A SINGULARLY PERTURBED LINEAR TWO - POINT BOUNDARY - VALUE PROBLEM

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ABSTRACT

We consider the following singularly perturbed linear two-point boundary-value problem:

$$\mathcal{L}\underset{\sim}{y}(x) \equiv \Omega(\epsilon) D_x \underset{\sim}{y}(x) - A(x,\epsilon)\underset{\sim}{y}(x) = \underset{\sim}{f}(x,\epsilon) \qquad 0 \le x \le 1 \qquad (1a)$$

$$\mathcal{B}\underset{\sim}{y} \equiv L(\epsilon)\underset{\sim}{y}(0) + R(\epsilon)\underset{\sim}{y}(1) = \underset{\sim}{g}(\epsilon) \qquad \epsilon \to 0^+ \qquad (1b)$$

Here $\Omega(\epsilon)$ is a diagonal matrix whose first m diagonal elements are 1 and last m elements are ϵ. Aside from reasonable continuity conditions placed on A, L, R, $\underset{\sim}{f}$, $\underset{\sim}{g}$, we assume the lower right $m \times m$ principle submatrix of A has no eigenvalues whose real part is zero. Under these assumptions a constructive technique is used to derive sufficient conditions for the existence of a unique solution of (1). These sufficient conditions are used to define when (1) is a regular problem. It is then shown that as $\epsilon \to 0^+$ the solution of a regular problem exists and converges on every closed subinterval of $(0,1)$ to a solution of the reduced problem. The reduced problem consists of the differential equation obtained by formally setting ϵ equal to zero in (1a) and initial conditions obtained from the boundary conditions (1b). Several examples of regular problems are also considered.

A similar technique is used to derive the properties of the solution of a particular difference scheme used to approximate (1). Under restrictions on the boundary conditions (1b) it is shown that for the stepsize much larger than ϵ the solution of the differ-

ence scheme, when applied to a regular problem, accurately represents the solution of the reduced problem.

Furthermore, the existence of a similarity transformation which block diagonalizes a matrix is presented as well as exponential bounds on certain fundamental solution matrices associated with the problem (1).

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0. INTRODUCTION

This thesis is concerned with the properites of the solution of a singularly perturbed linear two-point boundary-value problem. The form of this problem, called the general boundary-value problem, is:

$$D\begin{bmatrix} \underline{u}(x) \\ \epsilon\underline{v}(x) \\ \epsilon\underline{w}(x) \end{bmatrix} = \begin{bmatrix} A_{11}(x,\epsilon) & A_{12}(x,\epsilon) & A_{13}(x,\epsilon) \\ A_{21}(x,\epsilon) & A_{22}(x,\epsilon) & \epsilon A_{23}(x,\epsilon) \\ A_{31}(x,\epsilon) & \epsilon A_{32}(x,\epsilon) & A_{33}(x,\epsilon) \end{bmatrix}\begin{bmatrix} \underline{u}(x) \\ \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} + \begin{bmatrix} \underline{f}_1(x,\epsilon) \\ \underline{f}_2(x,\epsilon) \\ \underline{f}_3(x,\epsilon) \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} L_1(\epsilon) & L_2(\epsilon) & L_3(\epsilon) \end{bmatrix}\begin{bmatrix} \underline{u}(0) \\ \underline{v}(0) \\ \underline{w}(0) \end{bmatrix} + \begin{bmatrix} R_1(\epsilon) & R_2(\epsilon) & R_3(\epsilon) \end{bmatrix}\begin{bmatrix} \underline{u}(1) \\ \underline{v}(1) \\ \underline{w}(1) \end{bmatrix} = \underline{g}(\epsilon) \tag{2}$$

$$0 \leq x \leq 1 \qquad\qquad \epsilon \to 0^+$$

Here the square matrices A_{11}, A_{22}, A_{33} have the orders m, m_1, m_2 respectively, and there are $m+m_1+m_2$ linearly independent boundary conditions. In addition to reasonable assumptions about the continuity properties of the matrices A_{ij}, L_i, R_i and vectors $\underline{f}_i$, $\underline{g}$, we make the following:

Assumption: For some positive constants μ, ϵ_0 and each $(x,\epsilon) \in [0,1]\times[0,\epsilon_0]$ every eigenvalue of $A_{22}(x,\epsilon)$ $(A_{33}(x,\epsilon))$ has its real part less that $-\mu$ (greater than μ). (3)

It is possible, as demonstrated in chapter one, to transform a large class of singularly perturbed linear two-point boundary-value problems into problems of the form presented in (1, 2). In this transformation we use a nonsingular matrix whose existence and properties are developed in chapter four.

One consequence of the assumption presented in (3) is contained in the following:

Theorem: Suppose the matrices $A_{22}(x,\epsilon)$, $A_{33}(x,\epsilon)$ depend continuously on x and ϵ, for $(x,\epsilon)\,\varepsilon\,[0,1]\times[0,\epsilon_0]$, and satisfy assumption (3). Define the fundamental solution matrices $Y_2(x,\tau)$, $Y_3(x,\tau)$ by the following initial-value problems: (4)

$$\epsilon D_x Y_2(x,\tau) = A_{22}(x,\epsilon)\, Y_2(x,\tau) \qquad Y_2(\tau,\tau) = I$$
$$0 \le x,\tau \le 1$$
$$\epsilon D_x Y_3(x,\tau) = A_{33}(x,\epsilon)\, Y_3(x,\tau) \qquad Y_3(\tau,\tau) = I$$

Then there exist positive constants C_0, Δ, ϵ_1 such that for all $\epsilon\,\varepsilon\,(0,\epsilon_1]$:

$$\| Y_2(x,\tau) \| \le C_0 \exp\left\{-\frac{\Delta}{\epsilon}(x-\tau)\right\} \qquad 0 \le \tau \le x \le 1$$
$$\| Y_3(x,\tau) \| \le C_0 \exp\left\{-\frac{\Delta}{\epsilon}(\tau-x)\right\} \qquad 0 \le x \le \tau \le 1$$

Here the symbol $\|\cdot\|$ denotes the infinity norm. The proof of this theorem may be found in [7], and in chapter four we present a slightly modified version of the same proof.

As a result of the theorem presented in (4) it is possible to formulate a constructive proof that for all sufficiently small ϵ a solution of the differential equation (1) subject to the boundary conditions:

$$\underline{u}(0) = \underline{u}^0(\epsilon) \qquad \underline{v}(0) = \frac{1}{\epsilon}\underline{v}^0(\epsilon) \qquad \underline{w}(1) = \frac{1}{\epsilon}\underline{w}^1(\epsilon) \tag{5}$$

exists, is unique, and satisfies an a priori bound. Let it be understood that we always require the parameter ϵ to assume only positive values. The boundary-value problem described by equations (1) and (5) is called the special boundary-value problem.

Using the method of matched asymptotic expansions we derive a formal asymptotic solution of the special boundary-value problem accurate to order ϵ. This accuracy estimate is then shown to be rigorously correct through the use of the a priori bound satisfied by the exact solution of the special boundary-value problem.

At first it appears that we have gained little information about the solution of the general boundary-value problem by solving the special boundary-value problem. Fortunately, this is not true. From a result found in [2], and presented in Theorem 1.28, we can use the asymptotic expansion of the solution of the special boundary-value problem to state sufficient conditions for the existence of a unique solution of the general boundary-value problem. These sufficient conditions constitute the basis of our definition of a regular (general boundary-value) problem. In Corollary 2.69 we state that the solution of a regular problem exists, is unique, and converges to the solution of the reduced problem on every closed subinterval of $(0, 1)$ as $\epsilon \to 0^+$. The reduced problem corresponding to the general boundary-value problem consists of the differential equation obtained by formally setting ϵ equal to zero in (1) and an initial condition obtained from the boundary conditions (2). We apply this corollary to several examples which are presented at the end of chapter two.

Much of the work presented in this thesis was motivated by a desire to discover the properites of the solution of a difference scheme applied to the general problem (1, 2). The form of this difference scheme, called the general difference problem, is:

$$\frac{1}{h}\begin{bmatrix} \{\underline{u}^h(j+1) - \underline{u}^h(j)\} \\ \epsilon\{\underline{v}^h(j+1)-\underline{v}^h(j)\} \\ \epsilon\{\underline{w}^h(j+1)-\underline{w}^h(j)\}\end{bmatrix} = \begin{bmatrix} A^h_{11}(j) & A^h_{12}(j) & A^h_{13}(j) \\ A^h_{21}(j) & A^h_{22}(j) & \epsilon A^h_{23}(j) \\ A^h_{31}(j) & \epsilon A^h_{32}(j) & A^h_{33}(j)\end{bmatrix}\begin{bmatrix}\frac{1}{2}\{\underline{u}^h(j+1)+\underline{u}^h(j)\} \\ \underline{v}^h(j+1) \\ \underline{w}^h(j)\end{bmatrix} + \begin{bmatrix}\underline{f}^h_1(j)\\ \underline{f}^h_2(j) \\ \underline{f}^h_3(j)\end{bmatrix} \tag{6}$$

$$\begin{bmatrix} L_1(\epsilon) & L_2(\epsilon) & L_3(\epsilon)\end{bmatrix}\begin{bmatrix}\underline{u}^h(0) \\ \underline{v}^h(0) \\ \underline{w}^h(0)\end{bmatrix} + \begin{bmatrix} R_1(\epsilon) & R_2(\epsilon) & R_3(\epsilon)\end{bmatrix}\begin{bmatrix}\underline{u}^h(J) \\ \underline{v}^h(J) \\ \underline{w}^h(J)\end{bmatrix} = \underline{g}(\epsilon) \tag{7}$$

$$0 \le j \le J-1 \qquad\qquad \epsilon \to 0^+$$

Here we have defined:

$$h = \frac{1}{J} \qquad\qquad x_\alpha = \alpha h$$

$$A^h_{ij}(\ell) = A_{ij}(x_{\ell+\frac{1}{2}}, \epsilon) \qquad\qquad \underline{f}^h_i(\ell) = \underline{f}_i(x_{\ell+\frac{1}{2}}, \epsilon)$$

We recognize the difference scheme presented in (6) uses a mixture of the forward, centered, and backward Euler methods. The choice of what difference scheme to use was motivated by the desire to apply the following theorem, whose proof can be found in chapter four.

<u>Theorem</u>: Suppose the matrices $A_{22}(x,\epsilon), A_{33}(x,\epsilon)$ depend continuously on x and ϵ, for $(x,\epsilon) \in [0,1] \times [0,\epsilon_0]$, and satisfy assumption (3). Define the discrete versions of fundamental solution matrices $Y_2^h(j,k)$, $Y_3^h(j,k)$ by the following initial-value problems: (8)

$$\epsilon D_j^h Y_2^h(j,k) = A_{22}^h(j)\, Y_2^h(j+1,k) \qquad Y_2^h(k,k) = I$$
$$0 \le j \le J-1\ , \quad 0 \le k \le J$$
$$\epsilon D_j^h Y_3^h(j,k) = A_{33}^h(j)\, Y_3^h(j,k) \qquad Y_3(k,k) = I$$

Then there exist positive constants C_0, Δ, ϵ_1 such that for all $\epsilon \in (0,\epsilon_1]$:

$$\| Y_2^h(j,k) \| \le C_0 / (1+\Delta \tfrac{h}{\epsilon})^{j-k} \qquad 0 \le k \le j \le J$$

$$\| Y_3^h(j,k) \| \le C_0 / (1+\Delta \tfrac{h}{\epsilon})^{k-j} \qquad 0 \le j \le k \le J$$

We recognize theorem (8) is similar to theorem (4). As a result of this similarity it is possible to carry over many of the techniques used in the general boundary-value problem to determine the properties of the solution of the general difference problem. From Corollary 3.72 we find that under suitable restrictions the solution of the general difference scheme converges to the solution of the reduced problem on every closed subinterval of $(0,1)$ as $\epsilon, h \to 0^+$. The restrictions placed on the problem for this convergence to occur are that:

(a) the general boundary-value problem is regular,
(b) the boundary conditions (2) do not involve $\underset{\sim}{v}(1)$ or $\underset{\sim}{w}(0)$. (9)

The most important point of this convergence result is the fact that the solution of the reduced problem can be accurately determined for $h \gg \epsilon$. This fact is in sharp distinction to the usual convergence results obtained for difference schemes, the usual convergence results would require $h \ll \epsilon$.

By using the results presented in this thesis it is possible to modify the general difference scheme and improve the results obtained. One improvement eliminates the restriction (9b) by applying the general difference scheme on a nonuniform mesh. This nonuniform net has its mesh points concentrated near the boundaries $x = 0, 1$. Unfortunately, to retain a given degree of accuracy in the representation of the solution of the reduced problem as $\epsilon \to 0^+$ it is necessary to increase the total number of mesh point at a rate proportional to $\ln \frac{1}{\epsilon}$. Another improvement uses a modified version of the general difference scheme to improve the rate of convergence of the solution of the difference scheme to the solution of the reduced problem. Each of these improvements is possible because we have detailed knowledge of the behavior of the solution of the general difference scheme.

Singular perturbation problems of the general form presented in (1, 2) have been considered extensively in the literature, see for example [9, 10, 11, 12, 13]. The procedure used in this thesis to study the boundary-value problem (1, 2) differs from those presented in [9, 10, 11, 12, 13] in the fact that it is constructive. It is the constructive nature of this procedure which allows us to apply it almost directly to the study of the properties of the numerical scheme.

