Dirac Spectra, Summation Formulae, and the Spectral Action

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Abstract

Noncommutative geometry is a source of particle physics models with matter Lagrangians coupled to gravity. One may associate to any noncommutative space $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ its spectral action, which is defined in terms of the Dirac spectrum of its Dirac operator $\mathcal{D}$. When viewing a spin manifold as a noncommutative space, $\mathcal{D}$ is the usual Dirac operator. In this paper, we give nonperturbative computations of the spectral action for quotients of $SU(2)$, Bieberbach manifolds, and $SU(3)$ equipped with a variety of geometries. Along the way we will compute several Dirac spectra and refer to applications of this computation.

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Chapter 1

Introduction

Noncommutative geometry is a source of particle physics models with matter Lagrangians coupled to gravity [18]. A noncommutative geometry consists of a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, of an algebra $\mathcal{A}$ and a self-adjoint, typically unbounded operator $\mathcal{D}$ concretely represented on a Hilbert space $\mathcal{H}$. Such spaces are usually referred to as spectral triples. The associated action functional, which underlies these physics models, is defined as the trace of the cutoff of the Dirac operator by a test function, and is called the spectral action. More precisely, the spectral action is given by the expression

$$\mathrm{Tr} f(\mathcal{D}/\Lambda). \tag{1.1}$$

In this expression, Λ is a positive number, and f is a positive smooth even function decaying rapidly at infinity.

There is a general asymptotic expansion for the spectral action, and by studying the appropriate spectral triple, one can use this asymptotic expansion to recover the classical Lagrangian and Einstein-Hilbert action, along with additional terms [23].

Noncommutative cosmology aims to build cosmological models based on the spectral action [50]. To extrapolate these models to the recent universe, one cannot use the asymptotic form of the spectral action. The calculations below consist of more direct calculations of the spectral action which are valid in a broader regime than the asymptotic approach, and were undertaken in order to enable a further analysis of these cosmological models.

These calculations are possible in highly symmetric situations, where the spectrum of the Dirac operator is exactly known and decomposable into arithmetic progressions indexed by $\mathbb{Z}^k$ or $\mathbb{N}^k$, and the multiplicities are polynomials of the eigenvalues. In order to carry out the calculations one must first compute the Dirac spectrum. This is done by casting the problem into familiar problems in representation theory. In several of the cases considered below, the spectrum has been calculated explicitly by others. In other cases, formulae for generating the spectrum are known, and we simply apply these formulae to produce the explicit Dirac spectrum. In yet other cases, we determine for ourselves the formula for generating the spectrum. Once the Dirac spectrum has been computed, one then finds a suitable decomposition of the spectrum into arithmetic progressions and discovers the polynomials which describe the multiplicities of the eigenvalues. In this form, the spectral action may be usefully expressed by applying the Poisson summation formula or the Euler-Maclaurin formula when the spectrum is indexed by $\mathbb{Z}^k$ or $\mathbb{N}^k$ respectively. The first calculations of this sort were performed in [16], in the case of $SU(2)$ with the round metric and [17] in the case of $SU(2)$ equipped with the Robertson-Walker metric.

All of the spectral action computations below, with the exception of $SU(3)$, use the Poisson summation formula. Once the Dirac spectrum has been explicitly computed, there is a unifying structure to all of the calculations, which we presently review. Suppose that the spectrum has been computed and decomposed into one or several pieces as described above, where the multiplicities are described by one or more polynomials, P. Specifically, this means that the spectral action can be expressed as a sum of one or several terms of the form

$$\sum_{n\in\mathbb{Z}^k} P(n_0+n) f\left(\frac{n_0+n}{\Lambda}\right) = \sum_{n\in\mathbb{Z}^k} g(n), \tag{1.2}$$

where $g(n) = P(n_0+n) f\left(\frac{n_0+n}{\Lambda}\right)$.

Next, we apply the Poisson summation formula to g.

$$\sum_{n\in\mathbb{Z}^k} g(n) = \sum_{n\in\mathbb{Z}^k} \widehat{g}(n). \tag{1.3}$$

Since f is a Schwarz function, one has the estimates

$$\sum_{n\neq 0} |\widehat{f^{(j)}}((\Lambda n)| \leq C_k \Lambda^{-k}, \tag{1.4}$$

where $\widehat{f^{(j)}}$ denotes the Fourier transform of $|x|^j f(x)$.

As a result, as Λ goes to infinity, the sum of all terms $\widehat{g}(n)$ for $n \neq 0$ decays faster than

Λ^k for any $k > 0$. The final expression for the spectral action is

$$\mathrm{Tr} f(\mathcal{D}/\Lambda) = \widehat{g}(0) + O(\Lambda^{-k}), \tag{1.5}$$

for arbitrary $k > 0$.

As for $SU(3)$, we compute the spectral action using the Euler-Maclaurin formula. In one-dimension, the formula is

$$\sum_{k=a}^{b} g(k) = \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \sum_{j=2}^{m} \frac{B_j}{j!}(g^{(j-1)}(b) - g^{(j-1)}(a)) - R_m, \tag{1.6}$$

where B_j are the Bernoulli numbers, and the formula for the remainder R_m is

$$R_m = \frac{(-1)^m}{m!} \int_a^b g^{(m)}(x) B_m(x - [x])dx. \tag{1.7}$$

In the remainder formula, $B_m(x)$ are the Bernoulli polynomials. When we use the Euler-Maclaurin formula to compute the spectral action in this situation, we once again have $g(n) = P(n_0 + n) f\left(\frac{n_0+n}{\Lambda}\right)$, except here n will be summed over $\mathbb{N}$ instead of $\mathbb{Z}$. To compute the large Λ behavior of the spectral action, we take a Taylor expansion of the integrand, g, with respect to Λ, and estimate the remainder R_m for varying values of the parameter m. In the case of $SU(3)$, we need a two-dimensional version of the Euler-Maclaurin formula, but the analysis is completely analogous.

In chapter 2, we review the author's first calculations of the spectral action, on quaternionic space, the Poincaré homology sphere, and flat tori.

In chapter 3, we undertake a more systematic study of such calculations on Bieberbach manifolds. In this case, we use symmetries of the spectrum to obtain expressions of the spectra which are indexed over $\mathbb{Z}^k$.

In chapter 4 we undertake a more systematic study of such calculations on coset spaces of $SU(2)$. For these calculations, we apply general formulae to obtain explicit expressions for the Dirac spectra. Guided by the spectral action calculations, we uncover a mistake in the computation of the Dirac spectrum for lens spaces performed in [5].

In chapter 5 we study such calculations on twisted Dirac operators over coset spaces of $SU(2)$. These calculations were necessary in order to study the noncommutative cosmological models in the presence of matter.

In chapter 6 we study such calculations on a one-parameter family of Dirac operators over $SU(2)$ and $SU(3)$. In this chapter, we review formulae used to produce the Dirac spectrum developed by the author in collaboration with Alan Lai. To compute the spectral action of $SU(3)$ we apply a multivariate version of the Euler-Maclaurin formula.

Chapter 2

Quaternionic Space, Poincaré Homology Sphere, and Flat Tori

2.1 Introduction

To begin, we recall the first example of a nonperturbative calculation of the spectral action, undertaken by Chamseddine and Connes in [16]. The space considered here is the three-sphere, S^3. This calculation uses the same techniques as the one performed in the sequel and is somewhat simpler than the cases we will consider later on.

The basic tool used in this chapter is the Poisson summation formula. One version of this formula states that for a test function $h \in \mathcal{S}(\mathbb{R})$ in Schwartz space,

$$\sum_{n\in\mathbb{Z}} h(n) = \sum_{n\in\mathbb{Z}} \widehat{h}(n). \tag{2.1}$$

The notation $\widehat{h}$ denotes the Fourier transform,

$$\widehat{h}(x) = \int_{\mathbb{R}} h(u) e^{2\pi i u x} du.$$

For the calculations below, we need the slightly more general form of 2.1,

$$\sum_{n\in\mathbb{Z}} h(x+\lambda n) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} e^{\frac{2\pi i n x}{\lambda}} \widehat{h}(\frac{n}{\lambda}). \tag{2.2}$$

To compute the spectral action nonperturbatively the Poisson summation formula is applied to a function of the form $P(u)f(u/\Lambda)$, where $P(u)$ is a polynomial whose value at u is the multiplicity of the eigenvalue u in the Dirac spectrum, and f is a positive, smooth, even cutoff function of rapid decay. In the case of S^3, the eigenvalues are given by $\frac{1}{2}+n$, for $n \in \mathbb{Z}$, with multiplicities $n(n+1)$. Applying the Poisson summation formula gives the desired result:

$$\mathrm{Tr} f(\mathcal{D}/\Lambda) = \Lambda^3 \int_{\mathbb{R}} v^2 f(v) dv - \frac{1}{4}\Lambda \int_{\mathbb{R}} f(v) dv + O(\Lambda^{-k}),$$

as Λ goes to infinity, where k can be any positive integer.

2.2 The quaternionic cosmology and the spectral action

Let $Q8$ denote the group of quaternion units $\{\pm 1, \pm i, \pm j, \pm k\}$. It acts on the 3-sphere, with the latter identified with the group $SU(2)$.

2.2.1 The Dirac spectra for $SU(2)/Q8$

As we show here, the main reason why the case of $SU(2)/Q8$ can be treated with the same technique used in [16] for the sphere S^3 is because the Dirac spectrum is given in terms of arithmetic progressions indexed over the integers, so that one can again apply the same type of Poisson summation formula.

More precisely, we recall from [31] that one can endow the 3-manifold $SU(2)/Q8$ with a 3-parameter family of homogeneous metrics, depending on the parameters $a_i \in \mathbb{R}^*$, $i = 1, 2, 3$. The different possible spin structures ϵ_j on $SU(2)/Q8$ correspond to the four group homomorphisms $Q8 \to \mathbb{Z}/2\mathbb{Z}$ with $\epsilon_0 \equiv 1$ and $\mathrm{Ker}(\epsilon_j) = \{\pm 1, \pm\sigma_j\}$, with σ_j the Pauli matrices. The Dirac operator for each of these spin structures and its spectrum are computed explicitly in [31]. The case we are interested in here is the one where the metric has parameters $a_1 = a_2 = a_3 = 1$, for which $SU(2)/Q8$ is a spherical space form. For this case the Dirac spectrum was also computed in [4].

In this case, see Corollary 3.2 of [31], the Dirac spectrum for $SU(2)/Q8$ with the spherical metric $a_1 = a_2 = a_3 = 1$, is given in the case of the spin structure ϵ_0 by

$$\begin{cases} \frac{3}{2} + 4k & \text{with multiplicity} \quad 2(k+1)(2k+1) \\ \frac{3}{2} + 4k + 2 & \text{with multiplicity} \quad 4k(k+1) \\ -\frac{3}{2} - 4k - 1 & \text{with multiplicity} \quad 2k(2k+1) \\ -\frac{3}{2} - 4k - 3 & \text{with multiplicity} \quad 4(k+1)(k+2), \end{cases} \tag{2.3}$$

where k runs over $\mathbb{N}$. For all the other three spin structures ϵ_j, $j = 1, 2, 3$, the spectrum is given by

$$\begin{cases} \frac{3}{2} + 4k & \text{with multiplicity} \quad 2k(2k+1) \\ \frac{3}{2} + 4k + 2 & \text{with multiplicity} \quad 4(k+1)^2 \\ -\frac{3}{2} - 4k - 1 & \text{with multiplicity} \quad 2(k+1)(2k+1) \\ -\frac{3}{2} - 4k - 3 & \text{with multiplicity} \quad 4(k+1)^2, \end{cases} \tag{2.4}$$

again with $k \in \mathbb{N}$.

2.2.2 Trivial spin structure: nonperturbative spectral action

By replacing k with $-k-1$ in the third row and k with $-k-2$ in the fourth row, we rewrite the spectrum (2.3) in the form

$$\begin{cases} \frac{3}{2} + 4k & \text{with multiplicity} \quad 2(k+1)(2k+1) \\ \frac{3}{2} + 4k + 2 & \text{with multiplicity} \quad 4k(k+1), \end{cases} \tag{2.5}$$

where now k runs over the integers $\mathbb{Z}$. This expresses the spectrum in terms of two arithmetic progressions indexed over the integers. Now the condition that allows us to apply the Poisson summation formula as in [16] is the fact that the multiplicities can be expressed in terms of a smooth function of k. This is the case, since the multiplicites in (2.5) for an eigenvalue λ are given, respectively, by the functions $P_1(\lambda)$ and $P_2(\lambda)$ with

$$\begin{aligned} P_1(u) &= \frac{1}{4}u^2 + \frac{3}{4}u + \frac{5}{16} \\ P_2(u) &= \frac{1}{4}u^2 - \frac{3}{4}u - \frac{7}{16}. \end{aligned} \tag{2.6}$$

We then obtain an explicit nonperturbative calculation of the spectral action for $SU(2)/Q8$ as follows.

Theorem 2.2.1 *The spectral action on the 3-manifold $S = SU(2)/Q8$, with the trivial spin structure, is given by*

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{8}(\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{32}(\Lambda a)\widehat{f}(0) + \epsilon(\Lambda), \tag{2.7}$$

with a the radius of the 3-sphere $SU(2) = S^3$, with the error term satisfying $|\epsilon(\Lambda)| = O(\Lambda^{-k})$ for all $k > 0$, and with $\widehat{f}^{(k)}$ denoting the Fourier transform of $v^k f(v)$ as above. Namely, the spectral action for $SU(2)/Q8$ is $1/8$ of the spectral action for S^3.

Consider a test function for the Poisson summation formula which is of the form

$$h(u) = g(4u + \frac{s}{2}), \qquad \text{for some} \quad s \in \mathbb{Z}.$$

Then the Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}} g(4n + \frac{s}{2}) = \sum_{n\in\mathbb{Z}} \frac{1}{4}\exp(\frac{i\pi sn}{4})\,\widehat{g}(\frac{n}{4}), \tag{2.8}$$

which we apply to $g_i(u) = P_i(u)f(u/\Lambda)$, with P_i as in (2.6) and f the Schwartz function in the spectral action approximating a cutoff.

This gives an expression for the spectral action on $S = SU(2)/Q8$ with the trivial spin structure, and with the sphere $S^3 = SU(2)$ of radius one, which is of the form

$$\begin{aligned} \mathrm{Tr}(f(D/\Lambda)) = & \sum_{\mathbb{Z}} g_1(4n + \frac{3}{2}) + \sum_{\mathbb{Z}} g_2(4n + \frac{7}{2}) \\ = & \sum_{\mathbb{Z}} \frac{1}{4}\exp(\frac{3\pi i n}{4})\widehat{g}_1(\frac{n}{4}) + \sum_{\mathbb{Z}} \frac{1}{4}\exp(\frac{7\pi i n}{4})\widehat{g}_2(\frac{n}{4}). \end{aligned} \tag{2.9}$$

Assuming that f is a Schwartz function, then g_i is also Schwartz, hence so is $\widehat{g}_i$. Therefore, for each $k \in \mathbb{N}$, we get an estimate of the form

$$\sum_{n\neq 0} \frac{1}{4}|\widehat{g}_i(\frac{n}{4})| \leq C_k \Lambda^{-k}.$$

This shows that we can write the right hand side of (2.9) as the terms involving $\widehat{g}_i(0)$ plus an error term that is of order $O(\Lambda^{-k})$.

One then computes

$$\widehat{g}_1(0) = \frac{1}{4}\Lambda^3 \widehat{f}^{(2)}(0) + \frac{3}{4}\Lambda^2 \widehat{f}^{(1)}(0) + \frac{5}{16}\Lambda \widehat{f}(0). \tag{2.10}$$

Similarly, one has

$$\widehat{g}_2(0) = \frac{1}{4}\Lambda^3 \widehat{f}^{(2)}(0) - \frac{3}{4}\Lambda^2 \widehat{f}^{(1)}(0) - \frac{7}{16}\Lambda \widehat{f}(0), \tag{2.11}$$

so that one obtains for the spectral action in (2.9)

$$\begin{aligned} \mathrm{Tr}(f(D/\Lambda)) = & \ \tfrac{1}{4}\left(\widehat{g}_1(0) + \widehat{g}_2(0)\right) + O(\Lambda^{-k}) \\ = & \ \tfrac{1}{8}\Lambda^3 \widehat{f}^{(2)}(0) - \tfrac{1}{32}\Lambda \widehat{f}(0) + O(\Lambda^{-k}). \end{aligned} \tag{2.12}$$

The case with the 3-sphere $SU(2) = S^3$ of radius a is then analogous, with the spectrum scaled by a factor of a^{-1}, which is like changing Λ to Λa in the expressions above, so that one obtains (2.7).

2.2.3 Nontrivial spin structures: nonperturbative spectral action

The computation of the spectral action on $SU(2)/Q8$ in the case of the non-trivial spin structures ϵ_j with $j = 1, 2, 3$ is analogous. One starts with the Dirac spectrum (2.4) and writes it in the form of two arithmetic progressions indexed over the integers

$$\begin{cases} \frac{3}{2} + 4k & \text{with multiplicity} \quad 2k(2k+1) \\ \frac{3}{2} + 4k + 2 & \text{with multiplicity} \quad 4(k+1)^2. \end{cases} \tag{2.13}$$

In this case one again has polynomials interpolating the values of the multiplicities. They are of the form

$$\begin{aligned} P_1(u) & = \frac{1}{4}u^2 - \frac{1}{4}u - \frac{3}{16} \\ P_2(u) & = \frac{1}{4}u^2 + \frac{1}{4}u + \frac{1}{16}. \end{aligned} \tag{2.14}$$

We then obtain the following result.

Theorem 2.2.2 *The spectral action on the 3-manifold $S = SU(2)/Q8$, for any of the non-trivial spin structures ϵ_j, $j = 1, 2, 3$, is given by the same expression* (2.7) *as in the case of* *the trivial spin structure* ϵ_0.

It is enough to observe that the sum of the two polynomials (2.14) that interpolate the spectral multiplicities,

$$P_1(u) + P_2(u) = \frac{1}{2}u^2 - \frac{1}{8}$$

is the same as in the case (2.6) of the trivial spin structure. One then has the same value of

$$\frac{1}{4}\widehat{g}_1(0) + \frac{1}{4}\widehat{g}_2(0) = \frac{1}{4}\int_{\mathbb{R}} (P_1(u) + P_2(u))\, f(u/\Lambda)\, du,$$

which gives the spectral action up to an error term of the order of $O(\Lambda^{-k})$.

2.3 Poincaré homology sphere

The Poincaré homology sphere, which is the quotient of the 3-sphere S^3 by the binary icosahedral group Γ, is also commonly referred to as the dodecahedral space, due to the fact that the action of Γ on S^3 has a fundamental domain that is a dodecahedron. The dodecahedral space is obtained by gluing together opposite faces of a dodecahedron with the shortest clockwise twist that matches the faces.

2.3.1 Generating functions for spectral multiplicities

To explicitly compute the Dirac spectrum of the Poincaré homology sphere, we use a general result of Bär [4], which gives a formula for the generating function of the spectral multiplicities of the Dirac spectrum on space forms of positive curvature.

In the generality of [4], one considers a manifold M that is a quotient $M = S^n/\Gamma$ of an n-dimensional sphere, $n \geq 2$, with the standard metric of curvature one, and with $\Gamma \subset SO(n+1)$ a finite group acting without fixed points. It is shown in [4] that the classical Dirac operator on S^n has spectrum

$$\pm\left(\frac{n}{2} + k\right), k \geq 0, \qquad \text{with multiplicities} \quad 2^{[n/2]}\binom{k+n-1}{k}. \tag{2.15}$$

The eigenvalues of M are the same as the eigenvalues of S^n, but with smaller multiplicities. The spin structures of M are in 1-1 corrrespondence with homomorphisms $\epsilon : \Gamma \to \mathrm{Spin}(n+1)$, such that $\Theta \circ \epsilon = id_\Gamma$, where Θ is simply the double cover map from $\mathrm{Spin}(n+1)$ to

$SO(n+1)$. If D is the Dirac operator on M, then to specify the spectrum of M, for one of these spin structures, one just needs to know the multiplicities, $m(\pm(n/2+k))$, $k \geq 0$. These are encoded in two generating functions

$$F_+(z) = \sum_{k=0}^{\infty} m(\frac{n}{2} + k, D) z^k \tag{2.16}$$

$$F_-(z) = \sum_{k=0}^{\infty} m(-(\frac{n}{2} + k), D) z^k. \tag{2.17}$$

It is elementary to show that these power series have radii of convergence of at least 1 about $z = 0$.

Now denote the irreducible half spin representations of Spin($2m$) by

$$\rho^+ : \mathrm{Spin}(2m) \to \mathrm{Aut}(\Sigma_{2m}^+)$$

$$\rho^- : \mathrm{Spin}(2m) \to \mathrm{Aut}(\Sigma_{2m}^-),$$

where $\Sigma_{2m}^{\pm}$ are the positive and negative spinor spaces. Let $\chi^{\pm} : \mathrm{Spin}(2m) \to \mathbb{C}$ be the character of $\rho^{\pm}$. It is shown in [4] that the generating functions of the spectral multiplicities have the form

$$F_+(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^-(\epsilon(\gamma)) - z \cdot \chi^+(\epsilon(\gamma))}{\det(1_{2m} - z \cdot \gamma)}, \tag{2.18}$$

$$F_-(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^+(\epsilon(\gamma)) - z \cdot \chi^-(\epsilon(\gamma))}{\det(1_{2m} - z \cdot \gamma)}. \tag{2.19}$$

2.3.2 The Dirac spectrum of the Poincaré sphere

In order to compute explicitly the Dirac spectrum of the Poincaré homology sphere, it suffices then to compute the multiplicities by explicitly computing the generating functions (2.18) and (2.19).

Let Γ be the binary icosahedral group. To carry out our computations, we regard S^3 as the set of unit quaternions, and Γ as the following set of 120 unit quaternions:

- 24 elements are as follows, where the signs in the last group are chosen independently of one another:

$$\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}. \tag{2.20}$$

- 96 elements are either of the following form, or obtained by an even permutation of coordinates of the following form:

$$1/2(0 \pm i \pm \phi^{-1} j \pm \phi k), \tag{2.21}$$

  where ϕ is the golden ratio.

Then Γ acts on S^3 by left multiplication. Similarly, if S^3 is regarded as the unit sphere in $\mathbb{R}^4$, then $SO(4)$ acts on S^3 by left multiplication. In this way, we may identify $a+bi+cj+dk \in \Gamma$, with the following matrix in $SO(4)$:

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

2.3.3 The double cover $\mathrm{Spin}(4) \to SO(4)$

Let us recall some facts about the double cover $\mathrm{Spin}(4) \to SO(4)$. Let $S^3_L \simeq SU(2)$ be the group of left isoclinic rotations:

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix},$$

where $a^2 + b^2 + c^2 + d^2 = 1$. Similarly, let $S^3_R \simeq SU(2)$ be the group of right isoclinic rotations:

$$\begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix},$$

where $p^2 + q^2 + r^2 + s^2 = 1$. Then $\mathrm{Spin}(4) \simeq S^3_L \times S^3_R$, and the double cover $\Theta : \mathrm{Spin}(4) \to SO(4)$ is given by $(A, B) \mapsto A \cdot B$, where $A \in S^3_L$, and $B \in S^3_R$. The complex half-spin representation ρ^- is just the projection onto S^3_L, where we identify S^3_L with $SU(2)$ via

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \mapsto \begin{pmatrix} a-bi & d+ci \\ -d+ci & a+bi \end{pmatrix}.$$

The other complex half-spin representation ρ^+ is the projection onto S^3_R, where we identify S^3_R with $SU(2)$ via

$$\begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix}^t \mapsto \begin{pmatrix} p-qi & s+ri \\ -s+ri & p+qi \end{pmatrix}.$$

2.3.4 The spectral multiplicities

We define our spin structure $\epsilon : \Gamma \to \mathrm{Spin}(4)$ to simply be $A \mapsto (A, I_4)$. It is obvious that this map satisfies $\Theta \circ \epsilon = id_\Gamma$. Therefore, given $\gamma = a + bi + cj + dk \in \Gamma$, we see that

$$\chi^-(\epsilon(\gamma)) = 2a$$
$$\chi^+(\epsilon(\gamma)) = 2.$$

We then obtain the following result by direct computation of the expressions (2.18) and (2.19), substituting the explicit expressions for all the group elements.

Theorem 2.3.1 *Let $S = S^3/\Gamma$ be the Poincaré sphere, with the spin structure ϵ described here above. The generating functions for the spectral multiplicities of the Dirac operator are*

$$F_+(z) = -\frac{16(710647 + 317811\sqrt{5})G^+(z)}{(7+3\sqrt{5})^3(2207+987\sqrt{5})H^+(z)}, \tag{2.22}$$

where

$$\begin{aligned} G^+(z) = \; & 6z^{11} + 18z^{13} + 24z^{15} + 12z^{17} - 2z^{19} \\ - \; & 6z^{21} - 2z^{23} + 2z^{25} + 4z^{27} + 3z^{29} + z^{31}, \end{aligned}$$

and

$$\begin{aligned} H^+(z) = & -1 - 3z^2 - 4z^4 - 2z^6 + 2z^8 + 6z^{10} + 9z^{12} + 9z^{14} + 4z^{16} \\ & - 4z^{18} - 9z^{20} - 9z^{22} - 6z^{24} - 2z^{26} + 2z^{28} + 4z^{30} + 3z^{32} + z^{34}, \end{aligned}$$

and

$$F_-(z) = -\frac{1024(5374978561 + 2403763488\sqrt{5})G^-(z)}{(7+3\sqrt{5})^8(2207+987\sqrt{5})H^-(z)}, \tag{2.23}$$

where

$$\begin{aligned} G^-(z) = & \; 1 + 3z^2 + 4z^4 + 2z^6 - 2z^8 - 6z^{10} \\ & - 2z^{12} + 12z^{14} + 24z^{16} + 18z^{18} + 6z^{20}, \end{aligned}$$

and

$$\begin{aligned} H^-(z) = & -1 - 3z^2 - 4z^4 - 2z^6 + 2z^8 + 6z^{10} + 9z^{12} + 9z^{14} + 4z^{16} \\ & - 4z^{18} - 9z^{20} - 9z^{22} - 6z^{24} - 2z^{26} + 2z^{28} + 4z^{30} + 3z^{32} + z^{34}. \end{aligned}$$

We can then obtain explicitly the spectral multiplicities from the Taylor coefficients of $F_+(z)$ and $F_-(z)$, as in 2.16 and 2.17.

2.3.5 The spectral action for the Poincaré sphere

In order to compute the spectral action, we proceed as in the previous cases by identifying polynomials whose values at the points of the spectrum give the values of the spectral multiplicities. We obtain the following result.

