Möbius-like Groups of Homeomorphisms of the Circle

Thesis by

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In Partial Fulfilment of the Requirements
for the degree of
Doctor of Philosophy

California Institute of Technology

Pasadena, California

1994

(Submitted May 23, 1994)

Acknowledgements

I want to express special thanks to my advisor David Gabai who has contributed to this work in many ways. To begin with, he is the one who suggested the problem studied in this thesis. He has also provided many useful discussions about the issues I have encountered in my research. But, more than anything, I want to thank David for being stubbornly optimistic and encourageing at times when things were in a deadlock. I have learned from him how to approach problems with the faith that they can be solved if one is ready to put in a lot of hard thinking and self-discipline.

I also want to thank Ramin Naimi with whom I have had many useful conversations at the beginning stage of my research. He actually provided interesting examples which led to the right statements of some results.

N.K.

Abstract

A group $G \hookrightarrow Homeo_+(S^1)$ is a Möbius-like group if every element of G is conjugate in $Homeo(S^1)$ to a Möbius transformation. Our main result is: given a Mobus-like like group G which has at least one global fixed point, G is conjugate in $Homeo(S^1)$ to a Möbius group if and only if the limit set of G is all of S^1. Moreover, we prove that if the limit set of G is not S^1, then after identifying some closed subintervals of S^1 to points, the induced action of G is conjugate to an action of a Möbius group.

We also show that the above result does not hold in the case when G has no global fixed points. Namely, we construct examples of Möbius-like groups with limit set equal to S^1, but these groups cannot be conjugated to Möbius groups.

Contents

1. Introduction

Denote by $\mathcal{M}$ the group of orientation preserving Möbius transformations of the complex plane preserving the unit disc $\overline{D} = \{z \in C \mid |z| \leq 1\}$. Recall that the elements of $\mathcal{M}$ are transformations of the form

$$z \mapsto \frac{az + \bar{c}}{cz + \bar{a}},$$

where a and c are complex numbers such that $|a|^2 - |c|^2 = 1$. Viewing the open unit disc $D = \{z \in C \mid |z| < 1\}$ as the standard Poincare model for the two-dimensional hyperbolic space $\boldsymbol{H^2}$, the group $\mathcal{M}$ is $Isom_+(\boldsymbol{H^2})$, the group of orientation preserving isometries of $\boldsymbol{H^2}$. On the other hand, every element of $\mathcal{M}$ preserves the unit disc, so it also preserves its boundary, the unit circle $S^1 = \{z \in C \mid |z| = 1\}$. In other words, denoting the group of all orientation preserving homeomorphisms of S^1 by $Homeo_+(S^1)$, we have the following inclusion

$$\mathcal{M} \hookrightarrow Homeo_+(S^1)$$

$$f \longmapsto f|_{S^1}$$

where $f|_{S^1}$ denotes the restriction of f to S^1. This inclusion is faithful, i.e., two distinct Möbius transformations induce distinct actions on S^1. So one can view $\mathcal{M}$ as a subgroup of $Homeo_+(S^1)$, and this is the way $\mathcal{M}$ is viewed for most of this study, and in particular in the definition that follows.

Definition. *Given $f \in Homeo_+(S^1)$ we say that f is Möbius-like if there exists some homeomorphism g of S^1 such that $gfg^{-1} \in \mathcal{M}$. Accordingly, the group $G \hookrightarrow Homeo_+(S^1)$ is Möbius-like if every element of G is Möbius-like.*

There are two remarks to be made about the above definition. Firstly, the conjugating homeomorphism g need not be orientation preserving. Secondly, homeomorphisms conjugating distinct elements of G to elements of $\mathcal{M}$ are in general distinct. This second remark brings us directly to the heart of this study: we investigate when we can use the same conjugating homeomorphism for all elements of G. In other words we address the following question:

(*) Under which conditions is a Möbius-like group G actually Möbius up to conjugation, i.e., when can we find a single $g \in Homeo(S^1)$ so that $gGg^{-1} \hookrightarrow \mathcal{M}$?

Before announcing the main results of this study, let us review some definitions and known facts.

There are three types of elements of $\mathcal{M}$ which we distinguish by looking at their fixed points on S^1 : *hyperbolic* (transformations having two fixed points on S^1, one attractive, one repulsive; these two points correspond to the endpoints on S^1 of the hyperbolic axis of the transformation.); *parabolic* (transformations having one fixed point on S^1); *elliptic* (transformations having no fixed points on S^1; these are sometimes called hyperbolic rotations, because they can always be conjugated, within $\mathcal{M}$, to maps of the form $z \mapsto ze^{2\pi i\theta}$. If $\theta = p/q$ is rational, then the corresponding rotation has finite order q . If θ is irrational, then the corresponding rotation has infinite order.)

Orient S^1 counterclockwise. Recall that every homeomorphism f of S^1 lifts to a homeomorphism $\tilde{f}$ of $\boldsymbol{R}$ via

$$f(e^{2\pi i x}) = e^{2\pi i \tilde{f}(x)}, \quad x \in \boldsymbol{R}.$$

The three types of Möbius transformations mentioned above can easily be distinguished by looking at their lifts to $\boldsymbol{R}$, as suggested in Figure 1.1.

In order to get a clearer picture of Möbius-like homeomorphisms, we want to describe them topologicaly. The most immediate observation is: the number of fixed points is invariant under conjugation, so Möbius like homeomorphisms can have two, one or no fixed points on S^1. Accordingly we have the same characterization as for Möbius transformations: a Möbius-like homeomorphism of S^1 can be either hyperbolic, or parabolic or elliptic. However, there is more to the topology of Möbius like maps than just fixed points.

Topological characterization of Möbius-like maps.[1] Let $f \in Homeo_+(S^1)$, $f \neq id$. We want to give a method for deciding whether f is Möbius-like. We already know that f can have at most 2 fixed points, so we distuingish the following cases.

f has two fixed points on S^1. If one fixed point is attractive, usually denoted P_f, and the other is repulsive, usually denoted N_f, then f is Möbius-like hyperbolic.

f has one fixed point on S^1. Then f is Möbius-like parabolic.

f has no fixed points on S^1. Then things are more complicated. If f has finite order, i.e., $f^n = id$ for some n, then f is Möbius-like elliptic.

Now, if f has infinite order, then f needs to satisfy some additional condition in order to be Möbius-like, as demonstrated by Denjoy's construction below. For example, if the f-orbit of some point $x \in S^1$ is dense in S^1, then f is Möbius-like elliptic of infinite order.

Denjoy's construction.[2] Start with a genuine rotation f of infinite order. Say $f(z) = ze^{2\pi i\theta}$ with θ irrational. Choose any $x \in S^1$. Then the f-orbit of x, $o(x)$, is a

[1] A proof of this characterization can be found in e.g. [T]

[2] See [D].

countable dense set in S^1. Now construct a new, bigger circle $\overline{S^1}$ by inserting a closed interval at each point of $o(x)$, taking care that the total sum of the lengths of the inserted intervals is finite. See Figure 1.2. f induces a homeomorphism $\overline{f} : \overline{S^1} \to \overline{S^1}$ in the following way: if a point $z \in \overline{S^1}$ was untouched by the construction, i.e., no interval was inserted at z, set $\overline{f}(z) = f(z)$; to define $\overline{f}$ on the interval inserted at a point $z \in o(x)$, choose any orientation preserving homeomorphism mapping that interval to the interval inserted at the point $f(z) \in o(x)$. Now, $\overline{f}$ has no periodic points, so it cannot be conjugated to a finite order Möbius transformation. On the other hand, it cannot be conjugated to an irrational rotation of $\overline{S^1}$ either because there are $\overline{f}$-orbits which are not dense in $\overline{S^1}$, and that is a property which is invariant under conjugation.

There is another way of characterizing Möbius-like maps, namely using the notion of a convergence group.

Definition. (Ghering-Martin) *Let G be a subgroup of $Homeo_+(S^1)$. G is a convergence group if every sequence $\{f_n\}$ in G has a subsequence $\{f_{n_i}\}$ such that either:*

a) $\exists\ x, y \in S^1$ such that

$f_{n_i} \to y$ *uniformly on compact subsets of* $S^1 - \{x\}$

$f_{n_i}^{-1} \to x$ *uniformly on compact subsets of* $S^1 - \{y\}$, *or*

b) $\exists\ f \in Homeo_+(S^1)$ such that

$f_{n_i} \to f$ *uniformly on* S^1

$f_{n_i}^{-1} \to f^{-1}$ *uniformly on* S^1.

For example, $< \overline{f} >$, the group generated by the map $\overline{f}$ from Denjoy's construction is not a convergence group. Indeed, the original Möbius map f is an irrational rotation, so there exists a sequence $\{f^{n_i}\}$ of the powers of f which converges to the identity map. But the corresponding sequence $\{\overline{f^{n_i}}\}$ does not satisfy the convergence property. See Figure 1.3.

On the other hand, the group generated by any Möbius-like map h is a convergence group. Figure 1.4 illustrates this property in the case when h is hyperbolic. Actually, the next theorem expresses that the converse is true as well.

Theorem A. *$f \in Homeo_+(S^1)$ is Möbius like if and only if $< f >$, the group generated by f is a convergence group.*

The proof of Theorem A is not hard and can be found in e.g. [T]. So this theorem gives us another way of describing a Möbius-like map in topological terms. However, there is a much stronger result .

Theorem B. *A group $G \hookrightarrow Homeo_+(S^1)$ is conjugate in $Homeo(S^1)$ to a Möbius group if and only if G is a convergence group.*

Part ($\Rightarrow$) of Theorem B is straightforeward. On the other hand, the ($\Leftarrow$) part of the proof is highly nontrivial and was obtained over a number of years through the work of many mathematicians. It started with Nielsen [N], [3] and then Ziechang [Z] and Tukia [T] had generalized Nielsen's set up and resolved some special cases. They were mostly interested in discrete convergence groups. The case that was left was the hardest in some sense and was resolved independently, using quite different methods, by Gabai [G] on one hand and Casson and Jungreis [C-J] on the other. The main idea of the proof of Theorem B in the case when G is a discrete group

[3] Actually, notion of a convergence group did not exist in Nielsen's time. Instead, he was looking at subgroups of $\pi_0 Homeo(S)$, the group of isotopy classes of homeomorphisms of a closed surface S with negative Euler characteristic. He conjectured that every finite subgroup of $\pi_0 Homeo(S)$ can be realized as a group of isometries of a hyperbolic structure on S. This was known as Nielsen's realization problem.

can be summarized as follows: given a discrete convergence group G acting on S^1, extend its action to the whole unit disc $\overline{D}$ in such a way that G acts as a discrete convergence group on the disc as well. By looking at the quotient $\overline{D}/G$, one finds a reparametrization of $\overline{D}$ under which the action of G becomes Fuchsian.[4] Hinkkanen [H] proved the theorem for nondiscrete convergence groups.

Using theorems A and B we can now rephrase our main question (*) in the following way:

(**) Given a group $G \hookrightarrow Homeo_+(S^1)$ whose every element generates a convergence group, what can we say about the whole of G: is G a convergence group?

Before answering this question we need to make distinctions between three major cases: a group G can have two, one, or no global fixed points (i.e., points which are fixed by all elements of G). Then we have the following results.

Theorem. *Suppose G is a non-cyclic Möbius-like group with either one or two global fixed points . Then, G is a convergence group if and only if $L(G) = S^1$.*[5]

Moreover, if $L(G) \neq S^1$ then $L(G)$ is a Cantor set and $S^1 - L(G)$ is an infinite union of disjoint open intervals (x_i, y_i) so that G with the induced action on a new circle $S^1_\star$, which is obtaind from S^1 by identifying intervals $[x_i, y_i]$ to points, is a convergence group.

The above theorem actually puts together the contents of theorems 3.1, 3.4, 4.1, and 4.3. Now in the case when G is a Möbius-like group without global fixed points, we do not have such nice results. Namely, we demonstrate various examples of Möbius-like groups which are not convergence groups, although their limit sets

[4] Fuchsian group is a discrete subgroup of $\mathcal{M}$

[5] $L(G)$ denotes the limit set of G, see Definition 1.6.

are all of S^1 (see examples 5.3, 5.4, 5.5 and 5.6).

Remark. There is another way of expressing the above theorem in the case when G has exactely one fixed point. It was suggested to me by Francis Bonahon. Namely, one can view S^1 as $\boldsymbol{R} \cup \{\infty\}$ so that every element of G fixes ∞. With this in mind we have the following result.

Suppose G is a group of orientation preserving homeomorphisms of $\boldsymbol{R}$ such that every element of G can be conjugated (by a homeomorphism of $\boldsymbol{R}$) to an affine transformation (that is a map of the form $x \mapsto ax + b,\ \ a, b \in \boldsymbol{R}$). Then the whole group G can be conjugated by a single homeomorphism of $\boldsymbol{R}$ to an affine group (i.e., a group whose every element is an affine transformation).

The organization of this thesis is the following. In chapter 2 we review some basic definitions and facts and we prove some technical lemmas; also we explain some abuses of notation that appear later in the text. In chapter 3 we prove the characterization of the Möbius-like groups with two global fixed points as stated in the theorem above. In chapter 4 we do the same for Möbius-like groups with one global fixed point. Finally, in chapter 5 we give several examples of Möbius-like groups without fixed points which fail to be convergence groups in an "irreparable" way. In the end , we also give a conjecture on the characterization of Möbius-like groups without global fixed points. This is conjecture 5.7. It states that the set of well understood examples (5.3 through 5.6) basically describes the set of all possible ways the convergence property can be violated in a Möbius-like group.

2. Preliminary Definitions and Observations

Throughout this thesis we restrict ourselves to orientation preserving homeomorphisms of the circle only.

View S^1 as being oriented counterclockwise; therefore (x, y) denotes all the points counterclockwise from x to y. We define $[x, y)$, $(x, y]$, $[x, y]$ similarly. Given two open intervals (x, y) and (u, v) on S^1 we write $(x, y) \frown (u, v)$ if either $x < u < y < v < x$ or $u < x < v < y < x$, see Figure 2.1.

The universal cover of S^1 is $\boldsymbol{R}$ via covering map $x \mapsto e^{2\pi i x}$, $x \in \boldsymbol{R}$. By abuse of notation, we will denote all lifts to $\boldsymbol{R}$ of a point $x \in S^1$ by the same symbol x.

Given $f \in Homeo_+(S^1)$, it can be lifted to a homeomorphism $\tilde{f} : \boldsymbol{R} \to \boldsymbol{R}$ so that $f(e^{2\pi i x}) = e^{2\pi i \tilde{f}(x)}, \forall x \in \boldsymbol{R}$. Since f is orientation preserving, $\tilde{f}$ is a strictly increasing function. There are many different lifts of f, but we will most commonly take $\tilde{f}$ to be the lift of f whose graph is "the closest" to the line $y = x$. Actually, by abuse of notation we will use f for $\tilde{f}$.

Many arguments in this thesis deal with functions $S^1 \to S^1$ which are limits of sequences of homeomorphisms of S^1. It is therefore natural to introduce the following convention about drawing the graphs of such functions (i.e., their lifts to $\boldsymbol{R}$): if say a function f is a limit function of some sequence $\{f_n\}$ in $Homeo_+(S^1)$, and f has a jump discontinuity at some point x, then we will draw a vertical segment between points $(x, f_-(x))$ and $(x, f_+(x))$ as a part of the graph of f.

Definition 1.1. The group of all orientation preserving Möbius transformations of the complex plane $\boldsymbol{C}$, which preserve the unit disc $\overline{D} = \{z \in \boldsymbol{C} | \ |z| \leq 1\}$, is denoted by $\mathcal{M}$. Every element of $\mathcal{M}$ is of the form

$$z \mapsto \frac{az + \bar{c}}{cz + \bar{a}},$$

where a and c are complex numbers such that $|a|^2 - |c|^2 = 1$. The open unit disc $D = \{z \in \boldsymbol{C} | \ |z| < 1\}$ serves as a model for the hyperbolic plane $\boldsymbol{H}^2$. The group $\mathcal{M}$ is exactly the group of orientation preserving isometries of $\boldsymbol{H}^2$. Every element of $\mathcal{M}$ preserves the boundary S^1 of D, so we have the following faithful inclusion

$$\mathcal{M} \hookrightarrow Homeo_+(S^1)$$

$$f \longmapsto f|_{S^1}.$$

It is sometimes convenient to view S^1 as $\boldsymbol{R} \cup \{\infty\}$ and D as the upper half-plane $H = \{z \in \boldsymbol{C} | \ Im(z) > 0\}$. Then $\mathcal{M}$ becomes $PSL_2(\boldsymbol{R})$, i.e., the group of transformations of the form

$$z \mapsto \frac{az+b}{cz+d}, \quad z \in H,$$

where $a, b, c, d \in \boldsymbol{R}$, and $ad - bc = 1$. Accordingly, we have a faithful inclusion

$$PSL_2(\boldsymbol{R}) \hookrightarrow Homeo_+(\boldsymbol{R} \cup \{\infty\})$$

$$f \longmapsto f|_{\boldsymbol{R} \cup \{\infty\}}.$$

Definition 1.2. A group $G \hookrightarrow Homeo_+(S^1)$ is discrete if it is discrete in the compact-open topology of $Homeo_+(S^1)$. Equivalently, G is discrete if no sequence of elements of G converges uniformly to the identity map.

Definition 1.3. (Ghering-Martin)[6] Let G be a subgroup of $Homeo_+(S^1)$. G is a *convergence group* if every sequence $\{f_n\}$ in G has a subsequence $\{f_{n_i}\}$ such that either:

a) $\exists \ x, y \in S^1$ such that

$f_{n_i} \to y$ uniformly on compact subsets of $S^1 - \{x\}$

[6]In the original definition given in [G-M] G is assumed to be a subgroup of $Homeo(S^1)$

$f_{n_i}^{-1} \to x$ uniformly on compact subsets of $S^1 - \{y\}$, or

b) $\exists$, $f \in Homeo_+(S^1)$ such that

$f_{n_i} \to f$ uniformly on S^1

$f_{n_i}^{-1} \to f^{-1}$ uniformly on S^1.

In terms of lifts of f_{n_i}'s, condition a) means that the graphs of the f_{n_i}'s approach a sort of step function consisting of jumps and flats of length 1. See Figure 2.2.

Observation 1.4 In the above definition one can replace conditions a) and b) by:

a') $\exists x, y \in S^1$ such that

$f_{n_i} \to y$ pointwise on $S^1 - \{x\}$

$f_{n_i}^{-1} \to x$ pointwise on $S^1 - \{y\}$

b') $\exists f \in Homeo_+(S^1)$ such that

$f_{n_i} \to f$ pointwise on S^1

$f_{n_i}^{-1} \to f^{-1}$ pointwise on S^1.

Lemma 1.5 *Let $\{f_n\}$ be a sequence in $Homeo_+(S^1)$ such that $f_n \to f$ pointwise on S^1, where $f \in Homeo_+(S^1)$. Then $f_n \to f$ uniformly on S^1.*

Proof of Lemma 1.5. Consider the lifts of the f_n's and f on $[0,1]$. Given $\varepsilon > 0$, uniform continuity of f gives us $0 < \delta < \varepsilon/2$ such that

$$\forall x, y \in \boldsymbol{R}, \quad \text{if } |x - y| < \delta \text{ then } |f(x) - f(y)| < \varepsilon/2.$$

Partition interval $[0,1]$ by points $0 = x_0, x_1, \ldots, x_{n-1}, x_n = 1$, so that $|x_{i+1} - x_i| < \delta$ for all i. Then find n_0 such that $\forall n, \geq n_0$, $\forall i$, $|f_n(x_i) - f(x_i)| < \delta$. Consequently, for every $z \in (x_i, x_{i+1})$ we have

$$f(x_i) - \delta < f_n(x_i) < f_n(z) < f_n(x_{i+1}) < f(x_{i+1}) + \delta \text{ and } f(x_i) < f(z) < f(x_{i+1}),$$

because lifts of f_n's and f are increasing functions on $\boldsymbol{R}$. Thus

$$|f(z) - f_n(z)| < f(x_{i+1}) + \delta - f(x_i) + \delta = f(x_{i+1}) - f(x_i) + 2\delta < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Proof of Observation 1.4. b') $\Rightarrow$ b) is clear by Lemma 1.5.

a') $\Rightarrow$ a). In order to simplify the notation, assume $f_n \to y$ pointwise on $S^1 - \{x\}$ and $f_n^{-1} \to x$ pointwise on $S^1 - \{y\}$. We want to show that the first convergence is uniform on compact subsets of $S^1 - \{x\}$, so choose a closed interval $[a,b] \subset S^1 - \{x\}$. Given $\varepsilon > 0$ find n_0 such that $f_n(a), f_n(b) \in (y-\varepsilon, y+\varepsilon)$, for all $n \geq n_0$. It suffices to show that $f_n[a,b] \subset (y-\varepsilon, y+\varepsilon)$ for all but finitely many $n \geq n_0$.

