# Conditional independence in quantum many-body systems

Thesis by

Isaac Hyun Kim

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy


California Institute of Technology

Pasadena, California

2013

(Defended May 8th, 2013)

To my parents

# Acknowledgments

None of this work would have been possible without the help of people that I have met in IQIM. I would first like to thank my advisor John Preskill for his guidance and support throughout the past six years. Despite many of the mistakes I have made, he was always patient and understanding. His insightful comments and questions have always lead me to think twice about my research, as well as my life. I am afraid such an experience will come only occasionally once I leave this wonderful place.

I should also thank Robert König and Stephanie Wehner for my early exposures in quantum information. They undoubtedly had a huge influence in the early stages of my research. It was from the discussions that I had with them that I developed my first intuition in this field. I would like to thank Jeongwan Haah and Beni Yoshida for all the discussions about self-correcting quantum memory, and more. It was always a great fun to talk about this exciting subject, but it was even more so to talk about it with these amazing friends. I am greatly indebted to Spyridon Michalakis and Matthew Hastings for all the discussions about the rigorous results in quantum many-body systems. A big chunk of my thesis is based on their work, and none of this would have been possible without their feedbacks. I thank Jon Tyson and Mary Beth Ruskai for their great insights in matrix inequalities. I have truly learned a great deal from them. I would also like to thank Andreas Winter and Alexei Kitaev, for their insightful comments that lead to many of the works in this thesis. I also thank other former/current IQIM members, including Steve Flammia, Stephen Jordan, Norbert Schuch, Sergio Boixo, Salman Beigi, Liang Jiang, Alexey Gorshkov, Glen Evenbly, Zheng-Cheng Gu, David Poulin, Roger Mong, Gorjan Alagic, Prabha Mandayam, and Ersen Bilgin.

I would like to also thank David Politzer and Gil Refael, for the great experiences I had as a teaching assistant. I had a rare privilege of working as a TA for exciting and unconventional courses. I would also like to thank all of my students who made my work all the more enjoyable.

I want to thank my family, for always being there for me in good and bad times. My parents have always encouraged to pursue my dreams, and I have been always grateful for that. I want to thank my sister Hwayoung. I was most productive when she was visiting this area for her exchange student program, and I do not believe this was a mere coincidence.

# Abstract

In this thesis, I will discuss how information-theoretic arguments can be used to produce sharp bounds in the studies of quantum many-body systems. The main advantage of this approach, as opposed to the conventional field-theoretic argument, is that it depends very little on the precise form of the Hamiltonian. The main idea behind this thesis lies on a number of results concerning the structure of quantum states that are conditionally independent. Depending on the application, some of these statements are generalized to quantum states that are approximately conditionally independent. These structures can be readily used in the studies of gapped quantum many-body systems, especially for the ones in two spatial dimensions. A number of rigorous results are derived, including (i) a universal upper bound for a maximal number of topologically protected states that is expressed in terms of the topological entanglement entropy, (ii) a first-order perturbation bound for the topological entanglement entropy that decays superpolynomially with the size of the subsystem, and (iii) a correlation bound between an arbitrary local operator and a topological operator constructed from a set of local reduced density matrices. I also introduce exactly solvable models supported on a three-dimensional lattice that can be used as a reliable quantum memory.

# Contents

# List of Figures

# Chapter 1

# Introduction and preliminary materials

In this thesis, we shall study the generic properties of gapped quantum many-body systems by (i) constructing exactly solvable models and (ii) exploiting the tools from quantum information theory. In fact, we shall restrict our focus to a set of quantum many-body systems described by a local Hamiltonian. We say a Hamiltonian $H$ is local if it can be described by a sum of geometrically local terms with a bounded norm. Note that, while each of the geometrically local terms in the Hamiltonian have bounded norms, $H$ itself might have an unbounded norm for an infinite system. Two Hamiltonians $H$ and $H'$ are typically labelled to be *in the same phase* if they can be adiabatically connected to each other without closing the energy gap between the ground state and the rest of the spectrum.

One of the subtleties in defining a quantum phase lies on the fact that we are implicitly assuming an infinite volume limit of some sequence of Hamiltonians. After all, any generic finite dimensional Hamiltonian always has a gap between the first excited state and the ground state. Furthermore, as shown by Wen[1], there can be systems that have degenerate ground states in the thermodynamic limit even without any symmetry. Nowadays this is known as the topological ground state degeneracy.[1, 2, 3] The topological ground state degeneracy arises because an energy splitting between the different "ground state sectors" are suppressed exponentially in the system size. Hence, the energy splitting for a finite system may be nonzero due to the finite-size effect.

Even further complications may arise due to the existence of a gapless edge mode.[4, 5, 6] The gapless edge mode refers to a gapless excitation that is localized along a boundary of a finite system. For such systems, the energy spectrum of the excitations that are sufficiently far away from the boundary must be separated from the ground state by a constant that is independent of the system size. On the other hand, the energy spectrum of the excitations along the boundary is close to that of the ground state with an energy difference that approaches 0 in the thermodynamic limit. We shall call such systems to have a bulk gap and a gapless edge mode.

One of the approaches for understanding these phases is to study their trial wavefunctions. For example, Laughlin was able to construct a wavefunction that predicted the partially filled Landau level of a quantum Hall system as well as the existence of a quasi-particle with a fractional charge.[7] Laughlin's approach was subsequently vindicated by the discovery of an adiabatic path that interpolates between Laughlin's wavefunction and a realistic system with a Coulomb interaction.[8] Also, Moore and Read proposed a trial wavefunction for a $\nu = \frac{5}{2}$ fractional quantum Hall state, and predicted the existence of a quasi-particle that exhibits a non-Abelian statistics.[9]

Meanwhile, interesting developments were being made by several authors for the studies of one-dimensional quantum many-body systems. Partly inspired by Wilson's idea of the renormalization-group (RG) flow,[10, 11, 12] White introduced a powerful numerical tool known as the density matrix renormalization-group (DMRG).[13] Around the same time, Fannes et al. introduced a class of quantum states known as the finitely correlated states (FCS).[14] It eventually became clear that DMRG and FCS have an intimate connection. Several authors have studied the so called matrix product state (MPS) formalism, and such an approach was successfully used in understanding the structure of 1D gapped systems.[15, 16, 17, 18, 19] A justification for using the MPS formalism in such setting is based on the area law of 1D gapped system.[20] The area law states that the entanglement entropy $S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$ of a subsystem $A$ is bounded by the size of its area, as opposed to its volume. Hastings proved that this is the case for 1D gapped systems, and also showed that such states admit an efficient MPS description.[20]

The success of the MPS formalism was subsequently followed by the discovery other variational ansatzes, such as the projected entangled-pair states (PEPS)[21] and the multiscale entanglement renormalization ansatz(MERA)[22]. The motivation for studying these variational states are mainly twofold. First, by having a succinct description of the quantum many-body wavefunction, one can simulate their ground state properties efficiently. Second, one might be able to understand the generic structures that arise from these variational classes.

These variational states typically have a well-defined parent Hamiltonian.[23, 24, 25] If the parent Hamiltonian has a simple structure, it can substantially reduce the complexity of studying the properties of the quasi-particles. In particular, if the Hamiltonian consists of a sum of commuting terms, the underlying model is called as an exactly solvable model. Examples include the quantum double model and the string-net model.[3, 26][1] While it is hard to construct a physical system that realizes such a Hamiltonian *exactly*, the virtue of these models is that the properties of the phase are stable against a small enough perturbation.[28, 29, 30] For example, consider the toric code.[3] As noted by Kitaev,[3] a perturbation theory calculation of the ground state degeneracy splitting decays exponentially in the system size. Recently, this result was put on a rigorous ground by several authors.[28, 29, 30, 31] Once such a stability bound is obtained, one can formally use the

[1] It is important to note that there are examples which do not belong to such categories, see Ref.[27].

quasi-adiabatic continuation technique to obtain rigorous statements about the properties of the phase.[32, 33]

For example, a rather obvious consequence of the gap stability is the stability of the particle statistics and the logical operator that can map one of the ground states to another.[29] Under the adiabatic evolution, the quasi-particle excitations may spread out to a length that is comparable to its correlation length. Hence, the conventional braiding operation can be still described by a dressed string-like operator. Higher-dimensional analogues of these statements can be obtained quite straightforwardly. An important lesson that one can learn from these examples is that, once we are given a model with a protected energy gap, important properties of its phase can be rigorously proven to be stable under a generic perturbation that is sufficiently weak. Therefore, one can consider these exactly solvable models as representatives of each gapped quantum phases.[2]

For these exactly solvable models, there is a general tradeoff bound that constrains a number of topologically protected ground states and its ability to protect against creation and diffusion of the quasi-particles.[34, 35, 36, 37, 38] When applied to two-dimensional systems, these tradeoff bounds imply that there has to be a constant energy barrier to construct a map from one of the degenerate ground states to another ground state. Recall that Arrhenius' law states that a transition rate for such processes is of the order $e^{-\beta\Delta E}$, where $\beta$ is the inverse temperature and $\Delta E$ is the energy barrier. Since this expression alone does not account for the entropic contributions, it must not be considered as a mathematically rigorous result. However, this expression does cast doubt in using two-dimensional topologically ordered systems as a stable quantum memory without any active intervention. Indeed, there is a rigorous upper bound on the decoherence time for Kitaev's toric code that is independent of the system size.[39]

On the other hand, it is well-known that a variant of the toric code in four spatial dimensions can have an extensive energy barrier that grows with the system size.[40] Later Alicki et al. were able to obtain a rigorous lower bound on the decoherence time that scales exponentially with the system size.[41] An important open question was whether it is possible to have such a stable quantum memory in three spatial dimensions. A number of authors have introduced a possible generalization of two-dimensional exactly solvable models to three spatial dimensions, but they were all shown to have a constant energy barrier.[42, 43, 44, 45] Later we have obtained yet another variant of these models, which was motivated from the fact that the previously known models could be manifestly decomposed into the "electric" and the "magnetic" part, so that at least one of them has a constant energy barrier.[46] This new model did not have such a manifest decomposition, yet it shared all the qualitative features of the three-dimensional (3D) toric code.

Soon it was realized by Yoshida that there is a good reason behind why such conclusion was

[2] However, it is not clear if one can always obtain such exactly solvable models for *any* topologically ordered phase. For example, the exactly solvable models introduced by Levin and Wen in Ref.[26] is unable to reproduce the chiral gapless edge mode.

inevitable.[47] He showed that, given a three-dimensional system described by a stabilizer group formalism with a bounded number of encoded qubits, the energy barrier is always bounded by a constant. Roughly at the same time, Haah published his breakthrough result which seemed to have many counterintuitive properties.[48] One of the defining properties of Haah's model is that the quasi-particles cannot move freely without paying an extensive energy cost that grows with the length it travels.[49] Due to this reason, his model quickly became a candidate for a self-correcting quantum memory in 3D. However, it was later realized that the decoherence time only grows as $O(e^{\beta^2})$.[50] This bound does not grow with the size of the system, so Haah's model is not a self-correcting quantum memory in a strict sense. Nevertheless, it is interesting to see that one can have a substantially longer lifetime than ordinary two-dimensional quantum memories at a low temperature. The first part of the thesis will be about the results that hinge on these developments. More specifically, we will describe two exactly solvable models that are similar to (i) the 3D toric code and (ii) Haah's model. These shall be covered in Chapter 3.

The direction of the rest of the thesis shall be rather different in that we will be studying the *generic properties* of a gapped quantum many-body system from a rather small set of assumptions. More specifically, we shall assume that (i) the system is gapped and (ii) it satisfies a certain form of an area law. It is widely believed that the area law is true for a gapped quantum many-body system, but there are several reasons to be careful about such an assertion. For one thing, the area law has been only established for one-dimensional systems.[20, 51, 52, 53] Unfortunately, the techniques used in these works do not seem to have an easy generalization that is applicable to higher dimensional systems. Also, Michalakis was able to obtain a rigorous bound on the change of the entanglement entropy under an adiabatic evolution.[54] In this stability bound, a logarithmic divergence is present in any systems that are supported on a $d \geq 2$-dimensional lattice. These results suggest that, even if area law is true, rigorously proving it in $d \geq 2$ spatial dimensions is likely to be a difficult task.

Therefore, instead of attempting to prove the area law, we shall take it as an axiom and study its consequences. The key idea lies on an observation that (i) there is a special structure that arises for states that are conditionally independent and (ii) the RG fixed-point ground state wavefunction of a topologically ordered system has many subsystems with such a property.[55] These observations shall be later explained in more detail, but for the moment we would like to sketch the general principle behind this approach. A tripartite state $\rho_{ABC}$ is *conditionally independent* if its conditional mutual information $I(A:C|B) = S(AB)+S(BC)-S(B)-S(ABC)$ is equal to 0. If the state is conditionally independent, there are several different ways to reconstruct the global state $\rho_{ABC}$ from the local reduced density matrices, such as $\rho_{AB}$ and $\rho_{BC}$. There are several scenarios in which this property can be exploited.

For example, suppose we are given two quantum states, say $|\psi_1\rangle$ and $|\psi_2\rangle$ that are topologically ordered. In other words, these two states are indistinguishable from each other if the measurement is

restricted to their local subsystems. The local indistinguishability would imply that the local reduced density of these two states are identical. At least in this idealized setting, one can conclude that the conditional mutual information $I(A:C|B)$ for the two states cannot be equal to 0. Otherwise, one would be able to reconstruct the global state from the local reduced density matrices.[55] Since the local reduced density matrices were assumed to be identical, the reconstructed states for $|\psi_1\rangle$ and $|\psi_2\rangle$ must be identical to each other. However, such result contradicts the original assumption: that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal to each other.

The preceding argument is one of the many implications of the conditional independence. On one hand, this is encouraging in that we can obtain strong statements about quantum states without resorting to the properties of its parent Hamiltonian. On the other hand, a more careful analysis must be worked out. For example, a big open question in quantum information theory concerns a structure of states that are approximately conditionally independent. Realistic quantum states that arise as a ground state of a quantum many-body system will generically have a small conditional mutual information, rather than saturating its minimal value exactly. Hence, statements that are robust against such small deviation of the conditional independence condition is highly desirable.

An important tool that shall be used in conjunction with the preceding idea is the quasi-adiabatic continuation.[33, 56, 57, 58, 59, 60, 61, 29] Quasi-adiabatic continuation asserts that, given a set of approximately degenerate ground states $|\psi_i(s)\rangle_{i=1,\cdots N}$ that are sufficiently separated from the rest of the spectrum by a constant along an adiabatic path $s \in [0,1]$, there exists a unitary operation $U(s)$ such that

$$\sum_{i=1}^{N} |\psi_i(s)\rangle \langle\psi_i(s)| = U(s) \sum_{i=1}^{N} |\psi_i\rangle \langle\psi_i| U(s)^\dagger \tag{1.1}$$

for all $s \in [0,1]$. Further, the unitary operator $U(s)$ is generated by a sum of path-dependent quasi-local generators with a superpolynomially decaying tail. A similar statement can be obtained even if the system only preserves the bulk *mobility* gap alone, see Ref.[61].

Existence of such a quasi-local generator implies that one can use the so called Lieb-Robinson bound[33, 56, 57, 58, 59, 62, 63, 64, 65, 66, 60, 67, 68, 69, 70, 61, 29] to bound a speed at which information can propagate. Lieb-Robinson bound asserts that, given a time-dependent Hamiltonian $H(t)$ that consists of a sum of differentiable quasi-local bounded-norm terms with a tail that decays sufficiently fast, the dynamics generated by $H(t)$ has an effective light cone: the correlation between two observables that lie outside the lightcone is small. We shall show that these tools can be used to make quantitative statements about (i) several properties of the topological entanglement entropy[27, 71] and (ii) the structure of real-space entanglement spectrum.

### 1.0.1 How to read this thesis

Due to the scope of the thesis, we explain the necessary background materials to read each of the chapters. Chapter 2 concerns exactly solvable models that can be described by the stabilizer group formalism, which is briefly explained in Section 1.2. Therefore, the technical tools used in Chapter 3 will be irrelevant for the discussion. On the other hand, Chapter 4–6 will be based on the tools described in Chapter 3. In Chapter 4, we shall construct a set of inequalities between long-range entanglement and a topological ground state degeneracy. This result is based on the strong subadditivity of entropy alone, which is explained in Section 1.1. Chapter 5 studies a structure of the entanglement Hamiltonian in gapped quantum many-body systems. For this work, one would need Section 3.1,3.3, and 3.4. In Chapter 6, we establish a first-order perturbative stability of the topological entanglement entropy. All of the technical tools in Chapter 3 will be needed to understand the material. In Appendix A, we describe some of the technical tools that were developed in an attempt to attack the problems discussed in this thesis. These tools were superseded by the tools described in the main text of the thesis. Nevertheless, we list these results since they may be interesting in their own right.

## 1.1 Ground state properties of a topologically ordered system

In this section, we review some of the well-known facts about topologically ordered systems, mainly focusing on its ground state properties. Of course, it is not clear if the ground state wavefunction alone gives a sufficient information to completely determine its underlying phase. This is due to the fact that the properties of its quasi-particles may not be completely determined by the ground state wavefunction alone. For example, there exists a *gapless* Hamiltonian whose ground state subspace is exactly equal to that of the toric code Hamiltonian.[72] Such an example shows that one must impose a certain "naturalness" condition to discuss the properties of the quasi-particles. However, the situation may not be so bad in light of the result by Zhang et al.[73] They have shown that one can infer the elements of the topological $S$-matrix and $U$-matrix from the ground state entanglement alone. For certain systems, these data are sufficient to determine all the important properties of the topological phase, see Ref.[74].

### 1.1.1 Topological ground state degeneracy

Topological ground state degeneracy refers to a set of states that are locally indistinguishable from each other. It was first realized by Wen that a nontrivial ground state degeneracy can arise for a spin liquid system on a compact manifold.[1] Subsequently Niu and Wen discovered a similar

behavior for a fractional quantum Hall system.[2] The character of such a degeneracy is different from the degeneracy that arises from a symmetry breaking phenomenon. For example, consider an Ising model at zero temperature. The degenerate ground states can be distinguished by a local observable, namely the $\sigma_z$ operator. This is not the case for a topologically ordered system. No local observable can distinguish different ground state sectors. The local indistinguishability property is at the heart of the topological protection of quantum information. In order to disturb the quantum information that is encoded in the ground state subspace, a highly nonlocal operation must be carried out. Furthermore, under a generic condition that is believed to be satisfied by many systems, the degeneracy is protected by any local perturbation that is sufficiently weak.[28, 29, 31] We note in passing that there are other types of topologically protected degeneracy that may arise.[3, 75, 76, 77, 78] We believe our tool can be applied to these systems as well, but we leave that for the future work.

### 1.1.2 Topological entanglement entropy

It was first discovered by Hamma et al. for Kitaev's toric code model[79] and later generalized by Kitaev and Preskill[27] and Levin and Wen[71] that there exists a universal constant subcorrection term of the entanglement entropy that characterizes the phase. More precisely, given a simply connected subsystem $A$, its entanglement entropy $S(A)$ can be expressed as

$$S(A) = a|\partial A| - \gamma + O(e^{-|\partial A|/\xi}), \tag{1.2}$$

where $a$ is a nonuniversal constant, $|\partial A|$ is the boundary area of $A$, $\gamma$ is the topological entanglement entropy, and $\xi$ is the correlation length of the system. It was argued by these authors that $\gamma$ is an invariant, in that its value changes very little under an adiabatic evolution of the system that does not close the bulk gap. While a substantial amount of numerical work has confirmed their predictions,[80, 81, 74, 82] rigorously proving its stability still remains as an open problem. In fact, Bravyi has an unpublished model which can be adiabatically connected from a trivial state, yet has a nonzero amount of topological entanglement entropy defined as in Ref.[27, 71, 83].[83] Bravyi's counterexample shows that a constant subcorrection term of the entanglement entropies in such systems is not a stable invariant characterizing the phase. But then, what does $\gamma$ represent? The numerical examples give values that are close to what the ideal wavefunctions predict.[80, 81, 74, 82] Hence, it is natural to conclude that there exists an alternative definition of the topological entanglement entropy that evades Bravyi's counterexample.

Coming up with such a definition is an important problem, especially in light of the recent result by Cincio and Vidal.[74] Their conclusion, which was drawn from the result by Zhang et al.[73, 84], gives a complete information about the quasi-particles from the ground state wavefunction

alone. Since the quasi-particles in two spatial dimensions can be used to perform a fault-tolerant quantum computation,[3] it is important to have a correct definition of the ground state observables to characterize such phases from the microscopic Hamiltonian. We do not have a complete answer to this problem, but we shall propose an alternative definition that are in many ways natural. More specifically, we shall derive a universal inequality relating the number of topologically protected states and a certain linear combination of the entanglement entropies. This linear combination is reduced to the topological entanglement entropy for an idealized wavefunction that is a fixed-point of some RG flow. Furthermore, the inequality is saturated with an equality for Abelian anyon models, giving an automatic one-sided stability bound for this newly defined quantity. Of course, the one-sided stability result does not imply the stability of the topological entanglement entropy. Nevertheless, our result implies that it suffices to prove a rigorous upper bound for the topological entanglement entropy in order to prove its stability under an adiabatic evolution.

After the discovery of the topological entanglement entropy, several authors have attempted to find its finite-temperature generalizations. For a two-dimensional system, it was quickly realized that the topological entanglement entropy vanishes at any finite temperature.[85, 86] On the other hand, the topological entanglement entropy *does* survive at finite temperature for certain systems, see Ref.[43, 87]. There are some subtleties that are worth mentioning. For a three-dimensional variant of the toric code, there exists an order parameter that is analogous to the topological entanglement entropy.[85] The value of the order parameter vanishes at a sufficiently high temperature, but it attains a nonzero value even in the thermodynamic limit below a certain critical temperature. On the other hand, Hastings showed that the model can be mapped to a thermal state of a classical Hamiltonian under a finite-depth local quantum circuit.[88] These two results together imply that the topological entanglement entropy can be of a classical origin at a finite temperature. Therefore, it is not clear if it is a stable invariant under a small perturbation. We make a partial progress in showing the *perturbative* stability of this quantity. A similar technique shall be used to prove a first-order perturbative stability of the ground state topological entanglement entropy as well.

### 1.1.3 Entanglement Hamiltonian

Another surprising property of the gapped systems is the locality of the entanglement Hamiltonian. Formally, entanglement Hamiltonian is defined as a logarithm of a reduced density matrix of some region, say $A$. There is no *a priori* reason as to why such operator must have a special structure. However, Li and Haldane showed that the spectrum of the entanglement Hamiltonian of a $\nu = \frac{5}{2}$ variational FQHE wavefunction along the orbital cut resembles that of an one-dimensional local Hamiltonian.[89] Furthermore, the spectrum contains information about the conformal field

theory(CFT) that describes the FQHE wavefunction.[3]

Since the discovery of Li and Haldane, a number of authors followed up by investigating different variational states.[90, 91, 92, 93, 94, 95] One of the conclusions uniformly drawn from these works is that the spectrum of the entanglement Hamiltonian can be described by the spectrum of some local Hamiltonian. Further, this spectrum contains information about the phase. Unfortunately, such emergent local structure of the entanglement Hamiltonian was studied only by either investigating a class of variational states[95, 94] or performing numerical experiments.[90, 91, 92, 93]

In Chapter 5, we make a partial progress in understanding this local structure. More specifically, we shall show that a judiciously chosen linear combination of the entanglement Hamiltonian has a small correlation with almost all local observables, given that the ground state wavefunction obeys a certain form of an area law. Our result shows that the local structure of the entanglement Hamiltonian may be attributed to the area law of entanglement entropy[96] and the exponential clustering theorem[56, 57], which are believed to be the generic properties of a gapped quantum many-body system.

## 1.2 Information measures

A fundamental quantity in quantum information theory is the von Neumann entropy $S(\rho)$.

**Definition 1.**

$$S(\rho) := -Tr(\rho \log \rho). \tag{1.3}$$

The von Neumann entropy quantifies the amount of information that is present in a sequence of many copies of the state. Schumacher showed that $\rho^{\otimes n}$ can be compressed into $n(S(\rho) - \delta)$ qubits with an error that vanishes in $n \to \infty$ limit for any nonzero $\delta$.[97]

Entanglement entropy is a canonical measure for quantifying entanglement in a bipartite system. Given a quantum state $\rho$, its entanglement entropy of a subsystem $A$ is defined as follows.

**Definition 2.**

$$S(A) := -Tr(\rho_A \log \rho_A), \tag{1.4}$$

*where $\rho_A$ is the reduced density matrix over the subsystem $A$.*

Given a multipartite system, one can define a linear combination of the entanglement entropy. For example, mutual information is a measure of correlation that is present between two subsystems.

[3] In the literature, the spectrum of the entanglement Hamiltonian is called as the *entanglement spectrum*. We emphasize that the entanglement Hamiltonian contains more information than the entanglement spectrum, since one can simply read off the entanglement spectrum by computing the eigenvalues of the entanglement Hamiltonian.

**Definition 3.** *A mutual information between two subsystems $A$ and $B$ is*

$$I(A:B) = S(A) + S(B) - S(AB). \tag{1.5}$$

This also has an operational meaning, see Ref.[98]. We also define conditional mutual information.

**Definition 4.** *A conditional mutual information between $A$ and $C$ with respect to $B$ is*

$$I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC). \tag{1.6}$$

The conditional mutual information has an operational meaning in the context of a quantum state redistribution protocol, see Ref.[99]. We also define a quantum relative entropy, which is a quantum analogue of the Kullback-Leibler divergence.

**Definition 5.** *A relative entropy $D(\rho\|\sigma)$ between two quantum states $\rho$ and $\sigma$ is the following.*

$$D(\rho\|\sigma) := Tr(\rho(\log\rho - \log\sigma)). \tag{1.7}$$

The relative entropy appears in the context of quantum hypothesis testing, see Ref.[100, 101, 102, 103]. Another standard distance measure between quantum states is the Schatten $p$-norm. Given an operator $O$, its $p$-norm is defined as follows.

**Definition 6.**

$$|O|_p := (\sum_i e_i^p)^{1/p}, \tag{1.8}$$

*where $\{e_i\}$ is a set of eigenvalues of $|O| := (O^\dagger O)^{\frac{1}{2}}$.*

A special attention must be given to $p = 1$ and $p = \infty$ case. In particular, the Schatten $\infty$-norm is typically called as the *operator norm*. We shall denote such norm as follows:

$$\|O\| = |O|_\infty. \tag{1.9}$$

### 1.2.1 Inequalities

A linear inequality is an inequality that is linear in the von Neumann entropy and quantum relative entropy. One of the most basic linear inequalities is the concavity of the von Neumann entropy, which easily follows from the operator convexity of a function $f(x) = x\log x$.[104][4]

$$S(c\rho + (1-c)\sigma) \geq cS(\rho) + (1-c)S(\sigma), c \in [0,1]. \tag{1.10}$$

[4]In fact, this result can be proved from the convexity of $f(x)$ alone.

The concavity of the von Neumann entropy implies the nonnegativity of the quantum relative entropy $D(\rho\|\sigma) := \mathrm{Tr}(\rho(\log\rho - \log\sigma))$ between two quantum states $\rho$ and $\sigma$. This can be easily seen by dividing the both sides of Equation 1.10 by $c$ and taking the $c \to 0+$ limit.

There exists a class of inequalities that cannot be directly derived from the concavity of the von Neumann entropy. These are the descendants of the joint convexity of the quantum relative entropy:

$$cD(\rho_1\|\sigma_1) + (1-c)D(\rho_2\|\sigma_2) \geq D(c\rho_1 + (1-c)\rho_2\|c\sigma_1 + (1-c)\sigma_2), c \in [0,1], \tag{1.11}$$

where $\rho_1, \rho_2, \sigma_1, \sigma_2$ are some density matrices.[105] Equation 1.11 implies one of the most important results in quantum information theory, which is known as the monotonicity of the quantum relative entropy. The monotonicity of the quantum relative entropy asserts that the relative entropy between two quantum states does not increase under a quantum channel. That is,

$$D(\rho\|\sigma) \geq D(\Phi(\rho)\|\Phi(\sigma)) \tag{1.12}$$

for a completely positive trace-preserving map $\Phi$.

A rather straightforward consequence of Equation 1.12 is the strong subadditivity of entropy(SSA), which asserts that the conditional mutual information is nonnegative:

$$I(A:C|B) \geq 0. \tag{1.13}$$

Another useful inequality for the purpose of this thesis is Fannes' inequality, which holds for any density matrices $\rho$ and $\sigma$:

$$|S(\rho) - S(\sigma)| \leq 2\epsilon \log d - 2\epsilon \log 2\epsilon, \tag{1.14}$$

where $\epsilon = \frac{1}{2}|\rho - \sigma|_1$.[106] We note in passing that an optimal improvement of the Fannes' inequality was recently obtained by Audenaeart:

$$|S(\rho) - S(\sigma)| \leq \epsilon \log d + H(\epsilon, 1-\epsilon), \tag{1.15}$$

where $H(p, 1-p)$ is a binary entropy.[102]

$$H(p, 1-p) = -p\log p - (1-p)\log(1-p). \tag{1.16}$$

## 1.3 Quantum error-correcting code

Given a Hilbert space $\mathcal{H}$ and a set of operators $\mathcal{B}(\mathcal{H})$, one can formally define a quantum code to be a subspace $\mathcal{C} \subset \mathcal{H}$. We first start with several definitions.

**Definition 7.** *[107] A quantum code $\mathcal{C}$ detects an error $E \in \mathcal{B}(\mathcal{H})$ if there exists a constant $c(E)$ such that*

$$\forall \left|\psi_1\right\rangle, \left|\psi_2\right\rangle \in \mathcal{C}, \left\langle\psi_1\right| E \left|\psi_2\right\rangle = c(E) \left\langle\psi_1\right| \left|\psi_2\right\rangle. \tag{1.17}$$

The vectors of the quantum code shall be called as the *codewords*. The physical systems that are described by the quantum code typically consist of particles with bounded dimensions, say $d$. For such systems, there always exists a canonical basis for $\mathcal{B}(\mathcal{H})$ that is described by the operators in a tensor product form:

$$\forall O \in \mathcal{B}(\mathcal{H}), O = \sum_{i_1,\cdots,i_n} a_{i_1,\cdots i_n} U_{i_1} \otimes \cdots \otimes U_{i_n}, \tag{1.18}$$

where $U_{i_n}$ is an operator supported on the *local Hilbert space* describing the $n$th particle.[108] Furthermore, $\{U_i\}_{i=1,\cdots,d^2}$ can be chosen to be a complete set of unitary operators that are orthonormal.

$$\mathrm{Tr}(U_i U_j^\dagger) = d\delta_{ij}. \tag{1.19}$$

The operators do not necessarily have to be unitary, but it is a convenient choice for many of the analysis. For $d = 2$ case, one can actually get more structure: the basis set for the operators can be chosen to be Hermitian as well as unitary. The elements of this set is known as the Pauli operators:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X := \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad Z := \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.20}$$

Generally speaking, an operator of the tensor product form shall be called as a *weight-$w$* operator if $w$ of the local operators are not the identity operator. The *code distance* of a quantum code is the minimal weight of the operator which is not detectable.

If the system is subject to an interaction with its environment, the underlying physical process can be modeled by a quantum channel $\mathcal{E}$, see Ref.[109]. It is typically convenient to use a *Kraus representation* of a quantum channel $\{E_j\}$:

$$\mathcal{E}(O) = \sum_j E_j O E_j^\dagger. \tag{1.21}$$

One can formally define what it means for a quantum code to be able to correct errors from such processes.

**Definition 8.** *A quantum code $\mathcal{C}$ corrects errors from $\mathcal{E}$ if*

$$P E_i^\dagger E_j P = \alpha_{ij} P \tag{1.22}$$

*for some number $\alpha_{ij}$, where $P$ is the projector onto the codeword subspace and $\{E_j\}$ are the Kraus*

*operators of* $\mathcal{E}$.

An interesting class of quantum error-correcting code is the so called *topologically ordered systems*. Formally, we define a topological quantum order as follows.

**Definition 9.** *[59] A set of orthonormal states* $\{|\psi_i\rangle\}_{i=1,\cdots,N}$ *satisfies a* $(r,\epsilon)$*-TQO condition if they satisfy the following inequalities:*

$$\begin{aligned} |\langle\psi_i|O|\psi_i\rangle - \langle\psi_j|O|\psi_j\rangle| &\leq \|O\|\epsilon \\ |\langle\psi_i|O|\psi_j\rangle| &\leq 2\epsilon\|O\|, \end{aligned} \tag{1.23}$$

*where* $O$ *is a bounded operator that can be supported on a ball of radius* $r$.