Proposition 2.3.2 *There are polynomials $P_k(u)$, for $k = 0, \ldots, 59$, so that $P_k(3/2 + k + 60j) = m(3/2 + k + 60j, D)$ for all $j \in \mathbb{Z}$. The $P_k(u)$ are given as follows:*

$$\begin{aligned} P_k &= 0, \quad \text{whenever } k \text{ is even}, \\ P_1(u) &= \frac{1}{48} - \frac{1}{20}u + \frac{1}{60}u^2, \\ P_3(u) &= \frac{3}{80} - \frac{1}{12}u + \frac{1}{60}u^2, \\ P_5(u) &= \frac{13}{240} - \frac{7}{60}u + \frac{1}{60}u^2, \\ P_7(u) &= \frac{17}{240} - \frac{3}{20}u + \frac{1}{60}u^2, \\ P_9(u) &= \frac{7}{80} - \frac{11}{60}u + \frac{1}{60}u^2, \end{aligned}$$

$$P_{11}(u) = -\frac{19}{48} + \frac{47}{60}u + \frac{1}{60}u^2,$$
$$P_{13}(u) = \frac{29}{240} - \frac{1}{4}u + \frac{1}{60}u^2,$$
$$P_{15}(u) = \frac{11}{80} - \frac{17}{60}u + \frac{1}{60}u^2,$$
$$P_{17}(u) = \frac{37}{240} - \frac{19}{60}u + \frac{1}{60}u^2,$$
$$P_{19}(u) = -\frac{79}{240} + \frac{13}{20}u + \frac{1}{60}u^2,$$

$$P_{21}(u) = \frac{3}{16} - \frac{23}{60}u + \frac{1}{60}u^2,$$
$$P_{23}(u) = -\frac{71}{240} + \frac{7}{12}u + \frac{1}{60}u^2,$$
$$P_{25}(u) = \frac{53}{240} - \frac{9}{20}u + \frac{1}{60}u^2,$$
$$P_{27}(u) = \frac{19}{80} - \frac{29}{60}u + \frac{1}{60}u^2,$$
$$P_{29}(u) = -\frac{59}{240} + \frac{29}{60}u + \frac{1}{60}u^2,$$

$$P_{31}(u) = -\frac{11}{48} + \frac{9}{20}u + \frac{1}{60}u^2,$$
$$P_{33}(u) = \frac{23}{80} - \frac{7}{12}u + \frac{1}{60}u^2,$$
$$P_{35}(u) = -\frac{47}{240} + \frac{23}{60}u + \frac{1}{60}u^2,$$
$$P_{37}(u) = \frac{77}{240} - \frac{13}{20}u + \frac{1}{60}u^2,$$
$$P_{39}(u) = -\frac{13}{80} + \frac{19}{60}u + \frac{1}{60}u^2,$$

$$P_{41}(u) = -\frac{7}{48} + \frac{17}{60}u + \frac{1}{60}u^2,$$
$$P_{43}(u) = -\frac{31}{240} + \frac{1}{4}u + \frac{1}{60}u^2,$$
$$P_{45}(u) = \frac{31}{80} - \frac{47}{60}u + \frac{1}{60}u^2,$$
$$P_{47}(u) = -\frac{23}{240} + \frac{11}{60}u + \frac{1}{60}u^2,$$
$$P_{49}(u) = -\frac{19}{240} + \frac{3}{20}u + \frac{1}{60}u^2,$$

$$P_{51}(u) = -\frac{1}{16} + \frac{7}{60}u + \frac{1}{60}u^2,$$
$$P_{53}(u) = -\frac{11}{240} + \frac{1}{12}u + \frac{1}{60}u^2,$$
$$P_{55}(u) = -\frac{7}{240} + \frac{1}{20}u + \frac{1}{60}u^2,$$
$$P_{57}(u) = \frac{39}{80} - \frac{59}{60}u + \frac{1}{60}u^2,$$
$$P_{59}(u) = -\frac{119}{240} + \frac{59}{60}u + \frac{1}{60}u^2.$$

These are computed directly from the Taylor coefficients of the generating functions of the spectral multiplicities (4.100) and (4.101).

We then obtain the nonperturbative spectral action for the Poincaré sphere.

Theorem 2.3.3 *Let D be the Dirac operator on the Poincaré homology sphere $S = S^3/\Gamma$, with the spin structure $\epsilon : \Gamma \to Spin(4)$ with $A \mapsto (A, I_4)$. Then, for any Schwartz function, f, the spectral action is given by*

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{60}\left(\frac{1}{2}\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{8}\Lambda \widehat{f}(0)\right), \tag{2.24}$$

which is precisely 1/120 of the spectral action on the sphere.

The result follows by applying Poisson summation again, to the functions $g_j(u) = P_j(u) f(u/\Lambda)$. This gives, up to an error term which is of the order of $O(\Lambda^{-k})$ for any

$k > 0$, the spectral action in the form

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{60}\sum_{j=0}^{59}\widehat{g}_j(0) = \frac{1}{60}\int_{\mathbb{R}}\sum_j P_j(u)f(u/\Lambda)du.$$

It suffices then to notice that

$$\sum_{j=0}^{59} P_j(u) = \frac{1}{2}u^2 - \frac{1}{8}.$$

The result then follows as in the sphere case.

2.4 Flat tori

2.4.1 The spectral action on the flat tori

Let T^3 be the flat torus $\mathbb{R}^3/\mathbb{Z}^3$. The spectrum of the Dirac operator, denoted D_3, is given in Theorem 4.1 of [6] as

$$\pm 2\pi \parallel (m,n,p) + (m_0,n_0,p_0) \parallel, \tag{2.25}$$

where (m,n,p) runs through $\mathbb{Z}^3$. Each value of (m,n,p) contributes multiplicity 1. The constant vector (m_0,n_0,p_0) depends on the choice of spin structure.

Theorem 2.4.1 *The spectral action* $\mathrm{Tr}(f(D_3^2/\Lambda^2))$ *for the torus* $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ *is independent of the spin structure on* T^3 *and given by*

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2)du\,dv\,dw + O(\Lambda^{-k}), \tag{2.26}$$

for arbitrary $k > 0$.

By (2.25), we know the spectrum of D_3^2 is given by

$$4\pi^2 \parallel (m,n,p) + (m_0,n_0,p_0) \parallel^2,$$

where (m,n,p) runs through $\mathbb{Z}^3$, and each value of (m,n,p) contributes multiplicity 2.

Given a test function in Schwartz space, $f \in \mathcal{S}(\mathbb{R})$, the spectral action is then given by

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \sum_{(m,n,p)\in\mathbb{Z}^3} 2f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right),$$

In three dimensions, the Poisson summation formula is given by

$$\sum_{\mathbb{Z}^3} g(m,n,p) = \sum_{\mathbb{Z}^3} \widehat{g}(m,n,p),$$

where the Fourier transform is defined by

$$\widehat{g}(m,n,p) = \int_{\mathbb{R}^3} g(u,v,w) e^{-2\pi i(mu+nv+pw)} du dv dw.$$

If we define

$$g(m,n,p) = f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right), \tag{2.27}$$

and apply the Poisson summation formula, we obtain the following expression for the spectral action:

$$\begin{aligned}
\mathrm{Tr}(f(D_3^2/\Lambda^2)) &= 2 \sum_{(m,n,p)\in\mathbb{Z}^3} \widehat{g}(m,n,p) \\
&= 2\widehat{g}(0,0,0) + O(\Lambda^{-k}) \\
&= 2\int_{\mathbb{R}^3} f\left(\frac{4\pi^2((u+m_0)^2+(v+n_0)^2+(w+p_0)^2)}{\Lambda^2}\right) du\,dv\,dw \\
&+ O(\Lambda^{-k}) \\
&= \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2) du\,dv\,dw + O(\Lambda^{-k}).
\end{aligned}$$

The estimate $\sum_{(m,n,p)\neq 0} \widehat{g}(m,n,p) = O(\Lambda^{-k})$ for arbitrary $k > 0$ is elementary, using the fact that $f \in \mathcal{S}(\mathbb{R})$. We observe that the nonperturbative spectral action is independent of the choice of spin structure.

Chapter 3

Bieberbach Manifolds

3.1 Introduction

The simplest case of Bieberbach manifold is the flat torus T^3, which we considered in the previous chapter. In general, Bieberbach manifolds are quotients of the torus by a finite group action. In this section we give an explicit computation of the nonperturbative spectral action for all Bieberbach manifolds with the exception of one class.

Calculations of the spectral action for Bieberbach manifolds were simultaneously independently obtained in [60].

The Dirac spectrum of Bieberbach manifolds is computed in [63] for each of the six affine equivalence classes of three-dimensional orientable Bieberbach manifolds, and for each possible choice of spin structure and choice of flat metric. These classes are labeled $G1$ through $G6$, with $G1$ simply being the flat 3-torus.

In general, the Dirac spectrum for each space depends on the choice of spin structure. However, as in the case of the spherical manifolds, we show here that the nonperturbative spectral action is *independent of the spin structure.*

We follow the notation of [63], according to which the different possibilities for the Dirac spectra are indicated by a letter (e.g. $G2(a)$). Note that it is possible for several spin structures to yield the same Dirac spectrum.

The nonperturbative spectral action for $G1$ was computed in [50]. We recall here the result for that case and then we restrict our discussion to the spaces $G2$ through $G6$.

3.2 The structure of Dirac spectra of Bieberbach manifold

The spectrum of the Bieberbach manifolds generally consists of a symmetric component and an asymmetric component as computed in [63]. The symmetric components are parametrized by subsets $I \subset \mathbb{Z}^3$, such that the eigenvalues are given by some formula λ_x, $x \in I$, and the multiplicity of each eigenvalue, λ, is some constant times the number of $x \in I$ such that $\lambda = \lambda_x$. In the case of $G2$, $G4$, $G5$, $G6$ the constant is 1, while in the case of $G3$ the constant is 2.

The approach we use here to compute the spectral action nonperturbatively is to use the symmetries of λ_x, as a function of $x \in I$, to almost cover all of the points in $\mathbb{Z}^3$ and then apply the Poisson summation formula as used in [16]. By "almost cover", it is meant that it is perfectly acceptable if two-, one-, or zero-dimensional lattices through the origin are covered multiple times, or not at all.

The asymmetric component of the spectrum appears only some of the time. The appearance of the asymmetric component depends on the choice of spin structure. For those cases where it appears, the eigenvalues in the asymmetric component consist of the set

$$\mathcal{B} = \left\{ 2\pi \frac{1}{H} (k\mu + c) \, | \mu \in \mathbb{Z} \right\},$$

where c is a constant depending on the spin structure, and k is given in the following table:

Bieberbach manifold	k
$G2$	2
$G3$	3
$G4$	4
$G5$	6

.

For no choice of spin structure does $G6$ have an asymmetric component to its spectrum. Each of the eigenvalues in $\mathcal{B}$ has multiplicity 2. Using the Poisson summation formula as in [16], we see that the asymmetric component of the spectrum contributes to the spectral action

$$\frac{\Lambda H}{\pi k} \int_{\mathbb{R}} f(u^2) du. \tag{3.1}$$

The approach described here is effective for computing the nonperturbative spectral action for the manifolds labeled in [63] as $G2, G3, G4, G6$, but not for $G5$. Therefore, we do not consider the $G5$ case in this paper.

3.3 Recalling the torus case

We gave in Theorem 8.1 of [50] the explicit computation of the non-perturbative spectral action for the torus. We recall here the statement for later use.

Theorem 3.3.1 *Let $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ be the flat torus with an arbitrary choice of spin structure. The nonperturbative spectral action is of the form*

$$\mathrm{Tr}(f(D^2/\Lambda^2)) = \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw, \tag{3.2}$$

up to terms of order $O(\Lambda^{-\infty})$.

3.4 The spectral action for $G2$

The Bieberbach manifold $G2$ is obtained by considering a lattice with basis $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (T,S,0)$, with $H,L,S \in \mathbb{R}^*_+$ and $T \in \mathbb{R}$, and then taking the quotient $Y = \mathbb{R}^3/G2$ of $\mathbb{R}^3$ by the group $G2$ generated by the commuting translations t_i along these basis vectors a_i and an additional generator α with relations

$$\alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}. \tag{3.3}$$

Like the torus T^3, the Bieberbach manifold $G2$ has eight different spin structures, parameterized by three signs $\delta_i = \pm 1$, see Theorem 3.3 of [63]. Correspondingly, as shown in Theorem 5.7 of [63], there are four different Dirac spectra, denoted (a), (b), (c), and (d), respectively associated to the spin structures

	δ_1	δ_2	δ_3
(a)	± 1	1	1
(b)	± 1	-1	1
(c)	± 1	1	-1
(d)	± 1	-1	-1

.

We give the computation of the nonperturbative spectral action separately for each different spectrum and we will see that the result is independent of the spin structure and always a multiple of the spectral action of the torus.

3.4.1 The case of $G2(a)$

In this first case, we go through the computation in full detail. The symmetric component of the spectrum is given by the data ([63])

$$I = \{(k,l,m)|k,l,m \in \mathbb{Z}, m \geq 1\} \cup \{(k,l,m)|k,l \in \mathbb{Z}, l \geq 1, m = 0\}$$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}(k+\frac{1}{2})^2 + \frac{1}{L^2}l^2 + \frac{1}{S^2}(m - \frac{T}{L}l)^2},$$

We make the assumption that $T = L$. Set $p = m - l$. Then we have equivalently:

$$I = \{(k,l,p)|k,l,p \in \mathbb{Z}, p > -l\} \cup \{(k,l,p)|k,l \in \mathbb{Z}, l \geq 1, p = -l\} =: I_1 \cup I_2$$

$$\lambda^{\pm}_{klp} = \pm 2\pi \sqrt{\frac{1}{H^2}(k+\frac{1}{2})^2 + \frac{1}{L^2}l^2 + \frac{1}{S^2}p^2}.$$

Theorem 3.4.1 *Let $G2(a)$ be the Bieberbach manifold $\mathbb{R}^3/G2$, with $T = L$ and with a spin structure with $\delta_i = \{\pm 1, 1, 1\}$. The nonperturbative spectral action of the manifold $G2(a)$ is of the form*

$$\mathrm{Tr}(f(D^2/\Lambda^2)) = HSL\left(\frac{\Lambda}{2\pi}\right)^3 \int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw, \tag{3.4}$$

up to terms of order $O(\Lambda^{-\infty})$.

We compute the contribution to the spectral action due to I_1. Since $\lambda^{\pm}_{klp}$ is invariant under the transformation $l \mapsto -l$ and $p \mapsto -p$, we see that

$$\sum_{\mathbb{Z}^3} f(\lambda^2_{klp}/\Lambda^2) = 2\sum_{I_1} f(\lambda^2_{klp}/\Lambda^2) + \sum_{p=-l} f(\lambda^2_{klp}/\Lambda^2).$$

The decomposition of $\mathbb{Z}^3$ used to compute this contribution to the spectral action is displayed in figure 3.2. Applying the Poisson summation formula we get a contribution to the spectral action of

$$HSL\left(\frac{\Lambda}{2\pi}\right)^3 \int_{\mathbb{R}^3} f(u^2+v^2+w^2) - H\frac{LS}{\sqrt{L^2+S^2}}\left(\frac{\Lambda}{2\pi}\right)^2 \int_{R^2} f(u^2+v^2),$$

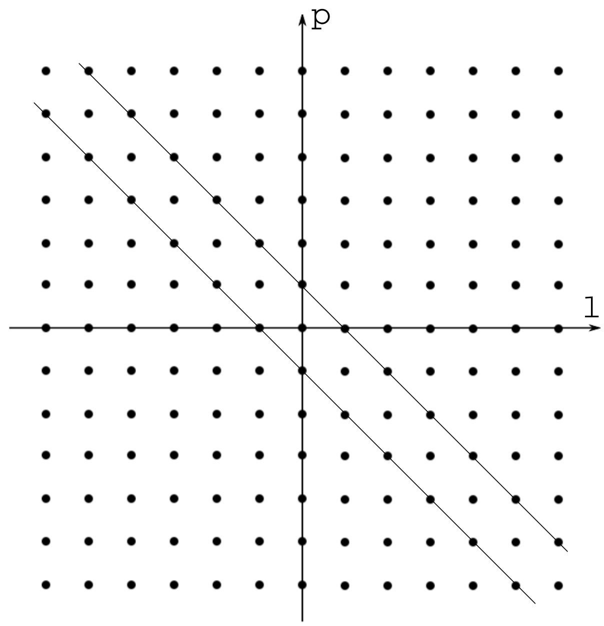

Figure 3.1: Lattice decomposition for the I_1 contribution to the spectral action of $G2(a)$. Two regions and the set $l = -p$

plus possible terms of order $O(\Lambda^{-\infty})$.

As for I_2 we again use the fact that the spectrum is invariant under the transformation $l \mapsto -l$, $p \mapsto -p$ to see that

$$\sum_{\mathbb{Z}^2} f(\lambda^2_{kl(-l)}/\Lambda^2) = 2\sum_{I_2} f(\lambda^2_{klp}/\Lambda^2) + \sum_{p=l=0} f(\lambda^2_{klp}/\Lambda^2).$$

The decomposition for this contribution to the spectral action is displayed in figure 3.1. We get a contribution to the spectral action of

$$H\frac{LS}{\sqrt{L^2+S^2}}\left(\frac{\Lambda}{2\pi}\right)^2\int_{\mathbb{R}^2} f(u^2+v^2) - H\left(\frac{\Lambda}{2\pi}\right)\int_{\mathbb{R}} f(u^2)$$

plus possible terms of order $O(\Lambda^{-\infty})$.

When we include the contribution (3.1) due to the asymmetric component we see that the spectral action of the space G2-(a) is equal to

$$\mathrm{Tr} f(D^2/\Lambda^2) = HSL\left(\frac{\Lambda}{2\pi}\right)^3\int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw$$

again up to possible terms of order $O(\Lambda^{-\infty})$.

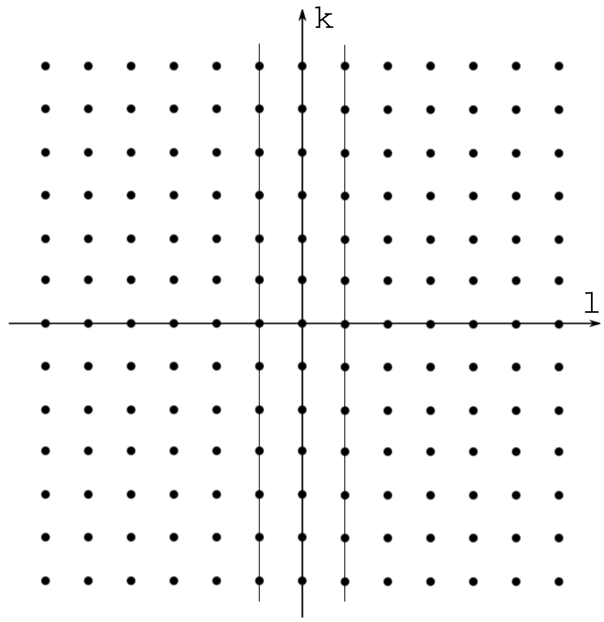

Figure 3.2: Lattice decomposition for the I_2 contribution to the spectral action of $G2(a)$ Two regions and the set $l = 0$.

3.4.2 The case of $G2(b)$ and $G2(d)$

The spectra of $G2(b)$ and $G2(d)$ have no asymmetric component. The symmetric component is given by

$$I = \{(k, l, m) | k, l, m \in \mathbb{Z}, l \geq 0\}$$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}(k + \frac{1}{2})^2 + \frac{1}{L^2}(l + \frac{1}{2})^2 + \frac{1}{S^2}(m + c - \frac{T}{L}(l + \frac{1}{2}))^2}.$$

Let us once again assume that $T = L$.

Theorem 3.4.2 *Let $G2(b)$ and $G2(d)$ be the Bieberbach manifolds $\mathbb{R}^3/G2$, with $T = L$ and with a spin structure with $\delta_i = \{\pm 1, -1, 1\}$ and $\delta_i = \{\pm 1, -1, -1\}$, respectively. The nonperturbative spectral action of the manifolds $G2(b)$ and $G2(d)$ is again of the form*

$$\mathrm{Tr}(f(D^2/\Lambda^2)) = HSL \left(\frac{\Lambda}{2\pi}\right)^3 \int_{\mathbb{R}^3} f(u^2 + v^2 + w^2) du dv dw, \tag{3.5}$$

up to terms of order $O(\Lambda^{-\infty})$.

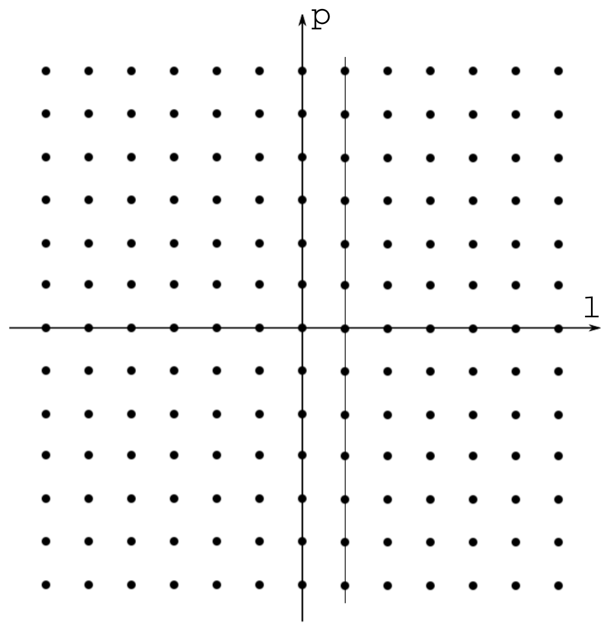

Figure 3.3: Lattice decomposition for $G2(b),(d)$ computation. Two regions.

With the assumption that $T = L$, and letting $p = m - l$, we can describe the spectrum equivalently by

$$I = \{(k,l,p)|k,l,p \in \mathbb{Z}, l \geq 0\}$$

$$\lambda^{\pm}_{klp} = \pm 2\pi \sqrt{\frac{1}{H^2}(k+\frac{1}{2})^2 + \frac{1}{L^2}(l+\frac{1}{2})^2 + \frac{1}{S^2}(p+c+\frac{1}{2})^2}.$$

Using the symmetry

$$l \mapsto -1 - l,$$

we cover $\mathbb{Z}^3$ exactly, (see figure 3.3) and we obtain the spectral action

$$\mathrm{Tr}(f(D^2/\Lambda^2)) = HSL\left(\frac{\Lambda}{2\pi}\right)^3 \int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw + O(\Lambda^{-\infty}).$$

3.4.3 The case of $G2(c)$

In this case, the symmetric component of the spectrum is given by

$$I = \{(k,l,m)|k,l,m \in \mathbb{Z}, m \geq 0\}$$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}(k+\frac{1}{2})^2 + \frac{1}{L^2}l^2 + \frac{1}{S^2}((m+1/2) - \frac{T}{L}l)^2}.$$

Again, we assume $T = L$.

Theorem 3.4.3 *Let $G2(c)$ be the Bieberbach manifolds $\mathbb{R}^3/G2$, with $T = L$ and with a spin structure with $\delta_i = \{\pm 1, 1, -1\}$. The nonperturbative spectral action of the manifold $G2(c)$ is again of the form*

$$\mathrm{Tr}(f(D^2/\Lambda^2)) = HSL\left(\frac{\Lambda}{2\pi}\right)^3 \int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw, \tag{3.6}$$

up to terms of order $O(\Lambda^{-\infty})$.

If we substitute $p = m - l$, we see that we may equivalently express the symmetric component with

$$I = \{(k,l,p)|k,l,p \in \mathbb{Z}, p \geq -l\}$$

$$\lambda^{\pm}_{klp} = \pm 2\pi\sqrt{\frac{1}{H^2}(k+\frac{1}{2})^2 + \frac{1}{L^2}l^2 + \frac{1}{S^2}((p+1/2)^2}.$$

Using the symmetry

$$l \mapsto -l \quad p \mapsto 1-p,$$

we cover $\mathbb{Z}^3$ exactly (see figure 3.4), and so the spectral action is again given by

$$\mathrm{Tr}f(D^2/\Lambda^2) = HSL\left(\frac{\Lambda}{2\pi}\right)^3 \int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw + O(\Lambda^{-\infty}).$$

3.5 The spectral action for $G3$

Consider the hexagonal lattice generated by vectors $a_1 = (0,0,H)$, $a_2 = (L,0,0)$ and $a_3 = (-\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, for H and L in $\mathbb{R}^*_+$. The Bieberbach manifold $G3$ is obtained by the quotient of $\mathbb{R}^3$ by the group $G3$ generated by commuting translations t_i along the vectors a_i and an additional generator α with relations

$$\alpha^3 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1}t_3^{-1}. \tag{3.7}$$

This has the effect of producing an identification of the faces of the fundamental domain with a turn by an angle of $2\pi/3$ about the z-axis, hence the "third-turn space" terminology.

As shown in Theorem 3.3 of [63], the Bieberbach manifold $G3$ has two different spin

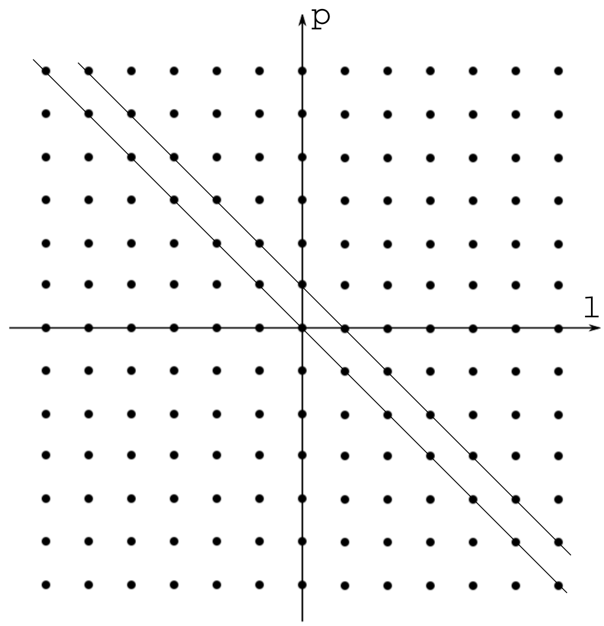

Figure 3.4: Lattice decomposition for $G2(c)$ computation. Two regions.

structures, parameterized by one sign $\delta_1 = \pm 1$. It is then shown in Theorem 5.7 of [63] that these two spin structures have different Dirac spectra, which are denoted as $G3(a)$ and $G3(b)$. We compute below the nonperturbative spectral action in both cases and we show that, despite the spectra being different, they give the same result for the nonperturbative spectral action, which is again a multiple of the action for the torus.

3.5.1 The case of $G3(a)$ and $G3(b)$

The symmetric component of the spectrum is given by

$$I = \{(k, l, m) | k, l, m \in \mathbb{Z}, l \geq 1, m = 0, \ldots, l-1\}, \tag{3.8}$$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}(k+c)^2 + \frac{1}{L^2} l^2 + \frac{1}{3L^2}(l-2m)^2}, \tag{3.9}$$

with $c = 1/2$ for the spin structure (a) and $c = 0$ for the spin structure (b).

The manifold $G3$ is unusual in that the multiplicity of $\lambda^{\pm}_{klm}$ is equal to twice the number of elements in I which map to it.

Theorem 3.5.1 *On the manifold G3 with an arbitrary choice of spin structure, the non-*

perturbative spectral action is given by

$$\mathrm{Tr}(f(D^2/\Lambda^2)) = \frac{1}{\sqrt{3}}\left(\frac{\Lambda}{2\pi}\right)^3 HL^2 \int_{\mathbb{R}^3} f\left(u^2+v^2+t^2\right) dudvdt \tag{3.10}$$

plus possible terms of order $O(\Lambda^{-\infty})$.

Notice that $\lambda^{\pm}_{klm}$ is invariant under the linear transformations R, S, T, given by

$$R(l) = -l$$
$$R(m) = -m$$

$$S(l) = m$$
$$S(m) = l$$

$$T(l) = l - m$$
$$T(m) = -m$$

Let $\tilde{I} = \{(k,l,m) | k,l,m \in \mathbb{Z}, l \geq 2, m = 1, \ldots, l-1\}$.

