Oscillatory Integral Operators Related To Pointwise Convergence of Schrödinger Operators

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To My Parents, With Love

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Abstract

In this thesis we consider smooth analogues of operators studied in connection with the pointwise convergence of the solution, $u(x,t)$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, of the free Schrödinger equation to the given initial data. Such operators are interesting examples of oscillatory integral operators with degenerate phase functions, and we develop strategies to capture the oscillations and obtain sharp $L^2 \to L^2$ bounds. We then consider, for fixed smooth $t(x)$, the restriction of u to the surface $(x, t(x))$. We find that $u(x, t(x)) \in L^2(\mathbb{D}^n)$ when the initial data is in a suitable L^2-Sobolev space $H^s(\mathbb{R}^n)$, where s depends on conditions on t.

Contents

Introduction

We begin this thesis by giving motivation for the objects we will study, placing them in their proper context.

§1. The Schrödinger Equation and Pointwise Convergence

Consider the initial value problem for the Schrödinger equation with no potential,

$$\text{(1)}\qquad \begin{cases} i\partial_t u(x,t) + \Delta_x u(x,t) = 0 & (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = f(x) \in L^2(\mathbb{R}^n). \end{cases}$$

Then

$$\text{(2)}\qquad u(x,t) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{f}(\xi)\, d\xi = (e^{it|\,\cdot\,|^2}\widehat{f}(\,\cdot\,))\check{}\,(x)$$

defines a (weak) solution of (1) such that $\lim_{t\to 0} u(x,t) = f(x)$ in the L^2 sense. When the integral in (2) is absolutely convergent, the limit is a pointwise limit; so for example, if f has continuous derivatives of order up to $s > n/2$ in L^2, then the limit exists pointwise. However, if f is an arbitrary L^2 function the integral in (2) may not be absolutely convergent, and we must take the right-hand side of (2) as the definition of $u(x,t)$. It is not self-evident that u converges pointwise to the initial data in this case, and in fact it sometimes does not. The question of what extra smoothness conditions on f will guarantee the existence pointwise a.e. of $\lim_{t\to 0} u(x,t)$ arises.

For a given $s \geq 0$ let $H^s(\mathbb{R}^n)$ denote the L^2-Sobolev space,

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2} < \infty \right\}.$$

In the context of L^2-Sobolev spaces the question of pointwise convergence to the initial data is completely understood when $n = 1$. It was shown by Carleson [C] that $\lim_{t\to 0} u(x,t) = f(x)$ whenever $f \in H^s(\mathbb{R})$, $s \geq 1/4$. Moreover Dahlberg and Kenig [DK] demonstrated that for all $s < 1/4$ there are functions $f \in H^s(\mathbb{R})$ such that $\overline{\lim_{t\to 0}}\, u(x,t) = \infty$ a.e..

The higher dimensional cases, $n \geq 2$, are not completely understood. For these cases Vega [V] and Sjölin [Sj] independently proved that the pointwise limit exists for all $f \in H^s(\mathbb{R}^n)$ provided $s > 1/2$, while there are counterexamples just as in the 1-dimensional case when $s < 1/4$. But the question of what happens when $1/4 \leq s \leq 1/2$ is in general unanswered. However, some progress has been made in the case when $n = 2$. In [B], Bourgain shows that there is an $\epsilon > 0$ such that $f \in H^{1/2-\epsilon}(\mathbb{R}^2)$ guarantees pointwise convergence to the initial data. The value of this ϵ, although in principle calculable, is not given (although $\epsilon \ll 1/4$). The point here is not what the value of ϵ is, but that there is some improvement of the above results when $n = 2$.

§2. The Schrödinger Maximal Operator and Oscillatory Integral Operators

The study of the pointwise behavior of $u(x,t)$ as $t \to 0$ involves the study of the corresponding maximal operator, the Schrödinger maximal operator,

$$u^*(x) = \sup_{|t|\leq 1} |u(x,t)| \tag{3}$$

with regard to its mapping properties—i.e., finding weak type or strong type inequalities for u^*. The idea in [C] and [B] is to replace the nonlinear operator u^* by a family of linear operators. For each measurable function t(x), defined say on

$\mathbb{D}^n$, the unit disk in $\mathbb{R}^n$, with the property that $|t(x)| \leq 1$, one considers the linear operator

$$f \longmapsto \int e^{i(x\cdot\xi + t(x)|\xi|^2)} \widehat{f}(\xi)\, d\xi = u(x, t(x)).$$

This is justified by the following.

Proposition. *Suppose that for some $s \geq 0$ there exists a constant C such that*

$$\|u(\,\cdot\,, t(\,\cdot\,))\|_{L^p(\mathbb{D}^n)} \leq C\, \|f\|_{H^s},$$

where $1 \leq p \leq \infty$ and C is uniform over all measurable functions t, with $|t(x)| \leq 1$. Then

$$\|u^*\|_{L^p(\mathbb{D}^n)} \leq C\, \|f\|_{H^s}.$$

In [B], L^2 estimates are considered. For a measurable function t look at integral operators of the form

$$R_k f(x) = \int e^{i(x\cdot y + t(x)|y|^2)} \theta_k(y) f(y)\, dy \qquad k = 1, 2, \ldots .$$

Here $\{\theta_k\}_0^\infty$ is a partition of unity such that $\operatorname{supp}(\theta_k) \subset \{\, y \,:\, 2^{k-1} \leq |y| \leq 2^{k+1} \,\}$ when $k \geq 1$.

Proposition. [1] *Suppose there is a C and $s_0 \geq 0$ such that*

$$\|R_k f\|_{L^2(\mathbb{D}^n)} \leq C 2^{s_0 k} \|f\|_2.$$

Then for any $s > s_0$ there is a C_s, depending on C and s, such that

$$\|u(\,\cdot\,, t(\,\cdot\,))\|_{L^2(\mathbb{D}^n)} \leq C_s\, \|f\|_{H^s}.$$

Thus we have reduced to the case of finding L^2 to L^2 estimates on a family of linear operators. This is a common task in harmonic analysis, and this particular

[1] See lemma 5.1.1 or [B].

one is aided by the similarites between the R_k's and a general class of operators, $\mathcal{T}_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, of the form

$$\mathcal{T}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a(x,y) f(y)\, dy.$$

Such operators, called oscillatory integral operators, are usually studied when the *amplitude* $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and the *phase function* $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and one is concerned with the behavior of $\|\mathcal{T}_\lambda\|$ as $\lambda \to \infty$.

There are two major differences, though, between $\mathcal{T}_\lambda$ and R_k that must be considered. Firstly, since the phase function in R_k is not homogeneous, we cannot do a change of variables $y \to 2^k y$ to get into the form of $\mathcal{T}_\lambda$. However, for the purpose of obtaining an ϵ-improvement in pointwise convergence results when $n = 2$, it is pointed out in [B] that it is sufficient to consider operators of the form

$$T_\lambda f(x) = \int_{\mathbb{R}^n} \exp\left(i\lambda \frac{|x-y|^2}{t(x) - \bar{t}(y)}\right) a(x,y) f(y)\, dy, \tag{4}$$

where $a \in C_0^\infty$ and t and $\bar{t}$ are measurable functions such that $1 \le |t(x) - \bar{t}(y)| \le 2$, and show that there exists an $\epsilon > 0$ such that

$$\|T_\lambda f\|_2 \le C\lambda^{-\epsilon}\, \|f\|_2\,, \qquad C \text{ independent of } t \text{ and } \bar{t}. \tag{5}$$

Such a result then implies an inequality as in (3) with $s_0 < 1/2$.

The second and more important difference is that the phase function in R_k (and T_λ) is not smooth. The main results about $\mathcal{T}_\lambda$ say that provided the derivitives of ϕ satisfy certain "non-degeneracy" conditions, $\|T_\lambda\| \le C\lambda^{-m}$ for some positive m depending on how non-degenerate ϕ is.[2] The techniques used in proving these results insist that ϕ is smooth. Nevertheless (5) is plausible due to the following theorem, whose proof is based on ideas in [B].

[2] See chapter 2 for a more detailed discussion.

Theorem 1. [3] *If T_λ is as above, then*

(6)
$$\|T_\lambda f\|_{L^2(\mathbb{R}^n)} \leq C\lambda^{-\frac{n-2}{4}} \|f\|_{L^2(\mathbb{R}^n)},$$

where C is uniform over all measurable t and $\bar{t}$ such that $1 \leq |t(x) - \bar{t}(y)| \leq 2$.

From this, when $n = 2$, we "almost" recover (5). The heart of [B] lies in dealing with the non-smoothness in the phase function of T_λ to get the estimate in (5), which is an ϵ improvement of (6).

§3. Smooth Analogues

In this thesis we discuss operators of the form T_λ and R_k when the functions t and $\bar{t}$ are assumed to be smooth. We begin by considering a special case of T_λ when $\bar{t} \equiv 0$.

Theorem 2. *Let $\phi(x,y) = \frac{|x-y|^2}{t(x)}$ where t is a smooth function such that $t \neq 0$.*

(I) *If $\dfrac{\nabla t(x)}{t(x)} \cdot (x - y) - 1 \neq 0$ on $\operatorname{supp}(a)$, then $\|T_\lambda f\|_2 \lesssim \lambda^{-n/2} \|f\|_2$. Moreover the exponent of λ is sharp.*

(II) *In general, $\|T_\lambda f\|_2 \lesssim \lambda^{-n/2+1/4} \|f\|_2$.*

(III) *For a given amplitude function $a \not\equiv 0$, there are functions $t \in C^\infty$ such that the exponent of λ in II is sharp.*

This result is interesting by itself for a number of reasons. Firstly, the bulk of our main ideas and techniques are illustrated in the proof of theorem 2. This proof serves as a template for the proofs of other results found in this thesis, most notably theorem 4 below. Second, a theorem which is analogous to theorem 2 in its statement and proof is given, which is then used to prove theorem 3.

[3] See the appendix for a proof.

Theorem 3. *Let* $\phi = \dfrac{|x-y|^2}{t(x)-\bar{t}(y)}$ *where* t *and* $\bar{t}$ *are smooth functions such that* $0 < |t(x) - \bar{t}(y)|$. *Then*

(I) $\|T_\lambda f\|_2 \lesssim \lambda^{-n/2+1/2}$.

(II) *For a given amplitude function* $a \not\equiv 0$, *there are* t *and* $\bar{t}$ *such that the bound in I is sharp.*

We find that it is possible to analyze t and $\bar{t}$ separately in order to prove interesting results about T_λ. Thus if we have an estimate as in part I of theorem 2 for t or $\bar{t}$ (or both), we may find correspondingly better estimates in part I of theorem 3. And in general, estimates which are better than those in part II of theorem 2 lead to improved estimates in theorem 3.

Finally we obtain some preliminary inequalities regarding R_k. Such inequalities then imply results of the form

$$\|u(\,\cdot\,, t(\,\cdot\,))\|_{L^2(\mathbb{D}^n)} \le C\,\|f\|_{H^s} \tag{7}$$

where s depends on conditions on the derivitives of t. The most notable of these is when ∇t is non-vanishing.

Theorem 4. *Suppose* $t \in C^\infty$ *is such that* $\nabla t(x) \neq 0\ \forall x \in \mathbb{D}^n$. *Then for any* $s > 0$,

$$\|u(\,\cdot\,, t(\,\cdot\,))\|_{L^2(\mathbb{D}^n)} \le C\,\|f\|_{H^s}\,,$$

where C *may depend on* s *and* t.

The organization of this thesis is as follows. Chapter 1 contains the tools that will be used in proving our main theorems. It contains results that are standard but are modified to suit our purposes. The next chapter on oscillatory integral operators also contains standard material on the subject. Again, a sharpening of some of these results is given in order to further our aims. Chapter 3 contains a proof of theorem 2 given in considerable detail. This proof is typical of others in this paper: namely the proof of a variant of theorem 2 used in proving theorem 3, and a proof of theorem 4. Chapter 4 is concerned with proving theorem 3 and a simple extension of it. Chapter 5 is about (7) in general and theorem 4 in particular. Finially the appendix has a proof of theorem 1, and, for the sake of completeness, a discussion of other issues raised throughout this thesis. Finially, we end the appendix with indications of further study along the lines taken up in these pages.

1. Preliminaries

This chapter contains a variety of results, most of them already known, which will be used in proving our main theorems. In general, the operators we are interested in are integral operators — $f \longmapsto \int K(x,y) f(y)\, dy$ — and §2 contains those results about such operators that we will exploit throughout this thesis. More specifically the kernels, $K(x,y)$, we consider are oscillatory and a careful examination of the method of stationary phase is crucial to our endeavors; this can be found in §3. Another result also in §3 is lemma 1.3.5, whose usefulness is evident in chapter 4. I am particularly indebted to my advisor T. Wolff for pointing out the main idea in this lemma to me. The last section in this chapter contains technical lemmas which are included here so as not to interrupt the flow of our other chapters.

§1. Notation

The following notation is used throughout.

x, y, z and ξ will denote variables in $\mathbb{R}^n$.

$x \cdot y$ is the inner product in $\mathbb{R}^n$: $x \cdot y = \sum_1^n x_i y_i$.

M^t denotes the transpose of the matrix M.

$\mathrm{H}\, f$ will denote the Hessian of f.

$\widehat{f}(\xi) = \int e^{-x\cdot\xi} a(x)\, dx$ is the Fourier transform of f.

$\check{f}(\xi) = (2\pi)^{-n} \int e^{ix\cdot\xi} f(x)\, dx$ is the inverse Fourier transform of f.

∂_j is the differential operator $\partial / \partial x_j$.

$\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class of functions on $\mathbb{R}^n$.

$B_r(p) = \{x \in \mathbb{R}^n : |x-p| < r\}$.

$\mathbb{D}^n$ denotes the unit ball in $\mathbb{R}^n$.

If $E \subset \mathbb{R}^n$ is measurable, then $|E|$ denotes the Lebesgue measure of E.

If $a(x,y)$ is a function of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then denote by $\operatorname{supp}_y(a)$ the projection onto the y-coordinates of the support of a. Let $\nabla_y a(x,y)$ denote the gradient of a as a function of y with x held fixed. Similarily $\Delta_y a(x,y) = \sum \partial^2/\partial y_j^2\, a(x,y)$.

The expression $x \lesssim y$ will mean that there is a constant C, which does not depend on quantities that are otherwise to be kept track of, such that $x \leq C\,y$. Dependence on such quantities will be explicitely noted.

§2. Integral Operators and Frozen Operators

Given x, $y \in \mathbb{R}^n$, write $x = (x', x_n)$ and $y = (y', y_n)$, where x' and y' are in $\mathbb{R}^{n-1}$. Let $K(x,y)$ be a given bounded measurable function, which for our purposes will be assumed to have compact support, and define an operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy,$$

whose adjoint is

$$T^* f(y) = \int_{\mathbb{R}^n} \overline{K(z,y)} f(z)\, dz.$$

For reference we quote the following result, Schur's lemma, a proof of which may be found in [St].

Theorem 1.2.1. *Suppose there exists positive constant C_1 and C_2 such that*

$$\sup_x \int_{\mathbb{R}^n} |K(x,y)|\, dy \leq C_1 \quad \text{and} \quad \sup_y \int_{\mathbb{R}^n} |K(x,y)|\, dx \leq C_2.$$

Then

$$\|Tf\|_2 \leq \sqrt{C_1 C_2}\, \|f\|_2 .$$

If we fix x_n and y_n and let $K_{x_n y_n}(x', y') = K(x', x_n, y', y_n)$, then we get the family of *frozen operators*, $T_{x_n y_n} : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ defined by

$$T_{x_n y_n} f(x') = \int_{\mathbb{R}^{n-1}} K_{x_n y_n}(x', y') f(y')\, dy'.$$

Note that there is no confusion between $(T^*)_{x_n y_n}$ and $(T_{x_n y_n})^*$ as the two are the same.

Lemma 1.2.2. *Suppose there exists a measurable function $\eta(x_n, y_n)$ such that*

$$\|T_{x_n y_n} f\|_{L^q(\mathbb{R}^{n-1})} \le \eta(x_n, y_n)\, \|f\|_{L^p(\mathbb{R}^{n-1})}, \tag{1.2.1}$$

$$\left\| \int \eta(x_n, y_n) h(y_n)\, dy_n \right\|_{L^q(\mathbb{R})} \le C\, \|h\|_{L^p(\mathbb{R})}. \tag{1.2.2}$$

Then

$$\|Tf\|_{L^q(\mathbb{R}^n)} \le C\, \|f\|_{L^p(\mathbb{R}^n)}.$$

For completeness we give a proof. The idea of frozen operators can be found in [S].