Suppose the contrary, that there exists a subsequence $\{f_{n_i}\}$ (with $n_i \geq n_0$ for all i) such that $f_{n_i}[a,b] \not\subset (y-\varepsilon, y+\varepsilon)$ (equivalently $f_{n_i}[b,a] \subset (y-\varepsilon, y+\varepsilon)$) for all i. By passing to a subsequence we can assume that sequences $\{f_{n_i}(a)\}$ and $\{f_{n_i}(b)\}$ are monotonic and such that either:

(i) $f_{n_i}[b,a] \supset f_{n_{i+1}}[b,a]$ for all i, or

(ii) $f_{n_i}[b,a] \cap f_{n_{i+1}}[b,a] = \emptyset$ for all i.

In both cases $\exists c \in \{a,b\}$ such that $f_{n_1}(c) \neq y$ for all i. Set $z = f_{n_1}(c)$.

In case (i) we have $f_{n_i}^{-1} f_{n_1}[b,a] \not\subset [b,a]$ for all i. In particular, $f_{n_i}^{-1}(z) \in [a,b]$ for all i. But this contradicts $f_n^{-1} \to x$ pointwise on $S^1 - \{y\}$.

In case (ii) $f_{n_i}[b,a] \cap f_{n_1}[b,a] = \emptyset$ for all i, or equivalently $f_{n_i}^{-1} f_{n_1}[b,a] \subset (a,b)$. So again we get $f_{n_i}^{-1}(z) \in (a,b)$ for all i. Contradiction.

Definiton 1.6. Given a Möbius-like group G, the limit set of G, denoted $L(G)$, is the set of all $x \in S^1$ such that G does not act properly discontinuously at x (i.e., for every neighborhood U of x there exist infinitely many elements $g \in G$ such that $g(U) \cap U \neq \emptyset$). It is easy to see that $L(G)$ is a closed G-invariant subset of S^1.

3. Groups With Two Global Fixed Points

Theorem 3.1. *Let $G \hookrightarrow Homeo_+(S^1)$ be a Möbius-like group with two global fixed points a and b. In other words, G is a purely hyperbolic group whose every nontrivial element is a hyperbolic Möbius-like map with $\{a, b\}$ as its axis. Assume also that G is not cyclic.* [7] *Then G is a convergence group if and only if $L(G) = S^1$.*

Lemma 3.2. *Suppose G is as above. If $L(G) = S^1$, then for every $x \in (a, b)$ (respectively (b, a)) the orbit of x under G is dense in $[a, b]$ (respectively $[b, a]$).*

Proof of Lemma 3.2. Let $x \in (a, b)$. Denote by $o(x)$ the G-orbit of x. Suppose $o(x)$ is not dense in $[a, b]$, i.e., $[a, b] - \overline{o(x)} \neq \emptyset$. Note that for every nontrivial $h \in G$, with say $N_h = a$ and $P_h = b$,

$$h^n(x) \to b \quad \text{and} \quad h^{-n}(x) \to a, \quad \text{as } n \to \infty.$$

Hence $a, b \in \overline{o(x)}$. Consequently, $[a, b] - \overline{o(x)}$ is an infinite union of disjoint open intervals $(x_i, y_i), i = 1, 2, \dots$ where $a < x_i < y_i < b$ and $x_i, y_i \in \overline{o(x)}$ for all i. See Figure 3.1.

Fix some i. $L(G) = S^1$ implies the existence of some $g \in G, g \neq id$, such that $g(x_i, y_i) \cap (x_i, y_i) \neq \emptyset$.

We can have neither $g(x_i, y_i) \subset (x_i, y_i)$ nor $g(x_i, y_i) \supset (x_i, y_i)$ since that would mean that g has at least one fixed point in $[x_i, y_i]$. Hence $g(x_i, y_i) \frown (x_i, y_i)$. But this contradicts the definition of x_i, y_i.

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[7] If G is cyclic then we know that G is a convergence group by Theorem A.

Proof of Theorem 3.1.

($\Rightarrow$) Suppose G is a convergence group. Then G is a Möbius group up to conjugation. We can actually assume that G is a genuine Möbius group since the condition $L(G){=}S^1$ is preserved under conjugation. It is an easy fact that a purely hyperbolic Möbius group with two global fixed points is discrete if and only if it is not cyclic, i.e., it cannot be generated by a single transformation. Since we have assumed that G is not cyclic, it follows that G is non-discrete, and hence $L(G){=}S^1$.

($\Leftarrow$) Suppose $L(G){=}S^1$, and let us prove that G is a convergence group. Note that given any two distinct elements $g, h \in G$, the graphs of g and h intersect at points a and b and are disjoint on $(a,b) \cup (b,a)$. Indeed, if they had an intersection at some point $x \neq a, b$, that would mean that $g(x) = h(x)$. Hence the transformation $g^{-1}h$ would have x as its third fixed point (besides a and b), which is a contradiction.

Let $\{f_n\}$ be a sequence of distinct elements of G. The graphs of the f_n's are pairwise disjoint on $(a,b) \cup (b,a)$. Therefore, after passing to a subsequence, we can assume that $\{f_n\}$ is strictly monotonic on both (a,b) and (b,a), only in the opposite sense. More precisely, if $\{f_n\}$ is increasing on one of these two intervals, then it is decreasing on the other. (If f_n increases on both (a,b) and (b,a), then $f_{n+1}^{-1}f_n$ is not Möbius-like hyperbolic. See Figure 3.2.) By replacing the f_n's by f_n^{-1}'s , if neccesary, we can assume that

$$\begin{aligned} &f_1 > f_2 > \dots \quad \text{on } (a,b) \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\Delta) \\ &f_1 < f_2 < \dots \quad \text{on } (b,a). \end{aligned}$$

Also we can pass to a subsequence so that all f_n's share the same repulsive point and therefore share the same attractive point. Assume for example that $P_{f_n} = a, N_{f_n} = b$, for all n. See Figure 3.3. Monotonicity of the sequences $\{f_n\}$ and $\{f_n^{-1}\}$ implies the existence of functions $f, f' : S^1 \to S^1$ so that

$$f_n \to f, \quad f_n^{-1} \to f' \quad \text{pointwise on } S^1.$$

In order to show that $\{f_n\}$ satisfies the convergence property we only have to show that: **(*) either both f and f' are continuous (so they are homeomorphisms) or else both f and f' are step functions consisting of jumps and flats of length 1.**

Note that both f and f' are nondecreasing functions (being limits of sequences of increasing functions), so the only type of discontinuity they can have is a jump. So assume **(*)** fails. That means that one of f, f' has a jump of length less than 1 at some point. Let us examine all the possibilities for such a jump and show that in each case one ends up with a contradiction.

1. f cannot have a jump at a since $P_{f_n} = a$ for all n. Similarly, f' has no jump at b.

2. f cannot have any jumps on $(a,b) \cup (b,a)$ (and the same is true for f'). Indeed, if f had a jump at some $x \in (a,b)$, i.e., $f_-(x) < f_+(x)$, see Figure 3.4, then by Lemma 3.2. we could find $g \in G$ such that $g(x) \in (f_-(x), f_+(x))$. But then the graph of g intersects the graph of f_n for n large enough, i.e., $g^{-1}f_n$ has more than two fixed points. Contradiction. Similarly for $x \in (b,a)$.

3. Finally, we show that f cannot have a jump of length < 1 at b (similarly, f' has no jump of length < 1 at a). Suppose the contrary, $f_-(b) \neq f_+(b)$. We distuingish the following cases:

   - $f_-(b) \in (a,b)$ or $f_+(b) \in (b,a)$ See Figure 3.5.

     Choose any $g \in G$ with $P_g = a, N_g = b$. Then for m large enough the graph of g^m intersects the jump of f at b and hence the graphs of f_n for n large enough. Contradiction.

Similarly, we cannot have $f'_+(a) \in (a,b)$ or $f'_-(a) \in (b,a)$.

- $f_-(b) = a,\ f_+(b) = b$ (The case $f_-(b) = b,\ f_+(b) = a$ is done similarly) Refer to Figure 3.6.

  Using 2. we conclude that f is a homeomorphism on $[b,a]$. But then again for any $g \in G$ such that $P_g = a, N_g = b$ the element $g^{-m} f_n$ is not hyperbolic for n, m large enough.

This finishes the proof under the assumptions that $P_{f_n} = a$ and $N_{f_n} = b$. Now we examine the case $P_{f_n} = b,\ N_{f_n} = a,\ \forall n$. Remember that our sequence $\{f_n\}$ satisfies the conditions (Δ). See Figure 3.7. In the same way as before we obtain functions f, f'. Clearly, f cannot have jump at b since $P_{f_n} = b$. Similarly, f' cannot have a jump at a. Now, the conditions (Δ) imply that f cannot have a jump at a either. Analogously, f' cannot have jump at b. So the only possibility for the jump of f or f' of length < 1 would be somwhere on $(a,b) \cup (b,a)$. But the same argument given in 2. works here as well. This finishes the proof that G is a convergence group.

Remark 3.3. The hypothesis $L(G) = S^1$ cannot be dropped from the above theorem, as shown by the following example which is derived from Denjoy's construction. Take some countable nondiscrete purely hyperbolic Möbius group G, all of whose elements fix points a and b (e.g., the group generated by $x \mapsto \alpha x, x \mapsto \beta x$, with $\frac{\log \alpha}{\log \beta} \notin \mathbf{Q}$, where we view S^1 as $\boldsymbol{R} \cup \{\infty\}$ and $a = 0, b = \infty$). Choose $x_0 \in (a,b)$. Then, at every point in the G-orbit of x_0 insert a closed interval. Since G is countable, there will be countably many such intervals, so we can make sure that the lengths of the inserted intervals when added up give a finite value. All this action takes place in (a,b). See Figure 3.8. This process gives a new, bigger circle $\overline{S^1}$.

Now we can define an induced action of G on $\overline{S^1}$ so that $G \hookrightarrow Homeo_+(\overline{S^1})$. Given $f \in G$, let $\overline{f}$ be a homeomorphism of $\overline{S^1}$ induced by f in the following way.

- If x is a point untouched by the construction, i.e., no interval was inserted at x, set $\overline{f}(x) = f(x)$.
- Given an interval $[u, v]$ which was inserted at the point $z = h(x_0)$ for some $h \in G$, define $\overline{f}$ on $[u, v]$ to be any orientation preserving homeomorphism which maps $[u, v]$ to the interval inserted at $f(z)$.

Note that the action of G on S^1 has the property that for any pair of distinct elements f_1, f_2 of G, if $z \neq a, b$ then $f_1(z) \neq f_2(z)$, so we will never encounter any problems when defining $\overline{f_1}, \overline{f_2}$ on the inserted intervals. Set

$$\overline{G} = \{id_{\overline{S}}\} \cup \{\overline{f} \mid f \in G - \{id_{S^1}\}\}.$$

It is easy to check that $\overline{G}$ is a subgroup of $Homeo_+(\overline{S^1})$ and that every element of $\overline{G}$ is conjugate in $Homeo(\overline{S^1})$ to a Möbius transformation on $\overline{S^1}$. [8] Also $\overline{G}$ is discrete because given any point z in $(a, b) \subset \overline{S^1}$, the $\overline{G}$-orbit of z is not dense in $[a, b]$. But $\overline{G}$ is not a convergence group acting on $\overline{S^1}$: since G is nondiscrete we can find sequence $\{f_n\}$ in G converging to id_{S^1} and that implies that the induced sequence $\{\overline{f_n}\}$ in $\overline{G}$ does not satisfy the convergence property. See Figure 3.9.

Note. In the above example we disturbed the action of G by inserting intervals at points of (a, b) only. We could have defined a new circle $\overline{S^1}$ by inserting intervals at points of the orbit of y_0 for some $y_0 \in (b, a)$ as well. The next theorem shows that the examples obtained in these two ways are essentially generic. As mentioned at the begining of this chapter, $G \hookrightarrow Homeo_+(S^1)$ is assumed to be a noncyclic Möbius-like group.

[8] Any homeomorphism of the circle that has two fixed points, one expanding and one contracting, is conjugate to a Möbius transformation. For a proof see [T].

Theorem 3.4 *Let $G \hookrightarrow Homeo_+(S^1)$ be a non-cyclic Möbius-like group with two global fixed points a and b. If $L(G) \neq S^1$ then $L(G)$ is a Cantor set, and $S^1 - L(G)$ is an infinite union of disjoint open intervals (x_i, y_i) so that G with the induced action on a new circle $S^1_\star$, which is obtaind from S^1 by identifying intervals $[x_i, y_i]$ to points, is a purely hyperbolic convergence group.*

Proof of Theorem 3.4. Suppose $L(G) \neq S^1$. Note that $a, b \in L(G)$. First assume that the interior of $L(G)$ is empty. Then $L(G)$ is a Cantor set on S^1, i.e.,

$$S^1 - L(G) = \bigcup_{i=1}^{\infty}(x_i, y_i)$$

where (x_i, y_i) are pairwise disjoint open intervals. Set

$$S^1_\star = S^1/[x_i, y_i] \sim point, \quad i = 1, 2, \ldots.$$

In other words, $S^1_\star$ is obtained from S^1 by identifying closed intervals $[x_i, y_i]$ to points. Clearly, $S^1_\star$ is homeomrphic to S^1. Since $\bigcup_{i=1}^{\infty}[x_i, y_i]$ is invariant under the action of G, we conclude that there is an induced action of G on $S^1_\star$. Write $G_\star$ when having this new action in mind. It is easy to see that $L(G_\star) = S^1_\star$, so by the Theorem 3.1 $G_\star$ is a convergence group.

Now if $intL(G) \neq \emptyset$ then there exists a closed interval $[c, d] \subset L(G)$ which is maximal in the sense that points c and d are approached by points of $S^1 - L(G)$. Then $\{c, d\} = \{a, b\}$. Indeed, if say $c \in (a, b)$ then find a small positive ε so that $(c, c + \varepsilon) \cap \{a, b\} = \emptyset$ and $(c, c + \varepsilon) \subset (c, d)$. Furthermore, $c \in L(G)$ so we can find $f \in G$ such that $f(c, c+\varepsilon) \frown (c, c+\varepsilon)$. Hence either $c \in f(c, c+\varepsilon)$ or $c \in f^{-1}(c, c+\varepsilon)$. In both cases we get a contradiction with the fact that c is approached by points of S^1-$L(G)$. Without loss of generality we can assume $[c, d] = [a, b]$. Then $L(G)$ consists of $[a, b]$ and some Cantor set on $[b, a]$. Therefore the action of G on $[a, b]$ is Möbius. In the same way as above one can identify appropriate intervals in

$[b, a]$ to points to obtain a new action $G_\star$ on a new circle (the action on $[a, b]$ stays unchanged). Then $G_\star$ is a convergence group.

4. Groups With One Global Fixed Point

We have the same theorem as in the case of groups with two global fixed points.

Theorem 4.1. *Let $G \hookrightarrow Homeo_+(S^1)$ be a Möbius-like group with one global fixed point a. In other words, every nontrivial element of G is either parabolic fixing a, or hyperbolic with a being one of its fixed points. Assume also that G is not cyclic. Then G is a convergence group if and only if $L(G) = S^1$.*

Lemma 4.2. *If $L(G) = S^1$, then for every $x \in S^1 - \{a\}$ the orbit of x under G is dense in S^1.*

Proof of Lemma 4.2. Suppose there exists some $x \in S^1 - \{a\}$ whose G-orbit $o(x)$ is not dense in S^1, i.e., $S^1 - \overline{o(x)} \neq \emptyset$. G does not have two global fixed points so $a \in \overline{o(x)}$ (To see this take any nontrivial $g \in G$ such that $g(x) \neq x$. If g is parabolic, then $g^n(x) \to a$. If g is hyperbolic, then either $g^n(x) \to a$ or $g^{-n}(x) \to a$.) Therefore we can find an open interval $(u, v) \subset S^1 - \{a\}$ such that $(u, v) \subset S^1 - \overline{o(x)}$ and $u, v \in \overline{o(x)}$.

$L(G)= S^1$ implies the existence of some $g \in G, g \neq id$ such that $g(u, v) \cap (u, v) \neq \emptyset$. $S^1 - \overline{o(x)}$ is G-invariant, so we cannot have $g(u, v) \frown (u, v)$. For the same reason neither $g(u, v) \subset (u, v)$ nor $g(u, v) \supset (u, v)$. Hence $g(u, v) = (u, v)$. But this gives g at least three fixed points. Contradiction.

Proof of Theorem 4.1. ($\Leftarrow$): Suppose G is a convergence group. Then G is a Möbius group up to conjugation by some homeomorphism of S^1. We can assume that G is a genuine Möbius group with one global fixed point. But every noncyclic Möbius group with one global fixed point is nondiscrete, so $L(G) = S^1$.

($\Rightarrow$): Suppose $L(G) = S^1$. We distinguish two cases.

1. G IS A PURELY PARABOLIC GROUP, i.e., every element of G is parabolic. The graphs of lifts to $\boldsymbol{R}$ of distinct elements of G are disjoint everywhere except at where they all intersect, so this case can be done in much the same way as in the proof of Theorem 2.1.

Namely, we start with an arbitrary sequence $\{f_n\}$ in G. After passing to a subsequence we can assume that $\{f_n\}$is monotonic. By replacing f_n's by f_n^{-1}'s if neccesary, we can assume that $\{f_n\}$ is increasing. Hence there exist limit functions $f, f' : S^1 \to S^1$ such that

$$f_n \to f, \quad f_n^{-1} \to f' \quad \text{pointwise on } S^1.$$

Also, we can pass to a subsequence so that all f_n's move the points of $S^1 - \{a\}$ in the same direction. First assume that all f_n's move the points of $S^1 - \{a\}$ in the counterclockwise direction. See Figure 4.1. As before, in order to prove that $\{f_n\}$ satisfies the convergence property we only need to show either both f and f' are continuous (so they are homeomorphisms) or else both f and f' are step functions consisting of jumps and flats of length 1.

So assume the contrary, that say f has a jump discontinuity of length less than 1. Figure 4.2 shows that such a jump cannot occur at the point a. Namely, we can take any parabolic $g \in G$ moving points of $S^1 - \{a\}$ in the counterclockwise direction. Then, for m large enough the graph of g^m intersects the graph of f at some point other than a and hence it also intersects the graph of f_n at some point other than a, for some large index n. But this contradicts our assumption that G is a purely parabolic group.

So now we only need to show that f cannot have a jump of length less than 1 at some point other than a either. If such a jump existed at some point x, then by Lemma 4.2. we could find some $g \in G$ such that $g(x) \in (f_-(x), f_+(x))$. But then the graph of g would have to intersect the graph of some f_n with high index n somwhere near x. That contradicts the fact that G is purely parabolic. This finishes the proof under the assumption that the f_n's move the points of $S^1 - \{a\}$ in the counterclockwise direction.