Then we may decompose $\mathbb{Z}^3$ as (see figure 3.5)

$$\mathbb{Z}^3 = I \sqcup R(I) \sqcup S(I) \sqcup RS(I) \sqcup T(\tilde{I}) \sqcup RT(\tilde{I}) \sqcup \{l = m\}. \tag{3.11}$$

Therefore, we have

$$\begin{aligned}\sum_{\mathbb{Z}^3} f(\lambda_{klm}^2/\Lambda^2) &= 4\sum_I f(\lambda_{klm}^2/\Lambda^2)\\ &\quad + 2\left(\sum_I f(\lambda_{klm}^2/\Lambda^2) - \sum_{m=0,\ l\geq 1} f(\lambda_{klm}^2/\Lambda^2)\right)\\ &\quad + \sum_{l=m} f(\lambda_{klm}^2/\Lambda^2)\\ &= 6\sum_I f(\lambda_{klm}^2/\Lambda^2) - \sum_{m=0} f(\lambda_{klm}^2/\Lambda^2)\\ &\quad + \sum_{m=0,\ l=0} f(\lambda_{klm}^2/\Lambda^2) + \sum_{l=m} f(\lambda_{klm}^2/\Lambda^2)\\ \sum_I f(\lambda_{klm}^2/\Lambda^2) &= \frac{1}{6}\left(\sum_{\mathbb{Z}^3} f(\lambda_{klm}^2/\Lambda^2) + \sum_{m=0} f(\lambda_{klm}^2/\Lambda^2)\right)\\ &\quad - \frac{1}{6}\left(\sum_{m=0,\ l=0} f(\lambda_{klm}^2/\Lambda^2) - \sum_{l=m} f(\lambda_{klm}^2/\Lambda^2)\right).\end{aligned}$$

Therefore the symmetric component of the spectrum contributes to the spectral action

$$\begin{aligned}&\frac{4}{6}(\left(\frac{\Lambda}{2\pi}\right)^3 HL^2 \int_{\mathbb{R}^3} f(u^2+v^2+\frac{1}{3}(v-2w)^2)\\ &+\left(\frac{\Lambda}{2\pi}\right)^2 HL \int_{\mathbb{R}^2} f(u^2+\frac{4}{3}v^2) - \left(\frac{\Lambda}{2\pi}\right) H \int_{\mathbb{R}} f(u^2)\\ &-\left(\frac{\Lambda}{2\pi}\right)^2 HL \int_{\mathbb{R}^2} f(u^2+\frac{4}{3}v^2)) + O(\Lambda^{-\infty})\\ &= \frac{4}{6}\left(\frac{\Lambda}{2\pi}^3 HL^2 \int_{\mathbb{R}^3} f(u^2+v^2+\frac{1}{3}(v-2w)^2) - \frac{\Lambda}{2\pi} H \int_{\mathbb{R}} f(u^2)\right)\\ &+ O(\Lambda^{-\infty}).\end{aligned}$$

Combining this with the asymmetric contribution (3.1), we see that the spectral action of spaces $G3(a)$ and $G3(b)$ is equal to

$$\frac{2}{3}\left(\frac{\Lambda}{2\pi}\right)^3 HL^2 \int_{\mathbb{R}^3} f\left(u^2+v^2+\frac{1}{3}(v-2w)^2\right) dudvdw + O(\Lambda^{-\infty}).$$

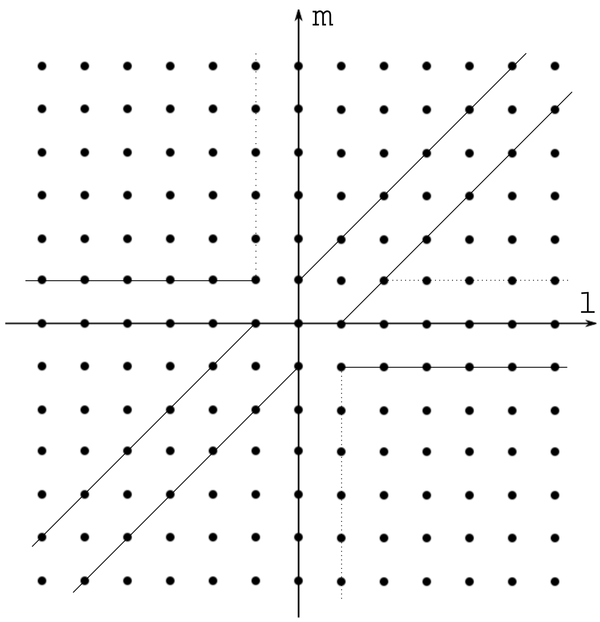

Figure 3.5: Lattice decomposition for $G3$ computation. Six regions and the set $l = m$. The dashed lines indicate one of the boundary lines which define the region $\tilde{I}$ along with its images under the symmetries of λ_{klm}. The other boundary line of $\tilde{I}$ overlaps with the boundary of I.

Now, if one makes the change of variables $(u, v, w) \mapsto (u, v, t)$, where

$$t = \frac{2w - v}{\sqrt{3}},$$

then the spectral action becomes

$$\frac{1}{\sqrt{3}} \left(\frac{\Lambda}{2\pi}\right)^3 HL^2 \int_{\mathbb{R}^3} f\left(u^2 + v^2 + t^2\right) dudvdt + O(\Lambda^{-\infty}).$$

Notice that, a priori, one might have expected a possibly different result in this case, because the Bieberbach manifold is obtained starting from a hexagonal lattice rather than the square lattice. However, up to a simple change of variables in the integral, this gives the same result, up to a multiplicative constant, as in the case of the standard flat torus.

3.6 The spectral action for $G4$

The Bieberbach manifold $G4$ is obtained by considering a lattice generated by the vectors $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, and $a_3 = (0, L, 0)$, with $H, L > 0$, and taking the quotient of

$\mathbb{R}^3$ by the group $G4$ generated by the commuting translations t_i along the vectors a_i and an additional generator α with the relations

$$\alpha^4 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1}. \tag{3.12}$$

This produces an identification of the sides of a fundamental domain with a rotation by an angle of $\pi/2$ about the z-axis. Theorem 3.3 of [63] shows that the manifold $G4$ has four different spin structures parameterized by two signs $\delta_i = \pm 1$. There are correspondingly two different forms of the Dirac spectrum, as shown in Theorem 5.7 of [63], one for $\delta_i = \{\pm 1, 1\}$, the other for $\delta_i = \{\pm 1, -1\}$, denoted by $G4(a)$ and $G4(b)$.

Again the nonperturbative spectral action is independent of the spin structure and equal in both cases to the same multiple of the spectral action for the torus.

3.6.1 The case of $G4(a)$

Theorem 3.6.1 *On the manifold G4 with a spin structure (a) with $\delta_i = \{\pm 1, 1\}$, the nonperturbative spectral action is given by*

$$\mathrm{Tr}(f(D^2/\Lambda^2)) = \frac{1}{2}\left(\frac{\Lambda}{2\pi}\right)^3 HL^2 \int_{\mathbb{R}^3} f(u^2+v^2+w^2)\,du\,dv\,dw \tag{3.13}$$

plus possible terms of order $O(\Lambda^{-\infty})$.

The symmetric component of the spectrum is given by

$$I = \{(k,l,m) | k,l,m \in \mathbb{Z}, l \geq 1, m = 0, \ldots, 2l-1\}$$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}(k+\frac{1}{2})^2 + \frac{1}{L^2}(l^2 + (m-l)^2)}.$$

First, we make the change of variables $p = m - l$. Then we use the symmetries

$$l \mapsto -l$$

$$l \mapsto p \quad p \mapsto l$$

$$l \mapsto p \quad p \mapsto -l$$

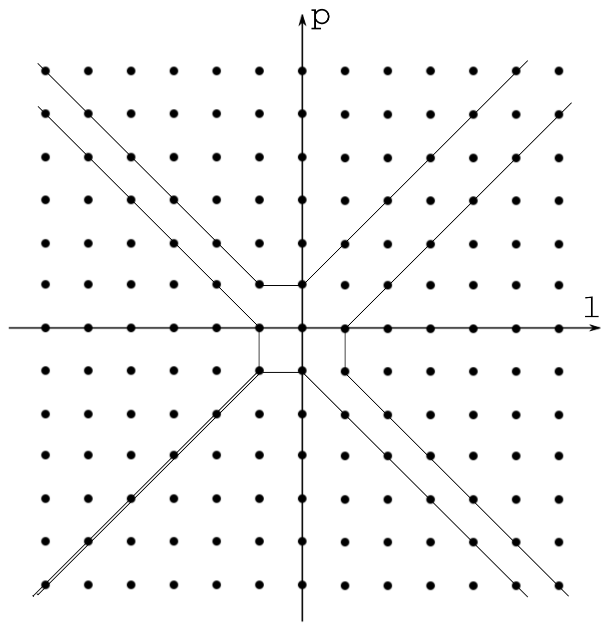

Figure 3.6: Lattice decomposition for $G4(a)$ computation. Four regions.

to cover all of $\mathbb{Z}^3$ except for the one-dimensional lattice $\{(k,l,p)|l=p=0\}$. This decomposition is depicted in figure 3.6. In the figure one sees that the points $l=p$ such that $l<0$ are covered twice, and the points $l=p$ such that $l>0$ are not covered at all, but via the transformation $(l,p)\mapsto -(l,p)$, this is the same as covering each of the points $l=p$, $l\neq 0$ once. Observations like this will be suppressed in the sequel. Then we see that the contribution from the symmetric component of the spectrum to the spectral action is

$$\frac{1}{2}\left(\frac{\Lambda}{2\pi}\right)^3 HL^2\int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw - \frac{1}{2}\left(\frac{\Lambda}{2\pi}\right) H\int_{\mathbb{R}} f(u^2)du, \tag{3.14}$$

up to terms of order $O(\Lambda^{-\infty})$. Combining this with the asymmetric component, we find that the spectral action is given by (3.13).

3.6.2 The case of $G4(b)$

In this case there is no asymmetric component in the spectrum. The symmetric component is given by the data

$$I=\{(k,l,m)|k,l,m\in\mathbb{Z},l\geq 1,m=0,\ldots,2l-2\}$$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}(k+\frac{1}{2})^2 + \frac{1}{L^2}((l-1/2)^2 + (m-l+1/2)^2)}.$$

We again obtain the same expression as in the $G4(a)$ case for the spectral action.

Theorem 3.6.2 *On the manifold $G4$ with a spin structure (b) with $\delta_i = \{\pm 1, -1\}$, the non-perturbative spectral action is also given by*

$$\mathrm{Tr}(f(D^2/\Lambda^2)) = \frac{1}{2}\left(\frac{\Lambda}{2\pi}\right)^3 HL^2 \int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw \tag{3.15}$$

up to possible terms of order $O(\Lambda^{-\infty})$.

We make the change of variables $p = m - l$. Using the symmetries

$$l \mapsto 1-l$$
$$l \mapsto p \quad p \mapsto l$$
$$l \mapsto p \quad p \mapsto 1-l,$$

we can exactly cover all of $\mathbb{Z}^3$, as shown in figure 3.7 and so the spectral action has the expression (3.15).

Remark 3.6.3 The technique we use here to sum over the spectrum to compute the non-perturbative spectral action does not appear to work in the case of the Bieberbach manifold $G5$, the quotient of $\mathbb{R}^3$ by the group $G5$ generated by commuting translations t_i along the vectors $a_1 = (0,0,H)$, $a_2 = (L,0,0)$ and $a_3 = (\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, $H, L > 0$, and an additional generator α with $\alpha^6 = t_1$, $\alpha t_2 \alpha^{-1} = t_3$ and $\alpha t_3 \alpha^{-1} = t_2^{-1} t_3$, which produces an identification of the faces of the fundamental domain with a $\pi/3$-turn about the z-axis. It is reasonable to expect that it will also give a multiple of the spectral action of the torus, with a proportionality factor of $HL^2/(4\sqrt{3})$.

3.7 The spectral action for $G6$

We analyze here the last remaining case of compact orientable Bieberbach manifold $G6$, the Hantzsche–Wendt space, according to the terminology followed in [66]. This is the quotient

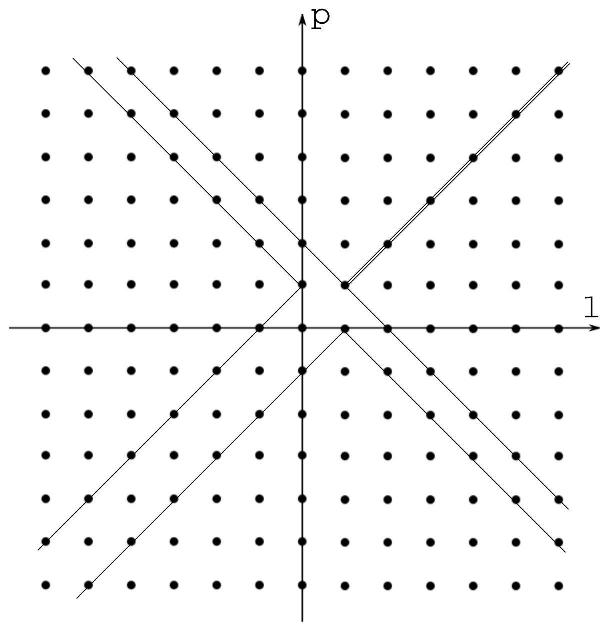

Figure 3.7: Lattice decomposition for $G4(b)$ computation. Four regions.

of $\mathbb{R}^3$ by the group $G6$ obtained as follows. One considers the lattice generated by vectors $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (0,S,0)$, with $H, L, S > 0$, and the group generated by commuting translations t_i along these vectors, together with additional generators α, β, and γ with the relations

$$\begin{gathered} \alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ \beta^2 = t_2, \quad \beta t_1 \beta^{-1} = t_1^{-1}, \quad \beta t_3 \beta^{-1} = t_3^{-1}, \\ \gamma^2 = t_3, \quad \gamma t_1 \gamma^{-1} = t_1^{-1}, \quad \gamma t_2 \gamma^{-1} = t_2^{-1}, \\ \gamma\beta\alpha = t_1 t_3. \end{gathered} \tag{3.16}$$

This gives an identification of the faces of the fundamental domain with a twist by an angle of π along each of the three coordinate axes.

According to Theorems 3.3 and 5.7 of [63], the manifold $G6$ has four different spin structures parameterized by three signs $\delta_i = \pm$ subject to the constraint $\delta_1\delta_2\delta_3 = 1$, but all of them yield the same Dirac spectrum, which has the following form.

The manifold $G6$ also has no asymmetric component to its spectrum, while the symmetric component is given by

$$I = \{(k,l,m) | k,l,m \in \mathbb{Z}, l \geq 0, k \geq 0\}$$

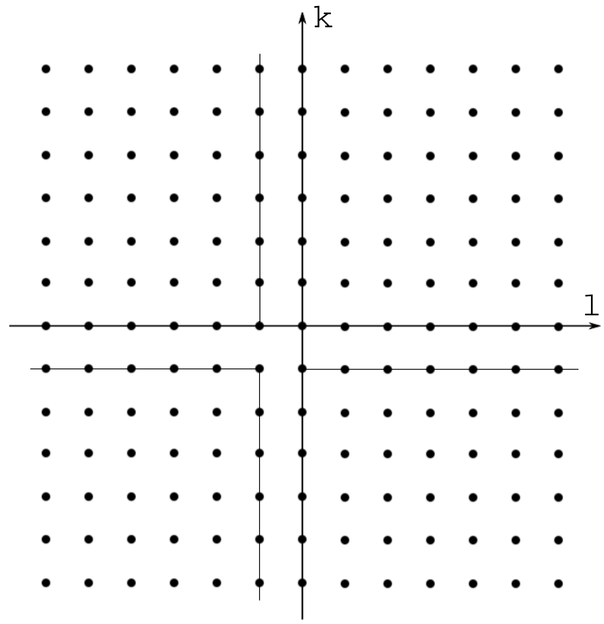

Figure 3.8: Lattice decomposition for $G6$ computation. Four regions.

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}(k+\frac{1}{2})^2 + \frac{1}{L^2}(l+\frac{1}{2})^2 + \frac{1}{S^2}(m+\frac{1}{2})^2}.$$

We then obtain the following result.

Theorem 3.7.1 *The Bieberbach manifold $G6$ with an arbitrary choice of spin structure has nonperturbative spectral action of the form*

$$\mathrm{Tr} f(D^2/\Lambda^2) = \frac{1}{2}\left(\frac{\Lambda}{2\pi}\right)^3 HLS \int_{\mathbb{R}^3} f(u^2+v^2+w^2)\,du\,dv\,dw \tag{3.17}$$

up to terms of order $O(\Lambda^{-\infty})$.

Using the three transformations

$$k \mapsto -k-1,$$
$$l \mapsto -l-1,$$
$$k \mapsto -k-1 \quad l \mapsto -l-1,$$

one exactly covers $\mathbb{Z}^3$, as seen in figure 3.8, and so we see that the nonperturbative spectral action is given by (3.17).

Chapter 4

Coset spaces of S^3

4.1 Introduction

The spectral action is a functional which is defined on spectral triples $(\mathcal{A}, \mathcal{H}, D)$ [15]. In this paper, we only consider the commutative case of compact Riemannian spin manifolds.

For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the spectral action is defined to be

$$\mathrm{Tr} f(D/\Lambda), \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a test function, and $\Lambda > 0$.

A compact Riemannian spin manifold, M, may be viewed as a spectral triple by taking $\mathcal{A}$ equal to $C^\infty(M)$, $\mathcal{H}$ equal to $L^2(M, \Sigma_n)$, the Hilbert space of L^2 spinor-valued functions on M, and D equal to the canonical Dirac operator. Since we are considering compact manifolds, the spectrum of the Dirac operator is discrete, and the meaning of the spectral action becomes simply

$$\mathrm{Tr} f(D/\Lambda) = \sum_{\lambda \in \mathrm{Spec} D} f(\lambda/\Lambda).$$

We will be content in each case to determine the spectral action up to an error term which is $O(\Lambda^{-k})$ for any $k > 0$. We will use the notation $O(\Lambda^{-\infty})$ to denote such a term.

There is an asymptotic expansion for the spectral action in terms of heat invariants, valid for large values of the parameter Λ, which is described in [16]. Since the heat invariants are local, it follows that the asymptotic series is multiplicative under quotients. That is, if D is the Dirac operator for a space X and D' is the Dirac operator for X/H, where H is a finite group acting freely on X, then the asymptotic expansion for $\mathrm{Tr} f(D'/\Lambda)$ is equal to $1/|H|$

times the asymptotic expansion for $\mathrm{Tr} f(D/\Lambda)$. In [16], Chamseddine and Connes obtain a nonperturbative expression for the spectral action of the round 3-sphere, $S^3 = SU(2)$. In the computation below we obtain a nonperturbative expression for the spectral action for $SU(2)/\Gamma$, where Γ is a finite subgroup of $SU(2)$ and this expression is a multiple of $1/|\Gamma|$ of the expression derived in [16], and so the nonperturbative expressions satisfy the same relation as the asymptotic expansions.

In this paper, we only consider one spin structure for each space, which we call the trivial spin structure. In general the Dirac spectrum depends on the choice of spin structure, so it would appear at first glance that the spectral action would also depend on the choice of spin structure. However, the asymptotic expansion of the spectral action does not depend on the choice of spin structure and so any such dependence must disappear as Λ goes to infinity.

The method used to compute the spectral action is a very slight modification to the one used in [16]. First, one computes the Dirac spectrum and decomposes the spectrum into a number of arithmetic progressions and finds a polynomial which describes the multiplicities for each arithmetic progression. Then one obtains a nonperturbative expression for the spectral action by using the Poisson summation formula.

This nonperturbative form of the spectral action of a three-dimensional space-like section of spacetime was used in the investigation, [50], on questions of cosmic topology. This application motivated the computations in this paper.

Up to conjugacy is it well-known that the finite subgroups of $SU(2)$ all lie in the following list.

- cyclic group, order N, $N = 1, 2, 3, \ldots$
- dicyclic group, order $4N$, $N = 2, 3, \ldots$
- binary tetrahedral group
- binary octahedral group
- binary icosahedral group.

The main result is the following:

Theorem 4.1.1 *Let Γ be any finite subgroup of $SU(2)$, then if D is the canonical Dirac operator on $SU(2)/\Gamma$ equipped with the round metric and trivial spin structure, the spectral action is given by*

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{|\Gamma|}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}). \tag{4.2}$$

In sections 4.4–4.7, we compute the Dirac spectrum for $SU(2)/\Gamma$ equipped with the Berger metric, and the trivial spin structure, and the spectral action for $SU(2)/\Gamma$ equipped with the round metric and trivial spin structure, where Γ is cyclic or dicyclic. In sections 4.2 and 4.3 we review the results and definitions needed to perform the computation, following the reference [5].

In sections 4.9, and 4.10 we compute the Dirac spectrum and spectral action in the case where Γ is the binary tetrahedral group and binary octahedral group respectively. For these two cases, we switch to the method of generating functions [4], because the representation theoretic calculations become difficult. This method gives us the spectrum for the round metric only. Again, we only consider the trivial spin structure. We review the key results needed for the computation in section 4.8.

In section 4.11 we compute the Dirac spectrum and spectral action in the case where Γ is the binary icosahedral group. We correct the expression of the Dirac spectrum found in [50]. The expression for the spectral action derived here is the same as the one found in [50].

There are two methods used to compute Dirac spectra, one which uses the representation theory of $SU(2)$ and one which uses generating functions. It is comforting to note that when both methods were used to compute the Dirac spectrum of a dicyclic space, they yielded the same answer.

In [31], the Dirac spectrum for $SU(2)/Q_8$ was computed for every possible choice of homogeneous Riemannian metric, and spin sructure. Q_8 is the quaternion group of order 8, which is the same thing as the dicyclic group with parameter $N = 2$.

In this paper, we take the least element of $\mathbb{N}$ to be zero.

4.2 Spin structures on homogeneous spaces

In this section we recall, for convenience, some facts about spin structures on homogeneous spaces appearing in [5].

In what follows, $M = G/H$ is an n-dimensional oriented Riemannian homogeneous space, where G is a simply connected Lie group.

In this case, the principal $SO(n)$-bundle of oriented orthonormal frames over M takes a simple form. Let V be the tangent space of $H \in G/H$, and let

$$\alpha : H \to SO(V)$$

be the isotropy representation induced by the action of H on G/H by left multiplication. If we choose an oriented orthonormal basis of V, then we obtain a representation of H into $SO(n)$, which we also denote by α. The bundle of oriented orthonormal frames may be identified with $G \times_\alpha SO(n)$, that is the space of pairs $[g, A]$, $g \in G$, $A \in SO(n)$, where one has the equivalence relation

$$[g, A] = [gh, \alpha(h^{-1})A], h \in H. \tag{4.3}$$

Let $\pi : G \to G/H$ be the projection map, and let $p = \pi(e)$. The identification of $G \times_\alpha SO(n)$ with the bundle of oriented frames is given by the formula

$$(g, A) \mapsto dg(p) \cdot b \cdot A, \tag{4.4}$$

where $b = (X_1, X_2, \ldots, X_n)$ is our chosen basis of V.

The spin structures of M are in one-to-one correspondence with the lifts $\alpha' : H \to$ Spin(n) satisfying

$$\Theta \circ \alpha' = \alpha,$$

where $\Theta :$ Spin(n) $\to$ SO(n) is the universal double covering map of $SO(n)$. One associates α' to the principal Spin(n)-bundle $G \times_{\alpha'}$ Spin(n). The right action of Spin(n) is given by

$$[g, \Lambda_1] \cdot \Lambda_2 = [g, \Lambda_1\Lambda_2], \tag{4.5}$$

and the covering map onto the frame bundle is given by

$$[g, \Lambda] \mapsto [g, \Theta(\Lambda)]. \tag{4.6}$$

In this paper we take $G = SU(2) \cong \mathrm{Spin}(3)$, and for any subgroup $\Gamma \subset SU(2)$, one always has the spin structure corresponding to the identity map $\iota : SU(2) \to \mathrm{Spin}(3)$, which lifts the isotropy homomorphism α. We call this spin structure the trivial spin structure.

4.3 Dirac operator on homogeneous spaces

In the case where Γ is the cyclic or dicyclic group, we shall compute the spectrum of the Dirac operator for the one-parameter family of Berger metrics. The key result that we use is the following.

Let $\Sigma_{\alpha'} M$ denote the spinor bundle corresponding to the spin structure α'. Let $\rho : \mathrm{Spin}(n) \to U(\Sigma_n)$ be the spinor representation. Let $\widehat{G}$ denote the set of irreducible representations, $\pi_\gamma : G \to GL(V_\gamma)$, of G up to equivalence. Let $\mathfrak{g}$ be the Lie algebra of G. We decompose $\mathfrak{g}$ as $\mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{p}$ is an $\mathrm{Ad_H}$-invariant subspace of $\mathfrak{g}$.

Theorem 4.3.1 ([5], Theorem 2 and Proposition 1) *The representation of the Dirac operator on $L^2(M, \Sigma_{\alpha'} M)$ is equivalent to*

$$\overline{\oplus_{\gamma \in \widehat{G}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)}.$$

Here, H acts on V_γ as the representation γ dictates, and on Σ_n via $\rho \circ \alpha'$. The Dirac operator acts on the summand $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$ as $\mathrm{id} \otimes D_\gamma$, where given $A \in \mathrm{Hom}_H(V_\gamma, \Sigma_n)$,

$$D_\gamma(A) := -\sum_{k=1}^{n} e_k \cdot A \circ (\pi_\gamma)_*(X_k) + \left(\sum_{i=1}^{n} \beta_i e_i + \sum_{i<j<k} \alpha_{ijk} e_i \cdot e_j \cdot e_k \right) \cdot A. \tag{4.7}$$

Here, e_i denotes the standard basis for $\mathbb{R}^n$, acting on spinors via Clifford multiplication,

$$\beta_i = \frac{1}{2} \sum_{j=1}^{n} \langle [X_j, X_i]_{\mathfrak{p}}, X_j \rangle, \tag{4.8}$$

$$\alpha_{ijk} = \frac{1}{4}(\langle [X_i, X_j]_{\mathfrak{p}}, X_k\rangle + \langle [X_j, X_k]_{\mathfrak{p}}, X_i\rangle + \langle [X_k, X_i]_{\mathfrak{p}}, X_j\rangle). \tag{4.9}$$

The notation $Y_{\mathfrak{p}}$ denotes the projection of $Y \in \mathfrak{g}$ onto $\mathfrak{p}$ with kernel $\mathfrak{h}$, and the angle brackets denote the pairing of tangent vectors via the Riemannian metric.

Let $V_n \in \widehat{SU(2)}$ be the $n+1$-dimensional irreducible representation of $SU(2)$ of complex homogeneous polynomials in two variables of degree n. When $G = SU(2)$, H is a finite subgroup of $SU(2)$, and G/H is equipped with the Berger metric corresponding to the parameter $T > 0$, 4.7 becomes (see [5], section 5)

$$D_n A = -\sum_k e_k \cdot A \cdot (\pi_n)_*(X_k) - \left(\frac{T}{2} + \frac{1}{T}\right).$$

Let D'_n denote the part

$$-\sum_k E_k \cdot A \cdot (\pi_n)_*(X_k). \tag{4.10}$$

Let $P_k \in V_n$ be the basis polynomial

$$P_k(z_1, z_2) = z_1^{n-k} z_2^k. \tag{4.11}$$

Now, we take $A_k,\ B_k,\ k = 0, 1, \ldots, n$ to be the basis for $\mathrm{Hom}_{\mathbb{C}}(V_n, \Sigma_3)$:

$$A_k(P_l) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{if } k = l,\ k \text{ is even} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \text{if } k = l,\ k \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

$$B_k(P_l) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \text{if } k = l,\ k \text{ is even} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{if } k = l,\ k \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

We have the following formulas for D'_n (see [5]):

$$D'_n A_k = \frac{1}{T}(n-2k)A_k + 2(k+1)A_{k+1}, \text{ k even}$$
$$D'_n A_k = 2(n+1-k)A_{k-1} + \frac{1}{T}(2k-n)A_k, \text{ k odd}$$
$$D'_n B_k = 2(n+1-k)B_{k-1} + \frac{1}{T}(2k-n)B_k, \text{ k even}$$
$$D'_n B_k = \frac{1}{T}(n-2k)B_k + 2(k+1)B_{k+1}, \text{ k odd.}$$

The formulas remain valid when $k = 0$ and $k = n$, provided that we take $A_{-1} = A_{n+1} = B_{-1} = B_{n+1} = 0$.

4.4 Dirac spectra for lens spaces with Berger metric

In this section we compute the Dirac spectrum on lens spaces equipped with the Berger metric and the trivial spin structure. This calculation corrects the corresponding one in [5].

To proceed, we need to determine which linear transformations, $f \in \mathrm{Hom}_{\mathbb{C}}(V_n, \Sigma_3)$ are $\mathbb{Z}_N$-linear. A $\mathbb{C}$-linear map f is $\mathbb{Z}_N$-linear if and only if f commutes with a generator of $\mathbb{Z}_N$. We take

$$B = \begin{pmatrix} e^{\frac{2\pi i}{N}} & 0 \\ 0 & e^{\frac{-2\pi i}{N}} \end{pmatrix}$$

to be our generator, and we define

$$\begin{pmatrix} f_{1k} \\ f_{2k} \end{pmatrix} := f(P_k). \tag{4.12}$$

Since we are considering the trivial spin structure corresponding to the inclusion map $\iota : \mathbb{Z}_N \to SU(2)$, f is $\mathbb{Z}_N$ linear if and only if

$$f \circ \pi_n(B) = \iota(B) \circ f,$$

which leads to the identity

$$\begin{pmatrix} f_{1k} \\ f_{2k} \end{pmatrix} = \begin{pmatrix} e^{2\pi i \frac{2k-n+1}{N}} f_{1k} \\ e^{2\pi i \frac{2k-n-1}{N}} f_{2k} \end{pmatrix}.$$

It follows that $\mathrm{Hom}_{\mathbb{Z}_N}(V_n, \Sigma_3)$ has the following basis:

$$\begin{aligned}
&\{A_k : k = \frac{mN+n-1}{2} \in \{0,1,\ldots,n\}, m \in \mathbb{Z}, k \text{ even}\} \\
&\cup \{B_k : k = \frac{mN+n-1}{2} \in \{0,1,\ldots,n\}, m \in \mathbb{Z}, k \text{ odd}\} \\
&\cup \{A_k : k = \frac{mN+n+1}{2} \in \{0,1,\ldots,n\}, m \in \mathbb{Z}, k \text{ odd}\} \\
&\cup \{B_k : k = \frac{mN+n+1}{2} \in \{0,1,\ldots,n\}, m \in \mathbb{Z}, k \text{ even}\}.
\end{aligned}$$

With the basis in hand, let us now compute the spectrum.

