On the Mumford-Tate Conjecture for Abelian Varieties with Reduction Conditions

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Abstract

In this thesis we study Galois representations corresponding to abelian varieties with certain reduction conditions. We show that these conditions force the image of the representations to be "big," so that the Mumford-Tate conjecture (:= MT) holds. We also prove that the set of abelian varieties satisfying these conditions is dense in a corresponding moduli space.

The main results of the thesis are the following two theorems.

Theorem A: Let A be an absolutely simple abelian variety, $\mathrm{End}^\circ(A) = k$: imaginary quadratic field, $g = \dim(A)$. Assume either $\dim(A) \leq 4$, or A has bad reduction at some prime $\wp$, with the dimension of the toric part of the reduction equal to $2r$, and $\gcd(r, g) = 1$, and $(r, g) \neq (15, 56)$ or $(m-1, \frac{m(m+1)}{2})$. Then MT holds.

Theorem B: Let M be the moduli space of abelian varieties with fixed polarization, level structure and a k-action. It is defined over a number field F. The subset of $M(\overline{\mathbb{Q}})$ corresponding to absolutely simple abelian varieties with a prescribed stable reduction at a large enough prime $\wp$ of F is dense in $M(\mathbb{C})$ in the complex topology. In particular, the set of simple abelian varieties having bad reductions with fixed dimension of the toric parts is dense.

Besides this we also established the following results:

(1) MT holds for some other classes of abelian varieties with similar reduction conditions. For example, if A is an abelian variety with $\mathrm{End}^\circ(A) = \mathbb{Q}$ and the dimension of the toric part of its reduction is prime to $\dim(A)$, then MT holds.

(2) MT holds for Ribet-type abelian varieties.

(3) The Hodge and the Tate conjectures are equivalent for abelian 4-folds.

(4) MT holds for abelian 4-folds of type II, III, IV (Theorem 5.0(2)) and some 4-folds of type I.

(5) For some abelian varieties either MT or the Hodge conjecture holds.

Contents

Introduction

The study of algebraic cycles on an algebraic variety yields deep consequences for algebraic geometry, as well as for arithmetic (cf. [T 0]). There are two famous conjectures due to Hodge and Tate related to the structure of the ring of algebraic cycles modulo homological equivalence. The Mumford-Tate conjecture suggests that for abelian varieties the two are essentially equivalent. The focus of my work is on the Mumford-Tate conjecture for special classes of abelian varieties.

One can see that the Poincaré duals of homological classes of algebraic cycles of codimension j on a (smooth projective) algebraic variety $X_{/\mathbb{C}}$ sit in the component $H^{j,j}(X)$ of the Hodge decomposition of $H^{2j}(X,\mathbb{C})$. More precisely, $\mathcal{A}^j(X) \subseteq \mathcal{H}^j(X) := H^{j,j}(X) \cap H^{2j}(X,\mathbb{Q})$. Hodge conjectured that all the Hodge cycles are algebraic: $\mathcal{A}^j(X) = \mathcal{H}^j(X)$. We denote $\mathcal{A} := \bigoplus_j \mathcal{A}^j$, and $\mathcal{H} := \bigoplus_j \mathcal{H}^j$ the rings of the Hodge and algebraic cycles respectively. As yet, the Hodge conjecture is neither proven nor disproven (cf. [Shi 0]). However, by the (1,1)-theorem of Lefschetz the Hodge conjecture holds for divisors, i.e., algebraic cycles of codimension 1. Let $\mathcal{D}(X)$ be the subring of $\mathcal{A}(X)$ generated by $\mathcal{A}^1(X)$. If for some X, $\mathcal{H}(X) = \mathcal{D}(X)$, then the theorem implies the Hodge conjecture for such an X.

On the other hand, for an algebraic variety defined over an algebraic number field, say K, one can consider ℓ-adic étale cohomology $H^*_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts continuously on $H^*_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$. Set $\mathfrak{g}$ to be the Lie algebra of the image of the Galois group in $\mathrm{End}_{\mathbb{Q}_\ell}(H^*_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell))$. Then it turns out that the (cohomology classes of) codimension j algebraic cycles $\mathcal{A}^j_\ell(X)$ are essentially invariant under the action of $\mathfrak{g}$, viz., they live in $\mathcal{T}^j_\ell(X) := H^{2j}_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)(j)^{\mathfrak{g}}$, where $H^{2j}_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)(j)$ is the j^{th} Tate twist of $H^{2j}_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$. Tate [T 0] conjectured that $\mathcal{A}^*_\ell(X) = \mathcal{T}^*_\ell(X)$. We denote $\mathcal{T}_\ell := \bigoplus_j \mathcal{T}^j_\ell$ the ring of the Tate cycles.

From now on, we restrict ourselves to the case of ***abelian varieties*** defined over number fields. On the one hand, this case is more concrete, and some progress has been made; on the other, it has important arithmetical applications.

Although not known in general, the analog of the (1,1)-theorem for the Tate cycles of codimension 1 for abelian varieties has been proved by Faltings [F]. Hence, as above, we can conclude that the Tate conjecture holds for an abelian variety A satisfying $\mathcal{T}_\ell(A) = \mathcal{D}(A)$. It is known that *generically*, but not always, the Hodge (resp. the Tate) cycles are all generated by divisors [Ma], [Ab].

The first (counter)example due to Mumford [Po] features a CM abelian 4-fold. Weil [W] has shown that the essential feature of Mumford's example causing $\mathcal{H} = \mathcal{D}$ to fail is an action in a special way of a quadratic imaginary field k on an abelian variety. Namely, consider a family of abelian varieties of even dimension, say $2d$, whose endomorphism algebra contains such a field k with the signature of the k-action (d, d). Generically for such a family, the ring of Hodge cycles is generated by divisors together with the exceptional (non-divisorial) cycle of codimension d. Recently, C. Schoen proved the Hodge conjecture for one family of abelian 4-folds of Weil type (with an action of $\mathbb{Q}(\mu_3)$).

In general, both conjectures seem to be very difficult in codimensions > 1.

Because of the existence of the comparison isomorphisms between the ℓ-adic and singular cohomology theories, carrying algebraic cycles in one theory to another, the Hodge and the Tate conjectures are describing essentially the same object. So, it is natural to ask if the two conjectures are equivalent in some sense. The precise statement in the case of abelian varieties constitutes the *Mumford-Tate conjecture*, which we denote by MT. It asserts that the Hodge and the Tate conjectures are equivalent for an abelian variety and all its self-products. Concretely, for an abelian variety A over a number field, say K, there exists a reductive algebraic group $Hg(A)$ over $\mathbb{Q}$ (resp. a reductive algebraic group $G_\ell(A)$ over $\mathbb{Q}_\ell$ for some prime number $\ell \in \mathbb{Z}$), such that the Hodge (resp. the Tate) cycles of codimension j are obtained as invariants in $H^{2j}(A,\mathbb{Q})$ (resp. in $H^{2j}_{\text{ét}}(A_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$) under $Hg(A)$ (resp. $G_\ell(A)$). Because of the comparison isomorphism between the two cohomology theories, $Hg_\ell(A) := Hg(A)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ acts on $H^{2p}_{\text{ét}}(A_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$. MT says that $Hg_\ell(A) = G_\ell(A)$. Note that $G_\ell(A)$ is ***not*** the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ in $\mathrm{GL}(H^*_{\text{ét}}(A_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell))$, but (the connected component of) the intersection of the (algebraic envelope of the) image of the Galois group with $\mathrm{SL}(H^*_{\text{ét}}(A_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell))$. That is why we do not Tate-twist the étale cohomology group.

Deligne, Piatetskii-Shapiro and Borovoi proved a "half" of MT, viz., $Hg_\ell(A) \supseteq G_\ell(A)$. Hence the Tate conjecture implies the Hodge conjecture.

MT for abelian varieties of CM-type is a consequence of the results of Shimura and Taniyama [ShT], [Po]. This must have been the motivating factor behind MT.

MT is proven in a few cases by imposing restrictions on the size of the endomorphism algebra and adding some divisibility conditions on the dimension of abelian varieties [S 0], [C 0]. In these cases the Hodge and the Tate cycles are all generated by divisors.

The main thrust of my work is to show that under suitable bad reduction conditions we can control the image of Galois; in particular, MT holds for a class of abelian varieties, including some Weil-type abelian varieties.

Note that if A is an absolutely simple abelian variety, $e = (\mathrm{End}^\circ(A) : \mathbb{Q})$ the degree over $\mathbb{Q}$ of its endomorphism algebra, and A has bad reduction at some prime $\wp$, then the e divides the dimension of the toric part of the reduction.

The first main result of my thesis is the following theorem (see Theorems 8.1, 9.10, 9.12).

Theorem A: Let A be an absolutely simple abelian variety, $\mathrm{End}^\circ(A) = k$: imaginary quadratic field, $g = \dim(A)$. Assume either $\dim(A) \leq 4$, or A has bad reduction at some prime $\wp$, with the dimension of the toric part of the reduction equal to $2r$, and $\gcd(r, g) = 1$, and $(r, g) \neq (15, 56)$ or $(m-1, \frac{m(m+1)}{2})$. Then MT holds.

This, in particular, implies the Tate conjecture for the class of Weil abelian 4-folds considered by Schoen.

Now we can ask whether the abelian varieties considered above exist, and if the answer is "yes," then how "big" this set is. Concretely, let M be the moduli space of abelian varieties with fixed polarization, level structure and a k-action (cf. 11.2). It is defined over a number field F. Then

Theorem B: The subset of $M(\overline{\mathbb{Q}})$ corresponding to absolutely simple abelian varieties with a prescribed stable reduction at a large enough prime $\wp$ of F is dense in $M(\mathbb{C})$ in the complex topology. In particular, the set of simple abelian varieties having bad reductions with fixed dimension of the toric parts is dense.

See §14, Theorem 14.1 for more precise formulation.

As mentioned above, MT is known to hold for CM abelian varieties. It is also known [ST] that such abelian varieties have good reduction at all primes, after possibly a finite base change. Although there exist non-CM abelian varieties with (potentially) good reduction everywhere (e.g., [So]), it is a very rare occasion. We have reasons to believe that the abelian varieties with *minimal* bad reduction (case $r = 1$ of Theorem A) are the "most typical" (see 15.2).

Along the way I established various other results. They are:

(1) MT holds for some other classes of abelian varieties with similar reduction conditions (Theorems 8.1, 9.11, 9.12). For example, if A is an abelian variety with $\mathrm{End}^\circ(A) = \mathbb{Q}$ and the dimension of the toric part of its reduction, which is not necessarily even in this case, is prime to $\dim(A)$, then MT holds.

(2) MT holds for Ribet-type abelian varieties (Theorem 1.2).

(3) The Hodge and the Tate conjectures are equivalent for abelian 4-folds (Theorem 5.0(1)).

(4) MT holds for abelian 4-folds of type II, III, IV (Theorem 5.0(2)) and some 4-folds of type I (Theorems 5.0(2), 8.2).

(5) For some abelian varieties, either MT or the Hodge conjecture holds (Theorem 10.1, Remark 10.2(4)).

Part I. Basics, generalities, first applications

§0. Basics and Notation

0.0 Let A be a *simple* abelian variety defined over some number field, say, $K \hookrightarrow \overline{\mathbb{Q}}$ (the embedding is fixed), $D := \mathrm{End}^\circ(A) := \mathrm{End}^\circ_{\overline{\mathbb{Q}}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, $V := H_1(A(\mathbb{C}), \mathbb{Q})$. Then $D \hookrightarrow \mathrm{End}_{\mathbb{Q}}(V)$.

0.1 Recall that $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ is given the complex structure induced by the natural isomorohism between $V_{\mathbb{R}}$ and the universal covering space of $A(\mathbb{C})$ (cf. [MAV]). Therefore we obtain a homomorphism of algebraic groups,

$$\varphi : T \to \mathrm{GL}(V),$$

defined over $\mathbb{R}$, where T is the compact one-dimensional torus over $\mathbb{R}$, i.e., $T_{\mathbb{R}} = \{z \in \mathbb{C} \mid |z| = 1\}$, by the formula

$$\varphi(e^{i\theta}) = \text{the element of } \mathrm{GL}(V), \text{ which is multiplication by } e^{i\theta}$$
$$\text{in the complex structure on } V_{\mathbb{R}}.$$

Note that there is a non-degenerate skew symmetric (Riemann) form $\Theta : V \times V \to \mathbb{Q}$ and that φ satisfies the Riemann conditions:

1. $\varphi(T) \subseteq \mathrm{Sp}(V, \Theta)$,
2. $\Theta(v, \varphi(i) \cdot v) > 0, \ \forall v \in V, \ x \neq 0$,

see [M 1].

0.1.1 **Definition** (Mumford): The *Hodge group* $Hg(A)$ of A is the smallest algebraic subgroup of $\mathrm{Sp}(V) := \mathrm{Sp}(V, \Theta)$ defined over $\mathbb{Q}$ which after extension of scalars to $\mathbb{R}$ contains the image of φ.

0.1.2 For the purpose of completeness and further reference, we give the following reformulation of the above construction and definition. The reference for what follows is [D 3], § 3.

Since $A(\mathbb{C})$ is a compact smooth Kähler manifold, $V_{\mathbb{C}} := H_1(A(\mathbb{C}), \mathbb{C})$ admits a Hodge decomposition

$$H_1(A(\mathbb{C}), \mathbb{C}) = H_{-1,0}(A) \oplus H_{0,-1}(A).$$

Thus we obtain a homomorphism

$$\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}(V)_{\mathbb{C}}$$

by defining $\mu(z)$, $\forall z \in \mathbb{C}^{\times}$, to be the automorphism of $V_{\mathbb{C}}$ which is multiplication by z on $H_{-1,0}(A)$ and by the identity on $H_{0,-1}(A)$.

0.1.3 **Definition:** The *Mumford-Tate group* $M(A)$ of A is the smallest algebraic subgroup of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$ which after extension of scalars to $\mathbb{C}$ contains the image of μ.

Clearly, over $\mathbb{C}$, $M(A)$ is the subgroup of $\mathrm{GL}(V)_{\mathbb{C}}$ generated by the conjugates ${}^{\sigma}\mu$, $\forall \sigma \in \mathrm{Aut}(\mathbb{C})$.

0.1.4 **Definition:** The *Hodge group* $Hg(A)$ of A (or the *special* Mumford-Tate group of A) is the connected component of the identity of the intersection $M(A) \cap \mathrm{SL}(V)$ in $\mathrm{GL}(V)$.

Remarks: 1. The construction of $M(A)$ furnishes it with a canonical character $\nu : M(A) \to \mathbb{G}_m$ defined over $\mathbb{Q}$ and characterized by the condition $\nu \circ \mu = \mathrm{id}_{\mathbb{G}_m}$. Then $Hg(A) = \mathrm{Ker}(\nu)$.

2. One can easily show that the two definitions of $Hg(A)$ are equivalent.

0.1.5 **Theorem** (Mumford): 1. $Hg(A)$ is a connected reductive group.

2. D (=End°(A)) $= \mathrm{End}_{Hg(A)}(V) = \mathrm{End}_{\mathfrak{h}}(A)$, where $\mathfrak{h} := \mathrm{Lie}(Hg(A))$.

We can improve on part 1 for abelian varieties of types I, II and III (in the Albert's classification of End°A).

Proposition (Tankeev, Zarkhin): $Hg(A)$ is semi-simple for an abelian variety A of type I, II or III.

Remarks: 1. $Hg(A^a \times B^b) \cong Hg(A \times B)$ for any abelian varieties A, B.

2. Part 2 of the previous theorem implies that A is simple if an only if V is $\mathfrak{h}$-simple if an only if (Schur's lemma) D is a division algebra.

0.1.6 Recall that the *Hodge classes* of A are classes of type (p,p) in the Hodge decomposition of homology of A.

0.1.7 The *Hodge conjecture* states that all the Hodge classes are algebraic.

0.1.8 By the Künneth formula $H_*(A) = \bigwedge^{*} H_1(A)$ (cf. [MAV]), hence $Hg(A)$ acts on $H_*(A)$. One can show (e.g., [M 2], [W]) that the Hodge classes of A are exactly those classes in $H_*(A)$ that are fixed by $Hg(A)$. In fact, the Hodge group has the following characteristic property.

Proposition (Mumford): The Hodge group $Hg(A)$ is the largest (reductive) subgroup of $\mathrm{GL}(V)$ fixing all the Hodge classes of A^s, $s \geq 1$.

0.1.9 By the Künneth formula $H_2(A^s) = \bigoplus_{i=1}^{s} H_2(A)$. Hence, in the view of the previous proposition, the Lefschetz (1,1)-theorem for abelian varieties takes the following form.

Theorem (Lefschetz, Mumford): Let $s \in \mathbb{N}$, $sV := V \oplus \ldots \oplus V (s$ times$)$, then the $\mathfrak{h}$-invariants $(\bigwedge_{\mathbb{Q}}^{2} sV)^{\mathfrak{h}}$ is exactly the ($\mathbb{Q}$-span of homological classes of) divisors on $A^s = A \times \ldots \times A$ (s times).

0.1.10 **Definition** (Murty, Ribet): The *Lefschetz group* $L(A)$ of A is the group of units of the connected component of the identity of the centralizer of $\mathrm{End}^\circ(A)$ in $\mathrm{End}_{\mathbb{Q}}(V)$.

The following are the main results about the Lefschetz group.

Theorem (Murty, Ribet, Hazama): 0. (i) $L(A)$ is a connected reductive algebraic group defined over $\mathbb{Q}$, $Hg(A) \subseteq L(A)$.

0. (ii) $L(A)$ is semi-simple for A of type I, II or III.

0. (iii) $L(A_1^{n_1} \times ... \times A_s^{n_s}) = L(A_1) \times ... \times L(A_s)$.

1. All the Hodge classes on A^s are divisorial if an only if $Hg(A) = L(A)$ and A is not of type III.

2. If A is of type III, then it has a non-divisorial Hodge class.

Remark: Let $\mathfrak{l} := \mathrm{Lie}(L(A))$, then $\mathfrak{h} \hookrightarrow \mathfrak{l} \hookrightarrow \mathfrak{sp}(V)$, $\mathfrak{h}^{ss} \hookrightarrow \mathfrak{l}^{ss}$, $C_{\mathfrak{h}} \hookrightarrow C_{\mathfrak{l}}$. Here $C_?$ is the center of ? and $?^{ss}$ is the semi-simple part of ?.