If $\epsilon$ is set to 0, the TQO-$(r,\epsilon)$ condition implies that the quantum code spanned by $\{|\psi_i\rangle\}_{i=1,\cdots,N}$ can detect any error that is supported on a ball of radius $r$. We note in passing that the code distance $d$ and the TQO-radius $r$ are not equal to each other in general. More specifically, their relation depends on the dimensionality of the underlying lattice. For example, consider Kitaev's toric code.[3] The code distance on a $L \times L$ lattice is $L$. On the other hand, the code satisfies a TQO-$([\frac{L}{2}]-1,0)$ condition. On the other hand, a 4D generalization of the toric code (i) has a code distance that grows as $\Theta(L^2)$ but (ii) satisfies the TQO condition with the TQO-radius $r = O(L)$.[40]

### 1.3.1 Stabilizer codes

An important class of quantum error-correcting code is the stabilizer code, which was first introduced by Gottesman.[110] One of the advantages of the stabilizer code is that it has an efficient description of the codewords in terms of a set of commuting operators. The quantum code can be described by a set of $n-k$ commuting operators, where $n$ is the number of qudits and $k$ is the number of encoded qudits. More specifically, the code is a set of states that are simultaneous $+1$ eigenstates of the *stabilizer group* elements.

The stabilizer group is defined as an Abelian subgroup of the Pauli group which does not contain $-I$ as its element. A similar construction can be carried out if the qubits are replaced with qudits

with a prime dimension.[5] The unitary generalizations of the $X$ and $Z$ operators are given as follows:

$$X_{ij} = 1 \quad j = i + 1 \mod d \tag{1.24}$$

$$= 0 \quad \text{otherwise.} \tag{1.25}$$

$$Z_{ij} = \omega^i \quad j = i \tag{1.26}$$

$$= 0 \quad \text{otherwise,} \tag{1.27}$$

where $\omega$ is the $d$th root of unity. We shall denote $X^{\alpha_1} Z^{\alpha_2}$ as $S_\alpha$ with a *symplectic pair* $\alpha = (\alpha_1, \alpha_2)$. As in the binary code case, $X$ and $Z$ satisfies a nontrivial commutation relation.

$$XZ = ZX\omega. \tag{1.28}$$

We define a *symplectic product*:

$$\langle \alpha, \beta \rangle = \alpha_1 \beta_2 - \alpha_2 \beta_1. \tag{1.29}$$

The symplectic product has two useful properties. First, a commutation relation of two generalized Pauli operators can be determined from a symplectic product of the symplectic pairs describing each of the generalized Pauli operators.

$$S_\alpha S_\beta = S_\beta S_\alpha \omega^{\langle \alpha, \beta \rangle}, \tag{1.30}$$

Second, in prime dimensions, a symplectic product of two symplectic pairs is 0 if and only if they are equivalent to each other up to a constant factor.

**Lemma 1.** *If $\alpha \neq (0, 0)$, and $d$ a prime number,*

$$\langle \alpha, \beta \rangle = 0 \tag{1.31}$$

*if and only if $\beta = a\alpha$ for some $a \in \mathbb{Z}_d$.*

For a *binary stabilizer code*, one can measure the *syndrome* of the code by measuring the stabilizer group elements. If the error rate is sufficiently low, one can make an intelligent guess on where the errors occurred by measuring these operations.[40] Alternatively, one can engineer a Hamiltonian whose ground state subspace is the code subspace of the stabilizer code:

$$H = -\sum_i s_i, \tag{1.32}$$

[5] The generalization can be straightforwardly carried out for any dimensions, but there are some subtleties that may arise for non-prime dimensions. In this thesis, the statements about the quantum error-correcting code will be translated to another statement about a vector space over a finite field. For prime dimensions, the corresponding finite field is easy to find: it is just $\mathbb{F}_d = \mathbb{Z}_d$. However, $\mathbb{Z}_d$ is not a field when $d$ is not a prime.

where $s_i$ is the generator of the stabilizer group. Of course, this construction works only if $s_i$ is Hermitian.

For a general qudit stabilizer code, one should hermitize the stabilizer generators in order to engineer a Hamiltonian whose ground state subspace is described by the code. If $d$ is an odd prime number,

$$P_\alpha(r) = \frac{1}{d}\sum_{m=0}^{d-1}(\omega^r S_\alpha)^m \tag{1.33}$$

is a complete set of orthogonal projections.[108] Therefore, a simultaneous $+1$ eigenstate of the following projector becomes the codeword of a qudit stabilizer code:

$$P_{s,r} = \frac{1}{d}\sum_{m=0}^{d-1} s^m \omega^{rm}, \tag{1.34}$$

where $s$ is the stabilizer generator. As in the standard stabilizer code, it does not matter which value of $r$ we choose for the error correction as long as the same convention is used throughout the analysis.

# Chapter 2

# Exactly solvable models

One of the simplest topologically ordered systems is Kitaev's toric code.[3] Consider a square lattice, where the qubits lie on the edges of the graph. Toric code can be formally defined as a set of degenerate ground states of the following Hamiltonian:

$$H = -\sum_s A_s - \sum_p B_p, \tag{2.1}$$

where $A_s$ is the "star" operator and $B_p$ is the "plaquette" operator. A star operator on a site $s$ is defined to be a tensor product of $X$ operators along the edges that are touching $s$. Similarly, a plaquette operator on a plaquette $p$ is defined to be a tensor product of $Z$ operators along the edges surrounding the plaquette. By the construction, one can easily see that all of the terms commute with each other. There are other properties of this system that can be easily verified, such as the braiding statistics and the ground state degeneracy, see Ref.[3].

While the toric code can reliably store a quantum information at zero temperature against a generic perturbation,[28, 29] it fails to do so at finite temperature.[39] This is due to the fact that it only takes a constant energy barrier to produce a logical operator, see FIG.2.2

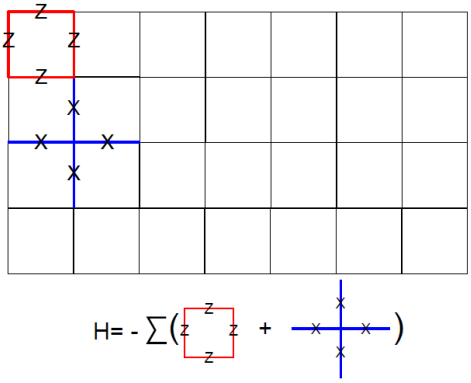
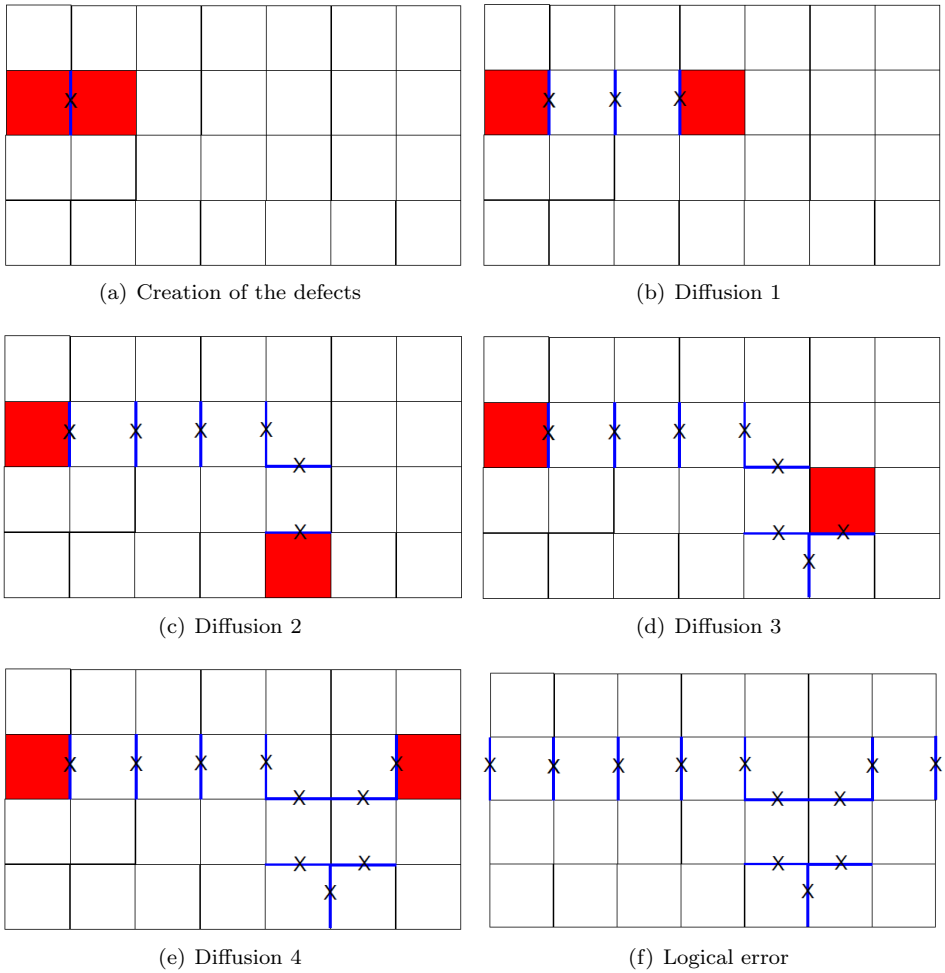



Figure 2.1: Kitaev's toric code



(a) Creation of the defects (b) Diffusion 1

(c) Diffusion 2 (d) Diffusion 3

(e) Diffusion 4 (f) Logical error

Figure 2.2: Defects can be created in pair, diffuse, and recombine to produce a logical operator. The energy barrier for this process is constant.

There is a straightforward generalization of the toric code to a three-dimensional system. This model is typically known as the three-dimensional (3D) toric code. Analogous to the construction of the toric code, 3D toric code is defined to be a set of degenerate ground states of the following Hamiltonian:

$$H = -\sum_{s} A_s - \sum_{p} B_p, \tag{2.2}$$

where $A_s$ and $B_p$ are defined similarly to the 2D toric code. More precisely, qubits reside on the edges of a cubic lattice. $A_s$ is a tensor product of the $\sigma_x$ operators meeting with a site $s$. $B_p$ is a tensor product of $\sigma_z$ operators surrounding a plaquette $p$. There is a general intuition that as the number of spatial dimensions increase, the order of the system becomes increasingly stable.[111] A similar intuition holds for toric code too. While the toric code loses its topological protection under a thermal noise, 3D toric code can store a *classical* information at a sufficiently low temperature.[43] Unfortunately, this is not enough to ensure the protection of the quantum information, since the

information encoded in the ground state cannot be protected against a phase flip error. An easy way to see this is to consider a sequence of $\sigma_z$ operators that makes a noncontractible loop. Once the defects are created, they can diffuse freely without paying any extra energy cost. There are other models that have similar properties to the 3D toric code, in that their Hamiltonian can be manifestly divided into two parts. One part is analogous to the star operators, responsible for the protection against the phase flip error. The other part is responsible for the protection against the bit flip error. Section 2.1 provides a model that evades such a natural decomposition. However, the conclusion is that these models also have a finite energy barrier.

## 2.1 XYZ-plaquette models

We place qubits on vertices of a 4-valent 3D lattice. The stabilizer group is generated by the following operators:

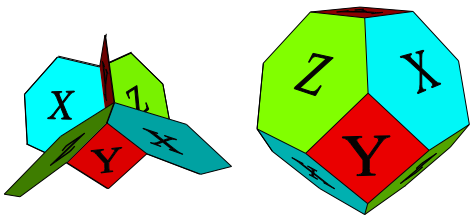
$$B_p^x = \Pi_{i \in p} X_i \tag{2.3}$$

$$B_p^y = \Pi_{i \in p} Y_i \tag{2.4}$$

$$B_p^z = \Pi_{i \in p} Z_i, \tag{2.5}$$

where $p$ is the plaquette and $\{i \in p\}$ denotes a set of vertices on a plaquette $p$. We shall partition a set of plaquettes into $P_x(P_y, P_z)$, which corresponds to a set of nontrivial supports for $B_p^x(B_p^y, B_p^z)$. We shall call elements of these sets as $X-(Y-, Z-)$plaquettes.

This model is inspired by the construction of the 3D topological color code.[42] For the topological color code, qubits reside on the vertices of a 3D lattice, and the lattice is 4-valent. The stabilizer generators are either a product of $X$s or a product of $Z$s, and they correspond to the unit cells of different dimensions; in one example, generators are either in cubic form or plaquette form. Our approach differs from theirs in a sense that we only allow plaquette operators as stabilizer generators.




(a) Vertex Figure (b) Unit Cell

Figure 2.3: The vertex figure and the unit cell of our model. Qubits reside on the vertices. One can see that $B_{p_x}^x$ meets with another $B_{p_x}^x$ at one vertex, whereas it meets with $B_{p_y}^y$ and $B_{p_z}^z$ at two vertices.

A local description of our model can be seen in Figure 2.3(a). At each vertices, there are

6 plaquette operators that share a nontrivial support. Each plaquette operators meet with the same kind of plaquette operator on each vertices and meet with 4 other plaquette operators on 2 vertices. Thus the assignment in Figure 2.3(a) guarantees the commutativity between the stabilizer generators. We must point out that not every lattice structure allows vertex figures like Figure 2.3(a). There are only 4 translationally invariant convex tessellations that have tetrahedral vertex figure: bitruncated qubic honeycomb, cantitruncated cubic honeycomb, omnitruncated cubic honeycomb, and cantitruncated alternated cubic honeycomb.[112] Only the first three admits an arrangement of plaquette operators similar to Figure 2.3(a) at every vertex. Here we mainly study the bitruncated qubic honeycomb model for its simplicity, but analogous results shall be discussed in full generality if possible. The unit cell is shown in Figure 2.3(b) and its tessellation is shown in Figure 2.4. The bitruncated cubic honeycomb is a space-filling tessellation made up of truncated octahedra. It has 14 faces, 36 edges, and 24 vertices. There are 6 square faces and 8 hexagonal faces. Without loss of generality, one can set the 6 square faces to be the $Y$ plaquette operators, 4 of the hexagonal faces to be the $X$-plaquette operators, and the remaining hexagonal faces to be the $Z$-plaquette operators. Hamiltonian is a sum over these plaquette operators.

$$H = -J\Big(\sum_{p_x \in P_x} B^x_{p_x} + \sum_{p_y \in P_y} B^y_{p_y} + \sum_{p_z \in P_z} B^z_{p_z}\Big). \tag{2.6}$$

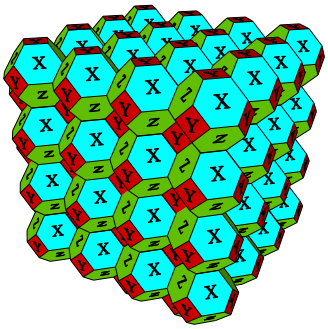


Figure 2.4: Arrangement of the stabilizer generators. The translation of unit cells form a tessellation.

### 2.1.1 Code subspace

Euler characteristic $\chi$ is defined as an alternating sum of $k_n$s, where $k_n$ denotes a number of cells of dimension $n$.

$$\chi = \sum_{i=0}^{d} k_i(-1)^i \tag{2.7}$$

One of the main ideas that we used here is that $\chi$ can be also written as an alternating sum of Betti number $b_i$s.

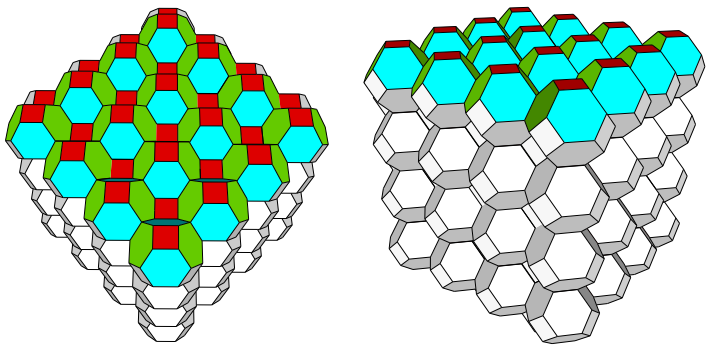
$$\chi = \sum_{i=0}^{d} b_i(-1)^i \tag{2.8}$$

$b_i$ is the rank of the $n$-th singular homotopy group. We briefly explain the Poincaré duality. Although it has different forms depending on the context, here we use the one originally introduced by Poincaré himself.

**Theorem 1.** *(Poincaré, 1895)* $b_k = b_{d-k}$ *for a closed orientable* $d$*-dimensional manifold.*

A number of encoded qubits can be computed from the size of the stabilizer group and the number of physical qubits. Since the plaquette operators are not independent to each other, we must count the number of independent relations. In such a pursuit, a geometrical interpretation of our model becomes useful. Note that multiplying all the plaquette operators on a unit cell reduces to an identity, see Figure 2.3(b). Since any contractible closed surface on the lattice can be represented as a union of unit cells, one can see that a multiplication of the plaquette operators on *any* contractible closed surface reduces to the identity. Therefore we have $C-1$ independent relations which generate smooth deformations, where $C$ is the number of unit cells. We must subtract 1 because multiplying all but one cell results in a relation for that very cell.

Let us consider a periodic boundary condition in all 3 directions. There exists a noncontractible surface that reduces to the identity as one can see in Figure 2.5(a) and Figure 2.5(b). Since there are 3 topologically distinct noncontractible surfaces, we have 3 independent relations, resulting in $C+2$ independent relations. Finally, multiplying all $X$-like operators adds one independent relation. One can check that multiplication of $Y$s and multiplication of $Z$s are implied by the previously mentioned relations.



(a) Top view (b) Side view

Figure 2.5: Representation of the nontrivial constraints between the stabilizer operators. One can see that the multiplication of all the plaquette operators on a noncontractible closed surface reduces to the identity. At each vertices, there are either 1) exactly one X, one Y, and one Z or 2) two $X$s and two $Z$s.

Accounting for these relations, the number of encoded qubits is $V - F + C + 3 = 3$ under a periodic boundary condition, where $V$ is the number of vertices, $F$ is the number of faces, and $C$ is the number of unit cells. The first two correspond to the number of qubits and the number of plaquette operators. The remaining terms represent a number of independent relations between plaquette operators. We shall show that the number of encoded qubits only depend on the second Betti number.

**Lemma 2.** *For a stabilizer group* $\{B^x_{p_x}, B^y_{p_y}, B^z_{p_z}\}$*, the number of encoded qubits is* $b_2$*.*

**Proof :** Let us consider the dual lattice. This can be constructed by replacing the $k$-dimensional object to a $(d-k)$-dimensional object. For instance, a vertex of the dual lattice resides on the center of the unit cells of the original lattice. A face on the dual lattice can be constructed by connecting the edges so that the resulting surface is perpendicular to the edges of the original lattice. The Euler characteristic is trivially 0 due to the Poincaré Duality. The unit cells of the resulting dual lattice is an irregular tetrahedron. Let us denote $k_i$s to be the number of $i$-dimensional cells on the dual lattice. The total number of vertices in the original lattice becomes $k_3$, which is the number of unit cells in the dual lattice. Similarly, $F$ is identical to $k_1$ and $C$ is identical to $k_0$. Note that $k_2 = k_3$, since each cell contains 4 faces and each faces meet with two tetrahedral cells. Therefore, we have

$$V - (F - C) = k_3 - k_1 + k_0 \tag{2.9}$$

$$= -k_3 + k_2 - k_1 + k_0 = 0. \tag{2.10}$$

Hence

$$k = V - (F - (C - 1 + 1 + b_2)) \tag{2.11}$$

$$= b_2, \tag{2.12}$$

where $b_2$ is the second Betti number of the manifold. One can also use this intuition to prove that the group generated by the plaquette operators does not contain $-I$.

**Lemma 3.** $\langle B^x_{p_x}, B^y_{p_y}, B^z_{p_z} \rangle$ *does not contain* $-I$*.*

**Proof:** Consider a product of plaquette operators that are proportional to the identity operator. Any such configuration can be generated by a product of all the $X$-plaquette operators, a product of all the $Y$-plaquette operators, a product of all the $Z$-plaquette operators, and a product of plaquettes along a closed surface. The first three are trivially $+I$. For a unit cell, we have 24 vertices at which $X, Y$, and $Z$ meet each other. Since all the generators commute with each other, we can arrange the

product in the following canonical form.

$$\Pi_{p_x} B^x_{p_x} \Pi_{p_y} B^y_{p_y} \Pi_{p_z} B^z_{p_z}. \tag{2.13}$$

Since $XYZ = i$, the product of plaquette operators on a unit cell is 1. Similarly, for the product of plaquette operators on a noncontractible surface described in Figure 2.5(a) and Figure 2.5(b), we have $4n$ vertices where $X, Y$, and $Z$ meet each other. Hence we arrive at the same conclusion. Since any product of the plaquette operators that results in a trivial operator can be constructed by these constraints, the group does not contain $-I$.

There are two logical operators that are reminiscent to the surface and the string operator of the 3D toric code. These are drawn in Figure 2.6. One can see the surface operator on the top of the lattice system, which is a product of $B^z_{p_y}$s on a layer of $Y$-plaquettes. The complementary logical operator is the string operator that has a sequence of $YZYXYZYXYZYX\cdots$ along the line perpendicular to the surface operator. This string winds around the torus and completes a noncontractible loop. These two operators anticommute with each other and both of them commute with the stabilizer generators. One can similarly define two sets of complementary operators in other

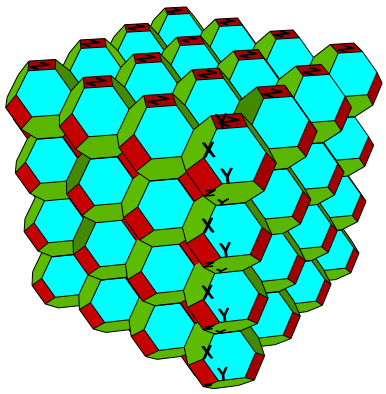



Figure 2.6: There is one surface operator and one string operator for each qubits. The surface operator corresponds to the product of $ZZZZ$ on $Y$-plaquettes. The string operator is the line perpendicular to this surface, with a sequence that goes as $YZYXYZYX\cdots$.

directions. One can easily check the expected commutation and anticommutation relations.

### 2.1.2 Low-energy excitation

Quasi-particle excitations in two-dimensional gapped systems are believed to have anyonic properties. For the case of the toric code, there are two types of particles that are named as the "electric" and the "magnetic" charge. If the electric charge winds around the magnetic charge, the wavefunction

attains a nontrivial phase. In 3D, one can always contract the trajectory of the loop to a point, unless the loop winds around the torus. Hence, one needs higher dimensional objects to attain a nontrivial braiding statistics. In 3D, there are closed string-like excitations and particle-like excitations.[113, 43] When the particle winds around the string, the system attains a nontrivial phase.

Despite the fact that our model is made up of only plaquette operators, it shows a similar behavior. A pair of particle-like excitations can be created from the vacuum by a constant energy. If we truncate a string-like logical operator, excitations form at the end points. When the particle-antiparticle pair is created, they can diffuse without any extra energy cost. The closed string-like excitations can be similarly thought as a truncated surface-like logical operator. There are excitations near the boundary of the constraint. Therefore, the energy cost grows linearly with the size of the surface. If a particle penetrates the closed string, we find that

$$|\psi_{Initial}\rangle = SP\,|\Phi\rangle \tag{2.14}$$

$$|\psi_{Final}\rangle = USP\,|\Phi\rangle = -\,|\psi_{Initial}\rangle\,, \tag{2.15}$$

where $S$ is the closed-string excitation, $P$ is the particle excitation, and $U$ is a trajectory of the particle. Therefore, the system picks up a phase of $e^{i\pi}$ in this process. This is illustrated in Figure 2.7. One can see that as the particle penetrates through the surface operator and returns to the original position, it meets with the surface operator at one vertex, giving the anticommutation relation.

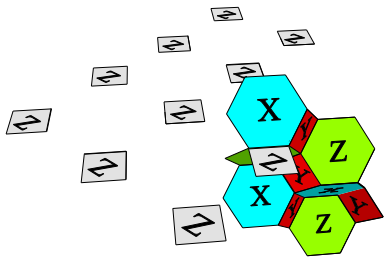



Figure 2.7: A representation of a particle penetrating through a string-like excitation. The truncated surface operator is a product of $Z$-plaquettes. The trajectory of the particle is the nontrivial support of the colored plaquette operators, which meets with the $Z$-surface at a point.

The low-energy excitation provides an intuitive picture for the thermal stability of the XYZ-plaquette model. Particles can be created out of vacuum in pair and propagate freely. They can diffuse and wind around the torus to produce a logical error. On the other hand, a closed string has an energy cost that is proportional to its perimeter. Given a closed string-like excitation as in Figure 2.7, the stabilizer generators anticommuting with the surface operator only reside near the

boundary of the surface. $Z$-plaquettes trivially commute with the surface operator. $X$-plaquettes commute with the surface operator since they meet at two vertices. However, there are $Y$-plaquettes meeting at exactly one vertex at the boundary. Hence we expect our system to be a stable classical memory.

### 2.1.3 Duality

Typically, a strong-weak duality relation relates a strong coupling limit of one model to a weak coupling limit of another model. We use a slightly different strategy here. We first show that our model can be mapped into an Ising gauge theory, from which we can use the Wegner-type duality relation with an Ising model.[114] Mapping from our model to the Ising gauge theory is not exact for a finite sized lattice, but this difference vanishes in the thermodynamic limit. Starting from the partition function of our model,

$$Z = \mathrm{Tr}(\exp(-\beta H)) \tag{2.16}$$

$$= \mathrm{Tr}(\Pi_{S_i \in S}(\cosh \beta J + S_i \sinh \beta J)), \tag{2.17}$$

where $S_i \in \{B^x_{p_x}, B^y_{p_y}, B^z_{p_z}\}$,

$$Z = (\cosh \beta J)^n tr(\Pi_i (1 + \alpha S_i)) \tag{2.18}$$

$$= (\cosh \beta J)^n tr(\sum_{\{k_i\}=0}^{1} \Pi_i \alpha^{k_i} S_i^{k_i}). \tag{2.19}$$

Note that there were two kinds of constraints: the constraints coming from the closed 2-manifold and the constraints coming from the space-filling products of $X$, $Y$s, and $Z$s. Therefore, we can write down the partition function in the following form:

$$Z = (2\cosh \beta J)^n (\sum_c \alpha^{A_c} + (1+\alpha^{n_x})(1+\alpha^{n_y})(1+\alpha^{n_z}) - 1 + C.T.). \tag{2.20}$$

Here $\sum_c$ is a sum over the configurations of the closed 2-manifolds. $A_c$ is the number of plaquettes for each configurations. $C.T.$ corresponds to the cross terms between the closed 2-manifolds and the space-filling products of $X$s, $Y$s, and $Z$s. $n_{x,y,z}$ corresponds to the number of $X, Y, Z-$plaquette operators. The main idea is that the partition function is dominated by the first term in the thermodynamic limit. The cross terms can be written as

$$C.T. = \sum_c \alpha^{A_c} \sum_{i \in \{x,y,z\}} \alpha^{n_i - 2n_i^c}, \tag{2.21}$$

where $n_x, n_y, n_z$ are the number of $X, Y, Z-$plaquettes and $n_x^c, n_y^c, n_z^c$ are the number of $X, Y, Z-$plaquettes

for a configuration $c$.

**Lemma 4.** *There exists an* $0 < \epsilon_{1,2} < 1$ *such that*

$$A_c + n_i - 2n_i^c \geq \epsilon_1 A_c + \epsilon_2 n_i \tag{2.22}$$

*for* $\forall c, i$.

**Proof :** Consider $i = x$. The left hand side of the inequality is

$$n_y^c + n_z^c - n_x^c + n_x \geq n_y^c + n_z^c - (1-\epsilon)n_x^c + (1-\epsilon)n_x \tag{2.23}$$

$$\geq (\frac{\epsilon}{2})A_c + (1-\epsilon)n_x \tag{2.24}$$

On the second line, we used the fact that the minimum is achieved for $n_y^c = 0$.

$$n_z^c = n_x^c = \frac{1}{2}A_c \tag{2.25}$$

The same logic can be applied to $i = z$. For $i = y$,

$$n_x^c + n_z^c - n_x^c + n_y \geq n_x^c + n_z^c - (1-\epsilon)n_y^c + (1-\epsilon)n_y \tag{2.26}$$

$$\geq (\frac{2}{5} - \frac{3}{5}(1-\epsilon))A_c + (1-\epsilon)n_y. \tag{2.27}$$

Similarly, we used the fact that the minimum is achieved if one of $n_x^c$ or $n_z^c$ is 0. Then we have a $2:3$ ratio between the $X-(Z-)$plaquettes and $Y-$plaquettes. Therefore, for $\epsilon > \frac{1}{3}$, we have such $(\epsilon_1, \epsilon_2)$.

**Lemma 5.**

$$\lim_{vol\to\infty} \frac{Z(\beta J)}{Z_{IG}(\beta J)} \to 1. \tag{2.28}$$

*, where* $Z_{IG}(\beta J)$ *is a partition function for the Ising gauge theory with a temperature* $\beta$ *and a coupling constant* $J$*. vol is the volume of the lattice.*

**Proof :**

We use the following equation:

$$\sum_c \alpha^{\epsilon_1 A_c} = \frac{(2\cosh\beta J')^n}{(2\cosh\beta J')^n} \sum_c \alpha'^{A_c} \tag{2.29}$$

$$= (\frac{1}{2\cosh\beta J'})^n Z_{IG}(\beta J'), \tag{2.30}$$

where

$$\tanh\beta J' = (\tanh\beta J)^{\epsilon_1}. \tag{2.31}$$

Thus the cross terms can be bounded by

$$Z_{IG}(\beta J')(\frac{\cosh \beta J}{\cosh \beta J'})^n \alpha^{\delta_i \epsilon_2 n}, . \tag{2.32}$$

where $\delta_i = \frac{n_i}{n}$, where $n$ is the total number of plaquettes. This expression becomes

$$Z_{IG}(\beta J')((\frac{1-t^2}{1-t^{\frac{2}{\epsilon_1}}})^{\frac{1}{2}} t^{\frac{\epsilon_2}{\delta \epsilon_1}})^n, \tag{2.33}$$

where $t = \tanh \beta J'$. One can show that $(\frac{1-t^2}{1-t^{\frac{2}{\epsilon_1}}})^{\frac{1}{2}} t^{\frac{\epsilon_2}{\epsilon_1 \delta}} < 1$ for $\beta J > 0$. Since the renormalized coupling constant $J'$ is larger than $J$, these correction terms become negligible in the thermodynamic limit. Therefore,

$$|\lim_{vol \to \infty} \frac{Z(\beta J) - Z_{IG}(\beta J)}{Z_{IG}(\beta J)}| \leq |\frac{Z_{IG}(\beta J')}{Z_{IG}(\beta J)} \lambda^n + O(\alpha^n)|, \tag{2.34}$$

where $J' > J$ and $0 < \lambda < 1$. In $n \to \infty$ limit, we get the desired result.

**Lemma 6.** *$Z - C.T. - (\alpha^{n_x} + \alpha^{n_y} + \alpha^{n_z}) = Z_{IG}(\beta J)$, where $Z_{IG}$ is a partition function of the Ising gauge theory on the same lattice with a temperature $\beta$ and a coupling constant $J$.*

**Proof :** Consider a mapping $B^x_{p_x} \to ZZZZZZ$, $B^y_{p_y} \to ZZZZ$, $B^z_{p_z} \to ZZZZZZ$, where $Z \cdots Z$ are products of $Z$ on the edges of each plaquettes. The resulting model is an Ising gauge theory on a bitruncated cubic honeycomb. The partition function is

$$Z_{IG} = tr(\exp(-\beta H)) \tag{2.35}$$

$$= (\cosh \beta J)^n tr(1 + \tanh \beta J S_i), \tag{2.36}$$

where $S_i$s are either $ZZZZZZ$ or $ZZZZ$ depending on the plaquette. Since the Pauli operators are traceless, a product of the plaquette operators survives only if the union of the plaquettes form a union of closed manifolds.

$$Z_{IG}(\beta J) = Z - C.T. - (\alpha^{n_x} + \alpha^{n_y} + \alpha^{n_z}). \tag{2.37}$$

Using the duality relation between the Ising gauge theory and the Ising model, we can map our model into an Ising model.

**Lemma 7.** *Ising gauge theory on the bitruncated cubic honeycomb is dual to the Ising model on its dual lattice.*

Proof:

$$Z = (\cosh \beta J)^n tr(\Pi_i (1 + \tanh \beta J S_i)) \tag{2.38}$$

$$= (\cosh \beta J)^n tr(\sum_{\{k_i\}=0}^{1} \Pi_i \alpha^{k_i} S_i^{k_i}) \tag{2.39}$$

$$= (2 \cosh \beta J)^n \sum_{\{k_i\}=0}^{1} \Pi_i \alpha^{k_i} \Pi_e \delta_2 (\sum_j k_{j;e}), \tag{2.40}$$

where $\Pi_e$ is a product over all the edges and $\sum_j k_{j;e}$ is a sum over $k_j$s that have nontrivial support on an edge $e$. There are three such $k_j$s. One can use $k_{j;e} = \frac{1}{2}(1 - ZZ)$, where $ZZ$ is a product of $Z$s on qubits that reside on the vertices of the dual lattice. For 8 spin configurations $(Z_1, Z_2, Z_3) = (-1,-1,-1)$, $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$, $(1,1,-1)$, $(-1,1,1)$, $(1,-1,1)$, they all satisfy the delta function. Furthermore, we have 2 combinations for $(k_1, k_2, k_3) = (0,0,0)$, 2 combinations for $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$. Plugging these relations in, we get

$$Z = (\cosh \beta J)^n \sum_{\{Z_i\}=0}^{1} \Pi_i \alpha^{1 - \frac{1}{2} Z_{i+\hat{n}_i} Z_{i-\hat{n}_i}}, \tag{2.41}$$

where $Z_{i \pm \hat{n}_i}$ is the $Z$ operator on the dual sites of the plaquette $i$. $\hat{n}_i$ is the unit normal vector to the plaquette. Therefore, up to a constant, the partition function is equal to the partition function of the Ising model with $\tilde{\beta J} = -\frac{1}{2} \ln \tanh \beta J$.