Now assume that the f_n's move the points of $S^1 - \{a\}$ in the clockwise direction. Recall that $\{f_n\}$ is increasing on (a, a). See Figure 4.3. We obtain functions f and f' in the same way as above.

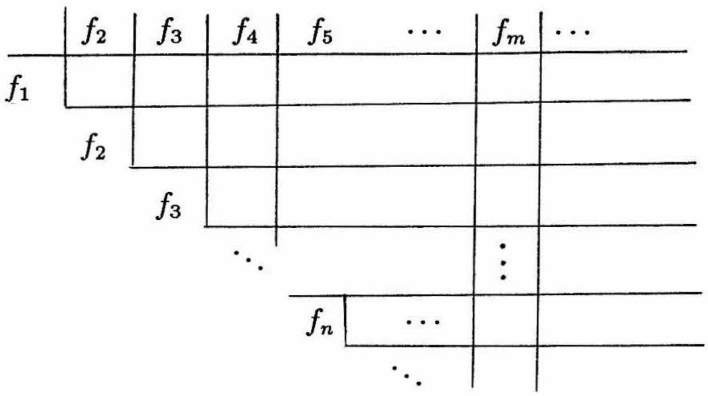
Again assume that say f has a jump discontinuity of length less than 1. Under the assumptions we made it is obvious that f cannot have any kind of jump at a. So a jump of length less than 1 has to occur at some point $x \in (a, a)$. Then the same argument as above shows that leads to a contradiction. This finishes the proof in the case when G is a purely parabolic group.

2. G HAS HYPERBOLIC ELEMENTS

By Lemma 4.2 it follows that L_0, the set of fixed points of all hyperbolic elements of G, is dense in S^1.

Let $\{f_n\}$ be a sequence of distinct elements in G. We want to show that $\{f_n\}$ satisfies convergence property.

First we show that after passing to a subsequence, we can get all the difference functions$f_m^{-1}f_n$'s with $m > n$, to be of the same type, i.e., all parabolic or all hyperbolic. This can be done in purely combinatorial way. To $\{f_n\}$ we can associate the following "matrix of relations":



where at the intersection of n-th row and m-th coloumn we put 1 if $f_m^{-1} f_n$ is parabolic, and we put 2 if $f_m^{-1} f_n$ is hyperbolic. One should memorize this rule in the following way: if the graphs of f_m and f_n intersect twice, we put 2; if they intersect once, we put 1. So the matrix corresponding to $\{f_n\}$ consists of 1's and 2's.

Combinatorial claim. There exists a subsequence of $\{f_n\}$ so that the matrix corresponding to that subsequence consists of either 1's only, or 2's only.

Proof of the combinatorial claim. Label every row of the matrix corresponding to $\{f_n\}$ by $\boxed{1}$,$\boxed{2}$ or $\boxed{12}$ according to the following rule:

label $\boxed{1}$ if the row contains only finitely many 2's

label $\boxed{2}$ if the row contains only finitely many 1's

label $\boxed{12}$ if the row contains infinitely many of both 1's and 2's.

Special case. Suppose that the matrix corresponding to $\{f_n\}$ has infinitely many rows labeled $\boxed{1}$. Then we can pass to a subsequence so that its matrix consists of 1's only.

Proof of the special case. Since there are infinitely many rows labeled $\boxed{1}$, we can first pass to a subsequence consisting of elements corresponding to these rows, i.e., we are getting rid of rows which are not labeled $\boxed{1}$. So now we have a new infinite sequence, call it $\{f_n\}$ again, whose matrix has the property that all its rows are labeled $\boxed{1}$. Set f_{n_1} to be the first element of this sequence. Then get rid of finitely many elements of $\{f_n\}$ which are producing 2's in the first row of the matrix of $\{f_n\}$. Let f_{n_2} be the next element among those that are left. Continue ... get rid of finitely many elements that produce 2's in the second row (note that the first row has 1's only), and take f_{n_3} to be the next among those that are left ... The sequence $\{f_{n_i}\}$ has the required property. This finishes the proof in the special case.

Now in general, given $\{f_n\}$ consider its matrix. If it contains infinitely many rows labeled $\boxed{1}$ or infinitely many rows labeled $\boxed{2}$, we are done by the special case. So assume all rows except finitely many are labeled $\boxed{12}$. Get rid of those finitely many elements whose rows are not labeled $\boxed{12}$. Now we have a sequence whose matrix has all rows labeled $\boxed{12}$.

After passing to a subsequence we can get the first row to consist of 1's only. Let f_{n_1} be the first element of this subsequence. Note that when we passed to this subsequence we obtained a new matrix with new labelings for the second, third, ... rows. This matrix can either have infinitely many rows labeled $\boxed{1}$, or infinitely many rows labeled $\boxed{2}$, or it can have all but finitely many rows labeled $\boxed{12}$. The first two possibilities put us in the special case, so we can assume that (after getting rid of finitely many elements) our matrix has all rows starting with the second one (remember the first row consists of 1's only) labeled $\boxed{12}$. Let f_{n_2} be the element corresponding to the second row of this last matrix. As before, throw out infinitely many elements of the sequence which produced 2's in the second row (i.e., pass to a subsequence). Now both the first and second row consist of 1's only, but the rest of the rows have new assignments. Again we are either done by the special case, or else we can continue our process and define f_{n_3} .... If this process ever stops, we are done by the special case, and if it never stops then it gives us an infinite subsequence $\{f_{n_i}\}$ with the required property.

End of proof of the combinatorial claim.

Now back to our initial sequence $\{f_n\}$. As shown above, after passing to a subsequence, we can assume that all $f_m^{-1}f_n$'s are of the same type.

Case II.1: $\forall m > n,\ f_m^{-1}f_n$ **is parabolic.**

In other words, the graphs of the f_n's are pairwise disjoint everywhere except at the point a where they all intersect. Therefore, after passing to a subsequence, $\{f_n\}$ is totally ordered, say $f_1 < f_2 < \cdots$ on (a, a). So there exist functions $f, f' : S^1 \to S^1$ such that $f_n \to f$ and $f_n^{-1} \to f'$ pointwise.

Suppose first that f has a jump of length less than 1 at some point. Figure 4.4 shows that such a jump cannot occur at a. Namely, one first notices the fact that $\{f_n\}$ is increasing on (a, a) and that $f_n(a) = a, \forall n$ imply $f_-(a) = a$. So, if $f_+(a) > a$, then choose any hyperbolic $h \in G$ such that $N_h = a$, $P_h \in (f_+(a), a)$. Clearly, for m large enough the graph of h^m will intersect the graph of f more than twice. Consequently, for n large enough $f_n^{-1}h^m$ has more than two fixed points. Contradiction.

So now assume a jump of length less than 1 occurs at some point $x \in (a, a)$. Denote $J = (f_-(x), f_+(x))$.

Claim. (Simplicity condition) There is no $h \in G$ such that $h(J) \frown J$.

Proof of claim. If such h exists, then by replacing h by h^{-1} we can assume that h moves J in a counterclockwise direction. Form a new sequence $\{hf_n\}$ and denote $hf = \lim_{n\to\infty} hf_n$. Then

$$(hf)_-(x) = h(f_-(x)),\ \ (hf)_+(x) = h(f_+(x)).$$

In other words, hf has a jump $h(J)$ at x. See Figure 4.5. Choose some high index n so that the graph of hf_n intersects the graph of f twice near x. Then for m large enough the graph of hf_n intersects twice the graph of f_m near x. This together with the global fixed point a gives three intersection points. A contradiction.

Now we will finish the proof by contradiction; i.e., we will prove that h as in the claim really exists, for any interval $J = (u, v)$ not containing a. Let p be any parabolic element of G which moves points of $S^1 - \{a\}$ in a clockwise direction. Refer to Figure 4.6. Let $k \in G$ be a hyperbolic element such that $N_k \in (u, v)$. Choose a large enough power m of k so that

$$k^m(v) > p^{-1}(v) > v.$$

Then $k^m(v) > pk^m(v) > v$, while $pk^m(u) < k^m(u)$. Hence $k^m(u,v) \frown pk^m(u,v)$. In other words,

$$(u, v) \frown k^{-m}pk^m(u, v),$$

so we can put $h = k^{-m}pk^m$.

So far we have proved that f cannot have jump discontinuities of length less than 1. Now, in order to prove the same for f', we first note that f' cannot have such jumps at points of (a, a) by the same argument which we used for f. So we only need to check whether f' can have a jump of length less than one at a. Recall that our initial assumption is that the sequence $\{f_n\}$ is increasing, i.e., $\{f_n^{-1}\}$ is decreasing. Hence $f'_+(a) = a$. Refer to Figure 4.7. If $f'_-(a) \in (a, a)$, then choose any hyperbolic $h \in G$ such that $N_h = a$, $P_h \in (a, f'_-(a))$. Clearly, for m large enough the graph of h^m will intersect the graph of f' more than twice. Contradiction.

Thus we had proved that the arbitrtary sequence $\{f_n\}$ in which all the difference functions $f_m^{-1}f_n$'s with $m > n$ are parabolic, satisfies the convergence property.

Case II.2: $\forall m > n,\ f_m^{-1} f_n$ **is hyperbolic.**

For every two elements f_m and f_n, $m > n$, either $P_{f_m^{-1} f_n} = a$ or $N_{f_m^{-1} f_n} = a$. By the same combinatorial argument as before (only here the "matrix of relations" has either P or N at the intersection of the n-th row and the m-th column, depending on whether $P_{f_m^{-1} f_n} = a$ or $N_{f_m^{-1} f_n} = a$), and by replacing $\{f_n\}$ by $\{f_n^{-1}\}$ if neccesary, we can assume that $P_{f_m^{-1} f_n} = a,\ \forall m > n$. We can also assume that after passing to a subsequence $\{N_{f_{n+1}^{-1} f_n}\}$ converges monotonicly to some point b.

(i) First assume $b \neq a$.

At this point our sequence $\{f_n\}$ is pretty "tame", i.e., it has the following property: given any $\varepsilon > 0$, there exists high enough index n so that

$$\left.\begin{array}{lll} f_n < f_{n+1} < \cdots & \text{on } (a, b-\varepsilon) & \text{and} \\ f_n > f_{n+1} > \cdots & \text{on } (b+\varepsilon, a) & \end{array}\right\} \quad (\Delta).$$

In other words, $\forall x \in S^1$ the sequence $\{f_n(x)\}$ becomes strictly monotonic after a large enough index, and the same is true for $\{f_n^{-1}(x)\}$. Therefore, there exist functions $f, f' : S^1 \to S^1$ such that $f_n \to f$ and $f_n^{-1} \to f'$ pointwise on S^1.

By the same arguments as in II.1, one can show that neither f nor f' can have a jump of length < 1 at a.

So let us show that f cannot have a jump of length < 1 at some $x \in S^1 - \{a, b\}$. Once that is established, we will only have to show that a jump of length less than 1 cannot occur at b. Suppose the contrary, that we have some $x \in (a, b) \cup (b, a)$ such that $f_-(x) \neq f_+(x)$. Set $J = (f_-(x), f_+(x))$. Note that $a \notin \overline{J}$, otherwise f' has a jump of length less than 1 at a.

We now list some consequences of the assumptions that we have made so far.

Claim 1. (Simplicity condition) $h(J) \frown J$ for no $h \in G$.

Proof of Claim 1. The same as in II.1.

Claim 2. For every $h \in G$ such that $b \notin \overline{h(J)}$, f is flat on $h(J)$.

Proof of Claim 2. If not, then f is non-constant on $h(J)$ so for large enough n we get $f_n(h(J)) \frown f_{n+1}(h(J))$, i.e., $h^{-1}f_{n+1}^{-1}f_nh(J) \frown J$. See Figure 4.8. Contradiction with Claim 1.

Set $A = \{y \in S^1 | \ f$ is flat in some neighborhood of $y\}$.

Claim 3. $L_0 - \{a, b\} \subset A$. [9]

Proof of Claim 3. Suppose $z \in L_0 - \{a, b\}$, i.e., $z = N_k, a = P_k$ for some hyperbolic element k of G. By Lemma 4.1, the G-orbit of z is dense in S^1, so $z \in h(J)$ for some $h \in G$. Then for sufficiently large m, the interval $I = k^{-m}(h(J))$ is small enough so that its closure contains neither a nor b. See Figure 4.9. By Claim 2 f is flat on I and $z \in I$, so $z \in A$.

As noted before L_0 is dense in S^1, so A is open and dense, and thus $S^1 - A$ is a Cantor set on S^1 containing a. Let c be a point in this Cantor set so that $b < c < a$. From now on we shall mostly concentrate on interval(c, a). Write

$$(c, a) \cap A = \bigcup_{i=1}^{\infty}(x_i, y_i),$$

where (x_i, y_i)'s are disjoint open intervals. From the definition of the intervals (x_i, y_i)'s it is clear that each one of them represents a maximal interval of flatness of the limit function f in the sense that if f is flat on an open interval $I \subset (c, a)$, then $I \subset (x_i, y_i)$ for some i.

Claim 4. $h(x_i, y_i) \frown (x_i, y_i)$ for no $h \in G$ and no i.

Proof of Claim 4. Since $\{f_n\}$ is monotonicly decreasing on (c, a) (equivalently, $\{f_n^{-1}\}$ is monotonicly increasing on (c, a)), and each (x_i, y_i) can be viewed as a jump of f', this claim is true for the same reason as Claim 1.

[9] Recall that L_0 denotes the set of all fixed points of hyperbolic elements of G.

Claim 5. If $h, k \in G$ are hyperbolic with $N_h, N_k \in (x_i, y_i)$, then h and k intersect within (x_i, y_i), i.e., $h^{-1}k$ is hyperbolic with one fixed point in (x_i, y_i).

Proof of Claim 5. If not, then $h(x_i, y_i) \frown k(x_i, y_i)$. See Figure 4.10. Eqivalently, $k^{-1}h(x_i, y_i) \frown (x_i, y_i)$ which contradicts Claim 4.

Claim 6. f is continuous at every y_i, $i = 1, 2, \ldots$.

Proof of claim 6. If not, then f has a jump at y_i, so f is strictly increasing on $[x_i, y_i]$. Thus for n large enough $f_n(x_i, y_i) \frown f_{n+1}(x_i, y_i)$. See Figure 4.11. Contradiction to Claim 4.

Claim 7. Let $g_n = f_{n+1}^{-1} f_n$. Then sequence $\{g_n\}$, when considered on the interval (c, a), converges to a function g whose graph consists of triangles arranged along the diagonal $y = x$, which are determined by the intervals (x_i, y_i). See Figure 4.12. More precisely,

$$\lim_{n \to \infty} g_n(x) = y_i, \quad x \in [x_i, y_i], \quad i = 1, 2, \ldots$$

After passing to a subsequence, $\{g_n\}$ converges to g on (c, a) in a strictly decreasing manner.

Proof of Claim 7. Fix i and let us see how does the sequence g_n behave on (x_i, y_i). Recall that the f_n's converge to f in a decreasing manner, f is flat on (x_i, y_i) and continuous at y_i. See Figure 4.12. So $f_{n+1}^{-1} f_n = g_n$ is an increasing function on (x_i, y_i) (actually, on all of (c, a)). This, together with Claim 4, implies that $x_i < y_i < g_n(x_i) < g_n(y_i)$. Moreover, continuity of f at y_i implies that as the index n increases, the intervals $g_n(x_i, y_i)$ become very small and very close to y_i. Therefore $g_n \to y_i$ on $[x_i, y_i]$. Since i was chosen arbitrarily, this proves that the limit function g has the right behavior. Now fix i again and pass to a subseqence so that $\{g_n\}$ is decreasing on (x_i, y_i). We claim that $\{g_n\}$ is then decreasing on all of (c, a), after a high enough index. Indeed, if this is not the case, then for arbitrarily large n, g_n and g_{n+1} intersect at some point $z \in (c, a)$. Let k be $g_{n+1}^{-1} g_n$ or ${g_n}^{-1} g_{n+1}$ depending on whether z comes before or after (x_i, y_i). In any case $z \in L_0 \subset A - \{a, b\}$, so z is contained in some (x_j, y_j). See Figure 4.13. By our choice of k, $z = N_k$, so k expands (x_j, y_j). But then $(x_j, y_j) \subset k(x_j, y_j) \subset (c, a)$, since we can take n as large as we want. By Claim 2 f has to be flat on $k(x_j, y_j)$ (think of (x_j, y_j) as a jump of f'). But this contradicts the definition of (x_j, y_j).

This finishes the list of technical claims that we need for the rest of the argument. Fix some $(x_i, y_i) \subset (c, a) \cap A$, and let h be any hyperbolic element of G such that $N_h \in (x_i, y_i)$. We now change our point of view. Namely, we view S^1 as $\boldsymbol{R} \cup \{\infty\}$ and G as a group acting on $\boldsymbol{R} \cup \{\infty\}$ whose every element fixes ∞. Strictly speaking we conjugate G by some homeomorphism $\phi : S^1 \to \boldsymbol{R} \cup \{\infty\}$ such that $\phi(a) = \infty$. Points on $\boldsymbol{R} \cup \{\infty\}$ we denote by the same symbols as their preimages under ϕ, and in general we abuse notation by denoting the maps of S^1 and their conjugates by ϕ, which are maps of $\boldsymbol{R}$, by the same symbols. After conjugating the whole group if necessary, we can assume that $N_h = 0$ and that $h(x) = x + 1$ on $(1, \infty)$.

Let $M = max\{1, y_i\}$. Note that the "triangles" defined by g are preserved under h on $[M, \infty)$, i.e., given any $(x_j, y_j), M < x_j$ we have $h[x_j, y_j] = [x_l, y_l] = [x_j + 1, y_j + 1]$, for some l. This comes as a consequence of the maximality of (x_j, y_j) as an interval of flatness of f.

Choose any hyperbolic $k_0 \in G$ such that $N_{k_0} \in (x_i, y_i), N_{k_0} > 0 = N_h$. By Claim 5 we know that the graph of k_0 intersects the graphs of all $h^p, p > 0$ within (x_i, y_i). Therefore the graphs of $k_0, h, h^2, \ldots$ are disjoint on $[M, \infty)$, and we can find $p \geq 0$ such that $h^p < k_0 < h^{p+1}$ on $[M, \infty)$. Set

$$k_n = h^{-n} k_0 h^n, \ \ n = 0, 1, 2, \ldots.$$

It is easy to see that:

(i) the graph of k_n on $[M, \infty)$ is just the graph of k_{n-1} on $[M + 1, \infty)$ shifted by the vector (-1, -1).

(ii) $N_{k_n} = h^{-n}(N_{k_0}) \to 0, n \to \infty$ and thus $N_{k_n} \in (x_i, y_i)$.

(iii) $h^p < k_n < h^{p+1}$ on $[M, \infty)$ for all n.

Now we have a sequence $\{k_n\}$ such that the graphs of all the k_n's on $[M, \infty)$ are contained in the strip between the lines $y = x + p$ and $y = x + p + 1$; moreover, the graphs of the k_n's are all disjoint so that we have

$$\underline{\text{e}}\text{ither} \quad k_0 > k_1 > \cdots \ \text{on}\,[M, \infty) \quad \underline{\text{o}}\text{r} \quad k_0 < k_1 < \cdots \quad \text{on}\,[M, \infty).$$

Let us examine the first possibility (the second one is done in an analoguous way). The sequence $\{k_n\}$ is decreasing on $[M, \infty)$, so let

$$k = \lim_{n \to \infty} k_n, \quad \text{on } [M, \infty).$$

See Figure 4.14. We first note that k has exactly the same intervals of flatness as g on $[M, \infty)$. Indeed, if say k was not flat on some (x_j, y_j), i.e., if it was strictly increasing, we would have $k_n(x_j, y_j) \frown k_{n+1}(x_j, y_j)$ for n large which contradicts Claim 4. On the other hand, if k was flat on some interval $I \supset (x_j, y_j)$, then $g_n(I) \frown g_{n+1}(I)$ for n large, which gives a contradiction in the same way as in the proof of the Simplicity condition.