Proof. We have by Minkowski's inequality that

$$\begin{aligned}
\|Tf\|_{L^q(\mathbb{R}^n)} &= \left(\int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} K(x', x_n, y', y_n) f(y', y_n)\, dy'\, dy_n \right|^q dx'\, dx_n \right)^{1/q} \\
&\le \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \|T_{x_n y_n} f(\cdot, y_n)\|_{L^q(\mathbb{R}^{n-1})}\, dy_n \right)^q dx_n \right)^{1/q} \\
&\le \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \eta(x_n, y_n)\, \|f(\cdot, y_n)\|_{L^p(\mathbb{R}^{n-1})}\, dy_n \right|^q dx_n \right)^{1/q} \quad \text{by (1.2.1)} \\
&\le C\, \|f\|_{L^p(\mathbb{R}^n)} \qquad \text{by (1.2.2)},
\end{aligned}$$

as stated. ■

Now consider

$$TT^*f(x) = \int f(z) \left(\int K(x,y)\overline{K(z,y)}\,dy \right) dz.$$

If $T_{x_n} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{n-1})$ is the operator

$$T_{x_n} f(x') = \int_{\mathbb{R}^n} K(x', x_n, y) f(y)\,dy,$$

and $T^*_{z_n} : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^n)$

$$T^*_{z_n} f(y) = \int_{\mathbb{R}^{n-1}} \overline{K(z', z_n, y)} f(z')\,dz'$$

is its adjoint, then clearly

$$(1.2.3) \qquad \begin{aligned} (TT^*)_{x_n z_n} &= T_{x_n} T^*_{z_n}, \\ \|T_{x_n}(T_{z_n})^* f\|_{L^2(\mathbb{R}^{n-1})} &\le \|T_{x_n}\|\,\|T^*_{z_n}\|\,\|f\|_2. \end{aligned}$$

Lemma 1.2.3.

$$(1.2.4) \qquad \left\|T^*_{z_n} f\right\|_{L^2(\mathbb{R}^n)} \le \|f\|_{L^2(\mathbb{R}^{n-1})} \left(\int_{-\infty}^{\infty} \|(T^*)_{z_n y_n}\|^2\,dy_n \right)^{1/2}.$$

$$(1.2.5) \qquad \|T_{x_n} f\|_{L^2(\mathbb{R}^{n-1})} \le \|f\|_{L^2(\mathbb{R}^n)} \left(\int_{-\infty}^{\infty} \|T_{x_n y_n}\|^2\,dy_n \right)^{1/2}.$$

Proof. For (1.2.4) we have that

$$\begin{aligned} \left\|T^*_{z_n} f\right\|_{L^2(\mathbb{R}^n)} &= \left(\int_{-\infty}^{\infty} \|(T^*)_{z_n y_n} f\|^2_{L^2(\mathbb{R}^{n-1})}\,dy_n \right)^{1/2} \\ &\le \|f\|_{L^2(\mathbb{R}^{n-1})} \left(\int_{-\infty}^{\infty} \|(T^*)_{z_n y_n}\|^2\,dy_n \right)^{1/2}. \end{aligned}$$

For (1.2.5) Minkowski's inequality gives that

$$\begin{aligned} \|T_{x_n} f\|_{L^2(\mathbb{R}^{n-1})} &\le \int_{-\infty}^{\infty} \|T_{x_n y_n} f(\cdot, y_n)\|_{L^2(\mathbb{R}^{n-1})}\,dy_n \\ &\le \int_{-\infty}^{\infty} \|T_{x_n y_n}\|\,\|f(\cdot, y_n)\|_{L^2(\mathbb{R}^{n-1})}\,dy_n \\ &\le \|f\|_{L^2(\mathbb{R}^n)} \left(\int_{-\infty}^{\infty} \|T_{x_n y_n}\|^2\,dy_n \right)^{1/2}, \end{aligned}$$

the last inequality following from Hölder's inequality. ∎

§3. Old Results Newly Modified

The following is a collection of already known theorems, some of which are modified to suit our purposes. The modifications consist mainly in our keeping track of constants that are usually ignored, as this is crucial to many of our arguments below.

We begin with a discussion of oscillatory integrals, $\int e^{i\lambda\phi(y)}a(y)\,dy$.

Theorem 1.3.1. *Let $a \in C_0^\infty(\mathbb{R}^n)$ and $\phi \in C^\infty(X)$, where X is a neighborhood of* $\operatorname{supp}(a)$. *Then for $N = 1, 2, \ldots$*

$$(1.3.1) \quad \left|\int e^{i\lambda\phi(y)}a(y)\,dy\right| \le C\lambda^{-N}|\operatorname{supp}(a)| \sum_{|\alpha|\le N} \sup |D^\alpha a||\nabla\phi|^{|\alpha|-2N} \qquad \lambda > 0,$$

where we may take $C = C(N,n)\,\|\phi\|^N_{C^{N+1}(X)}$.

This theorem appears almost verbatim in [H1]. Our version of the theorem includes a statement about $|\operatorname{supp}(a)|$, the volume of the support of a. See also [S], [St].

Proof. For $j = 1, 2 \ldots n$ let $U_j = \{y \in \operatorname{supp}(a) : |\partial_j\phi(y)| > (2n)^{-1/2}|\nabla\phi(y)|\}$. The U_j's cover the support of a, so let $a = \sum_{j=1}^{n} a_j$ be a partition of unity subordinate to $\{U_j\}_{j=1}^n$. Define operators

$$L_j = \frac{1}{i\lambda\partial_j\phi}\partial_j \quad \text{and} \quad L_j^* = \partial_j\left(\frac{1}{i\lambda\partial_j\phi}\right).$$

Then we have that for any $N = 1, 2, \ldots$

$$\int e^{i\lambda\phi(y)}a_j(y)\,dy = \int (L_j(e^{i\lambda\phi(y)}))^N a_j(y)\,dy = \int e^{i\lambda\phi(y)}(L_j^*(a_j(y)))^N\,dy.$$

Note that $(L_j^* a_j)^N$ is a sum of terms of the form

$$\left(\frac{1}{i\lambda}\right)^N \partial_j^{\alpha_0} a_j(y) \partial_j^{\alpha_1}\left(\frac{1}{\partial_j \phi}\right) \cdots \partial_j^{\alpha_N}\left(\frac{1}{\partial_j \phi}\right)(y) \qquad \text{where } \alpha_0 + \cdots \alpha_N = N.$$

Also $\partial_j^\beta(1/\partial_j\phi)$ is a sum of terms of the form

$$\frac{\partial_j^{\alpha_1}(\partial_j\phi)\ \cdots\ \partial_j^{\alpha_\beta}(\partial_j\phi)}{(\partial_j\phi)^{\beta+1}} \qquad \text{where } \alpha_1 + \cdots + \alpha_\beta = \beta.$$

After noticing that $|\partial_j\phi(y)|^{-1} \lesssim |\nabla\phi(y)|^{-1}$ when $y \in U_j$, good bookkeeping gives that

$$\begin{aligned}\left|\int e^{i\lambda\phi(y)}(L_j^*(a_j(y)))^N\,dy\right| &\le \int |(L_j^*(a_j(y)))^N|\,dy \\ &\le \lambda^{-N} C(N,n) \sum_{|\alpha|\le N} \sup|D^\alpha a||\nabla\phi|^{|\alpha|-2N}\,\|\phi\|_{N+1-|\alpha|}^{N-|\alpha|} \int_{\text{supp}(a_j)} dy\,.\end{aligned}$$

Then (1.3.1) is given by summing over j. ■

Theorem 1.3.2. *Suppose that $a \in \mathcal{S}(\mathbb{R}^n)$. Then for any positive integer k,*

(1.3.2)

$$\int_{\mathbb{R}^n} e^{i\lambda|y|^2} a(y)\,dy = \left(\frac{i\lambda}{\pi}\right)^{-n/2}\left(\sum_{j=0}^{k-1}(4i\lambda)^{-j}\Delta^j a(0)/j! + \int_{\mathbb{R}^n} r_k(i|\xi|^2/4\lambda)\check{a}(\xi)\,d\xi\right),$$

where $r_k(x)$ is the remainder of the k-th degree Taylor polynomial of e^x.

This well-known result, an example of the method of stationary phase, is not usually expressed in this form. We find it convenient to include a form of the remainder term in the asymptotic expansion of the left-hand side of (1.3.2) in powers of λ. See [H1], [St].

Proof. Since the Fourier transform of the function $e^{i\lambda|y|^2}$ is $(i\lambda/\pi)^{-n/2}e^{-i|\xi|^2/4\lambda}$, we have that

$$\begin{aligned}\int_{\mathbb{R}^n} e^{i\lambda|y|^2} a(y)\,dy &= \left(\frac{i\lambda}{\pi}\right)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{-i|\xi|^2}{4\lambda}}\check{a}(\xi)\,d\xi \\ &= \left(\frac{i\lambda}{\pi}\right)^{-n/2}\left(\sum_0^{k-1}(4i\lambda)^{-j}\Delta^j a(0)/j! + \int_{\mathbb{R}^n} r_k(-i|\xi|^2/4\lambda)\check{a}(\xi)\,d\xi\right)\end{aligned}$$

by the properties of the Fourier transform and by Taylor's formula. ∎

Remark 1.3.3. Note that $|r_k(x)| \le |x|^k/k!$ whenever $\operatorname{Re} x \le 0$. Then an application of the Cauchy-Schwartz inequality shows that for any integer $s > n/2$,

$$\left|\int_{\mathbb{R}^n} r_k(i|\xi|^2/4\lambda)\check{a}(\xi)\,d\xi\right| \lesssim \lambda^{-k} \sum_{|\alpha|\le 2k+s} \|\mathrm{D}^\alpha a\|_2\,. \tag{1.3.3}$$

A corollary of theorem 1.3.2 is needed in chapter 4, which is a variable parameter version of theorem 1.3.2.

Corollary 1.3.4. *Suppose that a is contained in a bounded subset X of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. Then for any multi-index α and any $\lambda \ge 1$ there is a constant $C = C(\alpha, X)$ such that*

$$\lambda^{n/2}\left|D_z^\alpha \int_{\mathbb{R}^n} e^{i\lambda|y|^2} a(y,z)\,dy\right| \le C. \tag{1.3.4}$$

Proof. We may differentiate under the integral sign in the left-hand side of (1.3.4). Combining (1.3.2) and (1.3.3) with $a(y)$ replaced by $\mathrm{D}_z^\alpha a(y,z)$, we find that for any given $s > n/2$ the left-hand side of (1.3.4) does not exceed

$$C\Big(\Delta_y \mathrm{D}_z^\alpha a(0,z) + \lambda^{-1} \sum_{|\beta|\le 2+s} \left\|\mathrm{D}_y^\beta \mathrm{D}_z^\alpha a\right\|_2\Big),$$

which is uniformly bounded given the hypothesis on a. ∎

We end this section with a lemma about C_0^∞ functions on the product space $\mathbb{R}^n \times \mathbb{R}^m$. It is at times convenient that such functions be of the form $\alpha(x)\beta(y)$, a so-called tensor, with $\alpha \in C_0^\infty(\mathbb{R}^n)$ and $\beta \in C_0^\infty(\mathbb{R}^m)$. While it is not possible that any given $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ function is a tensor, the next best thing is true, that such a function is an absolutely convergent sum of tensors. Lemma 1.3.5, a restatement of the absolute convergence of the Fourier series of a smooth function, is a quantitative assertion of this fact. A similar result may be found in [E].

Lemma 1.3.5. *Let $\psi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Then ψ is an absolutely convergent sum of C_0^∞ tensors. That is for $j = 1, 2, \ldots$ there exists functions α_j ,β_j and $a_j \in \mathbb{C}$ such that*

$$\alpha_j \in C_0^\infty(\mathbb{R}^n)\,, \beta_j \in C_0^\infty(\mathbb{R}^m)$$

$$\|\alpha\|_\infty \leq 1 \quad \|\beta\|_\infty \leq 1 \tag{1.3.5}$$

$$\sum_{j=1}^{\infty} |a_j| \leq C\Big(\|\psi\|_1 + \sum_{|\gamma|=s} \|D^\gamma \psi\|_2 \Big) \qquad s > (n+m)/2.$$

for which

$$\psi(x,y) = \sum_{j=1}^{\infty} a_j\, \alpha_j(x)\beta_j(y).$$

The constant C in (1.3.5) is bounded once ψ has bounded support.

Proof. Suppose first that $\operatorname{supp}(\psi) \subset [-1/4, 1/4]^{n+m}$, and consider its Fourier series on $\mathbb{T} = [-1/2, 1/2]^{n+m}$:

$$\psi(x,y) = \sum_{k\in\Lambda} a_k e^{2\pi i (x,y)\cdot(k_x,k_y)},$$

where Λ is the unit lattice in $\mathbb{R}^{n+m}$, $k_x = (k_1, \ldots k_n)$, $k_y = (k_{n+1} \ldots k_{m+n})$ and

$$a_k = \int_{\mathbb{T}} \psi(x,y) e^{-2\pi i (x,y)\cdot(k_x,k_y)}\, dx\, dy.$$

It is well known that for any $s > (n+m)/2$ (cf. [StW]),

$$\sum_{k\in\Lambda} |a_k| \leq \|\psi\|_1 + c_{n,s} \sum_{|\gamma|=s} \|\mathrm{D}^\gamma \psi\|_2\,.$$

Choose $g_1 \in C_0^\infty([-1/2, 1/2]^n)$ such that $g_1 \equiv 1$ on $[-1/4, 1/4]^n$, and choose $g_2 \in C_0^\infty([-1/2, 1/2]^m)$ such that $g_2 \equiv 1$ on $[-1/4, 1/4]^m$. Let $\alpha_k(x) = g_1(x) e^{2\pi i x\cdot k_x}$ and $\beta_k(y) = g_2(y) e^{2\pi i y\cdot k_y}$ to finish this special case.

In general choose $R > 1$ large enough so that $\operatorname{supp}(\psi) \subset [-R/4,\ R/4]^{n+m}$, and let $\psi_R(x,y) = \psi(Rx, Ry)$. Then $\operatorname{supp}(\psi_R) \subset [-1/4, 1/4]^{n+m}$. Clearly, by appealing to the special case we may finish the proof once we note that

$$\|\psi_R\|_1 + \sum_{|\gamma|=s} \|\mathrm{D}^\gamma \psi_R\|_2 \leq R^{-(n+m)} \|\psi\|_1 + R^{s-(n+m)/2} \sum_{|\gamma|=s} \|\mathrm{D}^\gamma \psi\|_2 ,$$

demonstration that C in (1.3.5) is bounded when ψ has bounded support. ∎

§4. Miscellany

We finish this section with a couple of lemmas about $n \times n$ matrices which are used in chapters 3 and 4.

Lemma 1.4.1. *Let M be an $n \times n$ matrix with entries $M_{ij} = \delta_i^j + a_i b_j$, where δ_i^j is the Kronecker delta and $a_i,\ b_j \in \mathbb{R}$. Then*

(1) $\det M = 1 + \sum_{i=1}^n a_i b_i$,

(2) $\operatorname{rank}(M) \geq n - 1$.

Proof. We see that M is of the form $M = (I + \widetilde{M})$ where I is the $n \times n$ identity matrix, and $\widetilde{M}$ is of the form $\widetilde{M}_{ij} = a_i b_j$ for real numbers a_i and b_j. Let $\mathrm{I}_1, \ldots, \mathrm{I}_n$ and $\widetilde{M}_1, \ldots, \widetilde{M}_n$ be the rows of I and $\widetilde{M}$ respectively. Consider $\det(\cdot)$ as an n-linear function of the rows of a matrix. Then

$$\det(M) = \det(\mathrm{I}_1 + \widetilde{M}_1, \ldots, \mathrm{I}_n + \widetilde{M}_n).$$

When the above is expanded out by multilinearity, terms with two or more rows of $\widetilde{M}$ appearing in the argument of det vanish. Thus

$$\begin{aligned}\det(M) = \det(\mathrm{I}_1, \ldots, \mathrm{I}_n) &+ \det(\mathrm{I}_1, \ldots, \mathrm{I}_{n-1}, \widetilde{M}_n) + \det(\mathrm{I}_1, \ldots, \mathrm{I}_{n-2}, \widetilde{M}_{n-1}, \mathrm{I}_n) \\ &+ \cdots + \det(\mathrm{I}_1, \widetilde{M}_2, \ldots, \mathrm{I}_n) + \det(\widetilde{M}_1, \mathrm{I}_2, \ldots, \mathrm{I}_n) \\ &= 1 + a_n b_n + a_{n-1} b_{n-1} + \cdots + a_2 b_2 + a_1 b_1.\end{aligned}$$

(2) If $\widetilde{M} = 0$ then (2) is obvious; otherwise it is clear that $\operatorname{rank}(\widetilde{M}) = 1$, and (2) follows immediately. ■

Lemma 1.4.2. *Let M be an $n \times n$ matrix of the form $M_{ij} = \delta^j_{i+1} + a_i b_j$. Then $\det(M) = a_n b_1$.*

Proof. Write $M = J + \widetilde{M}$ where J is an $n \times n$ Jordan block with zeros on the diagonal, and $\widetilde{M}_{ij} = a_i b_j$. Arguing as in lemma 1.4.1, we see that

$$\det(M) = \det(J_1 + \widetilde{M}_1, \ldots, J_n + \widetilde{M}_n) = \det(J_1, J_2 \ldots, \widetilde{M}_n) = a_n b_1,$$

as stated. ■

2. Oscillatory Integral Operators

We continue with preliminary material of a more specific nature than found in the previous chapter, and again we pay close attention to detail.