Throughout this thesis a consistent effort has been made to adhere to the following notational convention:

D ... the derivative operator. A subscript is added whenever the function differentiated has more than one argument.

D^h ... the forward difference operator. A subscript is added whenever the function differenced has more than one argument.

$\|\cdot\|$ ... the infinity vector norm or its induced matrix norm.

The numbering of equations and results is done consecutively throughout each chapter. When a reference is made to a number outside the present chapter it is preceded by the number of the chapter in which it occurs, i.e. a reference to 2.76 means equation seventy-six in chapter two.

1. BASIC CONCEPTS

1.1 The General Problem

We consider the following two-point boundary-value problem:

$$\begin{aligned} \tilde{\mathcal{L}}\tilde{y}(x) &\equiv \Omega(\epsilon)D\tilde{y}(x) - \tilde{A}(x,\epsilon)\tilde{y}(x) = \tilde{f}(x,\epsilon) \\ \tilde{\mathcal{B}}\tilde{y} &\equiv \tilde{L}(\epsilon)\tilde{y}(0) + \tilde{R}(\epsilon)\tilde{y}(1) = \tilde{g}(\epsilon) \\ & x \in I \equiv [0,1] \end{aligned} \tag{1}$$

where:

ϵ ... a small positive parameter.

I_ℓ ... $\ell \times \ell$ identity matrix.

$$\Omega(\epsilon) = \begin{bmatrix} I_m & 0 \\ 0 & \epsilon I_m \end{bmatrix} \qquad \tilde{y}(x) = \begin{bmatrix} \tilde{u}(x) \\ \tilde{z}(x) \end{bmatrix}$$

$$\tilde{A}(x,\epsilon) = \begin{bmatrix} \tilde{A}_{11}(x,\epsilon) & \tilde{A}_{12}(x,\epsilon) \\ \tilde{A}_{21}(x,\epsilon) & \tilde{A}_{22}(x,\epsilon) \end{bmatrix} \qquad \tilde{f}(x,\epsilon) = \begin{bmatrix} \tilde{f}_1(x,\epsilon) \\ \tilde{f}_2(x,\epsilon) \end{bmatrix} \tag{2}$$

$\Omega, \tilde{A}, \tilde{y}, \tilde{f}$ ... compatibly partitioned matrices and vectors.

We shall assume, for some $\epsilon_0 > 0$ and $E_0 = [0, \epsilon_0]$, that one of the following sets of continuity conditions holds:

(a) $\tilde{A}, \tilde{f}, \tilde{L}, \tilde{R}, \tilde{g}$ are infinitely differentiable functions of x and/or ϵ for $(x, \epsilon) \in I \times E_0$.

(b) $\tilde{A}, \tilde{f}, \tilde{L}, \tilde{R}, \tilde{g}, D_x\tilde{A}_{12}, D_x\tilde{A}_{22}, D_x^2\tilde{A}_{22}$ are continuous functions of x and/or ϵ for $(x, \epsilon) \in I \times E_0$. (3)

Clearly conditions (3b) are satisfied whenever conditions (3a) hold. Furthermore, we require the matrix $\tilde{A}_{22}$ to satisfy the following eigenvalue (E.V.) condition:

E.V. Condition: For each $(x, \epsilon) \epsilon I \times E_o$ no eigenvalue of $\tilde{A}_{22}(x,\epsilon)$ has its real part equal to zero. (4)

In the development of the theory which follows we shall see the eigenvalue condition (4) has two important consequences. The first consequence of (4) is contained in Theorem (1.29), while the second consequence is described in the following:

Theorem 1.5: Let the matrix $\tilde{A}_{22}(x,\epsilon)$ depend continuously on x and ϵ for $(x, \epsilon) \epsilon I \times E_o$. If $\tilde{A}_{22}(x,\epsilon)$ satisfies the eigenvalue condition (4) then there exists a nonsingular matrix $T(x,\epsilon)$ such that:

(a) $$T^{-1}(x,\epsilon)\tilde{A}_{22}(x,\epsilon)T(x,\epsilon) = \begin{bmatrix} \tilde{A}^{(1)}_{22}(x,\epsilon) & 0 \\ 0 & \tilde{A}^{(2)}_{22}(x,\epsilon) \end{bmatrix} \quad (x,\epsilon) \epsilon I \times E_o .$$

(b) For each $(x, \epsilon) \epsilon I \times E_o$ every eigenvalue of $\tilde{A}^{(1)}_{22}(x,\epsilon)$ $(\tilde{A}^{(2)}_{22}(x,\epsilon))$ has its real part negative (positive). (5)

(c) The continuity properties of $\tilde{A}_{22}(x,\epsilon)$ with respect to x and ϵ are also enjoyed by $T(x,\epsilon)$, $\tilde{A}^{(1)}_{22}(x,\epsilon)$, $\tilde{A}^{(2)}_{22}(x,\epsilon)$.

The proof of Theorem (1.5) may be found in chapter four. In this proof we show the existence of a positive constant $\tilde{\mu}$ (independent of x and ϵ) which bounds away from zero the magnitude of the real parts of the eigenvalues of $\tilde{A}_{22}(x,\epsilon)$. With this in mind we may interpret statements (5a, b, c) in the following manner. Statements (5a, b) tell us the number of eigenvalues of $\tilde{A}_{22}(x,\epsilon)$, counting multiplicities, with negative (positive) real part is independent of x and ϵ. Once this fact is known, one may construct for each (x,ϵ) a matrix $T(x,\epsilon)$ which "block diagonalizes" $\tilde{A}_{22}(x,\epsilon)$ in the manner shown in (5a, b). Finally, (5c) states the existence of at least one choice of the matrix $T(x,\epsilon)$ which has as many derivatives with respect to x and ϵ as does $\tilde{A}_{22}(x,\epsilon)$.

Note that the differential equation:

$$\epsilon D\tilde{\underline{z}}(x) = \tilde{A}_{22}(x,\epsilon)\,\tilde{\underline{z}}(x)$$

under the change of variables:

$$\tilde{\underline{z}}(x) = T(x,\epsilon)\,\underline{z}(x)$$

becomes:

$$\epsilon D\underline{z}(x) = T^{-1}(x,\epsilon)\left\{\tilde{A}_{22}(x,\epsilon)T(x,\epsilon) - \epsilon D_x T(x,\epsilon)\right\}\underline{z}(x)$$

Therefore, when $\tilde{A}_{22}$ satisfies the continuity conditions (3b) and the eigenvalue condition (4), we may use Theorem 1.5 to choose $T(x,\epsilon)$ such that:

$$\begin{aligned} T^{-1}(x,\epsilon)\left\{\tilde{A}_{22}(x,\epsilon)T(x,\epsilon) - \epsilon D_x T(x,\epsilon)\right\} &= \begin{bmatrix} A_{22}(x,\epsilon) & \epsilon A_{23}(x,\epsilon) \\ \epsilon A_{32}(x,\epsilon) & A_{33}(x,\epsilon) \end{bmatrix} \\ A_{22}(x,\epsilon) &= \tilde{A}_{22}^{(1)}(x,\epsilon) + \mathcal{O}(\epsilon) \qquad \epsilon \to 0^+ \\ A_{33}(x,\epsilon) &= \tilde{A}_{22}^{(2)}(x,\epsilon) + \mathcal{O}(\epsilon) \end{aligned} \tag{6}$$

In an exactly analogous manner we will make the change of variables:

$$\begin{bmatrix} \tilde{\underline{u}}(x) \\ \tilde{\underline{z}}(x) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & T(x,\epsilon) \end{bmatrix}\begin{bmatrix} \underline{u}(x) \\ \underline{z}(x) \end{bmatrix} \tag{7'}$$

$$\underline{z}(x) = \begin{bmatrix} \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} \qquad (7')$$

where $T(x,\epsilon)$ is the matrix chosen in (6), to transform problem (1, 2, 3, 4) into problem (8, 9, 10, 11):

$$\begin{gathered} \mathcal{L}\underline{y}(x) \equiv \Omega(\epsilon) D\underline{y}(x) - A(x,\epsilon)\underline{y}(x) = \underline{f}(x,\epsilon) \\ \mathcal{B}\underline{y} \equiv L(\epsilon)\underline{y}(0) + R(\epsilon)\underline{y}(1) = \underline{g}(\epsilon) \\ x \in I \equiv [0,1] \end{gathered} \qquad (8)$$

where:

ϵ ... a small positive parameter.

$0 < \epsilon_1 \leqslant \epsilon_0 \qquad E_1 = [0, \epsilon_1]$

$I_\ell \ldots \ell \times \ell$ identity matrix.

$$\Omega(\epsilon) = \begin{bmatrix} I_m & 0 & 0 \\ 0 & \epsilon I_{m_1} & 0 \\ 0 & 0 & \epsilon I_{m_2} \end{bmatrix} \qquad \underline{f}(x,\epsilon) = \begin{bmatrix} \underline{f}_1(x,\epsilon) \\ \underline{f}_2(x,\epsilon) \\ \underline{f}_3(x,\epsilon) \end{bmatrix}$$

$$A(x,\epsilon) = \begin{bmatrix} A_{11}(x,\epsilon) & A_{12}(x,\epsilon) & A_{13}(x,\epsilon) \\ A_{21}(x,\epsilon) & A_{22}(x,\epsilon) & \epsilon A_{23}(x,\epsilon) \\ A_{31}(x,\epsilon) & \epsilon A_{32}(x,\epsilon) & A_{33}(x,\epsilon) \end{bmatrix} \qquad \underline{y}(x) = \begin{bmatrix} \underline{u}(x) \\ \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} \qquad (9)$$

$\Omega, A, \underline{y}, \underline{f}$ ... compatibly partitioned matrices and vectors.

Under the change of variables (7) we lose some of the differentiability properties of the functions involved, therefore (3) becomes:

$$
\begin{aligned}
&\text{(a)} \quad A, f, L, R, g \text{ are infinitely differentiable functions of } x \text{ and/or } \epsilon \text{ for } (x,\epsilon) \in I \times E_1. \\
&\text{(b)} \quad A, f, L, R, g, D_x A_{12}, D_x A_{13}, D_x A_{22}, D_x A_{33} \text{ are continuous functions of } x \text{ and/or } \epsilon \text{ for } (x,\epsilon) \in I \times E_1.
\end{aligned}
\tag{10}
$$

As illustrated in (6), the eigenvalues of $A_{22}(A_{33})$ are perturbations of those of $\tilde{A}_{22}^{(1)}(\tilde{A}_{22}^{(2)})$. Therefore, by choosing ϵ_1 sufficiently small we deduce from (3, 4, 5c, 6):

<u>E. V. Condition:</u> There exists a positive constant μ such that for each $(x,\epsilon) \in I \times E_1$ every eigenvalue of $A_{22}(x,\epsilon)$ $(A_{33}(x,\epsilon))$ has its real part less that $-\mu$ (greater than $+\mu$). (11)

We call problem (8, 9, 10, 11) the general problem. Since the transformation of variables (7) is nonsingular, we recognize the boundary-value problems (1) and (8) are equivalent.

To illustrate the type of behavior we can expect of the solution of the general problem, we consider the following model problem:

$$
\begin{gathered}
\epsilon D^2 u(x) + D u(x) = f^0 \\
u(0) = u^0 \qquad u(1) = u^1 \\
0 \le x \le 1
\end{gathered}
\tag{12}
$$

When written as a first order system, problem (12) becomes:

$$
\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} D \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} + \begin{bmatrix} 0 \\ f^0 \end{bmatrix} \tag{13'}
$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ v(1) \end{bmatrix} = \begin{bmatrix} u^0 \\ u^1 \end{bmatrix} \tag{13'}$$

The exact solution of (13) is:

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} u^0 + (u^1 - u^0 - f^0)\,[1 - \exp(-\frac{x}{\epsilon})]/[1 - \exp(-\frac{1}{\epsilon})] + f^0 x \\ \frac{1}{\epsilon}(u^1 - u^0 - f^0)\,\exp(-\frac{x}{\epsilon})/[1 - \exp(-\frac{1}{\epsilon})] + f^0 \end{bmatrix} \tag{14}$$

As $\epsilon \to 0^+$ we find $u(x)$ is a bounded function on $[0,1]$ while $v(x)$ is bounded only on closed subintervals of $(0,1]$. Near the boundary $x=0$ we find both u and v make rapid transitions of an exponential nature from their value at $x=0$ to their value for x near zero; in fact $v(0)$ blows up like ϵ^{-1}.

Therefore, considering the extra complexity of (8) when compared to (13), we expect solutions of (8) to have u bounded on $[0,1]$ while v, w are only bounded on closed subintervals of $(0,1)$. Near the boundaries $x = 0,1$ we expect u, v, w to undergo rapid transitions of an exponential nature from their boundary values to their values near the boundary. In these regions of rapid transition, called boundary layers, we expect v and/or w to blow up like ϵ^{-1}.

In many cases of interest our expectations about the behavior of the solution of (8) will be correct. However, we hasten to point out that our expectations can be wrong. For example, if $u^0 = \epsilon^{-1}$ in (13) then both u and v blow up as $\epsilon \to 0^+$, while for $u^1 = u^0 + f^0$ no boundary layer occurs at $x=0$.

