $\boldsymbol{C}_0$, which are dense in MALG_μ, $\boldsymbol{D}_0$, resp., so extends uniquely to an isometry, also denoted by π, of MALG_μ with $\boldsymbol{D}_0$. Since $\pi(\varnothing) = \varnothing$, it follows that π is an isomorphism of the measure algebra MALG_μ with the measure algebra $\boldsymbol{D}_0$ (see Kechris [**Kec10**, pp. 1-2]). Fix row $\gamma \in \Gamma$. Then γ^a on MALG_μ is mapped by π to an automorphism $\pi(\gamma^a)$ of the measure algebra $\boldsymbol{D}_0$. Since $\pi(\gamma^a \cdot A) = \gamma^{a_{\mathcal{U}}} \cdot \pi(A)$, for $A \in \boldsymbol{A}_0$, it follows that $\pi(\gamma^a)|\boldsymbol{C}_0 = \gamma^{a_{\mathcal{U}}}|\boldsymbol{C}_0$, so since $\boldsymbol{C}_0$ generates $\boldsymbol{D}_0$, we have $\pi(\gamma^a) = \gamma^{a_{\mathcal{U}}}|\boldsymbol{D}_0$, i.e., π preserves the Γ-actions (a on MALG_μ and $a_{\mathcal{U}}$ on $\boldsymbol{D}_0$), thus $a \sqsubseteq b$.

Recall now that $a \in A(\Gamma, X, \mu)$ admits *non-trivial almost invariant sets* if there is a sequence (A_n) of Borel sets such that $\mu(A_n)(1 - \mu(A_n)) \not\to 0$ but $\forall\gamma(\lim_{n\to\infty} \mu(\gamma^a \cdot$

$A_n \Delta A_n) = 0)$. We call an action a *strongly ergodic* (or E_0-*ergodic*) if it does not admit non-trivial almost invariant sets. We now have:

PROPOSITION 5.6. Let $a \in A(\Gamma, X, \mu)$. Then a is strongly ergodic iff $\forall b \sim_w a$ (b is ergodic) iff $\forall b \prec a$ (b is ergodic).

Proof. Assume first that a is not strongly ergodic and let (A_n) be a sequence of Borel sets such that for some $\delta > 0$, $\delta \leq \mu(A_n) \leq 1 - \delta$ and $\forall \gamma (\lim_{n\to\infty} \mu(\gamma^a \cdot A_n \Delta A_n) = 0)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $A = [(A_n)]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}$. Then viewing A as an element of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ we have $\gamma^{a_{\mathcal{U}}} \cdot A = A, \forall \gamma \in \Gamma$, and $0 < \mu_{\mathcal{U}}(A) < 1$. Let $\boldsymbol{B}_0$ be a countable Boolean subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ closed under $a_{\mathcal{U}}$, the function $S_{\mathcal{U}}$ and containing $\boldsymbol{C}_0$ as before. Let b be the factor of $a_{\mathcal{U}}$ associated with $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$, so that $a \sim_w b$. Since $A \in \boldsymbol{B}$, clearly b is not ergodic.

Conversely assume $b \prec a$ and b is not ergodic. It follows easily then from the definition of weak containment that a is not strongly ergodic. $\dashv$

Finally we note the following fact that connects weak containment to factors.

PROPOSITION 5.7. Let $a, b \in A(\Gamma, X, \mu)$. Then the following are equivalent:

(i) $a \prec b$,

(ii) $\exists c \in A(\Gamma, X, \mu)(c \sim_w b \;\&\; a \sqsubseteq c)$.

Proof. (ii) clearly implies (i), since $a \sqsubseteq c \Rightarrow a \prec c$ and $\prec$ is transitive.

(i) $\Rightarrow$ (ii) Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. By 4.4, if $a \prec b$ then $a \sqsubseteq b_{\mathcal{U}}$. Then as in the first two paragraphs of §4, **(B)**, we can find an appropriate σ-subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ invariant under $b_{\mathcal{U}}$, so that if c is the corresponding factor, then $c \sim_w b$ (and in fact moreover $b \sqsubseteq c$) and $a \sqsubseteq c$. $\dashv$

6. Graph combinatorics of group actions

Let Γ be an infinite group with a finite set of generators $S \subseteq \Gamma$ for which *we assume throughout that* $1 \notin S$. We denote by $\mathrm{FR}(\Gamma, X, \mu)$ the set of free actions in $A(\Gamma, X, \mu)$. If

$a \in \mathrm{FR}(\Gamma, X, \mu)$ we associate with a the (simple, undirected) graph $G(S, a) = (X, E(S, a))$, where X is the set of vertices and $E(S, a)$, the set of edges, is given by

$$(x, y)E(S, a) \Leftrightarrow x \neq y \;\&\; \exists s \in S^{\pm 1}(s^a \cdot x = y),$$

where $S^{\pm 1} = \{s, s^{-1} \colon s \in S\}$. We also write $xE(s, a)y$ if $(x, y) \in E(S, a)$. As in Conley-Kechris [**CK13**], we associate with this graph the following parameters:

$$\chi_\mu(S, a) = \text{ the } \textit{measurable chromatic number,}$$

$$\chi_\mu^{ap}(S, a) = \text{ the } \textit{approximate chromatic number,}$$

$$i_\mu(S, a) = \text{ the } \textit{independence number,}$$

defined as follows:

- $\chi_\mu(S, a)$ is the smallest cardinality of a standard Borel space Y for which there is a ($\mu-$)measurable coloring $c \colon X \to Y$ (i.e., $xE(S, a)y \Rightarrow c(x) \neq c(y)$).
- $\chi_\mu^{ap}(S, a)$ is the smallest cardinality of a standard Borel space Y such that for each $\epsilon > 0$, there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \epsilon$ and a measurable coloring $c \colon A \to Y$ of the induced subgraph $G(S, a)|A = (A, E(S, A) \cap A^2)$.
- $i_\mu(S, a)$ is the supremum of the measures of Borel independent sets, where $A \subseteq X$ is *independent* if no two elements of A are adjacent.

Given a (simple, undirected) graph $G = (X, E)$, where X is the set of vertices and E the set of edges, a *matching* in G is a subset $M \subseteq E$ such that no two edges in M have a common point. We denote by X_M the set of matched vertices, i.e., the set of points belonging to an edge in M. If $X_M = X$ we say that M is a *perfect matching*.

For $a \in \mathrm{FR}(\Gamma, X, \mu)$ as before, we also define the parameter

$$m(S, a) = \text{ the } \textit{matching number,}$$

where $m(S,a)$ is 1/2 of the supremum of $\mu(X_M)$, with M a Borel (as a subset of X^2) matching in $G(S,a)$. If $m(S,a) = 1/2$ and the supremum is attained, we say that $G(S,a)$ admits an *a.e. perfect matching*.

Note that we can view a matching M in $G(S,a)$ as a Borel bijection $\varphi\colon A \to B$, with $A, B \subseteq X$ disjoint Borel sets and $xE(S,a)\varphi(x), \forall x \in A$. Then $X_M = A \cup B$ and so $\mu(A)$ is $1/2\mu(X_M)$. Thus $m(S,a)$ is equal to the supremum of $\mu(A)$ over all such φ.

It was shown in Conley-Kechris [**CK13**, 4.2, 4.3] that

$$a \prec b \Rightarrow i_\mu(S,a) \leq i_\mu(S,b), \chi_\mu^{ap}(S,a) \geq \chi_\mu^{ap}(S,b).$$

We note a similar fact about $m(S,a)$.

PROPOSITION 6.1. Let Γ be an infinite countable group and $S \subseteq \Gamma$ a finite set of generators. Then

$$a \prec b \Rightarrow m(S,a) \leq m(S,b).$$

Proof. Let $\varphi\colon A \to B$ be a matching for $G(S,a)$. Then there are Borel decompositions $A = \bigsqcup_{i=1}^n A_n, B = \bigsqcup_{i=1}^n B_n$, and $s_1, \dots, s_n \in S^{\pm 1}$ with $\varphi|A_i = s_i^a|A_i, \varphi(A_i) = B_i$. Fix $\delta > 0$. Since $a \prec b$, for any $\epsilon > 0$, we can find a sequence $C_1, \dots, C_n$ of pairwise disjoint Borel sets such that for any $\gamma \in \{1\} \cup (S^{\pm 1})^2$, $|\mu(\gamma^a \cdot A_i \cap A_j) - \mu(\gamma^b \cdot C_i \cap C_j)| < \epsilon$, for $i \leq i, j \leq n$. Since $s_i^a \cdot A_i \cap A_j = \varnothing$, for all $1 \leq i, j \leq n$, and $s_i^a \cdot A_i \cap s_j^a \cdot A_j = \varnothing$, for all $1 \leq i \neq j \leq n$, it follows that $|\mu(A_i) - \mu(C_i)| < \epsilon, 1 \leq i \leq n, \mu(s_i^b \cdot C_i \cap C_j) < \epsilon, 1 \leq i, j \leq n$, and $\mu(s_i^b \cdot C_i \cap s_j^b \cdot C_j) < \epsilon, 1 \leq i \neq j \leq n$. By disjointifying and choosing ϵ very small compared to δ, it is clear that we can find such pairwise disjoint $C_1, \dots, C_n$ with $s_i^b \cdot C_i \cap C_j = \varnothing, 1 \leq i, j \leq n, s_i^b \cdot C_i \cap s_j^b \cdot C_j = \varnothing, 1 \leq i \neq j \leq n$, and if $C = \bigsqcup_{i=1}^n C_i, D = \bigsqcup_{i=1}^n s_i^b \cdot C_i$, then $|\mu(C) - \mu(A)| < \delta$. Clearly $\psi\colon C \to D$ given by $\psi|C_i = s_i^b|C_i$ is a matching for $G(S,b)$ and $\mu(C) > \mu(A) - \delta$. Since δ was arbitrary this shows that $m(S,a) \leq m(S,b)$. $\dashv$

(B) The next result shows that, modulo weak equivalence, we can turn approximate parameters to exact ones.

THEOREM 6.2. Let Γ be an infinite countable group and $S \subseteq \Gamma$ a finite set of generators. Then for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is $b \in \mathrm{FR}(\Gamma, X, \mu)$ such that

(i) $a \sim_w b$ and $a \sqsubseteq b$,

(ii) $\chi_\mu^{ap}(S, a) = \chi_\mu^{ap}(S, b) = \chi_\mu(S, b)$,

(iii) $i_\mu(S, a) = i_\mu(S, b)$ and $i_\mu(S, b)$ is attained,

(iv) $m(S, a) = m(S, b)$ and $m(S, b)$ is attained.

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. The action b will be an appropriate factor of the ultrapower $a_\mathcal{U}$.

Let $k = \chi_\mu^{ap}(S, a)$. This is finite by Kechris-Solecki-Todorcevic [**KST99**, 4.6]. Let $i_\mu(S, a) = \iota \leq \frac{1}{2}$ and let $m(S, a) = m \leq \frac{1}{2}$. Then for each $n \geq 1$, find the following:

(a) A sequence $C_n^1, \dots, C_n^k$ of pairwise disjoint Borel sets such that $s^a \cdot C_n^i \cap C_n^i = \varnothing$, for $1 \leq i \leq k, s \in S^{\pm 1}$, and $\mu(\bigsqcup_{i=1}^k C_n^i) \geq 1 - \frac{1}{n}$.

(b) A Borel set I_n such that $s^a \cdot I_n \cap I_n = \varnothing, s \in S^{\pm 1}$, and $\mu(I_n) \geq \iota - \frac{1}{n}$.

(c) A pairwise disjoint family of Borel sets $(A_n^s)_{s \in S^{\pm 1}}$, such that $s^a \cdot A_n^s \cap A_n^t = \varnothing, s, t \in S^{\pm 1}, s^a \cdot A_n^s \cap t^a \cdot A_n^t = \varnothing, s, t \in S^{\pm 1}, s \neq t$, and

$$\mu(\bigsqcup_{s \in S^{\pm 1}} A_n^s) \geq m - \tfrac{1}{n}.$$

Consider now the ultrapower action $a_\mathcal{U}$ on $(X_\mathcal{U}, \mu_\mathcal{U})$ and the sets $C^i = [(C_n^i)]_n \in \boldsymbol{B}_\mathcal{U}^0, 1 \leq i \leq k, I = [(I_n)]_\mathcal{U} \in B_\mathcal{U}^0$ and $A^s = [(A_n^s)]_\mathcal{U} \in B_\mathcal{U}^0, s \in S^{\pm 1}$. Viewed as elements of $\mathrm{MALG}_{\mu_\mathcal{U}}$ they satisfy:

(a′) $C^i \cap C^j = \varnothing, 1 \leq i \neq j \leq k, s^{a_\mathcal{U}} \cdot C^i \cap C^i = \varnothing, 1 \leq i \leq k, s \in S^{\pm 1}; \mu_\mathcal{U}(\bigsqcup_{i=1}^k C^i) = 1$,

(b′) $s^{a_\mathcal{U}} \cdot I \cap I = \varnothing, s \in S^{\pm 1}; \mu_\mathcal{U}(I) \geq \iota$,

(c′) $A^s \cap A^t = \varnothing, s \neq t, s, t \in S^{\pm 1}; s^{a_\mathcal{U}} \cdot A^s \cap A^t = \varnothing, s, t \in S^{\pm 1}; s^{a_\mathcal{U}} \cdot A^s \cap t^{a_\mathcal{U}} \cdot A^t = \varnothing, s \neq t, s, t \in S^{\pm 1}; \mu(\bigsqcup_{s \in S^{\pm 1}} A^s) \geq m$.

Let now $\boldsymbol{B}_0$ be a countable Boolean subalgebra of $\mathrm{MALG}_{\mu_\mathcal{U}}$ closed under the action $a_\mathcal{U}$, the functions $S_\mathcal{U}, T_\mathcal{U}$ of §2, **(B)**, §3, **(B)**, resp., and containing the algebra $\boldsymbol{C}_0$ of §4,

(B) and also C^i $(1 \leq i \leq k), I, A^s$ $(s \in S^{\pm 1})$. Let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ and let b be the factor of $a_{\mathcal{U}}$ corresponding to $\boldsymbol{B}$. (We can of course assume that $b \in \mathrm{FR}(\Gamma, X, \mu)$.) Then by §4, **(B)** again, $a \sim_w b$ and $a \sqsubseteq b$. So, in particular, $\chi_\mu^{ap}(S,a) = \chi_\mu^{ap}(S,b) = k, i_\mu(S,a) = i_\mu(S,b) = \iota$ and $m(S,a) = m(S,b) = m$, since $a \sim_w b$. The sets $(C^i)_{i \leq k}$ give a measurable coloring of $G(S,b)|A$, for some A with $\mu(A) = 1$ and we can clearly color in a measurable way $G(S,b)| \sim A$ by ℓ colors, where ℓ is the chromatic number of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ of Γ, S. Since $\ell \leq k$, it follows that $\chi_\mu(S,b) \leq k$, so $\chi_\mu(S,b) = \chi_\mu^{ap}(S,b)$. Finally, (b′), (c′) show that $i_\mu(S,b) = \iota$ and $m(S,b) = m$ are attained. $\dashv$

7. Brooks' Theorem for group actions

(A) Brooks' Theorem for finite graphs asserts that for any finite graph G with degree bounded by d, the chromatic number $\chi(G)$ is $\leq d$, unless $d = 2$ and G contains an odd cycle or $d \geq 3$ and G contains a complete subgraph (clique) with $d+1$ vertices (and the chromatic number is always $\leq d+1$). In Conley-Kechris [**CK13**] the question of finding analogues of the Brooks bound for graphs of the form $G(S,a)$ is studied. Let $d = |S^{\pm 1}|$ be the degree of $\mathrm{Cay}(\Gamma, S)$. First note that by Kechris-Solecki-Todorcevic [**KST99**, 4.8], $\chi_\mu(S,a) \leq d+1$ (in fact this holds even for Borel instead of measurable colorings). A compactness argument using Brooks' Theorem also shows that $\chi(S,a) \leq d$, where $\chi(S,a)$ is the chromatic number of $G(S,a)$. It was shown in Conley-Kechris [**CK13**, 2.19, 2.20] that for any infinite Γ, $\chi_\mu^{ap}(S,a) \leq d$, for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, so one has a full "approximate" version of Brooks' Theorem. How about the full measurable Brooks bound $\chi_\mu(S,a) \leq d$? This is easily false for some action a (e.g., the shift action), when $\Gamma = \mathbb{Z}$ or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (with the usual sets of generators) and it was shown in Conley-Kechris [**CK13**, 5.12] that when Γ has finitely many ends and is not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, then one indeed has the Brooks' bound $\chi_\mu(S,a) \leq d$, for *any* $a \in \mathrm{FR}(\Gamma, X, \mu)$ (in fact even for Borel as opposed to measurable colorings). It is unknown if this still holds for Γ with infinitely many ends but 5.2 shows that one has the full analogue of the Brooks bound up to weak equivalence for any group Γ.

THEOREM 7.1. For any infinite group Γ and finite set of generators S with $d = |S^{\pm 1}|$, for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is $b \in \mathrm{FR}(\Gamma, X, \mu)$, with $b \sim_w a$ and $\chi_\mu(S, b) \leq d$.

This also leads to the solution of an open problem arising in probability concerning random colorings of Cayley graphs.

Let Γ be an infinite group with a finite set of generators S. Let $k \geq 1$. Consider the compact space k^Γ on which Γ acts by shift: $\gamma \cdot p(\delta) = p(\gamma^{-1}\delta)$. The set Col$(k, \Gamma, S)$ of colorings of Cay(Γ, S) with k colors is a closed (thus compact) invariant subspace of k^Γ. An *invariant, random k-coloring* of the Cayley graph Cay(Γ, S) is an invariant probability Borel measure on the space Col(k, Γ, S). Let d by the degree of Cay(Γ, S). In Aldous-Lyons [**AL07**, 10.5] the question of existence of invariant, random k-colorings is discussed and mentioned that Schramm (unpublished, 1997) had shown that for any Γ, S there is an invariant, random $(d+1)$-coloring (this also follows from the more general Kechris-Solecki-Todorcevic [**KST99**, 4.8]). They also point out that Brooks' Theorem implies that there is an invariant, random d-coloring when Γ is a sofic group (for the definition of sofic group, see, e.g., Pestov [**Pes08**]). The question of whether this holds for arbitrary Γ remained open. We show that 6.1 above provides a positive answer. First it will be useful to note the following fact:

PROPOSITION 7.2. Let Γ be an infinite group, S a finite set of generators for Γ and let $k \geq 1$. Then the following are equivalent:

(i) There is an invariant, random k-coloring,

(ii) There is $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $\chi_\mu(S, a) \leq k$.

Proof. (ii) $\Rightarrow$ (i). Let $c\colon X \to \{1, \dots, k\}$ be a measurable coloring of $G(S, a)$. Define $C\colon X \to k^\Gamma$ by $C(x)(\gamma) = c((\gamma^{-1})^a \cdot x)$. Then C is a Borel map from X to Col(k, Γ, S) that preserves the actions, so $C_*\mu$ is an invariant, random k-coloring.

(i) $\Rightarrow$ (ii). Let ρ be an invariant, random k-coloring. Consider the action of Γ on $Y = \mathrm{Col}(k, \Gamma, S)$ (by shift). Fix also a free action $b \in \mathrm{FR}(\Gamma, Z, \nu)$ (for some (Z, ν)). Let $X = Y \times Z, \mu = \rho \times \nu$. Then Γ acts freely, preserving μ on X by $\gamma \cdot (y, z) = (\gamma \cdot y, \gamma \cdot z)$.

Call this action a. We claim that $\chi_\mu(S,a) \leq k$. For this let $c\colon X \to \{1,\dots,k\}$ be defined by $c((y,z)) = y(1)$ (recall that $y \in \mathrm{Col}(k,\Gamma,S)$, so $y\colon \Gamma \to \{1,\dots,k\}$ is a coloring of Cay(Γ,S)). It is easy to check that this a measurable k-coloring of $G(S,a)$. ⊣

REMARK 7.1. From the proof of (ii) ⇒ (i) in 6.2, it is clear that if $a \in \mathrm{FR}(\Gamma,X,\mu)$ has $\chi_\mu(S,a) \leq k$, then there is an invariant, random k-coloring which is a factor of a.

We now have

COROLLARY 7.3. Let Γ be an infinite group and S a finite set of generators. Let $d = |S^{\pm 1}|$. Then there is an invariant, random d-coloring. Moreover, for each $a \in \mathrm{FR}(\Gamma,X,\mu)$ there is such a coloring which is weakly contained in a.

Proof. This is immediate from 6.1 and 6.3. ⊣

Lyons and Schramm (unpublished, 1997) raised the question (see Lyons-Nazarov [**LN11**, §5]) of whether there is, for any Γ, S, an invariant, random χ-coloring, where $\chi = \chi(\mathrm{Cay}(\Gamma,S))$ is the chromatic number of the Cayley graph. It is pointed out in this paper that the answer is affirmative for amenable groups (as there is an invariant measure for the action of Γ on Col(χ,Γ,S) by amenability) but the general question is open.

REMARK 7.2. One cannot in general strengthen the last statement in 6.4 to: For each $a \in \mathrm{FR}(\Gamma,X,\mu)$, there is an invariant, random d-coloring which is a factor of a. Indeed, this fails for $\Gamma = \mathbb{Z}$ or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (with the usual set of generators S for which $d = 2$) and a the shift action of Γ on 2^Γ, since then the shift action of Γ on $\mathrm{Col}(2,\Gamma,S)$ with this random coloring would be mixing and then as in (i) ⇒ (ii) of 6.2, by taking b to be also mixing, one could have a mixing action $a \in \mathrm{FR}(\Gamma,X,\mu)$ for which there is a measurable 2-coloring, which easily gives a contradiction. On the other hand, it follows from the result in [**CK13**, 5.12] that was mentioned earlier, that for any Γ with finitely many ends, except for $\Gamma = \mathbb{Z}$ or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, one indeed has for any $a \in \mathrm{FR}(\Gamma,X,\mu)$ an invariant, random d-coloring which is a factor of the action a. We do not know if this holds for groups with infinitely many ends.

(B) Let Γ, S be as before and let $G_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ be the automorphism group of the Cayley graph with the pointwise convergence topology. Thus $G_{\Gamma,S}$ is Polish and locally compact. The group $G_{\Gamma,S}$ acts continuously on $\mathrm{Col}(k, \Gamma, S)$ by: $\varphi \cdot c(\gamma) = c(\varphi^{-1}(\gamma))$. Clearly Γ can be viewed as a closed subgroup of $G_{\Gamma,S}$ identifying $\gamma \in \Gamma$ with the translation automorphism $\delta \mapsto \gamma\delta$. It will be notationally convenient below to denote this translation automorphism by $\langle\gamma\rangle$. One can now consider a stronger notion of invariant, random k-coloring by asking that the measure is now invariant under $G_{\Gamma,S}$ instead of Γ (i.e., $\langle\Gamma\rangle$). To distinguish the two notions let us call the stronger one a *$G_{\Gamma,S}$-invariant, random k-coloring*. We now note that the existence of an invariant, random k-coloring is equivalent to the existence of $G_{\Gamma,S}$-invariant, random k-coloring. In fact it follows from the following more general fact (applied to the special case of the action of $G_{\Gamma,S}$ on $\mathrm{Col}(k, \Gamma, S)$).

PROPOSITION 7.4. Let $G_{\Gamma,S}$ be as before and assume $G_{\Gamma,S}$ acts continuously on a compact, metrizable space X. Then there exists a Γ-invariant Borel probability measure on X iff there is a $G_{\Gamma,S}$-invariant Borel probability measure on X.

Proof. Denote by $R = R_{\Gamma,S} = \mathrm{Aut}_1(\mathrm{Cay}(\Gamma, S))$ the subgroup of $G = G_{\Gamma,S}$ consisting of all $\varphi \in G$ with $\varphi(1) = 1$ (we view this as the rotation group of Cay(Γ, S) around 1).

It is known that G is unimodular, i.e., there is a left and right invariant Haar measure (see Lyons-Peres [**LP05**, Ex. 7.3]), so fix such a Haar measure η. Since R is compact, open in G, $\infty > \eta(R) > 0$ and we normalize η so that $\eta(R) = 1$. Then $\rho = \eta|R$ is the Haar measure of R.

Next we note that $\Gamma \cap R = \{1\}$ and thus every $\varphi \in G$ can be written as

$$\varphi = \langle\gamma\rangle r = r'\langle\gamma'\rangle$$

for unique $\gamma, \gamma' \in \Gamma, r, r' \in R$. Indeed $\gamma = \varphi(1), r = \langle\gamma\rangle^{-1}\varphi$ and $\gamma' = (\varphi^{-1}(1))^{-1}, r' = \varphi\langle\gamma'\rangle^{-1} = \varphi\langle\varphi^{-1}(1)\rangle$. This gives a map $\alpha \colon \Gamma \times R \to R$ defined by $\alpha(\gamma, r) = r'$, where $\langle\gamma\rangle r = r'\langle\gamma'\rangle$. Thus

$$\alpha(\gamma, r) = \langle\gamma\rangle r\langle r^{-1}(\gamma^{-1})\rangle.$$

One can now easily verify that this is a continuous action of Γ on R and we will write

$$\gamma \cdot r = \alpha(\gamma, r) = \langle\gamma\rangle r \langle r^{-1}(\gamma^{-1})\rangle.$$

Moreover this action preserves the Haar measure ρ.

Indeed, fix $\gamma \in \Gamma$ and put $p_\gamma(r) = \gamma \cdot r$. We will show that $p_\gamma \colon R \to R$ preserves ρ. For $\delta \in \Gamma$, let $R_\delta = \{r \in R \colon r^{-1}(\gamma^{-1}) = \delta\}$. Then $R = \bigsqcup_{\delta \in \Gamma} R_\delta$ and $p_\gamma(r) = \langle\gamma\rangle r \langle\delta\rangle$ for $r \in R_\delta$, thus $p_\gamma | R_\delta$ preserves η and so p_γ preserves ρ.

Assume now that μ_Γ is a Borel probability measure on X which is Γ-invariant. We will show that there is a Borel probability measure μ_G on X which is G-invariant. Define

$$\mu_G = \int_R (r \cdot \mu_\Gamma) dr,$$

where the integral is over the Haar measure ρ on R, i.e., for each continuous $f \in C(X)$,

$$\mu_G(f) = \int_R (r \cdot \mu_\Gamma)(f) dr,$$

with $r \cdot \mu_\Gamma(f) = \mu_\Gamma(r^{-1} \cdot f), r^{-1} \cdot f(x) = f(r \cdot x)$. (As usual we put $\sigma(f) = \int f d\sigma$.) We will verify that μ_G is G-invariant.

Let $F \colon X \to X$ be a homeomorphism. For σ a Borel probability measure on X, let $F \cdot \sigma = F_* \sigma$ be the measure defined by

$$F \cdot \sigma(f) = \sigma(f \circ F^{-1}),$$

for $f \in C(X)$. Then we have

$$F \cdot \mu_G = \int_R F \cdot (r \cdot \mu_\Gamma) dr,$$

because for $f \in C(X)$,

$$\begin{aligned} F \cdot \mu_G(f) &= \mu_G(f \circ F^{-1}) \\ &= \int (r \cdot \mu_\Gamma)(f \circ F^{-1}) dr \\ &= \int F \cdot (r \cdot \mu_\Gamma) dr. \end{aligned}$$

We first check that μ_G is R-invariant. Indeed if $s \in R$,

$$\begin{aligned} s \cdot \mu_G &= \int s \cdot (r \cdot \mu_\Gamma) dr \\ &= \int (sr) \cdot \mu_\Gamma dr \\ &= \int (r \cdot \mu_\Gamma) dr \\ &= \mu_G \end{aligned}$$

by the invariance of Haar measure.

Finally we verify that μ_G is Γ-invariant (which completes the proof that μ_G is G-invariant as $G = \Gamma R$). Indeed, in the preceding notation

$$\begin{aligned} \langle \gamma \rangle \cdot \mu_G &= \int \langle \gamma \rangle \cdot (r \cdot \mu_\Gamma) dr \\ &= \int (\langle \gamma \rangle r) \cdot \mu_\Gamma dr \\ &= \int (\gamma \cdot r) \cdot (\langle \gamma' \rangle \cdot \mu_\Gamma) dr \\ &= \int (\gamma \cdot r) \cdot \mu_\Gamma dr \end{aligned}$$

(as $\langle \gamma' \rangle \cdot \mu_\Gamma = \mu_\Gamma$ for any $\gamma' \in \Gamma$).

But we have seen before that $r \mapsto \gamma \cdot r$ preserves the Haar measure of R, so

$$\begin{aligned}\langle\gamma\rangle \cdot \mu_G &= \int (\gamma \cdot r) \cdot \mu_\Gamma dr \\ &= \int (r \cdot \mu_\Gamma) dr \\ &= \mu_G\end{aligned}$$

$\dashv$

It is well known (see, e.g., Woess [**Woe00**, 12.12]) that if Γ is amenable, so is $G_{\Gamma,S}$. This also follows from 6.6.

(C) As was discussed in 6.5, for any Γ, S with finitely many ends, except $\Gamma = \mathbb{Z}$ or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, and any $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is an invariant, random d-coloring, where $d = |S^{\pm 1}|$, which is a factor of a. This is of particular interest in the case where a is the shift action s_Γ of Γ on $[0,1]^\Gamma$ (with the usual product measure). In that case $G_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ also acts via shift on $[0,1]^\Gamma$ via $\varphi \cdot p(\gamma) = p(\varphi^{-1}(\gamma))$ and one can ask whether there is actually a $G_{\Gamma,S}$-invariant, random d-coloring, which is a factor of the shift action of $G_{\Gamma,S}$ on $[0,1]^\Gamma$. We indeed have:

THEOREM 7.5. Let Γ be an infinite countable group, S a finite set of generators, and let $d = |S^{\pm 1}|$. If Γ has finitely many ends but is not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, and $G_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$, there is a $G_{\Gamma,S}$-invariant, random d-coloring which is a factor of the shift action of $G_{\Gamma,S}$ on $[0,1]^\Gamma$.

Proof. Put again $G = G_{\Gamma,S}$. Let X be the free part of the action of G on $[0,1]^\Gamma$, i.e.,

$$X = \{x \in [0,1]^\Gamma \colon \forall \varphi \in G \setminus \{1\} (\varphi \cdot x \neq x)\},$$

(where $\varphi \cdot x$ is the action of G on $[0,1]^\Gamma$).

If μ is the product measure on $[0,1]^\Gamma$, then $\mu(X) = 1$, since $X \supseteq \{x \in [0,1]^\Gamma \colon x \text{ is } 1-1\} = X_0$ and $\mu(X_0) = 1$. Moreover X is a G-invariant Borel subset of $[0,1]^\Gamma$.

Since $R = \mathrm{Aut}_1(\mathrm{Cay}(\Gamma, S))$ is compact, E^X_R, the equivalence relation induced by R on X, admits a Borel selector and

$$X_R = X/R = \{R \cdot x \colon x \in X\}$$

is a standard Borel space. Define the following Borel graph E on X_R

$$(R \cdot x)E(R \cdot y) \Leftrightarrow \exists s \in S^{\pm 1}(sR \cdot x \cap R \cdot y \neq \varnothing).$$

LEMMA 7.6. If $(R \cdot x)E(R \cdot y)$, then

$$(x_1, x_2) \in M_{R\cdot x, R\cdot y} \Leftrightarrow x_1 \in R \cdot x \ \&\ x_2 \in R \cdot y \ \&\ \exists s \in S^{\pm 1}(\langle s \rangle \cdot x_1 = x_2),$$

(is the graph of) a bijection between $R \cdot x, R \cdot y$ consisting of edges of the graph $G(S, s_\Gamma)$, i.e., it is a matching in this graph.

Proof. Fix $x_1^0 \in R \cdot x_1, x_2^0 \in R \cdot x_2$ and $s_0 \in S^{\pm 1}$ with $\langle s_0 \rangle \cdot x_1^0 = x_2^0$.

First we check that $M_{R\cdot x, R\cdot y}$ is a matching. Let $(x_1, x_2), (x_1, x_2') \in M_{R\cdot x, R\cdot y}$ and let $\langle s \rangle \cdot x_1 = x_2, \langle s' \rangle \cdot x_1 = x_2'$, for some $s, s' \in S^{\pm 1}$, and $r \cdot x_2 = x_2'$, for some $r \in R$. Then $r\langle s \rangle \cdot x_1 = \langle s' \rangle \cdot x_1$, so $r\langle s \rangle = \langle s' \rangle$, thus $r \in \Gamma$, so $r = 1$ and $x_2 = x_2'$. Similarly $(x_1, x_2), (x_1', x_2) \in M_{R\cdot x, R\cdot y}$ implies that $x_1 = x_1'$.

Next we verify that for every $x_1 \in R \cdot x$, there is an $x_2 \in R \cdot y$ with $(x_1, x_2) \in M_{R\cdot x, R\cdot y}$. Let $r_1 \in R$ be such that $r_1 \cdot x_1 = x_1^0$, so $\langle s_0 \rangle r_1 \cdot x_1 = x_2^0$. Now

$$\begin{aligned} \langle s_0 \rangle r_1 &= \big(\langle s_0 \rangle r_1 \langle r_1^{-1}(s^{-1}) \rangle\big) \langle r_1^{-1}(s^{-1}) \rangle^{-1} \\ &= r_2^{-1} \langle s' \rangle, \end{aligned}$$

where $r_2 \in R$ and $s' \in S^{\pm 1}$. Thus $r_2^{-1}\langle s' \rangle \cdot x_1 = x_2^0$, so $\langle s' \rangle \cdot x_1 = r_2 \cdot x_2^0 = x_2 \in R \cdot y$ and $(x_1, x_2) \in M_{R\cdot x, R\cdot y}$. Similarly for every $x_2 \in R \cdot y$ there is $x_1 \in R \cdot x$ with $(x_1, x_2) \in M_{R\cdot x, R\cdot y}$, and the proof is complete. $\dashv$

LEMMA 7.7. Let $x \in X$. Then the map

$$\gamma \mapsto R \cdot (\langle\gamma\rangle^{-1} \cdot x)$$

is an isomorphism of *Cay*(Γ, S) with the connected component of $R \cdot x$ in E.

Proof. Let $\gamma \in \Gamma$ and let $s_1, \dots, s_k \in S^{\pm 1}$ be such that $\gamma^{-1} = s_n \dots s_1$. Then $(R \cdot x)E(R \cdot (\langle s_1\rangle \cdot x))E \dots E(R \cdot (\langle\gamma\rangle^{-1} \cdot x))$, so $R \cdot (\langle\gamma\rangle^{-1} \cdot x)$ is in the connected component of $R \cdot x$. Conversely assume that $R \cdot y$ is in the connected component of $R \cdot x$ and say $(R \cdot x)E(R \cdot x_1)E(R \cdot x_2)E \dots E(R \cdot x_{n-1})E(R \cdot y)$. By Lemma 6.8, there are $s_1, \dots s_n \in S^{\pm 1}$ and $x'_1, \dots, x'_n$ such that $\langle s_1\rangle \cdot x = x'_1 \in R \cdot x_1, \langle s_2\rangle \cdot x'_1 = x'_2 \in R \cdot x_2, \dots, \langle s_n\rangle \cdot x'_{n-1} = x'_n \in R \cdot y$. Let $\gamma^{-1} = s_n \dots s_1$. Then $x'_n = \langle\gamma\rangle^{-1} \cdot x \in R \cdot y$, so $R \cdot (\langle\gamma\rangle^{-1} \cdot x) = R \cdot y$. Thus $\gamma \mapsto R \cdot (\langle\gamma\rangle^{-1} \cdot x)$ maps Γ onto the connected component of $R \cdot x$.

We next check that $\gamma \mapsto R \cdot (\langle\gamma\rangle^{-1} \cdot x)$ is 1-1. Indeed if $R \cdot (\langle\gamma\rangle^{-1} \cdot x) = R \cdot (\langle\delta^{-1}\rangle \cdot x)$, then $r\langle\gamma\rangle^{-1} \cdot x = \langle\delta\rangle^{-1} \cdot x$, for some $r \in R$, so as before $r = 1$ and $\gamma = \delta$.

Finally let $(\gamma, \gamma s)$ be an edge in the Cayley graph of Γ, S. Then clearly $R \cdot (\langle\gamma\rangle^{-1} \cdot x)ER \cdot \langle\gamma s\rangle^{-1} \cdot x) = R \cdot (\langle s\rangle^{-1}\langle\gamma\rangle^{-1} \cdot x)$. Conversely assume that $R(\langle\gamma\rangle^{-1} \cdot x)ER \cdot (\langle\delta\rangle^{-1} \cdot x)$, so that, by 6.8 again, there are $s \in S^{\pm 1}, r \in R$ with $\langle s\rangle\langle\gamma\rangle^{-1} \cdot x = r\langle\delta\rangle^{-1} \cdot x$, i.e., $\langle s\rangle\langle\gamma\rangle^{-1} = r\langle\delta\rangle^{-1}$. Then $r = 1$ and $\gamma s^{-1} = \delta$, so (γ, δ) is an edge in the Cayley graph. $\dashv$

The following will be needed in the next section, so we record it here.

Let $\pi \colon X \to X_R$ be the projection function: $\pi(x) = R \cdot x$. Let $\nu = \pi_* \mu$ be the image of μ.

LEMMA 7.8. E preserves the measure ν.

Proof. Let $\varphi \colon A \to B$ be a Borel bijection with A, B Borel subsets of X_R and graph$(\varphi) \subseteq E$. We will show that $\nu(A) = \nu(B)$.

We have $\nu(A) = \mu(\bigcup_{R \cdot x \in A} R \cdot x)$ and similarly for B. If $\varphi(R \cdot x) = R \cdot y$, then $M_{R \cdot x, R \cdot y}$ gives a Borel bijection of $R \cdot x, R \cdot y$ whose graph consists of edges of $G(S, s_\Gamma)$

and $\bigcup_{R\cdot x\in A} M_{R\cdot x,R\cdot y}$ gives the graph of a Borel bijection of $\bigcup_{R\cdot x\in A} R\cdot x$ with $\bigcup_{R\cdot x\in B} R\cdot x$, therefore $\nu(A)=\nu(B)$. $\dashv$

We now complete the proof of the proposition. Consider the graph (X_R, E). By 7.7, it is a vertex transitive Borel graph with degree $d=|S^{\pm 1}|$ and its connected components have finitely many ends. So by Conley-Kechris [**CK13**, 5.1, 5.7, 5.11] and Lemma 6.9, (X_R, E) has a Borel d-coloring. $C_R\colon X_R\to\{1,\ldots,d\}$. Define now $C\colon X\to\{1,\ldots,d\}$ by

$$C(x)=C_R(R\cdot x)$$

Then clearly C is a Borel d-coloring of $G(S,a)$. We use this as usual to define a random d-coloring of the Cayley graph. Define

$$\psi\colon X\to \mathrm{Col}(d,\Gamma,S)$$

by

$$\psi(x)(\gamma)=C(\langle\gamma\rangle^{-1}\cdot x).$$

and consider the measure $\psi_*\mu$ on $\mathrm{Col}(d,\Gamma,S)$. This will be G-invariant provided that ψ preserves the G-action, which we now verify.

First it is clear that ψ preserves the Γ-action. It is therefore enough to check that it preserves the R-action, i.e., $\psi(r\cdot x)=r\cdot\psi(x)$ for each $x\in X, r\in R$. Let $\gamma\in\Gamma$ in order to check that $\psi(r\cdot x)(\gamma)=(r\cdot\psi(x))(\gamma)$ or $C(\langle\gamma\rangle^{-1}r\cdot x)=\psi(x)(r^{-1}(\gamma))=C(\langle r^{-1}(\gamma)\rangle^{-1}\cdot x)$. But recall that

$$\langle\gamma\rangle^{-1}r=(\langle\gamma\rangle^{-1}r\langle r^{-1}(\gamma)\rangle)\langle r^{-1}(\gamma)\rangle^{-1},$$

so $\langle\gamma\rangle^{-1}r=r'\langle r^{-1}(\gamma)\rangle^{-1}$, for some $r'\in R$, therefore $R\cdot(\langle\gamma\rangle^{-1}r\cdot x)=R\cdot(\langle r^{-1}(\gamma)\rangle^{-1}\cdot x)$ and since $C(y)$ depends only on $R\cdot y$, this completes the proof. $\dashv$

(D) Fix an infinite group Γ and a finite set of generators S, let $G=G_{\Gamma,S}=\mathrm{Aut}(\mathrm{Cay}(\Gamma,S))$ and let $R=R_{\Gamma,S}=\mathrm{Aut}_1(\mathrm{Cay}(\Gamma,S))$ as in the proof of 6.6. Then the action $\gamma\cdot r$ of Γ on R defined there is an action by measure preserving homeomorphisms on the compact, metrizable group R. Provided that Γ, S have the property that R is uncountable, this may provide an interesting example of an action of Γ.

For instance, let $\Gamma = \mathbb{F}_2$, the free group with two generators, and let $S = \{a, b\}$ be a set of free generators. Then it is not hard to see that the action of Γ on R is free (with respect to the Haar measure ρ on R). Indeed, let $\Gamma_n = \{w \in \Gamma \colon |w| = n\}$ (where $|w|$ denotes word length in the generators a, b) and for $w, v \in \Gamma_n$, let $N_{w,v} = \{r \in R \colon r(w) = v\}$. If $v \neq v' \in \Gamma_n$, then $N_{w,v} \cap N_{w,v'} = \varnothing$ and since R acts transitively on Γ_n, there is $r \in R$ with $rv' = v'$, so $rN_{w,v} = N_{w,v'}$ and thus $\rho(N_{w,v}) = N_{w,v'}$. So

$$\rho(N_{w,v}) = \frac{1}{|\Gamma_n|}$$

for $w, v \in \Gamma_n$.

Let now $\gamma \in \Gamma \setminus \{1\}$ and assume that $r \in R$ is such that $\gamma^{-1} \cdot r = \langle\gamma\rangle^{-1} r \langle r^{-1}(\gamma)\rangle = r$ or $\langle\gamma\rangle r = r\langle r^{-1}(\gamma)\rangle$, so for all $\delta \in \Gamma$, $\gamma r(\delta) = r(r^{-1}(\gamma)\delta)$ or $r^{-1}(\gamma)\delta = r^{-1}(\gamma r(\delta))$ and letting $r(\delta) = \epsilon$, we have $r^{-1}(\gamma) r^{-1}(\epsilon) = r^{-1}(\gamma\epsilon)$. Since ϵ was arbitrary in Γ, this shows that $r^{-1}(\gamma^n) = (r^{-1}(\gamma))^n, \forall n \geq 1$. It is thus enough to show that for each $\gamma \in \Gamma \setminus \{1\}, \{r \in R \colon \forall n \geq 1 (r(\gamma^n) = (r(\gamma))^n)\}$ is null. Let $|\gamma^n| = a_n \to \infty$. Then if $\gamma \in \Gamma$, $\{r \in R \colon r(\gamma^n) = (r(\gamma))^n\} \subseteq \bigcup_{\epsilon \in \Gamma_k} \{r \in R \colon r(\gamma^n) = \epsilon^n\}$, so $\rho(\{r \in R \colon r(\gamma^n) = (r(\gamma))^n\}) \leq \sum_{\epsilon \in \Gamma_k} \rho(N_{\gamma^n, \epsilon^n}) \to 0$ as $n \to \infty$. Thus $\{r \in R \colon \forall n \geq 1 (r(\gamma^n) = (r(\gamma))^n)\}$ is null.

8. Matchings

(A) Let Γ be an infinite group and S a finite set of generators for Γ. For $a \in \mathrm{FR}(\Gamma, X, \mu)$, recall that $m(S, a)$ is the matching number of a, defined in §5. If $m(S, a) = \frac{1}{2}$ and the supremum in the definition of $m(S, a)$ is attained, we say that $G(S, a)$ admits an a.e. perfect matching.

Abért and collaborators (private communication) have shown that the Cayley graph $\mathrm{Cay}(\Gamma, S)$ admits a perfect matching.

Let $E_{\Gamma,S}$ be the set of edges of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ and consider the space $2^{E_{\Gamma,S}}$, which we can view as the space of all $A \subseteq E_{\Gamma,S}$. Denote by

$$M(\Gamma, S)$$

the closed subspace consisting of all $M \subseteq E_{\Gamma,S}$ that are perfect matchings of the Cayley graph. The group $G_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ acts on $2^{E_{\Gamma,S}}$ by shift: $\varphi \cdot x(\gamma, \delta) = x(\varphi^{-1}(\gamma), \varphi^{-1}(\delta))$ and so does the subgroup $\Gamma \leq G_{\Gamma,S}$. Clearly $M(\Gamma, S)$ is invariant under this action.

A *$G_{\Gamma,S}$-invariant, random perfect matching* of the Cayley graph is a shift invariant probability Borel measure on $M(\Gamma, S)$. If such a measure is only invariant under the shift action by Γ, we call it an *invariant, random perfect matching*.

Lyons and Nazarov [**LN11**] considered the question of the existence of invariant, random perfect matchings which are factors of the shift of Γ on $[0, 1]^\Gamma$ and showed the following result.

THEOREM 8.1. (Lyons-Nazarov [**LN11**]) Let Γ be a non-amenable group, S a finite set of generators for Γ and assume that $\mathrm{Cay}(\Gamma, S)$ is bipartite (i.e., has no odd cycles). Then there is a $G_{\Gamma,S}$-invariant, random perfect matching, which is a factor of the shift action of $G_{\Gamma,S}$ on $[0, 1]^\Gamma$.

Let us next note some facts that follow from earlier considerations in this paper.

PROPOSITION 8.2. Let Γ be an infinite group and S a finite set of generators for Γ. Then the following are equivalent:

(i) There is an invariant, random perfect matching.
(ii) There is $a \in \mathrm{FR}(\Gamma, X, \mu)$ such that $G(S, a)$ admits an a.e. perfect matching.
(iii) There is a sequence $a_n \in \mathrm{FR}(\Gamma, X, \mu)$ with $m(S, a_n) \to \frac{1}{2}$.

Proof. As in 6.2 and 5.2. $\dashv$

PROPOSITION 8.3. For Γ, S as in 7.2., if $a \in \mathrm{FR}(\Gamma, X, \mu)$ is such that the matching number $m(S, a) = \frac{1}{2}$, then there is $b \in \mathrm{FR}(\Gamma, X, \mu)$ with $b \sim_w a$ and $G(S, b)$ admitting an a.e. perfect matching, and there is an invariant, random perfect matching weakly contained in a.

Proof. As in 6.2 and the proof of 7.2. ⊣

PROPOSITION 8.4. Let $\Gamma, S, G_{\Gamma,S}$ be as before. Then there is an invariant, random perfect matching iff there is a $G_{\Gamma,S}$-invariant, random perfect matching.

Proof. By 6.6. ⊣

We now have

PROPOSITION 8.5. Let Γ be an infinite group and S a finite set of generators.

(i) If Γ is amenable or if S has an element of infinite order, then for any $a \in \mathrm{FR}(\Gamma, X, \mu), m(S, a) = \frac{1}{2}$.

(ii) If S has an element of even order, then for any $a \in \mathrm{FR}(\Gamma, X, \mu), G(S, a)$ admits an a.e. perfect matching.

Proof. i) When Γ is amenable, this follows from the result of Abért and collaborators that Cay(Γ, S) admits a perfect matching, using also the quasi-tiling machinery of Ornstein-Weiss [**OW80**], as in Conley-Kechris [**CK13**, 4.10, 4.11]. The second case follows immediately from Rokhlin's Lemma.

ii) This is obvious. ⊣

We do not know if $m(S, a) = \frac{1}{2}$ holds for *every* $\Gamma, S, a \in \mathrm{FR}(\Gamma, X, \mu)$. By 7.5 the only problematic case is when S consists of elements of odd order and Γ is not amenable. We will see below that the answer is affirmative for the group $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ and the usual set of generators $S = \{s, t\}$ with $s^3 = t^3 = 1$.

We also do not know if for every Γ, S, there is an invariant, random perfect matching (a question brought to our attention by Abért and also Lyons).

(B) We now consider some implications of the following result of Lyons-Nazarov [**LN11**];

THEOREM 8.6. (Lyons-Nazarov [**LN11**, 2.6]) Let (X, μ) be a non-atomic, standard measure space and $G = (X, E)$ a Borel locally countable graph which is bipartite and measure preserving (i.e., the equivalence relation it generates is measure preserving). If G is *expansive*, i.e., there is $c > 1$ such that for each Borel independent set $A \subseteq X, \mu(A') \geq c\mu(A)$, where $A' = \{x\colon \exists y Ex(y \in A)\}$, then G admits an a.e. perfect matching.

We use this to show

THEOREM 8.7. Let $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ with the usual set of generators $S = \{s, t\}$, with $s^3 = t^3 = 1$. Then for any $a \in \mathrm{FR}(\Gamma, X, \mu), G(S, a)$ admits an a.e. perfect matching.

Proof. Let $A = \langle s \rangle = \{1, s, s^2\}$ and $B = \langle t \rangle$. Let $X_A = X/A, X_B = X/B$ and let μ_A, μ_B be the corresponding quotient measures on X_A, X_B, normalized so that $\mu_A(X_A) = \mu_B(X_B) = \frac{1}{2}$. Let $Y = X_A \sqcup X_B, \nu = \mu_A + \mu_B$ and define the following graph F on Y:

$$y_1 F y_2 \Leftrightarrow y_1 \neq y_2 \text{ and } \exists x_1, x_2 \in X[y_1 = A \cdot x_1 \ \&\ y_2 = B \cdot x_2 \ \&\ y_1 \cap y_2 \neq \varnothing].$$

It is not hard to see that this graph satisfies the hypotheses of 7.6, so it admits an a.e. perfect matching, from which it follows that there is a Borel set $T \subseteq X$ that simultaneously meets every A-orbit in exactly one point and every B-orbit in exactly one point, modulo null sets.

Let $P = X \setminus T$ and consider the induced subgraph $G(S, a)|P$. Its connected components look like $\mathbb{Z}$-lines. Then we can find a Borel subset Q of P of very small measure such that it meets every such connected component and two points of Q in the same component ($\mathbb{Z}$-line) are at least 20 apart in this line. Call the elements of Q *markers*. Given two successive markers x, y in one such component, we can neglect points in the interval (x, y) in this line that are within distance at most 5 from x or y (since these have very small measure), so that the rest of this interval looks like a set of points $x_1, x_2, \dots, x_k$ where (x_1, x_2) is an s-edge (i.e., $x_2 = s^{\pm 1} \cdot x_1), (x_2, x_3)$ is a t-edge, (x_3, x_4) an s-edge, etc. Then consider the following edges: An s-edge (x_1, y_1), where $y_1 \in T$, (x_2, x_3), an s-edge (x_4, y_4), where $y_4 \in T$, an s-edge (x_5, y_5) where $y_5 \in T$, etc. This set of edges provides a Borel matching in $G(S, a)$ which covers all of X, except from an arbitrary small measure set, so $m(S, a) = \frac{1}{2}$. $\dashv$

Finally let us note that, using the argument in 6.7, one can show that Theorem 7.6 implies Theorem 7.1.

Proof that 7.6 $\Rightarrow$ 7.1. Using the notation of the proof of 6.7, we first show that the graph E defined there satisfies the hypotheses of 7.6.

LEMMA 8.8. (X_R, E) is bipartite.

Proof. By 6.9. $\dashv$

LEMMA 8.9. (X_R, E) is strictly expanding.

Proof. Let $A \subseteq X_R$ be an independent Borel set and $A' = \{x \in X_R \colon \exists y \in A(xEy)\}$. Since the group Γ is not amenable, the graph $G(S, s_\Gamma)$, where s_Γ is the shift action of Γ on $[0,1]^\Gamma$ is strictly expanding, so let $c > 1$ be the constant witnessing that. We will show that $\nu(A') \geq c\nu(A)$. This is immediate since $\bigcup_{R \cdot x \in A} R \cdot x$ is independent in $G(S, s_\Gamma)$ and $(\bigcup_{R \cdot x \in A} R \cdot x)' = \bigcup_{R \cdot x \in A'} R \cdot x$. $\dashv$

Thus by 7.6, there is an a.e. perfect matching for (X_R, E) which we denote by M_R. Using 6.8 this gives an a.e. perfect matching M for $G(S, s_\Gamma)$ defined by

$$(x, y) \in M \Leftrightarrow (R \cdot x, R \cdot y) \in M_R \ \& \ (x, y) \in M_{R \cdot x, R \cdot y}.$$

Define now

$$\varphi \colon [0,1]^\Gamma \to M(\Gamma, S)$$

by

$$(\gamma, \gamma s) \in \varphi(x) \Leftrightarrow (\langle \gamma \rangle^{-1} \cdot x, \langle s \rangle^{-1} \langle \gamma \rangle^{-1} \cdot x) \in M,$$

for $s \in S^{\pm 1}$. It is enough to show that φ preserves the $G_{\Gamma,S}$-action.

First we check that $\varphi(\langle\delta\rangle \cdot x) = \delta \cdot \varphi(x)$ for $\delta \in \Gamma$. Indeed $(\gamma, \gamma s) \in \varphi(\langle\delta\rangle \cdot x) \Leftrightarrow (\langle\gamma\rangle^{-1}\langle\delta\rangle \cdot x, \langle s\rangle^{-1}\langle\gamma\rangle^{-1}\langle\delta\rangle \cdot x) \in M \Leftrightarrow (\delta^{-1}\gamma, \delta^{-1}\gamma s) \in \varphi(x) \Leftrightarrow (\gamma, \gamma s) \in \delta \cdot \varphi(x)$.

Finally we verify that $\varphi(r \cdot x) = r \cdot \varphi(x)$, for $r \in R$, i.e., $(\gamma, \gamma s) \in \varphi(r \cdot x) \Leftrightarrow (\gamma, \gamma s) \in r \cdot \varphi(x)$. Now

$$(\gamma, \gamma s) \in \varphi(r \cdot x) \Leftrightarrow (\langle\gamma\rangle^{-1} r \cdot x, \langle s\rangle^{-1}\langle\gamma\rangle^{-1} r \cdot x) \in M$$

and

$$\begin{aligned}(\gamma, \gamma s) \in r \cdot \varphi(x) &\Leftrightarrow (r^{-1}(\gamma), r^{-1}(\gamma s)) \in \varphi(x)\\ &\Leftrightarrow \big(\langle r^{-1}(\gamma)\rangle^{-1} \cdot x, \langle s'\rangle^{-1}\langle r^{-1}(\gamma)\rangle^{-1} \cdot x\big) \in M,\end{aligned}$$

where $r^{-1}(\gamma s) = r^{-1}(\gamma)s'$, for some $s' \in S^{\pm 1}$. Now $\langle\gamma\rangle^{-1} r = p\langle\gamma'\rangle$, for some $p \in R$ and $\gamma' = (r^{-1}(\gamma))^{-1}$. We have therefore to show that

$$(p\langle\gamma'\rangle \cdot x, \langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x) \in M \Leftrightarrow (\langle\gamma'\rangle \cdot x, \langle s'\rangle^{-1}\langle\gamma'\rangle \cdot x) \in M.$$

Clearly $p\langle\gamma'\rangle \cdot x, \langle\gamma'\rangle \cdot x$ belong to the same R-orbit, so it is enough to show that $p' = \langle s\rangle^{-1} p\langle s'\rangle \in R$. Because then $\langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x = p'\langle s'\rangle^{-1}\langle\gamma'\rangle \cdot x$ and thus $R \cdot (p\langle\gamma'\rangle \cdot x) = R \cdot (\langle\gamma'\rangle \cdot x) = A$, $R \cdot (\langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x) = R \cdot (\langle s'\rangle^{-1}\langle\gamma'\rangle \cdot x) = B$ and $(p\langle\gamma'\rangle \cdot x, \langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x) \in M \Leftrightarrow (p\langle\gamma'\rangle \cdot x, \langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x) \in M_{A,B} \Leftrightarrow (\langle\gamma'\rangle \cdot x, \langle s'\rangle^{-1}\langle\gamma'\rangle \cdot x) \in M_{A,B}$ (by 6.8). Now $p' \in G_{\Gamma,S}$ and $p'(1) = s^{-1}p(s') = s^{-1}\big((\langle\gamma\rangle^{-1} r\langle\gamma'\rangle^{-1})\langle s'\rangle\big) = s^{-1}\big(\langle\gamma\rangle^{-1} r((\gamma')^{-1}s')\big) = s^{-1}\gamma^{-1} r(r^{-1}(\gamma)s') = s^{-1}\gamma^{-1} r(r^{-1}(\gamma s)) = s^{-1}\gamma^{-1}\gamma s = 1$, so $p' \in R$. $\dashv$

9. Independence numbers

Let Γ be an infinite group and S a finite set of generators. Consider the set

$$I(\Gamma, S) = \{i_\mu(S, a) \colon a \in \mathrm{FR}(\Gamma, X, \mu)\}$$

of independence numbers of actions of Γ. It was shown in Conley-Kechris [**CK13**, §4, **(C)**] that $I(\Gamma, S)$ is a closed interval $[i_\mu(S, s_\Gamma), i_\mu(S, a^{\mathrm{erg}}_{\Gamma,\infty})]$, where s_Γ is the shift action of Γ on

$[0,1]^\Gamma$ and $a^{\text{erg}}_{\Gamma,\infty}$ is the maximum, in the sense of weak containment, free ergodic action. Let

$$I^{\text{erg}}(\Gamma,S)=\{i_\mu(S,a)\colon a\in \mathrm{FR}(\Gamma,X,\mu),\ a \text{ ergodic}\}.$$

The question of understanding the nature of $I^{\text{erg}}(\Gamma,S)$ was raised in Conley-Kechris [**CK13**, §4, **(C)**]. We prove here the following result:

THEOREM 9.1. Let Γ be an infinite group and S a finite set of generators. If Γ has property (T), then $I^{\text{erg}}(\Gamma,S)$ is a closed set.

Proof. Since Γ has property (T), fix finite $Q\subseteq\Gamma$ and $\epsilon>0$ with the following property: If $a\in A(\Gamma,X,\mu)$ and there is a Borel set $A\subseteq X$ with

$$\forall\gamma\in Q(\mu(\gamma^a\cdot A\Delta A)<\mu(A)(1-\mu(A))),$$

then a is not ergodic (see, e.g., Kechris [**Kec10**, 12.6]).

Let now $\iota_n\in I^{\text{erg}}(\Gamma,S),\iota_\mu\to\iota$, in order to show that $\iota\in I^{\text{erg}}(\Gamma,S)$. Let $a_n\in \mathrm{FR}(\Gamma,X,\mu)$ be ergodic with $\iota_\mu(S,a_n)=\iota_n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the action $a=\prod_n a_n/\mathcal{U}$ on $(X_\mathcal{U},\mu_\mathcal{U})$. Then it is clear that there is no non-trivial Γ-invariant element in the measure algebra $\mathrm{MALG}_{\mu_\mathcal{U}}$. Because if $A=[(A_n)]_\mathcal{U}$ was Γ-invariant, with $\mu_\mathcal{U}(A)=\delta,0<\delta<1$, then $\mu_\mathcal{U}(\gamma^a\cdot A\Delta A)=0,\forall\gamma\in A$, so $\lim_{n\to\mathcal{U}}\mu(\gamma^{a_n}\cdot A_n\Delta A_n)=0$ and $\mu(A_n)\to\delta$, so for some n, and all $\gamma\in Q,\mu(\gamma^{a_n}\cdot A_n\Delta A)<\epsilon\mu(A_n)\mu(1-\mu(A_n))$, thus a_n is not ergodic, a contradiction.

Fix also independent sets $A_n\subseteq X$ for a_n with $|\mu(A_n)-\iota_n|<\frac{1}{n}$. Let $A=[(A_n)]_\mathcal{U}$. Then A is independent for a modulo null sets (i.e., $s^a\cdot A\cap A$ is $\mu_\mathcal{U}$-null, $\forall s\in S^{\pm1}$) and $\mu_\mathcal{U}(A)=\iota$. Consider now the factor b of a corresponding to the σ-algebra $\boldsymbol{B}=\sigma(B_0)$, where $\boldsymbol{B}_0$ is a countable Boolean subalgebra of $\mathrm{MALG}_{\mu_\mathcal{U}}$ closed under a, the functions $S_\mathcal{U},T_\mathcal{U}$ of §2, **(B)**, §3, **(B)**, resp., and containing A. We can view b as an element of $\mathrm{FR}(\Gamma,X,\mu)$. First note that b is ergodic, since $\mathrm{MALG}_{\mu_\mathcal{U}}$ and thus $\boldsymbol{B}$ has no Γ-invariant non-trivial sets. We now claim that $\iota_\mu(S,b)=\iota$, which completes the proof. Since $A\in\boldsymbol{B}$, it is clear that $\iota_\mu(S,b)\geq\mu_\mathcal{U}(A)=\iota$. So assume that $\iota_\mu(S,b)>\iota$ towards a contradiction, and let $B \in \mathrm{MALG}_{\mu_{\mathcal{U}}}$ be such that $s^a \cdot B \cap B = \varnothing, \forall s \in S^{\pm 1}$, and $\mu_{\mathcal{U}}(B) = \kappa > \iota$. We can assume of course that $B = [(B_n)]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}$, so $\lim_{n \to \mathcal{U}} \mu(B_n) = \kappa$ and $\lim_{n\to\mathcal{U}} \mu(s^{a_n} \cdot B_n \cap B_n) = 0, \forall s \in S^{\pm 1}$. Let $C_n = B_n \setminus s^{a_n} \cdot B_n$, so that $s^{a_n} \cdot C_n \cap C_n = \varnothing$ and $\mu(C_n) = \mu(B_n) - \mu(s^{a_n} \cdot B_n \cap B_n)$, thus $\lim_{n\to\mathcal{U}} \mu(C_n) = \lim_{n\to\mathcal{U}} \mu(B_n) = \kappa > \iota$. Since $\iota_n \to \iota$, for all large enough n, $\iota_n < \frac{\iota+\kappa}{2}$ and thus for some $U \in \mathcal{U}$, and any $n \in U, \mu(C_n) > \frac{\iota+\kappa}{2}$ but $\iota_\mu(S, a_n) = \iota_n < \frac{\iota+\kappa}{2}$. Since C_n is an independent set for a_n, this gives a contradiction. $\dashv$

Similar arguments show that the set of matching numbers $m(S, a), a \in \mathrm{FR}(\Gamma, X, \mu)$, is the interval $[m(S, s_\Gamma), m(S, a^{\mathrm{erg}}_{\Gamma,\infty})]$, and the set of matching numbers of the ergodic, free actions is a closed set, if Γ has property (T).

10. Sofic actions

(A) Recall that a group G is *sofic* if for every finite $F \subseteq G$ and $\epsilon > 0$, there is $n \geq 1$ and $\pi\colon F \to S_n$ (= the symmetric group on $n = \{0, \ldots, n-1\}$) such that (denoting by id_X the identity map on a set X):

(i) $1 \in F \Rightarrow \pi(1) = \mathrm{id}_n$,

(ii) $\gamma, \delta, \gamma\delta \in F \Rightarrow \mu_n(\{m\colon \pi(\gamma)\pi(\delta)(m) \neq \pi(\gamma\delta)(m)\}) < \epsilon$,

(iii) $\gamma \in F \setminus \{1\} \Rightarrow \mu_n(\{m\colon \pi(\gamma)(m) = m\}) < \epsilon$,

where μ_n is the normalized counting measure on n.

Elek-Lippner [**EL10**] have introduced a notion of soficity for equivalence relations. We give an equivalent definition due to Ozawa [**Oza**].

Let (X, μ) be a standard measure space and E a measure preserving, countable Borel equivalence relation on X. We let

$$[[E]] = \{\varphi\colon \varphi \text{ is a Borel bijection } \varphi\colon A \to B,$$
$$\text{where } A, B \text{ are Borel subsets of } X \text{ and}$$
$$xE\varphi(x), \mu\text{-a.e. } (x \in A)\}.$$

We identify φ, ψ as above if their domains are equal modulo null sets and they agree a.e. on their domains. We define the *uniform metric* on $[[E]]$ by

$$\delta_X(\varphi, \psi) = \mu(\{x \colon \varphi(x) \neq \psi(x)\}),$$

where

$$\varphi(x) \neq \psi(x)$$

means that

$$x \in \mathrm{dom}(\varphi) \Delta \mathrm{dom}(\psi)$$

or

$$x \in \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi) \ \& \ \varphi(x) \neq \psi(x).$$

If $\varphi \colon A \to B$ we put dom$(\varphi) = A$, rng$(\varphi) = B$. If $\varphi \colon A \to B, \psi \colon C \to D$ are in $[[E]]$, we denote by $\varphi\psi$ their composition with dom$(\varphi\psi) = C \cap \psi^{-1}(A \cap D)$ and $\varphi\psi(x) = \varphi(\psi(x))$ for $x \in \mathrm{dom}(\varphi\psi)$. If $(\varphi_i)_{i \in I}$, I countable, is a pairwise disjoint family of elements of $[[E]]$, i.e., dom$(\varphi_i), i \in I$, are pairwise disjoint and rng$(\varphi_i), i \in I$, are pairwise disjoint, then $\bigsqcup_{i \in I} \varphi_i \in [[E]]$, is the union of the $\varphi_i, i \in I$. If $\varphi \colon A \to B$ is in $[[E]]$, we denote by $\varphi^{-1} \colon B \to A$ the inverse function, which is also in $[[E]]$. Finally if $X = n$ and $\mu = \mu_n$ is the normalized counting measure, we denote by $[[n]]$ the set of all injections between subsets of n (thus $[[n]] = [[E]]$, where $E = n \times n$). We denote by δ_n the corresponding uniform (or Hamming) metric on $[[n]]$, so $\delta_n(\varphi, \psi) = \frac{1}{n}|\{m \colon \varphi(m) \neq \psi(m)\}|$.

DEFINITION 10.1. A measure preserving countable Borel equivalence relation E on a non-atomic standard measure space (X, μ) is *sofic* if for each finite $F \subseteq [[E]]$ and each $\epsilon > 0$, there is $n \geq 1$ and $\pi \colon F \to [[n]]$ such that

i) $\mathrm{id}_X \in F \Rightarrow \pi(\mathrm{id}_X) = \mathrm{id}_n$,

ii) $\varphi, \psi, \varphi\psi \in F \Rightarrow \delta_n(\pi(\varphi\psi), \pi(\varphi)\pi(\psi)) < \epsilon$,

iii) $\varphi \in F \Rightarrow \big|\mu(\{x \colon \varphi(x) = x\}) - \mu_n(\{m \colon \pi(\varphi)(m) = m\})\big| < \epsilon$.

We do not know if this definition is equivalent to the one in which $[[E]]$ is replaced by the *full group* $[E] = \{\varphi \in [[E]] \colon \mu(\text{dom}(\varphi)) = 1\}$ and $[[n]]$ by S_n, i.e., the soficity of the full group.

The following two facts, brought to our attention in a seminar talk by Adrian Ioana, can be proved by routine but somewhat cumbersome calculations.

PROPOSITION 10.2. For F, ϵ, n, π as in 9.1, if $\varphi, \psi \in F$ and $\delta_X(\varphi, \psi) < \epsilon$, then $\delta_n(\pi(\varphi), \pi(\psi)) < 10\epsilon$.

PROPOSITION 10.3. Let E be a measure preserving countable Borel equivalence relation on a non-atomic standard measure space (X, μ). Let F_m, $m \in \mathbb{N}$ be finite subsets of $[[E]]$ with $F_0 \subseteq F_1 \subseteq \dots F_m^{-1} = F_m$, $\varnothing \notin F_m$ (where $\varnothing$ is the empty function) and $\text{id}_{\text{dom}(\varphi)} \in F_m$ for any $\varphi \in F_m$. Let $\bigoplus F_m = \{\bigsqcup_{i=1}^k \varphi_i \colon \varphi_i \in F_m\} \subseteq [[E]]$. If $\bigcup_m(\bigoplus F_m)$ is dense in $[[E]]$ and for every m and every $\epsilon > 0$, 9.1 holds for $F = F_m F_m = \{\varphi\psi \colon \varphi, \psi \in F_m\}$ and $\epsilon > 0$, then E is sofic.

We next define sofic actions. For (X, μ) a non-atomic, standard measure space and Γ a countable group, for each $a \in A(\Gamma, X, \mu)$, denote by E_a the induced equivalence relation (defined modulo null sets)

$$xE_a y \Leftrightarrow \exists \gamma \in \Gamma(\gamma^a \cdot x = y).$$

DEFINITION 10.4. An action $a \in A(\Gamma, X, \mu)$ is *sofic* if E_a is sofic.

Let now $\boldsymbol{A}_0$ be any countable Boolean subalgebra of MALG_μ closed under an action $a \in A(\Gamma, X, \mu)$ and generating MALG_μ. Let $\Gamma = \{\gamma_n \colon n \in \mathbb{N}\}$, and let $(A_m)_{m \in \mathbb{N}}$ enumerate the elements of $\boldsymbol{A}_0$ of positive measure. Let $(\varphi_i^a)_{i \in \mathbb{N}}$ enumerate the family of elements of $[[E_a]]$ of the form $\gamma_n^a | A_m, n, m \in \mathbb{N}$. Then by 9.3 we have the following criterion. (Notice that if $F_m = \{\varphi_0^a, \dots, \varphi_m^a\} \cup \{(\varphi_0^a)^{-1}, \dots, (\varphi_m^a)^{-1}\}$, then $F_m F_m \subseteq \{\varphi_0^a, \varphi_1^a, \dots\}$ and $\bigcup_m(\bigoplus F_m)$ is dense in $[[E_a]]$.)

PROPOSITION 10.5. The action $a \in A(\Gamma, X, \mu)$ is sofic provided that for each m and $\epsilon > 0$, 9.1 holds for $F = \{\varphi_0^a, \dots, \varphi_m^a\}$ and ϵ.

We now have the following fact.

PROPOSITION 10.6. Let (X, μ) be a non-atomic standard measure space. Then the set of sofic actions in $\mathrm{FR}(\Gamma, X, \mu)$ is closed in $\mathrm{FR}(\Gamma, X, \mu)$. In particular, if $a, b \in \mathrm{FR}(\Gamma, X, \mu)$, b is sofic and $a \prec b$, then a is sofic.

Proof. Suppose $a_n, a \in \mathrm{FR}(\Gamma, X, \mu), a_n \to a$ and each a_n is sofic. We will show that a is sofic. Fix a countable Boolean algebra $\boldsymbol{A}_0$ which generates MALG_μ and is closed under all the $a_n, n \in \mathbb{N}$ and a. Let $(\gamma_n), (A_m), (\varphi_i^a)$ be as before for the action a, so that (φ_i^a) enumerates all $\gamma_n^a | A_m$. For $m, \epsilon > 0$ we want to verify 9.1 for $F = \{\varphi_0^a, \dots, \varphi_m^a\}, \epsilon > 0$. Say, for $i \leq m, \varphi_i^a = \delta_i^a | B_i$, where $\delta_i \in \Gamma, B_i \in \boldsymbol{A}_0$. Note that δ_i is uniquely determined by the freeness of the action a.

Choose N large enough so that $\mu(B_i \cap (\delta_j^{-1})^{a_N} \cdot B_i) \Delta (B_i \cap (\delta_j^{-1})^a \cdot B_i)) < \frac{\epsilon}{20}$, for $i, j \leq m$ and let $\psi_i = \delta_i^{a_N} | B_i, i \leq m$. Let then $\pi_N \colon \{\psi_0, \dots, \psi_m\} \to [[n]]$ satisfy 9.1 with $\frac{\epsilon}{20}$. Put $\pi(\varphi_i^a) = \pi_N(\psi_i)$. We will show that this satisfies i)-iii) of 9.1. It is clear that i) holds.

For iii): Given $\varphi_i, 1 \leq i \leq m$, note that $\mu(\{x \colon \varphi_i^a(x) = x\}) = \mu(B_i)$, if $\delta_i = 1$, and $\mu(\{x \colon \varphi_i^a(x) = x\}) = 0$, if $\delta_i \neq 1$. Thus $\mu(\{x \colon \varphi_i^a(x) = x\}) = \mu(\{x \colon \psi_i(x) = x\})$ and so iii) is clearly true.

For ii): Assume $i, j \leq m$ and for some $k \leq m, \varphi_i \varphi_j = \varphi_k$. Then

$$\begin{aligned}
\varphi_i \varphi_j &= \delta_i^a \delta_j^a | \big(B_j \cap (\delta_j^{-1})^a \cdot B_i\big) \\
&= (\delta_i \delta_j)^a | \big(B_j \cap (\delta_j^{-1})^a \cdot B_i\big) \\
&= \delta_k^a | B_k,
\end{aligned}$$

so $\delta_k = \delta_i \delta_j$ and $B_k = B_j \cap (\delta_j^{-1})^a \cdot B_i$. Then $\psi_i = \delta_i^{a_N} | B_i, \psi_j = \delta_j^{a_N} | B_j, \psi_i \psi_j = \delta_i^{a_N} \delta_j^{a_N} | B_j \cap (\delta_j^{-1})^{a_N} \cdot B_i, \psi_k = (\delta_i \delta_j)^{a_N} | B_j \cap (\delta_j^{-1})^a \cdot B_i$. Therefore $\delta_X(\psi_i \psi_j, \psi_k) < \frac{\epsilon}{20}$.

Then, by 9.2, $\delta_n(\pi_N(\psi_i\psi_j), \pi(\psi_k)) < \frac{\epsilon}{2}$. Therefore

$$\begin{aligned}
&\delta_n(\pi(\varphi_i\varphi_j), \pi(\varphi_i)\pi(\varphi_j)) \\
&\qquad = \delta_n(\pi(\varphi_k), \pi(\varphi_i)\pi(\varphi_j)) \\
&\qquad = \delta_n(\pi_N(\psi_k), \pi_N(\psi_i)\pi_N(\psi_j)) \\
&\qquad \leq \delta_n(\pi_N(\psi_k), \pi_N(\psi_i\psi_j)) + \delta_n(\pi_N(\psi_i\psi_j), \pi_N(\psi_i)\pi_n(\psi_j)) \\
&\qquad < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon
\end{aligned}$$

and the proof is complete. $\dashv$

(B) Consider now a sofic group Γ and fix an increasing sequence $1 \in F_0 \subseteq F_1 \subseteq \dots$ of finite subsets of Γ with $\bigcup_n F_n = \Gamma$. For each n, let X_n be a finite set of cardinality $\geq n$ with the normalized counting measure μ_n such that there is a map $\pi_n \colon F_n \to S_{X_n}$ (= the permutation group of X_n) such that

i) $\pi_n(1) = \mathrm{id}_{X_n}$,
ii) $\gamma, \delta, \gamma\delta \in F_n \Rightarrow \mu_n(\{x \colon \pi(\gamma)\pi(\delta)(x) \neq \pi(\gamma\delta)(x)\}) < \frac{1}{n}$,
iii) $\gamma \in F_n \setminus \{1\} \Rightarrow \mu_n(\{x \colon \pi(\gamma)(x) = x\}) < \frac{1}{n}$.

Define then $a_n \colon \Gamma \times X \to X$ by

$$a_n(\gamma, x) = \pi_n(\gamma)(x)$$

Then abbreviating $a_n(\gamma, x)$ by $\gamma \cdot_n x$ we have

i) $1 \cdot_n x = x$
ii) $\gamma, \delta, \gamma\delta \in F_n \Rightarrow \mu_n(\{x \colon \gamma\delta \cdot_n x \neq \gamma \cdot_n (\delta \cdot_n x)\}) < \frac{1}{n}$,
iii) $\gamma \in F_n \setminus \{1\} \Rightarrow \mu_n(\{x \colon \gamma \cdot_n x = x\}) < \frac{1}{n}$.

So we can view a_n as an "approximate" free action of Γ on X_n.

Fix now a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and let $X_{\mathcal{U}} = (\prod_n X_n)/\mathcal{U}$ and $\mu_{\mathcal{U}}$ the corresponding measure on the σ-algebra $\boldsymbol{B}_{\mathcal{U}}$ of $X_{\mathcal{U}}$. By 2.5 this is non-atomic. As in §3, we can also define an action $a_{\mathcal{U}}$ on Γ on $X_{\mathcal{U}}$ by

$$\gamma^{a_{\mathcal{U}}} \cdot [(x_n)]_{\mathcal{U}} = [(\gamma \cdot_n x_n)]_{\mathcal{U}}$$

(note that $\gamma \cdot_n x_n$ is well-defined for $\mathcal{U}$-almost all n). This action is measure preserving and, by iii) above, it is free, i.e., for $\gamma \in \Gamma \setminus \{1\}, \mu_{\mathcal{U}}(\{x \in X_{\mathcal{U}} \colon \gamma^{a_{\mathcal{U}}} \cdot x \neq x\}) = 0$ (see 3.2). So let $\boldsymbol{B}_0$ be a countable subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ closed under the action $a_{\mathcal{U}}$, the function $S_{\mathcal{U}}$ of §2, **(B)** and $T_{\mathcal{U}}$ of §3, **(B)**. Let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ and let b be the factor corresponding to $\boldsymbol{B}$. Then $b \in \mathrm{FR}(\Gamma, X, \mu)$, for a non-atomic standard measure space (X, μ).

We use this construction to give another proof of the following result:

THEOREM 10.7. (Elek-Lippner [**EL10**]). Let Γ be an infinite sofic group and let s_Γ be the shift action of Γ on $[0,1]^\Gamma$. Then s_Γ is sofic.

Proof. Consider the factor b as in the preceding discussion. By Abért-Weiss [**AW11**], $s_\Gamma \prec b$, thus using 9.6, it is enough to show that b is sofic. Using 9.5, it is clearly enough to show the following: For any $\gamma_1, \dots, \gamma_k \in \Gamma, [(A_n^1)]_{\mathcal{U}}, \dots, [(A_n^k)]_{\mathcal{U}} \in \boldsymbol{B}_0$ of positive measure and $\epsilon > 0$, letting $\varphi_i = \gamma^{a_{\mathcal{U}}} | [A_n^i)]_{\mathcal{U}}$, there is n and a map $\pi \colon \{\varphi_i \colon i \leq k\} \to [[X_n]]$ (the set of injections between subsets of X_n) such that

i) $\varphi_i = \mathrm{id}_X \Rightarrow \pi(\varphi_i) = \mathrm{id}_{X_n}$,

ii) If $i, j, \ell \leq k$ and $\varphi_i \varphi_j = \varphi_\ell$, then $\mu_n(\{x \colon \pi(\varphi_i)\pi(\varphi_j)(x) \neq \pi(\varphi_\ell)(x)\}) < \epsilon$,

iii) $|\mu(\{x \colon \varphi_i(x) = x\}) - \mu_n(\{x \colon \pi(\varphi_i)(x) = x\})| < \epsilon$.

Since $a_{\mathcal{U}}$ is free, note that $\varphi_i = \gamma_i^{a_{\mathcal{U}}} | [(A_n^i)]_{\mathcal{U}}$ uniquely determines γ_i. Choose now $n \in \mathcal{U}$ so that:

a) $\mu_n(\{x \colon \gamma_\ell \cdot_n x \neq \gamma_i \cdot_n (\gamma_j \cdot_n x)\}) < \frac{\epsilon}{2}$, if $\gamma_\ell = \gamma_i \gamma_j$ $(i, j, \ell \leq k)$,

b) $\mu_n(\{x \colon \gamma_i \cdot_n x = x\}) < \epsilon$, if $\gamma_i \neq 1$,

c) $\mu_n(A_n^\ell \Delta (A_n^j \cap \gamma_j^{-1} \cdot_n A_n^i)) < \frac{\epsilon}{2}$, if $\varphi_i \varphi_j = \varphi_\ell$ $(i, j, \ell \leq k)$.

Note that c) is possible since $[(A_n^\ell)]_{\mathcal{U}}$ is the domain of φ_ℓ, while $[(A_n^j)]_{\mathcal{U}} \cap (\gamma_j^{-1})^{a_{\mathcal{U}}} \cdot [(A_n^i)]_{\mathcal{U}}$ is the domain of $\varphi_i \varphi_j$, thus $0 = \mu_{\mathcal{U}}\big([(A_n^\ell)]_{\mathcal{U}} \Delta ([(A_n^j)]_{\mathcal{U}} \cap (\gamma_j^{-1})^{a_{\mathcal{U}}} \cdot [(A_n^i)]_{\mathcal{U}})\big) = \lim_{n \to \mathcal{U}} \mu_n(A_n^\ell \Delta (A_n^j \cap \gamma_j^{-1} \cdot_n A_n^i))$. Now define

1) $\pi(\varphi_i) = \mathrm{id}_{X_n}$, if $\varphi_i = \mathrm{id}_X$,

2) $\pi(\varphi_i) = \gamma_i^{a_n} | A_n^i$, otherwise,

where as usual $\gamma_i^{a_n}(x) = a_n(\gamma_i, x)$. We claim that this works. Clearly i) is satisfied. Also iii) is satisfied. This is trivial if $\varphi_i = \mathrm{id}_X$. Otherwise $\gamma_i \neq 1, \mu(\{x \colon \varphi_i(x) = x\}) = 0$ and $\mu_n(\{x \colon \pi(\varphi_i)(x) = x\}) \leq \mu_n(\{x \colon \gamma_i \cdot_n x = x\}) < \epsilon$. Finally for ii), assume $\varphi_i \varphi_j = \varphi_\ell$ $(i, j, \ell \leq k)$. Then $\gamma_i \gamma_j = \gamma_\ell$ and so

$$\mu_n(\{x \colon \gamma_\ell \cdot_n x \neq \gamma_i \cdot_n (\gamma_j \cdot_n x)\}) < \frac{\epsilon}{2},$$

thus

$$\mu_n(\{x \colon \pi(\varphi_\ell)(x) \neq \pi(\varphi_i)\pi(\varphi_j)(x)\}) \leq$$

$$\mu_n((A_n^\ell \Delta (A_n^j \cap \gamma_j^{-1} \cdot_n A_n^i))) + \mu_n(\{x \colon \gamma_\ell \cdot_n x \neq \gamma_i \cdot_n (\gamma_j \cdot_n x)\}) < \epsilon.$$

$\dashv$

(C) It is a well known problem whether every countable group is sofic. Elek-Lippner [**EL10**] also raised the question of whether every measure preserving, countable Borel equivalence relation on a standard measure space is sofic. They also ask the question of whether *every* free action $a \in \mathrm{FR}(\Gamma, X, \mu)$ of a sofic group Γ is sofic. They show that all treeable equivalence relations are sofic and thus every strongly treeable group (i.e., for which all free actions are treeable) has the property that all its free actions are sofic. These groups include the amenable and the free groups. Another class of groups that has this property is the class MD discussed in Kechris [**Kec12**]. A group Γ is in MD if it is residually finite and its finite actions (i.e., actions that factor through an action of a finite group) are dense in $A(\Gamma, X, \mu)$. These include residually finite amenable groups, free groups, and (Bowen) surface groups, and lattices in $\mathrm{SO}(3,1)$. Moreover MD is closed under subgroups and finite index extensions.

To see that every free action of a group in MD is sofic, note that by Kechris [**Kec12**, 4.8] if $a \in \mathrm{FR}(\Gamma, X, \mu)$, then $a \prec \iota_\Gamma \times p_\Gamma$, where ι_Γ is the trivial action of Γ on (X, μ)

and p_Γ the translation action of Γ on its profinite completion on $\hat{\Gamma}$. It is easy to check that $\iota_\Gamma \times p_\Gamma$ is sofic and thus a is sofic by 9.6.

We note that the fact that every free group Γ has MD and thus every free action of Γ is sofic can be used to give an alternative proof of the result of Elek-Lippner [**EL10**] that every measure preserving, treeable equivalence relation is sofic. Indeed it is a known fact that if E is such an equivalence relation on (X, μ), then there is $a \in \mathrm{FR}(\mathbb{F}_\infty, X, \mu)$ such that $E \subseteq E_a$. This follows, for example, by the method of proof of Conley-Miller [**CM10**, Prop. 8] or by using [**CM10**, Prop 9], that shows that $E \subseteq F$ where F is treeable of infinite cost, and then using Hjorth's result (see [**KM04**, 28.5]) that F is induced by a free action of $\mathbb{F}_\infty$. Since E_a is sofic and $[[E]] \subseteq [[E_a]]$, it immediately follows that E is sofic.

We do not know if every measure preserving treeable equivalence relation E is contained in some E_a, where $a \in \mathrm{FR}(\mathbb{F}_2, X, \mu)$.

Remark. For arbitrary amenable groups Γ, one can use an appropriate Følner sequence to construct a free action $a_\mathcal{U}$ on an ultrapower of finite sets as in §9, **(A)**. Then using an argument as in Kamae [**Kam82**], one can see that *every* action of Γ is a factor of this ultrapower (and thus as in 9.6 again every such action is sofic).

11. Concluding remarks

There are sometimes alternative approaches to proving some of the results in this paper using weak limits in appropriate spaces of measures instead of ultrapowers.

One approach is to replace the space of actions $A(\Gamma, X, \mu)$ by a space of invariant measures for the shift action of Γ on $[0,1]^\Gamma$ as in Glasner-King [**GK96**].

Let $R(X, \mu)$ be a non-atomic, standard measure space. Without loss of generality, we can assume that $X = [0,1], \mu = \lambda =$ Lebesgue measure on [0,1]. Denote by $\mathrm{SIM}_\mu(\Gamma)$ the compact (in the weak*-topology) convex set of probability Borel measures ν on $[0,1]^\Gamma$ which are invariant under the shift action s_Γ, such that the marginal $(\pi_1)_*\nu = \mu$ (where $\pi_1 \colon [0,1]^\Gamma \to [0,1]$ is defined by $\pi_1(x) = x(1)$). For $a \in A(\Gamma, X, \mu)$ let $\varphi^a \colon [0,1] \to [0,1]^\Gamma$

be the map $\varphi^a(x)(\gamma) = (\gamma^{-1})^a \cdot x$, and let $(\varphi^a)_*\mu = \mu_a \in \mathrm{SIM}_\mu(\Gamma)$. Then $\Phi(a) = \mu_a$ is a homeomorphism of $A(\Gamma, X, \mu)$ with a dense, G_δ subset of SIM$_\mu(\Gamma)$ (see [**GK96**]).

One can use this representation of actions to give another proof of Corollary 4.5.

If $a_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, is given, consider $\mu_n = \mu_{a_n} \in \mathrm{SIM}_\mu(\Gamma)$ as above. Then there is a subsequence $n_0 < n_1 < n_2 < \dots$ such that $\mu_{n_i} \to \mu_\infty \in \mathrm{SIM}_\mu(\Gamma)$ (convergence is in the weak*-topology of measures). Then μ_∞ is non-atomic, so we can find $a_\infty \in A(\Gamma, X, \mu)$ such that a_∞ on (X, μ) is isomorphic to s_Γ on $([0,1]^\Gamma, \mu_\infty)$. One can then show (as in the proof of (1) $\Rightarrow$ (3) in 4.3) that there are $b_{n_i} \cong a_{n_i}, b_{n_i} \in A(\Gamma, X, \mu)$ such that $b_{n_i} \to a_\infty$. (Similarly if we let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and $\mu^\mathcal{U} = \lim_{n\to\mathcal{U}} \mu_n$ and $a^\mathcal{U}$ in $A(\Gamma, X, \mu)$ is isomorphic to s_Γ on $([0,1]^\Gamma, \mu^\mathcal{U})$, then there are $b_n \in A(\Gamma, X, \mu), b_n \cong a_n$ with $\lim_{n\to\mathcal{U}} b_n = a^\mathcal{U}$.)

For other results, related to graph combinatorics, one needs to work with shift-invariant measures on other spaces. Let Γ be an infinite group with a finite set of generators S. We have already introduced in §6 the compact space Col(k, Γ, S) of k-colorings of Cay(Γ, S) and in §7 the compact space $M(\Gamma, S)$ of perfect matchings of Cay(Γ, S). On each one of these we have a canonical shift action of Γ and we denote by INV$_{\mathrm{Col}}(\Gamma, S)$, INV$_M(\Gamma, S)$ the corresponding compact spaces of invariant, Borel probability measures (i.e., the spaces of invariant, random k-colorings and invariant, random perfect matchings, resp.). Similarly, identifying elements of 2^Γ with subsets of Γ, we can form the space Ind(Γ, S) of all independent in Cay(Γ, S) subsets of Γ. This is again a closed subspace of 2^Γ which is shift invariant and we denote by INV$_{\mathrm{Ind}}(\Gamma, S)$ the compact space of invariant, Borel measures on Ind(Γ, S), which we can call *invariant, random independent sets*.

If $a \in \mathrm{FR}(\Gamma, X, \mu)$ and $A \subseteq X$ is a Borel independent set for $G(S, a)$, then we define the map

$$I_A \colon X \to \mathrm{Ind}(\Gamma, S),$$

given by

$$\gamma \in I_A(x) \Leftrightarrow (\gamma^{-1})^a \cdot x \in A.$$

This preserves the Γ-actions, so $(I_A)_*\mu = \nu \in \mathrm{INV}_{\mathrm{Ind}}(\Gamma, S)$. Moreover $\nu(\{B \in \mathrm{IND}(\Gamma, S) \colon 1 \in B\}) = \mu(A)$. If $i_\mu(S, a) = \iota$ and $A_n \subseteq X$ are Borel independent sets with $\mu(A_n) \to \iota$, let $\nu_n = (I_{A_n})_*\mu$. Then the shift action a_n on $(\mathrm{IND}(\Gamma, S), \mu_n)$ may not be free but one can still define independent sets for this action as being those C such that $s^{a_n} \cdot C \cap C = \varnothing$ (modulo null sets) and also the independence number $\iota_{\nu_n}(s, a_n)$ as before. We can also assume, by going to a subsequence, that $\nu_n \to \nu_\infty$. Denote by a_∞ the shift action for $(\mathrm{Ind}(\Gamma, S), \nu_\infty)$. Then $\{B \in \mathrm{IND}(\Gamma, S) \colon 1 \in B\}$ is independent for a_n and a_∞, so $\iota_{\nu_\infty}(S, a_n) \geq \iota$. But also $\iota_{\nu_n}(S, a_n) \leq \iota_\mu(S, a)$ and from this, it follows by a simple approximation argument that $\iota_{\nu_\infty}(S, a_\infty) \leq \iota$, so $\iota_{\nu_\infty}(S, a_\infty) = \iota$ and the sup is attained. This gives a weaker version of 5.2 (iii). Although one can check that $a_\infty \prec a$, it is not clear that a_∞ is free and moreover we do not necessarily have that $a \sqsubseteq a_\infty$. This would be remedied if we could replace a_∞ by $a_\infty \times a$, but it is not clear what the independence number of this product is. This leads to the following question: Let $a, b \in \mathrm{FR}(\Gamma, X, \mu)$ and consider $a \times b \in \mathrm{FR}(\Gamma, X^2, \mu^2)$. It is clear that $\iota_{\mu^2}(a \times b) \geq \max\{\iota_\mu(a), \iota_\mu(b)\}$. Do we have equality here?

Similar arguments can be given to prove weaker versions of 5.2 (iii), (iv).

However a weak limit argument as above (but for the space of colorings) can give an alternative proof of 6.4 using the "approximate" version of Brooks' Theorem in Conley-Kechris [**CK13**] (this was pointed out to us by Lyons). Indeed let $a \in \mathrm{FR}(\Gamma, S, \mu), d = |S^{\pm 1}|$. By Conley-Kechris [**CK13**, 2.9] and Kechris-Solecki-Todorcevic [**KST99**, 4.8], there is $k > d$ and for each n, a Borel coloring $c_n \colon X \to \{1, \dots, k\}$ such that $\mu(c_n^{-1}(\{d + 1, \dots, k\})) < \frac{1}{n}$. Let as usual $C_n \colon X \to \mathrm{Col}(k, \Gamma, S)$ be defined by $C_n(x)(\gamma) = c_n((\gamma^{-1})^a \cdot x)$. Let $(C_n)_*\mu = \nu_n$. Then $\nu_n(\{c \in \mathrm{Col}(k, \Gamma, S) \colon c(1) > d\}) = \mu(C_n^{-1}(\{d{+}1, \dots, k\})) < \frac{1}{n}$. By going to a subsequence we can assume that $\nu_n \to \nu$, an invariant, random k-coloring. Now $\nu(\{c \in \mathrm{Col}(k, \Gamma, S) \colon c(1) > d\}) = 0$, thus ν concentrates on $\mathrm{Col}(d, \Gamma, S)$ and thus is an invariant, random d-coloring. Moreover it is not hard to check that it is weakly contained in a.

A similar argument can be used to show that for every Γ, S except possibly non-amenable Γ with S consisting of elements of odd order, there is an invariant, random perfect matching (see 7.5).

Finally one can obtain by using weak limits in $\mathrm{INV}_{\mathrm{Ind}}(\Gamma, S)$ and the result in Glasner-Weiss [**GW97**], that if Γ has property (T) and $c_n \in I^{\mathrm{erg}}(\Gamma, S)$, $\iota_{\mu_n}(\Gamma, S) \to \iota$, then there is a measure $\nu \in \mathrm{INV}_{\mathrm{Ind}}(\Gamma, S)$ such that the shift action is ergodic relative to ν and has independence number equal to ι, but it is not clear that this action is free.

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Chapter 3

On a co-induction question of Kechris

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This note answers a question of Kechris: if $H < G$ is a normal subgroup of a countable group G, H has property MD and G/H is amenable and residually finite then G also has property MD. Under the same hypothesis we prove that for any action a of G, if b is a free action of G/H, and b_G is the induced action of G then $\mathrm{CInd}_H^G(a|H) \times b_G$ weakly contains a. Moreover, if $H < G$ is any subgroup of a countable group G, and the action of G on G/H is amenable, then $\mathrm{CInd}_H^G(a|H)$ weakly contains a whenever a is a Gaussian action.

1. Introduction

The Rohlin Lemma plays a prominent role in classical ergodic theory. Roughly speaking, it states that any aperiodic automorphism T of a standard non-atomic probability space (X, μ) can be approximated by periodic automorphisms. In [**OW80**], Ornstein and Weiss generalized the Rohlin Lemma to actions of amenable groups and used it to extend many classical ergodic theory results (such as Ornstein theory) to the amenable setting.

There is no analogue of the Rohlin Lemma for non-amenable groups. However, one can hope to understand more precisely how and why this is so. The concept of "weak containment" of actions, introduced by A. Kechris [**Kec10**], is a natural starting point. To be precise, let (X, μ), (Y, ν) be standard non-atomic probability spaces. Let $G \curvearrowright^a (X, \mu)$,

$G \curvearrowright^b (Y, \nu)$ be probability measure preserving (p.m.p.) actions. An *observable* ϕ for a is a measurable map $\phi : X \to \mathbb{N}$. For $F \subset G$, let $\phi_a^F : X \to \mathbb{N}^F = \{y : F \to \mathbb{N}\}$ be defined by

$$\phi_a^F(x)(f) = \phi(a(f)x).$$

Then a is said to be *weakly contained* in b (denoted $a \prec b$) if for every $\epsilon > 0$, every finite $F \subset G$, every observable ϕ for a, there is an observable ψ for b such that

$$\|\phi_*^F \mu - \psi_*^F \nu\|_1 \leq \epsilon.$$

The two actions are *weakly equivalent* if $a \prec b$ and $b \prec a$.