§1. Introduction

Having discussed the basic facts about integral operators and oscillatory integrals, we consider the family of operators $\mathcal{T}_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $\lambda > 1$, of the form

$$\mathcal{T}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a(x,y) f(y)\, dy, \tag{2.1.1}$$

where the *amplitude* $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and the real-valued *phase function* $\phi \in C^\infty(X)$ with X a neighborhood of $\operatorname{supp}(a)$. We call such operators *oscillatory integral operators* in view of the exponential factor in the kernel of $\mathcal{T}_\lambda$. Under suitable conditions the oscillations of the exponential factor give rise to cancellations, and $\|\mathcal{T}_\lambda\|$ tends to zero as λ tends to infinity. Just how rapidly $\|\mathcal{T}_\lambda\|$ decays depends, of course, on ϕ, and the relevant thing to consider, as we shall see in §2, is the *mixed Hessian* of ϕ, the $n \times n$ matrix $H_\phi(x,y)$ defined by

$$(H_\phi(x,y))_{i,j} = \frac{\partial^2 \phi}{\partial x_i \partial y_j}(x,y).$$

When H_ϕ is non-singular the decay of $\|\mathcal{T}_\lambda\|$ is as rapid as possible, and in such case we have the following theorem ([H2], [St]).

Theorem 2.1.1. *Suppose that H_ϕ is non-singular on* $\operatorname{supp}(a)$. *Then*

$$\|\mathcal{T}_\lambda f\|_2 \le C\lambda^{-n/2}\|f\|_2. \tag{2.1.2}$$

In §2 we explore theorem 2.1.1 in some detail, and again we are careful when keeping track of the constant C that appears in (2.1.2). In §3 we show how the estimate in (2.1.2) cannot be improved, and we outline a general strategy for showing how degeneracies of H_ϕ translate into slower rates of decay for $\|\mathcal{T}_\lambda\|$ — i.e., slower than in (2.1.2).

§2. The Main Estimate

The constant C in (2.1.2) may depend on the L^∞ norm of finitely many derivatives of a and ϕ, on the volume of supp(a) and on n. For the purposes of our subsequent work we are interested in the dependance of C on the properties of a. This leads to the following theorem, whose proof follows the outline of that in [H2].

Theorem 2.2.1. *Suppose that H_ϕ is non-singular on* supp(a) *and that the following quantities are uniformly bounded on* supp(a)*:*

(i) $\|H_\phi^{-1}(x,y)\|$

(ii) $\|\nabla_y D_x^\alpha \phi\|_{L^\infty(X)}$ *for all α with $|\alpha| = 2$*

(iii) $\|\nabla_x D_y^\alpha \phi\|_{L^\infty(X)}$ *for all α with $|\alpha| \le n+2$.*

Then if $M = \max\{1, |\operatorname{supp}_x(a)|\}$ and

$$M_a = \|a\|_\infty \left(M\,|\operatorname{supp}_y(a)| \left\{ \sum_{|\alpha|\le n+1} \sup_{xyz} |D_y^\alpha a(x,y)\overline{a(x,z)}| \right\}^{\frac{n}{n+1}} \right)^{1/2}, \tag{2.2.1}$$

then

$$\|\mathcal{T}_\lambda f\|_2 \le C M_a \lambda^{-n/2} \|f\|_2\,, \tag{2.2.2}$$

where C is bounded.

Proof. Assume without loss of generality that $\|a\|_\infty \le 1$. It is sufficient to consider $\mathcal{T}_\lambda \mathcal{T}_\lambda^*$ and show that

$$\|\mathcal{T}_\lambda \mathcal{T}_\lambda^* f\|_2 \le C M_a^2 \lambda^{-n} \|f\|_2\,. \tag{2.2.3}$$

We note that $\mathcal{T}_\lambda \mathcal{T}_\lambda^* f$ has the form $\mathcal{T}_\lambda \mathcal{T}_\lambda^* f(x) = \int_{\mathbb{R}^n} K_\lambda(x,z) f(z)\, dz$, where

$$K_\lambda(x,z) = \int_{\mathbb{R}^n} \exp\big(i\lambda\{\phi(x,y) - \phi(z,y)\}\big) a(x,y)\overline{a(z,y)}\, dy.$$

We will show that there is a constant C so that

$$|K_\lambda(x,z)| \le C M_a^2 (1 + \lambda|x-z|)^{-(n+1)}, \tag{2.2.4}$$

and in doing so we will keep careful track of C, showing that it is bounded once the quantities in (i), (ii) and (iii) are uniformly bounded. By theorem 1.2.1, (2.2.4) implies (2.2.3) since

$$\begin{aligned}\int_{\mathbb{R}^n} |K_\lambda(x,z)|\, dz &\le C M_a^2 \int_{\mathbb{R}^n} (1 + \lambda|x-z|)^{-(n+1)}\, dz \\ &= C M_a^2 \lambda^{-n} \int_{\mathbb{R}^n} (1+|z|)^{-(n+1)}\, dz = C M_a^2 \lambda^{-n}.\end{aligned}$$

Note that

$$|K_\lambda(x,z)| \le |\operatorname{supp}_y(a)|. \tag{2.2.5}$$

By Taylor's formula we have that

$$\begin{aligned}\nabla_y(\phi(x,y) - \phi(z,y)) &= H_\phi(z,y)(x-z) \\ &+ \sum_{ij}(x_i - z_i)(x_j - z_j) \int_0^1 \nabla_y \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)(tx + (1-t)z, y)(1-t)\, dt.\end{aligned} \tag{2.2.6}$$

Since H_ϕ is non-degenerate on supp(a), $|H_\phi(z,y)(x-z)| \ge \|H_\phi^{-1}\|^{-1}|x-z|$, while the remainder term in (2.2.6) is $\mathcal{O}(|x-z|^2)$ depending on (ii). If we were to assume that $|x-z| \lesssim \|H_\phi^{-1}\|^{-1}$ then (2.2.6) would guarantee that

$$\nabla_y\left(\frac{\phi(x,y) - \phi(z,y)}{|x-z|}\right) \gtrsim \|H_\phi^{-1}\|^{-1}. \tag{2.2.7}$$

If $\mathrm{supp}_x(a)$ is partitioned into small balls (whose size depends on H_ϕ) we may indeed assume that $|x-z|$ is as small as necessary by writing (2.1.1) as a sum of like terms; in doing so, by considerations of almost orthogonality[1], we gain the factor of $M^{1/2}$ in (2.2.1). Given assumption (iii) about the uniform boundedness of the derivitives of ϕ, by the mean value theorem,

$$\left|\frac{\mathrm{D}_y^\alpha\phi(x,y)-\mathrm{D}_y^\alpha\phi(z,y)}{|x-z|}\right|\leq\left\|\nabla_x\mathrm{D}_y^\alpha\phi\right\|_\infty\lesssim 1,$$

when $|\alpha|\leq n+2$. And so by (2.2.7) and theorem 1.3.1 (with $N=n+1$),

$$|K(x,z)|\lesssim(\lambda\,|x-z|)^{-(n+1)}|\,\mathrm{supp}_y(a)|\sum_{|\alpha|\leq n+1}\sup_y|D^\alpha a(x,y)\overline{a(z,y)}|, \tag{2.2.8}$$

the implicit constant in (2.2.8) being bounded given (i), (ii) and (iii). Clearly then, (2.2.5) and (2.2.8) give that

$$|K_\lambda(x,z)|\lesssim|\,\mathrm{supp}_y(a)|\left(1+\lambda\,|x-z|\left\{\sum_{|\alpha|\leq n+1}\sup_{xyz}|D^\alpha a(x,y)\overline{a(x,z)}|\right\}^{\frac{-1}{(n+1)}}\right)^{-(n+1)}$$

This is essentially (2.2.4) which gives the result. ■

Remark 2.2.2. When bounding $\|H_\phi^{-1}\|$ uniformly form above it is convenient to use the classical theorem for the inverse of a matrix, $H_\phi^{-1}=(\det H_\phi)^{-1}\operatorname{adj}H_\phi$. Then we need only bound $\det H_\phi$ uniformly from below and the entries of H_ϕ uniformly from above.

The situation when $\operatorname{rank}H_\phi<n$ is not completely understood. However, we may still say something about $\|\mathcal{T}_\lambda\|$ in this case, though we shall not do so in as much detail as in theorem 2.2.1.

Theorem 2.2.3. *Suppose that* $\operatorname{rank}H_\phi\geq k$ *on* $\mathrm{supp}(a)$. *Then*

$$\|\mathcal{T}_\lambda f\|_2\leq C\lambda^{-k/2}\|f\|_2. \tag{2.2.9}$$

This is a corollary of theorem 2.2.1 once we have the following change of variables lemma for H_ϕ, whose proof is an easy application of the chain rule and is omitted.

[1] See the appendix for a discussion of almost orthogonality.

Lemma 2.2.4. *Suppose that ϕ is a given C^∞ function, and let F, $G : \Omega \to \mathbb{R}^n$ be C^∞ diffeomorphisms on a domain $\Omega \subset \mathbb{R}^n$. If $\psi(x,y) = \phi(F(x), G(y))$, then*

$$H_\psi(x,y) = (DF(x))^t H_\phi(F(x), G(y)) DG(y). \tag{2.2.10}$$

Proof of theorem 2.2.3. After a linear change of variables, given (2.2.10), we may assume that

$$\det \left(\frac{\partial^2 \phi}{\partial x_i \partial y_j} \right)_{ij=1}^{k} \neq 0$$

on supp(a) . Write $x = (\bar{x}, \bar{\bar{x}})$, $y = (\bar{y}, \bar{\bar{y}})$ as a splitting of variables in $\mathbb{R}^k \times \mathbb{R}^{n-k}$. Then, since a has compact support

$$\begin{aligned} \|\mathcal{T}_\lambda f\|_2 &= \left(\int \int \left| \int \int e^{i\lambda\phi(x,y)} a(x,y) f(y)\, d\bar{y}\, d\bar{\bar{y}} \right|^2 d\bar{x}\, d\bar{\bar{x}} \right)^{1/2} \\ &\leq C \int \left(\int \left| \int e^{i\lambda\phi(x,y)} a(x,y) f(y)\, d\bar{y} \right|^2 d\bar{x} \right)^{1/2} d\bar{\bar{y}} \\ &\leq C \lambda^{-k/2} \|f\|_2, \end{aligned}$$

by theorem 2.2.1 applied to the k-dimensional case. ∎

§3. Lower Bounds

It is easy to show that the exponent of λ in (2.2.2) is sharp in the sense that $\lim_{\lambda \to \infty} \lambda^p \|\mathcal{T}_\lambda\| = \infty$ whenever $p > n/2$. On the other hand the exponent of λ in (2.2.9) is not necessarily the best possible. This first statement can be seen from the following well known result, whose proof is implicit in [H2]. The second one will be illustrated in chapters 3 and 4.

Theorem 2.3.1. *Let $\mathcal{T}_\lambda$ be as in (2.1.1). Suppose that there are measurable sets $R \subset \text{supp}_x(a)$ and $\widetilde{R} \subset \text{supp}_y(a)$ and measurable functions ϕ_1 and ϕ_2 such that*

$$\lambda |\phi(x,y) - \phi_1(x) - \phi_2(y)| < 1/2 \qquad \text{when } (x,y) \in R \times \widetilde{R}. \tag{2.3.1}$$

If $|a(x,y)| \geq c > 0$ when $(x,y) \in R \times \widetilde{R}$, and

$$\left| \int_{\widetilde{R}} a(x,y)\, dy \right| \geq 3/4 \int_{\widetilde{R}} |a(x,y)|\, dy \tag{2.3.2}$$

then there is a positive constant C such that

$$\|\mathcal{T}_\lambda\| \geq C\sqrt{|R|\,|\widetilde{R}|}.$$

Proof. Let $f(y) = e^{-i\lambda\phi_2(y)}\chi_{\widetilde{R}}(y)$; then $\|f\|_2 = |\widetilde{R}|^{1/2}$. When $x \in R$,

$$\begin{aligned} |\mathcal{T}_\lambda f(x)| &= |e^{-i\lambda\phi_1(x)}\mathcal{T}_\lambda f(x)| \\ &\geq \left| \int_{\widetilde{R}} a(x,y)\, dy \right| - \left| \int_{\widetilde{R}} \left(e^{i\lambda(\phi(x,y)-\phi_1(x)-\phi_2(y))} - 1 \right) a(x,y)\, dy \right| \\ &= I + II. \end{aligned}$$

By condition (2.3.1)

$$|II| \leq 1/2 \int_{\widetilde{R}} |a(x,y)|\, dy.$$

Thus (2.3.2) guarantees that

$$\int_R |\mathcal{T}_\lambda f(x)|^2\, dx \geq C|R|\,|\widetilde{R}|^2,$$

and dividing by $\|f\|_2^2$ gives the result. ∎

Now let (x_0, y_0) be a fixed point in supp(a). Suppose that we let $\phi_1(x) = \phi(x, y_0)$ and $\phi_2(y) = \phi(x_0, y) - \phi(x_0, y_0)$. If we expand $\Phi(x,y) = \phi(x,y) - \phi_1(x) - \phi_2(y)$ in a Taylor series about the point (x_0, y_0), we see that

$$\Phi(x,y) = H_\phi(x,y)(x - x_0)\cdot(y - y_0) + \mathcal{O}(|x - x_0|^2|y - y_0| + |x - x_0||y - y_0|^2).$$

Assuming that $|x - x_0| \lesssim \lambda^{-1/2}$, $|y - y_0| \lesssim \lambda^{-1/2}$ and that $a(x_0, y_0) \neq 0$, we may apply theorem 2.3.1 to see that for any given $\mathcal{T}_\lambda$ with $a \not\equiv 0$, $\|\mathcal{T}_\lambda\| \geq C\lambda^{-n/2}$. We shall use similar arguments in chapters 3 and 5.

3. Main Theorem, A Special Case

For a given fixed amplitude $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ we shall consider certain examples of oscillatory integral operators. In the first instance, in §1, we look at the special case of T_λ defined in (4) of the introduction when $\bar{t} \equiv 0$. Theorems 3.1.2, 3.1.6 and 3.1.7 below characterize the behavior of $||T_\lambda||$ in this specific case. The details of the proofs are fully given, and the ideas and techniques involved will be exploited throughout the rest of this thesis. The second example will involve a phase function that is similar to the one studied in §1. Its purpose is to provide a means of understanding T_λ in its full generality. The phase function involved, $\phi(x,y) = -2x \cdot y + t(x)|y|^2$, is more suited to this task from a technical standpoint. We could have just as easily applied theorem 3.1.6 to the more general case of T_λ, but the proofs of theorems 3.2.1, 3.2.2 and 3.2.3 follow closely those of their counterparts, theorems 3.1.2, 3.1.6 and 3.1.7, and we get these results almost for free.

§1. The Core Arguement

We let $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a fixed given function, and let $t \in C^\infty(\mathbb{R}^n)$ be such that $0 \neq t(x)$ on $\text{supp}_x(a)$. We shall consider oscillatory integral operators T_λ with phase function $\phi(x,y) = \dfrac{|x-y|^2}{t(x)}$ and catalogue the behavior of $||T_\lambda||$ below.

Remark 3.1.1. Since $\text{supp}_x(a)$ is compact and $0 \neq t(x)$ $\forall x \in \text{supp}_x(a)$, there is a constant $c > 0$ such that $c \leq |t(x)|$ $\forall x \in \text{supp}_x(a)$.

The best possible case is when H_ϕ is non-degenerate, and this situation is characterized in theorem 3.1.2 below.

Theorem 3.1.2. *If* $1-\frac{\nabla t(x)}{t(x)}\cdot(x-y)\neq 0$ *on* supp(a) *then* $\|T_\lambda f\|_2 \leq C\lambda^{-n/2}\|f\|_2$, *and the exponent of* λ *is sharp. Moreover if we fix* $c_1>0$, *then* $\exists\, c_2$ *(which depends on* c_1 *and* supp(a)*) such that the constant* C *above is uniform over the set*[1]

$$\Sigma=\Sigma(c_1,c_2)=\{\,t\in C^\infty\;:\;c_1\leq|t(x)|,\;|\nabla t(x)|,\;\|\mathrm{H}\,t(x)\|\leq c_2\;\forall x\in\mathrm{supp}(a)\}.$$

Proof. It is easy to calculate that

$$\frac{\partial^2\phi}{\partial x_i\partial y_j}(x,y)=\frac{-2}{t(x)}\left(\delta_{ij}-(x_j-y_j)\frac{\partial_i t(x)}{t(x)}\right). \tag{3.1.1}$$

Then we see that H_ϕ is of the form described in lemma 1.4.1, and we conclude that

$$\det H_\phi(x,y)=\left(\frac{-2}{t(x)}\right)^n\left(1-\frac{\nabla t(x)}{t(x)}\cdot(x-y)\right). \tag{3.1.2}$$

The first part of the theorem follows now from theorem 2.2.1; the second part will follow after a careful examination of the hypotheses in (i), (ii) and (iii) of theorem 2.2.1 as they relate to ϕ.