0.1.11 Let $k \hookrightarrow D$ be an imaginary quadratic field, $\mathrm{Gal}(k/\mathbb{Q}) = \{\sigma, \rho\}, \rho\ (= \sigma^2)$ is the fixed (identity) embedding $k \hookrightarrow \overline{\mathbb{Q}}$. In this case $V_{\mathbb{R}} := V \otimes \mathbb{R}$ has two complex structures. One is given by the isomorphism $V_{\mathbb{R}} = \mathrm{Lie}(A(\mathbb{C}))$, (cf. 0.1), and the other by the action of $k \otimes_{\mathbb{Q}} \mathbb{R}(\simeq \mathbb{C})$. Hence the splitting $V_{\mathbb{R}} = V^\sigma \oplus V^\rho$. The two complex structures coincide on one of the subspaces, say, V^ρ , and conjugate on the other (in this case, V^σ). If $m_\sigma = \dim_{\mathbb{C}}(V^\sigma)$, $m_\rho = \dim_{\mathbb{C}}(V^\rho)$, then (m_σ, m_ρ) is the *signature* of the k-action; $m_\sigma + m_\rho = g = \dim_{\mathbb{C}}(V_{\mathbb{R}}) = \dim(A)$.

0.1.12 Recall that the *Rosati involution* is the involution on $D = \mathrm{End}^\circ(A)$ induced by the Riemann form (cf. 0.1). The basic fact about the Rosati involution is that it is *positive*. Consequently, a field fixed by the involution is totally real (cf. [MAV]).

In the case $k \subseteq D$, we always assume that the Rosati involution preserves k. The positivity of the involution implies that it acts on k non-trivially. Hence this action coincides with (the complex conjugation) σ.

0.1.13 Since $\mathfrak{h}$ and $\mathfrak{l}$ centralize D (cf. 1.1.5 and 1.1.10)

$$\mathfrak{h} \hookrightarrow \mathfrak{l} \hookrightarrow \mathfrak{sp}(V)^k \hookrightarrow \mathfrak{sp}(V),$$

where $\mathfrak{sp}(V)^k$ is the centralizer of k in $\mathfrak{sp}(V)$. (0.1.11) & [D 3], Lemma 4.6, imply

$$\mathfrak{sp}(V)^k = \mathfrak{u}(V),$$

the Lie algebra of the unitary group of a k-Hermitian form on V viewed as the k-vector space. Extending scalars to k we get

$$\mathfrak{h}_k \hookrightarrow \mathfrak{l}_k \hookrightarrow \mathfrak{u}(V) \times \mathfrak{u}(V)^\sigma \hookrightarrow \mathfrak{sp}(V_k), \tag{0.1.13.1}$$

where $\mathfrak{h}_k := \mathfrak{h} \otimes_{\mathbb{Q}} k$, $\mathfrak{l}_k := \mathfrak{l} \otimes_{\mathbb{Q}} k$, $V_k := V \otimes_{\mathbb{Q}} k = V \oplus U$, U is the same V, but with the conjugate k-vector space structure.

0.1.13.2 The $\mathfrak{h}$-invariant k-Hermitian form referred to above is a non-degenerate element of $\check{V} \otimes \check{U}$ (cf. 0.1.11 & [D 3], Lemma 4.6), hence the isomorphism

$$U \cong \check{V}$$

of $\mathfrak{h}$-modules. Clearly the projection of $\mathfrak{h}_k$ to $\mathfrak{u}(V)$ is $\mathfrak{h}$, thus we can rewrite (0.1.13.1) as

$$\mathfrak{h} \hookrightarrow \mathfrak{l} \hookrightarrow \mathfrak{u}(V) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(V \oplus \check{V}),\ g \overset{\Delta}{\mapsto} \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}. \tag{0.1.13.3}$$

(**Note:** the embeddings above are considered over k.)

0.1.14 From this we get:

$$\bar{\mathfrak{h}} \hookrightarrow \bar{\mathfrak{l}} \hookrightarrow \mathfrak{gl}(W) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(W \oplus \check{W}),$$

where $\bar{\mathfrak{h}} := \mathfrak{h} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, $\bar{\mathfrak{l}} := \mathfrak{l} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, $W := V \otimes_{k,\rho} \overline{\mathbb{Q}}$, $\check{W}$: dual of W $(= W \otimes_{k,\sigma} \overline{\mathbb{Q}})$.

0.2 Let $V_\ell := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, where $T_\ell(A)$ is the ℓ-adic Tate module $(= H_1((A \times_K \overline{\mathbb{Q}})_{\text{ét}}, \mathbb{Z}_\ell) = \varprojlim_n {}_{\ell^n}A(\overline{\mathbb{Q}})$, where ${}_{\ell^n}A(\overline{\mathbb{Q}})$ = kernel of multiplication by ℓ^n : $A(\overline{\mathbb{Q}}) \to A(\overline{\mathbb{Q}}))$, $V = H_1(A(\mathbb{C}), \mathbb{Q})$ as above. Let $\tilde{G}_\ell$ be the algebraic envelope (= Zariski closure) of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ in $\mathrm{End}_{\mathbb{Q}_\ell}(V_\ell)$, K : the base field of A, $\tilde{\mathfrak{g}}_\ell := \mathrm{Lie}(\tilde{G}_\ell)$, $\mathfrak{g}_\ell := \tilde{\mathfrak{g}}_\ell \cap \mathfrak{sl}(V_\ell)$. It is known (e.g., [MAV]) that $\tilde{G}_\ell \subseteq \mathrm{GSp}(V_\ell)$, hence $\mathfrak{g}_\ell \subseteq \mathfrak{sp}(V)$. As $\mathbb{Q}_\ell \cdot 1_{V_\ell} \hookrightarrow \tilde{\mathfrak{g}}_\ell$ [Bo], $C_{\tilde{\mathfrak{g}}_\ell} = C_{\mathfrak{g}_\ell} \oplus \mathbb{Q}_\ell \cdot 1_{V_\ell}$, $\mathfrak{g}_\ell^{ss} = \tilde{\mathfrak{g}}_\ell^{ss}$.

Note: $\mathfrak{g}_\ell$ does not depend on finite extensions of K (cf. [S 2]).

Remark: $V_\ell \cong V_\ell(r)$, $\forall r \in \mathbb{Z}$, as $\mathfrak{g}_\ell$-modules, but not as $\tilde{\mathfrak{g}}_\ell$-modules.

0.2.0 The *Tate conjecture* states that the *Tate cycles*, i.e., the Galois invariants $H_*(A_{\text{ét}}, \mathbb{Q}_\ell)^{\mathfrak{g}_\ell} = (\bigwedge^{*} H_1(A_{\text{ét}}, \mathbb{Q}_\ell))^{\mathfrak{g}_\ell}$, are algebraic.

0.2.1 Faltings proved the analogs of the Theorem 0.1.5 of Mumford (a special case of the Tate conjecture) and the (1,1)-theorem for abelian varieties.

Theorem (Faltings):

1. Let $s \in \mathbb{N}$, $(\bigwedge^{2}_{\mathbb{Q}_\ell} sV_\ell)^{\mathfrak{g}_\ell}$ is exactly the ($\mathbb{Q}_\ell$-span of homological classes of) divisors of $A^s = A \times ... \times A$ (s times).
2. $\mathrm{End}_{\mathfrak{g}_\ell}(V_\ell) = D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.
3. $\mathfrak{g}_\ell$ is reductive.

0.2.2 $\mathfrak{h}_\ell := \mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell)$. The known relation between $\mathfrak{g}_\ell$ and $\mathfrak{h}_\ell$ is given by the following theorem.

Theorem (Deligne, Piatetskii-Shapiro, Borovoi):

$$\mathfrak{g}_\ell \hookrightarrow \mathfrak{h}_\ell \hookrightarrow \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell).$$

Remark: In this thesis I consider abelian varieties defined over *number fields*. However, this theorem holds for abelian varieties defined over arbitrary *finitely generated fields of characteristic 0* (cf. [D 3], § 2). Since any abelian variety over $\mathbb{C}$ has a model over a finitely generated field, the theorem implies that the Hodge conjecture is a consequence of the (suitably stated, loc. cit.) Tate conjecture.

0.2.3 The *Mumford-Tate conjecture* (=: MT) states $\mathfrak{g}_\ell = \mathfrak{h}_\ell$. Since $\mathfrak{g}_\ell$ and $\mathfrak{h}_\ell$ are reductive, it is the same as equivalence of the Hodge and the Tate conjectures for an abelian variety and all its self-products.

0.2.3.1 In order to prove MT it is enough to establish the conjecture for one ℓ ([LP], Theorem 4.3).

0.2.3.2 Moreover, it is enough to show $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell$, where $\overline{\mathfrak{g}}_\ell := \mathfrak{g}_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$, $\overline{\mathfrak{h}}_\ell := \mathfrak{h}_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ ([Z 2], §5, Key Lemma).

0.2.4 **Remark:** The (1,1)-theorems 0.1.9, 0.2.1(1) imply $(\bigwedge^2_{\mathbb{Q}_\ell} sV_\ell)^{\mathfrak{g}_\ell} = (\bigwedge^2_{\mathbb{Q}_\ell} sV_\ell)^{\mathfrak{h}_\ell}$.

0.2.5 The theorems 0.2.2 and 0.2.1(2) imply $\mathfrak{g}_\ell^{ss} \hookrightarrow \mathfrak{h}_\ell^{ss}$, $C_{\mathfrak{g}_\ell} \hookrightarrow C_{\mathfrak{h}_\ell}$.
In fact, $C_{\mathfrak{g}_\ell} = C_{\mathfrak{h}_\ell}$. This can be shown in a way similar to the proof of MT for CM abelian varieties (cf. [ShT], and also [D 1]).
So, in order to prove MT one must show that $\mathfrak{g}_\ell^{ss} = \mathfrak{h}_\ell^{ss}$.

0.2.6 Let $\mathfrak{a}$ be a semi-simple Lie algebra over an algebraically closed field of characteristic 0, and $\mathfrak{a} = \mathfrak{a}_1 \times ... \times \mathfrak{a}_n$ be the decomposition of $\mathfrak{a}$ into the product of its simple ideals. For any faithful irreducible representation U of $\mathfrak{a}$, U decomposes as a tensor product of irreducible representations U_i of $\mathfrak{a}_i$. Since U is faithful, none of the U_i's is trivial. Moreover, if the representation U admits a non-degenerate invariant bilinear form, then so does each U_i.

We say that the representation is *minuscule* if the highest weight of each U_i is minuscule, see [B], ch.VIII, §7.3. The following is the list of minuscule weights, [B], ch.VIII, §7.3 and Table 2:

type A_m $(m \geq 1)$: $\varpi_1, \varpi_2, \ldots, \varpi_m$; $\dim(\varpi_s) = \binom{m+1}{s}$;

type B_m $(m \geq 2)$: ϖ_1; $\dim(\varpi_1) = 2m+1$;

type C_m $(m \geq 2)$: ϖ_1; $\dim(\varpi_1) = 2m$;

type D_m $(m \geq 3)$: $\varpi_1, \varpi_{m-1}, \varpi_m$; $\dim(\varpi_1) = 2m$,

$\dim(\varpi_{m-1}) = \dim(\varpi_m) = 2^{m-1}$;

type E_6 : ϖ_1, ϖ_6; $\dim(\varpi_1) = \dim(\varpi_6) = 27$;

type E_7 : ϖ_7; $\dim(\varpi_7) = 56$;

there is no minuscule representations for the types E_8, F_4, G_2.

0.2.7 It is known that the representations of $\overline{\mathfrak{g}}_\ell^{ss}$, $\overline{\mathfrak{h}}_\ell^{ss}$ are minuscule (cf. [S ∗], [D 2]). It is also known that $\overline{\mathfrak{g}}_\ell^{ss}$ is not exceptional, see [S ∗], Theorem 7 (for the corresponding result for $\overline{\mathfrak{h}}_\ell^{ss}$ see [D 2], Remarque 1.3.10(i)).

0.2.8 Let again $k \hookrightarrow D$ and $k_\ell := k \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then, as in 0.1.13,

$$\mathfrak{g}_\ell \hookrightarrow \mathfrak{h}_\ell \hookrightarrow \mathfrak{sp}(V_\ell)^{k_\ell} \hookrightarrow \mathfrak{sp}(V_\ell).$$

If l splits in k, λ, λ' being the primes of k over l, $\lambda' = \lambda^\sigma$, then $k_\ell \cong \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$, $V_\ell = V_\lambda \oplus V_{\lambda'}$, $V_\lambda, V_{\lambda'}$: vector spaces over $k_\lambda \cong \mathbb{Q}_\ell$, $k_{\lambda'} \cong \mathbb{Q}_\ell$ respectively.

$$\mathfrak{g}_\ell \hookrightarrow \mathfrak{h}_\ell \hookrightarrow \mathfrak{gl}(V_\lambda) \oplus \mathfrak{gl}(V_{\lambda'}) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(V_\lambda \oplus V_{\lambda'}).$$

Since $\lambda' = \lambda^\sigma$, as in 0.1.13.2 we conclude $V_{\lambda'} \cong \check{V}_\lambda$ and can rewrite the above sequence as

$$\mathfrak{g}_\ell \hookrightarrow \mathfrak{h}_\ell \hookrightarrow \mathfrak{gl}(V_\lambda) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(V_\lambda \oplus \check{V}_\lambda).$$

Remark: If ℓ does *not* split in k, then k_ℓ is a field, $(k_\ell : \mathbb{Q}_\ell) = 2$ and the rest is identical to 0.1.13.

0.2.9 As in 0.1.13, by extending the scalars to $\overline{\mathbb{Q}}_\ell$ we get

$$\overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{gl}(W_\lambda) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda),$$

where $W_\lambda := V_\ell \otimes_{k_\ell, \rho_\lambda} \overline{\mathbb{Q}}_\ell$, $\rho_\lambda : k_\ell \to k_\lambda$ is the projection.

Remark: For ℓ non-split in k , the same holds (cf. 0.1.13, 0.1.14).

0.3 We shall need the following simple facts. We assume that $D = k$.

0.3.1 **Proposition:** The representations of $\overline{\mathfrak{g}}_\ell$ and $\overline{\mathfrak{h}}_\ell$ are *non-self-dual.*

Remarks: 1. This is true for any irreducible subrepresentation of W_λ for any type IV abelian variety (e.g., [Mu], [H]).

2. If the abelian variety is of type I (respectively II, respectively III), then the irreducible components are symplectic (respectively symplectic, respectively orthogonal) (loc.cit).

0.3.2 **Proposition:** $\overline{\mathfrak{g}}_\ell$ and $\overline{\mathfrak{h}}_\ell$ are semi-simple if an only if the signature of the k-action is (m, m). Further, if this is *not* the case, the centers $C_{\overline{\mathfrak{g}}_\ell}, C_{\overline{\mathfrak{h}}_\ell}$ are 1-dimensional.

Remarks: 1. If the signature of the k-action of a type IV abelian variety is (m, m), we call such an abelian variety a *Weil type* abelian variety (cf. [W]).

2. Since $\mathfrak{g}_\ell, \mathfrak{h}_\ell$ are semi-simple for abelian varieties of types II or III (0.1.5, 0.2.5), if $k \hookrightarrow \mathrm{End}^\circ(A)$, A : of type II or III, then the signature of the k-action is necessarily (m, m) (see the proof of the proposition 0.3.2 below).

0.3.3 **Proof** of 0.3.1: As was shown above, $k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \overset{Schur}{\Longrightarrow} V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ has 2 irreducible non-isomorphic components. But $V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell = W_\lambda \oplus \check{W}_\lambda$. □

Proof of 0.3.2: This is proved in [D 3], [W]. Let us briefly say why this holds and fix some notations.

Let $\mu : \mathbb{G}_{m,\mathbb{C}} \to GL(V_{\mathbb{R}})$ be the cocharacter defining the Hodge structure on V, then the map $h : \mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to GL(V_{\mathbb{C}})$ is given by $h(z) = \mu(z)$ on $V_{\mathbb{R}}$, (cf. 0.1.2) $h(z) = \overline{\mu}(z)$ on $U_{\mathbb{R}}$, where $U_{\mathbb{R}}$ is $V_{\mathbb{R}}$ with the conjugate $k \otimes \mathbb{R}$-action, $V_{\mathbb{C}} = V_{\mathbb{R}} \oplus U_{\mathbb{R}}$ (cf. 0.1.11). If the k-signature is (m_σ, m_ρ) then $V_{\mathbb{R}} = V_{\mathbb{R}}^\sigma \oplus V_{\mathbb{R}}^\rho$, $\dim_{\mathbb{C}}(V_{\mathbb{R}}^\sigma) = m_\sigma$, $\dim_{\mathbb{C}}(V_{\mathbb{R}}^\rho) = m_\rho$ (k acts by $\sigma(k)$ on $V_{\mathbb{R}}^\sigma$ and by $\rho(k)$ on $V_{\mathbb{R}}^\rho$). But $V_{\mathbb{R}} = H_{-1,0}$, hence the power of z by which $\mu(z)$ acts on $V_{\mathbb{R}}^\sigma$, $V_{\mathbb{R}}^\rho$ is 1. We shall call this power the *μ-weight*. Similarly, $V_{\mathbb{C}} = H_1(A(\mathbb{C}), \mathbb{C}) = V_{\mathbb{C}}^\sigma \oplus V_{\mathbb{C}}^\rho$. But also $V_{\mathbb{C}} = H_{-1,0} \oplus H_{0,-1} (= V_{\mathbb{R}} \oplus U_{\mathbb{R}})$ and these two decompositions commute, since the former is determined by $k \subseteq D$ and the Hodge group centralizes D in $\mathrm{End}_{\mathbb{Q}}(V)$. Hence we can write

$$\begin{aligned} V_{\mathbb{C}} &= V_{\mathbb{R}} \oplus U_{\mathbb{R}} \\ &= (V_{\mathbb{R}}^\sigma \oplus V_{\mathbb{R}}^\rho) \oplus (U_{\mathbb{R}}^\sigma \oplus U_{\mathbb{R}}^\rho) \\ &= (V_{\mathbb{R}}^\sigma \oplus U_{\mathbb{R}}^\sigma) \oplus (V_{\mathbb{R}}^\rho \oplus U_{\mathbb{R}}^\rho) \\ &= V_{\mathbb{C}}^\sigma \oplus V_{\mathbb{C}}^\rho, \end{aligned}$$

where $U_{\mathbb{R}}^\sigma$ (respectively $U_{\mathbb{R}}^\rho$) is the conjugate of $V_{\mathbb{R}}^\rho$ (respectively $V_{\mathbb{R}}^\sigma$). Thus $\dim_{\mathbb{C}}(U_{\mathbb{R}}^\sigma) = m_\rho$, $\dim_{\mathbb{C}}(U_{\mathbb{R}}^\rho) = m_\sigma$, so $\dim_{\mathbb{C}}(V_{\mathbb{C}}^\sigma) = m_\rho + m_\sigma = g = \dim_{\mathbb{C}}(V_{\mathbb{C}}^\rho)$.