1.2 Banach Spaces and Differential Equations

In the theory which follows we shall use the idea of a Banach Space (i.e. a complete, normed, linear space). Two examples of a Banach Space which we will use are:

Example 1: The Banach Space $\mathcal{C}^N_\infty[0,1]$. The linear space involved is the space $\mathcal{C}^N[0,1]$ of all continuous N-vector complex-valued functions defined on the interval $[0,1]$. The norm involved is the sup-norm:

$$\|\underline{f}\| \equiv \sup\{ \| \underline{f}(x) \| : x \in [0,1]\} \tag{15}$$

$$\| \underline{f} \| = \max\{ |f_j(x)| : 1 \leq j \leq N\}$$

Example 2: The Banach Space $\mathcal{D}^N_\infty[0,J]$. The linear space involved is the space $\mathcal{D}^N[0,J]$ of all N-vector complex-valued functions defined on the integers $(0, 1, \ldots, J)$. The norm involved is the sup-norm:

$$\|\underline{F}\| \equiv \sup\{ \| \underline{F}(j) \| : j \in (0,1,\ldots,J)\} \tag{16}$$

From a Banach Space $(X, \|\cdot\|)$, where X is the linear space and $\|\cdot\|$ the norm, one may construct a second Banach Space as follows. Define $\mathcal{L}(X)$ to be the space of all linear operators mapping X into itself. If $K \in \mathcal{L}(X)$ we say K is bounded iff the number:

$$\|K\| \equiv \sup\{ \|Kx\| : x \in X \text{ and } \|x\| = 1\} \tag{17}$$

is finite. If we define $\mathcal{BL}(X)$ to be the space of all bounded linear operators, then under the norm (17) $\mathcal{BL}(X)$ is a Banach Space.

Example 3: If $A(x,y)$ is a continuous $N \times N$ complex-valued matrix defined for $(x,y) \in [0,1] \times [0,1]$, then the mapping:

$$K\underline{f}(x) \equiv \underline{f}(x) + \int_0^x A(x,y)\, \underline{f}(y)\, dy \tag{18}$$

is in $\mathcal{B}\mathcal{L}(\mathcal{C}^{N}_{\infty}[0,1])$.

<u>Example 4:</u> If $A(i,j)$ is an $N \times N$ complex-valued matrix defined for $(i,j) \varepsilon (0,1,\ldots J) \times (0,1,\ldots J)$, then the mapping:

$$KF(j) \equiv F(j) + \frac{1}{J} \sum_{k}^{i-1} {}_{0} A(j,k) F(k) \tag{19}$$

is in $\mathcal{B}\mathcal{L}(\mathcal{D}^{N}_{\infty}[0,J])$.

With these ideas in mind let us prove two well known results:

<u>Theorem 1.20:</u> (Banach Lemma) Let $(X, \|\cdot\|)$ be a Banach Space and $K \varepsilon \mathcal{B}\mathcal{L}(X)$. If $\|K\| < 1$ the operator $I-K$ is nonsingular and:

$$\begin{aligned} &\text{(a)} \quad (I-K)^{-1} = \sum_{m=0}^{\infty} K^m \\ &\text{(b)} \quad \|(I-K)^{-1}\| \leq (1 - \|K\|)^{-1} \end{aligned} \tag{20}$$

Proof: If we define $L_N \equiv \sum_{m=0}^{N} K^m$, then $\{L_N\}_0^\infty$ is a Cauchy sequence in the Banach Space $\mathcal{B}\mathcal{L}(X)$. Therefore, for some $L \varepsilon \mathcal{B}\mathcal{L}(X)$ the sequence $\{L_N\}_0^\infty$ converges to L. Furthermore, since:

$$L_N(I-K) = I - K^{N+1}$$

we find:

$$L(I-K) = \lim_{N\to\infty} L_N(I-K) = I$$

Therefore $L = (I-K)^{-1}$. Since:

$$\|L_N\| \leq \sum_{m}^{N} {}_{0} \|K\|^m \leq (1 - \|K\|)^{-1}$$

then:

$$\|L\| = \lim_{N\to\infty} \|L_N\| \leq (1 - \|K\|)^{-1}$$

##

Theorem 1.21: Let $(X, \|\cdot\|)$ be a Banach Space. If K, $K^{-1}, L \in \mathcal{BL}(X)$ then for each positive $\epsilon_0 < \|K^{-1}L\|^{-1}$ and all $|\epsilon| \leq \epsilon_0$ the operator $K+\epsilon L$ is non-singular. Furthermore:

$$\begin{aligned} &(a) \quad (K+\epsilon L)^{-1} = (I+\epsilon K^{-1}L)^{-1}K^{-1} \\ &(b) \quad \|(K+\epsilon L)^{-1}\| \leq (1-\epsilon\|K^{-1}L\|)^{-1}\|K^{-1}\| \end{aligned} \qquad (21)$$

Proof: Since K is invertible we deduce:

$$K + \epsilon L = K(I + \epsilon K^{-1}L)$$

Therefore (21) follows because $(I + \epsilon K^{-1}L)$ is nonsingular by the Banach Lemma (20). ##

Since the general problem (8) is a linear differential equation, let us list the basic ideas underlying the solution of linear differential equations. Let $C(x)$ be a square matrix depending continuously on x for $x \in [0,1]$. Following Ince [1] we define another square matrix $Y(x,\tau)$, called the fundamental solution matrix (F.S.M.) for $C(x)$, by the initial-value problem:

$$\begin{aligned} D_x Y(x,\tau) &= C(x)\, Y(x,\tau) \\ Y(\tau,\tau) &= I \end{aligned} \qquad 0 \leq x,\tau \leq 1 \qquad (22)$$

Among the well known properties of $Y(x,\tau)$ are the following:

$$\begin{aligned} &(a) \quad Y(x,\tau) \text{ is a uniquely determined nonsingular matrix} \\ &(b) \quad Y(x,\tau)\, Y(\tau,x) = I \\ &(c) \quad D_\tau Y(x,\tau) = -Y(x,\tau)\, C(\tau) \\ &(d) \quad Y(x,\tau) = I + \int_\tau^x Y(x,s)\, C(s)\, ds \end{aligned} \qquad (23)$$

One obtains (23c) by differentiating (23b) with respect to τ and using (22, 23a). To obtain (23d) replace τ by s in (23c) and integrate the result from $s=\tau$ to $s=x$.

Consider the differential equation:

$$\mathcal{M}\underset{\sim}{y}(x) \equiv D\underset{\sim}{y}(x) - C(x)\,\underset{\sim}{y}(x) = \underset{\sim}{f}(x) \qquad 0 \leq x \leq 1 \tag{24}$$

If $\underset{\sim}{f}$ is a continuous function of x for $x \in [0, 1]$, then the unique solution of (24) satisfying the initial condition $\underset{\sim}{y} = \underset{\sim}{y}(s)$ at $x = s$ is:

$$\underset{\sim}{y}(x) = Y(x,s)\,\underset{\sim}{y}(s) + \int_s^x Y(x,\tau)\,\underset{\sim}{f}(\tau)\,d\tau \tag{25}$$

Formula (25) gives us a representation of the solution $\underset{\sim}{y}$ of (24) given its value at some point $s \in [0, 1]$, and it is called the variation of parameters (V.O.P.) formula.

Finally, consider the following pair of boundary-value problems:

BV(ν):

$$\begin{aligned} &\mathcal{M}\underset{\sim}{y}^{(\nu)}(x) = \underset{\sim}{f}(x) \\ &\mathcal{B}^{(\nu)}\underset{\sim}{y}^{(\nu)} \equiv L^{(\nu)}\underset{\sim}{y}^{(\nu)}(0) + R^{(\nu)}\underset{\sim}{y}^{(\nu)}(1) = \underset{\sim}{g} \\ &\nu = 0, 1 \qquad\qquad 0 \leq x \leq 1 \end{aligned} \tag{26}$$

We note BV(0) may differ from BV(1) only in the boundary conditions. One well known result is the following:

<u>Lemma</u>: BV(ν) has a unique solution for every $\underset{\sim}{f}, \underset{\sim}{g}$ iff the matrix $\mathcal{B}^{(\nu)}Y(\cdot, 0)$ is nonsingular.

The proof of this lemma rests on the fact that any solution $\underset{\sim}{y}^{(\nu)}(x)$ of BV(ν) has the unique representation:

$$\underset{\sim}{y}^{(\nu)}(x) = Y(x,0)\,\underset{\sim}{y}^{(\nu)}(0) + \int_0^x Y(x,\tau)\,\underset{\sim}{f}(\tau)\,d\tau \tag{27}$$

derived from the V.O.P. formula (25). Using (27) we argue there are as many solutions of BV(ν) as there are solutions $\underset{\sim}{y}^{(\nu)}(0)$ of the algebraic problem:

$$\{\mathcal{B}^{(\nu)} Y(\cdot,0)\}\, \underset{\sim}{y}^{(\nu)}(0) = \underset{\sim}{g} - R^{(\nu)} \int_0^1 Y(1,\tau)\, \underset{\sim}{f}(\tau)\, d\tau$$

Following Keller and White [2], we may use this lemma to prove the following:

Theorem 1.28: If BV(0) has a unique solution for every $\underset{\sim}{f}, \underset{\sim}{g}$ then there exists a unique matrix $Z(x)$ satisfying:

$$\mathcal{M}\, Z(x) = 0 \qquad\qquad \mathcal{B}^{(0)} Z = I \tag{28}$$

Furthermore, BV(1) has a unique solution for every $\underset{\sim}{f}, \underset{\sim}{g}$ iff $\mathcal{B}^{(1)} Z$ is nonsingular.

The proof of this theorem follows by noting $Z(x)$ has the representation:

$$Z(x) = Y(x,0)\, \{\mathcal{B}^{(0)} Y(\cdot,0)\}^{-1}$$

which, upon rearrangement , leads to the identity:

$$\mathcal{B}^{(1)} Y(\cdot,0) = \{\mathcal{B}^{(1)} Z\}\{\mathcal{B}^{(0)} Y(\cdot,0)\}$$

When we consider the general problem (8, 9, 10, 11) in the following chapter, we shall make extensive use of the following:

Theorem 1.29: (Exponential Dichotomy) Let $A_{22}(x,\epsilon)$, $A_{33}(x,\epsilon)$ be continuous functions of x and ϵ , for $(x,\epsilon) \in I \times E_1$, which satisfy the eigenvalue condition (11). If $Y_2(x,\tau), Y_3(x,\tau)$ are the F.S.M. for $\frac{1}{\epsilon} A_{22}(x,\epsilon)$, $\frac{1}{\epsilon} A_{33}(x,\epsilon)$ respectively, then there exist positive constants K_0, ϵ_2, Δ such that for all $0 < \epsilon \leq \epsilon_2$:

$$\begin{aligned} &\text{(a)} \quad \| Y_2(x,\tau) \| \leq K_0 \exp\{-\tfrac{\Delta}{\epsilon}(x-\tau)\} \qquad 0 \leq \tau \leq x \leq 1 \\ &\text{(b)} \quad \| Y_3(x,\tau) \| \leq K_0 \exp\{-\tfrac{\Delta}{\epsilon}(\;\;-x)\} \qquad 0 \leq x \leq \tau \leq 1 \end{aligned} \tag{29}$$

As mentioned in section 1.1, this theorem is the first important consequence of the continuity and eigenvalue conditions (3.4). The proof of Theorem 1.29 will be found in chapter four. We should note the norm used in (29) is the matrix norm induced by the vector norm $\| \cdot \|$.

1.3 A Perturbation Example

We illustrate the use of the basic principles outlined in the previous section by considering the following initial-value problem:

$$D^2 u(x) + D u(x) + \epsilon a(x) u(x) = f(x)$$

$$u(0) = 0 \qquad D u(0) = 0 \tag{30}$$

$$0 \leq x \leq 1$$

Here a, f are in $\mathcal{C}^1_\infty[0,1]$ and ϵ is a small parameter. By integrating equation (30) we find u must satisfy:

$$D u(x) = -u(x) + \int_0^x \{ f(\tau) - \epsilon a(\tau) u(\tau) \} d\tau \tag{31}$$

If we note the F.S.M. for -1 is:

$$Y(x,\tau) = \exp\{-(x-\tau)\}$$

then an application of the V.O.P. formula (25) with $s = 0$ allows us to deduce from (31):

$$u(x) = \int_0^x \exp\{-(x-\tau)\} \int_0^\tau \{ f(s) - \epsilon a(s) u(s) \} ds \, d\tau \tag{32}$$

If we introduce the operators $K, L \in \mathcal{BL}(\mathcal{C}^1_\infty[0,1])$ defined by:

$$K g(x) = \int_0^x \int_s^x \exp\{-(x-\tau)\}\, d\tau \, g(s)\, ds$$
$$L g(x) = a(x)\, g(x) \tag{33}$$

we note we may write (32) equivalently as:

$$(I + \epsilon KL)u = Kf \tag{34}$$

If we choose $\epsilon_0 > 0$ such that:

$$\epsilon_0 \|KL\|_\infty < 1$$

then an application of the Banach Lemma states:

$$M = -(I + \epsilon KL)^{-1} KL \qquad |\epsilon| \leq \epsilon_0$$

exists for $|\epsilon| \leq \epsilon_0$, and satisfies the bound:

$$\|M\|_\infty \leq M_\infty \equiv (1 - \epsilon_0 \|KL\|_\infty) \|KL\|_\infty \qquad |\epsilon| \leq \epsilon_0$$

Furthermore, we find:

$$(I + \epsilon KL)^{-1} = I + \epsilon M \qquad |\epsilon| \leq \epsilon_0 \tag{35}$$

Using the identity (35) we may write (34) as:

$$u = (I + \epsilon M) K f \qquad |\epsilon| \leq \epsilon_0 \tag{36}$$

We can interpret equation (36) in the following manner. First, it states a unique solution of (40) exists for all sufficiently small ϵ. Secondly, it states the unique solution of (30) satisfies the a priori bound:

$$\|u\|_\infty \leq (1 + \epsilon_0 M_\infty) \|K\|_\infty \|f\|_\infty \qquad |\epsilon| \leq \epsilon_0 \tag{37}$$

Furthermore, equation (36) implies:

$$u \sim Kf + \mathcal{O}(\epsilon) \qquad \text{as } |\epsilon| \to 0$$

where the $\sim$ sign and the symbol $\mathcal{O}$ are used in the following manner.