First note that if c_2 is chosen to be small enough, and $|\nabla t|\leq c_2$, then

$$\left|\frac{\nabla t(x)}{t(x)}\cdot(x-y)\right|\leq c_1^{-1}|\nabla t(x)|\,2\,\mathrm{diam}(\mathrm{supp}(a))\leq 1/2.$$

So if we assume that $t\in\Sigma$ for this choice of c_2 (and c_1), then $|\det H_\phi|\geq c_1^{-n}2^{n-1}$, and, given (3.1.1) and remark 2.2.2, $\|H_\phi^{-1}\|$ is uniformly bounded on supp(a) and over all $t\in\Sigma$. We claim also that $\left\|\nabla_x \mathrm{D}_y^\alpha\phi\right\|_{L^\infty}$ is uniformly bounded for all α with $|\alpha|\leq n+2$. In fact $|\alpha|\leq 2$ will suffice as all higher order derivatives vanish. The claim is evident from the form of ϕ as

$$\frac{\partial}{\partial x_j}\mathrm{D}_y^\alpha\phi(x,y)=-\frac{\partial_j t(x)}{(t(x))^2}D_y^\alpha(|x-y|^2),$$

[1] Here $\|\,\mathrm{H}\,t\|$ denotes the matrix norm of the $n\times n$ matrix H.

and $t \in \Sigma$. Finially we check that $\|\nabla_y D_x^\alpha \phi\|_{L^\infty}$ is also uniformly bounded when $\alpha = 2$ since

$$\frac{\partial^3 \phi(x,y)}{\partial x_i \partial x_j \partial y_k} = \frac{-2}{t(x)^2}\left(\partial_j t(x)\delta_i^k - \delta_j^k \partial_i t(x) \;-(x_k - y_k) + (x_k - y_k)\frac{\partial_i t(x)\partial_j t(x)}{t(x)}\right).$$

The hypotheses of theorem 2.2.1 being satisfied, the theorem is proven. ■

The more interesting case is when $\det H_\phi = 0$. Lemma 1.4.1 and theorem 2.2.3 readily give the estimate $\|T_\lambda f\| \lesssim \lambda^{-n/2+1/2}\, \|f\|_2$ in this case. But before proving a stronger estimate, a few remarks are in order.

Remark 3.1.3. If we cover $\mathrm{supp}_x(a)$ with balls of radius δ and take a partition of unity subordinate to these balls, we may assume that $\mathrm{diam}(\mathrm{supp}_x(a)) < \delta$ without any loss of generality if we provide that δ does not depend on λ. Then δ is chosen to be as small as necessary to assist in technical matters.

Remark 3.1.4. (3.1.2) says that we may assume (after a partition of unity) that on the support of a, $1 \lesssim |\nabla t(x)|$. Otherwise H_ϕ is non-singular, and we may again appeal to theorem 2.2.1.

Remark 3.1.5. If A is a rotation then the change of variables $(x,y) \to (Ax, Ay)$ "preserves" ϕ in the sense that $\phi(Ax, Ay) = \frac{|x-y|^2}{t\circ A}$ is of the same form as $\phi(x,y)$, the form of phase function presently under consideration, with t replaced by $t \circ A$. Similarily if A is a translation, ϕ is again "preserved" under the operation $\phi(x,y) \to \phi(Ax, Ay)$. Since such transformations are measure preserving, the norm of the oscillatory integral operator with phase function $\phi(x,y)$ and amplitude $a(x,y)$ is the same as the one with phase $\phi(Ax, Ay)$ and amplitude $a(Ax, Ay)$.

Theorem 3.1.6. *Let ϕ be as above. Then in general*

$$\|T_\lambda f\|_2 \leq C\lambda^{-n/2+1/4}\, \|f\|_2 .$$

Proof. Let $x \in \operatorname{supp}_x(a)$ be given. By remark 3.1.4 we may find a rotation A_x such that $\nabla t(x)A_x$ is parallel to the n-th unit vector e_n in $\mathbb{R}^n$. Let $z = A_x^{-1}x$ and $t_{A_x} = t \circ A_x$. Then $\nabla t_{A_x}(z)$ is parallel to e_n. Assuming that A_x is also a dialation, $\nabla t_{A_x}(z) = e_n$. Furthermore $|\nabla t_{A_x}| \geq 1$. Then there is a neighborhood U_x of z, a neighborhood V_x of x and a diffeomorphism $\rho_x : V_x \to U_x$ such that $t_{A_x} \circ \rho_x(w) = w_n$ for all $w \in V_x$. Moreover $\mathrm{D}\rho_x(z) = \mathrm{I}$, and we may assume that $\operatorname{diam}(U_x)$ and $\operatorname{diam}(V_x)$ are as small as necessary. Let $\widetilde{U}_x = A(U_x)$, and take a finite subcover $\{\widetilde{U}_{x_i}\}_{i=1}^m$ of $\operatorname{supp}_x(a)$. By remark 3.1.3 we may assume that $\operatorname{supp}_x(a) \subset \widetilde{U}_{x_1}$ for example. Since A_{x_1} is a rotation, by remark 3.1.5 we may assume therefore that there is a ball $B_\delta(x_0)$ and a diffoemorphism $\rho : B_\delta(x_0) \to \operatorname{supp}_x(a)$ such that $t\rho(x) = x_n$, and $\mathrm{D}\rho(x_0) = \mathrm{I}$.

Given this, it suffices after a change of variables to consider the operator

$$\widetilde{T}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\left(\frac{|\rho(x)-y|^2}{x_n}\right)} a(\rho(x), y) f(y)\, dy.$$

If $S_\lambda = \widetilde{T}_\lambda \widetilde{T}_\lambda^*$, then S_λ is an integral operator with kernel

$$K_\rho(x,z) = \int_{\mathbb{R}^n} \exp\left(i\lambda\left\{\frac{|\rho(x)-y|^2}{x_n} - \frac{|\rho(z)-y|^2}{z_n}\right\}\right) a(\rho(x),y)\,\overline{a(\rho(z),y)}\, dy.$$

We must show that

$$\|S_\lambda f\|_2 \lesssim \lambda^{-n+1/2}\, \|f\|_2\,. \tag{3.1.3}$$

Let $\psi \in C_0^\infty(B_1(0))$ be such that $\psi \equiv 1$ on $B_{1/2}(0)$, and let $\widetilde{\psi} = 1 - \psi$. Then

(3.1.4)
$$\begin{aligned} S_\lambda f(x) &= \int_{\mathbb{R}^n} f(z) K_\rho(x,z) \psi\left(\frac{x_n - z_n}{\epsilon}\right) dz + \int_{\mathbb{R}^n} f(z) K_\rho(x,z) \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) dz \\ &= S_\lambda^1 f(x) + S_\lambda^2 f(x), \end{aligned}$$

where ϵ is to be chosen. S_λ^1 represents when $|x_n - z_n|$ is small, and S_λ^2 represents when $|x_n - z_n|$ is large. Each operator is studied via differing strategies. S_λ^1 is considered as the composition of two operators, and we study the factors separately. S_λ^2 is found to be the sum of oscillatory integral operators, and an estimate is made for each term.

First consider S_λ^1 and its corresponding frozen operators

$$\text{(3.1.5)}\qquad \begin{aligned}(S_\lambda^1)_{x_n z_n} f(x') &= \psi\left(\frac{x_n - z_n}{\epsilon}\right) \int_{\mathbb{R}^{n-1}} f(z')(K_\rho)_{x_n z_n}(x', z')\, dz' \\ &= \psi\left(\frac{x_n - z_n}{\epsilon}\right) (\widetilde{T}_\lambda \widetilde{T}_\lambda^*)_{x_n z_n} f(x').\end{aligned}$$

Recalling (1.2.3) and (1.2.5), we consider $(\widetilde{T}_\lambda)_{x_n y_n}$ for fixed x_n and y_n. The $(n-1) \times (n-1)$ matrix

$$\left(\frac{\partial^2}{dx_i dy_j} |\rho(x) - y|^2\right)_{i,j=1}^{n-1} = \left(-2\mathrm{D}_i \rho_j(x)\right)_{i,j=1}^{n-1}$$

is the mixed Hessian for $(\widetilde{T}_\lambda)_{x_n y_n}$. As noted above, when $x = x_0$ this is -2 times the $(n-1) \times (n-1)$ identity matrix. So in a small neighborhood of x_0 the determinant of the above matrix does not vanish (see remark 3.1.3). Hence

$$\|(\widetilde{T}_\lambda)_{x_n y_n}\| \lesssim \lambda^{-(n-1)/2} \chi(x_n, y_n) \|f\|_2 ,$$

where χ is compactly supported. Then this along with (1.2.3) and (1.2.5) implies that

$$\left\|(S_\lambda^1)_{x_n y_n} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \lambda^{-n+1} \chi'(x_n, z_n) \psi\left(\frac{x_n - z_n}{\epsilon}\right) \|f\|_{L^2(\mathbb{R}^{n-1})} ,$$

where χ' is also compactly supported. By theorem 1.2.1 and lemma 1.2.2 we have

$$\text{(3.1.6)}\qquad \left\|S_\lambda^1 f\right\|_2 \lesssim \lambda^{-n+1}\epsilon \,\|f\|_2 .$$

Now consider S_λ^2 and its corresponding frozen operators $(S_\lambda^2)_{x_n z_n}$ for fixed x_n and z_n. Note that

(3.1.7)

$$\frac{|\rho(x) - y|^2}{x_n} - \frac{|\rho(z) - y|^2}{z_n} = \left(\frac{1}{x_n} - \frac{1}{z_n}\right) |y - F(x,z)|^2 - \frac{|\rho(x', x_n) - \rho(z', z_n)|^2}{x_n - z_n},$$

where

$$F(x,z) = \frac{z_n\rho(x) - x_n\rho(z)}{x_n - z_n}. \tag{3.1.8}$$

Let

$$\begin{aligned} A(x,z,y) &= a(\rho(x),y)\,\overline{a(\rho(z),y)}, \\ \mu &= \lambda\left(\frac{1}{x_n} - \frac{1}{z_n}\right). \end{aligned} \tag{3.1.9}$$

Then

$$\begin{aligned} K_\rho(x',z')_{x_n z_n} = \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) &\exp\left(i\frac{\lambda}{z_n - x_n}|\rho(x) - \rho(z)|^2\right) \\ &\times \int_{\mathbb{R}^n} e^{i\mu|y|^2} A(x,z,y+F(x,z))\,dy \end{aligned} \tag{3.1.10}$$

is the kernel of $(S_\lambda^2)_{x_n z_n}$. By theorem 1.3.2, for fixed N to be chosen,

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{i\mu|y|^2} A(x,z,y+F(x,z))\,dy \\ &\quad = \left(\frac{i\mu}{\pi}\right)^{-n/2}\left(\sum_{j=0}^{N-1}(4i\mu)^{-j}\Delta_y^j A(x,z,F(x,z))/j! \right. \\ &\qquad\qquad \left. + \int_{\mathbb{R}^n} r_N(i|\xi|^2/4\mu)e^{-i\xi\cdot F(x,z)}\check{A}(x,z,\xi)\,d\xi\right), \end{aligned} \tag{3.1.11}$$

where $\check{A}$ denotes the inverse Fourier transform in the last variable. In view of (3.1.10) and (3.1.11), $(S_\lambda^2)_{x_n z_n}$ is a sum of oscillatory integral operators

$$\left(\frac{i}{\pi}\right)^{-n/2}\widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right)\sum_{j=0}^{N}\frac{\mu^{-n/2-j}}{(4i)j!}R_{\lambda'}^j, \qquad \lambda' = \frac{\lambda}{x_n - z_n}, \tag{3.1.12}$$

where

(3.1.13)

$$R_{\lambda'}^j f(x') = \int_{\mathbb{R}^n} e^{i\lambda'|\rho(x)-\rho(z)|^2}(\Delta_y^j A)(x,z,F(x,z))f(z')\,dz' \qquad j = 1,\ldots,N-1$$

$$R_{\lambda'}^N f(x') = \int_{\mathbb{R}^n} e^{i\lambda'|\rho(x)-\rho(z)|^2}\mu^N\left(\int_{\mathbb{R}^n} r_N(i|\xi|^2/4\mu)e^{-i\xi\cdot F(x,z)}\check{A}(x,z,\xi)\,d\xi\right)f(z')\,dz'$$

Each $R^j_{\lambda'}$ has phase function of the form $|\rho(x',x_n)-\rho(z',z_n)|^2$ for fixed x_n and z_n. Look at the mixed Hessian:

$$\left(\frac{\partial^2}{dx_i dz_j}|\rho(x',x_n)-\rho(z',z_n)|^2\right)_{i,j=1}^{n-1} = \left(-2\sum_k \partial_i\rho_k(x)\partial_j\rho_k(z)\right)_{i,j=1}^{n-1}.$$

Since $\mathrm{D}\rho(x_0)=\mathrm{I}$,

$$\det\left(\frac{\partial^2}{dx_i dz_j}|\rho(x',x_n)-\rho(z',z_n)|^2\right)_{i,j=1}^{n-1} = (-2)^{n-1} \qquad \text{when } x=z=x_0.$$

So we may assume that this mixed Hessian is non-degenerate (see remark 3.1.3). Now for $j=1,\dots,N-1$, $R^j_{\lambda'}$ has amplitude $\Delta^j_y A(x,z,F(x,z))$. Since

$$|\mathrm{D}^\alpha_{z'}F(x,z)| \lesssim |x_n-z_n|^{-|\alpha|}, \tag{3.1.14}$$

we see that

$$\sup|\mathrm{D}^\alpha_{z'}\left(\Delta^j_y A(x,z,F(x,z))\,\overline{\Delta^j_y A(w,z,F(w,z))}\right)| \lesssim |x_n-z_n|^{-|\alpha|}. \tag{3.1.15}$$

To apply theorem 2.2.1 we must calculate, for fixed x_n and z_n, the volume of $\mathrm{supp}_{z'}(\Delta^j_y A(x,z,F(x,z)))$. Note that by the properties of supp(A) we must have that $F(x,z)\lesssim 1$ — i.e., $|x_n\rho(z)-z_n\rho(x)|\lesssim|x_n-z_n|$. This says that for fixed x, $\rho(z)$ is in the ball of radius $\dfrac{1}{x_n}|x_n-z_n|$ centered at $\dfrac{z_n}{x_n}\rho(x)$. Since $|x_n|$ is bounded from below (see remark 3.1.1) and ρ is a diffeomorphism, z lies in a set of diameter $\sim|x_n-z_n|$. So

$$|\,\mathrm{supp}_{z'}\,\Delta^j_y A(x,z,F(x,z))| \sim |x_n-z_n|^{n-1} \qquad \text{for } j=0,\dots,N-1. \tag{3.1.16}$$

Putting (3.1.15) and (3.1.16) into (2.2.1) (recalling that $\lambda'=\dfrac{\lambda}{(x_n-z_n)}$) gives that for $j=1,\dots,N-1$,

$$\left\|R^j_{\lambda'}f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \lambda^{\frac{-(n-1)}{2}}|x_n-z_n|^{\frac{(n-1)}{2}}\chi(x_n,z_n)\,\|f\|_{L^2(\mathbb{R}^{n-1})}, \tag{3.1.17}$$

where χ has compact support.

Coming to $R^N_{\lambda'}$, it has amplitude, call it $A_N(x,z)$, equal to

$$\mu^N \int_{\mathbb{R}^n} r_N(i|\xi|^2/4\mu)e^{-i\xi\cdot F(x,z)}\check{A}(x,z,\xi)\,d\xi.$$

This means that in view of the remark 1.3.3 and (3.1.14)

$$\sup_{z'} |D^\alpha_{z'} A_N(x,z)\,\overline{A_N(w,z)}| \lesssim |x_n - z_n|^{-|\alpha|}. \tag{3.1.18}$$

Then (3.1.18) implies that

$$\|R^N_{\lambda'} f\|_{L^2(\mathbb{R}^{n-1})} \lesssim \lambda^{\frac{-(n-1)}{2}} \chi(x_n,z_n)\,\|f\|_{L^2(\mathbb{R}^{n-1})}\,. \tag{3.1.19}$$

Looking at (3.1.12), (3.1.17) and (3.1.19) we see that

$$\left\|(S^2_\lambda)_{x_n z_n} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right)\chi(x_n,z_n)$$

$$\times \left(\lambda^{-n+1/2}|x_n - z_n|^{-1/2}.\right. \tag{3.1.20}$$

$$+ \sum_{j=1}^{N-1} \lambda^{-n+1/2-j}|x_n - z_n|^{-1/2-j} \tag{3.1.20'}$$

$$\left. + \lambda^{-n+1/2-N}|x_n - z_n|^{-n/2-N}\right) \|f\|_{L^2(\mathbb{R}^{n-1})} \tag{3.1.20''}$$

where χ is compactly supported. Now we apply lemma 1.2.2 to obtain

$$\left\|S^2_\lambda f\right\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

$$\times \left(\lambda^{-n+1/2}\right. \tag{3.1.21}$$

$$+ \sum_{j=1}^{N-1} \lambda^{-n+1/2-j}\epsilon^{1/2-j} \tag{3.1.21'}$$

$$\left. + \lambda^{-n+1/2-N}\epsilon^{-n/2-N+1}\right). \tag{3.1.21''}$$

In consideration of (3.1.6) and (3.1.21") set

$$\epsilon = \lambda^{-\frac{2N+1}{n+2N}}.$$

In this case

$$\lambda^{-n+1}\epsilon = \lambda^{-n+1/2-N}\epsilon^{-n/2-N+1} = \lambda^{-n+1/2}\lambda^{\frac{n-2(N+2)}{2(n+2N)}}.$$

Also, it is easy to check, in consideration of (3.1.21'), that

$$\lambda^{-n/2+1/2-j}\epsilon^{1/2-j} \leq \lambda^{-n+1/2}$$

for this choice of ϵ. Evidently (3.1.21) is the main term in the estimate of $\|S_\lambda\|$ provided that $n \leq 2(N+2)$. Since N can be arbitrarily large, (3.1.3) is demonstrated, and the proof of theorem 3.1.6 is complete. ■

Given the general nature of theorem 3.1.6, it is natural to ask whether or not the result is sharp. We shall find in the next theorem that we may not always be able to improve the exponent of λ in theorem 3.1.6.