If G is infinite and amenable, then as remarked in [**Kec12**], if a is a free action then a weakly contains every action of G. This is essentially equivalent to the Rohlin Lemma for amenable groups. However, when G is non-amenable then it may possess uncountably many free non-weakly equivalent actions [**AE11**]. It is unknown whether the same holds true for every non-amenable group.

It is natural to ask how weak equivalence behaves with respect to operations such as co-induction. To be precise, let $H < G$ be a subgroup. Let $H \curvearrowright^a (X, \mu)$ be a p.m.p. action. Let $Z = \{z \in X^G : \ a(h^{-1})z(g) = z(gh) \ \forall h \in H, g \in G\}$. Let $G \curvearrowright^b Z$ be the action $(b(g)z)(f) = z(g^{-1}f)$ for $g, f \in G, z \in Z$.

A *section* of H in G is a map $\sigma : G/H \to G$ such that $\sigma(gH) \subset gH$ for every $g \in G$. Let us assume $\sigma(H) = e$. Define $\Phi : Z \to X^{G/H}$ by $\phi(z)(gH) = z(\sigma(gH))$. This is a bijection. Define a measure ζ on Z by pulling back the product measure $\mu^{G/H}$ on $X^{G/H}$. Then $G \curvearrowright^b (Z, \zeta)$ is probability measure preserving. This action is called said to be *co-induced* from a and is denoted $b = \mathrm{CInd}_H^G(a)$.

Problem A.4. of [**Kec12**] asks the following.

PROBLEM 1.1. *Let G be a countable group with a subgroup $H < G$. Suppose the action of G on G/H is amenable. Is it true that for any p.m.p. action a of G on a standard non-atomic probability space, the co-induced action $\mathrm{CInd}_H^G(a|H)$ weakly contains a?*

A positive answer can be interpreted as providing a relative version of the Rohlin lemma. Note that the action of G on G/H being amenable is a necessary condition, since if we take a to be the trivial action τ_G of G on a standard non-atomic probability space (X, μ), then $\mathrm{CInd}_H^G(\tau_G|H)$ is isomorphic to the generalized Bernoulli shift action $s_{G,G/H,X}$ of G on $X^{G/H}$ (see section 5), and $s_{G,G/H,X}$ weakly containing τ_G is equivalent to the action of G on G/H being amenable by [**KT08**]. Also note that if replace the actions with unitary representations, then the analogous problem is known to have a positive answer (this is E.2.6 of [**BHV08**]).

Our main results solve Problem 1 in a number of cases and provide applications to property MD. To begin, we prove:

THEOREM 1.1. Let G be a countable group with normal subgroup H. Suppose that G/H is amenable and that $|G/H| = \infty$. Let b be any free p.m.p. action of G/H. Let b_G be the associated action of G (i.e., b_G is obtained by pre-composing b with the quotient map $G \to G/H$). Then for any p.m.p. action a of G on standard non-atomic probability space, the product action $\mathrm{CInd}_H^G(a|H) \times b_G$ weakly contains a.

Taking b to be the Bernoulli shift action of G/H over a standard non-atomic probability base space, we show that Theorem 1.1 implies (see 5.1 below)

$$a \prec \mathrm{CInd}_H^G((a \times \tau_G)|H)$$

where τ_G is the trivial action of G as above. In particular, if $a|H$ weakly contains $(a \times \tau_G)|H$, then $\mathrm{CInd}_H^G(a|H)$ weakly contains a. For instance, by [**AW11**] this is the case whenever a is an ergodic p.m.p. action of G that is not strongly ergodic. This also holds when a is a *universal* action of G, i.e., $b \prec a$ for every p.m.p. action b of G. That such actions exist for every countable group G is due to Glasner-Thouvenot-Weiss [**GTW06**] and, independently, to Hjorth (unpublished, see 10.7 of [**Kec10**]). This has the following consequence:

THEOREM 1.2. Let G and H be as in Theorem 1.1. If b is a universal action of H then $\mathrm{CInd}_H^G(b)$ is a universal action of G.

In section 6 we describe the Gaussian action construction. For every real positive definite function φ defined on a countable set T, a probability measure μ_φ on $\mathbb{R}^T$ is defined, and we call $(\mathbb{R}^T, \mu_\varphi)$ a *Gaussian probability space*. When G acts on T and φ is invariant for this action, then μ_φ will be an invariant measure for the shift action of G on $(\mathbb{R}^T, \mu_\varphi)$. A p.m.p. action a of G is called a *Gaussian action* if it is isomorphic to the shift action of G on some Gaussian probability space $(\mathbb{R}^T, \mu_\varphi)$ associated to an invariant positive definite function φ. We show that Problem 1 always has a positive answer for Gaussian actions.

THEOREM 1.3. Let G be a countable group with a subgroup $H < G$. Suppose the action of G on G/H is amenable. Then the co-induced action $\mathrm{CInd}_H^G(a|H)$ weakly contains a for every Gaussian action a of G.

Part of the motivation for posing Problem 1 above concerns a property of groups introduced by Kechris called property MD. To be precise, let G be a residually finite group, and let ρ_G be the canonical action of G on its profinite completion. Recall that τ_G is the trivial action of G on (X, μ), a standard non-atomic probability space. Then G has MD if and only if every p.m.p. action of G is weakly contained in the product action $\tau_G \times \rho_G$.

The property MD is an ergodic theoretic analogue of the property FD discussed in Lubotzky-Shalom [**LS04**] (see also Lubotzky-Zuk [**LZ03**]). This asserts that the finite unitary representations of G on an infinite-dimensional separable Hilbert space $\mathcal{H}$ are dense in the space of unitary representations of G in $\mathcal{H}$. It is not difficult to show that $MD \Rightarrow FD$ but the converse is unknown.

It is known (see [**Kec12**] for more details), that the following groups have MD: residually finite amenable groups, free products of finite groups, subgroups of MD groups, finite extensions of MD groups. On the other hand, various groups such as $SL_n(\mathbb{Z})$ for $n > 2$ are known not to have FD [**LS04**] [**LZ03**] and hence also do not have MD. It is an open question whether the direct product of two free groups has MD.

In [**Kec12**], Conjecture 4.14, Kechris conjectured the following:

THEOREM 1.4. Let N be an infinite, residually finite group satisfying MD. Let $N \lhd G$ with G residually finite. Assume that:

(1) For every $H \lhd N$ with $[N : H] < \infty$, there is $G' \lhd G$ such that $G' \subset H$ and $[N : G'] < \infty$.

(2) G/N is a residually finite, amenable group.

Then G satisfies MD.

As noted in [**Kec12**], this result implies that surface groups and the fundamental groups of virtually fibered closed hyperbolic 3-manifolds, (e.g., $SL_2(\mathbb{Z}[i])$) have property MD. This follows from the fact that free groups have property MD (proven in [**Kec12**] and in different terminology in [**Bow03**]). Kechris proved that an affirmative answer to Problem 1 above implies Theorem 1.4. Our proof follows his line of argument.

Note: If N is finitely generated then the first condition of Theorem 1.4 is automatically satisfied since if N is normal in G and $H < N$ has finite index, then for every $g \in G$, gHg^{-1} is a subgroup of N with the same index as H. Because N is finitely generated, this implies there are only finitely many different conjugates of H. The intersection of all these conjugates is a normal subgroup in G with finite index in N.

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2. The space of actions and proof of Theorem 1.4

Let (X, μ) denote a standard non-atomic probability space and $A(G, X, \mu)$ the set of all p.m.p. actions of G on (X, μ). This set is naturally identified with a subset of the product space $\mathrm{Aut}(X, \mu)^G$ where $\mathrm{Aut}(X, \mu)$ denotes the space of all automorphisms of (X, μ). We equip the $\mathrm{Aut}(X, \mu)$ with the weak topology, $\mathrm{Aut}(X, \mu)^G$ with the product topology, and $A(G, X, \mu)$ with the subspace topology (also called the weak topology). The group $\mathrm{Aut}(X, \mu)$ acts on $A(G, X, \mu)$ by $(Ta)(g) = Ta(g)T^{-1}$ for all $T \in \mathrm{Aut}(X, \mu)$, $a \in A(G, X, \mu)$ and $g \in G$. The orbit of a under this action is called its *conjugacy class*.

LEMMA 2.1. Let $a, b \in A(G, X, \mu)$. Then $a \prec b$ if and only if a is contained in the (weak) closure of the conjugacy class of b.

PROOF. This is Proposition 10.1 of [**Kec10**]. □

An action $a \in A(G, X, \mu)$ is *finite* if it factors through the action of a finite group. From lemma 2.1 it follows that for any $a \in A(G, X, \mu)$, $a \prec \tau_G \times \rho_G$ if and only if a is contained in the (weak) closure of the set of finite actions (this is implied by the proof of Proposition 4.8 [**Kec12**]).

We need the following lemmas.

LEMMA 2.2. Let a, b be actions of a countable group G. If a and b are weakly contained in $\tau_G \times \rho_G$ then $a \times b$ is weakly contained in $\tau_G \times \rho_G$.

PROOF. If a is a weak limit of finite actions a_i and b is a weak limit of finite actions b_i then $a \times b$ is the weak limit of $a_i \times b_i$. □

LEMMA 2.3. If $H < G$ is a normal subgroup, G/H is amenable and residually finite, and b is a p.m.p. action of G/H then the induced action b_G of G is weakly contained in $\tau_G \times \rho_G$.

PROOF. As noted in [**Kec12**], because G/H is residually finite and amenable, it has MD. Therefore, b is a weak limit of finite actions b_i of G/H. If $b_{G,i}$ are the induced actions of G, then the $b_{G,i}$ are also finite and $b_{G,i}$ converges weakly to b_G. □

PROOF OF THEOREM 1.4 FROM THEOREM 1.1. Let a be a p.m.p. action of G. In [**Kec12**] section 4, it is shown that $\mathrm{CInd}_N^G(a|N)$ is weakly contained in $\tau_G \times \rho_G$. Let b be a free p.m.p. action of G/N. Because G/N is amenable the previous lemmas imply $\mathrm{CInd}_N^G(a|N) \times b_G \prec \tau_G \times \rho_G$. So Theorem 1.1 implies $a \prec \mathrm{CInd}_N^G(a|N) \times b_G \prec \tau_G \times \rho_G$. Since a is arbitrary, G has MD. □

3. The Rohlin Lemma

The purpose of this section is to prove:

THEOREM 3.1. If G is a countably infinite amenable group then for every free p.m.p. action $G \curvearrowright^a (X,\mu)$, every finite $F \subset G$ and $\epsilon > 0$ there is a measurable map $J : X \to G$ such that

$$\mu(\{x \in X : \ J(a(f)x) = fJ(x)\ \forall f \in F\}) \geq 1 - \epsilon.$$

This will follow easily from the following version of the Rohlin Lemma due to Ollagnier [**Oll85**] Corollary 8.3.12 (see 2.2.8. for the definition of $M(D,\delta)$).

THEOREM 3.2. Let $G \curvearrowright (X,\mu)$ be as above. Then for every finite $F \subset G$, for every $\delta, \eta > 0$ there exists a finite collection $\{(\Lambda_i, A_i)\}_{i \in I}$ satisfying:

(1) for every $i \in I$, $\Lambda_i \subset G$ is finite and

$$\frac{|\{g \in \Lambda_i : \ \exists f \in F,\ fg \notin \Lambda_i\}|}{|\Lambda_i|} < \delta,$$

(2) each A_i is a measurable subset of X with positive measure,

(3) $a(\lambda_i)A_i \cap a(\lambda_j)A_j = \varnothing$ if $i \neq j$, $\lambda_i \in \Lambda_i$ and $\lambda_j \in \Lambda_j$,

(4) $a(\lambda)A_i \cap a(\lambda')A_i = \varnothing$ if $\lambda, \lambda' \in \Lambda_i$ and $\lambda \neq \lambda'$,

(5) $\mu\left(\cup_{i \in I} \cup_{\lambda \in \Lambda_i} a(\lambda)A_i\right) \geq 1 - \eta$.

PROOF OF THEOREM 3.1. Let $0 < \delta, \eta < \epsilon/2$. Without loss of generality, we assume $e \in F$. Let $\{(\Lambda_i, A_i)\}_{i \in I}$ be as in the theorem above. Define J by $J(x) = \lambda_j$ if there is a $j \in I$ and $\lambda_j \in \Lambda_j$ such that $x \in a(\lambda_j)A_j$. If x is not in $\cup_{i \in I} \cup_{\lambda \in \Lambda_i} a(\lambda)A_i$, then define $J(x)$ arbitrarily. For each i, let $\Lambda'_i = \cap_{f \in F} f^{-1}\Lambda_i$. The theorem above implies $|\Lambda'_i| \geq (1-\delta)|\Lambda_i|$. Observe that

$$\{x \in X : \ J(a(f)x) = fJ(x)\ \forall f \in F\} \supset \cup_{i \in I} \cup_{\lambda \in \Lambda'_i} a(\lambda)A_i.$$

Thus

$$\mu(\{x \in X : \ J(a(f)x) = fJ(x)\ \forall f \in F\}) \geq 1 - \eta - \delta \geq 1 - \epsilon.$$

$\square$

4. Proof of Theorem 1.1

Assume the hypotheses of Theorem 1.1. In particular, we assume that $G/H \curvearrowright^b (Y,\nu)$ is a free p.m.p. action of the infinite amenable group G/H. For simplicity, if $g \in G$ and $y \in Y$, let gy denote $b(gH)y$.

Let $F \subset G$ be finite and $\epsilon > 0$. Because G/H is amenable, Theorem 3.1 implies there exists a measurable function $J : Y \to G/H$ such that if

$$Y_0 = \{y \in Y : \ J(fy) = fJ(y) \, \forall f \in F\}$$

then $\nu(Y_0) \geq 1 - \epsilon$. Let $\sigma : G/H \to G$ be a section (i.e., $\sigma(gH) \in gH$ for all $g \in G$). Let $\tilde{J} : Y \to G$ be defined by $\tilde{J} = \sigma J$.

Recall that $G \curvearrowright^a (X,\mu)$ is a p.m.p. action, $Z = \{z \in X^G : \ a(h^{-1})z(g) = z(gh)\}$ and G acts on Z by $(gz)(f) = z(g^{-1}f)$ for $z \in Z, g, f \in G$. This action is $\mathrm{CInd}_H^G(a|H)$. It preserves the measure ζ on Z obtained by pulling back the product measure $\mu^{G/H}$ on $X^{G/H}$ under the map $\Phi : Z \to X^{G/H}$, $\Phi(z)(gH) = z(\sigma(gH))$.

For $(z,y) \in Z \times Y$, define $S_y(z) \in X$ by

$$S_y(z) = a(\tilde{J}(y))z(\tilde{J}(y)).$$

LEMMA 4.1. The map $(z,y) \in Z \times Y \mapsto S_y(z) \in X$ maps $\zeta \times \nu$ onto μ.

PROOF. For any $y \in Y$, if δ_y denotes the Dirac probability measure concentrated on y then it is easy to see that $(z,y) \mapsto S_y(z)$ maps $\zeta \times \delta_y$ onto μ. The lemma follows by integrating over y. $\square$

LEMMA 4.2. For every $(z,y) \in Z \times Y_0$ and $f \in F$, $S_{fy}(fz) = a(f)S_y(z)$.

PROOF. If $y \in Y_0$ then $J(fy) = fJ(y)$ for all $f \in F$. Therefore, for each $f \in F$ there is some $h \in H$ such that $\tilde{J}(fy) = f\tilde{J}(y)h$. Now

$$\begin{aligned} S_{fy}(fz) &= a(\tilde{J}(fy))(fz)(\tilde{J}(fy)) = a(f\tilde{J}(y)h)(fz)(f\tilde{J}(y)h) \\ &= a(f)a(\tilde{J}(y))a(h)z(\tilde{J}(y)h) = a(f)a(\tilde{J}(y))z(\tilde{J}(y)) = a(f)S_y(z). \end{aligned}$$

□

Now let $\phi : X \to \mathbb{N}$ be an observable. Define $\psi : Z \times Y \to \mathbb{N}$ by $\psi(z, y) = \phi(S_y(z))$. The lemma above implies that for all $(z, y) \in Z \times Y_0$, $\psi(fz, fy) = \phi(a(f)S_y(z))$ for all $f \in F$. Thus $\psi^F(z, y) = \phi^F(S_y(z))$ for $(z, y) \in Z \times Y_0$. Since $(z, y) \mapsto S_y(z)$ takes the measure $\zeta \times \nu$ to μ and $\nu(Y_0) \geq 1 - \epsilon$, it follows that

$$\|\psi^F_*(\zeta \times \nu) - \phi^F_* \mu\|_1 < \epsilon.$$

Because $F \subset G$, $\epsilon > 0$ and ϕ are arbitrary, this implies Theorem 1.1.

5. Consequences of Theorem 1.1

If K is a group acting on a countable set T, then for a measure space (X, μ) we denote the generalized shift action of K on (X^T, μ^T) (given by $(ky)(t) = y(k^{-1}t)$ for $k \in K, y \in X^T$, $t \in T$) by $s_{K,T,X}$.

COROLLARY 5.1. Let G be a countable group and let H be a normal subgroup of infinite index such that G/H is amenable. Then $a \prec \mathrm{CInd}_H^G((a \times \tau_G)|H)$ for every p.m.p. action a of G.

PROOF. Let (X, μ) be a standard non-atomic probability space. Let $s_{G/H,G/H,X}$ denote the shift of G/H on $X^{G/H}$, which is free. Let $s_{G,G/H,X}$ denote the generalized shift of G on $X^{G/H}$. Then $s_{G,G/H,X}$ is the action of G induced by $s_{G/H,G/H,X}$, i.e., $s_{G,G/H,X}$ factors through $s_{G/H,G/H,X}$. By Proposition A.2 of **[Kec12]** we have that $s_{G,G/H,X} \cong \mathrm{CInd}_H^G(s_{H,H/H,X})$. Now $s_{H,H/H,X} = \tau_H$ is just the identity action of H on X, and $\tau_H = \tau_G|H$ is the restriction of the identity action of G on X to H.

LEMMA 5.2. Let L be a subgroup of the countable group K. Let $a, b \in A(L, X, \mu)$. Then

$$\mathrm{CInd}_L^K(a) \times \mathrm{CInd}_L^K(b) \cong \mathrm{CInd}_L^K(a \times b)$$

PROOF. This is easy to see once we view $\mathrm{CInd}_L^K(a)$ as an action on the space $(X^{K/L}, \mu^{K/L})$ (using the bijection $\Phi : Z \to X^{K/L}$ defined in section 1), and similarly view $\mathrm{CInd}_L^K(b)$ and $\mathrm{CInd}_L^K(a \times b)$ as actions on $(X^{K/L}, \mu^{K/L})$ and $((X \times X)^{K/L}, (\mu \times \mu)^{K/L})$ respectively. □

Applying Theorem 1.1 we now obtain

$$a \prec \mathrm{CInd}_H^G(a|H) \times s_{G,G/H,X} \cong \mathrm{CInd}_H^G(a|H) \times \mathrm{CInd}_H^G(\tau_G|H) \cong \mathrm{CInd}_H^G((a \times \tau_G)|H),$$

so $a \prec \mathrm{CInd}_H^G((a \times \tau_G)|H)$. □

If in addition to the hypotheses in Corollary 5.1 we also have $(a \times \tau_G)|H \prec a|H$, then since co-inducing preserves weak containment (A.1 of [**Kec12**]) it will follow that

$$a \prec \mathrm{CInd}_H^G((a \times \tau_G)|H) \prec \mathrm{CInd}_H^G(a|H).$$

Recall that a p.m.p. action a of G on a standard non-atomic probability space is called a *universal* action of G if $b \prec a$ for every p.m.p. action b of G. We now have the following.

COROLLARY 5.3. Let G be a countable group and let H be a normal subgroup of infinite index such that G/H is amenable. Then any one of the following conditions on $a \in A(G, X, \mu)$ implies $a \prec \mathrm{CInd}_H^G(a|H)$:

(1) a is ergodic but not strongly ergodic;
(2) $a|H$ is ergodic but not strongly ergodic;
(3) a is a universal action of G;
(4) $a|H$ is a universal action of H;

In addition, the set of actions a of G for which $a \prec \mathrm{CInd}_H^G(a|H)$ is closed under taking products.

REMARK 5.1. The referee points out that condition *2* is in fact strictly stronger than condition *1*. That is, if G/H is amenable then $a|H$ being ergodic but not strongly ergodic implies that a itself is not strongly ergodic. This is a special case of [**Ioa06**] lemma 2.3.

PROOF OF 5.3. *3* and *4* are immediate from Corollary 5.1, and *1* and *2* follow from 5.1 along with Theorem 3 of [**AW11**] where they show that $a \times \tau_G \prec a$ holds for ergodic a that are not strongly ergodic. The last statement follows from 5.2 since if $a \prec \mathrm{CInd}_H^G(a|H)$ and $b \prec \mathrm{CInd}_H^G(b|H)$ then $a \times b \prec \mathrm{CInd}_H^G(a|H) \times \mathrm{CInd}_H^G(b|H) \cong \mathrm{CInd}_H^G((a \times b)|H)$. □

We can now prove Theorem 1.2

PROOF OF 1.2. Suppose b is a universal action of H. Let a be a universal action of G. It suffices to show that $a \prec \mathrm{CInd}_H^G(b)$. We have $a|H \prec b$ by universality of b, and so by *3* of Corollary 5.3 we have that $a \prec \mathrm{CInd}_H^G(a|H) \prec \mathrm{CInd}_H^G(b)$. □

REMARK 5.2. The assumption that G/H is amenable is in some cases necessary in order for CInd_H^G to preserve universality. That is, there are examples of groups $H \leq G$ with H infinite index in G such that G/H is *not* amenable, and such that $a \mapsto \mathrm{CInd}_H^G(a)$ does not map universal actions to universal actions. For example, if H is any subgroup of infinite index in a group G with property (T) (e.g., if $G = H \times K$ where both H and K are countably infinite with property (T)) then $\mathrm{CInd}_H^G(b)$ is weak mixing for every $b \in A(H, X, \mu)$ (see [**Ioa11**] lemma 2.2 (ii)), hence is *never* universal. Another example is when H is amenable and G/H is non-amenable (e.g., if $G = H \times K$ where H is any amenable group and K is any non-amenable group). This implies that G is non-amenable. If $s = s_{H,H,X}$ is the shift of H on (X^H, μ^H) then s is universal for H since H is amenable, but $\mathrm{CInd}_H^G(s) \cong s_{G,G,X}$ is not universal since G is non-amenable.

REMARK 5.3. In case H is finite index in G then we actually have the following form of Theorem 1.1. We do not assume that H is normal in G. Let b denote the action of G on G/H, where we view G/H as equipped with normalized counting measure ν. Then for any p.m.p. action a of G on a standard non-atomic probability space (X, μ), a is a *factor* of $\mathrm{CInd}_H^G(a|H) \times b$. One way to see this is to use the isomorphism $\mathrm{CInd}_H^G(a|H) \cong a^{G/H} \circledast s_{G,G/H,X}$ given by Proposition A.3 of [**Kec12**]. Here $a^{G/H} \circledast s_{G,G/H,X}$ is the p.m.p. action of G on $(X^{G/H}, \mu^{G/H})$ given by $a^{G/H} \circledast s_{G,G/H,X}(g) = a^{G/H}(g) \circ s_{G,G/H,X}(g)$ (note that the transformations $a^{G/H}(g)$ and $s_{G,G/H,X}(g)$ commute for all $g \in G$). Then

$(a^{G/H} \circledast s_{G,G/H,X}) \times b$ is an action on the space $(X^{G/H} \times G/H, \mu^{G/H} \times \nu)$, and the map $(f, gH) \mapsto f(gH) \in X$ factors this action onto a.

6. Gaussian actions

A (real) *positive definition function* $\varphi : I \times I \to \mathbb{R}$ on a countable set I is a real-valued function satisfying $\varphi(i,j) = \varphi(j,i)$ and $\sum_{i,j \in F} a_i a_j \varphi(i,j) \geq 0$ for all finite $F \subseteq I$ and reals a_i, $i \in F$.

THEOREM 6.1. If $\varphi : I \times I \to \mathbb{R}$ is a real-valued positive definite function on a countable set I, then there is a unique Borel probability measure μ_φ on $\mathbb{R}^I$ such that the projection functions $p_i : \mathbb{R}^I \to \mathbb{R}$, $p_i(x) = x(i)$ ($i \in I$), are centered jointly Gaussian random variables with covariance matrix φ. That is, μ_φ is uniquely determined by the two properties

(1) Every finite linear combination of the projection functions $\{p_i\}_{i \in I}$ is a centered Gaussian random variable on $(\mathbb{R}^I, \mu_\varphi)$;
(2) $\mathbb{E}(p_i p_j) = \varphi(i,j)$ for all $i, j \in I$.

For a finite $F \subseteq I$, let $p_F : \mathbb{R}^I \to \mathbb{R}^F$ be the projection $p_F(x) = x|F$. Then μ_φ can also be characterized as the unique Borel probability measure on $\mathbb{R}^I$ such that for each finite $F \subseteq I$ the measure $(p_F)_* \mu_\varphi$ on $\mathbb{R}^F$ has characteristic function

$$\widetilde{(p_F)_* \mu_\varphi}(u) = e^{-\frac{1}{2} \sum_{i,j \in F} u_i u_j \varphi(i,j)}.$$

We call μ_φ the *Gaussian measure associated to* φ and $(\mathbb{R}^I, \mu_\varphi)$ a *Gaussian probability space*. A discussion of this can be found in [**Kec10**] Appendix C and the references therein.

Let G be a countable group acting on I and suppose that the positive definite function $\varphi : I \times I \to \mathbb{R}$ is invariant for the action of G on I, i.e., $\varphi(g \cdot i, g \cdot j) = \varphi(i,j)$ for all $g \in G$, $i, j \in I$. Let s_φ denote the shift action of G on $(\mathbb{R}^I, \mu_\varphi)$

$$(s_\varphi(g)x)(i) = x(g^{-1} \cdot i).$$

Then invariance of φ implies that μ_φ is an invariant measure for this action. We call s_φ the *Gaussian shift associated to* φ.

Let π be an orthogonal representation of G on a separable real Hilbert space $\mathcal{H}_\pi$, and let $T \subseteq \mathcal{H}_\pi$ be a countable π-invariant set whose linear span is dense in $\mathcal{H}_\pi$. Then G acts on T via π, and we let $\varphi_T : T \times T \to \mathbb{R}$ be the G-invariant positive definite function given by $\varphi_T(t_1, t_2) = \langle t_1, t_2 \rangle$. We let $s_\pi = s_{\pi,T}$ be the corresponding Gaussian shift and call it the *Gaussian shift action associated to* π. It follows from Proposition 6.2 below that up to isomorphism this action does not depend on the choice of $T \subseteq \mathcal{H}_\pi$. For now, it is clear that an isomorphism θ of two representations π_1 and π_2 induces an isomorphism of the actions $s_{\pi_1,T}$ with $s_{\pi_2,\theta(T)}$.

By the GNS construction, every invariant real positive definite function φ on a countable G-set may be viewed as coming from an orthogonal representation in this way.

There is another way of obtaining an action on a Gaussian probability space from an orthogonal representation of G. Consider the product space $(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$, where μ is the $N(0,1)$ normalized, centered Gaussian measure on $\mathbb{R}$ with density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Let $p_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, $n \in \mathbb{N}$, be the projection functions $p_n(x) = x(n)$. The closed linear span $\langle p_n \rangle_{n \in \mathbb{N}} \subseteq L^2(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}}, \mathbb{R})$ has countable infinite dimension. Let $\mathcal{H} = \langle p_n \rangle_{n \in \mathbb{N}} \subseteq L^2(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}}, \mathbb{R})$ and let π be a representation of G on $\mathcal{H}$. Let $a(\pi)$ be the action of G on $(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$ given by

$$(a(\pi)(g)x)(n) = \pi(g^{-1})(p_n)(x).$$

This preserves the measure $\mu^{\mathbb{N}}$ by the characterization of $\mu^{\mathbb{N}}$ given in 6.1 since $\mu^{\mathbb{N}} = \mu_\varphi$, where $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is the positive definite function given by $\varphi(n,n) = 1$ and $\varphi(n,m) = 0$ for $n \neq m$.

It follows from the discussion in [**Kec10**] Appendix E that if π_1 and π_2 are isomorphic, then $a(\pi_1) \cong a(\pi_2)$. So if π is now an arbitrary orthogonal representation of G on an infinite-dimensional separable real Hilbert space $\mathcal{H}_\pi$, then by choosing an isomorphism θ of $\mathcal{H}_\pi$ with $\mathcal{H} = \langle p_n \rangle_{n \in \mathbb{N}}$ we obtain an isomorphic copy $\theta \cdot \pi$ of π, on $\mathcal{H}$, and the corresponding action $a(\theta \cdot \pi)$ is, up to isomorphism, independent of θ.

The construction of the actions $a(\pi)$ also works for representations on a finite-dimensional Hilbert space, replacing $\mathbb{N}$ above with $N = \dim(\mathcal{H}_\pi)$. The following proposition also holds in the finite-dimensional setting.

PROPOSITION 6.2. Let π be an orthogonal representation of G on $\mathcal{H} = \langle p_n \rangle_{n\in\mathbb{N}} \subseteq L^2(\mathbb{R}^\mathbb{N}, \mu^\mathbb{N}, \mathbb{R})$, let $T \subseteq \mathcal{H}$ be a countable π-invariant set of functions in $\mathcal{H}$ whose linear span is dense in $\mathcal{H}$, and let $s_{\pi,T}$ be the corresponding Gaussian shift on $(\mathbb{R}^T, \mu_{\varphi_T})$. Then the map $\Phi : (\mathbb{R}^\mathbb{N}, \mu^\mathbb{N}) \to (\mathbb{R}^T, \mu_{\varphi_T})$ given by

$$\Phi(x)(t) = t(x)$$

is an isomorphism of $a(\pi)$ with $s_{\pi,T}$. In particular, up to isomorphism, the action $s_{\pi,T}$ does not depend on the choice of T.

PROOF. Note that up to a $\mu^\mathbb{N}$-null set, Φ does not depend on the choice of representatives for the elements of T (viewing each $t \in T$ as an equivalence class of functions in $L^2(\mathbb{R}^\mathbb{N}, \mu^\mathbb{N}, \mathbb{R})$). This follows from T being countable. So Φ is well defined.

To see that $\Phi_*(\mu^\mathbb{N}) = \mu_{\varphi_T}$ we use 6.1. First, we show that if $f = \sum_{i=1}^k a_i p_{t_i}$ then f has a centered Gaussian distribution with respect to $\Phi_*(\mu^\mathbb{N})$. This is clear since $f_*\Phi_*(\mu^\mathbb{N}) = (f \circ \Phi)_*(\mu^\mathbb{N})$, and $f \circ \Phi = \sum_{i=1}^k a_i t_i$ has centered Gaussian distribution with respect to $\mu^\mathbb{N}$ by virtue of being in $\mathcal{H}$.

Second, we show that the covariance matrix of the projections $\{p_t\}_{t\in T}$ with respect $\Phi_*\mu^\mathbb{N}$ is equal to φ_T. We have

$$\begin{aligned}\int p_{t_1}(x)p_{t_2}(x)\, d(\Phi_*\mu^\mathbb{N}) &= \int \Phi(x)(t_1)\Phi(x)(t_2)\, d(\mu^\mathbb{N}) \\ &= \int t_1 t_2\, d(\mu^\mathbb{N}) = \langle t_1, t_2\rangle = \varphi(t_1, t_2).\end{aligned}$$

Next, we show that Φ takes the action a_π to the action $s_{\pi,T}$. We have, for $\mu^\mathbb{N}$-a.e. x,

$$\begin{aligned}\Phi(a(\pi)(g)x)(t) &= t(a(\pi)(g)x) = \textstyle\sum_n \langle t, p_n\rangle p_n(a(\pi)(g)x) = \sum_n \langle t, p_n\rangle \pi(g^{-1})(p_n)(x) \\ &= \pi(g^{-1})(\textstyle\sum_n \langle t, p_n\rangle p_n)(x) = \pi(g^{-1})(t)(x) = \Phi(x)(\pi(g^{-1})(t)) = s_{\pi,T}(g)(\Phi(x))(t).\end{aligned}$$

It remains to show that Φ is 1-1 on a $\mu^{\mathbb{N}}$-measure 1 set. Since the closed linear span of $\{t\}_{t\in T}$ in $\mathcal{H}$ contains each p_i, it follows that the σ-algebra generated by $\{t\}_{t\in T}$ is the Borel σ-algebra modulo $\mu^{\mathbb{N}}$-null sets, so there is a $\mu^{\mathbb{N}}$-conull set B such that $\{t|B\}_{t\in T}$ generates the Borel σ-algebra of B and thus $\{t|B\}$-separates points. It follows that Φ is 1-1 on B. $\square$

7. Induced representations and the proof of Theorem 1.3

We begin by briefly recalling the induced representation construction. Let H be a subgroup of the countable group G, and let $\sigma : G/H \to G$ be a selector for the left cosets of H in G with $\sigma(H) = e$. Let $\rho : G \times G/H \to H$ be defined by $\rho(g, kH) = \sigma(gkH)^{-1}g\sigma(kH) \in H$. Then ρ is a cocycle for the action of G on G/H, i.e., $\rho(g_0g_1, kH) = \rho(g_0, g_1kH)\rho(g_1, kH)$. (Note that this is the same as the cocycle ρ defined in the proof of Lemma 5.2.)

Let π be an orthogonal representation of H on the real Hilbert space $\mathcal{K}$. For each $gH \in G/H$ let $\mathcal{K}_{gH} = \mathcal{K} \times \{gH\} = \{(\xi, gH) : \xi \in \mathcal{K}\}$ be a Hilbert space which is a copy of $\mathcal{K}$. Then the induced representation $\mathrm{Ind}_H^G(\pi)$ of π is the representation of G on $\bigoplus_{g\in G/H} \mathcal{K}$, which we identify with the set of formal sums $\mathcal{K}' = \{\sum_{gH\in G/H}(\xi_{gH}, gH) \in \sum_{gH\in G/H} \mathcal{K}_{gH} : \sum_{gH\in G/H} ||\xi_{gH}||^2_{\mathcal{K}} < \infty\}$, that is given by

$$g_0 \cdot (\xi_{gH}, gH) = (\rho(g_0, gH) \cdot \xi_{gH},\ g_0gH) \in \mathcal{K}_{g_0gH}$$

for $(\xi_{gH}, gH) \in \mathcal{K}_{gH}$, and extending linearly.

LEMMA 7.1. Let H be a subgroup of the countable group G. Then

(1) $a(\pi|H) \cong a(\pi)|H$ for all orthogonal representations π of G.

(2) $\mathrm{CInd}_H^G(a(\pi)) \cong a(\mathrm{Ind}_H^G(\pi))$ for all orthogonal representations π of H.

PROOF. The first statement is clear. For the second, let $T \subseteq \mathcal{K}$ be a total, countable subset of $\mathcal{K}$ that is invariant under π. Then $T \times G/H \subseteq \mathcal{K}'$ is a total, countable subset of $\mathcal{K}'$ that is invariant under $\mathrm{Ind}_H^G(\pi)$. Let $\varphi : (T \times G/H) \times (T \times G/H) \to \mathbb{R}$ be the inner product determined by

$$\varphi((t_1, g_1H), (t_2, g_2H)) = \langle (t_1, g_1H), (t_2, g_2H)\rangle_{\mathcal{K}'} = \begin{cases} \langle t_1, t_2\rangle_{\mathcal{K}} & \text{if } g_1H = g_2H \\ 0 & \text{if } g_1H \neq g_2H. \end{cases}$$

Then the Gaussian shift action corresponding to $\mathrm{Ind}_H^G(\pi)$ is the action b of G on $(\mathbb{R}^{T\times G/H}, \mu_\varphi)$ given by

$$(b(g)\cdot x)((t, kH)) = x(g^{-1}\cdot (t, kH)) = x((\rho(g^{-1}, kH)\cdot t, g^{-1}kH)).$$

On the other hand, the Gaussian shift action corresponding to π is the action $s_\pi \cong a(\pi)$ of H on $(\mathbb{R}^T, \mu_{\varphi_T})$ given by $(s_\pi(h)\cdot w)(t) = w(h^{-1}\cdot t)$, and where $\varphi_T : T\times T \to \mathbb{R}$ is just the inner product $\varphi_T(t_1, t_2) = \langle t_1, t_2\rangle_{\mathcal{K}}$. The co-induced action $\mathrm{CInd}_H^G(s_\pi)$ is isomorphic to the action c of G on $((\mathbb{R}^T)^{G/H}, \mu_\varphi^{G/H})$ given by $(c(g)\cdot y)(kH) = s_\pi(\rho(g^{-1}, kH)^{-1})\cdot y(g^{-1}kH)$. Evaluating this at $t \in T$ gives

$$(c(g)\cdot y)(kH)(t) = (s_\pi(\rho(g^{-1}, kH)^{-1})\cdot y(g^{-1}kH))(t) = y(g^{-1}kH)(\rho(g^{-1}, kH)\cdot t).$$

It follows that the bijection $\Psi : \mathbb{R}^{T\times G/H} \to (\mathbb{R}^T)^{G/H}$ given by $\Psi(x)(kH)(t) = x((t, kH))$ takes the action b to the action c, and also takes the measure μ_φ to $\mu_{\varphi_T}^{G/H}$. So $b \cong c$ as was to be shown. $\square$

If π_1 and π_2 are orthogonal representations of G on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then we say π_1 is *weakly contained in* π_2 *in the sense of Zimmer* [**Zim84**] and write $\pi_1 \prec_Z \pi_2$ if for all $v_1, \ldots, v_n \in \mathcal{H}_1$, $\epsilon > 0$, and $F \subseteq G$ finite, there are $w_1, \ldots, w_n \in \mathcal{H}_2$ such that $|\langle \pi_1(g)(v_i), v_j\rangle - \langle \pi_2(g)(w_i), w_j\rangle| < \epsilon$ for all $g \in F$, $i, j \leq n$.

LEMMA 7.2. $\pi_1 \prec_Z \pi_2 \Rightarrow a(\pi_1) \prec a(\pi_2)$.

PROOF. This is the remark after Theorem 11.1 of [**Kec10**]. $\square$

LEMMA 7.3. Let G be a countable group with a subgroup $H < G$. Suppose the action of G on G/H is amenable. Then $\pi \prec_Z \mathrm{Ind}_H^G(\pi|H)$ for every orthogonal representation π of G.

PROOF. It is well known that the action of G on G/H being amenable is equivalent to the existence of a sequence u_n, $n \in \mathbb{N}$, of unit vectors in $l^2(G/H, \mathbb{R})$ that are asymptotically invariant for the quasi-regular representation $\lambda_{G/H}$ of G (given by $\lambda_{G/H}(g_0)(\delta_{g_1 H}) = \delta_{g_0 g_1 H}$ where $\delta_{gH} \in l^2(G/H, \mathbb{R})$ is the indicator of $\{gH\}$). This means that for every $g \in G$, $\langle \lambda_{G/H}(g)(u_n), u_n\rangle \to 1$ as $n \to \infty$.

Let $\mathcal{K}$ be the Hilbert space of π. The representation $\mathrm{Ind}_H^G(\pi|H)$ is isomorphic to $\pi \otimes \lambda_{G/H}$ (this is E.2.6 of [**BHV08**]); an isomorphism is given by (extending linearly) the map that sends $(\xi, gH) \in \mathcal{K}_{gH}$ to $\pi(\sigma(gH))(\xi) \otimes \delta_{gH} \in \mathcal{K} \otimes l^2(G/H, \mathbb{R})$. Given now $v_1, \ldots, v_n \in \mathcal{K}$, $\epsilon > 0$, and $F \subseteq G$ finite, we have that for all N sufficiently large

$$\begin{aligned}
&|\langle \pi(g)(v_i), v_j\rangle - \langle (\pi \otimes \lambda_{G/H})(g)(v_i \otimes u_N), v_j \otimes u_N\rangle| \\
&= \left|\langle \pi(g)(v_i), v_j\rangle \left(1 - \langle \lambda_{G/H}(g)(u_N), u_N\rangle\right)\right| < \epsilon
\end{aligned}$$

for each $g \in F$, $i, j \leq n$. So taking $w_i = v_i \otimes u_N$ for N sufficiently large shows that $\pi \prec_Z \pi \otimes \lambda_{G/H} \cong \mathrm{Ind}_H^G(\pi|H)$. □

PROOF OF THEOREM 1.3. Let π be an orthogonal representation of G such that $a \cong a(\pi)$. Then $\pi \prec_Z \mathrm{Ind}_H^G(\pi|H)$ by Lemma 7.3. Applying Lemma 7.2 and then Lemma 7.1 we obtain

$$a(\pi) \prec a(\mathrm{Ind}_H^G(\pi|H)) \cong \mathrm{CInd}_H^G(a(\pi|H)) \cong \mathrm{CInd}_H^G(a(\pi)|H). \qquad \square$$

REMARK 7.1. An alternative proof of Theorem 1.3 can be given that uses probability theory. For a Gaussian shift action s_φ on $(Y, \nu) = (\mathbb{R}^T, \mu_\varphi)$ one may identify $\mathrm{CInd}_H^G(s_\varphi|H)$ with the isomorphic action $b = s_\varphi^{G/H} \circledast s_{G,G/H,Y}$ (see A.3 of [**Kec12**]) on $(Y^{G/H}, \nu^{G/H})$. Using an appropriate Følner sequence $\{F_n\}$ for the action of G on G/H one defines the maps $p_n : Y^{G/H} \to Y$, $p_n(w) = |F_n|^{-1/2} \sum_{x \in F_n} w(x)$, each factoring the action $s_\varphi^{G/H}$ onto

s_φ. Then using arguments as in [**KT08**] it can be shown that for cylinder sets $A \subseteq Y$, the sequence $p_n^{-1}(A)$, $n \in \mathbb{N}$, is asymptotically invariant for $s_{G,G/H,Y}$, from which it follows that $s_\varphi \prec b$.

Chapter 4

Weak equivalence and non-classifiability of measure preserving actions

Robin D. Tucker-Drob

Abért-Weiss have shown that the Bernoulli shift s_Γ of a countably infinite group Γ is weakly contained in any free measure preserving action $\boldsymbol{a}$ of Γ. Proving a conjecture of Ioana we establish a strong version of this result by showing that $\boldsymbol{s}_\Gamma \times \boldsymbol{a}$ is weakly equivalent to $\boldsymbol{a}$. Using random Bernoulli shifts introduced by Abért-Glasner-Virag we generalized this to non-free actions, replacing $\boldsymbol{s}_\Gamma$ with a random Bernoulli shift associated to an invariant random subgroup, and replacing the product action with a relatively independent joining. The result for free actions is used along with the theory of Borel reducibility and Hjorth's theory of turbulence to show that the equivalence relations of isomorphism, weak isomorphism, and unitary equivalence on the weak equivalence class of a free measure preserving action do not admit classification by countable structures. This in particular shows that there are no free *weakly rigid* actions, i.e., actions whose weak equivalence class and isomorphism class coincide, answering negatively a question of Abért and Elek.

We also answer a question of Kechris regarding two ergodic theoretic properties of residually finite groups. A countably infinite residually finite group Γ is said to have property EMD* if the action $\boldsymbol{p}_\Gamma$ of Γ on its profinite completion weakly contains all ergodic measure preserving actions of Γ, and Γ is said to have property MD if $\boldsymbol{\iota} \times \boldsymbol{p}_\Gamma$ weakly contains all measure preserving actions of Γ, where $\boldsymbol{\iota}$ denotes the identity action on a standard non-atomic probability space. Kechris shows

that EMD* implies MD and asks if the two properties are actually equivalent. We provide a positive answer to this question by studying the relationship between convexity and weak containment in the space of measure preserving actions.

1. Introduction

By a *measure preserving action* of a countable group Γ we mean a triple $(\Gamma, a, (X, \mu))$, which we write as $\Gamma \curvearrowright^a (X, \mu)$, where (X, μ) is a standard probability space (i.e., a standard Borel space equipped with a Borel probability measure) and $a : \Gamma \times X \to X$ is a Borel action of Γ on X that preserves the Borel probability measure μ. In what follows all measures are probability measures unless explicitly stated otherwise and we will write $\boldsymbol{a}$ and $\boldsymbol{b}$ to denote the measure preserving actions $\Gamma \curvearrowright^a (X, \mu)$ and $\Gamma \curvearrowright^b (Y, \nu)$, respectively, when the group Γ and the underlying probability spaces (X, μ) and (Y, ν) are understood. Given measure preserving actions $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ and $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$, we say that $\boldsymbol{a}$ is *weakly contained* in $\boldsymbol{b}$, and write $\boldsymbol{a} \prec \boldsymbol{b}$, if for every finite partition $A_0, \dots, A_{k-1}$ of X into Borel sets, every finite subset $F \subseteq \Gamma$, and every $\epsilon > 0$, there exists a Borel partition $B_0, \dots, B_{k-1}$ of Y such that

$$|\mu(\gamma^a A_i \cap A_j) - \nu(\gamma^b B_i \cap B_j)| < \epsilon$$

for all $\gamma \in F$ and $0 \leq i, j < k$. We write $\boldsymbol{a} \sim \boldsymbol{b}$ if both $\boldsymbol{a} \prec \boldsymbol{b}$ and $\boldsymbol{b} \prec \boldsymbol{a}$, in which case $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be *weakly equivalent*. The notion of weak containment of measure preserving actions was introduced by Kechris [**Kec10**] as an ergodic theoretic analogue of weak containment for unitary representations.

Weak containment of unitary representations may be defined as follows (see [**BHV08**, Appendix F]). Let π and ρ be unitary representations of Γ on the Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$, respectively. Then π is *weakly contained* in ρ, written $\pi \prec \rho$, if for every unit vector ξ in $\mathcal{H}_\pi$, every finite subset $F \subseteq \Gamma$, and every $\epsilon > 0$, there exists a finite collection $\eta_0, \dots, \eta_{k-1}$ of unit vectors in $\mathcal{H}_\rho$ and nonnegative real numbers $\alpha_0, \dots, \alpha_{k-1}$ with $\sum_{i=0}^{k-1} \alpha_i = 1$ such that

$$|\langle \pi(\gamma)\xi, \xi\rangle - \sum_{i=0}^{k-1} \alpha_i \langle \rho(\gamma)\eta_i, \eta_i\rangle| < \epsilon$$

for all $\gamma \in F$. Each unit vector $\xi \in H_\pi$ gives rise to a normalized positive definite function on Γ defined by $\gamma \mapsto \langle \pi(\gamma)\xi, \xi\rangle$. We call such a function a *normalized positive definite function realized in* π and we may rephrase the definition of $\pi \prec \rho$ accordingly as: every normalized positive definite function realized in π is a pointwise limit of convex sums of normalized positive definite functions realized in ρ.

A similar rephrasing also applies to weak containment of measure preserving actions, as pointed out by Abért-Weiss [**AW11**]. If we view a finite Borel partition $A_0, \dots, A_{k-1}$ of X as a Borel function $\phi : X \to k = \{0, 1, \dots, k-1\}$ (where we view k as a discrete space) then, given a measure preserving action $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$, each partition $\phi : X \to k$ gives rise to a shift-invariant Borel probability measure $(\Phi^{\phi,a})_*\mu$ on k^Γ, where

$$\Phi^{\phi,a}(x)(\gamma) = \phi((\gamma^{-1})^a \cdot x).$$

The map $\Phi^{\phi,a}$ is equivariant between the action a and the shift action s on k^Γ given by $(\gamma^s \cdot f)(\delta) = f(\gamma^{-1}\delta)$, and one may show that the measures $(\Phi^{\phi,a})_*\mu$, as ϕ ranges over all Borel partitions of X into k-pieces, are precisely those shift-invariant Borel measures λ such that $\Gamma \curvearrowright^s (k^\Gamma, \lambda)$ is a factor of $\boldsymbol{a}$. In this language $\boldsymbol{a}$ being weakly contained in $\boldsymbol{b}$ means that for every natural number k, each shift-invariant measure on k^Γ that is a factor of $\boldsymbol{a}$ is a weak*-limit of shift-invariant measures that are factors of $\boldsymbol{b}$.

More precisely, given a compact Polish space K we equip K^Γ with the product topology, and we let $M_s(K^\Gamma)$ denote the convex set of shift-invariant Borel probability measures on K^Γ equipped with the weak*-topology so that it is also a compact Polish space. We define

$$E(\boldsymbol{a}, K) = \{(\Phi^{\phi,a})_*\mu \,:\, \phi : X \to K \text{ is Borel}\} \subseteq M_s(K^\Gamma).$$

Then Abért-Weiss characterize weak containment of measure preserving actions as follows: $\boldsymbol{a} \prec \boldsymbol{b}$ if and only if $E(\boldsymbol{a}, K) \subseteq \overline{E(\boldsymbol{b}, K)}$ for every finite K if and only if $E(\boldsymbol{a}, K) \subseteq \overline{E(\boldsymbol{b}, K)}$ for every compact Polish space K.

From this point of view one difference between the two notions of weak containment is apparent. While weak containment of representations allows for normalized positive definite functions realized in π to be approximated by *convex sums* of normalized positive definite functions realized in ρ, weak containment of measure preserving actions asks that shift-invariant factors of $\boldsymbol{a}$ be approximated by a *single* shift-invariant factor of $\boldsymbol{b}$ at a time. It is natural to ask for a characterization of the situation in which shift-invariant factors of $\boldsymbol{a}$ are approximated by convex sums of shift-invariant factors of $\boldsymbol{b}$. When this is the case we say that $\boldsymbol{a}$ is *stably weakly contained in* $\boldsymbol{b}$ and we write $\boldsymbol{a} \prec_s \boldsymbol{b}$. The relationship between weak containment and stable weak containment of measure preserving actions is analogous to the relationship between weak containment in the sense of Zimmer (see [**BHV08**, F.1.2.(ix)] and [**Kec10**, Appendix H.**(B)**]) and weak containment of unitary representations. Our first theorem is a characterization of this stable version of weak containment of measure preserving actions.

In what follows (X, μ) and (Y, ν) and (Z, η) always denote standard probability spaces. We let $\iota_\eta : \Gamma \times Z \to Z$ denote the trivial (identity) action of Γ on (Z, η), writing $\boldsymbol{\iota}_\eta$ for the corresponding triple $\Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$, and we write ι and $\boldsymbol{\iota}$ for ι_η and $\boldsymbol{\iota}_\eta$, respectively, when η is non-atomic. We show the following in §3.

THEOREM 1.1. *Let* $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ *be a measure preserving action of* Γ. *Then* $\overline{E(\boldsymbol{\iota} \times \boldsymbol{b}, K)} = \overline{co} E(\boldsymbol{b}, K)$ *for every compact Polish* K. *In particular, for any* $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ *we have that* $\boldsymbol{a} \prec \boldsymbol{\iota} \times \boldsymbol{b}$ *if and only if* $E(\boldsymbol{a}, K) \subseteq \overline{co} E(\boldsymbol{b}, K)$ *for every compact Polish space* K.

When $\boldsymbol{a}$ is ergodic, so that $E(\boldsymbol{a}, K)$ is contained in the extreme points of $M_s(K^\Gamma)$, we show that Theorem 1.1 implies the following direct analogue of the fact (see [**BHV08**, F.1.4]) that if π and ρ are representations of Γ, π is irreducible, and π is weakly contained in ρ, then every normalized positive definite function realized in π is actually a pointwise limit of normalized positive definite functions realized in ρ.

THEOREM 1.2. *Let* $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ *and* $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ *be measure preserving actions of* Γ *and suppose that* $\boldsymbol{a}$ *is ergodic. If* $\boldsymbol{a} \prec \boldsymbol{\iota} \times \boldsymbol{b}$ *then* $\boldsymbol{a} \prec \boldsymbol{b}$*.*

In Theorem 3.11 we show more generally that if $\boldsymbol{a}$ is an ergodic measure preserving action that is weakly contained in $\boldsymbol{d}$, then $\boldsymbol{a}$ is weakly contained in almost every ergodic component of $\boldsymbol{d}$. This may be seen as a weak containment analogue of the fact that if $\boldsymbol{a}$ is a factor of $\boldsymbol{d}$, then $\boldsymbol{a}$ is a factor of almost every ergodic component of $\boldsymbol{d}$ (see Proposition 3.8 below).

One consequence of Theorem 1.2 is that every non-amenable group has a free, non-ergodic weak equivalence class, and this in fact characterizes non-amenability (Corollary 4.2 below).

THEOREM 1.3. *If* $\boldsymbol{b}$ *a measure preserving action of* Γ *that is strongly ergodic, then* $\boldsymbol{\iota} \times \boldsymbol{b}$ *is not weakly equivalent to any ergodic action. In particular, if* Γ *is a non-amenable group and* $\boldsymbol{s}_\Gamma = \Gamma \curvearrowright^{s_\Gamma} ([0,1]^\Gamma, \lambda^\Gamma)$ *is the Bernoulli shift action of* Γ*, then* $\boldsymbol{\iota} \times \boldsymbol{s}_\Gamma$ *is a free action of* Γ *that is not weakly equivalent to any ergodic action.*

If $\mathcal{B}$ is a class of measure preserving actions of a countable group Γ and $\boldsymbol{a} \in \mathcal{B}$, then $\boldsymbol{a}$ is called *universal for* $\mathcal{B}$ if $\boldsymbol{b} \prec \boldsymbol{a}$ for every $\boldsymbol{b} \in \mathcal{B}$. When $\boldsymbol{a}$ is universal for the class of *all* measure preserving actions of Γ then $\boldsymbol{a}$ is simply called *universal*. In §4 we study the universality properties EMD, EMD*, and MD of residually finite groups introduced by Kechris [**Kec12**] (MD was also independently studied by Bowen [**Bow03**], but with different terminology), and defined as follows. Let Γ be a countably infinite group. Γ is said to have property EMD if the measure preserving action $\boldsymbol{p}_\Gamma$ of Γ on its profinite completion is universal. Γ is said to have property EMD* if $\boldsymbol{p}_\Gamma$ is universal for the class of all ergodic measure preserving actions of Γ. Γ is said to have property MD if $\boldsymbol{\iota} \times \boldsymbol{p}_\Gamma$ is universal.

Each of these properties imply that Γ is residually finite and it is clear that EMD implies both EMD* and MD. Kechris shows that EMD* implies MD and asks (Question 4.11 of [**Kec12**]) whether the converse is true. We provide a positive answer to this question.

THEOREM 1.4. *The properties MD and EMD* are equivalent.*

This implies (Corollary 4.7 below) that the properties EMD and MD are equivalent for all groups without property (T). We also show in Theorem 4.8 that the free product of groups with property MD has EMD and we give two reformulations of the problem of whether EMD and MD are equivalent in general (Theorem 4.10 below).

In §5 we discuss the structure of weak equivalence with respect to invariant random subgroups. A countable group Γ acts on the compact space $\mathrm{Sub}(\Gamma) \subseteq 2^\Gamma$ of all of its subgroups by conjugation. Following [**AGV12**], a conjugation-invariant Borel probability measure on $\mathrm{Sub}(\Gamma)$ will be called an *invariant random subgroup* (IRS) of Γ. We let $\mathrm{IRS}(\Gamma)$ denote the set of all invariant random subgroups of Γ. If $\boldsymbol{a} = \Gamma \curvearrowright^a (Y, \nu)$ is a measure preserving action of Γ then the stabilizer map $y \mapsto \mathrm{stab}_a(y) \in \mathrm{Sub}(\Gamma)$ is equivariant so that the measure $(\mathrm{stab}_a)_*\nu$ is an IRS of Γ which we call the *type* of $\boldsymbol{a}$, and denote $\mathrm{type}(\boldsymbol{a})$. It is shown in [**AE11**] that the type of a measure preserving action is an invariant of weak equivalence (we give a proof of this in 5.2 below).

In §5.2 we use the framework laid out in §3 to study the compact metric topology introduced by Abért-Elek [**AE11**] on the set $A_\sim(\Gamma, X, \mu)$ of weak equivalence classes of measure preserving actions of Γ. We show that the map $A_\sim(\Gamma, X, \mu) \to \mathrm{IRS}(\Gamma)$ sending each weak equivalence class to its type in $\mathrm{IRS}(\Gamma)$ is continuous when $\mathrm{IRS}(\Gamma)$ is equipped with the weak* topology.

In §5.3 we detail a construction, described in [**AGV12**], whereby, given a probability space (Z, η), one canonically associates to each $\theta \in \mathrm{IRS}(\Gamma)$ a measure preserving action $\boldsymbol{s}_{\theta,\eta}$ of Γ such that $\mathrm{type}(\boldsymbol{s}_{\theta,\eta}) = \theta$ when η is non-atomic. We call $\boldsymbol{s}_{\theta,\eta}$ the *θ-random Bernoulli shift of Γ over (Z, η)*. When $\boldsymbol{a}$ is free then $\mathrm{type}(\boldsymbol{a})$ is the point mass $\delta_{\langle e \rangle}$ on the trivial subgroup $\langle e \rangle$ of Γ and $\boldsymbol{s}_{\delta_{\langle e \rangle},\eta}$ is the usual Bernoulli shift action of Γ on (Z^Γ, η^Γ).

After establishing some properties of random Bernoulli shifts we show the following in §5.5.

THEOREM 1.5. *Let* $\boldsymbol{a} = \Gamma \curvearrowright^a (Y,\nu)$ *be a non-atomic measure preserving action of type* θ*, and let* $\boldsymbol{s}_{\theta,\eta}$ *be the* θ*-random Bernoulli shift over* (Z,η)*. Then the relatively independent joining of* $\boldsymbol{s}_{\theta,\eta}$ *and* $\boldsymbol{a}$ *over their common factor* $\Gamma \curvearrowright (\mathrm{Sub}(\Gamma),\theta)$ *is weakly equivalent to* $\boldsymbol{a}$*. In particular,* $\boldsymbol{s}_{\theta,\eta}$ *is weakly contained in every non-atomic action of type* θ*.*

When $\boldsymbol{a}$ is free then the relatively independent joining of $\boldsymbol{s}_{\delta_{\langle e\rangle},\eta}$ and $\boldsymbol{a}$ is simply the product of the Bernoulli shift with $\boldsymbol{a}$ and Theorem 1.5 proves a conjecture of Ioana, becoming the following strengthening of Abért-Weiss [**AW11**, Theorem 1]:

COROLLARY 1.6. *Let* $\boldsymbol{s}_\Gamma = \Gamma \curvearrowright^{s_\Gamma} ([0,1]^\Gamma,\lambda^\Gamma)$ *be the Bernoulli shift action of* Γ*, where* λ *denotes Lebesgue measure on* $[0,1]$*. Let* $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ *be a free measure preserving action of* Γ *on a non-atomic standard probability space* (X,μ)*. Then* $\boldsymbol{s}_\Gamma \times \boldsymbol{a}$ *is weakly equivalent to* $\boldsymbol{a}$*.*

Several invariants of measure preserving actions such as groupoid cost [**AW11**] ([**Kec10**] for the case of free actions) and independence number [**CK13**] are known to increase or decrease with weak containment (see also [**AE11**] and [**CKTD11**] for other examples). A consequence of Theorem 1.5 is that, for a finitely generated group Γ, among all non-atomic measure preserving actions of type θ, the groupoid cost attains its maximum and the independence number attains its minimum on $\boldsymbol{s}_{\theta,\lambda}$. Likewise, Corollary 1.6 implies that for any free measure preserving action $\boldsymbol{a}$ of Γ, both $\boldsymbol{a}$ and $\boldsymbol{s}_\Gamma \times \boldsymbol{a}$ have the same independence number, and the orbit equivalence relation associated to $\boldsymbol{a}$ and $\boldsymbol{s}_\Gamma \times \boldsymbol{a}$ have the same cost.

In §6 we address the question of how many isomorphism classes of actions are contained in a given weak equivalence class. We answer a question of Abért-Elek [**AE11**, Question 6.1], showing that the weak equivalence class of any free action always contains non-isomorphic actions. Our arguments show that there are in fact continuum-many isomorphism classes of actions in any free weak equivalence class, and from the perspective of Borel reducibility we can strengthen this even further. Let $A(\Gamma, X, \mu)$ denote the Polish space of measure preserving actions of Γ on (X, μ) and let $\boldsymbol{a}, \boldsymbol{b} \in A(\Gamma, X, \mu)$. Then $\boldsymbol{a}$ and $\boldsymbol{b}$ are called *weakly isomorphic*, written $\boldsymbol{a} \cong^{w} \boldsymbol{b}$, if both $\boldsymbol{a} \sqsubseteq \boldsymbol{b}$ and $\boldsymbol{b} \sqsubseteq \boldsymbol{a}$. We call $\boldsymbol{a}$ and $\boldsymbol{b}$ *unitarily equivalent*, written $\boldsymbol{a} \cong^{\mathcal{U}} \boldsymbol{b}$, if the corresponding Koopman representations $\kappa_0^{\boldsymbol{a}}$ and $\kappa_0^{\boldsymbol{b}}$ are unitarily equivalent. We let $\cong$ denote isomorphism of actions. Then $\boldsymbol{a} \cong \boldsymbol{b} \Rightarrow \boldsymbol{a} \cong^{w} \boldsymbol{b} \Rightarrow \boldsymbol{a} \cong^{\mathcal{U}} \boldsymbol{b}$. We now have the following.

THEOREM 1.7. *Let $\boldsymbol{a} = \Gamma \curvearrowright^{a} (X, \mu)$ be a free action of a countably infinite group Γ and let $[\boldsymbol{a}] = \{\boldsymbol{b} \in A(\Gamma, X, \mu) : \boldsymbol{b} \sim \boldsymbol{a}\}$ be the weak equivalence class of $\boldsymbol{a}$. Then isomorphism on $[\boldsymbol{a}]$ does not admit classification by countable structures. The same holds for both weak isomorphism and unitary equivalence on $[\boldsymbol{a}]$.*

Any two free actions of an infinite amenable group are weakly equivalent ([**FW04**], see also Remark 4.1 and Theorem 1.8 below), so for amenable Γ Theorem 1.7 follows from [**FW04**], [**Hjo97**] and [**Kec10**, 13.7, 13.8, 13.9] (see also [**KLP10**, 4.4]), while for non-amenable Γ there are continuum-many weak equivalence classes of free actions (see Remark 4.3 below), and Theorem 1.7 is therefore a refinement of the existing results. The proof of 1.7 uses the methods of [**Kec10**, 13.7] and [**KLP10**]. We fix an infinite-dimensional separable Hilbert space $\mathcal{H}$, and denote by $\mathrm{Rep}_{\lambda}(\Gamma, \mathcal{H})$ the Polish space of unitary representations of Γ on $\mathcal{H}$ that are weakly contained in the left regular representation λ_Γ of Γ. The conjugacy action of the unitary group $\mathcal{U}(\mathcal{H})$ on $\mathrm{Rep}_{\lambda}(\Gamma, \mathcal{H})$ is generically turbulent by [**KLP10**, 3.3], so Theorem 1.7 will follow by showing that unitary conjugacy on $\mathrm{Rep}_{\lambda}(\Gamma, \mathcal{H})$ is not generically $\cong |[\boldsymbol{a}]$-ergodic (and that the same holds for $\cong^{w}$ and $\cong^{\mathcal{U}}$ in place of $\cong$). For this we find a continuous homomorphism ψ from unitary conjugacy on $\mathrm{Rep}_{\lambda}(\Gamma, \mathcal{H})$ to isomorphism on $[\boldsymbol{a}]$ with the property that the inverse image of each $\cong^{\mathcal{U}}$-class is meager. The main new ingredient that is needed in the proof of Theorem 1.7 is Corollary 1.6, which shows that the homomorphism ψ we define takes values in $[\boldsymbol{a}]$.

In §7 we show that when Γ is amenable, $\mathrm{type}(\boldsymbol{a})$ completely determines the stable weak equivalence class (Definition 9.1) of a measure preserving action $\boldsymbol{a}$ of Γ.

THEOREM 1.8. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be two measure preserving actions of an amenable group* Γ. *Then*

(1) *type*$(\boldsymbol{a})$ = *type*$(\boldsymbol{b})$ *if and only if* $\boldsymbol{a} \sim_s \boldsymbol{b}$.

(2) *Suppose that type*$(\boldsymbol{a})$ = *type*$(\boldsymbol{b})$ *concentrates on the infinite index subgroups of* Γ. *Then* $\boldsymbol{a} \sim \boldsymbol{b}$.

Combining this with the results of §5.2 (in particular, Remark 5.8) shows that when Γ is amenable, the type map $[\boldsymbol{a}]_s \mapsto \text{type}(\boldsymbol{a})$, from the compact space $A_{\sim_s}(\Gamma, Y, \nu)$ of all stable weak equivalence classes of measure preserving actions of Γ, to the space $\text{IRS}(\Gamma)$, is a homeomorphism.

We end with two appendices, one on ultraproducts of measure preserving actions, and one on stable weak containment.

REMARK 1.9. After sending Gábor Elek a preliminary version of this paper, I was informed by him that he has independently obtained a version of Theorem 1.8. See [**Ele12**].

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2. Preliminaries and notation

Γ will always denote a countable group, and e will always denote the identity element of Γ.

2.1. Measure algebras and standard probability spaces. All measures will be probability measures unless explicitly stated otherwise. A *standard probability space* is a probability measure space $(X, \mu) = (X, \boldsymbol{B}(X), \mu)$ where X is a standard Borel space and μ is a probability measure on the σ-algebra $\boldsymbol{B}(X)$ of Borel subsets of X. In what follows, (X, μ), (Y, ν), and (Z, η) will always denote standard probability spaces. Though we mainly focus on standard probability spaces we will make use of nonstandard probability spaces arising as ultraproducts of standard probability spaces. We will write (W, ρ) for a probability space that may or may not be standard.

The *measure algebra* MALG_ρ of a probability space (W, ρ) is the σ-algebra of ρ-measurable sets modulo the σ-ideal of null sets, equipped with the measure ρ. We also equip MALG_ρ with the metric $d_\rho(A, B) = \rho(A \Delta B)$. We will sometimes abuse notation and identify a measurable set $A \subseteq W$ with its equivalence class in MALG_ρ when there is no danger of confusion.

2.2. Measure preserving actions. Let Γ be a countable group. A *measure preserving action* of Γ is a triple $(\Gamma, a, (X, \mu))$, which we write as $\Gamma \curvearrowright^a (X, \mu)$, where (X, μ) is a standard probability space and $a : \Gamma \times X \to X$ is a Borel action of Γ on X that preserves the probability measure μ. A measure preserving action $\Gamma \curvearrowright^a (X, \mu)$ will often also be denoted by a boldface letter such as $\boldsymbol{a}$ or $\boldsymbol{\mu}$ depending on whether we want to emphasize the underlying action or the underlying probability measure. When $\gamma \in \Gamma$ and $x \in X$ we write $\gamma^a \cdot x$ or $\gamma^a x$ for $a(\gamma, x)$. In what follows, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ and $\boldsymbol{d}$ will always denote measure preserving actions of Γ.

We will also make use of actions of Γ on nonstandard probability spaces. When (W, ρ) is a probability space and $o : \Gamma \times W \to W$ is a measurable action of Γ on W that preserves ρ then we will still use the notations $\boldsymbol{o} = \Gamma \curvearrowright^o (W, \rho)$, γ^o, etc., from above, *though we reserve the phrase "measure preserving action" for the case when the underlying probability space is standard*.

2.3. The space of measure preserving actions. We let $A(\Gamma, X, \mu)$ denote the set of all measure preserving actions of Γ on (X, μ) modulo almost everywhere equality. That is, two measure preserving actions $\boldsymbol{a}$ and $\boldsymbol{b}$ of Γ on (X, μ) are equivalent if $\mu(\{x \in X : \gamma^a x \neq \gamma^b x\}) = 0$ for all $\gamma \in \Gamma$. Though elements of $A(\Gamma, X, \mu)$ are equivalence classes of measure preserving actions we will abuse notation and confuse elements of $A(\Gamma, X, \mu)$ with their Borel representatives, making sure our statements and definitions are independent of the choice of representative when it is not obvious. We equip $A(\Gamma, X, \mu)$ with the *weak* *topology*, which is a Polish topology generated by the maps $\boldsymbol{a} \mapsto \gamma^a A \in \mathrm{MALG}_\mu$, with A ranging over MALG_μ and γ ranging over elements of Γ.

Notation. For $\boldsymbol{a} \in A(\Gamma, X, \mu)$ and $\boldsymbol{b} \in A(\Gamma, Y, \nu)$ we let $\boldsymbol{a} \sqsubseteq \boldsymbol{b}$ denote that $\boldsymbol{a}$ is a factor of $\boldsymbol{b}$ and we let $\boldsymbol{a} \cong \boldsymbol{b}$ denote that $\boldsymbol{a}$ and $\boldsymbol{b}$ are isomorphic. We let $\boldsymbol{\iota}_\eta \in A(\Gamma, Z, \eta)$ denote the trivial (identity) system $\Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$, and we write $\boldsymbol{\iota}$ for $\boldsymbol{\iota}_\eta$ when η is non-atomic. We call $\Gamma \curvearrowright^a (X, \mu)$ *non-atomic* if the probability space (X, μ) is non-atomic. If $T : X \to X$ then we let $\mathrm{supp}(T) = \{x \in X \, : \, T(x) \neq x\}$. For a $A \subseteq X$ we denote by $\mu|A$ the restriction of μ to A given by $(\mu|A)(B) = \mu(B \cap A)$ and we denote by μ_A the conditional probability measure $\mu_A(B) = \frac{\mu(B \cap A)}{\mu(A)}$ where we use the convention that $\mu_A \equiv 0$ when $A \subseteq X$ is null.

Convention. We will regularly neglect null sets when there is no danger of confusion.

3. Proofs of Theorems 1.1 and 1.2

3.1. Weak containment and shift-invariant factors. Let K be a compact Polish space and equip K^Γ with the product topology so that it is also a compact Polish space. Then Γ acts continuously on K^Γ by the shift action s, given by $(\delta^s f)(\gamma) = f(\delta^{-1}\gamma)$ for $\delta, \gamma \in \Gamma$, $f \in K^\Gamma$. Let (W, ρ) be a probability space and let $\boldsymbol{o} = \Gamma \curvearrowright^o (W, \rho)$ be a measurable action of Γ on W that preserves ρ. For each measurable function $\phi : W \to K$ we define $\Phi^{\phi,o} : W \to K^\Gamma$ by $\Phi^{\phi,o}(w)(\gamma) = \phi((\gamma^{-1})^o \cdot w)$, and we let

$$E(\boldsymbol{o}, K) = \{(\Phi^{\phi,o})_* \rho \, : \, \phi : W \to K \text{ is } \rho\text{-measurable}\}.$$

Each map $\Phi^{\phi,o}$ is a factor map from $\boldsymbol{o}$ to $\Gamma \curvearrowright^s (K^\Gamma, (\Phi^{a,\phi})_* \mu)$ since

$$\Phi^{\phi,o}(\delta^o \cdot w)(\gamma) = \phi((\gamma^{-1}\delta)^o \cdot w) = \phi(((\delta^{-1}\gamma)^{-1})^o \cdot w) = \Phi^{\phi,o}(w)(\delta^{-1}\gamma) = (\delta^s \cdot \Phi^{\phi,o}(w))(\gamma).$$

Conversely, given any measurable factor map $\psi : \Gamma \curvearrowright^o (W, \rho) \to \Gamma \curvearrowright^s (K^\Gamma, \pi_* \mu)$ the map $\phi(w) = \psi(w)(e)$ is also measurable, and for almost all $w \in W$ and all $\gamma \in \Gamma$ we have $\Phi^{\phi,o}(w)(\gamma^{-1}) = \phi(\gamma^a \cdot w) = \psi(\gamma^o \cdot w)(e) = (\gamma^s \cdot \psi(w))(e) = \psi(w)(\gamma^{-1})$ so that $\psi_* \rho = (\Phi^{\phi,o})_* \rho$. It follows that $E(\boldsymbol{o}, K)$ is the set of all shift-invariant Borel probability measures on K^Γ that are factors of $\boldsymbol{o}$. We let $M_s(K^\Gamma)$ denote the convex set of all shift-invariant Borel probability measures on K^Γ. Equipped with the weak* topology this is a compact metrizable subset of $C(K^\Gamma)^*$. If $E \subseteq M_s(K^\Gamma)$ we let co$E$ denote the convex hull of E and we let $\overline{\text{co}}E$ denote the closed convex hull of E. For $\gamma \in \Gamma$ we let $\pi_\gamma : K^\Gamma \to K$ denote the projection map $\pi_\gamma(f) = f(\gamma)$.

LEMMA 3.1. *Suppose that $\phi_n : W \to K$, $n \in \mathbb{N}$, is a sequence of measurable functions that converge in measure to the measurable function $\phi : W \to K$. Then $(\Phi^{\phi_n,o})_*\rho \to (\Phi^{\phi,o})_*\rho$ in $M_s(K^\Gamma)$.*

PROOF. ϕ_n converges to ϕ in measure if and only if for every subsequence $\{n_i\}$ there is a further subsequence $\{m_i\}$ such that $\phi_{m_i} \to \phi$ almost surely. If $\phi_{m_i} \to \phi$ almost surely then for all $\gamma \in \Gamma$, $\Phi^{\phi_{m_i},o}(w)(\gamma) \to \Phi^{\phi,o}(w)(\gamma)$ almost surely, and so $\Phi^{\phi_{m_i},o}(w) \to \Phi^{\phi,o}(w)$ almost surely. It follows that $\Phi^{\phi_n,o} \to \Phi^{\phi,o}$ in measure. Since convergence in measure implies convergence in distribution it follows that $(\Phi^{\phi_n,o})_*\rho \to (\Phi^{\phi,o})_*\rho$ in $M_s(K^\Gamma)$. $\square$

REMARK 3.2. We may form the space $L(W, \rho, K)$ of all measurable maps $\phi : W \to K$, where we identify two such maps if they agree ρ-almost everywhere. If $d \leq 1$ is a compatible metric for K then we equip $L(W, \rho, K)$ with the metric $\tilde{d}(\phi, \psi) = \int_W d(\phi(w), \psi(w))\, d\rho(w)$, and then $\phi_n \to \phi$ in this topology if and only if ϕ_n converges to ϕ in measure. Then Lemma 3.1 says that for each measure preserving action $\Gamma \curvearrowright^o (W, \rho)$, the map $\phi \mapsto (\Phi^{\phi,o})_*\rho$ from $L(W, \rho, K)$ to $M_s(K^\Gamma)$ is continuous. The metric $\tilde{d}$ is complete, and $\tilde{d}$ is separable when (W, ρ) is standard. We note for later use that the set of all $\phi \in L(W, \rho, K)$ with finite range is dense in $L(W, \rho, K)$ (this follows from d being separable). Proofs of these facts may be found in [**Kec10**, Section 19] and [**Moo76**] (these references assume that the space (W, ρ) is standard, but this assumption is not used to prove the facts mentioned here).

We will find the following generalization of weak containment useful.

DEFINITION 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of measure preserving actions of Γ. We say that $\mathcal{A}$ is *weakly contained in* $\mathcal{B}$, written $\mathcal{A} \prec \mathcal{B}$, if for every $\Gamma \curvearrowright^a (X, \mu) = \boldsymbol{a} \in \mathcal{A}$, for any Borel partition $A_0, \ldots, A_{k-1}$ of X, $F \subseteq \Gamma$ finite, and $\epsilon > 0$, there exists $\Gamma \curvearrowright^b (Y, \nu) = \boldsymbol{b} \in \mathcal{B}$ and a Borel partition $B_0, \ldots, B_{k-1}$ of Y such that

$$|\mu(\gamma^a A_i \cap A_j) - \nu(\gamma^b B_i \cap B_j)| < \epsilon$$

for all $i, j < k$ and $\gamma \in F$.

This is a generalization of weak containment in the sense that when $\mathcal{A} = \{\boldsymbol{a}\}$ and $\mathcal{B} = \{\boldsymbol{b}\}$ are both singletons then $\mathcal{A} \prec \mathcal{B}$ if and only if $\boldsymbol{a} \prec \boldsymbol{b}$ in the original sense defined in the introduction. We write $\boldsymbol{a} \prec \mathcal{B}$ for $\{\boldsymbol{a}\} \prec \mathcal{B}$, and $\mathcal{A} \prec \boldsymbol{b}$ for $\mathcal{A} \prec \{\boldsymbol{b}\}$. If both $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{A}$ then we put $\mathcal{A} \sim \mathcal{B}$. It is clear that $\prec$ is a reflexive and transitive relation on sets of actions. The arguments in 10.1 of [**Kec10**] show the following.

PROPOSITION 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be sets of non-atomic measure preserving actions of Γ. Then $\mathcal{A} \prec \mathcal{B}$ if and only if for every $\Gamma \curvearrowright^a (X, \mu) = \boldsymbol{a} \in \mathcal{A}$, there exists a sequence $\boldsymbol{a}_n \in A(\Gamma, X, \mu)$, $n \in \mathbb{N}$, converging to $\boldsymbol{a}$ such that each $\boldsymbol{a}_n$ is isomorphic to some $\boldsymbol{b}_n \in \mathcal{B}$. In particular, $\boldsymbol{a} \prec \mathcal{B}$ if and only if $\boldsymbol{a} \in \overline{\{\boldsymbol{d} \in A(\Gamma, X, \mu) \,:\, \exists \boldsymbol{b} \in \mathcal{B}\ \boldsymbol{d} \cong \boldsymbol{b}\}}$.*

We also have the corresponding generalization of [**AW11**, Lemma 8].

PROPOSITION 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be sets of measure preserving actions of Γ. Then the following are equivalent*

(1) *$\mathcal{A}$ is weakly contained in $\mathcal{B}$*
(2) $\bigcup_{\boldsymbol{d} \in \mathcal{A}} E(\boldsymbol{d}, K) \subseteq \overline{\bigcup_{\boldsymbol{b} \in \mathcal{B}} E(\boldsymbol{b}, K)}$ *for every finite K.*
(3) $\bigcup_{\boldsymbol{d} \in \mathcal{A}} E(\boldsymbol{d}, K) \subseteq \overline{\bigcup_{\boldsymbol{b} \in \mathcal{B}} E(\boldsymbol{b}, K)}$ *for every compact Polish K.*
(4) $\bigcup_{\boldsymbol{d} \in \mathcal{A}} E(\boldsymbol{d}, 2^{\mathbb{N}}) \subseteq \overline{\bigcup_{\boldsymbol{b} \in \mathcal{B}} E(\boldsymbol{b}, 2^{\mathbb{N}})}$.

PROOF. It suffices to show this for the case $\mathcal{A} = \{\boldsymbol{d}\}$ is a singleton. We let (X, μ) be the space of $\boldsymbol{d}$.

We begin with the implication (1)$\Rightarrow$(2). It suffices to show (2) for the case $K = k = \{0, 1, \ldots, k-1\}$ for some $k \in \mathbb{N}$. Fix a Borel function $\phi : X \to k$, let $\lambda = (\Phi^{\phi, d})_* \mu$, and let $A_i = \phi^{-1}(\{i\})$ for $i < k$. Fix an exhaustive sequence $e \in F_0 \subseteq F_1 \subseteq \cdots$ of finite subsets of Γ. For each finite $F \subseteq \Gamma$ and function $\tau : F \to k$ let $A_\tau = \bigcap_{\gamma \in F} \gamma^d A_{\tau(\gamma)}$. As $\boldsymbol{d} \prec \mathcal{B}$ we may find for each $n \in \mathbb{N}$ a measure preserving action $\boldsymbol{b}_n = \Gamma \curvearrowright^{b_n} (Y_n, \nu_n)$ in $\mathcal{B}$ along with Borel partitions $\{B^n_\tau\}_{\tau \in k^{F_n}}$ of Y_n such that

$$|\mu(\gamma^a A_{\tau_1} \cap A_{\tau_2}) - \nu_n(\gamma^{b_n} B^n_{\tau_1} \cap B^n_{\tau_2})| < \epsilon_n \tag{3.1}$$

for all $\tau_1, \tau_2 \in k^{F_n}$, and where ϵ_n is small depending on n, k, and $|F_n|$. Define $\psi_n : Y_n \to k$ by $\psi_n(y) = i$ if $y \in B^n_\tau$ for some $\tau \in k^{F_n}$ with $\tau(e) = i$, and let $\lambda_n = (\Phi^{\psi_n, b_n})_* \nu_n$. To show that $\lambda_n \to \lambda$ it suffices to show that $\lambda_n(A) \to \lambda(A)$ for every basic clopen set $A \subseteq k^\Gamma$ of the form $A = \bigcap_{\gamma \in F} \pi_\gamma^{-1}(\{i_\gamma\})$, where $e \in F \subseteq \Gamma$ is finite and $i_\gamma < k$ for each $\gamma \in F$. We let $\upsilon \in k^F$ be the function $\upsilon(\gamma) = i_\gamma$.

For $i < k$ let $B^n_i = \bigsqcup\{B_\tau : \tau \in k^{F_n} \text{ and } \tau(e) = i\}$. Let n_0 be so large that $F^2 \subseteq F_{n_0}$ and for all $n > n_0$ and each $\sigma \in k^J$, $J \subseteq F_n$, let $B^n_\sigma = \bigsqcup\{B_\tau : \tau \in k^{F_n} \text{ and } \sigma \sqsubseteq \tau\}$ and let $\tilde{B}^n_\sigma = \bigcap_{\gamma \in J} \gamma^d B^n_{\sigma(\gamma)}$. Then $B^n_i = \bigsqcup\{B^n_\sigma : \sigma \in k^F \text{ and } \sigma(e) = i\}$. For $\gamma \in \Gamma$, $J \subseteq \Gamma$ and $\sigma \in k^J$ let $\gamma \cdot \sigma \in k^{\gamma J}$ be given by $(\gamma \cdot \sigma)(\delta) = \sigma(\gamma^{-1}\delta)$ for all $\delta \in \gamma J$. For $\sigma \in k^F$ and $\gamma \in F$ we have $|\nu_n(\gamma^{b_n} B^n_\sigma \cap B^n_{\gamma\cdot\sigma}) - \mu(\gamma^d A_\sigma \cap A_{\gamma\cdot\sigma})| \leq \sum_{\{\tau \in k^{F_n} : \sigma \sqsubseteq \tau\}} \sum_{\{\tau' \in k^{F_n} : \gamma\cdot\sigma \sqsubseteq \tau'\}} |\nu_n(\gamma^{b_n} B^n_\tau \cap B^n_{\tau'}) - \mu(\gamma^d A_\tau \cap A_{\tau'})| \leq \epsilon_n k^{2|F_n|}$. Similarly, $|\nu_n(B^n_\sigma) - \mu(A_\sigma)| < \epsilon_n k^{2|F_n|}$ and $|\nu_n(B^n_{\gamma\cdot\sigma}) - \mu(A_{\gamma\cdot\sigma})| < \epsilon_n k^{2|F_n|}$. Since $\gamma^d A_\sigma = A_{\gamma\cdot\sigma}$ we obtain from this the estimate

$$d_{\nu_n}(\gamma^{d_n}(B^n_\sigma),\, B^n_{\gamma\cdot\sigma}) = \nu_n(B^n_\sigma) + \nu_n(B^n_{\gamma\cdot\sigma}) - 2\nu_n(\gamma^{d_n}(B^n_\sigma) \cap B^n_{\gamma\cdot\sigma}) < 3\epsilon_n k^{2|F_n|}. \tag{3.2}$$

Since $\{B^n_\tau\}_{\tau \in k^{F_n}}$ is a partition of Y_n and $F^2 \subseteq F_n$ we have the set identities

$$B^n_\upsilon = \bigsqcup_{\substack{\tau \in k^{F_n} \\ \upsilon \sqsubseteq \tau}} B^n_\tau = \bigcap_{\gamma \in F} \bigsqcup_{\substack{\sigma \in k^{\gamma F} \\ \sigma(\gamma) = \upsilon(\gamma)}} B^n_\sigma = \bigcap_{\gamma \in F} \bigsqcup_{\substack{\sigma \in k^F \\ \sigma(e) = \upsilon(\gamma)}} B^n_{\gamma\cdot\sigma}.$$

By (3.2) the d_{ν_n}-distance of this is no more than $3|F|\epsilon_n k^{3|F_n|}$ from the set

$$\bigcap_{\gamma \in F} \bigsqcup_{\substack{\sigma \in k^F \\ \sigma(e) = \upsilon(\gamma)}} \gamma^{d_n} B^n_\sigma = \bigcap_{\gamma \in F} \gamma^{d_n} B^n_{\upsilon(\gamma)} = \tilde{B}^n_\upsilon.$$

Thus $|\lambda_n(A)-\lambda(A)| = |\nu_n(\tilde{B}^n_{\mathfrak{v}})-\mu(A_{\mathfrak{v}})| \leq 3|F|\epsilon_n k^{3|F_n|}+|\nu_n(B^n_{\mathfrak{v}})-\mu(A_{\mathfrak{v}})| < 3|F|\epsilon_n k^{3|F_n|}+\epsilon_n k^{2|F_n|} \to 0$ by our choice of ϵ_n.

(2)⇒(3): Let K be a compact Polish space. It follows from Lemma 3.1 and Remark 3.2 that the set $E_f(\boldsymbol{d}, K)$ of all measures $\lambda \in E(\boldsymbol{d}, K)$ coming from Borel $\phi : X \to K$ with finite range is dense in $E(\boldsymbol{d}, K)$. By (2) we then have $E_f(\boldsymbol{d}, K) \subseteq \overline{\bigcup_{\boldsymbol{b}\in\mathcal{B}} E_f(\boldsymbol{b}, K)} \subseteq \overline{\bigcup_{\boldsymbol{b}\in\mathcal{B}} E(\boldsymbol{b}, K)}$, and (3) now follows.

The implication (3)⇒(4) is trivial. (4)⇒(1): Given a Borel partition $A_0, \ldots, A_{m-1}$ of X, $F \subseteq \Gamma$ finite, and $\epsilon > 0$, let $k_0, \ldots, k_{m-1} \in 2^{\mathbb{N}}$ be distinct and define the function $\phi : X \to 2^{\mathbb{N}}$ by $\phi(x) = i$ if $x \in A_i$. Then $\lambda = (\Phi^{\phi,d})_*\mu \in E(\boldsymbol{d}, 2^{\mathbb{N}})$ so by (4) there exists a sequence $\Gamma \curvearrowright^{b_n} (Y_n, \nu_n) = \boldsymbol{b}_n \in \mathcal{B}$, along with $\phi_n : Y_n \to 2^{\mathbb{N}}$ such that $\lambda_n \to \lambda$, where $\lambda_n = (\Phi^{\phi_n,b_n})_*\nu_n$. Let $C_0, \ldots, C_{m-1}$ disjoint clopen subsets of $2^{\mathbb{N}}$ with $k_i \in C_i$ and for each $n \in \mathbb{N}$ let $B^n_i = \phi_n^{-1}(C_i)$. Then for all $\gamma \in F$ we have

$$|\mu(\gamma^d A_i \cap A_j)-\nu_n(\gamma^{b_n} B^n_i \cap B^n_j)| = |\lambda(\pi_\gamma^{-1}(C_i) \cap \pi_e^{-1}(C_j)) - \lambda_n(\pi_\gamma^{-1}(C_i) \cap \pi_e^{-1}(C_j))| \to 0,$$

so for large enough n this quantity is smaller than ϵ. □

3.2. Convexity in the space of actions. The convex sum of measure preserving actions is defined as follows (see also [**Kec10**, 10.(**F**)]). Let $N \in \{1, 2, \ldots, \infty = \mathbb{N}\}$ and let $\boldsymbol{\alpha} = (\alpha_0, \alpha_1 \ldots) \in [0, 1]^N$ be a finite or countably infinite sequence of non-negative real numbers with $\sum_{i<N} \alpha_i = 1$. Given actions $\boldsymbol{b}_i = \Gamma \curvearrowright^{b_i} (X_i, \mu_i)$, $i < N$, we let $\sum_{i<N} X_i = \{(i, x) : i < N \text{ and } x \in X_i\}$ and we let $\tilde{\mu}_i$ be the image measure of μ_i under the inclusion map $X_i \hookrightarrow \sum_{i<N} X_i$, $x \mapsto (i, x)$. We obtain a measure preserving action $\sum_{i<N} \alpha_i \boldsymbol{b}_i = \Gamma \curvearrowright^{\sum_{i<N} b_i} (\sum_{i<N} X_i, \sum_{i<N} \alpha_i\tilde{\mu}_i)$ defined by $\gamma^{\sum_{i<N} b_i} \cdot (i, x) = (i, \gamma^{b_i} \cdot x)$. If $(X_i, \mu_i) = (X, \mu)$ for each $i < N$ then $(\sum_{i<N} X_i, \sum_{i<N} \alpha_i\tilde{\mu}_i) = (N \times X, \eta_{\boldsymbol{\alpha}} \times \mu)$ where $\eta_{\boldsymbol{\alpha}}$ is the discrete probability measure on N given by $\eta_{\boldsymbol{\alpha}}(\{i\}) = \alpha_i$. If furthermore $\boldsymbol{b}_i = \boldsymbol{b}$ for each $i < N$ then $\sum_{i<N} \alpha_i \boldsymbol{b}_i = \boldsymbol{\iota}_{\eta_{\boldsymbol{\alpha}}} \times \boldsymbol{b}$ is simply the product action.

LEMMA 3.6. *Let $\boldsymbol{b} \in A(\Gamma, X, \mu)$ and let $\boldsymbol{d} = \boldsymbol{\iota}_{\eta_{\boldsymbol{\alpha}}} \times \boldsymbol{b} = \sum_{i=0}^{n-1} \alpha_i \boldsymbol{b}$. Then $E(\boldsymbol{d}, K) \subseteq coE(\boldsymbol{b}, K) \subseteq E(\boldsymbol{\iota} \times \boldsymbol{b}, K)$ for every compact Polish K.*

PROOF. Given $\phi : n \times X \to K$, we want to show that $(\Phi^{\phi,d})_*(\eta_{\boldsymbol{\alpha}} \times \mu) \in \mathrm{co}E(\boldsymbol{b}, K)$. Let $\phi_i : X \to K$ be given by $\phi_i(x) = \phi(i, x)$. Then $(\Phi^{\phi,d})^{-1}(A) = \bigsqcup_{i=0}^{n-1} \{i\} \times (\Phi^{\phi_i,b})^{-1}(A)$ for $A \subseteq K^\Gamma$ and it follows that $(\Phi^{\phi,d})_*(\eta_{\boldsymbol{\alpha}} \times \mu) = \sum_{i=0}^{n-1} \alpha_i (\Phi^{\phi_i,b})_*\mu$, which shows the first inclusion.

Let the underlying space of $\boldsymbol{\iota}$ be (Z, η). Given Borel functions $\phi_0, \ldots, \phi_{n-1} : X \to K$ and $\alpha_0, \ldots, \alpha_{n-1} \geq 0$ with $\sum_{i=0}^{n-1} \alpha_i = 1$, we want to show that $\sum_{i=0}^{n-1} \alpha_i (\Phi^{\phi_i,b})_*\mu \in E(\boldsymbol{\iota} \times \boldsymbol{b}, K)$. Let $C_0, \ldots, C_{n-1}$ be a Borel partition of Z with $\eta(C_i) = \alpha_i$ for $i = 0, \ldots, n-1$. Define $i : Z \to n$ by $i(z) = i$ if $z \in C_i$ and let $\phi : Z \times X \to K$ be the map $\phi(z, x) = \phi_{i(z)}(x)$. Then

$$\Phi^{\phi,\iota\times b}(z, x)(\gamma) = \phi(\gamma^{\iota\times b} \cdot (z, x)) = \phi(z, \gamma^b \cdot x) = \phi_{i(z)}(\gamma^b \cdot x) = \Phi^{\phi_{i(z)},b}(x)(\gamma),$$

and so $(\Phi^{\phi,\iota\times b})^{-1}(A) = \bigsqcup_{i=0}^{n-1} C_i \times (\Phi^{\phi_i,b})^{-1}(A)$ for all $A \subseteq K^\Gamma$. It now follows that $\sum_{i=0}^{n-1} \alpha_i (\Phi^{\phi_i,b})_*\mu = (\Phi^{\phi,\iota\times b})_*(\eta \times \mu)$. $\square$

LEMMA 3.7. *Let $\boldsymbol{b} \in A(\Gamma, X, \mu)$, let $\boldsymbol{\alpha}(n) = (\frac{1}{n}, \ldots, \frac{1}{n}) \in [0,1]^n$, and let*

$$\mathcal{B}_1 = \{\boldsymbol{\iota}_{\eta_{\boldsymbol{\alpha}(n)}} \times \boldsymbol{b} \,:\, n \geq 1\}, \quad \mathcal{B}_2 = \left\{\boldsymbol{\iota}_{\eta_{\boldsymbol{\alpha}}} \times \boldsymbol{b} \,:\, n \geq 1,\ \boldsymbol{\alpha} \in [0,1]^n,\ \textstyle\sum_{i=0}^{n-1} \alpha_i = 1\right\}.$$

Then $\boldsymbol{\iota} \times \boldsymbol{b} \sim \mathcal{B}_1 \sim \mathcal{B}_2$.