**Theorem 2.** *The XYZ-plaquette model with a coupling constant $\beta J$ is dual to the classical Ising model on a dual lattice with a dual coupling constant $\tilde{\beta J} = -\frac{1}{2} \ln \tanh \beta J$.*

Since the Ising model undergoes a finite temperature phase transition, so does our model. This is analogous to the behavior of the 3D toric code under a temperature change. As in our model, one can show that the 3D toric code has a critical temperature by using the duality relation with the Ising model. Below the critical temperature, there is a symmetry breaking with respect to a surface-like logical operator. The symmetry associated to the string-like logical operator is broken only at the ground state.

One important difference though, is that the 3D toric code can be decomposed into two classical Hamiltonians without spoiling the phase transition: the Hamiltonian responsible for correcting the bit flip error is identical to the Ising gauge theory, which has a finite temperature phase transition. On the other hand, the Hamiltonian responsible for correcting the phase flip error does not have a phase transition. Therefore, one can intuitively understand that the 3D toric code can only correct bit flip errors but not phase flip errors under the influence of a thermal bath. Our model does not allow such a decomposition. Once we get rid of any of the $B^x_{p_x}, B^y_{p_y}$, or $B^z_{p_z}$, the partition function does not exhibit a phase transition anymore. This shows that non-CSS code with a finite

temperature phase transition in 3D does not necessarily provide a self-correcting quantum memory.

## 2.2 No-string rule

One of the defining properties of Haah's code is that it has no string-like logical operator.[48] Since we are dealing with a lattice system, one needs to precisely define what it means for an operator to be a string. Since this is an important concept, we first reiterate some of the definitions introduced by Haah.[48]

**Definition 10.** *(Haah 2011) A set of sites* $\{p_1, p_2, \cdots, p_n\}$ *is a* path *joining* $p_1$ *and* $p_n$ *if for each pair* $(p_i, p_{i+1})$ *of consecutive sites there exists a stabilizer generators that acts nontrivially on their pair simultaneously, for* $i = 1, \cdots, n-1$*. A set* $M$ *of sites is* connected *if every pair of sites in* $M$ *are joined by a path in* $M$*. A* connected Pauli operator *is a Pauli operator with connected support.*

**Definition 11.** *(Haah 2011) Let* $\Omega_1, \Omega_2$ *be congruent cubes consisting of* $w^3$ *sites, and* $O$ *be a finite Pauli operator. A triple* $\eta = (O, \Omega_1, \Omega_2)$ *is a logical string segment if every stabilizer generator that acts trivially on both* $\Omega_1$ *and* $\Omega_2$ *commutes with* $O$*. We call* $\Omega_{1,2}$ *the* anchor*. The* directional vector *of* $\eta$ *is the relative position of* $\Omega_1$ *to* $\Omega_2$*. The* length *is the* $l_1$*-length of the directional vector, and the* width *is* $w$*.*

**Definition 12.** *(Haah 2011) A logical string segment* $\eta = (O, \Omega_1, \Omega_2)$ *is* connected *if there exists two sites* $p_1 \in \Omega_1$*,* $p_2, \Omega_2$ *that can be joined by a path in* $supp(O) \cup \{p_1, p_2\}$*, where* $supp(O)$ *is a set of sites on which* $O$ *acts nontrivially. Two logical string segments* $(O, \Omega_1, \Omega_2)$*,* $(O', \Omega_1, \Omega_2)$ *are equivalent if* $O'$ *can be obtained from* $O$ *by multiplying finitely many stabilizer generators.* $\eta$ *is nontrivial if every equivalent logical string segment is connected.*

We say that a quantum code has no string if, given a bounded width $w$, the length $l$ of the logical string segment is bounded by a function of $w$. On the other hand, a quantum code has a string if such bound does not exist. Consider a toric code for an example. Given a set of defects, one can always move around the defects freely by applying a sequence of Pauli operators. Therefore, the length of the logical string segment is formally unbounded. On the other hand, consider a 4D generalization of the toric code.[40] For such models, one cannot move a defect without paying an extensive energy cost. For such models, the length of the logical string segment is $O(1)$.

Haah's code is special in a sense that the length of the logical string segment is bounded by a function that *grows* with the width $w$. In particular, one of his codes(Code I) has a bound that grows linearly with $w$. Further, this bound is tight in a sense that there exists a string segment that matches this lower bound with a multiplicative factor.[48] Bravyi and Haah were able to exploit this structure to obtain a rigorous energy lower bound for the logical error.[49]

The existence of Haah's code undoubtedly raises a lot of interesting questions. For example, an interesting question to ask is if there exists models that share the same properties of Haah's code. The energy lower bound of Bravyi and Haah is only based on the bound on the logical string segment length that grows linearly with its width. Once a family of models satisfying these conditions are found, the extensive energy barrier should follow trivially.

## 2.3 3D local qupit code

We consider a qudit stabilizer code that is supported on a 3D square lattice. The qudits are located at the vertices of the lattice. Recall that there were some complications that may arise when the dimensions of the particle is not a prime number. Due to this problem, we shall simply assume that the dimension is a prime number, hence the name qupit. In this setting, the stabilizer generator is described in Figure 2.8. The stabilizer group is generated by the translation of these generators in three different directions.

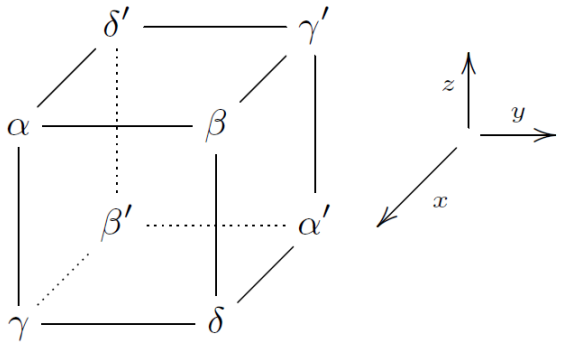



Figure 2.8: A stabilizer generator before enforcing any assumption

We assume that the stabilizer generators commute with each other. This assumption is necessary to use the stabilizer group formalism. Given a cube, set the middle of the cube to be the origin. Since two cubes can meet each other at a single vertex, the generalized Pauli operators located on the vertices that are diagonal to each other must commute with each other. Since

$$\langle \alpha, \beta \rangle = 0 \tag{2.42}$$

if and only if $\alpha = a\beta$ for some $b \in \mathbb{Z}_p$, the Pauli operators that are diagonally opposite with respect to the origin must be described by the same symplectic pair up to a multiplicative constant. The resulting code parameter is described in Figure 2.9. Similarly, two cubes can meet each other on an edge. Enforcing the commutation conditions on the edges leads to two types of stabilizer generators, see Figure 2.10.

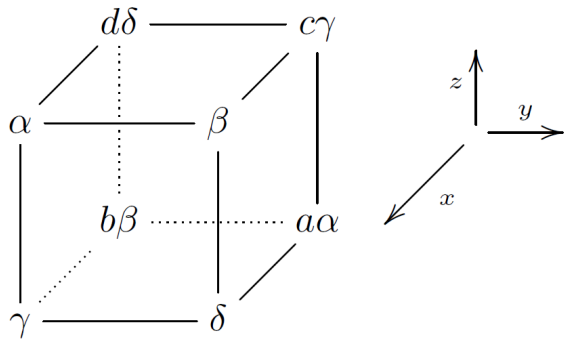



Figure 2.9: A stabilizer generator after enforcing the commutation relation at the vertices.

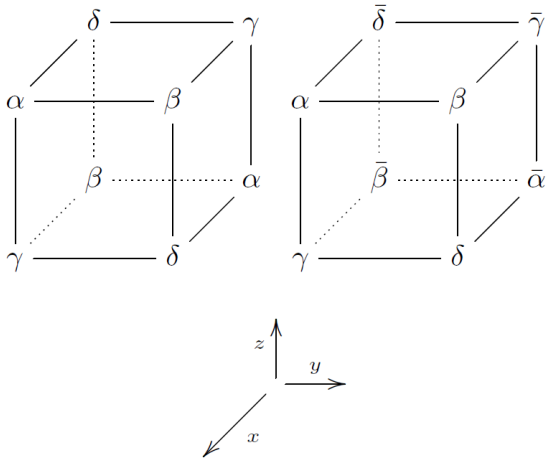



Figure 2.10: Stabilizer group generators for $\mathcal{C}_S^{\alpha\beta\gamma\delta}$ and $\mathcal{C}_A^{\alpha\beta\gamma\delta}$.

Under the aforementioned constraints, one can see that the quantum code is described by 4 symplectic pairs $\alpha, \beta, \gamma, \delta$ and the symmetric/antisymmetric nature of the code. We shall denote each of these codes as $\mathcal{C}_{A,S}^{\alpha\beta\gamma\delta}$, where $A$ stands for the antisymmetric code and $S$ stands for the symmetric code. Without loss of generality, we shall assume that the system has a periodic boundary condition, with a fixed length in all three directions equal to $L$.

To compare our code to Haah's code, Haah's code has two local stabilizer generators per cube, which corresponds to the generators responsible for the protection against the bit flip and the phase flip error. Our code has one stabilizer generators for each cube. Some of Haah's code is a CSS code, but all of our codes are non-CSS. Perhaps more importantly, the local particle dimension of Haah's code is $2^2$, while for our code it is a prime number $p$. We shall in fact see that $p = 2$ inevitably leads to an existence of string logical operator, which confirms the numerical result by Haah.[48] As we shall see throughout the rest of this chapter, the main difference comes from the structure of the base field: the base field for our code is GF[$p$], while it is GF[$2^2$] for Haah's code.

Our main contribution is a discovery of a simple sufficient condition for checking the absence

of any string logical operator. Given a stabilizer code with cubic local generators, no string rule is implied by a simple algebraic constraint on the parameters of the code over a finite field $\mathbb{F}_p$. The existence of the quantum code without string logical operator for $p \geq 5$ follows from this result.

Another important point to discuss is that the codes described by a different set of symplectic pairs may give rise to the same code. Of course, the actual codeword of the quantum code will be not identical. However, they may be equivalent to each other under a local unitary transformation.[1] If two quantum codes $\mathcal{C}_1, \mathcal{C}_2$ can be mapped to each other via such local unitary transformation, we shall denote their equivalence with the following notation:

$$\mathcal{C}_1 \cong \mathcal{C}_2. \tag{2.43}$$

Any two codes are equivalent to each other if they can be mapped by a lattice symmetry or a local unitary transformation. The lattice symmetry can be concisely represented as a permutation of the symplectic pairs $\alpha, \beta, \gamma, \delta$. The following lemma trivially follows from the definition of the code: the exchange of the symplectic pairs correspond to the rotation in the 3-space.

**Lemma 8.**

$$\mathcal{C}_{A,S}^{\alpha\beta\gamma\delta} \cong \mathcal{C}_{A,S}^{\sigma(\alpha)\sigma(\beta)\sigma(\gamma)\sigma(\delta)}, \tag{2.44}$$

*where $\sigma \in S_4$ over a set $\{\alpha, \beta, \gamma, \delta\}$.*

Any local Clifford transformation can be represented as an element of $SL(2, p)$.[115] One should also note that multiplying a nonzero element $a$ of GF[$p$] does not change the code. It corresponds to merely changing the stabilizer element $s$ into $s^a$.

**Lemma 9.** *If $\exists a \in \mathbb{F}_p, M \in SL(2, p)$ such that $aM\{\alpha, \beta, \gamma, \delta\} = \{\alpha, \beta, \gamma, \delta\}$*

$$\mathcal{C}_{A,S}^{\alpha\beta\gamma\delta} \cong \mathcal{C}_{A,S}^{\alpha'\beta'\gamma'\delta'}, \tag{2.45}$$

Finally, there is a subtle equivalence relation between the antisymmetric and the symmetric code. Instead of performing the same local unitary operation on all the qudits, one can imagine performing a unitary transformation on the even (or equivalently, odd) layer only, mapping $A \to -A$ for $A \in \{\alpha, \beta, \gamma, \delta\}$. This maps the symmetric code to the antisymmetric code and vice versa *in the bulk*. However, if the length in the direction normal to these layers is odd, such an operation is ill-defined. In other words, there exists a unitary operation that relates the antisymmetric and symmetric code in the bulk if $L$ is an even number.

Combining these equivalence relations together, we can get the following result.

[1] Typically, local unitary transformation in this setting is used in a much stronger sense. More specifically, a local unitary transformation is a quantum circuit with a bounded width and depth. Here we consider a simpler version, in that the width and the depth of the circuit are both 1.

**Lemma 10.** *For $d = 3$ code satisfying the deformability condition, there are two symmetric code and two antisymmetric code up to lattice symmetry and local Clifford operation. The parameters of the codes are $\{(1,0),(0,1),(1,1),(1,-1)\}$ and $\{(1,0),(0,1),(1,1),(-1,1)\}$.*

### 2.3.1 Sufficient condition for the no-string rule

Recall that the existence of a string logical operator is the *bulk property* of the code. That is, the formulation of the no-string rule does not involve anything about the boundary condition. Since there always exists an one-to-one correspondence between a symmetric code and an antisymmetric code that are described by the same symplectic pairs, it suffices to study only one of them for checking the absence of any string logical operator. If an antisymmetric code does not have any string logical operator, neither does the symmetric code with the same symplectic pairs. We shall obtain a sufficient condition for the code to not have any string logical operator for an antisymmetric code. The same statement for the symmetric code should follow trivially from this correspondence.

We first state our main result.

**Theorem 3.** *For $\mathcal{C}_{S,A}^{\alpha\beta\gamma\delta}$, the maximum length of a nontrivial string logical operator with a width $w$ is bounded by $2w$, if the following conditions are satisfied.*

- *Deformability condition :* $\langle A, B\rangle \neq 0 \ \forall A \neq B,\ A, B \in \{\alpha, \beta, \gamma, \delta\}$
- *Absence of minimal string :* $\det(T - T^{-1}) \neq 0$ *for* $T = T_{\delta\alpha}^{\gamma\beta}, T_{\alpha\beta}^{\delta\gamma}, T_{\alpha\delta}^{\gamma\beta}$.
- $\langle A, B\rangle^2 \neq \langle C, D\rangle^2 \ \forall A, B, C, D \in \{\alpha, \beta, \gamma, \delta\}$. *$A, B, C, D$ are distinct.*

The proof is quite technical, so we would like to give a brief overview. First, we note that there is a canonical way in which an arbitrary 3-dimensional logical string segment can be deformed into a quasi-2-dimensional surface. Such deformation procedure was originally used by Haah and Preskill[36], and it was also later used by Haah in finding his code.[48] Next, we write down a number of constraints that share a nontrivial support with the deformed quasi-2-dimensional surface. By counting the number of constraints that are independent to each other, we will be able to obtain the bound.

We note in passing that Haah's code is rather special in this regard. Given a string segment, deforming it into a canonical form of quasi-2-dimensional surface is a rather straightforward task. The main difficulty arises in the second step of the proof. Haah was able to show that for his code, the second part of the proof comes fairly easily as well. However, the same proof technique cannot be used to his other codes.[2]

[2] While there is only one code that is known as Haah's code, he actually proposed other codes without string logical operators in Ref.[48]. Given a finite anchor, the string segment of Haah's other codes are bounded by some function that grows superlinearly with the width of the anchor. On the other hand, Haah's code has a linear bound.

Suppose there exists a string logical operator whose support is confined on a cross section with a height $h$ and a width $w$. Certain models allow such string operators to be deformed to a flat surface. We will first explain the procedure, and then see what kind of condition is necessary for such a procedure to be possible. Suppose the logical operator is supported on a $h \times w \times l$ cylinder, where $l$ is the length in the direction perpendicular to the cross section. Pick one of the sites on the edge of the cube. Two stabilizer generators in the cylinder share a nontrivial support with this site. Multiply the logical operator with a combination of these stabilizer generators so that the resulting operator acts trivially on that site. This procedure is possible if

$$\det \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \beta_1 & -\beta_2 \end{pmatrix} \neq 0, \tag{2.46}$$

or alternatively, $\langle \alpha, \beta \rangle \neq 0$, where $\alpha, \beta$ are two symplectic pairs that share a nontrivial support with the site. Applying the same logic to other directions, we conclude that any two symplectic pairs $\alpha, \beta$ lying on a same edge must satisfy $\langle \alpha, \beta \rangle \neq 0$.

Since we started from a logical operator and multiplied it by the stabilizer group element, this operator must still commute with all the stabilizer generators. In particular, note that there are stabilizer generators that share a nontrivial support with the string logical operator at a single site. Using Lemma 1, one can see that the only way to get rid of the nontrivial support on these sites is to force the relation $\alpha = a\alpha'$ for some $a \in \mathbb{Z}_d$. Applying the same logic in all three directions, we obtain Figure 2.9.

Combining the commutation relation and the deformability condition, we arrive at the following conclusion. For a code $\mathcal{C}_{S,A}^{\alpha\beta\gamma\delta}$, if $\langle A, B \rangle \neq 0 \ \forall A \neq B$, where $A, B \in \{\alpha, \beta, \gamma, \delta\}$, the logical string segment can be mapped into an equivalent flat segment. An immediate consequence of this result is that if $d = 2$, it is impossible to come up with $\alpha, \beta, \gamma$, and $\delta$ satisfying the conditions introduced so far. Since there are only $2^2 - 1 = 3$ nontrivial symplectic pairs, there has to be at least two of these four pairs which are identical. Otherwise, one of them must be $(0, 0)$. In either case, there always exists a pair of symplectic pairs $\alpha, \beta$ such that $\langle \alpha, \beta \rangle = 0$. Hence, the minimal local particle dimension that can satisfy these conditions is $d = 3$.

For these codes, any string logical operator with a finite thickness can be deformed into an operator having a nontrivial support on a surface. In general, if we started with a string operator with a cross section width $w$ and a height $h$, such a logical operator can be confined in a surface with a kink, see Figure 2.12. Therefore, we arrive at an important conclusion: that if each of the symplectic pairs describing the quantum code have nonvanishing symplectic products with each other, the string logical operator segment can be deformed into a canonical form that is supported on a quasi-2-dimensional surface.

After deforming the string segment to a quasi-2-dimensional surface, we can enumerate all the

constraints and write it in a matrix form. Given a logical string segment described by a tensor product of generalized Pauli operators, the logical string segment must commute with the local stabilizer generators except at the anchor. Therefore, there are number of constraints that grows as the volume of the logical string segment. The main task is to determine the number of *linearly independent* constraints relative to the size of the string segment. If the number of constraints are too small, one will end up having a string logical operator as in the toric code case. On the other hand, if the number of constraints are large enough compared to the size of the string segment, there cannot be any string segment that can satisfy all of the constraints. In that case, one will end up having no string.

We introduce the following notation to formalize the preceding intuition.

**Definition 13.**

$$T_{\alpha\gamma} = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \gamma_1 & -\gamma_2 \end{pmatrix} \tag{2.47}$$

$$T^{\beta\delta}_{\alpha\gamma} = \begin{pmatrix} \beta_1 & -\beta_2 \\ \delta_1 & -\delta_2 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \gamma_1 & -\gamma_2 \end{pmatrix} \tag{2.48}$$

Roughly speaking, $T^{\beta\delta}_{\alpha\gamma}$ denotes a transition rule for a *minimal string segment.* As its name suggests, a minimal string segment refers to a string operator with a minimal width. The motivation for this transition rule comes from the following question: given symplectic pairs parameterizing the code, is it possible to have a string operator supported on a straight line in $x, y$, or $z$ direction? Obviously, this is a much simpler problem than proving the absence of an *arbitrary* string logical operator.

Given such a string operator, one can consider a length-2 string segment. There are four stabilizer generators sharing its nontrivial support with the string. At the same time, four elements in $\mathbb{Z}_p$ can completely specify the string segment. Therefore, if the linear constraints imposed by the stabilizer generators are all linearly independent to each other, there cannot exist any such string operator. Alternatively, given one of the symplectic pairs describing the string segment, one can infer the remaining symplectic pair from the condition imposed by two of the stabilizer generators.[3] In this sense $T^{\beta\delta}_{\alpha\gamma}$ is a transition matrix. One can construct another transition rule from the two remaining stabilizer generators, see Figure 2.13. From the construction of our code, the transition rule enforced by the other two generators are $(T^{\beta\delta}_{\alpha\gamma})^{-1}$. Therefore, a minimal string segment exists only if

$$\det(T^{\beta\delta}_{\alpha\gamma} - T^{\beta\delta}_{\alpha\gamma})^{-1} = 0. \tag{2.49}$$

[3]This is only true under the deformability condition.

To summarize, we have discussed so far the motivation behind the conditions listed in Theorem 3. We note in passing that the previously mentioned qutrit code satisfies first two of these conditions. However, these codes do not satisfy the third condition. This is not to say that these qutrit codes have a string logical operator. In fact, we have numerically checked the maximum length of a nontrivial string logical operators for these codes. Up to a width $w = 20$, the length was bounded by $w+1$. Hence we conjecture that these codes are free of string logical operators as well. In light of this numerical work, the third condition seems to be of a technical nature. The following equations are couple of useful facts about the transition matrix that will come handy throughout the analysis.

$$T^{AB}_{CD} = T^{BA}_{DC} \tag{2.50}$$

$$(T^{AB}_{CD})^{-1} = T^{CD}_{AB} \tag{2.51}$$

$$T^{AB}_{CD} T^{CD}_{EF} = T^{AB}_{EF} \tag{2.52}$$

Given a string segment, one can always enumerate all the constraints imposed by the local stabilizer generators. These constraints can be written as the rows of a larger matrix that we call as the *constraint matrix*. The objective is to show that, given a bounded string width $w$, there exists a length of the segment for which the number of independent constraints equals the number of variables that describe the string segment. If such condition is satisfied, these constraints can be formally written as:

$$Tv = 0, \tag{2.53}$$

where $T$ is the constraint matrix and $v$ is a vector describing the string segment. If the rank of $T$ is full, the only $v$ satisfying the constraint is the zero vector, which corresponds to a tensor product of identities on all sites. Hence, the problem of proving the absence of string logical operator reduces to the problem of bounding the rank of the constraint matrix. For a general matrix over a finite field, this is a hard problem. However, we can exploit the translational invariance of the system, which will give rise to a special structure.

We first define the relevant tools. Let $V$ be a $n$-dimensional vector space over a finite field $\mathbb{F}$. $T$ is a linear operator $T : V \to V$ and $v \in V$.

**Definition 14.** *$m_{T,v}(x)$ is a polynomial with a least degree which satisfies the relation*

$$m_{T,v}(T)v = 0. \tag{2.54}$$

*We call $m_{T,v}$ as a minimal polynomial of $T$ on $v$.*

**Definition 15.** *$m_T(x)$ is a polynomial with the least degree which satisfies the relation*

$$m_T(T)v = 0 \quad \forall v \in V \tag{2.55}$$

**Lemma 11.**

$$m_{T,v} | m_T \tag{2.56}$$

$$m_T | \chi_T, \tag{2.57}$$

*where $\chi_T$ is a characteristic polynomial of $T$.*

**Lemma 12.** *Let the degree of $m_{T,v}$ be $d$. Then $v, Tv, T^2v, \cdots, T^{d-1}v$ are linearly independent to each other.*

The sketch of the proof is as follows. Suppose we are given a logical string segment with a bounded anchor size. The segment must commute with the local stabilizer generators. As the length of the segment increases, the number of constraints increases, since more local stabilizer generators must commute. At the same time, the number of unknowns to specify the segment increases as well. One can show that the rate of the increase for the constraints is larger than that of the number of unknowns. Eventually the number of constraints overcome the number of unknowns. If the constraints are sufficiently independent to each other, the rank of the constraints becomes identical to the number of unknowns. In such cases, the logical string segment satisfying the commutation relation must be trivial.

The most general shape of the string segment is a quasi-2-dimensional surface with a kink. However, we first start with a case for a surface *without* a kink to illustrate the idea. Suppose the logical string segment can be supported on a flat surface which is normal to one of three $\hat{x}, \hat{y}, \hat{z}$ orthogonal directions. A logical string segment can be represented as a set of symplectic pairs on the vertices. Given a width $w$ and a length $l$ segment, we have total of $wl$ symplectic pairs, which results in $2wl$ unknowns over a field $\mathbb{F}$. We shall represent the logical operator as a $2wl$-dimensional vector over $\mathbb{F}$: the $i$th row, $j$th column, and the $k$th element of the symplectic pair is labeled by $2(j-1)n + 2(i-1) + k$, see Figure 2.14.

Within this convention, the constraints from the local stabilizer generators takes the following form:

$$c^T v_L = 0, \tag{2.58}$$

where $v_L$ is the vector representing the logical operator and $c$ is the constraint. For instance, consider $w = 1$, $l = 2$ string segment. The set of local constraints can be represented by the

following constraint matrix:

$$\begin{pmatrix} T_{\gamma\beta} & T_{\delta\alpha} \\ T_{\delta\alpha} & T_{\gamma\beta} \end{pmatrix}, \tag{2.59}$$

where each entries are $2 \times 2$ blocks. The rank of this matrix is preserved under a block Gaussian elimination. After some manipulation, the matrix can be transformed into a block upper triangular form.

$$\begin{pmatrix} I & T_{\delta\alpha}^{\gamma\beta} \\ 0 & T_{\gamma\beta}^{\delta\alpha} - T_{\delta\alpha}^{\gamma\beta} \end{pmatrix}. \tag{2.60}$$

This matrix is full rank if and only if $T_{\gamma\beta}^{\delta\alpha} - T_{\delta\alpha}^{\gamma\beta}$ is rank-2. Hence we arrive at the following conclusion.

**Lemma 13.** *A string logical operator with width* 1 *exists if and only if*

$$\det(T_{\gamma\beta}^{\delta\alpha} - T_{\delta\alpha}^{\gamma\beta}) \neq 0. \tag{2.61}$$

A pattern emerges as we increase the width of the segment. For instance, a constraint matrix for $w = 2$, $l = 3$ is the following.

$$\begin{pmatrix} T_{\gamma\beta} & 0 & T_{\delta\alpha} & 0 & 0 & 0 \\ T_{\alpha\delta} & T_{\gamma\beta} & T_{\beta\gamma} & T_{\delta\alpha} & 0 & 0 \\ 0 & T_{\delta\alpha} & 0 & T_{\gamma\beta} & 0 & 0 \\ 0 & 0 & T_{\gamma\beta} & 0 & T_{\delta\alpha} & 0 \\ 0 & 0 & T_{\alpha\delta} & T_{\gamma\beta} & T_{\beta\gamma} & T_{\delta\alpha} \\ 0 & 0 & 0 & T_{\delta\alpha} & 0 & T_{\gamma\beta} \end{pmatrix} \tag{2.62}$$

After a sequence of suitable block Gaussian elimination, we can arrive at the following canonical form. Here the width $w$ was set to $n$.

$$\begin{pmatrix} I_{2n} & \mathcal{T}_{2n} & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{2n} & \mathcal{T}_{2n} & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{2n} & \mathcal{T}_{2n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & I_{2n} & \mathcal{T}_{2n} \\ 0 & v_{2n} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & v_{2n} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & v_{2n} \end{pmatrix}, \tag{2.63}$$

where

$$\mathcal{T}_{2n} = \begin{pmatrix} T^{\gamma\beta}_{\delta\alpha} & 0 & \cdots & 0 \\ (-T^{\gamma\beta}_{\alpha\delta})x & T^{\gamma\beta}_{\delta\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ (-T^{\gamma\beta}_{\alpha\delta})^{n-1}x & (-T^{\gamma\beta}_{\alpha\delta})^{n-2}x & \cdots & T^{\gamma\beta}_{\delta\alpha} \end{pmatrix} \tag{2.64}$$

and

$$v_{2n} = \begin{pmatrix} (T^{\gamma\beta}_{\alpha\delta})^{n-1}x & (T^{\gamma\beta}_{\alpha\delta})^{n-2}x & \cdots & (T^{\gamma\beta}_{\alpha\delta})^{1}x & x \end{pmatrix}. \tag{2.65}$$

$I_{2n}$ is a $2n \times 2n$ identity matrix. Given a length $l$, the dimension of the constraint matrix is $2(w+1)(l-1) \times 2wl$. The rank of the constraint matrix can be bounded by $2w(l-1) + Rank(\mathcal{A}_{2n})$, where

$$\mathcal{A}_{2n} = \begin{pmatrix} v_{2n} \\ v_{2n}\mathcal{T}_{2n} \\ \vdots \\ v_{2n}(\mathcal{T}_{2n})^{2n-2} \\ v_{2n}(\mathcal{T}_{2n})^{2n-1} \end{pmatrix}. \tag{2.66}$$

**Lemma 14.** *Let $v^{1,2}_{2n}$ be the first and second column vector of $v_{2n}$.*

$$\max_{1,2}(\deg(m_{\mathcal{T}_{2n}, v^{1,2}_{2n}})) = 2n \tag{2.67}$$

*if the conditions in Theorem 3 is met.*

*Proof.*

$$\chi_{\mathcal{T}_{2n}} = \chi^{n}_{T^{\gamma\beta}_{\delta\alpha}}. \tag{2.68}$$

Under the conditions in Theorem 3,

$$\chi_{T^{\gamma\beta}_{\delta\alpha}}(\mathcal{T}_{2n}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ A & 0 & \cdots & 0 & 0 \\ B & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D & C & \cdots & A & 0 \end{pmatrix}, \tag{2.69}$$

where $\det(A) \neq 0$.

One can then show that

$$v_{2n}\chi_{T^{\gamma\beta}_{\delta\alpha}}(\mathcal{T}_{2n})^{n-1} \neq 0, \tag{2.70}$$

since the first $2 \times 2$ block of the matrix is of the form $(T^{\gamma\beta}_{\alpha\delta})^{n-1}xA^{n-1}$, and this is a product of invertible matrices. Hence the minimal polynomial of $\mathcal{T}_{2n}$ cannot have a degree of $2(n-1)$. If

$\chi_{T^{\gamma\beta}_{\delta\alpha}}(x)$ is irreducible, we are done. Otherwise, consider a nontrivial factor of $\chi_{T^{\gamma\beta}_{\delta\alpha}}(x)$ and denote it as $f(x) = x + a$. Consider the first $2 \times 2$ block of $v_{2n}\chi_{T^{\gamma\beta}_{\delta\alpha}}(\mathcal{T}_{2n})^{n-1}(\mathcal{T}_{2n} + a)$. It has the form of $xA^{n-1}A'$, where $A'$ is a rank one matrix. Hence, at least one of $v^{1,2}_{2n}$ has a minimal polynomial for $\mathcal{T}_{2n}$ with a degree $2n$. □

A similar idea is used for the most general form of the string segment. Given a set of constraints, one can bound the linear independence of these constraints by determining the minimal polynomial of a certain matrix. Under the same procedure, Equation 2.62 can be derived, but $\mathcal{T}_{2n}$ and $v_{2n}$ are changed. Their precise forms are not so concise, but for the proof only the following information is necessary. First, the first $2\times2$ block of $v_{2n}$ is $(T^{\gamma\alpha}_{\beta\delta})^{w-w_1}T_{int}(T^{\gamma\beta}_{\alpha\delta})^{w_1-1}x$, where $T_{int} = T^{\gamma\alpha}_{\delta 0} - T^{\gamma\alpha}_{\beta\delta}T^{\gamma\alpha}_{\alpha 0}$. An important property of $T_{int}$ is that $\det(T_{int}) = \langle\alpha,\gamma\rangle\langle\alpha,\delta\rangle\langle\delta,\alpha\rangle \neq 0$ due to the deformability condition. Second, $\mathcal{T}_{2n}$ is a $2 \times 2$-block lower triangular form with following entries.

$$(\mathcal{T}_{2n})_{ii} = T^{\gamma\beta}_{\delta\alpha} \qquad i < w_1 \tag{2.71}$$

$$= T^{\gamma\alpha}_{\delta\beta} \qquad i \geq w_1, \tag{2.72}$$

$$(\mathcal{T}_{2n})_{(i+1)i} = -T^{\gamma\beta}_{\alpha\delta}x \qquad i < w_1 \tag{2.73}$$

$$= -T^{\gamma\alpha}_{\beta\delta}x \qquad i > w_1, \tag{2.74}$$

where $w_1$ is the width before we encounter the corner. The rest of the entries can be computed as well, but they are irrelevant for the proof. Argument goes as follows.

$$\chi_{\mathcal{T}_{2n}}(x) = \chi_{T^{\gamma\beta}_{\delta\alpha}}(x)^{w_1-1}\chi_{T^{\gamma\alpha}_{\delta\beta}}(x)^{w-w_1+1}. \tag{2.75}$$

The minimal polynomial must contain the factor of $\chi_{T^{\gamma\beta}_{\delta\alpha}}(x)^{w_1-1}$. Otherwise, the first $2 \times 2$ block of $v_{2n}\chi^{\gamma\beta}_{\delta\alpha}(\mathcal{T}_{2n})$ is an invertible matrix. Consider a polynomial $g(x) = \chi^{\gamma\beta}_{\delta\alpha}(x)^{w_1-1}\chi_{T^{\gamma\alpha}_{\delta\beta}}(x)^{w-w_1}$. The $(w_1+1)$th $2 \times 2$ block of $v_{2n}g(\mathcal{T}_{2n})$ is nonzero, since it has the form of $(T^{\gamma\alpha}_{\beta\delta})^{n-1}xA^{w-w_1}$ for some invertible matrix $A$. If $\chi_{T^{\gamma\alpha}_{\delta\beta}}(x)$ is irreducible, we are done. Otherwise, use the argument used for the string segments without a kink.