The μ-weight of $U_{\mathbb{R}}$ is 0, hence the decomposition

$$V_{\mathbb{C}}^\sigma = V_{\mathbb{R}}^\sigma \oplus U_{\mathbb{R}}^\sigma$$

is according to μ-weights 1, 0. (This exactly corresponds to the Hodge-Tate decomposition of the λ-adic representation below.) Now the μ-weight (respectively the Hodge type) of

$$\bigwedge_{\mathbb{C}}^{g} V_{\mathbb{C}}^{\sigma} = \bigwedge_{\mathbb{C}}^{m_\sigma} V_{\mathbb{R}}^{\sigma} \otimes \bigwedge_{\mathbb{C}}^{m_\rho} U_{\mathbb{R}}^{\sigma} \subseteq \bigwedge^{g} V_{\mathbb{C}}^{\sigma}$$

is m_σ (respectively $(-m_\sigma, -m_\rho)$). Hence $\bigwedge_k^g V$ is a Hodge cycle (i.e., of Hodge type $(-\frac{g}{2}, -\frac{g}{2})$) if an only if $m_\sigma = m_\rho$. Hence it is fixed by $\mathfrak{h}$ if an only if $m_\sigma = m_\rho$, i.e., $\mathfrak{h} \hookrightarrow \mathfrak{sl}(V) \cap \mathfrak{u}(V)$ only in this case (here V is considered as a k-vector space). In other words, the center $C_{\mathfrak{h}}$ of $\mathfrak{h}$ kills the determinant $\bigwedge_k^g V$ (and hence $\neq \{0\}$) if an only if $m_\sigma \neq m_\rho$. Now, since $\overline{V} := V \otimes \overline{\mathbb{Q}} = W \oplus \check{W}$, W is irreducible, $\overline{\mathfrak{h}} \hookrightarrow \mathfrak{gl}(W) \overset{\Delta}{\to} \mathfrak{sp}(\overline{V})$. Summarizing,

$$\overline{\mathfrak{h}} := \overline{\mathfrak{h}}^{ss} \hookrightarrow \mathfrak{sl}(W) \overset{\Delta}{\to} \mathfrak{sp}(W \oplus \check{W}) \ \text{ if } m_\sigma = m_\rho,$$
$$\overline{\mathfrak{h}} = \overline{\mathfrak{h}}^{ss} \oplus \overline{\mathbb{Q}} \hookrightarrow \mathfrak{gl}(W) \overset{\Delta}{\to} \mathfrak{sp}(W \oplus \check{W}) \ \text{ if } m_\sigma \neq m_\rho.$$

Now, using 0.2.5 we can conclude

$$\overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{sl}(W_\lambda) \overset{\Delta}{\to} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda),$$
$$\overline{\mathfrak{g}} = \overline{\mathfrak{g}}^{ss}, \overline{\mathfrak{h}} = \overline{\mathfrak{h}}^{ss}, \ \text{if } m_\sigma = m_\rho,$$

$$\overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{gl}(W_\lambda) \overset{\Delta}{\to} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda),$$
$$C_{\overline{\mathfrak{g}}_\ell} = C_{\overline{\mathfrak{h}}_\ell} = \overline{\mathbb{Q}}_\ell, \ \text{if } m_\sigma \neq m_\rho.$$

□

Remark: As one can see from the proof, if $k \subseteq \operatorname{End}^\circ(A)$ (not necessarily equal), and $m_\sigma \neq m_\rho$, the center $C_{\mathfrak{h}}$ of $\mathfrak{h}$ must kill the determinant $\bigwedge_k^g V$, hence the center is non-trivial. However, if $k \neq \operatorname{End}^\circ(A)$, then the center can be non-trivial even if $m_\sigma = m_\rho$ (e.g., CM case).

§1. On the Hodge-Tate decomposition

1.0 We recall here certain basic facts on the Hodge-Tate decomposition and then give some applications. The classical/standard reference is [T 1].

1.0.1 According to Tate and Raynaud, $\overline{V}_\ell := V_\ell \otimes \mathbb{C}_\ell = H_1(\overline{A}_{\text{ét}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell$, $\mathbb{C}_\ell$: completion of $\overline{\mathbb{Q}}_\ell$, admits a decomposition

$$\overline{V}_\ell = \overline{V}_\ell(0) \oplus \overline{V}_\ell(1),$$

where $\overline{V}_\ell(i) := \overline{V}_\ell^{(i)} \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell$, $i = 1, 2$. The $\mathbb{Q}_\ell$-subspaces (but *not* $\mathbb{C}_\ell$-subspaces) $\overline{V}_\ell^{(i)}$'s of $\overline{V}_\ell$ are defined as follows:

$$\overline{V}_\ell^{(i)} := \{v \in \overline{V}_\ell \mid v^\sigma = \chi_\ell(\sigma)^i \cdot v,\ \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)\},\ i = 1, 2,$$

where $\chi_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \to \mathbb{Z}_\ell^\times$ is the cyclotomic character. Recall that the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ acts continously and semi-linearly on $\overline{V}_\ell$ (cf. [S 2], 1.2), and, clearly, the $\overline{V}_\ell^{(i)}$'s are Galois submodules of $\overline{V}_\ell$. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ acts on $\overline{V}_\ell(i)$ by the formula

$$(v \otimes c)^\sigma := v^\sigma \otimes c^\sigma,\ \forall v \in \overline{V}_\ell^{(i)},\ \forall c \in \mathbb{C}_\ell,\ \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell),$$

extended by linearity.

1.0.2 According to S. Sen ([Se], Section 4, Theorem 1), to the the Hodge-Tate decomposition on $\overline{V}_\ell$ one can associate a cocharacter

$$\phi : \mathbb{G}_{m,\mathbb{C}_\ell} \to \mathrm{GL}(V_\ell)_{\mathbb{C}_\ell},$$

by defining $\phi(z)$, $\forall z \in \mathbb{C}_\ell^\times$, to be the automorphism of $\overline{V}_\ell$ which is multiplication by z on $\overline{V}_\ell(1)$ and by the identity on $\overline{V}_\ell(0)$. This association is made in such a manner that $\tilde{G}_\ell$ (cf. 0.2) turns out to be the smallest algebraic group defined over $\mathbb{Q}_\ell$ which after extension of scalars to $\mathbb{C}_\ell$ contains the image of ϕ.

Remark: One can see that this cocharacter ϕ is completely analogous to the cocharacter μ associated to the Hodge decomposition on $V_{\mathbb{C}}$, $\tilde{G}_\ell$ is the analog of $M(A)$ and $\mathfrak{g}_\ell$ is the analog of $\mathfrak{h}$, see 0.1.2–0.1.4.

1.0.3 Before proceeding, recall that for a $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$-module X, the *Tate twist* $X(1)$ of X is defined to be $X \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)$ with the Galois structure of a tensor product of Galois modules (as in 1.0.1). Here $\mathbb{Q}_\ell(1)$ is the *Tate module:*

$$\mathbb{Q}_\ell(1) := (\varprojlim_n \mu_{\ell^n}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,\ \mu_{\ell^n} = \{\zeta \in \overline{\mathbb{Q}}_\ell \mid \zeta^{\ell^n} = 1\},$$

with the natural $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$-action by χ_ℓ:

$$\zeta^\sigma = \zeta^{\chi_\ell(\sigma)},\ \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell),\ \zeta \in \mu_{\ell^n},\ \text{for some } n.$$

1.0.4 The Hodge-Tate decomposition of $\overline{V}_\ell$ can be rewritten in the following explicit form ([T 1], § 4, Corollary 2, see also the Remark following that Corollary):

$$\overline{V}_\ell = \mathrm{Lie}(A^\vee_{\mathbb{C}_\ell})^\vee \oplus \mathrm{Lie}(A_{\mathbb{C}_\ell})(1),$$

where $\mathrm{Lie}(A^\vee_{\mathbb{C}_\ell})^\vee$ is the cotangent space of the dual abelian variety $A^\vee_{\mathbb{C}_\ell}$ at its origin and $\mathrm{Lie}(A_{\mathbb{C}_\ell})(1)$ is the tangent space of $A_{\mathbb{C}_\ell}$ at its origin Tate-twisted by χ_ℓ.

1.0.5 On the other hand, we have a Hodge decomposition on $V_{\mathbb{C}} = H_1(A(\mathbb{C}), \mathbb{C})$ (cf. 0.1.2):

$$\begin{aligned} H_1(A(\mathbb{C}), \mathbb{C}) &= H_1(A_{\mathbb{C}}, \mathcal{O}_{A_{\mathbb{C}}}) \oplus H_0(A_{\mathbb{C}}, \Omega^1_{A_{\mathbb{C}}}) \\ &= \mathrm{Lie}(A^\vee_{\mathbb{C}})^\vee \oplus \mathrm{Lie}(A_{\mathbb{C}}), \end{aligned}$$

or, in our notation,

$$V_{\mathbb{C}} = U_{\mathbb{R}} \oplus V_{\mathbb{R}},$$

see the proof of 0.3.2.

1.0.6 Fix an isomorphism $\mathbb{C}_\ell \cong \mathbb{C}$. Then the comparison isomorphism

$$c : H_1(\overline{A}_{\text{ét}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell \cong H_1(A(\mathbb{C}), \mathbb{C})$$

provides isomorphisms

$$c : V_{\mathbb{R}} \to \overline{V}_\ell(1), \;\; c : U_{\mathbb{R}} \to \overline{V}_\ell(0).$$

Note, that $\dim_{\mathbb{C}}(V_{\mathbb{R}}) = g = \dim_{\mathbb{C}_\ell}(\overline{V}_\ell(1))$, $\dim_{\mathbb{C}}(U_{\mathbb{R}}) = g = \dim_{\mathbb{C}_\ell}(\overline{V}_\ell(0))$.

1.0.7 On the other hand, both homology groups admit decompositions according to the action of $k \subseteq \mathrm{End}^\circ(A)$:

$$\overline{V}_\ell := \overline{V}_\lambda \oplus \overline{V}_{\lambda'}$$

where $\overline{V}_\lambda := W_\lambda \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}_\ell$, $\overline{V}_{\lambda'} := W_{\lambda'} \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}_\ell$, (cf. 0.2.9) and

$$V_{\mathbb{C}} := V_{\mathbb{C}}^\sigma \oplus V_{\mathbb{C}}^\rho.$$

These two types of splittings commute, and consequently $V_{\mathbb{C}}^\sigma, V_{\mathbb{C}}^\rho$ admit the Hodge decomposition, and $\overline{V}_\lambda$, $\overline{V}_{\lambda'}$ admit the Hodge-Tate decomposition. The map c respects these splittings, hence maps either $V_{\mathbb{C}}^\sigma$ to $\overline{V}_\lambda$, or $V_{\mathbb{C}}^\rho$ to $\overline{V}_\lambda$.

1.0.8 Let's *assume* that

$$c : V_{\mathbb{C}}^\sigma \xrightarrow{\sim} \overline{V}_\lambda,$$

hence $\dim_{\mathbb{C}}(V_{\mathbb{C}}^\sigma) = g = \dim_{\mathbb{C}_\ell}(\overline{V}_\lambda)$.

As a result we conclude

$$c : V_{\mathbb{R}}^{\sigma} \xrightarrow{\sim} \overline{V}_{\lambda}(1),$$

hence $\dim_{\mathbb{C}_\ell}(\overline{V}_\lambda(1)) = m_\sigma$, $\dim_{\mathbb{C}_\ell}(\overline{V}_\lambda(0)) = m_\rho$.

(**Remark:** If $c : V_{\mathbb{C}}^{\rho} \to \overline{V}_\lambda$, then m_σ and m_ρ exchange roles. This does not affect our results.)

1.1 For abelian varieties of type I, II and III $\mathfrak{g}_\ell$, $\mathfrak{h}_\ell$ are semi-simple and have the same invariants on $V_\ell \otimes_{\mathbb{Q}_\ell} V_\ell$ (cf. 0.2.4 and Remark 0.3.1(2)). In the case of type IV, the Lie algebras can be non-semi-simple. Since $\mathfrak{g}_\ell^{ss} \subseteq \mathfrak{h}_\ell^{ss}$, *a priori* $\mathfrak{g}_\ell^{ss}$ can have more invarints than $\mathfrak{h}_\ell^{ss}$ in the tensor powers of V_ℓ. However, we shall show that if $D = k$, $\mathfrak{g}_\ell^{ss}$ has "as many" invariants in $V_\ell \otimes_{\mathbb{Q}_\ell} V_\ell$ as $\mathfrak{h}_\ell^{ss}$ does.

Theorem: If $D = k$, then $\mathfrak{g}_\ell^{ss}$ is symplectic (respectively orthogonal, respectively non-self-dual) if an only if $\mathfrak{h}_\ell^{ss}$ is so.

Remark: If the abelian variety is of type IV, but the representations are semi-simple we get nothing new (cf. Proposition 0.3.1 and Remark 0.3.1(1)).

1.1.1 **Proof:** 0. If $\overline{\mathfrak{g}}_\ell^{ss}$ is symplectic or orthogonal, then so is $\overline{\mathfrak{g}}_\ell^{ss} \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}_\ell$. If $\overline{\mathfrak{h}}_\ell^{ss} \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}_\ell$ fixes a bilinear form on $\overline{V}_\ell$ coming from $V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ then $\overline{\mathfrak{h}}_\ell^{ss}$ fixes the form. So, we can extend scalars to $\mathbb{C}_\ell$ and shall use the same notation $\overline{\mathfrak{h}}_\ell$, $\overline{\mathfrak{g}}_\ell$ for the corresponding extentions of the Lie algebras.

1. Let's consider first the symplectic case.

Let $\chi \subseteq \bigwedge^2 \overline{V}_\lambda$ be a 1-dimensional $\overline{\mathfrak{g}}_\ell$-sub-representation. The Hodge-Tate decomposition implies

$$\bigwedge^2 \overline{V}_\lambda = \bigwedge^2 \overline{V}_\lambda(0) \oplus (\overline{V}_\lambda(1) \otimes \overline{V}_\lambda(0)) \oplus \bigwedge^2 \overline{V}_\lambda(1).$$

The Hodge-Tate weight of the terms on the right is 0, 1 and 2 respectively. Since $\dim_{\mathbb{C}_\ell}(\chi) = 1$, it is of *pure* Hodge-Tate weight 0, 1 or 2. $\chi^{g/2} = \det(\overline{V}_\lambda) := \bigwedge^{g}\overline{V}_\lambda = \bigwedge^{m_\sigma}\overline{V}_\lambda(1) \otimes \bigwedge^{m_\rho}\overline{V}_\lambda(0)$, the Hodge-Tate weight of the RHS is m_σ. Hence the Hodge- Tate weight of $\chi^{g/2}$ is m_σ. On the other hand, the weight of $\chi^{g/2}$ is $g/2$-times the weight of χ, hence is equal to 0, $g/2$ or g. So $m_\sigma = 0, g/2$ or g. The cases $m_\sigma = 0$ or g correspond to the k-signature $(0, g)$ or $(g, 0)$. In either case the abelian variety is isogenous to a product of CM elliptic curves ([Sh 1], Proposition 14). In the case $m_\sigma = g/2$ we have $m_\sigma = m_\rho$ and the Lie algebra $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{g}}_\ell^{ss}$ (as well as $\overline{\mathfrak{h}}_\ell = \overline{\mathfrak{h}}_\ell^{ss}$) is non-self-dual (cf. 0.3.1), hence this χ does not exist!

2. The orthogonal case is a direct consequence of the symplectic and 0.2.4 (take $s = 2$: $\mathrm{Sym}^2(\overline{V}_\lambda) \hookrightarrow \bigwedge^{2}(2\overline{V}_\lambda)$). □

1.1.2 **Remark:** One could have used instead an argument of Fontaine-Messing, [FM], 3.4, (see also our Remark in 0.2) to conclude the Theorem. Either way, the result is a consequence of the existence of the Hodge-Tate decomposition.

1.2 We can apply this consideration of the Hodge-Tate decomposition to abelian varieties with $D = k$ and k-signature (m_σ, m_ρ) such that $\gcd(m_\sigma, m_\rho) = 1$ (we call them ***Ribet-type*** abelian varieties, cf. [Ri]), then an argument of Serre ([S 1], §4) implies all the conjectures (Tate, Hodge, MT) in that case. Indeed, Ribet's proof of Theorem 3 loc.cit. *verbatim* provides the Tate cycles are generated by divisors and hence (cf. 0.2.2) the following theorem.

Theorem: If A is a Ribet-type abelian variety, then the Tate cycles (on the abelian variety, and all its self-products) are generated by divisors and hence the Tate, the Hodge and the Mumford-Tate conjectures hold.

1.2.1 **Sketch of proof:** The representation of $\mathfrak{g}_\ell$ on $\overline{V}_\lambda$ is irreducible (see the proof of 0.3.1 in 0.3.3) and $\overline{V}_\lambda = \overline{V}_\lambda(0) \oplus \overline{V}_\lambda(1)$, $\dim\overline{V}_\lambda(0) = m_\rho$, $\dim\overline{V}_\lambda(1) = m_\sigma$, (1.0.7). Since $\gcd(m_\rho, m_\sigma) = 1$ and $\mathrm{End}(\overline{V}_\lambda) = \mathbb{C}_\ell$, (cf. 0.3.3), we are in a position to apply [S 1], §4, Proposition 5, to conclude that the projection of $\mathfrak{g}_\ell$ on $\mathrm{End}(\overline{V}_\lambda)$ is surjective.

Furthermore, by replacing the cocharacter μ associated to the Hodge decomposition on $V_\mathbb{C}$ (cf. 0.1.2) by the cocharacter ϕ associated by the Sen's theorem (1.0.2) to the the Hodge-Tate decomposition on $\overline{V}_\ell$, and hence on $\overline{V}_\lambda$, we, as in [Ri], p.536, *Proof of Theorem 3*, conclude that $\bar{\mathfrak{g}}_\ell = \mathfrak{gl}(\overline{V}_\lambda) = \bar{\mathfrak{l}}_\ell$, (cf. 0.1.13-0.1.14). Here $\bar{\mathfrak{l}}_\ell := \mathrm{Lie}(L(A)) \otimes_\mathbb{Q} \mathbb{C}_\ell$, $L(A)$ is the Lefschetz group (0.1.10).