Theorem 3.1.7. *For a given amplitude function $a \not\equiv 0$, there are $t \in C^\infty$ such that the exponent of λ in theorem 3.1.6 is sharp.*

Proof. By assumption there is a point (x_0, y_0) such that $a(x_0, y_0) \neq 0$. After perhaps a translation and a rotation—in view of remark 3.1.5—we may assume that $x_0 = e_n$, the n-th unit vector in $\mathbb{R}^n$, and $y_0 = 0$. Then let $t(x) = x_n$, and note that we may assume $x_n \neq 0$ on supp(a). Let $f_\epsilon(y) = f(y')\tilde{f}(y_n)$, where $0 \leq f \in C_0^\infty(\mathbb{R}^{n-1})$,

$f \equiv 1$ on supp(a) and $\tilde{f}(y_n) = e^{i2\lambda y_n}\chi_{[-\epsilon,\epsilon]}(y_n)$. Then

$$\begin{aligned}
e^{-i\lambda x_n} T_\lambda f(x) &= \int_{\mathbb{R}^{n-1}} e^{\frac{\lambda}{x_n}|x'-y'|^2} f(y') \int_{-\epsilon}^{\epsilon} e^{i\lambda \frac{y_n^2}{x_n}} a(x,y',y_n)\, dy_n\, dy' \\
&= \left(\frac{i\lambda}{x_n \pi}\right)^{-(n-1)/2} 2\epsilon f(x') \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} a(x,x',y_n)\, dy_n \\
&+ \left(\frac{i\lambda}{x_n \pi}\right)^{-(n-1)/2} f(x') \int_{-\epsilon}^{\epsilon} \left(e^{i\lambda \frac{y_n^2}{x_n}} - 1\right) a(x,x',y_n)\, dy_n \\
&+ \left(\frac{i\lambda}{x_n \pi}\right)^{-(n-1)/2} 2\epsilon \int_{\mathbb{R}^{n-1}} r_1(ix_n|\xi'|^2/4\lambda) \\
&\quad \times \left(f(x'+\cdot\,)\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} e^{i\lambda \frac{y_n^2}{x_n}} a(x,\,\cdot+x',y_n)\, dy_n\right)\widehat{\ }(\xi')\, d\xi' \\
&= I + II + III.
\end{aligned}$$

Now it is easily seen that

$$|II| \lesssim \lambda^{-n/2+3/2}\epsilon^3,$$

and by (1.3.3)

$$\begin{aligned}
|III| &\lesssim \epsilon\lambda^{-n/2-1/2} \sum_{|\alpha|\le 2+n/2} \left\| \mathrm{D}_{y'}^{\alpha} f(x'+\cdot\,)\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} e^{i\lambda\frac{y_n^2}{x_n}} a(x,\,\cdot\,,y_n)\, dy_n \right\|_{L^2(\mathbb{R}^{n-1})} \\
&\lesssim \epsilon\lambda^{-n/2-1/2}.
\end{aligned}$$

Supposing that ϵ is small we have that

$$|I| \gtrsim \epsilon\lambda^{-n/2+1/2}\left|\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} a(x,x',y_n)\, dy_n\right| \gtrsim \epsilon\lambda^{-n/2+1/2}$$

on a set of positive measure in x-space. If we let $\epsilon = c\lambda^{-1/2}$ where c is a small constant independent of λ (i.e., let $\epsilon\lambda^{-n/2+1/2} = \epsilon^3\lambda^{-n/2+3/2}$), then clearly

$$\|T'_\lambda f_\epsilon\|_2 \gtrsim \lambda^{-n/2} \quad \text{and} \quad \|f_\epsilon\|_2 = \lambda^{-1/4}.$$

So

$$\frac{\|T'_\lambda f_\epsilon\|_2}{\|f_\epsilon\|_2} \gtrsim \lambda^{-n/2+1/4}$$

as desired. ∎

Remark 3.1.8. Using a result of Pan and Sogge [PS] it is a routine matter to show that if $Ht(x)(x-y)\cdot(x-y)\neq 0$ on supp(a) where $Ht(x)$ denotes the Hessian of t at x, then $\|T'_\lambda f\|_2 \le C\lambda^{-n/2+1/6}\|f\|_2$. See the appendix for details.

§2. A Variant

Here, as in §1, we let $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a given fixed amplitude function, but we let t denote any smooth function. Now we shall study oscillatory integral operator T_λ, with phase function $\phi(x,y) = -2\,x\cdot y + t(x)|y|^2$. Theorems 3.1.2, 3.1.6 and 3.1.7 find their analogues in theorems 3.2.1, 3.2.2 and 3.3.3 respectively. The proofs of theorems 3.2.1 and 3.2.2 follow the same pattern as their counterparts, and while being complete, they are brief. Theorem 3.2.3 is proved using theorem 2.3.1.

Theorem 3.2.1. *If* $1-\nabla t(x)\cdot y \neq 0$ *on* supp(a) *then* $\|T_\lambda f\|_2 \le C\lambda^{-n/2}\|f\|_2$, *and the exponent of* λ *is sharp. Moreover* $\exists\, c$ *such that the constant* C *is uniform over the set*

$$\Sigma = \Sigma(c) = \{\, t \in C^\infty \;:\; |\nabla t(x)|,\ \|\mathrm{H}\,t\| \le c\ \forall x \in \mathrm{supp}(a)\,\}.$$

Proof. We note that

$$\frac{\partial^2\phi}{\partial x_i \partial y_j}(x,y) = -2\big(\delta_{ij} - \partial_i t(x) y_j\big). \tag{3.2.1}$$

So lemma 1.4.1 guarantees that

$$\det H_\phi(x,y) = (-2)^n\,(1-\nabla t(x)\cdot y)\,. \tag{3.2.2}$$

Hence the bound for T_λ is as given.

The reasoning for the boundedness of C over Σ is the same as in theorem 3.2.1. If c is small enough and $|\nabla t| \leq c$, then

$$|\nabla t(x) \cdot y| \leq c\,\mathrm{diam}(\mathrm{supp}(a)_y) \leq 1/2.$$

This and (3.2.1) allow us to conclude that $\|H_\phi^{-1}\|$ is uniformly bounded on $\mathrm{supp}(a)$. Noting that the absolute value of

$$\frac{\partial}{\partial x_j} \mathrm{D}_y^\alpha \phi(x, z) = -2\mathrm{D}_y^\alpha y_j + \partial_j t(x) \mathrm{D}_y^\alpha |y|^2$$

on $\mathrm{supp}(a)$ for all $|\alpha| \leq 2$ depends only on $t \in \Sigma$ and $\mathrm{diam}(\mathrm{supp}(a))$, and that the same is true for

$$\frac{\partial^3 \phi(x, y)}{\partial x_i \partial x_j \partial y_k} = -\frac{\partial^2 t(x)}{\partial x_i \partial x_j} y_k,$$

the proof is complete. ∎

Theorem 3.2.2. *Let ϕ be as above. Then in general*

$$\|T_\lambda f\|_2 \leq C \lambda^{-n/2+1/4} \|f\|_2 .$$

Proof. Remark 3.1.3 still applies, (3.2.2) means that remark 3.1.4 is still in order, and remark 3.1.5 is still valid if we only allow A to be a rotation. Given this, as before, we may assume that there is a ball $B_\delta(x_0)$ and a diffeomorphism ρ : $B_\delta(x_0) \to \mathrm{supp}_x(a)$ such that $t \circ \rho(x) = x_n$ and $\mathrm{D}\rho(x_0) = \mathrm{I}$.

Again too, it suffices to consider

$$\widetilde{T}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\left(-2\rho(x)\cdot y + x_n |y|^2\right)} a(\rho(x), y) f(y)\, dy,$$

where $\mathrm{supp}(a)$ is sufficiently small. As in the proof of theorem 3.1.6 let $S_\lambda = \widetilde{T}_\lambda (\widetilde{T}_\lambda)^*$, an operator with kernal

$$K_\rho(x, z) = \int_{\mathbb{R}^n} \exp\Big(-2(\rho(x) - \rho(z) \cdot y) + (x_n - z_n)|y|^2\Big) a(\rho(x), y) \overline{a}(\rho(z), y)\, dy,$$

and write $S_\lambda = S_\lambda^1 + S_\lambda^2$ just as in (3.1.4).

First consider S_λ^1. For fixed x_n and y_n we readily find that

$$\|(\widetilde{T}_\lambda)_{x_n y_n} f\| \lesssim \lambda^{-(n-1)/2}\, \chi(x_n, y_n)\, \|f\|_2\,,$$

as $(\widetilde{T}_\lambda)_{x_n y_n}$ has mixed Hessian

$$\left(\frac{\partial^2}{dx_i dy_j} - 2\rho(x) + x_n|y|^2\right)_{i,j=1}^{n-1} = \left(-2\mathrm{D}_i\rho_j(x)\right)_{i,j=1}^{n-1},$$

which is non-degenerate on supp(a) . Hence, as in (3.1.6), we find after an application of theorem 2.1.1 and lemmas 1.2.2 and 1.2.3 that

$$\left\|S_\lambda^1 f\right\|_2 \lesssim \lambda^{-n+1}\epsilon\, \|f\|_2\,. \tag{3.2.3}$$

The similarities between the operators considered in theorem 3.1.6 and the ones considered here becomes apparent when we note that (cf. (3.1.7))

$$-2(\rho(x)-\rho(z))\cdot y+(x_n - z_n)|y|^2 = (x_n - z_n)|y - F(x,z)|^2 - \frac{|\rho(x', x_n) - \rho(z', z_n)|^2}{x_n - z_n},$$

where this time we let

$$\begin{aligned} F(x,z) &= \frac{\rho(x) - \rho(z)}{x_n - z_n}, \\ A(x,z,y) &= a(\rho(x), y)\,\overline{a(\rho(z), y)}, \\ \mu &= \lambda(x_n - z_n). \end{aligned} \tag{3.2.4}$$

Then, as in (3.1.10),

$$K_\rho(x', z')_{x_n z_n} = \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \exp\left(i\frac{\lambda}{x_n - z_n}|\rho(x) - \rho(z)|^2\right) \times \int_{\mathbb{R}^n} e^{i\mu|y|^2} A(x,z,y + F(x,z))\, dy$$

is the kernel of $(S_\lambda^2)_{x_n z_n}$. We see now that $(S_\lambda^2)_{x_n z_n}$ is virtually identical to its counterpart in the proof of theorem 3.1.6, the only (minor) differences being in the definitions of μ and F (cf (3.1.8) (3.1.9) (3.2.4)). Then we may proceed *exactly* as before taking care only when we must consider μ or F.

With μ and F as above we have that $(S_\lambda^2)_{x_n z_n}$ is a sum of oscillatory integral operators as in (3.1.12) with $R_{\lambda'}^j$, $j = 1 \cdots N$ as in (3.1.13). We know that each $R_{\lambda'}^j$ has the same non-degenerate phase function, and so we will have exactly the same estimates, (3.1.17) and (3.1.19), as before once we examine the amplitude function of each $R_{\lambda'}^j$ and find that they satisfy estimates like (3.1.15), (3.1.16) and (3.1.18). Estimates (3.1.15) and (3.1.18) follow from (3.1.14), and it is clear that the new F satisfies (3.1.14). So we need only calculate the volume of $\text{supp}_{z'}(\Delta_y^j A(x, z, F(x, z)))$ for fixed x_n and z_n. We argue only slightly differently than before; the only difference is that, unlike in theorem 3.1.6, we do not have to assume that $t(x) \neq 0$, and so we need not appeal to remark 3.1.1. By the properties of supp(A) we must have that $F(x, z) \lesssim 1$ — i.e. $|\rho(z) - \rho(x)| \lesssim |x_n - z_n|$. This says that for fixed x, $\rho(z)$ is in the ball of radius $|x_n - z_n|$ centered at $\rho(x)$. Since ρ is a diffeomorphism, z lies in a set of diameter $\sim |x_n - z_n|$. So (3.1.16) holds. Now it is clear that we have exactly the same bound for $(S_\lambda^2)_{x_n z_n}$ as in (3.1.20), (3.1.20') and (3.1.20"), and consequently $\|S_\lambda^2 f\|_2$ is bounded as in (3.1.21), (3.1.21') and (3.1.21"). The proof is completed by choosing ϵ as before. ∎

As expected we have the following result regarding the sharpness of the exponent of λ in theorem 3.2.2

Theorem 3.2.3. *For a given amplitude function $a \not\equiv 0$, there are $t \in C^\infty$ such that the exponent of λ in theorem 3.2.2 is sharp.*

Proof. We shall apply theorem 2.3.1. Let (x_0, y_0) be such that $a(x_0, y_0) \neq 0$. After a rotation we may assume that y_0 is parallel to e_n; let $y_0 = ke_n$ with $k > 0$. Take

$t(x) = x_n/k$ and notice that

$$-2x \cdot y + t(x)|y|^2 = -2(x' - x_0') \cdot y' + \frac{x_n}{k}|y'|^2 + \frac{x_n}{k}|y_n - k|^2 - kx_n - 2x_0' \cdot y'.$$

Take $\phi_1(x) = -kx_n$ and $\phi_2(y) = -2x_0' \cdot y$ and let

$$R = \{\, x \,:\, |x' - x_0'| \leq c\lambda^{-1/2},\ |x_n| \lesssim 1 \,\}$$
$$\widetilde{R} = \{\, y \,:\, |y'| \leq c\lambda^{-1/2},\ |y_n - k| \leq c\lambda^{-1/2} \,\},$$

where c is a small constant (independent of λ). If c is small enough then (2.3.1) and (2.3.2) hold, and

$$\|T_\lambda\| \gtrsim \sqrt{|R|\,|\widetilde{R}|} = \lambda^{-n/2+1/4}$$

as desired. ■

4. Main Theorem

We now turn our attention to the family of operators T_λ with phase function of the form $\phi(x,y) = \frac{|x-y|^2}{t(x)-\bar{t}(y)}$ where t and $\bar{t}$ are smooth functions such that $0 < |t(x) - \bar{t}(y)|$. In what follows we will always take a to be a fixed amplitude function with compact, connected support. Hence we may assume without loss of generality that there is a constant $c > 0$ such that $c \leq t(x) - \bar{t}(y)$ on $\operatorname{supp}(a)$. This will be our general setup throughout this chapter.

In §1 we shall find it to be possible that $\operatorname{rank} H_\phi = n-2$, and therefore the ideas in chapter 3 do not carry over into the analysis of $\|T_\lambda\|$. Instead, we would like to consider T_λ as a composition of operators whose factors are known to us, and as alluded to previously, these factors already have been studied in §2 of the previous chapter. We may then use the estimate in theorem 3.2.2 to get sharp results for T_λ. In actual fact, though, we will not realize T_λ directly as a composition, but nearly so. The composition of operators we consider has the same phase function as T_λ but has a different amplitude. The transition to T_λ in theorem 4.2.1 from theorem 3.2.2 is facilitated by lemma 4.1.2, which allows us to compare oscillatory integral operators with the same phase function but different amplitudes. We prove this lemma in §1. In §2 we prove sharp estimates for T_λ and we discuss further the idea of considering T_λ as a composition.

§1. Preparations

We begin with the following result about the singularities of H_ϕ.