### 2.3.2 Logical operators

The logical operators of the code is either a fractal or a noncontractible surface. We shall study the surface operators here. Depending on the system size, there are at least $1, 2$ or $4$ surface operators for each directions. Given a surface normal to one of the unit vectors $\hat{x}, \hat{y}, \hat{z}$, the number of distinct surface operators normal to the vector depends on the width and the height of the surface. If both of them are even, there are 4 surface operators. If one of them is even, there are 2. If none of them

are even, there is 1. The construction is quite straightforward.

If the width and the height of the surface are both even, all the periodic structures shown in Figure 2.15 are allowed. Otherwise, none of them are allowed. It is still possible, however, to construct logical operators by multiplying two of the logical operators in Figure 2.15. For instance, multiplying the first and the second or the third and the fourth in Figure 2.15 results in a periodic structure in the $x$ direction. Similarly, multiplying the first and the third or the second and the fourth results in a periodic structure in the $y$ direction. Similarly, when both the width and the height are odd numbers, multiplying all 4 of them results in a periodic structure in the $x$ and the $y$ directions. One can apply the same logic for all three different directions.

One final note we would like to point out is that the antisymmetric codes always have at least one encoded qudit. Given a system with $n$ qudits, there are $n$ cubic stabilizer generators. There is at least one nontrivial relation between the generators: multiplication of all the generators equals the identity. The same logic cannot be applied to the symmetric code.

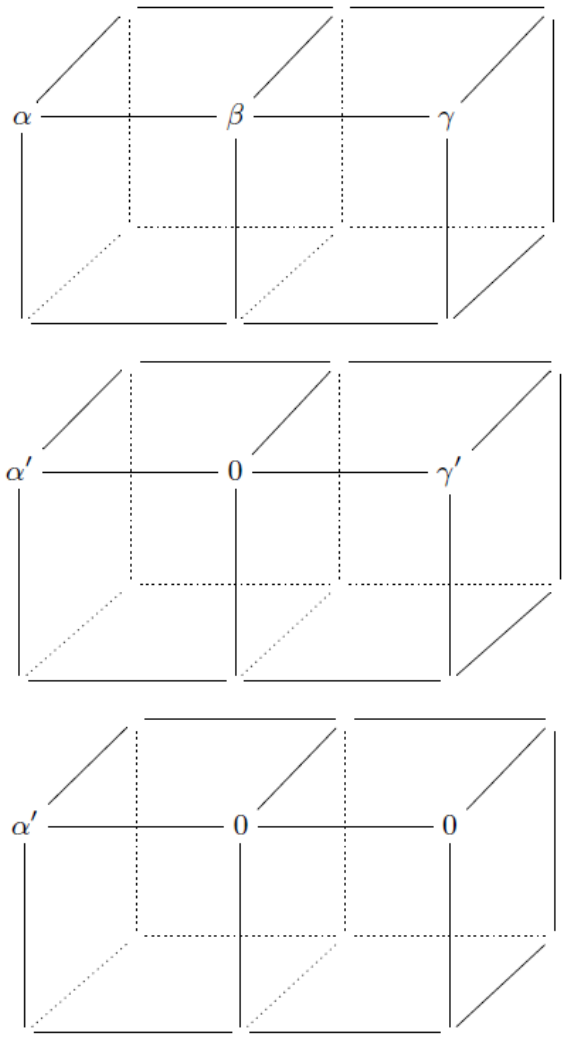



Figure 2.11: This diagram represents a deformation procedure. One can multiply a suitable choice of stabilizer elements so that the action of the string operator on $B$ is $0 = (0, 0)$. If two symplectic pairs on the diagonal line are linearly independent to each other, one can further deform $C'$ into 0. Repeat this procedure until we get rid of the entire line.

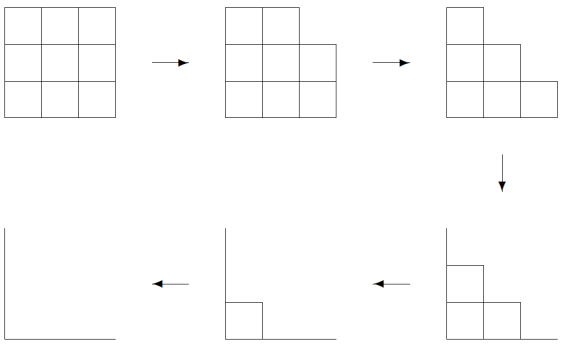


Figure 2.12: Deformation of the string logical operator viewed from the direction normal to its cross section.

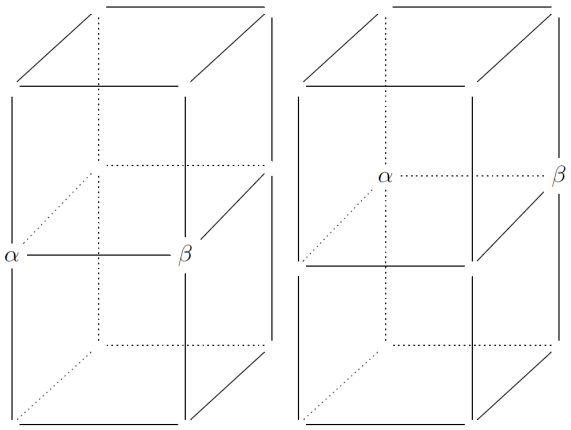



Figure 2.13: Two different kinds of boundary constraints. $A$ and $B$ are the unknown symplectic pairs and the cubes represent the stabilizer generators that share the support with the logical operator only at these two sites.

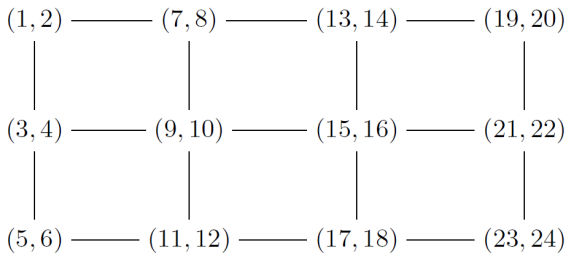



Figure 2.14: Notation for the basis vectors

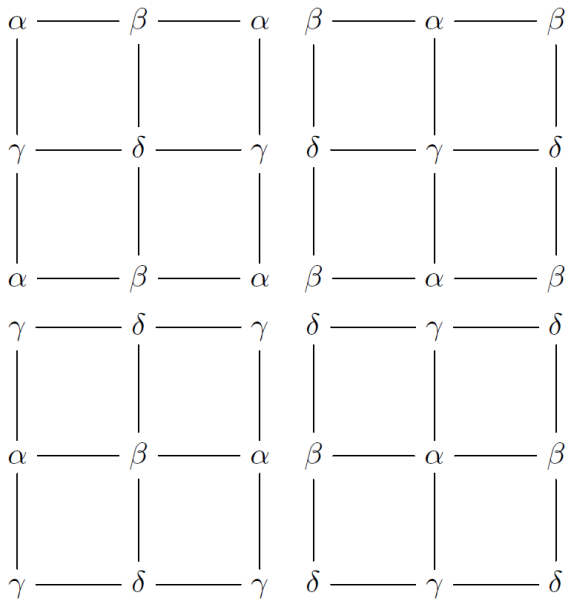



Figure 2.15: Construction of the logical operators on a plane. Each of them can be mapped into each other by a unit translation.

# Chapter 3

# Technical tools for studying generic quantum many-body systems

We describe four of the main technical tools used in this thesis. Each sections can be read independently. Section 3.1 deals with an operator extension of the strong subadditivity of entropy. This work also appears in [116]. It is worth noting that Ruskai subsequently extended some of these results to a much more general setting.[117] Section 3.2 deals with applications of the Lieb-Robinson bound that are relevant to this thesis. We would like to note the readers that there is a nice pedagogical introduction to this subject written by Hastings.[118] Unfortunately, the scope of Hastings' review article misses some of the results that are necessary for obtaining the main results of this thesis. Also, there has been a recent development in the subject by several authors which is particularly well-suited for the purpose of this thesis.[119] We shall describe some of the materials, focusing on the relevant results that shall be used here. Section 3.3 deals with a deformation technique developed by myself. Section 3.4 deals with a systematic regularization procedure for introducing a cutoff for an entanglement Hamiltonian. These works also appear in Ref.[120, 121]

We would also like to mention that each of the forthcoming chapters need different set of technical tools. For reader's convenience we list the relevant tools for each of the chapters. One may skip some of the sections accordingly depending on their interests. In Chapter 4, we shall construct a set of inequalities between long-range entanglement and a topological ground state degeneracy. None of the tools presented in this chapter is needed for this result. In fact, a judicious usage of SSA alone would lead to the main result. Chapter 5 studies a structure of the entanglement Hamiltonian in gapped quantum many-body systems. For this work, one would need Section 3.1,3.3, and 3.4. In Chapter 6, we establish a first-order perturbative stability of the topological entanglement entropy. All of the technical tools in this chapter will be needed to understand the materials.

## 3.1 Operator extension of strong subadditivity of entropy

A tripartite state saturating the equality condition of SSA has a very restrictive structure. Namely, the state forms a quantum Markov chain.[122, 123, 124] One desirable generalization of this result would be to understand the structure of states that are *approximately* conditionally independent. Unfortunately, only negative results exist in the literature. Ibinson, Linden, and Winter studied the following problem: given a tripartite quantum state $\rho_{ABC}$, what is its minimal distance from a set of quantum Markov chain?[125] More specifically, they have set the distance measure to be the relative entropy. Their motivation behind this choice comes from the fact that the distance becomes the conditional mutual information for a *classical state*. They have shown that, an inequality between the conditional mutual information and the distance has to be nonlinear as well as dimension-dependent.

Like any good no-go theorem, this result should be considered as a sign pointing away from the wrong direction. Note that the quantum Markov chain property is just one of the many properties that follows from the conditional independence. The most immediate consequence of the saturation of the SSA is the existence of a perfect recovery map.[122] This recovery map has been rediscovered many times throughout the literature under different names. To the best of author's knowledge, the discovery of this recovery map goes back as far as to the work of Accardi and Cecchini, where they named it as a generalized conditional expectation.[126] Later a more general recovery channel was discovered, which includes the generalized conditional expectation as a special case.[127] Petz later found the same recovery channel on a rather unrelated setting,[122] and this lead to the famous structure theorem discovered by Hayden et al.[123]

A relatively unknown result of Petz is the following necessary and sufficient condition for the conditional mutual information to be exactly 0.

**Theorem 4.** *(Petz 2003) Assume that $\rho_{ABC}$ is invertible. SSA holds with an equality if and only if the following equivalent conditions hold:*

$$(1)\rho_{ABC}^{it}\rho_{AB}^{-it} = \rho_{BC}^{it}\rho_{B}^{-it} \tag{3.1}$$

$$(2)\log\rho_{ABC} - \log\rho_{AB} = \log\rho_{BC} - \log\rho_{B}. \tag{3.2}$$

Some comment on the notation is in order. Here $\log\rho_{AB}$ is a short-hand notation for $\log\rho_{AB}\otimes I_C$. Similarly, whenever a logarithm of a reduced density matrix appears, a tensor product with the identity for the rest of the subsystems is implicitly assumed. While the work of Ibinson et al. precludes a possibility of constructing a linear inequality between conditional mutual information and a minimal relative entropy between $\rho_{ABC}$ and a quantum Markov chain state, a certain approximate version of Theorem 4 may be still true. This will be the direction that we shall pursue.

The main result of this section is the operator extension of the SSA.

**Theorem 5.**

$$Tr_{AB}(\rho_{ABC}(\hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC})) \geq 0. \tag{3.3}$$

Several remarks are in order. First, Theorem 5 implies SSA as a special case. This can be easily seen by taking a partial trace over $C$. Second, each terms in the left hand side of Equation 3.3 is Hermitian. Third, a similar inequality cannot be true if the partial trace is either restricted to $A$ or $B$ only: in that case, the resulting operators are not even Hermitian. Hence, if one considers a *general* quantum state, one cannot expect to come up with a linear operator inequality that includes Equation 3.3 as a special case.

### 3.1.1 Proof of Theorem 5

Since Lieb and Ruskai's original proof of SSA[128], alternative proofs have been introduced by several authors.[122, 129, 130, 131] In particular, Effros recently presented a proof based on the perspective of an operator convex function.[131] Effros basically extended the notion of perspective function from real numbers to operators. Given a function $f$, a perspective of $f$ is defined as

$$g(x,t) = f(x/t)t.$$

If $f(x)$ is convex, $g(x,t)$ is jointly convex in $x$ and $t$. The main insight of Effros is that a similar statement holds for a function $f$ that is *operator convex.* To be more precise, he proved the following statement.

**Theorem 6.** *(Effros 2009) If $f(x)$ is operator convex, and $[L,R]=0$, perspective*

$$g(L,R) = f(L/R)R \tag{3.4}$$

*is jointly convex in the sense that if $L = cL_1 + (1-c)L_2$ and $R = cR_1 + (1-c)R_2$ with $[L_i, R_i] = 0$ $(i=1,2)$, $0 \leq c \leq 1$,*

$$g(L,R) \leq cg(L_1,R_1) + (1-c)g(L_2,R_2). \tag{3.5}$$

Theorem 5 can be derived from a simple application of Theorem 6. Recall that Theorem 5 asserts the following inequality:

$$\mathrm{Tr}_{AB}(\rho_{ABC}(\hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC})) \geq 0. \tag{3.6}$$

*Proof.* We choose a matrix algebra $\mathcal{B}(\mathcal{H})$ with an inner product structure of $\langle X, Y\rangle = \mathrm{Tr}(XY^\dagger)$, where $X, Y$ are $n \times n$ matrices. Following Effros, we choose $L$ and $R$ to be superoperators that multiply a matrix from the left or the right. For $X \in \mathcal{B}(\mathcal{H})$, $L$ and $R$ are defined as follows.

$$\begin{aligned} LX &= \rho X \\ RX &= X\sigma. \end{aligned} \tag{3.7}$$

$L$ and $R$ commute with each other. One can also show the following relations.

$$\begin{aligned} \log(L)X &= \log(\rho)X \\ \log(R)X &= X\log(\sigma). \end{aligned} \tag{3.8}$$

Now we set $f(x) = x\log x$ and apply Theorem 6 with a judicious choice of operators. Since $f(x)$ is operator convex[104], $g(L,R) = L\log(L) - L\log(R)$ is jointly convex in $L$ and $R$. Therefore,

$$\langle g(L,R)(O), O\rangle = \mathrm{Tr}(\rho\log\rho OO^\dagger - \rho O\log\sigma O^\dagger) \tag{3.9}$$

is jointly convex in $L$ and $R$ for all $O \in \mathcal{B}(\mathcal{H})$.[1] Choose $\rho = \rho_{ABC}$, $\sigma = \rho_{AB} \otimes \frac{I_C}{d_C}$, $O = I_{AB} \otimes P_C$, where $P_C$ is an arbitrary projector supported on $C$ and $d_C$ is the dimension of $C$. Note

$$\frac{I_A}{d_A} \otimes \rho_{BC} = \frac{1}{d_A^2}\sum_{i=1}^{d_A^2} U_{A,i}\rho_{ABC}U_{A,i}^\dagger \tag{3.10}$$

for a set of unitaries $\{U_{A,i}\}$ that forms an orthogonal basis for $\mathcal{B}(\mathcal{H}_A)$. An example can be found in Ref. [108]. Using joint convexity,

$$\mathrm{Tr}(\frac{I_A}{d_A} \otimes \rho_{BC}(\hat{H}_B - \hat{H}_{BC})P_C) \leq \mathrm{Tr}(\rho_{ABC}(\hat{H}_{AB} - \hat{H}_{ABC})P_C). \tag{3.11}$$

The left hand side of the inequality is equal to $\mathrm{Tr}(\rho_{ABC}(\hat{H}_B - \hat{H}_{BC})P_C)$. Since the inequality holds for all $P_C$,

$$\mathrm{Tr}_{AB}(\rho_{ABC}(\hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC})) \geq 0. \tag{3.12}$$

□

An important application of Theorem 5 is the following result:

**Corollary 1.**

$$Tr(O_A\rho_{ABC}(\log\rho_{ABC} + \log\rho_B - \log\rho_{AB} - \log\rho_{BC})) \leq I(A:C|B)\|O_A\|. \tag{3.13}$$

[1] The joint convexity of Equation 3.9 was originally proved by Petz.[132]

*Proof.*

$$\begin{aligned} &\mathrm{Tr}(O_A \rho_{ABC}(\log \rho_{ABC} + \log \rho_B - \log \rho_{AB} - \log \rho_{BC})) \\ \leq\ &\mathrm{Tr}_A(O_A \mathrm{Tr}_{BC} \rho_{ABC}(\log \rho_{ABC} + \log \rho_B - \log \rho_{AB} - \log \rho_{BC})) \\ \leq\ &\|O_A\| |\mathrm{Tr}_{BC} \rho_{ABC}(\log \rho_{ABC} + \log \rho_B - \log \rho_{AB} - \log \rho_{BC})|_1 \\ =\ &\|O_A\| I(A:C|B) \end{aligned} \tag{3.14}$$

□

In comparison to Theorem 4, we have an interesting conclusion. Note that Petz's condition can be reformulated as:

$$\mathrm{Tr}(O \rho_{ABC}(\log \rho_{ABC} + \log \rho_B - \log \rho_{AB} - \log \rho_{BC})) = 0 \tag{3.15}$$

for any bounded operator $O$. Our result is weaker in that Equation 3.13 is a statement about an operator that is either supported on $A$ or $C$. On the other hand, our result is more general in that we do not have to assume the conditional mutual information is 0. Our result becomes particularly powerful for systems that have a small yet nonzero conditional mutual information. For such systems, the following operator analog of the conditional mutual information,

$$\log \rho_{ABC} + \log \rho_B - \log \rho_{AB} - \log \rho_{BC}, \tag{3.16}$$

has a small correlation with any local operator supported on $A$ or $C$.

We note in passing that a much stronger inequality can be obtained if one assumes the reduced density matrices to commute with each other.

**Lemma 15.** *If all the reduced density matrices commute with each other for a tripartite state $\rho_{ABC}$,*

$$Tr_A(\rho_{ABC}(\log \rho_{ABC} + \log \rho_B - \log \rho_{AB} - \log \rho_{BC})) \geq 0. \tag{3.17}$$

However, a similar inequality cannot be true for quantum states. The left hand side of Equation 3.17 is not even Hermitian in general.[2]

[2]We have also tested variants of Equation 3.17 that are manifestly Hermitian, such as $\mathrm{Tr}_A\{(\rho_{ABC}, (\log \rho_{ABC} + \log \rho_B - \log \rho_{AB} - \log \rho_{BC})\}$ and $\mathrm{Tr}_A(\rho_{ABC}^{\frac{1}{2}}(\log \rho_{ABC} + \log \rho_B - \log \rho_{AB} - \log \rho_{BC})\rho_{ABC}^{\frac{1}{2}})$. Numerical counterexamples were found for both of these conjectures.

## 3.2 Lieb-Robinson bound

Lieb-Robinson bound asserts that, given a Hamiltonian described by a sum of geometrically local bounded norm terms, the unitary evolution generated by the Hamiltonian has an effective light cone. As in the relativistic quantum mechanics, correlations outside the effective light cone is small.[3] We shall not give a proof of this statement, but rather assume the quantum many-body Hamiltonian satisfies the Lieb-Robinson bound and study its consequences. For a pedagogical introduction to the subject, we recommend Hastings' lecture note.[118] Given an observable $O_A(O_B)$ supported on $A(B)$, Lieb-Robinson bound can be formally stated as follows.

$$\|[O_A(t), O_B]\| \leq c\|O_A\|\|O_B\| \min(|A|, |B|)e^{c_1(vt-d(A,B))}, \tag{3.18}$$

where $0 < c, c_1, v < \infty$ are some constants that depend on the parameters of the Hamiltonian and $d(A, B)$ is a distance between $A$ and $B$. $O(t) = e^{-iHt}Oe^{iHt}$ is a time evolution of an operator $O$ under the Hamiltonian.

While Equation 3.18 is already useful for some applications,[133, 32] it is important to consider variants of the Lieb-Robinson bound in more general settings. A mathematical interest aside, such a generalization is absolutely necessary in constructing Hastings' quasi-adiabatic continuation technique. The quasi-adiabatic continuation asserts that, given a set of eigenstates of a Hamiltonian along some path $s \in [0, 1]$ that is separated from the rest of the spectrum, there exists a path-dependent "Hamiltonian" that generates a unitary flow between $s = 0$ and 1. Furthermore, if (i) the underlying Hamiltonian $H(s)$ consists of geometrically local, $s$-differentiable, bounded norm terms and (ii) those states are separated from the rest of the spectrum by a constant that is independent of the system size, the unitary flow can be generated by a sum of path-dependent quasi-local terms with a superpolynomially decaying tail. Therefore, the path-dependent quasi-local terms appear inevitably in the analysis of such systems. For the scope of this thesis, we shall not need to worry about the complications arising from the path-dependence.

An important concept in the application of the Lieb-Robinson bound is the so called filter function. Unfortunately the notation in the literature is not uniform, so we begin by defining the following superoperator:

$$\Phi_f(O) = \int_{-\infty}^{\infty} e^{-iHt}Oe^{iHt}f(t)dt. \tag{3.19}$$

It is worth noting that in the energy eigenbasis,

$$\Phi_f(O)|_{ij} = \tilde{f}(E_i - E_j)O_{ij}, \tag{3.20}$$

where $\tilde{f}(\omega)$ is an inverse Fourier transform of $f(t)$. By making a judicious choice of $f$, one can make

[3] Correlation is strictly 0 outside the light cone in a relativistic quantum mechanics.

(i) the off-diagonal terms of $\Phi_f(O)$ to decay sufficiently fast and (ii) $f(t)$ to decay sufficiently fast in $t$. Under such a choice of $f$, $\Phi_f(O)$ can be approximated by a diagonal matrix while maintaining its locality approximately.

More specifically, we can define a truncated superoperator $\Phi_f^T$ by introducing a cutoff $T$.

$$\Phi_f^T(O) = \int_{-T}^{T} e^{-iHt} O e^{iHt} f(t) dt. \tag{3.21}$$

A Lieb-Robinson type locality bound for $\Phi_f$ can be established as follows.

$$\|[\Phi_f(O_A), O_B]\| \leq \|[\Phi_f^T(O_A), O_B]\| + \|[\Delta_f^T(O_A), O_B]\|, \tag{3.22}$$

where $\Delta_f^T = \Phi_f - \Phi_f^T$. The first term can be bounded by

$$\int_{-T}^{T} |f(t)| dt \|O_A\| \|O_B\| c e^{c'(vT - d(A,B))} \tag{3.23}$$

from the Lieb-Robinson bound. The second term can be bounded by

$$\int_{\mathbb{R}\setminus[-T,T]} |f(t)| dt \|O_A\| \|O_B\|. \tag{3.24}$$

Depending on the function $f$, one can optimize the bound with a judicious choice of $T$. Many of the rigorous results using the Lieb-Robinson bound uses this line of logic.

### 3.2.1 Application to the finite-temperature systems

An important example of a filter function is the following function:

$$\tilde{f}_1^\beta(\omega) = \frac{\tanh(\beta\omega/2)}{\beta\omega/2}. \tag{3.25}$$

To the best of author's knowledge, $\tilde{f}_1^\beta$ was first used by Hastings to obtain the correlation decay bound for fermionic systems at a finite temperature.[133] Later a similar technique was used in justifying an approximation used in the quantum belief propagation(QBP) algorithm.[134] The motivation for using this function is mainly twofold. First, given a strictly local operator $v_i$, one can straightforwardly show that $\Phi_{\tilde{f}_1^\beta}(v_i)$ can be well approximated by a *strictly* local operator. More specifically, we have the following Lieb-Robinson type bound.

**Lemma 16.** *If $H$ satisfies Lieb-Robinson bound,*

$$\|[\Phi_{f_1^\beta}(O_A), O_B]\| \leq c\|O_A\|\|O_B\| \min(|A|, |B|) e^{-\frac{c' d(A,B)}{1 + c' v\beta/\pi}}, \tag{3.26}$$

*for some constant* $0 < c, c' < \infty$.

Subsequently, one can obtain the following local approximation of $\Phi_{f_1^\beta}(v_i)$.

**Corollary 2.**

$$\|\Phi_{f_1^\beta}(v_i) - [\Phi_{f_1^\beta}(v_i)]_{v_i(r)}\| \leq c'\|v_i\| e^{-\frac{c'r}{1+c'v\beta/\pi}}, \tag{3.27}$$

where $v_i(r)$ is a set of sites whose distance from the support of $v_i$ is less or equal to $r$. We have used the following notation introduced by Bravyi et al.[59] to approximate a quasi-local operator by a strictly local operator:

$$[O]_A = \frac{1}{\dim A^c} \mathrm{Tr}_{A^c}(O) \otimes I_{A^c}. \tag{3.28}$$

The proof follows a rather standard technique[59, 118] which is based on the "twirling" idea: that one can perform a partial trace operation by randomly applying a unitary over the Haar measure.

Secondly, $\Phi_{f_1^\beta}$ is a quantum channel that appears naturally when computing a directional derivative of a density matrix.[4]

**Lemma 17.** *For* $\rho(s) = \frac{e^{-\beta H(s)}}{Z}$,

$$\frac{d}{ds}\rho(s)|_{s=0} = \frac{\beta}{2}(\Phi_{f_1^\beta}(V)\rho_s + h.c.) - \beta\langle\Phi_{f_1^\beta}(V)\rangle, \tag{3.29}$$

*where h.c. is Hermitian conjugate.*

Therefore, provided the original Hamiltonian $H$ is local, the infinitesimal change to the Gibbs state under a local perturbation $V$ is generated by a sum of quasi-local terms.

### 3.2.2 Quasi-adiabatic continuation

The quasi-adiabatic continuation was originally conceived by Hastings in his proof of the higher dimensional generalization of the Lieb-Schultz-Mattis theorem.[32] This technique was later refined and generalized by a number of authors.[33, 56, 57, 58, 59, 60, 61, 29] Due to the vastness of the related literature, we will not be able to cover all of the details of this technique. Rather, we will give several important remarks that are relevant to this thesis.

The quasi-adiabatic continuation in its original form is formally defined as the following unitary operator[33]:

$$V(s) := \mathcal{S}' \exp(-\int_0^{s'}\int_0^\infty d\tau e^{-(\tau/t_q)^2/2}[\tilde{u}_{s'}^+(i\tau) - h.c.]), \tag{3.30}$$

[4] To see that $\Phi_{f_1^\beta}$ is a quantum channel, note that it has an integral Kraus representation. Furthermore, one can easily check from the normalization of $f_1^\beta$ that this channel is trace preserving.

where $u_s = \frac{d}{ds}H(s)$ and

$$\tilde{A}^+(i\tau) := \frac{1}{2\pi}\int dt\tilde{A}(t)\frac{1}{it+\tau}. \tag{3.31}$$

Here $\mathcal{S}'$ represents a path-ordered product in the variable $s'$ and $H(s)$ is a family of Hamiltonian parameterized by $s \in [0,1]$. $t_q$ is some parameter that will determine the "fidelity" of the quasi-adiabatic flow, in that $V(s)$ converges to the true adiabatic evolution in the $t_q \to 0$ limit. For certain applications, one can show that the quasi-adiabatic continuation simulates the true adiabatic evolution with a small error even if one chooses $t_q$ to be sufficiently small. For such small choices of $t_q$, one can approximate the generator of the flow by a sum of geometrically local bounded-norm terms.

However, a number of improved bounds were obtained recently. A particularly notable example is derived Ref.[29], where the authors have obtained the generator of the *exact* adiabatic evolution which consists of a sum of quasi-local bounded-norm terms with a superpolynomially decaying tail. For a continuous family of Hamiltonian $H(s) = H_0 + sV$, $s \in [0,1]$, the quasi-adiabatic continuation operator $\mathcal{D}_s$ is defined as follows:

$$\mathcal{D}_s := i\int dtF(t)e^{iH_s t}Ve^{-H_s t}, \tag{3.32}$$

where $F(t)$ has the following properties:

- $\tilde{F}(\omega) = -1/\omega$ for $|\omega| \geq \frac{1}{2}$
- $F(t) = -F(-t)$
- $F(t)$ is infinitely differentiable.

Then the unitary time evolution generated by $\mathcal{D}_s$ simulates the adiabatic evolution exactly. Note that we have not specified $F(t)$ yet. Bachman et al. essentially optimized the asymptotic behavior of this superpolynomial decay by making an intelligent choice of $F(t)$.[119] Their result shall be the version of the quasi-adiabatic continuation we use in this thesis. They showed that the generators is of the following form:

$$\mathcal{D}_s = \Phi_{W_\Gamma}(\frac{dH(s)}{ds}), \tag{3.33}$$

where $\Gamma = \min_{s\in[0,1]}\Gamma(s)$ and $W_\Gamma(t)$ is some superpolynomially decaying function. In our setting, $\frac{dH(s)}{ds} = V$. Each of the local terms $v_i$ in $V$ can be approximated as follows.[119]

$$\|\Phi_{W_\Gamma}(v_i) - [\Phi_{W_\Gamma}(v_i)]_{v_i(r)}\| \leq C\|v_i\|G^{(I)}(\frac{\Gamma r}{2v}), \tag{3.34}$$

where $v$ is the Lieb-Robinson velocity appearing in Equation 3.18, and $G^I(x)$ is a function that

satisfies the following property.

$$\begin{aligned} G^{(I)}(x) &= \frac{K}{\Gamma} && 0 \le x \le x_0 \\ &= 130e^2x^{10}u_{2/7}(x) && x > x_0. \end{aligned} \tag{3.35}$$

Estimates for the constants are $K \approx 14708$, $36057 < x_0 < 36058$.[119] Also, $u_a(x)$ is defined as follows.

$$u_a(x) = e^{-a\frac{x}{\ln^2 x}}. \tag{3.36}$$

## 3.3 Deformation moves

One of the basic properties of the quantum conditional mutual information is its chain rule. The chain rule is the following identity, which can be easily shown by a direct calculation:

$$I(A_1A_2 : C|B) = I(A_2 : C|B) + I(A_1 : C|A_2B). \tag{3.37}$$

Note that Equation 3.37 is merely a statement about a set. Therefore, the following operator, defined as the *conditional mutual spectrum* should also satisfy a similar relation:

$$\hat{H}_{A:C|B} = \hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC}, \tag{3.38}$$

where $\hat{H}_A$ is the entanglement spectrum of a subsystem $A$:

$$\hat{H}_A := -\log \rho_A \otimes I_{A^c}. \tag{3.39}$$

Again, one can easily check that the following *chain rule of the conditional mutual spectrum* holds:

$$\hat{H}_{A_1A_2:C|B} = \hat{H}_{A_2:C|B} + \hat{H}_{A_1:C|A_2B}. \tag{3.40}$$

*Deformation move* is a certain linear combination of Equation 6.5. Roughly, the main goal of the deformation move is to achieve the following specific task. Given a conditional mutual spectrum $\hat{H}_{A:C|B}$ and a local operator $O$, one would like to decompose $\hat{H}_{A:C|B}$ into a sum of the conditional mutual spectra $\hat{H}_{A_i:C_i|B_i}$ such that (i) $I(A_i : C_i|B_i)$ is small or (ii) $A_iB_iC_i$ is sufficiently far away from the support of $O$. We have not enforced any structure on the state so far, so one must first define what it means for the conditional mutual information to be small. Furthermore, we should also define what it means for the subsystems to be far away from each other.

Implicitly we are assuming the Hilbert space to have a tensor product structure:

$$\mathcal{H} = \otimes_i \mathcal{H}_i, \tag{3.41}$$

where $\mathcal{H}_i$ is the local Hilbert space with a bounded Hilbert space dimension. Here $i$ can be thought as a site on a lattice. Hence, one can define a metric that assigns a distance between each sites. Similarly, one can define a distance between subsystems to be the minimal distance between two sites that lie on each of the subsystems.