Recall now that by (0.2.2) $\mathfrak{g}_\ell \subseteq \mathfrak{h}_\ell$. From (0.1.10) it follows that if $\mathfrak{h}_\ell = \mathfrak{l}_\ell$, then in our case, i.e., $\mathrm{End}^\circ(A) = k$, all the Hodge classes on all the self-products of A are generated by divisors.

By putting all these results together we conclude the theorem. □

1.2.2 **Remark:** W. Chi [C 1] considered the Hodge-Tate decomposition in a very similar context. He proved the theorem (by *exactly* the method indicated above) in the case of abelian varieties of *prime* dimension. He stopped short of stating the result in the above form, even though the only thing he uses is the fact that $\gcd(m_\sigma, m_\rho) = 1$.

I learned of his proof after finishing this work.

§2. Abelian 4-folds of types II, III and IV

Now let A be a simple 4-dimensional abelian variety. We show that all such varieties of type II, III, IV (i.e., exactly those that admit an embedding $k \hookrightarrow D$) verify MT. We shall divide the proof in several steps according to what $D = \mathrm{End}^\circ(A)$ is. The possibilities are (cf. [MAV], §20):

1) type IV, $D =$ quaternion algebra over an imaginary quadratic field k;

2) type IV, $D = E$ a CM-field, $e = (E : \mathbb{Q})|2g = 8 \Rightarrow e = 2, 4, 8$;

3) type II (respectively III), $D =$ totally indefinite (respectively totally definite) quaternion algebra over $\mathbb{Q}$;

4) type II (respectively III), $D =$ totally indefinite (respectively totally definite) quaternion algebra over a real quadratic field E.

Note, that in the CM-case (i.e., type IV, $e = 8$) MT holds (cf. [Po]).

First we prove the case (1), then (2), $e = 2$ and conclude with cases (2), $e = 4$, (3) and (4) which are similar.

2.0 We shall need the following simple fact.

Lemma: $\mathfrak{a} \hookrightarrow \mathfrak{sl}(U)$ is a faithful irreducible minuscule representation of a semi-simple Lie algebra $\mathfrak{a}$ over an algebraically closed field of characteristic 0. If $\dim(U) = 4$ then $\mathfrak{a}$ is either $\mathfrak{sl}(U)(= \mathfrak{sl}_4)$, or $\mathfrak{sp}(U)(= \mathfrak{sp}_4)$, or $\mathfrak{so}(U)$ $(= \mathfrak{sl}_2 \times \mathfrak{sl}_2)$.

Hence,

$\mathfrak{a}$ is non-self-dual if and only if $\mathfrak{a} = \mathfrak{sl}(U)$,

$\mathfrak{a}$ is symplectic if and only if $\mathfrak{a} = \mathfrak{sp}(U)$,

$\mathfrak{a}$ is orthogonal if and only if $\mathfrak{a} = \mathfrak{so}(U)$.

Proof: Immediate from 0.2.6.

2.1 **Proposition:** If D is a quaternion algebra over an imaginary quadratic field k, then MT holds.

Proof: $V_\ell \otimes \overline{\mathbb{Q}}_\ell = W_\lambda \oplus \check{W}_\lambda$, $W_\lambda = V_\ell \otimes_{k,\lambda} \overline{\mathbb{Q}}_\ell$ (cf. 0.2.9), $\mathrm{End}_{\overline{\mathfrak{g}}_\ell}(W_\lambda) = \mathrm{End}_{\overline{\mathfrak{h}}_\ell}(W_\lambda) = D \otimes_{k,\lambda} \overline{\mathbb{Q}}_\ell \cong M_2(\overline{\mathbb{Q}}_\ell) \Rightarrow W = V_1 \oplus V_2, V_1 \cong V_2$, $V_1 : \overline{\mathfrak{g}}_\ell$-$, \overline{\mathfrak{h}}_\ell$-irreducible. Hence

$$\overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{gl}(W_\lambda) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda) \;\Rightarrow \overline{\mathfrak{g}}_\ell^{ss} \hookrightarrow \overline{\mathfrak{h}}_\ell^{ss} \hookrightarrow \mathfrak{sl}(W_\lambda) \cong \mathfrak{sl}_2$$

and, consequently, $\overline{\mathfrak{g}}_\ell^{ss}, \overline{\mathfrak{h}}_\ell^{ss}$ are either $\{0\}$ or $\mathfrak{sl}_2$. But $\overline{\mathfrak{g}}_\ell^{ss}, \overline{\mathfrak{h}}_\ell^{ss} \neq \{0\}$, since the abelian variety is *not* of CM-type $\Rightarrow \overline{\mathfrak{g}}_\ell^{ss} = \overline{\mathfrak{h}}_\ell^{ss} \cong \mathfrak{sl}_2$. □

2.2 **Proposition:** If $D = k$ is an imaginary quadratic field, then MT holds.

Proof: $V_\ell \otimes \overline{\mathbb{Q}}_\ell = W_\lambda \oplus \check{W}_\lambda$, $\dim_{\overline{\mathbb{Q}}_\ell}(W_\lambda) = 4$, $D \otimes \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell, \Rightarrow$ W_λ is 4-dimensional $\overline{\mathfrak{g}}_\ell$-$, \overline{\mathfrak{h}}_\ell$-irreducible representation. Hence $\overline{\mathfrak{g}}_\ell^{ss} \hookrightarrow \overline{\mathfrak{h}}_\ell^{ss} \hookrightarrow \mathfrak{sl}(W_\lambda) \cong \mathfrak{sl}_4$ are semi-simple 4-dimensional irreducible representations. If $\overline{\mathfrak{g}}_\ell^{ss} \neq \overline{\mathfrak{h}}_\ell^{ss}$ then $\overline{\mathfrak{h}}_\ell^{ss} \cong \mathfrak{sl}_4$, $\overline{\mathfrak{g}}_\ell^{ss}$: self-dual (cf. Lemma 2.0), in contradiction with Theorem 1.1 . □

Remark: If the k-signature is (2,2), then $\overline{\mathfrak{g}}_\ell$, $\overline{\mathfrak{h}}_\ell$ are semi-simple, non-self-dual (Propositions 0.3.1, 0.3.2), hence equal to $\mathfrak{sl}(W_\lambda) \cong \mathfrak{sl}_4$.

2.3 **Proposition:** If $D = E$ is a CM-field, $e = (E : \mathbb{Q}) = 4$, then MT holds.

Proof: $V_\ell \otimes \overline{\mathbb{Q}}_\ell = W_\lambda \oplus \check{W}_\lambda$, $\dim_{\overline{\mathbb{Q}}_\ell}(W_\lambda) = 4$, $D \otimes \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \Rightarrow W_\lambda \;= V_1 \oplus V_2$, $V_1 \ncong V_2$, V_i is a 2-dimensional $\overline{\mathfrak{g}}_\ell$-$, \overline{\mathfrak{h}}_\ell$-irreducible representation. Hence

$$\overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{gl}(V_1) \times \mathfrak{gl}(V_2) \hookrightarrow \mathfrak{gl}(W_\lambda) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda), \Rightarrow$$

$\overline{\mathfrak{g}}_\ell^{ss} \hookrightarrow \overline{\mathfrak{h}}_\ell^{ss} \hookrightarrow \mathfrak{sl}(V_1) \times \mathfrak{sl}(V_2) \hookrightarrow \mathfrak{sl}(W)$. Let $\mathfrak{h}_i := pr_i(\overline{\mathfrak{h}}_\ell^{ss}) \hookrightarrow \mathfrak{sl}(V_i) \cong \mathfrak{sl}_2$, $\mathfrak{g}_i := pr_i(\overline{\mathfrak{g}}_\ell^{ss}) \hookrightarrow \mathfrak{sl}(V_i) \cong \mathfrak{sl}_2$, $\mathfrak{g}_i \hookrightarrow \mathfrak{h}_i$, $i = 1, 2$.

2.3.1 Clearly, the only semi-simple subalgebras of $\mathfrak{sl}_2$ are $\{0\}$ and $\mathfrak{sl}_2$, hence

A) The only semi-simple subalgebras of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ are:

$\{0\} \times \{0\}$, $\{0\} \times \mathfrak{sl}_2$, $\mathfrak{sl}_2 \times \{0\}$, $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, graphs of automorphisms of $\mathfrak{sl}_2$.

B) $\mathfrak{h}_1 \cong \mathfrak{h}_2^\tau$, for some $\tau \in \mathrm{Gal}(E/k)$, since they are components of $\mathfrak{h}_\ell \otimes \overline{\mathbb{Q}}_\ell$ and $\mathfrak{h}_\ell$ is defined over $\mathbb{Q}_\ell$.

C) $\mathfrak{h}_1$ is not isomorphic to $\mathfrak{h}_2$ via an *inner* isomorphism: $\mathfrak{h}_1 \not\cong \mathfrak{h}_2^g$, $g \in \mathfrak{sl}_2$ (since $V_1 \not\cong V_2$).

D) $\mathfrak{g}_i$, $\mathfrak{h}_i \neq \{0\}$ simulteneously $\forall i$, since the abelian variety is *not* CM.

E) (A) & (C) $\Rightarrow \overline{\mathfrak{g}}_\ell^{ss}$, $\overline{\mathfrak{h}}_\ell^{ss} \neq \mathfrak{sl}_2 \times \mathfrak{sl}_2$.

F) (A) & (B) $\Rightarrow \overline{\mathfrak{g}}_\ell^{ss}$, $\overline{\mathfrak{h}}_\ell^{ss} \neq \{0\} \times \mathfrak{sl}_2$, $\mathfrak{sl}_2 \times \{0\}$.

G) (E) & (F) & (D) $\Rightarrow \overline{\mathfrak{g}}_\ell^{ss}$, $\overline{\mathfrak{h}}_\ell^{ss}$ are both graphs of automorphisms of $\mathfrak{sl}_2$. But there exists a unique 2-dimensional irreducible representation of $\mathfrak{sl}_2$ and $\overline{\mathfrak{g}}_\ell^{ss} \hookrightarrow \overline{\mathfrak{h}}_\ell^{ss} \Rightarrow \overline{\mathfrak{g}}_\ell^{ss} = \overline{\mathfrak{h}}_\ell^{ss}$ ($\cong \mathfrak{sl}_2$). □

2.4 **Note:** In the remaining cases $\mathfrak{g}_\ell$, $\mathfrak{h}_\ell$ are semi-simple (cf. Remark 0.3.2).

Proposition: If D is a (totally definite or totally indefinite) quaternion algebra over a totally real field E of degree $e = 1, 2$, then MT holds.

Proof: 1. If $e = 1$, (i.e., $E = \mathbb{Q}$), then $D \otimes \overline{\mathbb{Q}}_\ell \cong M_2(\overline{\mathbb{Q}}_\ell) \Rightarrow V_\ell \otimes \overline{\mathbb{Q}}_\ell = W_1 \oplus W_2$, $W_1 \cong W_2$, W_i is a 4-dimensional $\overline{\mathfrak{g}}_\ell$-, $\overline{\mathfrak{h}}_\ell$-irreducible representation. The representations of $\overline{\mathfrak{g}}_\ell, \overline{\mathfrak{h}}_\ell$ are symplectic (respectively orthogonal) if the abelian variety is of type II (respectively III). Lemma 2.0 provides $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell$.

2. If $e = 2$, (E : real quadratic), then $D \otimes \overline{\mathbb{Q}}_\ell \cong M_2(\overline{\mathbb{Q}}_\ell) \oplus M_2(\overline{\mathbb{Q}}_\ell) \Rightarrow V_\ell \otimes \overline{\mathbb{Q}}_\ell = W_1 \oplus W_2$, $W_1 \not\cong W_2$, $W_i = V_i^2$, V_i is a 2-dimensional $\overline{\mathfrak{g}}_\ell$-, $\overline{\mathfrak{h}}_\ell$-irreducible representation $\Rightarrow \overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{sl}(V_1) \times \mathfrak{sl}(V_2)$. Using the same argument as in 2.3.1 we conclude $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell$ ($\cong \mathfrak{sl}_2$). □

Remark: If $e = 2$ then $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell \cong \mathfrak{sl}_2$ and, hence, symplectic. So, this shows that there does not exist a type III *simple* abelian variety A with $\mathrm{End}^{\circ}(A)$ being totally definite quaternion algebra over a real quadratic field. In fact, such an abelian variety is isogenous to a product of 2 copies of a CM abelian variety ([Sh 1], Proposition 15).

§3. Abelian 4-folds of type I

For a simple abelian 4-fold A of type I, $D = \mathrm{End}^\circ(A) = E$ is a totally real field of degree $e|g$, so $g = 4 \Rightarrow e = 1, 2, 4$. We shall show that MT holds if $e = 2, 4$. If $e = 1$ and the abelian variety has bad, but not purely multiplicative reduction, then, as we show later, the conjecture also holds.

3.1 **Proposition:** If E is a real quadratic field, then MT holds.

Proof: $E \otimes \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \Rightarrow V_\ell \otimes \overline{\mathbb{Q}}_\ell = V_1 \oplus V_2$, $V_1 \not\cong V_2$, V_i is 4-dimensional symplectic $\overline{\mathfrak{g}}_\ell$-, $\overline{\mathfrak{h}}_\ell$-irreducible representation, $i = 1, 2 \Rightarrow$ (the projections) $\mathfrak{g}_i$, $\mathfrak{h}_i$ are 4-dimensional irreducible symplectic, hence isomorphic to $\mathfrak{sp}_4$ (Lemma 2.0). As in 2.3.1, $\overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{sp}(V_1) \times \mathfrak{sp}(V_2) \cong \mathfrak{sp}_4 \times \mathfrak{sp}_4$, are both graphs of automorphisms of $\mathfrak{sp}_4$, hence $\overline{\mathfrak{g}}_\ell \cong \mathfrak{sp}_4 \cong \overline{\mathfrak{h}}_\ell$.□

3.2 **Proposition:** If E is a totally real field of degree 4, then MT holds.

Proof: $E \otimes \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \Rightarrow V_\ell \otimes \overline{\mathbb{Q}}_\ell = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, $V_i \not\cong V_j$, $i \neq j$, V_i is a 2-dimensional $\overline{\mathfrak{g}}_\ell$-, $\overline{\mathfrak{h}}_\ell$-irreducible representation, hence $\overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{sl}(V_1) \times \mathfrak{sl}(V_2) \times \mathfrak{sl}(V_3) \times \mathfrak{sl}(V_4)$. Defining $\mathfrak{h}_{ij} := pr_{ij}(\overline{\mathfrak{h}}_\ell) \hookrightarrow \mathfrak{sl}(V_i) \times \mathfrak{sl}(V_j) \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{g}_{ij} := pr_{ij}(\overline{\mathfrak{g}}_\ell) \hookrightarrow \mathfrak{sl}(V_i) \times \mathfrak{sl}(V_j) \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{g}_{ij} \hookrightarrow \mathfrak{h}_{ij}$, we, as in 2.3.1, conclude $\mathfrak{g}_{ij} \cong \mathfrak{h}_{ij}$ are graphs of automorphisms of $\mathfrak{sl}_2$. This holds $\forall i, j \Rightarrow \overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell$ are both graphs of maps $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2 \Rightarrow \overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell$ ($\cong \mathfrak{sl}_2$). □

3.3 If $E = \mathbb{Q}$, then $V_\ell \otimes \overline{\mathbb{Q}}_\ell$ is a symplectic irreducible representation of $\overline{\mathfrak{g}}_\ell$, $\overline{\mathfrak{h}}_\ell$. The list 0.2.6 of minuscule representations contains only one such a *simple* Lie algebra, vis. $\mathfrak{sp}_8$. If the Lie algebra, say, $\mathfrak{a}$, is *not* simple then $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \hookrightarrow \mathfrak{sl}(V_1 \otimes V_2)$, $\mathfrak{a}_i \hookrightarrow \mathfrak{sl}(V_i)$: irreducible, $\dim(V_1) = 2$, $\dim(V_2) = 4$.

Hence $\mathfrak{a}_1 = \mathfrak{sl}_2$. Since both $\mathfrak{a}$ and $\mathfrak{a}_1$ are symplectic, $\mathfrak{a}_2$ must be orthogonal. Then Lemma 2.0 provides $\mathfrak{a} = \mathfrak{sl}_2 \times \mathfrak{sl}_2$. So, there are 2 choices for $\overline{\mathfrak{g}}_\ell$, $\overline{\mathfrak{h}}_\ell$: $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ and $\mathfrak{sp}_8$.

Remarks: 1. Generically $\overline{\mathfrak{h}}_\ell = \mathfrak{sp}_8$ and all the Hodge cycles are divisorial (cf. [Ma]).

2. Mumford in [M 2] constructed a simple abelian 4-fold with $D = \mathbb{Q}$ and $\overline{\mathfrak{h}}_\ell = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$. In this case MT holds, as well as the Hodge and the Tate conjectures (cf. Theorem 3.3.3).

3.3.1 Explicitly, if $\overline{\mathfrak{g}}_\ell \neq \overline{\mathfrak{h}}_\ell$, then one has $\overline{\mathfrak{g}}_\ell = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\overline{\mathfrak{h}}_\ell = \mathfrak{sp}_8$.

3.3.2 But even if this happens, the following lemma holds.

Lemma: Let $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_8 \hookrightarrow \mathfrak{sl}(V)$, $\dim(V) = 8$, be the irreducible representations over an algebraically closed field of characteristic zero, then

$$(\bigwedge^s V)^{\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2} = (\bigwedge^s V)^{\mathfrak{sp}_8}, \ \forall s \in \mathbb{N}.$$

Proof: From what was said above follows, that the only case to consider is $s = 4$. By calculating the formal character one can check that $(\bigwedge^4 V)^{\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2}$ is 1-dimensional (cf. [Ta], Lemma 4.10). On the other hand, $(\bigwedge^4 V)^{\mathfrak{sp}_8} \supseteq (\bigwedge^2 V)^{\otimes 2}$ thus non-zero and $(\bigwedge^4 V)^{\mathfrak{sp}_8} \subseteq (\bigwedge^4 V)^{\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2}$; we conclude that the invariants are the same and generated (in a clear sense) by the symplectic form. □

3.3.3 The Lemma applied to the abelian variety in question provides the following result.

Theorem: If A is a simple 4-dimensional abelian variety with $D = \mathbb{Q}$, then the rings of Tate cycles and Hodge cycles coincide and are generated by divisors. Hence the Hodge and the Tate conjectures hold. □

3.3.4 **Remark:** Similar to 3.3.2 calculations ([H 1], Lemma 5.2, (5.2.2)) show that $(\bigwedge^4(V \oplus V))^{\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2}$ is not generated by $(\bigwedge^2(V \oplus V))^{\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2}$. Hence the Hodge cycles on the "square" of the abelian 4-fold constructed by Mumford (3.3, Remark) are *not* all divisorial.