PROOF. $\mathcal{B}_1 \prec \mathcal{B}_2$ is trivial. $\mathcal{B}_2 \prec \boldsymbol{\iota} \times \boldsymbol{b}$ is clear (in fact, $\boldsymbol{d} \sqsubseteq \boldsymbol{\iota} \times \boldsymbol{b}$ for every $\boldsymbol{d} \in \mathcal{B}_2$). It remains to show that $\boldsymbol{\iota} \times \boldsymbol{b} \prec \mathcal{B}_1$. Let (Z, η) be the underlying non-atomic probability space of $\boldsymbol{\iota}$ and let $\lambda = \eta \times \mu$. Fix a partition $\mathcal{P}$ of $Z \times X$, $F \subseteq \Gamma$ finite and $\epsilon > 0$. We may assume without loss of generality that $\mathcal{P}$ is of the form $\mathcal{P} = \{A_i \times B_j \,:\, 0 \leq i < n,\ 0 \leq j < m\}$ where $\{A_i\}_{i=0}^{n-1}$ is a partition of Z, $\{B_j\}_{j=0}^{m-1}$ is a partition of X, and all the sets $A_0, \ldots, A_{n-1}$ have equal measure. Let $C_{i,j} = \{(i, x) \in n \times X \,:\, x \in B_j\}$. Then, letting $\boldsymbol{d} = \boldsymbol{\iota}_{\eta_{\boldsymbol{\alpha}(n)}} \times \boldsymbol{b}$, for all $\gamma \in F$ and $i, i' \leq n$, $j, j' \leq m$, if $i \neq i'$ we have

$\gamma^d C_{i,j} \cap C_{i',j'} = \varnothing = \gamma^{\iota\times b}(A_i \times B_j) \cap (A_{i'} \cap B_{j'})$, while if $i = i'$ we have

$$\begin{aligned}(\eta_{\boldsymbol{\alpha}(n)} \times \mu)(\gamma^d C_{i,j} \cap C_{i,j'}) &= \tfrac{1}{n}\mu(\gamma^b B_j \cap B_{j'}) \\ &= \eta(A_i)\mu(\gamma^b B_j \cap B_{j'}) = \lambda(\gamma^{\iota\times b}(A_i \times B_j) \cap (A_i \times B_{j'})),\end{aligned}$$

showing that $\boldsymbol{\iota} \times \boldsymbol{b} \prec \mathcal{B}_1$. □

PROOF OF THEOREM 1.1. We apply 3.5 and 3.7, then 3.6 to obtain

$$E(\boldsymbol{\iota} \times \boldsymbol{b}, K) \subseteq \overline{\bigcup_{n\geq 1} E(\boldsymbol{\iota}_{\eta_{\boldsymbol{\alpha}(n)}} \times \boldsymbol{b}, K)} \subseteq \overline{\mathrm{co}} E(\boldsymbol{b}, K) \subseteq \overline{E(\boldsymbol{\iota} \times \boldsymbol{b}, K)}$$

and so $\overline{E(\boldsymbol{\iota} \times \boldsymbol{b}, K)} = \overline{\mathrm{co}} E(\boldsymbol{b}, K)$. □

The proof of Theorem 1.2 now proceeds in analogy with the proof of the corresponding fact for unitary representations (see [**BHV08**, F.1.4]).

PROOF OF THEOREM 1.2. Suppose that $\boldsymbol{a}$ is ergodic and $\boldsymbol{a} \prec \boldsymbol{\iota} \times \boldsymbol{b}$. We want to show that $\boldsymbol{a} \prec \boldsymbol{b}$, or equivalently $E(\boldsymbol{a}, K) \subseteq \overline{E, (\boldsymbol{b}, K)}$ for every compact Polish K. By hypothesis we have that $E(\boldsymbol{a}, K) \subseteq \overline{E(\boldsymbol{\iota} \times \boldsymbol{b}, K)}$, so by Theorem 1.1, $E(\boldsymbol{a}, K) \subseteq \overline{\mathrm{co}} E(\boldsymbol{b}, K)$. Since every element of $E(\boldsymbol{a}, K)$ is ergodic, $E(\boldsymbol{a}, K)$ is contained in the extreme points of $M_s(K^\Gamma)$, and so *a fortiori* $E(\boldsymbol{a}, K)$ is contained in the extreme points of $\overline{\mathrm{co}} E(\boldsymbol{b}, K)$. Since in a locally convex space the extreme points of a given compact convex set are contained in every closed set generating that convex set (see, e.g., [**Phe01**, Proposition 1.5]), it follows that $E(\boldsymbol{a}, K) \subseteq \overline{E(\boldsymbol{b}, K)}$ as was to be shown. □

3.3. Ergodic decomposition and weak containment. We begin with the following observation about factors.

PROPOSITION 3.8. *Let* $\mathbf{d}$ *be a measure preserving action of* Γ *on* (Y, ν) *and suppose* $\pi : (Y, \nu) \to (Z, \eta)$ *is a factor map from* $\mathbf{d}$ *onto an identity action* $\Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$. *Let* $\nu = \int_z \nu_z \, d\eta$ *be the disintegration of* ν *with respect to* π *and let* $\mathbf{d}_z = \Gamma \curvearrowright^{d} (Y, \nu_z)$. *Suppose that* $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ *is an ergodic factor of* $\mathbf{d}$ *via the map* $\varphi : (Y, \nu) \to (X, \mu)$. *Then for* η*-almost every* $z \in Z$, $\boldsymbol{a}$ *is a factor of* $\mathbf{d}_z$ *via the map* φ.

PROOF. The map $\pi \times \varphi : (Y, \nu) \to (Z \times X, (\pi \times \varphi)_* \nu)$, $y \mapsto (\pi(y), \varphi(y))$, factors $\boldsymbol{d}$ onto a joining $\boldsymbol{b}$ of the identity action $\boldsymbol{\iota}_\eta$ and the ergodic action $\boldsymbol{a}$. Since ergodic and identity actions are disjoint ([**Gla03**, 6.24]) we have that $(\pi \times \varphi)_* \nu = \eta \times \mu$ and $\boldsymbol{b} = \boldsymbol{\iota}_\eta \times \boldsymbol{a}$. The measure $(\pi \times \varphi)_* \nu_z$ lives on $\{z\} \times X$ almost surely, and $\eta \times \mu = (\pi \times \varphi)_* \nu = \int_Z (\pi \times \varphi)_* \nu_z \, d\eta$, so by uniqueness of disintegration $(\pi \times \varphi)_* \nu_z = \delta_z \times \mu$ almost surely. Since $\mathrm{proj}_X \circ (\pi \times \varphi) = \varphi$ we have that $\varphi_* \nu_z = (\mathrm{proj}_X)_* (\delta_z \times \mu) = \mu$ almost surely. $\square$

COROLLARY 3.9. *If $\boldsymbol{a}$ is ergodic and φ factors $\mathbf{d}$ onto $\boldsymbol{a}$ then φ factors almost every ergodic component of $\mathbf{d}$ onto $\boldsymbol{a}$.*

Using ultraproducts of measure preserving actions (see Appendix 8) we can prove an analogous result for weak containment which generalizes Theorem 1.2. For the remainder of this section we fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and we also fix a compact Polish space K homeomorphic to $2^{\mathbb{N}}$. Let $\boldsymbol{a}_n = \Gamma \curvearrowright^{a_n} (Y_n, \nu)$, $n \in \mathbb{N}$, be a sequence of measure preserving actions of Γ and let $\boldsymbol{a}_{\mathcal{U}} = \Gamma \curvearrowright^{a_{\mathcal{U}}} (Y_{\mathcal{U}}, \mu_{\mathcal{U}})$ be the ultraproduct of the sequence $\boldsymbol{a}_n$ with respect to the nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $\phi_n : Y_n \to K$ be a sequence of Borel functions and let $\Phi_n = \Phi^{\phi_n, a_n} : Y_n \to K^\Gamma$. We let ϕ denote the ultralimit function determined by the sequence ϕ_n, i.e., $\phi : Y_{\mathcal{U}} \to K$ is the function given by

$$\phi([y_n]) = \lim_{n \to \mathcal{U}} \phi_n(y_n)$$

for $[y_n] \in Y_{\mathcal{U}}$. The function ϕ is $\boldsymbol{B}_{\mathcal{U}}$-measurable since $\phi^{-1}(V) = [\phi_n^{-1}(V)]$ whenever $V \subseteq K$ is open.

PROPOSITION 3.10. *Let $\Phi = \Phi^{\phi, a_{\mathcal{U}}}$. Then*

(1) $\Phi([y_n]) = \lim_{n \to \mathcal{U}} \Phi_n(y_n)$ *for all* $[y_n] \in Y_{\mathcal{U}}$;

(2) $\Phi_* \nu_{\mathcal{U}} = \lim_{n \to \mathcal{U}} (\Phi_n)_* \nu_n$;

(3) *For every $\boldsymbol{B}_{\mathcal{U}}$-measurable function $\psi : Y_{\mathcal{U}} \to K$ there exists a sequence $\varphi_n : Y_n \to K$ of Borel functions such that $\psi([y_n]) = \lim_{n \to \mathcal{U}} \varphi_n(y_n)$ for $\nu_{\mathcal{U}}$-almost every $[y_n] \in Y_{\mathcal{U}}$.*

PROOF. (1): For each $[y_n] \in Y_{\mathcal{U}}$ and $\gamma \in \Gamma$ we have $\Phi([y_n])(\gamma^{-1}) = \phi(\gamma^{a_{\mathcal{U}}}[y_n]) = \phi([\gamma^{a_n} y_n]) = \lim_{n\to\mathcal{U}} \phi(\gamma^{a_n} y_n) = \lim_{n\to\mathcal{U}} \Phi_n(y_n)(\gamma) = (\lim_{n\to\mathcal{U}} \Phi_n(y_n))(\gamma)$, the last equality following by continuity of the evaluation map $f \mapsto f(\gamma)$ on K^Γ.

(2): Let $\lambda = \lim_{n\to\mathcal{U}}(\Phi_n)_*\nu_n$. Then λ is the unique element of $M_s(K^\Gamma)$ such that $\lambda(C) = \lim_{n\to\mathcal{U}}((\Phi_n)_*\nu_n(C))$ for all clopen $C \subseteq K^\Gamma$. Part (1) implies that $\Phi^{-1}(C) = [\Phi_n^{-1}(C)]$ whenever $C \subseteq K^\Gamma$ is clopen, and so $\Phi_*\nu_{\mathcal{U}}(C) = \lim_{n\to\mathcal{U}} \nu_n(\Phi_n^{-1}(C)) = \lim_{n\to\mathcal{U}}((\Phi_n)_*\nu_n(C))$.

(3): We may assume $K = 2^{\mathbb{N}}$. For $m \in \mathbb{N}$ define $\psi_m : Y_{\mathcal{U}} \to K$ by $\psi_m([y_n]) = \psi([y_n])(m)$. For $i \in \{0,1\}$ let $A^{m,i} = \psi_m^{-1}(\{i\}) \in \boldsymbol{B}_{\mathcal{U}}$ and fix $[A_n^{m,i}] \in \boldsymbol{A}_{\mathcal{U}}$ such that $\nu_{\mathcal{U}}(A^{m,i}\Delta[A_n^{m,i}]) = 0$. For each $m,n \in \mathbb{N}$ let $B_n^{m,0} = A_n^{m,0} \setminus A_n^{m,1}$ and let $B_n^{m,1} = Y_n \setminus B_n^{m,0}$ so that $\{B_n^{m,0}, B_n^{m,1}\}$ is a Borel partition of Y_n. Then for each $m \in \mathbb{N}$ we have $\nu_{\mathcal{U}}(A^{m,0}\Delta[B_n^{m,0}]) = 0 = \nu_{\mathcal{U}}(A^{m,1}\Delta[B_n^{m,1}])$. Define $\varphi_n : Y_n \to K$ by taking $\varphi_n(y)(m) = i$ if and only if $y \in B_n^{m,i}$. Let $\varphi : Y_{\mathcal{U}} \to K$ be the ultralimit function $\varphi([y_n]) = \lim_{n\to\mathcal{U}} \varphi_n(y_n)$. Then for $i \in \{0,1\}$ we have

$$\varphi([y_n])(m) = i \;\Leftrightarrow\; \lim_{n\to\mathcal{U}}(\varphi_n(y_n)(m)) = i \;\Leftrightarrow\; \{n : y_n \in B_n^{m,i}\} \in \mathcal{U} \;\Leftrightarrow\; [y_n] \in [B_n^{m,i}],$$

and so φ is equal to ψ off the null set $\bigcup_{m\in\mathbb{N}, i\in\{0,1\}} A^{m,i}\Delta[B_n^{m,i}]$. $\square$

THEOREM 3.11. *Let $\boldsymbol{d}$ be a measure preserving action of Γ on (Y,ν) and suppose $\pi : (Y,\nu) \to (Z,\eta)$ is a factor map from $\boldsymbol{d}$ onto an identity action $\Gamma \curvearrowright^{\iota_\eta} (Z,\eta)$. Let $\nu = \int_z \nu_z \, d\eta$ be the disintegration of ν with respect to π and let $\boldsymbol{d}_z = \Gamma \curvearrowright^{d} (Y,\nu_z)$. Suppose that $\boldsymbol{a} = \Gamma \curvearrowright^{a} (X,\mu)$ is ergodic and is weakly contained in $\boldsymbol{d}$. Then $\boldsymbol{a}$ is weakly contained in $\boldsymbol{d}_z$ for almost all $z \in Z$.*

PROOF. Taking $K = 2^{\mathbb{N}}$ it suffices to show for each $\lambda \in E(\boldsymbol{a},K)$ that $\eta(\{z : \lambda \in \overline{E(\boldsymbol{d}_z,K)}\}) = 1$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ and let $\boldsymbol{d}_{\mathcal{U}} = \Gamma \curvearrowright^{d_{\mathcal{U}}} (Y_{\mathcal{U}},\nu_{\mathcal{U}})$ and $\boldsymbol{\iota}_{\mathcal{U}} = \Gamma \curvearrowright^{\iota_{\mathcal{U}}} (Z_{\mathcal{U}},\eta_{\mathcal{U}})$ be the ultrapowers of $\boldsymbol{d}$ and $\boldsymbol{\iota}_\eta$, respectively. The map $\pi_{\mathcal{U}} : Y_{\mathcal{U}} \to Z_{\mathcal{U}}$ defined by $\pi_{\mathcal{U}}([y_n]) = [\pi(y_n)]$ factors $\boldsymbol{d}_{\mathcal{U}}$ onto $\boldsymbol{\iota}_{\mathcal{U}}$.

Given any $\lambda \in E(\boldsymbol{a}, K)$, since $\boldsymbol{a} \prec \boldsymbol{d}$ there exists $\phi_n : Y \to K$ such that $(\Phi^{\phi_n, d})_* \nu \to \lambda$. Let $\phi : Y_{\mathcal{U}} \to K$ be the ultralimit of the functions ϕ_n, let $\Phi_n = \Phi^{\phi_n, d}$ and let $\Phi = \Phi^{\phi, d_{\mathcal{U}}} : Y_{\mathcal{U}} \to K^{\Gamma}$. By Proposition 3.10.(2), Φ factors $\boldsymbol{d}_{\mathcal{U}}$ onto $\Gamma \curvearrowright^s (K^{\Gamma}, \lambda)$.

Let $\rho = \sigma_* \nu_{\mathcal{U}}$, where $\sigma = \pi_{\mathcal{U}} \times \Phi : Y_{\mathcal{U}} \to Z_{\mathcal{U}} \times K^{\Gamma}$ is the map $\sigma([y_n]) = (\pi_{\mathcal{U}}([y_n]), \Phi([y_n]))$. Then $\rho = \eta_{\mathcal{U}} \times \lambda$ since each standard factor of $\boldsymbol{\iota}_{\mathcal{U}}$ is an identity action so is disjoint from $\boldsymbol{a}$. Let $\nu_{[z_n]} = \prod_n \nu_{z_n} / \mathcal{U}$, so that $\nu_{[z_n]}$ is a probability measure on $\boldsymbol{B}_{\mathcal{U}}(Y_{\mathcal{U}})$ for all $[z_n] \in Z_{\mathcal{U}}$.

CLAIM 1. *$\lim_{n \to \mathcal{U}} (\Phi_n)_* \nu_{z_n} = \lambda$ for $\eta_{\mathcal{U}}$-almost every $[z_n] \in Z_{\mathcal{U}}$.*

PROOF OF CLAIM. By Proposition 8.1, $\nu_{\mathcal{U}}(A) = \int_{[z_n]} \nu_{[z_n]}(A) \, d\eta_{\mathcal{U}}$ for all $A \in \boldsymbol{B}_{\mathcal{U}}(Y_{\mathcal{U}})$. As $\sigma_* \nu_{[z_n]}$ lives on $\{[z_n]\} \times K^{\Gamma}$ it follows for $D \in \boldsymbol{B}_{\mathcal{U}}(Z_{\mathcal{U}})$ and $C \subseteq K^{\Gamma}$ clopen that

$$\begin{aligned} \int_{[z_n] \in D} \lambda(C) \, d\eta_{\mathcal{U}} = \eta_{\mathcal{U}}(D) \lambda(C) = \rho(D \times C) = \int_{[z_n]} \sigma_* \nu_{[z_n]}(D \times C) \, d\eta_{\mathcal{U}} \\ = \int_{[z_n] \in D} \sigma_* \nu_{[z_n]}(Z_{\mathcal{U}} \times C) \, d\eta_{\mathcal{U}} = \int_{[z_n] \in D} \Phi_* \nu_{[z_n]}(C) \, d\eta_{\mathcal{U}}. \end{aligned}$$

Thus for each clopen $C \subseteq K^{\Gamma}$, $\Phi_* \nu_{[z_n]}(C) = \lambda(C)$ for $\eta_{\mathcal{U}}$ almost every $[z_n] \in Z_{\mathcal{U}}$. It follows that $\Phi_* \nu_{[z_n]} = \lambda$ for $\eta_{\mathcal{U}}$ almost every $[z_n] \in Z_{\mathcal{U}}$. By Proposition 3.10.(2), $\lim_{n \to \mathcal{U}} (\Phi_n)_* \nu_{z_n} = \lambda$ for $\eta_{\mathcal{U}}$ almost every $[z_n] \in Z_{\mathcal{U}}$. □[Claim]

If now V is any open neighborhood of λ in $M_s(K^{\Gamma})$ then let $B = \{z \in Z : E(\boldsymbol{d}_z, K) \cap V = \varnothing\}$. If $\eta(B) > 0$ then let $B_n = B$ for all n so that $[B_n] \in \boldsymbol{A}_{\mathcal{U}}(Z_{\mathcal{U}})$ and $\eta_{\mathcal{U}}([B_n]) > 0$. Thus, for some $[z_n] \in [B_n]$ we have $\lim_{n \to \mathcal{U}} (\Phi_n)_* \nu_{z_n} = \lambda$ and so $(\Phi_n)_* \nu_{z_n} \in E(\boldsymbol{d}_{z_n}, K) \cap V$ for some $n \in \mathbb{N}$. Since $z_n \in B_n = B$ this is a contradiction. Thus, $\eta(B) = 0$. It follows that $\lambda \in \overline{E(\boldsymbol{d}_z, K)}$ almost surely. □

THEOREM 3.12. *Let $\varphi : \Gamma \curvearrowright^b (X, \mu) \to \Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$ and $\psi : \Gamma \curvearrowright^d (Y, \nu) \to \Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$ be factor maps from $\boldsymbol{b}$ and $\boldsymbol{d}$ respectively onto $\boldsymbol{\iota}_\eta$. Let $\mu = \int_z \mu_z \, d\eta$ and $\nu = \int_z \nu_z \, d\eta$ be the disintegrations of μ and ν via φ and ψ respectively, and for each $z \in Z$ let $\boldsymbol{b}_z = \Gamma \curvearrowright^b (X, \mu_z)$ and let $\boldsymbol{d}_z = \Gamma \curvearrowright^d (Y, \nu_z)$. Then*

(1) *If $\boldsymbol{b}_z \prec \boldsymbol{d}_z$ for all $z \in Z$ then $\boldsymbol{b} \prec \boldsymbol{d}$.*

(2) *If $\boldsymbol{b} \prec \boldsymbol{d}_z$ for all $z \in Z$ the $\boldsymbol{\iota}_\eta \times \boldsymbol{b} \prec \boldsymbol{d}$ and if $\boldsymbol{b}_z \prec \boldsymbol{d}$ for all $z \in Z$ then $\boldsymbol{b} \prec \boldsymbol{\iota}_\eta \times \boldsymbol{d}$.*

(3) *If $\boldsymbol{b}_z \sim \boldsymbol{d}_z$ for all $z \in Z$ then $\boldsymbol{b} \sim \boldsymbol{d}$ and if $\boldsymbol{b} \sim \boldsymbol{d}_z$ for all $z \in Z$ then $\boldsymbol{\iota}_\eta \times \boldsymbol{b} \sim \boldsymbol{d}$.*

We also have the following version for stable weak containment (see Appendix 9):

(4) *If $\boldsymbol{b}_z \prec_s \boldsymbol{d}_z$ for all $z \in Z$ then $\boldsymbol{b} \prec_s \boldsymbol{d}$.*

(5) *If $\boldsymbol{b}_z \prec_s \boldsymbol{d}$ for all $z \in Z$ then $\boldsymbol{b} \prec_s \boldsymbol{d}$ and if $\boldsymbol{b}_z \prec_s \boldsymbol{d}$ for all $z \in Z$ then $\boldsymbol{b} \prec_s \boldsymbol{d}$.*

(6) *If $\boldsymbol{b}_z \sim_s \boldsymbol{d}_z$ for all $z \in Z$ then $\boldsymbol{b} \sim_s \boldsymbol{d}$ and if $\boldsymbol{b} \sim_s \boldsymbol{d}_z$ for all $z \in Z$ then $\boldsymbol{b} \sim_s \boldsymbol{d}$.*

PROOF. (1): Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable algebra of subsets of Y generating the Borel σ-algebra of Y. Fix a partition $A_0, \ldots, A_{k-1}$ of Borel subsets of X along with $F \subseteq \Gamma$ finite and $\epsilon > 0$. For each z there exists a k-tuple $(n_0, \ldots, n_{k-1}) \in \mathbb{N}^k$ such that the sets $B_{n_0}, \ldots, B_{n_{k-1}} \subseteq Y$ witness that $\boldsymbol{b}_z \prec \boldsymbol{d}_z$ with respect to the parameters $A_0, \ldots, A_{k-1}$, F, and ϵ. We let $n(z) = (n_0(z), \ldots, n_{k-1}(z))$ be the lexicographically least k-tuple that satisfies this for z. For each $j < k$ the set

$$D_j = \{y \in Y \,:\, \exists z \in Z(\psi(y) = z \text{ and } y \in B_{n_j(z)})\} = \bigsqcup_z (B_{n_j(z)} \cap \psi^{-1}(z))$$

is analytic and so is measurable. For all $z \in Z$, $\gamma \in \Gamma$, and $j < k$ we then have that $\gamma^d D_j \cap \psi^{-1}(z) = \gamma^{d_z} B_{n_j(z)} \cap \psi^{-1}(z)$ and it follows that $\nu_z(\gamma^d D_j \cap D_{j'}) = \nu_z(\gamma^{d_z} B_{n_j(z)} \cap B_{n_{j'}(z)})$, since ν_z concentrates on $\psi^{-1}(z)$. For $\gamma \in F$ and $i, j < k$ we then have

$$\begin{aligned}|\nu(\gamma^d D_i \cap D_j) - \mu(\gamma^b A_i \cap A_j)| &= \Big| \int_{z\in Z} \nu_z(\gamma^d D_i \cap D_j)\, d\eta(z) - \int_{z\in Z} \mu_z(\gamma^b A_i \cap A_j)\, d\eta(z) \Big| \\ &\leq \int_{z\in Z} |\nu_z(\gamma^{d_z} B_{n_i(z)} \cap B_{n_j(z)}) - \mu_z(\gamma^{b_z} A_i \cap A_j)|\, d\eta(z) \leq \eta(Z)\epsilon < \epsilon\end{aligned}$$

which finishes the proof of (1).

Statements (2) through (6) now follow from (1). $\square$

QUESTION 3.13. Is every measure preserving action $\boldsymbol{d}$ of Γ stably weakly equivalent to an action with countable ergodic decomposition?

A positive answer to Question 3.13 would be an ergodic theoretic analogue of the fact that every unitary representation of Γ on a separable Hilbert space is weakly equivalent to a countable direct sum of irreducible representations ([**Dix77**], this also follows from [**BHV08**, F.2.7]). We also mention the following related problem.

PROBLEM 3.14. *Describe the set* $ex(\overline{co}E(\boldsymbol{a},2^{\mathbb{N}}))$ *of extreme points of* $\overline{co}E(\boldsymbol{a},2^{\mathbb{N}})$ *for* $\boldsymbol{a}\in A(\Gamma,X,\mu)$.

4. Consequences of Theorem 1.2 and applications to MD and EMD

4.1. Free, non-ergodic weak equivalence classes. We can now prove Theorem 1.3.

PROOF OF THEOREM 1.3. If $\boldsymbol{a}$ is any ergodic action of Γ and $\boldsymbol{a}\prec\boldsymbol{\iota}\times\boldsymbol{b}$ then by Theorem 1.2 $\boldsymbol{a}\prec\boldsymbol{b}$, and so $\boldsymbol{a}$ is strongly ergodic. It follows that we cannot also have $\boldsymbol{\iota}\times\boldsymbol{b}\prec\boldsymbol{a}$, otherwise $\boldsymbol{a}$ would not be strongly ergodic. $\square$

REMARK 4.1. Foreman and Weiss [**FW04**, Claim 18] show that for any free measure preserving action $\boldsymbol{a}=\Gamma\curvearrowright^{a}(X,\mu)$ of an infinite amenable group $\boldsymbol{b}\prec\boldsymbol{a}$ for every $\boldsymbol{b}\in A(\Gamma,X,\mu)$. We note that a quick alternative proof of this follows from [**BTD11**, Theorem 1.2], which says that if Δ is a normal subgroup of a countably infinite group Γ and Γ/Δ is amenable, then $\boldsymbol{b}\prec\mathrm{CInd}_{\Delta}^{\Gamma}((\boldsymbol{\iota}\times\boldsymbol{b})|\Delta)$ for every $\boldsymbol{b}\in A(\Gamma,X,\mu)$. Taking Γ to be an infinite amenable group and $\Delta=\langle e\rangle$ the trivial group, the restriction $(\boldsymbol{\iota}\times\boldsymbol{b})|\langle e\rangle$ is trivial, so $\mathrm{CInd}_{\langle e\rangle}^{\Gamma}((\boldsymbol{\iota}\times\boldsymbol{b})|\langle e\rangle)$ is the Bernoulli shift action $\boldsymbol{s}_\Gamma$ of Γ. Thus, $\boldsymbol{b}\prec\boldsymbol{s}_\Gamma$. By [**AW11**, Theorem 1] (or alternatively, Corollary 1.6 below), since $\boldsymbol{a}$ is free, we have $\boldsymbol{s}_\Gamma\prec\boldsymbol{a}$ and so $\boldsymbol{b}\prec\boldsymbol{a}$.

Combining this with Theorem 1.3 gives a new characterization of (non-)amenability for a countable group Γ.

COROLLARY 4.2. *A countably infinite group* Γ *is non-amenable if and only if there exists a free measure preserving action of* Γ *that is not weakly equivalent to any ergodic action.*

REMARK 4.3. It is noted in [**CK13**, 4.(**C**)] that if Γ is a non-amenable group, and if $S\subseteq\Gamma$ is a set of generators for Γ such that the Cayley graph $\mathrm{Cay}(\Gamma,S)$ is bipartite, then there are continuum-many weak equivalence classes of free measure preserving actions of

Γ. Their method of using convex combinations of actions can be used to show that this holds for *all* non-amenable Γ, and in fact the proof shows that there exists a collection $\{\boldsymbol{a}_\alpha : 0 < \alpha \leq \frac{1}{2}\}$ with $\boldsymbol{a}_\alpha$ and $\boldsymbol{a}_\beta$ weakly incomparable when $\alpha \neq \beta$. Indeed, if $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ is any free strongly ergodic action of Γ (which exists when Γ is non-amenable), then for any $0 < \alpha < \beta \leq \frac{1}{2}$ the actions $\boldsymbol{a}_\alpha = \alpha\boldsymbol{a} + (1-\alpha)\boldsymbol{a}$ and $\boldsymbol{a}_\beta = \beta\boldsymbol{a} + (1-\beta)\boldsymbol{a}$ are weakly incomparable. To see this note that any action weakly containing $\boldsymbol{a}_\alpha$ has a sequence of asymptotically invariant sets with measures converging to α. Since $\boldsymbol{a}$ is strongly ergodic it is clear that no such sequence exists for $\boldsymbol{a}_\beta$, and so $\boldsymbol{a}_\alpha \not\prec \boldsymbol{a}_\beta$. Similarly, $\boldsymbol{a}_\beta \not\prec \boldsymbol{a}_\alpha$.

It is open whether every non-amenable group has continuum-many weak equivalence classes of free *ergodic* measure preserving actions. It is in fact unknown whether there exists a non-amenable group with just one free ergodic action up to weak equivalence (though it is shown in the fourth remark after 13.2 of [**Kec10**] that any such group must, among other things, have property (T) and cannot contain a non-abelian free group). Abért-Elek [**AE10**] show that Γ has continuum-many pairwise weakly incomparable (hence inequivalent) free ergodic actions when Γ is a finitely generated free group or a linear group with property (T). Their result also holds for stable weak equivalence in view of the following consequence of Theorem 1.2.

COROLLARY 4.4. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be ergodic measure preserving actions of Γ and let (Z,η) be a standard probability space. Then $\boldsymbol{a} \sim \boldsymbol{b}$ if and only if $\boldsymbol{\iota}_\eta \times \boldsymbol{a} \sim \boldsymbol{\iota}_\eta \times \boldsymbol{b}$. In particular $\boldsymbol{a} \sim \boldsymbol{b}$ if and only if $\boldsymbol{a} \sim_s \boldsymbol{b}$.*

PROOF. If $\boldsymbol{a} \sim \boldsymbol{b}$ then $\boldsymbol{\iota}_\eta \times \boldsymbol{a} \sim \boldsymbol{\iota}_\eta \times \boldsymbol{b}$ by continuity of the product operation. Conversely, if $\boldsymbol{\iota}_\eta \times \boldsymbol{a} \sim \boldsymbol{\iota}_\eta \times \boldsymbol{b}$ then $\boldsymbol{a} \prec \boldsymbol{\iota}_\eta \times \boldsymbol{a} \prec \boldsymbol{\iota}_\eta \times \boldsymbol{b}$ so that $\boldsymbol{a} \prec \boldsymbol{b}$ by Theorem 1.2. Likewise, $\boldsymbol{b} \prec \boldsymbol{a}$, so $\boldsymbol{a} \sim \boldsymbol{b}$. $\square$

I also do not know whether every non-amenable group has continuum-many stable weak equivalence classes of free measure preserving actions, or whether there exists a non-amenable group all of whose free measure preserving actions are stably weakly equivalent.

4.2. The properties MD and EMD.

DEFINITION 4.5. Let $\mathcal{B}$ be a class of measure preserving actions of a countable group Γ. If $\boldsymbol{a} \in \mathcal{B}$ then $\boldsymbol{a}$ is called *universal for* $\mathcal{B}$ if $\boldsymbol{b} \prec \boldsymbol{a}$ for every $\boldsymbol{b} \in \mathcal{B}$. When $\boldsymbol{a}$ is universal for the class of *all* measure preserving actions of Γ then $\boldsymbol{a}$ is simply called *universal*.

DEFINITION 4.6 ([**Kec12**]). Let Γ be a countably infinite group. Then Γ is said to have property EMD if the measure preserving action $\boldsymbol{p}_\Gamma$ of Γ on its profinite completion is universal. Γ is said to have property EMD* if $\boldsymbol{p}_\Gamma$ is universal for the class of all ergodic measure preserving actions of Γ. Γ is said to have property MD if $\boldsymbol{\iota} \times \boldsymbol{p}_\Gamma$ is universal.

If Γ has property EMD, EMD*, or MD, then $\boldsymbol{p}_\Gamma$ must be free (this follows, e.g., from the 5.3 below) and so Γ must be residually finite. It is also clear that EMD implies both EMD* and MD. We now show that EMD* and MD are equivalent.

PROOF OF THEOREM 1.4. The implication EMD* $\Rightarrow$ MD is shown in [**Kec12**], but also follows from Theorem 3.12 above. For the converse, suppose Γ has MD so that $\boldsymbol{\iota} \times \boldsymbol{p}_\Gamma$ is universal and let $\boldsymbol{a}$ be an ergodic action of Γ. Then $\boldsymbol{a} \prec \boldsymbol{\iota} \times \boldsymbol{p}_\Gamma$, so since $\boldsymbol{a}$ is ergodic, Theorem 1.2 implies $\boldsymbol{a} \prec \boldsymbol{p}_\Gamma$. Thus $\boldsymbol{p}_\Gamma$ is universal for ergodic actions of Γ, and so Γ has EMD*. $\square$

COROLLARY 4.7. *EMD and MD are equivalent for groups without property* (T).

PROOF. Suppose Γ has MD and does not have (T). Then $\boldsymbol{\iota} \times \boldsymbol{p}_\Gamma$ is universal and by Theorem 1.4, $\boldsymbol{p}_\Gamma$ is universal for ergodic measure preserving actions. Since Γ does not have property (T) there exists an ergodic $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ with $\boldsymbol{\iota} \prec \boldsymbol{a}$, and so $\boldsymbol{\iota} \prec \boldsymbol{a} \prec \boldsymbol{p}_\Gamma$. Since $\boldsymbol{p}_\Gamma$ is ergodic with $\boldsymbol{\iota} \prec \boldsymbol{p}_\Gamma$ it follows that $\boldsymbol{\iota} \times \boldsymbol{p}_\Gamma \prec \boldsymbol{p}_\Gamma$ (see [**AW11**, Theorem 3]) and so $\boldsymbol{p}_\Gamma$ is universal. $\square$

In what follows, if $\varphi : \Gamma \to \Delta$ is group homomorphism then for each $\boldsymbol{a} \in A(\Delta, X, \mu)$ we let $\boldsymbol{a}^\varphi \in A(\Gamma, X, \mu)$ denote the action that is the composition of $\boldsymbol{a}$ with φ, i.e., $\gamma^{a^\varphi} = \varphi(\gamma)^a$. Also, we note that for any two countable groups Γ_1, Γ_2, there is a natural equivariant homeomorphism from the diagonal action $\mathrm{Aut}(X, \mu) \curvearrowright A(\Gamma_1, X, \mu) \times A(\Gamma_2, X, \mu)$ to

$\mathrm{Aut}(X,\mu) \curvearrowright A(\Gamma_1 * \Gamma_2, X, \mu)$. We denote this map by $(\boldsymbol{a}_1, \boldsymbol{a}_2) \mapsto \boldsymbol{a}_1 * \boldsymbol{a}_2$. See [**Kec10**, 10.**(G)**]. We also refer to [**Kec10**, Appendix G] and [**Zim84**] for information about induced actions.

THEOREM 4.8. *Suppose Γ_1 and Γ_2 are nontrivial countable groups and that for each $i \in \{1,2\}$, Γ_i is either finite or has property MD. Then $\Gamma_1 * \Gamma_2$ has property EMD.*

PROOF. Let $(\boldsymbol{a}_1, \boldsymbol{a}_2) \in A(\Gamma_1, X, \mu) \times A(\Gamma_2, X, \mu)$ be given and let $U = U_1 \times U_2$ be an open neighborhood of $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ where U_i is an open neighborhood of $\boldsymbol{a}_i$ for $i = 1, 2$. By hypothesis, for each $i = 1, 2$ there exists a finite group $F_i \neq \{e\}$ along with a homomorphism $\varphi_i : \Gamma_i \to F_i$ and $\boldsymbol{b}_i \in A(F_i, X, \mu)$ such that the corresponding measure preserving action $\boldsymbol{b}_i^{\varphi_i}$ of Γ_i is in U_i. Let $\varphi = \varphi_1 * \varphi_2 : \Gamma_1 * \Gamma_2 \to F_1 * F_2$ and let $\boldsymbol{b} = \boldsymbol{b}_1 * \boldsymbol{b}_2$. Then $\boldsymbol{b}^{\varphi} = \boldsymbol{b}_1^{\varphi_1} * \boldsymbol{b}_2^{\varphi_2} \in U_1 \times U_2$. Let V_1, V_2 be open subsets about $\boldsymbol{b}_1 \in A(F_1, X, \mu)$ and $\boldsymbol{b}_2 \in A(F_2, X, \mu)$, respectively, such that $\{\boldsymbol{a}^{\varphi_i} : \boldsymbol{a} \in V_i\} \subseteq U_i$ for $i = 1, 2$ (this is possible since the map $\boldsymbol{a} \mapsto \boldsymbol{a}^{\varphi_i}$ is continuous). Then $\boldsymbol{b} \in V_1 \times V_2$ and for all $\boldsymbol{d} \in V_1 \times V_2$ we have $\boldsymbol{d}^{\varphi} \in U_1 \times U_2$.

There is a (possibly abelian) free subgroup $\mathbb{F} \leq F = F_1 * F_2$ of finite index (explicitly: $\mathbb{F} = \ker(\psi) = [F_1, F_2]$ where $\psi : F_1 * F_2 \to F_1 \times F_2$ is the natural projection map), and since $\mathbb{F}$ has EMD [**Kec12**, Theorem 1] we have $\boldsymbol{b}|\mathbb{F} \prec \boldsymbol{p}_{\mathbb{F}}$. Letting $\boldsymbol{a}_{F/\mathbb{F}}$ denote the action of F on $F/\mathbb{F}$ with normalized counting measure we now have

$$\boldsymbol{b} \sqsubseteq \boldsymbol{b} \times \boldsymbol{a}_{F/\mathbb{F}} \cong \mathrm{Ind}_{\mathbb{F}}^{F}(\boldsymbol{b}|\Gamma) \prec \mathrm{Ind}_{\mathbb{F}}^{F}(\boldsymbol{p}_{\mathbb{F}}).$$

The action $\boldsymbol{d} = \mathrm{Ind}_{\mathbb{F}}^{F}(\boldsymbol{p}_{\mathbb{F}})$ is a profinite action, and $\boldsymbol{d}$ is ergodic since $\boldsymbol{p}_{\mathbb{F}}$ is ergodic. As $\boldsymbol{b} \prec \boldsymbol{d}$ there exists an isomorphic copy $\boldsymbol{d}_0$ of $\boldsymbol{d}$ in $V_1 \times V_2$. Then $\boldsymbol{d}_0^{\varphi} \in U_1 \times U_2$ and $\boldsymbol{d}_0^{\varphi}$ is ergodic since $\boldsymbol{d}_0$ is ergodic. Thus $U_1 \times U_2$ contains an ergodic profinite action. $\square$

NOTE 4.9. The group $\Gamma_1 * \Gamma_2$ never has property (T) when Γ_1 and Γ_2 are nontrivial, so by Corollary 4.7 it would have been enough to show in the above proof that $\Gamma_1 * \Gamma_2$ has MD, and then EMD would follow.

THEOREM 4.10. *The following are equivalent*

(1) *MD and EMD are equivalent properties for any countably infinite group* Γ.

(2) *EMD passes to subgroups.*

(3) *MD is incompatible with property (T).*

PROOF. (1)⇒(2): Property MD passes to subgroups, so if MD and EMD are equivalent, then EMD passes subgroups. (2)⇒(1): If Γ is a countable group with MD then $\Gamma * \Gamma$ has EMD, so if EMD passes to subgroups then Γ actually has EMD. (1)⇒(3): EMD is incompatible with property (T) since if Γ is an infinite residually finite group with property (T) then $\boldsymbol{p}_\Gamma$ is strongly ergodic so that $\boldsymbol{\iota} \not\prec \boldsymbol{p}_\Gamma$. Thus, if MD and EMD are equivalent then MD is also incompatible with property (T). (3)⇒(1): This follows immediately from Corollary 4.7. □

Note also that Theorem 1.2 gives the following

PROPOSITION 4.11. *MD is incompatible with* $((\tau)$ *and* $\neg(\mathrm{T}))$. *That is, if a group* Γ *has both MD and property* (τ), *then* Γ *actually has property* (T).

PROOF. If Γ has MD then by 4.7, $\boldsymbol{p}_\Gamma$ is universal for ergodic actions, so if Γ does not have (T) then there exists an ergodic $\boldsymbol{a}$ with $\boldsymbol{\iota} \prec \boldsymbol{a}$. This implies $\boldsymbol{\iota} \prec \boldsymbol{p}_\Gamma$ so that Γ does not have property (τ). □

5. Weak equivalence and invariant random subgroups

5.1. Invariant random subgroups. We let $\mathrm{Sub}(\Gamma)$ denote the set of all subgroups of Γ. This is a compact subset of 2^Γ with the product topology, and is invariant under the left conjugation action of Γ, which is continuous, and which we denote by c, i.e., $\gamma^c \cdot H = \gamma H \gamma^{-1}$. *We will always view* Γ *as acting on* $\mathrm{Sub}(\Gamma)$ *by conjugation, though the underlying measure on* $\mathrm{Sub}(\Gamma)$ *will vary.* By an *invariant random subgroup* (IRS) of Γ we mean a conjugation-invariant Borel probability measure θ on $\mathrm{Sub}(\Gamma)$. Invariant random subgroups are studied in [**AGV12**] as a stochastic generalization of normal subgroups. See also [**AE11**], [**Bow10b**] and [**Ver12**]. We let $\mathrm{IRS}(\Gamma)$ denote the space of all invariant random subgroups of Γ. When $\theta \in \mathrm{IRS}(\Gamma)$ we will let $\boldsymbol{\theta}$ denote the measure preserving action $\Gamma \curvearrowright^c (\mathrm{Sub}(\Gamma), \theta)$. For a measure preserving action $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ we let type($\boldsymbol{a}$) denote the *type* of $\boldsymbol{a}$, which is defined to be the measure $(\mathrm{stab}_a)_*\mu$ on Sub(Γ), where $\mathrm{stab}_a : X \to \mathrm{Sub}(\Gamma)$ is the stabilizer map $x \mapsto \mathrm{stab}_a(x) = \Gamma_x = \{\gamma \in \Gamma : \gamma^a x = x\} \in \mathrm{Sub}(\Gamma)$. It is clear that type($\boldsymbol{a}$) is always an IRS of Γ. Types are studied in [**AE11**] in order to examine freeness properties of measure preserving actions.

5.2. The compact space of weak equivalence classes. Abért and Elek ([**AE11**]) define a compact Polish topology on the set of weak equivalence classes of measure preserving actions of Γ. We define this topology below and provide a variation of their proof showing that it is a compact Polish topology.

For this subsection we fix a standard probability space (X, μ) and a compact zero-dimensional Polish space K homeomorphic to Cantor space $2^{\mathbb{N}}$. We let $\mathcal{K} = \mathcal{K}(M_s(K^\Gamma))$ denote the space of all nonempty compact subsets of $M_s(K^\Gamma)$, equipped with the Vietoris topology τ_V which makes $\mathcal{K}$ into a compact Polish space. Since $M_s(K^\Gamma)$ is a compact metric space, convergence in this topology may be described as follows. A sequence $L_n \in \mathcal{K}$, $n \in \mathbb{N}$ converges if and only if the sets

$$\overline{\mathrm{Tlim}}_n L_n = \{\lambda \in M_s(K^\Gamma) \,:\, \exists (\lambda_n)\, [\forall n\, \lambda_n \in L_n, \text{ and } \lambda_n \to \lambda]\}$$

$$\underline{\mathrm{Tlim}}_n L_n = \{\lambda \in M_s(K^\Gamma) \,:\, \exists (\lambda_n)\, [\forall n\, \lambda_n \in L_n, \text{ and for some subsequence } (\lambda_{n_k}),\ \lambda_{n_k} \to \lambda]\}$$

are equal, in which case their common value is the limit of the sequence L_n (see, e.g., [**Kec95**, 4.F]).

Let $\Phi : A(\Gamma, X, \mu) \to \mathcal{K}$ be the map

$$\Phi(\boldsymbol{a}) = \overline{E(\boldsymbol{a}, K)}.$$

By Proposition 3.5, $\Phi(\boldsymbol{a}) = \Phi(\boldsymbol{b})$ if and only if $\boldsymbol{a} \sim \boldsymbol{b}$. We now have

THEOREM 5.1. *The image of Φ in $\mathcal{K}$ is a closed, hence compact subset of $(\mathcal{K}, \tau_V)$.*

PROOF. Let $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots$ be a sequence in $A(\Gamma, X, \mu)$ and suppose that $\Phi(\boldsymbol{a}_n)$ converges in $(\mathcal{K}, \tau_V)$ to the compact set $L \in \mathcal{K}$. We will show that there exists $\boldsymbol{a}_\infty \in$

$A(\Gamma, X, \mu)$ such that $\Phi(\boldsymbol{a}_\infty) = L$. Since $E(\boldsymbol{a}_n, K)$ is dense in $\Phi(\boldsymbol{a}_n)$ we may write L as

$$\begin{aligned} L &= \{\lambda \in M_s(K^\Gamma) \,:\, \exists(\lambda_n)\,[\forall n\, \lambda_n \in E(\boldsymbol{a}_n, K), \text{ and } \lambda_n \to \lambda]\} \\ &= \{\lambda \in M_s(K^\Gamma) \,:\, \exists(\lambda_n)\,[\forall n\, \lambda_n \in E(\boldsymbol{a}_n, K), \text{ and for some subsequence } (\lambda_{n_k}),\ \lambda_{n_k} \to \lambda]\}. \end{aligned}$$

Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, let $(X_\mathcal{U}, \mu_\mathcal{U})$ be the ultrapower of the measure space (X, μ), and let $\boldsymbol{a}_\mathcal{U} = \Gamma \curvearrowright^{a_\mathcal{U}} (X_\mathcal{U}, \mu_\mathcal{U})$ denote the ultraproduct $\prod_n \boldsymbol{a}_n/\mathcal{U}$ of the sequence $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$.

Claim 2. $L = E(\boldsymbol{a}_\mathcal{U}, K)$.

Proof of Claim. Let $\lambda \in L$ and let $\lambda_n \in E(\boldsymbol{a}_n, K)$, $n \in \mathbb{N}$, with $\lambda_n \to \lambda$. For each n there exists $\phi_n : X \to K$ such that $\lambda_n = (\Phi^{\phi_n, a_n})_*\mu$. Let $\phi : X_\mathcal{U} \to K$ be the ultralimit of the functions ϕ_n. By Proposition 3.10.(2) $(\Phi^{\phi, a_\mathcal{U}})_*\mu_\mathcal{U} = \lim_{n\to\mathcal{U}}(\Phi^{\phi_n, a_n})_*\mu = \lim_{n\to\mathcal{U}} \lambda_n = \lambda$. This shows $\lambda \in E(\boldsymbol{a}_\mathcal{U}, K)$, and thus $L \subseteq E(\boldsymbol{a}_\mathcal{U}, K)$.

Conversely, let $\lambda \in E(\boldsymbol{a}_\mathcal{U}, K)$, say $\lambda = (\Phi^{\psi, a_\mathcal{U}})_*\mu_\mathcal{U}$ for some $\boldsymbol{B}_\mathcal{U}$-measurable $\psi : X_\mathcal{U} \to K$. By Proposition 3.10.(3) we may find a sequence $\phi_n : X \to K$, $n \in \mathbb{N}$, of Borel functions such that, letting ϕ denote the ultralimit of the ϕ_n, $\mu_\mathcal{U}$-almost everywhere $\psi([x_n]) = \phi([x_n])$. Let $\Phi_n = \Phi^{\phi_n, a_n}$, let $\Phi = \Phi^{\phi, a_\mathcal{U}}$, and let $\lambda_n = (\Phi_n)_*\mu \in E(\boldsymbol{a}_n, K)$. Then $\Phi^{\psi, a_\mathcal{U}}([x_n]) = \Phi([x_n])$ almost everywhere, so by Proposition 3.10.(2) we have $\lambda = (\Phi^{\psi, a_\mathcal{U}})_*\mu_\mathcal{U} = \Phi_*\mu_\mathcal{U} = \lim_{n\to\mathcal{U}} \lambda_n$ so there exists a subsequence $n_0 < n_1 < \cdots$ such that $\lambda_{n_k} \to \lambda$. Hence $\lambda \in L$ and so $E(\boldsymbol{a}_\mathcal{U}, K) \subseteq L$. $\square$[Claim]

Let $D \subseteq L$ be a countable dense subset of $L = E(\boldsymbol{a}_\mathcal{U}, K)$. For each $\lambda \in D$ we choose some $\boldsymbol{B}_\mathcal{U}$-measurable $\phi_\lambda : X_\mathcal{U} \to K$ with $(\Phi^{\phi_\lambda, a_\mathcal{U}})_*\mu_\mathcal{U} = \lambda$, and we also choose a sequence $\phi_{\lambda,m} : X_\mathcal{U} \to K$, $m \in \mathbb{N}$, of functions converging in measure to ϕ_λ, such that each $\phi_{\lambda,m}$ is constant on some $\boldsymbol{B}_\mathcal{U}$-measurable finite partition $\mathcal{P}^{(\lambda,m)}$ of $X_\mathcal{U}$. By Theorem 8.3 there exists a countably generated standard factor $\boldsymbol{M}$ of $\mathrm{MALG}_{\mu_\mathcal{U}}$ containing $\bigcup_{\lambda\in D}\bigcup_{m\in\mathbb{N}} \mathcal{P}^{(\lambda,m)}$ that is isomorphic to MALG_μ. Let $\boldsymbol{a}_\infty$ be an action on (X, μ) corresponding to a point realization of the action of Γ on $\boldsymbol{M}$ by measure algebra automorphisms. It is clear that

$E(\boldsymbol{a}_\infty, K) \subseteq E(\boldsymbol{a}_\mathcal{U}, K) = L$. We show that $D \subseteq \overline{E(\boldsymbol{a}_\infty, K)}$. Given $\lambda \in D$, each of the functions $\phi_{\lambda,m}$ is $\boldsymbol{M}$-measurable, so $(\Phi^{\phi_{\lambda,m},a_\mathcal{U}})_*\mu_\mathcal{U} \in E(\boldsymbol{a}_\infty, K)$ for all m. Since $\phi_{\lambda,m} \to \phi_\lambda$ in measure it follows that $(\Phi^{\phi_{\lambda,m},a_\mathcal{U}})_*\mu_\mathcal{U} \to \lambda$, and thus $\lambda \in \overline{E(\boldsymbol{a}_\infty, K)}$. Thus $L = \overline{E(\boldsymbol{a}_\infty, K)}$. $\square$

For $\boldsymbol{a} \in A(\Gamma, X, \mu)$ let $[\boldsymbol{a}] \subseteq A(\Gamma, X, \mu)$ denote the weak equivalence class of $\boldsymbol{a}$ in $A(\Gamma, X, \mu)$. Let $A_\sim(\Gamma, X, \mu) = \{[\boldsymbol{a}] : \boldsymbol{a} \in A(\Gamma, X, \mu)\}$ be the set of all weak equivalence classes of elements of $A(\Gamma, X, \mu)$, and let τ denote the topology on $A_\sim(\Gamma, X, \mu)$ obtained by identifying $A_\sim(\Gamma, X, \mu)$ with a closed subset of $(\mathcal{K}, \tau_V)$ via Φ. This makes $A_\sim(\Gamma, X, \mu)$ into a compact metrizable space.

THEOREM 5.2.

(1) [**AE11**] *The type,* type$(\boldsymbol{a})$*, of a measure preserving action is an invariant of weak equivalence.*

(2) *The map* $[\boldsymbol{a}] \mapsto$ type$(\boldsymbol{a})$ *is a continuous map from the space* $(A_\sim(\Gamma, X, \mu), \tau)$ *of weak equivalence classes of measure preserving actions of* Γ *to the space* IRS(Γ) *of invariant random subgroups of* Γ *equipped with the weak*-topology.*

PROOF. Let $\boldsymbol{b}_n \in A(\Gamma, X, \mu)$, $n \in \mathbb{N}$, and suppose that $[\boldsymbol{b}_n] \to [\boldsymbol{b}]$ in τ, i.e., $\overline{E(\boldsymbol{b}_n, K)} \to \overline{E(\boldsymbol{b}, K)}$ in τ_V. In light of Proposition 8.4, both (1) and (2) will follow once we show that type$(\boldsymbol{a}_n) \to$ type$(\boldsymbol{a})$ for all $\boldsymbol{a}_n \in [\boldsymbol{b}_n]$ and $\boldsymbol{a} \in [\boldsymbol{b}]$. Let $\theta_n =$ type$(\boldsymbol{a}_n)$ and let $\theta =$ type$(\boldsymbol{a})$. Let $F, G \subseteq \Gamma$ be finite. We define $N_F = \{H \in \text{Sub}(\Gamma) : F \cap H = \varnothing\}$, $N_{F,G} = \{H \in \text{Sub}(\Gamma) : F \cap H = \varnothing \text{ and } G \subseteq H\}$ and

$$A_F^n = \bigcap_{\gamma \in F} \text{supp}(\gamma^{a_n}) \quad A_{F,G}^n = \bigcap_{\gamma \in F} \text{supp}(\gamma^{a_n}) \cap \bigcap_{\gamma \in G} X \setminus \text{supp}(\gamma^{a_n})$$
$$A_F = \bigcap_{\gamma \in F} \text{supp}(\gamma^{a}) \quad A_{F,G} = \bigcap_{\gamma \in F} \text{supp}(\gamma^{a}) \cap \bigcap_{\gamma \in G} X \setminus \text{supp}(\gamma^{a}).$$

Then $\theta_n(N_F) = \mu(A_F^n)$, $\theta_n(N_{F,G}) = \mu(A_{F,G}^n)$, $\theta(N_F) = \mu(A_F)$, and $\theta(N_{F,G}) = \mu(A_{F,G})$. We will be done once we show that $\mu(A_{F,G}^n) \to \mu(A_{F,G})$ for all finite $F, G \subseteq \Gamma$.

We first show that $\mu(A_F^n) \to \mu(A_F)$ for all finite $F \subseteq \Gamma$.

LEMMA 5.3. $\mu(A_F) \leq \liminf_n \mu(A_F^n)$ *for all finite* $F \subseteq \Gamma$.

PROOF. We may write A_F as a countable disjoint union $A_F = \bigsqcup_{m\geq 0} A_m$ where $\mu(\gamma^a A_m \cap A_m) = 0$ for all $\gamma \in F$ and $m \in \mathbb{N}$. Then for any $\epsilon > 0$ we can find M so large that $\sum_{m\geq M} \mu(A_m) < \frac{\epsilon}{2|F|}$. Since $[\boldsymbol{a_n}] \to [\boldsymbol{a}]$ in τ we have that $E(\boldsymbol{a}, K) \subseteq \mathrm{TLim}_n \overline{E(\boldsymbol{a}, K)}$ so by Proposition 3.5 $\boldsymbol{a} \prec \{\boldsymbol{a}_n : n \in I\}$ for any infinite $I \subseteq \mathbb{N}$. Thus there exists N such that for each $n > N$ we can find $A_0^n, \ldots, A_{M-1}^n$ such that for all $\gamma \in F \cup \{e\}$ and $i, j < M$ we have

$$|\mu(\gamma^a A_i \cap A_j) - \mu(\gamma^{a_n} A_i^n \cap A_j^n)| < \frac{\epsilon}{2M^2|F|}.$$

Then, fixing n with $n > N$, in particular we have $\mu(\gamma^{a_n} A_i^n \cap A_i^n) < \frac{\epsilon}{2M^2|F|}$ and $|\mu(A_i) - \mu(A_i^n)| < \frac{\epsilon}{2M^2|F|}$ for all $\gamma \in F$ and $i < M$, and $\mu(A_i^n \cap A_j^n) < \frac{\epsilon}{2M^2|F|}$ for all $i, j < M$, $i \neq j$. Define for $i < M$ the sets

$$B_i^n = A_i^n \setminus \big(\bigcup_{\gamma \in F} \gamma^{a_n} A_i^n \cup \bigcup_{j \neq i} A_j^n \big).$$

Then for $\gamma \in F$, $\gamma^{a_n} B_i^n \cap B_i^n = \varnothing$ and for $i \neq j$, $B_i^n \cap B_j^n = \varnothing$. Thus $\bigsqcup B_i^n \subseteq A_F^n$. Since $\mu(B_i^n) \geq \mu(A_i^n) - ((M-1) + |F|)\frac{\epsilon}{2M^2|F|} > \mu(A_i) - \frac{\epsilon}{2M}$ it follows that $\mu(A_F^n) \geq \sum_{i<M} \mu(B_i^n) > (\sum_{i<M} \mu(A_i)) - \frac{\epsilon}{2} > \mu(A_F) - \epsilon$. Since this holds for all $n > N$ and since $\epsilon > 0$ was arbitrary we are done. □[Lemma]

LEMMA 5.4. $\limsup_n \mu(A_F^n) \leq \mu(A_F)$ *for all finite* $F \subseteq \Gamma$.

PROOF. We may write each A_F^n as a countable disjoint union $A_F^n = \bigsqcup_{m=0}^{\infty} A_m^n$ where for all $n, m \in \mathbb{N}$, $\gamma^{a_n} \cdot A_m^n \cap A_m^n = \varnothing$. We also define $A_{-1}^n = X \setminus A_F^n$. Let $B_{-1}, B_0, B_1, B_2, \ldots$ be a sequence of disjoint nonempty clopen subsets of K, let $k_m \in B_m$, and define $\phi_n : X \to K$ by $\phi_n(x) = k_m$ for $x \in A_m^n$. The set

$$\begin{aligned} B_F &= \{f \in K^\Gamma : (\forall m \geq -1)\,[f(e) \in B_m \Rightarrow (\forall \gamma \in F)(f(\gamma) \notin B_m)]\} \\ &= K^\Gamma \setminus \bigcup_{m \geq -1} (\pi_e^{-1}(B_m) \cap \bigcup_{\gamma \in F} \pi_\gamma^{-1}(B_m)) \end{aligned}$$

is closed and contained in the open set $U_F = \{f : \forall\gamma \in F\ f(\gamma) \neq f(e)\}$. Fixing n, for each $m \geq 0$ we have that

$$(\Phi^{\phi_n,a_n})^{-1}(\pi_e^{-1}(B_m) \cap \bigcup_{\gamma\in F} \pi_\gamma^{-1}(B_m)) = A_m^n \cap \bigcup_{\gamma\in F} \gamma^{a_n} A_F^n = \varnothing,$$

while for $m = -1$ we have that $(\Phi^{\phi_n,a_n})^{-1}(\pi_e^{-1}(B_{-1}) \cap \bigcup_{\gamma\in F} \pi_\gamma^{-1}(B_{-1})) = A_{-1}^n$ since $A_{-1}^n \subseteq \bigcup_{\gamma\in F} \gamma^{a_n} A_{-1}^n$. It follows that $(\Phi^{\phi_n,a_n})^{-1}(B_F) = A_F^n$. Let $\lambda_n = (\Phi^{\phi_n,a_n})_*\mu \in E(\boldsymbol{a}_n, K)$. Take any convergent subsequence $\{\lambda_{n_k}\}$, and let $\lambda = \lim_k \lambda_{n_k}$. Since $\overline{E(\boldsymbol{a}_n, K)} \to \overline{E(\boldsymbol{a}, K)}$ we have that $\lambda \in \overline{E(\boldsymbol{a}, K)}$, so let $\rho_n = (\Phi^{\psi_n,a})_*\mu \in E(\boldsymbol{a}, K)$ be such that $\rho_n \to \lambda$. We now have

$$\begin{aligned}\limsup{}_k \mu(A_F^{n_k}) &= \limsup{}_k \lambda_{n_k}(B_F) \leq \lambda(B_F) \leq \lambda(U_F)\\ &\leq \liminf{}_n \rho_n(U_F) = \liminf{}_n \mu(\{x : \forall\gamma \in F\ \psi_n((\gamma^{-1})^a x) \neq \psi_n(x)\}) \leq \mu(A_F).\end{aligned}$$

Since the convergent subsequence (λ_{n_k}) was arbitrary we conclude that $\limsup_n \mu(A_F^n) \leq \mu(A_F)$. $\square$

It follows from the above two lemmas that $\mu(A_F) = \lim_n \mu(A_F^n)$ for all finite $F \subseteq \Gamma$. Now let $F, G \subseteq \Gamma$ be finite and note that $A_F^n = A_{F,G}^n \sqcup \bigcup_{\gamma\in G} A_{F\cup\{\gamma\}}^n$ and $A_F = A_{F,G} \sqcup \bigcup_{\gamma\in G} A_{F\cup\{\gamma\}}$. We have just shown that $\mu(A_F) = \lim_n \mu(A_F^n)$. By the inclusion-exclusion principle we have $\mu(\bigcup_{\gamma\in G} A_{F\cup\{\gamma\}}^n) = \sum_{k=1}^{|G|}(-1)^{k-1}\sum_{\{J\subseteq G:|J|=k\}} \mu(A_{F\cup J}^n)$, and since $\mu(A_{F\cup J}^n) \to \mu(A_{F\cup J})$ for each $J \subseteq G$ it follows after another application of inclusion-exclusion that $\mu(\bigcup_{\gamma\in G} A_{F\cup\{\gamma\}}^n) \to \mu(\bigcup_{\gamma\in G} A_{F\cup\{\gamma\}})$. Thus $\mu(A_{F,G}^n) \to \mu(A_{F,G})$. $\square$

COROLLARY 5.5 ([**AE11**]). *For each $\theta \in \mathrm{IRS}(\Gamma)$, $\{[\boldsymbol{a}] : \mathrm{type}(\boldsymbol{a}) = \theta\} \subseteq A_\sim(\Gamma, X, \mu)$ is compact in τ. In particular $\{[\boldsymbol{a}] : [\boldsymbol{a}]$ is free$\}$ is compact in τ.*

REMARK 5.6. The technique used in the proof of Theorem 5.2 can be used to show that combinatorial invariants of measure preserving actions such as independence number (see [**CK13**] and [**CKTD11**]) are continuous functions on $(A_\sim(\Gamma, X, \mu), \tau)$.

THEOREM 5.7. *Let Γ be a countable group.*

(1) *The map* $(A(\Gamma,X,\mu),w) \to (A_\sim(\Gamma,X,\mu),\tau)$, $\boldsymbol{a} \mapsto [\boldsymbol{a}]$, *is Baire class 1. In particular, for each* $\theta \in \mathrm{IRS}(\Gamma)$ *the space* $\{\boldsymbol{a} \in A(\Gamma,X,\mu) : \mathrm{type}(\boldsymbol{a}) = \theta\}$ *is a* G_δ *hence Polish subspace of* $(A(\Gamma,X,\mu),w)$.

(2) *The topology* τ *is a refinement of the quotient topology on* $A_\sim(\Gamma,X,\mu)$ *induced by* w. *If* (X,μ) *is not a discrete space and* $\Gamma \neq \{e\}$ *then the* τ *topology is strictly finer than the quotient topology.*

PROOF. We begin with (1). For this we show that $\boldsymbol{a} \mapsto \overline{E(\boldsymbol{a},K)} \in \mathcal{K}$ is Baire class 1. We observe that $\{\boldsymbol{a} : \overline{E(\boldsymbol{a},K)} \subseteq C\}$ is closed in $(A(\Gamma,X,\mu),w)$ whenever $C \subseteq M_s(K^\Gamma)$ is closed. This is because if $\boldsymbol{a}_n \in A(\Gamma,X,\mu)$, $n \in \mathbb{N}$, is such that $\overline{E(\boldsymbol{a},K)} \subseteq C$ and $\boldsymbol{a}_n \to \boldsymbol{a} \in A(\Gamma,X,\mu)$ in the weak topology then $E(\boldsymbol{a},K) \subseteq \overline{\bigcup_n E(\boldsymbol{a}_n,K)} \subseteq C$.

The topology τ_V on $\mathcal{K}$ is generated by the sets $\{L : L \subseteq U\}$ and $\{L : L \cap U \neq \varnothing\}$, where U ranges over all open subsets of $M_s(K^\Gamma)$. For any open $U \subseteq M_s(K^\Gamma)$ the above observation shows that $\{\boldsymbol{a} : \overline{E(\boldsymbol{a},K)} \cap U \neq \varnothing\}$ is open, and if we write $U = \bigcup_n C_n$ where each C_n is closed and $C_n \subseteq \mathrm{int}(C_{n+1})$ then $\{\boldsymbol{a} : \overline{E(\boldsymbol{a},K)} \subseteq U\} = \bigcup_n \{\boldsymbol{a} : \overline{E(\boldsymbol{a},K)} \subseteq C_n\}$, which is F_σ.

For the first part of (2) we note that the following are equivalent for a subset $\mathcal{B}$ of $A(\Gamma,X,\mu)$:

(i) $\mathcal{B}$ is weakly closed and for all $\boldsymbol{a},\boldsymbol{b} \in A(\Gamma,X,\mu)$, $\boldsymbol{a} \in \mathcal{B}$ and $\boldsymbol{b} \sim \boldsymbol{a}$ implies $\boldsymbol{b} \in \mathcal{B}$.

(ii) $\mathcal{B}$ is weakly closed and for all $\boldsymbol{a},\boldsymbol{b} \in A(\Gamma,X,\mu)$, $\boldsymbol{a} \in \mathcal{B}$ and $\boldsymbol{b} \cong \boldsymbol{a}$ implies $\boldsymbol{b} \in \mathcal{B}$.

(iii) For all $\boldsymbol{a} \in A(\Gamma,X,\mu)$, $\boldsymbol{a} \prec \mathcal{B}$ implies $\boldsymbol{a} \in \mathcal{B}$.

The implication (i)⇒(ii) is trivial, (ii)⇒(iii) follows from Proposition 3.5, and (iii)⇒(i) follows from the fact that if $\boldsymbol{a}_n \to \boldsymbol{a}$ in $A(\Gamma,X,\mu)$ then $\boldsymbol{a} \prec \{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$. To show the first part of (2) it suffices to show that if $\mathcal{B}$ satisfies the above equivalent properties, then $\mathcal{B}_\sim = \{[\boldsymbol{a}] : \boldsymbol{a} \in \mathcal{B}\}$ is closed in τ. Let $L = \overline{\bigcup_{\boldsymbol{a}\in\mathcal{B}} E(\boldsymbol{a},K)}$. Then $L \subseteq M_s(K^\Gamma)$ is closed and property (iii) tells us that $\mathcal{B}_\sim = \{[\boldsymbol{a}] \in A_\sim(\Gamma,X,\mu) : \overline{E(\boldsymbol{a},K)} \subseteq L\}$, which is exactly the definition of a basic closed set in τ_V.

Suppose that (X,μ) is not discrete and let $C \subseteq X$ be the continuous part of X so that $\mu(C) > 0$. Then (C,μ_C) is a standard non-atomic probability space so there exists a universal measure preserving action $\boldsymbol{a} = \Gamma \curvearrowright^{a} (C, \mu_C)$ weakly containing all other measure preserving actions of Γ. Let b be the action of Γ on (X, μ) whose restriction to C is equal to a and whose restriction to $X \backslash C$ is identity and let $\boldsymbol{b} = \Gamma \curvearrowright^{b} (X, \mu)$. As $\boldsymbol{\iota}_{\mu_C} \prec \boldsymbol{a}$ by Lemma 3.4 there exist isomorphic copies of $\boldsymbol{a}$ converging to $\boldsymbol{\iota}_{\mu_C}$ in $A(\Gamma, C, \mu_C)$. This yields isomorphic copies of $\boldsymbol{b}$ converging to $\boldsymbol{\iota}_{\mu}$ in $A(\Gamma, X, \mu)$. Thus $[\boldsymbol{\iota}_{\mu}]$ is in the closure of $\{[\boldsymbol{b}]\}$ in the quotient topology, but $[\boldsymbol{\iota}_{\mu}]$ is not in the τ topology closure of $\{[\boldsymbol{b}]\}$ since $\Gamma \neq \{e\}$ so that $[\boldsymbol{\iota}_{\mu}] \neq [\boldsymbol{b}]$. $\square$

REMARK 5.8. The map $\mathcal{K} \to \mathcal{K}$ sending $L \mapsto \overline{\text{co}}L$ is continuous in the Vietoris topology τ_V. Indeed, if $L_n \to L_\infty$ we show that $\underline{\text{Tlim}}_n \overline{\text{co}} L_n \subseteq \overline{\text{co}} L_\infty \subseteq \overline{\text{Tlim}}_n \overline{\text{co}} L_n$. Let $\lambda \in \underline{\text{TLim}}_n \overline{\text{co}} L_n$ so that there exists $\lambda_{n_k} \in \overline{\text{co}} L_{n_k}$ with $\lambda_{n_k} \to \lambda$. Then there exist probability measures μ_{n_k} on $M_s(K^\Gamma)$ supported on L_{n_k} with $\lambda_{n_k} = \int_{\rho \in M_s(K^\Gamma)} \rho \, d\mu_{n_k}$ and (after moving to a subsequence if necessary) we may assume that μ_{n_k} converges to some measure μ on $M_s(K^\Gamma)$. Then $\lambda = \int_{\rho \in M_s(K^\Gamma)} \rho \, d\mu$. Let $C_0 \supseteq C_1 \supseteq \cdots$ be a sequence of closed subsets of $M_s(K^\Gamma)$ with $L_\infty \subseteq \text{int}(C_m)$ for all m and $L_\infty = \bigcap_m C_m$. For each m the set $\{L \in \mathcal{K} : L \subseteq C_m\}$ is a neighborhood of L_∞ in $\mathcal{K}$ and so contains L_{n_k} for all large enough k. It follows that $\mu(C_m) \geq \liminf_k \mu_{n_k}(C_m) = 1$, and so $\mu(L_\infty) = \lim_m \mu(C_m) = 1$. Since μ is supported on L_∞ and has barycenter λ, it follows that $\lambda \in \overline{\text{co}} L_\infty$. For the second inclusion it is easy to see that $\text{co}L_\infty \subseteq \overline{\text{Tlim}}_n \overline{\text{co}} L_n$ and since the latter set is closed it follows that $\overline{\text{co}} L_\infty \subseteq \overline{\text{Tlim}}_n \overline{\text{co}} L_n$.

If now $\boldsymbol{a}$ is a measure preserving action of Γ and (Y, ν) is non-atomic then $\boldsymbol{a}$ is stably weakly equivalent to an action on (Y, ν) and we let $[\boldsymbol{a}]_s = \{\boldsymbol{b} \in A(\Gamma, Y, \nu) : \boldsymbol{b} \sim_s \boldsymbol{a}\}$ denote the stable weak equivalence class of $\boldsymbol{a}$ in (Y, ν) (see Definition 9.1). It follows that the space $A_{\sim_s}(\Gamma, Y, \nu) = \{[\boldsymbol{a}]_s : \boldsymbol{a} \text{ is a measure preserving action of } \Gamma\}$ of all stable weak equivalence classes of measure preserving actions of Γ may be viewed as a compact subset of $\mathcal{K}$ via the map $[\boldsymbol{a}]_s \mapsto \overline{\text{co}} E(\boldsymbol{a}, K)$. Since $\text{type}(\boldsymbol{a}) = \text{type}(\boldsymbol{\iota} \times \boldsymbol{a})$ it follows that $\text{type}(\boldsymbol{a})$ is an invariant of stable weak equivalence. The map $[\boldsymbol{a}] \mapsto \text{type}(\boldsymbol{a})$ then factors through $[\boldsymbol{a}] \mapsto [\boldsymbol{a}]_s$, and so Theorem 5.2 also holds for stable weak equivalence.

5.3. Random Bernoulli shifts. Given $\theta \in \mathrm{IRS}(\Gamma)$, one constructs a measure preserving action of Γ of type θ as follows (see [**AGV12**, Proposition 45]).

Fix a standard probability space (Z, η) and let $Z^{\leq\backslash\Gamma} = \bigsqcup_{H \in \mathrm{Sub}(\Gamma)} Z^{H\backslash\Gamma}$. Here, $H\backslash\Gamma$ denotes the collection of right cosets of H in Γ. We define the projection map $Z^{\leq\backslash\Gamma} \to \mathrm{Sub}(\Gamma)$, $f \mapsto H_f \in \mathrm{Sub}(\Gamma)$, where $H_f = H$ when $f \in Z^{H\backslash\Gamma}$. We endow $Z^{\leq\backslash\Gamma}$ with the standard Borel structure it inherits as a Borel subset of $Z^{\Gamma} \times \mathrm{Sub}(\Gamma)$ via the injection $f \mapsto ((\gamma \mapsto f(H_f\gamma)), H_f)$. The image of $Z^{\leq\backslash\Gamma}$ under this map is invariant under the product action $\tilde{s} \times c$ of Γ on $Z^{\Gamma} \times \mathrm{Sub}(\Gamma)$ (where $\tilde{s}$ denotes the shift action of Γ on Z^{Γ}), and we let s denote the corresponding action of Γ on $Z^{\leq\backslash\Gamma}$. We have that $H_{\gamma^s f} = \gamma H_f \gamma^{-1}$ for each $\gamma \in \Gamma$ and $f \in Z^{\leq\backslash\Gamma}$ and $(\gamma^s f)(\gamma H_f \gamma^{-1}\delta) = f(H_f\gamma^{-1}\delta)$. Let $\eta^{H\backslash\Gamma}$ denote the product measure on $Z^{H\backslash\Gamma} \subseteq Z^{\leq\backslash\Gamma}$, and observe that under this action we have $(\gamma^s)_*\eta^{H\backslash\Gamma} = \eta^{(\gamma H\gamma^{-1})\backslash\Gamma}$. It follows that the measure $\eta^{\theta\backslash\Gamma}$ on $Z^{\leq\backslash\Gamma}$ defined by

$$\eta^{\theta\backslash\Gamma} = \int_H \eta^{H\backslash\Gamma}\, d\theta(H)$$

is invariant under the action of Γ. We let $\boldsymbol{s}_{\theta,\eta}$ denote the measure preserving action $\Gamma \curvearrowright^{s} (Z^{\leq\backslash\Gamma}, \eta^{\theta\backslash\Gamma})$, and we call $\boldsymbol{s}_{\theta,\eta}$ the *θ-random Bernoulli shift of Γ over (Z, η)*. This action always contains $\boldsymbol{\theta}$ as a factor via the "projection" map $f \mapsto H_f$. When η is non-atomic then the stabilizer map $f \mapsto \Gamma_f$ of $\boldsymbol{s}_{\theta,\eta}$ coincides almost everywhere with this projection. Indeed, if η is non-atomic then for $\eta^{\theta\backslash\Gamma}$-almost every f the function $f : H\backslash\Gamma \to Z$ is injective. Since every $\gamma \in \Gamma_f$ satisfies $f(H\gamma^{-1}) = f(H)$, the inclusion $\Gamma_f \subseteq H_f$ is immediate for injective f, and as $H_f \subseteq \Gamma_f$ always holds we conclude that $\Gamma_f = H_f$ almost surely. In particular $\mathrm{type}(\boldsymbol{s}_{\theta,\eta}) = \theta$. We have thus shown the following.

PROPOSITION 5.9 ([**AGV12**, Proposition 45]). *Let Γ be a countable group. For every $\theta \in \mathrm{IRS}(\Gamma)$ there exists a measure preserving action of type θ. Namely, the θ-random Bernoulli shift $\boldsymbol{s}_{\theta,\eta}$ over a non-atomic base space (Z, η) has type θ.*

It is clear that an isomorphism $(Z_1, \eta_1) \cong (Z_2, \eta_2)$ of measure spaces induces an isomorphism $\boldsymbol{s}_{\theta,\eta_1} \cong \boldsymbol{s}_{\theta,\eta_2}$. The next proposition characterizes precisely when $\text{type}(\boldsymbol{s}_{\theta,\eta}) = \theta$ for various η. Below, we write $N(H)$ for the normalizer of a subgroup H of Γ.

PROPOSITION 5.10. *Let Γ be a countable group, let $\theta \in IRS(\Gamma)$, and let (Z, η) be a standard probability space.*

(1) *If η is non-atomic then $\Gamma_f = H_f$ almost surely;*
(2) *If η is a point mass then $\Gamma_f = N(H_f)$ almost everywhere and the map $f \mapsto H_f$ is an isomorphism $\boldsymbol{s}_{\theta,\eta} \cong \boldsymbol{\theta}$ so that* $\text{type}(\boldsymbol{s}_{\theta,\eta}) = \text{type}(\boldsymbol{\theta})$.
(3) *Suppose η is not a point mass. Then for each infinite index subgroup of $H \leq \Gamma$, $\Gamma_f = H_f$ for $\eta^{H\backslash\Gamma}$-almost every $f \in Z^{H\backslash\Gamma}$. Thus, if*
$$\theta(\{H : [\Gamma : H] < \infty \textit{ and } N(H) \neq H\}) = 0$$
then $\Gamma_f = H_f$ almost surely. In particular if θ concentrates on the infinite index subgroups of Γ then $\Gamma_f = H_f$ almost surely.
(4) *Suppose that η contains atoms. If*
$$\theta(\{H : [\Gamma : H] < \infty \textit{ and } N(H) \neq H\}) > 0$$
then $\text{type}(\boldsymbol{s}_{\theta,\eta}) \neq \theta$.

In particular, $\text{type}(\boldsymbol{s}_{\theta,\eta}) = \theta$ *if and only if $H_f = \Gamma_f$ almost surely.*

PROOF. We have already shown (1) in Proposition 5.9 and (2) is clear. For (3) fix an infinite index $H \leq \Gamma$ along with some $\gamma \notin H$ and inductively define an infinite sequence $\{\delta_n\}_{n\in\mathbb{N}}$ by taking $\delta_{n+1} \in \Gamma$ to be any element of the complement of $\bigcup_{i\leq n}(H\delta_i \cup H\gamma^{-1}\delta_i \cup (\gamma H\gamma^{-1})\delta_i \cup (\gamma H \gamma^{-1})(\gamma\delta_i))$ (we are using here the fact that the collection $\{H\delta : H \in \text{Sub}(\Gamma), \delta \in \Gamma, \text{ and } [\Gamma : H] = \infty\}$ of all right cosets of infinite index subgroups of Γ generates a proper ideal of Γ (see, e.g., the proof of Lemma 4.4 in [**Kec07**])). By construction all of the cosets $H\delta_0$, $H\gamma^{-1}\delta_0$, $H\delta_1$, $H\gamma^{-1}\delta_1, \ldots$ are distinct so, letting

$A \subseteq Z$ be any set with $0 < \eta(A) < 1$, it follows that

$$\begin{aligned}\eta^{H\backslash\Gamma}(\{f : \gamma \in \Gamma_f\}) &\leq \eta^{H\backslash\Gamma}(\{f : \forall\delta \in \Gamma\,(f(H\delta) = f(H\gamma^{-1}\delta))\}) \\ &\leq \eta^{H\backslash\Gamma}(\bigcap_{n\in\mathbb{N}}\{f : f(H\delta_n), f(H\gamma^{-1}\delta_n) \in A \text{ or } f(H\delta_n), f(H\gamma^{-1}\delta_n) \notin A\}) \\ &= \lim_{N\to\infty}(\eta(A)^2 + (1-\eta(A))^2)^N = 0.\end{aligned}$$

Thus $\gamma \notin \Gamma_f$ for $\eta^{H\backslash\Gamma}$-almost every f, and since this is true for each $\gamma \notin H$ we obtain $\Gamma_f \subseteq H$ for $\eta^{H\backslash\Gamma}$-almost every f.

We now prove (4). Let $\theta_{\boldsymbol{s}} = \text{type}(\boldsymbol{s}_{\theta,\eta})$. Let $z_0 \in Z$ be an atom for the measure η. The set $A = \{f \in Z^{\leq\backslash\Gamma} : [\Gamma : H_f] < \infty,\ N(H_f) \neq H_f \text{ and } \forall\gamma \in \Gamma\,(f(H_f\gamma) = z_0)\}$ is $\eta^{\theta\backslash\Gamma}$-non-null and $\Gamma_f = N(H_f) \neq H_f$ for each $f \in A$. Thus $[\Gamma : \Gamma_f] = [\Gamma : N(H_f)] < [\Gamma : H_f]$ for each $f \in A$. When $f \notin A$ we still have $[\Gamma : \Gamma_f] \leq [\Gamma : H_f]$. It follows that

$$\begin{aligned}\int_H \frac{1}{[\Gamma : H]}\,d\theta_{\boldsymbol{s}} &= \int_{f\in A} \frac{1}{[\Gamma : \Gamma_f]}\,d\eta^{\theta\backslash\Gamma} + \int_{f\notin A} \frac{1}{[\Gamma : \Gamma_f]}\,d\eta^{\theta\backslash\Gamma} \\ &> \int_{f\in A} \frac{1}{[\Gamma : H_f]}\,d\eta^{\theta\backslash\Gamma} + \int_{f\notin A} \frac{1}{[\Gamma : H_f]}\,d\eta^{\theta\backslash\Gamma} = \int_f \frac{1}{[\Gamma : H_f]}\,d\eta^{\theta\backslash\Gamma} = \int_H \frac{1}{[\Gamma : H]}\,d\theta\end{aligned}$$

and so $\theta_{\boldsymbol{s}} \neq \theta$, which finishes (4).

It is clear that $\Gamma_f = H_f$ almost everywhere implies $\text{type}(\boldsymbol{s}_{\theta,\eta}) = \theta$. Suppose now that $\Gamma_f \neq H_f$ for a non-null set of $f \in Z^{\leq\backslash\Gamma}$. Then (1) implies that η contains atoms and (3) implies that the set $J = \{f \in Z^{\leq\backslash\Gamma} : [\Gamma : H_f] < \infty \text{ and } \Gamma_f \neq H_f\}$ is non-null. The inclusions $H_f \subseteq \Gamma_f \subseteq N(H_f)$ holds for all $f \in Z^{\leq\backslash\Gamma}$ and so

$$\theta(\{H : [\Gamma : H] < \infty \text{ and } N(H) \neq H\}) \geq \eta^{\theta\backslash\Gamma}(J) > 0.$$

Part (4) now implies that $\text{type}(\boldsymbol{s}_{\theta,\eta}) \neq \theta$. □

THEOREM 5.11. *Let Γ be a countable group, let $\theta \in \text{IRS}(\Gamma)$, and let $\boldsymbol{s}_{\theta,\eta}$ be the θ-random Bernoulli shift over the standard measure space (Z, η). Let $p : Z^{\leq\backslash\Gamma} \to \text{Sub}(\Gamma)$ denote the projection $p(f) = H_f$ factoring $\boldsymbol{s}_{\theta,\eta}$ onto $\boldsymbol{\theta}$. Assume that η is not a point mass. Then the following are equivalent*

(1) *θ concentrates on the infinite index subgroups of Γ.*

(2) *The extension $p : \boldsymbol{s}_{\theta,\eta} \to \boldsymbol{\theta}$ is ergodic.*

(3) *The extension $p : \boldsymbol{s}_{\theta,\eta} \to \boldsymbol{\theta}$ is weak mixing.*

In particular, if $\boldsymbol{\theta}$ is infinite index then $\boldsymbol{s}_{\theta,\eta}$ is ergodic if and only if $\boldsymbol{\theta}$ is ergodic.

PROOF. (3)⇒(2) is trivial. (2)⇒(1): Suppose that $\theta(C) > 0$ where $C = \{H : [\Gamma : H] < \infty\}$ and let $A \subseteq Z$ be any measurable set with $0 < \eta(A) < 1$. Then the set $B = \{f \in Z^{\leq\backslash\Gamma} : H_f \in C \text{ and } \mathrm{ran}(f) \subseteq A\}$ is a nontrivial invariant set that is not p-measurable.

(1)⇒(3): We must show that the extension $\tilde{p} : \boldsymbol{s}_{\theta,\eta} \otimes_{\boldsymbol{\theta}} \boldsymbol{s}_{\theta,\eta} \to \boldsymbol{\theta}$ is ergodic, where

$$\boldsymbol{s}_{\theta,\eta} \otimes_{\boldsymbol{\theta}} \boldsymbol{s}_{\theta,\eta} = \Gamma \curvearrowright^{s\times s} \big(Z^{\leq\backslash\Gamma} \times Z^{\leq\backslash\Gamma}, \int_H \eta^{H\backslash\Gamma} \times \eta^{H\backslash\Gamma}\, d\theta\big)$$

and $\tilde{p}(f,g) = p(f)$. Let $(Y,\nu) = (Z \times Z, \eta \times \eta)$. Then we have the natural isomorphism $\varphi : \boldsymbol{s}_{\theta,\eta} \otimes_{\boldsymbol{\theta}} \boldsymbol{s}_{\theta,\eta} \cong \boldsymbol{s}_{\theta,\nu}$ such that $\tilde{p}(f,g) = p \circ \varphi(f,g)$ almost surely, so it suffices to show that the extension $p : \boldsymbol{s}_{\theta,\nu} \to \boldsymbol{\theta}$ is ergodic. If $\theta = \int_{w\in W} \theta(w)\, d\rho(w)$ is the ergodic decomposition of θ then $\boldsymbol{s}_{\theta,\nu}$ decomposes as $\boldsymbol{s}_{\theta,\nu} = \int_{w\in W} \boldsymbol{s}_{\theta_w,\nu}\, d\rho(w)$ and $p : Y^{\leq\backslash\Gamma} \to \mathrm{Sub}(\Gamma)$ factors $\boldsymbol{s}_{\theta_w,\nu}$ onto $\boldsymbol{\theta}_w$ almost surely. We may therefore assume that $\boldsymbol{\theta}$ is ergodic toward the goal of showing that $\boldsymbol{s}_{\theta,\nu}$ is ergodic as well.

Since $\boldsymbol{\theta}$ is ergodic, the index i of $N(H)$ in Γ is constant on a θ-conull set. If $i < \infty$ then the orbit of almost every H is finite and ergodicity implies that there exists an $H_0 \in \mathrm{Sub}(\Gamma)$ such that θ concentrates on the conjugates of H_0. Then H_0 is an infinite index normal subgroup of $K_0 = N(H_0)$ which implies that the generalized Bernoulli shift action $\boldsymbol{s} = K_0 \curvearrowright^s (Y^{H_0\backslash\Gamma}, \eta^{H_0\backslash\Gamma})$ is ergodic (see, e.g., [**KT08**]). Example 5.13 below then shows that $\boldsymbol{s}_{\theta,\nu} \cong \mathrm{Ind}_{K_0}^{\Gamma}(\boldsymbol{s})$, and so $\boldsymbol{s}_{\theta,\nu}$ is ergodic.

If $i = \infty$ then we proceed as follows. Let $(X,\mu) = (Y^{\leq\backslash\Gamma}, \nu^{\leq\backslash\Gamma})$ and suppose toward contradiction that $B \subseteq X$ is invariant and $0 < \mu(B) = r < 1$. The map $H \mapsto \nu^{H\backslash\Gamma}(B)$ is conjugation invariant so ergodicity of $\boldsymbol{\theta}$ implies that $\nu^{H\backslash\Gamma}(B) = \mu(B) = r$ almost surely. Let $\epsilon > 0$ be small depending on r. Fix some countable Boolean algebra $\boldsymbol{A}_0$ generating $\boldsymbol{B}(Y)$ and let $\boldsymbol{A}$ be the countable Boolean algebra of subsets of X generated by $\{\pi_\gamma^{-1}(D) : D \in \boldsymbol{A}_0 \text{ and } \gamma \in \Gamma\}$ where $\pi_\gamma(f) = f(H_f\gamma)$ for $f \in X$. Then for every $\epsilon > 0$ there exists $A_1, \ldots, A_n \in \boldsymbol{A}$ and a partition $C_0, \ldots, C_{n-1}$ of $\mathrm{Sub}(\Gamma)$ into non-null measurable sets such that $\mu(A \Delta B) < \epsilon^2$ where $A = \bigsqcup_{i<n}(A_i \cap p^{-1}(C_i))$. There exists a finite $F \subseteq \Gamma$ and a collection $\{D_\delta^{i,j} : \delta \in F,\ j < n_i,\ i < n\} \subseteq \boldsymbol{A}_0$ such that $A_i = \bigcup_{0 \leq j < n_i} \bigcap_{\delta \in F} \pi_\delta^{-1}(D_\delta^{i,j})$ for each $i < n$.

LEMMA 5.12. *Let $C \subseteq \mathrm{Sub}(\Gamma)$ be any non-null measurable set. Then for θ-almost every $H \in \mathrm{Sub}(\Gamma)$ there exists $\gamma \in \Gamma$ such that $\{H\alpha\}_{\alpha \in F} \cap \{H\gamma^{-1}\delta\}_{\delta \in F} = \varnothing$ and $\gamma H \gamma^{-1} \in C$.*

PROOF. Since $\boldsymbol{\theta}$ is ergodic and $[\Gamma : N(H)] = \infty$ almost surely, the intersection C^H, of C with the orbit of H, is almost surely infinite. Fix such an H with both $[\Gamma : N(H)] = \infty$ and C^H infinite. Since the set $FF^{-1} \cdot H = \{\delta\alpha^{-1}H\alpha\delta^{-1} : \alpha, \delta \in F\}$ is finite there exists $\gamma \in \Gamma$ with $\gamma H \gamma^{-1} \in C^H \setminus (FF^{-1} \cdot H)$. This γ works: $\gamma H \gamma^{-1} \notin FF^{-1} \cdot H$ is equivalent to $\gamma \notin \bigcup_{\alpha, \delta \in F} \delta\alpha^{-1}N(H)$, so if $\alpha, \delta \in F$ then $\gamma \notin \delta\alpha^{-1}N(H)$ and thus $H\alpha \neq H\gamma^{-1}\delta$. $\square$

Using this lemma and measure-theoretic exhaustion we may find a Borel function $\mathrm{Sub}(\Gamma) \to \Gamma$, $H \mapsto \gamma_H$, with $\{H\alpha\}_{\alpha \in F} \cap \{H\gamma_H^{-1}\delta\}_{\delta \in F} = \varnothing$ and $\gamma_H H \gamma_H^{-1} \in C_i$ for almost every $H \in C_i$, and such that the function $\psi : \mathrm{Sub}(\Gamma) \to \mathrm{Sub}(\Gamma)$, $H \mapsto \gamma_H H \gamma_H^{-1}$, is injective on a conull set. In particular, ψ is measure preserving. Let $\varphi : X \to X$ be given by $\varphi(f) = (\gamma_{H_f})^s \cdot f$ so that φ is also injective on a conull set and measure preserving.

For $H \leq \Gamma$ and $D \subseteq X$ let $D_H = D \cap Y^{H \backslash \Gamma}$. Then for each $i < n$ and almost every $H \in C_i$ we have $\gamma_H H \gamma_H^{-1} \in C_i$ and

$$\varphi(A)_{\gamma_H H \gamma_H^{-1}} = (\gamma_H)^s \cdot ((A_i)_H) = \bigcup_{j<n_i} \bigcap_{\alpha \in F} \{f \in Y^{\gamma_H H \gamma_H^{-1} \backslash \Gamma} : f(\gamma_H H \gamma_H^{-1} \gamma_H \alpha) \in D_\alpha^{i,j}\}$$

$$A_{\gamma_H H \gamma_H^{-1}} = (A_i)_{\gamma_H H \gamma_H^{-1}} = \bigcup_{j<n_i} \bigcap_{\delta \in F} \{f \in Y^{\gamma_H H \gamma_H^{-1} \backslash \Gamma} : f(\gamma_H H \gamma_H^{-1} \delta) \in D_\delta^{i,j}\}$$

By our choice of γ_H the sets $\{\gamma_H H \gamma_H^{-1} \gamma_H \alpha\}_{\alpha \in F}$ and $\{\gamma_H H \gamma_H^{-1} \delta\}_{\delta \in F}$ are almost surely disjoint and it follows that the sets A and $\varphi(A)$ are $\nu^{\gamma_H H \gamma_H^{-1} \backslash \Gamma}$-independent almost surely.

Since $H \mapsto \gamma_H H \gamma_H^{-1}$ is a measure preserving injection it follows that A and $\varphi(A)$ are $\nu^{H\backslash\Gamma}$-independent almost surely.

We have $\epsilon^2 > \mu(A\Delta B) = \int_H \nu^{H\backslash\Gamma}(A\Delta B)\, d\theta \geq \int_H |\nu^{H\backslash\Gamma}(A) - r|\, d\theta$ so that $\theta(\{H : |\nu^{H\backslash\Gamma}(A) - r| \leq \epsilon\}) \geq 1-\epsilon$ and since $\mu(A\Delta B) = \mu(\varphi(A)\Delta B)$ we also have $\theta(\{H : |\nu^{H\backslash\Gamma}(\varphi(A)) - r| \leq \epsilon\}) \geq 1-\epsilon$. Then

$$\begin{aligned} r = \mu(B) &\leq \mu(A\Delta B) + \mu(\varphi(A)\Delta B) + \mu(A\cap\varphi(A)) \\ &< 2\epsilon^2 + \int_H \nu^{H\backslash\Gamma}(A)\nu^{H\backslash\Gamma}(\varphi(A))\, d\theta \leq 2\epsilon^2 + 2\epsilon + (r+\epsilon)^2 \to_{\epsilon\to 0} r^2. \end{aligned}$$

This is a contradiction for small enough ϵ since $0 < r < 1$. $\square$

EXAMPLE 5.13. The simplest example of an ergodic $\theta \in \mathrm{IRS}(\Gamma)$ is a point mass $\theta = \delta_N$ on some normal subgroup $N \lhd \Gamma$. The corresponding random Bernoulli shift $\boldsymbol{s}_{\delta_N,\eta}$ is isomorphic to the usual generalized shift action of Γ on $(Z^{\Gamma/N}, \eta^{\Gamma/N})$.

Almost as simple is when $\theta \in \mathrm{IRS}(\Gamma)$ has the form $\theta = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\gamma_i H \gamma_i^{-1}}$ where $H \leq \Gamma$ is a subgroup with finitely many conjugates $\gamma_0 H\gamma_0^{-1}$, $\gamma_1 H\gamma_1^{-1}$, $\gamma_2 H\gamma_2^{-1}, \ldots \gamma_{n-1} H\gamma_{n-1}^{-1}$. Clearly $\boldsymbol{\theta}$ is ergodic. In this case the random Bernoulli shift $\boldsymbol{s}_{\theta,\eta}$ may be described as follows. The set $T = \{\gamma_i\}_{i<n}$ is a transversal for the left cosets of the normalizer $K = N(H)$ of H in Γ, and the natural action of Γ on T given by $\gamma \cdot t \in \gamma t K \cap T$ for $\gamma \in \Gamma$ and $t \in T$ preserves normalized counting measure ν_T on T. Since H is normal in K, the restriction to K of the action s leaves $Z^{H\backslash\Gamma}$ invariant and preserves the measure $\eta^{H\backslash\Gamma}$ so that $\boldsymbol{s} = K \curvearrowright^s (Z^{H\backslash\Gamma}, \eta^{H\backslash\Gamma})$ becomes the usual generalized Bernoulli shift. We let $\boldsymbol{b}$ denote the induced action $\boldsymbol{b} = \mathrm{Ind}_K^\Gamma(\boldsymbol{s})$, which is the measure preserving action $\Gamma \curvearrowright^b (Z^{H\backslash\Gamma} \times T,\, \eta^{H\backslash\Gamma} \times \nu_T)$ given by

$$\gamma^b(f,t) = (\rho(\gamma,t)^s f, \gamma\cdot t)$$

where $\rho : \Gamma \times T \to K$ is the cocycle given by $\rho(\gamma,t) = (\gamma\cdot t)^{-1}\gamma t$. The map $\pi : Z^{H\backslash\Gamma} \times T \to Z^{\leq\backslash\Gamma}$ given by $\pi(f,t) = t^s f \in Z^{tHt^{-1}\backslash\Gamma}$ is an isomorphism of $\boldsymbol{b}$ with $\boldsymbol{s}_{\theta,\eta}$.