Proposition 4.1.1. *Let ϕ be as above. Then* $\operatorname{rank} H_\phi(x,y) \geq n-2$, *and moreover* $\operatorname{rank} H_\phi(x,y) = n-2$ *if and only if*

(i) $1 - \dfrac{\nabla t(x)\cdot(x-y)}{t(x)-\bar{t}(y)} = 0.$

(ii) $1 - \dfrac{\nabla \bar{t}(y)\cdot(x-y)}{t(x)-\bar{t}(y)} = 0.$

(iii) $\nabla t(x)\cdot\nabla\bar{t}(y) = 0.$

Proof. We begin by noting that

(4.1.1)
$$\frac{\partial^2\phi(x,y)}{\partial x_i\partial y_j} = \frac{-2}{t(x)-\bar{t}(y)}\left(\delta_i^j - \frac{1}{t(x)-\bar{t}(y)}\left(\frac{\partial}{\partial y_j}\bar{t}(y)(x_i-y_i) + \frac{\partial}{\partial x_i}t(y)(x_j-y_j) + |x-y|^2\frac{\partial_i t(x)\partial_j\bar{t}(y)}{t(x)-\bar{t}(y)}\right)\right).$$

Suppose first that $\nabla\bar{t}(y)=0$. Then from (4.1.1),

$$(H_\phi(x,y))_{ij} = \frac{-2}{t(x)-\bar{t}(y)}\left(\delta_i^j - \frac{(x_j-y_j)}{t(x)-\bar{t}(y)}\partial_i t(x)\right),$$

and such a matrix has rank $\geq n-1$ by lemma 1.4.1. So we may assume that $\nabla\bar{t}(y)\neq 0$. In fact we may assume that $\nabla\bar{t}(y) \parallel e_n$. For let A be a rotation of $\mathbb{R}^n$ such that $\nabla\bar{t}(y)A \parallel e_n$. Consider $\phi_A(z,w) = \phi(Az,Aw) = \dfrac{|z-w|^2}{t_A(z)-\bar{t}_A(w)}$, $t_A = t\circ A$, $\bar{t}_A = \bar{t}\circ A$. Then $H_{\phi_A} = A^t H_\phi(Az,Aw)A$ by lemma 2.2.4. For given x and y, let $z=A^{-1}x$, $w=A^{-1}y$. Then clearly $\operatorname{rank} H_\phi(x,y) = \operatorname{rank} H_{\phi_A}(z,w)$. Moreover, it is routine to check that

(i') $1 - \dfrac{\nabla t(x)\cdot(x-y)}{t(x)-\bar{t}(y)} = 1 - \dfrac{\nabla t_A(z)\cdot(z-w)}{t_A(z)-\bar{t}_A(w)}.$

(ii') $1 - \dfrac{\nabla \bar{t}(x)\cdot(x-y)}{t(x)-\bar{t}(y)} = 1 - \dfrac{\nabla \bar{t}_A(w)\cdot(z-w)}{t_A(z)-\bar{t}_A(w)}.$

(iii') $\nabla t(x)\cdot\nabla\bar{t}(y) = \nabla t_A(z)\cdot\nabla\bar{t}_A(w).$

Since $\nabla\bar{t}(y)\parallel e_n$, we have that the $(n-1)\times(n-1)$ submatrix H'_ϕ of H_ϕ formed by deleting the last row and last column is of the form

$$(H'_\phi(x,y))_{ij} = \frac{-2}{t(x)-\bar{t}(y)}(\delta_i^j - (x_j-y_j)\partial_j t(x)) \qquad 1\leq i\leq n-1,\ 1\leq j\leq n-1.$$

This submatrix has rank $\geq n-2$ (by lemma 1.4.1); hence so does H_ϕ.

Suppose that (i), (ii) and (iii) are satisfied. Since $\nabla\bar{t}(y) \parallel e_n$ and (iii) holds, we have that $\partial_n t(x) = 0$. So by (ii),

$$H_\phi(x,y) = \left(\begin{array}{ccc|c} & & & * \\ & H'_\phi(x,y) & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & 0 \end{array}\right).$$

Since (i) (or (ii)) holds, $x - y \neq 0$. Clearly,

$$(0,0,\ldots,x_n - y_n)H_\phi(x,y) = 0,$$

while by (i)

$$(x' - y', 0)H_\phi(x,y) = 0.$$

So $\operatorname{rank} H_\phi(x,y) = n-2$.

Now suppose that $\operatorname{rank} H_\phi(x,y) = n-2$ (assuming again that $\nabla\bar{t}(y) \neq 0$). In particular, by lemma 1.4.1, H'_ϕ has rank $n-2$, and $\nabla_{x'}t(x)\cdot(x'-y') = t(x) - \bar{t}(y) \neq 0$. So we know that $x' - y' \neq 0$. Now it is clear that (i), (ii) and (iii) hold if we know that $\partial_n t(x) = 0$. The claim is that indeed $\partial_n t(x) = 0$, for suppose not. Consider the $(n-1)\times(n-1)$ submatrix of H_ϕ given by deleting the first row and n-th column. It is of the type described in lemma 1.4.2. Thus $(x_1 - y_1)\partial_n t(x) = 0$, and hence $(x_1 - y_1) = 0$. Now delete the second row and n-th column from H_ϕ. After switching two columns, we may again apply lemma 1.4.2 to obtain that $x_2 - y_2 = 0$, and continuing in this way we find that $x' - y' = 0$ which is a contradiction. ■

It is easy now to construct t and $\bar{t}$ such that (i), (ii) and (iii) of proposition 4.1.1 are satisfied, and we give a simple example to demonstrate this fact. Let $t(x) = x_1$ and $\bar{t}(y) = y_n$; then (i)–(iii) are satisfied on the set $\{(x,y) : x_1 = x_n,\, y_1 = y_n\}$. Suffice it to say that there are many t and $\bar{t}$ such that $\operatorname{rank}(H_\phi) = n-2$, and we reserve a more detailed discussion of this for a later writing prefering to move on to the main technical lemma of this chapter.[1]

It is the nature of the singularities of the phase function as reflected in the mixed Hessian that determine how rapidly the norm of an oscillatory integral operator tends to zero as the parameter λ tends to infinity, while the amplitude plays only an auxillary role, provided that it does not vanish when H_ϕ is singular, and that finitely many of its derivatives are bounded. Given two oscillatory integral operators with the same phase function and different amplitudes it is not unreasonable then to believe that a favorable estimate for the norm of one should imply a similar estimate for the other, given that the two amplitudes satisfy certain conditions. More generally, the norms of two different integral operators may be compared once their respective kernels satisfy relationships outlined in the following lemma, which makes use of lemma 1.3.5.

Lemma 4.1.2. *Let K_1, $K_2 \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and let*

$$T_j f(x) = \int_{\mathbb{R}^n} f(y) K_j(x,y)\, dy \qquad j = 1, 2.$$

Suppose that there is an open set $U \subset \mathbb{R}^n$ such that

$$\operatorname{supp}(K_2) \subset U \quad \text{and} \quad K_1 \neq 0 \text{ on } \bar{U}. \tag{4.1.2}$$

If there are constants C_1, C_2 and $C_3 > 0$ such that

$$\begin{gathered} \|K_2/K_1\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} \leq C_1, \\ \Big\| \sum_{|\alpha|=s} D^\alpha (K_2/K_1) \Big\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq C_2 \qquad \text{for some } s > n, \\ \operatorname{supp}(K_2) \subset B_{C_3}(0) \times B_{C_3}(0). \end{gathered} \tag{4.1.3}$$

[1] See however the appendix §4.

Then

$$\|T_2\| \le C\,\|T_1\|\,,$$

where $C = C(C_1, C_2, C_3)$ is bounded once C_1, C_2 and C_3 are bounded.

Proof. Note that $K_2(x,y) = \frac{K_2(x,y)}{K_1(x,y)} K_1(x,y)$ on $\operatorname{supp}(K_2)$. Clearly by (4.1.2) $K_2/K_1 \in C_0^\infty$, so

$$K_2(x,y) = K_1(x,y) \sum_{j=1}^{\infty} a_j\, \alpha_j(x)\beta_j(y)$$

where a_j, α_j and β_j are as in lemma 1.3.5. Then by (1.3.5),

$$\|T_2\| \le \sum_{j=1}^{\infty} |a_j|\ \|T_1\| \le C\|T_1\|,$$

as stated. ∎

If we let $K_1(x,y) = e^{i\lambda\phi(x,y)} b(x,y)$ and $K_2(x,y) = e^{i\lambda\phi(x,y)} a(x,y)$, the conditions (4.1.2) and (4.1.3) in this case translate as

(4.1.2') $$\exists\, U \subset \mathbb{R}^n \text{ open},\ \operatorname{supp}(a) \subset U \quad \text{and} \quad b \ne 0 \text{ on } \bar{U},$$

(4.1.3') $$\begin{gathered} \|a/b\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} \le C_1, \\ \Big\| \sum_{|\alpha|=s} \mathrm{D}^\alpha (a/b) \Big\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \le C_2 \qquad \text{for some } s > n, \\ \operatorname{supp}(a) \subset B_{C_3}(0) \times B_{C_3}(0). \end{gathered}$$

§2. The Main Theorem

From proposition 4.1.1 we readily obtain the result $\|T_\lambda f\|_2 \le C\lambda^{-(n-2)/2}\,\|f\|_2$ by considering theorem 2.2.3, and although it is possible to find functions which satisfy conditions (i), (ii) and (iii) of proposition 4.1.1 at a point, or on even larger varieties, it is not possible that H_ϕ should be so singular that such an estimate is sharp. This is the content of our main theorem.

Theorem 4.2.1. *Let T_λ be the oscillatory integral operator with phase function ϕ as above. Then*

$$\|T_\lambda f\|_2 \le C\lambda^{-(n-1)/2}\|f\|_2 \, .$$

Proof. Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Suppose with out loss of generality (after perhaps a dialation) that $\operatorname{supp}(a) \subset B_1 \times B_1$. Assume also (after a change of the parameter λ) that $1 \le t(x) - \bar{t}(y)$. Choose $\psi \in C_0^\infty(B_5)$ with $\psi \equiv 1$ on B_4. Consider the following operators:

$$S_\lambda^1 f(x) = \int_{\mathbb{R}^n} e^{i\lambda(-2x\cdot y + t(x)|y|^2)}\psi(x)\psi(y)f(y)\,dy, \tag{4.2.1}$$

$$S_\lambda^2 f(x) = \int_{\mathbb{R}^n} e^{i\lambda(-2x\cdot y + \bar{t}(x)|y|^2)}\psi(x)\psi(y)f(y)\,dy. \tag{4.2.2}$$

We know that

$$\|S_\lambda^j\| \lesssim \lambda^{-n/2+1/4} \qquad j = 1,2.$$

Thus

$$\|\lambda^{n/2}S_\lambda^1(S_\lambda^2)^*\| \lesssim \lambda^{-n/2+1/2}.$$

Now

$$\lambda^{n/2}S_\lambda^1(S_\lambda^2)^* f(x) = \int_{\mathbb{R}^n} e^{i\lambda\frac{|x-z|^2}{t(x)-\bar{t}(z)}} b_\lambda(x,z)f(z)\,dz,$$

where

(4.2.3)

$$b_\lambda(x,z) = \lambda^{n/2}\frac{\psi(x)\psi(z)}{(t(x)-\bar{t}(z))^{n/2}}\int_{\mathbb{R}^n} e^{i\lambda|y|^2}\psi^2\left(\frac{y}{(t(x)-\bar{t}(z))^{1/2}} + \frac{x-z}{t(x)-\bar{t}(z)}\right)dy.$$

We wish to apply lemma 4.1.2 to finish the proof; this amounts to checking (4.1.2') and(4.1.3') for a and b_λ. Let $U = B_2 \times B_2$, and suppose that $(x,z) \in \bar{U}$— i.e., $|x| \le 2$, $|z| \le 2$. We apply theorem 1.3.2 to the integral in (4.2.3) to obtain that

$$b_\lambda(x,z) = \pi^{\frac{n}{2}}e^{-i\frac{n}{4}\pi}\frac{\psi(x)\psi(z)}{(t(x)-\bar{t}(z))^{n/2}}\left(\psi^2\left(\frac{x-z}{t(x)-\bar{t}(z)}\right) + E(x,z)\right),$$

where $E(x,z)$ is the first-order remainder in (1.3.2). Now since

$$\left|\frac{x-z}{t(x)-\bar{t}(z)}\right| \leq 4 \qquad \text{when } (x,z) \in \bar{U},$$

then

$$\psi^2\left(\frac{x-z}{t(x)-\bar{t}(z)}\right) = 1 \qquad \text{when } (x,z) \in \bar{U}.$$

Moreover by 1.3.3,

$$|E(x,z)| \leq C\lambda^{-1} \sum_{|\alpha|\leq n/2+2} \left\|\mathrm{D}^\alpha \psi^2\right\|_2 \leq 1/2 \qquad \text{when } \lambda \gg 1.$$

So for large λ (depending only on ϕ), $|b_\lambda(x,z)| \geq 1/2$ on $\bar{U}$. We also see from corollary 1.3.4 that

$$\|\mathrm{D}^\alpha b_\lambda\|_\infty \lesssim 1 \qquad \text{for } |\alpha| \leq n+1.$$

Clearly then (4.1.2') and (4.1.3') are satisfied with C_1, C_2 and $C_3 \lesssim 1$. ■

Again, we may not make an improvement in theorem 4.2.1 as evidenced by the follwing.

Theorem 4.2.2. *For a given amplitude function $a \not\equiv 0$, we may find t and $\bar{t}$ such that the exponent of λ in theorem 4.2.1 is sharp.*

Proof. Let $t(x) = x_n$ and $\bar{t}(y) = y_n$. We may assume that on supp(a), $0 < x_n < 1$ and $2 < y_n < 3$. Choose $(x_0, y_0) \in$ supp(a) such that $a(x_0, y_0) \neq 0$, and without loss of generality assume that supp(a) is contained in a small neighborhood of (x_0, y_0). For fixed x_n and y_n define a family of operators $T(x_n, y_n) : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ by

$$T(x_n, y_n)f(x') = \lambda^{n/2-1/2} \int_{\mathbb{R}^{n-1}} e^{\frac{\lambda}{x_n - y_n}|x'-y'|^2} a(x', x_n, y', y_n) f(y')\, dy.$$

We know by the discussion after theorem 2.3.1 that there exists $g \in C_0^\infty(\mathbb{R}^{n-1})$ with $\|g\|_2 = 1$ such that

$$1 \lesssim \|T((x_0)_n, (y_0)_n)g\|_2 \,. \tag{4.2.4}$$

Since supp(a) is small, (4.2.4) is true for all x_n and y_n in supp(a) .

Let $f(y) = e^{i\lambda y_n} g(y')\chi_{[0,1]}(y_n)$; then $\|f\|_2 = 1$. Note that

$$\frac{|x-y|^2}{x_n - y_n} = \frac{|x'-y'|^2}{x_n - y_n} + x_n - y_n.$$

Then by the mean value theorem (for integrals) there exists $t \in \mathrm{supp}_{x_n}(a)$ and $s \in \mathrm{supp}_{y_n}(a)$ such that

$$\begin{aligned}\|T_\lambda f\|_2 = \Bigg(\int_0^1 \int_{\mathbb{R}^{n-1}} \Bigg| \int_2^3 \int_{\mathbb{R}^{n-1}} & e^{i\frac{\lambda}{x_n - y_n}|x'-y'|^2} \\ & \times a(x, x_n, y', y_n) g(y')\, dy'\, dy_n \Bigg|^2 dx'\, dx_n \Bigg)^{1/2} \\ = \lambda^{-n/2+1/2} \|T(t,s)g\|_2 \gtrsim \lambda^{-n/2+1/2}, &\end{aligned}$$

as desired. ■

We finish this section by noting that the estimate in theorem 4.2.1 may be improved when we have a better bound on one of the "factors" S_λ^1 or S_λ^2 of T_λ. If for example t and $\bar{t}$ satisfy the hypothesis of theorem 3.2.1 then infact $\|T_\lambda\| \lesssim \lambda^{-n/2}$. More precisely we have the following theorem whose proof is simply that of theorem 4.2.1.

Theorem 4.2.2. *Let t, $\bar{t}$ and ϕ be as above. For a given amplitude a with support contained in $B_1 \times B_1$, let*

$$T_\lambda f(x) = \int_{\mathbb{R}^n} \exp^{i\lambda \frac{|x-y|^2}{t(x) - \bar{t}(y)}} a(x,y) f(y)\, dy.$$

If S_λ^1 and S_λ^2 in (4.2.1) and (4.2.2) are such that

$$\|S_\lambda^1\| \lesssim \lambda^{-p} \quad \text{and} \quad \|S_\lambda^2\| \lesssim \lambda^{-q},$$

then

$$\|T_\lambda f\|_2 \lesssim \lambda^{-p-q+n/2} \|f\|_2 .$$

5. Applications to the Schrödinger Equation

§1. Introduction

In this chapter we consider, for a fixed smooth function $t(x)$, the solution to the Schrödinger equation (2) at time $t = t(x)$ — $u(x, t(x))$. As pointed out in the introduction, this is motivated by a desire to understand the Schrödinger maximal operator in (4). We obtain results of the form

$$\|u(\cdot\,, t(\cdot))\|_{L^2(\mathbb{D}^n)} \leq C\,\|f\|_{H^s}\,,$$

where s depends on conditions on the derivitives of t. We do this by first considering for $k = 0, 1, \ldots$ the operators $R_k : L^2(\mathbb{R}^n) \to L^2(\mathbb{D}^n)$ of the form,

$$R_k f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot y + t(x)|y|^2)} \theta_k(y) f(y)\, dy. \tag{5.1.1}$$

Here $\{\theta_k\}_{k=0}^{\infty}$ is a partition of unity subordinate to diadic intervals: $\theta_0 \in C_0^\infty(|y| \leq 2)$, $\theta_0(y) = 1$ when $|y| \leq 1$; $\theta_k(y) = \theta_0(2^{-k}y) - \theta_0(2^{1-k}y)$, when $k \geq 1$. Before stating our main results we prove a lemma which is the beginning step in all that follows (cf. [B]). Its purpose is to reduce H^s estimates to L^2 estimates.