Regarding the smallness of the conditional mutual information, we are implicitly assuming that the extensive terms of the entanglement entropy can be attributed to the local contributions. From a heuristic argument about the scaling properties of the entanglement entropy, one may argue that the entanglement entropy of a subsystem $A$ with a smooth boundary has the following form:

$$S(A) = \sum_{i=0}^{n} a_i l^{d-i}, \tag{3.42}$$

where $l$ is the linear size of the subsystem and $d$ is the spatial dimension of the system.[5] Further, we are assuming that the extensive terms can be canceled out by making a judicious choice of the subsystems.

Undoubtedly one of the most interesting physical systems that satisfy these properties is the ground state of a gapped system. While a rigorous proof does not exist yet, there is a reason to believe that the entanglement entropy of a two-dimensional gapped systems has the following form:

$$S(A) = a|\partial A| - \gamma, \tag{3.43}$$

where $a$ is a nonuniversal constant and $\gamma$ is the topological entanglement entropy. $|\partial A|$ is the boundary area of $A$.[27, 71] An important point is that $\gamma$ only depends on the topology of $A$. More precisely, it only depends on the number of connected components of the boundary of $A$.

Keeping this example in mind, we introduce three elementary deformation moves. While all of these deformation moves can be derived from the chain rule of the conditional mutual spectrum, it shall be easier to define the elementary moves to simplify the process. We would like to first point out a rather obvious property of the conditional mutual information $I(A:C|B)$: that it is invariant under the exchange of $A$ and $C$. This fact can be easily verified from the definition of the conditional mutual information. Therefore, it is natural to make a distinction between $B$ and the rest of the subsystems.

We shall call $A$ and $C$ as *target parties*, and $B$ as the *reference party*. In the diagrams, we shall

[5] It is a well known fact that such scaling relation fails in general. For example, one-dimensional critical systems have a logarithmic scaling law of entanglement entropy.[135] Therefore, one must be careful in using these assumptions.

represent the target parties with the "T" signs. Similarly, we shall represent the reference party with an "R" sign. Given an observable $O$ whose support is sufficiently small compared to the size of the subsystems, our strategy is to deform a given conditional mutual spectrum as follows.

1. Isolation move : If the observable has a support that overlaps with the reference party, deform the target as well as the reference party. After the deformation move, the support of the observable does not have any overlap with the new reference party.

2. Separation move : If the observable has a support that overlaps only with the target party, deform the target party. After the deformation move, the support of the observable does not have any overlap with the target party.

3. Absorption move : If the support of the observable does not overlap with any of the target or reference parties of a conditional mutual spectrum whose conditional mutual information is small, deform one of the target parties. After the deformation move, the support of the observable is contained in one of the target parties.

An astute reader may have noticed a somewhat conflicting agenda between the separation move and the absorption move. An important difference between these two moves is the conditional mutual information that is described by the conditional mutual spectrum that we are deforming. In the case of the separation move, there is no *a priori* assumption about the smallness of the conditional mutual information. On the other hand, we are explicitly assuming the conditional mutual information is small for the absorption move.

We have applied these moves to a two-dimensional gapped quantum many-body system. The isolation move is depicted in Figure 3.1. After applying the isolation move, the conditional mutual

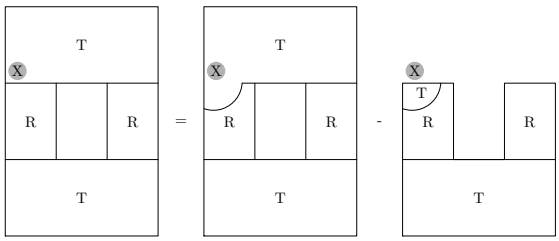



Figure 3.1: Isolation move

spectrum is deformed in such a way that (i) for the new conditional mutual spectrum, $X$ is sufficiently far away from the reference party, and (ii) the difference is a conditional mutual spectrum with a small conditional mutual information. One can see that the deformed reference party does not have any overlap with the support of the observable $X$. Further, there is a correction term that appears on the second part. More formally, this decomposition can be written in the following way:

$$\hat{H}_{A:C|B_1B_2} = \hat{H}_{AB_1:C|B_2} - \hat{H}_{B_1:C|B_2}. \tag{3.44}$$

One should note that

$$I(B_1 : C|B_2) \approx 0 \tag{3.45}$$

for two-dimensional gapped systems in general.

The separation move is depicted in Figure 3.2. In an algebraic form, the separation move can be

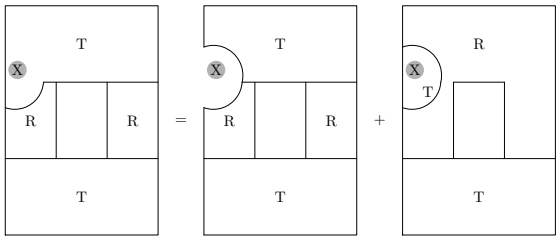



Figure 3.2: Separation move

written as follows:

$$\hat{H}_{A_1A_2:C|B} = \hat{H}_{A_2:C|B} + \hat{H}_{A_2:C|A_1B}. \tag{3.46}$$

After applying the separation move, the conditional mutual spectrum is deformed in such a way that (i) for the new conditional mutual spectrum, $X$ is sufficiently far away from both the reference and the target party (ii) the difference is a conditional mutual spectrum with a small conditional mutual information

By first applying the isolation move and then the separation move, one can always deform the configuration to be distance $\Omega(l)$ away from $X$, where $l$ is the linear size of the subsystems.[6]

Finally, the absorption move is depicted in Figure 3.3.

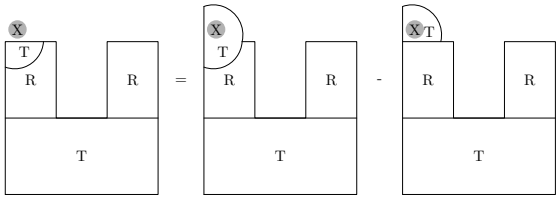



Figure 3.3: Absorption move

In an algebraic form, the absorption move can be written as follows:

$$\hat{H}_{A_1:C|B} = \hat{H}_{A_1A_2:C|B} - \hat{H}_{A_2:C|A_1B}. \tag{3.47}$$

As stated previously, the goal of the absorption move is to change the correction terms to a sum of the conditional mutual spectrum $\hat{H}_{A_i:C_i|B_i}$ such that (i) the support of $X$ is contained in either $A_i$ or $C_i$ and (ii) $I(A_i : C_i|B_i)$ is small.

[6]We are implicitly assuming that all of the subsystems are sufficiently smooth.

To summarize, given a conditional mutual spectrum $\hat{H}_{A:C|B}$ and an arbitrary operator $X$ that does not overlap with any of the boundaries, one can decompose the conditional mutual spectrum into a linear combination of the conditional mutual spectrum $\hat{H}_{A_i:C_i|B_i}$. Further, these conditional mutual spectrum can be classified into two types. The first type is the conditional mutual spectrum that is far away from the operator $X$. The second type is the conditional mutual spectrum $\hat{H}_{A_i:C_i|B_i}$ such that (i) it has a small conditional mutual information and (ii) only one of its target parties overlap with the support of $X$. For the terms of the second type, one can use Equation 3.13 to bound its value.

## 3.4 Regularization of the entanglement Hamiltonian

In this section, we introduce a systematic procedure for regularizing the entanglement Hamiltonian. The motivation behind this procedure comes from the following question. Suppose we are given a subsystem $A$, and we apply a unitary transformation that is generated by a quasi-local Hamiltonian which is centered around a region that is far away from $A$: while the Hamiltonian is nonlocal, it can be approximated by a strictly local Hamiltonian with a small tail that decays sufficiently fast.

Intuitively, one would expect the resulting change of the entanglement entropy would be small. More precisely, one can expect the unitary transformation $U$ to be approximated by another unitary $\tilde{U}$ that is strictly local, with the following bound on their difference:

$$\|\tilde{U} - U\| \leq \epsilon. \tag{3.48}$$

Infinitesimally, one can approximate $U \approx 1 + iHt$, where $H$ is the quasi-local Hamiltonian generates that the unitary evolution. Approximating $H$ by a strictly local operator $\tilde{H}$, one can bound the infinitesimal change of the entanglement entropy as

$$\frac{d}{ds}S(A) = -i\mathrm{Tr}([\rho, H]\log\rho_A) \tag{3.49}$$

$$= -i\mathrm{Tr}([\rho, \tilde{H}]\log\rho_A) - i\mathrm{Tr}([\rho, H-\tilde{H}]\log\rho_A), \tag{3.50}$$

using the standard perturbation theory technique. One can naïvely try to bound the correction term using the following inequality:

$$\mathrm{Tr}(AB) \leq |A|_1\|B\|. \tag{3.51}$$

However, we run into a problem: the operator norm of $\log\rho_A$ is formally unbounded.

A conventional remedy to this problem is to simply disregard the infinite eigenvalues of $\log\rho_A$. However, there is a physical reason why this approach will not be very successful in general. Consider the toric code for example. If one takes the logarithm of some reduced density matrix, it will have

infinite eigenvalues. These infinite eigenvalues correspond to the zero eigenvalues of the reduced density matrix. Once a perturbation is added to this system, these eigenvalues will be lifted to a finite value. Since everything is finite throughout this procedure, the zero eigenvalues will be continuously increase to a nonzero value. Therefore, simply getting rid of the infinite eigenvalues of $\log \rho_A$ will not be able to get rid of these eigenvalues that are large but not infinite.

Motivated from these examples, we define a regularized version of the entanglement Hamiltonian.

**Definition 16.** *Regularized entanglement Hamiltonian $\hat{H}_A^\Lambda$ with a cutoff $\Lambda$ is*

$$\hat{H}_A^\Lambda = -\sum_{p \geq 1/\Lambda} \log p_i \left|i\right\rangle \left\langle i\right|, \tag{3.52}$$

*where $|i\rangle$ is an eigenstate of $\rho_A$ with an eigenvalue $p_i$.*

Before we present the key lemma, we emphasize two points. First, the bound is determined from the cutoff value $\Lambda$ and the dimension of a *single subsystem A*. Secondly, by choosing the cutoff value to be $\Lambda = \Theta(d_A^6)$, one can make $|\rho_{AB}\Delta_A^\Lambda|_1$ arbitrarily small while bounding the operator norm of $\hat{H}_A^\Lambda$ by $O(\log d_A)$. Roughly speaking, this means that $\log \rho_A$ can be regarded as an operator with an operator norm $O(\log d_A)$ for bounding its correlation with other local observables.

**Lemma 18.**

$$|\rho_{AB}\Delta_A^\Lambda|_1 \leq \frac{d_A^3}{\Lambda^{\frac{1}{2}}} \log \Lambda, \tag{3.53}$$

*where $\Delta_A^\Lambda = \hat{H}_A - \hat{H}_A^\Lambda$.*

*Proof.* Purify $\rho_{AB}$ to $|\psi\rangle_{ABC}$. $|\psi\rangle_{ABC}$ admits the following Schmidt decomposition.

$$|\psi\rangle_{ABC} = \sum_{i=1}^{d_A} \sqrt{p_i} \left|i\right\rangle_A \left|i\right\rangle_{BC}, \tag{3.54}$$

where $p_i$s are the eigenvalues of $\rho_A$ and $|i\rangle_A$ ($|i\rangle_{BC}$) are the basis states for the Hilbert space $\mathcal{H}_A(\mathcal{H}_{BC})$.

For any operator $O \in \mathcal{B}(\mathcal{H}_{AB})$, it allows the following decomposition.

$$O = \sum_{i=1}^{d_A^2} \sum_{i=1}^{d_B^2} \frac{1}{d_A d_B} \mathrm{Tr}(U_{A,i} \otimes U_{B,j} O) U_{A,i}^\dagger \otimes U_{B,j}^\dagger, \tag{3.55}$$

where $U_{A,i}(U_{B,j})$ are unitary operators that are supported on $A(B)$ with appropriate normalization conditions.

$$\begin{aligned} \mathrm{Tr}(U_{A,i} U_{A,j}^\dagger) &= d_A \delta_{ij} \\ \mathrm{Tr}(U_{B,i} U_{B,j}^\dagger) &= d_B \delta_{ij}. \end{aligned} \tag{3.56}$$

In other words, $\{U_{A,i}/\sqrt{d_A}\}$ ($\{U_{B,i}/\sqrt{d_B}\}$) is a complete set of orthonormal basis for $\mathcal{B}(\mathcal{H}_A)$ ($\mathcal{B}(\mathcal{H}_B)$) under a Hilbert-Schmidt inner product $\langle O_1, O_2\rangle = \mathrm{Tr}(O_1^\dagger O_2)$. Such basis set always exists for a finite dimensional Hilbert space.[108] Equation 3.55 is equivalent to the following expression.

$$O = \sum_{i=1}^{d_A^2} O_{B,i} \otimes U_{A,i}^\dagger, \tag{3.57}$$

where

$$O_{B,i} = \frac{1}{d_A}\mathrm{Tr}_A(U_{A,i}O) \tag{3.58}$$

$$= \sum_{j=1}^{d_B} \frac{1}{d_A d_B}\mathrm{Tr}(U_{A,i}\otimes U_{B,j}O)U_{B,j}^\dagger. \tag{3.59}$$

Also, $O_{B,i}$ can be bounded as follows.

$$\|O_{B,i}\| = \frac{1}{d_A}\sup_{|\phi\rangle_{BC}} \sum_{i=1}^{d_A} \langle\phi|_{BC}\langle i|_A U_{A,i}O|i\rangle_A|\phi\rangle_{BC} \tag{3.60}$$

$$\leq \sum_{i=1}^{d_A}\frac{1}{d_A}\|U_{A,i}O\| = \|O\|. \tag{3.61}$$

Rewriting $\mathrm{Tr}(\rho_{AB}\Delta_A^\Lambda O_{B,i}\otimes U_{A,i}^\dagger)$ as $\langle\psi|_{ABC}\Delta_A^\Lambda O_{B,i}\otimes U_{A,i}^\dagger|\psi\rangle_{ABC}$,

$$\langle\psi|_{ABC}\Delta_A^\Lambda O_{B,i}\otimes U_{A,i}^\dagger|\psi\rangle_{ABC} = \mathrm{Tr}(\rho_A^{\frac{1}{2}}\Delta_A^\Lambda U_{A,i}^\dagger\rho_A^{\frac{1}{2}}\tilde{O}_{B,i}^T), \tag{3.62}$$

where $\tilde{O}_{B,i} = VO_{B,i}V^\dagger$ with an isometry $V = \sum_i |i\rangle_A\langle i|_{BC}$. $O^T$ is the transpose of $O$. Equation 3.62 can be bounded by

$$|\rho_A^{\frac{1}{2}}\Delta_A^\Lambda|_1\|U_i^\dagger\rho_A^{\frac{1}{2}}\tilde{O}_i^T\| \leq \frac{d_A}{\Lambda^{\frac{1}{2}}}\log\Lambda\|O_i\|. \tag{3.63}$$

Summing over all $i$, we get

$$|\mathrm{Tr}(\rho_{AB}\Delta_A^\Lambda O)| \leq \|O\|\frac{d_A^3}{\Lambda^{\frac{1}{2}}}\log\Lambda \tag{3.64}$$

□

One immediate consequence of this lemma is the following corollary.

**Corollary 3.**

$$|\,Tr(\rho_{AB}\log\rho_A O)| \leq 6\|O\|\log d_A \tag{3.65}$$

Using this bound, one can obtain an infinitesimal change of the entanglement entropy under a

unitary evolution generated by a quasi-local Hamiltonian.

$$|\frac{dS(A)}{ds}| = O(\epsilon \log d_A), \tag{3.66}$$

where $\epsilon = |H - \tilde{H}|$ is the error for approximating the quasi-local Hamiltonian $H$ by a strictly local Hamiltonian $\tilde{H}$.

It is even possible to obtain a similar bound for the connected correlation function.

**Corollary 4.** *Consider a connected correlation function* $\mathcal{C}(O_1, O_2) = \langle O_1 O_2 \rangle - \langle O_1 \rangle \langle O_2 \rangle$. *If* $\mathcal{C}(O_1, O_2) \leq \|O_1\| \|O_2\| \epsilon$ *for all* $O_1, O_2$,

$$|\mathcal{C}(\hat{H}_A, O)| \leq \epsilon \|O\| (18 \log d_A + 4 \log \frac{1}{\epsilon}). \tag{3.67}$$

While Lemma 18 is a bit technical, it has a succinct physical meaning: for the purpose of bounding a correlation between the entanglement Hamiltonian $\hat{H}_A$ and other local observables, one can regard it as a *bounded* operator with an operator norm that scales as the $O(\log d_A)$. In the case of the quantum many-body systems, $O(\log d_A) = O(V_A \log d)$, where $V_A$ is the volume of the subsystem and $d$ is the local particle dimension. This is a good sign, since in many cases correlation decays superpolynomially for the systems we are interested in. For example, it is known that correlation decays exponentially fast in a gapped quantum many-body system.[56, 57] Also, the nonlocal observables that appear in the theory of quasi-adiabatic continuation can be approximated by a strictly local operator with a superpolynomially decaying tail.[119] Indeed, this regularization technique will prove to be useful for analyzing the local properties of the entanglement Hamiltonian and the stability of the topological entanglement entropy, which are explained in Chapter 5 and 6.

# Chapter 4

# Long-range entanglement is necessary for a topological storage of quantum information

Entanglement entropy is a canonical measure for quantifying entanglement in a bipartite pure state.[136] One of the motivations behind these studies is that the entanglement entropy is a useful probe for detecting the phase of the quantum many-body system. For example, entanglement entropy in one-dimensional critical systems follows a universal logarithmic scaling law, and its prefactor is related to the conformal charge of the theory.[135] In two spatial dimensions, the quantum dimension of the topological quantum field theory describing the low-energy physics can be inferred from a constant subcorrection term of the entanglement entropy.[79, 27, 71]

Another important motivation comes from the numerical simulation of quantum many-body systems. Classes of variational ansatz such as the matrix product states(MPS),[14, 137] projected entangled pair states(PEPS),[21] and the multi-scale entanglement renormalization ansatz(MERA)[22] have certain entropy scaling laws. Since these variational states reproduce the entanglement scaling of gapped/critical systems, they are suitable for efficiently simulating the ground state properties of the quantum many-body systems. In particular, a rigorous argument can be made for 1D gapped systems, where an explicit scaling relation between the entanglement entropy and the MPS bond dimension is known.[20, 52, 53]

Recently another possibility has been explored by several authors.[55, 120, 116] Their approaches differ from the conventional ones in several aspects. For one thing, they explicitly use the entanglement entropy over multiple subsystems simultaneously. Moreover, the main objective of these works is not necessarily focused on obtaining an order parameter of the phase. Rather, they are interested in identifying a hidden structure of the quantum many-body phase that protects their universal properties.

This hidden structure by in large can be attributed to the structure of states that are ap-

proximately conditionally independent. A tripartite state $\rho_{ABC}$ is conditionally independent if its conditional mutual information $I(A:C|B) = S(AB)+S(BC)-S(B)-S(ABC)$ is equal to 0, where $S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$ is the entanglement entropy of the subsystem $A$. Similarly, a tripartite state is approximately conditionally independent if its conditional mutual information is close to 0. At least for gapped quantum many-body systems, there is a good reason to expect such subsystems to appear quite naturally.[55] The power of these approaches lies on the fact that their argument only relies on the generic properties of the ground state alone. Moreover, the structure that arises from the vanishing conditional mutual information is manifestly nonlocal: it gives a nontrivial constraint between the density matrices over large regions.[122, 123, 124]

Here we explore this structure further in the context of finding a fundamental limit on the information storage capacity of the quantum many-body system. There has been a number of results in recent years, where information storage tradeoff bounds for *local quantum codes* have been obtained.[34, 35, 36, 37, 38] A local quantum code refers to a set of degenerate ground states stabilized by a sum of geometrically local commuting projectors. Important examples include quantum double and Levin-Wen model.[3, 26] Bravyi et al. showed that the following bound holds for such systems

$$kd^{2/(D-1)} = O(n), \tag{4.1}$$

where $k$ is the number of qubits, $d$ is the code distance, $D$ is the spatial dimension of the lattice, and $n$ is the number of particles.[35]

Local quantum codes cover a rich array of systems. Indeed, it has been conjectured that Levin-Wen model can describe any non-chiral topologically ordered systems in two spatial dimensions.[26] However, none of these models give rise to the chiral gapless boundary excitations. Hence, the tradeoff bound cannot be applied to physical systems exhibiting the integer quantum Hall effect or the fractional quantum Hall effect(FQHE).[5, 6] Furthermore, there are models that are topologically ordered yet do not necessarily have a local commuting parent Hamiltonian. Examples include certain quantum dimer models,[138, 139] parent Hamiltonians of PEPS,[24, 25] and Kitaev's honeycomb model.[140]

To deal with such systems, we approach this problem by focusing only on the properties of the state. More precisely, we obtain an upper bound on the number of topologically protected qubits in terms of a linear combination of entanglement entropies over some local regions of a *single wavefunction*. Entanglement entropies are later related to the code parameters. At first, the preceding claim might seem oxymoron to some of the readers. How can one bound a number of degenerate ground states without knowing the parent Hamiltonian? Further, how can one even define an entanglement entropy of a subsystem if there are multiple states? Answers to both of these questions lie on an important difference between a classical error-correcting code and a quantum

error-correcting code.

The key property of the quantum error-correcting code that sets apart from its classical counterpart is the local indistinguishability property. While a more general version of this definition exists,[141, 142, 143] it will be convenient to use the concept of topological quantum order(TQO)[59] for our purpose. As we have covered in Chapter 1, we say that a set of states $\{|\psi_i\rangle\}_{i=1,\cdots,N}$ satisfies a TQO condition with $(r,\epsilon)$-error if

$$\begin{aligned} |\langle\psi_i|\,\phi\,|\psi_j\rangle| &\leq \|\phi\|\epsilon, \quad i \neq j \\ |\langle\psi_i|\,\phi\,|\psi_i\rangle - \langle\psi_j|\,\phi\,|\psi_j\rangle| &\leq \epsilon\|\phi\|, \end{aligned} \tag{4.2}$$

holds for any $\phi$ that is restricted to a ball of radius $r$, where $\|\cdots\|$ is the operator norm. If the approximation radius $r$ and the approximation error $\epsilon$ are obvious from the context, we shall simply say that the states are locally indistinguishable. An important consequence of Equation 4.2 is that the reduced density matrices of the locally indistinguishable states are close to each other in a trace norm. Therefore, one can unambiguously define the entanglement entropy of the aforementioned set of states up to a small error, so long as the subsystem can be contained in a ball of radius $r$. We shall call such subsystems to be *local*.[1]

Roughly speaking, each of the qubits in the quantum error-correcting code is entangled to each other so that the information can be stored nonlocally. As the number of encoded qubits increase, more entanglement is necessary to distribute the information sufficiently nonlocally. Hence, there has to be an inherent limit on the number of protected qubits if the entanglement across different subsystems are bounded.

Here we shall set this intuition on a rigorous ground. Our approach is purely information-theoretic: we only use the strong subadditivity of entropy as our main technical tool.[144] However, we believe it would be more instructive to discuss some of the related results to understand the motivation behind our construction. Strong subadditivity of entropy asserts that the conditional mutual information $I(A:C|B)$ of a tripartite quantum state $\rho_{ABC}$ is nonnegative. While this statement is true for *any* quantum states, a special structure arises when the inequality is satisfied with an equality. In particular, Petz[122] showed that a tripartite state $\rho_{ABC}$ can be reconstructed from $\rho_{AB}$ and $\rho_{BC}$[2] if the conditional mutual information vanishes:

$$\rho_{ABC} = \rho_{AB}^{\frac{1}{2}}\rho_B^{-\frac{1}{2}}\rho_{BC}\rho_B^{-\frac{1}{2}}\rho_{AB}^{\frac{1}{2}}. \tag{4.3}$$

Recursively applying Petz's result, Hastings and Poulin showed that quantum systems that

[1] We note in passing that classical error-correcting codes in general do not satisfy Equation 4.2, *e.g.,* a classical repetition code.

[2] Note that $\rho_B$ can be obtained from either $\rho_{AB}$ or $\rho_{BC}$.

saturate the equality condition for certain subsystems can be completely reconstructed from the local reduced density matrices alone.[55] A simple corollary of their result is that there cannot be another distinct state that is locally indistinguishable from the original state. To understand why, suppose there are two distinct states $\rho_1, \rho_2$ that are locally indistinguishable. Since both states have the same local reduced density matrices, the conditional mutual information computed from these local density matrices must be both 0. Due to the same reason, a recursive application of Equation 4.3 must yield a same recovered state for both $\rho_1$ and $\rho_2$. Since $\rho_1$ and $\rho_2$ were assumed to be distinct, this contradicts the original assumption. While this argument can be applied quite generally, it is not robust against a small deviation from the original assumptions: if the conditional mutual information is only approximately equal to 0, Equation 4.3 cannot be used anymore. Also, Hastings and Poulin's result applies only to tree graphs. We shall discuss how these issues can be circumvented by giving several concrete examples. Before we go into the details, we would also like to mention that some of the work also appears in Ref.[145].

## 4.1 1D system : correlation decay limits topological protection

We first apply our result to an one-dimensional system, see Figure 4.1. Without loss of generality, let us assume that there are $N$ states that satisfy the TQO condition with a sufficiently large approximation radius and zero approximation error. For a maximally mixed state over the $N$ states $\{|\psi_i\rangle\}$, apply the strong subadditivity of entropy:

$$I(A:C|B) \geq 0, \tag{4.4}$$

where $A, B$, and $C$ are three contiguous subsystems that partition the chain. Rearranging the entanglement entropies over the subsystems, we conclude:

$$S(AB) + S(BC) - S(B) \geq \log N, \tag{4.5}$$

where the underlying state is the maximal mixed state over the $N$ states. Since each of the states $|\psi_i\rangle$ are locally indistinguishable from each other, the inequality can be rewritten in the following form:

$$I(A:C) \geq \log N, \tag{4.6}$$

where the underlying state is *one of the topologically ordered states*. Here we have used the purity of the global state.

From this analysis, one can see that the mutual information between two different subsystems is

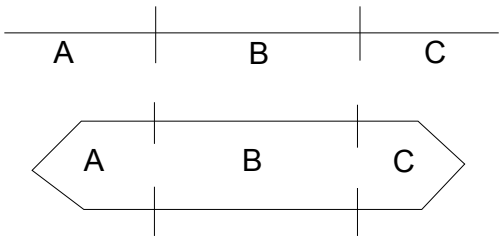



Figure 4.1: 1D chain with an open and closed boundary condition.

bounded from below by a constant if there exists at least two topologically ordered states. Therefore, any 1D system with an open boundary condition cannot have more than one locally indistinguishable ground states if correlation decays sufficiently fast. More precisely, consider a 1D chain with $n$ qubits, a code distance $d$, and a number of encoded qubits $k$. Equation 4.6 implies that one cannot have a code distance almost saturating $n$, *i.e.,* $d = n - O(1)$, if correlation decays asymptotically. The reason is that (i) $I(A:C)$ can be bounded in terms of $|\rho_{AC} - \rho_A \otimes \rho_C|_1$ using Fannes inequality, and (ii) the trace distance between $\rho_{AC}$ and $\rho_A \otimes \rho_C$ can be tightly bounded by a connected correlation function between two local observables[52]:

$$|\rho_{AC} - \rho_A \otimes \rho_C|_1 \leq \min(d_A, d_C) \max_{M_A, M_C \leq I} \langle M_A M_C \rangle - \langle M_A \rangle \langle M_C \rangle,$$

where $\langle O \rangle$ is a short-hand notation for the expectation value of $O$ on one of the ground states.

Our approach lets us bound even the storage capacity for critical systems. Despite the large amount of entanglement present in the critical state, there cannot be another locally indistinguishable state with a code distance $d = n - O(\log n)$ since correlation decays algebraically. The power of our approach comes from the fact that we do not need any specific information about the Hamiltonian for obtaining a bound on the number of topologically ordered states.

## 4.2 2D system : an inequality between topological entanglement entropy and topological degeneracy

We discuss a class of models in two spatial dimensions that satisfy a strict form of area law. Loosely speaking, we say a model satisfies a strict form of area law if the nonuniversal contribution to the entanglement entropy can be canceled out by taking a judicious choice of a linear combination. The

entanglement entropy of these models can be written as

$$S(A) = a|\partial A| - b_0(A)\gamma + O(e^{-|\partial A|/\xi}), \tag{4.7}$$

where $a$ is a nonuniversal constant, $|\partial A|$ is the number of particles along the boundary of $A$, $\xi$ is the correlation length, and $\gamma$ is the topological entanglement entropy.[27, 71] $b_0(A)$ is the number of connected of components of the boundary of $A$. The correlation length will be equal to 0 for a fixed-point wavefunction of some RG flow, but otherwise would remain nonzero.

Suppose we have $N$ states $\{|\psi_i\rangle\}_{i=1,\cdots,N}$ with $(cL, O(e^{-L/\xi}))$-TQO condition that satisfy Equation 4.7 for a local subsystem. Here we have assumed our system to be on a $L \times L$ torus. The specific value of the numerical constant $0 < c < 1$ is irrelevant for the analysis, as long as it is sufficiently close to 1. Our first main result gives a rigorous lower bound of $\gamma$ in terms of $N$, up to a small correction that vanishes in the thermodynamic limit.

$$\log N \leq 2\gamma + O(L^2 e^{-L/\xi}) \tag{4.8}$$

The idea for proving Equation 4.8 is to apply the Markov entropy decomposition(MED) to a maximally mixed state over the $N$ states.[55] More precisely, consider a sequence of subsystems $A_i, B_i, C_i$, $i = 1, \cdots, n$ such that (i) $A_iB_iC_i = A_{i+1}B_{i+1}$ (ii) $A_1B_1$ and $B_iC_i$ are *local* (iii) $A_nB_nC_n$ is the entire system. For such choice of subsystems, following linear combination of entanglement entropy is nonnegative due to the strong subadditivity of entropy:

$$\sum_{i=1}^{n} I(A_i : C_i|B_i) = S(A_1B_1) + \sum_{i=1}^{n} S(B_iC_i) - S(B_i) - S(A_nB_nC_n).$$

By choosing the global state to be a uniform mixture of the $N$ locally indistinguishable states, *i.e.,* $\sum_{i=1}^{N} \frac{1}{N} |\psi_i\rangle \langle\psi_i|$, we arrive at the following bound:

$$\log N \leq S(A_1B_1) + \sum_{i=1}^{n} S(B_iC_i) - S(B_i). \tag{4.9}$$

Since $A_1B_1$, $B_iC_i$, and $B_i$ are all local, their entanglement entropy can be replaced by an entanglement entropy of one of the states $|\psi_j\rangle$ with a small correction. The correction term can be estimated by using Fannes inequality[106] which holds for any quantum states $\rho, \sigma$ supported on a $d$-dimensional Hilbert space:

$$|S(\rho) - S(\sigma)| \leq \epsilon \log d - \epsilon \log \epsilon, \quad \epsilon := |\rho - \sigma|_1, \tag{4.10}$$

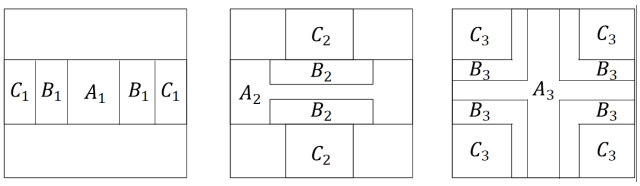



Figure 4.2: An example of the subsystem partition.

where $|\cdots|_1$ is the trace norm. Equation 4.8 can be derived by choosing an appropriate set of subsystems such that the boundary contributions cancel out, while the universal term survives. One choice of such subsystems is depicted in Figure 4.2. Each of the diagrams depict the subsystems $A_i, B_i, C_i$, $i = 1, 2, 3$ with a property $A_iB_iC_i = A_{i+1}B_{i+1}$. The only nonlocal contribution in the sum of the conditional mutual information $\sum_{i=1}^{3} I(A_i : C_i|B_i)$ is the entanglement entropy of $A_3B_3C_3$, which reduces to $\log N$ for a maximally mixed state over $N$ locally indistinguishable states. The rest of the contributions can be computed from the formula for the entanglement entropy of local subsystems, *i.e.,* Equation 4.7.