Indeed, π is equivariant since

$$\pi(\gamma^b(f,t)) = \pi(\rho(\gamma,t)^s f, \gamma\cdot t) = (\gamma\cdot t)^s \rho(\gamma,t)^s f = (\gamma t)^s f = \gamma^s t^s f = \gamma^s \pi(f,t)$$

and π is measure preserving since

$$\pi_*(\eta^{H\backslash\Gamma}\times\nu_T) = \frac{1}{n}\sum_{t\in T}\pi_*(\eta^{H\backslash\Gamma}\times\delta_t) = \frac{1}{n}\sum_{t\in T}\eta^{tHt^{-1}\backslash\Gamma} = \eta^{\theta\backslash\Gamma}.$$

It is also clear that π is injective since $t\mapsto tHt^{-1}$ is a bijection of T with the conjugates of H.

5.4. A sufficient condition for weak containment.

NOTATION. For sets A and B we let $A^{\subseteq B} = \bigcup_{C\subseteq B} A^C$. We identify $k\in\mathbb{N}$ with $k = \{0,1,\dots,k-1\}$. A *partition* of (X,μ) will always mean a finite partition of X into Borel sets. When $\mathcal{P}$ is a partition of (X,μ) we will often identify elements of $\mathcal{P}$ with their equivalence class in MALG_μ. We use the script letters $\mathcal{N},\mathcal{O},\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ to denote partitions, and the printed letters N, O, P, Q, R, S and T, respectively, to denote their corresponding elements. If $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of (X,μ) then we let $\mathcal{P}\vee\mathcal{Q} = \{P\cap Q : P\in\mathcal{P},\ Q\in\mathcal{Q}\}$ denote their join. We write $\mathcal{P}\leq\mathcal{Q}$ if $\mathcal{Q}$ is a refinement of $\mathcal{P}$, i.e., if every $Q\in\mathcal{Q}$ is contained, modulo null sets, in some $P\in\mathcal{P}$.

Suppose $\Gamma \curvearrowright^a (X,\mu)$ and $\mathcal{P} = \{P_0,\dots,P_{k-1}\}$ is a partition of X. If J is a finite subset of Γ and $\tau\in k^J$ then we define

$$P_\tau^a = \bigcap_{\gamma\in J}\gamma^a\cdot P_{\tau(\gamma)}.$$

We will write P_τ when the action a is understood. Note that $P_\varnothing = X$. We let Γ act on the set $\bigcup\{k^J : J\subseteq\Gamma \text{ is finite}\}$ by shift, i.e., $(\gamma\cdot\tau)(\delta) = \tau(\gamma^{-1}\delta)$. Then $\mathrm{dom}(\gamma\cdot\tau) = \gamma\mathrm{dom}(\tau)$.

The following lemma establishes a sufficient condition for a measure preserving action $\boldsymbol{a}$ to be weakly contained in $\mathcal{B}$ which will be used in the proof of Theorem 1.5. This lemma is inspired by [**AW11**, Lemma 5].

LEMMA 5.14. *Suppose $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ and $\mathcal{B}$ is a collection of measure preserving actions of Γ. Suppose $\mathcal{P}^{(0)} \leq \mathcal{P}^{(1)} \leq \cdots$ is a sequence of partitions of X such that the smallest $\boldsymbol{a}$-invariant measure algebra containing $\bigcup_n \mathcal{P}^{(n)}$ is all of MALG_μ. Then $\boldsymbol{a} \prec \mathcal{B}$ if for any n, writing $\mathcal{P}^{(n)} = \mathcal{P} = \{P_0, \ldots, P_{k-1}\}$, for all finite subsets $F \subseteq \Gamma$ and all $\delta > 0$, there exists some $\Gamma \curvearrowright^b (Y, \nu) = \boldsymbol{b} \in \mathcal{B}$ and a partition $\mathcal{Q} = \{Q_0, \ldots, Q_{k-1}\}$ of Y such that for all $\tau \in k^{\subseteq F}$, $|\mu(P_\tau) - \nu(Q_\tau)| < \delta$.*

PROOF. Suppose the condition is satisfied and let $A_1, \ldots, A_m \in \mathrm{MALG}_\mu$, $F_0 \subseteq \Gamma$ finite with $e \in F_0$, and $\epsilon > 0$ be given. Let $e \in G_0 \subseteq G_1 \subseteq \cdots$ be an increasing exhaustive sequence of finite subsets of Γ, and let $G_n \cdot \mathcal{P}^{(n)} = \bigvee_{\gamma \in G_n} \gamma^a \cdot \mathcal{P}^{(n)}$. Then $G_n \cdot \mathcal{P}^{(n)}$, $n = 0, 1, 2, \ldots$, is a sequence of finer and finer partitions of X and the algebra generated by $\bigcup_n G_n \cdot \mathcal{P}^{(n)}$ is dense in MALG_μ. There exists an N and $D_1, \ldots, D_m$ in the algebra generated by $G_N \cdot \mathcal{P}^{(N)}$ such that $\mu(A_i \Delta D_i) < \frac{\epsilon}{4}$ for all $i \leq m$. Let $G = G_N$ and $\mathcal{P} = \mathcal{P}^{(N)} = \{P_0, \ldots, P_{k-1}\}$.

We can express each D_i as a finite disjoint union of sets of the form P_σ, $\sigma \in k^T$, i.e., $D_i = \bigsqcup \{P_\sigma : \sigma \in I_i\}$ for some $I_i \subseteq k^G$. Applying the condition given by the lemma to $F = F_0 G$ and $0 < \delta < \frac{\epsilon}{2k^{|G|}}$ we obtain $\Gamma \curvearrowright^b (Y, \nu) = \boldsymbol{b} \in \mathcal{B}$ and a partition $\mathcal{Q} = \{Q_0, \ldots, Q_{k-1}\} \subseteq \mathrm{MALG}_\nu$ such that for all $\tau \in k^{\subseteq F_0 G}$, $|\mu(P_\tau) - \nu(Q_\tau)| < \delta$. For $i \leq m$ we let $B_i = \bigsqcup \{Q_\sigma : \sigma \in I_i\}$. Note that for $\gamma \in F_0$ and $\sigma, \sigma' \in k^G$ we have $\mathrm{dom}(\gamma \cdot \sigma) = \gamma G \subseteq F_0 G$ and

$$\gamma^a P_\sigma \cap P_{\sigma'} = P_{\gamma \cdot \sigma} \cap P_{\sigma'} = \begin{cases} P_{\gamma \cdot \sigma \cup \sigma'} & \text{if } \gamma \cdot \sigma \text{ and } \sigma' \text{ are compatible} \\ \varnothing & \text{otherwise.} \end{cases}$$

Similarly $\gamma^b \cdot Q_\sigma \cap Q_{\sigma'}$ equals either $Q_{\gamma \cdot \sigma \cup \sigma'}$ or $\varnothing$ depending on whether or not $\gamma \cdot \sigma$ and σ' are compatible partial functions. It then follows from our choice of F that $|\mu(\gamma^a P_\sigma \cap$

$P_{\sigma'}) - \nu(\gamma^b Q_\sigma \cap Q_{\sigma'})| < \delta$ for all $\sigma, \sigma' \in k^G$. We now have for $i, j \leq m$ and $\gamma \in F_0$ that

$$\begin{aligned} |\mu(\gamma^a A_i \cap A_j) - \mu(\gamma^b B_i \cap B_j)| &\leq \frac{\epsilon}{2} + |\mu(\bigsqcup_{\substack{\sigma \in I_i,\\ \sigma' \in I_j}} \gamma^a P_\sigma \cap P_{\sigma'}) - \nu(\bigsqcup_{\substack{\sigma \in I_i,\\ \sigma' \in I_j}} \gamma^b Q_\sigma \cap Q_{\sigma'})| \\ &\leq \frac{\epsilon}{2} + |I_i||I_j|\delta < \epsilon. \end{aligned}$$ □

5.5. Independent joinings over an IRS and the proof of Theorem 1.5. Let $\boldsymbol{a} = \Gamma \curvearrowright^a (Y, \nu)$ be a non-atomic measure preserving action of Γ, and let $\theta = \mathrm{type}(\boldsymbol{a})$. The stabilizer map $y \mapsto \Gamma_y$ factors $\boldsymbol{a}$ onto $\boldsymbol{\theta}$ and we let $\nu = \int_H \nu_H \, d\theta$ be the corresponding disintegration of ν over θ. Fix a standard probability space (Z, η) and let $\boldsymbol{s}_{\theta,\eta} = \Gamma \curvearrowright^s (Z^{\leq \backslash \Gamma}, \eta^{\theta \backslash \Gamma})$ be the θ-random Bernoulli shift over (Z, η). The map $f \mapsto H_f$ factors $\boldsymbol{s}_{\theta,\eta}$ onto $\boldsymbol{\theta}$ and the corresponding disintegration is given by $\eta^{\theta \backslash \Gamma} = \int_H \eta^{H \backslash \Gamma} \, d\theta$. The *relatively independent joining* of $\boldsymbol{s}_{\theta,\eta}$ and $\boldsymbol{a}$ over $\boldsymbol{\theta}$ is then the action $\Gamma \curvearrowright^{s \times a} (Z^{\leq \backslash \Gamma} \times Y, \eta^{\theta \backslash \Gamma} \otimes_{\boldsymbol{\theta}} \nu)$ where

$$\eta^{\theta \backslash \Gamma} \otimes_{\boldsymbol{\theta}} \nu = \int_H (\eta^{H \backslash \Gamma} \times \nu_H) \, d\theta = \int_H (\eta^{H \backslash \Gamma} \times \int_{\{y : \Gamma_y = H\}} \delta_y \, d\nu_H(y)) \, d\theta = \int_y (\eta^{\Gamma_y \backslash \Gamma} \times \delta_y) \, d\nu.$$

It is clear that $\eta^{\theta \backslash \Gamma} \otimes_{\boldsymbol{\theta}} \nu$ concentrates on the set $Z^{\leq \backslash \Gamma} \otimes_a Y = \{(f, y) \, : \, H_f = \Gamma_y\}$. We write $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ for $\Gamma \curvearrowright^{s \times a} (Z^{\leq \backslash \Gamma} \otimes_a Y, \eta^{\theta \backslash \Gamma} \otimes_{\boldsymbol{\theta}} \nu)$, so that $b = s \times a$, $X = Z^{\leq \backslash \Gamma} \otimes_a Y$, and

$$\mu = \int_{y \in Y} \eta^{\Gamma_y \backslash \Gamma} \times \delta_y \, d\nu(y).$$

Theorem 1.5 then says that $\boldsymbol{b}$ is weakly equivalent to $\boldsymbol{a}$.

PROOF OF THEOREM 1.5. It suffices to show that $\boldsymbol{b} \prec \boldsymbol{a}$. Let $\mathcal{N}^{(0)} \leq \mathcal{N}^{(1)} \leq \cdots$ and $\mathcal{R}^{(0)} \leq \mathcal{R}^{(1)} \leq \cdots$ be sequences of finite partitions of Z and Y, respectively, such that $\bigcup_n \mathcal{N}^{(n)}$ generates MALG_η and $\bigcup_n \mathcal{R}^{(n)}$ generates MALG_ν (for example, if $Z = Y = 2^{\mathbb{N}}$ then we can let $\mathcal{N}^{(n)} = \mathcal{R}^{(n)}$ consist of the rank n basic clopen sets). For each $\gamma \in \Gamma$ let $\pi_\gamma : X \to Z$ be the projection $\pi_\gamma(f, y) = f(\Gamma_y \gamma)$ and define the finite partitions $\mathcal{S}^{(0)} \leq \mathcal{S}^{(1)} \leq \cdots$ of X by

$$\mathcal{S}^{(n)} = \{\pi_e^{-1}(N) \, : \, N \in \mathcal{N}^{(n)}\}.$$

For $A \subseteq Y$ let $\tilde{A} \subseteq X$ denote the inverse image of A under the projection map $(f, y) \mapsto y \in Y$ and define

$$\tilde{\mathcal{R}}^{(n)} = \{\tilde{R} : R \in \mathcal{R}^{(n)}\}.$$

Then the smallest $\boldsymbol{b}$-invariant measure algebra containing the partitions $\mathcal{P}^{(n)} = \mathcal{S}^{(n)} \vee \tilde{\mathcal{R}}^{(n)}$, $n \in \mathbb{N}$ of X is all of MALG_{μ}. Fix n, define $\mathcal{N} = \mathcal{N}^{(n)} = \{N_0, \ldots, N_{d-1}\}$ and for $i < d$ define

$$S_i = \pi_e^{-1}(N_i)$$
$$\alpha_i = \mu(S_i) = \eta(N_i)$$

along with

$$\mathcal{S} = \mathcal{S}^{(n)} = \{S_0, \ldots, S_{d-1}\}$$
$$\mathcal{R} = \mathcal{R}^{(n)} = \{R_0, \ldots, R_{k-1}\}$$
$$\mathcal{P} = \mathcal{P}^{(n)} = \{P_{i,j} = S_i \cap \tilde{R}_j : i < d,\ j < k\}.$$

For $F \subseteq \Gamma$ finite we naturally identify $(d \times k)^{\subseteq F}$ with $\bigcup_{J \subseteq F} d^J \times k^J$. Under this identification, for $J \subseteq F$ and $(\tau, \sigma) \in d^J \times k^J$ we have

$$\begin{aligned} P^b_{(\tau,\sigma)} = \bigcap_{\gamma \in J} \gamma^{s \times a} P_{\tau(\gamma),\sigma(\gamma)} &= \bigcap_{\gamma \in J} \big(\gamma^{s \times a} S_{\tau(\gamma)} \cap \gamma^{s \times a} \tilde{R}_{\sigma(\gamma)}\big) \\ &= \big(\bigcap_{\gamma \in J} \gamma^{s \times a} S_{\tau(\gamma)}\big) \cap \big(\bigcap_{\gamma \in J} \gamma^{s \times a} \tilde{R}_{\sigma(\gamma)}\big) = S^b_{\tau} \cap \tilde{R}^b_{\sigma}. \end{aligned}$$

By Lemma 5.14, to show that $\boldsymbol{b} \prec \boldsymbol{a}$ it suffices to show that for every $F \subseteq \Gamma$ finite, and $\epsilon > 0$, there exists a partition $\mathcal{Q} = \{Q_{i,j} : i < d,\ j \leq k\}$ of Y such that for all $J \subseteq F$, $(\tau, \sigma) \in d^J \times k^J$

$$|\mu(S_\tau \cap \tilde{R}_\sigma) - \nu(Q_{(\tau,\sigma)})| < \epsilon.$$

Fix such an $F \subseteq \Gamma$ finite and $\epsilon > 0$. We will proceed by finding a partition $\mathcal{T} = \{T_0, \ldots, T_{d-1}\}$ of Y, and then take $Q_{i,j} = T_i \cap R_j$, in which case we will have $Q_{(\tau,\sigma)} =$

$(\bigcap_{\gamma\in J}\gamma^a T_{\tau(\gamma)}) \cap (\bigcap_{\gamma\in J}\gamma^a R_{\sigma(\gamma)}) = T_\tau \cap R_\sigma$. We are therefore looking for a partition $\mathcal{T}$ so that

$$\forall(\tau,\sigma)\in(d\times k)^{\subseteq F}\ \ |\mu(S_\tau\cap\tilde{R}_\sigma)-\nu(T_\tau\cap R_\sigma)|<\epsilon. \tag{5.1}$$

We first calculate the value of $\mu(S_\tau\cap\tilde{A})$ for $\tau\in d^J$ ($J\subseteq F$) and $A\subseteq Y$. Let $\mathcal{E}_J$ denote the finite collection of all equivalence relations on the set J. For $E\in\mathcal{E}_J$ let us say that $\tau\in d^J$ *respects* E, written $\tau\ll E$, if τ is constant on each E-equivalence class. For a subgroup $H\leq\Gamma$ let $E_J(H)\in\mathcal{E}_J$ denote the equivalence relation determined by $tE_J(H)s$ if and only if $Ht=Hs$ (if and only if $t^{-1}H=s^{-1}H$). We write $E_J(y)$ for $E_J(\Gamma_y)$. For each $E\in\mathcal{E}_J$ we fix a transversal $T_E\subseteq J$ for E. We then have

$$\begin{aligned}\mu(S_\tau\cap\tilde{A}) &= \int_{y\in A}\eta^{\Gamma_y\backslash\Gamma}\big(\{f\in Z^{\Gamma_y\backslash\Gamma}\,:\,\forall t\in J\ (f(\Gamma_y t)\in N_{\tau(t)})\}\big)\,d\nu(y)\\ &= \sum_{\{E\in\mathcal{E}_J\,:\,\tau\ll E\}}\int_{\{y\in A\,:\,E_J(y)=E\}}\eta^{\Gamma_y\backslash\Gamma}\big(\{f\in Z^{\Gamma_y\backslash\Gamma}\,:\,\forall t\in T_E\ (f(\Gamma_y t)\in N_{\tau(t)})\}\big)\,d\nu(y)\end{aligned}$$

(5.2)

$$= \sum_{\{E\in\mathcal{E}_J\,:\,\tau\ll E\}}\nu(A\cap\{y\,:\,E_J(y)=E\})\prod_{t\in T_E}\alpha_{\tau(t)}$$

We now proceed as in the proof of [**AW11**, Theorem 1]. Without loss of generality Y is a compact metric space with compatible metric $d_Y\leq 1$. Fix some $\epsilon_0>0$ such that $\epsilon_0^{1/2}<\frac{\epsilon}{2(dk)^{|F|/2}2^{|F|+1}}$. For $\delta\geq 0$ define the sets

$$D_\delta=\{y\in Y\,:\,\forall s,t\in F\ (t^{-1}y\neq s^{-1}y\Rightarrow d_Y(t^{-1}y,s^{-1}y)>\delta)\}$$

$$E_\delta=\{(y,y')\in D_\delta\times D_\delta\,:\,\forall s,t\in F\ (d_Y(s^{-1}y,t^{-1}y')>\delta)\}.$$

Then $\nu(D_0)=1$ by definition, and $\nu^2(E_0)=1$ since ν is non-atomic. Thus there exists $\delta>0$ such that $\nu(D_\delta)>1-\frac{\epsilon_0}{4|\mathcal{E}_F|}$ and $\nu^2(E_\delta)>1-\frac{\epsilon_0}{4|\mathcal{E}_F|^2}$.

Fix a finite Borel partition $\{O_m\,:\,1\leq m\leq M\}$ of Y with $\mathrm{diam}(O_m)<\delta$ for each m. For $y\in Y$ let $\alpha(y)=m$ if and only if $y\in O_m$. Let $(\Omega,\mathbb{P})=(d^M,\rho^M)$ and let $Y_m(\omega)=\omega(m)$, so that $\{Y_m\,:\,1\leq m\leq M\}$ are i.i.d. random variables. For $\omega\in\Omega$ and

$i = 0, \ldots, d-1$ define

$$T_i(\omega) = \{y \in Y \,:\, \omega(\alpha(y)) = i\}.$$

Then each $\omega \in \Omega$ defines the partition $\mathcal{T}(\omega) = \{T_0(\omega), \ldots, T_{d-1}(\omega)\}$ of Y. Let $T_i = \{(\omega, y) \,:\, y \in T_i(\omega)\}$ and let $T_\tau = \{(\omega, y) \in \Omega \times Y \,:\, y \in T_\tau(\omega) = \bigcap_{t \in J} t^a \cdot (T_{\tau(t)}(\omega))\}$, $\tau \in d^{\subseteq F}$. We view $\mathcal{T}$ as a "random partition" of Y. We let Γ act on Ω trivially so that, e.g., $\gamma \cdot (T_\tau(\omega)) = (\gamma \cdot T_\tau)(\omega)$, and for $B \subseteq \Omega \times Y$ and $y \in Y$ we let B^y denote the section $B^y = \{\omega \,:\, (\omega, y) \in B\}$. We show that $\mathcal{T}$ satisfies (5.1) with high probability.

Fix now some $A \subseteq Y$ and $\tau \in d^J$, $J \subseteq F$. Note that if $y \in Y$ and τ does not respect $E_J(y)$ then there exist $t, s \in J$ with $t^{-1}y = s^{-1}y$ and $\tau(t) \neq \tau(s)$, so that $(T_{\tau(t)})^{t^{-1}y} \cap (T_{\tau(s)})^{s^{-1}y} = \varnothing$ and thus $(T_\tau)^y = \bigcap_{t \in J} (t \cdot T_{\tau(t)})^y = \bigcap_{t \in J} (T_{\tau(t)})^{t^{-1}y} = \varnothing$. It follows that the expected measure of $T_\tau(\omega) \cap A$ is

$$\begin{aligned}
\mathbb{E}[\nu(T_\tau(\omega) \cap A)] &= \int_A \Big(\int_\Omega 1_{T_\tau}(\omega, y) \, d\,\mathbb{P}\,(\omega) \Big) \, d\nu(y) \\
&= \int_A \mathbb{P}((T_\tau)^y) \, d\nu(y) = \sum_{\{E \in \mathcal{E}_J \,:\, \tau \ll E\}} \int_{\{y \in A \,:\, E_J(y) = E\}} \mathbb{P}((T_\tau)^y) \, d\nu(y) \\
&= \sum_{\{E \in \mathcal{E}_J \,:\, \tau \ll E\}} \Big(\int_{\{y \in A \cap D_\delta \,:\, E_J(y) = E\}} \mathbb{P}((T_\tau)^y) \, d\nu \Big) + \int_{A \setminus D_\delta} \mathbb{P}((T_\tau)^y) \, d\nu. \qquad (5.3)
\end{aligned}$$

Fix some $E \in \mathcal{E}_J$ with $\tau \ll E$ and some $y \in D_\delta$ with $E_J(y) = E$. For $t, s \in J$, if t and s are not E-related then $t^{-1}y \neq s^{-1}y$ and so $d_Y(t^{-1}y, s^{-1}y) > \delta$. It follows that $O_{\alpha(t^{-1}y)} \neq O_{\alpha(s^{-1}y)}$ since each O_α has diameter smaller than δ. So as t ranges over T_E, the numbers $\alpha(t^{-1}y)$ are all distinct and the variables $Y_{\alpha(t^{-1}y)} : \omega \mapsto \omega(\alpha(t^{-1}y))$, $t \in T_E$, are therefore independent. We have $t^{-1}y \in T_{\tau(t)}(\omega)$ if and only if $\omega(\alpha(t^{-1}y)) = \tau(t)$, so the sets $(t \cdot T_{\tau(t)})^y = (T_{\tau(t)})^{t^{-1}y}$, $t \in T_E$, are all independent. If tEs then as $\tau \ll E$ we have that $(T_{\tau(t)})^{t^{-1}y} = (T_{\tau(s)})^{s^{-1}y}$. It follows that

$$\mathbb{P}((T_\tau)^y) = \mathbb{P}\big(\bigcap_{t \in J} (t \cdot T_{\tau(t)})^y \big) = \prod_{t \in T_E} \mathbb{P}((T_{\tau(t)})^{t^{-1}y}) = \prod_{t \in T_E} \alpha_{\tau(t)}. \qquad (5.4)$$

Continuing the computation, the second integral in (5.3) is no greater than $\nu(A \setminus D_\delta) < \frac{\epsilon_0}{4}$ and $\nu(A \cap D_\delta \cap \{y \,:\, E_J(y) = E\})$ is within $\frac{\epsilon_0}{4|\mathcal{E}_F|}$ of $\nu(A \cap \{y \,:\, E_J(y) = E\})$, so after summing over all $E \in \mathcal{E}_J$ we see that (5.3) is within $\frac{\epsilon_0}{2}$ of (5.2), i.e.,

$$\left| \mathbb{E}\left[\nu(T_\tau(\omega) \cap A)\right] - \mu(S_\tau \cap \tilde{A}) \right| < \frac{\epsilon_0}{2}. \tag{5.5}$$

Now we compute the second moment of $\nu(T_\tau(\omega) \cap A)$.

$$\begin{aligned} \mathbb{E}[\nu(T_\tau(\omega) \cap A)^2] &= \textstyle\int_\Omega \big(\int_{y \in A} 1_{T_\tau}(\omega, y)\, d\nu(y) \big) \big(\int_{y' \in A} 1_{B_\tau}(\omega, y')\, d\nu(y') \big)\, d\mathbb{P} \\ &= \textstyle\int_{(y,y') \in A \times A} \big(\int_\Omega 1_{T_\tau}(\omega, y) 1_{T_\tau}(\omega, y')\, d\,\mathbb{P} \big)\, d\nu^2 \\ &= \textstyle\int_{(y,y') \in A \times A} \mathbb{P}((T_\tau)^y \cap (T_\tau)^{y'})\, d\nu^2 \qquad (5.6) \end{aligned}$$

For $(y, y') \in E_\delta$, if $t, s \in J$ then $d_Y(t^{-1}y, s^{-1}y') > \delta$, so that $O_{\alpha(t^{-1}y)}$ and $O_{\alpha(s^{-1}x')}$ are disjoint. It follows that the two events $\{\omega : \forall t \in J\, (Y_{\alpha(t^{-1}y)}(\omega) = \tau(t))\} = \bigcap_{t \in J} (T_{\tau(t)})^{t^{-1}y} = (T_\tau)^y$ and $\{\omega : \forall s \in J\, (Y_{\alpha(s^{-1}y')}(\omega) = \tau(s))\} = \bigcap_{s \in J} (T_{\tau(s)})^{s^{-1}y} = (T_\tau)^{y'}$ are independent. We obtain that the part of (5.6) integrated over $(A \times A) \cap E_\delta$ is equal to

$$\begin{aligned} &\textstyle\int_{(y,y') \in (A \times A) \cap E_\delta} \mathbb{P}((T_\tau)^y \cap (T_\tau)^{y'})\, d\nu^2 = \int_{(y,y') \in (A \times A) \cap E_\delta} \mathbb{P}((T_\tau)^y)\, \mathbb{P}\left((T_\tau)^{y'}\right) d\nu^2 \\ &= \textstyle\sum_{\tau \ll E, E' \in \mathcal{E}_J} \nu^2((A \times A) \cap E_\delta \cap \{(y, y') : E_J(y) = E,\ E_J(y') = E'\}) \prod_{t \in T_E} \alpha_{\tau(t)} \prod_{s \in T_{E'}} \alpha_{\tau(s)} \end{aligned}$$

where we used the fact that $E_\delta \subseteq D_\delta \times D_\delta$ along with the known values from (5.3) and (5.4). The part of (5.6) integrated over $(A \times A) \setminus E_\delta$ is no greater than $\frac{\epsilon_0}{4}$, and for each pair $E, E' \in \mathcal{E}_J$ with $\tau \ll E, E'$, the value of $\nu^2((A \times A) \cap E_\delta \cap \{(y, y') : E_J(y) = E,\ E_J(y') = E'\})$ is within $\frac{\epsilon_0}{4|\mathcal{E}_F|^2}$ of $\nu(A \cap \{y : E_J(y) = E\})\nu(A \cap \{y' : E_J(y') = E'\})$. Summing over all such $E, E' \in \mathcal{E}_J$ we obtain that (5.6) is within $\frac{\epsilon_0}{2}$ of the square of (5.2), i.e.,

$$\left| \mathbb{E}\left[\nu(T_\tau(\omega) \cap A)^2\right] - \mu(S_\tau \cap \tilde{A})^2 \right| < \frac{\epsilon_0}{2}. \tag{5.7}$$

From (5.5) and (5.7) it follows that the variance of $\nu(T_\tau(\omega)\cap A)$ is no greater than ϵ_0. By Chebyshev's inequality we then have

$$\begin{aligned}&\mathbb{P}\big(|\nu(T_\tau(\omega)\cap A)-\mu(S_\tau\cap\tilde{A})|\geq\epsilon\big)\leq\mathbb{P}\big(|\nu(T_\tau(\omega)\cap A)-\mathbb{E}[\nu(T_\tau(\omega)\cap A)]|\geq\tfrac{\epsilon}{2}\big)\\&\leq\mathbb{P}\big(|\nu(T_\tau(\omega)\cap A)-\mathbb{E}[\nu(T_\tau(\omega)\cap A)]|\geq(kd)^{|F|/2}2^{|F|+1}\epsilon_0^{1/2}\big)\leq\tfrac{1}{(kd)^{|F|}2^{2|F|+2}}\end{aligned}$$

and since this is true for each $\tau\in d^{\subseteq F}$ and $|d^{\subseteq F}|\leq 2^{|F|}d^{|F|}$, we find that

$$\mathbb{P}\big(\exists\tau\in d^{\subseteq F}\ (|\nu(T_\tau(\omega)\cap A)-\mu(S_\tau\cap\tilde{A})|\geq\epsilon)\big)\leq\frac{1}{2^{|F|+2}k^{|F|}}.$$

Since $A\subseteq Y$ was arbitrary, this is in particular true for each $A=R_\sigma$, $\sigma\in k^{\subseteq F}$, so that

$$\mathbb{P}\big(\exists\tau\in d^{\subseteq F},\,\sigma\in k^{\subseteq F}\ (|\nu(T_\tau(\omega)\cap R_\sigma)-\mu(S_\tau\cap\tilde{R}_\sigma)|>\epsilon)\big)\leq\frac{1}{4}.$$

So taking any ω_0 in the complement of the above set, we obtain a partition $\mathcal{T}=\mathcal{T}(\omega_0)$ satisfying (5.1). $\square$

Theorem 1.5 shows that among all non-atomic weak equivalence classes of type θ there is a least, in the sense of weak containment. Namely $\boldsymbol{s}_{\theta,\lambda}$ where λ is Lebesgue measure on $[0,1]$. We note that there is also a greatest.

THEOREM 5.15. *Let $\theta\in\mathrm{IRS}(\Gamma)$. Then there exists a measure preserving action $\boldsymbol{a}_\theta$ of Γ with type$(\boldsymbol{a}_\theta)=\theta$ such that for all measure preserving actions $\boldsymbol{b}$ of Γ, if type$(\boldsymbol{b})=\theta$ then $\boldsymbol{b}\prec\boldsymbol{a}_\theta$.*

PROOF. Let (Y,ν) be a non-atomic standard probability space. If $\boldsymbol{b}$ is any measure preserving action of Γ of type θ then $\boldsymbol{\iota}\times\boldsymbol{b}$ is also of type θ, weakly contains $\boldsymbol{b}$, and is isomorphic to an element of $A(\Gamma,Y,\nu)$. It thus suffices to show there is an action $\boldsymbol{a}_\theta$ of type θ that weakly contains every element in the set $\mathcal{A}_\theta=\{\boldsymbol{a}\in A(\Gamma,Y,\nu)\,:\,\mathrm{type}(\boldsymbol{a})=\theta\}$.

Let $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathcal{A}_\theta$. For each n the stabilizer map $y\mapsto\mathrm{stab}_{a_n}(y)=\{\gamma\in\Gamma\,:\,\gamma^{a_n}y=y\}$ factors $\boldsymbol{a}_n$ onto $\boldsymbol{\theta}$. Let $\boldsymbol{a}_\theta$ denote the relatively independent joining of the actions $\boldsymbol{a}_0,\boldsymbol{a}_1,\boldsymbol{a}_2,\dots$ over the common factor $\boldsymbol{\theta}$, i.e., $\boldsymbol{a}_\theta=\Gamma\curvearrowright^{\prod_n a_n}(Y^{\mathbb{N}},\nu_\theta)$ where the measure ν_θ has each marginal equal to ν and concentrates on the set $\{(y_0, y_1, y_2, \dots) \in Y^{\mathbb{N}} : \forall n\, (\mathrm{stab}_{a_n}(y_n) = \mathrm{stab}_{a_0}(y_0))\}$. Then for ν_θ-almost every $(y_0, y_1, \dots) \in Y^{\mathbb{N}}$ we have $\mathrm{stab}_{\prod_n a_n}((y_0, y_1, \dots)) = \mathrm{stab}_{a_0}(y_0)$, from which it follows that $\mathrm{type}(\boldsymbol{a}_\theta) = \theta$. Since $\boldsymbol{a}_n \sqsubseteq \boldsymbol{a}_\theta$ for all n the set $\{\boldsymbol{a} \in \mathcal{A}_\theta : \boldsymbol{a} \prec \boldsymbol{a}_\theta\}$ is dense in $\mathcal{A}_\theta$ so by Lemma 3.4 $\boldsymbol{a}_\theta$ weakly contains every element of $\mathcal{A}_\theta$. $\square$

6. Non-classifiability

6.1. Non-classifiability by countable structures of $\cong$, $\cong^w$, and $\cong^{\mathcal{U}}$ on free weak equivalence classes.

DEFINITION 6.1. Let E and F be equivalence relations on the standard Borel spaces X and Y, respectively.

(1) A *homomorphism* from E to F is a map $\psi : X \to Y$ such that $xEy \Rightarrow \psi(x)F\psi(y)$.
(2) A *reduction* from E to F is a map $\psi : X \to Y$ such that $xEy \Leftrightarrow \psi(x)F\psi(y)$.
(3) E is said to *admit classification by countable structures* if there exists a countable language $\mathcal{L}$ and a Borel reduction from E to isomorphism $\cong_{\mathcal{L}}$ on $X_{\mathcal{L}}$, where $X_{\mathcal{L}}$ is the space of all $\mathcal{L}$-structures with universe $\mathbb{N}$.
(4) Suppose that the space X is Polish. We say that E is *generically F-ergodic* if for every Baire measurable homomorphism ψ from E to F, there exists some $y \in Y$ such that $\psi^{-1}([y]_F)$ is comeager.

The proof of the following lemma is clear.

LEMMA 6.2. *Let F_1 and F_2 be equivalence relations on the standard Borel spaces Y_1 and Y_2 respectively, and let E be an equivalence relation on the Polish space P. Suppose that E is generically F_2-ergodic and that there exists a Borel reduction from F_1 to F_2. Then E is generically F_1-ergodic.*

Since the orbit equivalence relation associated to a generically turbulent Polish group action is generically $\cong_{\mathcal{L}}$-ergodic for all countable languages $\mathcal{L}$ (**[Hjo00]**), Lemma 6.2 immediately implies the following.

LEMMA 6.3. *Let $\mathcal{G}$ be a Polish group and let P be a generically turbulent Polish $\mathcal{G}$-space with corresponding orbit equivalence relation $E^P_{\mathcal{G}}$. Let F be an equivalence relation on a standard Borel space Y and suppose that $E^P_{\mathcal{G}}$ is not generically F-ergodic. Then F does not admit classification by countable structures.*

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and let $\mathcal{U}(\mathcal{H})$ denote the unitary group of $\mathcal{H}$ which is a Polish group under the strong operator topology. The group $\mathcal{U}(\mathcal{H})$ acts on $\mathcal{U}(\mathcal{H})^\Gamma$ by conjugation on each coordinate and we may view the space $\mathrm{Rep}(\Gamma, \mathcal{H})$ of all unitary representations of Γ on $\mathcal{H}$ as an invariant closed subspace of $\mathcal{U}(H)^\Gamma$, so that it is a Polish $\mathcal{U}(\mathcal{H})$-space. We call the corresponding orbit equivalence relation on $\mathrm{Rep}(\Gamma, \mathcal{H})$ *unitary conjugacy* and if π_1 and π_2 are in the same unitary conjugacy class then we say that π_1 and π_2 are *unitarily conjugate* and write $\pi_1 \cong \pi_2$. Let $\lambda_\Gamma : \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ denote the left regular representation of Γ and let $\mathrm{Rep}_\lambda(\Gamma, \mathcal{H})$ be the set of unitary representations of Γ on $\mathcal{U}(\mathcal{H})$ that are weakly contained in λ_Γ. Then $\mathrm{Rep}_\lambda(\Gamma, \mathcal{H})$ is also a Polish $\mathcal{U}(\mathcal{H})$ space, being an invariant closed subspace of $\mathrm{Rep}(\Gamma, \mathcal{H})$.

The following lemma is proved in the same way as [**KLP10**, Lemma 2.4], using that the reduced dual $\hat{\Gamma}_\lambda$, which may be identified with the spectrum of the reduced C^*-algebra $C^*_\lambda(\Gamma)$, contains no isolated points ([**KLP10**, 3.2]).

LEMMA 6.4. *Let κ be a unitary representation of Γ on $\mathcal{H}$. Then the set $\{\pi \in \mathrm{Rep}_\lambda(\Gamma, \mathcal{H}) : \pi \perp \kappa\}$ is dense G_δ in $\mathrm{Rep}_\lambda(\Gamma, \mathcal{H})$.*

We are now ready to prove Theorem 1.7.

PROOF OF THEOREM 1.7. Given a free action $\boldsymbol{a}_0 \in A(\Gamma, X, \mu)$, we let $[\boldsymbol{a}_0] = \{\boldsymbol{b} \in A(\Gamma, X, \mu) : \boldsymbol{b} \sim \boldsymbol{a}_0\}$ denote its weak equivalence class in $A(\Gamma, X, \mu)$. Let $\mathcal{H} = \ell_2(\Gamma)$ and let $\boldsymbol{g} : \mathrm{Rep}(\Gamma, \mathcal{H}) \to A(\Gamma, X, \mu)$ be the continuous map assigning to each $\pi \in \mathrm{Rep}_\lambda(\Gamma, \mathcal{H})$ the corresponding Gaussian action $\boldsymbol{g}(\pi) \in A(\Gamma, X, \mu)$ (see [**Kec10**, Appendix E]). We have that $\boldsymbol{g}(\pi) \prec \boldsymbol{g}(\infty \cdot \lambda_\Gamma) \cong \boldsymbol{s}_\Gamma$ and so by Corollary 1.6, $\boldsymbol{a}_0 \times \boldsymbol{g}(\pi) \sim \boldsymbol{a}_0$. Fix some isomorphism $\varphi : X^2 \to X$ of the measure spaces (X^2, μ^2) and (X, μ) and denote by $\boldsymbol{b} \mapsto \varphi \cdot \boldsymbol{b}$ the corresponding homeomorphism of $A(\Gamma, X^2, \mu^2)$ with $A(\Gamma, X, \mu)$. Let ψ :

$\mathrm{Rep}_\lambda(\Gamma,\mathcal{H}) \to [\boldsymbol{a}_0]$ be the map $\pi \mapsto \varphi \cdot (\boldsymbol{a}_0 \times \boldsymbol{g}(\pi))$. This is a continuous homomorphism from unitary conjugacy on $\mathrm{Rep}_\lambda(\Gamma,\mathcal{H})$ to isomorphism on $[\boldsymbol{a}_0]$, and is therefore also a homomorphism to $\cong^w$ and to $\cong^{\mathcal{U}}$ on $[\boldsymbol{a}_0]$.

CLAIM 3. *The inverse image under ψ of each unitary equivalence class in $[\boldsymbol{a}_0]$ is meager. In particular the same is true for each isomorphism class and each weak isomorphism class.*

PROOF OF CLAIM. Let $\boldsymbol{c} \in [\boldsymbol{a}_0]$. By Lemma 6.4 the set $\{\pi \in \mathrm{Rep}_\lambda(\Gamma,\mathcal{H}) : \pi \perp \kappa_0^{\boldsymbol{c}}\}$ is comeager in $\mathrm{Rep}_\lambda(\Gamma,\mathcal{H})$. If $\psi(\pi) \cong^{\mathcal{U}} \boldsymbol{c}$ then $\pi \leq \kappa_0^{\boldsymbol{g}(\pi)} \leq \kappa_0^{\boldsymbol{a}_0\times\boldsymbol{g}(\pi)} \cong \kappa_0^{\boldsymbol{c}}$, so that $\pi \not\perp \kappa_0^{\boldsymbol{c}}$. □[Claim]

By [**KLP10**, 3.3], the conjugacy action of $\mathcal{U}(\mathcal{H})$ on $\mathrm{Rep}_\lambda(\Gamma,\mathcal{H})$ is generically turbulent. The homomorphism ψ witnesses that unitary conjugacy on $\mathrm{Rep}_\lambda(\Gamma,\mathcal{H})$ is not generically $F|[\boldsymbol{a}_0]$-ergodic when F is any of $\cong$, $\cong^w$, or $\cong^{\mathcal{U}}$. The theorem now follows from Lemma 6.3. □

REMARK 6.5. If the weak equivalence class $[\boldsymbol{a}_0]$ contains an ergodic (resp. weak mixing) action $\boldsymbol{b}_0$, then the action $\boldsymbol{b}_0 \times \boldsymbol{g}(\pi)$ is ergodic (resp. weak mixing) provided that the representation $\pi \in \mathrm{Rep}_\lambda(\Gamma,\mathcal{H})$ is weak mixing. Since the weak mixing π are dense G_δ in $\mathrm{Rep}_\lambda(\Gamma,\mathcal{H})$ ([**KLP10**, 3.6]) we conclude that isomorphism (and $\cong^w$ and $\cong^{\mathcal{U}}$) restricted to the ergodic (resp. weak mixing) elements of $[\boldsymbol{a}_0]$ does not admit classification by countable structures.

It also follows from the above arguments and [**HK95**, 2.2] that the equivalence relation E_0 of eventual agreement on $2^{\mathbb{N}}$ is Borel reducible to $F|[\boldsymbol{a}_0]$ when F is any of $\cong$, $\cong^w$, or $\cong^{\mathcal{U}}$ (and the same holds for $F|\{\boldsymbol{b} \in [\boldsymbol{a}_0] : \boldsymbol{b}$ is ergodic (resp. weak mixing)$\}$ when $[\boldsymbol{a}_0]$ contains ergodic (resp. weak mixing) elements).

6.2. Extending Theorem 1.7. It would be interesting to see an extension of Theorem 1.7 to weak equivalence classes of measure preserving actions that are not necessarily free. We outline here one possible generalization of the argument given in the proof of

Theorem 1.7 to measure preserving actions that almost surely have infinite orbits. Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ be such an action, and let $\theta = \text{type}(\boldsymbol{a})$, so that θ concentrates on the infinite index subgroups of Γ. In place of unitary conjugacy on $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ we work with the cohomology equivalence relation on a certain orbit closure in the Polish space $Z^1(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ of unitary cocycles of $\boldsymbol{\theta}$, where $\mathcal{H} = \ell^2(\mathbb{N})$. The cohomology equivalence relation on $Z^1(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ is the orbit equivalence relation generated by the action of the Polish group $\widetilde{\mathcal{U}(\mathcal{H})} = L(\text{Sub}(\Gamma), \theta, \mathcal{U}(\mathcal{H}))$ given by

$$(f \cdot \alpha)(\gamma, H) = f(\gamma H \gamma^{-1})\alpha(\gamma, H) f(H)^{-1} \in \mathcal{U}(\mathcal{H})$$

where $f \in \widetilde{\mathcal{U}(\mathcal{H})}$, $\alpha \in Z^1(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$, $\gamma \in \Gamma$, and $H \leq \Gamma$ (see [**Kec10**, Chapter III]). In place of the left regular representation λ of Γ we use a cocycle λ_θ associated to θ defined as follows. Identify right cosets of the infinite index subgroups $H \leq \Gamma$ with natural numbers by fixing a Borel map $n : \text{Sub}(\Gamma) \times \Gamma \to \mathbb{N}$ such that for each infinite index $H \leq \Gamma$ the map $\gamma \mapsto n(H, \gamma)$ is a surjection onto $\mathbb{N}$ and satisfies $n(H, \gamma) = n(H, \delta)$ if and only if $H\gamma = H\delta$. Let $\{e_n\}_{n \in \mathbb{N}}$ be the standard orthonormal basis for $\ell^2(\mathbb{N}) = \mathcal{H}$ and define $\lambda_\theta \in Z^1(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ by

$$\lambda_\theta(\gamma, H)(e_{n(H,\delta)}) = e_{n(\gamma H \gamma^{-1}, \gamma\delta)}$$

for all $\gamma \in \Gamma$ and $H \leq \Gamma$ of infinite index (recall that θ-almost every H is infinite index in Γ). Fix an isomorphism $T : \infty \cdot \mathcal{H} \to \mathcal{H}$ and let $\sigma \in Z^1(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ be the image of $\infty \cdot \lambda_\theta$ under T, i.e., $\sigma(\gamma, H) = T \circ (\infty \cdot \lambda_\theta)(\gamma, H) \circ T^{-1}$. Let $Z^1_\lambda(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ denote the orbit closure of σ in $Z^1(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$. Using the Gaussian map $U(\mathcal{H}) \to \text{Aut}(X, \mu)$ (see [**Kec10**, Appendix E] or [**BTD11**]), each $\alpha \in Z^1_\lambda(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ gives rise to a cocycle $g(\alpha) : \Gamma \times \text{Sub}(\Gamma) \to \text{Aut}(X, \mu)$ of $\boldsymbol{\theta}$ with values in the automorphism group $\text{Aut}(X, \mu)$ of a non-atomic probability space (X, μ). We obtain a skew product action $\boldsymbol{g}(\alpha) = (X, \mu) \ltimes_{g(\alpha)} \boldsymbol{\theta}$ on the measure space $(Y, \nu) = (X \times \text{Sub}(\Gamma), \mu \times \theta)$, which is an extension of $\boldsymbol{\theta}$. The action $\boldsymbol{g}(\lambda_\theta)$ is isomorphic to $\boldsymbol{s}_{\theta,\eta}$ (where η is non-atomic) and so the action $\boldsymbol{g}(\sigma)$ is isomorphic to $\boldsymbol{s}_{\theta,\eta^{\mathbb{N}}} \cong \boldsymbol{s}_{\theta,\eta}$ as well. Since $\alpha \in Z^1_\lambda(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ we have $\boldsymbol{g}(\alpha) \prec \boldsymbol{s}_{\theta,\eta}$ and thus the relatively independent joining $\boldsymbol{g}(\alpha) \otimes_{\boldsymbol{\theta}} \boldsymbol{a}$ is weakly equivalent to $\boldsymbol{a}$ by Theorem 1.5. The map $\psi_\theta(\alpha) := \varphi \cdot (\boldsymbol{g}(\alpha) \otimes_{\boldsymbol{\theta}} \boldsymbol{a})$ is then a homomorphism from the cohomology equivalence relation on $Z^1_\lambda(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ to isomorphism on $[\boldsymbol{a}]$, where $\varphi : Y \times X \to X$ is once again an isomorphism of measure spaces. The remaining ingredient that is needed is an analogue of the results from [**KLP10**].

QUESTION 6.6. Let θ be an ergodic IRS of Γ with infinite index. Is the action of $\widetilde{\mathcal{U}(\mathcal{H})}$ on the space $Z^1_\lambda(\boldsymbol{\theta}, \mathcal{U}(\mathcal{H}))$ generically turbulent? Is the preimage under ψ_θ of each $\cong^{\mathcal{U}}$-class meager?

Two ergodic theoretic analogues of the space $\mathrm{Rep}_\lambda(\Gamma, \mathcal{H})$ are the spaces $A_0(\Gamma, X, \mu) = \{\boldsymbol{a} \in A(\Gamma, X, \mu) : \boldsymbol{a} \prec \boldsymbol{s}_\Gamma\}$ and $A_1(\Gamma, X, \mu) = \{\boldsymbol{a} \in A(\Gamma, X, \mu) : \boldsymbol{a} \prec_s \boldsymbol{s}_\Gamma\}$, where (X, μ) is non-atomic. When Γ is amenable it follows from [**FW04**] that these spaces both coincide with $A(\Gamma, X, \mu)$ and the conjugacy action of $\mathrm{Aut}(X, \mu)$ on $A(\Gamma, X, \mu)$ is generically turbulent. For non-amenable Γ, the spaces $A_0(\Gamma, X, \mu)$, $A_1(\Gamma, X, \mu)$ and $A(\Gamma, X, \mu)$ do not all coincide.

QUESTION 6.7. Let Γ be a non-amenable group. Is conjugacy on either of $A_0(\Gamma, X, \mu)$ or $A_1(\Gamma, X, \mu)$ generically turbulent?

For all non-amenable Γ the set $A_0(\Gamma, X, \mu)$ is nowhere dense in $A_1(\Gamma, X, \mu)$ (by Theorem 1.3), so these two spaces may behave quite differently, generically (indeed, every action in $A_0(\Gamma, X, \mu)$ is ergodic, while the generic action in $A_1(\Gamma, X, \mu)$ has continuous ergodic decomposition). The question of generic turbulence of conjugacy on $\mathrm{ERG}(\Gamma, X, \mu) = \{\boldsymbol{a} \in A(\Gamma, X, \mu) : \boldsymbol{a} \text{ is ergodic}\}$ is discussed in [**Kec10**, §5 and §12].

7. Types and amenability

As noted in Remark 4.1, any two free measure preserving actions of an infinite amenable group Γ are weakly equivalent. In this section we prove Theorem 1.8, which extends this to actions that are not necessarily free.

7.1. The space COS(Γ). Let $\mathrm{COS}(\Gamma)$ be the space of all left cosets of all subgroups of Γ. Since $F \in \mathrm{COS}(\Gamma) \Leftrightarrow \forall \delta \in \Gamma\, (\delta \in F \Rightarrow \delta^{-1}F \in \mathrm{Sub}(\Gamma))$ it follows that $\mathrm{COS}(\Gamma)$ is a closed subset of 2^Γ. As every left coset of a subgroup $H \leq \Gamma$ is equal to a right coset of a conjugate of H and vice versa, $\mathrm{COS}(\Gamma)$ is also the space of all right cosets of subgroups of Γ and we have the equality $\mathrm{COS}(\Gamma) = \{\gamma H \delta^{-1} : H \leq \Gamma,\ \gamma, \delta \in \Gamma\} \subseteq 2^\Gamma$. We let ℓ denote the continuous action of Γ on $\mathrm{COS}(\Gamma)$ by left translation, $\gamma^\ell \cdot (H\delta) = \gamma H \delta$.

LEMMA 7.1. *Let Γ be a countable amenable group and let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ be a measure preserving action of Γ. Then for any finite $F \subseteq \Gamma$ and $\delta > 0$ there exists a measurable map $J : X \to \mathrm{COS}(\Gamma)$ such that*

$$\mu(\{x \in X : \forall \gamma \in F\ J(\gamma^a x) = \gamma^\ell \cdot J(x)\}) \geq 1 - \delta$$

and $J(x) \in \Gamma_x \backslash \Gamma$ for all x.

PROOF. We note that this is a generalized version of [**BTD11**, Theorem 3.1] which applies to the case in which $\boldsymbol{a}$ is free and which is an immediate consequence of the Rokhlin lemma for free actions of amenable groups. For the general case we use the Ornstein-Weiss Theorem [**OW80**, Theorem 6] which implies that the orbit equivalence relation E_a generated by $\boldsymbol{a}$ is hyperfinite when restricted to an invariant co-null Borel set $X' \subseteq X$. We may assume without loss of generality that $X' = X$ and E_a is hyperfinite. Then there exists an increasing sequence $E_0 \subseteq E_1 \subseteq \cdots$ of finite Borel sub-equivalence relations of E_a such that $E_a = \bigcup_{n=0}^\infty E_n$. Let F and $\delta > 0$ be given and find $N \in \mathbb{N}$ large enough so that $\mu(X_N) > 1 - \delta$ where $X_N = \{x : \gamma^a x \in [x]_{E_N} \text{ for all } \gamma \in F\}$. Fix a Borel selector $s : X \to X$ for E_N, i.e., for all x, $x E_N s(x)$ and $x E_N y \Rightarrow s(x) = s(y)$, and let $x \mapsto \gamma_x \in \Gamma$ be any Borel map such that $\gamma_x^a \cdot s(x) = x$ for all $x \in X$. Define $J : X \to \mathrm{COS}(\Gamma)$ by $J(x) = \gamma_x \Gamma_{s(x)}$. Then $J(x) \in \Gamma_x \backslash \Gamma$ since $\Gamma_x = \Gamma_{\gamma_x^a \cdot s(x)} = \gamma_x \Gamma_{s(x)} \gamma_x$. For each $x \in X_N$ and $\gamma \in F$ we have $\gamma^a x \in [x]_{E_N}$ so that $s(\gamma^a x) = s(x)$ and thus $(\gamma_{\gamma^a x})^a \cdot s(x) = \gamma^a x = (\gamma \gamma_x)^a \cdot s(x)$. It follows that

$$J(\gamma^a x) = \gamma_{\gamma^a x} \Gamma_{s(x)} = \gamma \gamma_x \Gamma_{s(x)} = \gamma^\ell \cdot J(x). \qquad \square$$

7.2. Proof of Theorem 1.8.

PROOF OF THEOREM 1.8.(1). Since type$(\boldsymbol{a})$ is an invariant of stable weak equivalence (see Remark 5.8), it remains to show the following:

$(*)$ If $\theta \in \mathrm{IRS}(\Gamma)$ and $\boldsymbol{a}$ and $\boldsymbol{d}$ are measure preserving actions of Γ both of type θ, then

$$\boldsymbol{a} \sim_s \boldsymbol{d}.$$

We first show that $(*)$ holds under the assumption that $\boldsymbol{a}$ and $\boldsymbol{d}$ are both ergodic. For this, by Theorem 1.5 it suffices to show that for any ergodic measure preserving action $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ of Γ, if type$(\boldsymbol{a}) = \theta$ then $\boldsymbol{a} \prec \boldsymbol{s}_{\theta,\eta}$ for some standard probability space (Z, η).

We will define a measure preserving action $\boldsymbol{b}$ containing $\boldsymbol{\theta}$ as a factor, and show that the relatively independent joining $\boldsymbol{b} \otimes_{\boldsymbol{\theta}} \boldsymbol{s}_{\theta,\eta}$ weakly contains $\boldsymbol{a}$ when η is a standard non-atomic probability measure. Then we will be done once we show $\boldsymbol{b} \otimes_{\boldsymbol{\theta}} \boldsymbol{s}_{\theta,\eta} \cong \boldsymbol{s}_{\theta,\eta}$.

Let $\mu = \int_H \mu_H \, d\theta$ be the disintegration of μ via $x \mapsto \mathrm{stab}_a(x)$, and define the measure ν on the space $Y = \bigsqcup_{H \in \mathrm{Sub}(\Gamma)} \{f \in X^{H\backslash\Gamma} : \mathrm{stab}_a(f(H\delta)) = H \text{ for all } \delta \in \Gamma\} \subseteq X^{\leq\backslash\Gamma}$ by the equation $\nu = \int_H \mu_H^{H\backslash\Gamma} \, d\theta$. Let $a^{\leq\backslash\Gamma}$ be the action on $X^{\leq\backslash\Gamma}$ that is equal to $a^{H\backslash\Gamma}$ on $X^{H\backslash\Gamma}$. Then $a^{\leq\backslash\Gamma}$ commutes with the shift action s on $X^{\leq\backslash\Gamma}$ and since $(\gamma^s)_*(\gamma^{a^{H\backslash\Gamma}})_*(\mu_H)^{H\backslash\Gamma} = \mu_{\gamma H\gamma^{-1}}^{(\gamma H\gamma^{-1})\backslash\Gamma}$ it follows from invariance of θ that the action $\gamma^b = \gamma^s\gamma^{a^{\leq\backslash\Gamma}}$ preserves the measure ν. We let $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$. Then $\boldsymbol{\theta}$ is a factor of $\boldsymbol{b}$ via the map $f \mapsto H_f$. Let (Z, η) be a standard non-atomic probability space, and let $\boldsymbol{b} \otimes_\theta \boldsymbol{s}_{\theta,\eta}$ denote the relatively independent joining of $\boldsymbol{b}$ and $\boldsymbol{s}_{\theta,\eta}$ over $\boldsymbol{\theta}$.

We now apply Lemma 7.1 to $\boldsymbol{s}_{\theta,\eta}$. Given $F \subseteq \Gamma$ finite and $\epsilon > 0$ there exists a measurable $J : Z^{\leq\backslash\Gamma} \to \mathrm{COS}(\Gamma)$ such that $\eta^{\theta\backslash\Gamma}(Z_0) \geq 1 - \epsilon$ where $Z_0 = \{g \in Z^{\leq\backslash\Gamma} : J(\gamma^s \cdot g) = \gamma^\ell \cdot J(g) \text{ for all } \gamma \in F\}$, and with $J(g) \in \Gamma_g\backslash\Gamma = H_g\backslash\Gamma$ for all $g \in Z^{\leq\backslash\Gamma}$. We let $\varphi : Y \times Z^{\leq\backslash\Gamma} \to X$ be the map defined $(\nu \otimes_{\boldsymbol{\theta}} \eta^{\leq\backslash\Gamma})$-almost everywhere by $\varphi(f, g) = f(J(g))$. Then for all $g \in Z_0$ and $\gamma \in F$ we have $\varphi(\gamma^{b\times s}(f, g)) = \gamma^a((\gamma^s f)(J(\gamma^s g))) =$

$\gamma^a(f(J(g))) = \gamma^a\varphi((f,g))$ and

$$\begin{aligned}\varphi_*(\nu \otimes_{\boldsymbol{\theta}} \eta^{\leq\backslash\Gamma}) &= \int_H \int_g \int_f \delta_{f(J(g))}\, d\mu_H^{H\backslash\Gamma}\, d\eta^{H\backslash\Gamma}\, d\theta \\ &= \int_H \sum_{t\in H\backslash\Gamma} \int_{\{g\,:\,J(g)=t\}} \mu_H\, d\eta^{H\backslash\Gamma}\, d\theta = \int_H \mu_H\, d\theta = \mu.\end{aligned}$$

It then follows that $\boldsymbol{a} \prec \boldsymbol{b}\otimes_{\boldsymbol{\theta}}\boldsymbol{s}_\theta$ since for any measurable partition $A_0,\dots,A_{k-1} \subseteq X$ of X, the sets $B_0 = \varphi^{-1}(A_0),\dots,B_{k-1} = \varphi^{-1}(A_{k-1})$ form a measurable partition of $Y \times X^{\leq\backslash\Gamma}$ satisfying $|\mu(\gamma^a A_i \cap A_j) - (\nu \otimes_{\boldsymbol{\theta}} \eta^{\leq\backslash\Gamma})(\gamma^{b\times s} B_i \cap B_j)| < \epsilon$ for all $\gamma \in F$.

By the Rokhlin skew-product theorem there exists a standard probability space (Z_1,η_1) and an isomorphism Ψ of $\boldsymbol{a}$ with a skew product action $\boldsymbol{d} = (Z_1,\eta_1) \ltimes \boldsymbol{\theta}$ on the space $(Z_1 \times \mathrm{Sub}(H), \eta_1 \times \theta)$. The isomorphism Ψ is of the form $\Psi(x) = (\Psi_0(x), \Gamma_x)$ and so the restriction Ψ_H of Ψ_0 to $X_H = \{x : \Gamma_x = H\}$ is an isomorphism of (X_H, μ_H) with (Z_1,η_1) almost surely. We now define an isomorphism $\Phi : Y \to Z_1^{\leq\backslash\Gamma}$ of $\boldsymbol{b}$ with $\boldsymbol{s}_{\theta,\eta_1}$ by taking $H_{\Phi(f)} = H_f$ and $\Phi(f)(H\gamma) = \Psi_{\gamma^{-1}H\gamma}((\gamma^{-1})^a(f(H\gamma)))$, where $H = H_f$. This is almost everywhere well-defined since $f(H\gamma) \in X_H$ almost surely, which ensures that $(\gamma^{-1})^a(f(H\gamma))$ is independent of our choice of representative for the coset $H\gamma$, and $(\gamma^{-1})^a(f(H\gamma)) \in X_{\gamma^{-1}H\gamma}$ so that we may apply $\Psi_{\gamma^{-1}H\gamma}$. The map Φ is equivariant since if $H_f = H$ then $H_{\delta^b f} = \delta H\delta^{-1}$ and $\Phi(\delta^b f)(\delta H\delta^{-1}\gamma) = \Psi_{\gamma^{-1}\delta H\delta^{-1}\gamma}((\gamma^{-1})^a(\delta^b f(\delta H\delta^{-1}\gamma)) = \Psi_{\gamma^{-1}\delta H(\gamma^{-1}\delta)^{-1}}((\gamma^{-1}\delta)^a(f(H\delta^{-1}\gamma))) = \Phi(f)(H\delta^{-1}\gamma) = (\delta^s\Phi(f))(\delta H\delta^{-1}\gamma)$. Finally, $\Phi_*\nu = \eta_1^{\theta\backslash\Gamma}$ since

$$\begin{aligned}\Phi_*\nu = \int_H \Phi_*\mu_H^{H\backslash\Gamma}\, d\theta &= \int_H \prod_{H\gamma\in H\backslash\Gamma} (\Psi_{\gamma^{-1}H\gamma})_*(\gamma^{-1})^a_*\mu_H\, d\theta \\ &= \int_H \prod_{H\gamma\in H\backslash\Gamma} (\Psi_{\gamma^{-1}H\gamma})_*\mu_{\gamma^{-1}H\gamma}\, d\theta = \int_H \eta_1^{H\backslash\Gamma}\, d\theta = \eta_1^{\theta\backslash\Gamma}\end{aligned}$$

and so $\boldsymbol{b} \cong \boldsymbol{s}_{\theta,\eta_1}$. Since $H_f = H_{\Phi(f)}$, this extends to an isomorphism of $\boldsymbol{b} \otimes_{\boldsymbol{\theta}} \boldsymbol{s}_{\theta,\mu}$ with $\boldsymbol{s}_{\theta,\eta_1} \otimes_{\boldsymbol{\theta}} \boldsymbol{s}_{\theta,\eta} \cong \boldsymbol{s}_{\theta,\eta_1\times\eta} \cong \boldsymbol{s}_{\theta,\eta}$, as was to be shown.

We next show that $(*)$ holds under the assumption that $\boldsymbol{\theta}$ is ergodic. Let $i \in \mathbb{N} \cup \{\infty\}$ be the index of θ. If i is finite then the orbit of almost every $H \in \mathrm{Sub}(\Gamma)$ is finite so by ergodicity of $\boldsymbol{\theta}$ there exists $H_0 \leq \Gamma$ of index i such that θ concentrates on the conjugates of H_0. Then for some spaces (Z_1, η_1) and (Z_2, η_2) we have $\boldsymbol{a} \cong \boldsymbol{\iota}_{\eta_1} \times \boldsymbol{a}_{\Gamma/H_0}$ and $\boldsymbol{d} \cong \boldsymbol{\iota}_{\eta_2} \times \boldsymbol{a}_{\Gamma/H_0}$ where $\boldsymbol{a}_{\Gamma/H_0}$ denotes the action of Γ on the left cosets of H_0 with normalized counting measure. Thus $\boldsymbol{a} \sim_s \boldsymbol{d}$. If $i = \infty$ then we let $\boldsymbol{a} = \int_Z \boldsymbol{a}_z \, d\eta$ and $\boldsymbol{d} = \int_W \boldsymbol{d}_w \, d\rho$ be the ergodic decompositions of $\boldsymbol{a}$ and $\boldsymbol{d}$, respectively. By Proposition 3.8, type$(\boldsymbol{a}_z) = \theta$ and type$(\boldsymbol{d}_w) = \theta$ almost surely, and $\boldsymbol{a}_z$ and $\boldsymbol{d}_w$ are non-atomic almost surely since θ is infinite index. Letting $\boldsymbol{b}$ be any non-atomic ergodic action of type θ the above case implies that $\boldsymbol{a} \sim_s \boldsymbol{b} \sim_s \boldsymbol{d}$.

Finally, we show that $(*)$ holds in general. Let $\theta = \int_{w\in W} \theta_w \, d\rho$ be the ergodic decomposition of θ. We then obtain corresponding decompositions $\boldsymbol{a} = \int_w \boldsymbol{a}_w \, d\rho$ and $\boldsymbol{d} = \int_w \boldsymbol{d}_w \, d\rho$ of $\boldsymbol{a}$ and $\boldsymbol{d}$ with type$(\boldsymbol{a}_w) = \theta_w =$ type$(\boldsymbol{d}_w)$ almost surely. The above cases imply that $\boldsymbol{a}_w \sim_s \boldsymbol{d}_w$ almost surely. Theorem 3.12 then implies $\boldsymbol{a} \sim_s \boldsymbol{d}$. $\square$

PROOF OF THEOREM 1.8.(2). Let $\theta =$ type$(\boldsymbol{a}) =$ type$(\boldsymbol{b})$. If $\boldsymbol{\theta}$ is ergodic then by Proposition 3.8 almost every ergodic component of $\boldsymbol{a}$ and $\boldsymbol{b}$ have type θ and so Theorem 1.8 and Corollary 4.4 imply that $\boldsymbol{a} \sim \boldsymbol{\iota}_{\eta_1} \times \boldsymbol{d}$ and $\boldsymbol{b} \sim \boldsymbol{\iota}_{\eta_2} \times \boldsymbol{d}$ for some ergodic $\boldsymbol{d}$ of type θ and some spaces (Z_1, η_1), (Z_2, η_2). Since Γ is amenable, $\boldsymbol{d}$ is not strongly ergodic, and since θ is infinite index, $\boldsymbol{d}$ is non-atomic, so by [**AW11**, Theorem 3] $\boldsymbol{d} \sim \boldsymbol{\iota} \times \boldsymbol{d}$ and thus $\boldsymbol{a} \sim \boldsymbol{b}$. The general case now follows by considering the ergodic decomposition of θ. $\square$

8. Ultraproducts of measure preserving actions

In this appendix we establish some properties of ultraproducts of measure spaces and actions.

Notation. We refer to [**CKTD11**] for background on ultraproducts of measure preserving actions and also [**ES07**] for background on ultraproducts of measure spaces. Our notation has some changes from that of [**CKTD11**] and is as follows. Given a sequence $\boldsymbol{a}_n = \Gamma \curvearrowright^{a_n} (X_n, \mu_n)$, $n \in \mathbb{N}$, of measure preserving actions of Γ and a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we denote by $(\prod_n \boldsymbol{a}_n)/\mathcal{U} = \Gamma \curvearrowright^{(\prod_n a_n)_\mathcal{U}} ((\prod_n X_n)/\mathcal{U}, (\prod_n \mu_n)/\mathcal{U})$, or simply $\boldsymbol{a}_{\mathcal{U}} = \Gamma \curvearrowright^{a_{\mathcal{U}}} (X_{\mathcal{U}}, \mu_{\mathcal{U}})$ when there is no danger of confusion, the corresponding ultraproduct of the sequence $(\boldsymbol{a}_n)$. We let $[x_n]$ denote the equivalence class of the sequence $(x_n) \in \prod_n X_n$ in $X_{\mathcal{U}}$ and we let $[B_n]$ denote the subset of $X_{\mathcal{U}}$ determined by the sequence $(B_n) \in \prod_n \boldsymbol{B}(X_n)$ of Borel sets. When $x_n = x$ for all n then we write $[x]$ for $[x_n]$ and when $B_n = B$ for all n we write $[B]$ for $[B_n]$. Then $\boldsymbol{A}_{\mathcal{U}} = \boldsymbol{A}_{\mathcal{U}}(X_{\mathcal{U}}) = \{[B_n] : (B_n) \in \prod_n \boldsymbol{B}(X_n)\}$ is an algebra of subsets of $X_{\mathcal{U}}$ and $\mu_{\mathcal{U}}$ is the unique measure on the σ-algebra $\boldsymbol{B}_{\mathcal{U}}(X_{\mathcal{U}}) = \sigma(\boldsymbol{A}_{\mathcal{U}})$ whose value on $[A_n] \in \boldsymbol{A}_{\mathcal{U}}$ is $\mu_{\mathcal{U}}([A_n]) = \lim_{n\to\mathcal{U}} \mu_n(A_n)$. We note that every element of $\boldsymbol{B}_{\mathcal{U}}$ is within a $\mu_{\mathcal{U}}$-null set of an element of $\boldsymbol{A}_{\mathcal{U}}$.

The following proposition deals with lifting measure disintegrations to ultraproducts.

PROPOSITION 8.1. *Suppose that for each $n \in \mathbb{N}$ the Borel map $\pi_n : (Y_n, \nu_n) \to (Z_n, \eta_n)$ factors $\boldsymbol{b}_n = \Gamma \curvearrowright^b (Y_n, \nu_n)$ onto $\boldsymbol{d}_n \curvearrowright^d (Z_n, \eta_n)$ and let $\nu_n = \int_{z\in Z_n} \nu_z^n \, d\eta_n(z)$ be the disintegration of ν_n over η_n with respect to π_n. Let $\boldsymbol{b}_{\mathcal{U}} = \Gamma \curvearrowright^{b_{\mathcal{U}}} (Y_{\mathcal{U}}, \nu_{\mathcal{U}})$ and $\boldsymbol{d}_{\mathcal{U}} = \Gamma \curvearrowright^{d_{\mathcal{U}}} (Z_{\mathcal{U}}, \eta_{\mathcal{U}})$ be the ultraproducts of the sequences $(\boldsymbol{b}_n)$ and $(\boldsymbol{d}_n)$, respectively. Then the map $\pi_{\mathcal{U}} : Y_{\mathcal{U}} \to Z_{\mathcal{U}}$ given by $\pi_{\mathcal{U}}([y_n]) = [\pi_n(y_n)]$ factors $\boldsymbol{b}_{\mathcal{U}}$ onto $\boldsymbol{d}_{\mathcal{U}}$. If for $[z_n] \in Z_{\mathcal{U}}$ we let $\nu_{[z_n]} = (\prod_n \nu_{z_n}^n)/\mathcal{U}$ then*

(I) *Each of the measures $\nu_{[z_n]}$ is a probability measure on $(Y_{\mathcal{U}}, \boldsymbol{B}_{\mathcal{U}}(Y_{\mathcal{U}}))$ and almost surely $\nu_{[z_n]}$ concentrates on $\pi_{\mathcal{U}}^{-1}([z_n])$.*

(II) *For each $D \in \boldsymbol{B}_{\mathcal{U}}(Y_{\mathcal{U}})$ the map $(Z_{\mathcal{U}}, \boldsymbol{B}_{\mathcal{U}}(Z_{\mathcal{U}})) \to ([0,1], \boldsymbol{B}([0,1]))$ sending $[z_n] \mapsto \nu_{[z_n]}(D)$ is measurable and $\nu_{\mathcal{U}}(D) = \int_{[z_n]\in Z_{\mathcal{U}}} \nu_{[z_n]}(D) \, d\eta_{\mathcal{U}}([z_n])$.*

(III) *If $[z_n] \mapsto \mu_{[z_n]}$ is another assignment satisfying* (I) *and* (II) *then for all $D \in \boldsymbol{B}_{\mathcal{U}}(Y_{\mathcal{U}})$ almost surely $\mu_{[z_n]}(D) = \nu_{[z_n]}(D)$.*

Additionally, for almost all $[z_n] \in Z_{\mathcal{U}}$ and every $\gamma \in \Gamma$ we have $(\gamma^{b_{\mathcal{U}}})_* \nu_{[z_n]} = \nu_{\gamma^{d_{\mathcal{U}}}[z_n]}$.

PROOF. It is clear that $\pi_{\mathcal{U}}$ factors $\boldsymbol{b}_{\mathcal{U}}$ onto $\boldsymbol{d}_{\mathcal{U}}$. Property (I) follows from the fact that for each n and $z \in Z_n$, each ν_z^n is a Borel probability measure on Y_n and almost surely ν_z^n concentrates on $\pi_n^{-1}(\{z\})$. Now let $\boldsymbol{D}$ be the collection of all subsets of $Y_{\mathcal{U}}$ satisfying (II). Given $[A_n] \in \boldsymbol{A}_{\mathcal{U}}$ and $V \subseteq [0,1]$ open we have $\nu_{[z_n]}(A_n) \in V$ if and only if $[z_n] \in [\{z :$

$\nu_z^n(A_n) \in V\}]$, so that $[z_n] \mapsto \nu_{[z_n]}([A_n])$ is measurable. As in [**ES07**, Lemma 2.2] we have

$$\begin{aligned}\int_{[z_n]} \nu_{[z_n]}(A_n)\, d\eta_{\mathcal{U}} &= \int_{[z_n]} \lim_{n\to\mathcal{U}} \nu_{z_n}^n(A_n)\, d\eta_{\mathcal{U}} \\ &= \lim_{n\to\mathcal{U}} \int_{z\in Z_n} \nu_z^n(A_n)\, d\eta_n = \lim_{n\to\mathcal{U}} \nu_n(A_n) = \nu_{\mathcal{U}}([A_n])\end{aligned}$$

which shows that $[A_n] \in \boldsymbol{D}$. Thus $\boldsymbol{A}_{\mathcal{U}} \subseteq \boldsymbol{D}$, and it is clear that $\boldsymbol{D}$ is a monotone class so $\boldsymbol{B}_{\mathcal{U}} \subseteq \boldsymbol{D}$, which shows (II). Suppose now that $[z_n] \mapsto \mu_{[z_n]}$ satisfies (I) and (II). Then for each $[B_n] \in \boldsymbol{A}_{\mathcal{U}}(Z_{\mathcal{U}})$ and $D \in \boldsymbol{B}_{\mathcal{U}}(Y_{\mathcal{U}})$ we have $\int_{[B_n]} \mu_{[z_n]}(D)\, d\eta_{\mathcal{U}} = \nu_{\mathcal{U}}(D \cap \pi_{\mathcal{U}}^{-1}([B_n])) = \int_{[B_n]} \nu_{[z_n]}(D)\, d\eta_{\mathcal{U}}$ so that $\mu_{[z_n]}(D) = \nu_{[z_n]}(D)$ almost surely, so that (III) holds.

For the last statement let $B_n \subseteq Z_n$ be an invariant η_n-conull set on which $(\gamma^{b_n})_*\nu_z^n = \nu_{\gamma^{d_n}z}^n$ for all $\gamma \in \Gamma$. Then for all $[z_n]$ in the $\eta_{\mathcal{U}}$-conull set $[B_n] \subseteq Z_{\mathcal{U}}$ we have for all $\gamma \in \Gamma$ and $[A_n] \in \boldsymbol{A}_{\mathcal{U}}(Y_{\mathcal{U}})$ that $(\gamma^{d_{\mathcal{U}}})_*\nu_{[z_n]}(A_n) = \lim_{n\to\mathcal{U}}(\gamma^{d_n})_*\nu_{z_n}^n(A_n) = \lim_{n\to\mathcal{U}} \nu_{\gamma^{d_n}z}^n(A_n) = \nu_{\gamma^{d_{\mathcal{U}}}[z_n]}([A_n])$ so that $(\gamma^{d_{\mathcal{U}}})_*\nu_{[z_n]} = \nu_{\gamma^{d_{\mathcal{U}}}[z_n]}$. $\square$

The next proposition describes the ultrapower of a standard probability space with atoms.

PROPOSITION 8.2. *Let (Z, η) be a standard probability space and let $A \subseteq Z$ be the set of atoms of (Z, η).*

(1) *If (Z, η) is discrete then $(\mathrm{MALG}_\eta, d_\eta)$ is a compact metric space homeomorphic to 2^A with the product topology, and the map $I_{\mathcal{U}} : \mathrm{MALG}_{\eta_{\mathcal{U}}} \to \mathrm{MALG}_\eta$ given by $I_{\mathcal{U}}([B_n]) = \lim_{n\to\mathcal{U}} B_n = \{z \in A : \{n : z \in B_n\} \in \mathcal{U}\}$ is a measure algebra isomorphism.*

(2) *In general $[A] = \{[z] : z \in A\} \subseteq Z_{\mathcal{U}}$ is the set of all atoms of $\eta_{\mathcal{U}}$ and the restriction $\eta|A$ of η to A is isomorphic as a measure space to the restriction $\eta_{\mathcal{U}}|[A]$ of $\eta_{\mathcal{U}}$ to $[A]$ via the map $z \mapsto [z]$. Under this isomorphism, letting $C = Z \setminus A$, we may identify $(Z_{\mathcal{U}}, \eta_{\mathcal{U}})$ with $([C] \sqcup A, (\eta|C)_{\mathcal{U}} + \eta|A)$.*

PROOF. First suppose that (Z, η) is discrete. Without loss of generality we may assume $Z = A$. As sets we may identify MALG_η with 2^A. Let $B_0, B_1, \ldots$ be a sequence in 2^A

converging in the product topology to some set $B \in 2^A$. Given $\epsilon > 0$ let $F \subseteq A$ be a finite set such that $\eta(A \setminus F) < \epsilon$. For all large enough n, B_n and B agree on F, so that $\eta(B_n \Delta B) < \eta(A \setminus F) < \epsilon$ and thus $d_\eta(B_n, B) \to 0$. This shows that the map $2^A \to \mathrm{MALG}_\eta$ is a continuous bijection from the compact Hausdorff space 2^A (with the product topology) to $(\mathrm{MALG}_\eta, d_\eta)$, so it is a homeomorphism. It is clear that the map φ taking $B \subseteq A$ to $[B] \subseteq [A]$ is an isometric embedding of MALG_η to $\mathrm{MALG}_{\eta_\mathcal{U}}$ that preserves all Boolean operations. If now $[B_n] \subseteq [A]$ and $\lim_{n\to\mathcal{U}} B_n = B$ then $d_{\eta_\mathcal{U}}([B_n], [B]) = \lim_{n\to\mathcal{U}} d_\eta(B_n, B) = 0$ so that $[B_n] = [B]$ and thus $\varphi^{-1} = I_\mathcal{U}$ which completes the proof of (1). Part (2) follows since $(Z_\mathcal{U}, \eta_\mathcal{U})$ decomposes as $([C] \sqcup [A], (\eta|C)_\mathcal{U} + (\eta|A)_\mathcal{U})$ and part (1) shows that $([A], (\eta|A)_\mathcal{U}) \cong (A, \eta)$. $\square$

THEOREM 8.3. *Let* $\boldsymbol{a}_0, \boldsymbol{a}_1, \dots$ *be a sequence of measure preserving actions of* Γ *on the standard probability space* (X, μ) *and let* $\boldsymbol{a}_\mathcal{U} = \Gamma \curvearrowright^{a_\mathcal{U}} (X_\mathcal{U}, \mu_\mathcal{U})$ *be their ultraproduct. Let* $\boldsymbol{M}_0 \subseteq \mathrm{MALG}_{\mu_\mathcal{U}}$ *be any subset such that* $(\boldsymbol{M}_0, d_{\mu_\mathcal{U}}|\boldsymbol{M}_0)$ *is separable. Then there exists an invariant measure sub-algebra* $\boldsymbol{M}$ *of* $\mathrm{MALG}_{\mu_\mathcal{U}}$ *containing* $\boldsymbol{M}_0$ *that is isomorphic as a measure algebra to* MALG_μ.

PROOF. Let $A \subseteq X$ be the collection of atoms of X and let $C = X \setminus A$. By Proposition 8.2.(2), $[A] \subseteq X$ is the discrete part of $\mu_\mathcal{U}$ and $x \mapsto [x]$ is an isomorphism $\mu|A \cong \mu_\mathcal{U}|[A]$. Define a function $S_\mathcal{U} : \mathrm{MALG}_{\mu_\mathcal{U}} \to \mathrm{MALG}_{\mu_\mathcal{U}}$ first on subsets $D \subseteq [C]$ by taking $S_\mathcal{U}(D)$ to be any subset of D satisfying $\mu_\mathcal{U}(S_\mathcal{U}(D)) = \frac{1}{2}\mu_\mathcal{U}(D)$, and then extending this to all of $\mathrm{MALG}_{\mu_\mathcal{U}}$ by taking $S_\mathcal{U}(D) = S_\mathcal{U}(D \cap [C]) \sqcup (D \cap [A])$. Fix a countable dense subset $\boldsymbol{M}_1$ of $\boldsymbol{M}_0$ and let $\boldsymbol{B}_0 \subseteq \mathrm{MALG}_{\mu_\mathcal{U}}$ be a countable Boolean algebra containing $\boldsymbol{M}_1 \cup \{\{[x]\} : x \in A\}$ and closed under the functions $S_\mathcal{U}$ and $\gamma^{a_\mathcal{U}}$ for all $\gamma \in \Gamma$. Then the σ-algebra $\boldsymbol{M} = \sigma(\boldsymbol{B}_0)$ equipped with $\mu_\mathcal{U}$ is an invariant countably generated measure sub-algebra of $\mathrm{MALG}_{\mu_\mathcal{U}}$ containing $\boldsymbol{M}_0$. Since $\boldsymbol{B}_0$ is closed under $S_\mathcal{U}$, the atoms of $\boldsymbol{B}_0$, and hence also those of $\boldsymbol{M}$, must be contained in $[A]$, and as $\boldsymbol{M}$ contains $\{[B] : B \subseteq A\}$, the discrete part of $\boldsymbol{M}$ is isomorphic to the discrete part of MALG_μ. It follows that $\boldsymbol{M} \cong \mathrm{MALG}_\mu$. $\square$

PROPOSITION 8.4. *Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ and $\boldsymbol{b} = \Gamma \curvearrowright^b (Y,\nu)$ be measure preserving actions of Γ. If $\boldsymbol{a}$ is weakly contained in $\boldsymbol{b}$ then then the measure space (X,μ) is a quotient of the measure space (Y,ν). If $\boldsymbol{a}$ and $\boldsymbol{b}$ are weakly equivalent then (X,μ) is isomorphic to (Y,ν). In particular, the identity actions $\boldsymbol{\iota}_{\eta_1}$ and $\boldsymbol{\iota}_{\eta_2}$ are weakly equivalent if and only if (Z_1,η_1) and (Z_2,η_2) are isomorphic measure spaces.*

PROOF. Suppose first that $\boldsymbol{a} \prec \boldsymbol{b}$. Let $\phi : X \to K = 2^{\mathbb{N}}$ be any Borel isomorphism and let $\lambda = (\Phi^{\phi,a})_*\mu$. Then $\boldsymbol{a} \cong \Gamma \curvearrowright^s (K^\Gamma,\lambda)$ and as $\boldsymbol{a} \prec \boldsymbol{b}$ there exists $\lambda_n = (\Phi^{\phi_n,b})_*\nu \in E(\boldsymbol{b},K)$ with $\lambda_n \to \lambda$. By Proposition 3.10 $\Gamma \curvearrowright^s (K^\Gamma,\lambda)$ is a factor of the ultrapower $\boldsymbol{b}_{\mathcal{U}}$ of $\boldsymbol{b}$ via $\Phi^{\phi,b_{\mathcal{U}}}$ where ϕ is the ultralimit of the ϕ_n. Thus $\boldsymbol{a}$ is also a factor of $\boldsymbol{b}_{\mathcal{U}}$ so by Theorem 8.3 this implies (X,μ) is a factor of (Y,ν).

Now suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are weakly equivalent. Then the measure spaces (X,μ) and (Y,ν) are factors of each other, say $\pi : (Y,\nu) \to (X,\mu)$ and $\varphi : (X,\mu) \to (Y,\nu)$. Let $A \subseteq X$ be the set of atoms of X and let $B \subseteq Y$ be the set of atoms of Y. If $\mu(A) = 0$ then we are done since this implies both (X,μ) and (Y,ν) are non-atomic. So suppose that $\mu(A) > 0$. It is clear that $A \subseteq \varphi^{-1}(B)$ and $B \subseteq \pi^{-1}(A)$, hence $\mu(A) = \nu(B)$. Additionally, $\mu(\varphi^{-1}(B) \setminus A) = 0$, otherwise $\nu(B) = \mu(\varphi^{-1}(B)) > \mu(A)$. Similarly $\nu(\pi^{-1}(A) \setminus B) = 0$. Thus $\varphi^{-1} : (\mathrm{MALG}_{\nu_B}, d_{\nu_B}) \to (\mathrm{MALG}_{\mu_A}, d_{\mu_A})$ and $\pi^{-1} : (\mathrm{MALG}_{\mu_A}, d_{\mu_A}) \to (\mathrm{MALG}_{\nu_B}, d_{\nu_B})$ are isometric embeddings of compact metric spaces (Proposition 8.2), so it follows that both π^{-1} and φ^{-1} are in fact isometric isomorphisms. Since these maps are also Boolean algebra homomorphisms it follows that both are measure algebra isomorphisms. This shows that the discrete parts of (X,μ) and (Y,ν) are isomorphic, from which it follows that (X,μ) and (Y,ν) are isomorphic. □

9. Stable weak containment

In this appendix we establish some basic properties of stable weak containment of measure preserving actions. Our development mirrors our development of weak containment of measure preserving actions.

DEFINITION 9.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of measure preserving actions of Γ. We say that $\mathcal{A}$ is *stably weakly contained in* $\mathcal{B}$, written $\mathcal{A} \prec_s \mathcal{B}$ if for every $\Gamma \curvearrowright^a (X,\mu) = \boldsymbol{a} \in \mathcal{A}$, for any Borel partition $A_0, \dots, A_{k-1}$ of X, $F \subseteq \Gamma$ finite, and $\epsilon > 0$, there exist nonnegative reals $\alpha_0, \dots, \alpha_{m-1}$ with $\sum_{i<m} \alpha_i = 1$ along with actions $\Gamma \curvearrowright^{b_i} (Y_i, \nu_i) = \boldsymbol{b}_i \in \mathcal{B}$, $i < m$, and a Borel partition $B_0, \dots, B_{k-1}$ of $\sum_{i<m} Y_i$ such that

$$|\mu(\gamma^a A_i \cap A_j) - (\textstyle\sum_{i<m} \alpha_i \tilde{\nu}_i)(\gamma^{\sum_{i<m} b_i} B_i \cap B_j)| < \epsilon$$

for all $i, j < k$ and $\gamma \in F$. (See §3.2 for notation.)

The relation $\prec_s$ is a reflexive and transitive relation on sets of measure preserving actions. We call $\mathcal{A}$ and $\mathcal{B}$ *stably weakly equivalent*, written $\mathcal{A} \sim_s \mathcal{B}$, if both $\mathcal{A} \prec_s \mathcal{B}$ and $\mathcal{B} \prec_s \mathcal{A}$. We write $\boldsymbol{a} \prec_s \mathcal{B}$, $\mathcal{A} \prec_s \boldsymbol{b}$, and $\boldsymbol{a} \prec_s \boldsymbol{b}$ for $\{\boldsymbol{a}\} \prec_s \mathcal{B}$, $\mathcal{A} \prec_s \{\boldsymbol{b}\}$ and $\{\boldsymbol{a}\} \prec_s \{\boldsymbol{b}\}$, respectively, and similarly with $\sim_s$ in place of $\prec_s$.

It is clear that $\boldsymbol{a} \prec_s \boldsymbol{b}$ if and only if $\boldsymbol{a} \prec \{\boldsymbol{\iota}_{\eta_{\boldsymbol{\alpha}}} \times \boldsymbol{b} : \boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{m-1}) \in [0,1]^m,\ \sum_{i<m} \alpha_i = 1,\ m \in \mathbb{N}\}$, so by Lemma 3.7 we have $\boldsymbol{a} \prec_s \boldsymbol{b}$ if and only if $\boldsymbol{a} \prec \boldsymbol{\iota} \times \boldsymbol{b}$ if and only if $\boldsymbol{\iota} \times \boldsymbol{a} \prec \boldsymbol{\iota} \times \boldsymbol{b}$. From this point of view Theorem 1.2 says that if $\boldsymbol{a}$ is ergodic then $\boldsymbol{a} \prec_s \boldsymbol{b}$ if and only if $\boldsymbol{a} \prec \boldsymbol{b}$. Theorem 1.1 implies that $\boldsymbol{a} \prec_s \boldsymbol{b}$ if and only if $E(\boldsymbol{a}, K) \subseteq \overline{\mathrm{co}} E(\boldsymbol{b}, K)$ for every compact Polish space K, and $\boldsymbol{a} \sim_s \boldsymbol{b}$ if and only if $\overline{\mathrm{co}} E(\boldsymbol{a}, K) = \overline{\mathrm{co}} E(\boldsymbol{b}, K)$ for every compact Polish space K. More generally, we have the following analogue of Proposition 3.5 which can be proved directly by using the same methods.

PROPOSITION 9.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be sets of measure preserving actions of Γ. Then the following are equivalent*

(1) $\mathcal{A} \prec_s \mathcal{B}$*;*
(2) $\bigcup_{\boldsymbol{d} \in \mathcal{A}} E(\boldsymbol{d}, K) \subseteq \overline{\mathrm{co}}(\bigcup_{\boldsymbol{b} \in \mathcal{B}} E(\boldsymbol{b}, K))$ *for every finite K;*
(3) $\bigcup_{\boldsymbol{d} \in \mathcal{A}} E(\boldsymbol{d}, K) \subseteq \overline{\mathrm{co}}(\bigcup_{\boldsymbol{b} \in \mathcal{B}} E(\boldsymbol{b}, K))$ *for every compact Polish K;*
(4) $\bigcup_{\boldsymbol{d} \in \mathcal{A}} E(\boldsymbol{d}, 2^{\mathbb{N}}) \subseteq \overline{\mathrm{co}}(\bigcup_{\boldsymbol{b} \in \mathcal{B}} E(\boldsymbol{b}, 2^{\mathbb{N}}))$.

Chapter 5

Mixing actions of countable groups are almost free

Robin D. Tucker-Drob

A measure preserving action of a countably infinite group Γ is called totally ergodic if every infinite subgroup of Γ acts ergodically. For example, all mixing and mildly mixing actions are totally ergodic. This note shows that if an action of Γ is totally ergodic then there exists a finite normal subgroup N of Γ such that the stabilizer of almost every point is equal to N. Surprisingly the proof relies on the group theoretic fact (proved by Hall and Kulatilaka as well as by Kargapolov) that every infinite locally finite group contains an infinite abelian subgroup, of which all known proofs rely on the Feit-Thompson theorem.

As a consequence we deduce a group theoretic characterization of countable groups whose non-trivial Bernoulli factors are all free: these are precisely the groups that possess no finite normal subgroup other than the trivial subgroup.

1. Introduction

Let Γ be a countably infinite discrete group and let $\boldsymbol{a}$ be a measure preserving action of Γ, i.e., $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ where X is a standard Borel space, μ is a Borel probability measure on X, and $a : \Gamma \times X \to X$ is a Borel action of Γ on X that preserves μ. In this note we examine how ergodicity and mixing properties of $\boldsymbol{a}$ can influence, and be influenced by, the freeness behavior of $\boldsymbol{a}$ and its factors. When $\boldsymbol{a}$ is not ergodic, for example, the ergodic decomposition of $\boldsymbol{a}$ directly exhibits a *non-trivial* action (i.e., with underlying measure not a point mass) that is a factor of $\boldsymbol{a}$ which is non-free.

More generally, if Γ contains some non-trivial normal subgroup N for which the restriction $\boldsymbol{a} \upharpoonright N$ of $\boldsymbol{a}$ to N is non-ergodic, then the action of Γ on the set Z of ergodic components of $\boldsymbol{a} \upharpoonright N$ corresponds to a non-trivial factor of $\boldsymbol{a}$ which is manifestly non-free. Indeed, this factor is not even faithful as N fixes all points in Z.

Working from the other direction, if $\pi : (X, \mu) \to (Y, \nu)$ factors $\boldsymbol{a}$ onto some non-trivial action $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ which is not faithful, then for any $B \subseteq Y$ with $0 < \nu(B) < 1$ the set $\pi^{-1}(B)$ will be a non-trivial subset of X witnessing that the kernel of $\boldsymbol{b}$ (i.e., the set of group elements fixing almost every point) does not act ergodically under the action $\boldsymbol{a}$. These observations are rephrased in the following proposition.

PROPOSITION 1.1. *The following are equivalent for a measure preserving action $\boldsymbol{a}$ of Γ:*

(1) *All non-trivial factors of $\boldsymbol{a}$ are faithful.*
(2) *All non-trivial normal subgroups of Γ act ergodically.*

Note that when Γ contains a finite normal subgroup N then no non-trivial action $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ of Γ can have the property (2) (and therefore (1)) of Proposition 1.1: if $\boldsymbol{a} \upharpoonright N$ is ergodic then X is finite, so the kernel of $\boldsymbol{a}$ is non-trivial and does not act ergodically. However, the observations preceding Proposition 1.1 also show the following:

PROPOSITION 1.2. *The following are equivalent for a measure preserving action $\boldsymbol{a}$ of Γ:*

(1) *All non-trivial factors of $\boldsymbol{a}$ have finite kernel.*
(2) *All infinite normal subgroups of Γ act ergodically.*

Propositions 1.1 and 1.2 express the equivalence of a freeness property on the one hand, and an ergodicity property on the other. By strengthening the ergodicity assumption on $\boldsymbol{a}$ it is shown below that an appropriately strong freeness results.

DEFINITION 1.3. A measure preserving action $\boldsymbol{a}$ of Γ is called *totally ergodic* if the restriction of $\boldsymbol{a}$ to every infinite subgroup of Γ is ergodic.

There are many examples of totally ergodic actions. All mildly mixing actions are totally ergodic, since the restriction of a mildly mixing action to an infinite subgroup is again mildly mixing and hence ergodic. In particular, all mixing actions are totally ergodic. The following theorem says that totally ergodic actions are, up to a finite kernel, always free.

THEOREM 1.4. *Let $\boldsymbol{a} = \Gamma \curvearrowright^{a} (X, \mu)$ be a non-trivial measure preserving action of the countably infinite group Γ. Suppose that $\boldsymbol{a}$ is totally ergodic. Then there exists a finite normal subgroup N of Γ such that the stabilizer of μ-almost every $x \in X$ is equal to N.*

COROLLARY 1.5. *All faithful totally ergodic actions of countably infinite groups are free. In particular, all faithful mildly mixing and all faithful mixing actions of countably infinite groups are free.*

A totally ergodic action of particular importance is the *Bernoulli shift* of Γ. This is the measure preserving action $\boldsymbol{s}_\Gamma$ of Γ on $([0,1]^\Gamma, \lambda^\Gamma)$ (where λ is Lebesgue measure) given by

$$(\gamma^{s_\Gamma} f)(\delta) = f(\gamma^{-1}\delta)$$

for $\gamma, \delta \in \Gamma$ and $f \in [0,1]^\Gamma$. By a *Bernoulli factor* of Γ we mean a factor of $\boldsymbol{s}_\Gamma$. One consequence of Theorem 1.4 is a particularly nice group theoretic characterization of groups all of whose non-trivial Bernoulli factors are free.

COROLLARY 1.6. *Let Γ be an infinite countable group. Then the following are equivalent*

(1) *Every non-trivial totally ergodic action of Γ is free.*

(2) *Every non-trivial mixing action of Γ is free.*

(3) *Every non-trivial Bernoulli factor of Γ is free.*

(4) *There exists a non-trivial measure preserving action* $\boldsymbol{a}$ *of* Γ *such that every non-trivial factor of* $\boldsymbol{a}$ *is free.*

(5) *There exists a non-trivial measure preserving action* $\boldsymbol{a}$ *of* Γ *such that every non-trivial factor of* $\boldsymbol{a}$ *is faithful.*

(6) Γ *contains no non-trivial finite normal subgroup.*

PROOF OF COROLLARY 1.6 FROM THEOREM 1.4. (6)⇒(1) follows immediately from Theorem 1.4. The implication (1)⇒(2) is clear. (2)⇒(3) holds since $\boldsymbol{s}_\Gamma$ is mixing and every factor of a mixing action is mixing. (3)⇒(4) and (4)⇒(5) are also clear. (5)⇒(6) follows from the discussion following Proposition 1.1 above. □

COROLLARY 1.7. *Let* Γ *be any infinite countable group that is either torsion free or ICC. Then every non-trivial totally ergodic action of* Γ *is free and in particular every non-trivial Bernoulli factor of* Γ *is free.*

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2. Definitions and notation

Γ will always denote a countably infinite discrete group and e will denote the identity element of Γ.

Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ be a measure preserving action of Γ. The *stabilizer* of a point $x \in X$ is the subgroup Γ_x of Γ given by $\Gamma_x = \{\gamma \in \Gamma : \gamma^a x = x\}$. For a subset $C \subseteq \Gamma$ we let

$$\mathrm{Fix}^a(C) = \{x \in X : \forall \gamma \in C \ \gamma^a x = x\}.$$

We write $\mathrm{Fix}^a(\gamma)$ for $\mathrm{Fix}^a(\{\gamma\})$. The *kernel* of $\boldsymbol{a}$ is the set $\ker(\boldsymbol{a}) = \{\gamma \in \Gamma : \mu(\mathrm{Fix}^a(\gamma)) = 1\}$. It is clear that $\ker(\boldsymbol{a})$ is a normal subgroup of Γ. The action $\boldsymbol{a}$ is called *(essentially) free* if the stabilizer of μ-almost every point is trivial, or equivalently, $\mu(\mathrm{Fix}^a(\gamma)) = 0$ for each $\gamma \in \Gamma \setminus \{e\}$. It is called *faithful* if $\ker(\boldsymbol{a}) = \{e\}$, i.e., $\mu(\mathrm{Fix}^a(\gamma)) < 1$ for each $\gamma \in \Gamma \setminus \{e\}$.

Let $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ be another measure preserving action of Γ. We say that $\boldsymbol{b}$ is a *factor* of $\boldsymbol{a}$ (or that $\boldsymbol{a}$ *factors onto* $\boldsymbol{b}$) if there exists a measurable map $\pi : X \to Y$ with $\pi_* \mu = \nu$ and such that for each $\gamma \in \Gamma$ the equality $\pi(\gamma^a x) = \gamma^b \pi(x)$ holds for μ-almost every $x \in X$. A measure preserving action $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ is called *trivial* if ν is a point mass. Otherwise, $\boldsymbol{b}$ is called *non-trivial*.

3. Proof of Theorem 1.4

PROOF OF THEOREM 1.4. We begin with a lemma also observed by Darren Creutz and Jesse Peterson [**CP12**].

LEMMA 3.1. *Let $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ be a non-trivial totally ergodic action of Γ.*

(i) *Suppose that $C \subseteq \Gamma$ is a subset of Γ such that $\nu(\{y \in Y \,:\, C \subseteq \Gamma_y\}) > 0$. Then the subgroup $\langle C \rangle$ generated by C is finite.*

(ii) *Γ_y is almost surely locally finite.*

PROOF OF LEMMA 3.1. Beginning with (i), the hypothesis tells us that the set $\mathrm{Fix}^b(C)$ is non-null. Since ν is not a point mass there is some $B \subseteq \mathrm{Fix}^b(C)$ with $0 < \nu(B) < 1$. The set B witnesses that $\boldsymbol{b} \restriction \langle C \rangle$ is not ergodic. As $\boldsymbol{b}$ is totally ergodic we conclude that $\langle C \rangle$ is finite.

For (ii), let $\mathcal{F}$ denote the collection of finite subsets F of Γ such that $\langle F \rangle$ is infinite and let NLF $\subseteq Y$ denote the set of points $y \in Y$ such that Γ_y is not locally finite. Then

$$\mathrm{NLF} = \bigcup_{F \in \mathcal{F}} \{y \in Y \,:\, F \subseteq \Gamma_y\}$$

By part (i), $\nu(\{y \in Y \,:\, F \subseteq \Gamma_y\}) = 0$ for each $F \in \mathcal{F}$. Since $\mathcal{F}$ is countable it follows that $\nu(\mathrm{NLF}) = 0$. □[Lemma]

Now let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ be a totally ergodic action as in the statement of Theorem 1.4. Let $N = \{\gamma \in \Gamma \,:\, \mu(\mathrm{Fix}^a(\gamma)) = 1\}$ denote the kernel of $\boldsymbol{a}$. Then N is a normal subgroup of Γ that is finite by Lemma 3.1.(i). Ignoring a null set, the action $\boldsymbol{a}$ descends to an action $\boldsymbol{b} = \Delta \curvearrowright^b (X, \mu)$ of the quotient group $\Delta = \Gamma/N$ that is still totally ergodic, and which is moreover faithful. Thus, after replacing Γ by Γ/N and $\boldsymbol{a}$ by $\boldsymbol{b}$ if necessary, we may assume that $\boldsymbol{a}$ is faithful toward the goal of showing that $\boldsymbol{a}$ is free.

For each $\gamma \in \Gamma$ let $C_\Gamma(\gamma)$ denote the centralizer of γ in Γ. Observe that $\mathrm{Fix}^a(\gamma)$ is an invariant set for $\boldsymbol{a} \restriction C_\Gamma(\gamma)$, for if $\delta \in C_\Gamma(\gamma)$ then $\delta^a \cdot \mathrm{Fix}^a(\gamma) = \mathrm{Fix}^a(\delta\gamma\delta^{-1}) = \mathrm{Fix}^a(\gamma)$. Thus for $\gamma \neq e$, if $C_\Gamma(\gamma)$ is infinite then $\boldsymbol{a} \restriction C_\Gamma(\gamma)$ is ergodic and the $\boldsymbol{a} \restriction C_\Gamma(\gamma)$-invariant set $\mathrm{Fix}^a(\gamma)$ must therefore be null since $\boldsymbol{a}$ is faithful. Letting C_∞ denote the collection of elements of $\Gamma \setminus \{e\}$ whose centralizers are infinite, this simply means that $\mu(\{x \in X \,:\, \gamma \in \Gamma_x\}) = 0$ for all $\gamma \in C_\infty$, and so

$$\mu(\{x \in X \,:\, \Gamma_x \cap C_\infty \neq \varnothing\}) = 0. \tag{3.1}$$

By Lemma 3.1.(ii), Γ_x is almost surely locally finite. By a theorem of Hall and Kulatilaka [**HK64**] and Kargapolov [**Kar63**], every infinite locally finite group contains an infinite abelian subgroup. In particular, each infinite locally finite subgroup of Γ intersects C_∞. It follows from this and (3.1) that Γ_x is almost surely finite.

Since there are only countably many finite subgroups of Γ there must be some finite subgroup $H_0 \leq \Gamma$ such that $\mu(A_0) > 0$ where

$$A_0 = \{x \in X \,:\, \Gamma_x = H_0\}.$$

Let N_0 denote the normalizer of H_0 in Γ. If T is a transversal for the left cosets of N_0 in Γ then $\{t^a A_0\}_{t \in T}$ are pairwise disjoint non-null subsets of X all of the same measure. It follows that T is finite and therefore N_0 is infinite and $\boldsymbol{a} \restriction N_0$ ergodic. The set A_0 is non-null and invariant for $\boldsymbol{a} \restriction N_0$, so $\mu(A_0) = 1$, i.e., $\Gamma_x = H_0$ almost surely. As $\boldsymbol{a}$ is faithful we conclude that $H_0 = \{e\}$ and that $\boldsymbol{a}$ is therefore free. $\square$

4. An example

In general, total ergodicity does not imply weak mixing, and weak mixing does not imply total ergodicity. For example, the action of $\mathbb{Z}$ corresponding to an irrational rotation of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ equipped with Haar measure is totally ergodic, but not weakly mixing. There are also many examples of weakly mixing measure preserving actions that lack total ergodicity. One such action is exhibited in 4.1 below. Example 4.1 also illustrates that total ergodicity of a measure preserving action is *not* necessary to ensure that each non-trivial factor of that action is free.

EXAMPLE 4.1. Here is an example of a free weakly mixing action $\boldsymbol{s}$ that is not totally ergodic, but that still has the property that every non-trivial factor of $\boldsymbol{s}$ is free: Let F denote the free group of rank 2 with free generating set $\{u, v\}$ and let $H = \langle u \rangle$ be the cyclic subgroup of F generated by u. The generalized Bernoulli shift action $\boldsymbol{s} = \boldsymbol{s}_{F,F/H} = F \curvearrowright^{s} ([0,1]^{F/H}, \lambda^{F/H})$ is weakly mixing (see [**KT08**]) but not totally ergodic since H fixes each set in the σ-algebra generated by the projection function $f \mapsto f(H)$. Given a subgroup $K \leq F$, if $\boldsymbol{s} \restriction K \cong \boldsymbol{s}_{K,F/H}$ is not ergodic then $K \curvearrowright F/H$ has a finite orbit (see [**KT08**]), say $K\gamma H$ is finite where $\gamma \in F$. Then for any $z \in K$ there is some $n > 0$ such that $z^n \in \gamma H \gamma^{-1}$, and therefore $z \in \gamma H \gamma^{-1}$. This shows that $K \subseteq \gamma H \gamma^{-1}$ so that K is cyclic. The restriction of $\boldsymbol{s}$ to each non-cyclic subgroup of F is therefore ergodic, so if $\boldsymbol{a} = F \curvearrowright^{a} (X, \mu)$ is any factor of $\boldsymbol{s}$ then $\boldsymbol{a}$ also has this property and, assuming $\boldsymbol{a}$ is non-trivial, an argument as in the proof of Lemma 3.1 shows that the stabilizer F_x of μ-almost every $x \in X$ is locally cyclic, hence cyclic. Arguing as in the last paragraph of the proof of Theorem 1.4 we see that there is some normal cyclic subgroup H_0 of F such that $F_x = H_0$ almost surely. The only possibility is that $H_0 = \{e\}$, and thus $\boldsymbol{a}$ is free.

5. A question

The proof of Theorem 1.4 relies on Hall, Kulatilaka, and Kargapolov's result, whose only known proofs make use of the Feit-Thompson theorem from finite group theory.

QUESTION 5.1. Is there a direct ergodic theoretic proof of Theorem 1.4?

Chapter 6

Shift-minimal groups, fixed Price 1, and the unique trace property

Robin Tucker-Drob

A countable group Γ is called shift-minimal if every non-trivial measure preserving action of Γ weakly contained in the Bernoulli shift $\Gamma \curvearrowright ([0,1]^\Gamma, \lambda^\Gamma)$ is free. We show that any group Γ whose reduced C^*-algebra $C^*_r(\Gamma)$ admits a unique tracial state is shift-minimal, and that any group Γ admitting a free measure preserving action of cost>1 contains a finite normal subgroup N such that Γ/N is shift-minimal. Any shift-minimal group in turn is shown to have trivial amenable radical. Recurrence arguments are used in studying invariant random subgroups of a wide variety of shift-minimal groups. We also examine continuity properties of cost in the context of infinitely generated groups and equivalence relations. A number of open questions are discussed which concern cost, shift-minimality, C^*-simplicity, and uniqueness of tracial state on $C^*_r(\Gamma)$.

1. Introduction

The *Bernoulli shift* of a countable discrete group Γ, denoted by $\boldsymbol{s}_\Gamma$, is the measure preserving action $\Gamma \curvearrowright^s ([0,1]^\Gamma, \lambda^\Gamma)$ (where λ denotes Lebesgue measure on $[0,1]$) of Γ given by

$$(\gamma^s \cdot f)(\delta) = f(\gamma^{-1}\delta)$$

for $\gamma, \delta \in \Gamma$ and $f \in [0,1]^\Gamma$. If Γ is infinite, then the Bernoulli shift may be seen as the archetypal free measure preserving action of Γ. This point of view is supported by Abért and Weiss's result [**AW11**] that s_Γ is weakly contained in every free measure preserving action of Γ. Conversely, it is well known that any measure preserving action weakly containing a free action must itself be free. A measure preserving action is therefore free if and only if it exhibits approximate Bernoulli behavior.

Inverting our point of view, the approximation properties exhibited by s_Γ itself have been shown to reflect the group theoretic nature of Γ. One example of this is Schmidt's characterization [**Sch81**] of amenable groups as exactly those groups Γ for which s_Γ admits a non-trivial sequence of almost invariant sets. An equivalent formulation in the language of weak containment is that Γ is amenable if and only if s_Γ weakly contains an action that is not ergodic. In addition, a direct consequence of Foreman and Weiss's work [**FW04**] is that amenability of Γ is equivalent to *every* measure preserving action of Γ being weakly contained in s_Γ. That each of these properties of s_Γ is necessary for amenability of Γ is essentially a consequence of the Ornstein-Weiss Theorem [**OW80**], while sufficiency of these properties may be reduced to the corresponding representation theoretic characterizations of amenability due to Hulanicki and Reiter (see [**Hul64, Hul66**], [**Zim84**, 7.1.8], [**BHV08**, Appendix G.3]): a group Γ is amenable if and only if its left regular representation λ_Γ weakly contains the trivial representation if and only if λ_Γ weakly contains every unitary representation of Γ.

This paper investigates further the extent to which properties of a group may be detected by its Bernoulli action. Roughly speaking, it is observed that even when a group is non-amenable, the manifestation (or lack thereof) of certain behaviors in the Bernoulli shift has implications for the extent of that group's non-amenability. Central to this investigation is the following definition.

DEFINITION 1.1. A countable group Γ is called *shift-minimal* if every non-trivial measure preserving action weakly contained in s_Γ is free.

The reader is referred to [**Kec10**] for background on weak containment of measure preserving actions. Note that by definition the trivial group $\{e\}$ is shift-minimal.

Shift-minimality, as with the above-mentioned ergodic theoretic characterizations of amenability, takes its precedent in the theory of unitary representations of Γ. It is well known that Γ is

C^**-simple* (i.e., its reduced C^*-algebra $C_r^*(\Gamma)$ is simple) if and only if every non-zero unitary representation of Γ weakly contained in the left regular representation λ_Γ is actually weakly equivalent to λ_Γ [**dlH07**]. Using the Abért-Weiss characterization of freeness it is apparent that Γ is shift-minimal if and only if every non-trivial m.p. action of Γ weakly contained in $\boldsymbol{s}_\Gamma$ is in fact weakly equivalent to $\boldsymbol{s}_\Gamma$. Apart from analogy, the relationship between shift-minimality and C^*-simplicity in general is unclear. However, we show in Theorem 5.15 that shift-minimality follows from a property closely related to C^*-simplicity. A group Γ is said to have the *unique trace property* if there is a unique tracial state on $C_r^*(\Gamma)$.

THEOREM 1.2. *Let* Γ *be a countable group. If* Γ *has the unique trace property then* Γ *is shift-minimal.*

In addition, a co-induction argument (Proposition 3.15) shows that shift-minimal groups have no non-trivial normal amenable subgroups, i.e., they have trivial amenable radical. This places shift-minimality squarely between two other properties whose general equivalence with C^*-simplicity remains an open problem. Indeed, it is open whether there are any general implications between C^*-simplicity and the unique trace property; in all concrete examples these two properties coincide. Furthermore, while the amenable radical of any C^*-simple group is known to be trivial [**PS79**], it is an open question - asked explicitly by Bekka and de la Harpe [**BdlH00**] - whether conversely, a group which is not C^*-simple always contains a non-trivial normal amenable subgroup. For shift-minimality in place of C^*-simplicity, a stochastic version of this question is shown to have a positive answer (Theorem 3.16).

THEOREM 1.3. *A countable group* Γ *is shift-minimal if and only if there is no non-trivial amenable invariant random subgroup of* Γ *weakly contained in* $\boldsymbol{s}_\Gamma$.

Here an *invariant random subgroup* (IRS) of Γ is a Borel probability measure on the compact space Sub_Γ of subgroups of Γ that is invariant under the conjugation action $\Gamma \curvearrowright \mathrm{Sub}_\Gamma$ of Γ. It is called *amenable* if it concentrates on the amenable subgroups of Γ. Invariant random subgroups generalize the notion of normal subgroups: if N is a normal subgroup of Γ then the Dirac measure δ_N on Sub_Γ is conjugation invariant. It is shown in [**AGV12**] that the invariant random subgroups of Γ are precisely those measures on Sub_Γ that arise as the stabilizer distribution of some measure preserving action of Γ (see §2.3).

Theorem 1.3 is not entirely satisfactory since it still seems possible that shift-minimality of Γ is equivalent to Γ having no non-trivial amenable invariant random subgroups whatsoever (see Question 7.4). In fact, the proof of Theorem 1.2 in §5.4 shows that this possibly stronger property is a consequence of the unique trace property.

THEOREM 1.4. *Let Γ be a countable group. If Γ has the unique trace property then Γ has no non-trivial amenable invariant random subgroups.*

The known general implications among all of the notions introduced thus far are expressed in Figure 1.

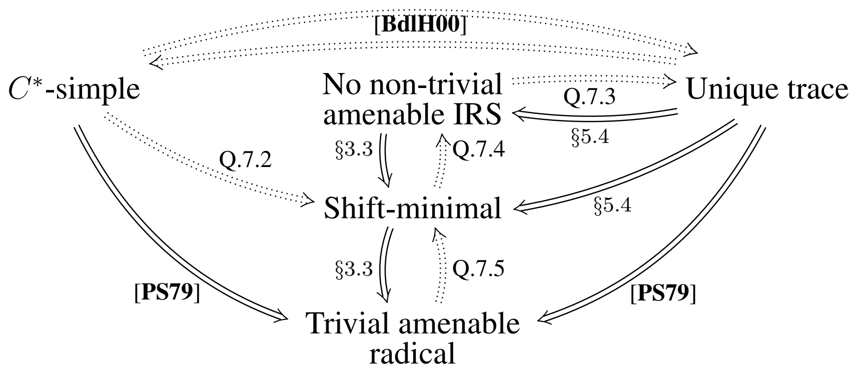

FIGURE 1. The solid lines indicate known implications and the dotted lines indicate open questions discussed in §7. Any implication which is not addressed by the diagram is open in general. However, these properties all coincide for large classes of groups, e.g., linear groups (see §5.3).

Our starting point in studying shift-minimality is the observation that if $\Gamma \curvearrowright^a (X, \mu)$ is a m.p. action that is weakly contained in $\boldsymbol{s}_\Gamma$ then every non-amenable subgroup of Γ acts ergodically. We call this property of a m.p. action *NA-ergodicity*. We show in Theorem 3.13 that when a m.p. action of Γ is NA-ergodic then the stabilizer of almost every point must be amenable.

§4 deals with permanence properties of shift-minimality by examining situations in which freeness of a m.p. action $\Gamma \curvearrowright^a (X, \mu)$ may be deduced from freeness of some acting subgroup. Many of the proofs in this section appeal to some form of the Poincaré Recurrence Theorem.

A wide variety of groups are known to have the unique trace property and Theorem 1.2 shows that all such groups are shift-minimal. Among these are all non-abelian free groups ([**Pow75**]), all Powers groups and weak Powers groups ([**dLH85**], [**BN88**]), groups with property $\mathrm{P_{nai}}$ [**BCdLH94**], all ICC relatively hyperbolic groups ([**AM07**]), and all ICC groups with a minimal non-elementary convergence group action [**MOY11**]. In §5 we observe that all of these groups share a common paradoxicality property we call (BP), abstracted from M. Brin and G. Picioroaga's proof that all weak Powers groups contain a free group (see [**dlH07**, following Question 15]). It is shown in Theorem 5.6 that any non-trivial ergodic invariant random subgroup of a group with property (BP) must contain a non-abelian free group almost surely. Recurrence once again plays a key role in the proof.

§6 studies the relationship between cost, weak containment, and invariant random subgroups. Kechris shows in [**Kec10**, Corollary 10.14] that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are free measure preserving actions of a finitely generated group Γ then $\boldsymbol{a} \prec \boldsymbol{b}$ implies $C(\boldsymbol{b}) \leq C(\boldsymbol{a})$ where $C(\boldsymbol{a})$ denotes the cost of $\boldsymbol{a}$ (i.e., the cost of the orbit equivalence relation generated by $\boldsymbol{a}$). This is deduced from the stronger fact [**Kec10**, Theorem 10.13] that the cost function $C : \mathrm{FR}(\Gamma, X, \mu) \to \mathbb{R}$, $\boldsymbol{a} \mapsto C(\boldsymbol{a})$, is upper semi-continuous for finitely generated Γ. In §6.2 we obtain a generalization which holds for arbitrary countable groups (Theorem 6.4 below). The consequences of this generalization are most naturally stated in terms of an invariant we call *pseudocost*.

If E is a m.p. countable Borel equivalence relation on (X, μ) then the pseudocost of E is defined as $PC_\mu(E) := \inf_{(E_n)} \liminf_n C_\mu(E_n)$, where $(E_n)_{n\in\mathbb{N}}$ ranges over all increasing sequences $E_0 \subseteq E_1 \subseteq \cdots$, of Borel subequivalence relations of E such that $\bigcup_n E_n = E$. The pseudocost of an action and of a group is then defined in analogy with cost (see Definition 6.6). It is immediate that $PC_\mu(E) \leq C_\mu(E)$, and while the pseudocost and cost coincide in most cases, including whenever E is treeable or whenever $C_\mu(E) < \infty$ (Corollary 6.8), it is unclear whether equality holds in general.

One of the main motivations for introducing pseudocost is the following useful continuity property (Corollary 6.20):

THEOREM 1.5. *Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ and $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ be measure preserving actions of a countable group Γ. Assume that $\boldsymbol{a}$ is free. If $\boldsymbol{a} \prec \boldsymbol{b}$ then $PC(\boldsymbol{b}) \leq PC(\boldsymbol{a})$.*

Combining Theorem 1.5 and [**AW11**, Theorem 1] it follows that, among all free m.p. actions of Γ, the Bernoulli shift $\boldsymbol{s}_\Gamma$ has the maximum pseudocost. Since pseudocost and cost coincide for m.p. actions of finitely generated groups, this generalizes the result of [**AW11**] that $\boldsymbol{s}_\Gamma$ has the greatest cost among free actions of a finitely generated group Γ. In Corollary 6.22 we use Theorem 1.5 to deduce general consequences for cost:

THEOREM 1.6. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be m.p. actions of a countably infinite group Γ. Assume that $\boldsymbol{a}$ is free and $\boldsymbol{a} \prec \boldsymbol{b}$.*

(1) *If $C(\boldsymbol{b}) < \infty$ then $C(\boldsymbol{b}) \leq C(\boldsymbol{a})$.*
(2) *If E_b is treeable then $C(\boldsymbol{b}) \leq C(\boldsymbol{a})$.*
(3) *If $C(\boldsymbol{a}) = 1$ then $C(\boldsymbol{b}) = 1$.*

This leads to a characterization of countable groups with fixed price 1 as exactly those groups whose Bernoulli shift has cost 1. This characterization was previously shown for finitely generated groups in [**AW11**].

THEOREM 1.7. *Let Γ be a countable group. The following are equivalent:*

(1) *Γ has fixed price 1*
(2) *$C(\boldsymbol{s}_\Gamma) = 1$*
(3) *$C(\boldsymbol{a}) = 1$ for some m.p. action $\boldsymbol{a}$ weakly equivalent to $\boldsymbol{s}_\Gamma$.*
(4) *$PC(\boldsymbol{a}) = 1$ for some m.p. action $\boldsymbol{a}$ weakly equivalent to $\boldsymbol{s}_\Gamma$.*
(5) *Γ is infinite and $C(\boldsymbol{a}) \leq 1$ for some non-trivial m.p. action $\boldsymbol{a}$ weakly contained in $\boldsymbol{s}_\Gamma$.*

We use this characterization to obtain a new class of shift-minimal groups in §6.5. In what follows, AR_Γ denotes the amenable radical of Γ (see Appendix 9). Gaboriau [**Gab00**, Theorem 3] showed that if Γ does not have fixed price 1 then AR_Γ is finite. We now have:

THEOREM 1.8. *Let Γ be a countable group that does not have fixed price 1. Then AR_Γ is finite and $\Gamma/\mathrm{AR}_\Gamma$ is shift-minimal.*

In Theorem 6.31 of §6.4 it is shown that if the hypothesis of Theorem 1.8 is strengthened to $C(\Gamma) > 1$, i.e., if *all* free m.p. actions of Γ have cost > 1, then the conclusion may be strengthened considerably. The following is an analogue of Bergeron and Gaboriau's result [**BG04**, §5] (see also [**ST10**, Corollary 1.6]) in which the statement is shown to hold for the first ℓ^2-Betti number in place of cost.

THEOREM 1.9. *Suppose that $C(\Gamma) > 1$. Let $\Gamma \curvearrowright^a (X, \mu)$ be an ergodic measure preserving action of Γ on a non-atomic probability space. Then exactly one of the following holds:*

(1) *Almost all stabilizers are finite;*

(2) *Almost every stabilizer has infinite cost, i.e.,* $C(\Gamma_x) = \infty$ *almost surely.*

In particular, AR_Γ *is finite and* $\Gamma/\mathrm{AR}_\Gamma$ *has no non-trivial amenable invariant random subgroups.*

The analysis of pseudocost in §6.2 is used in §6.3 to study the cost of generic actions in the Polish space $A(\Gamma, X, \mu)$ of measure preserving actions of Γ. For any group Γ there is a comeager subset of $A(\Gamma, X, \mu)$, consisting of free actions, on which the cost function $C : A(\Gamma, X, \mu) \to \mathbb{R} \cup \{\infty\}$ takes a constant value $C_{\text{gen}}(\Gamma) \in \mathbb{R}$ [**Kec10**]. Likewise, the pseudocost function $\boldsymbol{a} \mapsto PC(\boldsymbol{a})$ must be constant on a comeager set of free actions, and we denote this constant value by $PC_{\text{gen}}(\Gamma)$. Kechris shows in [**Kec10**] that $C_{\text{gen}}(\Gamma) = C(\Gamma)$ for finitely generated Γ and Problem 10.11 of [**Kec10**] asks whether $C_{\text{gen}}(\Gamma) = C(\Gamma)$ in general. The following is proved in Corollaries 6.28 and 6.27.

THEOREM 1.10. *Let* Γ *be a countably infinite group. Then*

(1) *The set* $\{\boldsymbol{a} \in A(\Gamma, X, \mu) \,:\, \boldsymbol{a} \textit{ is free and } PC(\boldsymbol{a}) = PC(\Gamma)\}$ *is dense* G_δ *in* $A(\Gamma, X, \mu)$.

(2) $PC_{\mathit{gen}}(\Gamma) = PC(\Gamma)$.

(3) *Either* $C_{\mathit{gen}}(\Gamma) = C(\Gamma)$ *or* $C_{\mathit{gen}}(\Gamma) = \infty$.

(4) *If* $PC(\Gamma) = 1$ *then* $C_{\mathit{gen}}(\Gamma) = C(\Gamma) = 1$.

(5) *The set*

$$\Big\{\boldsymbol{b} \in A(\Gamma, X, \mu) \ : \boldsymbol{b} \textit{ is free and } \exists \textit{aperiodic Borel subequivalence relations}$$
$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \textit{ of } E_b, \textit{ with } E_b = \bigcup_n E_n \textit{ and } \lim_n C_\mu(E_n) = C(\Gamma)\Big\}$$

is dense G_δ *in* $A(\Gamma, X, \mu)$.

(6) *If all free actions of* Γ *have finite cost then* $\{\boldsymbol{b} \in A(\Gamma, X, \mu) \,:\, \boldsymbol{b} \textit{ is free and } C(\boldsymbol{b}) = C(\Gamma)\}$ *is dense* G_δ *in* $A(\Gamma, X, \mu)$.

The only possible exception to the equality $C_{\text{gen}}(\Gamma) = C(\Gamma)$ would be a group Γ with $C(\Gamma) < \infty$ such that the set $\{\boldsymbol{a} \in A(\Gamma, X, \mu) \,:\, \boldsymbol{a} \text{ is free, } C(\boldsymbol{a}) = \infty \text{ and } E_a \text{ is not treeable}\}$ comeager in $A(\Gamma, X, \mu)$.

A number of questions are discussed in §7. The paper ends with two appendices, the first clarifying the relationship between invariant random subgroups and subequivalence relations. The second contains relevant results about the amenable radical of a countable group.

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2. Preliminaries

Throughout, Γ denotes a countable discrete group. The identity element of Γ is denoted by e_Γ, or simply e when Γ is clear from the context. All countable groups are assumed to be equipped with the discrete topology; a *countable group* always refers to a countable discrete group.

2.1. Group theory. Subgroups. Let Δ and Γ be countable groups. We write $\Delta \leq \Gamma$ to denote that Δ is a subgroup of Γ and we write $\Delta \triangleleft \Gamma$ to denote that Δ is a normal subgroup of Γ. The index of a subgroup $\Delta \leq \Gamma$ is denoted by $[\Gamma : \Delta]$. The *trivial* subgroup of Γ is the subgroup $\{e_\Gamma\}$ that contains only the identity element. For a subset $S \subseteq \Gamma$ we let $\langle S \rangle$ denote the subgroup generated by S. A group that is not finitely generated will be called *infinitely generated*.

Centralizers and normalizers. Let S be any subset of Γ and let H be a subgroup of Γ. The *centralizer* of S in H is the set

$$C_H(S) = \{h \in H \,:\, \forall s \in S\ (hsh^{-1} = s)\}$$

and the *normalizer* of S in H is the set

$$N_H(S) = \{h \in H \,:\, hSh^{-1} = S\}.$$

Then $C_H(S)$ and $N_H(S)$ are subgroups of H with $C_H(S) \triangleleft N_H(S)$. Clearly $C_H(S) = C_\Gamma(S) \cap H$ and $N_H(S) = N_\Gamma(S) \cap H$. The group $C_\Gamma(\Gamma)$ is called the *center* of Γ and is denoted by $Z(\Gamma)$. We say that a subset T of Γ *normalizes* S if $T \subseteq N_\Gamma(S)$. We call a subgroup H *self-normalizing* in Γ if $H = N_\Gamma(H)$.

Infinite conjugacy class (ICC) groups. A group Γ is called *ICC* if every $\gamma \in \Gamma \setminus \{e\}$ has an infinite conjugacy class. This is equivalent to $C_\Gamma(\gamma)$ having infinite index in Γ for all $\gamma \neq e$. Thus, according to our definition, the trivial group $\{e\}$ is ICC.

The Amenable Radical. We let AR_Γ denote the *amenable radical* of Γ. See Appendix 9 below.

2.2. Ergodic theory. Measure preserving actions. A *measure preserving (m.p.) action* of Γ is a triple $(\Gamma, a, (X,\mu))$, which we write as $\Gamma \curvearrowright^a (X,\mu)$, where (X,μ) is a standard probability space (possibly with atoms) and $a : \Gamma \times X \to X$ is a Borel action of Γ on X that preserves the probability measure μ. For $(\gamma, x) \in \Gamma \times X$ we let $\gamma^a x$ denote the image $a(\gamma, x)$ of the pair (γ, x) under a. We write $\boldsymbol{a}$ for $\Gamma \curvearrowright^a (X,\mu)$ when Γ and (X,μ) are clear from the context. A measure preserving action of Γ will also be called a Γ-*system* or simply a *system* when Γ is understood.

For the rest of this subsection let $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ and let $\boldsymbol{b} = \Gamma \curvearrowright^b (Y,\nu)$.

Isomorphism and factors. If $\varphi : (X,\mu) \to Y$ is a measurable map then we let $\varphi_*\mu$ denote the pushforward measure on Y given by $\varphi_*\mu(B) = \mu(\varphi^{-1}(B))$ for $B \subseteq Y$ Borel. We say that $\boldsymbol{b}$ is a *factor* of $\boldsymbol{a}$ (or that $\boldsymbol{a}$ *factors onto* $\boldsymbol{b}$), written $\boldsymbol{b} \sqsubseteq \boldsymbol{a}$, if there exists a measurable map $\pi : X \to Y$ with $\pi_*\mu = \nu$ and such that for each $\gamma \in \Gamma$ the equality $\pi(\gamma^a x) = \gamma^b \pi(x)$ holds for μ-almost every $x \in X$. Such a map π is called a *factor map* from $\boldsymbol{a}$ to $\boldsymbol{b}$. The factor map π is called an *isomorphism* from $\boldsymbol{a}$ to $\boldsymbol{b}$ if there exists a co-null subset of X on which π is injective. We say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are *isomorphic*, written $\boldsymbol{a} \cong \boldsymbol{b}$, if there exists some isomorphism from $\boldsymbol{a}$ to $\boldsymbol{b}$.

Weak containment of m.p. actions. We write $\boldsymbol{a} \prec \boldsymbol{b}$ to denote that $\boldsymbol{a}$ is weakly contained in $\boldsymbol{b}$ and we write $\boldsymbol{a} \sim \boldsymbol{b}$ to denote that $\boldsymbol{a}$ and $\boldsymbol{b}$ are weakly equivalent. The reader is referred to [**Kec10**] for background on weak containment of measure preserving actions.

Product of actions. The *product* of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the m.p. action $\boldsymbol{a} \times \boldsymbol{b} = \Gamma \curvearrowright^{a\times b} (X \times Y, \mu \times \nu)$ where $\gamma^{a\times b}(x,y) = (\gamma^a x, \gamma^b y)$ for each $\gamma \in \Gamma$ and $(x,y) \in X \times Y$.

Bernoulli shifts. Let $\Gamma \times T \to T$, $(\gamma, t) \mapsto \gamma \cdot t$ be an action of Γ on a countable set T. The *generalized Bernoulli shift* corresponding to this action is the system $\boldsymbol{s}_{\Gamma,T} = \Gamma \curvearrowright^s ([0,1]^T, \lambda^T)$, where λ is Lebesgue measure and where the action s is given by $(\gamma^s f)(t) = f(\gamma^{-1}\cdot t)$ for $\gamma \in \Gamma$, $f \in [0,1]^T$, $t \in T$. We write $\boldsymbol{s}_\Gamma$ for $\boldsymbol{s}_{\Gamma,\Gamma}$ when the action of Γ on itself is by left translation. The system $\boldsymbol{s}_\Gamma$ is called the *Bernoulli shift* of Γ.

The trivial system. We call $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ *trivial* if μ is a point mass. Otherwise, $\boldsymbol{a}$ is called *non-trivial*. Up to isomorphism, each group Γ has a unique trivial measure preserving action, which we denote by $\boldsymbol{i}_\Gamma$ or simply $\boldsymbol{i}$ when Γ is clear.

Identity systems. Let $\boldsymbol{\iota}_{\Gamma,\mu} = \Gamma \curvearrowright^{\iota} (X,\mu)$ denote the *identity system* of Γ on (X,μ) given by $\gamma^{\iota} = \mathrm{id}_X$ for all $\gamma \in \Gamma$. We write $\boldsymbol{\iota}_\mu$ when Γ is clear. Thus if μ is a point mass then $\boldsymbol{\iota}_\mu \cong \boldsymbol{i}$.

Strong ergodicity. A system $\boldsymbol{a}$ is called *strongly ergodic* if it is ergodic and does not weakly contain the identity system $\boldsymbol{\iota}_{\Gamma,\lambda}$ on $([0,1],\lambda)$.

Fixed point sets and free actions. For a subset $C \subseteq \Gamma$ we let

$$\mathrm{Fix}^b(C) = \{y \in Y \,:\, \forall \gamma \in C \ \gamma^b y = y\}.$$

We write $\mathrm{Fix}^b(\gamma)$ for $\mathrm{Fix}^b(\{\gamma\})$. The *kernel* of the system $\boldsymbol{b}$ is the set $\ker(\boldsymbol{b}) = \{\gamma \in \Gamma \,:\, \nu(\mathrm{Fix}^b(\gamma)) = 1\}$. It is clear that $\ker(\boldsymbol{b})$ is a normal subgroup of Γ. The system $\boldsymbol{b}$ is called *faithful* if $\ker(\boldsymbol{b}) = \{e\}$, i.e., $\nu(\mathrm{Fix}^b(\gamma)) < 1$ for each $\gamma \in \Gamma \setminus \{e\}$. The system $\boldsymbol{b}$ is called *(essentially) free* if the stabilizer of ν-almost every point is trivial, i.e., $\nu(\mathrm{Fix}^b(\gamma)) = 0$ for each $\gamma \in \Gamma \setminus \{e\}$.

2.3. Invariant random subgroups. The space of subgroups. We let $\mathrm{Sub}_\Gamma \subseteq 2^\Gamma$ denote the compact space of all subgroup of Γ and we let $c : \Gamma \times \mathrm{Sub}_\Gamma \to \mathrm{Sub}_\Gamma$ be the continuous action of Γ on Sub_Γ by conjugation.

Invariant random subgroups. An *invariant random subgroup* (IRS) of Γ is a conjugation-invariant Borel probability measures on Sub_Γ. The point mass δ_N at a normal subgroup N of Γ is an example of an invariant random subgroup. Let IRS_Γ denote the space of invariant random subgroups of Γ. We associate to each $\theta \in \mathrm{IRS}_\Gamma$ the measure preserving action $\Gamma \curvearrowright^c (\mathrm{Sub}_\Gamma, \theta)$. We also denote this system by $\boldsymbol{\theta}$.

Stabilizer distributions. Each measure preserving action $\boldsymbol{b} = \Gamma \curvearrowright^b (Y,\nu)$ of Γ gives rise to and invariant random subgroup $\theta_{\boldsymbol{b}}$ of Γ as follows. The *stabilizer* of a point $y \in Y$ under the action b is the subgroup Γ_y of Γ defined by

$$\Gamma_y = \{\gamma \in \Gamma \,:\, \gamma^b y = y\}.$$

The group Γ_y of course depends on the action b. The *stabilizer map* associated to b is the map $\mathrm{stab}_b : Y \to \mathrm{Sub}_\Gamma$ given by $\mathrm{stab}_b(y) = \Gamma_y$. The *stabilizer distribution* of $\boldsymbol{b}$, which we denote by $\theta_{\boldsymbol{b}}$ or $\mathrm{type}(\boldsymbol{b})$, is the measure $(\mathrm{stab}_b)_*\nu$ on Sub_Γ. It is clear that $\theta_{\boldsymbol{b}}$ is an invariant random subgroup of Γ. In [**AGV12**] it is shown that for any invariant random subgroup θ of Γ, there exists a m.p. action

$\boldsymbol{b}$ of Γ such that $\theta_{\boldsymbol{b}} = \theta$. Moreover, if $\boldsymbol{\theta}$ is ergodic then $\boldsymbol{b}$ can be taken to be ergodic as well. See [**CP12**].

Group theoretic properties of invariant random subgroups. Let θ be an invariant random subgroup of Γ. We say that a given property $\mathcal{P}$ of subgroups of Γ holds for θ if $\mathcal{P}$ holds almost everywhere. For example, θ is called *amenable* (or *infinite index*) if θ concentrates on the amenable (respectively, infinite index) subgroups of Γ.

The trivial IRS. By the *trivial* invariant random subgroup we mean the point mass at the trivial subgroup $\{e\}$ of Γ. We write δ_e instead of $\delta_{\{e\}}$ for the trivial invariant random subgroup. An invariant random subgroup not equal to δ_e is called *non-trivial*.

REMARK 2.1. We will often abuse terminology and confuse an invariant random subgroup θ with the measure preserving action $\boldsymbol{\theta} = \Gamma \curvearrowright^c (\mathrm{Sub}_\Gamma, \theta)$ it defines, stating, for example, that θ is ergodic or is weakly contained in $\boldsymbol{s}_\Gamma$ to mean that $\boldsymbol{\theta}$ is ergodic or is weakly contained in $\boldsymbol{s}_\Gamma$. When there is a potential for ambiguity we will make sure to distinguish between an invariant random subgroup and the measure preserving system to which it gives rise. We emphasize that "θ is non-trivial" will always mean that θ is not equal to the trivial IRS δ_e, whereas "$\boldsymbol{\theta}$ is non-trivial" will always mean that θ is not a point mass (at *any* subgroup).

3. Shift-minimality

3.1. Seven characterizations of shift-minimality. It will be useful to record here the main theorem of [**AW11**] which was already mentioned several times in the introduction.

THEOREM 3.1 ([**AW11**]). *Let Γ be a countably infinite group. Then the Bernoulli shift $\boldsymbol{s}_\Gamma$ is weakly contained in every free measure preserving action of* Γ.

We let $\mathrm{Aut}(X, \mu)$ denote the Polish group of measure preserving transformations of (X, μ), and we let $A(\Gamma, X, \mu)$ denote the Polish space of measure preserving actions of Γ on the measure space (X, μ). See [**Kec10**] for information on these two spaces. In the following proposition, let $[\boldsymbol{a}]$ denote the weak equivalence class of a measure preserving action $\boldsymbol{a}$ of Γ. Denote by $\boldsymbol{s}_{\Gamma,2}$ the full 2-shift of Γ, i.e., the shift action of Γ on $(2^\Gamma, \rho^\Gamma)$ where we identify 2 with $\{0, 1\}$ and where $\rho(\{0\}) = \rho(\{1\}) = 1/2$.

PROPOSITION 3.2. *Let Γ be a countable group and let (X,μ) be a standard non-atomic probability space. Then the following are equivalent.*

(1) *Γ is shift-minimal, i.e., every non-trivial m.p. action weakly contained in $\boldsymbol{s}_\Gamma$ is free.*

(2) *Every non-trivial m.p. action weakly contained in $\boldsymbol{s}_{\Gamma,2}$ is free.*

(3) *Among non-trivial m.p. actions of Γ, $[\boldsymbol{s}_{\Gamma,2}]$ is minimal with respect to weak containment.*

(4) *Either $\Gamma = \{e\}$ or, among non-trivial m.p. actions of Γ, $[\boldsymbol{s}_\Gamma]$ is minimal with respect to weak containment.*

(5) *Among non-atomic m.p. actions of Γ, $[\boldsymbol{s}_\Gamma]$ is minimal with respect to weak containment.*

(6) *The conjugation action of the Polish group $\mathrm{Aut}(X,\mu)$ on the Polish space $A_{\boldsymbol{s}}(\Gamma,X,\mu) = \{\boldsymbol{a}\in A(\Gamma,X,\mu) \,:\, \boldsymbol{a}\prec\boldsymbol{s}_\Gamma\}$ is minimal, i.e., every orbit is dense.*

(7) *For some (equivalently: every) non-principal ultrafilter $\mathcal{U}$ on the the natural numbers $\mathbb{N}$, every non-trivial factor of the ultrapower $(\boldsymbol{s}_\Gamma)_\mathcal{U}$ is free.*

PROOF. The equivalence (7)⇔(1) follows from [**CKTD11**, Theorem 1]. For the remaining equivalences, first note that if Γ is a finite group then $\boldsymbol{s}_\Gamma$ factors onto $\boldsymbol{\iota}_\mu$, so if $\Gamma\neq\{e\}$ then Γ does not satisfy (1), (4), (5) or (6). In addition, for $\Gamma\neq\{e\}$ finite, $\boldsymbol{s}_{\Gamma,2}$ factors onto a non-trivial identity system, which shows that Γ does not satisfy (2) or (3) either. This shows that the trivial group $\Gamma=\{e\}$ is the only finite group that satisfies any of the properties (1)-(6), and it is clear the trivial group satisfies all of these properties. We may therefore assume for the rest of the proof that Γ is infinite.

(1)⇒(2): This implication is clear since $\boldsymbol{s}_{\Gamma,2}$ is a factor of $\boldsymbol{s}_\Gamma$.

(2)⇒(3): Suppose that (2) holds. By hypothesis any $\boldsymbol{a}$ weakly contained in $\boldsymbol{s}_{\Gamma,2}$ is free and thus weakly contains $\boldsymbol{s}_\Gamma$ by Theorem 3.1. (3) follows since $\boldsymbol{s}_{\Gamma 2}$ is a factor of $\boldsymbol{s}_\Gamma$.

(3)⇒(4): Since we are assuming Γ is infinite, Theorem 3.1 implies $[\boldsymbol{s}_\Gamma] = [\boldsymbol{s}_{\Gamma,2}]$, and this implication follows. (4)⇒(5) is clear.

(5)⇒(6): Suppose (5) holds. By [**Kec10**, Proposition 10.1] the $\mathrm{Aut}(X,\mu)$-orbit closure of any $\boldsymbol{a}\in A(\Gamma,X,\mu)$ is equal to $\{\boldsymbol{b}\in A(\Gamma,X,\mu) \,:\, \boldsymbol{b}\prec\boldsymbol{a}\}$. Thus, if $\boldsymbol{a}$ is weakly equivalent to $\boldsymbol{s}_\Gamma$, then the orbit of $\boldsymbol{a}$ is dense in $A_{\boldsymbol{s}}(\Gamma,X,\mu)$. Since $[\boldsymbol{s}_\Gamma]$ is minimal with respect to weak containment, every element of $A_{\boldsymbol{s}}(\Gamma,X,\mu)$ is weakly equivalent to $\boldsymbol{s}_\Gamma$, so has dense orbit in $A_{\boldsymbol{s}}(\Gamma,X,\mu)$. Thus, the action $\mathrm{Aut}(X,\mu)\curvearrowright A_{\boldsymbol{s}}(\Gamma,X,\mu)$ is minimal.

(6)⇒(1): Suppose that every $\boldsymbol{a} \in A_{\boldsymbol{s}}(\Gamma, X, \mu)$ has dense orbit. If $\boldsymbol{\iota}_\mu \in A_{\boldsymbol{s}}(\Gamma, X, \mu)$ then, since $\boldsymbol{\iota}_\mu$ is a fixed point for the $\mathrm{Aut}(X, \mu)$ action, $\boldsymbol{\iota}_\mu = \boldsymbol{s}_\Gamma$ and thus $\Gamma = \{e\}$. Otherwise, if $\boldsymbol{\iota}_\mu \not\prec \boldsymbol{s}_\Gamma$ then the system $\boldsymbol{s}_\Gamma$ is strongly ergodic and the group Γ is therefore non-amenable. Let $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ be any non-trivial m.p. action of Γ weakly contained in $\boldsymbol{s}_\Gamma$. Then $\boldsymbol{b} \times \boldsymbol{b}$ is weakly contained in $\boldsymbol{s}_\Gamma \times \boldsymbol{s}_\Gamma \cong \boldsymbol{s}_\Gamma$ and therefore $\boldsymbol{b} \times \boldsymbol{b}$ is strongly ergodic since strong ergodicity is downward closed under weak containment (see, e.g., [**CKTD11**, Proposition 5.6]). In particular $\boldsymbol{b} \times \boldsymbol{b}$ is ergodic and it follows that the probability space (Y, ν) is non-atomic. The action $\boldsymbol{b}$ is then isomorphic to some action $\boldsymbol{a}$ on the non-atomic space (X, μ), and $\boldsymbol{a} \in A_{\boldsymbol{s}}(\Gamma, X, \mu)$ since $\boldsymbol{b} \prec \boldsymbol{s}_\Gamma$. By hypothesis $\boldsymbol{a}$ has dense orbit in $A_{\boldsymbol{s}}(\Gamma, X, \mu)$ so that $\boldsymbol{s}_\Gamma \sim \boldsymbol{a}$ by [**Kec10**, Proposition 10.1] and hence $\boldsymbol{a}$ is free, and thus $\boldsymbol{b}$ is free as well. □

Two more characterizations of shift-minimality are given in terms of amenable invariant random subgroups in Theorem 3.16 below.

3.2. NA-ergodicity.

DEFINITION 3.3. Let $\boldsymbol{a}$ be a measure preserving action of a countable group Γ. We say that $\boldsymbol{a}$ is *NA-ergodic* if the restriction of $\boldsymbol{a}$ to every non-amenable subgroup of Γ is ergodic. We say that $\boldsymbol{a}$ is *strongly NA-ergodic* if the restriction of $\boldsymbol{a}$ to every non-amenable subgroup of Γ is strongly ergodic.

EXAMPLE 3.4. The central example of an NA-ergodic (and in fact, strongly NA-ergodic) action is the Bernoulli shift action $\boldsymbol{s}_\Gamma$; if $H \leq \Gamma$ is non-amenable then $\boldsymbol{s}_\Gamma | H \cong \boldsymbol{s}_H$ is strongly ergodic. More generally, if Γ acts on a countable set T and the stabilizer of every $t \in T$ is amenable then the generalized Bernoulli shift $\boldsymbol{s}_T = \Gamma \curvearrowright^{s_T} ([0,1]^T, \lambda^T)$ is strongly NA-ergodic (see, e.g., [**KT08**]).

EXAMPLE 3.5. The action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright (\mathbb{T}^2, \lambda^2)$ by matrix multiplication, where λ^2 is Haar measure on $\mathbb{T}^2$, is another example of a strongly NA-ergodic action. A proof of this is given in [**Kec07**, 5.(B)].

EXAMPLE 3.6. I would like to thank L. Bowen for bringing my attention to this example. Let Γ be a countable group and let f be an element of the integral group ring $\mathbb{Z}\Gamma$. The left translation action of Γ on the discrete abelian group $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ is by automorphisms, and this induces an action of Γ by automorphisms on the dual group $\widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}$, which is a compact metrizable abelian group so that this action preserves normalized Haar measure μ_f. Bowen has shown that if the function f has an inverse in $\ell^1(\Gamma)$ then the system $\Gamma \curvearrowright (\widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}, \mu_f)$ is weakly contained in $\boldsymbol{s}_\Gamma$ and is therefore strongly NA-ergodic by Proposition 3.10 ([**Bow10a**, §5]; note that the hypothesis that Γ is residually finite is not used in that section so that this holds for arbitrary countable groups Γ).

REMARK 3.7. The actions from Examples 3.4, 3.5, and 3.6 share a common property: they are *tempered* in the sense of [**Kec07**]. A measure preserving action $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ is called *tempered* if the Koopman representation $\kappa_0^{\boldsymbol{a}}$ on $L_0^2(X,\mu) = L^2(X,\mu) \ominus \mathbb{C}1_X$ is weakly contained in the regular representation λ_Γ of Γ. Any tempered action $\boldsymbol{a}$ of a non-amenable group Γ has *stable spectral gap* in the sense of [**Pop08**] (this means $\kappa_0^{\boldsymbol{a}} \otimes \kappa_0^{\boldsymbol{a}}$ does not weakly contain the trivial representation), and this implies in turn that the product action $\boldsymbol{a} \times \boldsymbol{b}$ is strongly ergodic relative to $\boldsymbol{b}$ for every measure preserving action $\boldsymbol{b}$ of Γ (see [**Ioa06**]). In particular (taking $\boldsymbol{b} = \boldsymbol{i}_\Gamma$) a tempered action $\boldsymbol{a}$ of a non-amenable group is itself strongly ergodic. Since the restriction of a tempered action to a subgroup is still tempered it follows that every tempered action is strongly NA-ergodic. In [**Kec07**] it is shown that the converse holds for any action on a compact Polish group G by automorphisms (such an action necessarily preserves Haar measure μ_G):

THEOREM 3.8 (Theorem 4.6 of [**Kec07**]). *Let Γ be a countably infinite group acting by automorphisms on a compact Polish group G. Let $\hat{G}$ denote the (countable) set of all isomorphism classes of irreducible unitary representations of G and let $\hat{G}_0 = \hat{G} \setminus \{\hat{1}_G\}$. Then the following are equivalent:*

(1) *The action $\Gamma \curvearrowright (G, \mu_G)$ is tempered;*
(2) *Every stabilizer of the associated action of Γ on $\hat{G}_0$ is amenable.*
(3) *The action $\Gamma \curvearrowright (G, \mu_G)$ is NA-ergodic.*
(4) *The action $\Gamma \curvearrowright (G, \mu_G)$ is strongly NA-ergodic.*

Condition (2) of Theorem 3.8 should be compared with part (ii) of Lemma 3.11 below, although Lemma 3.11 deals with general NA-ergodic actions. It follows from [**Kec10**, Proposition 10.5] that any measure preserving action weakly contained in $\boldsymbol{s}_\Gamma$ is tempered. I do not know however whether the converse holds, although Example 3.6 and Theorem 3.8 suggest that this may be the case for actions by automorphisms on compact Polish groups.

QUESTION 3.9. Let Γ be a countable group acting by automorphisms on a compact Polish group G and assume the action is tempered. Does it follow that the action is weakly contained in $\boldsymbol{s}_\Gamma$? As a special case, is it true that the action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright (\mathbb{T}^2, \lambda^2)$ is weakly contained in $\boldsymbol{s}_{\mathrm{SL}_2(\mathbb{Z})}$?

We now establish some properties of general NA-ergodic actions.

PROPOSITION 3.10. *Any factor of an NA-ergodic action is NA-ergodic. Any action weakly contained in a strongly NA-ergodic action is strongly NA-ergodic.*

PROOF. The first statement is clear and the second is a consequence of strong ergodicity being downward closed under weak containment (see [**CKTD11**, Proposition 5.6]). $\square$

Part (ii) of the following lemma is one of the key facts about NA-ergodicity.

LEMMA 3.11. *Let $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ be any non-trivial NA-ergodic action of a countable group Γ.*

(i) *Suppose that $C \subseteq \Gamma$ is a subset of Γ such that $\nu(\{y \in Y : C \subseteq \Gamma_y\}) > 0$. Then the subgroup $\langle C \rangle$ generated by C is amenable.*
(ii) *The stabilizer Γ_y of ν-almost every $y \in Y$ is amenable.*

PROOF. We begin with part (i). The hypothesis tells us that $\nu(\mathrm{Fix}^b(C)) > 0$. Since ν is not a point mass there is some $B \subseteq \mathrm{Fix}^b(C)$ with $0 < \nu(B) < 1$. Then B witnesses that $\boldsymbol{b} \restriction \langle C \rangle$ is not ergodic, so $\langle C \rangle$ is amenable by NA-ergodicity of $\boldsymbol{b}$.

For (ii), let $\mathcal{F}$ denote the collection of finite subsets F of Γ such that $\langle F \rangle$ is non-amenable and let $\mathrm{NA} = \{y \in Y : \Gamma_y \text{ is non-amenable}\}$. Then

$$\mathrm{NA} = \bigcup_{F \in \mathcal{F}} \{y \in Y : F \subseteq \Gamma_y\}.$$

By part (i), $\nu(\{y \in Y : F \subseteq \Gamma_y\}) = 0$ for each $F \in \mathcal{F}$. Since $\mathcal{F}$ is countable it follows that $\nu(\mathrm{NA}) = 0$. $\square$[Lemma]

The function $N : \mathrm{Sub}_\Gamma \to \mathrm{Sub}_\Gamma$ sending a subgroup $H \leq \Gamma$ to its normalizer $N(H)$ in Γ is equivariant for the conjugation action $\Gamma \curvearrowright^c \mathrm{Sub}_\Gamma$. In [**Ver12**, §2.4] Vershik examines the following transfinite iterations of this function.

DEFINITION 3.12. Define $N^\alpha : \mathrm{Sub}_\Gamma \to \mathrm{Sub}_\Gamma$ by transfinite induction on ordinals α as follows.

$$\begin{aligned} N^0(H) &= H, \\ N^{\alpha+1}(H) &= N(N^\alpha(H)) \text{ is the normalizer of } N^\alpha(H) \\ N^\lambda(H) &= \bigcup_{\alpha<\lambda} N^\alpha(H) \text{ when } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Each N^α is equivariant with respect to conjugation. For each H the sets $H, N(H), \dots, N^\alpha(H), \dots$ form an increasing ordinal-indexed sequence of subsets of Γ. The least ordinal α_H such that $N^{\alpha_H+1}(H) = N^{\alpha_H}(H)$ is therefore countable. If $\theta \in \mathrm{IRS}_\Gamma$ then we let $\theta^\alpha = (N^\alpha)_*\theta$ for each countable ordinal $\alpha < \omega_1$. The net $\{\theta^\alpha\}_{\alpha<\omega_1}$ is increasing in the sense of [**CP12**, §3.5] (see also the paragraphs preceding Theorem 8.15 below), so by [**CP12**, Theorem 3.12] there is a weak*-limit θ^∞ such that $\theta^\alpha \le \theta^\infty$ for all α. Since IRS_Γ is a second-countable topological space there is a countable ordinal α such that $\theta^\beta = \theta^\infty$ for all $\beta \ge \alpha$. Thus $N_*\theta^\infty = \theta^\infty$, and it follows from [**Ver12**, Proposition 4] that θ^∞ concentrates on the self-normalizing subgroups of Γ.

THEOREM 3.13. *Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \nu)$ be a non-trivial measure preserving action of the countable group Γ. Suppose that $\boldsymbol{a}$ is NA-ergodic. Then the stabilizer Γ_x of μ-almost every $x \in X$ is amenable. In addition, at least one of the following is true:*

(1) *There exists a normal amenable subgroup $N \triangleleft \Gamma$ such that the stabilizer of μ-almost every $x \in X$ is contained in N.*

(2) *$\theta_{\boldsymbol{a}}^\infty$ is a non-atomic, self-normalizing, infinitely generated amenable invariant random subgroup, where $\theta_{\boldsymbol{a}}$ denotes the stabilizer distribution of $\boldsymbol{a}$.*

PROOF. Let $\theta = \theta_{\boldsymbol{a}}$. It is enough to show that either (1) or (2) is true. We may assume that Γ is non-amenable. There are two cases to consider.

Case 1: There is some ordinal α such that the measure θ^α has an atom. Let α_0 be the least such ordinal. Then $\boldsymbol{\theta}^{\alpha_0}$ is NA-ergodic, being a factor of $\boldsymbol{a}$, and thus the restriction of $\boldsymbol{\theta}^{\alpha_0}$ to every finite index subgroup of Γ is ergodic since Γ is non-amenable. Thus, θ^{α_0} having an atom implies that it is a point mass, so let $N \le \Gamma$ be such that $\theta^{\alpha_0} = \delta_N$. Then N is a normal subgroup of Γ and we show that N is amenable so that alternative (1) holds in this case. By definition of α_0, $\boldsymbol{a}$ and each $\boldsymbol{\theta}^\alpha$ for $\alpha < \alpha_0$ are non-trivial NA-ergodic actions. Lemma 3.11 then implies that the invariant random subgroups $\mathrm{type}(\boldsymbol{a}) = \theta^0$ and $\mathrm{type}(\boldsymbol{\theta}^\alpha) = \theta^{\alpha+1}$, for $\alpha < \alpha_0$, all concentrate on the amenable subgroups of Γ. If $\alpha_0 = 0$ or if α_0 is a successor ordinal then we see immediately that N is amenable. If α_0 is a limit ordinal then N is an increasing union of amenable groups and so is amenable in this case as well.

Case 2: The other possibility is that θ^∞ has no atoms. Thus $\boldsymbol{\theta}^\infty$ is a non-trivial NA-ergodic action with type$(\boldsymbol{\theta}^\infty) = N_*\theta^\infty = \theta^\infty$. This implies that θ^∞ is amenable by Lemma 3.11. Since θ^∞ is non-atomic and there are only countably many finitely generated subgroups of Γ, θ^∞ must concentrate on the infinitely generated subgroups. This shows that (2) holds. □

3.3. Amenable invariant random subgroups. We record a corollary of Theorem 3.13 which will be used in the proof of our final characterization of shift-minimality.

COROLLARY 3.14. *Any group Γ that is not shift-minimal either has a non-trivial normal amenable subgroup N, or has a non-atomic, self-normalizing, infinitely generated, amenable invariant random subgroup θ such that the action $\boldsymbol{\theta} = \Gamma \curvearrowright^c (\mathrm{Sub}_\Gamma, \theta)$ is weakly contained in $\boldsymbol{s}_\Gamma$.*

PROOF. Let Γ be a group that is not shift-minimal so that there exists some non-trivial $\boldsymbol{a}$ weakly contained in $\boldsymbol{s}_\Gamma$ which is not free. The action $\boldsymbol{a}$ is strongly NA-ergodic by 3.4 and 3.10, so $\boldsymbol{a}$ satisfies the hypotheses of Theorem 3.13. If (1) of Theorem 3.13 holds, say with witnessing normal amenable subgroup $N \leq \Gamma$, then N is non-trivial since $\boldsymbol{a}$ is non-free. If alternative (2) of Theorem 3.13 holds then taking $\theta = \theta^\infty_{\boldsymbol{a}}$ works. □

We also need

PROPOSITION 3.15. *If Γ is shift-minimal then Γ has no non-trivial normal amenable subgroups.*

PROOF. Suppose that Γ has a non-trivial normal amenable subgroup N. Amenability implies that $\boldsymbol{\iota}_N \prec \boldsymbol{s}_N$. Then since co-inducing preserves weak containment we have

$$\boldsymbol{s}_{\Gamma,\Gamma/N} \prec \mathrm{CInd}_N^\Gamma(\boldsymbol{\iota}_N) \prec \mathrm{CInd}_N^\Gamma(\boldsymbol{s}_N) \cong \boldsymbol{s}_\Gamma$$

which shows that $\boldsymbol{s}_{\Gamma,\Gamma/N} \prec \boldsymbol{s}_\Gamma$. The action $\boldsymbol{s}_{\Gamma,\Gamma/N}$ is not free since $N \subseteq \ker(\boldsymbol{s}_{\Gamma,\Gamma/N})$. This shows that Γ is not shift-minimal. □

The following immediately yields Theorem 1.3 from the introduction.

THEOREM 3.16. *The following are equivalent for a countable group Γ:*

(1) Γ *is not shift-minimal.*

(2) *There exists a non-trivial amenable invariant random subgroup θ of Γ that is weakly contained in $\boldsymbol{s}_\Gamma$.*

(3) *Either* AR_Γ *is finite and non-trivial, or there exists an infinite amenable invariant random subgroup θ of Γ that is weakly contained in $\boldsymbol{s}_\Gamma$.*

PROOF. (1)⇒(3): Suppose that Γ is not shift-minimal. If the second alternative of Corollary 3.14 holds then we are done. Otherwise, the first alternative holds and so AR_Γ is non-trivial. If AR_Γ is finite then (3) is immediate, and if AR_Γ is infinite then the point mass at AR_Γ shows that (3) holds.

(3)⇒(2) is clear. Now let θ be as in (2) and we will show that Γ is not shift-minimal. If θ is a point mass, say at $H \in \mathrm{Sub}(\Gamma)$, then H is normal and by hypothesis H is non-trivial and amenable so (1) then follows from Proposition 3.15. If θ is not a point mass then $\Gamma \curvearrowright^c (\mathrm{Sub}_\Gamma, \theta)$ is a non-trivial and non-free measure preserving action of Γ that is weakly contained in $\boldsymbol{s}_\Gamma$. This action then witnesses that Γ is not shift-minimal. □

Any group with no non-trivial normal amenable subgroups is ICC (see [**dlH07**, Appendix J] for a proof), so Proposition 3.15 also shows

PROPOSITION 3.17. *Shift-minimal groups are* ICC.

4. Permanence properties

This section examines various circumstances in which shift-minimality is preserved. §4.1 establishes a lemma which will be used to show that, in many cases, shift-minimality passes to finite index subgroups.

4.1. Invariant random subgroups with trivial intersection. For each invariant random subgroup θ of Δ define the set

$$P_\theta = \{\delta \in \Delta \,:\, \theta(\{H \,:\, \delta \in H\}) > 0\}.$$

We say that two invariant random subgroups θ and ρ *intersect trivially* if $P_\theta \cap P_\rho = \{e\}$. This notion comes from looking at freeness of a product action.

LEMMA 4.1. *If $\boldsymbol{a} = \Delta \curvearrowright^{a} (X, \mu)$ and $\boldsymbol{b} = \Delta \curvearrowright^{b} (Y, \nu)$ are measure preserving actions of Δ then $\boldsymbol{a} \times \boldsymbol{b}$ is free if and only if $\theta_{\boldsymbol{a}}$ and $\theta_{\boldsymbol{b}}$ intersect trivially.*

PROOF. For each $\delta \in \Delta$ we have $\mathrm{Fix}^{a\times b}(\delta) = \mathrm{Fix}^{a}(\delta) \times \mathrm{Fix}^{b}(\delta)$, and this set is $(\mu \times \nu)$-null if and only if either $\mathrm{Fix}^{a}(\delta)$ is μ-null or $\mathrm{Fix}^{b}(\delta)$ is ν-null. The lemma easily follows. $\square$

It is a straightforward group theoretic fact that if L and K are normal subgroups of Δ which intersect trivially then they commute. This generalizes to invariant random subgroups as follows.

LEMMA 4.2. *Let Δ be a countable group. Let $\theta, \rho \in \mathrm{IRS}_{\Delta}$ and suppose that θ and ρ intersect trivially. Suppose L and K are subgroups of Δ satisfying*

$$\theta(\{H \in \mathrm{Sub}_{\Delta} \,:\, L \leq H\}) > \tfrac{1}{m}$$

$$\rho(\{H \in \mathrm{Sub}_{\Delta} \,:\, K \leq H\}) > \tfrac{1}{n}$$

for some $n, m \in \mathbb{N}$. Then there exist commuting subgroups $L_0 \leq L$ and $K_0 \leq K$ with $[L : L_0] < n$ and $[K : K_0] < m$.

PROOF. Define the sets

$$Q_L = \{l \in L \,:\, \langle lKl^{-1} \cup K\rangle \subseteq P_{\rho}\}$$

$$Q_K = \{k \in K \,:\, \langle kLk^{-1} \cup L\rangle \subseteq P_{\theta}\}.$$

If $l \in Q_L$ then for any $k \in K$ we have $lkl^{-1}k^{-1} \in \langle lKl^{-1} \cup K\rangle \subseteq P_{\rho}$. Similarly, if $k \in Q_K$ then for any $l \in L$ we have $lkl^{-1}k^{-1} \in \langle kLk^{-1} \cup L\rangle \subseteq P_{\theta}$. Thus, if $l \in Q_L$ and $k \in Q_K$ then $lkl^{-1}k^{-1} \in P_{\rho} \cap P_{\theta} = \{e\}$ and so l and k commute. It follows that the groups $L_0 = \langle Q_L\rangle \leq L$ and $K_0 = \langle Q_K \rangle \leq K$ commute.

Suppose for contradiction that $[L : L_0] \geq n$ and let $l_0, \ldots, l_{n-1}$ be elements of distinct left cosets of L_0 in L, with $l_0 = e$. For each $i < n$ let $A_i = \{H \in \mathrm{Sub}_{\Delta} \,:\, l_i K l_i^{-1} \leq H\}$ so that $\rho(A_i) = \rho(l_i^c \cdot A_0) = \rho(A_0) > \frac{1}{n}$ by hypothesis. There must be some $0 \leq i < j < n$ with $\rho(A_i \cap A_j) > 0$. Let $l = l_j^{-1} l_i$. Then $\rho(l^c \cdot A_0 \cap A_0) = \rho(A_i \cap A_j) > 0$ and $l^c \cdot A_0 \cap A_0$ consists of those $H \in \mathrm{Sub}_{\Delta}$ such that $lKl^{-1} \cup K \leq H$. This shows that $\langle lKl^{-1} \cup K\rangle \subseteq P_{\rho}$ and thus $l \in Q_L \subseteq L_0$. But this contradicts that $l = l_j^{-1} l_i$ and $l_i L_0 \neq l_j L_0$. Therefore $[L : L_0] < n$. Similarly, $[K : K_0] < m$. $\square$[Lemma 4.2]

THEOREM 4.3. *Let $\theta, \rho \in \mathrm{IRS}_\Delta$, $L, K \leq \Delta$, and $n, m \in \mathbb{N}$ be as in Lemma 4.2, and assume in addition that L and K are finitely generated. Then there exist commuting subgroups N_L and N_K, both normal in Δ, with $[L : L \cap N_L] < \infty$ and $[K : K \cap N_K] < \infty$.*

PROOF. For a subgroup $H \leq \Delta$ and $i \in \mathbb{N}$ let $H(i)$ be the intersection of all subgroups of H of index strictly less than i. Then $L(n)$ is finite index in L, and $K(m)$ is finite index in K, since L and K are finitely generated. By Lemma 4.2 $L(n)$ and $K(m)$ commute. For any $\gamma, \delta \in \Delta$ the groups $\gamma L \gamma^{-1}$ and $\delta K \delta^{-1}$ satisfy the hypotheses of Lemma 4.2 hence the groups $(\gamma L \gamma^{-1})(n) = \gamma L(n) \gamma^{-1}$ and $(\delta K \delta^{-1})(m) = \delta K(m) \delta^{-1}$ commute. It follows that the normal subgroups $N_L = \langle \bigcup_{\delta \in \Delta} \delta L(n) \delta^{-1} \rangle$ and $N_K = \langle \bigcup_{\delta \in \Delta} \delta K(m) \delta^{-1} \rangle$ satisfy the conclusion of the theorem. $\square$

4.2. Finite index subgroups. The following is an analogue of a theorem of [**B9̀1**], and its proof is essentially the same as [**BdlH00**, Proposition 6].

PROPOSITION 4.4. *Let $\boldsymbol{a}$ be a measure preserving action of a countable group Γ and let N be a normal subgroup of Γ. If the restriction $\boldsymbol{a} \upharpoonright N$ of $\boldsymbol{a}$ to N is free then $\mu(\mathrm{Fix}^a(\gamma)) = 0$ for any $\gamma \in \Gamma$ satisfying*

$$|\{h\gamma h^{-1} : h \in N\}| = \infty. \tag{4.1}$$

Thus, if (4.1) holds for all $\gamma \notin N$ then a m.p. action of Γ is free if and only if its restriction to N is free.

For example, it is shown in [**B9̀1**] that (4.1) holds for all $\gamma \notin N$ whenever $C_\Gamma(N) = \{e\}$ and N is ICC.

PROOF OF PROPOSITION 4.4. Suppose $\gamma \in \Gamma \setminus \{e\}$ is such that $\mu(\mathrm{Fix}^a(\gamma)) > 0$ and $\{h\gamma h^{-1} : h \in N\}$ is infinite. It suffices to show that $\boldsymbol{a} \upharpoonright N$ is not free. The Poincaré recurrence theorem implies that there exist $h_0, h_1 \in N$ with $h_0 \gamma h_0^{-1} \neq h_1 \gamma h_1^{-1}$ and $\mu(h_0^a \cdot \mathrm{Fix}^a(\gamma) \cap h_1^a \cdot \mathrm{Fix}^a(\gamma)) > 0$. Let $h = h_1^{-1} h_0$ so that $h \in N$ and $h\gamma h^{-1} \neq \gamma$. Since $\mathrm{Fix}^a(\gamma) = \mathrm{Fix}^a(\gamma^{-1})$ we have

$$h^a \cdot \mathrm{Fix}^a(\gamma) \cap \mathrm{Fix}^a(\gamma) = \mathrm{Fix}^a(h\gamma h^{-1}) \cap \mathrm{Fix}^a(\gamma^{-1}) \subseteq \mathrm{Fix}^a(\gamma^{-1} h \gamma h^{-1}),$$

which implies $\mu(\mathrm{Fix}^a(\gamma^{-1}h\gamma h^{-1})) > 0$. This shows $\boldsymbol{a} \upharpoonright N$ is not free since $e \neq \gamma^{-1}(h\gamma h^{-1}) = (\gamma^{-1}h\gamma)h^{-1} \in N$ by our choice of h. $\square$

PROPOSITION 4.5. *Let K be a finite index subgroup of a countable* ICC *group Γ, and let $\boldsymbol{a}$ be a measure preserving action of Γ. If $\boldsymbol{a} \upharpoonright K$ is free, then $\boldsymbol{a}$ is free.*

PROOF. Let $N = \bigcap_{\gamma\in\Gamma} \gamma K \gamma^{-1}$ be the normal core of K in Γ. Then N is a normal finite index subgroup of Γ. Since Γ is ICC, the group $C_\Gamma(\gamma)$ is infinite index in Γ for any $\gamma \in \Gamma$, hence $C_\Gamma(\gamma) \cap N$ is infinite index in N. In particular $\{h\gamma h^{-1} : h \in N\}$ is infinite. If $\boldsymbol{a}$ is any m.p. action of Γ whose restriction to K is free, then the restriction of $\boldsymbol{a}$ to N is free, so by Proposition 4.4, $\boldsymbol{a}$ is free. $\square$

Proposition 4.5 can be used to characterize exactly when shift-minimality of Γ may be deduced from shift-minimality of one of its finite index subgroups.

PROPOSITION 4.6. *Let K be a finite index subgroup of the countable group Γ. Suppose that K is shift-minimal. Then the following are equivalent.*

(1) Γ *is shift-minimal.*
(2) Γ *is* ICC.
(3) Γ *has no non-trivial finite normal subgroups.*
(4) $C_\Gamma(N) = \{e\}$ *where* $N = \bigcap_{\gamma\in\Gamma} \gamma K \gamma^{-1}$.

PROOF. Since K is shift-minimal, it is also ICC by Proposition 3.15. The equivalence of (2), (3), and (4) then follows from [**Pré12**, Proposition 6.3]. It remains to show that (2)$\Rightarrow$(1). Suppose that Γ is ICC and that $\boldsymbol{a} \prec \boldsymbol{s}_\Gamma$ is non-trivial. Then $\boldsymbol{a} \upharpoonright K \prec \boldsymbol{s}_K$, so $\boldsymbol{a} \upharpoonright K$ is free by shift-minimality of K, and therefore $\boldsymbol{a}$ itself is free by Proposition 4.5. $\square$

Proposition 4.6 shows that, except for the obvious counterexamples, shift-minimality is inherited from a finite index subgroup. It seems likely that, conversely, shift-minimality passes from a group to each of its finite index subgroups. By Proposition 4.6 to show this it would be enough to show that shift-minimality passes to finite index *normal* subgroups (see the discussion following Question 7.11 in §7). Theorem 4.3 can be used to give a partial confirmation of this. Recall that a group is *locally finite* if each of its finitely generated subgroups is finite.

THEOREM 4.7. *Let N be a normal finite index subgroup of a shift-minimal group Γ. Suppose that N has no infinite locally finite invariant random subgroups that are weakly contained in $\boldsymbol{s}_N$. Then N is shift-minimal.*

COROLLARY 4.8. *Let Γ be a shift-minimal group. Then every finite index subgroup of Γ which is torsion-free is shift-minimal.*

PROOF OF COROLLARY 4.8. Let K be a torsion-free finite index subgroup of Γ. Note that K is ICC since the ICC property passes to finite index subgroups. The group $N := \bigcap_{\gamma\in\Gamma}\gamma K\gamma^{-1}$ is finite index in Γ and torsion-free, and it is moreover normal in Γ. By Theorem 4.7, N is shift-minimal, whence K is shift-minimal by Proposition 4.6. $\square$

Theorem 4.7 will follow from:

THEOREM 4.9. *Let Δ be a countable group with $\mathrm{AR}_\Delta = \{e\}$. Let θ and ρ be invariant random subgroups of Δ which are not locally finite. Suppose that $\boldsymbol{\rho}$ is* NA*-ergodic. Then θ and ρ have non-trivial intersection.*

We first show how to deduce 4.7 from 4.9.

PROOF OF THEOREM 4.7 FROM THEOREM 4.9. Let $\boldsymbol{a} = N \curvearrowright^a (X,\mu)$ be a non-trivial m.p. action of N weakly contained in $\boldsymbol{s}_N$. We will show that $\boldsymbol{a}$ is free.

The co-induced action $\boldsymbol{c} = \mathrm{CInd}_N^\Gamma(\boldsymbol{a})$ is weakly contained in $\boldsymbol{s}_\Gamma$, so $\boldsymbol{c}$ is free by shift-minimality of Γ. Let $T = \{t_0,\dots,t_{n-1}\}$ be a transversal for the left cosets of N in Γ. Then $\boldsymbol{c}\restriction N \cong \prod_{0\leq i<n}\boldsymbol{a}^{t_i}$ where for $\boldsymbol{b}\in A(N,X,\mu)$, $\boldsymbol{b}^t \in A(N,X,\mu)$ is given by $k^{b^t} = (t^{-1}kt)^b$ for each $k\in N$, $t\in T$ [**Kec10**, 10.(**G**)]. Observe that $\theta_{\boldsymbol{a}^t} = (\varphi_t)_*\theta_{\boldsymbol{a}}$ where $\varphi_t : \mathrm{Sub}_N \to \mathrm{Sub}_N$ is the conjugation map $H\mapsto tHt^{-1}$. In particular, for each $t\in T$, $\boldsymbol{a}^t$ is free if and only if $\boldsymbol{a}$ is free. It is easy to see that $(\boldsymbol{s}_N)^t \cong \boldsymbol{s}_N$ for each $t\in T$, so it follows that $\boldsymbol{c}\restriction N \cong \prod_{0\leq i<n}\boldsymbol{a}^{t_i} \prec \boldsymbol{s}_N$. For each $j<n$ let $\boldsymbol{c}_j = \prod_{j\leq i<n}\boldsymbol{a}^{t_i}$. We will show that $\boldsymbol{c}_j$ is free for all $0\leq j<n$, which will finish the proof since this will show that $\boldsymbol{c}_{n-1} = \boldsymbol{a}^{t_{n-1}}$ is free, whence $\boldsymbol{a}$ is free.

We know that $\boldsymbol{c}_0 = \boldsymbol{c}\restriction N$ is free. Assume for induction that $\boldsymbol{c}_{j-1}$ is free (where $j\geq 1$ is less than n) and we will show that $\boldsymbol{c}_j$ is free. Note the following:

(i) $\theta_{\boldsymbol{a}^{t_{j-1}}}$ and $\theta_{\boldsymbol{c}_j}$ intersect trivially. This follows from Lemma 4.1 because $\boldsymbol{c}_{j-1} = \boldsymbol{a}^{t_{j-1}}\times\boldsymbol{c}_j$ is free.

(ii) Both $\boldsymbol{\theta}_{\boldsymbol{a}^{t_{j-1}}}$ and $\boldsymbol{\theta}_{\boldsymbol{c}_j}$ are NA-ergodic, since they are both weakly contained in $\boldsymbol{s}_N$.

(iii) $\mathrm{AR}_N = \{e\}$. This is because Γ is shift-minimal, so that $\mathrm{AR}_\Gamma = \{e\}$ by Proposition 3.15, and N is normal in Γ so apply Proposition 9.1.

Theorem 4.9 along with (i), (ii), and (iii) imply that either $\theta_{\boldsymbol{a}^{t_{j-1}}}$ or $\theta_{\boldsymbol{c}_j}$ is locally finite. But N has no infinite locally finite invariant random subgroups weakly contained in $\boldsymbol{s}_N$ by hypothesis, and since $\mathrm{AR}_N = \{e\}$, N actually has no *non-trivial* locally finite invariant random subgroups weakly contained in $\boldsymbol{s}_N$. It follows that either $\theta_{\boldsymbol{a}^{t_{j-1}}}$ or $\theta_{\boldsymbol{c}_j}$ is trivial. If $\theta_{\boldsymbol{c}_j}$ is trivial then $\boldsymbol{c}_j$ is free, which is what we wanted to show. If $\theta_{\boldsymbol{a}^{t_{j-1}}}$ is trivial then $\boldsymbol{a}^{t_{j-1}}$ is free, so $\boldsymbol{a}^{t_i}$ is free for all $i < n$, and therefore $\boldsymbol{c}_j$ is free all the same. □

PROOF OF THEOREM 4.9. Suppose toward a contradiction that θ and ρ intersect trivially. By hypothesis θ is not locally finite, so the set of $H \in \mathrm{Sub}_\Delta$ that contain an infinite finitely generated subgroup is θ-non-null. As there are only countably many infinite finitely generated subgroups of Δ, there must be at least one - call it L - for which $\theta(\{H : L \subseteq H\}) > 0$. Similarly, there is an infinite finitely generated $K \leq \Delta$ with $\rho(\{H : K \leq H\}) > 0$. Then θ, ρ, L and K satisfy the hypotheses of Theorem 4.3 (for some n and m), so there exist normal subgroups $N_L, N_K \leq \Delta$ which commute, with $[L : L \cap N_L] < \infty$ and $[K : K \cap N_K] < \infty$. Since L and K are infinite, neither N_L nor N_K is trivial, and since $\mathrm{AR}_\Delta = \{e\}$, both N_L and N_K are non-amenable.

Pick some $k \neq e$ with $k \in K \cap N_K$. Since $k \in K$, the set $\{H : k \in H\}$ has positive ρ-measure, and it is N_L-invariant since N_L commutes with k. NA-ergodicity of $\boldsymbol{\rho}$ and non-amenability of N_L then imply that $\rho(\{H : k \in H\}) = 1$. On the other hand, the set

$$M_\rho = \{\delta \in \Delta : \rho(\{H : \delta \in H\}) = 1\}$$

is a normal subgroup of Δ which acts trivially under $\boldsymbol{\rho}$, so NA-ergodicity of $\boldsymbol{\rho}$ implies M_ρ is amenable, and as $\mathrm{AR}_\Delta = \{e\}$, we actually have $M_\rho = \{e\}$, which contradicts that $k \in M_\rho$. □

Question 7.11 below asks whether a finite index subgroup of a shift-minimal group is always shift-minimal.

4.3. Direct sums.

PROPOSITION 4.10. *Let $(\Gamma_i)_{i\in I}$ be a sequence of countable* ICC *groups and let $\boldsymbol{a}$ be a measure preserving action of $\Gamma = \bigoplus_{i\in I}\Gamma_i$. If $\boldsymbol{a} \upharpoonright \Gamma_i$ is free for each $i \in I$ then $\boldsymbol{a}$ is free. In particular, the direct sum of shift-minimal groups is shift-minimal.*

PROOF. We will show that if $\boldsymbol{a}$ is not free then $\boldsymbol{a} \upharpoonright \Gamma_i$ is not free for some $i \in I$. We give the proof for the case of the direct sum of two ICC groups - say Γ_1 and Γ_2 - since the proof for infinitely many groups is nearly identical. Let $\Gamma = \Gamma_1 \times \Gamma_2$ and let $(\gamma, \delta) \in \Gamma$ be such that $\mu\big(\mathrm{Fix}^a((\gamma,\delta))\big) > 0$ where $(\gamma,\delta) \neq e_\Gamma$. Suppose that $\delta \neq e$ (the case where $\gamma \neq e$ is similar). Since Γ_2 is ICC we have that $C_{\Gamma_2}(\delta)$ is infinite index in Γ_2 so by Poincaré recurrence there exists $\alpha \in \Gamma_2$, $\alpha \notin C_{\Gamma_2}(\delta)$ such that

$$\mu\big((e,\alpha)^a \cdot \mathrm{Fix}^a((\gamma,\delta)) \cap \mathrm{Fix}^a((\gamma,\delta))\big) > 0.$$

Thus $\mu\big(\mathrm{Fix}^a(\langle(\gamma, \alpha\delta\alpha^{-1}), (\gamma,\delta)\rangle)\big) > 0$ and in particular $\mu\big(\mathrm{Fix}^a((e, \alpha\delta\alpha^{-1}\delta^{-1}))\big) > 0$. Our choice of α implies that $\alpha\delta\alpha^{-1}\delta^{-1} \neq e$ and so $\boldsymbol{a} \upharpoonright \Gamma_2$ is non-free as was to be shown. ☐

4.4. Other permanence properties.

PROPOSITION 4.11. *Let $\boldsymbol{a}$ be a measure preserving action of Γ. Let N be a normal subgroup of Γ. Suppose that both N and $C_\Gamma(N)$ are* ICC*. Suppose that $\boldsymbol{a} \upharpoonright N$ and $\boldsymbol{a} \upharpoonright C_\Gamma(N)$ are both free. Then $\boldsymbol{a}$ is free.*

PROOF. Let $K = C_\Gamma(N)N$. Then K is normal in Γ since both N and $C_\Gamma(N)$ are normal. By hypothesis $C_\Gamma(N) \cap N = \{e\}$ so $K \cong C_\Gamma(N) \times N$. It follows that K is ICC, being a product of ICC groups. Proposition 4.10 then implies that $\boldsymbol{a} \upharpoonright K$ is free. Since $C_\Gamma(K) \leq C_\Gamma(C_\Gamma(N)) \cap C_\Gamma(N) = Z(C_\Gamma(N)) = \{e\}$, Proposition 4.4 implies that $\boldsymbol{a}$ is free. ☐

DEFINITION 4.12. A subgroup H of Γ is called *almost ascendant* in Γ if there exists a well-ordered increasing sequence $\{H_\alpha\}_{\alpha\leq\lambda}$ of subgroups of Γ, indexed by some countable ordinal λ, such that

(i) $H = H_0$ and $H_\lambda = \Gamma$.

(ii) For each $\alpha < \lambda$, either H_α is a normal subgroup of $H_{\alpha+1}$ or H_α is a finite index subgroup of $H_{\alpha+1}$.

(iii) $H_\beta = \bigcup_{\alpha<\beta} H_\alpha$ whenever β is a limit ordinal.

We call $\{H_\alpha\}_{\alpha\leq\lambda}$ an *almost ascendant series* for H in Γ. If H is almost ascendant in Γ and if there exists an almost ascendant series $\{H_\alpha\}_{\alpha\leq\lambda}$ for H in Γ such that H_α is normal in $H_{\alpha+1}$ for all $\alpha<\lambda$ then we say that H is *ascendant* in Γ and we call $\{H_\alpha\}_{\alpha\leq\lambda}$ an *ascendant series* for H in Γ.

PROPOSITION 4.13. *Let $\boldsymbol{a}=\Gamma\curvearrowright^a(X,\mu)$ be a measure preserving action of Γ.*

(1) *Suppose that L is an almost ascendant subgroup of Γ that is* ICC *and satisfies $C_\Gamma(L)=\{e\}$. Then $\boldsymbol{a}$ is free if and only if $\boldsymbol{a}\upharpoonright L$ is free. Thus, if L is shift-minimal then so is Γ.*
(2) *Suppose that L is an ascendant subgroup of Γ such that $\mathrm{AR}_L=\mathrm{AR}_{C_\Gamma(L)}=\{e\}$. Then $\boldsymbol{a}$ is free if and only if both $\boldsymbol{a}\upharpoonright L$ and $\boldsymbol{a}\upharpoonright C_\Gamma(L)$ are free.*

PROOF. (1): Assume that $\boldsymbol{a}\upharpoonright L$ is free. Let $\{L_\alpha\}_{\alpha\leq\lambda}$ be an almost ascendant series for L in Γ. Then $C_\Gamma(L_\alpha)=\{e\}$ for all $\alpha\leq\lambda$. By transfinite induction each L_α is ICC. Another induction shows that each $\boldsymbol{a}\upharpoonright L_\alpha$ is free: this is clear for limit α, and at successors, L_α is either normal or finite index in $L_{\alpha+1}$, so assuming $\boldsymbol{a}\upharpoonright L_\alpha$ is free it follows that $\boldsymbol{a}\upharpoonright L_{\alpha+1}$ is free by applying either Proposition 4.11 (Proposition 4.4 also works) or Proposition 4.5.

If now L is shift-minimal and $\boldsymbol{a}$ is a non-trivial m.p. action of Γ with $\boldsymbol{a}\prec\boldsymbol{s}_\Gamma$ then $\boldsymbol{a}\upharpoonright L\prec\boldsymbol{s}_L$ so that $\boldsymbol{a}\upharpoonright L$ is free and thus $\boldsymbol{a}$ is free.

(2): Assume that both $\boldsymbol{a}\upharpoonright L$ and $\boldsymbol{a}\upharpoonright C_\Gamma(L)$ are free. Let $\{L_\alpha\}_{\alpha\leq\lambda}$ be an ascendant series for L in Γ. Theorem 9.9 implies that $\mathrm{AR}_{L_\alpha}=\mathrm{AR}_{C_\Gamma(L_\alpha)}=\{e\}$ for all $\alpha\leq\lambda$. For each $\alpha\leq\lambda$ we have

$$\{e\}=\mathrm{AR}_{C_\Gamma(L_\alpha)}\cap L_{\alpha+1}=\mathrm{AR}_{C_\Gamma(L_\alpha)}\cap C_{L_{\alpha+1}}(L_\alpha)=\mathrm{AR}_{C_{L_{\alpha+1}}(L_\alpha)}$$

where the last equality follows from Corollary 9.4 since the series $\{C_{L_\beta}(L_\alpha)\}_{\beta\leq\lambda}$ is ascendant in $C_\Gamma(L_\alpha)$. It is clear that $C_{L_{\alpha+1}}(L_\alpha)\leq C_\Gamma(L)$, so by hypothesis $\boldsymbol{a}\upharpoonright C_{L_{\alpha+1}}(L_\alpha)$ is free for all $\alpha\leq\lambda$. We now show by transfinite induction on $\alpha\leq\lambda$ that $\boldsymbol{a}\upharpoonright L_\alpha$ is free. The induction is clear at limit stages. At successor stages, if we assume for induction that $\boldsymbol{a}\upharpoonright L_\alpha$ is free then all the hypotheses of Proposition 4.11 hold and it follows that $\boldsymbol{a}\upharpoonright L_{\alpha+1}$ is free. $\square$

PROPOSITION 4.14. *Let $\boldsymbol{a}=\Gamma\curvearrowright^a(X,\mu)$ be a measure preserving action of Γ. Let $K=\ker(\boldsymbol{a})$.*

(1) *Suppose that there exists a normal subgroup N of Γ such that $\boldsymbol{a}\upharpoonright N$ is free and such that every finite index subgroups of N acts ergodically. Then $\Gamma_x=K$ almost surely.*

(2) *Suppose that $\boldsymbol{a}$ is NA-ergodic and there exists a non-amenable normal subgroup N of Γ such that $\boldsymbol{a} \upharpoonright N$ is free. Then K is amenable and $\Gamma_x = K$ almost surely.*

PROOF. We begin with (1). Note that, by Proposition 4.4, if $\gamma \in \Gamma$ is such that the set $\{h\gamma h^{-1} : h \in N\}$ is infinite, then $\mu(\mathrm{Fix}^a(\gamma)) = 0$. It therefore suffices to show that if $\mu(\mathrm{Fix}^a(\gamma)) > 0$ and $\{h\gamma h^{-1} : h \in N\}$ is finite, then $\gamma \in K$. This set being finite means that the group $H = C_\Gamma(\gamma) \cap N$ is finite index in N, so $\boldsymbol{a} \upharpoonright H$ is ergodic by hypothesis. Since $H \leq C_\Gamma(\gamma)$, the set $\mathrm{Fix}^a(\gamma)$ is $\boldsymbol{a} \upharpoonright H$-invariant, so if it is non-null then it must be conull, i.e., $\gamma \in K$, by ergodicity.

For (2), amenability of K is immediate since $\boldsymbol{a}$ is non-trivial and NA-ergodic. NA-ergodicity also implies that every finite index subgroup of N acts ergodically, so (1) applies and we are done. □

The following Corollary replaces the hypothesis in Proposition 4.13.(1) that $C_\Gamma(L) = \{e\}$ with the hypotheses that $\mathrm{AR}_\Gamma = \{e\}$ and $\boldsymbol{a}$ is NA-ergodic.

COROLLARY 4.15. *Suppose* $\mathrm{AR}_\Gamma = \{e\}$. *Let $\boldsymbol{a}$ be any NA-ergodic action of Γ and suppose that there exists a non-trivial almost ascendant subgroup L of Γ such that the restriction $\boldsymbol{a} \upharpoonright L$ of $\boldsymbol{a}$ to L is free, then $\boldsymbol{a}$ itself is free.*

PROOF. Let $\{L_\alpha\}_{\alpha \leq \lambda}$ be an almost ascendant series for L in Γ. Since $\mathrm{AR}_\Gamma = \{e\}$, Corollary 9.4 implies that $\mathrm{AR}_{L_\alpha} = \{e\}$ for each $\alpha \leq \lambda$. Suppose for induction that we have shown that $\boldsymbol{a} \upharpoonright L_\alpha$ is free for all $\alpha < \beta$. If β is a limit then $L_\beta = \bigcup_{\alpha < \beta} L_\alpha$ so $\boldsymbol{a} \upharpoonright L_\beta$ is free as well. If $\beta = \alpha + 1$ is a successor then $\boldsymbol{a} \upharpoonright L_\alpha$ is free and L_α is either finite index or normal in L_β. If L_α is finite index in L_β then $\boldsymbol{a} \upharpoonright L_\beta$ is free by Proposition 4.5. If L_α is normal in L_β then $\boldsymbol{a} \upharpoonright L_\beta$ is free by Proposition 4.14.(2). It follows by induction that $\boldsymbol{a} \upharpoonright \Gamma$ is free. □

COROLLARY 4.16.

(1) *Let Γ be a countable group with* $\mathrm{AR}_\Gamma = \{e\}$. *If Γ contains a shift-minimal almost ascendant subgroup L then Γ is itself shift-minimal.*

(2) *Suppose that Γ is a countable group containing an ascendant subgroup L such that L is shift-minimal and* $\mathrm{AR}_{C_\Gamma(L)} = \{e\}$. *Then Γ is shift-minimal. In particular, if both L and $C_\Gamma(L)$ are shift-minimal then so is Γ.*

PROOF. Starting with (1), let L be a shift-minimal almost ascendant subgroup of Γ. Let $\boldsymbol{a}$ be a non-trivial measure preserving action of Γ weakly contained in $\boldsymbol{s}_\Gamma$. Then $\boldsymbol{a}$ is NA-ergodic and $\boldsymbol{a} \restriction L$ is free, so $\boldsymbol{a}$ is free by Corollary 4.15. Statement (2) is a special case of (1) since Theorem 9.9 shows that $\mathrm{AR}_\Gamma = \{e\}$. □

5. Examples of shift-minimal groups

Theorem 5.15 below shows that if the reduced C^*-algebra of a countable group Γ admits a unique tracial state then Γ is shift-minimal. We can also often gain more specific information by giving direct ergodic theoretic proofs of shift-minimality. These proofs often rely on an appeal to some form of the Poincaré recurrence theorem (several proofs of which may be found in [**Ber00**]).

5.1. Free groups. Since the argument is quite short it seems helpful to present a direct argument that free groups are shift-minimal.

THEOREM 5.1. *Let Γ be a non-abelian free group.*

(i) *If $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ is any non-trivial measure preserving action of Γ which is NA-ergodic then $\boldsymbol{a}$ is free.*
(ii) *Γ is shift-minimal.*

PROOF. For (i) we show that non-free actions of Γ are never NA-ergodic. Suppose that $\boldsymbol{a}$ is non-free so that $\mu(\mathrm{Fix}^a(\gamma)) > 0$ for some $\gamma \in \Gamma - \{e\}$. Fix any $\delta \in \Gamma - \langle\gamma\rangle$. By the Poincaré recurrence theorem there exists an $n > 0$ with $\mu(\delta^n \cdot \mathrm{Fix}^a(\gamma) \cap \mathrm{Fix}^a(\gamma)) > 0$. The group H generated by $\delta^n\gamma\delta^{-n}$ and γ is free on these elements and $\alpha^a \cdot x = x$ for every $\alpha \in H$ and $x \in \delta^n \cdot \mathrm{Fix}^a(\gamma) \cap \mathrm{Fix}^a(\gamma)$. In particular $\boldsymbol{a} \restriction H$ is not ergodic, whence $\boldsymbol{a}$ cannot be NA-ergodic.