Lemma 5.1.1. *Suppose there there is a C and an s_0 such that*

$$\|R_k f\|_{L^2(\mathbb{D}^n)} \leq C\, 2^{s_0 k}\, \|f\|_2\,.$$

Then for any $s > s_0$ there is a constant C_s depending on C and s such that

$$\|u(\cdot\,,t(\cdot))\|_{L^2(\mathbb{D}^n)} \le C_s\,\|f\|_{H^s}\,. \tag{5.1.2}$$

Proof. Note that $R_k\widehat{f} = R_k(\chi_{[\mathrm{supp}(\theta_k)]}\widehat{f})$. Hence

$$\begin{aligned}\|R_k\widehat{f}\|_{L^2(\mathbb{D}^n)} &\le C\,2^{s_0k}\Big(\int\limits_{\mathrm{supp}(\theta_k)} |\widehat{f}(y)|^2\,dy\Big)^{1/2}\\ &\lesssim (2^{-(s-s_0)})^k\Big(\int_{\mathbb{R}^n} |y|^{2s}|\widehat{f}(y)|^2\,dy\Big)^{1/2} \le C\,2^{-(s-s_0)k}\,\|f\|_{H^s}\,.\end{aligned}$$

Then by Minkowski's inequality,

$$\|u(\cdot\,,t(\cdot))\|_{L^2(\mathbb{D}^n)} \le \sum_{k=0}^{\infty}\|R_k\widehat{f}\|_{L^2(\mathbb{D}^n)} \le C\sum_{k=0}^{\infty}2^{-(s-s_0)k}\;\|f\|_{H^s} = C_s\,\|f\|_{H^s}\blacksquare$$

Remark 5.1.2. We may multiply R_k by a C_0^∞ function α which is unity on $\mathbb{D}^n$, if necessary, and all results about this "new" R_k will be the same as for that in (5.1.1). By abuse of notation R_k will denote either one.

§2. No Critical Points

Our first result is obtained by applying the techniques in the proof of theorem 3.1.6 to R_k. It involves the important special case when ∇t does not vanish.

Theorem 5.2.1. *Suppose $t \in C^\infty$ is such that $\nabla t(x) \ne 0\ \forall x \in \mathbb{D}^n$. Then for any $s > 0$,*

$$\|u(\cdot\,,t(\cdot))\|_{L^2(\mathbb{D}^n)} \le C\,\|f\|_{H^s}\,,$$

where C may depend on s and t.

This follows immediatly from Lemma 5.1.1 and the following.

Theorem 5.2.2. *Let t be as above. Then there is a constant C, which is independent of k, such that*

$$\|R_k f\|_{L^2(\mathbb{D}^n)} \leq C \, \|f\|_2 \, .$$

Proof. Since ∇t does not vanish, arguing as in theorem 3.1.6, we may assume that $\operatorname{supp}_x(a)$ is a small neighhborhood of a point x_0 such that there is a C^∞ diffeomorphism ρ, whose range lies in $\operatorname{supp}_x(a)$, and such that $t \circ \rho(x) = x_n$, and $\mathrm{D}\rho(x_0) = \mathrm{I}$. Let $\lambda = 2^k$. After making a change of variables ($x \to \rho(x)$, $y \to \lambda y$) it suffices to show that

$$\|\widetilde{R}_\lambda f\|_{L^2(\mathbb{D}^n)} \lesssim \lambda^{-n/2} \, \|f\|_2 \, , \tag{5.2.1}$$

where

$$\widetilde{R}_\lambda f(x) = \alpha(x) \int_{\mathbb{R}^n} e^{i(\lambda\rho(x)\cdot y + \lambda^2 x_n |y|^2)} \theta_1(y) f(y) \, dy.$$

Write $a(x,y) = \alpha(x)\theta_1(y)$ and proceed as in theorem 3.1.6. If

$$K(x,z) = \int_{\mathbb{R}^n} \exp\left(i\left\{\lambda(\rho(x)-\rho(z))\cdot y + \lambda^2(x_n - z_n)|y|^2\right\}\right) a(x,y)\overline{a(z,y)} \, dy,$$

then, if we let $S_\lambda = \widetilde{R}_\lambda \widetilde{R}_\lambda^*$,

$$\begin{aligned} S_\lambda f(x) &= \int_{\mathbb{R}^n} f(z) K(x,z) \psi\left(\frac{x_n - z_n}{\epsilon}\right) dz + \int_{\mathbb{R}^n} f(z) K(x,z) \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) dz \\ &= S^1_\lambda f(x) + S^2_\lambda f(x), \end{aligned}$$

where ψ is as in theorem 3.1.6.

Once again we find that the frozen operators $(S^1_\lambda)_{x_n z_n}$ have the form

$$(S^1_\lambda)_{x_n z_n} f(x') = \psi\left(\frac{x_n - z_n}{\epsilon}\right) (\widetilde{R}_\lambda \widetilde{R}^*_\lambda)_{x_n z_n} f(x').$$

For fixed x_n, since $x_n|y|^2$ is a function of y only, we may consider $(\widetilde{R}_\lambda)_{x_n y_n}$ as an oscillatory integral operator with phase function $\rho(x)\cdot y$. Clearly, by the construction of ρ, the mixed Hessian of this phase function is non-degenerate on $\operatorname{supp}(a)$.

Arguing as in theorem 3.1.6, we apply theorem 1.2.1 and lemmas 1.2.2 and 1.2.3 to see that

$$\left\|S_\lambda^1 f\right\|_2 \lesssim \lambda^{-n+1}\epsilon \left\|f\right\|_2 = \lambda^{-n}\left\|f\right\|_2, \tag{5.2.2}$$

if we take $\epsilon = \lambda^{-1}$. In what follows we shall take $\epsilon = \lambda^{-1}$, and in doing so we may assume, given the support properties of $\widetilde{\psi}$, that $\lambda|x_n - z_n| \gtrsim 1$.

Now we turn our attention to S_λ^2. Note that

$$\lambda(\rho(x)-\rho(z))\cdot y + \lambda^2(x_n - z_n)|y|^2 = \lambda^2(x_n - z_n)|y + F(x,z)|^2 - \frac{|\rho(x)-\rho(z)|^2}{4(x_n - z_n)},$$

where $F(x,z) = \dfrac{\rho(x)-\rho(z)}{2\lambda(x_n - z_n)}$. Let $A(x,z,y) = a(x,y)\overline{a(z,y)}$ and $\mu = \lambda^2(x_n - z_n)$. Then the kernal of S_λ^2 is

$$\widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right)\exp\left(-i\frac{|\rho(x)-\rho(z)|^2}{4(x_n - z_n)}\right)\int_{\mathbb{R}^n} e^{i\mu|y|^2}A(x,z,F(x,z))\,dy.$$

Here we have that

$$\int_{\mathbb{R}^n} e^{i\mu|y|^2}A(x,z,F(x,z))\,dy$$
$$= \left(\frac{i\mu}{\pi}\right)^{-n/2}\left(A(x,z,F(x,z)) + \int_{\mathbb{R}^n} r_1(i|\xi|^2/4\mu)e^{i\xi\cdot F(x,z)}\check{A}(x,z,\xi)\,d\xi\right)$$

where $\check{A}$ denotes the inverse Fourier transform in the last variable (cf. (3.1.11)). So $(S_\lambda^2)_{x_n z_n}$ is the sum of two terms, $(S_\lambda^2)'_{x_n z_n}$ and $(S_\lambda^2)''_{x_n z_n}$ having kernels $K'(x',z')$ and $K''(x',z')$ respectively. Since

$$K'(x',z') = \left(\frac{i\mu}{\pi}\right)^{-n/2}\widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right)\exp\left(i\lambda'|\rho(x)-\rho(z)|^2\right)A(x,z,F(x,z)),$$

where $\lambda' = \dfrac{1}{4x_n - z_n}$, we may treat $(S_\lambda^2)'_{x_n z_n}$ as an oscillatory integral operator with phase function $|\rho(x',x_n) - \rho(z',z_n)|^2$ and amplitude $A(x,z,F(x,z))$. And although this phase function does depend on λ, because $(\lambda|x_n - z_n|)^{-1} \lesssim 1$ we may uniformly bound finitely many z'-derivitives of A. Moreover, since $|x|$, $|z| \leq 2$, then $|\operatorname{supp}_{z'} A| \lesssim 1$. So by (2.2.2)

$$(5.2.3) \qquad \left\|(S_\lambda^2)'_{x_n z_n} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \lambda^{-n} |x_n - z_n|^{-1/2} \|f\|_{L^2(\mathbb{R}^{n-1})} .$$

Also $(S_\lambda^2)''_{x_n z_n}$ may be treated as an oscillatory integral operator as

$$K''(x', z') = (-i\pi)^{n/2} \mu^{-n/2-1} \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \exp(i\lambda' |\rho(x) - \rho(z)|^2)$$
$$\times \mu \int_{\mathbb{R}^n} r_1(i|\xi|^2/4\mu) e^{i\xi\cdot F(x,z)} \check{A}(x, z, \xi)\, d\xi.$$

The phase function is the same as in the previous case, but the amplitude is different. To apply thorem 2.2.1 we must consider z'-derivatives and the volume of the z'-support of this amplitude,

$$\mu \int_{\mathbb{R}^n} r_1(i|\xi|^2/4\mu) e^{i\xi\cdot F(x,z)} \check{A}(x, z, \xi)\, d\xi,$$

and find L^∞ bounds on these quantities which are independent of λ. Since $|x|$, $|z| \leq 2$ when this amplitude does vanish, and by consideration of (1.3.*) it suffices to show that for $s > n/2$

$$\sum_{|\alpha| \leq 2+s} \|\mathrm{D}_\xi^\alpha \mathrm{D}_{z'}^\beta (e^{i\xi\cdot F(x,z)} \check{A}(x, z, \xi)\, d\xi)\|_{L^2(d\xi)} \lesssim 1,$$

for all $|\beta| \leq n$, and this is easily seen to be so given that $|\lambda(x_n - z_n)| \geq 1$. Then (2.2.2) shows that

$$(5.2.4) \qquad \left\|(S_\lambda^2)''_{x_n z_n} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \lambda^{-n-2} |x_n - z_n|^{-3/2} \|f\|_{L^2(\mathbb{R}^{n-1})} .$$

Using lemma 1.2.2, (5.2.2), (5.2.3) and (5.2.4) we see that

$$\|S_\lambda f\|_2 \lesssim \lambda^{-n} \|f\|_2 ,$$

and this implies (5.2.1). ■

In some sense this represents a best possible case given the strength of the result in theorem 5.2.2— i.e., the constant C is independent of k. It is interesting to remark though that this constant must become unbounded as $|\nabla t|$ tends to zero, for otherwise the result in [DK] would be contradicted.

§3. Non-Degenerate Critical Points

The case when $\nabla t \neq 0$ represents the easiest to treat using the methods of theorem 3.1.6. When ∇t vanishes, the situation is more complicated. However the case when the Hessian of t is non-singular whenever ∇t vanishes—i.e., t has non-degenerate critical points—is treated below. We limit ourselves to the case when $n = 1$ or $n = 2$.

Theorem 5.3.1. *Suppose that t has only non-degenerate critical points. Then for any $s > 0$,*

$$\|u(\,\cdot\,, t(\,\cdot\,))\|_{L^2(\mathbb{D}^n)} \leq C\,\|f\|_{H^s}\,,$$

when $n = 1$ or $n = 2$.

This follows from

Theorem 5.3.2. *Suppose that t has only non-degenerate critical points. Then*

$$\|R_k f\| \leq C k^{n/2}\,\|f\|_2\,,$$

when $n = 1$ or $n = 2$.

Before giving the proof of theorem 5.3.2, we state a technical lemma whose proof is given at the end of this section.

Lemma 5.3.3. *Let $n = 1$ or $n = 2$, and suppose that* A *is an $n \times n$ diagonal matrix whose eigenvalues are ± 1. If* $\mathrm{A}(x)$ *denotes the quadratic form* $\mathrm{A}\,x \cdot x$*, then*

$$\sup_{|z| \leq 1} \int_{\mathbb{D}^n} \frac{dx}{(1 + \lambda^2 |\,\mathrm{A}(x) - \mathrm{A}(z)|)^{n/2}} \lesssim \left(\frac{\ln(\lambda)}{\lambda}\right)^n \tag{5.3.1}$$

Proof of theorem 5.3.2. We know that t only has finitely many isolated critical points in $\mathbb{D}^n$. Away from these critical points $|\nabla t| \geq c > 0$. Near a given critical point we may change variables in such a way that t is a quadratic form. After a partition of unity, an application of theorem 5.2.1 and a change of variables, we may assume that R_k is of the form

$$R_k f(x) = \lambda^{n/2} \int_{\mathbb{R}^n} \exp(i[\lambda\rho(x) \cdot y + \lambda^2 A(x)|y|^2]) a(x, y) f(y)\, dy,$$

where $\lambda = 2^k$, A is as in lemma 5.3.3, ρ is a C^∞ diffeomorphism and $a \in C_0^\infty(\mathbb{D}^n \times \mathbb{D}^n)$. As always $R_k R_k^*$ has a kernal K of the form

$$K(x, z) = \lambda^n \int_{\mathbb{R}^n} \exp(i[\lambda\rho(x) - \rho(z) \cdot y + \lambda^2(\mathrm{A}(x) - \mathrm{A}(z))|y|^2]) a(x, y)\overline{a(z, y)}\, dy.$$

In general $|K(x, z)| \lesssim \lambda^n$, while by stationary phase $|K(x, z)| \lesssim \lambda^n(\lambda^2(\mathrm{A}(x) - \mathrm{A}(z)))^{-n/2}$. Then an application of theorem 1.2.1 and lemma 5.3.3 yields the desired result. ■

We restrict ourselves to the case $n = 1, 2$ because the estimate in (5.3.1) is no longer valid for larger n. The estimate that one does get for $n \geq 3$ is not good enough to prove results that are better than those already found in [Sj] and [V].

Proof of lemma 5.3.3. We consider the cases of when $n = 1$ and $n = 2$ separately.

Case 1, $n = 1$.

After a change of variables, $x \mapsto x/\lambda$ it suffices to show that

$$\sup_{|z| \leq \lambda} \int_0^\lambda \frac{dx}{(1 + |x^2 - z^2|)^{1/2}} \lesssim \ln(\lambda).$$

We calculate, for fixed $|z| \le \lambda$, that

$$\int_0^\lambda \frac{dx}{(1+|x^2-z^2|)^{1/2}} = \int_0^{|z|} \frac{dx}{(1+z^2-x^2)^{1/2}} + \int_{|z|}^\lambda \frac{dx}{(1+x^2-z^2)^{1/2}}$$
$$= \arcsin\left(\frac{z}{\sqrt{1+z^2}}\right) + \ln\left(\frac{\lambda+\sqrt{1-z^2+\lambda^2}}{|z|+1}\right) \lesssim \ln(\lambda).$$

Case 2, $n = 2$ and $A = \pm \mathrm{I}$ (say $A = \mathrm{I}$).

Again we change variables as before, so it suffices to show that

$$\sup_{|z|\le\lambda} \int_0^\lambda \frac{r\,dr}{(1+|r^2-|z|^2|)} \lesssim \ln(\lambda)^2. \tag{5.3.2}$$

We make a further change of variables, $s = r^2$ so that the left-hand side of (5.3.2) is equal to (modulo a constant factor)

$$\int_0^{\lambda^2} \frac{dr}{1+|r-|z|^2|} = \int_0^{|z|^2} \frac{dr}{1+|z|^2-r} + \int_{|z|^2}^{\lambda^2} \frac{dr}{1+r-|z|^2}$$
$$= \ln(1+|z|^2) + \ln(1+\lambda^2-|z|^2) \lesssim \ln(\lambda).$$

Case 3, $n = 2$ and $A = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We must consider, where $c = Az \cdot z$,

$$\int_{B(0,1)} \frac{dx\,dy}{1+\lambda|x^2-y^2-c|}.$$

After the change of variables $u = x+y$, $v = x-y$ and a dilation, we may consider

$$\int_{-\lambda}^\lambda \int_{-\lambda}^\lambda \frac{dx\,dy}{1+|xy-c|} \qquad |c| \le \lambda^2.$$

In fact it is clear that we only have to consider

$$\int_1^\lambda \int_1^\lambda \frac{dx\,dy}{1+|xy-c|} \qquad |c| \le \lambda^2.$$

By changing variables the above is equal to

$$\int_1^{\lambda} \frac{1}{y}\Big(\int_y^{\lambda y} \frac{dx}{1+|x-c|}\Big) dy \lesssim \ln(\lambda)^2,$$

and this completes the proof. ∎

§4. Counterexamples

It is not possible that we may always get estimates as in theorems 5.2.1 and 5.3.1 for *all* $s > 0$ as the following 1-dimensional counterexample shows. The heart of the matter is found in theorem 5.4.2, but first a simple lemma is needed.