Let f, g be members of some Banach Space $(X, \|\cdot\|)$. We interpret the statements:

$$\begin{aligned} f &\sim g + \mathcal{O}(\epsilon^m) && \text{as } |\epsilon| \to 0 \\ f &\sim g + \mathcal{O}(\epsilon^m) && \text{as } \epsilon \to 0^+ \end{aligned} \tag{38}$$

as implying the existence of positive constants ϵ_1, K_1 satisfying the inequalities:

$$\begin{aligned} \| f - g \| &\leq K_1 |\epsilon|^m && \text{as } |\epsilon| \to 0 \\ \| f - g \| &\leq K_1 \epsilon^m && \text{as } \epsilon \to 0^+ \end{aligned} \tag{39}$$

Suppose the exact solution of (30) admitted the expansion:

$$\begin{aligned} u(x) &\sim w_N(x) + \mathcal{O}(\epsilon^N) \qquad \text{as } |\epsilon| \to 0 \\ w_N(x) &\equiv \sum_{\ell=0}^{N-1} \epsilon^\ell u_\ell(x) \end{aligned} \tag{40}$$

If we substitute w_N into (30), collect like powers of ϵ, and set the coefficient of ϵ^m (for $m = 0, 1, 2, \ldots, N-1$) equal to zero, then we are led to the following sequence of problems:

$$P_0: \quad \begin{aligned} &D^2 u_0(x) + D u_0(x) = f(x) \\ &u_0(0) = 0 \qquad D u_0(0) = 0 \end{aligned} \tag{41a}$$

$$P_n: \quad \begin{aligned} & D^2 u_m(x) + D u_m(x) = -a(x)\, u_{m-1}(x) \\ & u_m(0) = 0 \qquad D u_m(0) = 0 \end{aligned} \tag{41b}$$

We find (32) may be used to solve (41) with the result:

$$\begin{aligned} u_0(x) &= \int_0^x \int_s^x \exp\{-(x-\tau)\}\, d\tau\, f(s)\, ds \\ u_m(x) &= -\int_0^x \int_s^x \exp\{-(x-\tau)\}\, d\tau\, a(s)\, u_{m-1}(s)\, ds \qquad 1 \le m \le N-1 \end{aligned} \tag{42}$$

Let u denote the exact solution of (30), while w_N is defined by (40) and (42). We deduce from (30) and (41) that the error $e_N \equiv u - w_N$ satisfies:

$$\begin{aligned} & D^2 e_N(x) + D e_N(x) + \epsilon\, a(x)\, e_N(x) = \epsilon^N a(x)\, u_{N-1}(x) \\ & e_N(0) = 0 \qquad D e_N(0) = 0 \end{aligned} \tag{43}$$

By applying the a priori bound (37) to the exact solution of (43) we deduce:

$$\| e_N \|_\infty \le (1 + \epsilon_0 M_\infty) \| K \|_\infty \| a u_{N-1} \|_\infty |\epsilon|^N \qquad |\epsilon| \le \epsilon_0 \tag{44}$$

We recognize (44) as the rigorous justification of the statement that:

$$u(x) \sim w_N(x) + O(\epsilon^N) \qquad \text{as } |\epsilon| \to 0$$

In general, if $\underset{\sim}{y}$ satisfies the boundary-value problem:

$$\mathcal{L}\, \underset{\sim}{y}(x) = \underset{\sim}{f}(x) \qquad \mathcal{B}\, \underset{\sim}{y} = \underset{\sim}{g} \tag{45}$$

and we have shown:

$$\underset{\sim}{y}(x) \sim \underline{w}(x) + \mathcal{O}(\epsilon^N) \qquad \text{as } \epsilon \to 0$$

then we shall call $\underset{\sim}{w}$ an asymptotic (expansion of the exact) solution of the boundary-value problem (45).

2. THE GENERAL PROBLEM

2.0 Formulation of the General and Special Problems

We consider the general problem:

$$\begin{aligned} \mathcal{L}\underline{y}(x) &\equiv \Omega(\epsilon) D \underline{y}(x) - A(x,\epsilon)\underline{y}(x) = \underline{f}(x,\epsilon) \\ \mathcal{B}\underline{y} &\equiv L(\epsilon)\underline{y}(0) + R(\epsilon)\underline{y}(1) = \underline{g}(\epsilon) \\ x &\in I \equiv [0,1] \end{aligned} \tag{1}$$

where:

ϵ ... a small positive parameter.

$$\epsilon_1 > 0 \qquad E_1 = [0, \epsilon_1]$$

I_ℓ ... $\ell \times \ell$ identity matrix.

$$\Omega(\epsilon) = \begin{bmatrix} I_m & 0 & 0 \\ 0 & \epsilon I_{m_1} & 0 \\ 0 & 0 & \epsilon I_{m_2} \end{bmatrix} \qquad \underline{f}(x,\epsilon) = \begin{bmatrix} \underline{f}_1(x,\epsilon) \\ \underline{f}_2(x,\epsilon) \\ \underline{f}_3(x,\epsilon) \end{bmatrix} \tag{2}$$

$$A(x,\epsilon) = \begin{bmatrix} A_{11}(x,\epsilon) & A_{12}(x,\epsilon) & A_{13}(x,\epsilon) \\ A_{21}(x,\epsilon) & A_{22}(x,\epsilon) & \epsilon A_{23}(x,\epsilon) \\ A_{31}(x,\epsilon) & \epsilon A_{32}(x,\epsilon) & A_{33}(x,\epsilon) \end{bmatrix} \qquad \underline{y}(x) = \begin{bmatrix} \underline{u}(x) \\ \underline{v}(x) \\ \underline{w}(x) \end{bmatrix}$$

$\Omega, A, \underline{y}, \underline{f}$ ... compatibly partitioned matrices and vectors.

We assume one of the following sets of continuity conditions holds:

(a) $A, \underline{f}, L, R, \underline{g}$ are infinitely differentiable functions of x and/or ϵ for $(x,\epsilon) \epsilon I \times E_1$. (3)

(b) $A, \underline{f}, L, R, \underline{g}, D_x A_{12}, D_x A_{13}, D_x A_{22}, D_x A_{33}$ are continuous functions of x and ϵ for $(x,\epsilon) \epsilon I \times E_1$.

Furthermore, we place the following condition of the eigenvalues of the matrices A_{22}, A_{33} :

E.V. Condition: There exists a positive constant μ such that for each $(x,\epsilon) \epsilon I \times E_1$ every eigenvalue of $A_{22}(x,\epsilon)$ ($A_{33}(x,\epsilon)$) has its real part less than $-\mu$ (greater than $+\mu$). (4)

As mentioned in section 1.2 , the eigenvalue condition (4) leads to the following:

Theorem 2.5: (Exponential Dichotomy) Let $A_{22}(x,\epsilon)$, $A_{33}(x,\epsilon)$ be continuous functions of x and ϵ , for $(x,\epsilon) \epsilon I \times E_1$, which satisfy the eigenvalue condition (4). If $Y_2(x,\tau)$, $Y_3(x,\tau)$ are the F.S.M. for $\frac{1}{\epsilon} A_{22}(x,\epsilon)$ $\frac{1}{\epsilon} A_{33}(x,\epsilon)$ respectively, then there exist positive constants C_0, ϵ_2, Δ such that for all $0 < \epsilon \leq \epsilon_2$:

$$\text{(a)} \quad \| Y_2(x,\tau) \| \leq C_0 \exp\{-\tfrac{\Delta}{\epsilon}(x-\tau)\} \qquad 0 \leq \tau \leq x \leq 1 \tag{5}$$
$$\text{(b)} \quad \| Y_3(x,\tau) \| \leq C_0 \exp\{-\tfrac{\Delta}{\epsilon}(\tau-x)\} \qquad 0 \leq x \leq \tau \leq 1$$

We also consider the following special problem:

$$\mathcal{L}\underline{y}(x) = \underline{f}(x,\epsilon) \tag{6}$$
$$\mathcal{B}^*\underline{y} = L^*(\epsilon)\underline{y}(0) + R^*(\epsilon)\underline{y}(1) = \underline{g}^*(\epsilon)$$

where:

$$\underline{L}^*(\epsilon) = \begin{bmatrix} I & 0 & 0 \\ 0 & \epsilon I & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{R}^*(\epsilon) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon I \end{bmatrix} \quad \underline{g}^*(\epsilon) = \begin{bmatrix} \underline{u}^0(\epsilon) \\ \underline{v}^0(\epsilon) \\ \underline{w}^1(\epsilon) \end{bmatrix} \tag{7}$$

$\underline{y}, \underline{L}^*, \underline{R}^*, \underline{g}^* \ldots$ compatibly partitioned matrices and vectors.

We note the boundary-value problems (1) and (6) may differ only in the boundary conditions.

2.1 Formal Asymptotic Solution of the Special Problem

In this section we will derive, by the method of matched asymptotic expansions, a formal asymptotic solution of the special problem:

$$\begin{aligned} \mathcal{L}\, \underline{y}(x) &= \underline{f}(x,\epsilon) \\ \mathcal{B}^*\, \underline{y} &= \underline{g}^*(\epsilon) \end{aligned} \qquad x \in I \tag{8}$$

We assume the continuity conditions (3a) hold in addition to the eigenvalue condition (4). Before we apply the perturbation method let us define the following matrix and vector functions:

$$\begin{aligned}
a(x,\epsilon) &\equiv A_{11}(x,\epsilon) - A_{12}(x,\epsilon) A_{22}^{-1}(x,\epsilon) A_{21}(x,\epsilon) - A_{13}(x,\epsilon) A_{33}^{-1}(x,\epsilon) A_{31}(x,\epsilon) \\
\underline{F}(x,\epsilon) &\equiv \underline{f}_1(x,\epsilon) - A_{12}(x,\epsilon) A_{22}^{-1}(x,\epsilon) \underline{f}_2(x,\epsilon) - A_{13}(x,\epsilon) A_{33}^{-1}(x,\epsilon) \underline{f}_3(x,\epsilon) \\
A_{ij}(x) &\equiv A_{ij}(x,0) \qquad A_{ij}^L \equiv A_{ij}(0) \qquad A_{ij}^R \equiv A_{ij}(1) \\
\underline{f}_i(x) &\equiv \underline{f}_i(x,0) \qquad \underline{f}_i^L \equiv \underline{f}_i(0) \qquad \underline{f}_i^R \equiv \underline{f}_i(1) \\
a(x) &\equiv a(x,0) \qquad \underline{F}(x) = \underline{F}(x,0)
\end{aligned} \tag{9'}$$

$$\left.\begin{aligned} C_{22}^{L,R}(x) &\equiv x D_x A_{22}(x_0,0) + D_\epsilon A_{22}(x_0,0) \\ C_{33}^{L,R}(x) &\equiv x D_x A_{33}(x_0,0) + D_\epsilon A_{33}(x_0,0) \end{aligned}\right\} \quad \begin{aligned} x_0 &= 0 \text{ for } L \\ x_0 &= 1 \text{ for } R \end{aligned}$$

$$Y_1(x,\tau) \ldots \text{ F.S.M. for } a(x)$$

$$Y_2^{L,R}(x,\tau) \equiv \exp\{(x-\tau)A_{22}^{L,R}\} \ldots \text{ F.S.M. for } A_{22}^{L,R} \tag{9'}$$

$$Y_3^{L,R}(x,\tau) \equiv \exp\{(x-\tau)A_{33}^{L,R}\} \ldots \text{ F.S.M. for } A_{33}^{L,R}$$

$$Y(x) \equiv Y(x,0) \qquad Y_2^{L,R}(x) = Y_2^{L,R}(x,0) \qquad Y_3^{L,R}(x) = Y_3^{L,R}(x,0)$$

Since the eigenvalue condition (4) holds for the constant matrices $A_{22}^{L,R}$, $A_{33}^{L,R}$ we find the following:

<u>Theorem 2.10</u>: There exist positive constants C_0, Δ such that:

$$\begin{aligned} &\text{(a)} \quad \| Y_2^{L,R}(x,\tau) \| \le C_0 \exp\{-\Delta(x-\tau)\} \qquad 0 \le x-\tau \\ &\text{(b)} \quad \| Y_3^{L,R}(x,\tau) \| \le C_0 \exp\{-\Delta(\tau-x)\} \qquad 0 \le \tau-x \end{aligned} \tag{10}$$

The proof of this theorem may be found in chapter four.