N even

First let us consider the case $N \equiv 0 \bmod 4$.

In this case, $\frac{mN+n-1}{2}$ is an integer precisely when n is odd. In particular, this means that $\mathrm{Hom}_{\mathbb{Z}_N}(V_n, \Sigma_3)$ is trivial if n is even.

If $n \equiv 1 \bmod 4$, and m is an integer, satisfying

$$-n \leq mN - 1 < n, \tag{4.13}$$

then

$$k = \frac{mN+n-1}{2} \tag{4.14}$$

is an even integer between 0 and $n-1$, inclusive. Since k is strictly less than n, A_{k+1} is not equal to 0. Therefore A_k and A_{k+1} lie in $\mathrm{Hom}_{\mathbb{Z}_N}(V_n, \Sigma_3)$, and span an invariant two-dimensional subspace of D'_n. With respect to these two vectors, D'_n has the matrix expression

$$\begin{pmatrix} \frac{1}{T}(n-2k) & 2(n+1-(k+1)) \\ 2(k+1) & \frac{1}{T}(2(k+1)-n) \end{pmatrix}, \tag{4.15}$$

which has eigenvalues

$$\lambda = \frac{1}{T} \pm \sqrt{(1+n)^2 + m^2N^2\left(\frac{1}{T^2} - 1\right)}. \tag{4.16}$$

Now let us consider the case $n \equiv 3 \bmod 4$. In this case, if 4.13 and 4.14 hold then k is an odd integer between 0 and $n-1$ inclusive, B_k and B_{k+1} lie in $\mathrm{Hom}_{\mathbb{Z}_N}(V_n, \Sigma_3)$, and span an invariant subspace of D_n. B_{k+1} is not equal to 0, and with respect to these two vectors D'_n once again has the matrix expression given by Equation 4.15, with eigenvalues given by Equation 4.16.

If

$$mN - 1 = n, \quad m = 1, 2, \dots, \tag{4.17}$$

then B_0, and B_n are eigenvectors of D'_n with eigenvalue

$$\lambda = -\frac{n}{T} = \frac{1 - mN}{T}. \tag{4.18}$$

In the case $N \equiv 2 \bmod 4$, the analysis proceeds exactly as when $N \equiv 0 \bmod 4$, except for a few minor changes which do not alter the spectrum. Namely, for $n \equiv 1 \bmod 4$, it is B_k, B_{k+1} which span an invariant subspace of D'_n, not A_k, A_{k+1}, and for $n \equiv 3 \bmod 4$, A_k, A_{k+1} span an invariant subspace of D'_n, not B_k, B_{k+1}.

To determine the spectrum of D we just need to add $-\frac{T}{2} - \frac{1}{T}$ to D'_n, which just shifts the eigenvalues, and then tensor with id_{V_n} which just multiplies the multiplicities by $n+1$.

To summarize we have the following.

Theorem 4.4.1 *If N is even, then the Dirac operator on the lens space $\mathcal{L}_N$ equipped with the Berger metric corresponding to parameter $T > 0$, and the trivial spin structure has the following spectrum:*

λ	*multiplicity*
$\{-\frac{T}{2} \pm \sqrt{(1+n)^2 + m^2N^2\left(\frac{1}{T^2} - 1\right)} \mid n \in 2\mathbb{N}+1, m \in \mathbb{Z}, -n \le mN - 1 < n\}$	$n+1$
$\{-\frac{T}{2} - \frac{mN}{T} \mid m \in \mathbb{N}\}$	$2mN$

.

Note that the second row of the table corresponds to the case $n = mN - 1$, in which case, $n + 1 = mN$, which accounts for the factor of mN in the multiplicity.

N odd

In contrast to the case where N is even, $\mathrm{Hom}_{\mathbb{Z}_N}(V_n, \Sigma_3)$ may be nontrivial whether n is even or odd.

As in the case when N is even, if 4.13 and 4.14 hold, then one of A_k, A_{k+1}, or B_k, B_{k+1}

spans a two-dimensional invariant subspace of D'_n, where D'_n has matrix expression (4.15) and eigenvalues (4.16).

When n is even, k is an integer if and only if m is odd. On the other hand, when n is odd, k is an integer if and only if m is even.

If $n = mN - 1$, where m is a positive integer, then B_0 and either B_n or A_n, (depending on whether n is even or odd) are eigenvectors of D'_n each with eigenvalue $-\frac{n}{T} = \frac{1-mN}{T}$.

We have shown the following.

Theorem 4.4.2 *If N is odd, then the Dirac operator on the lens space $\mathcal{L}_N$ equipped with the Berger metric corresponding to parameter $T > 0$ and the trivial spin structure has the following spectrum:*

λ	*multiplicity*
$\{-\frac{T}{2} \pm \sqrt{(1+n)^2 + m^2N^2\left(\frac{1}{T^2} - 1\right)}\,\vert$ $(n \in 2\mathbb{N}+1, m \in 2\mathbb{Z})$ or $(n \in 2\mathbb{N}, m \in 2\mathbb{Z}+1),\ -n \leq mN - 1 < n\}$	$n+1$
$\{-\frac{T}{2} - \frac{mN}{T} \vert m \in \mathbb{N}\ \}$	$2mN$

.

4.5 Spectral action of round lens spaces

The Berger metric corresponding to $T = 1$ is the round metric. By substituting $T = 1$ into Theorems 4.4.1 and 4.4.2, we obtain the following expressions for the Dirac spectrum.

If N is even, then the Dirac operator on the lens space $\mathcal{L}_N$ equipped with the round metric has the following spectrum:

λ	multiplicity
$\{-\frac{3}{2} - n, \frac{1}{2} + n\vert$ $n \in 2\mathbb{N}+1, m \in \mathbb{Z}, -n \leq mN - 1 < n\}$	$n+1$
$\{-\frac{1}{2} - mN \vert m \in \mathbb{N}\}$	$2mN$

. (4.19)

If N is odd, then the Dirac operator on the lens space $\mathcal{L}_N$ equipped with the round metric has the following spectrum:

λ	multiplicity
$\{-\frac{3}{2}-n, \frac{1}{2}+n\|$ $(n \in 2\mathbb{N}+1, m \in 2\mathbb{Z})$ or $(n \in 2\mathbb{N}, m \in 2\mathbb{Z}+1)$, $-n \leq mN-1 < n\}$	$n+1$
$\{-\frac{1}{2}-mN\|m \in \mathbb{N}\}$	$2mN$

. (4.20)

However, these are not the simplest expressions for the spectra. In this special case, the eigenvalues in the first row of the spectrum no longer depend on m, so we should count the values of m which satisfy the inequality as a function of n in order to eliminate the dependence of the spectrum on m.

4.5.1 Round metric, $T = 1$

N even

Let us write $n = kN + 2s + 1$, for $s \in \{0, 1, 2, \ldots \frac{N-2}{2}\}$, and $k \in \mathbb{N}$ (recall that n is always odd in this case). Then we may replace the inequality

$$-n \leq mN - 1 < n$$

by the inequality

$$-kN \leq mN \leq kN,$$

where $-kN$ and kN are respectively the minimum and maximum values of mN which satisfy the inequalities. From these new inequalities, it is clear that there are $2k+1$ values of m satisfying them.

We now have the following form of the Dirac spectrum, which is still not quite the definitive form.

λ	multiplicity
$\{\frac{3}{2}+kN+2s\|k \in \mathbb{N}, s \in \{0,1,\ldots,\frac{N-2}{2}\}\}$	$(2k+1)(kN+2s+2)$
$\{-\frac{5}{2}-k'N-2s'\|k' \in \mathbb{N}, s' \in \{0,1,\ldots,\frac{N-2}{2}\}\}$	$(2k'+1)(k'N+2s'+2)$
$\{-\frac{1}{2}-mN\|m \in \mathbb{N}\}$	$2mN$

. (4.21)

The definitive form of the spectrum of the lens space $\mathcal{L}_N$ equipped with the round metric, with N even, is obtained when one realizes that the first row of table 4.21 already completely describes the spectrum as soon as one lets the parameter k take values in all of $\mathbb{Z}$ as opposed to just in $\mathbb{N}$. To see that this is indeed the case, one absorbs the third row into the second row by making the substitution $m = k' + 1$, which affects the multiplicity of the second row only in the case $s = \frac{N-2}{2}$. Next, one shows that when k is allowed to take values in all of $\mathbb{Z}$, one combines the parts of the spectra corresponding to s and s', when $s + s' = \frac{N-4}{2}$ if s and s' are less than $\frac{N-4}{2}$ and when $s = s' = \frac{N-2}{2}$ otherwise. As a result we have the following corollary.

Corollary 4.5.1 *If N is even then the Dirac operator on the lens space $\mathcal{L}_N$ equipped with the round metric has the following spectrum:*

λ	*multiplicity*
$\{\frac{3}{2} + kN + 2s \mid k \in \mathbb{Z}, s \in \{0, 1, \ldots, \frac{N-2}{2}\}\}$	$(2k+1)(kN + 2s + 2)$

.

N odd

The corresponding expression in the case N odd is only slightly more complicated. Here, we need to divide our analysis according to whether n is even/odd, and k is even/odd. We write

$$n = kN + j, \quad j \in \{0, 1, 2, \ldots, N-1\}, \quad k \in \mathbb{N}. \tag{4.22}$$

Suppose n is odd. Then if k is even, one can see that there are $k+1$ even values of m satisfying the inequalities 4.13. If k is odd, then there are k such values of m.

If n is even, then when k is even there are k odd values of m satisfying the inequalities, and if k is odd, there are $k+1$ such values of m.

Therefore, we have the following expression for the Dirac spectrum in the round, odd case.

If N is odd, then the Dirac operator on the lens space $\mathcal{L}_N$ equipped with the round metric has the following spectrum:

λ	multiplicity
$\{\frac{-5-4aN-4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-3}{2}\}\}$	$(2a+1)(2aN+2b+2)$
$\{\frac{3+4aN+4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-3}{2}\}\}$	$(2a+1)(2aN+2b+2)$
$\{\frac{-3-(4a+2)N-4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-1}{2}\}\}$	$(2a+1)((2a+1)N+2b+1)$
$\{\frac{1+(4a+2)N+4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-1}{2}\}\}$	$(2a+1)((2a+1)N+2b+1)$
$\{\frac{-3-4aN-4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-1}{2}\}\}$	$2a(2aN+2b+1)$
$\{\frac{1+4aN+4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-1}{2}\}\}$	$2a(2aN+2b+1)$
$\{\frac{-5-(4a+2)N-4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-3}{2}\}\}$	$(2a+2)((2a+1)N+2b+2)$
$\{\frac{-5-(4a+2)N-4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-3}{2}\}\}$	$(2a+2)((2a+1)N+2b+2)$
$\{\frac{3+(4a+2)N+4b}{2} \vert a\in\mathbb{N},\, b\in\{0,\dots\frac{N-3}{2}\}\}$	$(2a+2)((2a+1)N+2b+2)$
$\{\frac{-1}{2}-mN \vert m\in\mathbb{N}\}$	$2mN$

. (4.23)

Just as in the even case, we can simplify the expression 4.23 by combining rows. The last row can be split into two parts and combined with the third and fifth rows, altering the multiplicity in each case for $b = \frac{N-1}{2}$. Then, the first and fourth rows, second and third rows, fourth and eight rows, and fifth and sixth rows may be combined by letting the parameter a run over all of $\mathbb{Z}$ instead of just $\mathbb{N}$. The definitive form of the spectrum in the odd case is given by the following corollary.

Corollary 4.5.2 *If N is odd then the Dirac operator on the lens space $\mathcal{L}_N$ equipped with the round metric has the following spectrum:*

λ	*multiplicity*
$\{\frac{3+4aN+4b}{2} \vert a\in\mathbb{Z},\, b\in\{0,\dots\frac{N-3}{2}\}\}$	$(2a+1)(2aN+2b+2)$
$\{\frac{1+(4a+2)N+4b}{2} \vert a\in\mathbb{Z},\, b\in\{0,\dots\frac{N-1}{2}\}\}$	$(2a+1)((2a+1)N+2b+1)$
$\{\frac{1+4aN+4b}{2} \vert a\in\mathbb{Z},\, b\in\{0,\dots\frac{N-1}{2}\}\}$	$2a(2aN+2b+1)$
$\{\frac{3+(4a+2)N+4b}{2} \vert a\in\mathbb{Z},\, b\in\{0,\dots\frac{N-3}{2}\}\}$	$(2a+2)((2a+1)N+2b+2)$

.

4.5.2 Computing the spectral action

First we consider the case where N is even. For $s\in\{0,1,2,\dots\frac{N-2}{2}\}$, define

$$P_s(u) = \frac{-3+N-4s-4u+2Nu-8su+4u^2}{2N}.$$

Then, $P_s(\lambda)$ equals the multiplicity of

$$\lambda = 3/2 + kN + 2s. \tag{4.24}$$

Moreover, we have the following identity:

$$\sum_{s=0}^{\frac{N-2}{2}} P_s(u) = -\frac{1}{4} + u^2. \tag{4.25}$$

Now to compute the spectral action, we proceed as in [16], and use the Poisson summation formula, except here we sum over $\frac{N-2}{2}$ arithmetic progressions instead of just one.

The Poisson summation formula is given by

$$\sum_{\mathbb{Z}} h(k) = \sum_{\mathbb{Z}} \widehat{h}(x), \tag{4.26}$$

where our choice of the Fourier transform is

$$\widehat{h}(x) = \int_{\mathbb{R}} h(u) e^{-2\pi i u x} du. \tag{4.27}$$

In each instance of a spectral action computation below, we encounter the situation described by the following lemma:

Lemma 4.5.3 *Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwarz function, and let $P(u) = \sum_{j=0}^{n} c_k u^j$ be a polynomial. Define $g(u) = P(u) f(u/\Lambda)$, $h(u) = g(a + uM)$, for real constants a and M, then*

$$\sum_{\mathbb{Z}} h(u) = \frac{1}{M} \sum_{j=0}^{n} \Lambda^{j+1} c_j \widehat{f}^{(j)}(0) + O(\Lambda^{-\infty}),$$

where $\widehat{f}^{(j)}$ is the Fourier transform of $v^j f(v)$.

Furthermore,

$$
\begin{aligned}
\widehat{h}(k) &= \int h(x)e^{-2\pi ixk}dx \\
&= \int g(a+xM)e^{-2\pi ixk}dx \\
&= \frac{1}{M}(e^{\frac{2\pi ia}{M}})^k \int g(v)e^{2\pi i\frac{vk}{M}}dv \\
&= \frac{1}{M}(e^{\frac{2\pi ia}{M}})^k \widehat{g}(k/M).
\end{aligned}
$$

Since $f \in \mathcal{S}(\mathbb{R})$, so too are the functions $\widehat{f}^{(j)}$ and so we have the estimate as Λ approaches plus infinity,

$$
\begin{aligned}
\sum_{k\neq 0} |\widehat{h}(k)| &= \sum_{k\neq 0} \frac{1}{M}|\widehat{g}(k/M)| \\
&\leq \sum_{j=0}^{n} \left(|c_j|\Lambda^{j+1} \sum_{k\neq 0} |\widehat{f}^{(j)}(k\Lambda/M)| \right) \\
&= O(\Lambda^{-\infty}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\widehat{h}(0) = \frac{1}{M}\widehat{g}(0) &= \frac{1}{M}\int P(v)f(\frac{v}{\Lambda})dv \\
&= \frac{1}{M}\sum_{j=0}^{n} c_j \int v^j f(\frac{v}{\Lambda})dv \\
&= \frac{1}{M}\sum_{j=0}^{n} \Lambda^{j+1} c_j \widehat{f}^{(j)}(0).
\end{aligned}
$$

Now one applies Lemma 4.5.3 and the identity 4.25 to compute the spectral action of the round lens spaces for N even:

$$
\begin{aligned}
\mathrm{Tr}(f(D/\Lambda)) &= \sum_{s=0}^{\frac{N-2}{2}} \sum_{k\in\mathbb{Z}} P_s(\frac{3}{2}+kN+s)f((\frac{3}{2}+kN+s)/\Lambda) \\
&= \frac{1}{N}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}).
\end{aligned}
$$

In the case where N is odd, the interpolating polynomials are collected in the following table.

$P_b(u) = \frac{-3-4b+2N-4u-8bu+4Nu+4u^2}{4N},\ b \in \{0, 1, \dots \frac{N-3}{2}\}$
$Q_b(u) = \frac{-1-4b-8bu+4u^2}{4N},\ b \in \{0, 1, \dots \frac{N-1}{2}\}$
$R_b(u) = \frac{-1-4b-8bu+4u^2}{4N},\ b \in \{0, 1, \dots \frac{N-1}{2}\}$
$S_b(u) = \frac{-3-4b+2N-4u-8bu+4Nu+4u^2}{4N},\ b \in \{0, 1, \dots \frac{N-3}{2}\}$

.

With these polynomials in hand, we obtain the identity,

$$\sum_{j=0}^{\frac{N-3}{2}} P_j + \sum_{j=0}^{\frac{N-1}{2}} Q_j + \sum_{j=0}^{\frac{N-1}{2}} R_j + \sum_{j=0}^{\frac{N-3}{2}} S_j = -\frac{1}{2} + 2u^2. \tag{4.28}$$

Then using Lemma 4.5.3 and equation 4.28 we see that

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{2N}\left(2\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{2}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}) \tag{4.29}$$

$$= \frac{1}{N}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}). \tag{4.30}$$

We have shown the following:

Theorem 4.5.4 *For each $N = 1, 2, 3, \dots$ the spectral action on the round lens space $\mathcal{L}_N$, with the trivial spin structure is given by*

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{N}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}).$$

4.6 Dirac spectra for dicyclic spaces with Berger metric

Here we consider the space forms S^3/Γ, where $\Gamma \subset SU(2)$ is the binary dihedral group, or dicyclic group, concretely generated by the elements B and C, where

$$B = \begin{pmatrix} e^{\frac{\pi i}{N}} & 0 \\ 0 & e^{\frac{-\pi i}{N}} \end{pmatrix},$$

and

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

First we consider the trivial spin structure corresponding to the inclusion $\iota : \Gamma \to SU(2)$. Therefore, a linear map $f \in \mathrm{Hom}_{\mathbb{C}}(V_n, \Sigma_3)$ is Γ-linear, if f in addition satisfies the conditions

$$f \circ \pi_n(B) = \iota(B) \circ f, \tag{4.31}$$

and

$$f \circ \pi_n(C) = \iota(C) \circ f. \tag{4.32}$$

We once again use the notation of Equation 4.12, whence the Equations 4.31 and 4.32 become the set of conditions

$$f_{1k} = e^{\frac{i\pi}{N}(2k-n+1)} f_{1k}, \tag{4.33}$$

$$f_{2k} = e^{\frac{i\pi}{N}(2k-n-1)} f_{2k}, \tag{4.34}$$

$$f_{1k} = (-1)^{n-k+1} f_{2(n-k)}, \tag{4.35}$$

$$f_{2k} = (-1)^{n-k} f_{1(n-k)}. \tag{4.36}$$

These conditions imply that for $k \in \{0, 1, \ldots, n\}$, $f_{1k} = 0$ unless $\frac{2k-n+1}{2N}$ is an integer and $f_{2k} = 0$ unless $\frac{2k-n-1}{2N}$ is an integer.

When performing our analysis for the dicyclic group of order $4N$, we need to break up our analysis into the cases N even, and N odd.

N even

Suppose

$$\frac{2k-n+1}{2N} = m \in \mathbb{Z}. \tag{4.37}$$

Then

$$k = \frac{2mN+n-1}{2}. \tag{4.38}$$

k is an integer precisely when n is odd. Therefore, we only need to consider the cases $n \equiv 1, 3 \mod 4$.

First, if $n \equiv 1 \mod 4$, one deduces from conditions 4.33 – 4.36 that for each integer m

such that

$$-n \leq 2mN - 1 < -1, \tag{4.39}$$

$$v_1 = A_k + A_{n-k} \tag{4.40}$$

and

$$v_2 = A_{k+1} + A_{n-k-1} \tag{4.41}$$

span an invariant two-dimensional subspace of D'_n. With respect to the ordered pair (v_1, v_2), D'_n has the familiar matrix expression 4.15, which gives the eigenvalues

$$\frac{1}{T} \pm \sqrt{(1+n)^2 + 4m^2N^2\left(\frac{1}{T^2} - 1\right)}, \tag{4.42}$$

which are slightly different from those given in equation 4.16, the difference being due to the fact that the relationship between k and m is slightly different. When $2mN - 1 = -1$, i.e. when $m = 0$, then

$$k = \frac{2mN + n - 1}{2} = \frac{n-1}{2}, \tag{4.43}$$

and $A_k + A_{n-k}$ is an eigenvector with eigenvalue

$$\lambda = \frac{1}{T} + n + 1. \tag{4.44}$$

Now suppose $n \equiv 3 \bmod 4$. This case is very similar to the case $n \equiv 1 \bmod 4$. In this case, for each integer m such that 4.39 holds,

$$v_1 = B_k - B_{n-k} \tag{4.45}$$

and

$$v_2 = B_{k+1} - B_{n-k-1} \tag{4.46}$$

form an invariant two-dimensional subspace of D'_n. Once again, with respect to the pair (v_1, v_2), D'_n has the matrix expression 4.15. When

$$2mN - 1 = -1, \tag{4.47}$$

$B_k - B_{n-k}$ is an eigenvector of D'_n with eigenvalue $\frac{1}{T} - (n+1)$.

The only remaining case is when

$$n = 2mN - 1, \tag{4.48}$$

in which case $k = n$, and $B_n - B_0$ is an eigenvector of eigenvalue $\frac{-n}{T}$.

As in the lens space case, to determine the spectrum of the Dirac operator, we simply shift the spectrum of D'_n by $-\frac{T}{2} - \frac{1}{T}$ and multiply the multiplicities by $n+1$.

Therefore we see that if N is even, then the Dirac operator on the dicyclic space S^3/Γ equipped with the Berger metric corresponding to parameter $T > 0$, and the trivial spin structure has the following spectrum:

λ	multiplicity
$\{-\frac{T}{2} \pm \sqrt{(1+n)^2 + 4m^2N^2\left(\frac{1}{T^2} - 1\right)}\,\vert\; n \in 2\mathbb{N}+1, m \in \mathbb{Z}, -n \leq 2mN - 1 < -1\}$	$n+1$
$\{-\frac{T}{2} + n + 1 \vert n \in \mathbb{N}, n \equiv 1(4)\}$	$n+1$
$\{-\frac{T}{2} - (n+1) \vert n \in \mathbb{N}, n \equiv 3(4)\}$	$n+1$
$\{-\frac{T}{2} - \frac{2mN}{T} \vert m \in \mathbb{N}\}$	$2mN$

(4.49)

N odd

Now let us consider the case where N is odd. Unlike the case of lens spaces, the expression for the spectrum is the same whether N is even or odd. As in the case where N is even, k is an integer only when n is odd, which means that $\mathrm{Hom}_\Gamma(V_n, \Sigma_3)$ is trivial unless n is odd. So suppose n is odd. For every integer m such that 4.39 holds either the set $\{A_k + A_{n-k}, A_{k+1} + A_{n-k-1}\}$ or the set $\{B_k - B_{n-k}, B_{k+1} - B_{n-k-1}\}$ span an invariant two-dimensional subspace for D'_n, when k is even or odd respectively. The eigenvalues of each two dimensional subspace are given once again by expression 4.42. Exactly as in the case when N is even, for each $n \equiv 1(4)$,

$$A_{\frac{n-1}{2}} + A_{\frac{n+1}{2}} \tag{4.50}$$

is an eigenvector of eigenvalue $\frac{1}{T} + n + 1$, and for each $n \equiv 3(4)$,

$$B_{\frac{n-1}{2}} - B_{\frac{n+1}{2}} \tag{4.51}$$

is an eigenvector of eigenvalue $\frac{1}{T} - (n+1)$. For each $m \in \mathbb{N}$, $B_n - B_0$ is an eigenvector of eigenvalue $\frac{1}{T} - \frac{n+1}{T}$. These eigenvectors form a basis of $\mathrm{Hom}_\Gamma(V_n, \Sigma_3)$, and we see that the spectrum has the same expression as when N is even.

Theorem 4.6.1 *Let Γ be the dicyclic group of order $4N$. The Dirac operator on the dicyclic space S^3/Γ equipped with the Berger metric corresponding to parameter $T > 0$, and the trivial spin structure has the following spectrum:*

λ	*multiplicity*
$\{-\frac{T}{2} \pm \sqrt{(1+n)^2 + 4m^2N^2\left(\frac{1}{T^2} - 1\right)} \mid n \in 2\mathbb{N}+1, m \in \mathbb{Z}, -n \le 2mN - 1 < -1\}$	$n+1$
$\{-\frac{T}{2} + n + 1 \mid n \in \mathbb{N}, n \equiv 1(4)\}$	$n+1$
$\{-\frac{T}{2} - (n+1) \mid n \in \mathbb{N}, n \equiv 3(4)\}$	$n+1$
$\{-\frac{T}{2} - \frac{2mN}{T} \mid m \in \mathbb{N}\}$	$2mN$

.

4.7 Spectral action of round dicyclic space

4.7.1 Round metric, T=1

Substituting $T = 1$ into Theorem 4.6.1, we obtain the spectrum for dicyclic space equipped with the round metric:

λ	multiplicity
$\{-\frac{3}{2} - n \mid n \in 2\mathbb{N}+1, m \in \mathbb{Z}, -n \le 2mN - 1 < -1\}$	$n+1$
$\{\frac{1}{2} + n \mid n \in 2\mathbb{N}+1, m \in \mathbb{Z}, -n \le 2mN - 1 < -1\}$	$n+1$
$\{\frac{1}{2} + n \mid n \in \mathbb{N}, n \equiv 1(4)\}$	$n+1$
$\{-\frac{3}{2} - n \mid n \in \mathbb{N}, n \equiv 3(4)\}$	$n+1$
$\{-\frac{1}{2} - 2mN \mid m = 1, 2, 3, \ldots\}$	$2mN$

.

Now, we may write $n \in 2\mathbb{N}+1$ uniquely as

$$n = 2kN + 2s + 1, \quad k \in \mathbb{N}, s \in \{0, 1, 2, \ldots N-1\}. \tag{4.52}$$

Then the inequality 4.39 becomes

$$-2kN - 2s - 1 \leq 2mN - 1 < -1$$

$$-2kN - 2s \leq 2mN < 0$$

$$-2kN \leq 2mN < 0$$

$$-k \leq m < 0,$$

whence we see that there are k integer values of m satisfying the inequality. Therefore we may rewrite the spectrum as follows:

λ	multiplicity
$\{\frac{3}{2} + 2kN + 2s \mid k \in \mathbb{N}, s \in \{0, 1, \ldots, N-1\}\}$	$(2kN + 2s + 2)k$
$\{-\frac{5}{2} - 2kN - 2s \mid k \in \mathbb{N}, s \in \{0, 1, \ldots, N-1\}\}$	$(2kN + 2s + 2)k$
$\{\frac{1}{2} + n \mid n \in \mathbb{N}, n \equiv 1(4)\}$	$n + 1$
$\{-\frac{3}{2} - n \mid n \in \mathbb{N}, n \equiv 3(4)\}$	$n + 1$
$\{-\frac{1}{2} - 2mN \mid m = 1, 2, 3, \ldots\}$	$2mN$

. (4.53)

In order to find the definitive form of the spectrum, we must first analyze the rows into commensurable parts. To do the analysis we need to consider the cases where N is odd and N is even separately.