Thus, setting $w_n = k_{n+1}^{-1} k_n, \ n = 1, 2, \ldots$ we get

$$\lim_{n \to \infty} w_n = g \ \text{ on } [M, \infty).$$

Note. As we move closer to ∞, the graph of w_0 gets closer to the limit function g. More precisely, since $w_n = h^{-n} w_0 h^n$ the graph of w_0 on $(x_j + n, y_j + n)$ is just the shift of the graph of w_n on (x_j, y_j). As n grows large the graph of w_n gets closer to the graph of g on (x_j, y_j), which means that the graph of w_0 gets to be very close to the graph of g on $(x_j + n, y_j + n)$. Refer to Figure 4.15.

Actually, this same property holds for every $w_0^m, m > 0$ as well. Indeed, consider some (x_{j_1}, y_{j_1}). Given $\varepsilon > 0$ chose $m-1$ small intervals $(x_{j_2}, y_{j_2}), \ldots, (x_{j_m}, y_{j_m})$ so that they all fit in $(y_{j_1}, y_{j_1} + \varepsilon)$. See Figure 4.16. Choose n large enough so that

$$w_n < x_{j_{s+1}} \quad \text{on } [x_{j_s}, y_{j_s}] \ , s = 1, \ldots, m-1$$

and

$$w_n < y_{j_1} + \varepsilon \text{ on } [x_{j_m}, y_{j_m}].$$

Then it is not hard to see that

$$w_n^m < y_{j_1} + \varepsilon \text{ on } [x_{j_1}, y_{j_1}].$$

Equivalently,

$$w_0^m < y_{j_1} + n + \varepsilon \quad \text{on } [x_{j_1} + n, y_{j_1} + n].$$

In particular, the graph of w_0^m has to lie between h and id if we are sufficiently close to ∞.

On the other hand, $N_{w_0} \in (x_i, y_i)$ so w_0^m intersects h within (x_i, y_i) by Claim 5, for all $m > 0$. But for m large enough $w_0^m(y_i) > h(y_i)$. Contradiction.

Thus, we have proved that a jump of length less than 1 cannot occur at any point in $S^1 - \{a, b\}$, so the only case that is left is if such a jump occurs at b. If f had such a jump at b, then f would be a homeomorphism away from the point b. This would mean that some non-discreteness phenomena occurs (i.e., the sequence $\{f_{n+1}^{-1} f_n\}$ would converge to the identity map on some non-empty interval). But that easily leads to a contradiction.

(ii) Assume $a = b$. Figure 4.17 shows the two ways the sequence $\{f_n\}$ can behave, depending on from which side do the points $N_{f_{n+1}^{-1} f_n}$ approach the point $b = a$. Obviously, we can use the same argument as in (i) to show that the limit functions f and f' of the sequences $\{f_n\}$ and $\{f_n^{-1}\}$, respectivly, have no jumps at points of (a, a). Showing that they do not have jumps at a is also done in a familliar way.

Theorem 4.3. *If* $L(G) \neq S^1$ *then* $L(G)$ *is a Cantor set and* S^1-$L(G)$ *is an infinite union of disjoint open intervals* (x_i, y_i) *so that* G *with the induced action on a new circle* $S^1_\star$*, which is obtained from* S^1 *by identifying intervals* $[x_i, y_i]$ *to points, is a convergence group.*

Proof of Theorem 4.3. Clearly $a \in L(G)$. We cannot have $\{a\} = L(G)$ because that would imply that G is cyclic. Can $L(G)$ have non-empty interior? If so, then choose an interval $[u, v] \subset L(G)$ which is maximal in the sense that both u and v are approached by the points of $S^1 - L(G)$. Clearly $a \notin [u, v]$ because of the maximality of $[u, v]$. Choose $\varepsilon > 0$ small enough so that $u < u + \varepsilon < v$. See Figure 4.18. Thus $(u, u + \varepsilon) \subset L(G)$, so there exists some nontrivial element $g \in G$ such that $g(u, u+\varepsilon) \cap (u, u+\varepsilon) \neq \emptyset$. By looking at how $g(u, u+\varepsilon)$ can look with respect to $(u, u+\varepsilon)$, one sees a contradiction with the choice of u. So the conclusion is that $L(G)$ has empty interior.

Thus $L(G)$ is a Cantor set . Once this is established, the rest of the proof goes in the exact same way as the corresponding part of the proof of the Theorem 3.4.

5. Groups with no global fixed points

As announced in the introduction, for groups without global fixed points we do not have the nice results which we have for groups with global fixed points. However, in some simple special cases we do have such results. They are expressed in the next two propositions.

Proposition 5.1. *Suppose $G \hookrightarrow Homeo_+(S^1)$ is a non-cyclic purely elliptic Möbius-like group. Then G is a convergence group if and only if $L(G) = S^1$.*

Proposition 5.2. *Suppose $G \hookrightarrow Homeo_+(S^1)$ is a Möbius-like group of the following type: there exist two points $a, b \in S^1$ such that every element of G is either an elliptic element of order 2 permuting $\{a, b\}$, or a hyperbolic element with $\{a, b\}$ as its unoriented axis. Then G is a convergence group if and only if $L(G) = S^1$.*

Both of the above propositions can be proved by repeating the familiar arguments from the previous chapters.

Now, we want to present an example of a Möbius-like group G such that $L(G) = S^1$, but G is not a convergence group.

Example 5.3. Start with some nondiscrete purely elliptic Möbius group H whose every element has finite order (e.g., the group generated by rotations $z \mapsto ze^{2\pi/n}$, $n = 1, 2, \ldots$). Choose two points $x, y \in S^1$ with disjoint H-orbits. Now insert closed intervals into points of the orbits of both x and y. As usual, make sure that the lengths of all the inserted intervals sum to a finite value. This gives us a new, bigger, circle which we denote by $\overline{S^1}$. Then extend the action of H onto $\overline{S^1}$ as follows.

Given $f \in H$, let $\overline{f}$ be a homeomorphism of $\overline{S^1}$ induced by f in the following way.

- If w is a point untouched by the construction, i.e., no interval was inserted at w, set $\overline{f}(w) = f(w)$.
- Given an interval $[a, b]$ which was inserted at the point $z = h(x)$ for some $h \in H$, define $\overline{f}$ on $[a, b]$ to be any orientation preserving homeomorphism mapping $[a, b]$ to the interval inserted at $f(z)$.
- Given an interval $[c, d]$ which was inserted at the point $t = h(y)$ for some $h \in H$, define $\overline{f}$ on $[c, d]$ to be any orientation preserving homeomorphism mapping $[c, d]$ to the interval inserted at $f(t)$.

Set

$$\overline{H} = \{id_S\} \cup \{\overline{f} \mid f \in G - \{id_{\overline{S^1}}\}\}.$$

As explained in the similar situation of remark 3.4, $\overline{H}$ is a well defined group. Moreover, $\overline{H} \hookrightarrow Homeo_+(\overline{S^1})$, and $\overline{H}$ is Möbius-like.

Now draw the geodesic lines X and Y of $\boldsymbol{H}^2$ corresponding to the intervals inserted at x and y. See Figure 5.1. Let g be a genuine hyperbolic Möbius transformation on $\overline{S^1}$ such that $g(X) = Y$ and the axis of g crosses both X and Y. Define $G \hookrightarrow Homeo_+(\overline{S^1})$ to be the group generated by $\overline{H}$ and g.

$$G = <\overline{H}, g>$$

We now need to see how G acts on $\overline{S^1}$, and for that purpose let us denote by C the region bounded by the $\overline{H}$-orbits of X and Y (the shaded region in Figure 5.1). Then we can say that $\overline{H}$ fixes C, i.e., $h(C) = C$ for every $h \in \overline{H}$. Although, strictly speaking, the last statement does not make sense because $\overline{H}$ acts only on $\overline{S^1}$ and not on the whole disc, it should be clear what is the corresponding rigorous way of expressing it. The same applies for the considerations which follow. Our main guideline is the idea that we should first completely understand how G acts on the region C; once that is accomplished, we should be able to understand how G acts on $\overline{S^1}$.

Before stating some observations of a general type, let us consider an example. Let $w_0 = h_2 g^{-2} h_1 g$, where h_1, h_2 are some nontrivial elements of $\overline{H}$. Figure 5.2 shows how to trace down $w_0(C) = h_2(g^{-2}(h_1(g(C))))$.

Now every element w of G can be represented in the form:

$$w = h_k g^{p_k} \ \ldots \ h_2 g^{p_2} h_1 g^{p_1}, \qquad (*)$$

where $h_1, \ldots, h_{k-1}$ are nontrivial elements of $\overline{H}$ and $p_2, \ldots, p_k \neq 0$. The above example makes it clear how to trace down $w(C)$ for any given w. It should be noted that the fact that X and Y come from different orbits of $\overline{H}$ implies that $w(C) \neq C$, unless $w \in \overline{H}$. When presenting the words in G in the form (*), we notice that as the length of presentation grows larger (i.e., k increases), $w(C)$ gets moved "further" [10] from C. Thus we have the following two claims.

Claim 1. Stabilizer of C, the subgroup of G consisting of elements which fix C, is exactly equal to $\overline{H}$.

Claim 2. G is a free product of $\overline{H}$ and $< g >$, the group generated by g.

So the group G defines a sort of tiling of the disc, the tiles being translates of the region C by the elements of G. Well, not exactly: we need to check whether the

[10]We can measure the distance between C and $w(C)$ by counting the least possible number of translates of C, by the elements of G, one has to pass through on the way from C to $w(C)$. E.g., in the above example $dist(C, g^{-2}(C)) = 1$, $dist(C, w_0(C)) = 2$

whole disc gets tiled. And indeed that does not have to be the case, but we have the following claim.

Claim 3. We can assume that G tiles the whole disc.

Proof of Claim 3. The only type of "anomaly" that stops us from the tiling the whole disc is: there exists some interval $[u, v] \subset \overline{S^1}$, with corresponding geodesic line Z, such that some sequence of translates of C limits onto Z as in Figure 5.3. Then no translate of C lives in the region enclosed by Z and $[u, v]$. Clearly, the family $\mathcal{F}$ of all intervals with the same property as $[u, v]$ is a G-invariant subset of $\overline{S^1}$. It is also clear that the intervals of $\mathcal{F}$ are pairwise disjoint. Therefore we can define yet another circle :

$$\overline{S^1_\star} = \overline{S^1}/[u, v] \sim \text{point}, \quad [u, v] \in \mathcal{F},$$

with the induced action of G on it. Clearly, the action of G on $\overline{S^1_\star}$ has all the properties of the action of G on $\overline{S^1}$ except that every point of $\overline{S^1_\star}$ is being approached by the translates of C. That is exactly what we wanted.

So let us then assume that given any $z \in \overline{S^1}$, there are translates of C which are arbitrarily close to z.

Claim 4. G is Möbius-like.

Proof of Claim 4. Orient the X-axis. Let f be an arbitrary nontrivial element of G. We want to show that f is Möbius-like, so we distinguish the following cases.

f has one fixed point on $\overline{S^1}$. Then f is Möbius-like as mentioned in the Introduction.

f has more than one fixed points on $\overline{S^1}$. The set of all fixed points of f is closed in $\overline{S^1}$, so we can find some maximal open interval (u, v) in its complement so that f fixes both u and v and has no fixed points in (u, v). Let

A be the geodesic line corresponding to (u, v); so we can think of f as fixing A. Then some translate of X has to cross A. Indeed, if that were not the case, than A would have to be contained in $h(C)$ for some $h \in G$ (because the whole disc is tiled), or possibly A could be in the boundary of $h(C)$. In either case, f fixes a subset of $h(C)$, so $f \in Stab(h(C))$. But, by Claim 1, $Stab(h(C)) = h\overline{H}h^{-1}$, which is a contradiction because all elements of $h\overline{H}h^{-1}$ are fixed-point free.

So A crosses $k(X)$ for some $k \in G$. Refer to Figure 5.4. From the definition of (u, v), it follows that one endpoint of $k(X)$, namely the one lying in (u, v), gets moved under f. Then, by the tiling property, the whole region $k(C)$ gets moved by f. So clearly f cannot have any fixed points on (v, u) either. It is also clear that f has the right dynamics, i.e., one of its fixed points is attractive, while the other is repulsive. Therefore f is Möbius-like hyperbolic.

f has no fixed points on $\overline{S^1}$. Let $h(C)$ be the translate of C which minimizes $dist(h(C), f(h(C))$ (see the last footnote). Since conjugation does not change the fact that a map is Möbius-like, we can conjugate f so that $dist(C, f(C)) = d$ is minimal. If $d = 0$ then $f \in Stab(C) = \overline{H}$ so f is Möbius-like. If $d > 0$ then $f(C) \neq C$. Assume first that $f(C)$ is not adjacent to C. Let Z be the boundary component of C which is the closest to $f(C)$. See Figure 5.5. f has no fixed points so $f(Z)$ must be as in the Figure 5.5. But then there exists some translate of C living in the "gap" between Z and $f(Z)$, which decreases d. Contradiction. So assume now that $f(C)$ is adjacent to C, i.e., they share one boundary component, call it Z. See Figure 5.6. Then f^2 fixes both endpoints of Z. Consequently $f^2 = id$ (if not, then from above it follows that f^2 would have to be hyperbolic, which is impossible). So f is Möbius-like as stated in the Introduction.[11]

So we have proved Claim 4 and hence accomplished our goal: G is a Möbius-like group, but it is not a convergence group because $\overline{H} \hookrightarrow G$ is not a convergence group. Figure 5.7 ilustrates how the convergence property fails in G i.e., $\overline{H} \hookrightarrow G$.

Examples 5.4. It is clear that the above example can be varied by changing H; we can do the exact same construction starting with any nondiscrete elementary group H. Care should be taken when "blowing up" the orbits of H: we should not insert intervals at points which are fixed points of some element of H. In all of these cases the convergence property fails in such a way that there is a sequence whose limit function consists of infinitely many jumps and flats.

Example 5.5. Consider the following Möbius transformations on $\boldsymbol{R} \cup \{\infty\} = S^1$:

$$f_n(x) = \frac{\frac{1}{n}x + n}{\frac{1}{n^2}x + (n+1)}, \quad n = 1, 2, \ldots.$$

The sequence $\{f_n\}$ satisfies the convergence property in the following way:

$$f_n \to 1 \quad on\ \boldsymbol{R}$$

$$f_n^{-1} \to \infty \quad on\ \boldsymbol{R} \cup \{\infty\}.$$

Figure 5.8 illustrates the behavior of $\{f_n\}$ (thinking of $\boldsymbol{R} \cup \{\infty\}$ as S^1). One should pay attention to the fact that $f_n(\infty) \to \infty$. It is not hard to see that one can find a subsequence $\{f_{n_i}\}$ such that the group H generated by all the f_{n_i}'s is a discrete Möbius group.[12]

Claim. The point ∞ is not fixed by any hyperbolic element of H.

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[11] Actually, $f^2 = id$ cannot really happen because of Claim 1.

[12] Discrete Möbius groups are called Fuchsian groups.

Proof of claim. Suppose the contrary, that $\{\infty, a\}$ is the axis of some hyperbolic element h of H. Then $f_{n_i}(\infty) \to \infty$, while $f_{n_i}(a) \to 1$. In other words, the axes of the elements of the sequence $f_{n_i} h f_{n_i}^{-1}$, of conjugates of h, limit onto the axis $\{\infty, 1\}$. This contradicts the discreteness of H. Namely, it is a well known fact that in a Fuchsian group, given two disjoint open intervals I and J on S^1, there can be only finitely many conjugates of a hyperbolic element h with the property that their axes start in I and terminate in J.[13]

Now choose some point $x \in \boldsymbol{R}$ which is not fixed by any element of H and whose H-orbit does not contain ∞. We want to "blow" up the orbits of x and ∞ and proceed with the construction as in the example 5.3. In other words, we want to define $\overline{H}$, add a new element g, and then consider the group $G =< \overline{H}, g >$. This could certanly be done if we knew that ∞ is not a fixed point of any element of H. But what if some parabolic element p of H fixes ∞? The answer is: the construction can still be carried out. Namely, when "blowing" up the orbit of ∞, define the induced action of p on a new circle $\overline{S^1}$ to be hyperbolic so that its axis is determined by the endpoints of the interval I inserted at the point ∞.[14] Refer to Figure 5.9. Now, defining the induced action on I for the rest of the elements of H is done in the most natural way: given some point y from the H-orbit of ∞, let f be any element of H such that $f(\infty) = y$. Then, the set of all elements of H which map ∞ to y is exactely $\{fp^n \mid n \in \boldsymbol{Z}\}$. If J is the interval inserted at y, define f on I to be any orientation preserving homeomorphism mapping I onto J. The rest is clear.

So we define the group G acting on $\overline{S^1}$ just as before, and show that it defines a tiling of the disc. Thus G is Möbius-like. But G is not a convergence group because

[13]For proof see e.g. [T].

[14]In the original group H, the subgroup of all elements fixing ∞ has to be cyclic (otherwise H would not be discrete). So p should be a generator of this subgroup.

the convergence property is violated in $\overline{H}$, as illustrated in Figure 5.10. Just look at the limit function of the seqauence $\{\overline{f_{n_i}}\}$: it consists of two jumps and two flats.

Examples 5.6. Example 5.5 can be varied by starting with any Fuchsian group H in which there exists a sequence $\{f_n\}$ satisfying the convergence property in the following way:

$f_{n_i} \to y$ pointwise on $S^1 - \{x\}$, but $f_{n_i}(x)$ converges to some point other than y
$f_{n_i}^{-1} \to x$ pointwise on $S^1 - \{y\}$.