We note the boundary conditions for the special problem may be written as:

$$\begin{bmatrix} \underline{u}(0) \\ \underline{v}(0) \\ \underline{w}(1) \end{bmatrix} = \begin{bmatrix} \underline{u}^0(\epsilon) \\ \frac{1}{\epsilon}\underline{v}^0(\epsilon) \\ \frac{1}{\epsilon}\underline{w}^1(\epsilon) \end{bmatrix} \sim \frac{1}{\epsilon}\begin{bmatrix} \underline{0} \\ \underline{v}_{-1}^0 \\ \underline{w}_{-1}^1 \end{bmatrix} + \begin{bmatrix} \underline{u}_0^0 \\ \underline{v}_0^0 \\ \underline{w}_0^1 \end{bmatrix} + O(\epsilon) \quad \epsilon \to 0^+ \tag{11}$$

We now apply the method of matched asymptotic expansions to derive a formal asymptotic solution of (8). Consistent with the expected behavior of the solution of problem (8), as discussed in section 1.1, we assume the following expansions are valid:

Outer Solution: In each closed subinterval of $(0, 1)$ we expect $\underline{u}, \underline{v}, \underline{w}$ to remain bounded as $\epsilon \to 0^+$. We try in this, the outer region:

$$\begin{bmatrix} \underline{u}(x) \\ \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} \sim \sum_{m\,0}^{\infty} \epsilon^m \begin{bmatrix} \underline{u}_m(x) \\ \underline{v}_m(x) \\ \underline{w}_m(x) \end{bmatrix} \quad \text{as } \epsilon \to 0^+ \tag{12}$$

Left Boundary Layer: (L. B. L.) Near the left boundary we expect $\underline{u}, \underline{v}, \underline{w}$ to undergo rapid transitions of an exponential nature from their values at $x = 0$ to their values for x near zero. In this region of rapid transition we expect $\underline{u}$ to remain bounded while $\underline{v}$ and/or $\underline{w}$ may blow up like ϵ^{-1} as $\epsilon \to 0^+$. We try in the L. B. L.:

$$\begin{bmatrix} \underline{u}(x) \\ \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} \sim \sum_{m\,-1}^{\infty} \epsilon^m \begin{bmatrix} \underline{u}_m^L(s) \\ \underline{v}_m^L(s) \\ \underline{w}_m^L(s) \end{bmatrix} \quad \text{as } \epsilon \to 0^+ \tag{13}$$

$$\underline{u}_{-1}^L(s) \equiv 0 \qquad s = \frac{x}{\epsilon} \qquad s \geqslant 0$$

Right Boundary Layer: (R. B. L.): Near the right boundary we expect $\underline{u}, \underline{v}, \underline{w}$ to undergo rapid transitions similar to those encountered in the L. B. L. We try in the R. B. L.:

$$\begin{bmatrix} \underline{u}(x) \\ \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} \sim \sum_{m\,-1}^{\infty} \epsilon^m \begin{bmatrix} \underline{u}_m^R(r) \\ \underline{v}_m^R(r) \\ \underline{w}_m^R(r) \end{bmatrix} \quad \text{as } \epsilon \to 0^+ \tag{14}$$

$$\underline{u}_{-1}^R(r) \equiv 0 \qquad r = \frac{x-1}{\epsilon} \qquad r \leqslant 0$$

Note that the three expansions (12, 13, 14) have the form of a power series in ϵ . The important difference between these expansions lies in the choice of the independent variable x, s, or r .

Once the expansions used in each region have been chosen, the procedure for recursively determining the unknown functions is straightforward. The steps one might follow are:

1. Choose one of the expansions (12, 13, 14).
2. Change the independent variable, if necessary, to the one used in the expansion.
3. Substitute the expansion into the differential equation, expand A and $\underline{f}$ with respect to ϵ , and collect like powers of ϵ .

After these steps have been performed, one is left with an equation of the form:

$$0 = \mathcal{L}\underline{y}(\cdot) - \underline{f}(\cdot,\epsilon) \sim \sum_m \epsilon^m P_m(\cdot) \quad \text{as } \epsilon \to 0^+ \qquad (15)$$

$$(\cdot) \ldots \; x, s, \text{ or } r$$

Since ϵ varies independently of x, s, or r, one then argues that (15) will hold for $\epsilon \to 0^+$ iff:

$$P_m(\cdot) = 0 \qquad \text{for every } m \qquad (16)$$

$$(\cdot) \ldots \; x, s, r$$

If we wish to find the solution in each region to order ϵ , we must solve the following sequence of problems:

Outer: $0 \le x \le 1$

$$P_0: \quad D_x \begin{bmatrix} \underline{u}_0(x) \\ \underline{0} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} A_{11}(x) & A_{12}(x) & A_{13}(x) \\ A_{21}(x) & A_{22}(x) & 0 \\ A_{31}(x) & 0 & A_{33}(x) \end{bmatrix} \begin{bmatrix} \underline{u}_0(x) \\ \underline{v}_0(x) \\ \underline{w}_0(x) \end{bmatrix} + \begin{bmatrix} \underline{f}_1(x) \\ \underline{f}_2(x) \\ \underline{f}_3(x) \end{bmatrix} \tag{17}$$

L. B. L.: $s \geq 0$

$$P_{-1}: \quad D_s \begin{bmatrix} \underline{v}_{-1}^L(s) \\ \underline{w}_{-1}^L(s) \end{bmatrix} = \begin{bmatrix} A_{22}^L & 0 \\ 0 & A_{22}^R \end{bmatrix} \begin{bmatrix} \underline{v}_{-1}^L(s) \\ \underline{w}_{-1}^L(s) \end{bmatrix}$$

$$P_0: \quad D_s \begin{bmatrix} \underline{u}_0^L(s) \\ \underline{v}_0^L(s) \\ \underline{w}_0^L(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ A_{21}^L & A_{22}^L & 0 \\ A_{31}^L & 0 & A_{33}^L \end{bmatrix} \begin{bmatrix} \underline{u}_0^L(s) \\ \underline{v}_0^L(s) \\ \underline{w}_0^L(s) \end{bmatrix} + \begin{bmatrix} A_{12}^L \underline{v}_{-1}^L(s) + A_{13}^L \underline{w}_{-1}^L(s) \\ C_{22}^L(s) \underline{v}_{-1}^L(s) + A_{23}^L \underline{w}_{-1}^L(s) + \underline{f}_2^L \\ A_{31}^L \underline{v}_{-1}^L(s) + C_{33}^L(s) \underline{w}_{-1}^L(s) + \underline{f}_3^L \end{bmatrix} \tag{18}$$

R. B. L.: $r \leq 0$

$$P_{-1}: \quad D_r \begin{bmatrix} \underline{v}_{-1}^R(r) \\ \underline{w}_{-1}^R(r) \end{bmatrix} = \begin{bmatrix} A_{22}^R & 0 \\ 0 & A_{33}^R \end{bmatrix} \begin{bmatrix} \underline{v}_{-1}^R(r) \\ \underline{w}_{-1}^R(r) \end{bmatrix}$$

$$P_0: \quad D_r \begin{bmatrix} \underline{u}_0^R(r) \\ \underline{v}_0^R(r) \\ \underline{w}_0^R(r) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ A_{21}^R & A_{22}^R & 0 \\ A_{31}^R & 0 & A_{33}^R \end{bmatrix} \begin{bmatrix} \underline{u}_0^R(r) \\ \underline{v}_0^R(r) \\ \underline{w}_0^R(r) \end{bmatrix} + \begin{bmatrix} A_{12}^R \underline{v}_{-1}^R(r) + A_{13}^R \underline{w}_{-1}^R(r) \\ C_{22}^R(r) \underline{v}_{-1}^R(r) + A_{23}^R \underline{w}_{-1}^R(r) + \underline{f}_2^R \\ A_{32}^R \underline{v}_{-1}^R(r) + C_{33}^R(r) \underline{w}_{-1}^R(r) + \underline{f}_3^R \end{bmatrix} \tag{19}$$

By using the V.O.P. formula (1.25) we may solve problems (17, 18, 19). Since the representations of the solution in (13, 14) are valid at the boundaries $x = 0, 1$, it is reasonable to demand these representations satisfy the appropiate boundary conditions listed in (11). We therefore deduce:

Outer: $0 < x < 1$

$$\begin{aligned} \underline{u}_0(x) &= Y_1(x)\, \underline{u}_0(0) + \int_0^x Y_1(x,\tau)\, \underline{F}(\tau)\, d\tau \\ \underline{v}_0(x) &= - A_{22}^{-1}(x) \left\{ A_{21}(x)\, \underline{u}_0(x) + \underline{f}_2(x) \right\} \end{aligned} \tag{20'}$$

$$\underline{w}_0(x) = -A_{33}^{-1}(x)\left\{ A_{31}(x)\,\underline{u}_0(x) + \underline{f}_3(x) \right\} \tag{20'}$$

L. B. L.: $s \geq 0$

$$\begin{aligned}
\underline{v}^L_{-1}(s) &= Y^L_2(s)\,\underline{v}^0_{-1} \qquad\qquad \underline{w}^L_{-1}(s) = Y^L_3(s)\,\underline{w}^L_{-1}(0) \\
\underline{u}^L_0(s) &= \underline{u}^0_0 + \int_0^s \left\{ A^L_{12}\,\underline{v}^L_{-1}(\tau) + A^L_{13}\,\underline{w}^L_{-1}(\tau) \right\} d\tau \\
\underline{v}^L_0(s) &= Y^L_2(s)\,\underline{v}^0_0 + \int_0^s Y^L_2(s,\tau)\left\{ A^L_{21}\,\underline{u}^L_0(\tau) + C^L_{22}(\tau)\,\underline{v}^L_{-1}(\tau) \right. \\
&\qquad \left. + A^L_{23}\,\underline{w}^L_{-1}(\tau) + \underline{f}^L_2 \right\} d\tau \\
\underline{w}^L_0(s) &= Y^L_3(s)\,\underline{w}^L_0(0) + \int_0^s Y^L_3(s,\tau)\left\{ A^L_{31}\,\underline{u}^L_0(\tau) + A^L_{32}\,\underline{v}^L_{-1}(\tau) \right. \\
&\qquad \left. + C^L_{33}(\tau)\,\underline{w}^L_{-1}(\tau) + \underline{f}^L_3 \right\} d\tau
\end{aligned} \tag{21}$$

R. B. L.: $r \leq 0$

$$\begin{aligned}
\underline{v}^R_{-1}(r) &= Y^R_2(r)\,\underline{v}^R_{-1}(0) \qquad\qquad \underline{w}^R_{-1}(r) = Y^R_3(r)\,\underline{w}^1_{-1} \\
\underline{u}^R_0(r) &= \underline{u}^R_0(0) + \int_0^r \left\{ A^R_{12}\,\underline{v}^R_{-1}(\tau) + A^R_{13}\,\underline{w}^R_{-1}(\tau) \right\} d\tau \\
\underline{v}^R_0(r) &= Y^R_2(r)\,\underline{v}^R_0(0) + \int_0^r Y^R_2(r,\tau)\left\{ A^R_{21}\,\underline{u}^R_0(\tau) + C^R_{22}(\tau)\,\underline{v}^R_{-1}(\tau) \right. \\
&\qquad \left. + A^R_{23}\,\underline{w}^R_{-1}(\tau) + \underline{f}^R_2 \right\} d\tau \\
\underline{w}^R_0(r) &= Y^R_3(r)\,\underline{w}^1_0 + \int_0^r Y^R_3(r,\tau)\left\{ A^R_{31}\,\underline{u}^R_0(\tau) + A^R_{32}\,\underline{v}^R_{-1}(\tau) \right. \\
&\qquad \left. + C^R_{33}(\tau)\,\underline{w}^R_{-1}(\tau) + \underline{f}^R_3 \right\} d\tau
\end{aligned} \tag{22}$$

The solutions determined in (20, 21, 22) involve the as yet undetermined constant vectors:

$$\underline{u}_0(0)\,,\ \underline{w}^L_{-1}(0)\,,\ \underline{w}^L_0(0)\,,\ \underline{v}^R_{-1}(0)\,,\ \underline{u}^R_0(0)\,,\ \underline{v}^R_0(0) \tag{23}$$

To determine these unknowns we employ the following elementary version of a matching principle:

Matching Principle:

$$\begin{aligned}
&\text{(a)} \quad \lim_{s\to\infty} \{\text{L.B.L. Solution}\} = \lim_{x\to 0} \{\text{Outer Solution}\} \\
&\text{(b)} \quad \lim_{r\to -\infty} \{\text{R.B.L. Solution}\} = \lim_{x\to 1} \{\text{Outer Solution}\}
\end{aligned} \qquad (24)$$

If we note:

$$Y_3^L(s,\tau) = Y_3^L(s)\, Y_3^L(0,\tau) \qquad \lim_{s\to\infty} Y_3^L(s) = \infty$$

$$Y_2^R(r,\tau) = Y_2^R(r)\, Y_2^R(0,\tau) \qquad \lim_{r\to -\infty} Y_2^R(r) = \infty$$

then the limits as $s\to\infty$ and $r\to-\infty$ occuring in (24) will exist iff:

$$\underline{w}_{-1}^L(0) = \underline{0} \qquad\qquad \underline{v}_{-1}^R(0) = \underline{0}$$

$$\underline{w}_0^L(0) + \int_0^{\infty} Y_3^L(0,\tau)\{A_{31}^L\, \underline{u}_0^L(\tau) + A_{32}^L\, \underline{v}_{-1}^L(\tau) + \underline{f}_3^L\}\,d\tau = \underline{0}$$

$$\underline{v}_0^R(0) + \int_0^{-\infty} Y_2^R(0,\tau)\{A_{21}^R\, \underline{u}_0^R(\tau) + A_{23}^R\, \underline{w}_{-1}^R(\tau) + \underline{f}_2^R\}\,d\tau = \underline{0}$$