N even

In this case, the third and fourth rows of 4.53 may be decomposed into $N/2$ parts and written as

λ	multiplicity
$\{\frac{3}{2} + 2kN + 2s \mid k \in \mathbb{N}, s \in \{0, 1, \ldots, N-1\}, s \text{ even}\}$	$2kN + 2s + 2$
$\{-\frac{5}{2} - 2kN - 2s \mid k \in \mathbb{N}, s \in \{0, 1, \ldots, N-1\}, s \text{ odd}\}$	$2kN + 2s + 2$

. (4.54)

The fifth row of 4.53 can be combined with the case of $s = N - 1$ in the second row. Combining the rows together yields the following expression for the spectrum in the even case:

λ	multiplicity
$\{\frac{3}{2}+2kN+2s \mid k\in\mathbb{N}, s\in\{0,2,\ldots,N-2\}\}$	$(2kN+2s+2)(k+1)$
$\{\frac{3}{2}+2kN+2s \mid k\in\mathbb{N}, s\in\{1,3,\ldots,N-1\}\}$	$(2kN+2s+2)k$
$\{-\frac{5}{2}-2kN-2s \mid k\in\mathbb{N}, s\in\{0,2,\ldots,N-2\}\}$	$(2kN+2s+2)k$
$\{-\frac{5}{2}-2kN-2s \mid k\in\mathbb{N}, s\in\{1,3,\ldots,N-3\}\}$	$(2kN+2s+2)(k+1)$
$\{-\frac{1}{2}-2kN-2N \mid k\in\mathbb{N}\}$	$(2kN+2N)(k+2)$

. (4.55)

At this point it is easy to check that the first two rows of 4.55 describe the entire spectrum if k takes values in all of $\mathbb{Z}$, in which case the second row accounts for the third row, and the first row accounts for the fourth and fifth rows. Writing s alternately as $2t$ and $2t+1$ we obtain the following definitive form of the spectrum in the even case.

Corollary 4.7.1 *If N is even then the Dirac operator on the dicyclic space $SU(2)/\Gamma$, where Γ is the dicyclic group of order $4N$, $N \geq 2$, equipped with the round metric and trivial spin structure has the following spectrum:*

λ	*multiplicity*
$\{\frac{3}{2}+2kN+4t \mid k\in\mathbb{Z}, t\in\{0,1,\ldots,\frac{N-2}{2}\}\}$	$(2kN+4t+2)(k+1)$
$\{\frac{7}{2}+2kN+4t \mid k\in\mathbb{Z}, t\in\{0,1,\ldots,\frac{N-2}{2}\}\}$	$(2kN+4t+4)k$

.

N odd

By writing k alternately as $2a$, and $2a+1$, and also by writing s alternately as $2t$ and $2t+1$, the first two rows of 4.53 may be written respectively as

λ	multiplicity
$\{\frac{3+8aN+8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-1}{2}\}\}$	$(4aN+4t+2)(2a)$
$\{\frac{7+8aN+8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$(4aN+4t+4)(2a)$
$\{\frac{3+8aN+4N+8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-1}{2}\}\}$	$(4aN+2N+4t+2)(2a+1)$
$\{\frac{7+8aN+4N+8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$(4aN+2N+4t+4)(2a+1)$
$\{\frac{-5-8aN-8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$(4aN+4t+2)(2a)$
$\{\frac{-1-8aN-4N}{2} \mid a \in \mathbb{N}\}$	$4a(1+2a)N$
$\{\frac{-9-8aN-8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$(4aN+4t+4)(2a)$
$\{\frac{-5-8aN-4N-8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$(4aN+2N+4t+2)(2a+1)$
$\{\frac{-1-8aN-8N}{2} \mid a \in \mathbb{N}\}$	$4(1+a)(1+2a)N$
$\{\frac{-9-8aN-4N-8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$(4aN+2N+4t+4)(2a+1)$

. (4.56)

We have separated out the case $t = \frac{N-1}{2}$ from the fifth and eighth rows and given them their own rows to make it clear how this case combines with the other rows.

Next, we analyze the third and fourth rows of 4.53 each into N arithmetic progressions, and then separate each set of progressions into two groups. Doing this we obtain the table

λ	multiplicity
$\{\frac{3+8aN+8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-1}{2}\}\}$	$4aN+4t+2$
$\{\frac{7+8aN+4N+8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$4aN+2N+4t+4$
$\{\frac{-9-8aN-8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$4aN+4t+4$
$\{\frac{-5-8aN-4N-8t}{2} \mid a \in \mathbb{N}, t \in \{0,\ldots,\frac{N-3}{2}\}\}$	$4aN+2N+4t+2$
$\{\frac{-1-8aN-8N}{2} \mid a \in \mathbb{N}\}$	$4aN+4N$

. (4.57)

Again, we separated out the case $t = \frac{N-1}{2}$ from the fourth row so that the rows combine simply.

The fifth row of 4.53 can be decomposed into two parts yielding

λ	multiplicity
$\{\frac{-1-8aN-4N}{2} \mid a \in \mathbb{N}\}$	$4aN+2N$
$\{\frac{-1-8aN-8N}{2} \mid a \in \mathbb{N}\}$	$4aN+4N$

(4.58)

At this point, the rows have been decomposed into commensurable parts. Rows with the same value of λ can be combined by summing the multiplicities together. Once this is done, it is easy to check that rows coming from the positive spectrum combine perfectly with rows coming from the negative spectrum just as in the case of lens spaces. One checks this by making the variable substitutions $a = -a' - 1$, $t + t' = \frac{N-3}{2}$, (with the case $t = t' = \frac{N-1}{2}$ being a special case which is also easy to check), and allowing the variable a to run through all of $\mathbb{Z}$.

We now have the definitive form of the spectrum in the odd case.

Corollary 4.7.2 *If N is odd then the Dirac operator on the dicyclic space $SU(2)/\Gamma$, where Γ is the dicyclic group of order $4N$, $N \geq 2$, equipped with the round metric and trivial spin structure, has the following spectrum:*

λ	*multiplicity*
$\{\frac{3+8aN+8t}{2} \mid a \in \mathbb{Z}, t \in \{0, \ldots, \frac{N-1}{2}\}\}$	$(4aN + 4t + 2)(2a + 1)$
$\{\frac{7+8aN+8t}{2} \mid a \in \mathbb{Z}, t \in \{0, \ldots, \frac{N-3}{2}\}\}$	$(4aN + 4t + 4)(2a)$
$\{\frac{3+8aN+4N+8t}{2} \mid a \in \mathbb{Z}, t \in \{0, \ldots, \frac{N-1}{2}\}\}$	$(4aN + 2N + 4t + 2)(2a + 1)$
$\{\frac{7+8aN+4N+8t}{2} \mid a \in \mathbb{Z}, t \in \{0, \ldots, \frac{N-3}{2}\}\}$	$(4aN + 2N + 4t + 4)(2a + 2)$

. (4.59)

4.7.2 Computing the spectral action

To compute the spectral action, in the even case, one observes that the two rows of Corollary 4.7.1 can respectively be interpolated by the polynomials

$$P_t(u) = \frac{1}{2} - \frac{3}{8N} - \frac{t}{N} + u - \frac{u}{2N} - \frac{2tu}{N} + \frac{u^2}{2N}$$
$$Q_t(u) = \frac{-7}{8N} - \frac{t}{N} - \frac{3u}{2N} - \frac{2tu}{N} + \frac{u^2}{2N}.$$

One has the identity

$$\sum_{t=0}^{\frac{N-1}{2}} P_t(u) + Q_t(u) = -\frac{1}{8} + \frac{u^2}{2}. \tag{4.60}$$

In the odd case, the rows of Corollary 4.7.2 are interpolated respectively by the polynomials

$$P_t(u) = \frac{1}{2} - \frac{3}{8N} - \frac{t}{N} + u - \frac{u}{2N} - \frac{2tu}{N} + \frac{u^2}{2N}$$
$$Q_t(u) = \frac{-7}{8N} - \frac{t}{N} - \frac{3u}{2N} - \frac{2tu}{N} + \frac{u^2}{2N}$$
$$R_t(u) = -\frac{3}{8N} - \frac{t}{N} - \frac{u}{2N} - \frac{2tu}{N} + \frac{u^2}{2N}$$
$$S_t(u) = \frac{1}{2} - \frac{7}{8N} - \frac{t}{N} + u - \frac{3u}{2N} - \frac{2tu}{N} + \frac{u^2}{2N}.$$

We have the identity

$$\sum_{t=0}^{\frac{N-1}{2}} P_t(u) + \sum_{t=0}^{\frac{N-3}{2}} Q_t(u) + \sum_{t=0}^{\frac{N-1}{2}} R_t(u) + \sum_{t=0}^{\frac{N-3}{2}} S_t(u) = -\frac{1}{4} + u^2. \tag{4.61}$$

Therefore using identities 4.60 and 4.61, and Lemma 4.5.3 we have:

Theorem 4.7.3 *The spectral action for round dicyclic space with the trivial spin structure is given for each $N \geq 2$ by*

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{4N}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}). \tag{4.62}$$

4.8 Generating function method

When Γ is the binary tetrahedral, binary octahedral, or binary icosahedral group, it becomes difficult to determine $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$, so we turn to another method to compute the Dirac spectrum, which we presently review. The key results, taken from [4], are presented here for convenience. A similar discussion was presented in [50].

In this case, we only consider the round metric on S^n. Let Γ be a finite fixed point free subgroup of $SO(n+1)$, acting as usual on $S^n \subset \mathbb{R}^{n+1}$. The spin structures of S^n/Γ are in one-to-one correspondence with homomorphisms

$$\epsilon : \Gamma \to \mathrm{Spin}(n+1) \tag{4.63}$$

which lift the inclusion

$$\iota : \Gamma \to SO(n+1) \tag{4.64}$$

with respect to the double cover

$$\Theta : \mathrm{Spin}(n+1) \to SO(n+1). \tag{4.65}$$

That is, homomorphisms ϵ such that $\iota = \Theta \circ \epsilon$.

Let $M = S^n/\Gamma$, equipped with spin structure ϵ. Note that we may assume that n is odd, since when n is even, the only nontrivial possibility for M is $\mathbb{RP}^n$, which is not a spin manifold. Let D be the Dirac operator on M. The Dirac spectrum of S^n equipped with the round metric is the set

$$\{\pm(n/2+k) | k \in \mathbb{N}\}. \tag{4.66}$$

The spectrum of D is a subset of the spectrum of S^n, and the multiplicities of the eigenvalues are in general smaller. Let $m(a, D)$ denote the multiplicity of $a \in \mathbb{R}$ of D. One defines formal power series $F_+(z)$, $F_-(z)$ according to

$$F_+(z) = \sum_{k=0}^{\infty} m\left(\frac{n}{2}+k, D\right) z^k, \tag{4.67}$$

$$F_-(z) = \sum_{k=0}^{\infty} m\left(-\left(\frac{n}{2}+k\right), D\right)\right) z^k. \tag{4.68}$$

Using the fact that the multiplicities of D are majorized by the multiplicities of the Dirac spectrum of S^n, one may show that these power series converge absolutely for $|z| < 1$.

The complex spinor representation of $\mathrm{Spin}(2m)$ decomposes into two irreducible representations, ρ_+, ρ_- called the half-spin representations. Let $\chi^{\pm}$ be the characters of these two representations. The key result is the following.

Theorem 4.8.1 ([4], Theorem 2) *With the notation as above, we have the identities*

$$F_+(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^-(\epsilon(\gamma)) - z \cdot \chi^+(\epsilon(\gamma))}{\mathrm{Det}(\mathrm{I_{2m} - z \cdot \gamma})} \tag{4.69}$$

$$F_-(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^+(\epsilon(\gamma)) - z \cdot \chi^-(\epsilon(\gamma))}{\mathrm{Det}(\mathrm{I_{2m} - z \cdot \gamma})}. \tag{4.70}$$

One may identify $SU(2)$ with the set of unit quaternions, and choose $\{1, i, j, k\}$ to be an ordered basis of $\mathbb{R}^4$, then via the action of Γ on $SU(2)$ by left multiplication one may identify the unit quaternion

$$a + bi + cj + dk \in \Gamma \tag{4.71}$$

with the matrix in $SO(4)$

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}. \tag{4.72}$$

4.8.1 The double cover $\mathrm{Spin}(4) \to SO(4)$

The text in this section is reproduced with slight modification from [50].

Let us recall some facts about the double cover $\mathrm{Spin}(4) \to SO(4)$. Let $S^3_L \simeq SU(2)$ left isoclinic rotations:

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix},$$

where $a^2 + b^2 + c^2 + d^2 = 1$. Similarly, let $S^3_R \simeq SU(2)$ be the group of right isoclinic rotations:

$$\begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix},$$

where

$$p^2 + q^2 + r^2 + s^2 = 1. \tag{4.73}$$

Then

$$\mathrm{Spin}(4) \simeq S^3_L \times S^3_R, \tag{4.74}$$

and the double cover

$$\Theta : \mathrm{Spin}(4) \to SO(4) \tag{4.75}$$

is given by

$$(A, B) \mapsto A \cdot B, \tag{4.76}$$

where $A \in S^3_L$, and $B \in S^3_R$. The complex half-spin representation ρ^+ is just the projection onto S^3_L, where we identify S^3_L with $SU(2)$ via

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \mapsto \begin{pmatrix} a-bi & d+ci \\ -d+ci & a+bi \end{pmatrix}.$$

The other complex half-spin representation ρ^- is the projection onto S^3_R, where we identify S^3_R with $SU(2)$ via

$$\begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix}^t \mapsto \begin{pmatrix} p-qi & s+ri \\ -s+ri & p+qi \end{pmatrix}.$$

In this paper, when Γ is the binary tetrahedral group, binary octahedral group, or binary icosahedral group, we choose the spin structure corresponding to

$$\epsilon : \Gamma \to \mathrm{Spin}(4), \tag{4.77}$$

$$A \mapsto (A, I_4), \tag{4.78}$$

and we call this the trivial spin structure. It is obvious that ϵ lifts the identity map, and hence that it corresponds to a spin structure.

4.9 Dirac spectrum of round binary tetrahedral coset space

Let $2T$ denote the binary tetrahedral group of order 24. Concretely, as a set of unit quaternions, this group may be written as

$$\left\{ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k) \right\}, \tag{4.79}$$

where every possible combination of signs is used in the final term.

Theorem 2 of [4] provides formulae for generating functions whose Taylor coefficients about $z = 0$ give the multiplicities for the Dirac spectra of spherical space forms. Using these formulae we obtain the following generating functions for the Dirac spectra of $S^3/2T$:

$$F_+(z) = -\frac{2(1 + z^2 - z^4 + z^6 + 7z^8 + 3z^{10})}{(-1 + z^2)^3(1 + 2z^2 + 2z^4 + z^6)^2},$$

$$F_-(z) = -\frac{2z^5(3 + 7z^2 + z^4 - z^6 + z^8 + z^{10})}{(-1 + z^2)^3(1 + 2z^2 + 2z^4 + z^6)^2}.$$

The kth Taylor coefficient of $F_\pm(z)$ at $z = 0$ equals the multiplicity of the eigenvalue

$$\lambda = \pm\left(\frac{3}{2} + k\right) \tag{4.80}$$

of the Dirac operator of the coset space $S^3/2T$.

The Taylor coefficients of a rational function satisfy a recurrence relation. Using this recurrence relation, one may show by induction that the multiplicity of

$$u = 3/2 + k + 12n, \quad n \in \mathbb{Z}, \tag{4.81}$$

is given by $P_k(u)$, where P_k, $k = 0, 1, 2, \ldots, 11$ are the polynomials

$$\begin{aligned}
P_k(u) &= 0, \text{if } k \text{ is odd} \\
P_0(u) &= \frac{7}{16} + \frac{11}{12}u + \frac{1}{12}u^2 \\
P_2(u) &= -\frac{7}{48} - \frac{3}{12}u + \frac{1}{12}u^2 \\
P_4(u) &= -\frac{11}{48} - \frac{5}{12}u + \frac{1}{12}u^2 \\
P_6(u) &= \frac{9}{48} + \frac{5}{12}u + \frac{1}{12}u^2 \\
P_8(u) &= \frac{5}{48} + \frac{1}{4}u + \frac{1}{12}u^2 \\
P_{10}(u) &= -\frac{23}{48} - \frac{11}{12}u + \frac{1}{12}u^2.
\end{aligned}$$

Let

$$F_+(z) = \sum_{k=0}^{\infty} a_k z^k \tag{4.82}$$

be the series expansion for F_+ about $z = 0$. Clearly,

$$P_k(3/2 + k + 12n) = a_{k+12n} \tag{4.83}$$

for each n if and only if

$$P_k(3/2 + k + 12(n+1)) - P_k(3/2 + k + 12n) = a_{k+12(n+1)} - a_{k+12n} \tag{4.84}$$

for each n and

$$P_k(3/2 + k + 12n) = a_{k+12n} \tag{4.85}$$

for some n. Now, let

$$\sum_{j=m}^{P} b_j z^j, \sum_{j=0}^{M} c_j z^j \tag{4.86}$$

be the numerator and denominator respectively of the rational function $F_+(z)$. Then for each k, one has the recurrence relation

$$b_k = \sum_{j=0}^{M} a_{k-j} c_j. \tag{4.87}$$

In particular, for each $k > P$, we have

$$\sum_{j=0}^{M} a_{k-j} c_j = 0, \tag{4.88}$$

and hence also

$$\sum_{j=0}^{M} (a_{k+12-j} - a_{k-j}) c_j = 0. \tag{4.89}$$

We don't need to worry about the smaller values of k since we can simply check those by hand. Therefore to verify that

$$P_k(3/2 + k + 12n) = a_{k+12n} \tag{4.90}$$

for each k and n, one simply verifies that the values $P_k(3/2 + k + 12n)$ satisfy the same recurrence relation as a_{k+12n} and checks that these two values are equal for small values of n. The recurrence relation is given by

$$P_{[k]}(3/2+k+12(n+1)) - P_{[k]}(3/2+k+12n) =$$
$$\frac{-1}{c_0}\sum_{j=1}^{M}(P_{[k-j]}(3/2+k+12(n+1)-j) - P_{[k-j]}(3/2+k+12n-j))c_j,$$

for each k from 0 to 11, and this can be checked by direct evaluation of the polynomials. Once one also verifies that the polynomials interpolate the spectrum for small eigenvalues, then by induction (12 inductions in parallel) one has shown that the polynomials interpolate the part of the spectrum in the positive reals. Here $[a]$ is the integer from 0 to 11 to which a is equivalent modulo 12.

The polynomials P_k also interpolate that part of the spectrum in the negative reals. To verify this, one checks a recurrence relation much like the above except derived from F_- instead of F_+. Namely, if the numerator and denominator of F_- are given respectively by

$$\sum_{j=m}^{P} b_j z^j, \sum_{j=0}^{M} c_j z^j \tag{4.91}$$

then the recurrence relation for the polynomials P_k that must be checked is

$$P_{[k]'}(-3/2-k-12(n+1)) - P_{[k]'}(-3/2-k-12n) =$$
$$\frac{-1}{c_0}\sum_{j=0}^{M}(P_{[k-j]'}(-3/2-k+j-12(n+1)) - P_{[k-j]'}(-3/2-k+j-12n))c_j,$$

which again can be checked by direct evaluation. Here $[a]'$ is the number between 0 and 11 such that $[a]+[a]'+3$ is a multiple of 12. This ensures that we have the set equality

$$\{-3/2-k'-12n | n \in \mathbb{Z}\} = \{3/2+k+12n | n \in \mathbb{Z}\}, \tag{4.92}$$

which is clearly the condition we need to have in order for the polynomials P_k to interpolate the entire spectrum.

By this procedure we have:

Proposition 4.9.1 *For the round binary tetrahedral space with the trivial spin structure the spectrum of the canonical Dirac operator D is contained in the set $\{\pm(3/2+k) | k \in \mathbb{N}\}$.*

The multiplicity of $3/2+k+12t$*, where* $k \in \{0,1,\dots 11\}$ *and* $t \in \mathbb{Z}$*, is equal to*

$$P_k(3/2+k+12t). \tag{4.93}$$

Now observe that

$$\sum_{k=0}^{11} P_k(u) = \frac{1}{2}(u^2 - 1/4). \tag{4.94}$$

Therefore by Lemma 4.5.3 we have computed the spectral action of the binary tetrahedral coset space.

Theorem 4.9.2 *The spectral action of the binary tetrahedral coset space is given by*

$$\frac{1}{24}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}). \tag{4.95}$$

4.10 Dirac spectrum of round binary octahedral coset space

Let $2O$ be the binary octahedral group of order 48. Binary octahedral space is the space $SU(2)/2O$. It consists of the 24 elements of the binary tetrahedral group, (4.79), as well as the 24 elements obtained from

$$\frac{1}{\sqrt{2}}(\pm 1 \pm i + 0j + 0k), \tag{4.96}$$

by permuting the coordinates and taking all possible sign combinations.

The generating functions are

$$F_+(z) = -\frac{2(1+z^2+z^4-z^6+2z^8+2z^{10}+10z^{12}+4z^{14}+4z^{16})}{(-1+z^2)^3(1+2z^2+3z^4+3z^6+2z^8+z^{10})^2},$$

and

$$F_-(z) = -\frac{2z^7(4+4z^2+10z^4+2z^6+2z^8-z^{10}+z^{12}+z^{14}+z^{16})}{(-1+z^2)^3(1+2z^2+3z^4+3z^6+2z^8+z^{10})^2}.$$

We define polynomials $P_k(u)$, $k = 0, 1, 2, \ldots, 23$, where

$$
\begin{aligned}
P_k(u) &= 0, \text{if } k \text{ is odd}, \\
P_0(u) &= \frac{15}{32} + \frac{23}{24}u + \frac{1}{24}u^2, \\
P_2(u) &= -\frac{7}{96} - \frac{1}{8}u + \frac{1}{24}u^2, \\
P_4(u) &= -\frac{11}{96} - \frac{5}{24}u + \frac{1}{24}u^2, \\
P_6(u) &= -\frac{5}{32} - \frac{7}{24}u + \frac{1}{24}u^2, \\
P_8(u) &= \frac{29}{96} + \frac{5}{8}u + \frac{1}{24}u^2, \\
P_{10}(u) &= -\frac{23}{96} - \frac{11}{24}u + \frac{1}{24}u^2, \\
P_{12}(u) &= \frac{7}{32} + \frac{11}{24}u + \frac{1}{24}u^2, \\
P_{14}(u) &= -\frac{31}{96} - \frac{5}{8}u + \frac{1}{24}u^2, \\
P_{16}(u) &= \frac{13}{96} + \frac{7}{24}u + \frac{1}{24}u^2, \\
P_{18}(u) &= \frac{3}{32} + \frac{5}{24}u + \frac{1}{24}u^2, \\
P_{20}(u) &= \frac{5}{96} + \frac{1}{8}u + \frac{1}{24}u^2, \\
P_{22}(u) &= -\frac{47}{96} - \frac{23}{24}u + \frac{1}{24}u^2.
\end{aligned}
$$

One uses the procedure of section 4.9 to show the following.

Proposition 4.10.1 *For the round binary octahedral space with the trivial spin structure the spectrum of the canonical Dirac operator D is contained in the set $\{\pm(3/2 + k) | k \in \mathbb{N}\}$. The multiplicity of $3/2 + k + 12t$, where $k \in \{0, 1, \ldots 11\}$ and $t \in \mathbb{Z}$, is equal to*

$$P_k(3/2 + k + 24t). \tag{4.97}$$

The sum of the polynomials is

$$\sum_{k=0}^{23} P_k(u) = \frac{1}{2}(u^2 - 1/4). \tag{4.98}$$

By Lemma 4.5.3, we have the following.

Theorem 4.10.2 *The spectral action of the binary octahedral coset space is given by*

$$\frac{1}{48}\left(\Lambda^3\widehat{f}^{(2)}(0)-\frac{1}{4}\Lambda\widehat{f}(0)\right)+O(\Lambda^{-\infty}). \tag{4.99}$$

4.11 Dirac spectrum of round Poincaré homology sphere

When Γ is the binary icosahedral group the space $SU(2)/\Gamma$ is known as the Poincaré homology sphere.

This case was discussed in [50]. Unfortunately, the expressions for the generating functions $F_+(z)$, $F_-(z)$, and the interpolating polynomials P_k in [50] are incorrect. They are necessarily incorrect because they imply that the spectrum of the Poincaré homology sphere is not a subset of the spectrum of binary tetrahedral space, which is a contradiction, since the binary tetrahedral group is a subgroup of the binary icosahedral group. However, the expression for the spectral action in [50] is correct, which was the only thing that was used in the rest of [50], and so the rest of the paper is unaffected. The correct expressions are found below.

Let $S=SU(2)/\Gamma$ be the Poincaré homology sphere, with the spin structure ϵ described here above. The generating functions for the spectral multiplicities of the Dirac operator are

$$F_+(z)= \\ -\frac{2(1+3z^2+4z^4+2z^6-2z^8-6z^{10}-2z^{12}+12z^{14}+24z^{16}+18z^{18}+6z^{20})}{(-1+z^2)^3(1+2z^2+2z^4+z^6)^2(1+z^2+z^4+z^6+z^8)^2} \tag{4.100}$$

and

$$F_-(z)= \\ -\frac{2z^{11}(6+18z^2+24z^4+12z^6-2z^8-6z^{10}-2z^{12}+2z^{14}+4z^{16}+3z^{18}+z^{20})}{(-1+z^2)^3(1+2z^2+2z^4+z^6)^2(1+z^2+z^4+z^6+z^8)^2}. \tag{4.101}$$

In order to compute the spectral action, we proceed as in the previous cases by finding interpolating polynomials. Using the procedure of section 4.9 we obtain the following result.

Proposition 4.11.1 *There are polynomials $P_k(u)$, for $k=0,\dots,59$, so that $P_k(3/2+k+$*

$60j) = m(3/2+k+60j, D)$ *for all* $j \in \mathbb{Z}$*. The* $P_k(u)$ *are given as follows:*

$$P_k = 0, \quad \text{whenever } k \text{ is odd},$$
$$P_0(u) = \frac{39}{80} + \frac{59}{60}u + \frac{1}{60}u^2,$$
$$P_2(u) = -\frac{7}{240} - \frac{1}{20}u + \frac{1}{60}u^2,$$
$$P_4(u) = -\frac{11}{240} - \frac{1}{12}u + \frac{1}{60}u^2,$$
$$P_6(u) = -\frac{1}{16} - \frac{7}{60}u + \frac{1}{60}u^2,$$
$$P_8(u) = -\frac{19}{240} - \frac{3}{20}u + \frac{1}{60}u^2,$$

$$P_{10}(u) = -\frac{23}{240} - \frac{11}{60}u + \frac{1}{60}u^2,$$
$$P_{12}(u) = \frac{31}{80} + \frac{47}{60}u + \frac{1}{60}u^2,$$
$$P_{14}(u) = -\frac{31}{240} - \frac{1}{4}u + \frac{1}{60}u^2,$$
$$P_{16}(u) = -\frac{7}{48} - \frac{17}{60}u + \frac{1}{60}u^2,$$
$$P_{18}(u) = -\frac{13}{80} - \frac{19}{60}u + \frac{1}{60}u^2,$$

$$P_{20}(u) = \frac{77}{240} + \frac{13}{20}u + \frac{1}{60}u^2,$$
$$P_{22}(u) = -\frac{47}{240} - \frac{23}{60}u + \frac{1}{60}u^2,$$
$$P_{24}(u) = \frac{23}{80} + \frac{7}{12}u + \frac{1}{60}u^2,$$
$$P_{26}(u) = -\frac{11}{48} - \frac{9}{20}u + \frac{1}{60}u^2,$$
$$P_{28}(u) = -\frac{59}{240} - \frac{29}{60}u + \frac{1}{60}u^2,$$

$$P_{30}(u) = \frac{19}{80} + \frac{29}{60}u + \frac{1}{60}u^2,$$
$$P_{32}(u) = \frac{53}{240} + \frac{9}{20}u + \frac{1}{60}u^2,$$
$$P_{34}(u) = -\frac{71}{240} - \frac{7}{12}u + \frac{1}{60}u^2,$$
$$P_{36}(u) = \frac{3}{16} + \frac{23}{60}u + \frac{1}{60}u^2,$$
$$P_{38}(u) = -\frac{79}{240} - \frac{13}{20}u + \frac{1}{60}u^2,$$

$$P_{40}(u) = -\frac{37}{240} + \frac{19}{60}u + \frac{1}{60}u^2,$$
$$P_{42}(u) = \frac{11}{80} + \frac{17}{60}u + \frac{1}{60}u^2,$$
$$P_{44}(u) = \frac{29}{240} + \frac{1}{4}u + \frac{1}{60}u^2,$$
$$P_{46}(u) = -\frac{19}{48} - \frac{47}{60}u + \frac{1}{60}u^2,$$
$$P_{48}(u) = \frac{7}{80} + \frac{11}{60}u + \frac{1}{60}u^2,$$

$$P_{50}(u) = \frac{17}{240} + \frac{3}{20}u + \frac{1}{60}u^2,$$
$$P_{52}(u) = \frac{13}{240} + \frac{7}{60}u + \frac{1}{60}u^2,$$
$$P_{54}(u) = \frac{3}{80} + \frac{1}{12}u + \frac{1}{60}u^2,$$
$$P_{56}(u) = \frac{1}{48} + \frac{1}{20}u + \frac{1}{60}u^2,$$
$$P_{58}(u) = -\frac{119}{240} - \frac{59}{60}u + \frac{1}{60}u^2.$$

These are computed directly from the Taylor coefficients of the generating functions of the spectral multiplicities (4.100) and (4.101). Notice that

$$\sum_{j=0}^{59} P_j(u) = \frac{1}{2}u^2 - \frac{1}{8}. \tag{4.102}$$

Once again, using Lemma 4.5.3 we obtain the nonperturbative spectral action for the Poincaré homology sphere.