Equation 4.8 confirms the intuition that an amount of long-range entanglement limits the topological ground state degeneracy. For 2D models supporting anyons, this result is already known. The ground state degeneracy of a topologically ordered system is $n_p^2$, where $n_p$ is the number of particle types.[2] On the other hand, the topological entanglement entropy is related to the total quantum dimension of the system:

$$\gamma = \log \sqrt{\sum_a d_a^2},$$

where $d_a$ is the quantum dimension of a particle with a topological charge $a$.[27, 71] Since non-Abelian charges can have a quantum dimension that is strictly larger than 1, topological entanglement entropy has to be always larger or equal to $\frac{1}{2}\log N$.[27] However, we emphasize that we did not assume anything about the Hamiltonian at all. Equation 4.8 was derived only from the property of the states. Our result shows that even the topological ground state degeneracy - typically thought as a property inherited from the Hamiltonian - is already strongly constrained by the structure of the state alone. Admittedly our premise about the form of the entanglement entropy is quite restrictive. However, we note that there are number of models whose ground state entanglement entropy can be either analytically computed[3, 26, 27, 71, 146, 25] or computed with a good numerical precision[147, 148, 149, 150, 151].

It is also important to note that the following expression gives an alternative definition of the topological entanglement entropy that is inherently robust in one direction, at least for certain models.

$$\gamma := \frac{1}{2} \min_{A_i,B_i,C_i} [S(A_1B_1) + \sum_{i=1}^{n} S(B_iC_i) - S(B_i)]. \tag{4.11}$$

An advantage of Equation 4.11 over the conventional definition - constant subcorrection term of the entanglement entropy- is that it is well-defined even for systems that are not translationally invariant. For fixed-point Hamiltonians supporting Abelian anyons, Equation 4.8 is saturated with an equality. Since the ground state degeneracy is protected from a generic local perturbation, the lower bound for $\gamma$ defined as in Equation 4.11 remains stable under an adiabatic evolution. Hence, one can obtain an one-sided stability bound for the topological entanglement entropy from the stability property of the ground states alone. For such models, the stability of the topological entanglement entropy would follow by giving a tight upper bound matching the lower bound of Equation 4.8 in the thermodynamic limit. The adiabatic evolution can be simulated by a unitary evolution generated by a path-dependent quasi-local "Hamiltonian" with a superpolynomially decaying tail.[33] Hence, locally indistinguishable states become approximately locally indistinguishable with a correction decaying superpolynomially in the system size. These observations suggest that, in order to prove the stability of topological entanglement entropy for Abelian anyon models, it suffices to obtain a rigorous upper bound under the adiabatic evolution.

There are several assumptions that we have implicitly assumed in the preceding analysis. First, we have assumed that the entanglement entropy of a region that can be contained in a radius of $cL$ for a sufficiently large value of $0 < c < 1$ is expressed as Equation 4.7. However, we did not specify the value of $c$ explicitly. We have also assumed that the corrections to the area law and the local indistinguishability property decays exponentially in the system size. Here we show that these conditions can be relaxed significantly in general.

More precisely, we obtain an inequality analogous to Equation 4.8 under the following set of assumptions.

- The system satisfies the $(cL, \epsilon)$-TQO condition with $0 < c < 1$, with $\epsilon = O(\frac{1}{L^{\alpha_1}})$, $\alpha_1 > 0$.
- A number of locally indistinguishable states satisfying the preceding TQO condition is $N$
- The correction to the area law decays algebraically, *i.e.,* $S(A) = a|\partial A| - \gamma + O(\frac{1}{|A|^{\alpha_2}})$, $\alpha_2 > 0$.
- Local dimension of the particles is $d$.

We begin by bounding a difference between the entanglement entropy of a single wavefunction, say $|\psi_1\rangle$, and the entanglement entropy of a mixed state $\rho = \frac{1}{N}\sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i|$. Without loss of generality, we shall assume that Equation 4.7 is the entanglement entropy formula for $|\psi_1\rangle$. Denoting the entanglement entropy of the single wavefunction as $S^1(A)$ and the entanglement entropy of the mixed state $\rho$ as $S^\rho(A)$, we obtain the following bound.

**Lemma 19.** *Let $A$ be a region that can be contained in a ball of radius $cL$.*

$$|S^1(A) - S^\rho(A)| \leq \epsilon(d|A| - \log \epsilon) \tag{4.12}$$

This result follows from a simple application of the Fannes inequality, see Equation 4.10.

As we did previously, consider a sequence of subsystems $A_i, B_i$, and $C_i$ such that $A_iB_iC_i = A_{i+1}B_{i+1}$. By choosing the subsystems $A_1B_1$ and $B_iC_i$ to be small enough to be contained in a ball of radius $cL$,

$$\log N \leq S^{\rho}(A_1B_1) + \sum_{i=1}^{n} S^{\rho}(B_iC_i) - S^{\rho}(B_i) \tag{4.13}$$

$$\leq S^{1}(A_1B_1) + \sum_{i=1}^{n} S^{1}(B_iC_i) - S^{1}(B_i) + \epsilon_{LA}, \tag{4.14}$$

where $\epsilon_{LA}$ is the approximation error from Equation 4.12. One can choose the size of the subsystems $A_1B_1$ and $B_iC_i$ to be $\Theta(c^2L^2)$, in which case $n$ becomes $n = \Theta(\frac{1}{c^2})$. Therefore,

$$\epsilon_{LA} = O(\epsilon d L^2). \tag{4.15}$$

Also,

$$S^{1}(A_1B_1) + \sum_{i=1}^{n} S^{1}(B_iC_i) - S^{1}(B_i) = 2\gamma + O(\frac{c^{-2(1+\alpha_2)}}{L^{2\alpha_2}}). \tag{4.16}$$

Combining these bounds together, for a constant $c$ and $d$, we conclude that

$$\log N \leq 2\gamma + O(L^{2-\alpha_1}) + O(L^{-2\alpha_2}). \tag{4.17}$$

Therefore, for a sufficiently large system size, our main result holds for any $\alpha_1 > 2$, $\alpha_2 > 0$. The significance of this bound comes from the fact that a conservative estimate for the area law correction term gives $\alpha_2 = \frac{1}{2}$.[27, 152] The preceding analysis shows the robustness of our bound. Even under an algebraic correction to the ideal wavefunction, the inequality remains intact.

We would also like to explain the intuition behind why a factor of 2 appears in front of $\gamma$. It should be clear from the construction that the area terms cancel out. Therefore, one only needs to be concerned about the topology of the subsystems $A_1B_1$, $B_iC_i$, and $B_i$. The general idea is that for the construction in Figure 4.2, $A_1B_1$ and $B_1C_1$ are simply connected, yet $B_1$ is a union of two simply connected subsystems. Hence the topological contribution must be $\gamma$ for $A_1B_1$ and $B_1C_1$, and $2\gamma$ for $B_1$. Similarly, $B_2C_2$ gives a contribution of $\gamma$, while $B_2$ gives $2\gamma$. $B_3C_3$ is simply connected, so it gives a contribution of $\gamma$. $B_3$ is an annulus, hence it gives $2\gamma$. Combining these results together, the topological contribution to our bound becomes $2\gamma$. A similar analysis can be carried out for other choice of subsystems as well. What is important is that Equation 4.13 holds for *any* choice of subsystems. Therefore, one can always optimize over different set of subsystems to obtain the best bound.

## 4.3 Higher-dimensional systems

The preceding argument is based on the observation that the entanglement entropy of the subsystem can be decomposed into two parts. The first part is the local entanglement, which can be canceled out by making a judicious choice of the subsystems. More precisely, we expect the entanglement entropy of a subsystem $A$ to have the following form:

$$S(A) = S_{local}(A) + S_{topo}(A),$$

where $S_{local}(A)$ is an entanglement that can be decomposed into strictly local contributions.[152] The rest of the contributions, including the finite-size effects, are included in $S_{topo}(A)$. For such systems, the maximal number of topologically protected states depends on the scaling law of $S_{topo}(A)$. Alternatively, a quantum error-correcting code with an extensive ground state degeneracy would imply that there is an extensive subcorrection term for the entanglement entropy. Interesting examples include Chamon's model and Haah's cubic code, which are known to have a ground state degeneracy that increases as $N = 2^{\Theta(L)}$ for certain choices of $L$.[153, 154, 48] Therefore, the entanglement entropy of these models must have a subcorrection term that grows as $\Omega(L)$. Also, one may argue on a physical ground that the subcorrection term to the area law for gapped systems typically scales as $|A|^{D-2}$.[152] For such systems, the maximal number of topologically protected qubits is bounded by $O(L^{D-2})$.

## 4.4 Bounds for more generic systems

In general, one cannot expect the leading terms of the entanglement entropy to be canceled out by choosing an appropriate set of subsystems, especially for critical systems and ground states of nonlocal Hamiltonian. We show that, even for such generic systems, a nontrivial tradeoff bound can be obtained. Using the subadditivity of entropy, *i.e.,* $I(A : B) = S(A) + S(B) - S(AB) \geq 0$, Equation 4.9 yields the following inequality:

$$k \leq \sum_i S(X_i), \quad |X_i| < d \tag{4.18}$$

for a quantum code with a code distance $d$ and a number of encoded qubits $k = \log N$, where $\{X_i\}$ is a partition of the system. Dividing both sides of the inequality by the number of particles $n$, we find that the rate of a quantum error-correcting code $\frac{k}{n}$ is bounded by the average entanglement entropy per volume over any partitions $\{X_i\}$, $|X_i| < d$. Our result shows that studying the entanglement properties of a quantum error-correcting code is a relevant problem for understanding its fundamental limit. In particular, Equation 4.18 gives a necessary condition for a quantum error-correcting code

to have a nonvanishing rate: its average entanglement entropy over subsystems smaller than the code distance must satisfy a strict volume law.

Equation 4.18 can be used to obtain a tradeoff bound for quantum codes satisfying a subvolume law of entanglement entropy. If the entanglement entropy satisfies a subvolume law,

$$S(A) = O(|A|^{\alpha}),$$

we can obtain the following tradeoff bound:

$$kd^{1-\alpha} = O(n). \tag{4.19}$$

There are several important differences between Equation 4.1 and Equation 4.19. On one hand, Equation 4.19 is more general than Equation 4.1 in that it does not require any structure about the parent Hamiltonian. On the other hand, Equation 4.19 provides weaker tradeoff bound than Equation 4.1 does.

## 4.5 Stability of the lower bound

Based on the idea that long-range entanglement is necessary for a topological storage of quantum information, we obtain a rigorous lower bound for the topological entanglement entropy that remains stable under a local unitary transformation. More precisely, the existence of the topological entanglement entropy can be attributed to the existence of a set of states that are locally indistinguishable. Since an adiabatic evolution that does not close the energy gap can be simulated by a unitary generated by a sum of quasi-local Hamiltonians, we can use the Lieb-Robinson bound technique to formalize this statement.

We show that a set states with a $(r, \epsilon)$-TQO condition remains to satisfy a $(r', \epsilon')$-TQO condition with some modified constants $r'$ and $\epsilon'$ under a unitary evolution generated by a local Hamiltonian. Suppose the unitary evolution generated by the Hamiltonian satisfies the Lieb-Robinson bound:

$$\|[O_A(t), O_B]\| \leq c\|O_A\|\|O_B\| \min(|A|, |B|)e^{c_1(vt - d(A,B))}. \tag{4.20}$$

$$|\langle\psi_i(t)| O_A |\psi_i(t)\rangle - \langle\psi_j(t)| O_A |\psi_j\rangle (t)| = |\langle\psi_i(0)| U^{\dagger}(t) O_A U(t) |\psi_i(0)\rangle - \langle\psi_j(0)| U^{\dagger}(t) O_A U(t) |\psi_j\rangle (0)| \tag{4.21}$$

$$\leq \|O_A\|\epsilon + \|O_A\||A|e^{c_1(vt-x)}, \tag{4.22}$$

where $x$ is chosen to be the largest distance such that any operator that is supported on a set of points that are distance $x$ or more away from $A$ is locally indistinguishable for any of the states $\{|\psi_i(0)\rangle\}$. Therefore, $(r, \epsilon)$-TQO condition becomes a $(r - x, \epsilon + O(r^D e^{c_1(vt-x)}))$-TQO condition, where $x$ is a free parameter. For a constant value of $t$, one can set $x$ to be $\Theta(\log r)$ to ensure that the approximation error vanishes in the thermodynamic limit. Therefore, under a unitary evolution generated by a local Hamiltonian, the TQO condition remains stable. Since the quasi-adiabatic continuation can be generated by a sum of quasi-local bounded operators, we expect a similar treatment should be possible for a generic adiabatic evolution as well.

# Chapter 5

# Structure of the entanglement Hamiltonian

It is commonly believed that the gapped phase of quantum many-body systems exhibits area law: entanglement entropy of a simply connected subsystem increases with the area of the boundary.[96] An overwhelming amount of evidences supporting this statement has been suggested, including the explicit proof for a ground state of a one-dimensional gapped system[20], exactly solvable models[26], and variational wavefunctions[155]. The constant subcorrection to the entanglement entropy - also known as the topological entanglement entropy - can be extracted by judiciously choosing a set of subsystems that cancel out the boundary contributions.[27, 71] The topological entanglement entropy is believed to be a universal constant characterizing the phase of the quantum many-body system.

Li and Haldane(LH) were the first to realize that the spectrum of the reduced density matrix may reveal an information about the phase that cannot be inferred from the entanglement entropy alone.[89, 90] While LH studied reduced density matrix in the orbital cuts, one may study its spectrum along a real-space partition and arrive at a similar conclusion.[91, 92, 93] In particular, it has been recently suggested by several authors that the entanglement Hamiltonian along a real-space partition has a low-lying part that can be described by a local field theory.[94, 95]

Topological entanglement entropy can be obtained from a real-space entanglement Hamiltonian of variational wavefunctions, similar to the way it is extracted from the entanglement entropy.[95] Consequently, the corresponding linear combination of the entanglement Hamiltonian is "topological," in a sense that (i) it does not interact with any local observable and (ii) it is equal to the topological entanglement entropy.

Here we claim that the existence of such topological operator can be attributed to an approximate conditional independence of these quantum states. A tripartite state $\rho_{ABC}$ is conditionally independent if conditional mutual information $I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC}$ is equal to 0. A state is approximately conditionally independent if 0 is replaced by a small number $\epsilon > 0$.

To the best of author's knowledge, Hastings and Poulin were the first to point out that there can be configurations that are conditionally independent even in a quantum many-body system with long-range entanglement.[55] To illustrate their idea, suppose entanglement entropy satisfies an area law with a *universal* constant subcorrection term.

$$S_A = a|\partial A| - \gamma, \tag{5.1}$$

One can show that $I(A:C|B) = 0$ for a choice of $A, B, C$ such that (i) $AB, BC, B, ABC$ are all simply connected and (ii) $A$ and $C$ do not share a boundary.

A state that is conditionally independent saturates the equality condition of the strong subadditivity of entropy.[128] Such state forms a quantum Markov chain, and the structure of the reduced density matrix is vastly restricted compared to an arbitrary state.[122, 123, 124] It is important to note that one cannot directly use these results for a generic quantum many-body system, since the conditional independence condition is unlikely to hold exactly. Still, one may hope for these properties to hold approximately for a sufficiently small conditional mutual information. This is precisely the key idea behind this paper. More specifically, we shall use the recently discovered operator extension of the strong subadditivity of entropy as our main technical tool.[120]

The rest of the paper is structured as follows. In Section 5.1, we shall briefly review several information-theoretic inequalities. In Section 5.2, we shall introduce a diagrammatic trick that leads to the main result of this paper. Its physical interpretation shall be given in Section 5.3.

## 5.1 Approximately conditionally independent states

Strong subadditivity of entropy is one of the most widely used tools in quantum information theory. Its importance stems from the fact that there exists a variety of nontrivial structure theorems that relate the reduced density matrix of different subsystems if the inequality is saturated with an equality condition.[122, 123, 124] In particular, Petz showed that the following relation holds if and only if the conditional mutual information $I(A:C|B)$ is equal to 0.[122]

$$\hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC} = 0, \tag{5.2}$$

where $\hat{H}_A = -I_{A^c} \otimes \log \rho_A$ is a formal definition of the entanglement Hamiltonian. From now on, we denote the left hand side of the equation as $\hat{H}_{A:C|B}$ and refer to it as a *conditional mutual spectrum* of $ABC$. It follows that

$$\mathcal{C}(\hat{H}_{A:C|B}, X) = 0, \tag{5.3}$$

where $\mathcal{C}(\hat{H}_{A:C|B}, X) = \langle \hat{H}_{A:C|B} X \rangle - \langle \hat{H}_{A:C|B} \rangle \langle X \rangle$ is a connected correlation function between the conditional mutual spectrum and an arbitrary operator $X$. $\langle \cdots \rangle$ denotes ground state expectation value.

While such operator trivially has zero correlation with any local operator, exact conditional independence is rarely satisfied by any realistic physical systems. Motivated by this observation, we have obtained an operator extension of the strong subadditivity of entropy, see Equation 3.3. We rewrite the inequality for the reader's convenience:

$$\mathrm{Tr}_{BC}(\rho_{ABC} \hat{H}_{A:C|B}) \geq 0. \tag{5.4}$$

We should again emphasize the important consequence of this inequality: that Equation 3.3 reproduces a statement similar to Equation 5.2 when the conditional mutual information is 0.

$$|\mathrm{Tr}_{ABC}(\rho_{ABC} \hat{H}_{A:C|B} O_A)| \leq \|O_A\| I(A:C|B), \tag{5.5}$$

where $\| \cdots \|$ is $l_\infty$ norm.

If the conditional mutual information vanishes, the corresponding conditional mutual spectrum has zero correlation with any operator supported on $A$. Furthermore, since both $\hat{H}_{A:C|B}$ and $I(A:C|B)$ are symmetric under the exchange of $A$ and $C$, the same statement holds for an operator supported on $C$ as well. Secondly, Equation 3.3 is satisfied by any quantum states. Therefore, unlike Equation 5.2, it can be applied to quantum states that *approximately* saturate the strong subadditivity of entropy.

## 5.2 Correlation bound for the entanglement Hamiltonian

The main goal of this section is to obtain a statement that resembles Equation 5.3 when the global state is a ground state of a gapped quantum many-body system. Such correlation bound can be easily obtained in certain cases using Equation 3.13 alone, but there are also important caveats. For example, there are choices of subsystems that yield a nonzero value of the conditional mutual information even at a fixed point of some renormalization-group flow.[27, 71] Furthermore, Lemma Equation 3.13 alone cannot produce any bound on the correlation between the conditional mutual spectrum $\hat{H}_{A:C|B}$ and an operator supported on $B$. We shall show that, despite these shortcomings, it is still possible to obtain a bound analogous to Equation 5.3 under a reasonable set of assumptions.

We postulate the following modified formula for the entanglement entropy to account for the deviations from the ideal area law.

$$S_A = a|\partial A| - \gamma + \epsilon_A. \tag{5.6}$$

$$S_A + S_B - S_{AB} = \epsilon_{A:B}. \tag{5.7}$$

For a large enough subsystem size, we expect $\epsilon_A$ to approach 0. $\epsilon_{A:B}$ denotes a long-range correlation of the ground state. Due to the exponential clustering theorem, we expect $\epsilon_{A:B}$ to scale as $\min(|A|, |B|)^2 e^{-\frac{2l}{\xi}}$, where $\xi$ is the correlation length and $|A|$ is the volume of the subsystem $A$.[1]

To simplify the analysis, we assume that each of the subsystems are sufficiently smooth and their boundary lengths are $O(l)$. We assume that the support of $X$ is sufficiently small compared to the size of the subsystems. We also assume that $X$ is supported on only one of the subsystems that partitions the system.

### 5.2.1 Modified form of exponential clustering theorem

Before we explain the details of our analysis, we would like to present a technical background about the subject. Exponential clustering theorem states that

$$|\mathcal{C}(O_A, O_B)| \leq c\|O_A\|\|O_B\| \min(|A|, |B|) e^{-\frac{d(A,B)}{\xi}} \tag{5.8}$$

for two spatially separated operator $O_A$ and $O_B$, provided there is a gapped parent Hamiltonian that consists of a sum of geometrically local bounded-norm terms.[56, 57] Since the spectrum of $\hat{H}_A$ is formally unbounded, one cannot directly apply exponential clustering theorem. We circumvent this problem by regularizing the entanglement Hamiltonian and bounding the error from the regularization procedure.

Recall that the regularized entanglement Hamiltonian $\hat{H}_A^\Lambda$ with a cutoff $\Lambda$ is defined as follows:

$$\hat{H}_A^\Lambda = -\sum_{p \geq 1/\Lambda} \log p_i \, |i\rangle \langle i| \, . \tag{5.9}$$

A simple consequence of this construction is that the operator norm is bounded, *i.e.*, $\|\hat{H}_A^\Lambda\| \leq \log \Lambda$. The correction from the regularization can be bounded using the following lemma.

**Lemma 20.**

$$Tr(\rho_{AB} \Delta_A^\Lambda O_B) \leq \|O_B\| \frac{\log \Lambda}{\Lambda} d_A \tag{5.10}$$

*for* $\Lambda \geq 2$, *where* $\Delta_A^\Lambda = \hat{H}_A - \hat{H}_A^\Lambda$.

*Proof.* Purify $\rho_{AB}$ to $|\psi\rangle_{ABC}$. Rewrite the formula as $\mathrm{Tr}(\rho_{AB}\Delta_A^\Lambda O_B) = \langle\psi|_{ABC} \Delta_A^\Lambda O_B |\psi\rangle_{ABC}$. Note that $|\psi\rangle_{ABC}$ admits a Schmidt decomposition $|\psi\rangle_{ABC} = \sum_i \sqrt{p_i} |i\rangle_A |i\rangle_{BC}$, where $\rho_A =$

[1] The volume factor was chosen in such a way that the bound on connected correlation function from mutual information in Ref. [156] yields the exponential clustering theorem in Ref.[57].

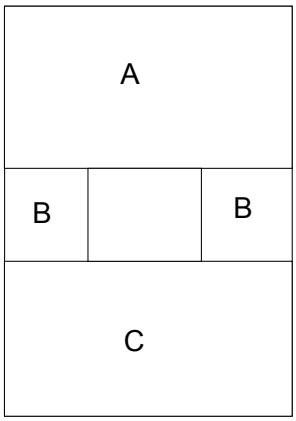



Figure 5.1: Levin-Wen configuration

$\sum_i p_i |i\rangle_A \langle i|_A$. This in turn can be expressed as

$$\sum_{p_i \leq 1/\Lambda} -p_i \log p_i \langle i|_{BC} O_B |i\rangle_{BC} . \tag{5.11}$$

Using $-p_i \log p_i \leq \frac{1}{\Lambda} \log \Lambda$ and $| \langle i| O_B |i\rangle | \leq \|O_B\|$, one can complete the proof. □

### 5.2.2 Derivation of the correlation bound

Consider a configuration proposed by Levin and Wen, see Figure 5.1.[71] From the area law of entanglement entropy, one can see that

$$I(A : C|B) = 2\gamma + o(1). \tag{5.12}$$

Given an operator $X$ that does not have any overlap with the boundary, the objective is to bound a connected correlation function between $X$ and $\hat{H}_{A:C|B}$. If $X$ is sufficiently far away from $ABC$, one can simply use the modified form of the exponential clustering theorem to conclude that their correlation is small. There are three nontrivial cases. First, the support of $X$ is located in a region that is distance $\Theta(1)$ away from $ABC$. Second, the support of $X$ is located in one of the target parties. Third, the support of $X$ is located in the reference party. We shall show that the correlation is small for all of these cases. Further, we show that the bound can be obtained in a sequential manner: one can reduce the third case to the second case, and the second case can be reduced to the first case.

Keeping this reduction in mind, we deal with the third case first. That is, we assume the support

of $X$ is contained in $B$. In this setting, we apply the isolation move:

$$\mathcal{C}(\hat{H}_{A:C|B}, X) = \mathcal{C}(\hat{H}_{AB_1:C|B_2}, X) - \mathcal{C}(\hat{H}_{B_1:C|B_2}, X) \tag{5.13}$$

$$\leq \mathcal{C}(\hat{H}_{AB_1:C|B_2}, X) + 2\|X\| I(B_1 : C|B_2), \tag{5.14}$$

where $B_1 \subset B$ is a subsystem that contains $X$ and $B_2 = B \setminus B_1$, see Figure 3.1. We have used the isolation move and Equation 3.13 from the first line to the second line.

Notice that the support of $X$ is contained in the target party of the deformed subsystem $AB_1$. Now we study the second case. Since the conditional mutual spectrum is invariant under the exchange of two target parties, we assume that the support of $X$ is contained in one of the target parties, say $A$. We apply the separation move in this setting:

$$\mathcal{C}(\hat{H}_{A:C|B}, X) = \mathcal{C}(\hat{H}_{A_2:C|B}, X) + \mathcal{C}(\hat{H}_{A_2:C|A_1B}, X) \tag{5.15}$$

$$\leq \mathcal{C}(\hat{H}_{A_2:C|B}, X) + 2\|X\| I(A_2 : C|A_1B), \tag{5.16}$$

where $A_2 \subset A$ is a subsystem that contains $X$ and $A_1 = A \setminus A_2$, see Figure 3.2

For a sufficiently large subsystem size, the deformed subsystems can be sufficiently separated from $X$. Therefore, one can use the modified form of the exponential clustering theorem. There is a subtle issue that was left out in the preceding discussion. For example, the support of $X$ can be contained in the target party and has a distance $\Theta(1)$ to the reference party. One can still apply the separation move so that the deformed subsystem does not contain $X$. However, the distance will not be small in general. In order to circumvent this issue, one must first apply the isolation move, so that the support of $X$ is sufficiently far away from the reference party. The correction term from this move can be bounded by the absorption move, see Figure 3.3.

## 5.3 Physical interpretation

Setting $\Lambda = d_{ABC} e^{O(l)/\xi}$, we arrive at the following conclusion.

$$|\mathcal{C}(\hat{H}_{A:C|B}, X)| \leq \|X\|(\epsilon_1(l) + \epsilon_2(l))l^2, \tag{5.17}$$

where $\epsilon_1$ represents a deviation from the ideal area law, and $\epsilon_2$ represents an error from the long-range correlation. As $l \to \infty$, the conditional mutual spectrum has vanishing correlation with any local operator, provided that (i) $X$ is supported on one of $A, B, C$, or $(ABC)^c$ and (ii) both $\epsilon_1$ and $\epsilon_2$ decays sufficiently fast. In $l \to \infty$ limit, we have

$$\langle \hat{H}_{A:C|B} X \rangle = I(A : C|B)\langle X \rangle. \tag{5.18}$$

We conclude that the operator $\hat{H}_{A:C|B}$ is topological, in a sense that (i) it has vanishing correlation with any operator that is localized in one of the subsystems and (ii) its eigenvalues contain information about the phase. A set of assumptions to conclude so was that (i) correlation decays exponentially, (ii) the extensive terms of the entanglement entropy cancel out each other, and (iii) the deformation procedure separating $X$ from $ABC$ does not change the topology of the configuration.

We emphasize that the derivation of our result is not necessarily restricted to a pure state. At finite-temperature, entanglement entropy obtains volume contributions, but one may be able to show that those contributions can be canceled out as well. In particular, we expect these conditions to be met for quantum many-body systems at sufficiently high temperature.

In the large volume limit, it seems the local contribution of the reduced density matrices cancel out each other, at least when $I(A : C|B) = o(\frac{1}{l^2})$. We do not have a definitive proof for this statement, but we argue as follows. If $\hat{H}_{A:C|B}$ contains a localized term, one could have chosen $X$ to be an operator supported nearby so as to have a large correlation with the local term. Such terms will violate Equation 5.18. Our result suggests a decomposition of the entanglement Hamiltonian into (i) terms that can be canceled out by a suitable choice of subsystems and (ii) terms that cannot be canceled out and have a small correlation with almost any local operators. It would be interesting if the terms of the first kind can be shown to be quasi-local.

To summarize, We have presented a general argument as to why certain linear combination of entanglement Hamiltonian allows a cancelation of its local degrees of freedom, owing in part to a recently discovered information-theoretic inequality. While our formulation is not as precise as the ones described by the variational wavefunction,[94, 91, 95] it has an advantage of being applicable to a more general class of quantum states. Indeed, we have only used an approximate form of the area law and the exponential clustering theorem, which are strongly believed to be generic properties of a gapped phase.

It would be interesting if the approximate conditional independence can be shown to hold in other systems. There are evidences suggesting that models based on BF theory should satisfy such a condition[43], yet no studies have been performed for exotic models in three dimensions such as Haah's code.[48] As for the finite-temperature states, the approximate conditional independence is one of the key ideas of the quantum belief propagation(QBP) algorithm.[134] Success of the QBP indicates that our result may be applicable to finite-temperature quantum states as well.[157]

On the other hand, we wish to find a deeper insight as to why the conditional independence arises in these systems. In particular, exactly solvable models that satisfy exact conditional independence can be thought as a fixed point of some renormalization-group procedure.[158] Does conditional mutual information of topologically trivial configurations monotonically decrease under such flow?

# Chapter 6

# Perturbative analysis of topological entanglement entropy

Topological order is a new kind of order that cannot be described by Landau's symmetry breaking paradigm. Properties of these exotic phases include a ground state degeneracy that depends on the manifold, anyonic statistics, and long-range entanglement.[3, 26, 27, 71] Such phases are expected to be stable against a generic perturbation if its strength is sufficiently weak and its interaction range is bounded. Indeed, it was shown by several authors that the spectral stability follows under a set of reasonable assumptions.[28, 29, 31]

If the energy gap remains open under the perturbation, one can adiabatically continue from the ground state of the original Hamiltonian to the ground state of the perturbed Hamiltonian.[33] Since the generator of this flow consists of quasi-local terms which decay almost exponentially, the perturbed Hamiltonian has similar properties to the unperturbed Hamiltonian.[33, 64, 29] For example, one can define local operators that create defects with well-defined energies and string operators that can move around the defects. One may argue that the long-range entanglement in the ground state can be preserved in a similar vein, although one must define precisely what the long-range entanglement is.

Long range entanglement in a 2D system refers to the nontrivial constant subcorrection term of the entanglement entropy, also known as the topological entanglement entropy.[79, 27, 71] While a proof with a full mathematical rigor has not been established to the best of author's knowledge, it is widely accepted by now that the topological entanglement entropy is a universal constant that characterizes the phase of the gapped quantum many-body system. If one accepts the topological quantum field theory description of the low-energy physics, there is a simple explanation as to why the topological entanglement entropy remains stable against a generic perturbation.[27] There are also mounting numerical evidences suggesting its stability.[159, 80, 81]

The presence of the long-range entanglement can be interpreted as a consequence of some non-trivial nonlocal constraint. For example, in the ground state of a 2D gapped system supporting

anyonic quasi-particles, the total charge enclosed in some region must add up to be a trivial charge. However, the existence of the constant subcorrection term alone does not necessarily imply that the nature of the constraint is quantum. 3D toric code at a finite-temperature has nonlocal contributions to the entanglement entropy[43], yet such state can be mapped to a Gibbs state of a classical Hamiltonian under a local unitary transformation.[88] We wish to understand if this nonlocal contribution to the entanglement entropy is an invariant of the phase. We would also like to understand the mechanism behind their stability, instead of arguing on the ground of an effective field theory. In such pursuit, we introduce a property of these states that has apparently been unnoticed so far with few notable exceptions.

As we have been alluding in the previous chapters, we shall exploit the properties of the conditionally independent states by using the deformation move. However, the aforementioned deformation move suffers from a problem: that it is only applicable to an observable that does not have an overlap with the boundaries of the subsystems. In order to circumvent this shortcoming, we will need to introduce a variant of the deformation move. Unfortunately, this can be achieved only by enforcing a rather strong assumption: that the state satisfies a $c_0$-boundedness condition. We shall explain what this condition is later. For the moment, we would just like to note that the states in this family include (i) the ground state of the quantum double or Levin-Wen model and (ii) a finite-temperature Gibbs state of a stabilizer Hamiltonian.

## 6.1 The setup

The Hilbert space has a tensor product structure $\otimes_i \mathcal{H}_i$ where $\mathcal{H}_i$ corresponds to the local Hilbert space located at vertices of a square lattice. Local Hilbert space dimension is $d$. We assume a periodic boundary condition with a sufficiently large system size. We define a set of operators having nontrivial support on $\mathcal{H}_A$ as $\mathcal{B}(\mathcal{H}_A)$. The boundary of subsystem $A$ is denoted as $\partial A$. $|A|$ represents the volume of $A$ and similarly $|\partial A|$ is the boundary area of $A$. We set the size of the subsystems to be $\mathcal{O}(l)$ unless specified otherwise.

We consider a family of Hamiltonian $H(s) = H_0 + sV$ and study its behavior in the vicinity of $s = 0$. Both the original Hamiltonian $H_0 = \sum_i h_i$ and the perturbation $V = \sum_i v_i$ consists of a sum of terms that are supported on a ball of radius $r_0$ and the interaction strength is uniformly bounded by $J$, *i.e.,* $\|h_i\|, \|v_i\| \leq J$. $\|\cdots\|$ is $l_\infty$ norm. We denote the spectral gap as $\Gamma(s)$.

Following Bravyi et al.'s construction[59], we define an approximation of a quasi-local operator as follows.

$$[O]_A = \frac{1}{\dim A^c} \mathrm{Tr}_{A^c}(O) \otimes I_{A^c} \tag{6.1}$$

This approximation is motivated from the fact that a correlation generated by a local Hamiltonian falls off exponentially outside an effective light cone. The quasi-local operators generated by such

time evolution can be approximated by a local operator supported on a ball of a finite radius $R$, with the correction term decreasing superpolynomially with $R$.