Statement (ii) now follows since any non-trivial action weakly contained in $\boldsymbol{s}_\Gamma$ is strongly NA-ergodic, hence free by (i). □

Another proof of part (i) of Theorem 5.1 follows from Theorem 3.13 (see also [**AGV12**, Lemma 24]). Indeed, alternative (2) of Theorem 3.13 can never hold since a non-abelian free group has only countably many amenable subgroups. So if $\boldsymbol{a}$ is any non-trivial NA-ergodic action of a non-abelian free group Γ then (1) of Theorem 3.13 holds, and so $\boldsymbol{a}$ is free since the only normal amenable subgroup of Γ is the trivial group $N = \{e\}$.

5.2. Property (BP).

DEFINITION 5.2. Let Γ be a countable group.

(1) Γ is said to be a *Powers group* ([**dLH85**]) if $\Gamma \neq \{e\}$ and for every finite subset $F \subseteq \Gamma \setminus \{e\}$ and every integer $N > 0$ there exists a partition $\Gamma = C \sqcup D$ and elements $\alpha_1, \ldots, \alpha_N \in \Gamma$ such that

$$\gamma C \cap C = \varnothing \text{ for all } \gamma \in F$$

$$\alpha_j D \cap \alpha_k D = \varnothing \text{ for all } j, k \in \{1, \ldots, N\},\ j \neq k.$$

Γ is said to be a *weak Powers group* ([**BN88**]) if Γ satisfies all instances of the Powers property with F ranging over finite subsets of mutually conjugate elements of $\Gamma \setminus \{e\}$. We define Γ to be a *weak* Powers group* if Γ satisfies all instances of the Powers property with F ranging over singletons in $\Gamma \setminus \{e\}$.

(2) Γ has *property P_{nai}* ([**BCdLH94**]) if for any finite subset F of Γ there exists an element $\alpha \in \Gamma$ of infinite order such that for each $\gamma \in F$, the canonical homomorphism from the free product $\langle \gamma \rangle * \langle \alpha \rangle$ onto the subgroup $\langle \gamma, \alpha \rangle$ of Γ generated by γ and α is an isomorphism.

If Γ satisfies the defining property of $\mathrm{P_{nai}}$ but with F only ranging over singletons, then we say that Γ has *property P^*_{nai}*.

(3) Γ is said to have property (PH) ([**Pro93**]) if for all nonempty finite $F \subseteq \Gamma \setminus \{e\}$ there exists some ordering $F = \{\gamma_1, \ldots, \gamma_m\}$ of F along with an increasing sequence $e \in Q_1 \subseteq \cdots \subseteq Q_m$ of subsets of Γ such that for all $i \leq m$, all nonempty finite $M \subseteq Q_i$ and all $n > 0$ we may find $\alpha_1, \ldots, \alpha_n \in Q_i$ and $T_1, \ldots, T_n$ pairwise disjoint such that

$$(\alpha_j \delta)\gamma_i(\alpha_j \delta)^{-1}(\Gamma \setminus T_j) \subseteq T_j$$

for all $\delta \in M$ and $1 \leq j \leq n$.

Examples of groups with these properties may be found in [**AM07, dlHP11, MOY11, PT11**] along with the references given in the above definitions. For our purposes, what is important is a common consequence of these properties.

DEFINITION 5.3. A countable group Γ is said to have property (BP) if for all $\gamma \in \Gamma \setminus \{e\}$ and $n \geq 2$ there exists $\alpha_1, \dots, \alpha_n \in \Gamma$, a subgroup $H \leq \Gamma$, and pairwise disjoint subsets $T_1, \dots, T_n \subseteq H$ such that

$$\alpha_j \gamma \alpha_j^{-1}(H \setminus T_j) \subseteq T_j$$

for all $j = 1, \dots, n$.

Note that when γ, H, $\alpha_1, \dots, \alpha_n$, and $T_1, \dots, T_n$ are as above, then $\alpha_j \gamma \alpha_j^{-1} \in H$ and $T_j \neq \varnothing$ for all $j \leq n$.

We show in Theorem 5.6 that groups with property (BP) satisfy a strong form of shift-minimality. The definition of property (BP) (as well as its name) is motivated by an argument of M. Brin and G. Picioroaga showing that all weak Powers groups contain a free group. Their proof appears in [**dlH07**] (see the remark following Question 15 in that paper), though we also present a version of their proof in Theorem 5.4 since we will need it for Theorem 5.6.

THEOREM 5.4 (Brin, Picioroaga [**dlH07**]).

(1) *All weak* Powers groups have property* (BP).

(2) *Property* $\mathrm{P}^*_{\mathrm{nai}}$ *implies property* (BP).

(3) *Property* (PH) *implies property* (BP).

(4) *Groups with property* (BP) *contain a free group.*

PROOF. (1): given $\gamma \in \Gamma \setminus \{e\}$ and $n \geq 1$ by the weak* Powers property there exists $\alpha_1, \dots, \alpha_n$ and a partition $\Gamma = C \sqcup D$ of Γ with $\gamma C \cap C = \varnothing$ and $\alpha_i D \cap \alpha_j D = \varnothing$ for all $1 \leq i, j \leq n$, $i \neq j$. Take $H = \Gamma$ and for each $1 \leq j \leq n$ let $T_j = \alpha_j D$ so that the sets $T_1, \dots, T_n$ are pairwise disjoint and

$$\alpha_j \gamma \alpha_j^{-1}(\Gamma \setminus T_j) = \alpha_j \gamma(\Gamma \setminus D) = \alpha_j \gamma C \subseteq \alpha_j(\Gamma \setminus C) = \alpha_j D = T_j$$

thus verifying (BP).

(2): Let $\gamma \in \Gamma \setminus \{e\}$. By property $\mathrm{P}^*_{\mathrm{nai}}$ there exists an element $\alpha \in \Gamma$ of infinite order such that the subgroup $H = \langle \gamma, \alpha \rangle$ of Γ is canonically isomorphic to the free product $\langle \gamma \rangle * \langle \alpha \rangle$. Let T_n denote the set of elements of H whose reduced expression starts with $\alpha^n \gamma^k$ for some $k \in \mathbb{Z}$ with $\gamma^k \neq e$. Then the sets T_n, $n \in \mathbb{N}$, are pairwise disjoint and $\alpha^n \gamma \alpha^{-n}(H \setminus T_n) \subseteq T_n$.

(3): Assume that Γ has property (PH) and fix any $\gamma \in \Gamma \setminus \{e\}$ and $n \geq 1$ toward the aim of verifying property (BP). Taking $F = \{\gamma\}$ we obtain a set $Q = Q_1 \subseteq \Gamma$ from the above definition of (PH) with $e \in Q$. Taking $M = \{e\}$, the defining property of Q produces $\alpha_1, \dots, \alpha_n \in Q$ and pairwise disjoint $T_1, \dots, T_n \subseteq \Gamma$ with

$$\alpha_j \gamma \alpha_j^{-1}(\Gamma \setminus T_j) \subseteq T_j,$$

so taking $H = \Gamma$ confirms this instance of property (BP).

Statement (4) is a consequence of the following Lemma, which will be used in Theorem 5.6 below.

LEMMA 5.5 (Brin, Picioroaga). *Suppose that $x_1, \dots x_4$ are elements of a group H and that $T_1, \dots, T_4$ are pairwise disjoint subsets of H such that*

$$x_j(H \setminus T_j) \subseteq T_j$$

for each $j \in \{1, \dots, 4\}$. Then the group elements $u = x_1x_2$ and $v = x_3x_4$ freely generate a non-abelian free subgroup of H.

PROOF OF LEMMA 5.5. The hypothesis $x_j(H \setminus T_j) \subseteq T_j$ implies that also $x_j^{-1}(H \setminus T_j) \subseteq T_j$. For distinct $i, j \in \{1, \dots, 4\}$ it then follows that

$$x_ix_j(H \setminus T_j) \subseteq x_iT_j \subseteq x_i(H \setminus T_i) \subseteq T_i$$
$$\text{and} \quad (x_ix_j)^{-1}(H \setminus T_i) \subseteq x_j^{-1}T_i \subseteq x_j^{-1}(H \setminus T_j) \subseteq T_j$$

so for $u = x_1x_2$ and $v = x_3x_4$ we have

$$u(H \setminus T_2) \subseteq T_1 \quad u^{-1}(H \setminus T_1) \subseteq T_2$$
$$v(H \setminus T_4) \subseteq T_3 \quad v^{-1}(H \setminus T_3) \subseteq T_4.$$

A ping pong argument now shows that u and v freely generate a non-abelian free subgroup of H. □[Lemma 5.5]

If now Γ has property (BP) then taking any $\gamma \in \Gamma \setminus \{e\}$ and $n = 4$ we obtain $\alpha_1, \dots, \alpha_4 \in \Gamma$, $H \leq \Gamma$ and $T_1, \dots, T_4 \subseteq H$ as in the definition of property (BP). Lemma 5.5 now applies with $x_j = \alpha_j \gamma \alpha_j^{-1}$ for $j \in \{1, \dots, 4\}$. □[Theorem 5.4]

Lemma 5.5 can be used to show that any non-trivial ergodic invariant random subgroup of a group with property (BP) contains a free group.

THEOREM 5.6. *Let Γ have property* (BP) *and let $\boldsymbol{a} = \Gamma \curvearrowright^a (Y,\nu)$ be an ergodic measure preserving action of Γ. Suppose that $\boldsymbol{a}$ is non-free. Then the stabilizer of ν-almost every $y \in Y$ contains a non-abelian free group. In particular, all groups with property* (BP) *are shift-minimal.*

PROOF. Since $\boldsymbol{a}$ is non-free there exists an element $\gamma \in \Gamma \setminus \{e\}$ such that $\nu(A) = r > 0$ where $A = \mathrm{Fix}^a(\gamma)$. By the Poincaré recurrence theorem, for all large enough n (depending on r), if $A_1, \ldots, A_n \subseteq Y$ is any sequences of measurable subsets of Y each of measure r, then there exist distinct $i_1, \ldots, i_4 \leq n$ with $\nu(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}) > 0$. Pick such an n with $n \geq 4$. By property (BP) there exists $\alpha_1, \ldots, \alpha_n \in \Gamma$, $H \leq \Gamma$, and pairwise disjoint $T_1, \ldots, T_n \subseteq H$ such that $\alpha_i \gamma \alpha_i^{-1}(H \setminus T_i) \subseteq T_i$ for all $i \in \{1, \ldots, n\}$. By our choice of n there must exist distinct $i_1, \ldots, i_4 \leq n$ such that

$$\nu(\alpha_{i_1}^a A \cap \alpha_{i_2}^a A \cap \alpha_{i_3}^a A \cap \alpha_{i_4}^a A) > 0. \tag{5.1}$$

For $j = 1, \ldots, 4$ let $x_j = \alpha_{i_j} \gamma \alpha_{i_j}^{-1}$. Lemma 5.5 (applied to $x_1, \ldots x_4$ and $T_1, \ldots T_4$) shows that $\langle x_1, \ldots, x_4 \rangle$ contains a free group. Additionally, (5.1) shows that $\nu(\mathrm{Fix}^a(\langle x_1, \ldots, x_4 \rangle)) > 0$ since

$$\mathrm{Fix}^a(\langle x_1, \ldots, x_4 \rangle) \supseteq \bigcap_{j=1}^{4} \mathrm{Fix}^a(x_i) = \bigcap_{j=1}^{4} \alpha_{i_j}^a A.$$

The event that Γ_y contains a free group is therefore non-null. This event is also $\boldsymbol{a}$-invariant, so ergodicity now implies that almost every stabilizer contains a free group.

If now $\boldsymbol{b}$ is any non-trivial measure preserving action of Γ weakly contained in $\boldsymbol{s}_\Gamma$ then $\boldsymbol{b}$ is ergodic and by Lemma 3.11 almost every stabilizer is amenable hence does not contain a free group. Then $\boldsymbol{b}$ is essentially free by what we have already shown. Therefore Γ is shift-minimal. $\square$

In [**B\`91**] Bèdos defines a group Γ to be an *ultraweak Powers group* if it has a normal subgroup N that is a weak Powers group such that $C_\Gamma(N) = \{e\}$. Let us say that Γ is an *ultraweak* Powers group* if it has a normal subgroup N that is an weak* Powers group such that $C_\Gamma(N) = \{e\}$.

THEOREM 5.7. *Let Γ be a countable group.*

(1) *Suppose that Γ contains an almost ascendant subgroup L with property* (BP) *such that $C_\Gamma(L) = \{e\}$. Then for every ergodic m.p. action $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ of Γ, either $\boldsymbol{a}$ is free or $\Gamma_x \cap L$ contains a non-abelian free group almost surely.*

(2) *Suppose that* Γ *contains an ascendant subgroup* L *such that both* L *and* $C_\Gamma(L)$ *have property* (BP). *Then for every ergodic m.p. action* $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ *of* Γ*, either* $\boldsymbol{a}$ *is free,* $\Gamma_x \cap L$ *contains a non-abelian free group almost surely, or* $\Gamma_x \cap C_\Gamma(L)$ *contains a non-abelian free group almost surely.*

(3) *Every non-trivial ergodic invariant random subgroup of an ultraweak*-Powers group contains a non-abelian free group almost surely.*

PROOF. (1) Since L has property (BP) it is ICC, so if $\boldsymbol{a} \restriction L$ is free then $\boldsymbol{a}$ itself is free by part (1) of Proposition 4.13. Suppose then that $\boldsymbol{a} \restriction L$ is non-free. Let $\pi : (X,\mu) \to (Z,\eta)$ be the ergodic decomposition map for $\boldsymbol{a} \restriction L$ and let $\mu = \int_z \mu_z \, d\eta(z)$ be the disintegration of μ with respect to η. Since $\boldsymbol{a} \restriction L$ is non-free then the set $A \subseteq Z$, consisting of of all $z \in Z$ such that $L \curvearrowright^a (X,\mu_z)$ is non-free, is η-non-null. If $z \in A$ then $\mu_z(\{x : L_x \text{ contains a non-abelian free group}\}) = 1$ by Theorem 5.6. The event that L_x contains a non-abelian free group is therefore μ-non-null. This event is Γ-invariant (a subgroup contains a free group if and only if any of its conjugates contains one), so ergodicity implies that L_x contains a free group almost surely. Since $L_x = \Gamma_x \cap L$ we are done.

The proof of (2) is similar, using part (2) of Proposition 4.13. (3) is immediate from (1) and the definitions. $\square$

We note also that (BP) is preserved by extensions.

PROPOSITION 5.8. *Let* N *be a normal subgroup of* Γ*. If* N *and* Γ/N *both have property* (BP) *then* Γ *also has property* (BP).

PROOF. Let $\gamma \in \Gamma \setminus \{e\}$ and $n \geq 1$ be given.

If $\gamma \in N$ then property (BP) for N implies that there exists $\alpha_1, \ldots, \alpha_n \in N$, $H \leq N$ and pairwise disjoint $T_1, \ldots, T_n \subseteq H$ as in the definition of (BP) for N. These also satisfy this instance of property (BP) for Γ.

If $\gamma \notin N$ then the image of γ in Γ/N is not the identity element so property (BP) for Γ/N implies that there exist cosets $\alpha_1 N, \cdots \alpha_n N \in \Gamma/N$, a subgroup $K \leq \Gamma$ containing N, and pairwise disjoint $T_1, \ldots, T_n \subseteq K/N$ as in the definition of (BP) for Γ/N. Then $\alpha_1, \ldots, \alpha_n$, K, and the sets $T_i' = \bigcup T_i$, $i = 1, \ldots, M$, verify this instance of property (BP) for Γ. $\square$

REMARK 5.9. If a group Γ has property (BP) then it has the unique trace property. A quick proof of this follows [**BCdLH94**]. The proof of this is almost exactly as in [**BCdLH94**, Lemma 2.2] with just a minor adjustment to the first part of their proof which we now describe. One first shows for any $\gamma \in \Gamma \setminus \{e\}$ and any $n \geq 2$, if $\alpha_1, \dots, \alpha_n$, H, and $T_1, \dots, T_n$ are as in the definition of (BP) then for all $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we have

$$||\sum_{j=1}^{n} z_j \lambda_\Gamma(\alpha_j \gamma \alpha_j^{-1})|| \leq 2||z||_2. \tag{5.2}$$

Let $x_j = \alpha_j \gamma \alpha_j^{-1}$ so that $x_j \in H$ and $x_j(H \setminus T_j) \subseteq T_j$ for all $j = 1, \dots, n$. Let 1_A denote the indicator function of a subset $A \subseteq H$. For $f, g \in \ell^2(H)$ we then have

$$\begin{aligned} |\langle \lambda_H(x_j)f, g\rangle| &\leq |\langle \lambda_H(x_j)(1_{T_j}f), g\rangle| + |\langle \lambda_H(x_j)(1_{H\setminus T_j}f), g\rangle| \\ &= |\langle \lambda_H(x_j)(1_{T_j}f), g\rangle| + |\langle 1_{x_j(H\setminus T_j)}\lambda_H(x_j)(f), 1_{T_j}g\rangle| \leq ||1_{T_j}f||\,||g|| + ||f||\,||1_{T_j}g||. \end{aligned}$$

The remainder of the proof of (5.2) now proceeds as in [**BCdLH94**, Lemma 2.2] using that the T_j are pairwise disjoint. It now follows as in the paragraph following [**BCdLH94**, Definition 1] that $C_r^*(\Gamma)$ has a unique tracial state.

5.3. Linear groups. In the case that Γ is a countable linear group, a theorem of Y. Glasner [**Gla12**] shows that the existence of a non-trivial normal amenable subgroup is the only obstruction to shift-minimality: Glasner shows that every amenable invariant random subgroup of a linear group Γ must concentrate on the subgroups of the amenable radical of Γ. Along with Proposition 3.15 this implies that a countable linear group Γ is shift-minimal if and only if Γ contains no non-trivial normal amenable subgroups. Another way to deduce these results is to use Theorem 5.14 below along with the following Theorem of Poznansky.

THEOREM 5.10 (Theorem 1.1 of [**Poz09**]). *Let Γ be a countable linear group. Then the following are equivalent*

(1) Γ *is C^*-simple.*
(2) Γ *has the unique trace property.*
(3) Γ *contains no non-trivial normal amenable subgroups, i.e.,* $\mathrm{AR}_\Gamma = \{e\}$.

COROLLARY 5.11. *Let Γ be a countable linear group. The properties (1), (2), and (3) of Theorem 5.10 are equivalent to each of the following properties:*

(4) Γ *is shift-minimal.*

(5) Γ *has no non-trivial amenable invariant random subgroups.*

PROOF. The implication (2)⇒(5) follows from Theorem 5.14, the implication (5)⇒(4) is Corollary 3.14, and (4)⇒(3) follows from Proposition 3.15. The remaining implications follow from Poznansky's Theorem 5.10. □

5.4. Unique tracial state on $C_r^*(\Gamma)$. We write $C_r^*(\Gamma)$ for the reduced C^*-algebra of Γ. This is the C^*-algebra generated by $\{\lambda_\Gamma(\gamma) : \gamma \in \Gamma\}$ in $\mathcal{B}(\ell^2(\Gamma))$, where λ_Γ denotes the left regular representation of Γ. Let $1_e \in \ell^2(\Gamma)$ denote the indicator function of $\{e\}$. We obtain a tracial state τ_Γ, called the *canonical trace* on $C_r^*(\Gamma)$, given by $\tau_\Gamma(a) = \langle a(1_e), 1_e\rangle$.

Let ρ be a probability measure on Sub_Γ and define the function $\varphi_\rho \in \ell^\infty(\Gamma)$ by

$$\varphi_\rho(\gamma) = \rho(\{H : \gamma \in H\}).$$

It is shown in [**IKT09**] (see also Theorem 8.16) and [**Ver11**] that φ_ρ is a positive definite function on Γ. It will be useful here to identify φ_ρ as the diagonal matrix coefficient of a specific unitary representation of Γ described below.

Consider the field of Hilbert spaces $\{\ell^2(\Gamma/H) : H \in \mathrm{Sub}_\Gamma\}$. For $\gamma \in \Gamma$ denote by $x^\gamma \in \prod_H \ell^2(\Gamma/H)$ the vector field $x^\gamma_H = 1_{\gamma H}$ where $1_{\gamma H} \in \ell^2(\Gamma/H)$ is the indicator function of the singleton set $\{\gamma H\} \subseteq \Gamma/H$. Then $\{x^\gamma\}_{\gamma\in\Gamma}$ determines a fundamental family of measurable vector fields and we let $\mathcal{H}_\rho = \int_H^\oplus \ell^2(\Gamma/H)\, d\rho$ denote the corresponding Hilbert space consisting of all square integrable measurable vector fields. The inner product on $\mathcal{H}_\rho$ is given by $\langle x, y\rangle = \int_H \langle x_H, y_H\rangle_{\ell^2(\Gamma/H)}\, d\rho$. Define the unitary representation λ_ρ of Γ on $\mathcal{H}_\rho$ by

$$\lambda_\rho = \int_H^\oplus \lambda_{\Gamma/H}\, d\rho,$$

i.e., $\lambda_\rho(\gamma)(x)_H = \lambda_{\Gamma/H}(\gamma)(x_H)$, where $\lambda_{\Gamma/H}$ denotes the quasi-regular representation of Γ on $\ell^2(\Gamma/H)$. We then have

$$\begin{aligned}\langle\lambda_\rho(\gamma)(x^e), x^e\rangle &= \int_H \langle\lambda_\rho(\gamma)(x^e)_H, x^e_H\rangle_{\ell^2(\Gamma/H)}\, d\rho \\ &= \int_H \langle\lambda_{\Gamma/H}(\gamma)(1_H), 1_H\rangle_{\ell^2(\Gamma/H)}\, d\rho = \rho(\{H : \gamma\in H\}) = \varphi_\rho(\gamma).\end{aligned}$$

We have shown the following.

PROPOSITION 5.12. $(\mathcal{H}_\rho, \lambda_\rho, x^e)$ *is the* GNS *triple associated with the positive definite function* φ_ρ *on* Γ.

It is clear that if ρ is conjugation invariant (i.e., if ρ is an invariant random subgroup) then φ_ρ will be constant on each conjugacy class of Γ.

LEMMA 5.13. *If H is an amenable subgroup of Γ then $\lambda_{\Gamma/H}$ is weakly contained in λ_Γ. Thus, for all $f\in\ell^1(\Gamma)$ we have $||\lambda_{\Gamma/H}(f)|| \le ||\lambda_\Gamma(f)||$.*

PROOF. H being amenable implies that the one-dimensional unit representation 1_H of H is weakly contained in the left regular representation λ_H of H ([**BHV08**, Theorem G.3.2]). Thus by [**BHV08**, Theorem F.3.5] we have $\lambda_{\Gamma/H} \cong \mathrm{Ind}_H^\Gamma(1_H) \prec \mathrm{Ind}_H^\Gamma(\lambda_H) \cong \lambda_\Gamma$. The second statement follows immediately from [**BHV08**, F.4.4]. $\square$

THEOREM 5.14. *If ρ is any measure on* Sub_Γ *concentrating on the amenable subgroups then λ_ρ is weakly contained in the left regular representation λ_Γ of Γ.*

Therefore, if θ is an amenable invariant random subgroup of Γ then φ_θ extends to a tracial state on $C^*_r(\Gamma)$ which is distinct from the canonical trace τ_Γ whenever θ is non-trivial.

PROOF. By [**BHV08**, F.4.4] to show that $\lambda_\rho \prec \lambda_\Gamma$ it suffices to show that $||\lambda_\rho(f)|| \le ||\lambda_\Gamma(f)||$ for all $f\in\ell^1(\Gamma)$. Using that ρ concentrates on the amenable subgroups and Lemma 5.13 we have for $f \in \ell^1(\Gamma)$ and $x, y \in \mathcal{H}_\rho$

$$
\begin{aligned}
|\langle \lambda_\rho(f)x, y\rangle| &= \Big| \int_H \langle \lambda_{\Gamma/H}(f)(x_H), y_H\rangle_{\ell^2(\Gamma/H)}\, d\rho \Big| \\
&\leq \int_H ||\lambda_{\Gamma/H}(f)||\, ||x_H||\, ||y_H||\, d\rho \\
&\leq ||\lambda_\Gamma(f)|| \int_H ||x_H||\, ||y_H||\, d\rho \\
&\leq ||\lambda_\Gamma(f)||\, ||x||\, ||y||
\end{aligned}
$$

from which we conclude that $||\lambda_\rho(f)|| \leq ||\lambda_\Gamma(f)||$.

Suppose now θ is an amenable invariant random subgroup of Γ. Since λ_θ is weakly contained in λ_Γ, λ_θ extends to a representation of $C^*_r(\Gamma)$ and φ_θ extends to a state on $C^*_r(\Gamma)$ via $a \mapsto \langle \lambda_\theta(a)(x^e), x^e\rangle$. Since φ_θ is conjugation invariant this is a tracial state. If θ is non-trivial then there is some $\gamma \in \Gamma \setminus \{e\}$ with $\varphi_\theta(\gamma) = \theta(\{H : \gamma \in H\}) > 0$ showing that this is distinct from the canonical trace. □

COROLLARY 5.15. *Let* Γ *be a countable group with the unique trace property. Then* Γ *has no non-trivial amenable invariant random subgroups. It follows that every non-trivial NA-ergodic action of* Γ *is free and* Γ *is shift-minimal.*

PROOF. That Γ has no non-trivial amenable invariant random subgroups follow from Theorem 5.14. If $\boldsymbol{a}$ is a non-trivial NA-ergodic action of Γ then the invariant random subgroup $\theta_{\boldsymbol{a}}$ is amenable by Theorem 3.13, and thus $\theta_{\boldsymbol{a}} = \delta_e$, i.e., $\boldsymbol{a}$ is free. Since every m.p. action weakly contained in $\boldsymbol{s}_\Gamma$ is NA-ergodic, Γ is also shift-minimal. □

REMARK 5.16. The positive definite function φ_θ associated to an invariant random subgroup θ is also realized in the Koopman representation $\kappa_0^{\boldsymbol{s}_\theta}$ corresponding to the θ-random Bernoulli shift $\boldsymbol{s}_{\theta,\eta}$ of Γ with a non-atomic base space (Z, η) (see [**TD12c**] for the definition of the θ-random Bernoulli shift). Indeed, take $Z = \mathbb{R}$ and take η to be the standard Gaussian measure (with unit variance). Let $p_\gamma : \mathbb{R}^{\leq\backslash\Gamma} \to \mathbb{R}$ be the function $p_\gamma(f) = f(H_f\gamma)$. Then $p_\gamma \in L^2_0(\eta^{\theta\backslash\Gamma})$ and each p_γ is a unit vector. In addition we have $\kappa_0^{\boldsymbol{s}_{\theta,\eta}}(\gamma)(p_e) = p_\gamma$ and

$$
\langle p_\gamma, p_e\rangle = \int_H \int_{f \in \mathbb{R}^{H\backslash\Gamma}} f(H\gamma)f(H)\, d\eta^{H\backslash\Gamma}\, d\theta(H) = \int_H 1_{\{H\,:\,H\gamma = H\}}\, d\theta = \varphi_\theta(\gamma) \tag{5.3}
$$

and so $(L^2_0(\eta^{\theta\backslash\Gamma}), \kappa_0^{\boldsymbol{s}_{\theta,\eta}}, p_e)$ is a triple realizing φ_θ.

6. Cost

6.1. Notation and background. See [**Gab00**] and [**KM04**] for background on the theory of cost of equivalence relations and groups. We recall the basic definitions to establish notation and terminology.

DEFINITION 6.1. Let (X, μ) be a standard non-atomic probability space.

(i) By an *L-graphing* on (X, μ) we mean a countable collection $\Phi = \{\varphi_i : A_i \to B_i\}_{i \in I}$ of partial Borel automorphism of X that preserve the measure μ. The *cost* of the L-graphing Φ is given by

$$C_\mu(\Phi) = \sum_{i \in I} \mu(A_i).$$

In (ii)-(vi) below Φ denotes an L-graphing on (X, μ).

(ii) We denote by $\mathcal{G}_\Phi$ the graph on X associated to Φ, i.e., for $x, y \in X$, $(x, y) \in \mathcal{G}_\Phi$ if and only if $x \neq y$ and $\varphi^{\pm 1}(x) = y$ for some $\varphi \in \Phi$. We let $d_\Phi : X \times X \to \mathbb{N} \cup \{\infty\}$ denote the graph distance corresponding to $\mathcal{G}_\Phi$, i.e., for $x, y \in X$,

$$d_\Phi(x, y) = \inf\{m \in \mathbb{N} : \exists \varphi_0, \dots, \varphi_{m-1} \in \Phi^* \ (\varphi_{m-1}^{\pm 1} \circ \cdots \circ \varphi_1^{\pm 1} \circ \varphi_0^{\pm 1}(x) = y)\}$$

where $\Phi^* = \Phi \cup \{\mathrm{id}_X\}$ and $\mathrm{id}_X : X \to X$ is the identity map.

(iii) We let E_Φ denote the equivalence relation on X generated by Φ, i.e., $xE_\Phi y \Leftrightarrow d_\Phi(x, y) < \infty$. Then E_Φ is a countable Borel equivalence relation that preserves the measure μ.

(iv) Let E be a measure preserving countable Borel equivalence relation on (X, μ). We say that Φ is an *L-graphing of* E if there is a conull set $X_0 \subseteq X$ such that $E_\Phi \upharpoonright X_0 = E \upharpoonright X_0$. This is equivalent to the condition that $[x]_{E_\Phi} = [x]_E$ for μ-almost every $x \in X$. The *cost* of E is defined as

$$C_\mu(E) = \inf\{C_\mu(\Psi) : \Psi \text{ is an L-graphing of } E\}.$$

(v) Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ be a measure preserving action of Γ. Let Q be a subset of Γ and let $A : Q \to \mathrm{MALG}_\mu$ be a function assigning to each $\delta \in Q$ a measurable subset A_δ of X. Then a and A define an L-graphing $\Phi^{a,A} = \{\varphi_\delta^{a,A} : \delta \in Q\}$, where $\varphi_\delta^{a,A} = \delta^a \upharpoonright A_\delta$, i.e.,

$\mathrm{dom}(\varphi_\delta^{a,A}) = A_\delta$ and $\varphi_\delta^{a,A}(x) = \delta^a x$ for each $x \in A_\delta$. It is clear that $E_{\Phi^{a,A}} \subseteq E_a$ and

$$C_\mu(\Phi^{a,A}) = \sum_{\delta \in Q} \mu(A_\delta)$$

so that $C_\mu(\Phi^{a,A})$ only depends on the assignment A and not on the action a.

(vi) As a converse to (v), whenever $E_\Phi \subseteq E_a$ we may find a function $A = A^{a,\Phi} : \Gamma \to \mathrm{MALG}_\mu$ such that $\mathcal{G}_{\Phi^{a,A}} = \mathcal{G}_\Phi$ and $C_\mu(\Phi^{a,A}) \leq C_\mu(\Phi)$. Indeed, for each $\varphi \in \Phi$ there exists a measurable partition $X = \bigsqcup_{\delta \in \Gamma} A_\delta^{a,\varphi}$ such that $\varphi \upharpoonright A_\delta^{a,\varphi} = \delta^a \upharpoonright A_\delta^{a,\varphi}$. Then taking $A_\delta = \bigcup_{\varphi \in \Phi} A_\delta^{a,\varphi}$ works.

For a measure preserving action $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ of Γ denote by E_a the orbit equivalence relation generated by a. The cost of $\boldsymbol{a}$ is defined by $C(\boldsymbol{a}) = C_\mu(E_a)$. Denote by $C(\Gamma)$ the cost of the group Γ, i.e., $C(\Gamma)$ is the infinimum of costs of free m.p. actions of Γ.

By ”subequivalence relation” we will always mean ”Borel subequivalence relation.”

6.2. Cost and weak containment in infinitely generated groups. Lemma 6.2 together with Theorem 6.4 provide a generalization of [**Kec10**, Theorem 10.13]. The purpose of Lemma 6.2 is to isolate versions of a few key observations from Kechris's proof.

LEMMA 6.2. *Let $F \subseteq \Gamma$ be finite and let $r \in \mathbb{R} \cup \{\infty\}$. Then the following are equivalent for a measure preserving action $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ of Γ:*

(1) *There exists a sub-equivalence relation E of E_a such that $E_{a \upharpoonright \langle F \rangle} \subseteq E \subseteq E_a$ and $C_\mu(E) < r$.*
(2) *There exists a finite $Q \subseteq \Gamma$ containing F and a sub-equivalence relation E of E_a such that $E_{a \upharpoonright \langle F \rangle} \subseteq E \subseteq E_{a \upharpoonright \langle Q \rangle}$ and $C_\mu(E) < r$.*
(3) *There exists a finite $Q \subseteq \Gamma$ containing F, an assignment $A : Q \to \mathrm{MALG}_\mu$, and a natural number $M \in \mathbb{N}$ such that*

$$C_\mu(\Phi^{a,A}) + \sum_{\gamma \in F} \mu(\{x \,:\, d_{\Phi^{a,A}}(x, \gamma^a x) > M\}) < r.$$

PROOF OF LEMMA 6.2. We begin with the implication (3)⇒(2). If such an $A : Q \to \mathrm{MALG}_\mu$ and $M \in \mathbb{N}$ exist then define $B : Q \to \Gamma$ by taking $B \upharpoonright Q \setminus F = A \upharpoonright Q \setminus F$ and for $\gamma \in F$ taking

$$B_\gamma = A_\gamma \cup \{x \,:\, d_{\Phi^{a,A}}(x, \gamma^a x) > M\}.$$

Let $E = E_{\Phi^{a,B}}$. Then $C_\mu(E) \leq C_\mu(\Phi^{a,B}) < r$ and $E_{\Phi^{a,B}} \subseteq E_{a\restriction\langle Q\rangle}$. In addition we have $E_{a\restriction\langle F\rangle} \subseteq E_{\Phi^{a,B}}$ since for each $\gamma \in F$ and $x \in X$, either $d_{\Phi^{a,A\restriction Q}}(x, \gamma^a x) \leq M$ so that $(x, \gamma^a x) \in E_{\Phi^{a,A}} \subseteq E_{\Phi^{a,B}}$, or $d_{\Phi^{a,A\restriction Q}}(x, \gamma^a x) > M$, in which case $x \in \mathrm{dom}(\varphi_\gamma^{a,B})$ and so $(x, \gamma^a x) \in E_{\Phi^{a,B}}$.

(2)⇒(1) is obvious, and it remains to show (1)⇒(3). Let E be as in (1) and let Φ be an L-graphing of E with $C_\mu(\Phi) = s < r$. Since $E \subseteq E_a$ we may by 6.1.(vi) assume without loss of generality that $\Phi = \Phi^{a,B}$ for some $B : \Gamma \to \mathrm{MALG}_\mu$, $\gamma \mapsto B_\gamma$. Let $\epsilon > 0$ be such that $s + \epsilon < r$.

We have $E_{a\restriction\langle F\rangle} \subseteq E = E_{\Phi^{a,B}}$ so, as F is finite, if we take a large enough finite set $Q \subseteq \Gamma$ containing F, we can ensure that

$$\sum_{\gamma \in F} \mu(\{x \,:\, d_{\Phi^{a,B\restriction Q}}(x, \gamma^a x) = \infty\}) < \epsilon.$$

So if we take $M \in \mathbb{N}$ large enough then

$$\sum_{\gamma \in F} \mu(\{x \,:\, d_{\Phi^{a,B\restriction Q}}(x, \gamma^a x) > M\}) < \epsilon.$$

It follows that $A = B \restriction Q$ and M satisfy the desired properties. □[Lemma 6.2]

DEFINITION 6.3. For each finite $F \subseteq \Gamma$ and $r \in \mathbb{R} \cup \{\infty\}$ let $A_{F,r} = A_{F,r}(\Gamma, X, \mu)$ denote the set of $\boldsymbol{a} \in A(\Gamma, X, \mu)$ that satisfy any - and therefore all - of the equivalent properties (1)-(3) of Lemma 6.2.

It is clear that the set $A_{F,r}(\Gamma, X, \mu)$ is an isomorphism-invariant (and in fact, orbit-equivalence-invariant) subset of $A(\Gamma, X, \mu)$. In what follows, we let $\mathrm{FR}(\Gamma, X, \mu)$ denote the subset of $A(\Gamma, X, \mu)$ consisting of all free actions.

THEOREM 6.4. *Let Γ be an infinite countable group. For each finite $F \subseteq \Gamma$ and $r \in \mathbb{R} \cup \{\infty\}$ the set $A_{F,r}(\Gamma, X, \mu) \cap \mathrm{FR}(\Gamma, X, \mu)$ is contained in the interior of $A_{F,r}(\Gamma, X, \mu)$. In particular, $A_{F,r}(\Gamma, X, \mu) \cap \mathrm{FR}(\Gamma, X, \mu)$ is open in $\mathrm{FR}(\Gamma, X, \mu)$.*

PROOF. Let $\boldsymbol{a} \in A_{F,r}$ be free and let $Q \subseteq \Gamma$, $A : Q \to \mathrm{MALG}_\mu$ and $M \in \mathbb{N}$ be given by Lemma 6.2.(3). For each $\gamma \in F$ let $s_\gamma^a = \mu(\{x \,:\, d_{\Phi^{a,A}}(x, \gamma^a x) > M\})$. Let $s = C_\mu(\Phi^{a,A}) + \sum_{\gamma \in F} s_\gamma^a$. By hypothesis we have $s < r$. Let $\epsilon > 0$ be small enough so that $s + |F|\epsilon < r$. Since the number $C_\mu(\Phi^{a,A}) = \sum_{\delta \in Q} \mu(A_\delta)$ is independent of $\boldsymbol{a}$, if we can show for each $\gamma \in F$ that the set

$$\{\boldsymbol{b} \in A(\Gamma, X, \mu) \,:\, \mu(\{x \,:\, d_{\Phi^{b,A}}(x, \gamma^b x) > M\}) < s_\gamma^a + \epsilon\} \tag{6.1}$$

contains an open neighborhood of $\boldsymbol{a}$, then the intersection of these sets as γ ranges over F will by Lemma 6.2 be a subset of $A_{F,r}$ containing an open neighborhood of $\boldsymbol{a}$ and we will be done.

Fix then $\gamma \in F$, let $Q^* = Q \cup \{e\}$ and let Σ be the collection

$$\Sigma = \{((\delta_{M-1}, \ldots, \delta_0), (\epsilon_{M-1}, \ldots, \epsilon_0)) \, : \, \delta_j \in Q^* \text{ and } \epsilon_j \in \{-1, 1\} \text{ for } j = 0, \ldots, M-1\}.$$

For each $\boldsymbol{b} \in A(\Gamma, X, \mu)$ and $\sigma \in \Sigma$, writing σ as

$$\sigma = ((\delta_{M-1}, \ldots, \delta_0), (\epsilon_{M-1}, \ldots, \epsilon_0)) \tag{6.2}$$

(where $\delta_j \in Q^*$ and $\epsilon_j \in \{-1, 1\}$ for $j = 0, \ldots, M-1$), we define

$$\varphi_\sigma^b := (\varphi_{\delta_{M-1}}^{b,A})^{\epsilon_{M-1}} \circ \cdots \circ (\varphi_{\delta_0}^{b,A})^{\epsilon_0}.$$

Let $\Sigma(\gamma)$ denote the set of all $\sigma \in \Sigma$ with the property that $\delta_{M-1}^{\epsilon_{M-1}} \cdots \delta_0^{\epsilon_0} = \gamma$. Observe that for $\sigma \in \Sigma(\gamma)$ and $\boldsymbol{b} \in A(\Gamma, X, \mu)$, if $x \in \mathrm{dom}(\varphi_\sigma^b)$ then $\varphi_\sigma^b(x) = \gamma^b x$ and so $d(x, \gamma^b x) \leq M$. It follows that

$$\{x \, : \, d_{\Phi^{b,A}}(x, \gamma^b x) > M\} \subseteq \bigcap_{\sigma \in \Sigma(\gamma)} X \setminus \mathrm{dom}(\varphi_\sigma^b). \tag{6.3}$$

If we assume further that $\boldsymbol{b}$ is (essentially) free then, ignoring a null set, the set containment (6.3) becomes an equality. Indeed, restricting to a co-null set X_0 on which b is free we have, for $x \in X_0$, if $d_{\Phi^{b,A}}(x, \gamma^b x) \leq M$ then there exists some $\sigma \in \Sigma$ such that $x \in \mathrm{dom}(\varphi_\sigma^b)$ and $\varphi_\sigma^b(x) = \gamma^b x$. Writing σ as in (6.2), this means that $(\delta_{M-1}^{\epsilon_{M-1}} \cdots \delta_0^{\epsilon_0})^b x = \gamma^b x$. Since $\boldsymbol{b}$ is free on X_0 this implies $\delta_{M-1}^{\epsilon_{M-1}} \cdots \delta_0^{\epsilon_0} = \gamma$ and therefore $\sigma \in \Sigma(\gamma)$.

Now, for each $\sigma \in \Sigma$ and $\boldsymbol{b} \in A(\Gamma, X, \mu)$ we see from the definition of φ_σ^b that the set $\mathrm{dom}(\varphi_\sigma^b)$ is an element of the Boolean algebra $\mathcal{A}^b$ generated by

$$\{\alpha^b A_\delta \, : \, \delta \in Q \text{ and } \alpha \in (Q^* \cup Q^{-1})^M\}$$

where $(Q^* \cup Q^{-1})^M = \{\delta_{M-1} \cdots \delta_1 \delta_0 \, : \, \delta_j \in Q^* \cup Q^{-1} \text{ for } j = 0, \ldots, M-1\}$. The algebra $\mathcal{A}^b$ is finite since Q is finite. The Boolean operations are continuous on MALG_μ, so if $\eta > 0$ is small enough (depending on ϵ, Q, and A) then every $\boldsymbol{b}$ in the open neighborhood U_η of $\boldsymbol{a}$ given by

$$U_\eta = \{\boldsymbol{b} \in A(\Gamma, X, \mu) \, : \, \forall \alpha \in (Q^* \cup Q^{-1})^M \, \forall \delta \in Q \; (\mu(\alpha^b A_\delta \Delta \alpha^a A_\delta) < \eta)\}$$

satisfies

$$\mu\Big(\bigcap_{\sigma\in\Sigma(\gamma)} X\setminus \mathrm{dom}(\varphi^b_\sigma)\Big) < \mu\Big(\bigcap_{\sigma\in\Sigma(\gamma)} X\setminus \mathrm{dom}(\varphi^a_\sigma)\Big) + \epsilon = s^a_\gamma + \epsilon$$

where the equality follows from the paragraph following (6.3) since $\boldsymbol{a}$ is free. By (6.3) we then have for such η and $\boldsymbol{b}\in U_\eta$ that

$$\mu(\{x\,:\,d_{\Phi^b,A}(x,\gamma^b x) > M\}) < s^a_\gamma + \epsilon$$

which shows that the open neighborhood U_η of $\boldsymbol{a}$ is contained in the set (6.1). □

Note that if $\boldsymbol{a}\in A(\Gamma,X,\mu)$ and $C_\mu(E_a) < r$, then $E = E_a$ witnesses that $\boldsymbol{a}$ satisfies property (1) of Lemma 6.2 and therefore $\boldsymbol{a}\in A_{F,r}(\Gamma,X,\mu)$ for all finite $F\subseteq\Gamma$. It is immediate that if Γ is generated by a finite set F_0 then $A_{F_0,r}(\Gamma,X,\mu) = \{\boldsymbol{a}\in A(\Gamma,X,\mu)\,:\,C(\boldsymbol{a}) < r\}$, so we recover (a slightly stronger formulation of) [**Kec10**, Theorem 10.13] in the following Corollary.

COROLLARY 6.5 (Kechris, [**Kec10**]). *Let Γ be an infinite, finitely generated group. Then the cost function $C : A(\Gamma,X,\mu)\to\mathbb{R}$ is upper semicontinuous at each $\boldsymbol{a}\in \mathrm{FR}(\Gamma,X,\mu)$, i.e.,*

$$\limsup_{\boldsymbol{b}\to\boldsymbol{a}} C(\boldsymbol{b}) \leq C(\boldsymbol{a}).$$

For general groups, Theorem 6.4 has several consequences for cost and weak containment. It will be helpful to introduce the following notation and definitions.

DEFINITION 6.6. Let $E_0, E_1, E_2, \dots$, and E be m.p. countable Borel equivalence relations on (X,μ). The sequence $(E_n)_{n\in\mathbb{N}}$ is called an *exhaustion of* E, denoted $(E_n)_{n\in\mathbb{N}}\uparrow E$, if $E_0\subseteq E_1\subseteq \cdots$, and $E = \bigcup_n E_n$. The *pseudocost* of E, denoted $PC_\mu(E)$, is defined by

$$PC_\mu(E) = \inf\{\liminf_n C_\mu(E_n)\,:\,(E_n)_{n\in\mathbb{N}}\uparrow E\}.$$

If $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ is a m.p. action of a countable group Γ then define the pseudocost of $\boldsymbol{a}$ by $PC(\boldsymbol{a}) := PC_\mu(E_a)$. Finally, define the pseudocost of Γ by $PC(\Gamma) := \inf\{PC(\boldsymbol{a})\,:\, \boldsymbol{a}$ is a free m.p. action of $\Gamma\}$.

It is shown in Corollary 6.17 below that the infimum in the definition of $PC_\mu(E)$ is always attained. If E is aperiodic then $PC_\mu(E)\geq 1$ by [**KM04**, 20.1 and 21.3]. We have $PC_\mu(E)\leq$

$C_\mu(E)$ as witnessed by the constant sequence $(E_n)_{n\in\mathbb{N}}$ given by $E_n = E$ for all n. In many cases we actually have the equality $PC_\mu(E) = C_\mu(E)$ as we now show. Recall that a countable Borel equivalence relation E on a standard Borel space X is called *treeable* if there exists an acyclic Borel graph $\mathcal{T} \subseteq X \times X$ whose connected components are the equivalence classes of E. Such a $\mathcal{T}$ is called a *treeing* of E, and we say that E is *treed* by $\mathcal{T}$ to mean that $\mathcal{T}$ is a treeing of E. A theorem of Gaboriau (Theorem 1 of [**Gab00**]) states that if μ is an E-invariant measure on X and if $\mathcal{T}$ is a treeing of E then $C_\mu(E) = C_\mu(\mathcal{T}) = \frac{1}{2}\int_x \deg_{\mathcal{T}}(x)\, d\mu$. This will be used implicitly below.

PROPOSITION 6.7. *Let E be a m.p. countable Borel equivalence relation on (X,μ) and let $(E_n)_{n\in\mathbb{N}}$ be an exhaustion of E.*

(1) *Suppose that $C_\mu(E) < \infty$. Then $C_\mu(E) \leq \liminf_n C_\mu(E_n)$.*
(2) *Suppose that E is treeable. Then $C_\mu(E) \leq \liminf_n C_\mu(E_n)$.*
(3) *(Gaboriau [**Gab00**]) Suppose that $\lim_n C_\mu(E_n) = 1$. Then $C_\mu(E) = 1$.*

In terms of pseudocost vs. cost this implies

COROLLARY 6.8. *Let E be a m.p. countable Borel equivalence relation on (X,μ).*

(1) *If $C_\mu(E) < \infty$ then $PC_\mu(E) = C_\mu(E)$.*
(2) *If E is treeable then $PC_\mu(E) = C_\mu(E)$.*
(3) *$PC_\mu(E) = 1$ if and only if $C_\mu(E) = 1$.*

PROOF OF PROPOSITION 6.7. (1): Let $r = \liminf_n C_\mu(E_n)$ and fix $\epsilon > 0$. We may assume that $r < \infty$. Let $\Phi = \{\varphi_i\}_{i=0}^\infty$ be an L-graphing of E with $C_\mu(\Phi) = \sum_{i\geq 0}\mu(\mathrm{dom}(\varphi_i)) < \infty$. Let N be so large that $\sum_{i>N}\mu(\mathrm{dom}(\varphi_i)) < \epsilon$. If $M_0 \in \mathbb{N}$ is large enough then for any $n > M_0$ we have $\sum_{i\leq N}\mu(\{x \in \mathrm{dom}(\varphi_i) : (x,\varphi_i(x)) \notin E_n\}) < \epsilon$. Since $r = \liminf_n C_\mu(E_n)$ we can find some $n > M_0$ with $C_\mu(E_n) < r+\epsilon$. Let Ψ be an L-graphing of E_n with $C_\mu(\Psi) < r+\epsilon$. Then

$$\Psi \sqcup \{\varphi_i\}_{i>N} \sqcup \{\varphi_i \restriction \{x \in \mathrm{dom}(\varphi_i) : (x,\varphi_i(x)) \notin E_n\}\}_{i\leq N}$$

is an L-graphing of E with cost strictly less than $r + 3\epsilon$.

(2): Let $\mathcal{T}$ be a treeing of E and let $\mathcal{T}_n = \mathcal{T} \cap E_n$. Then $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$ and $\mathcal{T} = \bigcup_n \mathcal{T}_n$ so $\lim_n C_\mu(\mathcal{T}_n) = C_\mu(\mathcal{T})$. Let R_n be the equivalence relation generated by $\mathcal{T}_n$. Then $R_n \subseteq E_n$ and $R_n \cap \mathcal{T} = \mathcal{T}_n$. We need the following lemma which is due to Clinton Conley.

LEMMA 6.9 (C. Conley). *Let F be a countable Borel equivalence relation treed by $\mathcal{T}_F$ and let $R \subseteq F$ be a subequivalence relation treed by $\mathcal{T}_R \subseteq \mathcal{T}_F$ (so that $\mathcal{T}_R = R \cap \mathcal{T}_F$). Then any equivalence relation R' with $R \subseteq R' \subseteq F$ has a treeing $\mathcal{T}_{R'}$ with $\mathcal{T}_R \subseteq \mathcal{T}_{R'}$.*

PROOF. Proposition 3.3.(iii) of [**JKL02**] shows how to obtain a treeing $\mathcal{T}_{R'}$ of R' from the given treeing $\mathcal{T}_F$ of F. It is clear from their construction that if an edge of $\mathcal{T}_F$ connects two R'-equivalent points, then that edge remains in $\mathcal{T}_{R'}$. Hence, every edge in $\mathcal{T}_R$ remains in $\mathcal{T}_{R'}$. □[Lemma 6.9]

Apply Lemma 6.9 to $F = E$, $R = R_n$, and $R' = E_n$, along with $\mathcal{T}_F = \mathcal{T}$ and $\mathcal{T}_R = \mathcal{T}_n$, to obtain a treeing $\mathcal{T}'_n$ of E_n with $\mathcal{T}_n \subseteq \mathcal{T}'_n$. Then $\liminf_n C_\mu(E_n) = \liminf_n C_\mu(\mathcal{T}'_n) \geq \liminf_n C_\mu(\mathcal{T}_n) = C_\mu(\mathcal{T})$.

(3): Since the E_n are increasing and $\lim_n C_\mu(E_n) = 1$ we have $|[x]_{E_n}| \to \infty$ almost surely (see [**KM04**, 22.1]), and so E is aperiodic. It follows that $PC_\mu(E) = 1$, so by Corollary 6.17 there is an exhaustion $(E'_n)_{n\in\mathbb{N}}$ of E with $C_\mu(E'_n) \to 1$ such that E'_n is aperiodic for all n. It follows from [**KM04**, Proposition 23.5] that $C_\mu(E) = 1$. □

REMARK 6.10. One may also deduce (2) of Proposition 6.7 by using the equality $C_\mu(E) - 1 = \beta_1(E) - \beta_0(E)$ for treeable E [**Gab02**, Corollary 3.23] along with [**Gab02**, Corollary 5.13].

COROLLARY 6.11. *If E is a m.p. treeable equivalence relation on (X, μ) of infinite cost then any increasing sequence $E_0 \subseteq E_1 \subseteq \cdots$, with $E = \bigcup_n E_n$ satisfies $C_\mu(E_n) \to \infty$.*

PROOF. Immediate from (2) of Proposition 6.7. □

REMARK 6.12. Corollary 6.11 may be seen as a generalization of a theorem of Takahasi.

COROLLARY 6.13 (Takahasi [**Tak50**]). *Suppose $H_0 \subseteq H_1 \subseteq \cdots$ is an ascending chain of subgroups of a free group F, and assume that the H_n have rank uniformly bounded by some natural number $r < \infty$. Then all H_n coincide for n sufficiently large.*

PROOF. Suppose that infinitely many H_n are distinct. Then $H = \bigcup_n H_n$ has infinite rank, so Corollary 6.11 implies that for any free m.p. action $H \curvearrowright^a (X, \mu)$ we have $C_\mu(E_{a\restriction H_n}) \to \infty$, contradicting that $\sup_n C_\mu(E_{a\restriction H_n}) \leq \sup_n \mathrm{rank}(H_n) \leq r$. □

We will use another characterization of pseudocost in order to show that it respects weak containment. In what follows, a sequence $(Q_n)_{n\in\mathbb{N}}$ of subsets of a countable group Γ is called an *exhaustion of* Γ if $Q_0 \subseteq Q_1 \subseteq \cdots$ and $\bigcup_n Q_n = \Gamma$. A sequence $(Q_n)_{n\in\mathbb{N}}$ is called a *finite exhaustion of* Γ if $(Q_n)_{n\in\mathbb{N}}$ is an exhaustion of Γ and Q_n is finite for all $n \in \mathbb{N}$.

LEMMA 6.14. *Let E be a m.p. countable Borel equivalence relation on (X, μ) and let $r \in \mathbb{R} \cup \{\infty\}$. Then the following are equivalent:*

(1) *There exists an exhaustion $(E_n)_{n\in\mathbb{N}}$ of E with $\limsup_n C_\mu(E_n) \leq r$.*

(2) *For any countable group Γ and any m.p. action $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ with $E = E_b$, and any sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Γ, there exists a finite exhaustion $(Q_n)_{n\in\mathbb{N}}$ of Γ along with an exhaustion $(E_n)_{n\in\mathbb{N}}$ of E such that $F_n \subseteq Q_n$ and $E_{b\restriction\langle Q_n\rangle} \subseteq E_n \subseteq E_{b\restriction\langle Q_{n+1}\rangle}$ for all $n \in \mathbb{N}$, and $\limsup_n C_\mu(E_n) \leq r$.*

(3) *For any countable group Γ, any m.p. action $\Gamma \curvearrowright^b (X, \mu)$ with $E = E_b$, and any sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Γ, there exists an exhaustion $(E_n)_{n\in\mathbb{N}}$ of E satisfying $E_{b\restriction\langle F_n\rangle} \subseteq E_n$ for all n and $\limsup_n C_\mu(E_n) \leq r$.*

(4) *For any countable group Γ and any m.p. action $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ with $E = E_b$, we have $\boldsymbol{b} \in A_{F,r+\epsilon}$ for all finite $F \subseteq \Gamma$ and all $\epsilon > 0$.*

(5) *There exists a countable group Γ and a m.p. action $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ with $E = E_b$ such that $\boldsymbol{b} \in A_{F,r+\epsilon}$ for all finite $F \subseteq \Gamma$ and all $\epsilon > 0$.*

(6) *There exists a countable group Γ and a m.p. action $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ with $E = E_b$, along with an exhaustion $(Q_n)_{n\in\mathbb{N}}$ of Γ and a (not necessarily increasing) sequence $(E_n)_{n\in\mathbb{N}}$ of subequivalence relations of E such that $E_{b\restriction\langle Q_n\rangle} \subseteq E_n$ and $\limsup_n C_\mu(E_n) \leq r$.*

REMARK 6.15. It is clear that each of the conditions (1), (2), (3), and (6) of Lemma 6.14 are equivalent to their counterparts in which "$\limsup$" is replaced with "$\liminf$" or with "$\lim$."

PROOF OF 6.14. (1)$\Rightarrow$(4): Assume that $(E_n)_{n\in\mathbb{N}}$ is a sequence as in (1). Let Γ and $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ with $E = E_b$ be given. Fix a finite $F \subseteq \Gamma$ and $\epsilon > 0$. Let $n \in \mathbb{N}$ be large enough so that $C_\mu(E_n) < r + \epsilon/2$ and $\sum_{\gamma\in F} \mu(\{x : \gamma^b x \notin [x]_{E_n}\}) < \epsilon/2$. Let $\Phi = \big\{\gamma^b \restriction \{x : \gamma^b x \notin [x]_{E_n}\}\big\}_{\gamma\in F}$. Then $R := E_n \vee E_\Phi$ is a subequivalence relation of E containing $E_{b\restriction\langle F\rangle}$ with $C_\mu(R) \leq C_\mu(E_n) + C_\mu(\Phi) < r + \epsilon/2 + \epsilon/2 = r + \epsilon$. Then R witnesses that $\boldsymbol{b} \in A_{F,r+\epsilon}(\Gamma, X, \mu)$. This shows that (4) holds.

(4)$\Rightarrow$(2): Assume (4) holds. Let Γ and $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ with $E = E_b$ be given along with a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Γ. We may assume without loss of generality that $(F_n)_{n\in\mathbb{N}}$

is a finite exhaustion of Γ. Fix some sequence of real numbers $\epsilon_n > 0$ with $\epsilon_n \to 0$. We proceed by induction to construct sequences $(Q_n)_{n\in\mathbb{N}}$ and $(E_n)_{n\in\mathbb{N}}$ as in (2). Define $Q_0 = F_0$. Suppose for induction that we have constructed finite subsets $Q_0 \subseteq Q_1 \subseteq \cdots Q_k$ of Γ and equivalence relations $E_0, \ldots, E_{k-1}$ with $F_i \subseteq Q_i$ for all $i \leq k$ and $E_{b\restriction\langle Q_i\rangle} \subseteq E_i \subseteq E_{b\restriction\langle Q_{i+1}\rangle}$ for all $i < k$. By (4) we have $\boldsymbol{b} \in A_{Q_k\cup F_{k+1}, r+\epsilon_k}$, so by Lemma 6.2 there exists a finite $Q_{k+1} \subseteq \Gamma$ containing $Q_k \cup F_{k+1}$ and a subequivalence relation E_k of E_b with $E_{b\restriction\langle Q_k\rangle} \subseteq E_k \subseteq E_{b\restriction\langle Q_{k+1}\rangle}$ and $C_\mu(E_k) < r + \epsilon_k$. Then Q_{k+1} and E_k extend the induction to the next stage. We obtain from this inductive procedure sequences (Q_n) and (E_n) which satisfy (2) by construction.

(2)⇒(3) is clear. (3)⇒(6) holds since there always exists some countable group Γ and some m.p. action $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ with $E = E_b$ (see [**FM77**]). (6)⇒(5) is routine. Finally, the proof of $(4) \Rightarrow (2)$ shows that $(5) \Rightarrow (1)$. □

REMARK 6.16. If the equivalence relation E in Lemma 6.14 is aperiodic then condition (1) implies the stronger statement (1^*) in which the equivalence relations E_n are additionally required to be aperiodic. Indeed, assume that E is aperiodic and that (1) holds. Then (3) holds as well. By [**Kec10**, 3.5] there is an aperiodic $T \in [E]$. Take any countable subgroup $\Gamma \leq [E]$ that generates E and with $T \in \Gamma$. Then Γ naturally acts on (X, μ) as a subgroup of $[E]$. Take some finite exhaustion $\{F_n\}_{n\in\mathbb{N}}$ of Γ with $T \in F_0$. Now apply (3) of Lemma 6.14 to this sequence $\{F_n\}_{n\in\mathbb{N}}$ to obtain the desired aperiodic sequence satisfying (1^*).

Similarly, if E is aperiodic then (3), and (6) of Lemma 6.14 are each equivalent to their counterparts (3*), and (6*), in which the equivalence relations E_n are each required to be aperiodic.

COROLLARY 6.17. *Let E be a m.p. countable Borel equivalence relation on (X, μ). There exists an exhaustion $(E_n)_{n\in\mathbb{N}} \uparrow E$ with $\lim_n C_\mu(E_n) = PC_\mu(E)$. In other words, the infimum in the definition of pseudocost is always attained. In addition, if E is aperiodic then such an exhaustion $(E_n)_{n\in\mathbb{N}}$ exists with E_n aperiodic for all n.*

PROOF. Let $s = PC_\mu(E)$. By definition of $PC_\mu(E)$, for any $\delta > 0$ there exists a sequence $(E_n^\delta)_{n\in\mathbb{N}} \uparrow E$ with $\limsup_n C_\mu(E_n^\delta) < s + \delta/2$. By [**FM77**] there is a countable group Γ and some action $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ of Γ such that $E = E_b$. Now, E satisfies (1) of Lemma 6.14 with respect to the parameter $r = s + \delta/2$, so by (1)⇒(4) of Lemma 6.14 we have $\boldsymbol{b} \in A_{F, s+\delta/2+\epsilon}$ for all finite $F \subseteq \Gamma$ and $\epsilon > 0$. Taking $\epsilon = \delta/2$ shows that $\boldsymbol{b} \in A_{F, r+\delta}$ for all finite $F \subseteq \Gamma$. Since

$\delta > 0$ was arbitrary this shows that $\boldsymbol{b}$ satisfies (5) of Lemma 6.14 with respect to the parameter s, so by (5)⇒(1) Lemma 6.14 there exists a sequence $(E_n)_{n\in\mathbb{N}} \uparrow E$ with $\limsup_n C_\mu(E_n) \leq s$. Since $s = PC_\mu(E) \leq \liminf_n C_\mu(E_n)$ this shows that in fact $\lim_n C_\mu(E_n) = PC_\mu(E)$. By remark 6.16 if E is aperiodic then we can choose such a sequence $(E_n)_{n\in\mathbb{N}}$ with E_n aperiodic for all n. □

COROLLARY 6.18. *Let E be an aperiodic m.p. countable Borel equivalence relation on (X, μ). Assume that E is ergodic. Then for any exhaustion $(R_n)_{n\in\mathbb{N}}$ of E satisfying $C_\mu(R_n) < \infty$ for all $n \in \mathbb{N}$, there exists an exhaustion $(E_n)_{n\in\mathbb{N}}$ of E with $R_n \subseteq E_n$ for all $n \in \mathbb{N}$ and $\lim_n C_\mu(E_n) = PC_\mu(E)$.*

PROOF. Let $(R_n)_{n\in\mathbb{N}}$ be an exhaustion of E with $C_\mu(R_n) < \infty$ for all n. Since E is ergodic we many apply [**KM04**, Lemma 27.7] to obtain, for each $n \in \mathbb{N}$, a finitely generated group Γ_n and a m.p. action $\boldsymbol{b}_n = \Gamma_n \curvearrowright^{b_n} (X, \mu)$ with $R_n = R_{b_n}$. There is a unique action $\boldsymbol{b} = \Gamma \curvearrowright^{a} (X, \mu)$ of the free product Γ of $\{\Gamma_n\}_{n\in\mathbb{N}}$ satisfying $\boldsymbol{b} \upharpoonright \Gamma_n = \boldsymbol{b}_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let F_n be a finite generating set for Γ_n. By Corollary 6.17 there exists an exhaustion $(E'_n)_{n\in\mathbb{N}}$ of E with $\lim_n C_\mu(E'_n) = r$ where $r = PC_\mu(E)$. This shows that E satisfies (1) of Lemma 6.14, so, by applying (3) of Lemma 6.14 to the action $\boldsymbol{b}$ and the sequence $(F_n)_{n\in\mathbb{N}}$, we obtain an exhaustion $(E_n)_{n\in\mathbb{N}}$ of E with $R_n = E_{b\upharpoonright\Gamma_n} \subseteq E_n$ and $\limsup_n C_\mu(E_n) \leq r$. Since $r = PC_\mu(E)$ it follows that $\lim_n C_\mu(E_n) = PC_\mu(E)$. □

COROLLARY 6.19. *Let $\boldsymbol{a} = \Gamma \curvearrowright^{a} (X, \mu)$ be a m.p. action of Γ. Then $PC(\boldsymbol{a}) \leq r$ if and only if $\boldsymbol{a} \in A_{F,r+\epsilon}$ for every finite $F \subseteq \Gamma$ and $\epsilon > 0$.*

PROOF. This follows from the equivalence (1)⇔(4) from Lemma 6.14. □

COROLLARY 6.20. *Let $\boldsymbol{a} = \Gamma \curvearrowright^{a} (X, \mu)$ and $\boldsymbol{b} = \Gamma \curvearrowright^{b} (Y, \nu)$ be measure preserving actions of a countable group Γ. Assume that $\boldsymbol{a}$ is free. If $\boldsymbol{a} \prec \boldsymbol{b}$ then $PC(\boldsymbol{b}) \leq PC(\boldsymbol{a})$.*

PROOF. Let $r = PC(\boldsymbol{a})$. Fix $F \subseteq \Gamma$ finite and $\epsilon > 0$. Since $PC(\boldsymbol{a}) = r$ we have $\boldsymbol{a} \in A_{F,r+\epsilon}(\Gamma, X, \mu)$ by Corollary 6.19. Since $\boldsymbol{a}$ is free, Theorem 6.4 implies that $\boldsymbol{a}$ is contained in the interior of $A_{F,r+\epsilon}(\Gamma, X, \mu)$, so by [**Kec10**, Proposition 10.1] there exists some $\boldsymbol{c} \in A_{F,r+\epsilon}(\Gamma, X, \mu)$ which is isomorphic to $\boldsymbol{b}$. Hence $\boldsymbol{b} \in A_{F,r+\epsilon}(\Gamma, Y, \nu)$ and therefore $PC(\boldsymbol{b}) \leq r$ by Corollary 6.19. □

COROLLARY 6.21. *Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ and $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ be measure preserving actions of a countably infinite group Γ. Assume that $\boldsymbol{a}$ is free and is weakly contained in $\boldsymbol{b}$. Then there exists an exhaustion $(E_n)_{n\in\mathbb{N}}$ of E with $\lim_n C_\mu(E_n) \leq C(\boldsymbol{a})$ and E_n aperiodic for all $n \in \mathbb{N}$.*

PROOF. Corollary 6.20 tells us that $PC(\boldsymbol{b}) \leq PC(\boldsymbol{a})$, so by 6.17 we can find an exhaustion $(E_n)_{n\in\mathbb{N}}$ of E, with $\lim_n C_\mu(E_n) \leq PC(\boldsymbol{a})$ and E_n aperiodic for all $n \in \mathbb{N}$. Since $PC(\boldsymbol{a}) \leq C(\boldsymbol{a})$ we are done. □

COROLLARY 6.22. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be m.p. actions of a countably infinite group Γ. Assume that $\boldsymbol{a}$ is free and $\boldsymbol{a} \prec \boldsymbol{b}$.*

(1) *If $C(\boldsymbol{b}) < \infty$ then $C(\boldsymbol{b}) \leq C(\boldsymbol{a})$.*
(2) *If E_b is treeable then $C(\boldsymbol{b}) \leq C(\boldsymbol{a})$.*
(3) *If $C(\boldsymbol{a}) = 1$ then $C(\boldsymbol{b}) = 1$.*

PROOF. (1) and (2): Suppose $C(\boldsymbol{b}) < \infty$ or E_b is treeable. Then by Corollary 6.8 and Corollary 6.20 we have $C(\boldsymbol{b}) = PC(\boldsymbol{b}) \leq PC(\boldsymbol{a}) \leq C(\boldsymbol{a})$.

Similarly, if $C(\boldsymbol{a}) = 1$ then by Corollary 6.20 we have $PC(\boldsymbol{b}) \leq PC(\boldsymbol{a}) \leq C(\boldsymbol{a}) = 1$, so $PC(\boldsymbol{b}) = 1$ and thus $C(\boldsymbol{b}) = 1$ by Corollary 6.8. □

DEFINITION 6.23. A group Γ is said to have *fixed price* 1 if $C(\boldsymbol{a}) = 1$ for every free measure preserving action $\boldsymbol{a}$ of Γ.

In [**AW11**], Abért and Weiss combine their theorem on free actions (stated above in Theorem 3.1) with [**Kec10**, Theorem 10.13] to characterize finitely generated groups Γ with fixed price 1 in terms of the Bernoulli shift $\boldsymbol{s}_\Gamma$. We can now remove the hypothesis that Γ is finitely generated.

COROLLARY 6.24. *Let Γ be a countable group. Then the following are equivalent:*

(1) *Γ has fixed price 1*
(2) *$C(\boldsymbol{s}_\Gamma) = 1$*
(3) *$C(\boldsymbol{a}) = 1$ for some m.p. action $\boldsymbol{a}$ weakly equivalent to $\boldsymbol{s}_\Gamma$.*
(4) *$PC(\boldsymbol{a}) = 1$ for some m.p. action $\boldsymbol{a}$ weakly equivalent to $\boldsymbol{s}_\Gamma$.*
(5) *Γ is infinite and $C(\boldsymbol{a}) \leq 1$ for some non-trivial m.p. action $\boldsymbol{a}$ weakly contained in $\boldsymbol{s}_\Gamma$.*

PROOF. (1)⇒(2) holds since $\boldsymbol{s}_\Gamma$ is free. (2)⇒(3) is clear. (3) ⇔ (4) follows from Corollary 6.8. Suppose that (3) holds and we will prove (1). Let $\boldsymbol{a}$ be weakly equivalent to $\boldsymbol{s}_\Gamma$ with $C(\boldsymbol{a}) = 1$.

This implies $\boldsymbol{a}$ is free. If $\boldsymbol{b}$ is another free measure preserving action of Γ then $\boldsymbol{a} \prec \boldsymbol{b}$ by Theorem 3.1, so Corollary 6.22 shows that $C(\boldsymbol{b}) = 1$. Thus Γ has fixed price 1. This shows that properties (1), (2), and (3) are equivalent. The implication (3)$\Rightarrow$(5) is clear.

The proof of the remaining implication (5)$\Rightarrow$(3) uses Lemma 6.34, proved in §6.5 below. Assume that (5) holds. Let $\boldsymbol{a} = \Gamma \curvearrowright^{a} (X, \mu)$ be a non-trivial action weakly contained in $\boldsymbol{s}_\Gamma$ with $C(\boldsymbol{a}) \leq 1$. Let $\theta = \theta_{\boldsymbol{a}}$. If Γ is amenable then (1) holds, so we may assume that Γ is non-amenable. Then $\boldsymbol{s}_\Gamma$ is strongly ergodic, hence both $\boldsymbol{a}$ and $\boldsymbol{\theta}$ are weakly mixing. It follows that θ is either a point mass at some finite normal subgroup N of Γ, or θ concentrates on the infinite subgroups of Γ.

Case 1: θ is a point mass at some finite normal subgroup $N \leq \Gamma$. Then $C(\boldsymbol{a}) = 1$ since E_a is aperiodic. By [**CKTD11**, Proposition 4.7] there is some $\boldsymbol{b} = \Gamma \curvearrowright^{b} (Y, \nu)$ weakly equivalent to $\boldsymbol{s}_\Gamma$ such that $\boldsymbol{a}$ is a factor of $\boldsymbol{b}$, say via the factor map $\pi : Y \to X$. Let Y_0 be a Borel transversal for the orbits of $N \curvearrowright^{b} (Y, \nu)$ and let $\sigma : Y \to Y_0$ be the corresponding selector. Let ν_0 denote the normalized restriction of ν to Y_0 and let $\boldsymbol{b}_0$ be the action of Γ on (Y_0, ν_0) given by $\gamma^{b_0} y = \sigma(\gamma^{b} y)$. Then π factors $\boldsymbol{b}_0$ onto $\boldsymbol{a}$. Since $\theta_{\boldsymbol{a}} = \theta_{\boldsymbol{b}_0} = \delta_N$, the actions $\boldsymbol{a}$ and $\boldsymbol{b}_0$ descend to free actions $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}_0$, respectively, of Γ/N, and π factors $\tilde{\boldsymbol{b}}_0$ onto $\tilde{\boldsymbol{a}}$. Then $C(\tilde{\boldsymbol{a}}) = C(\boldsymbol{a}) = 1$, so $C(\tilde{\boldsymbol{b}}_{\boldsymbol{0}}) = 1$ by Corollary 6.22. Since $E_{b_0} = E_b \restriction Y_0$ we have $C_{\nu_0}(E_b \restriction Y_0) = 1$, so $C(\boldsymbol{b}) = C_\nu(E_b) = 1$ by [**KM04**, Theorem 25.1] ([**KM04**, Theorem 21.1] also works). This shows that (3) holds.

Case 2: θ is infinite. We have $\boldsymbol{a} \prec \boldsymbol{s}_\Gamma$, so $\boldsymbol{a}$ is NA-ergodic and therefore θ is amenable by Theorem 3.13. Then $C(\boldsymbol{\theta}_{\boldsymbol{a}} \times \boldsymbol{s}_\Gamma) = 1$ by Lemma 6.34, and $\boldsymbol{\theta}_{\boldsymbol{a}} \times \boldsymbol{s}_\Gamma$ is weakly equivalent to $\boldsymbol{s}_\Gamma$, so (3) holds. $\square$

NOTE 6.25. Similar to [**Kec10**, Corollary 10.14], one may strengthen Corollaries 6.20, 6.21, and 6.22 by replacing the hypothesis $\boldsymbol{a} \prec \boldsymbol{b}$ their statements with the weaker hypothesis that

$$\boldsymbol{a} \in \overline{\{\boldsymbol{c} \in A(\Gamma, X, \mu) \, : \, E_c \text{ is orbit equivalent to } E_b\}} \tag{6.4}$$

where (X, μ) is the underlying space of $\boldsymbol{a}$. The proofs remain the same. Note that (6.4) is actually slightly weaker than the hypothesis $\boldsymbol{a} \preceq \boldsymbol{b}$ from [**Kec10**, Corollary 10.14], since the action $\boldsymbol{c}$ from (6.4) ranges over all of $A(\Gamma, X, \mu)$ and not just $\mathrm{FR}(\Gamma, X, \mu)$. Specializing to the case where Γ is finitely generated, we recover a somewhat strengthened version of the first statement of [**Kec10**, Corollary 10.14].

6.3. The cost of a generic action. The results of the previous section have consequences for generic properties (with respect to the weak topology) in $\mathrm{FR}(\Gamma, X, \mu)$ related to cost. We begin by proving analogues of Corollaries 6.17 and 6.8 for groups. Recall that a countable group Γ is called *treeable* if it admits a free measure preserving action $\boldsymbol{a}$ such that E_a is treeable.

PROPOSITION 6.26. *Let Γ be a countably infinite group.*

(1) *Suppose that $C(\Gamma) < \infty$. Then for any free m.p. action $\boldsymbol{b} = \Gamma \curvearrowright^b (X, \mu)$ of Γ, and any exhaustion $(E_n)_{n\in\mathbb{N}}$ of E_b, we have $\liminf_{n\to\infty} C_\mu(E_n) \geq C(\Gamma)$. Hence $PC(\Gamma) = C(\Gamma)$.*
(2) *Suppose that Γ is treeable. Then $PC(\Gamma) = C(\Gamma)$.*
(3) *$PC(\Gamma) = 1$ if and only if $C(\Gamma) = 1$.*
(4) *$PC(\Gamma)$ is attained by some free m.p. action of Γ. In fact, if $\boldsymbol{a} \in FR(\Gamma, X, \mu)$ has dense conjugacy class in $(FR(\Gamma, X, \mu), w)$ then $PC(\boldsymbol{a}) = PC(\Gamma)$.*

PROOF. (1): Let $\boldsymbol{b}$ be a free m.p. action of Γ. It suffices to show that $PC(\boldsymbol{b}) \geq C(\Gamma)$. Let $\boldsymbol{a}$ be a free m.p. action of Γ with $C(\boldsymbol{a}) = C(\Gamma) < \infty$ and let $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. Then by the remark at the bottom of p. 78 in [**Kec10**] we have $C(\boldsymbol{c}) \leq C(\boldsymbol{a}) = C(\Gamma)$, hence $C(\boldsymbol{c}) = C(\Gamma) < \infty$. Since $C(\boldsymbol{c}) < \infty$ we have $PC(\boldsymbol{c}) = C(\boldsymbol{c})$ by (1) of Corollary 6.8. In addition, $\boldsymbol{b} \prec \boldsymbol{c}$ and $\boldsymbol{b}$ is free, so Corollary 6.20 implies $PC(\boldsymbol{b}) \geq PC(\boldsymbol{c}) = C(\boldsymbol{c}) = C(\Gamma)$.

(2): Let $\boldsymbol{b}$ be a free m.p. action of Γ. Once again it suffices to show $PC(\boldsymbol{b}) \geq C(\Gamma)$. Let $\boldsymbol{a}$ be a free m.p. action of Γ with E_a treeable and let $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. By [**KM04**, Proposition 30.5] E_c is treeable and $C(\boldsymbol{c}) = C(\boldsymbol{a}) = C(\Gamma)$. Then (2) of Corollary 6.8 implies that $PC(\boldsymbol{c}) = C(\boldsymbol{c})$, so, as $\boldsymbol{b} \prec \boldsymbol{c}$, Corollary 6.20 implies that $PC(\boldsymbol{b}) \geq PC(\boldsymbol{c}) = C(\boldsymbol{c}) = C(\Gamma)$.

(3): This is immediate from (3) of Corollary 6.8.

(4): If $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ has dense conjugacy class this means that $\boldsymbol{b} \prec \boldsymbol{a}$ for every m.p. action $\boldsymbol{b}$ of Γ [**Kec10**, Proposition 10.1] (also note that such an $\boldsymbol{a}$ exists by [**Kec10**, Theorem 10.7]). Corollary 6.20 then shows that $PC(\boldsymbol{a}) \leq \inf\{PC(\boldsymbol{b}) : \boldsymbol{b} \in \mathrm{FR}(\Gamma, X, \mu)\} = PC(\Gamma)$, hence $PC(\boldsymbol{a}) = PC(\Gamma)$. $\square$

By [**Kec10**, Proposition 10.10] the cost function $\boldsymbol{a} \mapsto C(\boldsymbol{a})$ is constant on a dense G_δ subset of $\mathrm{FR}(\Gamma, X, \mu)$. Let $C_{\mathrm{gen}}(\Gamma) \in [0, \infty]$ denote this constant value. Similarly, the pseudocost function $\boldsymbol{a} \mapsto PC(\boldsymbol{a})$ is constant on a dense G_δ subset of $\mathrm{FR}(\Gamma, X, \mu)$. Denote this constant value by

$PC_{\text{gen}}(\Gamma)$. Problem 10.11 of [**Kec10**] asks whether $C_{\text{gen}}(\Gamma) = C(\Gamma)$ holds for every countably infinite group Γ, and [**Kec10**, Corollary 10.14] shows that the equality holds whenever Γ is finitely generated.

COROLLARY 6.27. *Let Γ be a countably infinite group. Then*

(1) *The set* $\text{MINPCOST}(\Gamma, X, \mu) = \{\boldsymbol{a} \in \text{FR}(\Gamma, X, \mu) : PC(\boldsymbol{a}) = PC(\Gamma)\}$ *is dense G_δ in $A(\Gamma, X, \mu)$. In particular, $PC_{gen}(\Gamma) = PC(\Gamma)$.*

(2) *Either $C_{gen}(\Gamma) = C(\Gamma)$ or $C_{gen}(\Gamma) = \infty$.*

(3) *If $PC(\Gamma) = 1$ then $C_{gen}(\Gamma) = C(\Gamma) = 1$.*

PROOF. (1): Let $r = PC(\Gamma)$. Corollary 6.19 shows that

$$\text{MINPCOST}(\Gamma, X, \mu) = \bigcap \{A_{F,r+1/n}(\Gamma, X, \mu) \cap \text{FR}(\Gamma, X, \mu) : F \subseteq \Gamma \text{ is finite and } n \in \mathbb{N}\}.$$

To show this set is dense G_δ in $A(\Gamma, X, \mu)$ it therefore suffices to show that $A_{F,r+\epsilon}(\Gamma, X, \mu) \cap \text{FR}(\Gamma, X, \mu)$ is dense G_δ for each $F \subseteq \Gamma$ finite and $\epsilon > 0$. By [**Kec10**, Theorem 10.8], the set $\text{FR}(\Gamma, X, \mu)$ is dense G_δ in $A(\Gamma, X, \mu)$. Theorem 6.4 shows that $A_{F,r+\epsilon}$ is relatively open in $\text{FR}(\Gamma, X, \mu)$, so it only remains to show that it is dense. By Proposition 6.26 we have $PC(\boldsymbol{a}) = PC(\Gamma)$ whenever $\boldsymbol{a} \in \text{FR}(\Gamma, X, \mu)$ has a dense conjugacy class. Since the set of actions with dense conjugacy class is dense G_δ in $\text{FR}(\Gamma, X, \mu)$ the result follows.

(2): Suppose that $C_{\text{gen}}(\Gamma) = r < \infty$. This means the generic $\boldsymbol{a} \in \text{FR}(\Gamma, X, \mu)$ has $C(\boldsymbol{a}) = r$. Since $r < \infty$ it follows from Corollary 6.8 that $C(\boldsymbol{a}) = r \Rightarrow C(\boldsymbol{a}) = PC(\boldsymbol{a})$. Thus the generic free action $\boldsymbol{a}$ satisfies $PC(\boldsymbol{a}) = r = C(\boldsymbol{a})$ and by part (1) we therefore have $C(\Gamma) \geq PC(\Gamma) = PC_{\text{gen}} = C_{\text{gen}}(\Gamma) \geq C(\Gamma)$, which shows that $C_{\text{gen}}(\Gamma) = C(\Gamma)$.

(3) follows from (1) along with Corollary 6.8. $\square$

Let $\text{MINCOST}(\Gamma, X, \mu) = \{\boldsymbol{a} \in \text{FR}(\Gamma, X, \mu) : C(\boldsymbol{a}) = C(\Gamma)\}$.

COROLLARY 6.28. *Let Γ be a countably infinite group. Then the set*

$$D = \Big\{\boldsymbol{b} \in \text{FR}(\Gamma, X, \mu) : \exists \textit{aperiodic subequivalence relations} \\ E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \textit{ of } E_b, \textit{ with } E_b = \bigcup_n E_n \textit{ and } \lim_n C_\mu(E_n) = C(\Gamma)\Big\}$$

is dense G_δ in $A(\Gamma, X, \mu)$. Additionally, if $C(\Gamma) < \infty$ then we have the equality of sets

$$\text{MINCOST}(\Gamma, X, \mu) = D \cap \{\boldsymbol{b} \in \text{FR}(\Gamma, X, \mu) \,:\, C(\boldsymbol{b}) < \infty\}. \tag{6.5}$$

In particular, if all free actions of Γ have finite cost then $\text{MINCOST}(\Gamma, X, \mu) = D$ *is dense G_δ.*

PROOF. We begin by showing D is dense G_δ. By [**Kec10**, Theorem 10.8], $\text{FR}(\Gamma, X, \mu)$ is dense G_δ in $A(\Gamma, X, \mu)$. If $C(\Gamma) = \infty$ then $D = \text{FR}(\Gamma, X, \mu)$ and we are done, so we may assume that $C(\Gamma) < \infty$. Then $C(\Gamma) = PC(\Gamma)$ by Proposition 6.26, so it follows from Corollary 6.17 that $D = \{\boldsymbol{a} \in \text{FR}(\Gamma, X, \mu) \,:\, PC(\boldsymbol{a}) = PC(\Gamma)\} = \text{MINPCOST}(\Gamma, X, \mu)$, and therefore D is dense G_δ by Corollary 6.27.

For the second statement of the theorem, suppose that $C(\Gamma) < \infty$. Then $C(\Gamma) = PC(\Gamma)$ by Proposition 6.26. The inclusion from left to right in (6.5) is clear. If $\boldsymbol{b}$ has finite cost and $\boldsymbol{b} \in D$ then, $PC(\boldsymbol{b}) \leq C(\Gamma) = PC(\Gamma)$, hence $PC(\boldsymbol{b}) = PC(\Gamma) = C(\Gamma)$, i.e., $\boldsymbol{b} \in \text{MINCOST}(\Gamma, X, \mu)$. □

6.4. Cost and invariant random subgroups. Equip each of the spaces Γ^Γ and 2^Γ with the pointwise convergence topology.

LEMMA 6.29. *There exists a continuous assignment* $\text{Sub}_\Gamma \to \Gamma^\Gamma$*, $H \mapsto \sigma_H$, with the following properties:*

(i) *For each $H \in \text{Sub}_\Gamma$, $\sigma_H : \Gamma \to \Gamma$ is a selector for the right cosets of H in Γ, i.e., $\sigma_H(\delta) \in H\delta$ for all $\delta \in \Gamma$, and σ_H is constant on each right coset of H.*

(ii) *$\sigma_H(h) = e$ whenever $h \in H$.*

(iii) *The corresponding assignment of transversals* $\text{Sub}_\Gamma \to 2^\Gamma$*, $H \mapsto T_H := \sigma_H(\Gamma)$, is continuous.*

PROOF. Fix a bijective enumeration $\Gamma = \{\gamma_m\}_{m\in\mathbb{N}}$ of Γ with $\gamma_0 = e$, and define $\sigma_H(\gamma_m) = \gamma_i$ where i is least such that $\gamma_m\gamma_i^{-1} \in H$. This is continuous and (i) and (ii) are clearly satisfied, and (iii) follows from continuity of $H \mapsto \sigma_H$, since the map $\Gamma^\Gamma \to 2^\Gamma$ sending $f : \Gamma \to \Gamma$ to its set of fixed points is continuous. □

Define the set

$$A(\text{Sub}_\Gamma, X, \mu) := \{(H, \boldsymbol{a}) \,:\, H \in \text{Sub}_\Gamma, \text{ and } \boldsymbol{a} \in A(H, X, \mu)\}.$$

This set has a natural Polish topology in which $(H_n, \boldsymbol{a}_n) \to (H, \boldsymbol{a})$ if and only if $H_n \to H$ and $\boldsymbol{a}_n \to \boldsymbol{a}$ pointwise. We make this precise by taking $*$ to be some point isolated from $\mathrm{Aut}(X, \mu)$ and then defining $\gamma^b = *$ whenever $H \leq \Gamma$, $\boldsymbol{b} \in A(H, X, \mu)$, and $\gamma \notin H$. Then $(H_n, \boldsymbol{a}_n) \to (H, \boldsymbol{a})$ means that $\gamma^{a_n} \to \gamma^a$ for every $\gamma \in \Gamma$.

LEMMA 6.30. *For any $r \in \mathbb{R}$ the sets*

$$S_r = \{H \in \mathrm{Sub}_\Gamma \,:\, C(H) < r\}$$

$$A_r = \{(H, \boldsymbol{a}) \in A(\mathrm{Sub}_\Gamma, X, \mu) \,:\, \boldsymbol{a} \text{ is free and } C(\boldsymbol{a}) < r\}$$

are analytic. In particular, the map $H \mapsto C(H)$ is universally measurable.

PROOF. It suffices to show that A_r is analytic since S_r is the image of A_r under projection onto Sub_Γ which is continuous. We may assume that $X = 2^{\mathbb{N}}$ and that μ is the uniform product measure.

Let $\Gamma \curvearrowright^s X^\Gamma$ denote the left shift action given by $(\gamma^s \cdot f)(\delta) = f(\gamma^{-1}\delta)$ for $f \in X^\Gamma$. Let $H \mapsto \sigma_H$ and $H \mapsto T_H \subseteq \Gamma$ be a continuous assignment of selectors and transversals given by Lemma 6.29. For $(H, \boldsymbol{a}) \in A(\mathrm{Sub}_\Gamma, X, \mu)$ define the map $\Phi_{H,\boldsymbol{a}} : X \to X^\Gamma$ by $\Phi_{H,\boldsymbol{a}}(x)(ht) = (h^{-1})^a x$ for $h \in H$, $t \in T_H$, $x \in X$. Then $\Phi_{H,\boldsymbol{a}}$ is injective and equivariant from $H \curvearrowright^a X$ to the shift action $H \curvearrowright^s X^\Gamma$ and so the measure $\mu_{H,\boldsymbol{a}} := (\Phi_{H,\boldsymbol{a}})_*\mu$ is $H \curvearrowright^s X^\Gamma$ invariant, and the systems $H \curvearrowright^a (X, \mu)$ and $H \curvearrowright^s (X^\Gamma, \mu_{H,\boldsymbol{a}})$ are isomorphic. Let P denote the space of Borel probability measures on X^Γ equipped with the weak*-topology.

CLAIM 4. *The map $A(\mathrm{Sub}_\Gamma, X, \mu) \to P$, $(H, \boldsymbol{a}) \mapsto \mu_{H,\boldsymbol{a}}$ is continuous.*

PROOF OF CLAIM. Suppose that $(H_n, \boldsymbol{a}_n) \to (H_\infty, \boldsymbol{a}_\infty)$ in $A(\mathrm{Sub}_\Gamma, X, \mu)$. Letting $\mu_n = \mu_{H_n,\boldsymbol{a}_n}$, it suffices to check that $\mu_n(A) \to \mu_\infty(A)$ whenever $A \subseteq X^\Gamma$ is of the form $A = \{f \in X^\Gamma \,:\, \forall \gamma \in F \ (f(\gamma) \in A_\gamma)\}$ where $F \subseteq \Gamma$ is finite and $A_\gamma \subseteq X$ is Borel. For $\gamma \in F$ write $\gamma = h_\gamma t_\gamma$ where $t_\gamma \in T_{H_\infty}$ and $h_\gamma \in H_\infty$. By continuity of $H \mapsto \sigma_H$ and $H \mapsto T_H$, for all large enough n, $h_\gamma \in H_n$ and $t_\gamma \in T_{H_n}$ for all $\gamma \in F$. Then $\mu_n(A) = \mu(\bigcap_{\gamma \in F} h_\gamma^{a_n}(A_\gamma)) \to \mu(\bigcap_{\gamma \in F} h_\gamma^{a}(A_\gamma)) = \mu_\infty(A)$ since $\boldsymbol{a}_n \to \boldsymbol{a}$. $\square$[Claim]

Now let E_H denote the orbit equivalence relation on X^Γ generated by $H \curvearrowright^s X^\Gamma$. The set

$$B = \{(H, \nu) \in \mathrm{Sub}_\Gamma \times P \,:\, \nu \text{ is } E_H\text{-invariant and } H \curvearrowright^s (X^\Gamma, \nu) \text{ is essentially free}\}$$

is Borel so by the proof of [**KM04**, Proposition 18.1] the set $D = \{(H,\nu) \in B \, : \, C_\nu(E_H) < r\}$ is analytic. We have $(H, \boldsymbol{a}) \in A_r$ if and only if $(H, \mu_{H,\boldsymbol{a}}) \in D$, which shows that A_r is analytic. $\square$

It follows that for any ergodic invariant random subgroup θ of Γ there is an $r \in \mathbb{R} \cup \{\infty\}$ such that $C(H) = r$ for almost all $H \leq \Gamma$. The following is an analogue of [**BG04**, §5] for cost. I would like to thank Lewis Bowen for a helpful discussion related to this.

THEOREM 6.31. *Let θ be an invariant random subgroup of Γ and suppose that θ concentrates on the infinite subgroups of Γ which have infinite index in Γ. If $\theta(\{H \, : \, C(H) < \infty\}) \neq 0$ then $C(\Gamma) = 1$.*

Thus, if $C(\Gamma) > 1$ then for any ergodic non-atomic m.p. action $\Gamma \curvearrowright^a (X,\mu)$, either Γ_x is finite almost surely, or $C(\Gamma_x) = \infty$ almost surely.

PROOF. To see that the second statement follows from the first observe that an ergodic non-atomic m.p. action cannot have stabilizers which are finite index. We now prove the first statement. By decomposing θ into its ergodic components we may assume without loss of generality that θ is ergodic and there is an $r \in \mathbb{R}$ such that $C(H) < r$ almost surely.

By Lemma 6.30 the set $A_r = \{(H, \boldsymbol{a}) \in A(\mathrm{Sub}_\Gamma, X, \mu) \, : \, \boldsymbol{a} \text{ is free and } C(\boldsymbol{a}) < r\}$ is an analytic subset of $A(\mathrm{Sub}_\Gamma, X, \mu)$. Since $C(H) < r$ almost surely, we may measurably select for each $H \in \mathrm{Sub}_\Gamma$ a free action $\boldsymbol{a}_H \in \mathrm{FR}(H, X, \mu) \subseteq \mathrm{A}(H, X, \mu)$ of H such that almost surely $C(\boldsymbol{a}_H) < r$ (we are applying [**Kec95**, 18.1] to the flip of the graph of the projection function $A_r \to \mathrm{Sub}_\Gamma$, $(H, \boldsymbol{a}) \mapsto H$). A co-inducing process can now be used to obtain an action $\boldsymbol{b}$ of Γ from the selection $H \mapsto \boldsymbol{a}_H \in A(H, X, \mu)$ as follows.

Let $H \mapsto \sigma_H$ be as in Lemma 6.29. Let $\mathrm{COS}_\Gamma \subseteq 2^\Gamma$ denote the closed subspace of all right cosets of subgroups of Γ, on which Γ acts continuously by left translation $\gamma^\ell \cdot H\delta = \gamma H \delta$. The function $\rho : \Gamma \times \mathrm{COS}_\Gamma \to \Gamma$ defined by

$$\rho(\gamma, H\delta) = (\sigma_{\gamma H \gamma^{-1}}(\gamma\delta))^{-1} \gamma \sigma_H(\delta)$$

is a continuous cocycle of this action with values in Γ. It is clear that $\rho(\gamma, H\delta) \in \delta^{-1} H \delta$, so the map $(\gamma, H\delta) \mapsto \rho(\gamma, H\delta)^{a_{\delta^{-1}H\delta}}$ is a well-defined measurable cocycle with values in $\mathrm{Aut}(X,\mu)$. We therefore obtain an action b of Γ on the space $W = \{(H, f) \, : \, H \leq \Gamma \text{ and } f : H\backslash\Gamma \to X\}$

given by $\gamma^b(H,f) = (\gamma H\gamma^{-1}, \gamma^{b_H} f)$ where $\gamma^{b_H} f : \gamma H\gamma^{-1}\backslash\Gamma \to X$ is given by

$$(\gamma^{b_H} f)(\gamma H\delta) = \rho(\gamma, H\delta)^{a_{\delta^{-1}H\delta}}(f(H\delta)).$$

This action preserves the measure $\kappa = \int_H (\delta_H \times \mu^{H\backslash\Gamma})\, d\theta(H)$ since

$$\gamma^b_*\kappa = \int_H (\delta_{\gamma H\gamma^{-1}} \times \gamma^{b_H}_* \mu^{H\backslash\Gamma})\, d\theta = \int_H \Big(\delta_{\gamma H\gamma^{-1}} \times \prod_{\gamma H\delta\in\gamma H\gamma^{-1}\backslash\Gamma} (\rho(\gamma, H\delta)^{a_{\delta^{-1}H\delta}})_*\mu\Big)\, d\theta$$
$$= \int_H \Big(\delta_{\gamma H\gamma^{-1}} \times \prod_{\gamma H\delta\in\gamma H\gamma^{-1}\backslash\Gamma} \mu\Big)\, d\theta = \int_H \big(\delta_{\gamma H\gamma^{-1}} \times \mu^{\gamma H\gamma^{-1}\backslash\Gamma}\big)\, d\theta = \int_H \delta_H \times \mu^{H\backslash\Gamma} d\theta = \kappa.$$

LEMMA 6.32.