Theorem 4.11.2 *Let D be the Dirac operator on the Poincaré homology sphere $S = S^3/\Gamma$, with the trivial spin structure and round metric. Then the spectral action is given by*

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{120}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}). \tag{4.103}$$

Chapter 5

Twisted Dirac Operators

5.1 Introduction

Following the method developed in [16] and [50], [73], we compute the spectral action of the quotient spaces S^3/Γ equipped with the twisted Dirac operator corresponding to a finite-dimensional representation α of Γ as follows. We define a finite set of polynomials labeled P_m^+, and P_m^- which describe the multiplicities of, respectively, the positive and negative eigenvalues of the twisted Dirac operator, in the sense that $P_m^{\pm}(u)(\lambda)$ equals the multiplicity of the eigenvalue

$$\lambda = -1/2 \pm (k+1),\ \ k \geq 1 \tag{5.1}$$

whenever $k \equiv m \bmod \mathrm{c}_\Gamma$, where c_Γ is the exponent of the group Γ, the least common multiple of the orders of the elements in Γ.

The main technical result we will prove is the following relation between these polynomials:

$$\sum_{m=1}^{c_\Gamma} P_m^+(u) = \sum_{m=0}^{c_\Gamma - 1} P_m^-(u) = \frac{N c_\Gamma}{\#\Gamma}\left(u^2 - \frac{1}{4}\right). \tag{5.2}$$

Since the polynomial on the right-hand-side is a multiple of the polynomial for the spectral multiplicities of the Dirac spectrum of the sphere S^3 (see [16]), we will obtain from this the relation between the non-perturbative spectral action of the twisted Dirac operator D_α^Γ on S^3/Γ and the spectral action on the sphere, see Theorem 5.1.1 below.

Furthermore, we shall show that the polynomials $P_m^+(u)$ match up perfectly with the polynomials $P_m^-(u)$, so that the polynomials $P_m^+(u)$ alone describe the entire spectrum by allowing the parameter k in equation 5.1 to run through all of $\mathbb{Z}$. Namely, what we need to show is that

$$P_m^+(u) = P_{m'}^-(u), \tag{5.3}$$

where for each m, m' is the unique number between 0 and $c_\Gamma - 1$ such that $m + m' + 2$ is a multiple of c_Γ. To be more precise,

$$m' = \begin{cases} c_\Gamma - 2 - m, & \text{if } 1 \le m \le c_\Gamma - 2 \\ c_\Gamma - 1, & \text{if } m = c_\Gamma - 1 \\ c_\Gamma - 2 & \text{if } m = c_\Gamma \end{cases}. \tag{5.4}$$

Define

$$g_m(u) = P_m^+(u) f(u/\Lambda). \tag{5.5}$$

Now, we apply the Poisson summation formula, to obtain,

$$\begin{aligned} \mathrm{Tr}(f(D/\Lambda)) &= \sum_m \sum_{l \in \mathbb{Z}} g_m(1/2 + c_\Gamma l + m + 1) \\ &= \frac{N}{\#\Gamma} \sum_m \widehat{g_m}(0) + O(\Lambda^{-\infty}) \\ &= \frac{N}{\#\Gamma} \left(\int_{\mathbb{R}} u^2 f(u/\Lambda) - \frac{1}{4} \int_{\mathbb{R}} f(u/\Lambda) \right) + O(\Lambda^{-\infty}) \\ &= \frac{N}{\#\Gamma} \left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4} \Lambda \widehat{f}(0) \right) + O(\Lambda^{-\infty}), \end{aligned}$$

and so we have the main result.

Theorem 5.1.1 *Let Γ be a finite subgroup of S^3, and let α be a N-dimensional representation of Γ. Then the spectral action of S^3/Γ equipped with the twisted Dirac operator is*

$$\mathrm{Tr} f(D/\Lambda) = \frac{N}{|\Gamma|} \left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4} \Lambda \widehat{f}(0) \right) + O(\Lambda^{-\infty}). \tag{5.6}$$

Here $\widehat{f}^{(2)}$ denotes the Fourier transform of $u^2 f(u)$.

Similar computations of the spectral action have also been performed in [50], [51], and [73].

In the sequel we describe how to obtain equation (5.2), by explicitly analyzing the cases of the various spherical space forms: lens spaces, dicyclic group, and binary tetrahedral, octahedral, and icosahedral groups. In all cases we compute explicitly the polynomials of the spectral multiplicities and check that (5.2) is satisfied. Our calculations are based on a result of Cisneros-Molina, [19], on the explicit form of the Dirac spectra of the twisted Dirac operators D_α^Γ, which we recall here below.

5.2 Twisted Dirac spectra of spherical space forms

The spectra of the twisted Dirac operators on the quotient spaces are derived in [19]. Let us recall the notation and the main results.

Let E_k denote the $k+1$-dimensional irreducible representation of $SU(2)$ on the space of homogeneous complex polynomials in two variables of degree k. By the Peter–Weyl theorem, one can decompose $C^\infty(S^3,\mathbb{C}) = \oplus_k E_k \otimes E_k^*$ as a sum of irreducible representations of $SU(2)$. This gives that, on $C^\infty(S^3,\mathbb{C}^2\otimes\mathbb{C}^N) = \oplus_k E_k \otimes E_k^* \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$, the Dirac operator $D \otimes id_{\mathbb{C}^N}$ decomposes as $\oplus_k \mathrm{id}_{E_k} \otimes D_k \otimes id_{\mathbb{C}^N}$, with $D_k : E_k^* \otimes \mathbb{C}^2 \to E_k^* \otimes \mathbb{C}^2$. Upon identifying $C^\infty(S^3,\mathbb{C}^2\otimes\mathbb{C}^N)^\Gamma = \oplus_k E_k \otimes \mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2\otimes\mathbb{C}^N)$, one sees that, as shown in [19], the multiplicities of the spectrum of the twisted Dirac operator D_α^Γ are given by the dimensions $\dim_{\mathbb{C}} \mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2\otimes\mathbb{C}^N)$, which in turn can be expressed in terms of the pairing of the characters of the corresponding Γ-representation, that is, as $\langle \chi_{E_k}, \chi_{\sigma\otimes\alpha}\rangle_\Gamma$. One then obtains the following:

Theorem 5.2.1 (Cisneros-Molina, [19]) *Let $\alpha : \Gamma \to GL_N(\mathbb{C})$ be a representation of Γ. Then the eigenvalues of the twisted Dirac operator D_α^Γ on S^3/Γ are*

$$
\begin{aligned}
-\frac{1}{2} - (k+1) \ &\textit{with multiplicity } \langle \chi_{E_{k+1}}, \chi_\alpha\rangle_\Gamma (k+1), \qquad && k \geq 0,\\
-\frac{1}{2} + (k+1) \ &\textit{with multiplicity } \langle \chi_{E_{k-1}}, \chi_\alpha\rangle_\Gamma (k+1), \qquad && k \geq 1.
\end{aligned}
$$

Proposition 5.2.2 (Cisneros-Molina, [19]) *Let $k = c_\Gamma l + m$ with $0 \leq m < c_\Gamma$.*

1. If $-1 \in \Gamma$, then

$$\langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma = \begin{cases} \frac{c_\Gamma l}{|\Gamma|}(\chi_\alpha(1) + \chi_\alpha(-1)) + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma & \textit{if } k \textit{ is even} \\ \frac{c_\Gamma l}{|\Gamma|}(\chi_\alpha(1) - \chi_\alpha(-1)) + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma & \textit{if } k \textit{ is odd.} \end{cases}$$

2. If $-1 \notin \Gamma$, then

$$\langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma = \frac{N c_\Gamma l}{\#\Gamma} + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma.$$

5.3 Lens spaces, odd order

In this section we consider $\Gamma = \mathbb{Z}_n$, where n is odd. When n is odd, $-1 \notin \Gamma$, which affects the expression for the character inner products in Proposition 5.2.2.

For $m \in \{1, \ldots, n\}$, we introduce the polynomials,

$$P_m^+(u) = \frac{N}{n}u^2 + (\beta_m^\alpha - \frac{mN}{n})u + \frac{\beta_m^\alpha}{2} - \frac{mN}{2n} - \frac{N}{4n},$$

where

$$\beta_m^\alpha = \langle \chi_{E_{m-1}}, \chi_\alpha \rangle_\Gamma,$$

and m takes on values in $\{1, 2, \ldots, n\}$.

Using Theorem 5.2.1 and Proposition 5.2.2, it is easy to see that the polynomials $P_m^+(u)$ describe the spectrum on the positive side of the real line, in the sense that $P_m^+(u)(\lambda)$ equals the multiplicity of the eigenvalue

$$\lambda = -1/2 + (k+1), \ k \geq 1$$

whenever $k \equiv m$ mod n.

For the negative eigenvalues, the multiplicities are described by the polynomials

$$P_m^-(u) = \frac{N}{n}u^2 + \left(\frac{2N}{n} + \frac{mN}{n} - \gamma_m^\alpha\right)u + \frac{3N}{4n} + \frac{mN}{2n} - \frac{\gamma_m^\alpha}{2},$$

$m \in \{0, 1, \ldots n-1\}$, in the sense that $P_m^-(u)(\lambda)$ equals the multiplicity of the eigenvalue

$$\lambda = -1/2 - (k+1), \ k \geq 0$$

whenever $k \equiv m$ mod n. Here γ_m^α is defined by

$$\gamma_m^\alpha = \langle \chi_{E_{m+1}}, \chi_\alpha \rangle_\Gamma.$$

Let us denote the irreducible representations of $\mathbb{Z}_n$ by χ_t, sending the generator to $\exp(\frac{2\pi i t}{N})$. Here t is a residue class of integers modulo n.

For the sake of computation, we take $\mathbb{Z}_n$ to be the group generated by

$$B = \begin{bmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{bmatrix}.$$

Then in the representation E_k, B acts on the basis polynomials $P_j(z_1, z_2)$, $j \in \{0, 1, \dots k\}$ as follows:

$$\begin{aligned} B \cdot P_j(z_1, z_2) &= P_j\left((z_1, z_2)B\right) \\ &= P_j(e^{\frac{2\pi i}{n}} z_1, e^{-\frac{2\pi i}{n}} z_2) \\ &= (e^{\frac{2\pi i}{n}} z_1)^{k-j}(e^{-\frac{2\pi i}{n}} z_2)^j \\ &= e^{\frac{2\pi i}{n}(k-2j)} P_j(z_1, z_2). \end{aligned}$$

Hence, B is represented by a diagonal matrix with respect to this basis, and we have:

Proposition 5.3.1 *The irreducible characters χ_{E_k} of the irreducible representations of $SU(2)$ restricted to $\mathbb{Z}_n$, n odd, are decomposed into the irreducible characters $\chi_{[t]}$ of $\mathbb{Z}_n$ by the equation*

$$\chi_{E_k} = \sum_{j=0}^{j=k} \chi_{[k-2j]}. \tag{5.7}$$

Here, $[t]$ denotes the number from 0 to $n-1$ to which t is equivalent mod n.

In the case where $-1 \notin \Gamma$, that is to say, when $\Gamma = \mathbb{Z}_n$ where n is odd, by equating coefficients of the quadratic polynomials P_m^+ and $P_{m'}^-$, the condition 5.3 is replaced by one that may be simply checked.

Lemma 5.3.2 *Let Γ be any finite subgroup of $SU(2)$ such that $-1 \notin \Gamma$ the condition 5.3 is* *equivalent to the condition*

$$\beta^{\alpha}_{m} + \gamma^{\alpha}_{m'} = \begin{cases} \chi_{\alpha}(1), & \textit{if } 1 \leq m \leq c_{\Gamma} - 2 \\ 2\chi_{\alpha}(1), & \textit{if } m = c_{\Gamma} - 1, c_{\Gamma} \end{cases}, \tag{5.8}$$

where α *is an irreducible representation of* Γ. *Furthermore this condition holds in all cases.*

Using proposition 5.3.1, it is a simple combinatorial matter to see that

$$\sum_{m=1}^{n} \langle \chi_{E_{m-1}}, \chi_{\alpha} \rangle_{\Gamma} = N \frac{n+1}{2}, \tag{5.9}$$

for any representation α of $\mathbb{Z}_n$.

For the argument to go through, one also needs to check the special case

$$P^{+}_{c_{\Gamma}}(1/2) = 0.$$

By direct evaluation one can check that this indeed holds.

For the negative side, we see that

$$\sum_{m=1}^{n} \langle \chi_{E_{m+1}}, \chi_{\alpha} \rangle_{\Gamma} = N \frac{n+3}{2}, \tag{5.10}$$

for any representation α of $\mathbb{Z}_n$, and so:

Proposition 5.3.3 *Let* Γ *be cyclic with* $\#\Gamma$ *odd, and let* α *be a* N*-dimensional representation of* Γ. *Then*

$$\sum_{m=1}^{n} P^{+}_{m}(u) = \sum_{m=0}^{n-1} P^{-}_{m}(u) = Nu^2 - \frac{N}{4}.$$

Note that in the statement of Theorem 5.2.1, the first line holds even if we take $k = -1$, since the multiplicity for this value evaluates to zero. Therefore, we automatically have

$$P^{-}_{c_{\Gamma}-1}(-1/2) = 0,$$

which we still needed to check.

5.4 Lens spaces, even order

When n is even, we have $-1 \in \mathbb{Z}_n$. When $-1 \in \Gamma$, from Theorems 5.2.1 and 5.2.2 it follows that the multiplicity of the eigenvalue

$$\lambda = 1/2 + lc_\Gamma + m,\ \ l \in \mathbb{N}$$

is given by, P_m^+, $m \in \{1, 2, \ldots, c_\Gamma\}$,

$$\begin{aligned} P_m^+(u) = & \frac{1}{|\Gamma|}(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1))u^2 + \\ & \left(\beta_m^\alpha - \frac{1}{\#\Gamma}(m(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1))\right)u \\ & + \frac{1}{2}\beta_m^\alpha - \frac{1}{4\#\Gamma}(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1)) \\ & - \frac{1}{2\#\Gamma}m(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1))). \end{aligned}$$

The one case that is not clear is $\lambda = 1/2$. It is not an eigenvalue of the twisted Dirac operator. However, it is not clear from Theorems 5.2.1 and 5.2.2 that

$$P_{c_\Gamma}^+(1/2) = 0, \tag{5.11}$$

and this needs to hold in order for the argument using the Poisson summation formula to go through. Evaluating equation (5.11), we see that one needs to check that

$$\langle \chi_{E_{c_\Gamma - 1}}, \chi_\alpha \rangle = \frac{c_\Gamma}{\#\Gamma}(\chi_\alpha(1) + (-1)^{c_\Gamma + 1}\chi_\alpha(-1)), \tag{5.12}$$

and indeed it holds for each subgroup Γ and irreducible representation α.

Proposition 5.4.1 *For any subgroup $\Gamma \subset S^3$ of even order, the sum of the polynomials*

P_m^+ *is*

$$\sum_{m=1}^{c_\Gamma} P_m^+(u) = \frac{c_\Gamma}{\#\Gamma}\chi_\alpha(1)u^2 + \left(-\frac{c_\Gamma^2\chi_\alpha(1)}{2\#\Gamma} - \frac{c_\Gamma(\chi_\alpha(1)-\chi_\alpha(-1))}{2\#\Gamma} + \sum_{m=1}^{c_\Gamma}\beta_m^\alpha\right)u - \frac{c_\Gamma\chi_\alpha(1)}{2\#\Gamma} - \frac{c_\Gamma^2\chi_\alpha(1)}{4\#\Gamma} + \frac{c_\Gamma}{4\#\Gamma}\chi_\alpha(-1) + \frac{1}{2}\sum_{m=1}^{c_\Gamma}\beta_m^\alpha.$$

Since the coefficients of the polynomial are additive with respect to direct sum, it suffices to consider only irreducible representations.

In the case of lens spaces, $c_\Gamma = \#\Gamma$, and $\chi_t(-1) = (-1)^t$. As a matter of counting, one can see that:

Proposition 5.4.2

$$\sum_{m=1}^{c_\Gamma}\beta_m^t = \begin{cases} \frac{n+2}{2} & \text{if } t \text{ is even} \\ \frac{n}{2} & \text{if } t \text{ is odd} \end{cases}$$

Putting this all into the expression of proposition 5.4.1, we have, for an N-dimensional representation, α,

$$\sum_{m=1}^{c_\Gamma} P_m^+(u) = N\left(u^2 - \frac{1}{4}\right). \tag{5.13}$$

The negative eigenvalues are described by the polynomials

$$P_m^-(u) = \frac{1}{\#\Gamma}(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1))u^2 + \left(\frac{2+m}{\#\Gamma}(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1)) - \gamma_m^\alpha\right)u \; \frac{3+2m}{4|\Gamma|}(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1)) - \frac{1}{2}\gamma_m^\alpha,$$

$m \in \{0, 1, \ldots c_\Gamma - 1\}$. And so, we have the following proposition.

Proposition 5.4.3 *For any subgroup $\Gamma \subset S^3$ of even order, the sum of the polynomials*

P_m^- *is*

$$\sum_{m=1}^{c_\Gamma} P_m^-(u) = \frac{c_\Gamma}{\#\Gamma}\chi_\alpha(1)u^2 + \left(\frac{\chi_\alpha(1)c_\Gamma^2}{2\#\Gamma} + \frac{3\chi_\alpha(1)c_\Gamma}{2\#\Gamma} + \frac{\chi_\alpha(-1)c_\Gamma}{2\#\Gamma} - \sum_{m=0}^{c_\Gamma-1}\gamma_m^\alpha\right)u + \frac{\chi_\alpha(1)c_\Gamma}{2\#\Gamma} + \frac{\chi_\alpha(1)c_\Gamma^2}{4\#\Gamma} + \frac{\chi_\alpha(-1)c_\Gamma}{4\#\Gamma} - \frac{1}{2}\sum_{m=0}^{c_\Gamma-1}\gamma_m^\alpha.$$

By counting, one can see that

$$\sum_{m=0}^{c_\Gamma-1}\gamma_m^t = \begin{cases} \frac{n+4}{2} & \text{if } t \text{ is even} \\ \frac{n+2}{2} & \text{if } t \text{ is odd} \end{cases}. \tag{5.14}$$

To complete the computation of the spectral action one still needs to verify the condition (5.3). We have the following lemma, which is obtained by equating the coefficients of P_m^+ and $P_{m'}^-$, and it covers the cases of the binary tetrahedral, octahedral and icosahedral groups as well.

Lemma 5.4.4 *Let* Γ *be any finite subgroup of* $SU(2)$ *such that* $-1 \in \Gamma$ *the condition* (5.3) *is equivalent to the condition*

$$\beta_m^\alpha + \gamma_{m'}^\alpha = \begin{cases} \chi_\alpha(1)(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1)), & \text{if } 1 \le m \le c_\Gamma - 2 \\ 2\chi_\alpha(1)(\chi_\alpha(1) + \chi_\alpha(-1)), & \text{if } m = c_\Gamma - 1 \\ 2\chi_\alpha(1)(\chi_\alpha(1) - \chi_\alpha(-1)), & \text{if } m = c_\Gamma \end{cases}, \tag{5.15}$$

where α *is an irreducible representation of* Γ. *Furthermore this condition holds in all cases.*

5.5 Dicyclic group

The character table for the dicyclic group of order $4r$ is, for r odd,

Class	1_+	1_-	2_l	r_0	r_1
ψ_t	2	$2(-1)^t$	$\zeta_{2r}^{lt}+\zeta_{2r}^{-lt}$	0	0
χ_1	1	1	1	1	1
χ_2	1	-1	$(-1)^l$	i	$-i$
χ_3	1	1	1	-1	-1
χ_4	1	-1	$(-1)^l$	$-i$	i

,

and for r even,

Class	1_+	1_-	2_l	r_0	r_1
ψ_t	2	$2(-1)^t$	$\zeta_{2r}^{lt}+\zeta_{2r}^{-lt}$	0	0
χ_1	1	1	1	1	1
χ_2	1	-1	$(-1)^l$	i	$-i$
χ_3	1	1	1	-1	-1
χ_4	1	-1	$(-1)^l$	$-i$	i

.

Here $\zeta_{2r} = e^{\frac{\pi i}{r}}$, $1 \le t \le r-1$, $1 \le l \le r-1$. The notation for the different conjugacy classes can be understood as follows. The number indicates the order of the conjugacy class. A sign in the subscript indicates the sign of the traces of the elements in the conjugacy class as elements of $SU(2)$.

For the dicyclic group of order $4r$, the exponent of the group is

$$c_\Gamma = \begin{cases} 2r & \text{if } r \text{ is even} \\ 4r & \text{if } r \text{ is odd} \end{cases}.$$

One can decompose the characters χ_{E_k} into the irreducible characters by inspection, and with some counting obtain the following propositions.

Proposition 5.5.1 *Let Γ be the dicyclic group of order $4r$, where r is even.*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} \frac{r}{2} & \chi_\alpha \in \{\chi_1, \chi_2, \chi_3, \chi_4\} \\ r & \chi_\alpha = \psi_t,\ t \text{ is even} \\ r+1 & \chi_\alpha = \psi_t,\ t \text{ is odd} \end{cases}$$

$$\sum_{m=0}^{c_\Gamma - 1} \gamma_m^\alpha = \begin{cases} \frac{r}{2}+1 & \chi_\alpha \in \{\chi_1, \chi_2, \chi_3, \chi_4\} \\ r+2 & \chi_\alpha = \psi_t, \ t \text{ is even} \\ r+1 & \chi_\alpha = \psi_t, \ t \text{ is odd} \end{cases}.$$

Proposition 5.5.2 *Let Γ be the dicyclic group of order $4r$, where r is odd. The following equations hold:*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} 2r & \chi_\alpha \in \{\chi_1, \chi_3\} \\ 2r+1 & \chi_\alpha \in \{\chi_2, \chi_4\} \\ 4r & \chi_\alpha = \psi_t, \ t \text{ is even} \\ 4r+2 & \chi_\alpha = \psi_t, \ t \text{ is odd} \end{cases}$$

$$\sum_{m=0}^{c_\Gamma - 1} \gamma_m^\alpha = \begin{cases} 2r+2 & \chi_\alpha \in \{\chi_1, \chi_3\} \\ 2r+1 & \chi_\alpha \in \{\chi_2, \chi_4\} \\ 4r+4 & \chi_\alpha = \psi_t, \ t \text{ is even} \\ 4r+2 & \chi_\alpha = \psi_t, \ t \text{ is odd} \end{cases}$$

5.6 Binary tetrahedral group

The binary tetrahedral group has order 24 and exponent 12. The character table of the binary tetrahedral group is

Class	1_+	1_-	4_{a+}	4_{b+}	4_{a-}	4_{b-}	6
Order	1	2	6	6	3	3	4
χ_1	1	1	1	1	1	1	1
χ_2	1	1	ω^2	ω	ω	ω^2	1
χ_3	1	1	ω	ω^2	ω^2	ω	1
χ_4	2	-2	1	1	-1	-1	0
χ_5	2	-2	ω^2	ω	$-\omega$	$-\omega^2$	0
χ_6	2	-2	ω	ω^2	$-\omega^2$	$-\omega$	0
χ_7	3	3	0	0	0	0	-1

.

Here, $\omega = e^{\frac{2\pi i}{3}}$.

For the remaining three groups, we can use matrix algebra to decompose the characters χ_{E_k}.

Let χ_j, x_j, $j = 1, 2 \ldots, d$ denote the irreducible characters, and representatives of the conjugacy classes of the group Γ. Then since every character decomposes uniquely into the irreducible ones, we have a unique expression for χ_{E_k} as the linear combination

$$\chi_{E_k} = \sum_{j=0}^{d} c_j^k \chi_j.$$

If we let $b = (b_j)$ $j = 1, \ldots, d$ be the column with $b_j = \chi_{E_k}(x_j)$, and let $A = (a_{ij})$ be the $d \times d$ matrix where $a_{ij} = \chi_j(x_i)$ and let $c = (c_j^k)$ $j = 1, \ldots d$ be another column, then we have

$$b = Ac.$$

A is necessarily invertible by the uniqueness of the coefficient column c, and so c is given by

$$c = A^{-1}b.$$

By this method, we obtain the following proposition.

Proposition 5.6.1 *Let Γ be the binary tetrahedral group. The following equations hold:*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} 3, & \chi_\alpha \in \{\chi_1, \chi_2, \chi_3\} \\ 7, & \chi_\alpha \in \{\chi_4, \chi_5, \chi_6\} \\ 9, & \chi_\alpha = \chi_7 \end{cases}$$

$$\sum_{m=0}^{c_\Gamma - 1} \gamma_m^\alpha = \begin{cases} 4, & \chi_\alpha \in \{\chi_1, \chi_2, \chi_3\} \\ 7, & \chi_\alpha \in \{\chi_4, \chi_5, \chi_6\} \\ 12, & \chi_\alpha = \chi_7 \end{cases}$$

5.7 Binary octahedral group

The binary octahedral group has order 48 and exponent 24. The character table of the binary octahedral group is

Class	1_+	1_-	6_+	6_0	6_-	8_+	8_-	12
Order	1	2	8	4	8	6	3	4
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	-1	1	1	-1
χ_3	2	2	0	2	0	-1	-1	0
χ_4	2	-2	$\sqrt{2}$	0	$-\sqrt{2}$	1	-1	0
χ_5	2	-2	$-\sqrt{2}$	0	$\sqrt{2}$	1	-1	0
χ_6	3	3	-1	-1	-1	0	0	1
χ_7	3	3	1	-1	1	0	0	-1
χ_8	4	-4	0	0	0	-1	1	0

.

Proposition 5.7.1 *Let* Γ *be the binary octahedral group.*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} 6, & \chi_\alpha \in \{\chi_1, \chi_2\} \\ 12, & \chi_\alpha = \chi_3 \\ 13, & \chi_\alpha =\in \{\chi_4, \chi_5\} \\ 18, & \chi_\alpha \in \{\chi_6, \chi_7\} \\ 26, & \chi_\alpha = \chi_8 \end{cases}$$

$$\sum_{m=0}^{c_\Gamma - 1} \gamma_m^\alpha = \begin{cases} 7, & \chi_\alpha \in \{\chi_1, \chi_2\} \\ 14, & \chi_\alpha = \chi_3 \\ 13, & \chi_\alpha =\in \{\chi_4, \chi_5\} \\ 21, & \chi_\alpha \in \{\chi_6, \chi_7\} \\ 26, & \chi_\alpha = \chi_8 \end{cases}$$

5.8 Binary icosahedral group

The binary icosahedral group has order 120 and exponent 60. The character table of the binary icosahedral group is

Class	1_+	1_-	30	20_+	20_-	12_{a+}	12_{b+}	12_{a-}	12_{b-}
Order	1	2	4	6	3	10	5	5	10
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	-2	0	1	-1	μ	ν	$-\mu$	$-\nu$
χ_3	2	-2	0	1	-1	$-\nu$	$-\mu$	ν	μ
χ_4	3	3	-1	0	0	$-\nu$	μ	$-\nu$	μ
χ_5	3	3	-1	0	0	μ	$-\nu$	μ	$-\nu$
χ_6	4	4	0	1	1	-1	-1	-1	-1
χ_7	4	-4	0	-1	1	1	-1	-1	1
χ_8	5	5	1	-1	-1	0	0	0	0
χ_9	6	-6	0	0	0	-1	1	1	-1

.