Conjecture. *Suppose G is a Möbius-like group acting on S^1 without global fixed points. Also assume that the G-orbit of every point on the circle is dense. If G is not a convergence group, then there exiasts a subgroup G' of G such that G' is exactely one of the examples described above in 5.3, 5.4, 5.5, 5.6.*

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Table of Figures

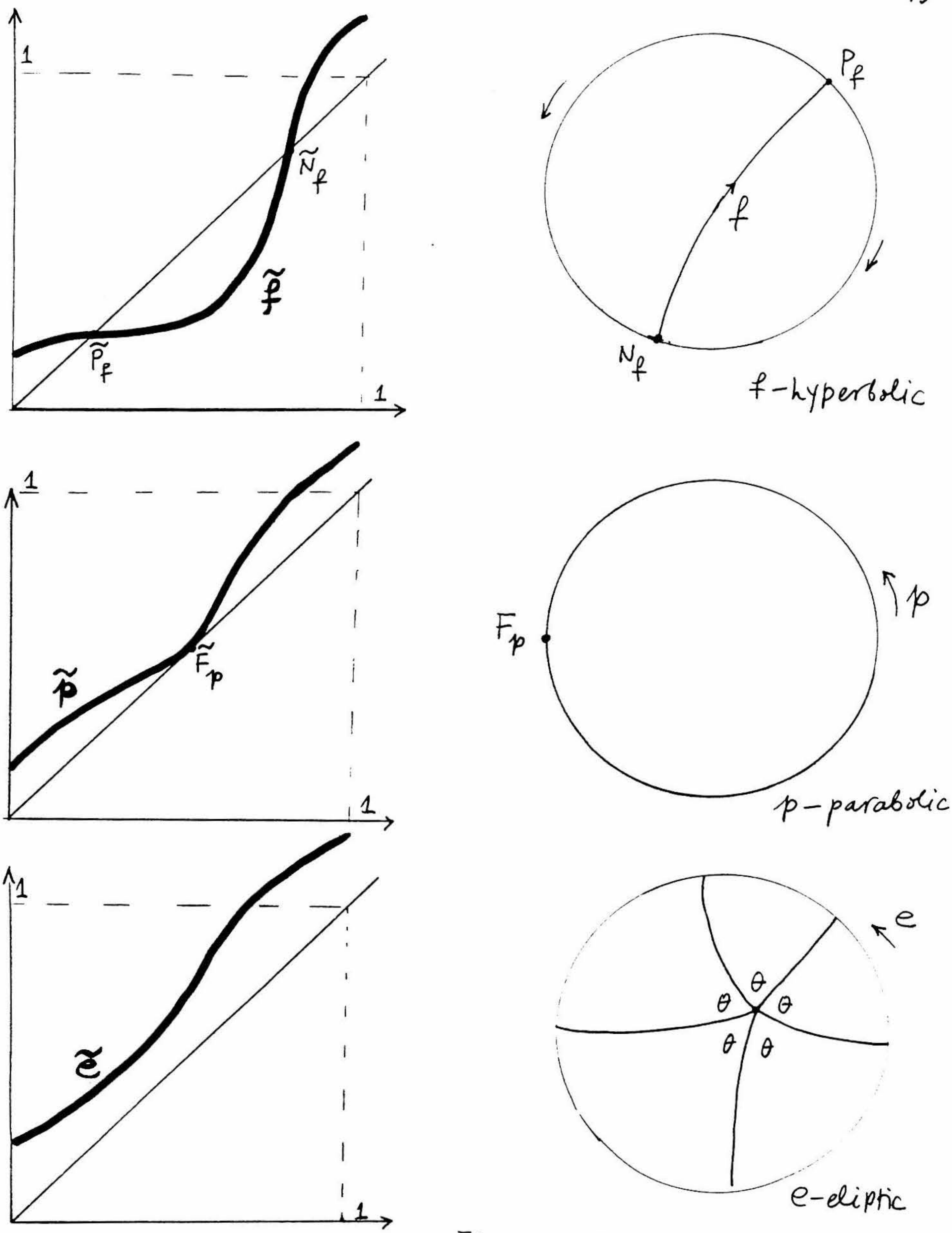

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Figure 1.1

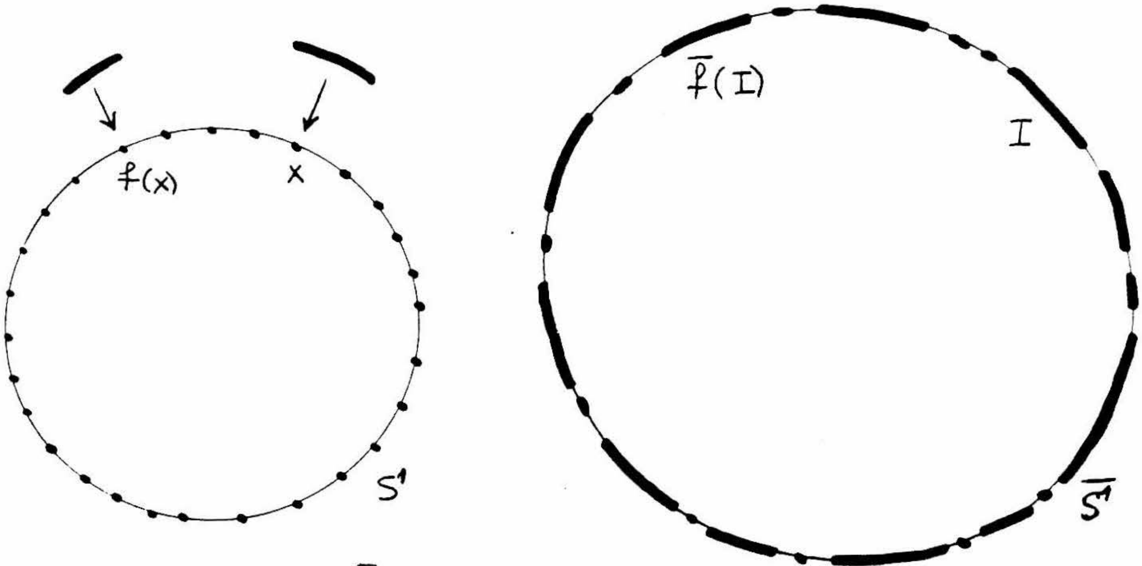


Figure 1.2

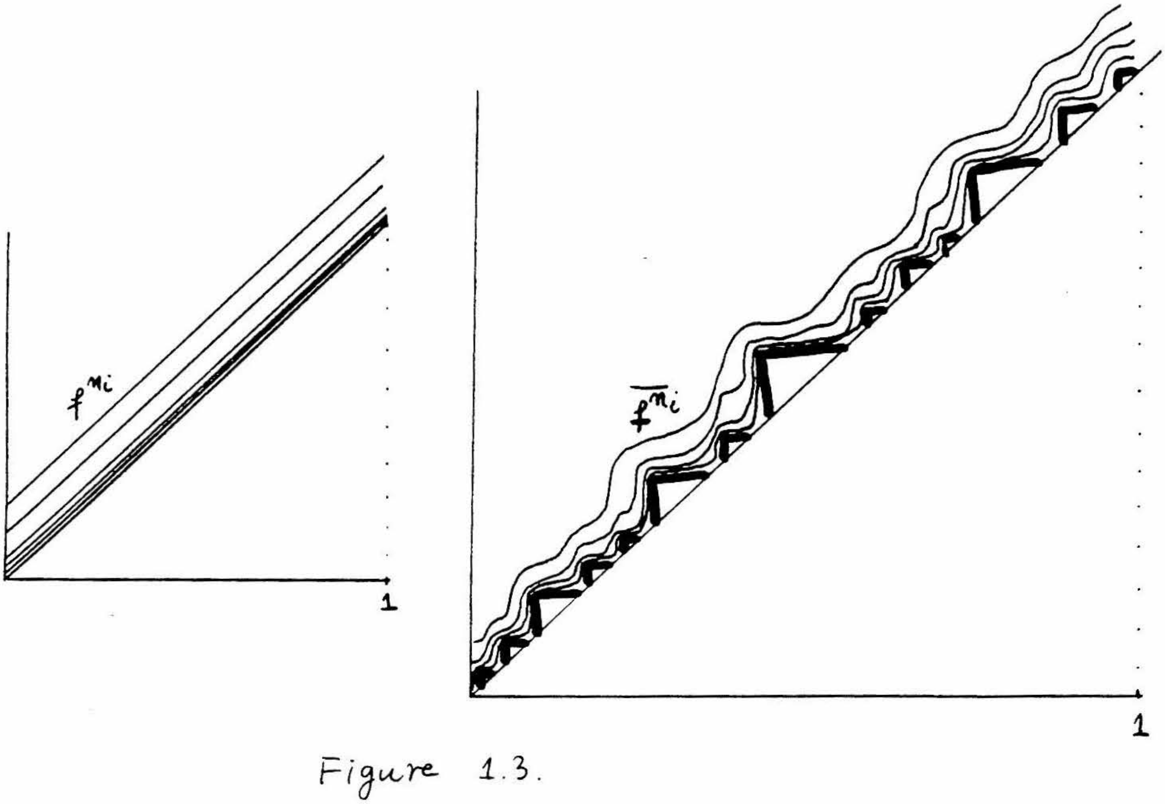


Figure 1.3.

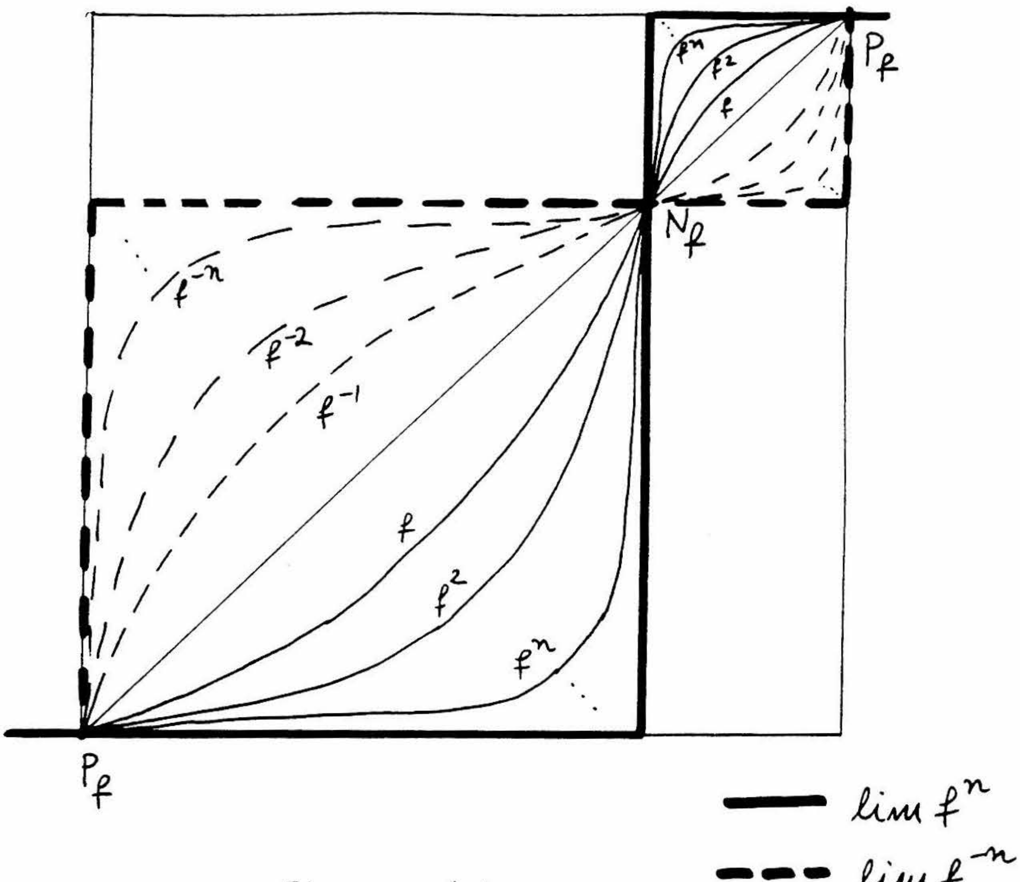


Figure 1.4.

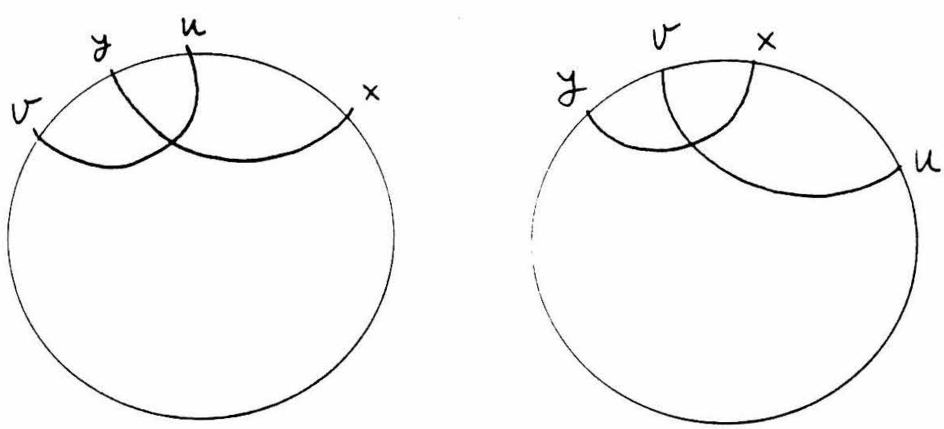


Figure 2.1.

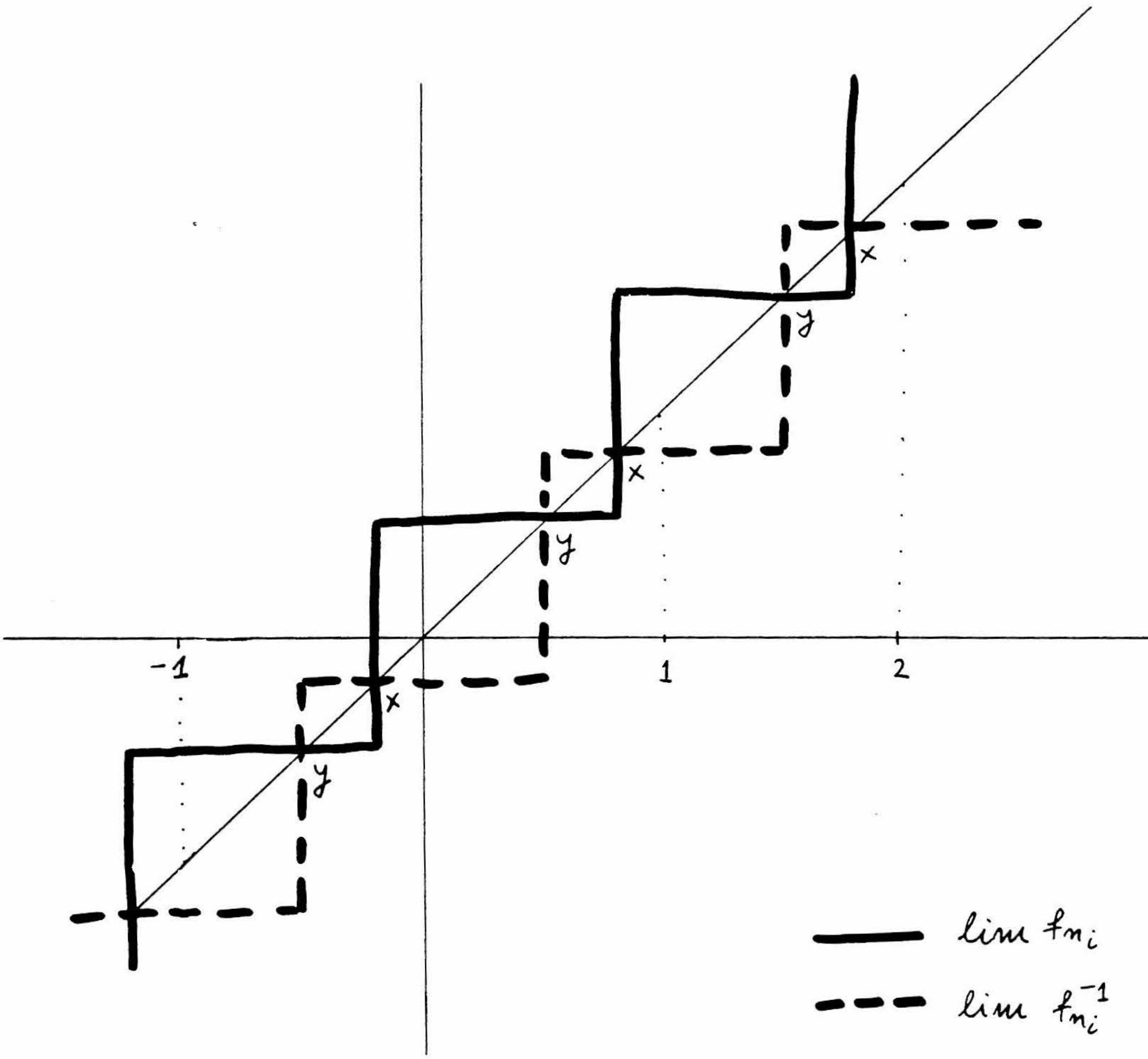


Figure 2.2.

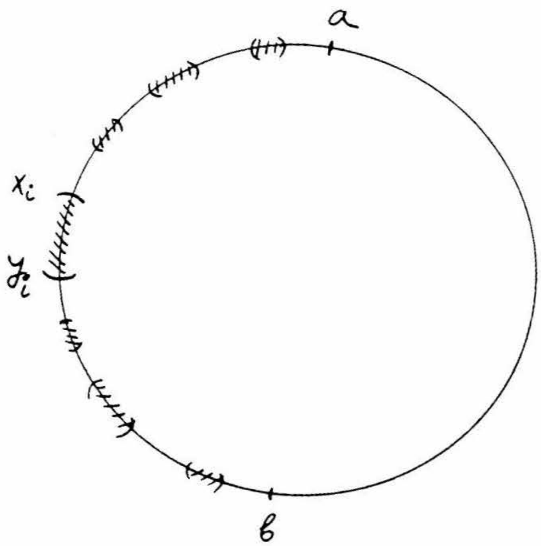
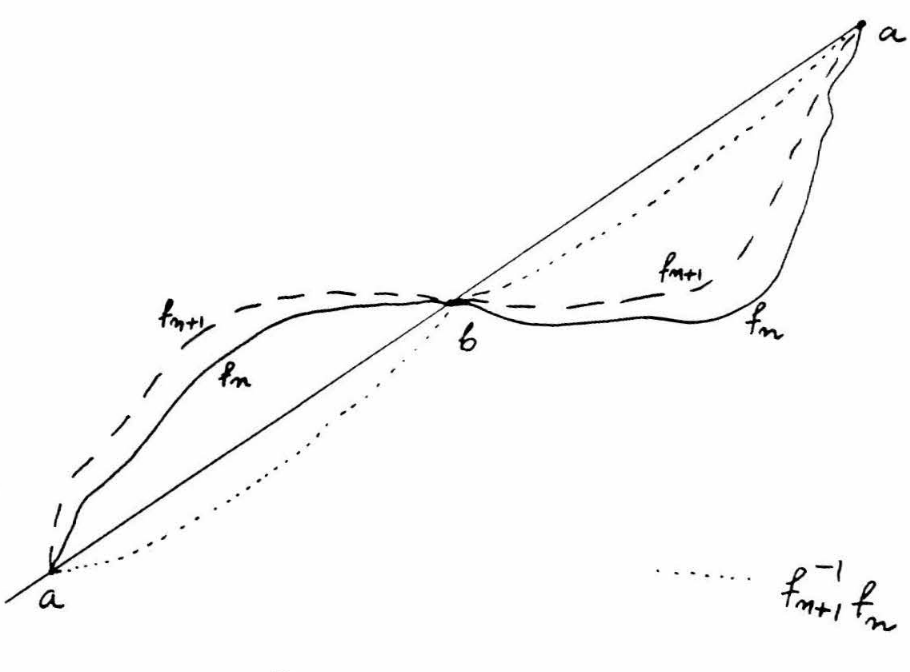


Figure 3.1



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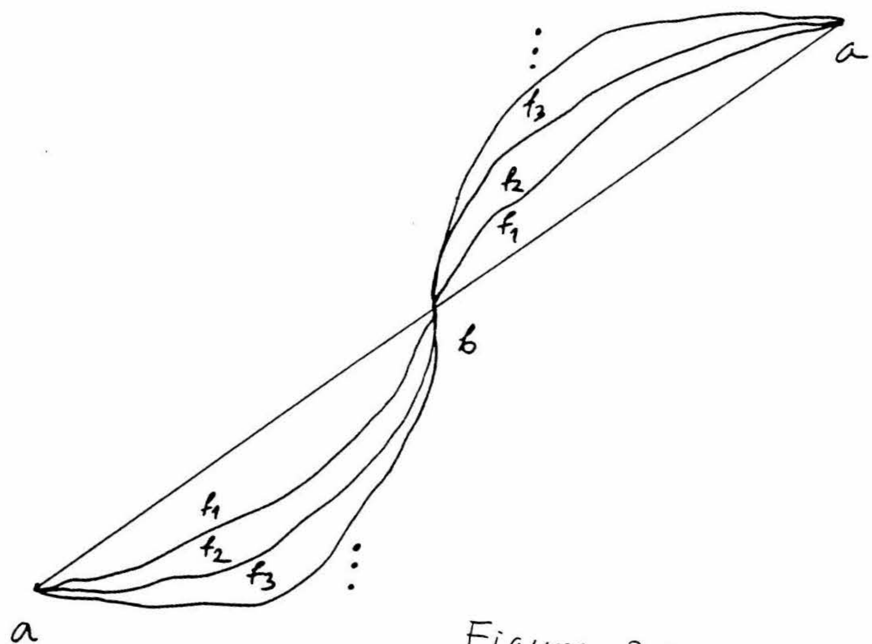