(1) *For each $(H,f) \in W$, and $h \in H$ we have $(h^{b_H} f)(H) = h^{a_H}(f(H))$ and thus the map $X^{H\backslash\Gamma} \to X$, $f \mapsto f(H)$ factors*

$$\boldsymbol{b}_H = H \curvearrowright^{b_H} (X^{H\backslash\Gamma}, \mu^{H\backslash\Gamma})$$

onto $\boldsymbol{a}_H$.

(2) *(Analogue of* [**Ioa11**, Lemma 2.1]*) For almost all $H \leq \Gamma$ and every $\gamma \in \Gamma \setminus \{e\}$ the sets*

$$W^H_\gamma = \{f \in X^{H\backslash\Gamma} \,:\, \gamma H\gamma^{-1} = H \text{ and } (\gamma^{b_H} f)(H) = f(H)\}$$

*are $\mu^{H\backslash\Gamma}$-null. In particular, **b** is essentially free.*

PROOF. (1) is clear from the definition of b_H. For (2), If $f \in W^H_\gamma$ then $\rho(\gamma, H)^{a_H}(f(H\gamma^{-1})) = f(H)$ by definition of b_H. So for each H with $\boldsymbol{a}_H$ essentially free, if $\gamma \in H \setminus \{e\}$ then $f \in W^H_\gamma$ if and only if $\gamma^{a_H}(f(H)) = f(H)$, so that W^H_γ is null, while if $\gamma \in \Gamma \setminus H$ then $W^H_\gamma \subseteq \{f \in X^{H\backslash\Gamma} \,:\, \rho(\gamma, H)^{a_H}(f(H\gamma^{-1})) = f(H)\}$, which is null since $H\gamma^{-1} \neq H$ and μ is non-atomic. Since almost all $\boldsymbol{a}_H$ are essentially free we are done. $\square$[Lemma 6.32]

We now apply a randomized version of an argument due to Gaboriau (see [**KM04**, Theorem 35.5]). There is another measure preserving action $\boldsymbol{s} = \Gamma \curvearrowright^s (W,\kappa)$ of Γ on (W,κ) given by $\gamma^s(H,f) = (\gamma H\gamma^{-1}, \gamma^{s_H} f)$ where $(\gamma^{s_H} f)(\gamma H\delta) = f(H\delta)$ (this is the random Bernoulli shift determined by θ [**TD12c**, §5.3]). The projection map $W \to \mathrm{Sub}_\Gamma$, $(H,f) \mapsto H$ factors both $\boldsymbol{b}$ and $\boldsymbol{s}$ onto $\boldsymbol{\theta}$. We let $\boldsymbol{a}$ denote the corresponding relatively independent joining of $\boldsymbol{b}$ and $\boldsymbol{s}$ over $\boldsymbol{\theta}$, i.e.,

$\boldsymbol{a}$ is the measure preserving action of Γ on

$$(Z,\eta) = \big(\{(H,f,g) \,:\, f,g \in X^{H\backslash\Gamma}\},\ \int_H (\delta_H \times \mu^{\Gamma/H} \times \mu^{\Gamma/H})\, d\theta\big)$$

given by $\gamma^a(H,f,g) = (\gamma H\gamma^{-1}, \gamma^{b_H} f, \gamma^{s_H} g)$ where $(\gamma^{s_H} g)(\gamma H\delta) = g(H\delta)$. This action is free since it factors onto $\boldsymbol{b}$.

Let $p : Z \to W$ denote the projection map $p((H,f,g)) = (H,g)$. For each $(H,g) \in W$ the set $p^{-1}((H,g))$ is $a \restriction H$-invariant, and we let $E_{(H,g)}$ denote the orbit equivalence relation on $p^{-1}((H,g))$ generated by $\boldsymbol{a} \restriction H$, i.e., $(H,f_1,g)E_{(H,g)}(H,f_2,g)$ if and only if there is some $h \in H$ such that $h^{b_H} f_1 = f_2$. Define the equivalence relation E on Z by $E = \bigsqcup_{(H,g)\in W} E_{(H,g)}$, i.e.,

$$(H_1,f_1,g_1)E(H_2,f_2,g_2) \Leftrightarrow (H_1,g_1) = (H_2,g_2) \text{ and } \exists h \in H_1\, (h^{b_H} f_1 = f_2).$$

Recall that if $F \subseteq R$ are countable Borel equivalence relations on a standard Borel space Y, then F is said to be *normal* in R if there exists some countable group Δ of Borel automorphisms of Y which generates R and satisfies $xFy \Rightarrow \delta(x)F\delta(y)$ for all $\delta \in \Delta$.

LEMMA 6.33. *E is a normal subequivalence relation of E_a that is almost everywhere aperiodic and with $C_\eta(E) < r$.*

PROOF OF LEMMA 6.33. It is clear that E is an equivalence relation and that E is contained in E_a. Also, E is almost everywhere aperiodic since θ concentrates on the infinite subgroups of Γ by hypothesis. Let $\gamma \in \Gamma$ and let $(H,f,g), (H,f',g) \in \mathrm{Sub}_\Gamma \times X$ be E-related so that $h^{b_H} f = f'$ for some $h \in H$. To show E is normal in E_a we must show that $\gamma^a(H,f,g)$ and $\gamma^a(H, h^{b_H} f, g)$ are E-related as well, i.e., we must find some $k \in \gamma H \gamma^{-1}$ such that $(k\gamma)^{b_H} f_1 = \gamma^{b_H}(h^{b_H} f_1)$. The element $k = \gamma h \gamma^{-1}$ works.

If we disintegrate η via the E-invariant map $p : Z \to W$, then for each $(H,g) \leq \Gamma$, the equivalence relation $E_{(H,g)}$ on $(p^{-1}((H,g)), \eta_{(H,g)})$ is isomorphic to the orbit equivalence relation generated by $b_H \restriction H$ on $(X^{H\backslash\Gamma}, \mu^{H\backslash\Gamma})$. By Lemma 6.32.(1), $\boldsymbol{b}_H$ factors onto $\boldsymbol{a}_H$, so for θ-almost every H we have $r \leq C_{\eta_{(H,g)}}(E_{(H,g)}) = C(\boldsymbol{b}_H) \leq C(\boldsymbol{a}_H) < r$ by [**Kec10**, bottom of p. 78]. Then by [**KM04**, Proposition 18.4] we have

$$C_\eta(E) = \int_{H,g} C_{\eta_{(H,g)}}(E_{(H,g)})\, d\theta(H) < r. \qquad \square\text{[Lemma 6.33]}$$

Since H is almost surely infinite index, the equivalence relation E_s on W generated by $\boldsymbol{s}$ is aperiodic. By [**Kec10**] the full group $[E_s]$ contains an aperiodic transformation $T : W \to W$. Let $B : \Gamma \to \mathrm{MALG}_\kappa$, $\gamma \mapsto B_\gamma$, be a partition of W such that $T \restriction B_\gamma = \gamma^s \restriction B_\gamma$. Then $A : \Gamma \to \mathrm{MALG}_\kappa$ given by $A_\gamma = p^{-1}(B_\gamma)$ is a partition of Z, and determines the L-graphing $\Phi^{a,A} = \{\varphi_\gamma^{a,A}\}_{\gamma\in\Gamma}$ where $\varphi_\gamma^{a,A} \restriction A_\gamma = \gamma^a \restriction A_\gamma$.

Fix $\epsilon > 0$ and find by Lemma 6.33 a graphing $\{\varphi_i\}_{i\in\mathbb{N}}$ of $E \subseteq Z$ of finite cost $\sum_i C_\eta(\varphi_i) < \infty$. Let M be so large that $\sum_{i>M} C_\eta(\varphi_i) < \epsilon/2$. Let $Y_0 \subseteq W$ be a Borel complete section for E_T with $\kappa(Y_0) < \epsilon/(2M)$, and let $Y = p^{-1}(Y_0)$. Then $\eta(Y) = \kappa(Y_0) < \epsilon/M$, and Y is E-invariant so that $\{\varphi_i \restriction Y\}_{i\in\mathbb{N}}$ is an L-graphing of $E \restriction Y$. It follows that

$$C_{\eta\restriction Y}(E \restriction Y) \leq \sum_{i\in\mathbb{N}} C_\eta(\{\varphi_i \restriction Y\}) \leq M \cdot \eta(Y) + \sum_{i\geq M} C_\eta(\{\varphi_i\}) < \epsilon.$$

CLAIM 5. $E \subseteq E \restriction Y \vee E_{\Phi^{a,A}}$.

PROOF. Suppose $(H,f,g)E(H,f',g)$. Since Y_0 is a complete section for E_T there exists $\gamma_1,\ldots,\gamma_k$ and $\epsilon_1,\ldots,\epsilon_k \in \{-1,1\}$ such that $(\varphi_{\gamma_k}^{s,B})^{\epsilon_k} \circ \cdots (\circ\varphi_{\gamma_1}^{s,B})^{\epsilon_1}((H,g)) \in Y_0$. Let $\gamma = \gamma_k^{\epsilon_k}\cdots\gamma_1^{\epsilon_1}$ and let $(H_0,g_0) = \gamma^s((H,g)) \in Y_0$. It follows that

$$\gamma^a(H,f,g) = (\gamma_k^{\epsilon_k})^a \cdots (\gamma_1^{\epsilon_1})^a(H,f,g) = (\varphi_{\gamma_k}^{a,A})^{\epsilon_k} \circ \cdots \circ (\varphi_{\gamma_1}^{a,A})^{\epsilon_1}(H,f,g)$$
$$\gamma^a(H,f',g) = (\gamma_k^{\epsilon_k})^a \cdots (\gamma_1^{\epsilon_1})^a(H,f',g) = (\varphi_{\gamma_k}^{a,A})^{\epsilon_k} \circ \cdots \circ (\varphi_{\gamma_1}^{a,A})^{\epsilon_1}(H,f',g).$$

This shows that $(H,f,g)E_{\Phi^{a,A}}\gamma^a(H,f,g)$ and $\gamma^a(H,f',g)E_{\Phi^{a,A}}(H,f',g)$. As $\gamma^a(H,f,g) = (H_0,\gamma^{b_H}f,g_0) \in Y$ and $\gamma^a(H,f',g) = (H_0,\gamma^{b_H}f',g_0) \in Y$ we will be done if we can show these two points are E-related. Let $h \in H$ be such that $h^{b_H}f = f'$ and let $k = \gamma h\gamma^{-1}$. Then $k \in \gamma H\gamma^{-1} = H_0$ and

$$k^a(H_0,\gamma^{b_H}f,g_0) = (k\gamma)^a(H,f,g) = (\gamma h)^a(H,f,g) = \gamma^a(H,f',g) = (H_0,\gamma^{b_H}f',g_0)$$

which shows that $(H_0,\gamma^{b_H}f,g_0)E_{(H_0,g_0)}(H_0,\gamma^{b_H}f',g_0)$. □[Claim 5]

We have $C_\eta(E \restriction Y \vee E_{\Phi^{a,A}}) \leq 1+\epsilon$. Since we have shown that $E \subseteq E \restriction Y \vee E_{\Phi^{a,A}}$ and that E is an aperiodic normal subequivalence relation of E_a, it follows from [**KM04**, 24.10] that $C_\eta(E_a) \leq C_\eta(E \restriction Y \vee E_{\Phi^{a,A}}) \leq 1+\epsilon$. As $\epsilon > 0$ was arbitrary it follows that $C_\eta(E_a) = 1$ and therefore $C(\Gamma) = 1$. □

6.5. Fixed price 1 and shift-minimality. The following lemma will be needed for Theorem 6.36.

LEMMA 6.34. *Let θ be an invariant random subgroup of a countable group Γ that concentrates on the infinite amenable subgroups of Γ. Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ be a free measure preserving action of Γ and let*

$$\boldsymbol{\theta} \times \boldsymbol{a} = \Gamma \curvearrowright^{c \times a} (\mathrm{Sub}_\Gamma \times X, \theta \times \mu)$$

be the product Γ-system. Then $C_{\theta \times \mu}(E_{c \times a}) = 1$.

REMARK 6.35. The proof shows that the hypothesis that θ is amenable can be weakened to the hypothesis that θ concentrates on groups of fixed price 1.

PROOF. The proof is similar to that of Lemma 6.33. Since $E_{c \times a}$ is aperiodic it suffices to show that $C_{\theta \times \mu}(E_{c \times a}) \leq 1$. For each $H \in \mathrm{Sub}_\Gamma$ let $E_{a \restriction H}$ denote the orbit equivalence relation on X generated by $\boldsymbol{a} \restriction H = H \curvearrowright^a (X, \mu)$. Define the subrelation $E \subseteq E_{c \times a}$ on $\mathrm{Sub}_\Gamma \times X$ by $E = \{((H, x), (H, y)) \, : \, x E_{a \restriction H} y\}$, i.e.,

$$(H, x) E (L, y) \;\Leftrightarrow\; H = L \text{ and } (\exists h \in H)\, (h^a \cdot x = y).$$

Then E is a normal sub-equivalence relation of $E_{c \times a}$. Since θ concentrates on the infinite subgroups of Γ, E is aperiodic on a $(\theta \times \mu)$-conull set. By [**KM04**, 24.10] and then [**KM04**, Proposition 18.4] we therefore have

$$C_{\theta \times \mu}(E_{c \times a}) \leq C_{\theta \times \mu}(E) = \int_H C_\mu(E_{a \restriction H})\, d\theta(H) = 1$$

where the last equality follows from [**KM04**, Corollary 31.2] since θ-almost every H is infinite amenable. □

THEOREM 6.36. *Let Γ be a countably infinite group that contains no non-trivial finite normal subgroup. If Γ is not shift-minimal then Γ has fixed price 1.*

PROOF. Suppose that Γ is not shift-minimal. By Corollary 3.14 either Γ has a non-trivial normal amenable subgroup N that is necessarily infinite by our hypothesis on Γ, or there is an infinitely generated amenable invariant random subgroup θ of Γ that is weakly contained in $\boldsymbol{s}_\Gamma$. In the first case define $\theta = \delta_N$, so that in either case θ concentrates on the infinite amenable subgroups of Γ, and $\boldsymbol{\theta} \prec \boldsymbol{s}_\Gamma$.

Let (X, μ) denote the underlying measure space of $\boldsymbol{s}_\Gamma$ and consider the product Γ-system

$$\boldsymbol{\theta} \times \boldsymbol{s}_\Gamma = \Gamma \curvearrowright^{c \times s} (\mathrm{Sub}_\Gamma \times X, \theta \times \mu).$$

By Lemma 6.34 we have $C(\boldsymbol{\theta} \times \boldsymbol{s}_\Gamma) = 1$. The action $\boldsymbol{\theta}$ is weakly contained in $\boldsymbol{s}_\Gamma$, so $\boldsymbol{\theta} \times \boldsymbol{s}_\Gamma$ is weakly equivalent to $\boldsymbol{s}_\Gamma$. This implies that Γ has fixed price 1 by (3)⇒(1) of Corollary 6.24. □

COROLLARY 6.37. *Suppose that* Γ *does not have fixed price* 1*. Then the following are equivalence*

(1) Γ *is shift-minimal.*

(2) Γ *contains no non-trivial finite normal subgroups.*

(3) AR_Γ *is trivial.*

PROOF. (3)⇒(2) is obvious. (2)⇒(1) is immediate from Theorem 6.36 by our assumption that Γ does not have fixed price 1. (1)⇒(3) holds in general with no assumptions on Γ. □

COROLLARY 6.38. *Let* Γ *be any group that does not have fixed price* 1*. Then* AR_Γ *is finite and* $\Gamma/\mathrm{AR}_\Gamma$ *is shift-minimal.*

PROOF. Any group containing an infinite normal amenable subgroup has fixed price 1 [**KM04**, Proposition 35.2]. Therefore $N = \mathrm{AR}_\Gamma$ is finite. Let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ be a free measure preserving action of Γ of cost $C_\mu(E_{\boldsymbol{a}}) > 1$. The measure preserving action $\boldsymbol{b}$ of Γ/N on the ergodic components of $\boldsymbol{a} \restriction N$ is free, and since N is finite we have $C(\boldsymbol{b}) \geq C(\boldsymbol{a}) > 1$. Thus, Γ/N does not have fixed price 1, and $\mathrm{AR}_{\Gamma/N} = \{e\}$ by Proposition 9.1. Corollary 6.37 now shows that Γ/N is shift-minimal. □

7. Questions

7.1. General implications. A countable group Γ is called C^**-simple* if the reduced C^*-algebra of Γ is simple, i.e., $C^*_r(\Gamma)$ has no non-trivial closed two-sided ideals. As observed in the introduction, there is a strong parallel between shift-minimality and C^*-simplicity. The following characterization of C^*-simplicity of a countable group Γ may be found in [**dlH07**]. Let λ_Γ denote the left regular representation of Γ on $\ell^2(\Gamma)$.

PROPOSITION 7.1. *Let* Γ *be a countable group. Then* Γ *is* C^**-simple if and only if* $\pi \prec \lambda_\Gamma$ *implies* $\pi \sim \lambda_\Gamma$ *for all nonzero unitary representations* π *of* Γ.

In this characterization of C^*-simplicity we may actually restrict our attention to irreducible representations of Γ. That is, Γ is C^*-simple if and only if every irreducible unitary representation π of Γ that is weakly contained in λ_Γ is actually weakly equivalent to λ_Γ. See [**BdlH00**]. See also [**BHV08**, Appendix F] and [**Dix77**] for more on weak containment of unitary representations.

Characterization (6) of shift-minimality from Proposition 3.2 also has an analogue for C^*-simplicity. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and let $\mathrm{Irr}_\lambda(\Gamma, \mathcal{H})$ denote the Polish space of irreducible representation of Γ on $\mathcal{H}$ that are weakly contained in λ_Γ (see [**Dix77**]). Let $\mathcal{U}(\mathcal{H})$ be the Polish group of all unitary operators on $\mathcal{H}$. Then Γ is C^*-simple if and only if Γ is ICC and the conjugation action of $\mathcal{U}(\mathcal{H})$ on $\mathrm{Irr}_\lambda(\Gamma, \mathcal{H})$ is minimal (i.e., every orbit is dense). See [**Kec10**, Appendix H.(**C**)].

Consider now the following properties of a countable group Γ:

- (UT) Γ has the unique trace property.
- (CS) Γ is C^*-simple.
- (SM) Γ is shift-minimal.
- (UIRS_0) Γ has no non-trivial amenable invariant random subgroup that is weakly contained in $\boldsymbol{s}_\Gamma$.
- (UIRS) Γ has no non-trivial amenable invariant random subgroups.
- (AR_e) Γ has no non-trivial amenable normal subgroups, i.e., the amenable radical AR_Γ of Γ is trivial.

All of the known implications (besides (SM)$\Leftrightarrow$(UIRS_0)) are depicted in Figure 1 in the introduction. It is known that (UT) and (CS) imply (AR_e) ([**PS79**], see also [**BdlH00**, Proposition 3]), though it is an open question whether there are any other implications among the properties (UT), (CS), and (AR_e) in general [**BdlH00**]. The following questions concern some of the remaining implications.

The implication (UT)$\Rightarrow$(SM) was shown in Theorem 5.15. One of the most pressing questions is:

QUESTION 7.2. Does (CS) imply (SM)? That is, are C^*-simple groups shift-minimal?

For a positive answer to Question 7.2 it would suffices by Corollary 3.14 to show that if θ is a non-atomic self-normalizing amenable IRS of a countable group Γ that is weakly contained in $\boldsymbol{s}_\Gamma$ then the tracial state on $C_r^*(\Gamma)$ extending φ_θ from the proof of Theorem 5.14 is not faithful.

The implication from (UT) to (UIRS) is quite direct. The converse would mean that a tracial state on $C_r^*(\Gamma)$ different from τ_Γ somehow gives rise to a non-trivial amenable invariant random subgroup of Γ. This is addressed by the following question:

QUESTION 7.3. Does (UIRS) imply (UT)? That is, if Γ does not have any non-trivial amenable invariant random subgroups then does $C_r^*(\Gamma)$ have a unique tracial state?

We know from Theorem 3.16 that (SM) and (UIRS_0) are equivalent. The equivalence of (SM) and (UIRS) is open however (clearly though (UIRS)$\Rightarrow$(UIRS_0))

QUESTION 7.4. Does (UIRS_0) imply (UIRS)?

To obtain a positive answer to Question 7.4 it would be enough to show the following: ($\star$) Every ergodic amenable invariant random subgroup of a countable group Γ that is not almost ascendant is weakly contained in $\boldsymbol{s}_\Gamma$.

Indeed, assume that ($\star$) holds and suppose that Γ does not have (UIRS), i.e., there is an amenable invariant random subgroup θ of Γ other than $\delta_{\langle e\rangle}$. By moving to an ergodic component of θ we may assume without loss of generality that θ is ergodic. If θ is not almost ascendant then ($\star$) implies that $\boldsymbol{\theta}$ is weakly contained in $\boldsymbol{s}_\Gamma$, which shows that Γ does not have (UIRS_0). On the other hand, if θ is almost ascendant then, by Corollary 9.4, θ concentrates on the subgroups of AR_Γ, and in particular AR_Γ is non-trivial, so $\delta_{\mathrm{AR}_\Gamma}$ witnesses that Γ does not have (UIRS_0).

The implication (SM)$\Rightarrow$(AR_e) is shown in Proposition 3.15 above. The converse is a tantalizing question:

QUESTION 7.5. Does (AR_e) imply (SM)? That is, if Γ has no non-trivial amenable normal subgroup then is every non-trivial m.p. action that is weakly contained in $\boldsymbol{s}_\Gamma$ free?

To obtain a positive answer to Question 7.5 by Corollary 3.14 it would be enough to show that if θ is a non-atomic self-normalizing invariant random subgroup weakly contained in $\boldsymbol{s}_\Gamma$ then θ concentrates on subgroups of the amenable radical of Γ. (Note that θ does indeed concentrate on the amenable subgroups of Γ by NA-ergodicity.)

7.2. Cost and pseudocost. In the infinitely generated setting it appears that pseudocost, rather than cost, may be a more useful way to define an invariant. In addition to the properties exhibited in §6.2, pseudocost enjoys many of the nice properties already known to hold for cost. For instance, pseudocost respects ergodic decomposition, and $PC(\Gamma) \leq PC(N)$ whenever N is an infinite normal subgroup of Γ. (The proofs are routine: for the first statement one uses the corresponding fact about cost along with basic properties of pseudocost, and the proof of the second is nearly identical to the corresponding proof for cost.)

QUESTION 7.6. Is there an example of a m.p. countable Borel equivalence relation E such that $PC_\mu(E) < C_\mu(E)$?

By Corollary 6.8.(1) the equality $PC_\mu(E) = C_\mu(E)$ holds whenever $C_\mu(E) < \infty$, so the question is whether it is possible to have $PC_\mu(E) < \infty$ and $C_\mu(E) = \infty$. Equivalently: does there exist an increasing sequence $E_0 \subseteq E_1 \subseteq \cdots$, of m.p. countable Borel equivalence relations on (X, μ) with $\sup_n C_\mu(E_n) < \infty$ and $C_\mu(\bigcup_n E_n) = \infty$? If such a sequence $(E_n)_{n\in\mathbb{N}}$ exists then, letting $E = \bigcup_n E_n$, Corollary 6.8.(2) implies that E could not be treeable. In addition, E would provide an example of strict inequality $\beta_1(E) + 1 < C_\mu(E)$. This follows from [**Gab02**, 5.13, 3.23]. Gaboriau has shown that any aperiodic m.p. countable Borel equivalence R satisfies $\beta_1(R) + 1 \leq C_\mu(R)$ [**Gab02**], although it is open whether this inequality can ever be strict. Note that a positive answer to 7.6 would not necessarily provide a counterexample to the fixed price conjecture, even if the equivalence relation E comes from a free action of some group Γ; at this time there is no way to rule out the possibility that such a Γ has fixed cost ∞ while at the same time admitting various free actions with finite pseudocost.

QUESTION 7.7. Suppose that a countable group Γ has some free action $\boldsymbol{a}$ with $C_\mu(\boldsymbol{a}) = \infty$. Does it follow that $C_\mu(\boldsymbol{s}_\Gamma) = \infty$?

By Corollary 6.20, $\boldsymbol{s}_\Gamma$ attains the maximum pseudocost among free actions of Γ. Corollary 6.22 implies that

$$C(\boldsymbol{s}_\Gamma) \geq \sup\{C(\boldsymbol{b}) \,:\, \boldsymbol{b} \in \mathrm{FR}(\Gamma, X, \mu) \text{ and either } C(\boldsymbol{b}) < \infty \text{ or } E_b \text{ is treeable}\}.$$

This is not enough to conclude that $\boldsymbol{s}_\Gamma$ always attains the maximum cost among free actions of Γ. A positive answer to Question 7.7 would imply that $\boldsymbol{s}_\Gamma$ always attains this maximum cost.

It would be just as interesting if $\boldsymbol{s}_\Gamma$ could detect whether $C(\Gamma) < \infty$.

QUESTION 7.8. Suppose that a countable group Γ has some free action $\boldsymbol{a}$ with $C_\mu(\boldsymbol{a}) < \infty$. Does it follow that $C_\mu(\boldsymbol{s}_\Gamma) < \infty$?

At this time it appears that one cannot rule out any combination of answers to Questions 7.7 and 7.8. A positive answer to both questions would amount to showing that no group has both free actions of infinite cost and free actions of finite cost - this would essentially affirm a special case of the fixed price conjecture!

7.3. Other questions. It is shown in [**TD12a**] that the natural analogue of Question 7.5, where "amenable" is replaced by "finite" and "weakly contained in" is replaced by "is a factor of," has a positive answer:

THEOREM 7.9 (Corollary 1.6 of [**TD12a**]). *Let Γ be a countable group. If Γ has no non-trivial finite normal subgroups then every non-trivial totally ergodic action of Γ is free.*

In particular, if Γ has no non-trivial finite normal subgroups then every non-trivial factor of $\boldsymbol{s}_\Gamma$ is free.

Here, a measure preserving action of Γ is called *totally ergodic* if all infinite subgroups of Γ act ergodically. Theorem 7.9 motivates the following question concerning strong NA-ergodicity.

QUESTION 7.10. Let $\Gamma \curvearrowright^a (X, \mu)$ be a non-trivial measure preserving action of a countable group Γ. Suppose that for each non-amenable subgroup $\Delta \leq \Gamma$ the action $\Delta \curvearrowright^a (X, \mu)$ is strongly ergodic. Does it follow that the stabilizer of almost every point is contained in the amenable radical of Γ?

A positive answer to 7.10 would imply a positive answer to 7.5 by Proposition 3.10.

The following question concerns the converse of Proposition 4.6:

QUESTION 7.11. Suppose Γ is shift-minimal. Is it true that every finite index subgroup of Γ is shift-minimal?

Question 7.11 is equivalent to the question of whether every finite index *normal* subgroup N of a shift-minimal group Γ is shift-minimal. Indeed, suppose the answer is positive for normal subgroups and let K be a finite index subgroup of a shift-minimal group Γ. Then K is ICC, since the ICC property passes to finite index subgroups. Since the group $N = \bigcap_{\gamma\in\Gamma}\gamma K\gamma^{-1}$ is finite index and normal in Γ, it is shift-minimal by our assumption. Proposition 4.6 then implies that K is shift-minimal.

Corollary 4.8 provides a positive answer to Question 7.11 for finite index subgroups which are torsion-free. Theorem 4.7 gives a positive answer for finite index normal subgroups N of Γ for which there is no infinite locally finite invariant random subgroup that is weakly contained in $\boldsymbol{s}_N$. Note that a positive answer to the analogue of Question 7.11 for C^*-simplicity was demonstrated in [**BdlH00**] (and likewise for the unique trace property).

The results from §6.2 and §6.5 suggest that the following may have a positive answer:

QUESTION 7.12. If an infinite group Γ has positive first ℓ^2-Betti number then is it true that $C^*_r(\Gamma/\mathrm{AR}_\Gamma)$ is simple and has a unique tracial state?

There are already partial results in this direction: Peterson and Thom [**PT11**] have shown a positive answer under the additional assumptions that Γ is torsion free and that every non-trivial element of $\mathbb{Z}\Gamma$ acts without kernel on $\ell^2\Gamma$.

Finally, we record here a question raised earlier in this paper.

(Question 3.9). Let Γ be a countable group acting by automorphisms on a compact Polish group G and assume the action is tempered. Does it follow that the action is weakly contained in $\boldsymbol{s}_\Gamma$? As a special case, is it true that the action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright (\mathbb{T}^2, \lambda^2)$ is weakly contained in $\boldsymbol{s}_{\mathrm{SL}_2(\mathbb{Z})}$?

8. Appendix: Invariant random subgroups as subequivalence relations

This first appendix studies *invariant random partitions* of Γ which are a natural generalization of invariant random subgroups. In §8.1 it is shown that every invariant random partition of Γ comes from a pair $(\boldsymbol{a}, F)$ where $\boldsymbol{a}$ is a free m.p. action of Γ and F is a (Borel) subequivalence relation of E_a. It is shown in §8.2 that for an invariant random subgroup any such pair $(\boldsymbol{a}, F)$ will have the property that F is *normalized* by $\boldsymbol{a}$, i.e., γ^a is in the normalizer of the full group of F for every $\gamma\in\Gamma$.

Many of the ideas here are inspired by (and closely related to) the notion of a *measurable subgroup* developed by Bowen-Nevo [**BN09**] and Bowen [**Bow12a**]. See also Remark 8.14.

8.1. Invariant random partitions. By a *partition* of Γ we mean an equivalence relation on Γ. The set $\mathcal{P}_\Gamma$ of all partitions of Γ is a closed subset of $2^{\Gamma\times\Gamma}$ and Γ acts continuously on $\mathcal{P}_\Gamma$ by left translation $\Gamma \curvearrowright^{\ell} \mathcal{P}_\Gamma$, i.e.,

$$(\alpha,\beta)\in\gamma P \;\Leftrightarrow\; (\gamma^{-1}\alpha,\gamma^{-1}\beta)\in P$$

for each $\gamma,\alpha,\beta\in\Gamma$ and $P\in\mathcal{P}_\Gamma$. For $P\in\mathcal{P}_\Gamma$ and $\alpha\in\Gamma$ let $[\alpha]_P=\{\beta\,:\,(\alpha,\beta)\in P\}$ denote the P-class of α. Then it is easy to check that $\gamma[\alpha]_P=[\gamma\alpha]_{\gamma P}$ for all $\gamma\in\Gamma$.

DEFINITION 8.1. An *invariant random partition* of Γ is a translation-invariant Borel probability measure on $\mathcal{P}_\Gamma$.

REMARK 8.2. Let IRP_Γ denote the space of all invariant random partitions of Γ. This is a convex set that is compact and metrizable in the weak*-topology. Similarly, let IRS_Γ denote the compact convex set of all invariant random subgroups of Γ. There is a natural embedding Φ : $\mathrm{Sub}_\Gamma \hookrightarrow \mathcal{P}_\Gamma$ that assigns to each $H\in\mathrm{Sub}_\Gamma$ the partition of Γ determined by the right cosets of H, i.e., $[\delta]_{\Phi(H)}=H\delta$ for $\delta\in\Gamma$. Observe that this embedding is Γ-equivariant between the conjugation action $\Gamma\curvearrowright^{c}\mathrm{Sub}_\Gamma$ and the translation action $\Gamma\curvearrowright^{\ell}\mathcal{P}_\Gamma$. We thus obtain an embedding $\Phi_*:\mathrm{IRS}_\Gamma\hookrightarrow\mathrm{IRP}_\Gamma$, $\theta\mapsto\Phi_*\theta$.

Suppose now that $F\subseteq X\times X$ is a measure preserving countable Borel equivalence relation on (X,μ) and $\boldsymbol{a}=\Gamma\curvearrowright^{a}(X,\mu)$ is a m.p. action of Γ. Each point $x\in X$ determines a partition $P^a_F(x)$ of Γ given by

$$P^a_F(x)=\{(\alpha,\beta)\in\Gamma\,:\,\beta^{-1}xF\alpha^{-1}x\}.$$

Note that $P^a_F(x)=P^a_{F\cap E_a}(x)$ for all $x\in X$, so if we are only concerned with properties of P^a_F then we might as well assume that $F\subseteq E_a$.

PROPOSITION 8.3. *The map $x\mapsto P^a_F(x)$ is equivariant and therefore $(P^a_F)_*\mu$ is an invariant random partition of* Γ.

PROOF. For any $\gamma\in\Gamma$ and $x\in X$ we have

$$(\alpha,\beta)\in P^a_F(\gamma x)\Leftrightarrow\alpha^{-1}\gamma xF\beta^{-1}\gamma x\Leftrightarrow(\gamma^{-1}\alpha,\gamma^{-1}\beta)\in P^a_F(x)\Leftrightarrow(\alpha,\beta)\in\gamma^{\ell}\cdot P^a_F(x).\quad\square$$

Proposition 8.3 has a converse in a strong sense: given an invariant random partition ρ of Γ there is a *free* m.p. action $\boldsymbol{b}=\Gamma\curvearrowright^{b}(Y,\nu)$ of Γ and a subequivalence relation $\mathcal{F}$ of E_b with $(P^b_{\mathcal{F}})_*\nu=\rho$.

In fact, $\mathcal{F}$ and b can be chosen independently of ρ, with only ν depending on ρ, as we now show. Let $\boldsymbol{\rho}$ denote the m.p. action $\Gamma \curvearrowright^{\ell} (\mathcal{P}_{\Gamma}, \rho)$ and let $\boldsymbol{b} = \boldsymbol{\rho} \times \boldsymbol{s}_{\Gamma}$ (any free action of Γ will work in place of $\boldsymbol{s}_{\Gamma}$) so that $(Y, \nu) = (\mathcal{P}_{\Gamma} \times [0,1]^{\Gamma}, \rho \times \lambda^{\Gamma})$. Define $\mathcal{F} \subseteq Y \times Y$ by

$$(8.1) \qquad (P,x)\mathcal{F}(Q,y) \;\Leftrightarrow\; \exists \gamma \in \Gamma\; (\gamma^{-1} \in [e]_P \text{ and } (\gamma P, \gamma x) = (Q,y)).$$

THEOREM 8.4. *Let ρ be an invariant random partition of Γ and write $\boldsymbol{b} = \Gamma \curvearrowright^{b} (Y,\nu)$ for the action $\boldsymbol{\rho} \times \boldsymbol{s}_{\Gamma}$. Let $\mathcal{F}$ be given by* (8.1)*. Then $\mathcal{F}$ is an equivalence relation contained in the equivalence relation E_b generated by the b, and $P^{b}_{\mathcal{F}}((P,x)) = P$ for ν-almost every $(P,x) \in Y$. In particular, $(P^{a}_{\mathcal{F}})_*\nu = \rho$.*

PROOF OF THEOREM 8.4. It is clear that $\mathcal{F} \subseteq E_b$. We show that $\mathcal{F}$ is an equivalence relation: It is clear that $\mathcal{F}$ is reflexive. To see $\mathcal{F}$ is symmetric, suppose $(P,x)\mathcal{F}(Q,y)$, as witnessed by $\gamma^{-1} \in [e]_P$ with $\gamma P = Q$ and $\gamma x = y$. Then $\gamma \in [e]_{\gamma P} = [e]_Q$ and $(\gamma^{-1}Q, \gamma^{-1}y) = (P,x)$, so $(Q,y)\mathcal{F}(P,x)$. For transitivity, if $(P,x)\mathcal{F}(Q,y)\mathcal{F}(R,z)$ as witnessed by $\gamma^{-1} \in [e]_P$ with $(\gamma P, \gamma x) = (Q,y)$ and $\delta^{-1} \in [e]_Q$ with $(\delta Q, \delta y) = (R,z)$ then $\gamma^{-1} \in [e]_P$ and $\gamma P = Q$ implies $[e]_Q = [e]_{\gamma P} = \gamma[e]_P$. Therefore $\delta^{-1} \in \gamma[e]_P$, i.e., $(\delta\gamma)^{-1} \in [e]_P$ and $(\delta\gamma P, \delta\gamma x)(\delta Q, \delta y) = (R,z)$.

Fix now $(P,x) \in Y$. We show that $P^{b}_{\mathcal{F}}((P,x)) = P$. For each $\alpha, \beta \in \Gamma$ we have by definition

$$\begin{aligned} (\alpha,\beta) \in P^{a}_{\mathcal{F}}((P,x)) &\Leftrightarrow (\alpha^{-1}P, \alpha^{-1}x)F(\beta^{-1}P, \beta^{-1}x) \\ (8.2) \qquad &\Leftrightarrow \exists \gamma \in \Gamma \left(\gamma^{-1} \in [e]_{\alpha^{-1}P} \text{ and } (\gamma\alpha^{-1}P, \gamma\alpha^{-1}x) = (\beta^{-1}Q, \beta^{-1}x)\right). \end{aligned}$$

Therefore, if $(\alpha,\beta) \in P^{b}_{\mathcal{F}}((P,x))$ as witnessed by some γ as in (8.2) then $\gamma\alpha^{-1}x = \beta^{-1}x$ so freeness of $\boldsymbol{a}$ implies $\gamma = \beta^{-1}\alpha$. Then $\alpha^{-1}\beta = \gamma^{-1} \in [e]_{\alpha^{-1}P}$, i.e., $(\alpha^{-1}\beta, e) \in \alpha^{-1}P$, which is equivalent to $(\beta,\alpha) \in P$. This shows that $P^{b}_{\mathcal{F}}((P,x)) \subseteq P$. For the reverse inclusion, if $(\alpha,\beta) \in P$ then $\gamma = \beta^{-1}\alpha$ satisfies (8.2) and thus $(\alpha,\beta) \in P^{b}_{\mathcal{F}}((P,x))$. $\square$

DEFINITION 8.5. Let $\boldsymbol{a} = \Gamma \curvearrowright^{a} (X,\mu)$ be a m.p. action of Γ and let F be a subequivalence relation of E_a. If ρ is an invariant random partition of Γ then the pair $(\boldsymbol{a}, F)$ is called a *realization* of ρ if $(P^{a}_{F})_*\mu = \rho$. If θ is an invariant random subgroup of Γ then $(\boldsymbol{a}, F)$ is called a *realization* of θ if it is a realization of $\Phi_*\theta$, where $\Phi_* : \mathrm{IRS}_{\Gamma} \to \mathrm{IRP}_{\Gamma}$ is the embedding defined in Remark 8.2. A realization $(\boldsymbol{a}, F)$ is called *free* if $\boldsymbol{a}$ is free.

The following is a straightforward consequence of Theorem 8.4 and the definitions.

COROLLARY 8.6. *Every invariant random partition admits a free realization.*

The remainder of this subsection works toward a characterization of the set $\Phi_*(\mathrm{IRS}_\Gamma)$. Let K be a metrizable compact space and consider the set $\mathcal{P}_\Gamma \otimes K$ of all pairs (P, f) where $f : P^* \to K$ is a function with $\mathrm{dom}(f) = P^* = \{[\alpha]_P : \alpha \in \Gamma\}$ and taking values in K. The set $\mathcal{P}_\Gamma \otimes K$ has a natural compact metrizable topology coming from its identification with the closed set

$$\widetilde{\mathcal{P}_\Gamma \otimes K} = \{(P, g) \in \mathcal{P}_\Gamma \times K^\Gamma \,:\, g \text{ is constant on each } P\text{-class}\} \subseteq \mathcal{P}_\Gamma \times K^\Gamma$$

via the injection $(P, f) \mapsto (P, \tilde{f})$ where $\tilde{f}(\alpha) = f([\alpha]_P)$ for $\alpha \in \Gamma$. Observe that $\widetilde{\mathcal{P}_\Gamma \otimes K}$ is invariant in $\mathcal{P}_\Gamma \times K^\Gamma$ with respect to the product action $\ell \times s$ of Γ (where s denotes the shift action $\Gamma \curvearrowright^s K^\Gamma$), so we obtain a continuous action $\Gamma \curvearrowright^{\ell\otimes s} \mathcal{P}_\Gamma \otimes K$. Explicitly, this action is given by $\gamma \cdot (P, f) = (\gamma P, \gamma^{s_P} f)$ where $\gamma^{s_P} f : (\gamma P)^* \to K$ is the function

$$(\gamma^{s_P} f)([\alpha]_{\gamma P}) = f(\gamma^{-1}[\alpha]_{\gamma P}) = f([\gamma^{-1}\alpha]_P).$$

There is a natural equivalence relation $\mathcal{R} = \mathcal{R}_K$ on $\mathcal{P}_\Gamma \otimes K$ given by

$$(P, f)\mathcal{R}(Q, g) \;\Leftrightarrow\; \exists \gamma \in [e]_P \; (\gamma^{-1}(P, f) = (Q, g)).$$

It is clear that $\mathcal{R}$ is an equivalence relation that is contained in $E_{\ell\otimes s}$.

LEMMA 8.7. $P \subseteq P^{\ell\otimes s}_{\mathcal{R}}((P, f))$ *for every* $(P, f) \in \mathcal{P}_\Gamma \otimes K$.

PROOF. Suppose that $(\alpha, \beta) \in P$. Then $\beta^{-1}\alpha \in [e]_{\beta^{-1}P}$ so for any $f \in K^{P^*}$, from the definition of $\mathcal{R}$ we have

$$(\beta^{-1}P, \beta^{-1}f)\,\mathcal{R}\,(\beta^{-1}\alpha)^{-1}(\beta^{-1}P, \beta^{-1}f) = (\alpha^{-1}P, \alpha^{-1}f),$$

i.e., $\beta^{-1}(P, f)\mathcal{R}\alpha^{-1}(P, f)$. This means that $(\alpha, \beta) \in P^{\ell\otimes s}_{\mathcal{R}}((P, f))$ by definition. $\square$

If ρ is an invariant random partition and μ is a Borel probability measure on K then the measure $\rho \otimes \mu$ on $\mathcal{P}_\Gamma \otimes K$ given by

$$\rho \otimes \mu = \int_P (\delta_P \times \mu^{P^*})\, d\rho$$

is $\ell \otimes s$-invariant.

THEOREM 8.8. *Let ρ be an invariant random partition of Γ, let μ be any atomless measure on K, and let $\mathcal{R} = \mathcal{R}_K$. Then the following are equivalent:*

(1) $\rho \in \Phi_*(\mathrm{IRS}_\Gamma)$

(2) $(\rho \otimes \mu)$*-almost every* $\mathcal{R}$*-class is trivial.*

PROOF. (1)⇒(2): Suppose that (1) holds. It follows that $(\rho\otimes\mu)$ concentrates on pairs $(\Phi(H), f) \in \mathcal{P}_\Gamma \otimes K$ with $H \in \mathrm{Sub}_\Gamma$. It therefore suffices to show that the $\mathcal{R}$-class of such a pair $(\Phi(H), f)$ is trivial. If $(\Phi(H), f)\mathcal{R}(Q, g)$ then there is some $\gamma \in [e]_{\Phi(H)} = H$ with $\gamma^{-1}\Phi(H) = Q$ and $\gamma^{-1}f = Q, g$. But $\gamma^{-1}\Phi(H) = \Phi(\gamma^{-1}H\gamma) = \Phi(H)$ (since $\gamma \in H$) so that $Q = \Phi(H)$. In addition, for each $\delta \in \Gamma$ we have $\gamma[\delta]_{\Phi(H)} = \gamma H\delta = H\delta = [\delta]_{\Phi(H)}$ since $\gamma \in H$. Therefore $g([\delta]_{\Phi(H)}) = (\gamma^{-1}f)([\delta]_{\Phi(H)}) = f(\gamma[\delta]_{\Phi(H)}) = f([\delta]_{\Phi(H)})$, showing that $g = f$.

(2)⇒(1): Suppose that (2) holds. Since μ is non-atomic, for each $P \in \mathcal{P}_\Gamma$ the set $\{f \in K^{P^*} : f \text{ is injective}\}$ is μ^{P^*}-conull. This along with (2) implies that there is a Γ-invariant $(\rho \otimes \mu)$-conull set $Y \subseteq \mathcal{P}_\Gamma \otimes K$ on which $\mathcal{R}$ is trivial and such that $f : P^* \to K$ is injective whenever $(P, f) \in Y$. The projection $Y_0 = \{P \in \mathcal{P}_\Gamma : \exists f\ (P, f) \in Y\}$ is then ρ-conull so it suffices to show that $Y_0 \subseteq \Phi(\mathrm{Sub}_\Gamma)$. Fix $P \in Y_0$ and an $f : P^* \to K$ with $(P, f) \in Y$.

CLAIM 6. *Let $\alpha, \beta \in \Gamma$. Then $(\alpha, \beta) \in P$ if and only if $\beta\alpha^{-1} \in [e]_P$.*

PROOF OF CLAIM. Suppose $(\alpha, \beta) \in P$. Lemma 8.7 implies $(\alpha, \beta) \in P^{a}_{\mathcal{R}}(P, f)$ so as the relevant $\mathcal{R}$-classes are trivial this implies $\alpha^{-1}(P, f) = \beta^{-1}(P, f)$ and thus $\alpha\beta^{-1}P = P$ and $\alpha\beta^{-1}f = f$. Then $f([e]_P) = (\alpha\beta^{-1}f)([e]_P) = f([\beta\alpha^{-1}]_P)$ so injectivity of f shows that $[\beta\alpha^{-1}]_P = [e]_P$, i.e., $\beta\alpha^{-1} \in [e]_P$.

Conversely, suppose $\beta\alpha^{-1} \in [e]_P$. Then $(\beta\alpha)^{-1}(P, f)\mathcal{R}(P, f)$ by definition of $\mathcal{R}$, and since the $\mathcal{R}$-classes are trivial this implies $(\beta\alpha)^{-1}(P, f) = (P, f)$ and thus $\beta^{-1}(P, f) = \alpha^{-1}(P, f)$. Therefore $f([\beta]_P) = (\beta^{-1}f)([e]_{\beta^{-1}P}) = (\alpha^{-1}f)([e]_{\alpha^{-1}P}) = f([\alpha]_P)$. Since f is injective we conclude that $[\beta]_P = [\alpha]_P$, i.e., $(\alpha, \beta) \in P$. □[Claim]

It is immediate from the claim that $[e]_P$ is a subgroup of Γ and that P is the partition determined by the right cosets of $[e]_P$, i.e., $P = \Phi([e]_P)$. □

8.2. Normalized subequivalence relations. As in the previous section let $F \subseteq X \times X$ be a m.p. countable Borel equivalence relation on (X, μ) and let $\boldsymbol{a} = \Gamma \curvearrowright^a (X, \mu)$ be a m.p. action of Γ.

DEFINITION 8.9. F is said to be *normalized* by $\boldsymbol{a} = \Gamma \curvearrowright^a (X,\mu)$ if there is a conull set $X_0 \subseteq X$ such that

$$xFy \;\Rightarrow\; \gamma xF\gamma y$$

for all $\gamma \in \Gamma$ and $x, y \in X_0$. Equivalently, F is normalized by $\boldsymbol{a}$ if the image of Γ in $\mathrm{Aut}(X,\mu)$ is contained in the normalizer of the full group of F. A realization $(\boldsymbol{a}, F)$ of an invariant random partition ρ of Γ is called *normal* if F is normalized by $\boldsymbol{a}$.

Note that if F is normalized by $\boldsymbol{a}$ then $F \cap E_a$ is normalized by $\boldsymbol{a}$ and $P^a_{F\cap E_a}(x) = P^a_F(x)$ so it makes sense once again to restrict our attention to the case where $F \subseteq E_a$. Define now

$$\Gamma^a_F(x) = \{\gamma \in \Gamma \,:\, \gamma^{-1}xFx\}$$

It follows from the definitions that $\Gamma^a_F(x) = [e]_{P^a_F(x)}$.

PROPOSITION 8.10. *Let F be a subequivalence relation of E_a. Then the following are equivalent*

(1) *F is normalized by $\boldsymbol{a}$.*
(2) *For almost all x, $\Gamma^a_F(x)$ is a subgroup of Γ and $P^a_F(x)$ is the partition of Γ determined by the right cosets of $\Gamma^a_F(x)$, i.e.,*

$$(\alpha,\beta) \in P^a_F(x) \;\Leftrightarrow\; \Gamma^a_F(x)\alpha = \Gamma^a_F(x)\beta.$$

for all $\alpha, \beta \in \Gamma$.

(3) *$\Gamma^a_F(\gamma x) = \gamma\Gamma^a_F(x)\gamma^{-1}$ for almost all $x \in X$ and all $\gamma \in \Gamma$.*
(4) *The set $[e]_P$ is a subgroup of Γ for $(P^a_F)_*\mu$-almost every $P \in \mathcal{P}_\Gamma$ and the map $P \mapsto [e]_P$ is an isomorphism from $\Gamma \curvearrowright^\ell (\mathcal{P}_\Gamma, (P^a_F)_*\mu)$ to $\Gamma \curvearrowright^c (\mathrm{Sub}_\Gamma, (\Gamma^a_F)_*\mu)$.*

PROOF. (1)$\Rightarrow$(2): Suppose (1) holds. By ignoring a null set we may assume without loss of generality that $xFy \;\Rightarrow\; \gamma xF\gamma y$ for all $x, y \in X$ and $\gamma \in \Gamma$. We have that $e \in \Gamma^a_F(x)$ for all x. If $\gamma \in \Gamma^a_F(x)$ then $\gamma^{-1}xFx$ so by normality we have $xF\gamma x$ and thus $\gamma^{-1} \in \Gamma^a_F(x)$. If in addition $\delta \in \Gamma^a_F(x)$ then $\delta^{-1}xFxF\gamma x$ so that $\delta^{-1}xF\gamma x$ which by normality implies $\gamma^{-1}\delta^{-1}xFx$, i.e., $\delta\gamma \in \Gamma^a_F(x)$. This shows that $\Gamma^a_F(x)$ is a subgroup. It remains to show that $[\delta]_{P^a_F(x)} = \Gamma^a_F(x)\delta$. We have $\gamma \in [\delta]_{P^a_F(x)}$ if and only if $\delta^{-1}xF\gamma^{-1}x$ which by normality is equivalent to $(\delta\gamma^{-1})xFx$, i.e., $\gamma \in \Gamma^a_F(x)\delta$.

(2)⇒(3): Suppose (2) holds. Then for almost all x and all $\gamma, \delta \in \Gamma$ we have

$$\delta \in \Gamma_F^a(\gamma x) \;\Leftrightarrow\; \delta^{-1}\gamma x F \gamma^a x \;\Leftrightarrow\; \gamma^{-1}\delta^{-1}\gamma x F x \;\Leftrightarrow\; \delta \in \gamma \Gamma_F^a(x)\gamma^{-1}.$$

(3)⇒(1): Suppose that (3) holds. Let $X_0 \subseteq X$ be an E_a-invariant conull set such that $\Gamma_F^a(\gamma x) = \gamma\Gamma_F^a(x)\gamma^{-1}$ for all $x \in X_0$ and $\gamma \in \Gamma$. Then for any $x, y \in F$, if xFy then xE_ay so that $y = \delta x$ for some $\delta \in \Gamma$. This means that $\delta^{-1} \in \Gamma_F^a(x)$ and, so for all $\gamma \in \Gamma$ we have $\gamma\delta^{-1}\gamma^{-1} \in \Gamma_F^a(\gamma x)$ and thus

$$\gamma y = (\gamma\delta^{-1}\gamma^{-1})^{-1}(\gamma x)F\gamma x.$$

This shows that F is normalized by $\boldsymbol{a}$.

(2)+(3)⇒(4): Assume (2) and (3) hold. Then the measure $(P_F^a)_*\mu$ concentrates on $\Phi(\mathrm{Sub}_\Gamma)$. It follows that $P \mapsto [e]_P$ is injective on a $(P_F^a)_*\mu$-conull set. By (3) this map is equivariant on a conull set. Since the composition $x \mapsto P_F^a(x) \mapsto [e]_{P_F^a(x)}$ is the same as $x \mapsto \Gamma_F^a(x)$ this map is measure preserving.

Finally, the implication (4)⇒(3) is clear. □

The following corollary is immediate.

COROLLARY 8.11. *If F is normalized by $\boldsymbol{a}$ then $(\Gamma_F^a)_*\mu$ is an invariant random subgroup of Γ.*

Theorem 8.4 also implies a converse to Corollary 8.11. Let θ be an invariant random subgroup of Γ and let $\rho = \Phi_*\theta$. Let $\boldsymbol{b}$ and $\mathcal{F}$ be defined as in Theorem 8.4. Let $\boldsymbol{a} = \boldsymbol{\theta} \times \boldsymbol{s}_\Gamma$ so that $(X, \mu) = (\mathrm{Sub}_\Gamma \times [0,1]^\Gamma, \theta \times \lambda)$. Then the map $\Psi : (H, x) \mapsto (\Phi(H), x)$ is an isomorphism of $\boldsymbol{a}$ with $\boldsymbol{b}$. Letting $\mathcal{F}_0 = (\Psi \times \Psi)^{-1}(\mathcal{F})$, we have that

$$(H, x)\mathcal{F}_0(L, y) \;\Leftrightarrow\; H = L \text{ and } (\exists h \in H)(h^a x = y). \tag{8.3}$$

COROLLARY 8.12. *$\mathcal{F}_0$ is a subequivalence relation of E_a on X which is normalized by $\boldsymbol{a}$ and satisfies $\Gamma_{\mathcal{F}_0}^a(H, x) = H$ for $\theta \times \mu$-almost-every $(H, x) \in X$. Thus $(P_{\mathcal{F}_0}^a)_*\mu = \Phi_*\theta$. It follows that every invariant random subgroup of Γ admits a normal, free realization.*

PROOF. All that needs to be checked is that $\mathcal{F}_0$ is normalized by $\boldsymbol{\theta} \times \boldsymbol{a}$. If $(H, x)\mathcal{F}_0(L, y)$ then $H = L$ and $h^a x = y$ for some $h \in H$. Then for any $\gamma \in \Gamma$ we must show that $\gamma \cdot (H, x)\,\mathcal{F}_0\,\gamma \cdot (H, h^a x)$. Now, $\gamma \cdot (H, x) = (\gamma H \gamma^{-1}, \gamma^a x)$, so as $\gamma h \gamma^{-1} \in \gamma H \gamma^{-1}$ the definition (8.3) of $\mathcal{F}_0$

shows that

$$(\gamma H\gamma^{-1}, \gamma^a x)\,\mathcal{F}_0\,\gamma h\gamma^{-1} \cdot (\gamma H\gamma^{-1}, \gamma^a x) = \gamma \cdot (H, h^a x) \qquad \square$$

REMARK 8.13. In Corollary 8.12, if θ concentrates on the amenable subgroups of Γ then $\mathcal{F}_0$ will always be an amenable equivalence relation. For other properties of θ, a judicious choice of free action $\boldsymbol{d}$ in place of $\boldsymbol{s}_\Gamma$ in the definition of $\boldsymbol{a}$ may ensure that properties of θ are reflected by the equivalence relation F. For example, if θ concentrates on subgroups of cost r then the proof of Theorem 6.31 above shows that $\boldsymbol{d}$ can be chosen so that the corresponding equivalence relation $\mathcal{F}_0$ has cost r. Similarly, if θ concentrates on treeable subgroups then $\mathcal{F}_0$ can be made a treeable equivalence relation.

REMARK 8.14. Following [**BN09**, §2.2] let $2^\Gamma_e = \{L \in 2^\Gamma \,:\, e \in L\}$ and define the equivalence relation $\mathcal{R}_e \subseteq 2^\Gamma_e \times 2^\Gamma_e$ by

$$(L, K) \in \mathcal{R}_e \Leftrightarrow \exists \gamma \in L \;\, \gamma^{-1}L = K.$$

Then any $\mathcal{R}_e$-invariant Borel probability measure on 2^Γ_e is called a a *measurable subgroup* of Γ (see [**BN09**] and [**Bow12a**]). If ρ is any invariant random partition of Γ then the image of ρ under $P \mapsto [e]_P$ is a measurable subgroup of Γ. I do not know whether every measurable subgroup of Γ comes from an invariant random partition in this way.

Creutz and Peterson [**CP12**] define the *subgroup* partial order on $(\mathrm{IRS}_\Gamma, \leq)$ as follows: Let $\theta_1, \theta_2 \in \mathrm{IRS}_\Gamma$. Then θ_1 is called a *subgroup* of θ_2 (written $\theta_1 \leq \theta_2$) if there exists a joining of θ_1 and θ_2 that concentrates on the set $\{(H, L) \in \mathrm{Sub}_\Gamma \,:\, H \leq L\}$. It is shown in [**CP12**] that this is a partial order on IRS_Γ. The same idea can be used to define a notion of refinement for invariant random partitions.

For partitions $P, Q \in \mathcal{P}_\Gamma$, P is said to *refine* Q, written $P \leq Q$, if P is a subset of Q. Equivalently $P \leq Q$ means that $[\alpha]_P \subseteq [\alpha]_Q$ for every $\alpha \in \Gamma$. If ρ_1 and ρ_2 are invariant random partitions of Γ then ρ_1 *refines* ρ_2, written $\rho_1 \leq \rho_2$, if there exists a joining of ρ_1 and ρ_2 that concentrates on the set $\{(P, Q) \in \mathcal{P}_\Gamma \times \mathcal{P}_\Gamma \,:\, P \leq Q\}$. It is clear that the restriction of the refinement relation on $\mathcal{P}_\Gamma$ (respectively, IRP_Γ) to Sub_Γ (respectively, IRS_Γ) is the subgroup relation.

The point of view developed in this section can be used to give a characterization of the partial orders $(\mathrm{IRS}_\Gamma, \leq)$ and $(\mathrm{IRP}_\Gamma, \leq)$ in terms of subequivalence relations of free actions of Γ.

THEOREM 8.15. *Let $\rho_1, \rho_2 \in \mathrm{IRP}_\Gamma$. Then the following are equivalent*

(1) $\rho_1 \leq \rho_2$

(2) *There exists a free m.p. action* $\Gamma \curvearrowright^a (X, \mu)$ *of* Γ *and equivalence relations* $F_1 \subseteq F_2 \subseteq E_a$ *with* $(P^a_{F_1})_*\mu = \rho_1$ *and* $(P^a_{F_2})_*\mu = \rho_2$.

If $\theta_1, \theta_2 \in \mathrm{IRS}_\Gamma$ *then then following are equivalent*

(1') $\theta_1 \leq \theta_2$.

(2') *There exists a free m.p. action* $\Gamma \curvearrowright^a (X, \mu)$ *of* Γ *and normalized equivalence relations* $F_1 \subseteq F_2 \subseteq E_a$ *with* $(\Gamma^a_{F_1})_*\mu = \theta_1$ *and* $(\Gamma^a_{F_2})_*\mu = \theta_2$.

PROOF. Suppose (2) holds and let $P^a_{F_1} \times P^a_{F_2} : X \to \mathcal{P}_\Gamma \times \mathcal{P}_\Gamma$ be the map $x \mapsto (P^a_{F_1}(x), P^a_{F_2}(x))$. Then $(P^a_{F_1} \times P^a_{F_2})_*\mu$ is a joining of ρ_1 and ρ_2 with the desired property.

Assume that (1) holds and let ν be a joining of ρ_1 and ρ_2 witnessing that $\rho_1 \leq \rho_2$. Let $X = \mathcal{P}_\Gamma \times \mathcal{P}_\Gamma \times [0,1]^\Gamma$, let $\mu = \nu \times \lambda^\Gamma$, and let $a = \ell \times \ell \times s$. Then we define the equivalence relations F_1 and F_2 on X by

$$(P_1, P_2, x) F_1 (Q_1, Q_2, y) \Leftrightarrow \exists \gamma \in \Gamma(\gamma^{-1} \in [e]_{P_1} \text{ and } \gamma^a \cdot (P_1, P_2, x) = (Q_1, Q_2, y))$$

$$(P_1, P_2, x) F_2 (Q_1, Q_2, y) \Leftrightarrow \exists \gamma \in \Gamma(\gamma^{-1} \in [e]_{P_2} \text{ and } \gamma^a \cdot (P_1, P_2, x) = (Q_1, Q_2, y)).$$

Then as in the proof of Theorem 8.4, F_1 and F_2 are equivalence relations that are contained in E_a and $(\boldsymbol{a}, F_i)$ is a realization of F_i for each $i \in \{1, 2\}$. The defining property of ν also ensures that $F_1 \subseteq F_2$.

The equivalence of (1') and (2') then follows from the equivalence of (1) and (2) along with Proposition 8.10. $\square$

Finally, we note the following (observed by Vershik [**Ver11**] in the case of invariant random subgroups), which is a consequence of [**IKT09**, §1].

THEOREM 8.16. *Let* ρ *be an invariant random partition of* Γ. *Then the function*

$$\varphi_\rho(\gamma) = \rho(\{P : \gamma \in [e]_P\})$$

is a positive definite function on Γ.

PROOF. By Corollary 8.6 there is a free m.p. action $\boldsymbol{b} = \Gamma \curvearrowright^b (Y, \nu)$ of Γ and a subequivalence relation F of E_b such that $(P^b_F)_*\nu = \rho$. Thus

$$\varphi_\rho(\gamma) = \nu(\{y : \gamma^{-1}yFy\}).$$

This is a positive definite function by [**IKT09**]. □

9. Appendix: The amenable radical of a countable group

Every countable discrete group Γ contains a largest normal amenable subgroup called the *amenable radical* of Γ (see, e.g., [**Zim84**, 4.1.12]). We write AR_Γ for the amenable radical of Γ. We present in this appendix some facts concerning AR_Γ for countable Γ.

9.1. Basic properties of AR_Γ.

PROPOSITION 9.1. *Let Γ be a countable group.*

(1) AR_Γ *is an amenable characteristic subgroup of* Γ *which contains every normal amenable subgroup of* Γ.
(2) *Suppose $\varphi : \Gamma \to \Delta$ is a group homomorphism and that ker(φ) is amenable. Then* $\varphi(\mathrm{AR}_\Gamma) = \mathrm{AR}_{\varphi(\Gamma)}$. *In particular, the amenable radical of the quotient group* $\Gamma/\mathrm{AR}_\Gamma$ *is trivial.*
(3) *If H is normal in Γ then* AR_H *is a normal subgroup of* AR_Γ *with* $\mathrm{AR}_H = \mathrm{AR}_\Gamma \cap H$.
(4) *If H is finite index in Γ then* AR_H *is a finite index subgroup of* AR_Γ *with* $\mathrm{AR}_H = \mathrm{AR}_\Gamma \cap H$.

PROOF. For (1) see [**Zim84**]. For (2), let $N = \ker(\varphi)$. It is clear that $\varphi(\mathrm{AR}_\Gamma)$ is a normal amenable subgroup of $\varphi(\Gamma)$, so that $\varphi(\mathrm{AR}_\Gamma) \leq \mathrm{AR}_{\varphi(\Gamma)}$ by (1). The group $K = \varphi^{-1}(\mathrm{AR}_{\varphi(\Gamma)})$ is normal in Γ and K is amenable since both N and $K/N \cong \mathrm{AR}_{\varphi(\Gamma)}$ are amenable. Hence $K \leq \mathrm{AR}_\Gamma$ and so $\mathrm{AR}_{\varphi(\Gamma)} \leq \varphi(K) \leq \varphi(\mathrm{AR}_\Gamma)$.

We now prove (3). Suppose that H is normal in Γ. It is clear that $\mathrm{AR}_\Gamma \cap H$ is normal in AR_Γ, so it suffices to show that $\mathrm{AR}_\Gamma \cap H = \mathrm{AR}_H$. Conjugation by any element of Γ is an automorphism of H, so fixes (setwise) the characteristic subgroup AR_H. This shows that AR_H is normal in Γ, and since it is amenable it must be contained in AR_Γ. Thus $\mathrm{AR}_H \leq \mathrm{AR}_\Gamma \cap H$. In addition, $\mathrm{AR}_\Gamma \cap H$ is a normal amenable subgroup of H, so $\mathrm{AR}_\Gamma \cap H \leq \mathrm{AR}_H$. This proves (3).

We need the following Lemma for (4):

LEMMA 9.2. *Suppose that K is an amenable subgroup of Γ whose normalizer $N_\Gamma(K)$ is finite index in Γ. Then $K \leq \mathrm{AR}_\Gamma$.*

PROOF OF LEMMA 9.2. Suppose first that K is finite. $N_\Gamma(K)$ being finite index means K has only finitely many conjugates in Γ, so as K itself is finite this implies that every element of K has a finite conjugacy class in Γ. Thus, $K \subseteq \mathrm{FC}_\Gamma \subseteq \mathrm{AR}_\Gamma$, where FC_Γ is the amenable characteristic subgroup of Γ consisting of all elements of Γ with finite conjugacy classes (see, e.g., [**dlH07**, Appendix J]).

Suppose now that K is infinite. The normal core $N = \bigcap_{\gamma\in\Gamma} \gamma N_\Gamma(K)\gamma^{-1}$ of $N_\Gamma(K)$ in Γ is a normal finite index subgroup of Γ. Thus, letting $H = K \cap N$, we have $[K : H] = [KN : N] \leq [\Gamma : N] < \infty$, and so H is finite index in K. It is clear that H is normal in N, and H is an amenable group since it is a subgroup of K. Thus $H \leq \mathrm{AR}_N$. In addition, AR_N is normal in Γ since AR_N is characteristic in N and N is normal in Γ. Therefore

$$H \leq \mathrm{AR}_N \leq \mathrm{AR}_\Gamma.$$

Now, H is finite index in K, and $H \leq \mathrm{AR}_\Gamma$, so the image $p(K)$ of K in $\Gamma/\mathrm{AR}_\Gamma$ under the quotient map p is a finite subgroup of Γ. So if $p(K)$ were non-trivial then $\Gamma/\mathrm{AR}_\Gamma$ would have non-trivial amenable radical, contrary to part (2). □[Lemma 9.2]

We can now show (4). If H is finite index in Γ, then AR_H is an amenable subgroup of Γ whose normalizer $N_\Gamma(\mathrm{AR}_H)$ contains H. Therefore $N_\Gamma(\mathrm{AR}_H)$ is finite index in Γ, so $\mathrm{AR}_H \leq \mathrm{AR}_\Gamma$ by Lemma 9.2, and thus $\mathrm{AR}_H \leq \mathrm{AR}_\Gamma \cap H$. The group AR_Γ is normal in Γ, so $\mathrm{AR}_\Gamma \cap H$ is normal in H and since it is an amenable group we have the other inclusion $\mathrm{AR}_\Gamma \cap H \leq \mathrm{AR}_H$. □

LEMMA 9.3. *Let Γ be a countable group and let $\{H_\alpha\}_{\alpha\leq\lambda}$ be an almost ascendant series in Γ (Definition 4.12). Then $\{\mathrm{AR}_{H_\alpha}\}_{\alpha\leq\lambda}$ is an almost ascendant series in AR_Γ. The same holds if we replace "almost ascendant" by "ascendant."*

PROOF. We show by transfinite induction on ordinals α (with $\alpha \leq \lambda$) that $\{\mathrm{AR}_{H_\beta}\}_{\beta\leq\alpha}$ is an almost ascendant series in AR_{H_α}. If $\alpha = \beta + 1$ is a successor ordinal then by hypothesis H_β is either normal or finite index in $H_{\beta+1}$. Proposition 9.1 then implies that AR_{H_β} is either normal or finite index in $\mathrm{AR}_{H_{\beta+1}}$.

Suppose now that α is a limit ordinal and let $K = \bigcup_{\beta<\alpha} \mathrm{AR}_{H_\beta}$. We must show that $\mathrm{AR}_{H_\alpha} = K$. By the induction hypothesis the groups AR_{H_β}, $\beta < \alpha$, are increasing with β, so K is amenable, being an increasing union of amenable groups. Additionally, K is normal in H_α as we now show. For each $h \in H_\alpha$ there is some $\beta_0 < \alpha$ such that $h \in H_{\beta_0}$. Therefore $h \in H_\beta$ for all $\beta_0 \leq \beta < \alpha$. Thus h normalizes AR_{H_β} for all $\beta_0 \leq \beta < \alpha$, and since the AR_{H_β} are increasing we have

$$hKh^{-1} = \bigcup_{\beta_0 \leq \beta < \alpha} h\mathrm{AR}_{H_\beta}h^{-1} = \bigcup_{\beta_0 \leq \beta < \alpha} \mathrm{AR}_{H_\beta} = K.$$

It follows that $K \leq \mathrm{AR}_{H_\alpha}$. We have the equality $K = \mathrm{AR}_{H_\alpha}$ since $\mathrm{AR}_{H_\alpha} = \bigcup_{\beta<\alpha}(\mathrm{AR}_{H_\alpha} \cap H_\beta) \leq \bigcup_{\beta<\alpha} \mathrm{AR}_{H_\beta} = K$. □

COROLLARY 9.4. *Let* Γ *be a countable group and let* H *be an almost ascendant subgroup of* Γ. *Then*

$$\mathrm{AR}_H = \mathrm{AR}_\Gamma \cap H,$$

In particular, AR_H *is contained in* AR_Γ, *and* AR_Γ *contains every almost ascendant amenable subgroup of* Γ.

PROOF. The containment $\mathrm{AR}_H \leq \mathrm{AR}_\Gamma \cap H$ is immediate from Lemma 9.3. We have equality since $\mathrm{AR}_\Gamma \cap H$ is an amenable normal subgroup of H. □

COROLLARY 9.5. *Let* Γ *be a countable group and let* $\gamma \in \Gamma$. *If the centralizer* $C_\Gamma(\gamma)$ *of* γ *is almost ascendant in* Γ *then* $\gamma \in \mathrm{AR}_\Gamma$. *Thus, if* AR_Γ *is trivial then the centralizer of any non-trivial element of* Γ *is not almost ascendant.*

PROOF. The group $\langle\gamma\rangle$ is a normal amenable subgroup of $C_\Gamma(\gamma)$, so if $C_\Gamma(\gamma)$ is almost ascendant then $\langle\gamma\rangle \leq \mathrm{AR}_{C_\Gamma(\gamma)} \leq \mathrm{AR}_\Gamma$ by 9.4. □

9.2. Groups with trivial amenable radical.

LEMMA 9.6. *Let* N *be a normal subgroup of* Γ. *Then* AR_Γ *is trivial if and only if both* AR_N *and* $\mathrm{AR}_{C_\Gamma(N)}$ *are trivial.*

PROOF. Since N is normal in Γ, $C_\Gamma(N)$ is normal in Γ as well. Thus, if AR_Γ is trivial it follows from Proposition 9.1 that both AR_N and $\mathrm{AR}_{C_\Gamma(N)}$ are trivial.

Suppose now that AR_N and $\mathrm{AR}_{C_\Gamma(N)}$ are trivial. We have

$$\mathrm{AR}_\Gamma \cap N = \mathrm{AR}_N = \{e\}$$

and thus AR_Γ and N must commute, being normal subgroups of Γ with trivial intersection. This means that $\mathrm{AR}_\Gamma \leq C_\Gamma(N)$ and so

$$\mathrm{AR}_\Gamma = \mathrm{AR}_\Gamma \cap C_\Gamma(N) = \mathrm{AR}_{C_\Gamma(N)} = \{e\}. \qquad \square$$

LEMMA 9.7. *Suppose $\{H_\alpha\}_{\alpha\leq\lambda}$ is an ascendant series of length λ and suppose $\Gamma = H_\lambda$ has trivial amenable radical. Then $\mathrm{AR}_{C_\Gamma(H_\alpha)} = \{e\}$ for all $\alpha \leq \lambda$.*

PROOF. We proceed by transfinite induction on λ. By Corollary 9.4 we know that $\mathrm{AR}_{H_\alpha} = \{e\}$ for all $\alpha \leq \lambda$.

Limit stages: Suppose first that λ is a limit ordinal. Fix $\alpha \leq \lambda$ and let $H = H_\alpha$. By intersecting each term of the ascendant series $\{H_\beta\}_{\beta\leq\lambda}$ with $C_\Gamma(H)$ we obtain the series $\{C_{H_\beta}(H)\}_{\beta\leq\lambda}$ which is ascendant in $C_\Gamma(H)$. Lemma 9.3 implies that $\{\mathrm{AR}_{C_{H_\beta}(H)}\}_{\beta\leq\lambda}$ is an ascendant series in $\mathrm{AR}_{C_\Gamma(H)}$ and so

$$\mathrm{AR}_{C_\Gamma(H)} = \bigcup_{\alpha\leq\beta<\lambda} \mathrm{AR}_{C_{H_\beta}(H)} \tag{9.1}$$

where the union is increasing. For each β with $\alpha \leq \beta < \lambda$ the series $\{H_\xi\}_{\xi\leq\beta}$ has length strictly less than λ, so by the induction hypothesis we have

$$\mathrm{AR}_{C_{H_\beta}(H)} = \{e\}.$$

Since this holds for each β with $\alpha \leq \beta < \lambda$, equation (9.1) shows that $\mathrm{AR}_{C_\Gamma(H)} = \{e\}$ as was to be shown.

Successor stages: Suppose now that $\lambda = \mu + 1$ is a successor ordinal. Fix for the moment some $\alpha < \lambda$ and let $H = H_\alpha$. Applying the induction hypothesis to the ascendant series $\{H_\beta\}_{\beta\leq\mu}$ in H_μ we obtain that $\mathrm{AR}_{C_{H_\mu}(H)} = \{e\}$. Since H_μ is normal in Γ, $C_{H_\mu}(H)$ is normal in $C_\Gamma(H)$, so it follows from Proposition 9.1.(3) that

$$\mathrm{AR}_{C_\Gamma(H)} \cap H_\mu = \mathrm{AR}_{C_\Gamma(H)} \cap C_{H_\mu}(H) = \mathrm{AR}_{C_{H_\mu}(H)} = \{e\}. \tag{9.2}$$

Since α was an arbitrary ordinal satisfying $\alpha < \lambda$, (9.2) holds for all $\alpha < \lambda$. We use this to show the following.

CLAIM 7. *Let ξ and β be ordinals with $\xi \leq \beta < \lambda$. Then*

$$\mathrm{AR}_{C_\Gamma(H_\xi)} \leq \mathrm{AR}_{C_\Gamma(H_\beta)}$$

PROOF OF CLAIM 7. We show by transfinite induction on $\beta < \lambda$ that $\{\mathrm{AR}_{C_\Gamma(H_\xi)}\}_{\xi \leq \beta}$ is increasing in ξ. If $\beta = 0$ this is trivial. If $\beta = \alpha+1$ is a successor ordinal then the induction hypothesis tells us that $\{\mathrm{AR}_{C_\Gamma(H_\xi)}\}_{\xi \leq \alpha}$ is increasing with ξ and we must show that $\mathrm{AR}_{C_\Gamma(H_\alpha)} \leq \mathrm{AR}_{C_\Gamma(H_{\alpha+1})}$.

Since H_α is normal in $H_{\alpha+1}$, Proposition 9.1.(2) shows that $H_{\alpha+1}$ normalizes $\mathrm{AR}_{C_\Gamma(H_\alpha)}$. Thus, for $\delta \in H_{\alpha+1}$ and $\gamma \in \mathrm{AR}_{C_\Gamma(H_\alpha)}$ we have

$$\begin{aligned}
&(\delta\gamma\delta^{-1})\gamma^{-1} \in \mathrm{AR}_{C_\Gamma(H_\alpha)} \\
&\delta(\gamma\delta^{-1}\gamma^{-1}) \in H_\mu(\gamma H_\mu \gamma^{-1}) = H_\mu \\
\text{so that} \quad &\delta\gamma\delta^{-1}\gamma^{-1} \in \mathrm{AR}_{C_\Gamma(H_\alpha)} \cap H_\mu = \{e\}
\end{aligned}$$

by (9.2) (we use in the second line that $H_{\alpha+1} \leq H_\mu$ and $H_\mu \lhd \Gamma$). This shows that the groups $\mathrm{AR}_{C_\Gamma(H_\alpha)}$ and $H_{\alpha+1}$ commute, and so $\mathrm{AR}_{C_\Gamma(H_\alpha)}$ is a subgroup of $C_\Gamma(H_{\alpha+1})$. As $C_\Gamma(H_{\alpha+1})$ is contained in $C_\Gamma(H_\alpha)$ we conclude that $\mathrm{AR}_{C_\Gamma(H_\alpha)}$ is normal in $C_\Gamma(H_{\alpha+1})$ and therefore $\mathrm{AR}_{C_\Gamma(H_\alpha)} \leq \mathrm{AR}_{C_\Gamma(H_{\alpha+1})}$.

Now suppose β is a limit ordinal. The induction hypothesis tells us that $\{\mathrm{AR}_{C_\Gamma(H_\xi)}\}_{\xi<\beta}$ is increasing with $\xi < \beta$ and we must show that $\mathrm{AR}_{C_\Gamma(H_\xi)} \leq \mathrm{AR}_{C_\Gamma(H_\beta)}$ for all $\xi < \beta$. Fix $\xi < \beta$. For each α with $\xi \leq \alpha < \beta$ we have that $\mathrm{AR}_{C_\Gamma(H_\xi)} \leq \mathrm{AR}_{C_\Gamma(H_\alpha)} \leq C_\Gamma(H_\alpha)$. Intersecting this over all such α shows

$$\mathrm{AR}_{C_\Gamma(H_\xi)} \leq \bigcap_{\xi \leq \alpha < \beta} C_\Gamma(H_\alpha) = C_\Gamma\big(\bigcup_{\xi \leq \alpha < \beta} H_\alpha\big) = C_\Gamma(H_\beta).$$

Since $C_\Gamma(H_\beta) \leq C_\Gamma(H_\xi)$ we actually have $\mathrm{AR}_{C_\Gamma(H_\xi)} \lhd C_\Gamma(H_\beta)$ and so $\mathrm{AR}_{C_\Gamma(H_\xi)} \leq \mathrm{AR}_{C_\Gamma(H_\beta)}$, which finishes the proof of the claim. □[Claim 7]

Given now any $\alpha < \lambda$ we have shown that $\mathrm{AR}_{C_\Gamma(H_\alpha)} \leq \mathrm{AR}_{C_\Gamma(H_\mu)}$. But H_μ is normal in Γ and $\mathrm{AR}_\Gamma = \{e\}$, so Lemma 9.6 shows that $\mathrm{AR}_{C_\Gamma(H_\mu)} = \{e\}$ and therefore $\mathrm{AR}_{C_\Gamma(H_\alpha)} = \{e\}$ as was to be shown. □[Lemma 9.7]

LEMMA 9.8. *Let* $\{H_\alpha\}_{\alpha\leq\lambda}$ *be an ascendant series of length* λ *with* $H_0 = H$ *and* $H_\lambda = \Gamma$. *Suppose that* $\mathrm{AR}_{C_\Gamma(H)} = \mathrm{AR}_H = \{e\}$. *Then* $\mathrm{AR}_\Gamma = \{e\}$.

PROOF. We proceed by transfinite induction on the length λ of the series.

Limit stages: Suppose first that λ is a limit ordinal. By intersecting each group in the series $\{H_\alpha\}_{\alpha\leq\lambda}$ with $C_\Gamma(H)$ we obtain the series $\{C_{H_\alpha}(H)\}_{\alpha\leq\lambda}$, which is ascendant in $C_\Gamma(H)$. Applying Lemma 9.3 to the series $\{C_{H_\alpha}(H)\}_{\alpha\leq\lambda}$ we obtain

$$\bigcup_{\alpha<\lambda} \mathrm{AR}_{C_{H_\alpha}(H)} = \mathrm{AR}_{C_\Gamma(H)}.$$

Since $\mathrm{AR}_{C_\Gamma(H)} = \{e\}$ we conclude that $\mathrm{AR}_{C_{H_\alpha}(H)} = \{e\}$ for all $\alpha < \lambda$. In addition we have $\mathrm{AR}_H = \{e\}$ so it follows from the induction hypothesis (applied to each series $\{H_\xi\}_{\xi<\alpha}$ for $\alpha < \lambda$) that $\mathrm{AR}_{H_\alpha} = \{e\}$ for all α. Another application of Lemma 9.3 now shows that $\mathrm{AR}_\Gamma = \bigcup_{\alpha<\lambda} \mathrm{AR}_{H_\alpha} = \{e\}$.

Successor stages: Now assume that $\lambda = \mu + 1$ is a successor ordinal. Since H_μ is normal in $H_{\mu+1} = \Gamma$ we have $C_{H_\mu}(H) \triangleleft C_\Gamma(H)$. It follows that $\mathrm{AR}_{C_{H_\mu}(H)} \leq \mathrm{AR}_{C_\Gamma(H)} = \{e\}$ and so

$$\mathrm{AR}_{C_{H_\mu}(H)} = \{e\}.$$

By assumption $\mathrm{AR}_H = \{e\}$ so the induction hypothesis applied to $\{H_\alpha\}_{\alpha\leq\mu}$ implies that

$$\mathrm{AR}_{H_\mu} = \{e\}. \tag{9.3}$$

Since H_μ is normal in Γ, $C_\Gamma(H_\mu)$ is normal in Γ as well. In addition, $C_\Gamma(H_\mu)$ is contained in $C_\Gamma(H)$, so in fact $C_\Gamma(H_\mu) \triangleleft C_\Gamma(H)$. It follows that

$$\mathrm{AR}_{C_\Gamma(H_\mu)} \leq \mathrm{AR}_{C_\Gamma(H)} = \{e\}. \tag{9.4}$$

We see from (9.3) and (9.4) that the normal subgroup H_μ of Γ satisfies the hypotheses of Lemma 9.6 and so $\mathrm{AR}_\Gamma = \{e\}$. This completes the induction. $\square$

THEOREM 9.9. *Let* H *be an ascendant subgroup of a countable group* Γ. *Then* $\mathrm{AR}_\Gamma = \{e\}$ *if and only if* $\mathrm{AR}_H = \{e\}$ *and* $\mathrm{AR}_{C_\Gamma(H)} = \{e\}$.

Chapter 7

Appendix: Mixing via filters and Gaussian actions

1. Milding mixing = IP*-mixing for groups

Let G be an infinite, countable group. Let βG denote the space of ultrafilters on G (topologized as a subspace of $2^{(2^{\mathbb{N}})}$ with the product topology).

DEFINITION 1.1. For any sequence $(g_i) = (g_i)_{i\in\mathbb{N}}$ of (not necessarily distinct elements) define

$$\mathrm{FP}((g_i)) := \{g_{i_1}g_{i_2}\cdots g_{i_k} \,:\, i_1 < i_2 < \cdots < i_k \ \ (k\in\mathbb{N})\}.$$

The following Proposition is similar to [**HS98**, 5.11].

PROPOSITION 1.2. *Let $A \subseteq G$. The following are equivalent:*

(1) *FP$((g_i)) \subseteq A$ for some sequence (g_i) in G with the property that for each $g \in$ FP$((g_i))$ there is a unique finite sequence $i_1 < \cdots < i_k$ in $\mathbb{N}$ such that $g = g_{i_1}\cdots g_{i_k}$;*
(2) *FP$((g_i)) \subseteq A$ for some injective sequence (g_i) in G with $e \notin$ FP$((g_i))$;*
(3) *FP$((g_i)) \subseteq A$ for some sequence (g_i) in G with $g_i \to \infty$.*
(4) *FP$((g_i)) \subseteq A$ for some sequence (g_i) in G taking infinitely many values.*
(5) *There exists a nonprincipal idempotent ultrafilter $p \in \beta G \setminus G$ with $A \in p$.*

PROOF. Note that if $(g_i)_{i\in\mathbb{N}}$ witnesses that (1) holds then $e \notin \mathrm{FP}((g_i))$, otherwise, say $e = g_{i_1}\cdots g_{i_k}$, then $g_{i_k+1} = g_{i_1}\cdots g_{i_k}g_{i_k+1}$ contradicting uniqueness. This shows (1)⇒(2). It is clear that (2)⇒(3)⇒(4). We show (4)⇒(1). Suppose that $\{g_i\}_{i\in\mathbb{N}}$ takes infinitely many values and FP$((g_i)) \subseteq A$. Define a subsequence $(h_i) \subseteq (g_i)$ as follows. Choose $h_1 \in \{g_i\}_{i\in\mathbb{N}} \setminus \{e\}$. Suppose for induction that $h_1, \ldots, h_n$ have been chosen with $e \notin$ FP$((h_1, \ldots, h_n))$ and suppose that for each $h \in$ FP$((h_1, \ldots, h_n))$ there exists a unique $i_1 < \cdots < i_k \leq n$ with $h = h_{i_1} \cdots h_{i_k}$. Note that for any $h \in G$ we have FP$((h_1, \ldots, h_n, h)) =$ FP$((h_1, \ldots, h_n)) \cup$ FP$((h_1, \ldots, h_n))h \cup \{h\}$. Now define h_{n+1} to be any element of $(g_i)_{i\in\mathbb{N}}$ with $h_{n+1} \notin$ FP$((h_1, \ldots, h_n))^{-1} \cup \{e\}$ and $h_{n+1} \notin$ FP$((h_1, \ldots, h_n))^{-1}FP((h_1, \ldots, h_n))$. This can be done since each of these sets is finite whereas $\{g_i\}_{i\in\mathbb{N}}$ is infinite by hypothesis. The second condition implies that

$$(*) \qquad \text{FP}((h_1, \ldots, h_n))h_{n+1} \cap \text{FP}((h_1, \ldots, h_n)) = \varnothing.$$

Now, the induction hypothesis, along with the condition $h_{n+1} \notin$ FP$((h_1, \ldots, h_n))^{-1} \cup \{e\}$ ensures that FP$((h_1, \ldots, h_n, h_{n+1}))$ does not contain e. We show that this choice of h_{n+1} carries the induction hypothesis to the next stage. Suppose that for some $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$ (and $k, l \geq 1$) we have

$$(**) \qquad h = h_{i_1} \cdots h_{i_k} = h_{j_1} \cdots h_{j_l} \in \text{FP}((h_1, \ldots, h_{n+1}))$$

and we will show the expressions are the same. We cannot have both $i_k = n+1$ and $j_l < n+1$ because this would contradict (*). Similarly, we cannot have $j_l = n+1$ and $i_k < n+1$. If $i_k, j_l < n+1$ then by uniqueness at stage n the expressions are the same, and if $i_k = j_l = n+1$ then after multiplying $(**)$ on the right by h_{n+1}^{-1} the induction hypothesis implies that the remaining expressions are the same and so both expressions of h in (**) are the same as well. This finishes the induction and the proof of (4)⇒(1).

We now show (2)⇒(5). It suffices to show that if $(g_i)_{i\in\mathbb{N}}$ is an injective sequence with $e \notin$ FP$((g_i))$, then FP$((g_i)) \in p$ for some idempotent $p \in \beta G \setminus G$. Let $C = \bigcap_{n=1}^{\infty} \overline{\text{FP}((g_i)_{i=n}^{\infty})}$. Here the closure is taken in βG. By compactness, C is nonempty and is itself compact.

Claim: C is a (compact) subsemigroup of βG. *Proof:* Suppose $p, q \in C$. We want to show that $p \cdot q \in C$, where $A \in p \cdot q$ iff $\{g \in G : g^{-1}A \in q\} \in p$. Note that $r \in C \Leftrightarrow \forall n,$ FP$((g_i)_{i=n}^{\infty}) \in r$. So fix n, and we show that FP$((g_i)_{i=n}^{\infty}) \in p \cdot q$, i.e., that $A := \{g \in G : g^{-1}\text{FP}((g_i)_{i=n}^{\infty}) \in q\} \in p$. Note that if $g \in$ FP$((g_i)_{i=n}^{\infty})$, say $g = g_{i_1} \cdots g_{i_k}$ $(n \leq i_1 < \cdots < i_k)$, then g^{-1}FP$((g_i)_{i=n}^{\infty}) \supseteq$ FP$((g_i)_{i=i_k+1}^{\infty}) \in q$, whence $g \in A$. Thus $A \supseteq$ FP$((g_i)_{i=n}^{\infty}) \in p$, and so $A \in p$, as was to be shown.

Since C is a compact left-topological semigroup, C has an idempotent. Let $p \in C$ be an idempotent. Note that $e \notin C$ since $e \notin$ FP$((g_i))$, so $p \neq e$ and therefore $p \in \beta G \setminus G$. It follows that $p \in \overline{\text{FP}((g_i))}$, i.e., FP$((g_i)) \in p$.

Now assume that (5) holds and we prove (2). Let $p \in \beta G \setminus G$ be any idempotent ultrafilter, and let $A \in p$. By idempotence the set $A \cap \{g \in G : g^{-1}A \in p\}$ is in p, and in particular it is infinite. So there exists $g_1 \in A \setminus \{e\}$ with $B_1 := g_1^{-1}A \in p$. By idempotence again $A \cap g_1^{-1}A \cap \{g : g^{-1}(A \cap g_1^{-1}A) \in p\} \in p$. So since $A \cap g_1^{-1}A \in p$ (and hence is infinite) there exists $g_2 \in (A \cap g_1^{-1}A) \setminus \{e, g_1, g_1^{-1}\}$ such that $g_2^{-1}A \cap g_2^{-1}g_1^{-1}A \in p$. Therefore $B_2 \in p$ where $B_2 := g_1^{-1}A \cap g_2^{-1}A \cap g_2^{-1}g_1^{-1}A$, and also $g_1, g_2, g_1g_2 \in A$ (since $g_2 \in g_1^{-1}A$) with $g_1 \neq g_2$ and $e \notin$ FP$((g_1, g_2)) \subseteq A$. Assume for induction that distinct $g_1, \ldots, g_n$ have been chosen with $e \notin$ FP$((g_i)_{i=1}^n) \subseteq A$ and with $B_n := \bigcap_{g \in \text{FP}((g_i)_{i=1}^n)} g^{-1}A \in p$. Then by idempotence $A \cap B_n \cap \{g : g^{-1}(A \cap B_n) \in p\} \in p$ so there exists $g_{n+1} \in (A \cap B_n) \setminus \big(\{e\} \cup \{g_1, \ldots, g_n\} \cup \text{FP}((g_i)_{i=1}^n)^{-1}\big)$ such that $g_{n+1}^{-1}(A \cap B_n) \in p$. It follows that the set $B_{n+1} := B_n \cap g_{n+1}^{-1}(A \cap B_n)$ is in p, and that FP$((g_i)_{i=1}^{n+1}) \subseteq A$ since $g_{n+1} \in B_n$ and FP$((g_i)_{i=1}^{n+1}) =$ FP$((g_i)_{i=1}^n) \cup$ FP$((g_i)_{i=1}^n)g_{n+1} \cup \{g_{n+1}\}$. This shows that the induction hypothesis is satisfied for the next step, and completes the proof. $\square$

DEFINITION 1.3. A subset A of G is called an IP$_G$ set if it satisfies any of the equivalent conditions (1)-(5) of Proposition 1.2

Then next Corollary is an immediate consequence of condition (5) of Proposition 1.2 and the definition of an ultrafilter.

COROLLARY 1.4. *If an IP$_G$ set is partitioned into finitely many sets, then one of the pieces of the partition is an IP$_G$ set.*

We may write IP$_G = \bigcup\{p \in \beta G \setminus G : p \text{ is idempotent}\}$. Define

$$\text{IP}_G^* = \{B \subseteq G : \forall A \in \text{IP}_G \; B \cap A \neq \varnothing\}$$

Then IP$_G^* = \bigcap\{p \in \beta G \setminus G : p \text{ is idempotent}\}$ since $B \notin p$ for some idempotent $p \in \beta G \setminus G$ if and only if $G \setminus B \in p$ for some idempotent $p \in \beta G \setminus G$ iff some A disjoint from satisfies $A \in$ IP$_G$ and $B \cap A = \varnothing$. This shows that the IP$_G^*$ sets form a filter on G.

DEFINITION 1.5. Let Z be a compact metric space and let G be a discrete group acting on Z by homeomorphisms. A point $z_0 \in Z$ is called *recurrent* (with respect to the action of G) if there exists a sequence $(g_i) \subseteq G$ with $g_i \to \infty$ and $g_i(z) \to z$ as $i \to \infty$. Equivalently, for every $\delta > 0$, the set $\{g : d(g(z_0), z_0) < \delta\}$ is infinite.

DEFINITION 1.6 ([**BdJ07**]). Let $\mathcal{F}$ be the collection of finite subsets of $\mathbb{N} = \{1, 2, \dots\}$. For $\alpha, \beta \in \mathcal{F}$ write $\alpha < \beta$ if $\max \alpha < \min \beta$. $\mathcal{F}$ is a semigroup with respect to the union operation.

If X is any set, an **$\mathcal{F}$-sequence** in X is a map $\varphi : \mathcal{F} \to X$. We will sometimes write this as $\{x_\alpha\}_{\alpha \in \mathcal{F}} \subseteq X$. The sequence $\varphi' : \mathcal{F} \to X$ is a **subsequence** of the sequence φ if there exists $\alpha_1 < \alpha_2 < \cdots$ in $\mathcal{F}$ such that $\varphi'(\{i_1, \dots, i_k\}) = \varphi(\alpha_{i_1} \cup \cdots \cup \alpha_{i_k})$.

If G is a semigroup, then an $\mathcal{F}$-sequence $\varphi : \mathcal{F} \to G$ is a **homomorphism** if $\varphi(\{i_1 < \cdots < i_k\}) = \varphi(\{i_1\}) \cdots \varphi(\{i_k\})$. Equivalently, $\varphi(\alpha \cup \beta) = \varphi(\alpha)\varphi(\beta)$ whenever $\alpha < \beta$.

A subsequence φ', as above, of a homomorphism φ is itself a homomorphism since

$$\varphi'(\{i_1 < \cdots < i_k\}) = \varphi(\alpha_{i_1} \cup \cdots \cup \alpha_{i_k}) = \varphi(\alpha_{i_1}) \cdots \varphi(\alpha_{i_k}) = \varphi'(\{i_1\}) \cdots \varphi'(\{i_k\})$$

(the second equality is justified since $\alpha_p < \alpha_q$ for $p < q$ by definition of a subsequence). A homomorphism $\varphi : \mathcal{F} \to G$ is completely determined by the values of $\varphi(\{i\}) := g_i^\varphi$ for $i \in \mathbb{N}$. We have that $\varphi(\mathcal{F}) = \mathrm{FP}((g_i^\varphi)_{i \in \mathbb{N}})$.

DEFINITION 1.7. Let G be a group. A homomorphism $\varphi : \mathcal{F} \to G$ is called *non-trivial* when there exists a subhomomorphism φ' of φ such that the set $\{g_i^{\varphi'}\}_{i \in \mathbb{N}}$ is infinite. In this case $\varphi(\mathcal{F})$ is an IP_G set.