Here, $\mu = \frac{\sqrt{5}+1}{2}$, and $\nu = \frac{\sqrt{5}-1}{2}$.

Proposition 5.8.1 *Let Γ be the binary icosahedral group.*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} 15, & \chi_\alpha = \chi_1 \\ 31, & \chi_\alpha \in \{\chi_2, \chi_3\} \\ 45, & \chi_\alpha \in \{\chi_4, \chi_5\} \\ 60, & \chi_\alpha = \chi_6 \\ 62, & \chi_\alpha = \chi_7 \\ 75, & \chi_\alpha = \chi_8 \\ 93, & \chi_\alpha = \chi_9 \end{cases}$$

$$\sum_{m=0}^{c_\Gamma - 1} \gamma_m^\alpha = \begin{cases} 16, & \chi_\alpha = \chi_1 \\ 31, & \chi_\alpha \in \{\chi_2, \chi_3\} \\ 48, & \chi_\alpha \in \{\chi_4, \chi_5\} \\ 64, & \chi_\alpha = \chi_6 \\ 62, & \chi_\alpha = \chi_7 \\ 80, & \chi_\alpha = \chi_8 \\ 93, & \chi_\alpha = \chi_9 \end{cases}$$

5.9 Sums of polynomials

If we input the results of propositions 5.4.2, 5.5.1, 5.5.2, 5.6.1, 5.7.1, 5.8.1 into propositions 5.4.1, 5.4.3 and also recalling proposition 5.3.3 we obtain the following.

Proposition 5.9.1 *Let Γ be any finite subgroup of $SU(2)$ and let α be an N-dimensional representation of Γ. Then the sums of the polynomials P_m^+ and P_m^- are given by*

$$\sum_{m=1}^{c_\Gamma} P_m^+(u) = \sum_{m=0}^{c_\Gamma - 1} P_m^-(u) = \frac{N c_\Gamma}{\#\Gamma} \left(u^2 - \frac{1}{4} \right).$$

Chapter 6

One-Parameter Family of Dirac Operators, $SU(2)$ and $SU(3)$

6.1 Introduction

6.2 One-parameter family of Dirac operators $\mathcal{D}_t$

Let us recall the one-parameter family of Dirac operators constructed by Agricola [2] on a Lie group G. Instead of considering just the Levi-Civita connection, one may consider a whole family of connections,

$$\nabla^t_X := \nabla^0_X + t[X, \cdot],$$

where ∇^0 is the connection induced by left-multiplication.

One checks that ∇^t is a metric $\mathfrak{so}(\mathfrak{g})$ connection. The torsion, $T(X,Y) = (2t-1)[X,Y]$, vanishes when $t = 1/2$ and so we see that $\nabla^{1/2}_X$ is the Levi-Civita connection.

The $\mathfrak{so}(\mathfrak{g})$ connection ∇^t lifts to a metric $\mathfrak{spin}(\mathfrak{g})$ connection $\widehat{\nabla^t}$ [62] given by the formula

$$\widehat{\nabla^t}_X = \nabla^0_X + t\frac{1}{4}\sum_{k,l}\langle X, [X_k, X_l]\rangle X_k X_l.$$

Let $\mathbb{C}\mathrm{l}(\mathfrak{g})$ denote the Clifford algebra generated by $\mathfrak{g}$ with the relation $XY + YX = -2\langle X, Y\rangle$ for $X, Y \in \mathfrak{g}$. Let $\{X_i\}$ denote the set of orthonormal basis of $\mathfrak{g}$ with respect to the metric $\langle \cdot, \cdot \rangle$. The Dirac operator in $\mathbb{C}\mathrm{l}(\mathfrak{g}) \otimes \mathcal{U}(g)$ induced by $\widehat{\nabla^t}$ is given by the formula

$$\mathcal{D}_t := \sum_i X_i \otimes X_i + tH \in \mathbb{C}\mathrm{l}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}), \tag{6.1}$$

where

$$H := \frac{1}{4}\sum_{j,k,l} X_j X_k X_l \otimes \langle X_j, [X_k, X_l]\rangle \in \mathbb{C}\mathrm{l}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}).$$

6.3 Spectrum of $\mathcal{D}_t^2$

In this section we review a general analysis of the spectrum of $\mathcal{D}_t^2$ given in [43].

The operator $\mathcal{D}_t^2$ can be written in terms of the Casimir operator Cas. This is useful because the action of Cas on irreducible components of a Lie algebra representation is well-known.

Let us recall the calculation for $\mathcal{D}_t^2$, done in [2].

Proposition 6.3.1 ([2]) *Let $\{X_i\}$ be the set of orthonormal basis of $\mathfrak{g}$, $\mathcal{D}_t$ to be defined as in Equation* (6.1). *Then*

$$\mathcal{D}_t^2 \quad = \quad 1 \otimes \mathrm{Cas} + (1-3t)\frac{1}{2}\sum_{k,l} X_k X_l \otimes [X_k, X_l] + 9t^2|\rho|^2.$$

Now, we write the degree one term in terms of the Casimir as well, using the homomorphism,

Theorem 6.3.2 ([40]) *The map $\pi : \mathfrak{g} \to \mathbb{C}\mathrm{l}(\mathfrak{g})$ given in the orthonormal basis $X_k \in \mathfrak{g}$ by*

$$\pi(X_i) := \frac{1}{4}\sum_{k,l} \langle X_i, [X_k, X_l]\rangle X_k X_l$$

is a Lie algebra homomorphism.

For a semi-simple Lie algebra, π is an injection. Extend π to $\pi : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}\mathrm{l}(\mathfrak{g})$. Now write the degree one term in $\mathcal{D}_t^2$ as

$$\begin{aligned} \frac{1}{2}\sum_{k,l} X_k X_l \otimes [X_k, X_l] &= \sum_i \frac{1}{2}\sum_{k,l}\langle [X_k, X_l], X_i\rangle X_k X_l \otimes X_i \\ &= 2\sum_i \pi(X_i) \otimes X_i. \end{aligned}$$

Let $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ denote the co-multiplication given by the diagonal embedding

$$\Delta(X_i) := X_i \otimes 1 + 1 \otimes X_i \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}).$$

Then

$$\begin{aligned}\Delta\,\mathrm{Cas} &= -\sum_i (X_i\otimes 1 + 1\otimes X_i)^2\\ &= 1\otimes \mathrm{Cas} + \mathrm{Cas}\otimes 1 - 2\sum_i X_i\otimes X_i.\end{aligned}$$

Thus,

$$-\frac{1}{2}\sum_{k,l} X_kX_l\otimes [X_k,X_l] \;=\; (\pi\otimes 1)\left(\Delta\,\mathrm{Cas} - 1\otimes\mathrm{Cas} - \mathrm{Cas}\otimes 1\right).$$

We obtain

Theorem 6.3.3 *For $\mathcal{D}_t$ defined as in Equation* (6.1)*, $\mathcal{D}_t^2$ can be written as*

$$\mathcal{D}_t^2 = (\pi\otimes 1)T_t$$

where

$$\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g}) \ni T_t := 1\otimes\mathrm{Cas} + (3t-1)(\Delta\,\mathrm{Cas} - 1\otimes\mathrm{Cas} - \mathrm{Cas}\otimes 1) + 9t^2|\rho|^2.$$

Since π is injective, the action of $\mathcal{D}_t^2$ is determined by the action of T_t which in turn is written as a combination of $1\otimes\mathrm{Cas}$, $\mathrm{Cas}\otimes 1$, and $\Delta\,\mathrm{Cas}$.

Lemma 6.3.4 ([38]) *Let V_λ be an irreducible representation of $\mathfrak{g}$ with highest weight λ. Then* Cas *acts as a scalar on V_λ, and the scalar is given by*

$$|\lambda+\rho|^2 - |\rho|^2 = \langle \lambda + 2\rho, \lambda\rangle.$$

Suppose that $\mathcal{S}$ is a Clifford $\mathbb{C}\mathrm{l}(\mathfrak{g})$-module so that $\mathbb{C}\mathrm{l}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ acts on $\mathcal{S}\otimes L^2(G)$, where $\mathbb{C}\mathrm{l}(\mathfrak{g})$ acts on $\mathcal{S}$ via the Clifford action and $\mathcal{U}(\mathfrak{g})$ acts on $L^2(G)$ as left-invariant differentiation.

By the Peter-Weyl theorem, $L^2(G)$ decomposes as the norm closure of

$$\bigoplus_{\lambda\in\widehat{G}} V_\lambda\otimes V_\lambda^*,$$

where λ ranges over the irreducible representations $\widehat{G}$ of G.

Since $\mathcal{U}(\mathfrak{g})$ acts as left invariant differential operators on $L^2(G)$, it acts as the identity on the dual components V_λ^*.

Theorem 6.3.5 ([40]) *Let $\mathcal{S}$ be any $\mathbb{C}\mathrm{l}(\mathfrak{g})$-module. Then the $\mathfrak{g}$-representation on $\mathcal{S}$ defined by composition with π is a direct sum of ρ-representations, where ρ is the half sum of all positive roots, the Weyl vector.*

The theorem implies that as a $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ representation, $\mathcal{S} \otimes L^2(G)$ decomposes as

$$\mathcal{S} \otimes L^2(G) = \bigoplus_\rho \bigoplus_{\lambda \in \widehat{G}} V_\rho \otimes V_\lambda \otimes V_\lambda^*.$$

The sum $\bigoplus_\rho$ and the dual components V_λ^* only change the multiplicity of the action. We will for the moment ignore them and only look at the $V_\rho \otimes V_\lambda$ part.

We can read off how $\mathrm{Cas} \otimes 1$ and $1 \otimes \mathrm{Cas}$ act on $V_\rho \otimes V_\lambda$ already by directly applying Lemma 6.3.4. However to know how T_t, hence $\mathcal{D}_t^2$, acts on $V_\rho \otimes V_\lambda$, we need to know the action of $\Delta\,\mathrm{Cas}$, which can be obtained if one knows the direct sum decomposition of $V_\rho \otimes V_\lambda$ into irreducible components, the so-called Clebsch-Gordan decomposition. In effect, we are reducing the study of the spectrum of $\mathcal{D}_t^2$ to the Clebsch-Gordan decomposition of $V_\rho \otimes V_\lambda$.

Suppose that $V_\rho \otimes V_\lambda$ decomposes as

$$V_\rho \otimes V_\lambda = \bigoplus_\gamma V_{\lambda \pm \gamma} \tag{6.2}$$

for some weights γ.

We examine on it the action of

$$T_t = 1 \otimes \mathrm{Cas} + (3t-1)(\Delta\,\mathrm{Cas} - 1 \otimes \mathrm{Cas} - \mathrm{Cas} \otimes 1) + 9t^2|\rho|^2.$$

One has

$$\begin{aligned} \left(-(3t-1)\,\mathrm{Cas} \otimes 1 + 9t^2|\rho|^2\right)\Big|_{V_\rho \otimes V_\lambda} &= -(3t-1)\langle \rho + 2\rho, \rho\rangle + 9t^2|\rho|^2 \\ &= (3t-1)(3t-2)|\rho|^2 + |\rho|^2. \end{aligned}$$

and

$$\begin{aligned}((3t-1)\Delta\operatorname{Cas} - 1\otimes\operatorname{Cas})\Big|_{V_\rho\otimes V_\lambda} &= (3t-1)\Big(\langle\lambda\pm\gamma+2\rho,\lambda\pm\gamma\rangle - \langle\lambda+2\rho,\lambda\rangle\Big)\\ &= 2(3t-1)\langle\lambda+\rho,\pm\gamma\rangle + (3t-1)|\gamma|^2\\ &= 2\langle\lambda+\rho,\pm(3t-1)\gamma\rangle + |(3t-1)\gamma|^2 - (3t-1)(3t-2)|\gamma|^2.\end{aligned}$$

Thus, the action of T_t on $V_\rho\otimes V_\lambda$ is given by the following.

Theorem 6.3.6 *Let γ denote the weights in the Clebsch-Gordan decomposition of $V_\rho\otimes V_\lambda$ as in Equation* (6.2). *Then*

$$T_t\Big|_{V_\rho\otimes V_\lambda} = |\lambda+\rho\pm\gamma(3t-1)|^2 + (3t-1)(3t-2)\left(|\rho|^2-|\gamma|^2\right). \tag{6.3}$$

Notice that for $t=1/3$, $T_{1/3}$ acts on $V_\rho\otimes V_\lambda$ as $|\lambda+\rho|^2$, obviating the Clebsch-Gordan decomposition.

By the injectivity of $\pi\otimes 1$, one obtains as well the action of $\mathcal{D}_t^2$ on $V_\rho\otimes V_\lambda$.

6.4 Spectral action for $SU(2)$

Using the results of the previous section, we can now compute the spectral action for the one-parameter family of Dirac operators on $SU(2)$.

In the case of $SU(2)$, let V_m denote the irreducible representation of $SU(2)$ of dimension $m+1$, $m\in\mathbb{N}$. The Weyl vector is given by $\rho=1$, and the tensor product decomposition that we need for our calculation is

$$V_1\otimes V_m = V_{m+1}\oplus V_{m-1}.$$

For $m=0$, we ignore V_{-1} and the equation reads $V_1\otimes V_0 = V_1$. In this case, the Clifford module $\mathcal{S}$ equals just a single copy of V_1.

Plugging $\lambda=m$, $\rho=1$, and $\gamma=1$ into Equation (6.3), one obtains the action of T_t is $m+3t$ on V_{m+1} with multiplicity $(m+2)(m+1)$ for $m\geq 0$; and $m+2-3t$ on V_{m-1} with multiplicity $m(m+1)$ for $m\geq 1$, which can alternatively be written as $-n-3t$ with multiplicity $(n+2)(n+1)$ for $n\leq -1$ by the change of indices $m+2=-n$. Hence, the spectrum of $\mathcal{D}_t^2$ is given by $(n+3t)^2$ for $n\in\mathbb{Z}$ with multiplicity $(n+2)(n+1)$, and tthe spectrum action $\mathrm{Tr}\, f\left(\frac{\mathcal{D}_t^2}{\Lambda^2}\right)$ is

$$\mathrm{Tr}\, f\left(\frac{\mathcal{D}_t^2}{\Lambda^2}\right) = \sum_{n\in\mathbb{Z}}(n+2)(n+1)f\left(\frac{(n+3t)^2}{\Lambda^2}\right). \tag{6.4}$$

Now we follow the analysis of Chamseddine and Connes [16]. Let $g(u) = (u+2)(u+1)f\left(\frac{(u+3t)^2}{\Lambda^2}\right)$. Its Fourier transform, denoted by $\widehat{g}(x)$, is

$$\begin{aligned}
\widehat{g}(x) &= \int_{\mathbb{R}}(u+2)(u+1)f\left(\frac{(u+3t)^2}{\Lambda^2}\right)e^{-2\pi i x u}du \\
&= \int_{\mathbb{R}}(\Lambda y-(3t-1))(\Lambda y-(3t-2))f(y^2)e^{-2\pi i x(\Lambda y-3t)}\Lambda dy \\
&= \Lambda^3 e^{-2\pi i x(-3t)}\int_{\mathbb{R}}y^2 f(y^2)e^{-2\pi i x\Lambda y}dy \\
&\quad -\Lambda^2 e^{-2\pi i x(-3t)}\int_{\mathbb{R}}3(2t-1)y f(y^2)e^{-2\pi i x\Lambda y}dy \\
&\quad +\Lambda\int_{\mathbb{R}}(3t-1)(3t-2)f(y^2)e^{-2\pi i x\Lambda y}dy.
\end{aligned}$$

Now let $\widehat{f}^{(m)}$ denote the Fourier transform of $y^m f(y^2)$. By the Poisson summation formula, $\sum_{\mathbb{Z}} g(n) = \sum_{\mathbb{Z}}\widehat{g}(x)$, the spectral action $\mathrm{Tr}\, f\left(\frac{\mathcal{D}_t^2}{\Lambda^2}\right)$ (6.4) then becomes

$$\begin{aligned}
\mathrm{Tr}\, f\left(\frac{\mathcal{D}_t^2}{\Lambda^2}\right) &= \sum_{\mathbb{Z}}\widehat{g}(n) \\
&= \Lambda^3\sum_{\mathbb{Z}}e^{-2\pi i n(-3t)}\widehat{f}^{(2)}(\Lambda n) \\
&\quad -\Lambda^2\sum_{\mathbb{Z}}3(2t-1)e^{-2\pi i n(-3t)}\widehat{f}^{(1)}(\Lambda n) \\
&\quad +\Lambda\sum_{\mathbb{Z}}(3t-1)(3t-2)e^{-2\pi i n(-3t)}\widehat{f}(\Lambda n).
\end{aligned}$$

By taking f to be a Schwartz function, $\widehat{f}^{(m)}$ has rapid decay, thus for all k

$$\left|\widehat{f}^{(m)}(\Lambda n)\right| < C_k(\Lambda n)^{-k}$$

and

$$\left|\sum_{n\neq 0}\widehat{f}^{(m)}(\Lambda n)\right| < C_k'\Lambda^{-k}.$$

As a result,

$$\sum_{n\neq 0} \widehat{g}(n) \in O(\Lambda^{-\infty}).$$

Finally, we obtain:

Theorem 6.4.1 *The spectral action of* $\mathcal{D}_t$ *for* $SU(2)$ *is*

$$\begin{aligned} \operatorname{Tr} f\left(\frac{\mathcal{D}_t^2}{\Lambda^2}\right) &= \widehat{g}(0) + O(\Lambda^{-\infty}) \\ &= \Lambda^3 \widehat{f^2}(0) - \Lambda^2 \widehat{f^1}(0) 3(2t-1) + \Lambda \widehat{f}(0)(3t-1)(3t-2) + O(\Lambda^{-\infty}) \\ &= \Lambda^3 \int_{\mathbb{R}} y^2 f(y^2) dy + \Lambda(3t-1)(3t-2) \int_{\mathbb{R}} f(y^2) dy + O(\Lambda^{-\infty}). \end{aligned}$$

The result of Theorem 6.4 coincides with that of [16] for $t = 1/2$, where $\mathcal{D}_{1/2}$ is the spin-Dirac operator.

6.5 Spectrum of Dirac Laplacian of $SU(3)$

First, let us summarize our results for the spectrum of the Dirac Laplacian of $SU(3)$.

Theorem 6.5.1 *In each row of the table below, for each pair* (p, q) *with* p, q *in the set of parameter values displayed, the Dirac Laplacian* $\mathcal{D}_t^2$ *of* $SU(3)$ *has an eigenvalue in the first column of the multiplicity listed in the center column.*

Let

$$\lambda(u, v) = u^2 + v^2 + uv,$$

and

$$m(a, b) = \frac{(p+1)(q+1)(p+q+2)(p+1+a)(q+1+b)(p+q+2+a+b)}{4}.$$

We denote by $\mathbb{N}^{\geq a}$, *the set* $\{n \in \mathbb{N} : n \geq a\}$, *and we take* $\mathbb{N}$ *to be the set of integers greater than or equal to zero.*

Eigenvalue	Multiplicity	Parameter Values
$\lambda(p+3t, q+3t)$	$m(1,1)$	$p \in \mathbb{N}, q \in \mathbb{N}$
$\lambda(p+2-3t, q-1+6t)$	$m(-1,2)$	$p \in \mathbb{N}, q \in \mathbb{N}$
$\lambda(p+1, q+1) + 3(3t-1)(3t-2)$	$m(0,0)$	$p \in \mathbb{N}, q \in \mathbb{N}, (p,q) \neq (0,0)$
$\lambda(p+3-6t, q+3t)$	$m(-2,1)$	$p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}$
$\lambda(p-1+6t, q+2-3t)$	$m(2,-1)$	$p \in \mathbb{N}, q \in \mathbb{N}$
$\lambda(p+1, q+1) + 3(3t-1)(3t-2)$	$m(0,0)$	$p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}^{\geq 1}$
$\lambda(p+3t, q+3-6t)$	$m(1,-2)$	$p \in \mathbb{N}, q \in \mathbb{N}^{\geq 1}$
$\lambda(p+2-3t, q+2-3t)$	$m(-1,-1)$	$p \in \mathbb{N}, q \in \mathbb{N}$

There is some flexibility in the set of parameter values. For instance in the second line, we could have used instead $p \in \mathbb{N}^{\geq 1}$ since for that line the multiplicity is zero whenever $p = 0$.

6.5.1 Spectrum for $t = 1/3$

In the case of $t = 1/3$, the expression of the spectrum becomes much simpler, as we no longer need to take the Clebsch-Gordan decomposition into account.

Theorem 6.5.2 *The spectrum for the Dirac Laplacian $\mathcal{D}^2_{1/3}$ of SU(3) is given in the following table.*

Eigenvalue	Multiplicity	Parameter Values
$p^2 + q^2 + pq$	$2p^2q^2(p+q)^2$	$p \in \mathbb{N}, q \in \mathbb{N}$

We will later apply the Poisson summation formula to the result of Theorem 6.5.2, and we will make use of the nice property that the multiplicities of $(p, 0)$ and $(0, q)$ are zero for $p, q \in \mathbb{N}$.

6.5.2 Derivation of the spectrum

The pairing $\langle \cdot, \cdot \rangle$ is in general the dual pairing on the weight space of a nondegenerate symmetric bilinear form on the Cartan subalgebra of $\mathfrak{g}$. Such a nondegenerate symmetric bilinear form is necessarily a constant multiple of the Killing form, which for $SU(3)$ is given by

$$\kappa(X,Y) = 6\,\mathrm{Tr}(\mathrm{ad}_X\,\mathrm{ad}_Y), \tag{6.5}$$

where the trace and multiplication are taken in the natural representation of X, Y as 3×3 matrices. One may identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by identifying $\lambda \in \mathfrak{g}^*$ with the unique X_λ such that $\langle X_\lambda, Y\rangle = \lambda(Y)$, for all $Y \in \mathfrak{g}$. This is possible due to the nondegeneracy of the pairing on $\mathfrak{g}$. In this way, one defines the dual pairing on $\mathfrak{g}^*$. The particular pairing which occurs depends on the normalization of the Riemannian metric.

Henceforth, we assume that the metric is normalized so that $\langle \rho, \rho\rangle = 3$. This leads to the simplest expressions for the spectrum.

In order to derive the spectrum of the Dirac Laplacian, one must first analyze the pairing of weights. We take for our basis of the Cartan subalgebra, $\mathfrak{h}$, the set $\{H_1, H_2\}$,

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.6}$$

We identify weights concretely using this basis, i.e. for $\lambda \in \mathfrak{h}^*$, we identify λ with $(\lambda(H_1), \lambda(H_2))$.

In terms of Theorem 6.3.6, in the case of $SU(3)$ we have $\lambda = (p, q) \in \mathbb{N} \times \mathbb{N}$, $\rho = (1, 1)$, $\gamma = (a, b) = (0, 0), (1, 1), (2, -1)$, or $(-1, 2)$; and its multiplicity is $\frac{1}{4}(p + 1 \pm a)(p + 1)(q + 1 \pm b)(q + 1)(p + q + 2 \pm (a + b))(p + q + 2)$. And $\mathcal{S} = V_{(1,1)}$.

The weights $\lambda_1 = (1, 0)$ and $\lambda_2 = (0, 1)$ form an $\mathbb{N}$-basis of the highest weights of irreducible representations of $SU(3)$. The pairing of weights can be determined up to normalization, using duality, and the Killing form, from which one deduces the relations

$$\langle \lambda_1, \lambda_1\rangle = \langle \lambda_2, \lambda_2\rangle = 2\langle \lambda_1, \lambda_2\rangle. \tag{6.7}$$

From these relations and Lemma 6.3.4, we immediately obtain the following lemma.

Lemma 6.5.3 *On the irreducible representation of highest weight (p, q), $p, q \in \mathbb{N}$, the Casimir element acts by the scalar*

$$\mathrm{Cas}\Big|_{V_{(p,q)}} = (p^2 + q^2 + 3p + 3q + pq)\langle \lambda_1, \lambda_1\rangle. \tag{6.8}$$

For the normalization that we are considering, we have $\langle \lambda_1, \lambda_1\rangle = 1$.

We have listed the irreducible representations of $SU(3)$ as well as the action of the Casimir operator on them. To write down the spectrum of the Dirac Laplacian the only obstacle now is to understand the term $\Delta\,\mathrm{Cas}$ in Theorem 6.3.6; i.e. we need to know the Clebsch-Gordan coefficients of the tensor products $V_\rho \otimes V_{(p,q)}$. These were computed in [61]. We recall the Clebsch-Gordan coefficients that we will need below.

Lemma 6.5.4 ([61]) *The decomposition of $V_\rho \otimes V_{(p,q)}$ into irreducible representations is*

$$V_\rho \otimes V_{(p,q)} = \oplus_\mu V_\mu, \tag{6.9}$$

where the summands V_μ appearing in the direct sum are given by the following table:

Summand	Parameter Values
$V_{(p+1,q+1)}$	$p \in \mathbb{N}, q \in \mathbb{N}$
$V_{(p-1,q+2)}$	$p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}$
$V_{(p,q)}$	$p \in \mathbb{N}, q \in \mathbb{N}, (p,q) \neq (0,0)$
$V_{(p-2,q+1)}$	$p \in \mathbb{N}^{\geq 2}, q \in \mathbb{N}$
$V_{(p+2,q-1)}$	$p \in \mathbb{N}, q \in \mathbb{N}^{\geq 1}$
$V_{(p,q)}$	$p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}^{\geq 1}$
$V_{(p+1,q-2)}$	$p \in \mathbb{N}, q \in \mathbb{N}^{\geq 2}$
$V_{(p-1,q-1)}$	$p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}^{\geq 1}$

. (6.10)

Each summand in the left column appears once if (p,q) lies in the set of parameter values listed on the right column. For instance for $(p,q) = (1,1)$, the summand $V_{(p,q)} = V_{(1,1)}$ appears twice in the direct sum decomposition, since $V_{(p,q)}$ appears twice in the left column, and (p,q) is in the set of parameter values in each of the two rows.

By combining Theorem 6.3.6, Lemma 6.5.3, and Lemma 6.5.4, we obtain Theorem 6.5.1. The multiplicities are obtained using the Weyl dimension formula

$$\dim V_{(p,q)} = \frac{1}{2}(p+1)(q+1)(p+q+2). \tag{6.11}$$

When $t = 1/3$, the formula for the Dirac Laplacian in Theorem 6.3.6 simplifies to

$$\mathcal{D}^2_{1/3} = 1 \otimes \mathrm{Cas} + 3. \tag{6.12}$$

Therefore, we no longer need to decompose any tensor products into irreducible components, and using just Lemma 6.5.3, one obtains Theorem 6.5.2.

6.6 Spectral action for $SU(3)$

In this section, we compute the spectral action, $\mathrm{Tr}\, f(\mathcal{D}_t^2/\Lambda^2)$. In the case $t = 1/3$, one may apply the Poisson summation formula as in [16] to quickly obtain the full asymptotic expansion for the spectral action. For general t however, this approach no longer works. An expansion can however still be generated using a two variable generalization of the Euler-Maclaurin formula [37]. However, this requires more work to produce, and produces the full expansion of the spectral action if the test function f is taken to be "flat" at the origin. The flatness assumption of f is natural as the role of f is to act as a cut-off function. Here, we compute the spectral action to order Λ^0.