Lemma 5.4.1. *Let t be a fixed C^∞ function and let R_k be as in (5.1.1). Suppose there exists constants C and ρ, independent of λ, such that as an operator form $L^2(\mathbb{R}^n) \to L^2(\mathbb{D}^n)$,*

$$\|R_k\| \geq C2^{\rho k}. \tag{5.4.1}$$

Then the map

$$Sf(x) = u(x, t(x)) \tag{5.4.2}$$

is not a bounded map from $H^s(\mathbb{R}^n) \to L^2(\mathbb{D}^n)$ for any $s < \rho$.

Proof. The condition (5.4.1) means that for each $k = 1, 2, \ldots$ there exists a function $f_k \in L^2$ such that $\|f_k\|_2 = 1$, $\text{supp}(f_k) \in \text{supp}(\theta_k)$ and for which

$$\|R_k f_k\|_{L^2(\mathbb{D}^n)} \geq C2^{\rho k}.$$

Choose $g_k \in L^2$ such that $\widehat{g} = f_k\theta_k$. Fix an s for which S in (5.4.2) is a bounded map from $H^s(\mathbb{R}^n) \to L^2(\mathbb{D}^n)$. On the one hand, by (5.4.1),

$$\|Sg_k\|_{L^2(\mathbb{D}^n)} = \|R_k f_k\|_{L^2(\mathbb{D}^n)} \geq C2^{\rho k}.$$

On the other hand, by the choice of s,

$$\|Sg_k\|_{L^2(\mathbb{D}^n)} \leq C'\|g\|_{H^s} = C'\left(\int |f(x)|^2(1+|x|^2)^s\,dx\right)^{1/2}$$
$$\leq C''2^{sk}\|f\|_2 = C''2^{sk}.$$

Hence $2^{(\rho-s)k} \lesssim 1$, which is only possible when $\rho \leq s$. ■

This lemma is the counterpart to lemma 5.1.1 and justifies our reduction form H^s norms to L^2 norms. The only drawback to this scheme is that it does not give endpoint results—i.e., when $\rho = s$ in lemmas 5.1.1 and 5.4.1.

Theorem 5.4.2. *Let $t(x) = \frac{-1}{2}x^m$ where $m \geq 2$. Then*

$$\|R_k\| \geq C(2^k)^{\frac{m-2}{6(m-1)}}.$$

Proof. Let $\lambda = 2^k$. Then $\|R_k\| = \lambda^{1/2}\|\widetilde{R}_k\|$, where

$$\widetilde{R}_k f(x) = \int_{-\infty}^{\infty} \epsilon^{\lambda(xy-\frac{1}{2}\lambda x^m|y|^2)}\theta_1(y)f(y)\,dy.$$

We will apply theorem 2.2.2 to $\widetilde{R}_k$, with $\phi(x,y) = xy - \frac{1}{2}\lambda x^m|y|^2$. Let $(x_0,y_0) = ((m\lambda)^{\frac{-1}{m-1}},1)$. We expand $\Phi(x,y) = \phi(x,y) - \phi(x_0,y) - \phi(x,y_0) + \phi(x_0,y_0)$ in a Taylor series around the point (x_0,y_0) to obtain

$$\Phi(x,y) = -(m\lambda)^{\frac{1}{m-1}}(x-x_0)^2(y-y_0) - (x-x_0)(y-y_0)^2$$
$$+\,\mathcal{O}(|x-x_0|^3|y-y_0| + |x-x_0||y-y_0|^3 + |x-x_0|^2|y-y_0|^2).$$

Now assume that $|x-x_0| \leq c\lambda^{-p}$ and $|y-y_0| \leq c\lambda^{-q}$, where $p = \frac{m+1}{3(m-1)}$, $q = \frac{m-2}{3(m-1)}$ and c is a small constant independent of λ. On this rectangle, $\lambda|\Phi(x,z)| \leq 1/2$, and an application of theorem 2.3.1 gives the desired result. ■

6. Appendix

In this final chapter we discuss those issues previously mentioned, not central to our main ideas, which have not been adequately discussed. In §1 we give a proof of theorem 1 of the introduction. The idea of almost orthogonality as it was used in chapter 2 is discussed in §2. In §3 we continue the discussion of oscillatory integral operators taken up in chapter 2 and segue into §4 where we remark on questions of further research.

§1. A Proof of Theorem 1

In the introduction we stated the following theorem.

Theorem 6.1.1. *Let*

$$T_\lambda f(x) = \int_{\mathbb{R}^n} \exp\left(i\lambda \frac{|x-y|^2}{t(x) - \bar{t}(y)} \right) a(x,y) f(y)\, dy,$$

where $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *and* t *and* $\bar{t}$ *are measurable functions defined on* $\text{supp}_x(a)$ *and* $\text{supp}_y(a)$ *respectively such that* $1 \leq |t(x) - \bar{t}(y)| \leq 2$. *Then*

$$\|T_\lambda f\|_2 \leq C\lambda^{-\frac{n-2}{4}} \|f\|_2\,, \tag{6.1.1}$$

where C *is a universal constant independent of all such* t *and* $\bar{t}$.

Although this theorem does not appear in [B], some of the ideas in the proof may be found there. It is interesting to note that in spite of the non-smoothness of the phase function, $\|T_\lambda\| \to 0$ as $\lambda \to \infty$ when $n > 2$.

Proof. For ϵ to be chosen, let

$$U_j = \{\, x \in \mathrm{supp}_x(a) \,:\, j\epsilon \le t(x) < (j+1)\epsilon \,\}$$

$$\bar{U}_k = \{\, x \in \mathrm{supp}_y(a) \,:\, k\epsilon \le \bar{t}(y) < (k+1)\epsilon \,\}$$

Letting $\chi_j(x)$ and $\bar{\chi}_k(y)$ be the characteristic functions of U_j and $\bar{U}_k$ respectively, we see that $T_\lambda f(x) =$

$$\begin{aligned}
&\sum_{j\,k} \int_{\mathbb{R}^n} \exp\left(i\lambda \frac{|x-y|^2}{t(x)-\bar{t}(y)} \right) \chi_j(x) a(x,y) \bar{\chi}_k(y) f(y)\, dy \\
&= \sum_{j\,k} \int_{\mathbb{R}^n} \exp\left(i\lambda \frac{|x-y|^2}{t_j - \bar{t}_k} \right) \chi_j(x) a(x,y) \bar{\chi}_k(y) f(y)\, dy \\
&+ \int_{\mathbb{R}^n} \sum_{j\,k} \left(\exp\left(i\lambda \frac{|x-y|^2}{t(x)-\bar{t}(y)} \right) - \exp\left(i\lambda \frac{|x-y|^2}{t_j - \bar{t}_k} \right) \right) \chi_j(x) a(x,y) \bar{\chi}_k(y) f(y)\, dy \\
&= T_{\lambda 1} f(x) + T_{\lambda 2} f(x),
\end{aligned}$$

where $t_j \in [j\epsilon,\ (j+1)\epsilon)$ and $\bar{t}_k \in [k\epsilon,\ (k+1)\epsilon)$. When $x \in U_j$ and $y \in \ \bar{U}_k$ it is clear that

$$\left| \exp\left(i\lambda \frac{|x-y|^2}{t(x)-\bar{t}(y)} \right) - \exp\left(i\lambda \frac{|x-y|^2}{t_j - \bar{t}_k} \right) \right| \lesssim \lambda\epsilon.$$

Then by theorem 1.2.1

$$\|T_{\lambda 1} f\|_2 \lesssim \lambda\epsilon \, \|f\|_2 \, . \tag{6.1.2}$$

Now we estimate $\|T_{\lambda 2}\|$ by duality. Let $g \in L^2$ be such that $\|g\|_2 = 1$. First notice that the number of indices j or $k \sim \epsilon^{-1}$, and let $T_{j\,k}$ be the oscillatory integral operator with phase function $\dfrac{|x-y|^2}{t_j-\bar{t}_k}$ and amplitude $\chi_j(x)a(x,y)\bar{\chi}_k(y)$. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T_{\lambda 2} f(x) g(x)\, dx\right| \\
&= \left|\sum_{j\,k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(i\lambda\frac{|x-y|^2}{t_j-\bar{t}_k}\right) g(x)\chi_j(x)a(x,y)\bar{\chi}_k(y)f(y)\,dx\,dy\right| \\
&\leq \sum_{j\,k} \|g\bar{\chi}_k\|_2\, \|T_{j\,k}(f\chi_j)\|_2 \leq \sum_j \Big(\sum_k \|g\bar{\chi}_k\|_2^2\Big)^{1/2} \Big(\|T_{j\,k}(f\chi_j)\|_2^2\Big)^{1/2} \\
&\lesssim \|g\|_2\, \|f\|_2\, \lambda^{-n/2}\epsilon^{-1} \qquad \text{by theorem 2.1.1.}
\end{aligned}$$

So

$$\|T_{\lambda 2} f\|_2 \lesssim \lambda^{-n/2}\epsilon^{-1}\, \|f\|_2\,. \tag{6.13}$$

Choosing $\epsilon = \lambda^{-n/4-1/2}$ in (6.1.2) and (6.1.3) yields (6.1.1). ∎

§2. Almost Orthogonality

We consider a collection of operators $\{T_j\}_{j=1}^N$ on $L^2(\mathbb{R}^n)$, or more generally any Hilbert space, and we wish to improve upon the estimate

$$\left\|\sum_{j=1}^N T_j\right\| \leq \sum_{j=1}^N \|T_j\|. \tag{6.2.1}$$

If the T_j's were mutually orthogonal, that is $T_jT_k^* = 0 = T_j^*T_k$ when $j \neq k$, then we may bound the left-hand side of (6.2.1) by $\max\{\|T_1\|, \ldots, \|T_N\|\}$. The next best thing is when the T_j's are almost orthogonal, when we may favorably bound the number of compositions $T_jT_k^*$, $T_j^*T_k$ which are not identically zero. This is the subject of Cotlar's lemma, which appears below as theorem 6.2.1; our proof is adapted to fit our needs in chapter 2 form the one found in [St].

Theorem 6.2.1. *Let* $\{T_j\}_{j=1}^N$ *be operators on a Hilbert Space. Let*

$$M = \max\{\,\|T_1\|, \dots, \|T_N\|\,\}$$

$$A_1 = \max_{1\le j\le N} |\{\,k : T_jT_k^* \ne 0\,\}|$$

$$A_2 = \max_{1\le j\le N} |\{\,k : T_j^*T_k \ne 0\,\}|.$$

Then

$$\left\|\sum_{j=1}^N T_j\right\| \le M\sqrt{A_1A_2}.$$

Proof. Let $T = \sum_{j=1}^N T_j$. Since $\|T\| = \|T^*T\|^{1/2}$, then for any n,

$$\|T\| = \|(T^*T)^n\|^{1/2n},$$

and we have that

$$\|(T^*T)^n\| \le \sum_{j_1=1}^N \sum_{k_1=1}^N \cdots \sum_{j_n=1}^N \sum_{k_n=1}^N \|T_{j_1}^*T_{k_1}\cdots T_{j_n}^*T_{k_n}\|. \tag{6.2.2}$$

A typical term in (6.2.2) is bounded by M^{2n}, and if we count the number of non-vanishing terms we find that there are $NA_1^{n-1}A_2^n$ of them. Thus

$$\|T\| = \|(T^*T)^n\|^{1/2n} \le M\sqrt{A_1A_2}(NA_1)^{1/2n},$$

and letting $n \to \infty$ gives the result. ∎

As an application of this theorem, we consider what happens when we partition the support of the kernel of an integral operator. As in §2 of chapter 1, let $K \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, and let $Tf(x) = \int K(x,y)f(y)\,dy$. Suppose that $\{\alpha_i\}_{i=1}^{N_1}$, $\{\beta_j\}_{i=1}^{N_2}$ are partitions of unity subordinate to covers of $\operatorname{supp}_x(K)$ and $\operatorname{supp}_y(K)$ respectively such that the number of α_i's and β_j's with over-lapping support is a fixed constant independent of N_1 and N_2. Then $T = \sum_{ij} T_{ij}$ where T_{ij} is the integral operator with kernel $\alpha_i(x)K(x,y)\beta_j(y)$, and we find that $A_1 \sim N_1$, $A_2 \sim N_2$. The theorem then says that $\|T\| \lesssim M\sqrt{N_1 N_2}$ which is an improvment over the estimate in (6.2.1), $\|T\| \leq MN_1N_2$. So if for example the α_i's are all supported in balls of radius ~ 1, and say $N_2 = 1$, then $\|T\| \lesssim |\operatorname{supp}_x(K)|^{1/2}M$. This is the situation occurring in the proof of theorem 2.2.1.

§3. A Non-Folding Canonical Relation

We recall the situation in §1 of chapter 3. We are given an oscillatory integral operator with phase function $\phi = \dfrac{|x-y|^2}{t(x)}$, where $t \in C^\infty, t \neq 0$. We mentioned in remark 3.1.8 that if

$$\mathrm{H}\, t(x)(x-y)\cdot(x-y) \neq 0 \qquad \text{on } \operatorname{supp}(a), \tag{6.3.1}$$

then we have an even stronger result than in theorem 3.1.6, namely

$$\|T_\lambda f\|_2 \leq C\lambda^{-n/2+1/6}\,\|f\|_2\,. \tag{6.3.2}$$

This follows from a result in [PS] regarding oscillatory integral operators in general.

For a given phase function ϕ let C_ϕ denote the manifold

$$C_\phi = \{(x, \phi'_x(x,y), y, -\phi'_y(x,y))\},$$

and let $\Pi_j : C_\phi \to \mathbb{R}^n$ (j=1, 2) be the projections

$$\Pi_1(x,\xi,y,\eta) = (x,\xi) \qquad \Pi_2(x,\xi,y,\eta) = (y,\eta).$$

If $\mathcal{T}_\lambda$ is as in (2.1.1) and if the Π_j's have at most folding singularities[1], then (6.3.2) is true of $\mathcal{T}_\lambda$. The condition on t given in (3.1.6) is just a restatement of the conditions on the Π_j's.

[1] See [H1, Vol III] for the definition of a folding singularity.

In fact, if we consider the case of T_λ as above we easily find that Π_1 has only folding singularities, while Π_2 may or may not depending on whether or not the condition in (6.3.1) is satisfied. Nevertheless we find that the inequality in theorem 6.3.1 is always satisfied. This is no coincidence as we have recently found the following result in [GS].

Theorem 6.3.1. *Let $\mathcal{T}_\lambda$ be as in (2.1.1). Suppose that one of the Π_j's has at most folding singularities. Then*

$$\|\mathcal{T}_\lambda\| \leq C\lambda^{-n/2+1/4}.$$

Our result in theorem 3.1.7 shows that in general this result cannot be improved. It is interesting to consider intermediate cases—i.e., when say Π_1 has fold singularities, and the singularities of Π_2 are such that $\|\mathcal{T}_\lambda\| \lesssim \lambda^{-n/2+r}$, where $1/4 < r < 1/6$—and apply this to the operator in §1 of chapter 3, but in the context of pointwise convergence this does not seem fruitful.

§4. Further Directions

What does seem fruitful is a study of the dependence of $\|T_\lambda\|$ and $\|R_k\|$ on the C^∞ data of t and $\bar{t}$. As we are originally interested in the pointwise convergence of Schrödinger operators, as noted in the introduction, the case when t and $\bar{t}$ are only assumed to be measurable is really our main concern. On the one hand we have an estimate on $\|T_\lambda\|$ as in theorem 6.1.1 which does not depend at all on the C^∞ data of t and $\bar{t}$, while on the other hand the estimate in theorem 4.2.1, although much better in terms of the expenent of λ, assumes that t and $\bar{t}$ are smooth. In dimension $n = 2$ theorem 6.1.1 is not good enough to prove a pointwise convergence result, and it would be too much to that theorem 4.2.1 be valid when t and $\bar{t}$ are only measurable. How much can we relax the smoothness assumption on t and $\bar{t}$

and still prove a theorem better than theorem 6.1.1?

The question of smoothness also relates to counterexamples—i.e., finding when pointwise convergence results are not valid for *all* H^s functions. Perhaps it is possible to find sequences of functions $\{t_j\}$, $\{\bar{t}_j\}$ whose C^∞ data becomes unbounded in such a way that in the limit theorem 3.1.6 is no longer valid.[2] In dimension $n = 2$, at least, this may provide counterexamples. Also, *á la* theorem 5.4.2, we may find counterexamples by considering smooth functions $\{t_j\}$ whose limit is "bad."

Of course, we do not have a full understanding of the R_k's even when t is an arbitrary smooth function. It would be interesting to find a "good" estimate on $\|R_k\|$ which is valid for any smooth t. One approach is as in theorems 5.2.1 and 5.3.1 where we deal directly with R_k. Another approach is along the lines pursued in [B] where we reduce to the case of T_λ in (4). The argument there does not lend itself very easily to sharp results about $\|R_k\|$, and it would be interesting to develop strategies to deduce results about $\|R_k\|$ from those about $\|T_\lambda\|$, which we now understand.

[2] This approach was suggested to me by T. Wolff.

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