§4. Remarks

4.1 3.3 and 3.3.2 imply that if $E = \mathbb{Q}$, then either MT holds (if $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell$) or the Hodge cycles on the self-products of the abelian variety are all divisorial. Indeed, in this case the (Lie algebra of the) Hodge group is equal to the (Lie algebra of the) Lefschetz group $\overline{\mathfrak{h}} = \overline{\mathfrak{l}} = \mathfrak{sp}_8$ (cf. 0.1.10).

4.2 In the considered in §2 and §3.1–3.2 cases of abelian varieties of types I and II one can easily see, that the Galois and Hodge groups coincide with the Lefschetz group and hence all the Hodge and the Tate cycles are generated by divisors (although this is not true in general for type III (cf. 0.1.10), the Hodge conjecture holds in that case too). Thus the Hodge and the Tate conjectures hold (cf. 0.1.9(Lefschetz) and 0.2.1(Faltings)).

4.3 On abelian varieties with $D = k$ (type IV) and the signature of the k-action (1,3) or (3,1) all the Hodge (and Tate) cycles are divisorial (cf. 1.2).

4.4 However, in the Weil case (cf. Remark 0.3.2(1)), generically, the ring of Hodge cycles is not generated by divisors ([W], see also [MZ]). So, if there are any doubts about the Hodge conjecture (and hence the Tate conjecture), the Weil abelian varieties are the ones to look at. Recently, C. Schoen ([Sc], see also [vG]) succeded in proving the Hodge conjecture for one family of Weil 4-folds admitting an action of $\mathbb{Q}(\mu_3)$.

4.5 2.2 answers a question of Tate (cf. [T 2], p. 82) on whether the Tate conjecture is true for the Schoen family. This has been a motivation for this work.

§5. The Mumford-Tate conjecture for abelian 4-folds

Summarizing the above discussion we can state the theorem.

5.0 **Theorem:** 1. If A is any 4-dimensional abelian variety, then the rings of the Tate cycles and the Hodge cycles coincide (hence, the Hodge and the Tate conjectures for this variety are equivalent).
2. If, additionally, $\mathrm{End}^{\circ}(A) \neq \mathbb{Q}$, then MT holds. □

Remarks: 1. Later (Theorem 8.2) we shall see that even when $\mathrm{End}^{\circ}(A) = \mathbb{Q}$, MT holds under some reduction conditions.
2. Recall (0.2.3), that MT implies that the Hodge and the Tate conjectures are equivalent for an abelian variety and *all its self-products.*

Proof: If A is *simple*, these results were proved in §§ 2, 3.
If A is a *non*-simple abelian 4-fold, say A is isogenous to $A_1 \times A_2$, then $\dim A_i \leq 3$. Hence, by the (1,1)-theorems (0.1.9), (0.2.1) and duality, all the Hodge cycles on A_i are divisorial. The embeddings $\mathfrak{g}_\ell^{ss} \hookrightarrow \mathfrak{h}_\ell^{ss} \hookrightarrow \mathfrak{sp}(V_\ell)$ factor through the sub-representations corresponding to the simple components of A. The dimensions of the sub-representations are ≤ 6, and arguing as in §§ 2, 3, we conclude that $\mathfrak{g}_\ell^{ss} = \mathfrak{h}_\ell^{ss}$, hence MT holds. □

Note: If A is non-simple, then the second hypothesis of the theorem is satisfied.

5.1 Let me indicate what is the situation regarding the Hodge and the Tate conjectures for *non-simple* abelian 4-folds. As above, let A be a non-simple abelian 4-fold, $A \approx A_1 \times A_2$, and all the Hodge and the Tate cycles on the A_i's are divisorial.

5.1.1 We can also say that all the Hodge cycles (and hence the Tate cycles) on A are generated by divisors in the following cases:

1. Neither of the A_i's is of type IV ([H 2], Theorem 0.1).

2. A_1 is not of type IV, A_2 is of CM-type (loc. cit., Proposition 3.1).

3. If the A_i's are *non*-CM, type IV abelian surfaces, then according to [Sh 1], Theorem 5, Propositions 17, 19, the A_i's are products of CM elliptic curves. Hence so is $A = A_1 \times A_2$ and for such abelian varieties the result stated above is known ([Im]; [H 1], Theorem 2.7).

4. If the A_i's are *isogenous* CM surfaces, then by remark 0.1.5(1) and 0.1.10, $Hg(A) = Hg(A_1)$, $L(A) = L(A_1)$. By 0.1.10(1) $L(A_1) = Hg(A_1)$, hence $L(A) = Hg(A)$, and applying 0.1.10(1) once again we conlude the result.

5.1.2 For the remaining case, viz., both the A_i's are *non-isogenous* CM abelian varieties, let me just mention that Shioda constructed an example of a product of a simple CM 3-fold, say A_1, with a CM elliptic curve, say A_2, such that on $A = A_1 \times A_2$ there are exceptional, non-divisorial, Hodge cycles, [Shi 1], Example 6.1. In this example, however, the Hodge (hence the Tate) conjecture holds.

Part II. Abelian varieties with reduction conditions

§6. Bad reduction and inertia action

6.0 Let A be an abelian variety defined over a number field K. Assume A has bad reduction at a prime $\wp$ of $\mathcal{O}_K$, let $\bar{A}$ be the identity component of the special fiber of the Néron model of A. Then $\bar{A}$ is semi-abelian:

$$0 \to H \to \bar{A} \to B \to 0,$$

where H is the affine subgroup of $\bar{A}$, B is the abelian quotient.

6.0.1 Since we are concerned with the Lie algebra of (the image of) Galois, we can pass to a finite extension of K (cf. Note 0.2). So, according to the semi-stable reduction theorem, by extending the base field if necessary, we can assume that the reduction is *stable* (i.e., H is a torus) and *split* (i.e., H is split: $H \cong \mathbb{G}_m^r$).

The dimension r of H we call the *toric rank* of (the reduction of) A.

6.0.2 $D = \mathrm{End}^\circ(A)$ as before, there is a homomorphism $D \to \mathrm{End}^\circ(H)$, $1_A \mapsto 1_H$. But $\mathrm{End}^\circ(\mathbb{G}_m^r) = M_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$, hence $(D : \mathbb{Q}) | r$.

6.1 Consider the corresponding specialization sequence (cf. [SGA 7_{I}], Exp. IX, Proposition 3.5)

$$0 \to \mathcal{W} \to V_\ell(A) \xrightarrow{sp} V_\ell(\bar{A}) \to 0,$$

where $\mathcal{W}$ is the module of vanishing cycles $(\otimes \mathbb{Q}_\ell)$, $V_\ell(\bar{A})$ is the Tate module $(\otimes \mathbb{Q}_\ell)$ of $\bar{A}$. We have $\dim_{\mathbb{Q}_\ell}(V_\ell(A)) = 2g$, $\dim_{\mathbb{Q}_\ell}(V_\ell(\bar{A})) = 2g - r$, $\dim_{\mathbb{Q}_\ell}(\mathcal{W}) = r$.

6.2 Let $I := I(\wp) \hookrightarrow G(\wp)$ be the inertia and the decomposition groups at $\wp$. Then the above sequence is a sequence of $G(\wp)$-modules, hence of I-modules. The I-action is called *local monodromy* action.

6.3 $V_\ell(\bar{A})$ is a trivial I-module. Let V_ℓ^I be the I-invariants of $V_\ell(A)$, then V_ℓ^I is isomorphic to $V_\ell(\bar{A})$ via the reduction map [ST], $\mathcal{W} \hookrightarrow V_\ell^I$ and $\mathcal{W} \overset{red}{\cong} V_\ell(H) \cong \mathbb{Q}_\ell(1)^r$ ("*red*" is the reduction map; cf. [I], [O]).

6.4 The monodromy action on $V_\ell(A)$ is, in general, quasi-unipotent (e.g., [ST], [O]). However, since (we assumed that) the reduction of A is *stable* and *split*, this action is, in fact, *unipotent*. Repeat, that $V_\ell^I \overset{red}{\cong} \mathrm{im}(V_\ell(A) \overset{sp}{\to} V_\ell(\bar{A})) = V_\ell(A)/\mathcal{W}$. So, picking some vector subspaces U, T of $V_\ell(A)$ specializing to $V_\ell(H)$, $V_\ell(B)$ respectively, we get the matrix form of the monodromy action:

$$\begin{matrix} V_\ell^I \left\{ \begin{matrix} \mathcal{W}\{ \\ T\{ \end{matrix} \right. \\ U\{ \end{matrix} \begin{pmatrix} 1_r & 0 & *_r \\ 0 & 1_{2g-2r} & 0 \\ 0 & 0 & 1_r \end{pmatrix}.$$

Passing to the Lie algebra $\mathfrak{i} := \mathrm{Lie}(I)$, we conclude the existence of nilpotents, say $\tau \in \mathfrak{i} \hookrightarrow \mathfrak{g}_\ell$, of order 2 ($\tau^2 = 0$) and rank (with respect to V_ℓ) $\mathrm{rk}_{V_\ell}(\tau) \leq r$, where $\mathrm{rk}_{V_\ell}(\tau) := \dim_{\mathbb{Q}_\ell}(\tau V_\ell) =$ rank of the matrix of $\tau \in \mathfrak{gl}(V_\ell)$.

Note: The Neron-Ogg-Shafarevich criterion ensures that $\exists \tau \neq 0$, since A has bad reduction.

Remark: If N is given by the above matrix, then $\tau = N - 1_{2g} =$ logarithm of the monodromy, and the sequence in 6.1 corresponds to the *weight filtration*.

6.5 By extending scalars to $\overline{\mathbb{Q}}_\ell$ we get the corresponding nilpotents (of the same order) in each irreducible component of $V_\ell \otimes \overline{\mathbb{Q}}_\ell$ with the sum of the ranks with respect to each of the components being equal to the rank with respect to V_ℓ.

§7. Minimal reduction

7.0 **Definition:** An abelian variety A over a number field has *minimal* bad reduction at a prime $\wp$ of this field (or, just minimal reduction, for short) if the reduction is bad and the rank of the toric part H of $\bar{A}$ (cf. 6.0) is the minimal possible.

7.1 Let us go back to the case $D = k$, in which $\overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{gl}(W_\lambda) \overset{\Delta}{\to} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda)$ (cf. 0.2.9). The toric rank should be even (6.0.2), say, $2r$. If $\tau' = \Delta(\tau) \in \Delta(\overline{\mathfrak{g}}_\ell)$ is a nilpotent of rank $\mathrm{rk}_{V_\ell \otimes \overline{\mathbb{Q}}_\ell}(\tau') \leq 2r$, then $\tau^2 = \tau'^2 = 0$, $\mathrm{rk}_{W_\lambda}(\tau) \leq r$.

7.1.2 6.0.2 implies that in our case, i.e., when $D = k$, the minimal rank of H is 2.

7.1.3 Hence in the minimal reduction case $\exists \tau \in \overline{\mathfrak{g}}_\ell \hookrightarrow \overline{\mathfrak{h}}_\ell \hookrightarrow \mathfrak{gl}(W_\lambda)$, such that $\tau^2 = 0$, $\mathrm{rk}_{W_\lambda}(\tau) = 1$. The same, clearly, holds if we replace the Lie algebras with their semi-simple components, since all nilpotents live in these components.

Let us denote $\overline{\mathfrak{g}}_\ell^{ss}$ just by $\mathfrak{g}$, $\overline{\mathfrak{h}}_\ell^{ss}$ by $\mathfrak{h}$ (this shall not cause any confusion with $\mathfrak{h} = \mathrm{Lie}(Hg) \subseteq \mathfrak{gl}(V)$), W_λ by W. Then we rewrite the above as:

$$\tau \in \mathfrak{g} \hookrightarrow \mathfrak{h} \hookrightarrow \mathfrak{sl}(W), \ \tau^2 = 0, \ \mathrm{rk}_W(\tau) = 1,$$

$$\mathfrak{g}, \ \mathfrak{h} \ : \ \text{semi-simple irreducible representations.}$$

Such an element τ of rank 1 and order 2 is called a *transvection.*

7.2 It is a very restrictive condition for a semi-simple irreducible representation to contain a transvection.

Lemma: If $\mathfrak{a} \hookrightarrow \mathfrak{sl}(W)$ is a semi-simple irreducible representation, $\tau \in \mathfrak{a}$, $\tau^2 = 0$, $\mathrm{rk}_W(\tau) = 1$, then $\mathfrak{a}$ is simple and, moreover, it is either $\mathfrak{sp}(W)$ or $\mathfrak{sl}(W)$.

Proof: This is proved in [McL] (cf. also [PS]).

We prove here the first part, since we shall need a consequence of the proof. If $\mathfrak{a}$ is *non*-simple, let $\tau = \tau_1 \times \tau_2 \in \mathfrak{a}_1 \times \mathfrak{a}_2 = \mathfrak{a}$, $\mathfrak{a}_i \hookrightarrow \mathfrak{sl}(W_i)$: semi-simple, $i = 1, 2$, $W = W_1 \otimes W_2$; $\tau_1 \times \tau_2$ acts on $W_1 \otimes W_2$ as $\tau_1 \otimes 1 + 1 \otimes \tau_2$, i.e., $(\tau_1 \times \tau_2)(v_1 \otimes v_2) = \tau_1(v_1) \otimes v_2 + v_1 \otimes \tau_2(v_2)$.

<u>*Case 1:*</u> $\ker(\tau_1) \neq \{0\}$.

$\exists v_1^\circ \in V_1 \setminus \{0\}$, such that $\tau_1(v_1^\circ) = 0$, then $0 = (\tau_1 \times \tau_2)^2(v_1^\circ \otimes v_2) = v_1^\circ \otimes \tau_2^2 v_2 \Rightarrow \tau_2^2 v_2 = 0$, $\forall v_2 \in V_2 \Rightarrow \tau_2^2 = 0 \Rightarrow \ker(\tau_2) \neq \{0\}$. So, $\ker(\tau_1) \neq \{0\} \iff \ker(\tau_2) \neq \{0\}$, and in this case one of τ_i's is a zero map. Indeed, $\tau_1^2 = 0 = \tau_2^2$, and if $\tau_1 \neq 0$, then $\exists v_1$ such that $\tau_1(v_1) =: v_1' \neq 0$, and $0 = (\tau_1 \times \tau_2)^2(v_1 \otimes v_2) = 2\tau_1(v_1) \otimes \tau_2(v_2) = 2v_1' \otimes \tau_2(v_2)$, $\forall v_2 \in W_2$, $\Rightarrow \tau_2(v_2) = 0 \; \forall v_2 \Rightarrow \tau_2 = 0$.

<u>*Case 2:*</u> $\ker(\tau_1) \neq \{0\} \neq \ker(\tau_2)$.

Fix an eigenvalue, say $\alpha \neq 0$, of τ_1 and let β be any eigenvalue of τ_2; let v_i, $i = 1, 2$, be the corresponding eigenvectors. Then

$$(\tau_1 \times \tau_2)(v_1 \otimes v_2) = \alpha v_1 \otimes v_2 + v_1 \otimes \beta v_2.$$

Hence $\tau^2 = 0 \iff (\alpha + \beta)^2 = 0 \iff \beta = -\alpha$, i.e., all the eigenvalues of τ_2 are the same and equal to $-\alpha \Rightarrow$ all the eigenvalues of τ_1 are the same and equal to α, i.e., $\tau_1 = \alpha + \mu_1$, $\tau_2 = -\alpha + \mu_2$, and $\mu_i \in \mathfrak{gl}(W_i)$ are nilpotents. But then $(\tau_1 \times \tau_2)(v_1 \otimes v_2) = (\mu_1 \times \mu_2)(v_1 \otimes v_2)$ and $\ker(\mu_i) \neq \{0\}$. So, we are in the case 1 with μ_i's replacing τ_i's. Hence one of μ_i's is a zero map, say $\mu_2 = 0$, and $\tau_2 = -\alpha \in C_{\mathfrak{a}_2} = \{0\}$ ($\mathfrak{a}$ is semi-simple) $\Rightarrow \alpha = 0$ contrary to our assumption.

So, $\tau_1 \in \mathfrak{a}_1$ is an order 2 nilpotent in $\mathfrak{sl}(W_1)$, $\tau_2 = 0 \Rightarrow \mathrm{rk}_W(\tau) = \dim(W_2) \cdot \mathrm{rk}_{W_1}(\tau_1) = 1 \Rightarrow \dim(W_2) = 1$ & $\mathrm{rk}_{W_1}(\tau_1) = 1$. But $\mathfrak{a}_2$ is faithful and semi-simple thus $\mathfrak{a}_2 = 0 \Rightarrow \mathfrak{a}$ is simple. □

Remark: One can convince oneself that any "non-standard" representation (i.e., the highest weight is not ϖ_1) of a simple Lie algebra cannot contain elements of rank 1. But even in the "standard" representation of an orthogonal algebra there is no such elements, since the rank of a quadratic nilpotent in such an algebra is even (cf. [St], ch. IV, 2.19).

7.3 As a consequence of the above proof we get

Corollary: If $\mathrm{rk}_W(\tau)$ is prime to $\dim(W)$, then $\mathfrak{a}$ is simple.

§8. Applications of minimal reduction

An immediate application of 7.1.3 (existence of rank 1 quadratic nilpotents in $\mathfrak{g}$) and Lemma 7.2 is the following theorem.

8.1 **Theorem:** If A is a simple abelian variety with $D \subseteq k$, having minimal reduction, then MT holds.
Moreover, if $D = \mathbb{Q}$, then all the Hodge and the Tate cycles are divisorial, hence the Hodge and the Tate conjectures hold.