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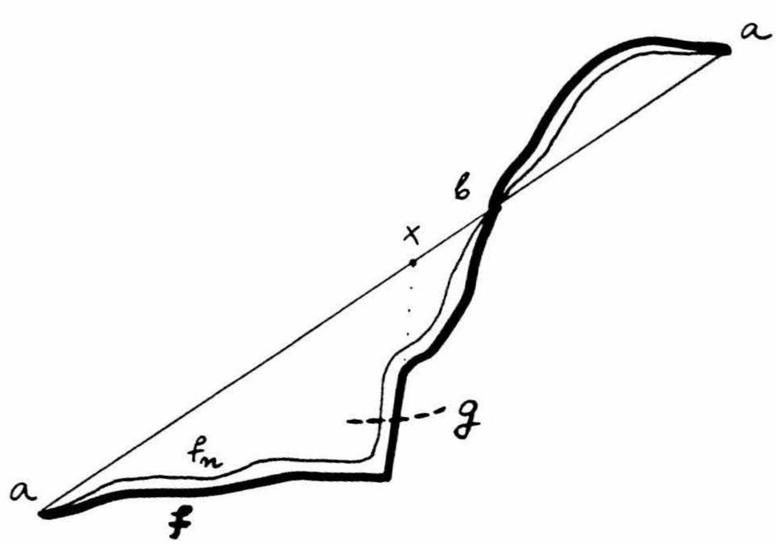

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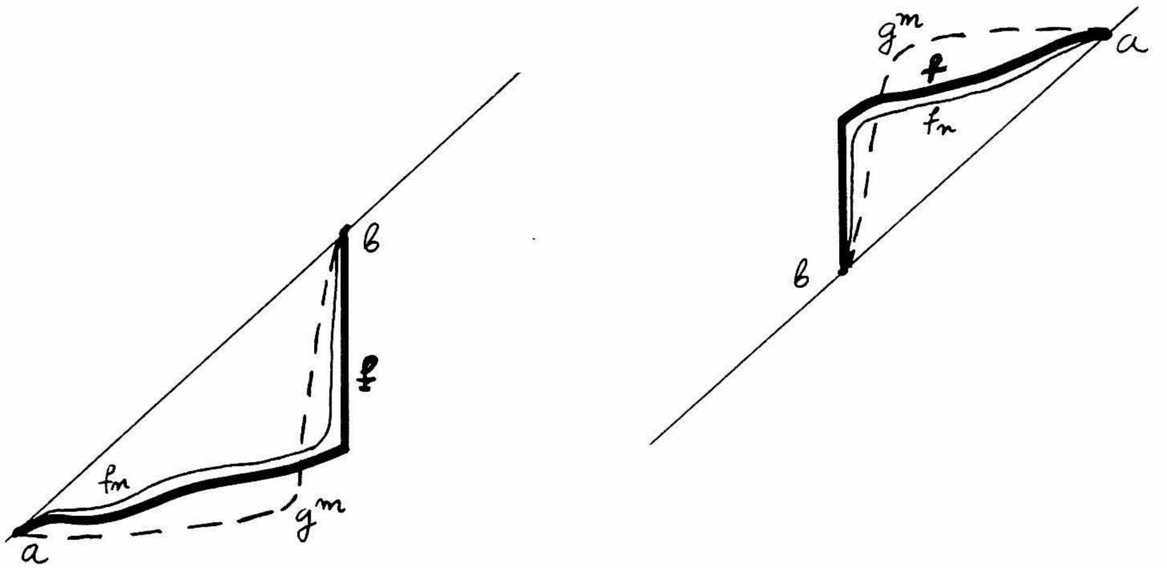

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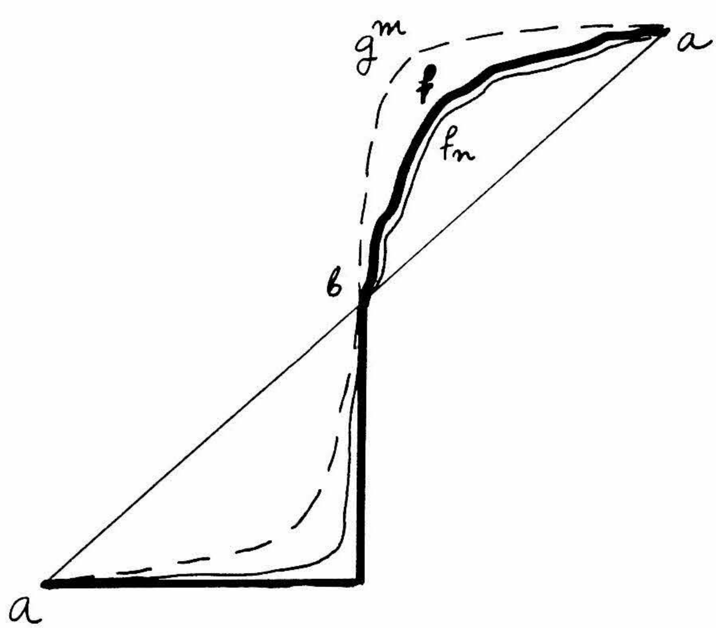

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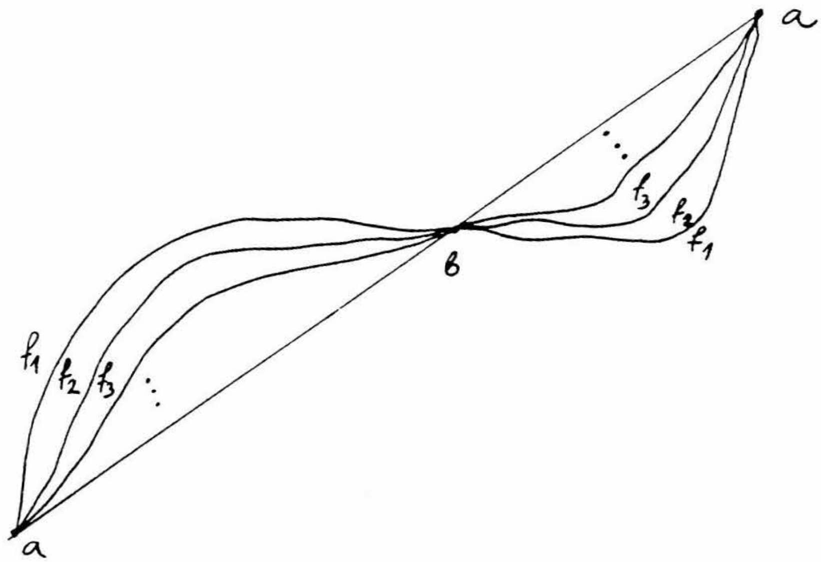

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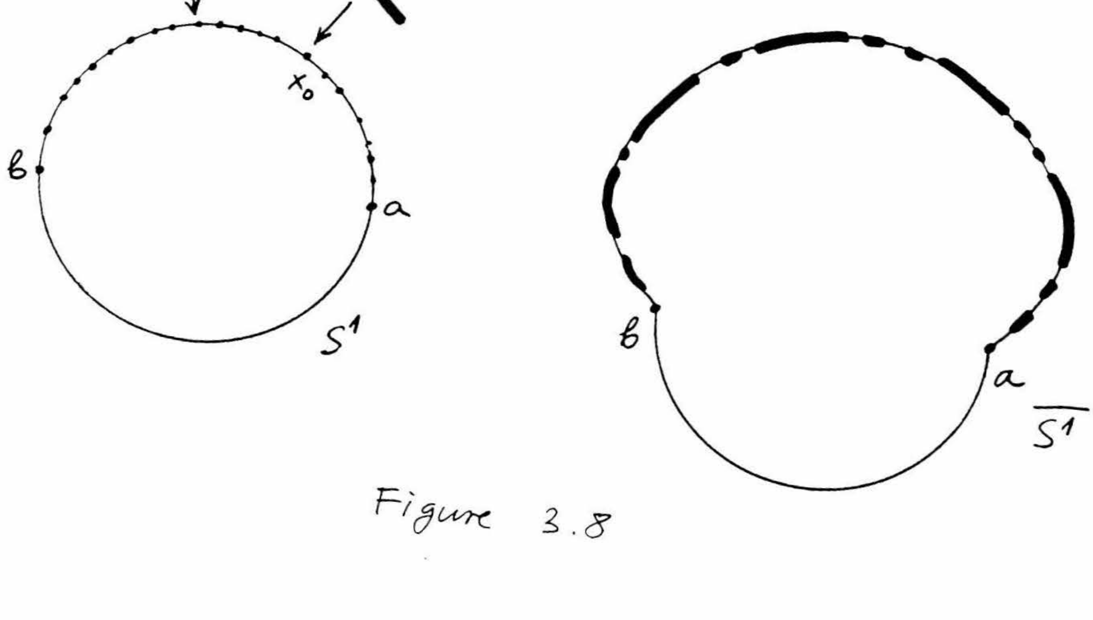

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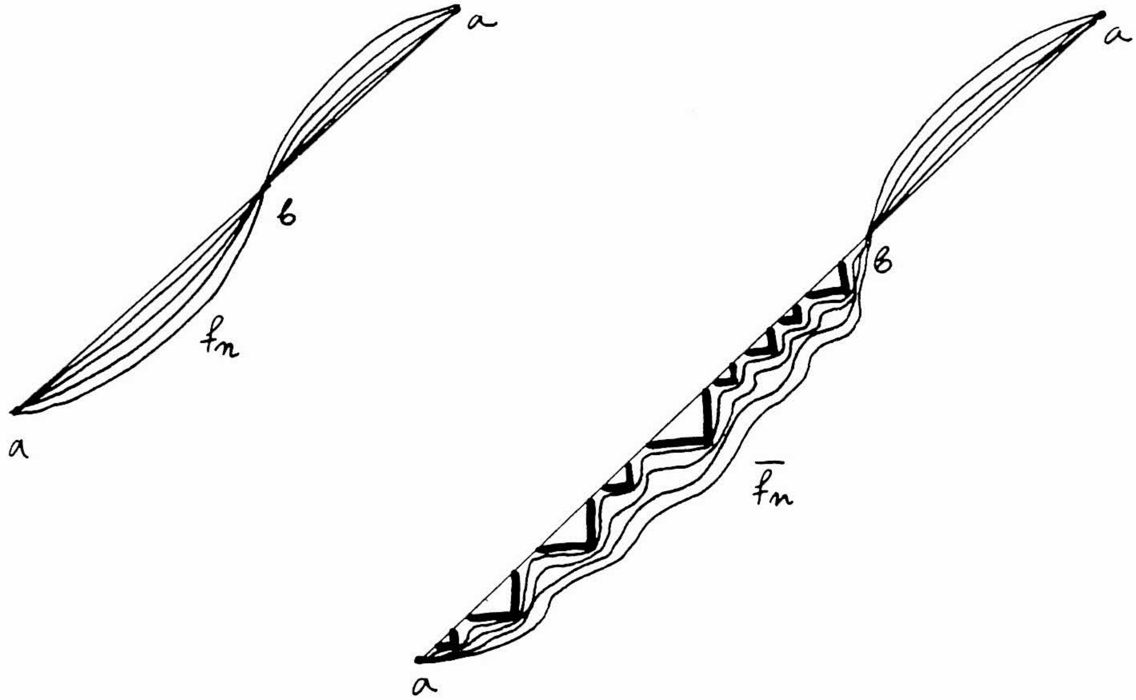

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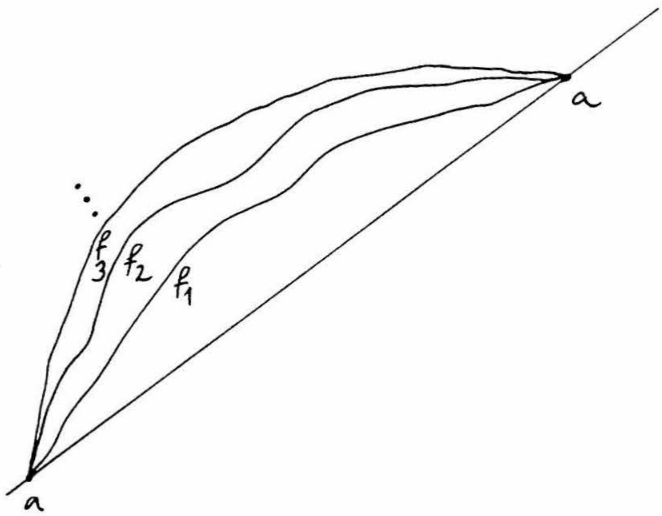
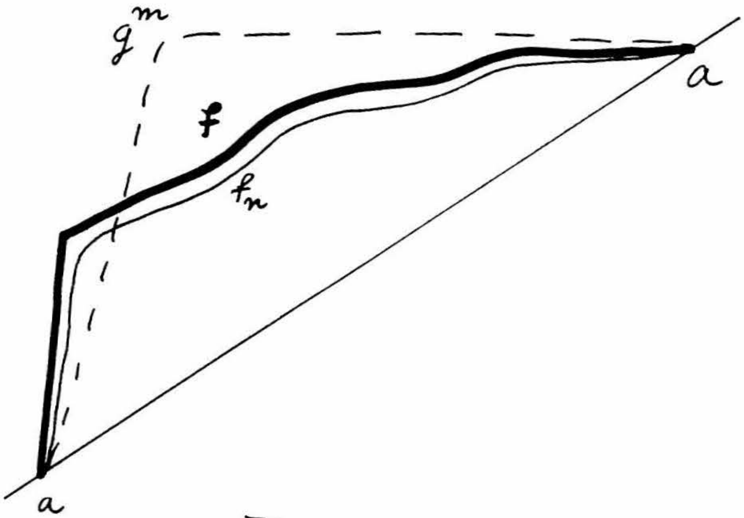


Figure 4.1



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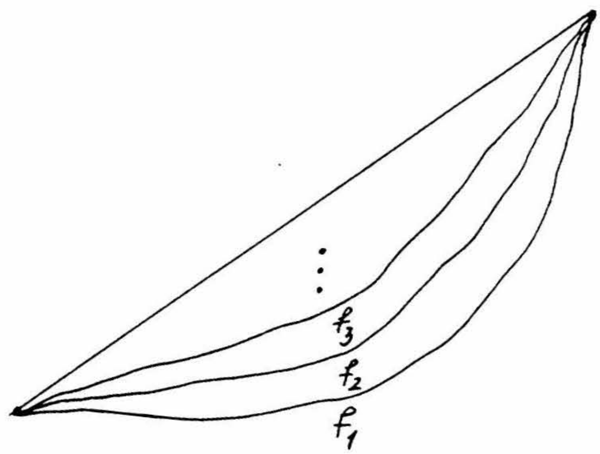

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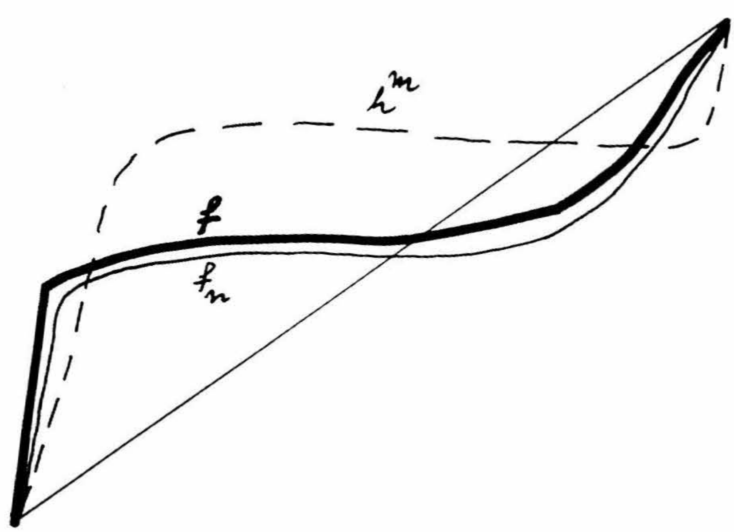

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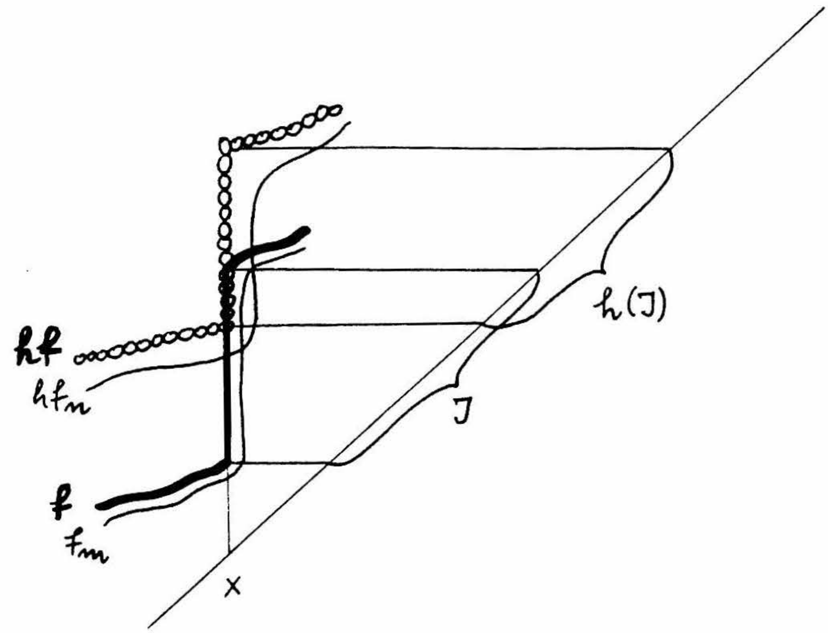

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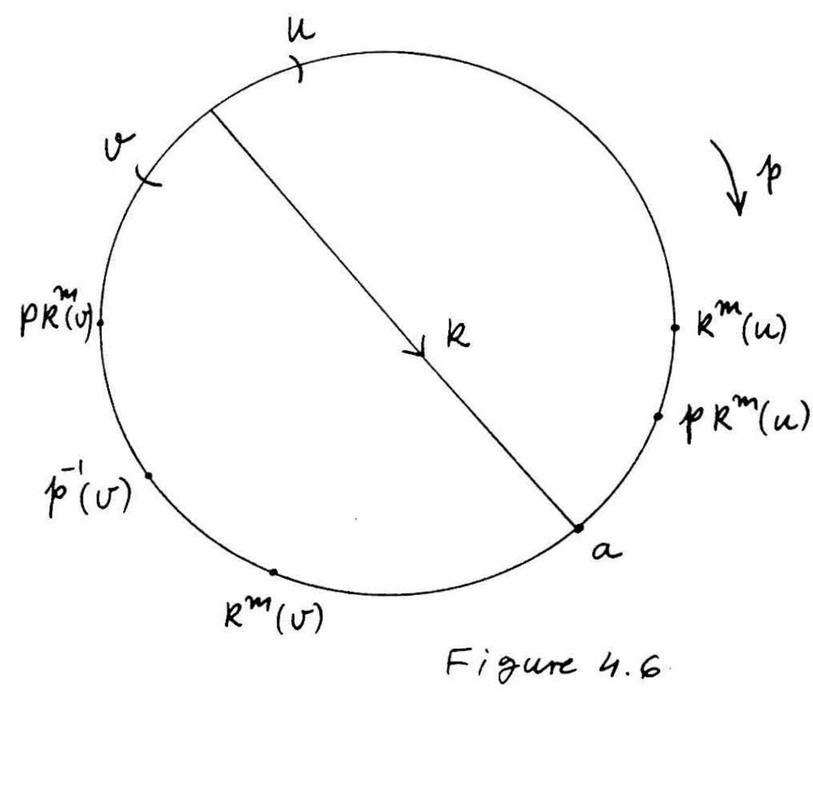