6.6.1 t = 1/3

Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwarz function. By Theorem 6.5.2, the spectral action of $SU(3)$, for $t = 1/3$ is given by

$$\mathrm{Tr}\, f(\mathcal{D}_{1/3}^2/\Lambda^2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2p^2q^2(p+q)^2 f\left(\frac{p^2+q^2+pq}{\Lambda^2}\right). \tag{6.13}$$

In order to apply the Poisson summation formula, one needs to turn this sum into a sum over $\mathbb{Z}^2$. For this purpose, one takes advantage of the fact that the expressions for the eigenvalues and multiplicities are both invariant under a set of transformations of $\mathbb{N}^2$ which together cover $\mathbb{Z}^2$. The linear transformations of $\mathbb{N}^2$ which together cover $\mathbb{Z}^2$ are

$$\begin{aligned}
T_1(p,q) &= (p,q),\\
T_2(p,q) &= (-p,p+q),\\
T_3(p,q) &= (-p-q,p),\\
T_4(p,q) &= (-p,-q),\\
T_5(p,q) &= (p,-p-q),\\
T_6(p,q) &= (p+q,-p).
\end{aligned}$$

Each of the transformations is injective on $\mathbb{N}^2$. The union of the images is all of $\mathbb{Z}^2$. The six images of $\mathbb{N}^2$ overlap on the sets $\{(p,q) : p = 0\}$ and $\{(p,q) : q = 0\}$. However, the multiplicity is equal to zero at these points, and so this overlap is of no consequence. Therefore, we may now write the spectral action as a sum over $\mathbb{Z}^2$ as

$$\operatorname{Tr} f(\mathcal{D}_{1/3}^2/\Lambda^2) = \frac{1}{6}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} 2p^2q^2(p+q)^2 f\left(\frac{p^2+q^2+pq}{\Lambda^2}\right). \tag{6.14}$$

For a sufficiently regular function, the Poisson summation formula (in two variables) is

$$\sum_{\mathbb{Z}^2} g(p,q) = \sum_{\mathbb{Z}^2} \widehat{g}(x,y). \tag{6.15}$$

Applying Equation (6.15) to Equation (6.14), and applying the argument used in [16] we get the following result.

Theorem 6.6.1 *Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwarz function. For $t = 1/3$, the spectral action of SU(3) is*

$$\operatorname{Tr} f(\mathcal{D}_{1/3}^2/\Lambda^2) = \frac{1}{3}\iint_{\mathbb{R}^2} x^2y^2(x+y)^2 f(x^2+y^2+xy)dxdy\ \Lambda^8 + O(\Lambda^{-k}), \tag{6.16}$$

for any integer k.

6.6.2 General t and the Euler-Maclaurin formula

The one-variable Euler-Maclaurin formula was used in [17] to compute the spectral action of $SU(2)$ equipped with the Robertson-Walker metric. A two-variable Euler-Maclaurin formula may be applied here to compute the spectral action on $SU(3)$ for all values of t.

Let m be a positive integer. Let g be a function on $\mathbb{R}^2$ with compact support. One instance of the two-variable Euler-Maclaurin formula is [37]

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}{}' g(p,q) = L^{2k}(\frac{\partial}{\partial h_1})L^{2k}(\frac{\partial}{\partial h_2})\int_{h_1}^{\infty}\int_{h_2}^{\infty} g(p,q)dpdq\Big|_{h_1=0,h_2=0} + R_m^{st}(g). \tag{6.17}$$

The notation $\sum\sum'$ indicates that terms of the form $g(0,q)$, $q \neq 0$, and $g(p,0)$, $p \neq 0$ have a coefficient of $1/2$, $g(0,0)$ has a coefficient of $1/4$, and the rest of the terms are given the usual coefficient of 1. The operator $L^{2k}(S)$ is defined to be

$$L^{2k}(S) = 1 + \frac{1}{2!}b_2S^2 + \ldots + \frac{1}{(2k)!}b_{2k}S^{2k}, \tag{6.18}$$

where b_j is the jth Bernoulli number. The number k is defined by $k = \lfloor m/2 \rfloor$. The remainder $R_m^{st}(g)$ is

$$R_m^{st}(g) \tag{6.19}$$
$$= \sum_{I\subsetneq\{1,2\}}(-1)^{(m-1)(2-|I|)}\prod_{i\in I}L^{2k}(\frac{\partial}{\partial h_i})\int_{h_1}^{\infty}\int_{h_2}^{\infty}\prod_{i\notin I}P_m(x_i)\prod_{i\notin I}\left(\frac{\partial}{\partial x_i}\right)^m g(x_1,x_2)dx_1dx_2\Bigg|_{h=0}.$$

Equation (6.17) is proved in an elementary way in [37], by casting the one-variable Euler-Maclaurin formula in a suitable form, and then iterating it two times.

Using Theorem 6.5.1, one may write the spectral action in terms of eight summations of the form

$$\sum_{(p,q)\in\mathbb{N}^2} g_i(p,q),$$

where

$$g_i(p,q) = f\left(\frac{\lambda_i(p,q)}{\Lambda^2}\right)m_i(p,q), \quad i = 1,\ldots,8.$$

The notations $\lambda_i(p,q)$ and $m_i(p,q)$ denote the eigenvalues and multiplicities of the spectrum in Theorem 6.5.1.

One then applies the two-variable Euler-Maclaurin formula to each of the eight summations to replace the sums with integrals. Then to obtain an asymptotic expression in Λ, one controls the remainder, $R_m^{st}(g)$, to arbitrary order in Λ by taking m to be sufficiently large, and computes the big-O behavior of the other integrals to arbitrary order in Λ by applying the multivariate Taylor's theorem to a large enough degree. The terms in the Taylor expansions of the integrals yield the asymptotic expansion of the spectral action.

6.6.3 Analysis of remainders

Let us consider in detail the case $I = \{\}$, of the remainder, (6.19). The functions $P_m(x_i)$ are periodic, and hence bounded. Furthermore, they are independent of Λ. Therefore to study the big-O behavior with respect to Λ of the remainder, (6.19), we only need to estimate the integral

$$\iint \left| \frac{\partial^m}{\partial p^m} \frac{\partial^m}{\partial q^m} f(s\lambda(p,q))m(p,q) \right|. \tag{6.20}$$

The integration happens over $(\mathbb{R}^+)^2 = [0,\infty) \times [0,\infty)$, and $m(p,q)$ is the multiplicity polynomial. The differentiated function is a sum of terms, whose general term is given by

$$Cs^i f^{(i)}(t\lambda)\lambda^{(a_1,b_1)}(p,q)\dots\lambda^{(a_i,b_i)}(p,q)m^{(j,k)}(p,q), \tag{6.21}$$

where C is a combinatorial constant, $s = \Lambda^{-2}$, and where j and k are less than or equal to m and $0 \le i \le 2m - j - k$ and

$$\sum (a_i, b_i) = (m-j, m-k).$$

Since m is degree 4 in both p and q, we know that $j \le 4$ and $k \le 4$. Since λ is degree 2 in both p and q we know that each of the coefficients a_k, b_k is less than or equal to 2. Therefore, one has the estimate

$$2i \ge \sum a_i = m - j \ge m - 4, \tag{6.22}$$

and so

$$i \geq \frac{m-4}{2}. \tag{6.23}$$

It is not too hard to see that the integral

$$\iint f^{(i)}(s\lambda)\lambda^{(a_1,b_1)}(p,q)\ldots\lambda^{(a_i,b_i)}(p,q)m^{(j,k)}(p,q)dpdq \tag{6.24}$$

is uniformly bounded as s approaches zero, and therefore we have that the integral has a big-O behavior of $O(s^{\frac{m-4}{2}})$ as s goes to zero.

The same argument gives the same estimate for the terms in the cases $I = \{1\}$ and $I = \{2\}$. Therefore we have shown:

Lemma 6.6.2 *The remainder $R_m^{st}(g)$ behaves like $O(\Lambda^{-(m-4)})$ as Λ approaches infinity.*

Since the sum in the Euler-Maclaurin formula, (6.17), gives only a partial weight to terms on the boundary, and since the functions $g_i(p,q)$, are at times nonzero on the boundary, $\{p=0\} \cup \{q=0\}$ even when there are no eigenvalues there, we must compensate at the boundary in order to obtain an accurate expression for the spectral action.

In doing so, one considers sums of the form

$$\sum_{p=0}^{\infty} g_i(p,0) \quad \text{and} \quad \sum_{q=0}^{\infty} g_i(0,q).$$

One treats these sums using the usual one-variable Euler-Maclaurin formula, which for a function, h, with compact support is

$$\sum_{p=0}^{\infty} h(p) = \int_0^{\infty} h(x)dx + \frac{1}{2}h(0) - \sum_{j=1}^{m} \frac{b_{2j}}{(2j)!} h^{(2j-1)}(0) + R_m(h), \tag{6.25}$$

where the remainder is given by

$$R_m(h) = \int_0^{\infty} P_m(x) \left(\frac{\partial}{\partial x}\right)^m h(x)dx. \tag{6.26}$$

The necessary estimate for the remainder (6.26) is as follows.

Lemma 6.6.3 *$R_m(g(p,\cdot))$ and $R_m(g(\cdot,q))$ behave as $O(\Lambda^{-m+4})$ as Λ approaches infinity.*

To prove this, we observe that since the polynomial $P_m(x)$ is bounded and independent of x, we only need to estimate for instance

$$\left| \int_0^\infty \left(\frac{\partial}{\partial x} \right)^m g_i(x,0) dx \right| .$$

The function $g_i(x,0)$ is of the form

$$f\left(\frac{ax^2 + bx + c}{\Lambda^2} + d \right) m(x,0), \tag{6.27}$$

where a, b, c are independent of Λ and x, and d is independent of x. The polynomial $m(x,0)$ is of degree 4 in x. Therefore, when one expands the derivative of (6.27) using the product rule, the derivatives of $f(\frac{ax^2+bx+c}{\Lambda^2} + d)$ are all of order $j \geq m - 4$. A simple inductive argument shows that the expansion of $(\partial/\partial x)^j f(\frac{ax^2+bx+c}{\Lambda^2} + d)$ under the chain rule the terms are all of the form

$$\frac{1}{\Lambda^k} f^{(i)} \left(\frac{ax^2 + bx + c}{\Lambda^2} + d \right) \alpha(x),$$

where $k \geq j$, and $\alpha(x)$ is a polynomial. Finally we conclude the proof of the lemma by observing that

$$\int_0^\infty \left(\frac{\partial}{\partial x} \right)^j f\left(\frac{ax^2 + bx + c}{\Lambda^2} + d \right) \alpha(x) dx \tag{6.28}$$

is uniformly bounded as Λ goes to infinity.

6.6.4 Analysis of main terms

With the remainders taken care of, one still needs to work out the big-O behavior of the spectral action with respect to Λ of the remaining terms coming from the two-variable and one-variable Euler-Maclaurin formulas.

The calculation required is lengthy, but the technique is elementary. One changes variables to remove (most of) the Λ dependence from the argument of the test function f. Then, one uses Taylor's theorem to remove the Λ dependence from the limits of integration, and whatever Λ dependence remains in the argument of f. In this way, one can obtain the big-O behavior of the spectral action with respect to Λ to any desired order. We have performed the computation up to constant order in Λ. If one assumes that the test function f has all derivatives equal to zero at the origin, then one obtains the asymptotic expansion to all orders in Λ.

To give a better idea of how the calculation proceeds, let us consider in detail a couple of terms coming from the Euler-Maclaurin formulas.

One term that appears upon application of the Euler-Maclaurin formula is

$$\int_0^\infty \int_0^\infty g_1(p,q)dpdq, \tag{6.29}$$

where

$$g_1(p,q) = f\left(\frac{(p+3t)^2+(q+3t)^2+(p+3t)(q+3t)}{\Lambda^2}\right)\frac{(p+1)(q+1)(p+q+2)(p+2)(q+2)(p+q+4)}{4}.$$

First, one performs on (6.29) the change of variables,

$$x = \frac{p+3t}{\Lambda} \quad \text{and} \quad y = \frac{q+3t}{\Lambda},$$

whereby one obtains

$$\frac{1}{4}\int_{3t/\Lambda}^\infty \int_{3t/\Lambda}^\infty f(x^2+y^2+xy)(1-3t+x\Lambda)(2-3t+x\Lambda)\times$$
$$\times(1-3t+y\Lambda)(2-3t+y\Lambda)(2-6t+x\Lambda+y\Lambda)(4-6t+x\Lambda+y\Lambda)\Lambda^2 dxdy.$$

Next, one does a Taylor expansion on the two lower limits of integration about 0. The first term in this Taylor series is obtained by setting the limits of integration to zero.

$$\frac{1}{4}\int_0^\infty \int_0^\infty f(x^2+y^2+xy)(1-3t+x\Lambda)(2-3t+x\Lambda)\times \tag{6.30}$$
$$\times(1-3t+y\Lambda)(2-3t+y\Lambda)(2-6t+x\Lambda+y\Lambda)(4-6t+x\Lambda+y\Lambda)\Lambda^2 dxdy.$$

Remarkably, if one sums the analog of (6.30) for $g_1,\ldots,g_8$ one obtains the complete spectral action to constant order. All of the other terms which appear in the computation (of which there are many) cancel out, to constant order in Λ, in an intricate manner.

The end result of the calculation is the following.

Theorem 6.6.4 *Let f be a real-valued function on the real line with compact support. To* *constant order, the spectral action,* $\mathrm{Tr}\, f(\mathcal{D}_t^2/\Lambda^2)$ *of* $SU(3)$ *is equal to*

$$\begin{aligned}
&\mathrm{Tr}\, f(\mathcal{D}_t^2/\Lambda^2)\\
&= 2\iint_{(\mathbb{R}^+)^2} f(x^2+y^2+xy)x^2y^2(x+y)^2 dxdy\ \Lambda^8\\
&\quad +3(3t-1)(3t-2)\iint_{(\mathbb{R}^+)^2} f(x^2+y^2+xy)(x^4+2x^3y+3x^2y^2+2xy^3+y^4)dxdy\ \Lambda^6\\
&\quad +9(3t-1)^2(3t-2)^2\iint_{(\mathbb{R}^+)^2} f(x^2+y^2+xy)(x^2+xy+y^2)dxdy\ \Lambda^4\\
&\quad +6(3t-1)^3(3t-2)^3\iint_{(\mathbb{R}^+)^2} f(x^2+y^2+xy)dxdy\ \Lambda^2+O(\Lambda^{-1}).
\end{aligned}$$

Here, the integrals are taken over the set $(\mathbb{R}^+)^2=[0,\infty)\times[0,\infty)$. *When* f *is taken to be a cut-off function so that it is flat at the origin, this expression gives the full asymptotic expansion of the spectral action.*

The linear transformations, $T_1 \dots T_6$ in Subsection 6.6.1, are all unimodular, and the images of $(\mathbb{R}^+)^2$ cover $\mathbb{R}^2$, up to a set of measure zero. Therefore, in the case of $t = 1/3$, integrating over $\mathbb{R}^2$ multiplies the result by a factor of 6, and we see that Theorem 6.6.4 agrees with Theorem 6.6.1.

When computing the asymptotic expansion of the spectral action using the Euler-Maclaurin formula, as a result of the chain rule, the negative powers, Λ^{-j} appear only with derivatives $f^{(k)}(0)$, $k \geq j$. This is why the terms of the asymptotic expansion vanish for negative powers of Λ, when the derivatives of f vanish at zero.

6.7 Details of the Calculations

Since the Dirac Laplacian spectrum of $SU(3)$, Theorem 6.5.1 is divided into eight pieces, the spectral action also naturally divides into eight pieces of the form

$$\sum_{p,q} g(p,q),$$

where

$$g(p,q) = f(t\lambda_i(p,q))m_i(p,q), \quad i = 1,\dots,8,$$

and the index values taken on by p and q are determined by the expression of the spectrum, Theorem 6.5.1.

By the Euler-Maclaurin formula, we have

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}{}' g(p,q) = L^{2k}(\frac{\partial}{\partial h_1})L^{2k}(\frac{\partial}{\partial h_2})\int_{h_1}^{\infty}\int_{h_2}^{\infty} g(p,q)dpdq\Big|_{h_1=0,h_2=0} + R_k(g).$$

The apostrophe in the double sum indicates that the terms $g(p,0)$, $p \neq 0$, and $g(0,q)$, $q \neq 0$ are taken with weight $1/2$, and the term $g(0,0)$ is taken with weight $1/4$. We compensate for these weights, and arrive at the equation

$$\begin{aligned}\sum_{(p,q)} g(p,q) &= L^{2k}(\frac{\partial}{\partial h_1})L^{2k}(\frac{\partial}{\partial h_2})\int_{h_1}^{\infty}\int_{h_2}^{\infty} g(p,q)dpdq\Big|_{h_1=0,h_2=0} + R_k(g)\\ &\quad +\alpha\sum_{q=0}^{\infty} g(0,q) + \beta\sum_{p=0}^{\infty} g(p,0) + \gamma g(0,0),\end{aligned}$$

where the indices taken on by p and q are the appropriate ones as determined by the spectrum in Theorem 6.5.1, and the constants α, β lie in the set $\{-1/2, 0, 1/2\}$, as determined by the spectrum. The term $\gamma g(0,0)$ is there to ensure one has the correct term at the corner $(p,q) = (0,0)$. One then applies the one dimensional Euler-Maclaurin formula to these boundary sums, to get our final formula

$$\sum_{(p,q)} g(p,q) = L^{2k}(\frac{\partial}{\partial h_1})L^{2k}(\frac{\partial}{\partial h_2})\int_{h_1}^{\infty}\int_{h_2}^{\infty} g(p,q)dpdq\Big|_{h_1=0,h_2=0} \tag{6.31}$$

$$+\alpha L^{2k}(\frac{\partial}{\partial h})\int_{h}^{\infty} g(0,q)dq + \beta L^{2k}(\frac{\partial}{\partial h})\int_{h}^{\infty} g(p,0)dp \tag{6.32}$$

$$+\gamma g(0,0) + R_k(g). \tag{6.33}$$

We have collected the remainders coming from the two-dimensional Euler-Macularin formula and the two instances of the one-dimensional Euler-Maclaurin formula into a single remainder, $R_k(g)$. We already demonstrated that $R_k(g)$ can be made to behave like $O(\Lambda^{-s})$ for any s, so long as k is chosen to be sufficiently large.

Now let us analyze this final formula term by term, and demonstrate how to transform these terms into asymptotic expressions in Λ.

6.7.1 The Identity Term

First we consider the term

$$\int_{h_1}^{\infty}\int_{h_2}^{\infty} g(p,q)dpdq\Big|_{h_1=0,h_2=0}, \tag{6.34}$$

which is in a class of its own. Let us work out one concrete example. We take

$$\begin{aligned} &g(p,q) \\ &= \; f\left(\frac{(p+1)^2+(q+1)^2+(p+1)(q+1)+3(3t-1)(3t-2)}{\Lambda^2}\right)\frac{1}{4}(p+1)^2(q+1)^2(p+q+2)^2. \end{aligned}$$

We perform the change of variables $(p+1)/\Lambda = u$, $(q+1)/\Lambda = v$. Then the integral (6.34) becomes

$$\Lambda^2\int_{\frac{1}{\Lambda}}^{\infty}\int_{\frac{1}{\Lambda}}^{\infty} f\left(u^2+v^2+uv+\frac{3(3t-1)(3t-2)}{\Lambda^2}\right)P(u,v,\Lambda)dudv. \tag{6.35}$$

Here and below, P denotes a generic polynomial. To move the remaining Λ dependence outside of f, we make the Taylor expansion replacement

$$f(x+y) = f(x) + f'(x)y + f''(x)y^2/2! + f'''(y^3)/3! + f''''(y^4)/4! + O(y^5),$$

where

$$x = u^2+v^2+uv \quad \text{and} \quad y = \frac{3(3t-1)(3t-2)}{\Lambda^2}.$$

Here, y^5 is $O(\Lambda^{-10})$, which is enough to suppress the positive powers of Λ in the remaining part of the expression, if one is working up to constant order.

Next, we perform a Taylor expansion in the two lower limits of integration. This process leads to three classes of terms.

If we let $h(x,y)$ denote the double integral, where x and y are the two lower limits of integration, then the expansion is of the form

$$h(x,y) = \sum_{i+j\le s} h^{(i,j)}(0,0)\frac{x^i y^j}{i!j!} + O(\Lambda^{-s}).$$

This expansion naturally leads to three classes of terms

Class 1: i=j=0.

This is simply the double integral (6.35) with lower limits set to zero

$$\Lambda^2 \int_0^\infty \int_0^\infty (\cdots) P(u, v, \Lambda) du dv.$$

If one collects this term for each of the eight pieces of the spectrum one obtains all of the terms which contribute to the expansion of the spectral action.

Class 2: Exactly one of i, j equals zero.

Suppose for instance that $i = 0$, then the terms in this class look like

$$\Lambda^2 \int_0^\infty (-1) \left(\frac{\partial}{\partial u}\right)^{j-1} \left((\cdots) P(u, v, \Lambda)\right)\Big|_{u=0} dv \left(\frac{1}{\Lambda}\right)^j \frac{1}{j!}. \tag{6.36}$$

This class of terms has non-vanishing terms up to order Λ^5.

Class 3: Neither i nor j equals zero.

This class of terms is very straightforward to compute. They are of the form

$$\Lambda^2 \left(\frac{\partial}{\partial u}\right)^{j-1} \left(\frac{\partial}{\partial v}\right)^{i-1} \left((\cdots) P(u, v, \Lambda)\right)\Big|_{u=0, v=0} \left(\frac{1}{\Lambda}\right)^i \left(\frac{1}{\Lambda}\right)^j \frac{1}{i!j!}. \tag{6.37}$$

This class of terms only has non-vanishing terms to constant order or lower in Λ. The constant order term is a degree 8 polynomial in t.

When working to constant order, only small values of i and j are needed. For large values of i and j the powers of $\frac{1}{\Lambda}$ suppress the powers of Λ appearing in the remainder of the expression.

6.7.2 The terms $\frac{b_{2i}}{(2i)!} \left(\frac{\partial}{\partial h}\right)^{2i} \iint g(p, q) dp dq$

Here b_{2i} are the even Bernoulli numbers. The ones we need to compute up to constant order in Λ are $b_0 = 1$, $b_2 = 1/6$, $b_4 = -1/30$, $b_6 = 1/42$, $b_8 = -1/30$. We now consider the next set of terms in (6.31). Performing the partial derivative in h gives

$$\left(\frac{\partial}{\partial h}\right)^{2i} \int_0^\infty \int_h^\infty g(p, q) dp dq\Big|_{h=0} = \int_0^\infty (-1) \frac{\partial}{\partial p}^{2i-1} g(p, q)\Big|_{p=0} dq. \tag{6.38}$$

Once again, let us work out one case concretely. We let

$$g(p,q) = f\left(\frac{(p+3t)^2 + (q+3t)^2 + (p+3t)(q+3t)}{\Lambda^2}\right) \text{mult(p, q)}. \tag{6.39}$$

In this case, the expression (6.38) becomes

$$\int_0^\infty (-1)P\left(f^{(b)}\left(\frac{(3t)^2 + (q+3t)^2 + (3t)(q+3t)}{\Lambda^2}\right), q, t, \Lambda^a\right) dq. \tag{6.40}$$

Here and below, the argument $f^b(x)$ in the polynomial indicates that f and some of its derivatives evaluated at x are variables of the polynomial. Next, one does the variable substitution $(q+3t)/\Lambda = v$, to get

$$\Lambda \int_{3t/\Lambda}^\infty (-1)P\left(f^{(b)}\left(v^2 + \frac{(3t)^2 + 3t\Lambda v}{\Lambda^2}\right), v, t, \Lambda^a\right) dv. \tag{6.41}$$

Finally, one makes the replacement

$$f^{(b)}(x+y) = f^{(b)}(x) + f^{(b+1)}(x)y \ldots + f^{(b+8)}(x)\frac{y^8}{8!} + O(y^9), \tag{6.42}$$

where $x = v^2$ and $y = ((3t)^2 + 3t\Lambda v)/\Lambda^2$. Since y^9 is $O(\Lambda^{-9})$ this is enough to suppress the other powers of Λ when working to constant order.

The final expression will be of the form

$$\Lambda \int_{3t/\Lambda}^\infty (-1)P\left(f^{(b)}(v^2), v, t, \Lambda^a\right) dv. \tag{6.43}$$

Next we do the Taylor expansion in the lower limit of the integral:

$$h(x) = \sum_{j=0} h^{(j)}(0)\frac{x^j}{j!}.$$

This leads to two classes of terms

Class 1: $j = 0$ Here one simply sets the lower limit of integration to zero.

$$\frac{b_{2i}}{(2i)!}\Lambda \int_0^\infty (-1)P\left(f^{(b)}(v^2), v, t, \Lambda^a\right) dv.$$

This class of terms has non-vanishing contributions up to order Λ^5.

Class 2: $j \neq 0$ These terms are of the form

$$\frac{b_{2i}}{(2i)!}\Lambda(-1)\frac{\partial}{\partial v}^{(j-1)} P\left(f^{(b)}(v^2), v, t, \Lambda^a\right)\Big|_{v=0}\left(\frac{3t}{\Lambda}\right)^j \frac{1}{j!}.$$

This class of terms only has non-vanishing contributions no higher than constant order in Λ. This constant order term is a polynomial in t of degree 5.

6.7.3 The terms $\frac{b_{2i}}{(2i)!}\frac{b_{2j}}{(2j)!}\left(\frac{\partial}{\partial h_1}\right)^{2i}\left(\frac{\partial}{\partial h_2}\right)^{2j}\iint g(p,q)dpdq$

These terms generate just a single class of terms, which are easy to handle. They are of the form

$$\frac{b_{2i}}{(2i)!}\frac{b_{2j}}{(2j)!}\frac{\partial}{\partial p}^{j-1}\frac{\partial}{\partial q}^{i-1} g(p,q)\Big|_{p=0,q=0}.$$

This works out to an expression of the form

$$\frac{b_{2i}}{(2i)!}\frac{b_{2j}}{(2j)!}P\left(f^{(b)}\left(\frac{\cdots}{\Lambda^2}\right)\right). \tag{6.44}$$

In order to get an asymptotic expansion to constant order, one simply replaces the arguments of all of the $f^{(b)}$ with zero.

The resulting expression has non-vanishing contributions no higher than constant order, and this term is constant with respect to t.

6.7.4 Boundary Term $\int g(0,q)dq$, $\int g(p,0)dp$

In the two-dimensional Euler-Maclaurin formula, the terms corresponding to the boundary, $p = 0$ and $q = 0$ are not given full weight. In addition, there may or may not be eigenvalues with positive multiplicity at the boundary, depending on which of the eight pieces of the spectrum one is considering. Therefore, one must fill in or take away the sum at the boundary in order to obtain the full spectral action. One can do this by applying the one-dimensional Euler-Maclaurin formula.

Now let us consider the terms that come when compensating for the boundary. The first terms are of the form

$$\int_0^\infty g(0,q)dq,$$

and

$$\int_0^\infty g(p,0)dp,$$

by symmetry in p and q, it is sufficient to consider only one of these cases. Let us consider the case $p = 0$, and take

$$g(p,q) = f\left(\frac{(p+3t)^2 + (q+3t)^2 + (p+3t)(q+3t)}{\Lambda^2}\right)\text{mult(p, q)}. \tag{6.45}$$

Then

$$\int_0^\infty g(0,q)dq = \int_0^\infty f\left(\frac{(3t)^2 + (q+3t)^2 + (3t)(q+3t)}{\Lambda^2}\right)\text{mult}(0,\text{q})dq.$$

We perform the variable substitution $(q+3t)/\Lambda = v$ so now we have

$$\Lambda \int_{3t/\Lambda}^\infty f\left(v^2 + \frac{(3t)^2 + 3t\Lambda v}{\Lambda^2}\right) P(\Lambda, v, t)dv.$$

Next we remove the Λ dependence out of f using a Taylor expansion to get

$$\Lambda \int_{3t/\Lambda}^\infty P(f^{(b)}(v^2), \Lambda^a, v, t)dv.$$

Performing the Taylor expansion in the lower limit of integration we are led to two classes of terms

$$h(x) = \sum_{j=0} h^{(j)}(0)\frac{x^j}{j!}.$$

Class 1: $j = 0$.

Simply set the lower limit to zero, and use a Taylor expansion. This class of terms has non-vanishing contributions up to order Λ^5.

Class 2: $j \neq 0$.

These terms have non-vanishing contributions no higher than constant order in Λ. The constant order term in Λ is a polynomial in t of degree 5.

6.7.5 The Boundary Terms $\frac{b_{2i}}{(2i)!}\left(\frac{\partial}{\partial h}\right)^{2i}\int g(0,q)dq$

These terms are straightforward to handle and are of the form

$$-\frac{b_{2i}}{(2i)!}\left(\frac{\partial}{\partial q}\right)^{2i-1} g(0,q)\Big|_{q=0}.$$

These terms contribute no higher than constant order in Λ. The constant order term in Λ is constant in t.

6.7.6 Corner Term

Finally we have the term $\gamma g(0,0)$. This contributes no higher than constant order in Λ, and the constant order term in Λ is constant in t.

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