These results lead to the cancellation of the terms $\nearrow$ indicated in (21, 22) and a further simplification of $\underline{w}_0^L$ and $\underline{v}_0^R$. Using these results we find (20, 21, 22) become:

Outer: $\quad 0 < x < 1$

$$\underline{u}_0(x) = Y_1(x)\, \underline{u}_0(0) + \int_0^x Y_1(x,\tau)\, \underline{F}(\tau)\,d\tau \qquad (25')$$

$$\underline{v}_0(x) = -A_{22}^{-1}(x)\{A_{21}(x)\, \underline{u}_0(x) + \underline{f}_2(x)\}$$

$$\underline{w}_0(x) = -A_{33}^{-1}(x)\left\{A_{31}(x)\underline{u}_0(x) + \underline{f}_3(x)\right\} \tag{25'}$$

L. B. L. : $\quad s \geq 0$

$$\begin{aligned}
\underline{v}_{-1}^L(s) &= Y_2^L(s)\,\underline{v}_{-1}^0 \qquad\qquad \underline{w}_{-1}^L(s) = 0 \\
\underline{u}_0^L(s) &= \underline{u}_0^0 + \int_0^s A_{12}^L \underline{v}_{-1}^L(\tau)\,d\tau \\
\underline{v}_0^L(s) &= Y_2^L(s)\,\underline{v}_0^0 + \int_0^s Y_2^L(s,\tau)\left\{A_{21}^L \underline{u}_0^L(\tau) + C_{22}^L(\tau)\underline{v}_{-1}^L(\tau) + \underline{f}_2^L\right\}d\tau \\
\underline{w}_0^L(s) &= -\int_s^\infty Y_3^L(s,\tau)\left\{A_{31}^L \underline{u}_0^L(\tau) + A_{32}^L \underline{v}_{-1}^L(\tau) + \underline{f}_3^L\right\}d\tau
\end{aligned} \tag{26}$$

R. B. L. : $\quad r \leq 0$

$$\begin{aligned}
\underline{v}_{-1}^R(r) &= 0 \qquad\qquad \underline{w}_{-1}^R(r) = Y_3^R(r)\,\underline{w}_{-1}^1 \\
\underline{u}_0^R(r) &= \underline{u}_0^R(0) + \int_0^r A_{13}^R \underline{w}_{-1}^R(\tau)\,d\tau \\
\underline{v}_0^R(r) &= \int_r^{-\infty} Y_2^R(r,\tau)\left\{A_{21}^R \underline{u}_0^R(\tau) + A_{23}^R \underline{w}_{-1}^R(\tau) + \underline{f}_2^R\right\}d\tau \\
\underline{w}_0^R(r) &= Y_3^R(r)\,\underline{w}_0^1 + \int_0^r Y_3^R(r,\tau)\left\{A_{31}^R \underline{u}_0^R(\tau) + C_{33}^R(\tau)\underline{w}_{-1}^R(\tau) + \underline{f}_3^R\right\}d\tau
\end{aligned} \tag{27}$$

Using the functions (25, 26, 27) one finds the matching principle (24) is satisfied iff:

$$\begin{aligned}
\underline{u}_0(0) &\equiv \underline{u}_0^0 + \int_0^\infty A_{12}^L \underline{v}_{-1}^L(\tau)\,d\tau = \underline{u}_0^0 - A_{12}^L A_{22}^{L^{-1}} \underline{v}_{-1}^0 \\
\underline{u}_0(1) &\equiv \underline{u}_0^R(0) + \int_0^{-\infty} A_{13}^R \underline{w}_{-1}^R(\tau)\,d\tau = \underline{u}_0^R(0) - A_{13}^R A_{33}^{R^{-1}} \underline{w}_{-1}^1
\end{aligned} \tag{28}$$

From the properties of the F. S. M. as described in section 1.2, and the relations (28) obtained from matching, we may partially integrate (26, 27) with the result:

$$\underline{u}_0^L(s) = \underline{u}_0(0) + A_{12}^L A_{22}^{L^{-1}} Y_2^L(s)\, \underline{v}_{-1}^0$$

$$\underline{v}_0^L(s) = \underline{v}_0(0) + Y_2^L(s)\left[\underline{v}_0^0 - \underline{v}_0(0)\right] + \int_0^s Y_2^L(s,\tau)\left\{ A_{21}^L A_{12}^L A_{22}^{L^{-1}} + C_{22}^L(\tau) \right\} Y_2^L(\tau)\, d\tau\, \underline{v}_{-1}^0$$

$$\underline{w}_0^L(s) = \underline{w}_0(0) - \int_s^{\infty} Y_3^R(s,\tau)\left\{ A_{31}^L A_{12}^L A_{22}^{L^{-1}} + A_{32}^L \right\} Y_2^R(\tau)\, d\tau\, \underline{v}_{-1}^0$$

$$\underline{u}_0^R(r) = \underline{u}_0(1) + A_{13}^R A_{33}^{R^{-1}} Y_3^R(r)\, \underline{w}_{-1}^1 \tag{29}$$

$$\underline{v}_0^R(r) = \underline{v}_0(1) - \int_r^{-\infty} Y_2^R(r,\tau)\left\{ A_{21}^R A_{13}^R A_{33}^{R^{-1}} + A_{23}^R \right\} Y_3^R(\tau)\, d\tau\, \underline{w}_{-1}^1$$

$$\underline{w}_0^R(r) = \underline{w}_0(1) + Y_3^R(r)\left[\underline{w}_0^1 - \underline{w}_0(1)\right] + \int_0^r Y_3^R(r,\tau)\left\{ A_{31}^R A_{13}^R A_{33}^{R^{-1}} + C_{33}^R(\tau) \right\} Y_3^R(\tau)\, d\tau\, \underline{w}_{-1}^1$$

Note that the first term on the right of each equality in (29) represents the common term shared by both the outer solution and each boundary layer solution. If the expansions (12, 13, 14) are added and these common terms subtracted, one may obtain the following composite expansion:

$$\underline{y}(x) \sim \hat{\underline{y}}(x) + O(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

$$\hat{\underline{y}}(x) \equiv \begin{bmatrix} \hat{\underline{u}}(x) \\ \hat{\underline{v}}(x) \\ \hat{\underline{w}}(x) \end{bmatrix} \tag{30}$$

where:

$$\hat{\underline{u}}(x) \equiv \underline{u}_0(x) + A_{12}^L A_{22}^{L^{-1}} Y_2^L(\tfrac{x}{\epsilon})\, \underline{v}_{-1}^0 + A_{13}^R A_{33}^{R^{-1}} Y_3^R(\tfrac{x-1}{\epsilon})\, \underline{w}_{-1}^1$$

$$\hat{\underline{v}}(x) \equiv \underline{v}_0(x) + \tfrac{1}{\epsilon} \mathcal{P}_1(\tfrac{x}{\epsilon}) \underline{v}_{-1}^0 + Y_2^L(\tfrac{x}{\epsilon}) \left[\underline{v}_0^0 - \underline{v}_0(1) \right] + \mathcal{P}_2(\tfrac{x-1}{\epsilon})\, \underline{w}_{-1}^1$$

$$\hat{\underline{w}}(x) \equiv \underline{w}_0(x) + \mathcal{P}_3(\tfrac{x}{\epsilon}) \underline{v}_{-1}^0 + \tfrac{1}{\epsilon} \mathcal{P}_4(\tfrac{x-1}{\epsilon})\, \underline{w}_{-1}^1 + Y_3^R(\tfrac{x-1}{\epsilon}) \left[\underline{w}_0^1 - \underline{w}_0(1) \right]$$

$$\underline{u}_0(x) \equiv Y_1(x)\, \underline{u}_0(0) + \int_0^x Y_1(x,\tau)\, \underline{F}(\tau)\, d\tau$$

$$\underline{v}_0(x) \equiv - A_{22}^{-1}(x) \left\{ A_{21}(x)\, \underline{u}_0(x) + \underline{f}_2(x) \right\}$$

$$\underline{w}_0(x) \equiv - A_{33}^{-1}(x) \left\{ A_{31}(x)\, \underline{u}_0(x) + \underline{f}_3(x) \right\} \tag{31}$$

$$\underline{u}_0(0) \equiv \underline{u}_0^0 - A_{12}^L A_{22}^{L^{-1}} \underline{v}_{-1}^0$$

$$\underline{u}_0(1) \equiv \underline{u}_0^R(0) - A_{13}^R A_{33}^{R^{-1}} \underline{w}_{-1}^1$$

$$\mathcal{P}_1(s) \equiv Y_2^L(s) + \epsilon \int_0^s Y_2^L(s,\tau) \left\{ A_{21}^L A_{12}^L A_{22}^{L^{-1}} + C_{22}^L(\tau) \right\} Y_2^L(\tau)\, d\tau$$

$$\mathcal{P}_2(r) \equiv - \int_r^{-\infty} Y_2^R(r,\tau) \left\{ A_{21}^R A_{13}^R A_{33}^{R^{-1}} + A_{23}^R \right\} Y_3^R(\tau)\, d\tau$$

$$\mathcal{P}_3(s) \equiv - \int_s^{\infty} Y_3^L(s,\tau) \left\{ A_{31}^L A_{12}^L A_{22}^{L^{-1}} + A_{32}^L \right\} Y_2^L(\tau)\, d\tau$$

$$\mathcal{P}_4(r) \equiv Y_3^R(r) + \epsilon \int_0^r Y_3^R(r,\tau) \left\{ A_{31}^R A_{13}^R A_{33}^{R^{-1}} + C_{33}^R(\tau) \right\} Y_3^R(\tau)\, d\tau$$

2.2 Existence of a Solution for the Special Problem

In this section we will prove the special problem (6), subject to the continuity conditions (3b) and eigenvalue condition (4), has a unique solution. Furthermore, we will derive an a priori bound which this unique solution satisfies. Before we state the theorem, let us define the following matrices and a norm on $\mathcal{C}^N[0,1]$:

$$a(x,\epsilon) = A_{11}(x,\epsilon) - A_{12}(x,\epsilon)A_{22}^{-1}(x,\epsilon)A_{21}(x,\epsilon) - A_{13}(x,\epsilon)A_{33}^{-1}(x,\epsilon)A_{31}(x,\epsilon)$$

$$\begin{array}{lll} Y_1(x,\tau) \ldots \text{ F.S.M. for } a(x,\epsilon) & & Y_1(x) = Y_1(x,0) \\ Y_2(x,\tau) \ldots \text{ F.S.M. for } \frac{1}{\epsilon}A_{22}(x,\epsilon) & & Y_2(x) = Y_2(x,0) \\ Y_3(x,\tau) \ldots \text{ F.S.M. for } \frac{1}{\epsilon}A_{33}(x,\epsilon) & & Y_3(x) = Y_3(x,1) \end{array} \tag{32}$$

$$\| \underline{f} \|_1 \equiv \int_0^1 \| \underline{f}(x) \| \, dx$$

Theorem 2.33: Under the continuity and eigenvalue conditions (3b, 4) the special boundary-value problem (6):

$$\mathcal{L}\underline{y}(x) = \underline{f}(x,\epsilon) \qquad \mathcal{B}^*\underline{y} = \underline{g}(\epsilon)$$

has, for all sufficiently small ϵ, a unique solution. Furthermore, for all sufficiently small ϵ, this unique solution satisfies the bound:

$$\begin{aligned} \|\underline{u}\|_\infty &\le C_1\{\|\underline{u}(0)\| + \epsilon\|\underline{v}(0)\| + \epsilon\|\underline{w}(1)\| + \|\underline{f}_1\|_1 + \|\underline{f}_2\|_\infty + \|\underline{f}_3\|_\infty\} \\ \|\underline{v}\|_\infty &\le C_1\{\|\underline{u}(0)\| + \|\underline{v}(0)\| + \epsilon\|\underline{w}(1)\| + \|\underline{f}_1\|_1 + \|\underline{f}_2\|_\infty + \|\underline{f}_3\|_\infty\} \\ \|\underline{w}\|_\infty &\le C_1\{\|\underline{u}(0)\| + \epsilon\|\underline{v}(0)\| + \|\underline{w}(1)\| + \|\underline{f}_1\|_1 + \|\underline{f}_2\|_\infty + \|\underline{f}_3\|_\infty\} \end{aligned} \tag{33}$$

Here C_1 is a positive constant independent of ϵ.