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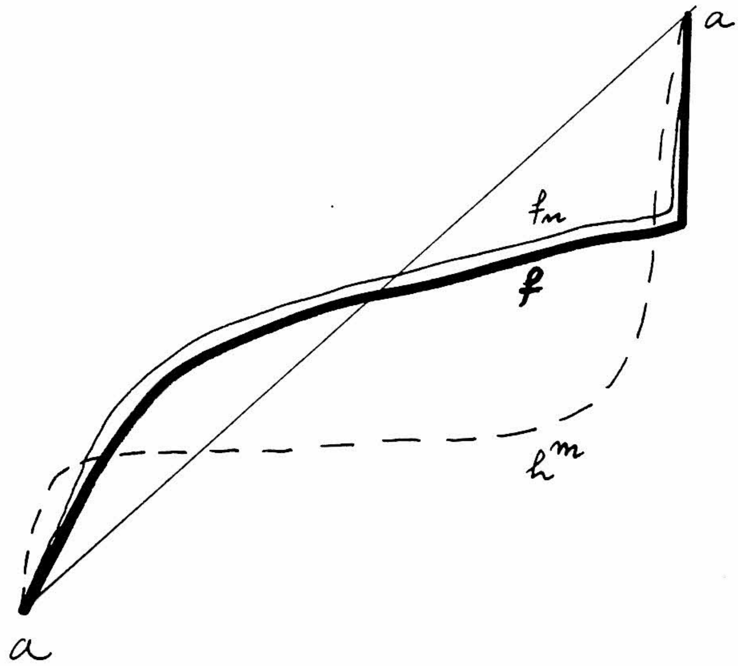

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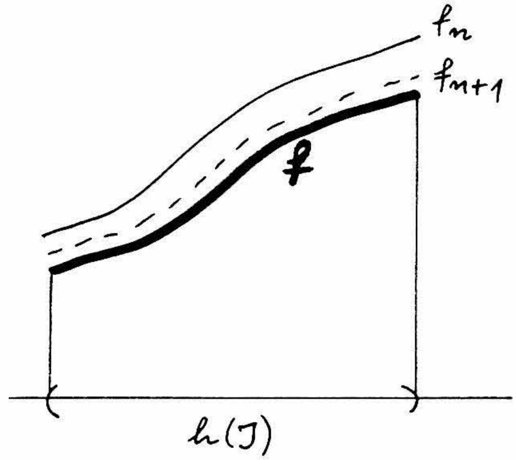

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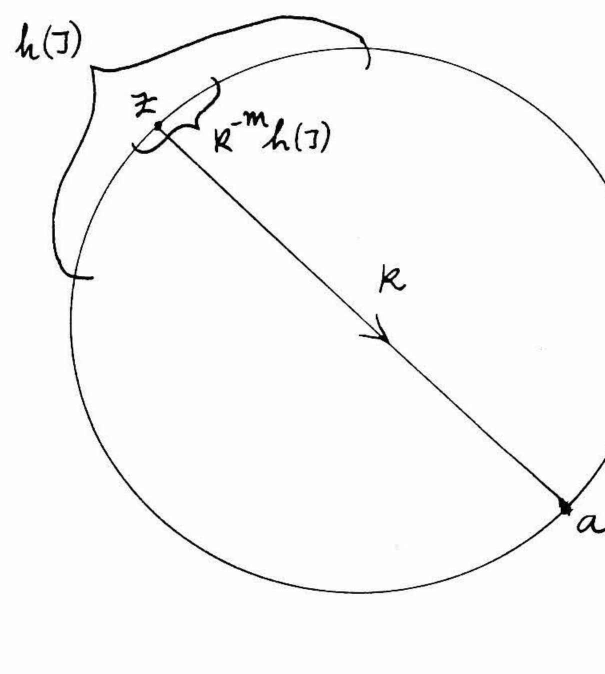

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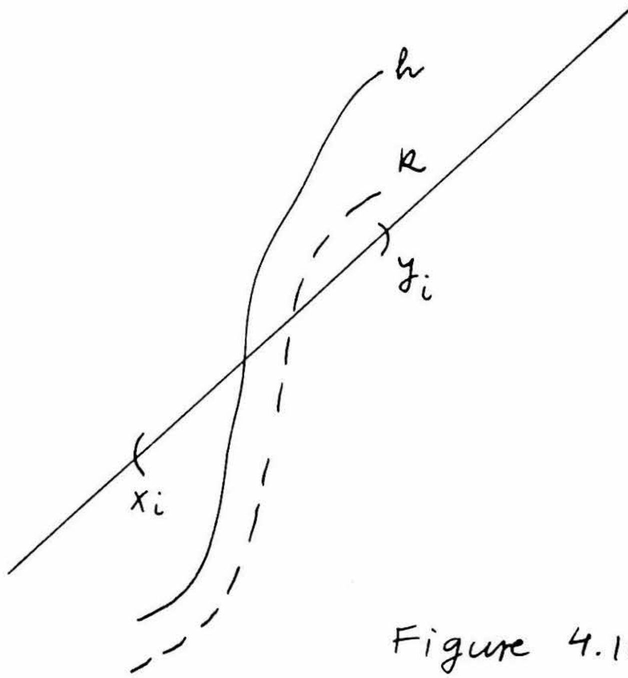

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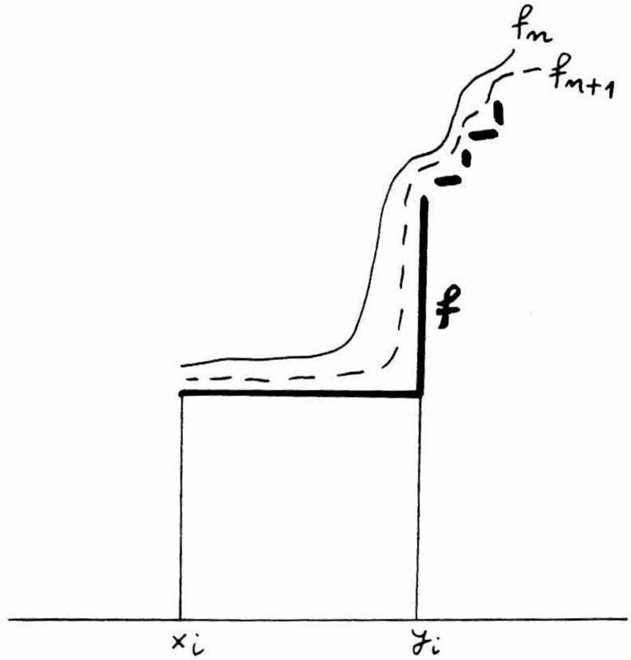

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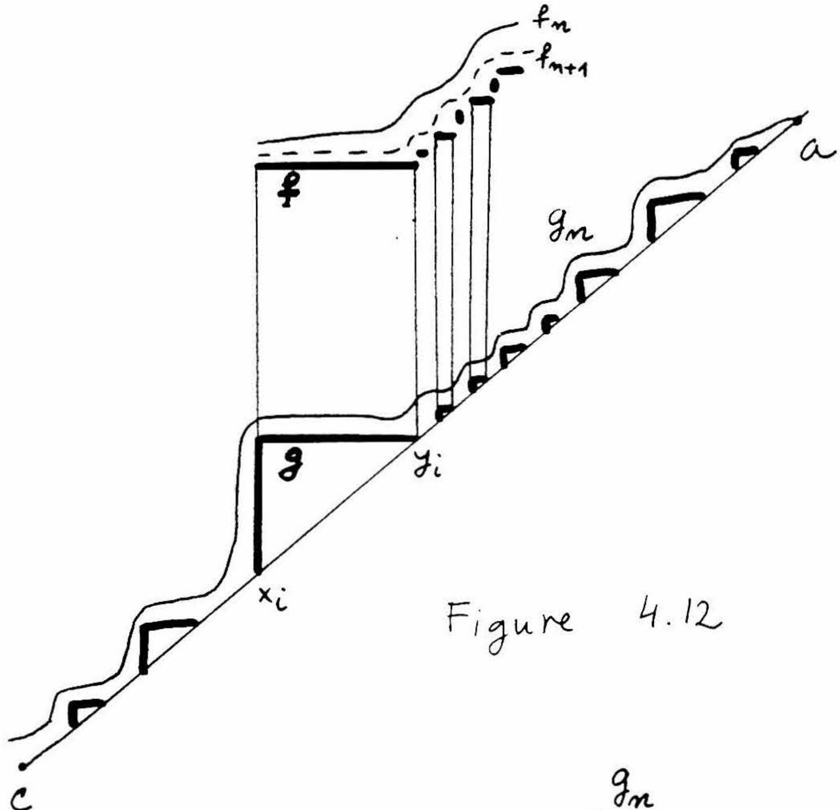

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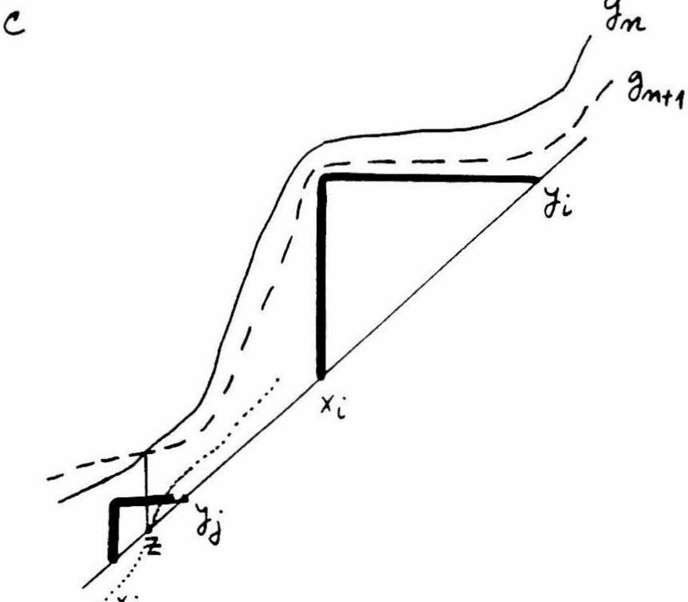

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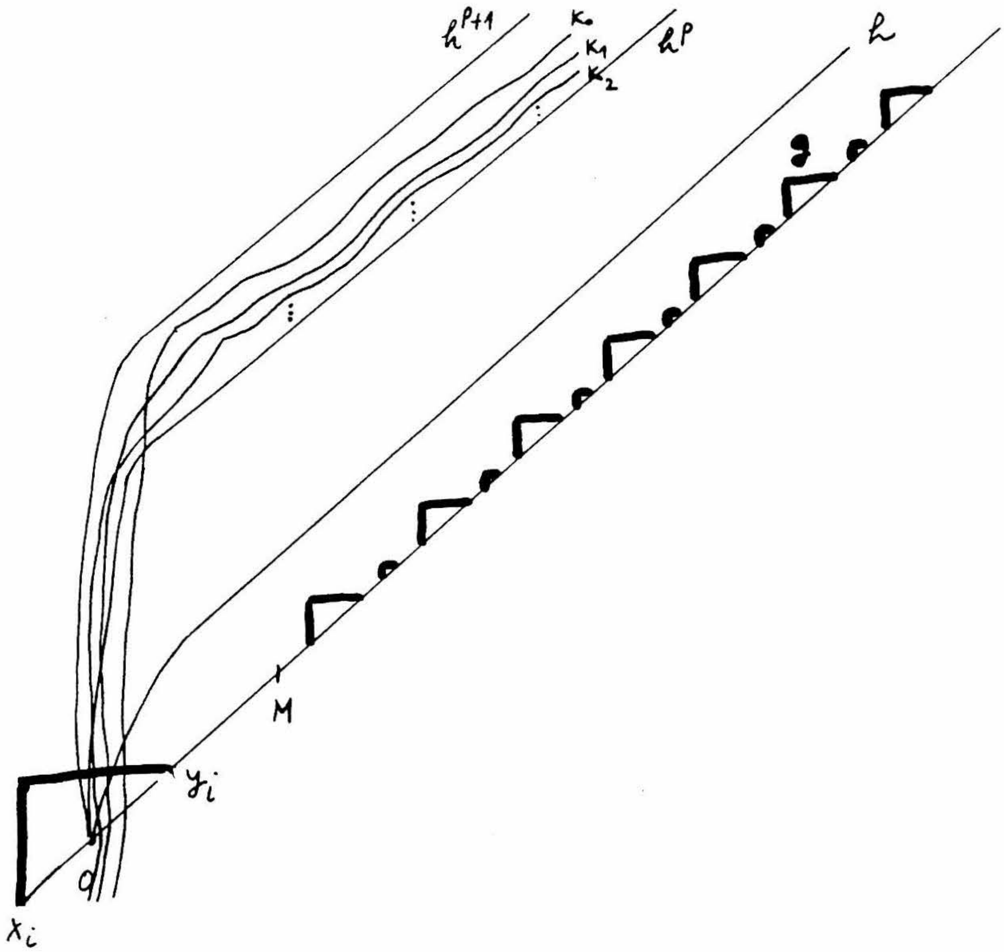

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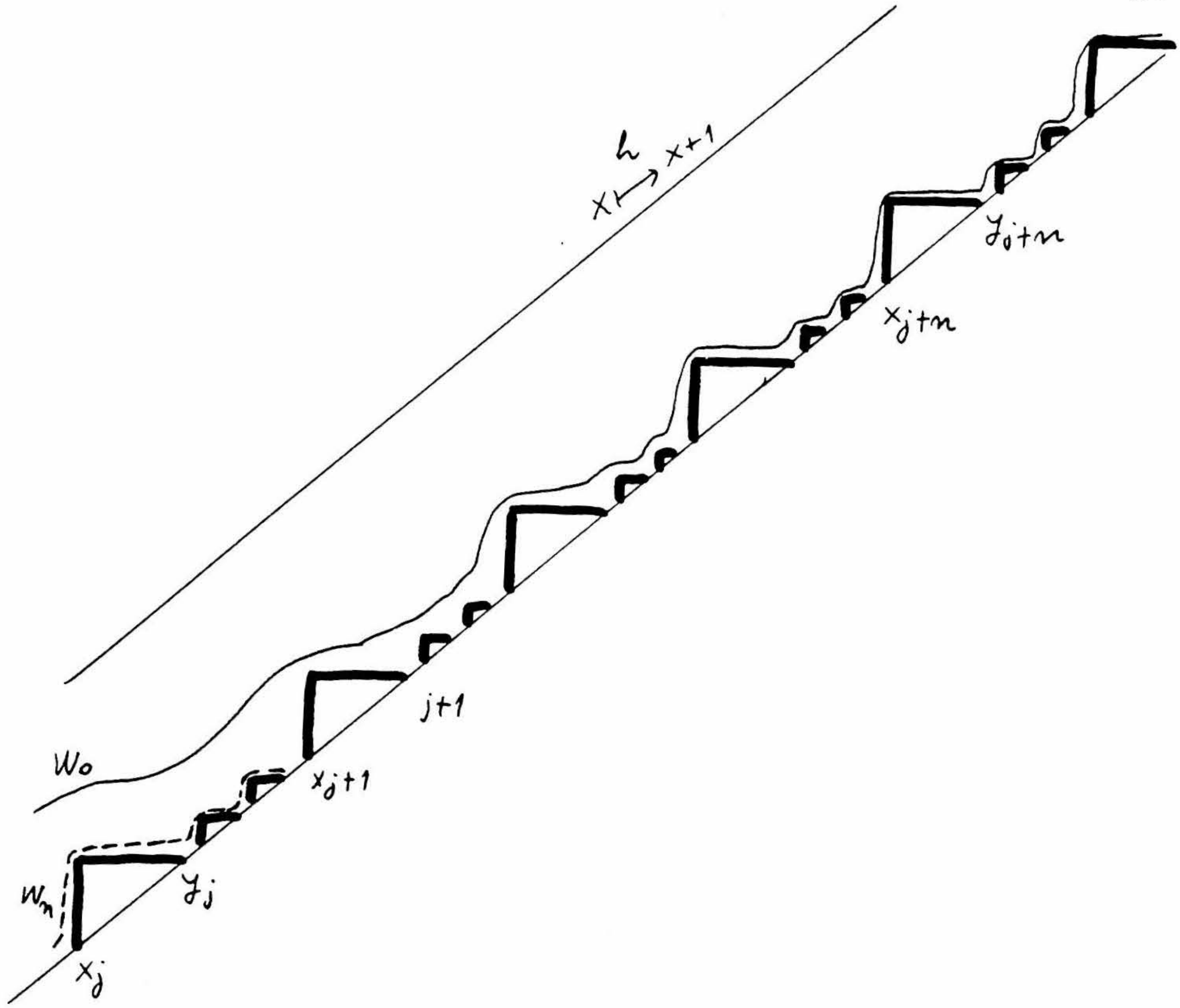

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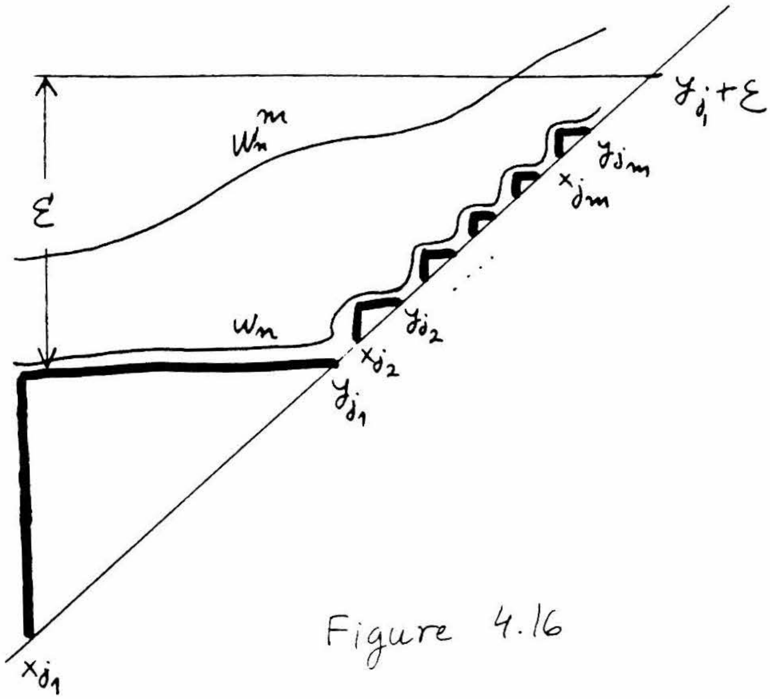

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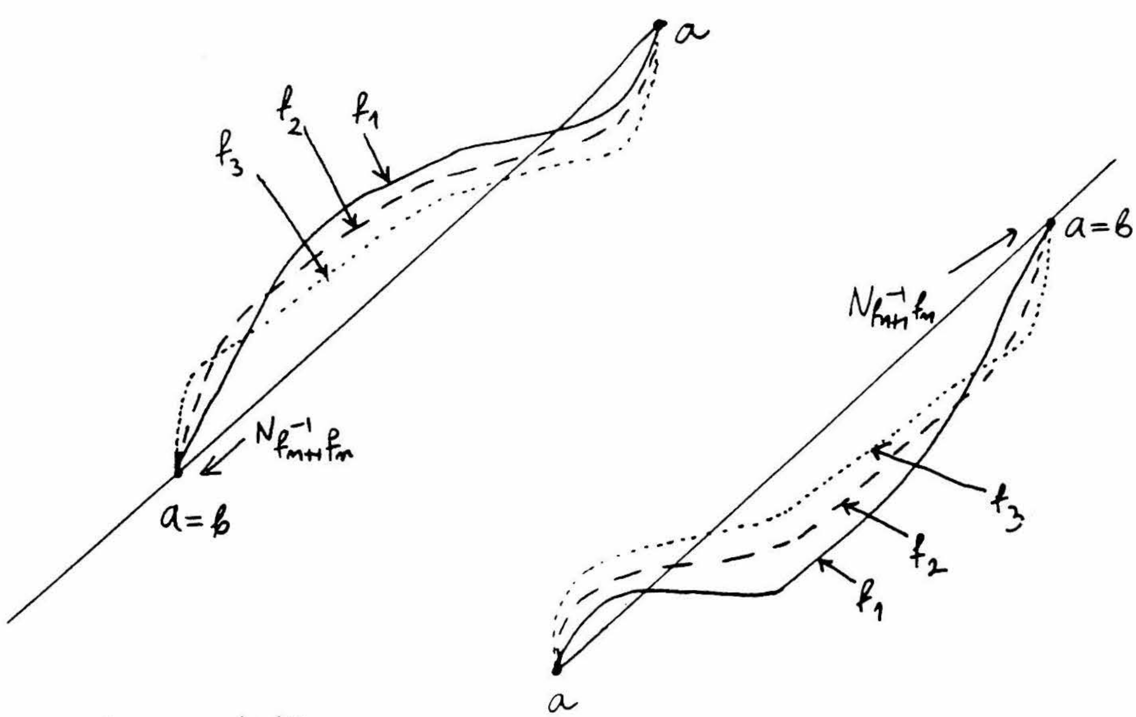