The proof of (4)⇒(1) of Proposition 1.2 shows that if (g_i) takes infinitely many values then there is a subsequence $i_1 < i_2 < \cdots$ such that every $g \in \mathrm{FP}((g_{i_j})_{j \in \mathbb{N}})$ can be uniquely expressed as a product $g_{i_{j_1}} \cdots g_{i_{j_m}}$ for some $j_1 < \cdots < j_m$. Therefore, if φ is non-trivial as witnessed by the subhomomorphism φ', then by moving to a further subhomomorphism φ'' we can ensure that $g \in \mathrm{FP}((g_i^{\varphi''}))$ has a unique expression of the form $g = g_{i_1}^{\varphi''} \cdots g_{i_k}^{\varphi''}$ with $i_1 < \cdots < i_k$. This is equivalent to injectivity of φ''. This shows that a homomorphism φ is non-trivial if it has an injective subhomomorphism. Note that every subhomomorphism of an injective homomorphism is itself injective, hence non-trivial.

The set $\mathcal{F}$ is directed under $<$. If $\{x_\alpha\}_{\alpha \in \mathcal{F}}$ is an $\mathcal{F}$-sequence, we say that $\mathcal{F} \lim_\alpha x_\alpha = x$ if $x_\alpha \to x$ as a net on the directed set $(\mathcal{F}, <)$. This means that for every open set U containing x, there exists α such that $\beta > \alpha$ implies $x_\beta \in U$.

The following is an analogue of [**Fur81**, Theorem 2.17] for general countablly infinite groups G. The proof is identical.

THEOREM 1.8. *If $z_0 \in Z$ is a recurrent point with respect to the action of an infinite countable group G by homeomorphisms on the compact metric space Z, then for every $\delta > 0$ the set $R_\delta = \{g : d(g(z_0), z_0) < \delta\}$ contains an IP$_G$ set.*

THEOREM 1.9 (8.12 of [**Fur81**]). *If $\mathcal{F}$ is partitioned into finitely many sets, $\mathcal{F} = C_1 \uplus \cdots \uplus C_r$, then there exists $\alpha_1 < \alpha_2 < \cdots \in \mathcal{F}$ such that one of the C_j's contains $\{\alpha_{i_1} \cup \cdots \cup \alpha_{i_k} : i_1 < \cdots < i_k\}$.*

THEOREM 1.10 (8.14 of [**Fur81**]). *If $\{x_\alpha\}$ is an $\mathcal{F}$-sequence with values in a compact metric space, then there exists an $\mathcal{F}$-subsequence $\{x_\alpha\}$ which converges as an $\mathcal{F}$-sequence.*

LEMMA 1.11 (8.15 of [**Fur81**]). *Let G be a semigroup. If $\varphi : \mathcal{F} \to G$ is a homomorphism and if G acts on the compact metric space X by homeomorphisms, and $x \in X$, then there exists a subhomomorphism $\phi : \mathcal{F} \to G$ of φ such that $\phi(\alpha)(x)$ converges as an $\mathcal{F}$-sequence to a point $y \in X$, and at the same time $\mathcal{F} \lim_\alpha \phi(\alpha)(y) = y$.*

Note that for general semigroups we have not defined a notion of a non-trivial homomorphism. For groups, where this notion has been defined, the above lemma is true for non-trivial homomorphisms in place of homomorphisms. That is:

LEMMA 1.12. *Let G be an infinite countable group. If $\varphi : \mathcal{F} \to G$ is a non-trivial homomorphism and if G acts on the compact metric space X by homeomorphisms, and $x \in X$, then there exists a non-trivial subhomomorphism ϕ of φ such that $\phi(\alpha)(x)$ $\mathcal{F}$-converges to a point $y \in X$, and at the same time $\mathcal{F} \lim_\alpha \phi(\alpha)(y) = y$.*

PROOF. Let φ' be an injective subhomomorphism of φ and apply the previous lemma to obtain a subhomomorphism ϕ of φ'. Then ϕ is a non-trivial subhomomorphism of φ. $\square$

The following as an analogue of [**Fur81**, Theorem 9.20]

LEMMA 1.13. *Let G be an infinite countable group. X be a G-dynamical system with X a compact metric space and assume that there exists a unique point $x_0 \in X$ that is recurrent for G.*

Then

$$(*) \qquad \lim_{g\to IP^*_G} g(x) = x_0$$

for every $x \in X$. Conversely, if (*) holds for every $x \in X$, then x_0 is the unique recurrent point of X.

PROOF. Assume (*) holds. By 1.8, if x_1 is a recurrent point, then for every neighborhood U of x_1 $\{g : g(x_1) \in U\}$ is an IP$_G$ set. Thus, if $x_1 \neq x_0$ and if U and V are disjoint neighborhoods of x_1 and x_0 then $\{g : g(x_1) \in U\}$ is IP$_G$, but is disjoint from $\{g : g(x_0) \in V\}$, which is IP$^*_G$ by (*), a contradiction.

Suppose that x_0 is the unique recurrent point. If (*) did not hold, then for some neighborhood V of x_0 and for some $x \in X$ and some sequence $g_i \to \infty$ we have $gx \notin V$ for $g \in$ FP$((g_i))$. Let $\varphi : \mathcal{F} \to G$ be the (non-trivial) homomorphism corresponding to (g_i), i.e, with $\varphi(\{i\}) = g_i$, and by 1.12 there is a non-trivial subhomomorphism ϕ such that $\phi(\alpha)x \to y$ and $\phi(\alpha)y \to y$. Thus, y is recurrent and so $y = x_0$, but this contradicts that $\phi(\alpha)x \to y = x_0$ since $\phi(\alpha) \in$ FP$((g_i))$ for all α and so $\phi(\alpha)x \notin V$. $\square$

Now let H be a separable Hilbert space and let π be a unitary representation of the group G, and let $X = X_r$ =the ball of radius r in H, with the weak topology. Then X is compact metrizable. 0 is a recurrent point of (X, π), and in general, x is recurrent iff $\exists g_i \to \infty$ with $\pi(g_i)(x) \to x$ weakly. Since $||\pi(g_i)(x)|| = ||x||$ this implies $\pi(g_i)(x) \to x$ in norm. The following is an analogue of [**Fur81**, 9.21].

LEMMA 1.14. *Let H be a separable Hilbert space and π a unitary representation of the infinite countable group G. If 0 is the only recurrent vector of H for π, then for every $u, v \in H$,*

$$\lim_{g\to IP^*_G} \langle \pi(g)u, v\rangle = 0.$$

Conversely, if the above holds for all $u, v \in H$, then 0 is a the unique recurrent vector.

PROOF. ($\Rightarrow$): If 0 is the unique recurrent vector, then for any $u \in H$, we have that $u \in X_{||u||}$ and 0 is the unique recurrent point of this compact system. Hence $\lim_{g\to \mathrm{IP}^*_G} \pi(g)(u) = 0$ for all $u \in H$. The limit is taken in the weak topology, so this means precisely that $\lim_{g\to \mathrm{IP}^*_G} \langle \pi(g)u, v\rangle = 0$ for all $v \in X_{||u||}$, hence for any $v \in H$ by scaling (i.e., write $v = c \cdot v'$ with $v' \in X_{||u||}$). ($\Leftarrow$):

Conversely, the condition implies that 0 is the unique recurrent point in each X_r, hence 0 is the unique recurrent point in H. $\square$

DEFINITION 1.15. The representation π is called *mildly mixing* if it has no nonzero recurrent points. Equivalently, $\liminf_{\gamma\to\infty} ||\pi(\gamma)x - x|| > 0$ for all $x \neq 0$.

A measure preserving action a is called *mildly mixing* if the Koopman representation κ_0^a on L_0^2 is mildly mixing. That is, zero is the only rigid function $f \in L_0^2(X,\mu)$, where a function $f \in L^2(X)$ is rigid if for some sequence $g_n \to \infty$, $\kappa^a(g_n)(f) \to f$ in $L^2(X)$.

Note that there exists a sequence $g_n \to \infty$ with $\kappa^a(g_n)(f) \to f$ in the norm topology if and only if such a sequence exists for the weak topology, if and only if such a sequence exists such that the convergence is μ-almost everywhere if and only if such a sequence exists such that the convergence takes place in measure. Proof: If it is true in the weak topology then since $||\kappa^a(g_n)(f)||_2 = ||f||_2$, the convergence also takes place in the norm topology. This implies convergence in measure which implies convergence of a subsequence almost everywhere, which in turn implies convergence in L^2 (i.e., the norm topology) of this subsequence since the measure space is finite and $||\kappa^a(g_{n_k})(f)||_2 = ||f||_2$.

PROPOSITION 1.16. *$a \in A(G,X,\mu)$ is mild mixing if and only if for all $f,h \in L^2(X,\mu)$*

$$(1) \qquad \lim_{g\to IP_G^*} \int f(g^{-1}x)h(x)\,d\mu = \Big(\int f\,d\mu\Big)\Big(\int h\,d\mu\Big),$$

or, equivalently, for all measurable $A,B \subseteq X$

$$(2) \qquad \lim_{g\to IP_G^*} \mu(g(A)\cap B) = \mu(A)\mu(B).$$

PROOF. Mild mixing implies (1) by applying the previous lemma to $f_0 = f - \int f\,d\mu, h_0 = h - \int h\,d\mu \in L_0^2(X,\mu)$. The previous lemma also shows that (1) implies mild mixing. It is clear that (1) implies (2). For the converse, suppose that (2) holds. We only need to show that (1) holds for simple functions f,h (Proof: note that

$$|\langle f, hg^{-1}\rangle - \int f\,d\mu \int h\,d\mu| \leq$$

$$|\langle f-f', hg^{-1}\rangle| + |\langle f', hg^{-1} - h'g^{-1}\rangle| + |\langle f', h'g^{-1}\rangle - \int f' \int h'| + |\int f' \int h' - \int f \int h|.$$

So given $\epsilon > 0$ if f' and h' are simple functions chosen close enough to f and h so that the first two summands and the last summand above is $\leq \epsilon/4$, then $|\langle f', h'g^{-1}\rangle - \int f'\,d\mu \int h'\,d\mu| < \epsilon/4$ implies $|\langle f, hg^{-1}\rangle - \int f\,d\mu \int h\,d\mu| < \epsilon$, hence $\{g : |\langle f', h'g^{-1}\rangle - \int f' \int h'| < \epsilon/4\} \subseteq \{g : |\langle f, hg^{-1}\rangle - \int f \int h| < \epsilon\}$ and, assuming the former is IP^*_G, the latter is IP^*_G as well.)

Thus we show, assuming (2), that (1) holds for simple functions. We compute

$$\begin{aligned}\int \sum_{i=1}^{n} a_i 1_{gA_i}(x) \sum_{j=1}^{m} b_j 1_{B_j}(x) d\mu(x) &= \sum_{i,j} a_i b_j \int 1_{gA_i \cap B_j}(x)\,d\mu \\ &\to_{g\to\mathrm{IP}^*_G} \sum_{i,j} a_i b_j \mu(A_i)\mu(B_j) \\ &= \int \sum_i a_i 1_{A_i}\,d\mu \int \sum_j b_j 1_{B_j}\,d\mu. \qquad \square\end{aligned}$$

COROLLARY 1.17. *The countable product of mild mixing actions is mild mixing.*

2. $\mathcal{F}$-mixing

DEFINITION 2.1. Let Γ be a countable group. Let $\mathcal{F}$ be a proper filter on Γ (i.e., containing the Fréchet filter). Let H be a separable complex Hilbert space and $\pi \in \mathrm{Rep}(\Gamma, H)$ a unitary representation. π is call *$\mathcal{F}$-mixing* if for every $u, v \in H$,

$$\lim_{\gamma\to\mathcal{F}} \langle \pi(\gamma)u, v\rangle = 0.$$

For a given representation π, for each $u, v \in H$ we let $f^{\pi}_{u,v} : \Gamma \to \mathbb{C}$ be the matrix coefficient of π given by $f^{\pi}_{u,v}(\gamma) := \langle \pi(\gamma)u, v\rangle$. When π is understood we simply write $f_{u,v}$. Also, we put f_u for $f_{u,u}$. In these terms, π being $\mathcal{F}$-mixing simply means that every matrix coefficient $f^{\pi}_{u,v}(\gamma) = \langle \pi(\gamma)u, v\rangle$ vanishes as $\gamma \to \mathcal{F}$. We define $\mathcal{F}$-mixing analogously for orthogonal representations of Γ on a real Hilbert space.

DEFINITION 2.2. A measure preserving action $a \in A(\Gamma, X, \mu)$ is called *$\mathcal{F}$-mixing* if the Koopoman representation κ^a_0 on $L^2_0(X, \mu)$ is $\mathcal{F}$-mixing.

Let $\mathcal{F}\mathrm{MIX}(\Gamma, X, \mu) \subseteq A(\Gamma, X, \mu)$ denote the subspace of $\mathcal{F}$-mixing actions. We show it suffices to check that the diagonal coefficients vanish.

PROPOSITION 2.3. *The representation π is $\mathcal{F}$ mixing if and only if for every $w \in H$ the diagonal matrix coefficient $f_w(\gamma) = \langle \pi(\gamma)w, w\rangle$ vanishes as $\gamma \to \mathcal{F}$.*

PROOF. ($\Rightarrow$) is trivial. ($\Leftarrow$) Assume each $f_w(\gamma) \to 0$ as $\gamma \to \mathcal{F}$. Note that the map $(u,v) \mapsto f_{u,v}$ is bilinear in the sense that

- $f_{\alpha u,v} = \alpha f_{u,v}$ and $f_{u,\alpha v} = \overline{\alpha} f_{u,v}$.
- $f_{u_1+u_2,v} = f_{u_1,v} + f_{u_2,v}$, and $f_{u,v_1+v_2} = f_{u,v_1} + f_{u,v_2}$;

The map is only conjugate symmetric up to an inverse, i.e., $f_{u,v}(\gamma) = \overline{f_{v,u}(\gamma^{-1})}$. Thus we have the polarization identity:

$$\begin{aligned}
(f_{u+v} - f_{u-v} + i f_{u+iv} - i f_{u-iv}) = \\
&= (f_u + f_v + f_{u,v} + f_{v,u}) - (f_u + f_v - f_{u,v} - f_{v,u}) \\
&\quad + i(f_u + f_v - i f_{u,v} + i f_{v,u}) - i(f_u + f_v + i f_{u,v} - i f_{v,u}) \\
&= 2 f_{u,v} + 2 f_{v,u} + (f_{u,v} - f_{v,u}) + (f_{u,v} - f_{v,u}) \\
&= 4 f_{u,v}.
\end{aligned}$$

It follows that $f_{u,v}(\gamma) \to 0$ as $\gamma \to \mathcal{F}$, since $f_{u,v}$ is a linear combination of diagonal matrix coefficients. $\square$

NOTE 2.4. As in [**BD08**], if $\mathcal{F}$ is a filter on Γ, then let $\mathcal{F}_\bullet$ denote the "hull" of $\mathcal{F}$ – that is $\mathcal{F}_\bullet$ consists of those elements of $\mathcal{F}$ all of whose left shifts are in $\mathcal{F}$:

$$\mathcal{F}_\bullet = \{A \subseteq \Gamma \,:\, \forall \gamma \in \Gamma\,(\gamma \cdot A \in \mathcal{F})\} = \bigcap_{\gamma \in \Gamma} \gamma^{-1} \cdot \mathcal{F}.$$

Then $\mathcal{F}_\bullet \subseteq \mathcal{F}$ is clearly a filter contained in $\mathcal{F}$, and so $\mathcal{F}_\bullet$-mixing implies $\mathcal{F}$-mixing. On the other hand, suppose π is $\mathcal{F}$-mixing. This means that for any $u, v \in H_\pi$, $\gamma \in \Gamma$, the set $Q^\epsilon_{u,v} := \{\gamma \,:\, |\langle \pi(\gamma)u, v\rangle| < \epsilon\} \in \mathcal{F}$. We show $Q^\epsilon_{u,v} \in \mathcal{F}_\bullet$. For $\delta \in \Gamma$ we have that

$$\begin{aligned}
\delta \cdot Q^\epsilon_{u,v} &= \{\gamma \,:\, |\langle \pi(\delta^{-1}\gamma)u, v\rangle| < \epsilon\} \\
&= \{\gamma \,:\, |\langle \pi(\gamma)u, \pi(\delta)v\rangle| < \epsilon\} = Q^\epsilon_{u,\pi(\delta)v} \in \mathcal{F}
\end{aligned}$$

since π is $\mathcal{F}$-mixing. Hence π is $\mathcal{F}_\bullet$-mixing. Thus, we lose no generality by restricting our attention to (left) shift-invariant filters. In fact, we have

$$\begin{aligned}\delta_1 \cdot Q^\epsilon_{u,v} \cdot \delta_2^{-1} &= \{\gamma \,:\, |\langle \pi(\gamma)(\pi(\delta_2)u), \pi(\delta_1)v\rangle| < \epsilon\} = Q^\epsilon_{\pi(\delta_2)u,\pi(\delta_1)v} \\ (Q^\epsilon_{u,v})^{-1} &= \{\gamma \,:\, |\langle \pi(\gamma^{-1})u, v\rangle| < \epsilon\} \\ &= \{\gamma \,:\, |\langle \pi(\gamma)v, u\rangle| < \epsilon\} = Q^\epsilon_{v,u}.\end{aligned}$$

It follows that if π is $\mathcal{F}$-mixing, then it is is also $\hat{\mathcal{F}}$-mixing, where

$$\hat{\mathcal{F}} = \{A \in \mathcal{F} \,:\, \forall \gamma, \delta \in \Gamma \; (\gamma A \delta \in \mathcal{F} \text{ and } \gamma A^{-1}\delta \in \mathcal{F})\} = \bigcap_{\gamma,\delta\in\Gamma} \gamma^{-1} \cdot (\mathcal{F} \cap \mathcal{F}^{-1}) \cdot \delta^{-1}.$$

$\hat{\mathcal{F}}$ is a filter since the intersection of filters is a filter. It is clear that $\hat{\hat{\mathcal{F}}} = \hat{\mathcal{F}}$. The filter $\hat{\mathcal{F}}$ is two-sided invariant and symmetric (i.e., $A \in \hat{\mathcal{F}} \Leftrightarrow A^{-1} \in \hat{\mathcal{F}}$). In studying mixing properties no generality is lost if we restrict our attention to filters which are two-sided invariant and symmetric.

If $\mathcal{F}$ is the Fréchet filter, then $\mathcal{F}$-mixing is just the standard definition of mixing. When $\mathcal{F}$ is the IP*-filter this corresponds to mild mixing, and when $\mathcal{F}$ is the $\mathcal{C}^*$-filter (where $\mathcal{C}^*$ is the intersection of all minimal idempotent nonprincipal ultrafilters on Γ) then this corresponds to weak mixing [**BG05**].

For any filter $\mathcal{F}$, an $\mathcal{F}$-mixing representation is ergodic since if v is an invariant vector then $||v||^2 = \langle v, v\rangle = \langle \pi(\gamma)v, v\rangle \to 0$ as $\gamma \to \mathcal{F}$, so since $\varnothing \notin \mathcal{F}$, it follows that $||v||^2 < \epsilon$ for all ϵ, hence $||v||^2 = 0$, $v = 0$. In fact, for any $\mathcal{F}$, an $\mathcal{F}$-mixing representation is weakly mixing [**BR88**].

LEMMA 2.5. *If the representations π_n are $\mathcal{F}$-mixing on H_n for all n, then their direct sum $\pi = \bigoplus_n \pi_n$ is $\mathcal{F}$-mixing on $H_\pi = \bigoplus_n H_n$.*

PROOF. We must check, for a dense set of $v \in H_\pi$, that $\langle \pi(\gamma)v, v\rangle \to 0$ as $\gamma \to \infty$. Vectors of the form $v = \oplus_{n=1}^N v_n$, where $v_n \in H_n$, are dense. We compute

$$\begin{aligned}\Big\langle \bigoplus_n \pi_n(\gamma)(\oplus_{n=1}^N v_n), \oplus_{m=1}^N v_m\Big\rangle &= \langle \oplus_{n=1}^N (\pi_n(\gamma)(v_n)), \oplus_{m=1}^N v_m\rangle \\ &= \sum_{n,m\le N} \langle \pi_n(\gamma)(v_n), v_m\rangle = \sum_{n=1}^N \langle \pi_n(\gamma)(v_n), v_n\rangle\end{aligned}$$

which vanishes as $\gamma \to \mathcal{F}$ since each $\langle \pi_n(\gamma)(v_n), v_n\rangle$ vanishes as $\gamma \to \mathcal{F}$. $\square$

PROPOSITION 2.6. *Let Γ be an infinite countable group. Then $a \in A(\Gamma, X, \mu)$ is $\mathcal{F}$-mixing if and only if for all $f, h \in L^2(X, \mu)$*

$$(1) \qquad \lim_{\gamma \to \mathcal{F}} \int f(\gamma^{-1}x)h(x)\,d\mu = \Big(\int f\,d\mu\Big)\Big(\int h\,d\mu\Big),$$

or, equivalently, for all measurable $A, B \subseteq X$

$$(2) \qquad \lim_{\gamma \to \mathcal{F}} \mu(\gamma(A) \cap B) = \mu(A)\mu(B).$$

PROOF. (1) clearly implies that a is $\mathcal{F}$-mixing. Assume a is $\mathcal{F}$-mixing. We show (1) holds. Let $f, h \in L^2(X, \mu)$. Then $f_0 = f - \int f$ and $h_0 = h - \int h$ are in $L^2_0(X, \mu)$. We have

$$\begin{aligned}\langle \kappa_0^a(\gamma) f_0, \overline{h_0}\rangle &= \langle f \circ \gamma^{-1}, \overline{h}\rangle - \langle f \circ \gamma^{-1}, \overline{\int h}\rangle - \langle \int f \circ \gamma^{-1}, \overline{h}\rangle + \langle \int f \circ \gamma^{-1}, \overline{\int h}\rangle \\ &= \langle f \circ \gamma^{-1}, \overline{h}\rangle - \int f \int h = \int f(\gamma^{-1}x)h(x)\,d\mu - \int f(x)\,d\mu \int h(x)\,d\mu\end{aligned}$$

which gives us (1). It is clear that (1) implies (2). For the converse, suppose that (2) holds. It suffices to show that (1) holds for simple functions, since for any $f, h \in L^2(X, \mu)$ we have

$$|\langle f\gamma^{-1}, h\rangle - \int f\,d\mu \int h\,d\mu| \leq$$
$$|\langle f\gamma^{-1}, h - h'\rangle| + |\langle f\gamma^{-1} - f'\gamma^{-1}, h'\rangle| + |\langle f'\gamma^{-1}, h'\rangle - \int f' \int h'| + |\int f' \int h' - \int f \int h|.$$

So given $\epsilon > 0$ if f' and h' are simple functions chosen close enough to f and h so that the first two summands and the last summand above is $\leq \epsilon/4$, then $|\langle f'\gamma^{-1}, h'\rangle - \int f'\,d\mu \int h'\,d\mu| < \epsilon/4$ implies $|\langle f\gamma^{-1}, h\rangle - \int f\,d\mu \int h\,d\mu| < \epsilon$, hence $\{\gamma : |\langle f'\gamma^{-1}, h'\rangle - \int f' \int h'| < \epsilon/4\} \subseteq \{\gamma : |\langle f\gamma^{-1}, h\rangle - \int f \int h| < \epsilon\}$ and, assuming the former is in $\mathcal{F}$, the latter is in $\mathcal{F}$ as well.

Thus we show, assuming (2), that (1) holds for simple functions. We compute

$$\begin{aligned}\int \sum_{i=1}^n a_i 1_{\gamma A_i}(x) \sum_{j=1}^m b_j 1_{B_j}(x) d\mu(x) &= \sum_{i,j} a_i b_j \int 1_{\gamma A_i \cap B_j}(x)\,d\mu \\ &\to_{\gamma \to \mathcal{F}} \sum_{i,j} a_i b_j \mu(A_i)\mu(B_j) \\ &= \int \sum_i a_i 1_{A_i}\,d\mu \int \sum_j b_j 1_{B_j}\,d\mu.\end{aligned}$$

$\square$

COROLLARY 2.7. *If Γ is an infinite countable group and $a_n \in A(\Gamma, X, \mu)$ are $\mathcal{F}$-mixing for all n, then $\prod_n a_n$ is $\mathcal{F}$-mixing.*

PROOF. It suffices to show that (2) of Proposition 2.6 holds for $\prod_n a_n \in A(\Gamma, X^{\mathbb{N}}, \mu^{\mathbb{N}})$ when A and B are taken from a dense set in the measure algebra of $\mu^{\mathbb{N}}$. Since cylinder sets of the form $A_1 \times \cdots \times A_N \times X \times X \times \cdots$ are dense, this comes down to showing that (2) of Proposition 2.6 holds on finite products. For this, it suffices to show that (2) holds for $a \times b \in A(\Gamma, X^2, \mu^2)$ whenever $a, b \in A(\Gamma, X, \mu)$ are $\mathcal{F}$-mixing, since then a trivial induction takes care of the general finite case. This is clear since $\gamma^{a \times b}((A_1 \times A_2) \cap (B_1 \times B_2)) = (\gamma^a(A_1) \cap B_1) \times (\gamma^b(A_2) \cap B_2)$. Hence the measure of this set converges, as $\gamma \to \mathcal{F}$ to $\mu(A_1)\mu(B_1)\mu(A_2)\mu(B_2) = \mu(A_1 \times B_1)\mu(A_2 \times B_2)$. $\square$

NOTE 2.8. The descriptive complexity of $\mathcal{F}$ gives a bound on the descriptive complexity of the set $\mathcal{F}\mathrm{MIX}(\Gamma, X, \mu) \subseteq A(\Gamma, X, \mu)$ as follows.

Let $\{A_n\}_{n \in \mathbb{N}}$ be dense in the measure algebra of μ. Let $\varphi_{n,m,\epsilon} : A(\Gamma, X, \mu) \to 2^\Gamma = \mathcal{P}(\Gamma)$ send the Γ-action a to the set $\varphi_{n,m,\epsilon}(a) = \{\gamma : |\mu(\gamma^a A_n \cap A_m) - \mu(A_n)\mu(A_m)| < \epsilon\}$. Note that if $\gamma \in \varphi_{n,m,\epsilon}(a)$, say $|\mu(\gamma^a A_n \cap A_m) - \mu(A_n)\mu(A_m)| < \delta < \epsilon$, and if b is so close a that $|\mu(\gamma^a A_n \cap A_m) - \mu(\gamma^b A_n \cap A_m)| < \epsilon - \delta$, then $\gamma \in \varphi_{n,m,\epsilon}(b)$. Thus $\varphi_{n,m,\epsilon}^{-1}(U_\gamma)$ is open, where $U_\gamma = \{C \subseteq \Gamma : \gamma \in C\}$. On the other hand, if $\gamma \notin \varphi_{n,m,\epsilon}(a_k)$ then $\epsilon \leq |\mu(\gamma^{a_k} A_n \cap A_m) - \mu(A_n)\mu(A_m)|$, so if $a_k \to a$, then since $\mu(\gamma^{a_k} A_n \cap A_m) \to_{k \to \infty} \mu(\gamma^a A_n \cap A_m)$ we get hat $\gamma \notin \varphi_{n,m,\epsilon}(a)$. Thus $\varphi_{n,m,\epsilon}^{-1}(\hat{U}_\gamma)$ is closed, where $\hat{U}_\gamma = \{C : \gamma \notin C\}$.

We have that a is $\mathcal{F}$-mixing if and only if $\forall n \forall m \forall k \left(a \in \varphi_{n,m,\frac{1}{k}}^{-1}(\mathcal{F})\right)$. So if $\mathcal{F}$ is $\mathbf{\Sigma}^0_\alpha$ then $\varphi_{n,m,\epsilon}^{-1}$ is $\mathbf{\Sigma}^0_{\alpha+1}$ (since $\varphi_{n,m,\epsilon}$ is Baire class 1) and so $\mathcal{F}\mathrm{MIX}$ is $\mathbf{\Pi}^0_{\alpha+2}$. If $\mathcal{F}$ is $\mathbf{\Pi}^0_\alpha$ then $\varphi_{n,m,1/k}^{-1}(\mathcal{F})$ is $\mathbf{\Pi}^0_{\alpha+1}$, hence so is $\mathcal{F}\mathrm{MIX}$.

In particular, if $\mathcal{F}$ is Borel, then so is $\mathcal{F}\mathrm{MIX}$.

DEFINITION 2.9. Γ has HAP($\mathcal{F}$) iff there is a unitary representation π of Γ that is $\mathcal{F}$-mixing, and with $1_\Gamma \prec \pi$. That is, there is a seqeuence of non-zero almost invariant vectors, i.e., a sequence $\{v_n\}$ of unit vectors such that $||\pi(\gamma)(v_n) - v_n|| \to 0$ for all $\gamma \in \Gamma$.

For example, Γ has HAP(IP*) if and only if there is a mildly mixing unitary representation π of Γ with $1_\Gamma \prec \pi$. Also, since Γ does not have property (T) if and only if there is a weakly mixing representation π of Γ with $1_\Gamma \prec \pi$, so that the negation of property (T) is equivalent to HAP($\mathcal{C}^*$).

LEMMA 2.10 (Analogue of p. 79 [**Kec10**]). Γ *has HAP($\mathcal{F}$) if and only if there is an orthogonal representation* $\pi : \Gamma \to O(H)$ *on a real Hilbert space, which has non-0 almost invariant vectors and is $\mathcal{F}$-mixing.*

PROOF. ($\Leftarrow$): Suppose $\pi : \Gamma \to O(H)$ is $\mathcal{F}$-mixing with non-0 almost invariant vectors $\{v_n\}$. Since $O(H)$ is a closed subgroup of $U(H_{\mathbb{C}})$ via the identification $T \mapsto T_{\mathbb{C}} = T + i \cdot T$, we get a unitary representation $\pi_{\mathbb{C}} : \Gamma \to U(H_{\mathbb{C}})$. We must check that it is $\mathcal{F}$-mixing and has almost invariant vectors. For the same sequence $\{v_n\}$, but now considered as a subset of $H_{\mathbb{C}}$ we have $||\pi_{\mathbb{C}}(\gamma)(v_n) - v_n||_{H_{\mathbb{C}}} = ||\pi(\gamma)(v_n) - v_n||_H \to 0$ as $n \to \infty$. Also,

$$\langle \pi(\gamma)v + i\pi(\gamma)w, r + is\rangle_{H_{\mathbb{C}}} = \langle \pi(\gamma)v, r\rangle_H + \langle \pi(\gamma)w, s\rangle_H + i\langle \pi(\gamma)w, r\rangle_H - i\langle \pi(\gamma)v, s\rangle_H$$

which goes to zero as $\gamma \to \mathcal{F}$, since π is $\mathcal{F}$-mixing.

($\Rightarrow$): Suppose $\{v_n\}$ are non-0 almost invariant vectors for the $\mathcal{F}$-mixing unitary representation $\pi : \Gamma \to U(H)$. Let $\varphi_n(\gamma) = \langle \pi(\gamma)(v_n), v_n\rangle$. Then φ_n is positive-definite, $\varphi_n(1) = 1$ (since $||v_n|| = 1$), and $\lim_{\gamma\to\mathcal{F}} \varphi_n(\gamma) = 0$. As

$$0 = \lim_n ||\pi(\gamma)(v_n) - v_n||^2 = \lim_n \left(2||v_n||^2 - 2\mathrm{Re}\langle \pi(\gamma)(v_n), v_n\rangle\right) = 2 - \lim_n 2\mathrm{Re}\langle \pi(\gamma)(v_n), v_n\rangle$$

we also have that $\mathrm{Re}\langle \pi(\gamma)v_n, v_n\rangle \to 1$ as $n \to \infty$. Letting $\psi_n = \mathrm{Re}\varphi_n$, then ψ_n is real positive-definite: it is real-valued and symmetric since

$$\begin{aligned}\psi_n(\gamma_j^{-1}\gamma_i) &= \mathrm{Re}\langle \pi(\gamma_j^{-1}\gamma_i)v_n, v_n\rangle = \mathrm{Re}\langle \pi(\gamma_i)v_n, \pi(\gamma_j)(v_n)\rangle \\ &= \mathrm{Re}\langle \pi(\gamma_j)(v_n), \pi(\gamma_i)(v_n)\rangle = \mathrm{Re}\langle \pi(\gamma_i^{-1}\gamma_j)(v_n), v_n\rangle \\ &= \psi_n(\gamma_i^{-1}\gamma_j),\end{aligned}$$

and given $c_1, \dots, c_n \in \mathbb{R}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ we have

$$\sum_{j,k=1}^{n} c_j c_k \mathrm{Re}\langle \pi(\gamma_k)(v_n), \pi(\gamma_j)(v_n)\rangle = ||\sum_k c_k \pi(\gamma_k)v_n||^2 \geq 0.$$

Also, $\psi_n(1_\Gamma) = \mathrm{Re}||v_n||^2 = 1$, $\lim_{\gamma\to\mathcal{F}} \psi_n(\gamma) = 0$, and $\psi_n(\gamma) \to 1$ as $n \to \infty$. Let (ρ_n, H_n, w_n) be the *orthogonal* representation given by the GNS construction for ψ_n, so that w_n is a cyclic vector and $\langle \rho_n(\gamma)(w_n), w_n\rangle = \psi_n(\gamma)$ (so in particular, since $\psi_n(1_\Gamma) = 1$, w_n is a unit vector). Let $\rho = \bigoplus_n \rho_n$, $H_\rho = \bigoplus_n H_n$. Then $\langle \rho(\gamma)(w_n), w_n\rangle = \psi_n(\gamma) \to 1$ as $n \to \infty$, and so $||\rho(\gamma)(w_n) -$

$w_n|| = 2-2\text{Re}\langle\rho(\gamma)(w_n), w_n\rangle \to 0$ as $n \to \infty$. Also, we need to show that $\lim_{\gamma\to\mathcal{F}}\langle\rho(\gamma)(v), v\rangle = 0$ for all $v \in H_\rho$. As usual it suffices to show this for a dense subset since then for arbitrary $v \in H_\rho$ we have

$$|\langle\rho(\gamma)v, v\rangle| \leq |\langle\rho(\gamma)v, v-u\rangle| + |\langle\rho(\gamma)(v-u), u\rangle| + |\langle\rho(\gamma)u, u\rangle|$$
$$\leq ||v|| \cdot ||v-u|| + ||v-u|| \cdot ||u|| + |\langle\rho(\gamma)u, u\rangle|,$$

so if $||v-u|| < \min\{\frac{\epsilon}{3(||v||+1)}, 1\}$ then $\{\gamma : \langle\rho(\gamma)v, v\rangle < \epsilon\} \supseteq \{\gamma : \langle\rho(\gamma)u, u\rangle < \epsilon/3\} \in \mathcal{F}$.

Since $H = \bigoplus_{n=1}^{\infty} H_n$, any $u \in H_n$ can be approximated to an arbitrary degree by some finite linear combination $u' = \sum_{n=1}^{N} u_n$ where $u_n \in H_n$. But if u_1 and u_2 are in different cyclic components, then

$$\langle\rho(\gamma)(u_1+u_2), u_1+u_2\rangle = \langle\rho(\gamma)u_1, u_1\rangle + \langle\rho(\gamma)u_2, u_2\rangle + \langle\rho(\gamma)u_1, u_2\rangle + \langle\rho(\gamma)u_2, u_1\rangle$$

and the last two terms are zero (since u_1 and u_2 are in invariant subspaces which are orthogonal) hence $f^\rho_{u_1+u_2} = f^\rho_{u_1} + f^\rho_{u_2}$. So if $f^\rho_{u_1}$ and $f^\rho_{u_2}$ vanish as $\gamma \to \mathcal{F}$ then so does $f^\rho_{u_1+u_2} = f^\rho_{u_1} + f^\rho_{u_2}$. Thus, it suffices to check that $\lim_{\gamma\to\mathcal{F}} f_u(\gamma) = 0$ for u of the form $u = \sum_{i=1}^{k} c_i\rho(\gamma_i)w_n$, since the linear span of $\{\rho(\gamma)w_n\}_{\gamma\in\Gamma}$ is dense in H_n. We have

$$\Big\langle\rho(\gamma)\Big(\sum_{i=1}^{k} c_i\rho(\gamma_i)w_n\Big), \sum_{j=1}^{k} c_j\rho(\gamma_j)w_n\Big\rangle = \sum_{i,j\leq k} c_i\overline{c_j}\langle\rho((\gamma_j)^{-1}\gamma\gamma_i)w_n, w_n\rangle$$
$$= \sum_{i,j\leq k} c_i\overline{c_j}\psi_n(\gamma_j^{-1}\gamma\gamma_i)$$

which vanishes as $\gamma \to \mathcal{F}$ since $\gamma_j^{-1}\gamma\gamma_i \to \mathcal{F}$ as $\gamma \to \mathcal{F}$ (since we may assume that $\mathcal{F}$ is a two-sided shift invariant filter) and $\psi_n(\delta) \to 0$ has $\delta \to \mathcal{F}$. $\square$

THEOREM 2.11 (Analogue of Theorem 11.1 [**Kec10**]). *Let Γ be an infinite countable group. TFAE:*

(1) Γ *has HAP($\mathcal{F}$).*

(2) Γ *has a measure preserving, $\mathcal{F}$-mixing action which is not E_0-ergodic.*

(3) Γ *has a free, measure preserving, $\mathcal{F}$-mixing action which is not E_0-ergodic.*

In particular, Γ does not have HAP($\mathcal{F}$) iff $\mathcal{F}$MIX$(\Gamma, X, \mu) \subseteq E_0RG(\Gamma, X, \mu)$.

PROOF. (2)⇒(3): If $a \in A(\Gamma, X, \mu)$ satisfies (2), then let $b \in A(\Gamma, X, \mu)$ be any free, $\mathcal{F}$-mixing action (e.g., the shift of Γ on 2^Γ). Then $a \times b$ is free and $\mathcal{F}$-mixing. Also, if A_n are non-trivial almost invariant sets for a, then $\mu(\gamma^a A_n \times \gamma^b X \Delta A_n \times X) = \mu(\gamma^a A_n \Delta A_n) \to 0$ as $n \to \infty$, so $A_n \times X$ are non-trivial (since $\mu(A_n \times X) = \mu(A_n)$) almost invariant sets for $a \times b$.

(3)⇒(1): Let $a \in A(\Gamma, X, \mu)$ satisfy (3). Then κ_0^a is $\mathcal{F}$-mixing, and a is not E_0-ergodic, so $1_\Gamma \prec \kappa_0^a$.

(1)⇒(2): First we show that if π is $\mathcal{F}$-mixing on H then $\pi^{\odot n}$ is $\mathcal{F}$-mixing on $H^{\odot n}$. Linear combinations of vectors of the form $\odot_{i=1}^n v_i$, with $v_i \in H$, are dense in $H^{\odot n}$. We have that

$$\Big\langle \pi^{\odot n}(\gamma)\Big(\sum_{k=1}^m c_k \odot_{i=1}^n v_i^{(k)}\Big), \sum_{l=1}^m c_l \odot_{j=1}^n v_j^{(l)}\Big\rangle = \Big\langle \sum_{k=1}^m c_k \odot_{i=1}^n \pi(\gamma)(v_i^{(k)}), \sum_{l=1}^m c_l \odot_{j=1}^n v_j^{(l)}\Big\rangle$$

$$\sum_{k,l\le m} c_k \overline{c_l} \Big\langle \odot_{i=1}^n \pi(\gamma)(v_i^{(k)}), \odot_{j=1}^n v_j^{(l)}\Big\rangle = \frac{1}{n!}\sum_{k,l\le m} c_k\overline{c_l} \sum_{\sigma\in S_n}\prod_{j=1}^n \langle \pi(\gamma)(v_i^{(k)}), v_{\sigma(i)}^{(l)}\rangle$$

and since each product and sum is finite, and each term $\langle \pi(\gamma)v, w\rangle \to 0$ as $\gamma \to \mathcal{F}$, this does as well.

Now, by the previous theorem, Γ having HAP($\mathcal{F}$) means that there is an orthogonal $\mathcal{F}$-mixing representation π of Γ which has non-0 almost invariant vectors $\{v_n\}$. By replacing π by infinitely many copies of it, we can assume that $\{v_n\}$ is orthonormal and that H is infinite-dimensional. Let $(X, \nu) = (\mathbb{R}^\mathbb{N}, \mu^\mathbb{N})$ be the product space with μ normalized Gaussian measure on $\mathbb{R}$. Without loss of generality $H = H^{:1:} = \langle p_n\rangle_{n\in\mathbb{N}} \subseteq L_0^2(X, \nu, \mathbb{R})$ (where $p_n : \mathbb{R}^\mathbb{N} \to \mathbb{R}$ the n-th projection). Let $a = a_\pi$ be the Gaussian action associated to π. Then $\kappa_0^a \cong \bigoplus_{n=1}^\infty \pi^{\odot n}$ and each $\pi^{\odot n}$ is $\mathcal{F}$-mixing, hence so is κ_0^a, and therefore so is a. As usual we have $1_\Gamma \prec \pi \Rightarrow i_\Gamma \prec a$, so we are done. □

THEOREM 2.12 (Analogue of Theorem 12.7 of [**Kec10**]). *Let Γ be an infinite countable group. Then Γ does not have HAP($\mathcal{F}$) iff* $\overline{\mathcal{F}\mathrm{MIX}(\Gamma, X, \mu)} \subseteq \mathrm{ERG}(\Gamma, X, \mu)$.

PROOF. (⇒): If Γ does not have HAP($\mathcal{F}$), then

$$\mathcal{F}\mathrm{MIX} \subseteq \{a \in A(\Gamma, X, \mu) : 1_\Gamma \not\prec \kappa_0^a\}.$$

Assume towards contradiction that $a_n \in \mathcal{F}\mathrm{MIX}(\Gamma, X, \mu)$ and $a_n \to a \notin \mathrm{ERG}(\Gamma, X, \mu)$. Let $b = \prod_n a_n$. Then $a_n \prec b$ for all n, so $a \prec b$ and thus $\kappa_0^a \prec \kappa_0^b$. Since a is not ergodic, $1_\Gamma \le \kappa_0^a$, so $1_\Gamma \prec \kappa_0^b$, contradicting that b, being the product of $\mathcal{F}$-mixing actions, is $\mathcal{F}$-mixing, and that Γ does not have HAP($\mathcal{F}$).

($\Leftarrow$): Assume now that Γ has HAP($\mathcal{F}$). Then there is an action $a_0 \in \mathcal{F}$MIX $\setminus$ E$_0$RG(Γ, X, μ). Let $\{A_n\}$ be a sequence of almost invariant Borel sets in X with $\mu(A_n) = \frac{1}{2}$. Then $b := \frac{1}{2}a_0 + \frac{1}{2}a_0 \prec a_0 \times a_0$ (Proof is same as p. 85 of [**Kec10**], just uses the fact that $a_0 \notin$ E$_0$RG). Since b is not ergodic (having two ergodic components, each of measure $\frac{1}{2}$) and $a_0 \times a_0$ is $\mathcal{F}$-mixing we have $b \in \overline{\mathcal{F}\text{MIX}} \setminus$ ERG. □

One may show that the main result of [**Hjo09**] goes through for $\mathcal{F}$-mixing in place of mixing. The proof is nearly identical.

THEOREM 2.13 (Analogue of [**Hjo09**]). *Let Γ be a countable group with HAP($\mathcal{F}$). Let (X, μ) be an atomless standard Borel probability space. Then the $\mathcal{F}$-mixing actions are dense in $A(\Gamma, X, \mu)$.*

COROLLARY 2.14. *The countable group Γ has* HAP($\mathcal{F}$) *if and only if the set of $\mathcal{F}$-mixing actions are dense in $A(\Gamma, X, \mu)$.*

3. Permanence properties of $\mathcal{F}$-mixing

DEFINITION 3.1. Let $C_{\mathcal{F}}(\Gamma) \subseteq l^\infty(\Gamma)$ denote the set of functions $\varphi \in l^\infty(\Gamma)$ with $\lim_{\gamma \to \mathcal{F}} \varphi(\gamma) = 0$.

PROPOSITION 3.2 (Analogue of Ch. 2 of [**CCJ$^+$01**]). *Γ has HAP($\mathcal{F}$) iff there there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of positive definite functions in $C_{\mathcal{F}}(\Gamma)$ with $\varphi_n(e) = 1$ for all n and $\varphi_n \to 1$ pointwise as $n \to \infty$.*

PROOF. ($\Rightarrow$): Let $\{v_n\}$ be a sequence of almost invariant unit vectors for the $\mathcal{F}$-mixing representation π of Γ. Let $\varphi_n = \langle \pi(\cdot)v_n, v_n\rangle$. Then each φ_n is positive definite with $\varphi_n(e) = ||v_n|| = 1$, and

$$|\varphi_n(\gamma) - 1| = |\langle \pi(\gamma)v_n, v_n\rangle - \langle v_n, v_n\rangle| \leq ||\pi(\gamma)v_n - v_n|| \to_{n\to\infty} 0$$

since the $\{v_n\}$ are almost invariant. The representation π is $\mathcal{F}$-mixing, so $\varphi_n(\gamma) = \langle \pi(\gamma)v_n, v_n\rangle \to_{\gamma\to\mathcal{F}} 0$.

($\Leftarrow$): Conversely, if the $\varphi_1, \varphi_2, \ldots$ are functions in $C_{\mathcal{F}}(\Gamma)$ with $\varphi_n(e) = 1$ for all n, and $\varphi_n \to 1$ as $n \to \infty$, then let (H_n, π_n, w_n) be the GNS triple associated to φ_n, so that w_n is a cyclic unit vector for π_n and $\varphi_n(\gamma)\langle \pi(\gamma)w_n, w_n\rangle$. Let $\pi = \oplus_n \pi_n$ be the representation of Γ on

$H = \bigoplus_n H_n$. Then $\langle \pi(\gamma)(w_n), w_n \rangle = \varphi_n(\gamma) \to 1$ as $n \to \infty$ and so $||\pi(\gamma)(w_n) - w_n|| = 2 - 2\mathrm{Re}\langle \pi(\gamma)(w_n), w_n \rangle \to 0$ as $n \to \infty$, so the $\{w_n\}$ are almost invariant vectors. Also, we need to show that $\lim_{\gamma \to \mathcal{F}} \langle \pi(\gamma)(v), v \rangle = 0$ for all $v \in H$. It suffices to show this for a dense set of v, and since the $\{w_n\}$ are pairwise orthogonal it actually suffices to show this for v of the form $v = \sum_{i=1}^k c_i \pi(\gamma_i) w_n$. Note that we may assume without loss of generality that $\mathcal{F}$ is two-sided invariant. We then have

$$\Big\langle \pi(\gamma)\Big(\sum_{i=1}^k c_i \pi(\gamma_i) w_n\Big), \sum_{j=1}^k c_j \pi(\gamma_j) w_n \Big\rangle = \sum_{i,j \le k} c_i c_j \langle \pi(\gamma_j^{-1} \gamma \gamma_i) w_n, w_n \rangle$$
$$= \sum_{i,j \le k} c_i c_j \varphi_n(\gamma_j^{-1} \gamma \gamma_i) \to 0$$

as $\gamma \to \mathcal{F}$ since $\mathcal{F}$ is two-sided invariant. □

PROPOSITION 3.3 (Analogue of 2.1.1 of [**CCJ$^+$01**]). *The countable group Γ has HAP($\mathcal{F}$) iff there is a $\psi : \Gamma \to \mathbb{R}^+$ such that $\varphi^{-1}(K) \in \mathcal{I}$ (where $\mathcal{I}$ is the dual ideal to $\mathcal{F}$) for every compact $K \subseteq \mathbb{R}^+$, and which is conditionally negative definite, i.e., $\psi(e) = 0$, $\psi(\gamma) = \psi(\gamma^{-1})$ for all $\gamma \in \Gamma$, and for all $\gamma_1, \ldots, \gamma_n \in \Gamma$, $a_1, \ldots, a_n \in \mathbb{C}$ with $\sum a_i = 0$,*

$$\sum_{i,j} \overline{a_i} a_j \psi(\gamma_i^{-1} \gamma_j) \le 0.$$

PROOF. ($\Rightarrow$) Write Γ as an increasing union of finite subsets $\Gamma = \bigcup_{n \ge 1} F_n$, $F_n \subseteq F_{n+1}$. Let $(\alpha_n)_{n \ge 1}$ be an increasing sequence in $\mathbb{R}^+$ tending to ∞ and let $(\epsilon_n)_{n \ge 1}$ decrease to 0 be such that $\sum_n \alpha_n \epsilon_n$ converges. Let $(\varphi_n)_{n \ge 1}$ be a sequence of positive definite functions in $C_{\mathcal{F}}(\Gamma)$ with $\varphi_n(e) = 1$ for all n and $\varphi_n \to 1$ pointwise as $n \to \infty$. Let n_1 be so large that $n \ge n_1$ implies $|\varphi_n(\gamma) - 1| \le \epsilon_1$ for all $\gamma \in F_1$. Let $n_m \ge n_{m-1}$ be so large that $n \ge n_m$ implies $|\varphi_n(\gamma) - 1| \le \epsilon_m$ for all $\gamma \in F_m$. So WoLOG (after moving to the subsequence (φ_{n_m}) if necessary) we may assume that for all $n \ge 1$

$$\sup_{\gamma \in F_n} |\varphi_n(\gamma) - 1| \le \epsilon_n.$$

Now, since $1 = \varphi_n(e) = \sup_{\gamma \in \Gamma} |\varphi_n(\gamma)|$, by replacing φ_n by $|\varphi_n|^2$ if necessary we may assume that $0 \le \varphi_n \le 1$ for all n. For $\gamma \in \Gamma$ let

$$\psi(\gamma) = \sum_{n \ge 1} \alpha_n (1 - \varphi_n(\gamma)).$$

This is conditionally negative definite on Γ since if $\sum_{i\leq m} a_i = 0$ then

$$\sum_{i,j\leq m} \overline{a_i} a_j \alpha_n (1-\varphi_n(\gamma_i^{-1}\gamma_j)) = -\alpha_n \sum_{i,j\leq m} \overline{a_i} a_j \varphi_n(\gamma_i^{-1}\gamma_j) \leq 0.$$

Given a $K \in \mathbb{R}^+$ let n be so large that $\alpha_n \geq 2K$. Let $A \in \mathcal{F}$ be such that $|\varphi_n(\gamma)| < 1/2$ for $\gamma \in A$. Then $\psi(\gamma) \leq K$ implies $(1-\varphi_n(\gamma)) \leq 1/2$ and so $\varphi_n(\gamma) \geq 1/2$, whence $\gamma \notin A$, i.e., $\{\gamma : \psi(\gamma) \leq K\} \subseteq \Gamma \setminus A \in \mathcal{I}$.

($\Leftarrow$) Conversely, suppose ψ is conditionally negative definite with $\psi^{-1}(K) \in \mathcal{I}$ for compact $K \subseteq \mathbb{R}^+$, as in the statement of the proposition. Then by Schoenberg's theorem [BdHV C.4.1.9] $e^{-t\psi}$ is positive definite for all $t \geq 0$. So if $\varphi_n(\gamma) = e^{-n\psi(\gamma)}$ then $\varphi_n \to 1$ pointwise and $\varphi_n(e) = e^0 = 1$. For fixed n and $0 < \epsilon < 1$ we have that $\varphi_n(\gamma) < \epsilon$ iff $e^{-n\psi(\gamma)} < \epsilon$ iff $n\psi(\gamma) > -\log(\epsilon)$ iff $\gamma \notin \{\gamma : \psi(\gamma) \leq -\frac{1}{n}\log(\epsilon)\} \in \mathcal{I}$. This implies that $\lim_{\gamma\to\mathcal{F}} \varphi_n(\gamma) = 0$. $\square$

DEFINITION 3.4. Call a positive definite function $\varphi \in C(G)$ *normalized* if $\varphi(e_\Gamma) = 1$.

PROPOSITION 3.5. *Suppose G is a countable group and that G is the increasing union of a sequence $(G_n)_{n\geq 1}$ of infinite subgroups. Suppose that $\mathcal{F}$ is a filter on G and suppose for each n that $\mathcal{F}_n$ is a filter on G_n with the property that $G_n \setminus A \in \mathcal{F}_n \Rightarrow G \setminus A \in \mathcal{F}$ whenever $A \subseteq G_n$. If G_n has $\mathcal{HAP}(\mathcal{F}_n)$ for all n Then G has $\mathcal{HAP}(\mathcal{F})$.*

NOTE 3.6. If we let $\mathcal{I}_n = \{G_n \setminus A : A \in \mathcal{F}_n\}$ and $\mathcal{I} = \{G \setminus A : A \in \mathcal{F}\}$ be the ideals corresponding to the $\mathcal{F}_n$'s and $\mathcal{F}$, respectively, then the above hypotheses on $\mathcal{F}_n$ and $\mathcal{F}$ is equivalent to $\mathcal{I} \supseteq \bigcup_n \mathcal{I}_n$.

PROOF. For each n let $\{\varphi_k^n\}_{k\in\mathbb{N}} \subseteq C_{\mathcal{F}_n}(G_n)$ be a sequence of normalized positive definite functions such that $\varphi_k^n \to 1$ pointwise as $k \to \infty$, as in Proposition 3.5. Let $\tilde{\varphi}_k^n$ be the extension of φ_k^n to G such that $\tilde{\varphi}_k^n | G \setminus G_n \equiv 0$. Then $\tilde{\varphi}$ are normalized positive definition functions, and for any n, k and $\epsilon > 0$ we have that

$$\{g \in G : |\tilde{\varphi}_k^n(g)| < \epsilon\} = G \setminus G_n \cup \{g \in G_n : |\varphi_k^n(g)| < \epsilon\} \in \mathcal{F}$$

since $\varphi_k^n \in C_{\mathcal{F}_n}(G_n)$ and by the hypotheses on $\mathcal{F}$. Hence $\tilde{\varphi}_k^n \in C_{\mathcal{F}}(G)$. Now, enumerate $G = \{\gamma_1, \gamma_2, \dots\}$, and let $n(m)$ and $k(m)$ be increasing sequences such that for each m, $n(m)$ is so large that $\gamma_1, \dots, \gamma_m \in G_{n(m)}$, and $k(m)$ is so large that $\varphi_{k(m)}^{n(m)}(\gamma_i) < 2^{-m}$ for $i = 1, 2, \dots, m$. Then

$\{\tilde{\varphi}^{n(m)}_{k(m)}\}_{m\in\mathbb{N}} \subseteq C_{\mathcal{F}}(G)$ is a sequence of normalized positive definite functions on G converging pointwise to 1. So G has HAP($\mathcal{F}$). $\square$

Let Δ a subgroup of Γ. The co-induced action is defined as follows (see [**Kec10**]). Fix a transversal T for the left cosets of Δ, with $1 \in T$. Let Γ act on T by defining $\gamma \cdot t$ to be the unique element of $T \cap \gamma t\Delta$, and let $\rho : \Gamma \times T \to \Delta$ be the cocycle defined by $\rho(\gamma, t) = (\gamma \cdot t)^{-1}\gamma t$. Given an action $a \in A(\Delta X, \mu)$ we define $b = \mathrm{CInd}^{\Gamma}_{\Delta}(a) \in A(\Gamma, X^T, \mu^T)$ by

$$\gamma^b((x_s)_{s\in T})(t) = (\rho(\gamma^{-1}, t)^{-1})^a(x_{\gamma^{-1}\cdot t}).$$

Let $\mathrm{CInd}^{\Gamma}_{\Delta} : A(\Delta, X, \mu) \to A(\Gamma, X^T, \mu^T)$ be the co-inducing map. This map is continuous (in the weak topologies of these spaces), and $a \cong b \Rightarrow \mathrm{CInd}^{\Gamma}_{\Delta}(a) \cong \mathrm{CInd}^{\Gamma}_{\Delta}(b)$. It follows that $a \prec b \Rightarrow \mathrm{CInd}^{\Gamma}_{\Delta}(a) \prec \mathrm{CInd}^{\Gamma}_{\Delta}(b)$. We show that this map preserves $\mathcal{F}$-mixing in some cases.

LEMMA 3.7 (Analogue of Ioana [**Ioa11**]). *Suppose that a is $\mathcal{F}$-mixing. Let $\mathcal{I} = \{\Delta \setminus A : A \in \mathcal{F}\}$. Let $\mathcal{I}'$ be the Γ-invariant ideal generated by $\mathcal{I}$ in Γ, and let $\mathcal{F}'$ be the corresponding filter on Γ. Then $b = \mathit{CInd}^{\Gamma}_{\Delta}(a)$ is $\mathcal{F}'$ mixing.*

PROOF. It suffices to show that for a dense set of $f, h \in L^2_0(X^T, \mu^T)$ we have $\langle \kappa^b_0(\gamma)(f), h\rangle \to 0$ as $\gamma \to \mathcal{F}'$. We show this for f, h of the form $f = \otimes_{t\in A} f_t$, $h = \otimes_{s\in B} h_s$, where $f_t, h_s \in L^\infty_0(X, \mu)$ and $A, B \subseteq T$ are finite. This means that for $(x_t)_{t\in T} \in X^T$ we have $f((x_t)_{t\in T}) = \prod_{t\in A} f_t(x_t)$, and similarly for h. Then

$$\kappa^b_0(\gamma)(f)((x_t)_{t\in T}) = f((\gamma^{-1})^b((x_t)_{t\in T})) = \prod_{t\in A} f_t((\rho(\gamma, t)^{-1})^a(x_{\gamma\cdot t}))$$

so that $\langle \kappa^b_0(\gamma)(f), h\rangle =$

$$= \Big(\prod_{\gamma\cdot t\in\gamma\cdot A\setminus B} \int f_t \, d\mu\Big)\Big(\prod_{s\in B\setminus\gamma\cdot A} \int \overline{h_s} \, d\mu\Big)\Big(\prod_{\gamma\cdot t = s\in\gamma\cdot A\cap B} \int f_t((t^{-1}\gamma^{-1}s)^a(x))\overline{h_s(x)} \, d\mu(x)\Big)$$

and so $\langle \kappa^b_0(\gamma)(f), h\rangle = 0$ unless $|A| = |B|$ and $\gamma \cdot A = B$. In this case, we have that

$$\langle \kappa^b_0(\gamma)(f), h\rangle = \prod_{t\in A} \langle ((\gamma \cdot t)^{-1}\gamma t)^a(f_t), h_{\gamma\cdot t}\rangle.$$

Now, there are only finitely many bijections $\pi : A \to B$ with $\pi(t)^{-1}\gamma t \in \Delta$ for all $t \in A$. For each such π, let $\Gamma_\pi = \{\gamma \in \Gamma : \forall t \in A\ \pi(t)^{-1}\gamma t \in \Delta\}$. It suffices to show for each such

π and every $t \in A$ and $\epsilon > 0$ that $\{\gamma \in \Gamma_\pi : |\langle(\pi(t)^{-1}\gamma t)^a(f_t), h_s\rangle| \geq \epsilon\} \in \mathcal{I}'$ since the (finite) union of these sets as t varies over A and π varies over all bijections $A \to B$ contains $\{\gamma \in \Gamma : |\langle\kappa_0^b(\gamma)(f), h\rangle| \geq \epsilon\}$.

So let π and $\epsilon > 0$ be given. Fix some $t \in A$ and some $\lambda \in \Gamma_\pi$ (if $\Gamma_\pi = \varnothing$ we are done). Then $t^{-1}\lambda^{-1}\pi(t) \in \Delta$, and we let

$$S = \{\delta \in \Delta : |\langle\delta^a(f_t), (t^{-1}\lambda^{-1}\pi(t))^a(h_{\pi(t)})\rangle| \geq \epsilon\} \in \mathcal{I}.$$

Now, if $\gamma \in \Gamma_\pi$ then $t^{-1}\lambda^{-1}\gamma t = (\pi(t)^{-1}\lambda t)^{-1}(\pi(t)^{-1}\gamma t) \in \Delta$, and

$$\begin{aligned}\langle(\pi(t)^{-1}\gamma t)^a(f_t), h_{\pi(t)}\rangle &= \langle(\pi(t)^{-1}\lambda t)^a(t^{-1}\lambda^{-1}\gamma t)^a(f_t), h_{\pi(t)}\rangle \\ &= \langle(t^{-1}\lambda^{-1}\gamma t)^a(f_t), (t^{-1}\lambda^{-1}\pi(t))^a h_{\pi(t)})\rangle\end{aligned}$$

so that

$$\begin{aligned}&\{\gamma : |\langle(\pi(t)^{-1}\gamma t)^a(f_t), h_{\pi(t)}\rangle| \geq \epsilon\} \\ &\quad = \{\gamma : |\langle(t^{-1}\lambda^{-1}\gamma t)^a(f_t), (t^{-1}\lambda^{-1}\pi(t))^a h_{\pi(t)})\rangle| \geq \epsilon\} \\ &\quad = \{\gamma : t^{-1}\lambda^{-1}\gamma t \in S\} = \lambda t S t^{-1} \in \mathcal{I}'\end{aligned}$$

as was to be shown. □

It follows that if Δ is an infinite index subgroup of Γ, then $b = \mathrm{CInd}_\Delta^\Gamma(a)$ is mixing with respect to the invariant Γ-ideal generated by Δ. Since this is a proper ideal when Δ has infinite index, b is weak mixing. Another consequence of the above lemma is that co-induction preserves mild mixing.

THEOREM 3.8. *Let $\Delta \subseteq \Gamma$ be countably infinite groups and $a \in A(\Delta, X, \mu)$. Then a is mildly mixing if and only if $b =$ CInd$_\Delta^\Gamma(a) \in A(\Gamma, X, \mu)$ is mildly mixing.*

PROOF. ($\Leftarrow$) If b is mildly mixing then for every $A \in \mathrm{MALG}_\mu$ we have $\liminf_{\gamma\to\infty} \mu(\gamma^b A \Delta A) > 0$, i.e., there is some finite $F \subseteq \Gamma$ and $\epsilon > 0$ such that $\mu(\gamma^b A\Delta A) > \epsilon$ for $\gamma \notin F$. Then $\liminf_{\delta\to\infty,\, \delta\in\Delta} \mu(\delta^b A\Delta A) > 0$ since the for $\delta \in \Delta \setminus F$ the value is greater than ϵ. So $b|\Delta$ is mildly mixing. Since a is a factor of $b|\Delta$ it follows that a is mildly mixing.

($\Rightarrow$) Let $\mathrm{IP}^*_\bullet(\Delta)$ be the two-sided invariant filter generated by $\mathrm{IP}^*(\Delta)$ (so $D \in \mathrm{IP}^*_\bullet(\Delta)$ if and only if every two-sided translate of D is in IP$^*(\Delta)$). Let $\mathcal{I}_\Delta$ be the ideal associated to IP$^*_\bullet(\Delta)$, i.e., $A \in \mathcal{I}_\Delta$ if and only if $\Delta \setminus A \in$ IP$^*_\bullet(\Delta)$.

The action a being mildly mixing implies that a is $\mathcal{I}_\Delta$-mixing.

Let $\mathcal{I}$ be the ideal on Γ corresponding to IP$^*_\Gamma$:

$$B \in \mathcal{I} \;\Leftrightarrow\; \Gamma \setminus B \in \mathrm{IP}^*_\Gamma \quad (B \subseteq \Gamma).$$

Then the action $b = \mathrm{CInd}_\Delta^\Gamma(a)$ is mildly mixing if and only if b is $\mathcal{I}$-mixing. Let $\mathcal{I}'$ be the ideal on Γ generated by all the two-sided Γ-shifts of elements of $\mathcal{I}_\Delta$. By Lemma 3.7, b is $\mathcal{I}'$-mixing. To show that b is mild mixing it therefore suffices to show that $\mathcal{I}' \subseteq \mathcal{I}$, since this will imply that b is $\mathcal{I}$-mixing. So let $B \in \mathcal{I}'$. Then

$$B = \gamma_1^1 A_1 \gamma_1^2 \cup \gamma_2^1 A_2 \gamma_2^2 \cup \gamma_3^1 A_3 \gamma_3^2 \cup \cdots \cup \gamma_n^1 A_n \gamma_n^2,$$

for some $A_i \in \mathcal{I}_\Delta$ and $\gamma_i^1, \gamma_i^2 \in \Gamma$, $i = 1, \ldots, n$. To show $B \in \mathcal{I}$ it suffices to show that each $\gamma_i^1 A_i \gamma_i^2 \in \mathcal{I}$. So fix i and let $A = A_i \in \mathcal{I}_\Delta$, let $A' = \Gamma \setminus A$, and let $s_1 = \gamma_i^1$, $s_2 = \gamma_i^2 \in \Gamma$. Then every Δ-shift of $\Delta \setminus A$ intersects every IP$_\Delta$ set. If we can show that the set

$$\Gamma \setminus (s_1 A s_2) = s_1 A' s_2 = s_1(\Gamma \setminus \Delta \cup \Delta \setminus A) s_2$$

intersects every IP$_\Gamma$ set, then we will be done since this means that $s_1 A s_2 \in \mathcal{I}$. Let $F \in$ IP$_\Gamma$, say $F \supseteq$ FP$((\gamma_i)_{i=1}^\infty)$. If $A' \cap s_1^{-1} F s_2^{-1} = \varnothing$ then as $A' \supseteq \Gamma \setminus \Delta$ it must be that $s_1^{-1} F s_2^{-1} \subseteq \Delta$, FP$((\gamma_i)_{i=1}^\infty) \subseteq F \subseteq s_1 \Delta s_2$. As $\gamma_1, \gamma_2, \gamma_1\gamma_2 \in F \subseteq s_1 \Delta s_2$ let $\delta_1, \delta_2, \delta_{1,2} \in \Delta$ be such that

$$\gamma_1 = s_1 \delta_1 s_2$$

$$\gamma_2 = s_1 \delta_2 s_2$$

$$\gamma_1 \gamma_2 = s_1 \delta_{1,2} s_2.$$

Then

$$s_1 \delta_1 s_2 s_1 \delta_2 s_2 = s_1 \delta_{1,2} s_2$$

$$\delta_1 s_2 s_1 \delta_2 = \delta_{1,2}$$

$$s_2 s_1 = \delta_1^{-1} \delta_{1,2} \delta_2^{-1} \in \Delta.$$

Let $\delta = s_2 s_1 \in \Delta$. Then

$$\varnothing = \delta A' \cap \delta s_1^{-1} F s_2^{-1} = \delta A' \cap s_2 F s_2^{-1}.$$

Since $A' \supseteq \Delta \setminus A$ it follows that $\delta A'$ intersects every IP$_\Delta$ set. Now

$$s_2 F s_2^{-1} \supseteq s_2 \mathrm{FP}((\gamma_i)_i) s_2^{-1} = \mathrm{FP}((s_2 \gamma_i s_2^{-1})_i).$$

Since $\delta A' \supseteq \Gamma \setminus \Delta$ it must be that no $s_2 \gamma_i s_2^{-1}$ is in $\Gamma \setminus \Delta$. But then FP$((s_2\gamma_i s_2^{-1})_i)$ is an IP$_\Delta$ set, so intersects $\delta A'$, a contradiction. □

COROLLARY 3.9. *Let Δ be a subgroup of Γ such that the action of Γ on the homogeneous space Γ/Δ is amenable. If Δ has HAP($\mathcal{F}$) then Γ has HAP($\mathcal{F}'$) where the $\mathcal{F}'$-small sets are generated by the (left and right) shifts of the $\mathcal{F}$-small sets in Γ.*

PROOF. Let $a \in A(\Delta, X, \mu)$ be an $\mathcal{F}$-mixing action which is not E_0-ergodic, i.e., $i_\Delta \prec a$. Let $b = \mathrm{CInd}_\Delta^\Gamma(a)$. Then b is $\mathcal{F}'$-mixing, and $\mathrm{CInd}_\Delta^\Gamma(i_\Delta) \prec b$. The action $s_{\Gamma/\Delta} = \mathrm{CInd}_\Delta^\Gamma(i_\Delta)$ is the action of Γ by shift on $X^{\Gamma/\Delta}$ and by [**KT08**] $i_\Gamma \prec s_{\Gamma/\Delta}$ is implied by the action of Γ on Γ/Δ being amenable. Thus $i_\Gamma \prec s_{\Gamma/\Delta} \prec b$, and so Γ has HAP($\mathcal{F}'$). □

4. Gaussian actions

For an orthogonal representation π of Γ we let $a(\pi)$ denote the corresponding Gaussian measure preserving action of Γ. See [**Kec10**] for the definition.

PROPOSITION 4.1. *The map ORep$(\Gamma, H) \to A(\Gamma, X, \mu)$ sending $\pi \mapsto a(\pi) = a_\pi$ is continuous.*

PROOF. Suppose $\pi_n \to \pi$. We have to check that $a_{\pi_n} \to a_\pi$. This is equivalent to showing that the Koopman representations converge: $\kappa^{a_{\pi_n}} \to \kappa^{a_\pi}$.

So it suffices to show that $\pi_n \to \pi$ then $\pi^{\odot\infty} = \kappa^{a_{\pi_n}} \to \kappa^{a_\pi} = \pi^{\odot\infty}$. Here $\pi^{\odot\infty} = \bigoplus_{n=0}^\infty \pi^{\odot n}$ is a representation on $H^{\odot\infty} = \bigoplus_{n\geq 0} H^{\odot n}$. If $\pi_n \to \pi$ then we show $\pi_n^{\odot m} \to_{n\to\infty} \pi^{\odot m}$: linear combination of vectors of the form $f_1 \odot \cdots \odot f_m$, $f_i \in H$ are dense in $H^{\odot m}$, so by the triangle inequality we only need to show convergence on vectors of the form $f_1 \odot \cdots \odot f_m$. We have

$$\pi^{\odot m}(\gamma)(f_1 \odot \cdots \odot f_m) = \pi(\gamma)(f_1) \odot \cdots \odot \pi(\gamma)(f_m) = \tfrac{1}{\sqrt{m!}} \textstyle\sum_{\sigma \in S_m} \otimes_{i=1}^m \pi(\gamma)(f_{\sigma(i)})$$

So it suffices to show that $|| \otimes_{i=1}^m \pi_n(\gamma)(g_i) - \otimes_{i=1}^m \pi(\gamma)(g_i)|| \to 0$ as $n \to \infty$ for $g_1, \ldots, g_m \in H$. We want to show that if $g_i^{(n)} \to g_i$ for each $i \leq m$ then $\otimes_{i=1}^m g_i^{(n)} \to \otimes_{i=1}^m g_i$. This is true by multilinearity and definition of the norm and inner product in $H^{\otimes m}$. Since $\bigcup_m H^{\odot m}$ is dense in $H^{\odot\infty}$ we are done. $\square$

If $U_n \cdot \pi_n \to \pi$ then $T_n \cdot a_{\pi_n} \to a_\pi$ where $T_n = a_{U_n}$. This will follow from continuity once we show:

PROPOSITION 4.2. *Let T be the Gaussian $\mathbb{Z}$-action (i.e., mpt) coming from the unitary operator U. Then for any representation π we have $T \cdot a_\pi = a_{U \cdot \pi}$. So $\pi \mapsto a_\pi$ is equivariant.*

PROOF. If $f \in H^{:1:}$ we show that $\kappa^{T \cdot a_\pi}(\gamma)|H^{:1:} = (U \cdot \pi)(\gamma)$. Let V be the Koopman operator associated to T, so $V|H^{:1:} = U$.

$$\begin{aligned}\kappa^{T \cdot a_\pi}(\gamma)(f) &= f \circ (T(\gamma^{-1})^{a_\pi} T^{-1}) = V(\kappa^{a_\pi}(\gamma)((V^{-1}f))) \\ &= U\pi(\gamma)U^{-1}(f) = (U \cdot \pi)(\gamma)(f)\end{aligned}$$ $\square$

PROPOSITION 4.3. *Let π be an orthogonal representation of Γ on the real Hilbert space H. If a_π is ergodic then π is weak mixing.*

PROOF. Suppose π is not weak mixing so that there is a finite-dimensional invariant subspace of H, say H_0, and let $s_1, \ldots, s_k$ be an orthonormal basis for H_0. We show that a_π is not ergodic. Note that by invariance of $H_0 = \langle s_1, \ldots, s_k \rangle$, for any $\gamma \in \Gamma$ and $i \leq k$ we can write

$$\pi(\gamma)s_i = \sum_{j=1}^{k} \alpha_{i,j} s_j,$$

where $\alpha_j = \langle \pi(\gamma)s_i, s_j \rangle \in \mathbb{R}$. Let $T \subseteq H$ be countable π-invariant set containing $s_1, \ldots, s_k$, such that the linear span of T is dense in H. We have that a_π is isomorphic to the shift on $(\mathbb{R}^T, \mu_\varphi)$. Let $S = \{s_1, \ldots, s_k\}$ and let $A \subseteq \mathbb{R}^S$ be a spherically symmetric subset of measure $0 < \mu^S(A) < 1$ (e.g., a ball) where μ is the normalized $N(0,1)$ Gaussian measure on $\mathbb{R}$. Then

$$B = \{c \in \mathbb{R}^T \,:\, c|S \in A\}$$

has measure $\mu_{\varphi|S}(A) = \mu^S(A)$ since the s_i are orthonormal, so that $\varphi|S$ is the identity covariance matrix (and hence the corresponding measure is product measure). For $t \in T$ let $p_t : \mathbb{R}^T \to \mathbb{R}$

denote the projection $p(c) = c(t)$. Then the map from $H^{:1:}(L^2(\mu_\varphi)) \to H$ sending $p_t \mapsto t \in H$ extends to an isomorphism and takes $\kappa_0^{a_\pi}|H^{:1:}$ to π, since $\kappa_0^{a_\pi}(\gamma) \cdot p_t = p_{\pi(\gamma)(t)}$. Thus, for any $i \leq k$, the equality $\pi(\gamma)s_i = \sum_{j=1}^k \alpha_{i,j}s_j$ implies

$$p_{\pi(\gamma)s_i} = \sum_{j=1}^k \alpha_{i,j}p_{s_j}$$

where the equality is in L^2, so these functions are equal almost everywhere. Thus, for μ_φ-almost every $c \in \mathbb{R}^T$ we have that

$$c(\pi(\gamma)s_i) = \sum_{j=1}^k \alpha_{i,j}c(s_j).$$

Since π is an orthogonal transformation, the matrix $M = (\alpha_{i,j})_{i,j\leq k}$ is an orthogonal $k \times k$ matrix. In particular, the set $A \subseteq \mathbb{R}^S \cong \mathbb{R}^k$ is invariant under M. Suppose $c \in B$. Then $c|S = (c(s_1), \ldots, c(s_k)) \in A$. We have

$$\begin{aligned}(\gamma^{-1} \cdot c)|S = c|(\pi(\gamma)S) &= (c(\pi(\gamma)s_1), \ldots, c(\pi(\gamma)s_k)) \\ &= (\textstyle\sum_{j=1}^k \alpha_{1,j}c(s_j), \ldots, \sum_{j=1}^k \alpha_{k,j}c(s_j)) \\ &= M \cdot (c(s_1), \ldots, c(s_k)) \in A\end{aligned}$$

since A is invariant under M. $\square$

Compare the above proof with (i)⇒(ii) of [**KT08**, Proposition 2.1]. We can also give an alternative proof (in the spirit of (iii)⇒(ii) of [**KT08**, Proposition 2.1] that if π is weak mixing then a_π is weak mixing (the usual proof just uses that a_π is weak mixing iff $\kappa_0^{a_\pi} \cong \oplus_n \pi^{\odot n}$ is weak mixing). We use a condition equivalent to weak mixing that one should compare with [**KT08**, Proposition 2.2]. (which says that all orbits are infinite iff for all $F_1, F_2 \subseteq X$, there exists $\gamma \in \Gamma$ such that $\gamma \cdot F_1 \cap F_2 = \varnothing$).

π is weak mixing iff for all $\epsilon > 0$ and finite $F_1, F_2 \subseteq H_\pi$ there exists a $\gamma \in \Gamma$ such that $\langle \pi(\gamma)u, v\rangle < \epsilon$ for all $u \in F_1$ and $v \in F_2$.

We can think of this is saying that $\pi(\gamma)(F_1)$ and F_2 are within ϵ of being orthogonal. For Gaussian actions given by Γ-invariant positive definite functions $\varphi : T \times T \to \mathbb{R}$, the condition becomes

$(*)$ $\forall\epsilon > 0$ and $F_1, F_2 \subseteq T$ finite, there exists $\gamma \in \Gamma$ such that $\forall x \in F_1, y \in F_2$, $\varphi(\gamma \cdot x, y) < \epsilon$.

PROPOSITION 4.4. *The condition (*) implies that the Gaussian action a_φ corresponding to φ is weak mixing.*

PROOF. We view a_φ as an action by shift on $(\mathbb{R}^T, \mu_\varphi)$. For $x \in T$ we let $p_x : \mathbb{R}^T \to \mathbb{R}$ be the projection $p_x(c) = c(x)$. It suffices to show that $a_\varphi \times a_\varphi$ on $(\mathbb{R}^{T \sqcup T}, \mu_{\varphi'} = \mu_\varphi \times \mu_\varphi)$ is ergodic, where φ' restricted to each diagonal copy of $T \times T$ is equal to φ and is zero everywhere else. We let $\mu := \mu_{\varphi'}$. So suppose not, i.e., suppose there is some $A \subseteq \mathbb{R}^{T \sqcup T}$ invariant with $0 < \mu(A) < 1$. Then we can find a finite $F \subseteq T \sqcup T$, $F = \{x_1, \ldots, x_k\}$ and a set $B \subseteq \mathbb{R}^{T \sqcup T}$ only depending on the coordinates in F, such that for some $\epsilon > 0$, $\mu(B \Delta A) < \epsilon/4$ and $\mu(B) - \mu(B)^2 > \epsilon$. By condition $(*)$ we have that for all $n \in \mathbb{N}$ we can find $\gamma_n \in \Gamma$ such that $\varphi(\gamma_n \cdot x, y) < \frac{1}{n}$ for all $x, y \in F$. Let $p_F : \mathbb{R}^{T \times T} \to \mathbb{R}^F \cong \mathbb{R}^k$ take $x \mapsto x|F \in \mathbb{R}^F$ and let $\tilde{B} = p_F(B) \in \mathbb{R}^k$. Consider the random vector

$$Z_n = (p_{\gamma_n \cdot F}, p_F)$$

in $\mathbb{R}^k \times \mathbb{R}^k$ with distribution measure $\mu_n = \mu_{\varphi' |(F \cup \gamma_n \cdot F)}$. This is a centered Gaussian random vector with characteristic function

$$\psi_n(u) = \exp(-\frac{1}{2}\langle u, M_n u\rangle)$$

where M_n is a block matrix of the form $M_n = \begin{pmatrix} \varphi'|F & A_n \\ A_n^T & \varphi'|F \end{pmatrix}$, and every entry of A_n is smaller than $1/n$. It is clear that the characteristic functions of the Z_n converge pointwise to the function $\psi(u) = \exp(-\frac{1}{2}\langle u, Mu\rangle)$, where $M = \begin{pmatrix} \varphi'|F & 0 \\ 0 & \varphi'|F \end{pmatrix}$, which is the characteristic function of a normal random vector on $\mathbb{R}^k \times \mathbb{R}^k$ distributed like $\mu_{\varphi'|F} \times \mu_{\varphi'|F}$. Since pointwise convergence of characteristic functions implies convergence in distribution it follows that the sequence of measures μ_n converge weakly to $\mu_{\varphi'|F} \times \mu_{\varphi'|F}$ weakly. Thus,

$$\mu_{\varphi'}(\gamma_n B \cap B) = \mu_n(\tilde{B} \times \tilde{B}) \to \mu_{\varphi'|F}(\tilde{B})^2 = \mu_{\varphi'}(B)^2$$

(note that the marginals of μ_n on both the left and right $\mathbb{R}^k$-factors are each $\mu_{\varphi'|F}$) so if n is large enough then $|\mu_{\varphi'}(\gamma_n \cdot B \cap B) - \mu_{\varphi'}(B)^2| < \frac{\epsilon}{4}$. But this implies

$$\begin{aligned}|\mu(B) - \mu(B)^2| \leq |\mu(B) - \mu(A)| + |\mu(A) - \mu(\gamma_n A \cap A)| \\ + |\mu(\gamma_n A \cap A) - \mu(\gamma_n B \cap B)| + |\mu(\gamma_n B \cap B) - \mu(B)^2| \\ < \frac{\epsilon}{4} + 0 + 2\frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon\end{aligned}$$

a contradiction. $\square$

Let $\mathcal{H}$ be a separable complex Hilbert space. Let $B(\mathcal{H})$ be the space of bounded operators on $\mathcal{H}$.

The Hilbert-Schmidt norm of an operator A is given by

$$||A||_{HS}^2 = \sum_{n=0}^{\infty} ||Ae_n||^2$$

where $\{e_n\}_{n\geq 0}$ is any orthonormal Basis for $\mathcal{H}$. A is called a Hilbert-Schmidt operator if this norm is finite. Let $HS(\mathcal{H})$ denote the set of Hilbert-Schmidt operators.

The trace norm of $A \in B(\mathcal{H})$ is given by

$$||A||_{Tr} = \sum_{n=0}^{\infty} \langle |A| e_n, e_n \rangle.$$

PROPOSITION 4.5 (Powers-Størmer inequality). *Let A and B be positive self-adjoint operators on a Hilbert space $\mathcal{H}$. Then*

$$||A^{\frac{1}{2}} - B^{\frac{1}{2}}||_{HS}^2 \leq ||A - B||_{Tr}.$$

Note that, taking $A = T^*T = |T|^2$ and $B = S^*S = |S|^2$ we get

$$||\,|T| - |S|\,||_{HS}^2 \leq ||(T^*T) - (S^*S)||_{Tr}$$

and since

$$\begin{aligned}(T+S)^*(T-S) + (T-S)^*(T+S) &= T^*T - T^*S + S^*T - S^*S + T^*T + T^*S - S^*T - S^*S \\ &= 2T^*T - 2S^*S\end{aligned}$$

the inequality becomes

$$\begin{aligned}
||\,|T|-|S|\,||_{HS}^2 &\le ||\tfrac{1}{2}\big[(T+S)^*(T-S)+(T-S)^*(T+S)\big]||_{Tr}\\
&\le \tfrac{1}{2}||(T+S)^*(T-S)||_{Tr}+\tfrac{1}{2}||(T-S)^*(T+S)||_{Tr}\\
&\le \tfrac{1}{2}||(T+S)^*||_{HS}||T-S||_{HS}+\tfrac{1}{2}||(T-S)^*||_{HS}||T+S||_{HS}\\
&= ||T+S||_{HS}||T-S||_{HS}.
\end{aligned}$$

If $\mathcal{H}$ is a complex Hilbert space then the conjugate Hilbert space is the space $\overline{\mathcal{H}}$ with underlying set the same as $\mathcal{H}$ (we denote the copy of $\xi \in \mathcal{H}$ that is in $\overline{\mathcal{H}}$ by ξ^*), and with scalar multiplication defined by

$$\lambda \cdot \xi^* = (\overline{\lambda} \cdot \xi)^*$$

and with inner product defined by

$$[\xi, \eta]_{\overline{\mathcal{H}}} = \langle \eta, \xi \rangle_{\mathcal{H}}.$$

If ρ is a unitary representation of Γ on $\mathcal{H}$ then conjugate representation $\overline{\rho}$ is defined to be the representation on $\overline{H}$ such that $\overline{\rho}(\gamma)$ is the same underlying set map as $\rho(\gamma)$ for each $\gamma \in \Gamma$.

In general we identify the space $\mathcal{H} \otimes \mathcal{K}$ with $HS(\overline{\mathcal{K}}, \mathcal{H})$ the space of Hilbert-Schmidt operators from $\overline{\mathcal{K}}$ to $\overline{\mathcal{H}}$, via $\xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K} \ \mapsto S_{\xi\otimes\eta}$ where

$$S_{\xi\otimes\eta}(\zeta^*) = \langle \eta, \zeta \rangle_{\mathcal{K}} \xi.$$

If we now take $\mathcal{K} = \overline{\mathcal{H}}$ then $\mathcal{H} \otimes \overline{\mathcal{H}}$ is isomorphic to the space of Hilbert-Schmidt operators on $\mathcal{H}$. The adjoint of the operator $S_{\xi\otimes\eta^*}$ is the operator $S_{\eta\otimes\xi^*}$. If we let $\mathcal{H} \odot \overline{\mathcal{H}}$ denote the subspace of $\mathcal{H} \otimes \overline{\mathcal{H}}$ that coincides with $\mathcal{H} \odot \mathcal{H}$ as a set, i.e., generated by elements of the form $\xi \odot \eta^* = \frac{1}{\sqrt{2}}(\xi \otimes \eta^* + \eta \otimes \xi^*)$, then this subspace coincides with the subspace generated by the self-adjoint Hilbert-Schmidt operators.

If π is a representation of Γ on $\mathcal{H}$, and ρ a representation of Γ on $\mathcal{K}$, $\pi \otimes \rho$ is isomorphic to the representation on $HS(\overline{\mathcal{K}}, \mathcal{H})$ given by

$$(\pi \otimes \rho)(\gamma)(S) = \pi(\gamma) S \overline{\rho}(\gamma^{-1}).$$

Similarly, we view $\pi \odot \overline{\pi}$ as a representation on the space generated by the self-adjoint Hilbert-Schmidt operators (i.e., just the restriction of $\pi \otimes \overline{\pi}$ to this subspace).

The conjugate $\overline{\pi_{\mathbb{C}}}$ of the complexification of π is canonically isomorphic to $\pi_{\mathbb{C}}$ itself, the isomorphism given by the conjugation map $\Phi : \mathcal{H}_{\mathbb{C}} \to \overline{\mathcal{H}_{\mathbb{C}}}$ defined by

$$\Phi(x + i \cdot y) = x - i \cdot y.$$

Denote this $\Phi(\xi) = \overline{\xi}$. This is linear since $\overline{\alpha \cdot \xi + \eta} = \overline{\alpha} \cdot \overline{\xi} + \overline{\eta}$ (recall how scalar multiplication was defined for conjugate spaces). It preserves the inner product since

$$\begin{aligned}
[\Phi(x_1 + i \cdot y_1), \Phi(x_2 + i \cdot y_2)]_{\overline{\mathcal{H}_{\mathbb{C}}}} &= \langle x_2 - i \cdot y_2, x_1 - i \cdot y_1 \rangle_{\mathcal{H}_{\mathbb{C}}} \\
&= \langle x_2, x_1 \rangle_{\mathcal{H}} + \langle y_2, y_1 \rangle_{\mathcal{H}} + i \langle x_2, y_1 \rangle_{\mathcal{H}} - i \langle y_2, x_1 \rangle_{\mathcal{H}} \\
&= \langle x_1, x_2 \rangle_{\mathcal{H}} + \langle y_1, y_2 \rangle_{\mathcal{H}} + i \langle y_1, x_2 \rangle_{\mathcal{H}} - i \langle x_1, y_2 \rangle_{\mathcal{H}} \\
&= \langle x_1 + i \cdot y_1, x_2 + i \cdot y_2 \rangle_{\mathcal{H}_{\mathbb{C}}}
\end{aligned}$$

and it takes $\pi_{\mathbb{C}}$ to $\overline{\pi_{\mathbb{C}}}$ since

$$\overline{\pi_{\mathbb{C}}}(\gamma)\Phi(x + i \cdot y) = \pi(\gamma)(x) - i \cdot \pi(\gamma)(y) = \Phi(\pi(\gamma)(x) + i \cdot \pi(y)) = \Phi(\pi_{\mathbb{C}}(\gamma)(x + i \cdot y)).$$

For $\xi = x + i \cdot y \in \mathcal{H}_{\mathbb{C}}$ we will use the notation $\overline{\xi}$ to refer to $x - i \cdot y$.

Using this isomorphism, we obtain an isomorphism $\mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}} \otimes \overline{\mathcal{H}_{\mathbb{C}}}$ given by the map $\xi \otimes \eta \mapsto \xi \otimes \overline{\eta}$, and also an isomorphism $\mathcal{H}_{\mathbb{C}} \odot \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}} \odot \overline{\mathcal{H}_{\mathbb{C}}}$ via $\xi \odot \eta \mapsto \xi \odot \overline{\eta}$.

THEOREM 4.6 (Popa?). *Let π be an orthogonal representation. Then the following are equivalent:*

(1) $1 \prec \pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}$ $(\cong \pi_{\mathbb{C}} \otimes \pi_{\mathbb{C}})$
(2) $1 \prec \pi_{\mathbb{C}} \odot \overline{\pi_{\mathbb{C}}}$ $(\cong \pi_{\mathbb{C}} \odot \pi_{\mathbb{C}})$
(3) $1 \prec \kappa_0^{a_\pi}$
(4) $1 \prec \kappa_0^{a_\pi} \otimes \kappa_0^{a_\pi}$ $(\cong \kappa_0^{a_\pi} \otimes \overline{\kappa_0^{a_\pi}})$

where $\kappa_0^{a_\pi} \cong \bigoplus_{n \geq 1} \pi_{\mathbb{C}}^{\odot n}$ is the unitary Koopman representation of the Gaussian action a_π.

PROOF. We proceed to show $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

(1)⇒(2): Given $F \subseteq \Gamma$ finite and $\epsilon > 0$ let $T \in HS(\mathcal{H}_{\mathbb{C}})$ be such that $||T||_{HS} = 1$ and for all

$\gamma \in F$

$$\epsilon > ||\pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}(\gamma)(T) - T||_{HS} = ||\pi_{\mathbb{C}}(\gamma)T - T\pi_{\mathbb{C}}(\gamma)||_{HS}.$$

We have that $|T| = (T^*T)^{1/2} \in \mathcal{H}_{\mathbb{C}} \odot \overline{\mathcal{H}_{\mathbb{C}}}$ is a positive self-adjoint Hilbert-Schmidt operator. Let $S = \pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}(\gamma)(T)$. Then

$$\begin{aligned} S^*S &= (\pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}(\gamma)(T))^*(\pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}(\gamma)(T)) = \pi_{\mathbb{C}}(\gamma)T^*T\pi_{\mathbb{C}}(\gamma^{-1}) \\ &= \pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}(\gamma)(|T|)^2 = \pi_{\mathbb{C}} \odot \overline{\pi_{\mathbb{C}}}(\gamma)(|T|)^2 \end{aligned}$$

so $|S| = \pi_{\mathbb{C}} \odot \overline{\pi_{\mathbb{C}}}(\gamma)(|T|)$. By the Powers-Størmer inequality we obtain for $\gamma \in F$

$$\begin{aligned} ||\,|T| - \pi_{\mathbb{C}} \odot \overline{\pi_{\mathbb{C}}}(\gamma)(|T|)\,||_{HS}^2 &= ||\,|T| - |S|\,||_{HS}^2 \\ &\le ||T + S||_{HS}||T - S||_{HS} \\ &\le (||T||_{HS} + ||S||_{HS})||T - \pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}(\gamma)(T)||_{HS} \\ &= 2||T - \pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}(\gamma)(T)||_{HS} < 2\epsilon \end{aligned}$$

so that $|T|$ is almost invariant for $\pi_{\mathbb{C}} \odot \overline{\pi_{\mathbb{C}}}$.

(2)⇒(3): This is obvious since $\pi_{\mathbb{C}} \odot \pi_{\mathbb{C}}$ is a subrepresentation of $\kappa_0^{a_\pi}$.

(3)⇒(4): This is also obvious.

(4)⇒(1): By Lemma 3.2 of [**Pop08**] we have that $1 \prec \pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}$ if and only if there exists some representation ρ with $1 \prec \pi_{\mathbb{C}} \otimes \rho$. So assume $1 \prec \kappa_0^{a_\pi} \otimes \kappa_0^{a_\pi}$. We have

$$\begin{aligned} 1 \prec \kappa_0^{a_\pi} \otimes \kappa_0^{a_\pi} &\cong (\bigoplus_{n\ge1} \pi_{\mathbb{C}}^{\odot n}) \otimes (\bigoplus_{m\ge1} \pi_{\mathbb{C}}^{\odot m}) \\ &\le (\bigoplus_{n\ge1} \pi_{\mathbb{C}}^{\otimes n}) \otimes (\bigoplus_{m\ge1} \pi_{\mathbb{C}}^{\otimes m}) \\ &\cong \bigoplus_{n,m\ge1} \pi_{\mathbb{C}}^{\otimes n+m} \cong \pi_{\mathbb{C}} \otimes (\bigoplus_{n,m\ge1} \pi_{\mathbb{C}}^{\otimes n+m-1}) \end{aligned}$$

so the $1 \prec \pi_{\mathbb{C}} \otimes (\bigoplus_{n,m\ge1} \pi_{\mathbb{C}}^{\otimes n+m-1})$. Applying Popa's Lemma, we get $1 \prec \pi_{\mathbb{C}} \otimes \overline{\pi_{\mathbb{C}}}$. □

This has the following implication. It is known that if $\pi \cong \lambda_I$ is a real quasi-regular representation of G on $l^2(I, \mathbb{R})$, then $\pi_{\mathbb{C}}$ has almost invariant vectors iff $\pi_{\mathbb{C}}$ is amenable iff a_π has non-trivial almost invariant sets iff $\kappa_0^{a_\pi}$ has almost invariant vectors iff the action of G on I is amenable. Theorem 4.6 shows that in general this does not hold, since there are examples of π which are amenable but that do not have almost invariant vectors.

In [**ET10**] an action a of a group Δ is constructed which is not anti-modular, and such that the Koopman representation κ_0^a of a does not *weakly* contain any finite-dimensional representations of Δ. In particular κ_0^a does not have non-trivial almost invariant vectors. If κ_0^a were non-amenable then, by the main result of [**ET10**], a would be anti-modular, which is not the case. Hence κ_0^a is amenable.

It is unclear whether a_π having non-trivial almost invariant sets is equivalent to $\kappa_0^{a_\pi}$ having non-trivial almost invariant vectors. Clearly the former implies the latter, but the does the reverse implication hold? Note that Theorem 4.6 shows that $1 \prec \kappa_0^{a_\pi}$ implies $1 \prec \pi^{\odot 2}$, so the question is whether $\pi^{\odot 2}$ having almost invariant vectors implies a_π having non-trivial almost invariant sets.

One implication that we can rule out is a_π having almost invariant sets implies $1 \prec \pi$. Assume toward a contradiction that this implication holds and let π be an amenable representation which does not weakly contain $1_{\mathbb{C}}$. Then by Theorem 4.6 we have $1_{\mathbb{C}} \prec \kappa_0^{a_\pi} \leq \kappa_0^{a_\pi^{\mathbb{N}}}$. Since the commutator of $a_\pi^{\mathbb{N}}$ in $\mathrm{Aut}(X^{\mathbb{N}}, \mu^{\mathbb{N}})$ acts ergodically, Lemma 10 of [**CI10**] implies that $1 \prec a_\pi^{\mathbb{N}}$, i.e., $a_\pi^{\mathbb{N}}$ has almost invariant sets. But $a^{\mathbb{N}} \cong a_{\bigoplus_n \pi}$, so by assumption this implies $1_{\mathbb{C}} \prec \bigoplus_n \pi$, which is equivalent to $1_{\mathbb{C}} \prec \pi$, contradicting our choice of π.

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