Proof: 1. If $D = \mathbb{Q}$, then the Tate module $V_\ell(A)$ is absolutely irreducible and symplectic. The minimality of reduction implies that the rank of a correspondent nilpotent is 1. The result follows from 7.2 and 1.1.
2. If $D = k$, the result follows from 7.1.3 and 1.1. □

Remarks: 1. As we show later, in part III, such abelian varieties exist and, moreover, form a subset dense in the complex topology in the corresponding moduli space.
2. The importance of the Weil type abelian varieties is not limited to the fact that they (may) have non-divisorial Weil cycles (cf. 4.4). Proving the algebraicity of the Weil cycles is a critical ingredient in proving the Tate conjecture (cf. [D 3], §§ 4-6).

8.2 Now we can extend Theorem 5.0 in the following way.

Theorem: If A is a simple abelian 4-fold with $D = \mathbb{Q}$, admitting bad, but not purely multiplicative reduction, then all the Hodge and the Tate cycles are divisorial, hence the Hodge conjecture, the Tate conjecture and MT hold.

Proof: The possible values of the toric rank in this case are 1, 2, 3 (4 corresponds to the purely multiplicative reduction). Theorem 8.1 takes care of the rank 1 case.

From the proof of Lemma 7.2 we see that $r = \mathrm{rk}_W(\tau) = \dim(W_2)\cdot \mathrm{rk}_{W_1}(\tau_1)$ (notations as in 7.2). Hence, if $r = 2$ (respectively 3), then either $\dim(W_2) = 1$ & $r_1 = \mathrm{rk}_{W_1}(\tau_1) = 2$ (respectively 3), or $\dim(W_2) = 2$ (respectively 3) & $r_1 = 1$. In the former case $\mathfrak{g}$, $\mathfrak{h}$ are simple, hence isomorphic to $\mathfrak{sp}_8$ (7.2 & 3.3). In the latter case, $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, where $\mathfrak{g}_1$ is simple by Lemma 7.2 and $\mathfrak{g}_2$ is simple since $\dim(W_2)$ is a prime number. But the only "non-simple" possibility for $\mathfrak{g}$ is $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ (3.3) which does not work. So, $\mathfrak{g} = \mathfrak{h} \cong \mathfrak{sp}_8$ in these cases and 0.1.10 finishes the proof. □

Remark: G. Mustafin [Mus] considered abelian varieties with purely multiplicative reduction, but in geometric setting (i.e., with the action of the geometric monodromy). It appears, a suitable adaptation of his methods could be used to prove MT in that case.

8.3 **Remarks:** 1. The idea of using special element(s) in the representation of the Hodge group has been used before. However, to my knowledge, in those earlier cases the element was semi-simple (e.g., [Z 1]) of rank 1 or 2 and the results then follow from a theorem of Kostant [Ko] (cf. also [Z 3]).

2. Katz used special unipotent elements to show that certain monodromy groups are large. However, the unipotents he considered were of the maximal possible rank, i.e., having only *one* Jordan block ([Ka 1], ch. 7). He was also using semi-simple elements, [Ka 2].

§9. Another type of bad reduction

9.0 Now we want to show the result analogous to Theorem 1.2 for another type of bad reduction, although not minimal, but satisfying the conditions of Corollary 7.3. In this case $\mathfrak{g}$, $\mathfrak{h}$ (notation as in 7.1.3) are simple and contain nilpotents τ of rank prime to the dimension of the representations. To achieve this goal we use the results of Premet-Suprunenko [PS] on classification of quadratic elements (i.e., nilpotents of order 2) and quadratic modules (i.e., representations containing non-trivial quadratic elements) of simple Lie algebras. (This terminology is apparently standard in the finite groups theory, cf. [Th].)

We use the fact that the representations of $\mathfrak{g}$, $\mathfrak{h}$ are minuscule and $\mathfrak{g}$ is not exceptional (0.2.7).

We assume that the dimension of the representations is > 4.

9.1 **Theorem:** If $\mathfrak{a} \hookrightarrow \mathfrak{b} \hookrightarrow \mathfrak{sl}(W)$, $\mathfrak{a} \neq \mathfrak{b}$, both Lie algebras are simple and the representations are irreducible and minuscule, then $\mathfrak{b}$ is classical and (its highest weight is) ϖ_1.

Proof: Since any minuscule representation is quadratic (cf. [B], ch VIII, §7.3), we can apply [PS], Theorem 3, and exclude non-minuscule representations. □

9.2 Now, we, case by case, consider all the possibilities of $\mathfrak{g} \subsetneq \mathfrak{h}$. Denote $r = \mathrm{rk}_W(\tau)$, $n = \dim(W)$. The standing assumptions are: $\gcd(r,n) = 1$, $\mathfrak{h}$: classical, ϖ_1.

9.3 First we exclude the cases $\mathfrak{g} = (D_m, \varpi_{m-1})$, (D_m, ϖ_m) for $m > 4$.

Lemma: If $\tau \in \mathfrak{g} \hookrightarrow \mathfrak{sl}(W)$, $\mathfrak{g} = (D_m, \varpi_{m-1})$, (D_m, ϖ_m), τ is a quadratic element, then $\gcd(r, n) > 2$.

Proof: [PS], Lemma 21 & Note 2 & Lemma 17 $\Rightarrow r = 2^{m-3}$ or 2^{m-2}, while $n = 2^{m-1}$. □

9.4 $\mathfrak{g} \neq (D_4, \varpi_3)$, (D_4, ϖ_4) either. It is enough to show this for ϖ_4, since they are (graph) isomorphic.

Proposition: $(D_4, \varpi_4) \not\hookrightarrow$ (classical, ϖ_1).

Proof: Note that $n = 8$ here.

1. $(D_4, \varpi_4) \not\hookrightarrow (B_\bullet, \varpi_1)$, since the dimension of the RHS is odd.
2. $(D_4, \varpi_4) \not\hookrightarrow (C_\bullet, \varpi_1)$, since the LHS is orthogonal while the RHS is symplectic (cf. [B], table 1).
3. $(D_4, \varpi_4) \not\hookrightarrow (D_4, \varpi_1)$ (e.g., [Z 2], §5, Key lemma, although that is an overkill).
4. $\mathfrak{g} = (D_4, \varpi_4)$, $\mathfrak{h} = (A_\bullet, \varpi_1)$ is impossible by Theorem 0.4, since the LHS is orthogonal, the RHS is not. □

9.5 **Proposition:** If $\mathfrak{g} = (D_m, \varpi_1)$, then $\mathfrak{g} = \mathfrak{h}$.

Proof: Assume $\mathfrak{g} = (D_m, \varpi_1)$.

1. $\mathfrak{h} = A_\bullet$, $C_\bullet$ are excluded by Theorem 0.4: $\mathfrak{g}$ is orthogonal, $\mathfrak{h}$ is not.
2. $\mathfrak{h} = B_\bullet$ is excluded by a dimensional reason. □

9.6 **Proposition:** If $\mathfrak{g} = (C_m, \varpi_1)$, then $\mathfrak{g} = \mathfrak{h}$.

Proof: $\mathfrak{h} = A_\bullet$, $B_\bullet$, $D_\bullet$ are not symplectic ... □

9.7 **Proposition:** If $\mathfrak{g} = (B_m, \varpi_1)$, then $\mathfrak{g} = \mathfrak{h}$.

Proof: $\mathfrak{h} = A_\bullet,\ C_\bullet$ are not orthogonal...

$(D_\bullet, \varpi_1)$ is even-dimensional... □

9.8 **Proposition:** If $\mathfrak{g} = (A_m, \varpi_s)$, then $\mathfrak{h}$ must be (A_{n-1}, ϖ_1).

Proof: First note, that the only self-dual representation of A_m is ϖ_s with $s = \frac{m+1}{2}$ (there is no such a representation if m is even). So, if $\mathfrak{h}$ is self-dual, then so is $\mathfrak{g}$ (Theorem 1.1) and we may assume $s = \frac{m+1}{2}$, m : odd. But then $\dim(A_m, \varpi_s) = \binom{2s}{s}$: even, thus $\mathfrak{h} \neq B_\bullet$. To exclude the other cases (i.e., $C_\bullet,\ D_\bullet$) we use the fact that $r = \mathrm{rk}(\tau) = \binom{m-1}{s-1}$ ([PS], §2 & Lemma18): $r = \binom{2(s-1)}{s-1}$, $n = \binom{2s}{s} = \binom{2(s-1)}{s-1}\frac{(2s-1)2s}{s^2} = r\frac{2(2s-1)}{s}$; $\gcd(n,r) = 1 \Rightarrow r|s$ which is not true. Since $n > 4$, this follows from

Claim : $\binom{2t}{t} > t+1$, for $t \geqq 2$.

Indeed, for $t = 2$, $\binom{2t}{t} = \binom{4}{2} = 6 > t+1 = 3$. If $\binom{2t}{t} > t+1$, then $\binom{2(t+1)}{t+1} = \binom{2t}{t}\frac{2(2t+1)}{t+1} > 2(2t+1) > t+2$. Hence the claim and the proposition. □

Remark: If the k-signature of the abelian variety is (m_σ, m_ρ) with $m_\sigma \neq m_\rho$, then, in general (i.e., when $\mathfrak{g}$ is not necessarily simple), simple components of $\mathfrak{g}$ are of type A (cf. [Y]). In our case, $\mathfrak{g}$ is of type A itself.

9.9 So, the only possibility for $\mathfrak{g} \subsetneqq \mathfrak{h}$ is $\mathfrak{g} = (A_m, \varpi_s)$ for some s, $\mathfrak{h} = (A_{n-1}, \varpi_1)$. In this case we can say the following.

Proposition: Let $\mathfrak{g} = (A_m, \varpi_s) \hookrightarrow \mathfrak{h} = (A_{n-1}, \varpi_1) = \mathfrak{sl}(W)$ (fix the isomrphism), $2 \leq s < \frac{m+1}{2}$, $\tau \in \mathfrak{g}$, $r = \mathrm{rk}_W(\tau)$, $\gcd(n,r) = 1$. Then one of the following holds:

1. $s = 3,\ m = 7$,
2. $s = 2,\ m \not\equiv 3 \pmod 4$.

Proof: $r = \binom{m-1}{s-1}$ ([PS], §2 & Lemma 18), $n = \binom{m+1}{s} = \binom{m-1}{s-1}\frac{m(m+1)}{s(m+1-s)}$, $\gcd(r,n) = 1 \Rightarrow r \mid s(m+1-s)$ and a half of the result follows from the following lemma.

Lemma: $r = \binom{m-1}{s-1}|s(m+1-s)$ if and only if $(m,s) = (7,3)$ or $s = 2$.

Proof: 1. Claim: $\binom{m-1}{s-1} > s(m+1-s)$ for $m \geqq 8,\ 3 \leq s < \frac{m+1}{2}$.

Indeed, this is true for $m = 8$.

If this is true for m, then $\binom{m}{s-1} = \binom{m-1}{s-1}\frac{m}{m+1-s} > s(m+1-s)\frac{m}{m+1-s} = ms > s(m+s-2)$, since $s \geqq 3$. Hence the claim.

2. If $m \leq 7,\ 3 \leq s < \frac{m+1}{2}$, then the calculations show that only for $(m,s) = (7,3)$, $\binom{m-1}{s-1}|s(m+1-s)$. □

Now, back to the proof of the proposition. Assume $s = 2$, then $r = \binom{m-1}{s-1} = m-1$, $n = \binom{m+1}{s} = \frac{m(m+1)}{2}$. Note that $\gcd(m, m-1) = 1$, $\gcd(m-1, m+1) = 1$ or 2. So, $\gcd(r,n) = 1$ if and only if m : even or $m \equiv 1 \pmod 4$ $\iff m \not\equiv 3 \pmod 4$. □

9.10 So, we have the following result.

Theorem: If A is a simple abelian variety with $D = k,\ g = \dim(A)$ having bad reduction with the toric rank $2r$ and r is prime to g, then MT holds if (g,r) is neither (56, 15) nor of the form $(\frac{m(m+1)}{2}, m-1)$. □

Remark: According to (the proof of) Proposition 9.9, $\gcd(\frac{m(m+1)}{2}, m-1) = 1$ if and only if $m \not\equiv 3 \pmod 4$.

9.11 If $D = \mathbb{Q}$, then $\mathfrak{g}_\ell,\ \mathfrak{h}_\ell$ are symplectic (cf. 0.3.1(2)), hence the above theorem holds without any exceptions (cf. 9.1 – 9.9).

Theorem: If A is an abelian variety with $D = \mathbb{Q},\ g = \dim(A)$ having bad reduction with the toric rank r prime to $2g$, then MT holds. □

9.12 Using the same methods one can handle the case of quadratic elements of rank $r = 2$. Namely, the following result holds.

Theorem: Let A be a simple abelian variety, $D \subseteq k$. If A has bad reduction with toric rank $r = 2 \cdot (D : \mathbb{Q})$, then MT holds. □

9.13 All the varieties considered in §9 exist and dense in the corresponding moduli spaces (cf. Theorem 14.1).

§10. A strange result

Let's mention another application of Theorem 9.1.

10.1 **Theorem:** If A is a simple abelian variety with (semi-simple parts of) $\mathfrak{h}$, $\mathfrak{g}_\ell$ simple, satisfying one of the following conditions:

1. the variety is of type I or II,
2. $D = k$, the variety is of non-Weil type (i.e., the k-signature is (m_σ, m_ρ) with $m_\sigma \neq m_\rho$),

then either MT holds or the Hodge cycles are all generated by divisor classes.

Proof: As we mentioned in 9.0, the representations of $\mathfrak{h}_\ell$, $\mathfrak{g}_\ell$ are minuscule, hence quadratic (cf. [B], ch VIII, §7.3, Proposition 7), then so is $\mathfrak{l}_\ell := \mathfrak{l} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ (cf. 0.1.10) and $\overline{\mathfrak{g}}_\ell^{ss} \hookrightarrow \overline{\mathfrak{h}}_\ell^{ss} \hookrightarrow \overline{\mathfrak{l}}_\ell^{ss} \hookrightarrow \mathfrak{sl}(W)$ (cf. 0.1.14). If $\overline{\mathfrak{g}}_\ell^{ss} \subsetneqq \overline{\mathfrak{h}}_\ell^{ss}$ (i.e., MT is wrong), then by Theorem 9.1 $\overline{\mathfrak{h}}_\ell^{ss} \hookrightarrow \mathfrak{sl}(W)$ is classical and ϖ_1. Thus $\overline{\mathfrak{l}}_\ell^{ss}$ is also simple, classical and ϖ_1. We want to show that in this case $\overline{\mathfrak{h}}_\ell = \overline{\mathfrak{l}}_\ell$ and the theorem shall follow from Theorem 0.1.10.

Consider first the case of an abelian variety of type I or II. The Lie algebras $\overline{\mathfrak{h}}_\ell$ and $\overline{\mathfrak{l}}_\ell$ are both symplectic (cf. [Mu], Lemma 2.3), simple (cf. 0.1.10(0(ii))), classical and ϖ_1. Hence $\overline{\mathfrak{h}}_\ell = \mathfrak{sp}(W) = \overline{\mathfrak{l}}_\ell$, and the result follows.

If $D = k$, then $C_{\overline{\mathfrak{h}}_\ell}$ is 1-dimensional (0.3.2), hence $C_{\overline{\mathfrak{h}}_\ell} = C_{\overline{\mathfrak{l}}_\ell}$ and it is enough to show $\overline{\mathfrak{h}}_\ell^{ss} = \overline{\mathfrak{l}}_\ell^{ss}$. It is known that in this case $\overline{\mathfrak{l}}_\ell^{ss} = \mathfrak{sl}(W)$ (cf. loc. cit.). Since $\overline{\mathfrak{h}}_\ell^{ss}$ is simple and ϖ_1, it is enough to show that $\overline{\mathfrak{h}}_\ell^{ss}$ is non-self-dual.

Yamagata [Y] has shown that in this case, i.e., $m_\sigma \neq m_\rho$, $\overline{\mathfrak{g}}_\ell^{ss}$ is of type A. One can use his argument to prove the same for $\overline{\mathfrak{h}}_\ell^{ss}$. We know that $\overline{\mathfrak{g}}_\ell^{ss}$ and $\overline{\mathfrak{h}}_\ell^{ss}$ are self-dual or non-self-dual simultaneously (Theorem 1.1). The only self-dual representation of an algebra of type A is (A_{2s-1}, ϖ_s). Hence in this case $\overline{\mathfrak{g}}_\ell^{ss} = \overline{\mathfrak{h}}_\ell^{ss}$. But we assumed $\overline{\mathfrak{g}}_\ell^{ss} \neq \overline{\mathfrak{h}}_\ell^{ss}$, so $\overline{\mathfrak{h}}_\ell^{ss}$ (as well as $\overline{\mathfrak{g}}_\ell^{ss}$) is non-self-dual. The theorem follows. □

10.2 **Remarks:** 1. Abelian varieties with bad reduction as in 9.10 have simple (semi-simple parts of) Hodge and Galois groups.

2. For the Weil type varieties $C_{\overline{\mathfrak{h}}_\ell} = \{0\} \neq C_{\overline{\mathfrak{l}}_\ell}$ *generically.* This is the main result of [W].

3. One can consider the *motivic Galois* group (cf. Deligne-Milne [DM], [J]). It is reductive and contains the Hodge group (it "sits between" the Hodge and the Lefschetz groups if an abelian variety is not of type III, since V. K. Murty [Mu] has shown that the invariants of $L(A)$ are exactly the divisorial cycles if A is not of type III, cf. 0.1.10(1)). The Hodge (respectively the Tate) conjecture follows from equality of the Hodge (respectively the Galois) group to the motivic Galois.

Suppose the (semi-simple parts of) Hodge and Galois groups are simple, then we can conclude that either MT holds or the semi-simple parts of the Hodge and motivic Galois groups coincide. So, the Hodge conjecture in that case is equivalent to the equality of the centers of these two groups.

4. Also, the consideration of the motivic Galois implies (as in the proof of Theorem 10.1) that for type III abelian varieties with simple Hodge and Galois groups either MT or the Hodge conjecture holds.

Part III. Moduli spaces and reductions

§11. Motivations. Moduli space interpretation

11.0 Now I would like to address the problem of existence of abelian varieties considered in Part II, and to know how "big" the set of such abelian varieties is (or, how "typical" are the properties we required in 8.1, 9.10 – 9.12).