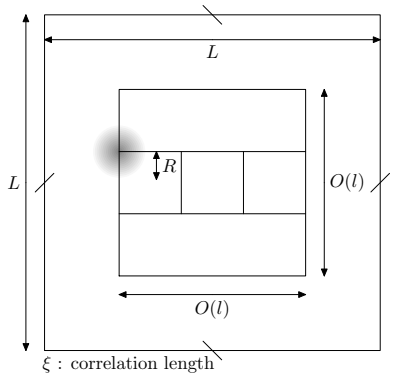



Figure 6.1: The shaded region represents an effect of the perturbation that is smeared out in space. We shall approximate this effect by a strictly local operator with a finite radius $R$. The correction decreases superpolynomially with $R$.

Entanglement spectrum of a subsystem $A$ is defined as $\hat{H}_A = -I_{A^c} \otimes \log \rho_A$, where $\rho_A$ is the reduced density matrix of $A$. We define conditional mutual spectrum as $\hat{H}_{A:C|B} = \hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC}$. Note that

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}) = I(A:C|B). \tag{6.2}$$

We also define $\langle \cdots \rangle = \mathrm{Tr}(\rho \cdots)$ as an expectation value. Throughout this chapter, constants $c$ and $c'$ denote numerical constants, and their exact values may be different in each contexts.

## 6.2 Deformation move for a $c_0$-bounded states

Here we construct a variant of the deformation move that is applicable to $c_0$-bounded states. As in Ref.[120], the statement concerns a correlation bound between $\hat{H}_{A:C|B}$ and an arbitrary operator $O$. The main difference is that here we relax the condition on the support of $O$: $O$ is allowed to be located anywhere, as long as its support is sufficiently small compared to the subsystem. The price we have to pay is that we must impose a condition on the reduced density matrices.

**Definition 17.** *$\rho_{ABC}$ is $c_0$-bounded if*

$$|\mathit{Tr}_C(\rho_{ABC}\hat{H}_{A:C|B})|_1 \leq c_0 I(A:C|B). \tag{6.3}$$

Note that all classical states are 1-bounded. Reduced density matrices of a finite-temperature Gibbs state for the so called "stabilizer models" are also 1-bounded. A detailed explanation about these states shall be presented in Section 6.5. If $I(A:C|B) = 0$, conditional mutual spectrum is

1-bounded by Petz's theorem.[122] More specifically, Petz showed that

$$\hat{H}_{A:C|B} = 0 \tag{6.4}$$

if and only if $I(A:C|B) = 0$.[1]

Following the previous conventions, given a conditional mutual spectrum $\hat{H}_{A:C|B}$, we shall refer $B$ as a *reference party.* $A$ and $C$ shall be referred as *target parties.* Diagrammatically, the reference party will be denoted with an "R" sign and the target parties will be denoted with the "T" signs.

We reiterate the key idea behind the deformation move. For any local operator $O$, one can decompose $\hat{H}_{A:C|B}$ into $\hat{H}_{A_i:C_i|B_i}$ such that either (i) $I(A_i:C_i|B_i) = o(1)$ or (ii) $O$ is sufficiently far away from $A_iB_iC_i$. Such a decomposition can be expressed as a linear combination of the following chain rule, which can be verified easily.

$$\hat{H}_{A_1A_2:C|B} = \hat{H}_{A_2:C|B} + \hat{H}_{A_1:C|A_2B}, \tag{6.5}$$

Similar to the previous approach outlined in Chapter 3, we define three elementary deformation moves.

The first step in the deformation procedure is to apply an *isolation move.* Goal of the isolation move is to deform the boundary between the reference and the target party so that the support of $O$ is sufficiently separated from the reference party, see Figure 6.2 Applying the isolation move, the conditional mutual spectrum is deformed in such a way that (i) for the new conditional mutual spectrum, $O$ is sufficiently far away from the reference party and (ii) the difference is a conditional mutual spectrum with a small conditional mutual information.

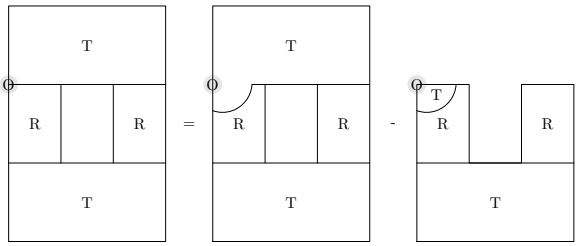



Figure 6.2: Isolation move for a $c_0$-bounded state

Once the support of $O$ is isolated from the reference party, we can apply a *separation move*, which separates the support of $O$ from the target parties, see Figure 6.3. Applying the separation move, the conditional mutual spectrum is deformed in such a way that (i) for the new conditional mutual spectrum, $O$ is sufficiently far away from both the reference and the target parties and (ii) the difference is a conditional mutual spectrum with a small conditional mutual information.

[1] Here the value of the constant actually does not matter, since both sides of the inequality is 0.

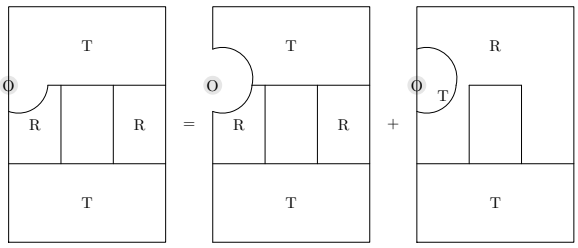



Figure 6.3: Separation move for a $c_0$-bounded state

The last step is to apply an *absorption move*. Absorption move enables us to write the correction terms as a linear combination of $\hat{H}_{A_i:C_i|B_i}$ such that (i) the support of $O$ is contained in either $A_iB_i$ or $B_iC_i$ and (ii) $I(A_i : C_i|B_i) = o(1)$, see Figure 6.4. Applying the absorption move, the conditional mutual spectrum is expressed in terms of a linear combination of the conditional mutual spectrum $\hat{H}_{A_i:C_i|B_i}$ such that (i) the support of $O$ is contained in either $A_iB_i$ or $B_iC_i$ and (ii) $I(A_i : C_i|B_i)$ is small.

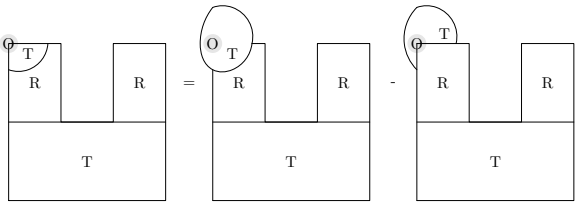



Figure 6.4: Absorption move for a $c_0$-bounded state

To summarize, given a local operator $O$, one can decompose the conditional mutual spectrum $\hat{H}_{A:C|B}$ into $\hat{H}_{A':C'|B'}$ and correction terms with the following properties. First, the distance between $A'B'C'$ and the support of $O$ is $\mathcal{O}(l)$. Second, the correction term consists of a sum of the conditional mutual spectrum such that the support of $O$ is contained in the reference party and one of the target parties. Third, the conditional mutual spectra in the correction term have a small conditional mutual information for the ground state of the topologically ordered system.

In Section 6.3 and 6.5, we shall frequently encounter terms of the following form.

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A_i:C_i|B_i}O), \tag{6.6}$$

where $O$ is an operator whose support is contained in $A_iB_i$. If $\rho_{ABC}$ is $c_0$-bounded, this term can be bounded as follows.

$$\begin{aligned}\mathrm{Tr}(\rho_{ABC}\hat{H}_{A_i:C_i|B_i}O) &= \mathrm{Tr}_{A_iB_i}\mathrm{Tr}_{C_i}(\rho_{ABC}\hat{H}_{A_i:C_i|B_i}O)\\ &\le |\mathrm{Tr}_{C_i}(\rho_{ABC}\hat{H}_{A_i:C_i|B_i})|_1\|O\|\\ &\le c_0 I(A:C|B)\|O\|. \end{aligned}\tag{6.7}$$

## 6.3 Ground state of exactly solvable models

The exact formula for the entanglement entropy is known for quantum double and Levin-Wen models.[79, 71, 160] If the subsystem is simply connected, the entanglement entropy satisfies the area law.

$$S_A = a|\partial A| - \gamma, \tag{6.8}$$

where $\gamma$ is the topological entanglement entropy. These systems have zero correlation length, so the density matrices of two nonoverlapping regions factorize, *i.e.,* $\rho_{AB} = \rho_A \otimes \rho_B$. Therefore, the following formula holds for the entanglement entropy:

$$S_{AB} = S_A + S_B, \tag{6.9}$$

where $A \cap B = \emptyset$.

Using the standard perturbation theory, for a family of quantum states $\rho(s)$ that are differentiable with respect to $s$,

$$\frac{dS_A}{ds} = \mathrm{Tr}(\frac{d\rho}{ds}\hat{H}_A). \tag{6.10}$$

Therefore,

$$\begin{aligned}\frac{dI(A:C|B)}{ds} &= \mathrm{Tr}(\frac{d\rho}{ds}\hat{H}_{A:C|B}) \\ &= i\sum_j \mathrm{Tr}([\Phi_{W_\Gamma}(v_j), P_0]\hat{H}_{A:C|B}),\end{aligned} \tag{6.11}$$

where $P_0$ is a projector onto the ground state.

Without loss of generality, let us consider terms $v_j$ that are distance $al$ or less away from $ABC$, where $a > 0$ is some constant. Using the deformation moves,

$$\hat{H}_{A:C|B} = \hat{H}_{A':C'|B'} + \sum_i a_i \hat{H}_{A_i:C_i|B_i}, \tag{6.12}$$

where $d(v_j, A'B'C') = \mathcal{O}(l)$ and $I(A_i : C_i|B_i) = 0$. By Petz's theorem, $\hat{H}_{A_i:C_i|B_i} = 0$. Now approximate $\Phi_{W_\Gamma}(v_j)$ by $[\Phi_{W_\Gamma}(v_j)]_{v_j(cl)}$ for some $c > 0$ such that the support of $[\Phi_{W_\Gamma}(v_j)]_{v_j(cl)}$ does not overlap with $A'B'C'$. This implies the following relation.

$$\mathrm{Tr}([[\Phi_{W_\Gamma}(v_j)]_{v_j(cl)}, P_0]\hat{H}_{A':C'|B'}) = 0. \tag{6.13}$$

To see this, consider an operator $O$ that is supported on one of $A', B', C'$, or $D = (A'B'C')^c$.

$$\begin{aligned} i\mathrm{Tr}([O, P_0]\hat{H}_{A':C'|B'}) &= \frac{d}{dt}\mathrm{Tr}(e^{iOt}P_0e^{-iOt}\hat{H}_{A':C'|B'}) \\ &= \frac{d}{dt}I(A' : C'|B'), \end{aligned} \tag{6.14}$$

where the infinitesimal generator generates a unitary transformation supported on $(A'B'C')^c$. Since the entanglement entropy is invariant under a local unitary transformation, this is 0. The correction terms are of the following form.

$$i\mathrm{Tr}([\Phi_{W_\Gamma}(v_j) - [\Phi_{W_\Gamma}(v_j)]_{v_j(cl)}, P_0]\hat{H}_{A':C'|B'}). \tag{6.15}$$

Using Equation 3.65 and 3.34, we conclude that the effect of each terms are bounded by $cJG^{(I)}(c'\frac{\Gamma l}{2v})l^2d$ for some constant $c$ and $c'$. Since there are $O(l^2)$ terms that are distance $al$ or less away from the configurations, the local contributions from this region scales as $O(JG^{(I)}(c'\frac{\Gamma l}{2v})l^4d)$.

Terms that are distance $al$ or more away from $ABC$ can be bounded by approximating $\Phi_{W_\Gamma}(v_j)$ as $[\Phi_{W_\Gamma}(v_j)]_{v_j(R)}$, where $R$ is the distance between $v_j$ and $ABC$. There are some subtleties that are worth mentioning. If the approximation radius is set to a constant for all the terms, the bound does not converge in the thermodynamic limit. However, by setting the approximation radius to be the distance between $v_j$ and $ABC$, the approximation errors from each of these terms scales as $O(JG^{(I)}(c'\frac{\Gamma R}{2v}))l^2d$. Recall that $G^{(I)}(x)$ is a superpolynomially decaying function. Therefore,

$$\int_{al}^{\infty} G^{(I)}(x)xdx = H^{(I)}(al), \tag{6.16}$$

where $H^{(I)}(x)$ is some superpolynomially decaying function. Combining all of these contributions together, we arrive at the following bound:

$$\frac{d\gamma}{ds}|_{s=0} \leq cJ(\frac{\Gamma l}{v})^{10}l^4u_{2/7}(c'\frac{\Gamma l}{v}), \tag{6.17}$$

where we have assumed $l$ to be sufficiently large. One can see that the bound diverges for gapless systems. For certain systems, we can assume that the Lieb-Robinson velocity $v$ to be approximately equal to $J$ up to some constant that depends on the range of the Hamiltonian. For such systems,

$$|\frac{d\gamma}{ds}_{s=0}| = O(l^{14}\Delta^{10}u_{2/7}(\Omega(l\Delta))), \tag{6.18}$$

where $\Delta$ is the gap when $J$ is set to 1. Setting the correlation length $\xi$ as $1/\Delta$, we get

$$|\frac{d\gamma}{ds}_{s=0}| = O(l^4(\frac{l}{\xi})^{10}u_{2/7}(\Omega(l/\xi))). \tag{6.19}$$

We note in passing that the same technique can be applied to topologically trivial configurations, *i.e.,* $I(A:C|B)=0$. Under a general perturbation that consists of a sum of short-range bounded-norm terms, conditionally independent configurations become approximately conditionally independent. One may wish to establish a bootstrapping argument that recursively uses the approximate conditional independence of these configurations. The main difficulty of this approach lies on proving the $c_0$-boundedness.

## 6.4 Higher-dimensional deformation move

We emphasize that the deformation technique only exploits (i) the large distance between the target parties and (ii) the smoothness of the subsystems. Therefore, there is no reason why this technique should only work in two spatial dimensions. Here we apply the same ideas to a three-dimensional system to demonstrate the generality of these arguments. Of course, we will have to make some assumptions about the state to ensure that certain subsystems are conditionally independent to each other. A suitable example that fits this description is the ground state of the models that are described by the BF theory, *e.g.,* 3D toric code.

In the 3D toric code, one can define the topological entanglement entropy to be the conditional mutual information $I(A:C|B)$, where each of the subsystems are labeled in Figure 6.5.

As for the case of the two-dimensional topologically ordered system, one can bound the infinitesimal change of the topological entanglement entropy by applying the isolation move, the separation move, and the absorption move sequentially. For example, given a local operator that is localized near the reference party, one can apply the isolation move depicted in Figure 6.6 so that the new reference party is sufficiently separated from the local operator. Repeating the same analysis, we have the following bound:

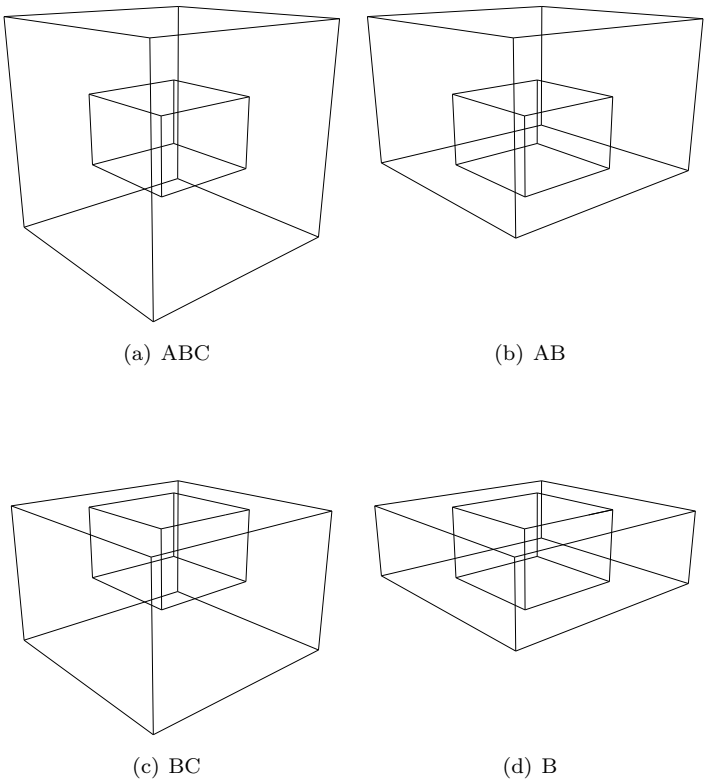
$$\left|\frac{dI(A:C|B)}{ds}_{s=0}\right| = O(l^6(\frac{l}{\xi})^{10}u_{2/7}(\Omega(l/\xi))). \tag{6.20}$$

Of course, the preceding argument can be generalized to any conditional mutual information that is invariant under a deformation of the subsystems in any dimensions. For a $D$-dimensional system, the bound becomes the following:

$$\left|\frac{dI(A:C|B)}{ds}_{s=0}\right| = O(l^{2D}(\frac{l}{\xi})^{10}u_{2/7}(\Omega(l/\xi))). \tag{6.21}$$

## 6.5 Stabilizer models at finite-temperature

Unlike the ground state of the exactly solvable models, the exact formula for the entanglement entropy of a finite-temperature system is not known except for few special cases.[85, 43, 86] To



(a) ABC (b) AB

(c) BC (d) B

Figure 6.5: Subsystems involved in the calculation of the topological entanglement entropy.

cope with this difficulty, we make a nontrivial but natural assumption: that the corrections from the deformation moves consist of the conditional mutual spectrum with a small conditional mutual information. For the 3D toric code, topological entanglement entropy does not depend on the size of the subsystem for a sufficiently large subsystem.[43] We shall denote the conditional mutual information in the correction terms as $\epsilon(l)$ and study how the first order perturbation effect depends on it.[2] We shall also assume that the correlation decays exponentially.

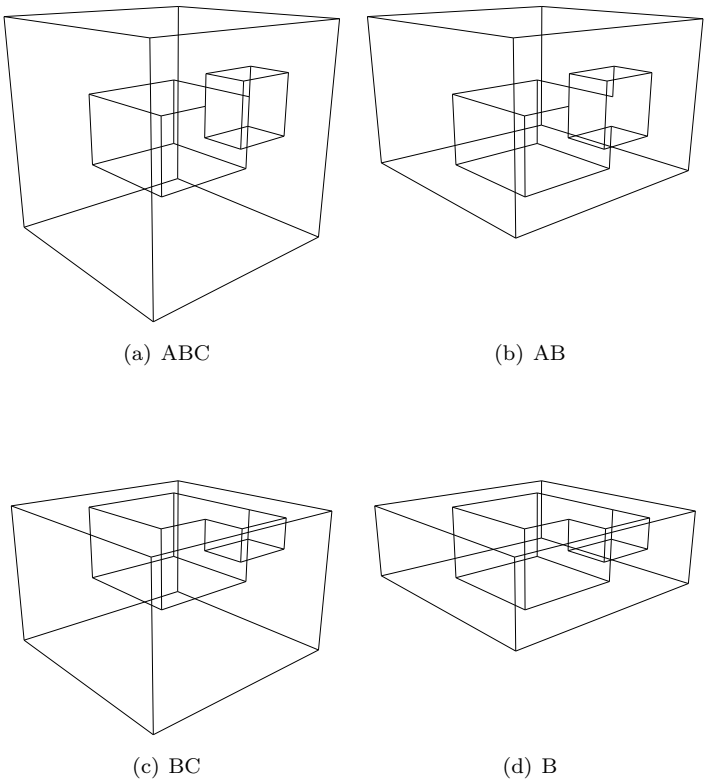
$$\mathcal{C}(O_A, O_B) \leq \|O_A\|\|O_B\| \min(|A|, |B|) e^{-d(A,B)/\xi}. \tag{6.22}$$

Stabilizer model refers to a Hamiltonian of the following form

$$H = -\sum_i J_i s_i, \tag{6.23}$$

where $J_i > 0$ is the coupling constant and $s_i$ is the element of the stabilizer group. Important property of the stabilizer models is that their reduced density matrices commute with each other.

[2]Something that one must be careful about is the invariance of the topological entanglement entropy under an *arbitrary* deformation. Castelnovo and Chamon proved size independence in Ref.[43], but that does not necessarily imply the invariance under an arbitrary small deformation. In this paper, we have implicitly assumed the invariance under arbitrary deformation.



(a) ABC (b) AB

(c) BC (d) B

Figure 6.6: Deformed subsystems after applying the isolation move.

**Lemma 21.** $\rho_A = \sum_{S_i \in S(A)} c_i S_i$ *for some coefficients* $\{c_i\}$.

*Proof.* $\rho$ can be expanded as a sum of the stabilizer group elements. After taking the partial trace, any operator that has a nontrivial support on $A^c$ vanishes. Any stabilizer group element that has a nontrivial support only on $A$ survives. These terms are generated from the generator of the stabilizer group, so they are again elements of the stabilizer group. □

It trivially follows that for the Gibbs state of the stabilizer Hamiltonian, reduced density matrices commute with each other. Therefore, any reduced density matrix $\rho_{ABC}$ for the stabilizer model is 1-bounded. To see this, recall that the following inequality holds

$$D_1(\ln D_1 - \ln D_2) \geq D_1 - D_2 \tag{6.24}$$

for positive semidefinite operators $D_1, D_2$ which commute with each other. Setting $D_1 = \rho_{ABC}$ and $D_2 = \rho_{AB}\rho_B^{-1}\rho_{BC}$ and taking a partial trace over $C$, we conclude that $\mathrm{Tr}_C(\rho_{ABC}\hat{H}_{A:C|B})$ is a positive semidefinite operator. Since $l_1$ norm is equal to the trace for a positive semidefinite operator, $\rho_{ABC}$ is 1-bounded.

Consider terms $v_j$ that are distance $al$ or less away from $ABC$. Using the deformation moves,

$\hat{H}_{A:C|B} = \hat{H}_{A':C'|B'} + \sum_i a_i \hat{H}_{A_i:C_i|B_i}$, where $d(v_j, A'B'C') = \mathcal{O}(l)$ and $I(A_i : C_i|B_i) = \epsilon(l)$. Choose an approximation radius $R$ such that $\Phi_{f_1^\beta}(v_j)$ is approximated by $[\Phi_{f_1^\beta}(v_j)]_{v_j(R)}$. The first order effect of $v_j$ can be divided into three parts: the connected correlation between $[\Phi_{f_1^\beta}(v_j)]_{v_j(R)}$ and $\hat{H}_{A':C'|B'}$, the approximation error of $\Phi_{f_1^\beta}(v_j)$, and the corrections from the deformation moves. Terms that are distance $al$ or more away from $ABC$ can be similarly bounded by using the exponential correlation decay and making a judicious choice for the approximation radius $R$. All of these effects combined together results in the following bound.

$$\frac{1}{\beta J}\frac{d\gamma}{ds}|_{s=0} \leq \mathcal{O}(l^{2D}(e^{-c_1 l/\xi}) + \mathcal{O}(l^{2D}e^{-c_2 l/\beta})) + \mathcal{O}(l^D \epsilon(l)), \tag{6.25}$$

where $D$ is the number of spatial dimensions and $c_1, c_2, c_3$ are some numerical constants.

## 6.6 Higher order terms

A close inspection of the first order bound reveals that the $c_0$-boundedness plays a pivotal role in the derivation. For example, consider a perturbed ground state of the topologically ordered system which satisfies the area law approximately. Equation 6.17 is only modified by including the area law correction terms, provided the $c_0$-boundedness condition holds.

It turns out that the $c_0$-boundedness in a finite neighborhood of $s$ implies a nontrivial bound for the higher order terms as well. The key idea is that Equation 6.17 can be applied to the topologically trivial configuration as well as the topologically nontrivial configuration. Since Equation 6.17 relied on the fact that the conditional mutual information of a topologically trivial configuration is small, we can bootstrap this argument to bound the higher order terms.

Assuming the $c_0$-boundedness for $s \in [0, s_0)$, the following inequality holds.

$$|\frac{d}{ds} I(A : C|B)_s| \leq \delta_s(l) + \sum_i a_i I(A_i : C_i|B_i)_s, \tag{6.26}$$

where $\delta_s(l)$ is a function that decreases superpolynomially with $l$, and $a_i$ is a finite number that is uniformly bounded for $s \in [0, s_0]$. $I(A_i : C_i|B_i)_s$ is a conditional mutual information appearing in the correction terms of the deformation moves.

If the energy gap remains open for $s \in [0, s_0)$, $\delta_s(l)$ can be uniformly bounded by some $\delta(l)$ that decays superpolynomially in $l$. As a result, one can obtain the following recursive bound.

$$\begin{aligned} |\gamma_s - \gamma_0| &\leq \int_0^s \delta(l) + \sum_i a_i I(A_i : C_i|B_i)_{s'} ds' \\ &= s\delta(l) + \sum_i a_i \int_0^s \int_0^{s'} \frac{dI(A_i : C_i|B_i)_{s''}}{ds''} ds'' ds', \end{aligned} \tag{6.27}$$

Here we used the fact that the conditional mutual information arising from the deformation move is 0 at $s = 0$. Recursively applying this logic, the second order term can be bounded by $\mathcal{O}(l^2\delta(l))$. Higher order terms can be obtained in a similar manner.

To investigate the validity of the $c_0$-boundedness for a general quantum many-body system, we have generated random density matrices and studied a relationship between both sides of Equation 6.3. We have first performed this numerical test over a random mixed state. We have randomly generated the eigenvalues of the density matrix from a uniform distribution over $[0, 1]$, normalized, and applied a random unitary operation drawn from the Haar measure. The result is plotted in Figure 6.7. As one can see, the observed value of $c_0$ does not deviate too much from 1, but this

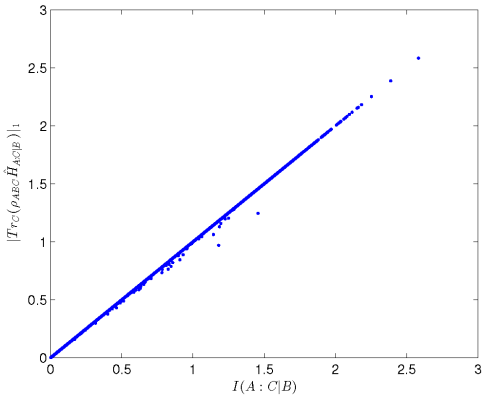



Figure 6.7: We have numerically computed $I(A : C|B)$ and $|\mathrm{Tr}_C \rho_{ABC} \hat{H}_{A:C|B}|_1$ for $10^6$ randomly generated mixed states. The largest observed ratio $|\mathrm{Tr}_C \rho_{ABC} \hat{H}_{A:C|B}|_1 / I(A : C|B)$ was 1.08.

could be an artifact of the low Hilbert space dimension. Furthermore, the result does not look as promising for randomly generated *pure states*, see Figure 6.8. For pure states, we have applied a random unitary from Haar measure. It seems that for certain states that have a small conditional mutual information, the smallest value of $c_0$ increases significantly. For this reason, we urge the readers to be careful in using this condition in general. This difficulty can be circumvented for stabilizer models against stabilizer perturbations, since the commutativity of the reduced density matrices is preserved. However, it remains to be seen if the correction terms from the higher order deformation moves are small.

We would like to make two remarks about the $c_0$-boundedness. First, the first order bound is only modified by a polynomial factor of the subsystem size if the value of $c_0$ has a dimension dependence that grows polylogarithmically.[3] Such a contribution will only add a factor that grows polynomially with the subsystem size, which can be controlled by the superpolynomially decaying function $u_{2/7}(x)$. However, even if the $c_0$-boundedness condition holds, the higher order perturbation expansion obtained from Equation 6.27 may not converge. This is due to the fact that at the $n$th

[3]More precisely, it should depend only on the dimension of one of the target parties and the reference party.

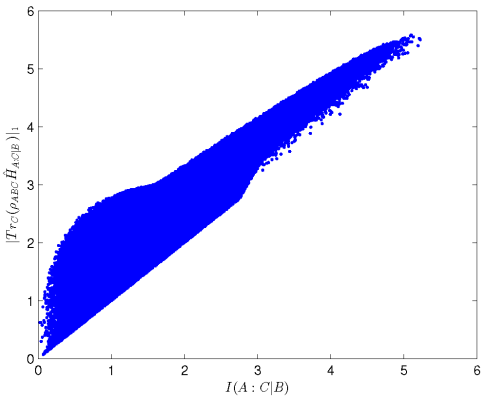



Figure 6.8: We have numerically computed $I(A:C|B)$ and $|\mathrm{Tr}_C \rho_{ABC} \hat{H}_{A:C|B}|_1$ for $10^6$ randomly generated pure states. Largest observed ratio $|\mathrm{Tr}_C \rho_{ABC} \hat{H}_{A:C|B}|_1 / I(A:C|B)$ was 24.2.

order of the perturbation series, there are $O(l^n)$ local terms. Naïvely bounding these terms will result in the factor of $e^{O(l)}$, which cannot be controlled by the function $u_{2/7}(x)$. In higher dimensional systems, the situation is even worse. There the local contributions from a naïve counting argument gives a factor of $O(l^{n(D-1)})$ at the $n$-th order of perturbation series. Such a bound cannot be controlled even if $u_{2/7}$ is replaced by an exponentially decaying function.

To summarize, we have demonstrated that the conditional independence condition strongly constrains the structure of quantum many-body system so as to ensure the first order perturbative stability of the topological entanglement entropy. Admittedly, our technique gives bounds in limited settings where (i) an exact conditional independence is achieved or (ii) the reduced density matrices commute with each other. However, once these conditions are met, the argument can be applied quite generally. In particular, we expect our method to be applicable to the studies of Chamon's model and Haah's model.[153, 48] These models satisfy the topological quantum order conditions introduced by Bravyi et al., and their Hamiltonian consists of a sum of frustration-free commuting projectors.[29] Therefore, the energy gap is protected against a generic perturbation that consists of a sum of short-range bounded-norm terms.

There are compelling reasons to believe that these models are not described by the BF theory or multiple stacks of Chern-Simons theory: the movement of the quasi-particles are constrained in a peculiar manner, and their ground state degeneracy is determined by some number-theoretic function that depends on the size of the system.[154, 49] It would be interesting if one can apply our method to find a linear combination of the entanglement entropy that allows the first order perturbative stability.

We have also shown that our method can be extended to higher orders of perturbation series if the $c_0$-boundedness holds in a finite neighborhood of $s$, but such statement seems unlikely to hold for general quantum states. It would be very interesting if one can find an alternative technique

that relies on the operator extension of SSA.

As for the the finite-temperature topological entanglement entropy in 3D, we needed two non-trivial assumptions to bound the first order perturbation effect. First, the connected correlation function between two observables decay exponentially. Second, the correction terms from the deformation moves can be expressed as a sum of small conditional mutual information. We emphasize that neither of these assumptions were explicitly proved. Further studies in explicitly bounding both of these terms are necessary.

# Appendix A

# Special quantum channels in quantum statistical mechanics

Here we list some of the technical tools that were developed in an attempt to solve the problems discussed in this thesis, but was superseded by the alternative tools discussed in the previous chapters. Nevertheless, we list these results because they may be interesting in their own right. While many of these results are already known, they are unfortunately scattered around the literature in different contexts. The aim of this chapter is to put these materials in the context of quantum statistical mechanics.

We first define the following quantum cannel $\Phi^{\rho}_{f(\omega)}$:

$$\Phi^{\rho}_{f(\omega)}(\sigma) = \int_{-\infty}^{\infty} \rho^{it} \sigma \rho^{-it} \tilde{f}(t) dt, \tag{A.1}$$

where $\tilde{f}(t)$ is the normalized inverse Fourier transform of $f(\omega)$, $\rho$ is an invertible density matrix, and $\tilde{f}(t) \geq 0$. Clearly, $\Phi^{\rho}_{f(t)}$ is a unital quantum channel. Denote the eigenbasis of $\rho$ as $\{|i\rangle\}$. In this basis, one can easily check that

$$\Phi^{\rho}_{f(\omega)}(\sigma)_{ij} = \sigma_{ij} f(\log \rho_i - \log \rho_j), \tag{A.2}$$

where $\tilde{f}(\omega)$ is the fourier transform of $f(t)$ and $\rho_i = \langle i| \rho |i\rangle$.

In the quantum stiatistical mechanics, we are interested in the Gibbs state:

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}, \tag{A.3}$$

where $\beta$ is the inverse temperature and $H$ is the Hamiltonian describing the system. The goal is to develop a machinery that produces sensible answers in the limit $\|H\| \to \infty$. The main difficulty

comes from the fact that the following expression is unbounded in general:

$$e^{-\beta H} V e^{\beta H}, \tag{A.4}$$

even if $V$ is bounded. One may think that the problem may be resolved if one imposes that (i) $V$ is a strictly local term and (ii) $H$ consists of a sum of geometrically local, bounded-norm terms. Unfortunately, such a statement is not known. In fact, to the best of author's knowledge, the only quantitative result in such direction was pursued by Araki.[161] Unfortunately, Araki's result only concerns one-dimensional systems.