Proof: Consider the differential equation $\mathcal{L}\underline{y} = \underline{f}$. Integrate the equation for $\underline{u}$ from $x = 0$ and apply the V.O.P. formula (1.25) with $s=0$ ($s=1$) to integrate the equation for $\underline{v}$ ($\underline{w}$). The result may be written as:

$$\Theta \underset{\sim}{y} = \underline{H} \tag{34}$$

$$\Theta = \Theta_1 - \epsilon K_4 = \begin{bmatrix} I - K_0 A_{11} & -K_0 A_{12} & -K_0 A_{13} \\ -K_2 A_{21} & I_{m_1} & 0 \\ -K_3 A_{31} & 0 & I_{m_2} \end{bmatrix} - \epsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K_2 A_{23} \\ 0 & K_3 A_{32} & 0 \end{bmatrix}$$

$$\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \\ \underset{\sim}{w} \end{bmatrix} \qquad \underline{H} = \begin{bmatrix} \underset{\sim}{u}(0) + K_0 \underset{\sim}{f}_1 \\ Y_2 \cdot \underset{\sim}{v}(0) + K_2 \underset{\sim}{f}_2 \\ Y_3 \cdot \underset{\sim}{w}(1) + K_3 \underset{\sim}{f}_3 \end{bmatrix}$$

$$K_0 \underset{\sim}{z}(x) \equiv \int_0^x \underset{\sim}{z}(\tau)\, d\tau \qquad \underset{\sim}{z} \in \mathcal{C}^N[0,1] \tag{35}$$

$$K_1 \underset{\sim}{z}(x) \equiv \int_0^x Y_1(x,\tau)\, \underset{\sim}{z}(\tau)\, d\tau \qquad Y_i \underset{\sim}{z}(x) \equiv Y_i(x)\, \underset{\sim}{z}(x)$$

$$K_2 \underset{\sim}{z}(x) \equiv \frac{1}{\epsilon} \int_0^x Y_2(x,\tau)\, \underset{\sim}{z}(\tau)\, d\tau \qquad A_{ij} \underset{\sim}{z}(x) \equiv A_{ij}(x,\epsilon)\, \underset{\sim}{z}(x)$$

$$K_3 \underset{\sim}{z}(x) \equiv \frac{1}{\epsilon} \int_1^x Y_3(x,\tau)\, \underset{\sim}{z}(\tau)\, d\tau \qquad \mathcal{a}\, \underset{\sim}{z}(x) \equiv \mathcal{a}(x,\epsilon)\, \underset{\sim}{z}(x)$$

I_N . . . the identity operator in $\mathcal{C}^N[0,1]$

By using the continuity conditions (3b) and the exponential dichotomy (5), one may prove the linear operators K_i, Y_i , A_{ij} , $\mathcal{a}$ are bounded. These bounds, as well as those which follow, may be chosen to hold uniformly in ϵ for all ϵ sufficiently small.

Note that for suitable matrices α , β , γ :

$$\begin{bmatrix} \alpha & -\gamma \\ -\beta & I \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I \\ \beta \end{bmatrix} Q^{-1} [\, I \;\; \gamma \,] \tag{36}$$

$$Q = \alpha - \gamma\beta$$

Therefore, by comparing the forms (35, 36) one deduces the inverse of θ_1 (if it exists) has the form:

$$\theta_1^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{m_1} & 0 \\ 0 & 0 & I_{m_2} \end{bmatrix} + \begin{bmatrix} I_m \\ K_2 A_{21} \\ K_3 A_{31} \end{bmatrix} \mathcal{Q}^{-1} \begin{bmatrix} I & K_0 A_{12} & K_0 A_{13} \end{bmatrix} \tag{37}$$

$$\mathcal{Q} = I - K_0 A_{11} - K_0 A_{12} K_2 A_{21} - K_0 A_{13} K_3 A_{31}$$

For some ϵ_3 in $(0, \epsilon_2]$ we will prove:

$$\mathcal{Q}^{-1} = (I + \epsilon K_8)(I + K_1 \mathcal{a}) \qquad 0 < \epsilon \leq \epsilon_3 \tag{38}$$

where K_8 is a bounded linear operator. Therefore the identity (37) will be valid, and θ_1^{-1} will be a bounded linear operator for $0 < \epsilon \leq \epsilon_3$. To establish the identity (38) we first prove:

$$\left.\begin{aligned} K_0 A_{12} K_2 A_{21} &= -K_0 A_{12} A_{22}^{-1} A_{21} + \epsilon K_5 \\ K_0 A_{13} K_3 A_{31} &= -K_0 A_{13} A_{33}^{-1} A_{31} + \epsilon K_6 \end{aligned}\right\} \quad 0 < \epsilon \leq \epsilon_2 \tag{39}$$

where K_5, K_6 are bounded linear operators.

Let us establish in detail the second identity:

$$K_0 A_{13} K_3 A_{31} \, \underline{z}(x) \equiv \int_0^x A_{13}(s,\epsilon) \cdot \frac{1}{\epsilon} \int_1^s Y_3(s,\tau) A_{13}(\tau,\epsilon) \, \underline{z}(\tau) \, d\tau \, ds$$

$$= -\frac{1}{\epsilon} \int_0^x A_{13}(s,\epsilon) \int_s^1 Y_3(s,\tau) A_{13}(\tau,\epsilon) \, \underline{z}(\tau) \, d\tau \, ds$$

... split the τ integral.

(39)

$$= -\frac{1}{\epsilon}\int_0^x A_{13}(s,\epsilon)\int_s^x Y_3(s,\tau)A_{13}(\tau,\epsilon)\underline{z}(\tau)\,d\tau\,ds + \bar{K}_6\underline{z}(x)$$

... change the order of integration.

$$= -\frac{1}{\epsilon}\int_0^x\int_0^\tau A_{13}(s,\epsilon)Y_3(s,\tau)ds\; A_{13}(\tau,\epsilon)\underline{z}(\tau)d\tau + \bar{K}_6\underline{z}(x)$$

... $\frac{1}{\epsilon}Y_3(s,\tau) = A_{33}^{-1}(s,\epsilon)D_sY_3(s,\tau)$, integrate the s integral by parts.

$$= -\int_0^x A_{13}(\tau,\epsilon)A_{33}^{-1}(\tau,\epsilon)A_{31}(\tau,\epsilon)\underline{z}(\tau)d\tau + \epsilon K_6\underline{z}(x)$$

$$= -K_0A_{13}A_{33}^{-1}A_{31}\underline{z}(x) + \epsilon K_6\underline{z}(x)$$

Note that for $0<\epsilon\le\epsilon_2$:

$$\bar{K}_6\underline{z}(x) \equiv -\frac{1}{\epsilon}\int_0^x A_{13}(s,\epsilon)\int_x^1 Y_3(s,\tau)A_{31}(\tau,\epsilon)\underline{z}(\tau)\,d\tau\,ds$$

$$K_6\underline{z}(x) \equiv \frac{1}{\epsilon}\bar{K}_6\underline{z}(x) + \frac{1}{\epsilon}\int_0^x\Big[A_{13}(0,\epsilon)A_{33}^{-1}(0,\epsilon)Y_3(0,\tau) + \int_0^\tau D_s\{A_{13}(s,\epsilon)A_{33}^{-1}(s,\epsilon)\}Y_3(s,\tau)\,ds\Big]A_{31}(\tau,\epsilon)\underline{z}(\tau)d\tau$$

By the continuity conditions (3b) and the exponential dichotomy (5) satisfied by Y_3 we find K_6 is a bounded linear operator for $0<\epsilon\le\epsilon_2$. To establish the first identity in (39) we note:

$$K_0A_{12}K_2A_{21}\underline{z}(x) = \int_0^x A_{12}(s,\epsilon)\cdot\frac{1}{\epsilon}\int_0^s Y_2(s,\tau)A_{21}(\tau,\epsilon)\underline{z}(\tau)\,d\tau\,ds$$

... change the order of integration.

$$= \frac{1}{\epsilon}\int_0^x \int_\tau^x A_{12}(s,\epsilon)\, Y_2(s,\tau)\, ds\, A_{21}(\tau,\epsilon)\, \underline{z}(\tau)\, d\tau$$

$\ldots \quad \frac{1}{\epsilon} Y_2(s,\tau) = A_{22}^{-1}(s,\epsilon)\, D_s Y_2(s,\tau)$, integrate the s integral by parts.

$$= -\int_0^x A_{12}(\tau,\epsilon)\, A_{22}^{-1}(\tau,\epsilon)\, A_{21}(\tau,\epsilon)\, \underline{z}(\tau)\, d\tau + \epsilon K_5\, \underline{z}(x)$$

$$= -K_0 A_{12} A_{22}^{-1} A_{21}\, \underline{z}(x) + \epsilon K_5\, \underline{z}(x)$$

where:

$$K_5\, \underline{z}(x) \equiv \frac{1}{\epsilon}\int_0^x \Big[A_{12}(x,\epsilon)\, A_{22}^{-1}(x,\epsilon)\, Y_2(x,\tau) - \int_\tau^x D_s\{ A_{21}(s,\epsilon)\, A_{22}^{-1}(s,\epsilon)\}\, Y_2(s,\tau)\, ds \Big] A_{21}(\tau,\epsilon)\, \underline{z}(\tau)\, d\tau$$

From the continuity conditions (3b) and the exponential dichotomy satisfied by Y_2 we find K_5 is a bounded linear operator for $0<\epsilon\le\epsilon_2$. By using the identities (39) we may write:

$$\mathcal{Q} = I - K_0 \mathcal{a} - \epsilon K_7$$
$$K_7 \equiv K_5 + K_6 \tag{40}$$

where, for $0<\epsilon\le\epsilon_2$, K_7 is a bounded linear operator. Note that:

$$K_1\, \mathcal{a}\, K_0 = K_1 - K_0 \tag{41}$$

because:

$$K_1 a K_0 \underline{z}(x) = \int_0^x Y_1(x,s)\, a(s,\epsilon) \int_0^s \underline{z}(\tau)\, d\tau\, ds$$

... change the order of integration.

$$= \int_0^x \int_\tau^x Y_1(x,s)\, a(s,\epsilon)\, ds\, \underline{z}(\tau)\, d\tau$$

... use the identity (1.23d).

$$= \int_0^x \{ Y_1(x,\tau) - I \} \underline{z}(\tau)\, d\tau$$

$$= K_1 \underline{z}(x) - K_0 \underline{z}(x)$$

Therefore , we find:

$$(I + K_1 a)(I - K_0 a) = I + K_1 a - K_0 a - K_1 a K_0 a = I$$

from which we deduce:

$$(I - K_0 a)^{-1} = I + K_1 a \tag{42}$$

From (35, 40) we have:

$$\begin{aligned} \theta &= \theta_1 [I - \epsilon\, \theta_1^{-1} K_4] \\ Q &= (I - K_0 a)[I - \epsilon (I + K_1 a) K_7] \end{aligned} \tag{43}$$

If we define:

$$\begin{aligned} \epsilon_3 &= \tfrac{1}{2} \min \{ \| [I + K_1 a] K_7 \|_\infty^{-1},\ \| \theta_1^{-1} K_4 \|_\infty^{-1},\ 2\epsilon_2 \} \\ K_8 &= (I - \epsilon [I + K_1 a] K_7)^{-1} [I + K_1 a] K_7 \\ K_9 &= (I - \epsilon\, \theta_1^{-1} K_4)\, \theta_1^{-1} K_4 \end{aligned} \tag{44}$$

then by the Banach Lemma (1.20) :

$$\begin{aligned} \| K_8 \|_\infty &\le 2 \| [I + K_1 a] K_7 \|_\infty \\ \| K_9 \|_\infty &\le 2 \| \theta_1^{-1} K_4 \|_\infty \end{aligned} \qquad 0 < \epsilon \le \epsilon_3$$

Therefore, we conclude the operators K_8, K_9 are bounded independently of ϵ for $0 < \epsilon \le \epsilon_3$. By an application of Theorem (1.21) we deduce from (37, 43, 44):

$$\begin{aligned} (I - \epsilon \theta_1^{-1} K_4)^{-1} &= I + \epsilon K_9 \\ (I - \epsilon [I + K_1 a] K_7)^{-1} &= I + \epsilon K_8 \\ \theta^{-1} &= (I + \epsilon K_9) \theta_1^{-1} \\ \tilde{Q}^{-1} &= (I + \epsilon K_8)(I + K_1 a) \end{aligned} \qquad 0 < \epsilon \le \epsilon_3 \qquad (45)$$

If we collect the results contained in (34, 35, 37, 45) we find for some bounded linear operators K_8, K_9 :

$$\begin{aligned} \underline{S} &= \theta^{-1} \underline{H} \\ \theta^{-1} &= (I + \epsilon K_9) \theta_1^{-1} \\ \theta_1^{-1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{m_1} & 0 \\ 0 & 0 & I_{m_2} \end{bmatrix} + \begin{bmatrix} I \\ K_2 A_{21} \\ K_3 A_{31} \end{bmatrix} \tilde{Q}^{-1} \begin{bmatrix} I & K_0 A_{12} & K_0 A_{13} \end{bmatrix} \\ \tilde{Q}^{-1} &= (I + \epsilon K_8)(I + K_1 a) \end{aligned} \qquad 0 < \epsilon \le \epsilon_3 \qquad (46)$$

From (46) we deduce the result that the special boundary-value problem has, for all sufficiently small ϵ (i.e. for $0<\epsilon\leq\epsilon_3$), a unique solution. The derivation of the a priori bound also follows from our knowledge of the detailed form of θ^{-1}. To derive this a priori bound let us note the identities:

$$(I + K_1 a) K_0 = K_1$$
$$(I + K_1 a)\cdot \underline{u}(0) = Y_1 \cdot \underline{u}(0) \tag{47}$$

The first identity follows from (41), while the second identity may be deduced from (1.23d). As a result we find:

$$\mathbb{Q}^{-1}[I \;\; K_0A_{12} \;\; K_0A_{13}]\,\underline{H} = [I+\epsilon K_8][Y_1\cdot\underline{u}(0) + \tag{48}$$
$$K_1\underline{f}_1 + K_1A_{12}(Y_2\cdot\underline{v}(0) + K_2\underline{f}_2) + K_1A_{13}(Y_3\cdot\underline{w}(1) + K_3\underline{f}_3)]$$

From the continuity conditions (3b) and the exponential dichotomy (5) satisfied by Y_2, Y_3, we deduce from (48):

$$\|\mathbb{Q}^{-1}[I \;\; K_0A_{12} \;\; K_0A_{13}]\,\underline{H}\,\|_\infty \leq C_2[\,|\underline{u}(0)| + \epsilon\,|\underline{v}(0)| + \tag{49}$$
$$\epsilon\,|\underline{w}(1)| + \|\underline{f}_1\|_1 + \|\underline{f}_2\|_\infty + \|\underline{f}_3\|_\infty\,]$$

for some constant C_2 and all ϵ in $(0,\epsilon_3]$. Combining this bound with the result contained in (46) we obtain the a priori bound (33), valid for all $0<\epsilon\leq\epsilon_3$. ##

2.3 Asymptotic Solution of the Special Problem

In section 2.1 we derived a formal asymptotic solution (30, 31) of the special problem (6). In section 2.2 we deduced that the special problem (6) had, for all sufficiently small ϵ, a unique solution. This unique solution also satisfied the a priori bound (33). We will now use this a priori bound to rigorously jusitfy (30) as an asymptotic solution of the special problem.