11.1 The strategy is the following. Consider the boundary component of a (smoothly) compactified moduli space (over the residue field of some prime) corresponding to semi-abelian varieties with the needed toric rank. Describe the set of ($\overline{\mathbb{Q}}$-)points of the moduli space that get reduced modulo this prime to a chosen point in the boundary component. Show that the subset of the points corresponding to simple abelian varieties with a prescribed endomorphism ring($\otimes\mathbb{Q}$) is dense in the (set of $\mathbb{C}$-points of the) moduli space in the complex topology.

11.2 Let $\overset{\circ}{M}$ be a connected component of the moduli space of (suitably rigidified, cf. [Mi], Remark 1.4(b), p. 171) abelian varieties, say T, of dimension g with some (fixed) polarization, level N structure (for some $N \gg 1$), embedding $k \hookrightarrow \mathrm{End}^\circ(T)$ such that k is stable with respect to the Rosati involution (cf. 0.1.12) and the signature (cf. 0.1.11) of the k-action is (m_σ, m_ρ), $m_\sigma + m_\rho = g$ (cf. [D 1], [Sh 2]).

Remark: The choices of the level structure and the polarization are necessary in order to get a nice moduli space, but irrelevant for our purposes.

11.3 Let M be a smooth compactification of $\overset{\circ}{M}$. It is known that M (as well as $\overset{\circ}{M}$) has a model over a number field, say F, which is of finite type over F ([D 1], [Sh 2], [Pi]) and that the universal abelian scheme A over $\overset{\circ}{M}(K)$ extends to a semi-abelian scheme over M (cf. [FC], ch. IV, §§ 5,6), which we again denote by A. Hence we can find a finite set of primes of the ring of integers $\mathcal{O}_F$ of F, such that $A_{/M}$ admits an integral model over $\mathcal{O} := \mathcal{O}_F[S^{-1}]$, where S is the product of these primes. We denote the integral models of A by $\mathcal{A}$ and of M by $\mathcal{M}$.

11.3.1 Let K be a finite extension of F, R be the integral closure of $\mathcal{O}$ in K. Let x be a K-point of $\overset{\circ}{M}$ which is the generic fiber of an R-point $\tilde{x}$ of $\mathcal{M}$

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} \overset{\circ}{M} \hookrightarrow & M \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}) & \xrightarrow{\quad \tilde{x} \quad} & \mathcal{M} \end{array}$$

We also assume that the reduction of the abelian variety A_x (corresponding to $x \in \overset{\circ}{M}(K)$) at a prime $\wp$ of K, $\wp \nmid S$, is stable (cf. 6.0.1). Then the fiber $\mathcal{A}_{\tilde{x}}$ over $\tilde{x}$ is isomorphic to the identity component of the Néron model of A_x, (cf. [BLR], 7.4.3).

11.3.2 Let y be the generic fiber of some other R-point $\tilde{y}$ of $\mathcal{M}$, with $y \in \overset{\circ}{M}(K)$, and such that the special fibers

$$\overline{x}_\wp : \mathrm{Spec}(\kappa(\wp)) \to \mathrm{Spec}(R) \xrightarrow{\tilde{x}} \mathcal{M}$$

and

$$\overline{y}_\wp : \mathrm{Spec}(\kappa(\wp)) \to \mathrm{Spec}(R) \xrightarrow{\tilde{y}} \mathcal{M}$$

coincide. Then the reductions modulo $\wp$ of the abelian varieties A_x, A_y corresponding to the points x, y of $\overset{\circ}{M}$, are the same. Indeed, the fibers $\mathcal{A}_{\tilde{x}}$, $\mathcal{A}_{\tilde{y}}$ are naturally isomorphic to the identity components of the Néron models of A_x, A_y respectively (cf. [BLR], 7.4.3), and the condition that $\overline{x}_\wp = \overline{y}_\wp$ exactly means that the special fibers of the identity components of the Néron models are the same, i.e., A_x, A_y have the same reductions modulo $\wp$.

11.3.3 Clearly, if $\overline{y}_\wp \in \mathcal{M}(R_L)$, where L is a finite extension of K, R_L is the integral closure of R in L, and $\wp_L$ is some prime of L over $\wp$, then, by considering $\tilde{x}$ to be an R_L-point via the natural embedding $\mathcal{M}(R) \hookrightarrow \mathcal{M}(R_L)$, we conclude that if $\tilde{x}$, $\tilde{y}$ have the same reductions modulo $\wp_L$, then the corresponding abelian varieties (over L) have the same reductions modulo $\wp_L$.

Remark: Since A_x has *stable* reduction, the *identity component* of the Néron model behaves nicely with respect to base change (cf. [BLR], §7).

11.3.4 Let $r_{\wp_L} : R_L \to \kappa(\wp_L) := \mathcal{O}_L/\wp_L$ be the reduction map, which induces the map $r_{\wp_L} : \mathcal{M}(R_L) \to \mathcal{M}(\kappa(\wp_L))$. Let $\overline{x} \in \mathcal{M}(\kappa(\wp)) \hookrightarrow \mathcal{M}(\kappa(\wp))$ be the reduction of the fixed point x and $B_{L,\wp_L} := r_{\wp_L}^{-1}(\overline{x})$. It follows from the argument in 11.3.2–11.3.3 that the abelian varieties corresponding to the image $B_{L,\wp_L}(\hookrightarrow \mathcal{M}(R_L)) \hookrightarrow M(L)$ have the same reductions at $\wp_L$ as A_x.

11.3.5 If the point x we started with in 11.3.1 corresponds to an abelian variety with bad reduction modulo $\wp$ of some toric rank, say r, then all the (abelian varieties corresponding to the) points in $B_{L,\wp_L}$ have the same toric rank in their reductions modulo the prime $\wp_L$ over $\wp$.

We want to show that the set of abelian varieties defined over number fields (containing K) having the same reduction as x is "big". So, we have to study the sets $B_{L,\wp_L}$ for all algebraic extensions L over K.

11.3.6 Finally, an abelian variety with a prescribed toric rank of a reduction certainly exists, e.g., one can take $T_0 = T_1 \times T_2$, where $\dim(T_1) = r$,

$\dim(T_2) = g - r$ and T_1 has purely multiplicative reduction at $\wp$, T_2 has good reduction at $\wp$ and the conditions on the endomorphism ring, polarization, level structure, etc... are satisfied for both T_1 and T_2. We may also assume that $\wp \nmid S$ (cf. 11.3.1). Any such an abelian variety can be taken for the point x.

§12. Localization of the problem

12.0 The problem of describing the set of all abelian varieties with the same reduction as x (cf. 11.3.5) is local at $\wp$. Indeed, suppose we can find an abelian variety over a local field, say L, which is an extension of the localization of K at $\wp$. This abelian variety is defined over a finitely generated field, and this field is a subfield of $\overline{\mathbb{Q}}$, since L is. Hence this abelian variety is defined over a number field and has the prescribed reduction modulo $\wp$. So we may replace K with its localization at $\wp$, and R with $\mathcal{O}_K$. We use the same notation for $r_{\wp_L} : \mathcal{M}(\mathcal{O}_L) \to \mathcal{M}(\kappa(\wp_L))$, $B_{L,\wp_L} := r_{\wp_L}^{-1}(\overline{x})$ for any extension (of local fields) L/K.

12.1.1 Let $\widehat{L}$ be the completion of L at $\wp_L$, $\widehat{\mathcal{O}}_L$ be the corresponding ring of integers, $r_{\wp_L}$, $\widehat{r}_{\wp_L}$ be the reduction-modulo-$\wp_L$ maps of $\mathcal{O}_L$, $\widehat{\mathcal{O}}_L$ respectively, $B_{\widehat{L},\wp_L} := \widehat{r}_{\wp_L}^{-1}(\overline{x})$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} M(L) & \hookleftarrow & \mathcal{M}(\mathcal{O}_L) & \xrightarrow{r_{\wp_L}} & \mathcal{M}(\kappa(\wp_L)) \ni \overline{x} \\ \downarrow & & \downarrow & & \uparrow \widehat{r}_{\wp_L} \\ M(\widehat{L}) & \hookleftarrow & \mathcal{M}(\widehat{\mathcal{O}}_L) & \longleftarrow & B_{\widehat{L},\wp_L} \end{array}$$

12.1.2 The Hensel's lemma ensures that the "$\wp_L$-adic ball" $B_{\widehat{L},\wp_L}$ is non-empty.

Lemma: (i) $B_{\widehat{L},\wp_L} \neq \varnothing$.

(ii) $B_{\widehat{L},\wp_L}$ is an open set in $M(\widehat{L})$ in the topology induced by the $\wp_L$-adic valuation on $\widehat{L}$.

Proof: Part (i) is just the Hensel's lemma (cf. "Newton's Lemma" in [Gr] and also [Ba], Theorem 1, Corollary 1). Part (ii) is immediate from the definition of the $\wp_L$-adic topology. $\square$

12.1.3 Clearly, $B_{L,\wp_L} = B_{\widehat{L},\wp_L} \cap \mathcal{M}(\mathcal{O}_L) \hookrightarrow \mathcal{M}(\widehat{\mathcal{O}}_L)$. However, unless we impose some extra conditions on L, there is no reason for $B_{L,\wp_L}$ to be non-trivial. This will be the case though if $M(L) \hookrightarrow M(\widehat{L})$ is $\wp_L$-adically dense.

12.1.4 **Note:** Let L be a number field or a local field, $\widehat{L}$ be its completion at a prime $\wp_L$. Then the set $M(L)$ has an induced topology via the embedding $M(L) \hookrightarrow M(\widehat{L})$, where $M(\widehat{L})$ is taken with the topology induced by the valuation on $\widehat{L}$. We say that a subset of $M(L)$ is *$\wp_L$-open*, or *$\wp_L$-dense* if this subset has this properties in the induced topology on $M(L)$.

On the other hand, if $L \hookrightarrow \overline{\mathbb{Q}}$ is a *fixed* embedding, then $M(L) \subseteq M(\widehat{L}) \subseteq M(\mathbb{C})$ has an induced *complex* topology, and we use similarly the notions of *$\mathbb{C}$-openness*, *$\mathbb{C}$-density*, etc...

Also, if $\overline{\wp}$ is a prime of $\overline{\mathbb{Q}}$, we consider in $M(\overline{\mathbb{Q}})$ *$\wp$-opens*, etc...; and also *$\mathbb{C}$-opens*, etc...

12.1.5 **Remark:** We are interested in algebraic points of the moduli space which are $\wp$-adically close to x, hence having the same reduction mod $\wp$. So we are free to consider any extensions of K as long as they live in $\overline{\mathbb{Q}}$.

12.2 Let K^h be the *Henselization* of K (see [N] , ch. VII, § 43 or [R], ch. VIII for definitions); K^h is the largest extension of K in $\overline{\mathbb{Q}}$ in which $\wp$ splits completely (cf. [N]). Denote $\widehat{K^h}$ the completion of K^h. Although K^h is a fairly large extension of K, here is the main reason why we are considering it.

12.2.1 **Theorem:** $M(K^h) \hookrightarrow M(\widehat{K^h})$ is $\wp$-adically dense.

Proof: See [BLR], 3.6, Corollary 10 or [Gr], § 3, Lemmas 1, 2 (cf. also [A], Theorem 1.10). □

12.2.2 **Corollary:** $B_{K^h,\wp} = B_{\widehat{K^h},\wp} \cap M(K^h)$ is a non-empty $\wp$-open subset of $M(K^h)$.

Proof: $B_{\widehat{K^h},\wp} \hookrightarrow M(\widehat{K^h})$ is a $\wp$-open. □

12.2.3 **Remarks:** 1. $B_{K^h,\wp}$ is $\wp$-dense in $B_{\widehat{K^h},\wp}$.

2. $B_{K^{sh},\wp}$ is *Zariski*-dense in $M(K^{sh})$, where K^{sh} is the *strict* Henselization of K (= maximal subfield of $\overline{\mathbb{Q}}$ unramified at $\wp$, cf. [R], [N]).

12.3 **Proposition:** $M(\overline{\mathbb{Q}}) \hookrightarrow M(\overline{\mathbb{Q}}_{\bar{\wp}})$ is $\bar{\wp}$-dense.

Remark: Here $\overline{\mathbb{Q}}$ is the algebraic closure of K, $\overline{\mathbb{Q}}_{\bar{\wp}}$ is the algebraic closure of $\widehat{K}$.

Proof: Let $U \hookrightarrow M(\overline{\mathbb{Q}}_{\bar{\wp}})$ be a $\bar{\wp}$-open set, pick any point $t \in U$, then $\exists \widehat{L} \subseteq \overline{\mathbb{Q}}_{\bar{\wp}}$: finite extension of $\widehat{K}$, such that $t \in \widehat{L}$. This $\widehat{L}$ is, in fact, a completion of some finite extension L of K (e.g., [BGR], 3.4.2 Proposition 5), thus $U_{\widehat{L}} := U \cap M(\widehat{L}) \neq \varnothing$. So, if L^h is the Henselization of L, $\widehat{L^h}$ is the completion of L^h, then $U_{\widehat{L^h}} := U \cap M(\widehat{L^h})$ is non-empty, $\wp$-open in $M(\widehat{L^h})$. But by the theorem $M(L^h) \hookrightarrow M(\widehat{L^h})$ is $\bar{\wp}$-dense (the topology on $M(\widehat{L^h})$ is induced from $M(\overline{\mathbb{Q}}_{\bar{\wp}})$), hence $U_{L^h} \cap M(L^h) \hookrightarrow U \cap M(\overline{\mathbb{Q}}) \neq \varnothing$. □

§13. Global picture. First density theorem

13.0 Now we would like to return to the global setting. All previous considerations (12.0–12.3) correspond to fixing *one* prime $\bar{\wp}$ of $\overline{\mathbb{Q}}$ over $\wp$ and ignoring all other such primes.

13.0.1 First, for a fixed $\bar{\wp}$ in $\overline{\mathbb{Q}}$, if $L \subseteq L'$, then $B_{\widehat{L},\bar{\wp}\cap\widehat{L}} \subseteq B_{\widehat{L'},\bar{\wp}\cap\widehat{L'}}$, $B_{L,\bar{\wp}\cap L} \subseteq B_{L',\bar{\wp}\cap L'}$. So we can define $B_{\overline{\mathbb{Q}}_{\bar{\wp}},\bar{\wp}} := \bigcup_L B_{\widehat{L},\bar{\wp}\cap\widehat{L}} \subseteq M(\overline{\mathbb{Q}}_{\bar{\wp}})$, $B_{\overline{\mathbb{Q}},\bar{\wp}} := \bigcup_L B_{L,\bar{\wp}\cap L} \subseteq M(\overline{\mathbb{Q}})$. Clearly, $B_{\overline{\mathbb{Q}},\bar{\wp}} = B_{\overline{\mathbb{Q}}_{\bar{\wp}},\bar{\wp}} \cap M(\overline{\mathbb{Q}})$ (in $M(\overline{\mathbb{Q}}_{\bar{\wp}})$). Let me stress the fact that $B_{\overline{\mathbb{Q}},\bar{\wp}}$ is *non-empty* by 12.2.2.

Now combine together these $\bar{\wp}$-adic balls in $M(\overline{\mathbb{Q}})$ for all $\bar{\wp}|\wp$: $B_{\overline{\mathbb{Q}},\wp} := \bigcup_{\bar{\wp}|\wp} B_{\overline{\mathbb{Q}},\bar{\wp}} \subseteq M(\overline{\mathbb{Q}})$.

13.1 We would like to prove the following theorem.

Theorem: $B_{\overline{\mathbb{Q}},\bar{\wp}} \subseteq M(\overline{\mathbb{Q}})$ is $\mathbb{C}$-dense.

We will prove the theorem in several steps.

13.2 **Step 0. Preliminary reductions.**

Let U be a non-empty $\mathbb{C}$-open subset of $M(\overline{\mathbb{Q}})$. We have to show that $U \cap B_{\overline{\mathbb{Q}},\bar{\wp}} \neq \varnothing$, and we may clearly shrink U as much as necessary. Since $M(\overline{\mathbb{Q}})$ is covered by Zariski affine opens $\{V_\alpha\}$, U must meet some V_α. Then, since the analytic topology is stronger than the Zariski topology, $U \cap V_\alpha$ is a *non-empty* open set in V_α. Hence we may effectively assume that M itself is affine. Moreover, suppose there exists a birational map $\varphi : M \to W$. Then there exists a Zariski open $V \subseteq M$ such that φ is an isomorphism on V. Since the complement of V has positive codimension in M, $U \cap V \neq \varnothing$. So we may replace U by $U \cap V$ and even $\varphi(U \cap V)$, and replace M by W. Note also that by the same reasons for any subset of

M the properties to be $\wp$-open or $\wp$-dense, etc... hold if we restrict to an affine open in M or apply a birational isomorphism (the adic topology is stonger than Zariski topology; no Zariski closed set can contain a $\wp$-open). Further, any irreducible variety is birationally isomorphic to a hypersurface (cf. [Shaf], ch. I, § 3, Theorem 6), so we may assume that M is a hypersurface $V(f(\mathbf{t}, z)) \hookrightarrow \mathbb{A}^{n+1}$, $\mathbf{t} = (t_1, t_2, ..., t_n)$, $f \in K[\mathbf{t}, z]$: irreducible. Then we have the natural projection $\mathbb{A}^{n+1} \xrightarrow{pr} \mathbb{A}^n$ (corresponding to the embedding $K[\mathbf{t}] \overset{pr^*}{\hookrightarrow} K[\mathbf{t}, z]$), which, when restricted to M, gives a finite map $\pi : M \to \mathbb{A}^n$, $\deg(\pi) = \deg_z f(\mathbf{t}, z) =: d$. Let $B := B_{\overline{\mathbb{Q}},\wp}$, $D := \pi B$. So we have the following commutative diagram:

$$\begin{array}{ccccc} B & \hookrightarrow & M(\overline{\mathbb{Q}}) = V(f(\mathbf{t}, z))(\overline{\mathbb{Q}}) & \hookrightarrow & \mathbb{A}^{n+1}(\overline{\mathbb{Q}}) \\ \pi \downarrow & & \pi \downarrow & & pr \downarrow \\ D & \hookrightarrow & \mathbb{A}^n(\overline{\mathbb{Q}}) & \xrightarrow{\quad = \quad} & \mathbb{A}^n(\overline{\mathbb{Q}}) \end{array}$$

Further, we may assume $x = 0 \in \mathbb{A}^{n+1}$. Then

$$B = \{y' = (y_1, ..., y_{n+1}) \in M(\overline{\mathbb{Q}}) \subseteq \mathbb{A}^{n+1}(\overline{\mathbb{Q}}) | \forall i,\ \exists \bar{\wp}_i | \wp \text{ a prime of } \overline{\mathbb{Q}},\ y_i \in \bar{\wp}_i\},$$

i.e., by definition, $B = \{y' \in M(\overline{\mathbb{Q}}) \mid y'^*(t_i), y'^*(z) \in \bar{\wp}_i,\ \forall i,\ \bar{\wp}_i | \wp\}$. Hence $D = \pi B = \{y = (y_1, ..., y_n) = pr(y') = pr((y_1, ..., y_n, y_{n+1})) \mid y' \in B\}$. Denote $f_y(z) := f(y_1, ..., y_n, z)$, $\forall y = (y_1, ..., y_n) \in \pi M$. Note that a finite map is surjective (cf. [Shaf], ch. I, § 5, Theorem 4), thus $\pi M = \mathbb{A}^n$. Clearly, $y = \pi(y') \iff f_y(y'_{n+1}) = 0$. Now, $f_y(z) = z^d + \sum_{r=0}^{d-1} a_r(y) z^r$, $a_r(\mathbf{t}) = \sum_{|\mathbf{s}|=0}^{m_r} \alpha_{r\mathbf{s}} \mathbf{t}^{\mathbf{s}}$, $\mathbf{s} = (s_1, ..., s_n) \in \mathbb{N}^n$, $|\mathbf{s}| := s_1 + ... + s_n$, $\mathbf{t}^{\mathbf{s}} := t_1^{s_1} ... t_n^{s_n}$.