Therefore, we would like to avoid Equation A.4 at all costs. We would like to also get rid of the normalization condition to simplify the analysis. We shall consider a perturbation of the following form:

$$\rho \to e^{\log \rho + \epsilon H}. \tag{A.5}$$

Note that the normalization condition can be restored by shifting $H$ by some constant. We define a directional derivative that generates the infinitesimal change as follows:

$$\partial_H f(\rho) := \lim_{\epsilon \to 0} \frac{f(e^{\log \rho + \epsilon H}) - f(\rho)}{\epsilon}. \tag{A.6}$$

Therefore,

$$\partial_H \log \rho = H. \tag{A.7}$$

As it was discussed in Section 3.2.1, we obtain the following formula:

**Lemma 22.**

$$\partial_H \rho = \frac{1}{2}\{\rho, \Phi^{\rho}_{f_1(x)}(H)\}, \tag{A.8}$$

*where*

$$f_1(\omega) = \frac{\tanh(\omega/2)}{\omega/2}. \tag{A.9}$$

*Proof.* The identity can be easily checked by comparing both sides in the eigenbasis. The nontrivial part is showing that $\tilde{f}_1(t)$ is nonnegative. We resort to the following identity:

$$\frac{\tanh z}{z} = \sum_{k=0}^{\infty} \frac{8}{4z^2 + (2k+1)^2 \pi^2}. \tag{A.10}$$

Since the Fourier transform of $\frac{1}{1+\omega^2}$ is nonnegative, this completes the proof. □

## A.1 Construction of quantum channels from positive definite functions

$f_1(\omega)$ is an example of a *positive definite function.*

**Definition 18.** *[162]$f(x)$ is positive definite if the following $n \times n$ matrix $F$ is nonnegative for any choice of $\lambda_i \in \mathbb{R}$.*

$$F_{ij} = f(\lambda_i - \lambda_j). \tag{A.11}$$

The following result is a special case of the Bochner's theorem, which gives a useful characterization of the positive definite function.[162]

**Theorem 7.** *$f(x)$ is positive definite if and only if its Fourier transform is nonnegative.*

*Proof.* Without loss of generality, consider a vector $|v\rangle = \sum_i a_i |i\rangle$.

$$\langle v| F |v\rangle = \sum_{j,k} a_j a_k^* f(\lambda_j - \lambda_k) \tag{A.12}$$

$$= \sum_{j,k} a_j a_k^* \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega(\lambda_j - \lambda_k)} \tag{A.13}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) | \sum_j a_j e^{i\omega\lambda_j} |^2. \tag{A.14}$$

□

Therefore, we conclude the following:

**Corollary 5.** *These three statements are equivalent.*

- *$f$ is a positive definite function.*
- *$\Phi^{\rho}_{f(\omega)}$ is a unital quantum channel for any invertible density matrix $\rho$.*
- *Fourier transform of $f$ is nonnegative.*

We define two functions that shall be extensively used.

**Definition 19.**

$$f_2(\omega) := \frac{\omega/2}{\sinh(\omega/2)}. \tag{A.15}$$

$$f_3(\omega) := \frac{1/}{\cosh(\omega/2)}. \tag{A.16}$$

**Lemma 23.** *$f_2$ and $f_3$ is positive definite.*

*Proof.*

$$\int_{-\infty}^{\infty} \frac{1}{\cosh t} e^{i\omega t} dt = \pi \frac{1}{\cosh(\frac{\pi\omega}{2})} \geq 0. \tag{A.17}$$

$$\int_{-\infty}^{\infty} \frac{t}{\sinh t} e^{i\omega t} dt = \frac{\pi^2}{2} \frac{1}{\cosh^2(\frac{\pi\omega}{2})} \geq 0. \tag{A.18}$$

□

**Lemma 24.**

$$\rho^{\frac{1}{2}} V \rho^{\frac{1}{2}} = \frac{1}{2}(\{\rho, \Phi^{\rho}_{f_3(\omega)}\}(V)). \tag{A.19}$$

*Proof.* Since $\rho$ was assumed to be invertible, the statement is equivalent to the following:

$$V = \frac{1}{2}(\rho^{-\frac{1}{2}} \Phi^{\rho}_{f_3(\omega)}(V) \rho^{\frac{1}{2}} + \rho^{\frac{1}{2}} \Phi^{\rho}_{f_3(\omega)}(V) \rho^{-\frac{1}{2}}). \tag{A.20}$$

The identity can be verified in the eigenbasis of $\rho$. □

These results can be used to obtain an infinitesimal change of the entanglement Hamiltonian. More precisely, given a density matrix $\rho$, we obtain a closed-form expression for $\partial_H \log \rho_A$.

**Lemma 25.**

$$\partial_H \log \rho_A = \frac{1}{2} \rho_A^{-\frac{1}{2}} Tr_{A^c}[\Phi^{\rho_A}_{f_2(\omega)}(\{\rho, \Phi^{\rho}_{f_1(\omega)}(H)\})] \rho_A^{-\frac{1}{2}} \tag{A.21}$$

*Proof.* First note that

$$\partial_H \mathrm{Tr}_{A^c} \rho = \partial_H e^{\log \rho_A} \tag{A.22}$$

$$= \rho_A^{\frac{1}{2}} \tilde{\Phi}^{\rho_A}_{f_2(\omega)}(\partial_H \log \rho_A) \rho_A^{\frac{1}{2}}, \tag{A.23}$$

where we have used Duhamel's formula:

$$\frac{d}{ds} e^{H+sV}|_{s=0} = \int_0^1 e^{tH} V e^{(1-t)H} dt \tag{A.24}$$

$$= e^{\frac{1}{2}H} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-tH} V e^{tH} e^{\frac{1}{2}H} \tag{A.25}$$

$$= e^{\frac{1}{2}H} \tilde{\Phi}^{e^H}_{f_2(\omega)}(V) e^{\frac{1}{2}H}, \tag{A.26}$$

and $\tilde{\Phi}^{\rho}_{f(\omega)}$ is defined as

$$\langle i| \tilde{\Phi}^{\rho}_{f(\omega)}(V) |j\rangle = V_{ij} \frac{1}{f(\log \rho_i - \log \rho_j)}. \tag{A.27}$$

Note that

$$\tilde{\Phi}^{\rho}_{f(\omega)} \circ \Phi^{\rho}_{f(\omega)} = \Phi^{\rho}_{f(\omega)} \circ \tilde{\Phi}^{\rho}_{f(\omega)} = \mathcal{I}, \tag{A.28}$$

where $\mathcal{I}$ is the identity element in $\mathcal{B}(\mathcal{H})$. Rearranging the terms, we can derive the result. □

While the statement of Lemma 25 is rather convoluted, we emphasize that both $\Phi^{\rho_A}_{f_2(\omega)}$ and $\Phi^{\rho}_{f_1(\omega)}$ are unital quantum channels. More importantly, these superoperators are norm-nonincreasing. In particular, consider a scenario in which $H$ commutes with $\rho$. For example, a physical example would be the change of the Gibbs state under an infinitesimal temperature change. Using $\frac{1}{2}\{\rho, \Phi^{\rho}_{f_1(\omega)}(H)\} = \rho^{\frac{1}{2}} H \rho^{\frac{1}{2}}$, the expression can be reduced to the following:

$$\Phi^{\rho_A}_{f_2(\omega)} \circ \Phi_{A^c|A} \circ \Phi^{\rho}_{f_1(\omega)}(H), \tag{A.29}$$

where

$$\Phi_{A^c|A}(O) = \rho_A^{-\frac{1}{2}} \mathrm{Tr}_{A^c}(\rho^{\frac{1}{2}} O \rho^{\frac{1}{2}}) \rho_A^{-\frac{1}{2}} \tag{A.30}$$

is the conditional expectation channel. This channel was first introduced by Accardi and Cecchini in Ref.[126], see also Ref.[124]. Unfortunately, this approach is only limited to the case that the perturbation commutes with the density matrix. Nevertheless, it is important to note that the effect of the perturbation can be expressed as a composition of norm-nonincreasing maps. While these tools are primitive at this stage, one can see that the imaginary time evolution may be avoided in the analysis of quantum statistical mechanics if the "right" questions are asked.

# Bibliography

[1] X. G. Wen. Vacuum degeneracy of chiral spin states in compactified space. *Phys. Rev. B*, 40:7387–7390, Oct 1989.

[2] X. G. Wen and Q. Niu. Ground-state degeneracy of the fractional quantum hall states in the presence of a random potential and on high-genus Riemann surfaces. *Phys. Rev. B*, 41:9377–9396, May 1990.

[3] A. Y. Kitaev. Fault-tolerant quantum computation by anyons. *Annals Phys.*, 303:2–30, 2003.

[4] R. B. Laughlin. Quantized hall conductivity in two dimensions. *Phys. Rev. B*, 23:5632–5633, May 1981.

[5] B. I. Halperin. Quantized hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. *Phys. Rev. B*, 25:2185–2190, Feb 1982.

[6] X. G. Wen. Theory of the edge states in fractional quantum hall effects. *Int. J. Mod. Phys. B*, 6:1711, 1992.

[7] R. B. Laughlin. Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.*, 50:1395–1398, May 1983.

[8] F. D. M. Haldane and E. H. Rezayi. Finite-size studies of the incompressible state of the fractionally quantized hall effect and its excitations. *Phys. Rev. Lett.*, 54:237–240, Jan 1985.

[9] G. Moore and N. Read. Nonabelions in the fractional quantum hall effect. *Nuclear Physics B*, 360(2):362–396, 1991.

[10] K. G. Wilson. Renormalization group and critical phenomena. i. renormalization group and the kadanoff scaling picture. *Phys. Rev. B*, 4:3174–3183, Nov 1971.

[11] K. G. Wilson. Renormalization group and critical phenomena. ii. phase-space cell analysis of critical behavior. *Phys. Rev. B*, 4:3184–3205, Nov 1971.

[12] K. G. Wilson. The renormalization group and critical phenomena. *Rev. Mod. Phys.*, 55:583–600, Jul 1983.

[13] S. R. White. Density matrix formulation for quantum renormalization groups. *Phys. Rev. Lett.*, 69:2863–2866, Nov 1992.

[14] M. Fannes, B. Nachtergaele, and R. F. Werner. Finitely correlated states on quantum spin chains. *Comm. Math. Phys.*, 144:443–490, 1992.

[15] F. Verstraete, J. I. Cirac, and V. Murg. Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems. *Adv. Phys.*, 57:143, July 2009.

[16] X. Chen, Z. C. Gu, and X. G. Wen. Classification of gapped symmetric phases in 1d spin systems. *Phys. Rev. B*, 83:035107, August 2010.

[17] N. Schuch, D. Perez-Garcia, and J. I. Cirac. Classifying quantum phases using matrix product states and peps. *Phys. Rev. B*, 84:165139, October 2010.

[18] A. Turner, F. Pollmann, and E. Berg. Topological phases of one-dimensional fermions: An entanglement point of view. *Phys. Rev. B*, 83:075102, August 2010.

[19] L. Fidkowski and A. Y. Kitaev. Topological phases of fermions in one dimension. *Phys. Rev. B*, 83:075103, Feb 2011.

[20] M. B. Hastings. An area law for one-dimensional quantum systems. *JSTAT, P*, 08024, May 2007.

[21] F. Verstraete and J. I. Cirac. Renormalization algorithms for quantum-many body systems in two and higher dimensions. 2007.

[22] G. Vidal. A class of quantum many-body states that can be efficiently simulated. *Phys. Rev. Lett.*, 101:110501, 2008.

[23] M. Aguado and G. Vidal. Entanglement renormalization and topological order. *Phys. Rev. Lett.*, 100,:070404, December 2007.

[24] David Pérez-García, Frank Verstraete, J. Ignacio Cirac, and Michael M. Wolf. PEPS as unique ground states of local Hamiltonians. *Quant. Inf. Comp.*, 8:0650–0663, July 2007.

[25] N. Schuch, D. Perez-Garcia, and J. I. Cirac. Peps as ground states: degeneracy and topology. *Annals of Physics*, 325:2153, January 2010.

[26] M. Levin and X. G. Wen. String-net condensation: A physical mechanism for topological phases. *Phys.Rev. B*, 71:045110, 2005.

[27] A. Y. Kitaev and J. Preskill. Topological entanglement entropy. *Phys.Rev.Lett.*, 96:110404, 2006.

[28] I. Klich. On the stability of topological phases on a lattice. *Annals of Physics, Volume*, 325:2120–2131, December 2009.

[29] S. Bravyi, M. B. Hastings, and S. Michalakis. Topological quantum order: stability under local perturbations. *J. Math. Phys.*, 51:093512, January 2010.

[30] S. Bravyi and M. B. Hastings. A short proof of stability of topological order under local perturbations. January 2010.

[31] S. Michalakis and J. Pytel. Stability of frustration-free Hamiltonians. September 2011.

[32] M. B. Hastings. Lieb-Schultz-Mattis in higher dimensions. *Phys. Rev. B*, 69:104431, Mar 2004.

[33] M. B. Hastings and X. G. Wen. Quasi-adiabatic continuation of quantum states: The stability of topological ground state degeneracy and emergent gauge invariance. *Phys.Rev. B*, 72:045141, 2005.

[34] S. Bravyi and B. Terhal. A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes. *New J. Phys.*, 11:043029, October 2008.

[35] S. Bravyi, D. Poulin, and B. Terhal. Tradeoffs for reliable quantum information storage in 2d systems. *Phys. Rev. Lett.*, 104:050503, September 2009.

[36] J. Haah and J. Preskill. Logical operator tradeoff for local quantum codes. *Phys. Rev. A*, 86:032308, November 2010.

[37] O. Landon-Cardinal and D. Poulin. Local topological order inhibits thermal stability in 2d. September 2012.

[38] N. Delfosse. Tradeoffs for reliable quantum information storage in surface codes and color codes. January 2013.

[39] R. Alicki, M. Fannes, and M. Horodecki. On thermalization in Kitaev's 2d model. *J. Phys. A: Math. Theor.*, 42:065303, October 2008.

[40] E. Dennis, A. Y. Kitaev, A. Landahl, and J. Preskill. Topological quantum memory. *J. Math. Phys.*, 43:4452–4505., 2002.

[41] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki. On thermal stability of topological qubit in Kitaev's 4d model. *Open Syst. Inf. Dyn.*, 17:1, November 2008.

[42] H. Bombin and M. A. Martin-Delgado. Exact topological quantum order in $d = 3$ and beyond: Branyons and brane-net condensates. *Phys. Rev. B*, 75:075103, Feb 2007.

[43] C. Castelnovo and C. Chamon. Topological order in a 3d toric code at finite temperature. *Phys. Rev. B*, 78:155120, April 2008.

[44] Z. Nussinov and G. Ortiz. Autocorrelations and thermal fragility of anyonic loops in topologically quantum ordered systems. *Phys. Rev. B*, 77:064302, Feb 2008.

[45] Z. Nussinov and G. Ortiz. A symmetry principle for topological quantum order. *Annals of Physics*, 324(5):977 – 1057, 2009.

[46] I. H. Kim. Local non-CSS quantum error-correcting code on a three-dimensional lattice. *Phys. Rev. A*, 83:052308, May 2011.

[47] B. Yoshida. Feasibility of self-correcting quantum memory and thermal stability of topological order. *Annals of Physics*, 326:2566–2633, March 2011.

[48] J. Haah. Local stabilizer codes in three dimensions without string logical operators. *Phys. Rev. A*, 83:042330, January 2011.

[49] S. Bravyi and J. Haah. On the energy landscape of 3d spin Hamiltonians with topological order. *Phys. Rev. Lett.*, 107:150504, May 2011.

[50] S. Bravyi and J. Haah. Analytic and numerical demonstration of quantum self-correction in the 3d cubic code. December 2011.

[51] I. Arad, Z. Landau, and U. Vazirani. An improved 1d area law for frustration-free systems. *Phys. Rev. B*, 85:195145, November 2011.

[52] F. G. S. L. Brandao and M. Horodecki. Exponential decay of correlations implies area law. June 2012.

[53] I. Arad, A. Y. Kitaev, Z. Landau, and U. Vazirani. An area law and sub-exponential algorithm for 1d systems. January 2013.

[54] S. Michalakis. Stability of the area law for the entropy of entanglement. June 2012.

[55] D. Poulin and M. B. Hastings. Markov entropy decomposition: a variational dual for quantum belief propagation. *Phys. Rev. Lett.*, 106:080403, December 2010.

[56] M. B. Hastings and T. Koma. Spectral gap and exponential decay of correlations. *Commun.Math.Phys.*, 265:781–804, 2006.

[57] B. Nachtergaele and R. Sims. Lieb-Robinson bounds and the exponential clustering theorem. *Commun. Math. Phys.,*, 265:119–130, 2006.

[58] B. Nachtergaele, Y. Ogata, and R. Sims. Propagation of correlations in quantum lattice systems. *J. Stat. Phys.*, 124:1–13, 2006.

[59] S. Bravyi, M. B. Hastings, and F. Verstraete. Lieb-Robinson bounds and the generation of correlations and topological quantum order. *Phys. Rev. Lett.*, 97:050401, 2006.

[60] M. B. Hastings and S. Michalakis. Quantization of hall conductance for interacting electrons without averaging assumptions. November 2009.

[61] M. B. Hastings. Quasi-adiabatic continuation for disordered systems: Applications to correlations, Lieb-Schultz-Mattis, and Hall conductance. January 2010.

[62] B. Nachtergaele and R. Sims. A multi-dimensional Lieb-Schultz-Mattis theorem. *Commun. Math. Phys.*, 276:437–472, 2007.

[63] C. K. Burrell and T. J. Osborne. Bounds on information propagation in disordered quantum spin chains. *Phys. Rev. Lett.*, 99,:167201, 2007.

[64] T. J. Osborne. Simulating adiabatic evolution of gapped spin systems. *Phys. Rev. A*, 75:032321, 2007.

[65] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims. Lieb-Robinson bounds for harmonic and anharmonic lattice systems. *Commun. Math. Phys.*, 286:1073–1098, December 2007.

[66] L. Erdos and B. Schlein. Quantum dynamics with mean field interactions: a new approach. April 2008.

[67] E. Hamza, S. Michalakis, B. Nachtergaele, and R. Sims. Approximating the ground state of gapped quantum spin systems. *J. Math. Phys.*, 50:095213, April 2009.

[68] I. Prémont-Schwarz, A. Hamma, I. Klich, and F. Markopoulou-Kalamara. Lieb-Robinson bounds for commutator-bounded operators. *Physical Review A Vol.*, 81,:No.4, December 2009.

[69] B. Nachtergaele, B. Schlein, R. Sims, S. Starr, and V. Zagrebnov. On the existence of the dynamics for anharmonic quantum oscillator systems. *Rev. Math. Phys.,*, 22:207–231, September 2009.

[70] D. Poulin. Lieb-Robinson bound and locality for general Markovian quantum dynamics. *Phys. Rev. Lett.*, 104:190401, March 2010.

[71] M. Levin and X. G. Wen. Detecting topological order in a ground state wave function. *Phys. Rev. Lett.*, 96:110405, 2006.

[72] C. Fernández-González, N. Schuch, M. M.Wolf, J. I. Cirac, and D. Pérez-García. Gapless Hamiltonians for the toric code using the projected entangled pair state formalism. *Phys. Rev. Lett.*, 109:260401, Dec 2012.

[73] Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath. Quasi-particle statistics and braiding from ground state entanglement. *Phys. Rev. B*, 85:235151, November 2011.

[74] L. Cincio and G. Vidal. Characterizing topological order by studying the ground states of an infinite cylinder. *Phys. Rev. Lett.*, 110,:067208, August 2012.

[75] N. Read and D. Green. Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries, and the fractional quantum hall effect. *Phys.Rev.B*, 61:10267, 2000.

[76] D. A. Ivanov. Non-Abelian statistics of half-quantum vortices in p-wave superconductors. *Phys. Rev. Lett.*, 86:268, 2001.

[77] H. Bombin. Topological order with a twist: Ising Anyons from an abelian model. *Phys.Rev.Lett.*, 105:030403, April 2010.

[78] M. Barkeshli and X. L. Qi. Topological nematic states and non-Abelian lattice dislocations. *Phys. Rev. X*, 2:031013, December 2011.

[79] A. Hamma, R. Ionicioiu, and P. Zanardi. Ground state entanglement and geometric entropy in the Kitaev's model. *Phys.Lett. A*, 337:22, 2005.

[80] S. V. Isakov, M. B. Hastings, and R. G. Melko. Topological entanglement entropy of a Bose-Hubbard spin liquid. *Nature Physics*, 7:772, February 2011.

[81] H. C. Jiang, Z. Wang, and L/ Balents. Identifying topological order by entanglement entropy. *Nature Physics*, 8:902–905, May 2012.

[82] A. Selem, C. M. Herdman, and K. B. Whaley. Entanglement entropy at generalized rk points of quantum dimer models. November 2012.

[83] M. B. Hastings. Private communication. 2012.

[84] T. Grover. Quantum entanglement and detection of topological order in numerics. December 2011.

[85] C. Castelnovo and C. Chamon. Entanglement and topological entropy of the toric code at finite temperature. *Phys. Rev. B*, 76:184442, April 2007.

[86] S. Iblisdir, D. Pérez-García, M. Aguado, and J. Pachos. Thermal states of Anyonic systems. *Nuclear Physics B*, 829(3):401 – 424, 2010.

[87] D. Mazac and A. Hamma. Topological order, entanglement, and quantum memory at finite temperature. *Ann. Phys.*, 327:2096, December 2011.

[88] M. B. Hastings. Topological order at non-zero temperature. *Phys. Rev. Lett.*, 107:210501, June 2011.

[89] H. Li and F. D. M. Haldane. Entanglement spectrum as a generalization of entanglement entropy: Identification of topological order in non-abelian fractional quantum hall effect states. *Phys. Rev. Lett.*, 101:010504, Jul 2008.

[90] R. Thomale, A. Sterdyniak, N. Regnault, and B. A. Bernevig. Entanglement gap and a new principle of adiabatic continuity. *Phys. Rev. Lett.*, 104:180502, May 2010.

[91] A. Sterdyniak, A. Chandran, N. Regnault, B. A. Bernevig, and P. Bonderson. Real-space entanglement spectrum of quantum hall states. *Phys. Rev. B*, 85:125308, 2011.

[92] J. Dubail, N. Read, and E. H. Rezayi. Real-space entanglement spectrum of quantum hall systems. *Phys. Rev. B*, 85:115321, November 2011.

[93] I. D. Rodríguez, S. H. Simon, and J. K. Slingerland. Evaluation of ranks of real space and particle entanglement spectra for large systems. *Phys. Rev. Lett.*, 108:256806, Jun 2012.

[94] J. I. Cirac, D. Poilblanc, N. Schuch, and F. Verstraete. Entanglement spectrum and boundary theories with projected entangled-pair states. *Phys. Rev. B*, 83:245134, March 2011.

[95] J. Dubail, N. Read, and E. H. Rezayi. Edge-state inner products and real-space entanglement spectrum of trial quantum hall states. *Phys. Rev. B*, 86:245310, Dec 2012.

[96] J. Eisert, M. Cramer, and M. B. Plenio. Area laws for the entanglement entropy - a review. *Rev. Mod. Phys.*, 82,:277, August 2008.

[97] B. Schumacher. Quantum coding. *Phys. Rev. A*, 51:2738–2747, Apr 1995.

[98] C. H. Bennett, P. Shor, J. Smolin, and A. V. Thapliyal. Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem. *IEEE Trans. Info. Theory*, 48:10::26372655, 2002.

[99] I. Devetak and J. Yard. Exact cost of redistributing multipartite quantum states. *Phys. Rev. Lett.*, 100:230501, Jun 2008.

[100] F. Hiai and D. Petz. The proper formula for relative entropy and its asymptotics in quantum probability. *Comm. Math. Phys.*, 143:99–114, 1991.

[101] T. Ogawa and H. Nagaoka. Strong converse and steins lemma in the quantum hypothesis testing. *IEEE Trans. Inf. Theory*, 46:7:2428–2433, 2000.

[102] K. M. R. Audenaert, J. Calsamiglia, R. Mu noz Tapia, E. Bagan, L. Masanes, A. Acin, and F. Verstraete. Discriminating states: The quantum chernoff bound. *Phys. Rev. Lett.*, 98:160501, Apr 2007.

[103] M. Nussbaum and A. Szkola. The Chernoff lower bound for symmetric quantum hypothesis testing. *Ann. Statist.*, 37:2:1040–1057, 2009.

[104] R. Bhatia. *Matrix Analysis*. Graduate Texts in Mathematics. Springer, 1997.

[105] E. H. Lieb. Convex trace functions and the WignerYanaseDyson conjecture. *Adv. Math.*, 11:267–288, 1973.

[106] M. Fannes. A continuity property of the entropy density for spin lattice systems. *Comm. Math. Phys.*, 31:291–294, 1973.

[107] A. Y. Kitaev, A. H A. H. Shen, and M. N. Vyalyi. *Classical and Quantum Computation*. American Mathematical Society, Boston, MA, USA, 2002.

[108] A. O. Pittenger and M. H. Rubin. Separability and Fourier representations of density matrices. *Phys. Rev. A*, 62:032313, 2000.

[109] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[110] D. Gottesman. Stabilizer codes and quantum error correction. *Caltech Ph.D. Thesis*, 1997.

[111] N. D. Mermin and H. Wagner. Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic Heisenberg models. *Phys. Rev. Lett.*, 17:1133–1136, Nov 1966.

[112] G. Branko. Uniform tilings of 3-space. *Geombinatorics*, 4:49–56, 1994.

[113] H. Bombin and M. A. Martin-Delgado. Topological quantum distillation. *Phys.Rev.Lett.*, 97:180501, 2006.

[114] J. B. Kogut. An introduction to lattice gauge theory and spin systems. *Rev. Mod. Phys.*, 51:659–713, Oct 1979.

[115] D. M. Appleby. SIC-POVMs and the extended Clifford group. *J. Math. Phys.*, 46:052107, 2005.

[116] I. H. Kim. Operator extension of strong subadditivity of entropy. *J. Math. Phys.*, 53:122204, October 2012.

[117] M. B. Ruskai. Remarks on Kim's strong subadditivity matrix inequality: Extensions and equality conditions. November 2012.

[118] M. B. Hastings. Locality in quantum systems. *arXiv:1008.5137*, August 2010.

[119] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims. Automorphic equivalence within gapped phases of quantum lattice systems. *Commun. Math. Phys.*, 309:835–871, February 2011.

[120] I. H. Kim. Determining structure of real-space entanglement spectrum from approximate conditional independence. *Phys. Rev. B*, 87:155120, October 2013.

[121] I. H. Kim. Perturbative analysis of topological entanglement entropy from conditional independence. *Phys. Rev. B*, 86:245116, October 2012.

[122] D. Petz. Monotonicity of quantum relative entropy revisited. *Rev. Math. Phys.*, 15:79, 2003.

[123] P. Hayden, R. Jozsa, D. Petz, and A. Winter. Structure of states which satisfy strong subadditivity of quantum entropy with equality. *Commun. Math. Phys.*, 246:359–374, 2004.

[124] M. Leifer and D. Poulin. Quantum graphical models and belief propagation. *Ann. Phys.*, 323:1899, August 2007.

[125] B. Ibinson, N. Linden, and A. Winter. Robustness of quantum Markov chains. *Commun. Math. Phys.*, 277:289–304, 2008.

[126] L. Accardi and C. Cecchini. Conditional expectations in von Neumann algebras and a theorem of Takesaki. *J. Funct. Anal.*, 45:245–273, 1982.

[127] H. Barnum and E. Knill. Reversing quantum dynamics with near-optimal quantum and classical fidelity. *J. Math. Phys.*, 43:2097, 2002.

[128] E. H. Lieb and M. B. Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *J. Math. Phys.*, 14:1938, 1973.

[129] M. A. Nielsen and D. Petz. A simple proof of the strong subadditivity inequality. 2007.

[130] M. B. Ruskai. Another short and elementary proof of strong subadditivity of quantum entropy. *Reports on Mathematical Physics*, 60:1–12., 2007.

[131] E. G. Effros. A matrix convexity approach to some celebrated quantum inequalities. *Proc. Natl. Acad. Sci. USA*, 106:1006, 2009.

[132] D. Petz. Quasi-entropies for finite quantum systems. *Rep. Math. Phys.*, 23:57–65, 1986.

[133] M. B. Hastings. Decay of correlations in Fermi systems at nonzero temperature. *Phys. Rev. Lett.*, 93:126402, Sep 2004.

[134] M. B. Hastings. Quantum belief propagation. *Phys. Rev. B Rapids*, 76:201102, June 2007.

[135] P. Calabrese and J. Cardy. Entanglement entropy and quantum field theory. *J.Stat.Mech.*, page P06002, 2004.

[136] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. *Rev.Mod.Phys.*, 81:865–942, 2009.

[137] D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac. Matrix product state representations. *Quantum Inf. Comput.*, 7,:401, 2007.

[138] R. Moessner and S. L. Sondhi. An rvb phase in the triangular lattice quantum dimer model. *Phys. Rev. Lett*, 86:1881, 2001.

[139] R. Moessner and S. L. Sondhi. Three-dimensional resonating valence bond liquids and their excitations. *Phys. Rev. B*, 68:184512, 2003.

[140] A. Y. Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321:2–111, 2006.

[141] E. Knill and R. Laflamme. Theory of quantum error-correcting codes. *Phys. Rev. A*, 55:900–911, Feb 1997.

[142] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge Series on Information and the Natural Sciences. Cambridge University Press, 2000.

[143] D. W. Kribs, R. Laflamme, D. Poulin, and M. Lesosky. Operator quantum error correction. *Quant. Inf. & Comp.*, 6:383–399, 2006.

[144] E. H. Lieb and M. B. Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *J. Math. Phys.*, 14:1938, 1973.

[145] I. H. Kim. Long-range entanglement is necessary for a topological storage of quantum information. April 2013.

[146] H. Yao and X. L. Qi. Entanglement entropy and entanglement spectrum of the Kitaev model. *Phys. Rev. Lett.,*, 105,:080501, January 2010.

[147] M. Haque, O. Zozulya, and K. Schoutens. Entanglement entropy in fermionic Laughlin states. *Phys. Rev. Lett.*, 98:060401, Feb 2007.

[148] O. S. Zozulya, M. Haque, J. Schoutens, and E. H. Rezayi. Bipartite entanglement entropy in fractional quantum hall states. *Phys. Rev. B*, 76:125310, Sep 2007.

[149] S. Furukawa and G. Misguich. Topological entanglement entropy in the quantum dimer model on the triangular lattice. *Phys. Rev. B*, 75:214407, Jun 2007.

[150] Y. Zhang, T. Grover, and A. Vishwanath. Topological entanglement entropy of z2 spin liquids and lattice Laughlin states. *Phys.Rev.B*, 84:075128, June 2011.

[151] D. Poilblanc, N. Schuch, D. Pérez-García, and J. I. Cirac. Topological and entanglement properties of resonating valence bond wavefunctions. *Phys. Rev. B*, 86:014404, February 2012.

[152] T. Grover, A. M. Turner, and A. Vishwanath. Entanglement entropy of gapped phases and topological order in three dimensions. *Phys. Rev. B*, 84:195120, August 2011.

[153] C. Chamon. Quantum glassiness. *Phys. Rev. Lett.*, 94:040402, 2005.

[154] S. Bravyi, B. Leemhuis, and B. M. Terhal. Topological order in an exactly solvable 3d spin model. *Ann. Phys.*, 326:839, June 2010.

[155] F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac. Criticality, the area law, and the computational power of peps. *Phys. Rev. Lett.*, 96:220601., 2006.

[156] M. M. Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac. Area laws in quantum systems: mutual information and correlations. *Phys. Rev. Lett.*, 100,:070502, April 2007.

[157] E. Bilgin and D. Poulin. Coarse grained belief propagation for simulation of interacting quantum systems at all temperatures. *Phys. Rev. B*, 81:054106, October 2009.

[158] R. Koenig, B. W. Reichardt, and G. Vidal. Exact entanglement renormalization for string-net models. *Phys. Rev. B*, 79:195123, June 2008.

[159] A. Hamma, W. Zhang, S. Haas, and D. A. Lidar. Entanglement, fidelity and topological entropy in a quantum phase transition to topological order. *Phys. Rev. B*, 77:155111, May 2007.

[160] S. T. Flammia, A. Hamma, T. L. Hughes, and X. G. Wen. Topological entanglement renyi entropy and reduced density matrix structure. *Phys. Rev. Lett.*, 103:261601, September 2009.

[161] H. Araki. Gibbs states of a one-dimensional quantum lattice. *Comm. Math. Phys.*, 14:120–157, 1969.

[162] M. Reed and B. Simon. *Methods of Modern Mathematical Physics, Vol. II: Fourier Analysis, Self Adjointness.* Academic Press, 1975.