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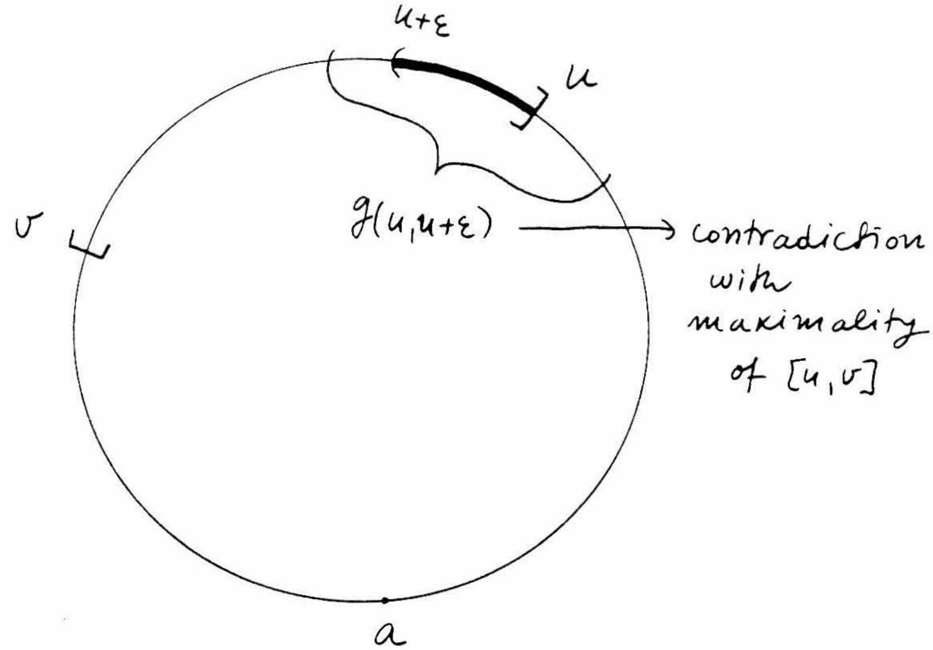

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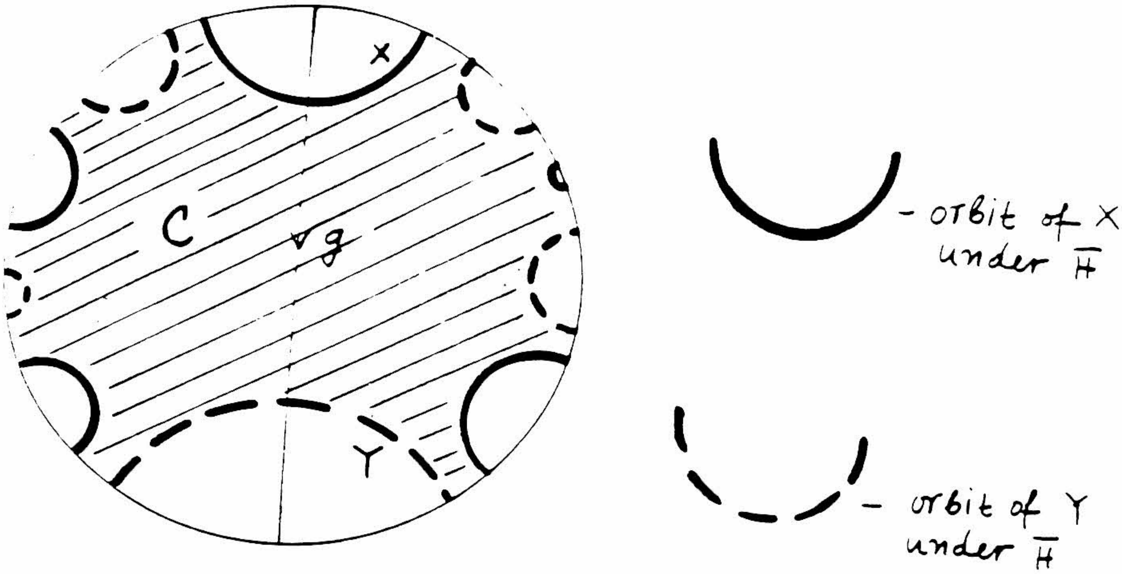
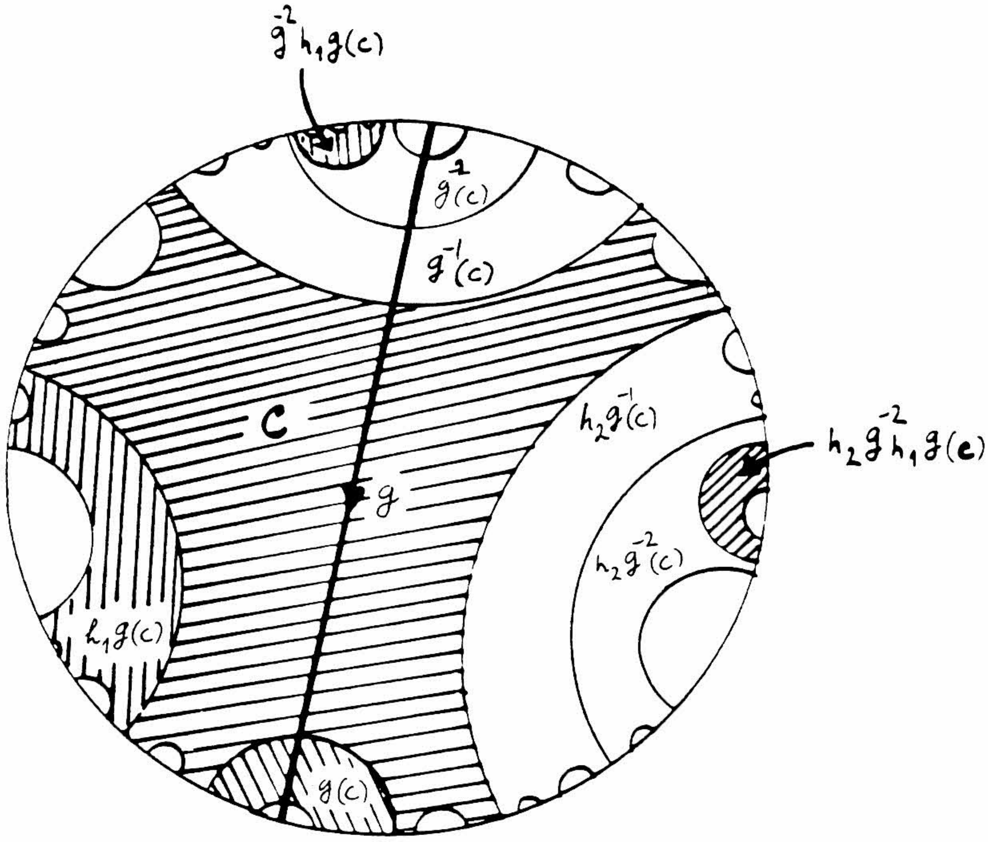


Figure 5.1



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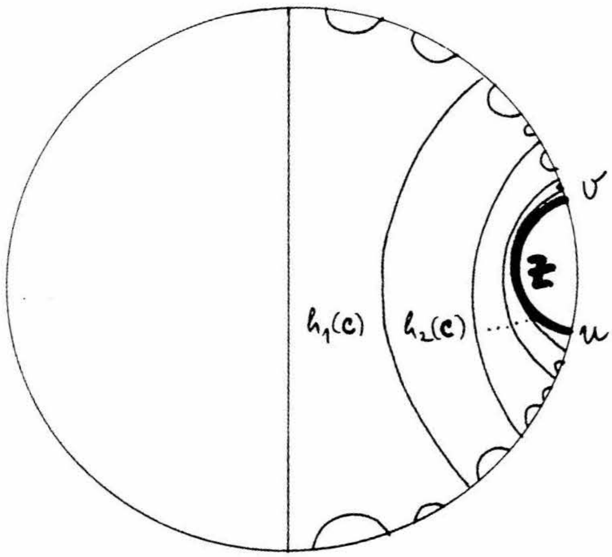

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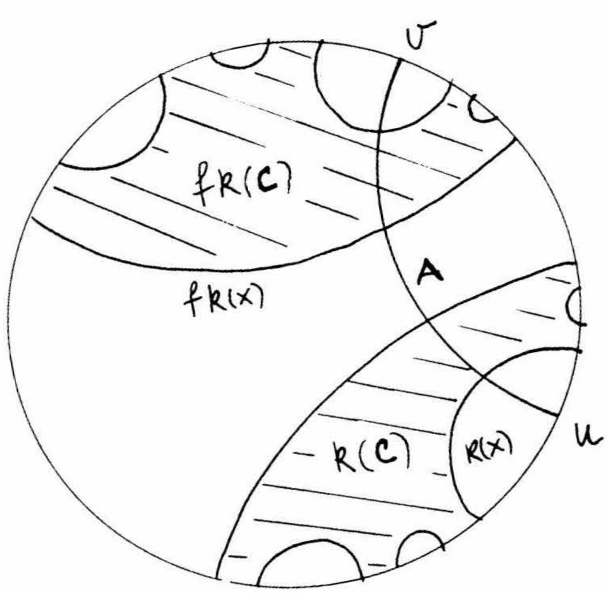

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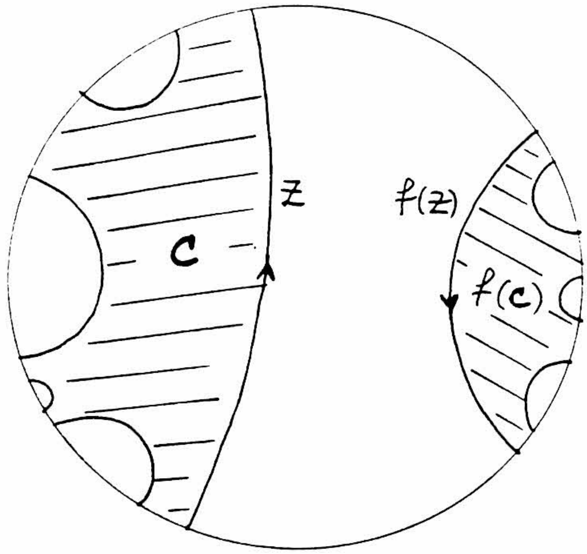

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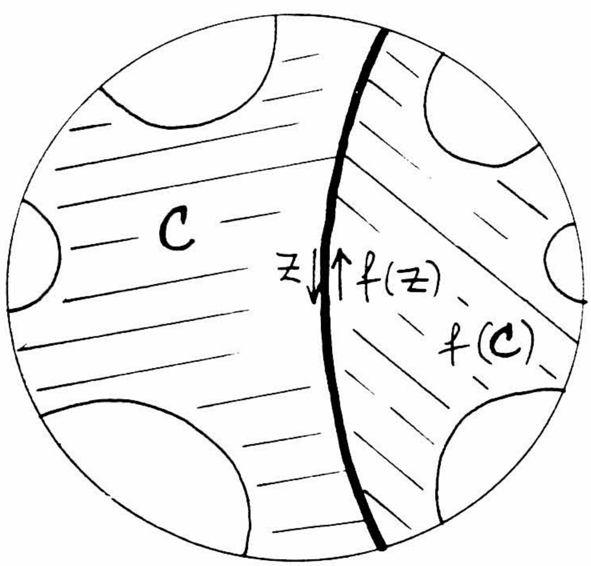

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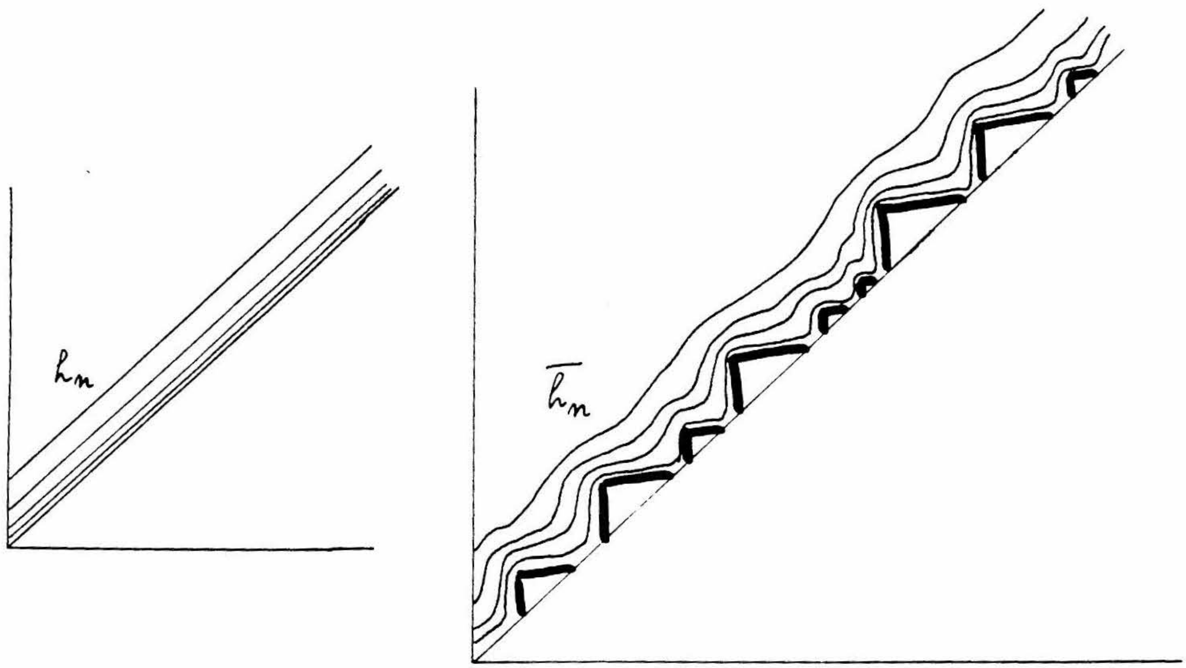

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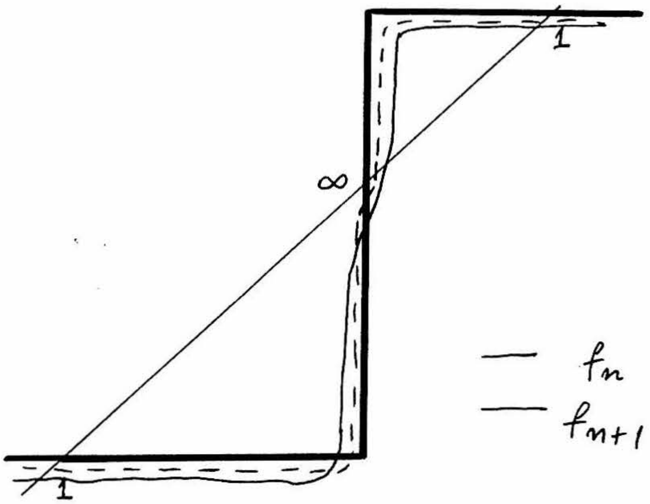

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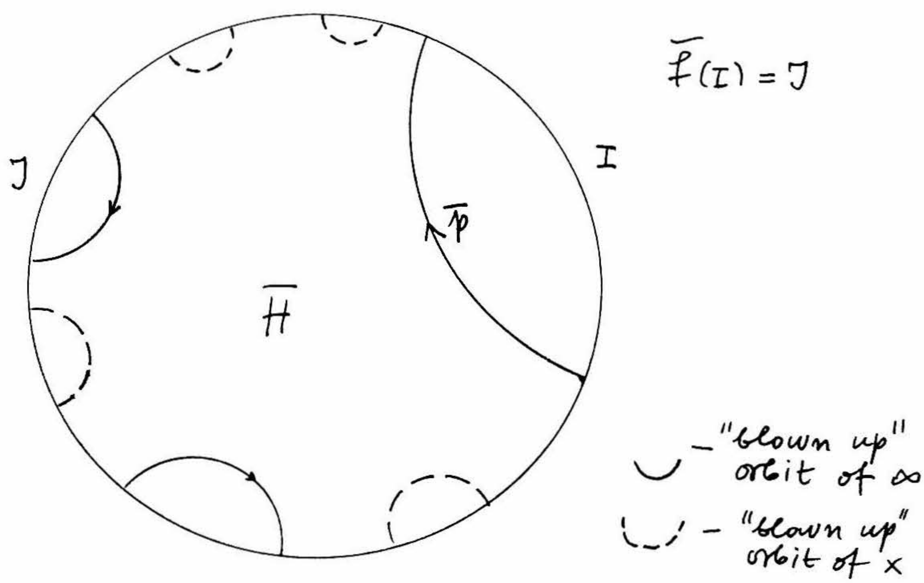

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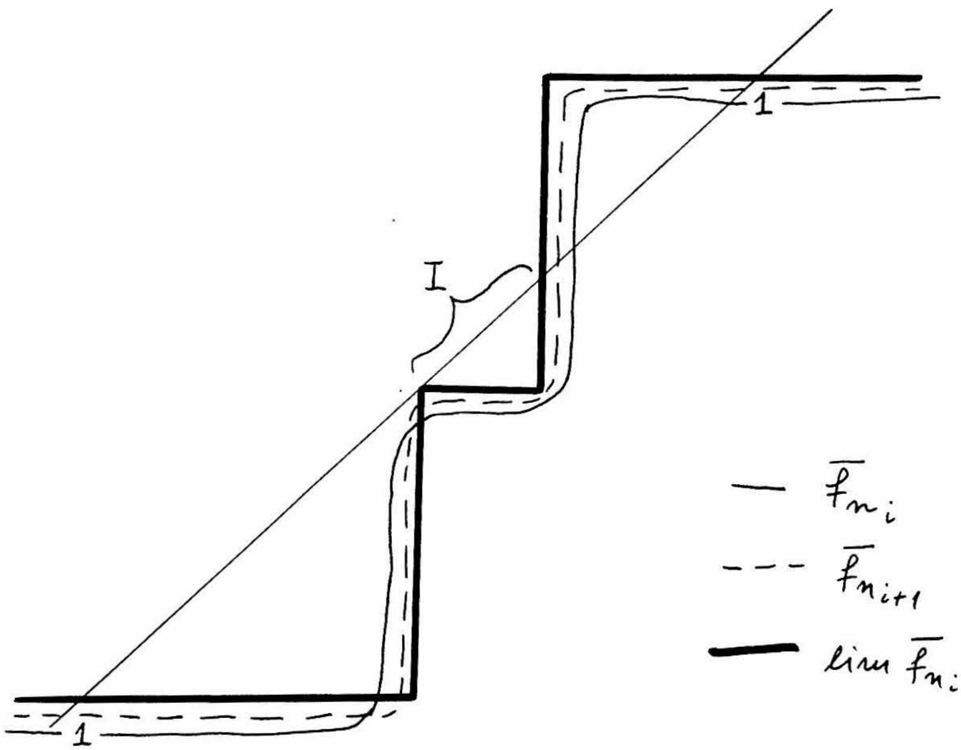

Figure 5.10.