For a polynomial $h(u) = \sum_i h_i u^i \in L[u]$, L : an extension of K, $\wp_L | \wp$: prime, define its $\wp_L$-*Gauss norm* as $|h|_{\wp_L} := \max_i\{|h_i|_{\wp_L}\}$ (cf. [BGR], 1.4.1).

13.3 **Step 1. Reduction to $\mathbb{A}^n$.**

Next proposition reduces the theorem 13.1 to showing that $D \subseteq \mathbb{A}^n(\overline{\mathbb{Q}})$ is $\mathbb{C}$-dense.

13.3.1 **Proposition:** $\pi^{-1}D = B$.

Indeed, this result implies the following.

13.3.2 **Proposition:** $D \subseteq \mathbb{A}^n(\overline{\mathbb{Q}})$ is $\mathbb{C}$-dense if and only if $B \subseteq M(\overline{\mathbb{Q}})$ is.

Proof: The "if" part is trivial, so assume $D \subseteq \mathbb{A}^n(\overline{\mathbb{Q}})$ is $\mathbb{C}$-dense.
Let $U \subseteq M(\overline{\mathbb{Q}})$ be $\mathbb{C}$-open, then πU is $\mathbb{C}$-open in $\mathbb{A}^n(\overline{\mathbb{Q}})$, because π being a finite map is open in $\mathbb{C}$-topology (local uniformization, implicit function theorem). Since $D \subseteq \mathbb{A}^n(\overline{\mathbb{Q}})$ is assumed to be $\mathbb{C}$-dense, $D \cap U \neq \varnothing$. If U was choosen sufficiently small, then $\pi^{-1}U = \{U_i\}_{1 \leq i \leq \deg(\pi)}$ and $B = \pi^{-1}D$ (by proposition 13.3.1) intersects every sheet U_i, i.e., $B \subseteq M(\overline{\mathbb{Q}})$ is $\mathbb{C}$-dense. □

13.3.3 **Proof** of Proposition 13.3.1 : Let $y' \in B \subseteq M(\overline{\mathbb{Q}})$, $y := \pi y'$. Set $L := K(y) := K(y_1, ..., y_n)$: field of rationality of y, $L' := L(y'_{n+1}) = K(y')$. Then

$$\begin{aligned}\pi^{-1}y &= M \underset{\mathbb{A}^n, y}{\times} \mathrm{Spec}(L) \\ &= \mathrm{Spec}(K[\mathbf{t}, z]/(f(\mathbf{t}, z)) \underset{K[\mathbf{t}], y^*}{\otimes} L) \\ &= \mathrm{Spec}(L[z]/(f_y(z)) \underset{L}{\otimes} L) \\ &= \mathrm{Spec}(L' \underset{L}{\otimes} L) \\ &= \underset{\sigma \in \mathrm{Aut}(L'/L)}{\oplus} \mathrm{Spec}(L'^{\sigma}).\end{aligned}$$

Therefore, we can rewrite this as

$$\begin{aligned}\pi^{-1}y &= \{y'' \in M(\overline{\mathbb{Q}}) \mid f_y(y''_{n+1}) = 0\} \\ &= \{y'^{\sigma} = (y_1, ..., y_n, {y'}^{\sigma}_{n+1}) \mid \sigma \in \mathrm{Aut}(L'/L)\}.\end{aligned}$$

However, $y'_{n+1} \in \wp'$, for some prime $\wp'$ of L' over $\wp$ $\Rightarrow \forall \sigma \in \mathrm{Aut}(L'/L)$, $y'^{\sigma} \in \wp'^{\sigma}$: prime of L'^{σ} $\Rightarrow y'' \in B$ (cf. the description of B in 13.2). □

13.4 **Step 2. Proof for $\mathbf{A}^n$.**

Let $B_{\bar{\wp}}$ $(= B_{\overline{\mathbb{Q}},\bar{\wp}}) \subset B$ be the $\bar{\wp}$-part of B (cf. 13.0.1) for some prime $\bar{\wp}|\wp$ of $\overline{\mathbb{Q}}$, and $D_{\bar{\wp}} := \pi B_{\bar{\wp}} \subseteq \mathbf{A}^n(\overline{\mathbb{Q}})$. We show that

1. $D_{\bar{\wp}}$ is $\bar{\wp}$-open in $\mathbf{A}^n(\overline{\mathbb{Q}})$,

2. any $\bar{\wp}$-open subset of $\mathbf{A}^n(\overline{\mathbb{Q}})$ is dense in the complex topology.

This will finish the proof of the theorem 13.1.

13.4.1 **Proposition:** $D_{\bar{\wp}}$ is $\bar{\wp}$-open in $\mathbb{A}^n(\overline{\mathbb{Q}})$.

Proof: Let $y^{(1)} \in D_{\bar{\wp}}$, δ a small positive number, and let $y^{(2)} \in \mathbb{A}^n(\overline{\mathbb{Q}})$ a point satisfying $|y_i^{(1)} - y_i^{(2)}|_{\bar{\wp}} < \delta$, $\forall i$. Such $y^{(2)}$'s exist, since $\bar{\wp}$-opens in $\mathbf{A}^n(\overline{\mathbb{Q}})$ are "big," by the same reasons as in 12.3: $\mathbf{A}^n(\overline{\mathbb{Q}}) \subseteq \mathbf{A}^n(\overline{\mathbb{Q}}_{\bar{\wp}})$ is $\bar{\wp}$-dense, hence a $\bar{\wp}$-adic δ-neighborhood of $y^{(1)} \in \mathbb{A}^n(\overline{\mathbb{Q}})$ has many $\overline{\mathbb{Q}}$-points. Alternatively, it can be seen as a consequence of the weak approximation for number fields. In any case, $f_{y^{(1)}}(z)$ is close to $f_{y^{(2)}}(z)$ in $\bar{\wp}$-Gauss norm:

$$|f_{y^{(1)}}(z) - f_{y^{(2)}}(z)|_{\bar{\wp}} = \max_r\{|a_r(y^{(1)}) - a_r(y^{(2)})|_{\bar{\wp}}\},$$
$$|a_r(y^{(1)}) - a_r(y^{(2)})|_{\bar{\wp}} = |\sum_{|\mathbf{s}|=0}^{m_r} \alpha_{r\mathbf{s}}(y^{(1)^{\mathbf{s}}} - y^{(2)^{\mathbf{s}}})|_{\bar{\wp}},$$
$$\text{and } |y^{(1)^{\mathbf{s}}} - y^{(2)^{\mathbf{s}}}|_{\bar{\wp}} < \delta, \ \forall \mathbf{s} \in \mathbb{N}^n$$
$$\Rightarrow |f_{y^{(1)}}(z) - f_{y^{(2)}}(z)|_{\bar{\wp}} < \delta.$$

By the "$\bar{\wp}$-adic continuity of roots" ([BGR], 3.4.1, Proposition 1), this implies that the roots of $f_{y^{(1)}}(z)$ are $\bar{\wp}$-close to the roots of $f_{y^{(2)}}(z)$: for each root y'_{n+1} of $f_{y^{(1)}}(z)$ there exists a root y''_{n+1} of $f_{y^{(2)}}(z)$ such that

$$|y'_{n+1} - y''_{n+1}|_{\bar{\wp}} < |f_{y^{(1)}}(z) - f_{y^{(2)}}(z)|_{\bar{\wp}}^{1/d} \cdot |f|_{\bar{\wp}},$$

where $d = \deg f_y(z)$ $(\forall y)$. But $B_{\bar{\wp}}$ is $\bar{\wp}$-open (cf. 12.1.2), hence the fiber over $y^{(2)}$ is in $B_{\bar{\wp}} \Rightarrow y^{(2)} \in D_{\bar{\wp}} \subset D$. □

13.4.2 **Remarks:** 1. This is a down-to-earth proof of the openness of a finite map in the adic topology. It would not have been necessary if I could have found a "standard" reference for this fact.

2. This proof is neither the shortest, nor the easiest, nor, probably, the most elementary.

13.4.3 **Proposition:** $D_{\bar{\wp}}$ is $\mathbb{C}$-dense in $\mathbb{A}^n(\overline{\mathbb{Q}})$.

Proof: Since $D_{\bar{\wp}}$ is $\bar{\wp}$-open, it contains a $\bar{\wp}$-neighborhood U_y of each of its points y. It suffices to show that U_y is $\mathbb{C}$-dense. Let $y \in D_{\bar{\wp}}$, $K' := K(y)$, $\wp' := \bar{\wp} \cap K'$, and consider U_y to be a $\bar{\wp}$-neighborhood of y of the form "$y + \wp'^a \overline{\mathbb{Z}}^n$" $= \{y' \in \mathbb{A}^n(\overline{\mathbb{Q}}) \mid (y_i - y'_i) \in \wp'^a \overline{\mathbb{Z}},\ \forall i\}$, where $a \in \mathbb{N}$, $\overline{\mathbb{Z}} \subseteq \overline{\mathbb{Q}}$ is the ring of algebraic integers. To prove that this $\bar{\wp}$-neighborhood of y is $\mathbb{C}$-dense in $\mathbb{A}^n(\overline{\mathbb{Q}})$ it is enough to do this for $n = 1$ ("coordinate-wise approximation"). But the case $n = 1$ is a consequence of the following two simple lemmas.

13.4.4 **Lemma:** Let $A \subseteq \overline{\mathbb{Q}}$ be a subset. Then A is $\mathbb{C}$-dense if and only if $\alpha + \beta A := \{\alpha + \beta a \mid a \in A\}$ is dense $\forall \alpha, \beta \in \overline{\mathbb{Q}},\ \beta \neq 0$.

Proof: Fix some α, β then $\forall t \in \overline{\mathbb{Q}},\ |t - (\alpha + \beta a)| < \epsilon \iff |(t - \alpha)\beta^{-1} - a| < |\beta^{-1}|\epsilon$. □

13.4.5 **Lemma:** $\overline{\mathbb{Z}}$ is dense in $\mathbb{C}$.

Proof: First, find some subset $\mathcal{R} \subseteq \overline{\mathbb{Z}} \cap \mathbb{R}$ dense in $\mathbb{R}$. (Everyone has his/her own favorite choice; here is a couple of examples:

1. integers in a real quadratic field, e.g., $\mathbb{Z}[\sqrt{2}] := \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\} \subseteq \mathbb{R}$ is dense ("irrational wrapping"!),

2. $\{m + n(2^{1/r} - 1) \mid m, n \in \mathbb{Z},\ r \in \mathbb{N}\}$.)

Then "spread" this set over the whole $\mathbb{C}$ by roots of 1 : $\mu_\infty\mathcal{R} := \{\zeta r \mid r \in \mathcal{R},\ \zeta^m = 1 \text{ for some } m \in \mathbb{N}\} \subseteq \overline{\mathbb{Z}}$ is dense in $\mathbb{C}$. Here is a prescription for approximation of some $z = \varrho e^{i\theta}$: first, "turn" it by some ζ to make it close to $\varrho \in \mathbb{R}$ and then approximate ϱ by an $\alpha \in \mathcal{R}$: $\forall\epsilon > 0$, let $m, n \in \mathbb{Z}$ be such that $\frac{m}{n} \leq \frac{\theta}{2\pi} < \frac{m+1}{n}$, $2\pi\frac{\varrho}{n} < \frac{\epsilon}{2}$; let $z' := ze^{-i2\pi\frac{m}{n}}$, choose $\alpha \in \mathcal{R}$ such that $|\alpha - \varrho| < \frac{\epsilon}{2}$, then

$$\begin{aligned}|z' - \alpha| &\leq |z' - \varrho| + |\varrho - \alpha| \\ &< \varrho(\theta - 2\pi\frac{m}{n}) + \frac{\epsilon}{2} \\ &< \varrho 2\pi\frac{1}{n} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon.\end{aligned}$$

Now $|z - \alpha\zeta| = |\zeta^{-1}||z' - \alpha| < \epsilon$. □

13.4.6 To conclude the proof of Proposition 13.4.3 choose a uniformizer for $\wp'$, say ϖ. By lemma 13.4.4, $y_i + \varpi^a\overline{\mathbb{Z}}$ is dense in $\overline{\mathbb{Q}}$ if and only if $\overline{\mathbb{Z}}$ is. Lemma 13.4.5 finishes the proof of the Proposition and hence of Theorem 13.1. □

§14. A second density theorem:
Existence of *simple* abelian varieties with prescribed reduction

14.0 So, we have found a subset of the moduli space (since the boundary components of the compactification are of positive codimensions, we can safely forget about them), which is dense in the usual $\mathbb{C}$-topology and each point of the set corresponds to an abelian variety defined over a number field (depending on this abelian variety) with the prescribed reduction. But for applications in Part II of this thesis we need simple abelian varieties with endomorphism ring($\otimes\mathbb{Q}$) exactly equal to the imaginary quadratic field k (cf. 8.1, 9.10).

14.1 Let us change notation slightly and set M to be a connected component of the moduli space of (suitably rigidified) abelian varieties, say T, of even dimension $g > 2$ with some (fixed) polarization, level structure, embedding $k \hookrightarrow \mathrm{End}^\circ(T)$ such that k is stable with respect to the Rosati involution and the signature of the k-action is (m_σ, m_ρ), $m_\sigma \cdot m_\rho \neq 0$, (cf. 11.2).

Theorem: There exists $N > 0$, such that $\forall \wp$ with the absolute norm $\mathrm{N}\wp > N$, the set of *simple* abelian varieties in M with a prescribed stable reduction modulo $\wp$ and the endomorphism ring($\otimes\mathbb{Q}$) equal to k is $\mathbb{C}$-dense in M. In particular, the set of abelian varieties having bad reductions with fixed toric rank is dense.

Proof: 1. First notice, that because of the Poincaré reducibility theorem and Schur's lemma, if an abelian variety A has $\mathrm{End}^\circ(A) = k$, then A is simple.

2. The number N is the upper bound of norms of "bad" primes dividing S (cf. 11.3).

3. There is an analytic open subset U of M (which is a complement of a countable union of subvarieties of positive codimension (cf. [M 1]; [An], Lemma 4) for each point of which the corresponding abelian variety has the "smallest possible" endomorphism ring. Since $m_\sigma \cdot m_\rho \neq 0$ and $g > 2$, by [Sh 1], Theorem 5, this "smallest" ring($\otimes\mathbb{Q}$) is exactly k. Since the set of our abelian varieties is $\mathbb{C}$-dense, its intersection with such an analytic open set U is $\mathbb{C}$-dense in U . □

§15. Comments

15.1 **Remarks:** 1. We can assume that abelian varieties from the analytic subset of the moduli space considered in the proof of Theorem 14.1 have exceptional Hodge classes (cf. [W]).

1′. Theorems 8.1, 9.11–9.12 deal also with abelian varieties with endomorphism ring($\otimes\mathbb{Q}$) = $\mathbb{Q}$. Clearly, the density results hold for such abelian varieties as well.

2. For the first density theorem, instead of (compactification of) the moduli space we could have considered any family of abelian varieties over a base which is of finite type over a number field. If there is a $\mathbb{C}$-open subset of the base, over which the fibers are simple, then the proof of the (part of the) second density theorem (regarding density of simple varieties) goes through as well.

3. We did not need the point x we started with. We could have taken any *smooth* point $\overline{x}$ corresponding to a semi-abelian variety with a fixed toric rank.

4. One can prove similar density results for abelian varieties with prescribed reduction modulo any finite powers of any finite set of primes. The basic idea is the same: reduce the problem to an affine space case where it is easy.

5. One can apply the previous remark to prove density of *simple* abelian varieties with a prescribed reduction regardless of the endomorphisms. Just impose an extra condition that at some auxiliary prime the reduction is a simple abelian variety. If there are simple abelian varieties in the family (cf. 2. above) this is certainly possible.

6. There is another way of proving the existence of an abelian variety with prescribed reduction (using deformation theory à la Faltings-Chai, Artin). By a trick of Oort and van der Put ([OP], Lemma 3.1) we, in fact, could have guaranteed that the conditions on the endomorphism ring are also satisfied. The way we have chosen is somewhat similar, but all the deformation theory is hidden in moduli and the existence of compactifications that we have assumed.

7. There is another way of proving $\mathbb{C}$-density of simple abelian varieties with reductions of needed toric rank, once we have at least one such abelian variety. Namely, use [Bor] (cf. also [D 1], Proposition 5.2) to show that the set of abelian varieties isogenous to the given one is dense in the corresponding Shimura variety.

15.2 I would like to mention separately that abelian varieties admitting minimal reduction (cf. 7.0) at some prime of their field of definition are, in some very vague sense, the "most typical." These abelian varieties are "liftings" of the biggest boundary components of moduli (from all positive characteristics), since *"minimal toric rank* $\Longleftrightarrow$ *maximal dimension of the abelian quotient."* So, for each prime, "most" of the abelian varieties with bad reduction at this prime have minimal reduction. Since there are "very few" abelian varieties having good reduction everywhere, "most" abelian varieties have minimal reduction (at some prime).

Is there a sensible way to describe the set of such abelian varieties (more) precisely?

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