Corollary 1.50: Let the continuity conditions (3a) and the eigenvalue condition (4) hold for the special problem (6). Then, for all sufficiently small ϵ, the unique solution of the special problem has the asymptotic expansion (30, 31). (50)

Proof: The steps we perform in the proof of this corollary are the same as those presented in section 1.3, where we justified (1.40). From Theorem 2.33 we know, for all sufficiently small ϵ, the special problem (6) has a unique solution $\underset{\sim}{y}(x)$. Let $\hat{\underset{\sim}{y}}(x)$ be the formal asymptotic solution (30, 31) of the special problem. Define:

$$\underline{e}(x) \equiv \hat{\underset{\sim}{y}}(x) - \underset{\sim}{y}(x)$$

$$\underline{e}(x) = \begin{bmatrix} \underline{u}^e(x) \\ \underline{v}^e(x) \\ \underline{w}^e(x) \end{bmatrix}$$

We immediately find that:

$$\mathcal{B}^{*}\underline{e} = \begin{bmatrix} \underline{u}^{e}(0) \\ \underline{v}^{e}(0) \\ \underline{w}^{e}(1) \end{bmatrix} \sim \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^{+} \tag{51}$$

Let us define:

$$\underline{f}^{e}(x,\epsilon) \equiv \mathcal{L}\hat{\underline{y}}(x) - \underline{f}(x,\epsilon)$$

$$\underline{f}^{e}(x,\epsilon) = \begin{bmatrix} \underline{f}^{e}_{1}(x,\epsilon) \\ \underline{f}^{e}_{2}(x,\epsilon) \\ \underline{f}^{e}_{3}(x,\epsilon) \end{bmatrix}$$

and note:

$$\mathcal{L}\underline{e}(x) = \underline{f}^{e}(x,\epsilon)$$

By using (31) we find:

$$\begin{aligned}
\underline{f}^{e}_{1}(x,\epsilon) = {} & \overset{\textcircled{1}}{D\underline{u}_{0}(x)} + \frac{1}{\epsilon} A^{L}_{12} \overset{\textcircled{3}}{Y^{L}_{2}\left(\frac{x}{\epsilon}\right)} \underline{v}^{0}_{-1} + \frac{1}{\epsilon} A^{R}_{13} \overset{\textcircled{4}}{Y^{R}_{3}\left(\frac{x-1}{\epsilon}\right)} \underline{w}^{1}_{-1} \\
& - A_{11}(x,\epsilon)\left\{ \overset{\textcircled{1}}{\underline{u}_{0}(x)} + A^{L}_{12} A^{L^{-1}}_{22} \overset{\textcircled{2}}{Y^{L}_{2}\left(\frac{x}{\epsilon}\right)} \underline{v}^{0}_{-1} + A^{R}_{13} A^{R^{-1}}_{33} \overset{\textcircled{2}}{Y^{R}_{3}\left(\frac{x-1}{\epsilon}\right)} \underline{w}^{1}_{-1} \right\} \\
& - A_{12}(x,\epsilon)\left\{ \overset{\textcircled{1}}{\underline{v}_{0}(x)} + \frac{1}{\epsilon} \overset{\textcircled{3}}{P_{1}\left(\frac{x}{\epsilon}\right)} \underline{v}^{0}_{-1} + Y^{L}_{2}\left(\frac{x}{\epsilon}\right) \overset{\textcircled{2}}{\left[\underline{v}^{0}_{0} - \underline{v}_{0}(0)\right]} + \overset{\textcircled{2}}{P_{2}\left(\frac{x-1}{\epsilon}\right)} \underline{w}^{1}_{-1} \right\} \\
& - A_{13}(x,\epsilon)\left\{ \overset{\textcircled{1}}{\underline{w}_{0}(x)} + \overset{\textcircled{2}}{P_{3}\left(\frac{x}{\epsilon}\right)} \underline{v}^{0}_{-1} + \frac{1}{\epsilon} \overset{\textcircled{4}}{P_{4}\left(\frac{x-1}{\epsilon}\right)} \underline{w}^{1}_{-1} + Y^{R}_{3}\left(\frac{x-1}{\epsilon}\right) \overset{\textcircled{2}}{\left[\underline{w}^{1}_{0} - \underline{w}_{0}(1)\right]} \right\} \\
& - \overset{\textcircled{1}}{\underline{f}_{1}(x,\epsilon)}
\end{aligned}$$

If we note:

① ... from the continuity properties of A_{1j}, $\underline{f}_{1}$ with respect to ϵ, and the relations between $\underline{u}_{0}$, $\underline{v}_{0}$, and $\underline{w}_{0}$, these terms cancel to order ϵ.

(2) ... due to the exponential dichotomy $Y_2^{L,R}$ and $Y_3^{L,R}$ satisfy, these terms each make at most an order ϵ contribution to the 1-norm of $\underline{f}_1^e$.

(3) ... from the continuity properties and exponential dichotomies:

$$\left\| \frac{1}{\epsilon}\left(A_{12}^L - A_{12}(x,\epsilon)\right) Y_2^L\left(\frac{x}{\epsilon}\right) \underline{v}_{-1}^0 \right\|_1 \sim \mathcal{O}\left(\frac{1}{\epsilon}\int_0^1 (x+\epsilon)\exp\left(-\frac{\Delta}{\epsilon}x\right) dx\right) \sim \mathcal{O}(\epsilon)$$

$$\left\| \int_0^{\frac{x}{\epsilon}} Y_2^L\left(\frac{x}{\epsilon},\tau\right)\left\{ A_{21}^L A_{12}^L A_{22}^{L^{-1}} + C_{22}^L(\tau)\right\} Y_2^L(\tau)\, d\tau\, \underline{v}_{-1}^0 \right\|_1 \sim$$

$$\mathcal{O}\left(\int_0^1 \exp\left(-\frac{\Delta}{\epsilon}x\right)\int_0^{\frac{x}{\epsilon}} (1+\tau)\, d\tau\, dx\right) \sim \mathcal{O}(\epsilon)$$

and so the 1-norm of the difference of the terms (3) is of order ϵ .

(4) ... by reasoning analogous to that used in (3) above, the 1-norm of the difference of the terms (4) is of order ϵ .

then we deduce:

$$\|\underline{f}_1^e\|_1 \sim \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{52}$$

Next, consider $\underline{f}_2^e$. We find from (31):

$$\begin{aligned}
\underline{f}_2^e(x,\epsilon) = {} & \overset{(1)}{\epsilon D \underline{v}_0(x)} + \frac{1}{\epsilon}\overset{(3)}{P_1'\left(\frac{x}{\epsilon}\right)\underline{v}_{-1}^0} + \overset{(4)}{A_{22}^L Y_2^L\left(\frac{x}{\epsilon}\right)\left[\underline{v}_0^0 - \underline{v}_0(0)\right]} + \overset{(5)}{P_2'\left(\frac{x-1}{\epsilon}\right)\underline{w}_{-1}^1} \\
& - A_{21}(x,\epsilon)\left\{ \overset{(2)}{\underline{u}_0(x)} + \overset{(3)}{A_{12}^L A_{22}^{L^{-1}} Y_2^L\left(\frac{x}{\epsilon}\right)\underline{v}_{-1}^0} + \overset{(5)}{A_{13}^R A_{33}^{R^{-1}} Y_3^R\left(\frac{x-1}{\epsilon}\right)\underline{w}_{-1}^1} \right\} \\
& - A_{22}(x,\epsilon)\left\{ \overset{(2)}{\underline{v}_0(x)} + \overset{(3)}{\frac{1}{\epsilon}P_1\left(\frac{x}{\epsilon}\right)\underline{v}_{-1}^0} + \overset{(4)}{Y_2^L\left(\frac{x}{\epsilon}\right)\left[\underline{v}_0^0 - \underline{v}_0(0)\right]} + \overset{(5)}{P_2\left(\frac{x-1}{\epsilon}\right)\underline{w}_{-1}^1} \right\} \\
& - \epsilon A_{23}(x,\epsilon)\left\{ \overset{(1)}{\underline{w}_0(x)} + \overset{(1)}{P_3\left(\frac{x}{\epsilon}\right)\underline{v}_{-1}^0} + \overset{(5)}{\frac{1}{\epsilon}P_4\left(\frac{x-1}{\epsilon}\right)\underline{w}_{-1}^1} + \overset{(1)}{Y_3^R\left(\frac{x-1}{\epsilon}\right)\left[\underline{w}_0^1 - \underline{w}_0(1)\right]} \right\} \\
& - \overset{(2)}{\underline{f}_2(x,\epsilon)}
\end{aligned}$$

If we note:

① ... each of these terms is of order ϵ .

② ... from the continuity properties of A_{21} , $\underline{f}_2$ with respect to ϵ and the relation between $\underline{u}_0$, $\underline{v}_0$, these terms cancel to order ϵ .

③ ... from (31):

$$\mathcal{P}_1'(s) = A_{22}^L \mathcal{P}_1(s) + \epsilon \{ A_{21}^L A_{12}^L A_{22}^{L^{-1}} + C_{22}^L(s) \} Y_2^L(s)$$

the continuity properties of A_{22}, A_{21} :

$$\left.\begin{aligned} A_{22}(x,\epsilon) &\sim A_{22}^L + \epsilon C_{22}^L(\tfrac{x}{\epsilon}) + \mathcal{O}(x^2 + \epsilon x + \epsilon^2) \\ A_{21}(x,\epsilon) &\sim A_{21}^L + \mathcal{O}(x+\epsilon) \end{aligned}\right\} \quad \text{as } \epsilon \to 0$$

and the estimate:

$$x^n \exp\{-\tfrac{\Delta}{\epsilon} x\} \sim \mathcal{O}(\epsilon^n) \qquad \text{for } x \geq 0 \text{ as } \epsilon \to 0^+ \tag{53}$$

we deduce the terms ③ cancel to order ϵ .

④ ... from the continuity properties of A_{22} and the estimate (53) the terms ④ cancel to order ϵ .

⑤ ... from (31):

$$\mathcal{P}_2'(r) = A_{22}^R \mathcal{P}_2(r) + \{ A_{21}^R A_{13}^R A_{33}^{R^{-1}} + A_{23}^R \} Y_3^R(r)$$

$$\mathcal{P}_4(r) \sim Y_3^R(r) + \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

the continuity properties of A_{2j} :

$$A_{2j}(x,\epsilon) \sim A_{2j}^R + \mathcal{O}((1-x) + \epsilon) \qquad \text{as } \epsilon \to 0$$

and the estimate:

$$(1-x)^m \exp\left\{-\frac{\Delta}{\epsilon}(1-x)\right\} \sim \mathcal{O}(\epsilon^m) \qquad \text{for } x \leq 1 \text{ as } \epsilon \to 0^+$$

we deduce the terms ⑤ cancel to order ϵ.

Therefore, we find:

$$\| \underline{f}_2^e \|_\infty \sim \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{54}$$

In a manner analogous to the argument yielding (54) we estimate:

$$\| \underline{f}_3^e \|_\infty \sim \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{55}$$

By using the estimates (51, 52, 54, 55) in the a priori bound (33) satisfied by $\underline{e}$ we deduce, for all sufficiently small ϵ :

$$\| \underline{e} \|_\infty \sim \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{56}$$

The estimate (56) is the rigorous justification of the error estimate made in (30), in other words $\underline{y}$ and $\hat{\underline{y}}$ agree to order ϵ , uniformly for $x \in [0, 1]$, as $\epsilon \to 0^+$. ##

2.4 Fundamental Matrices of the Special Problem

In this section we define two fundamental matrices related to the special problem (6).

Definition: Let $Z_0(x)$ be that matrix which satisfies the boundary-value problem:

$$\mathcal{L} Z_0(x) = 0 \qquad \mathcal{B}^* Z_0 = I \tag{57}$$

Let $Z_1(x)$ be the matrix defined by:

$$Z_1(x) = Z_0(x)\, T_0 \tag{58}$$

where:

$$T_0 = \begin{bmatrix} I & A_{12}^L A_{22}^{L^{-1}} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{59}$$

Under the continuity conditions (3b) and the eigenvalue condition (4) we find Theorems 1.28 and 2.33 guarantee the existence and uniqueness of the matrix Z_0 for all sufficiently small ϵ. Of great interest to us will be the values Z_0, Z_1 assumes at $x = 0, 1$.

Corollary 1.60: Under the continuity conditions (3a) and the eigenvalue condition (4) the matrix Z_0 admits the asymptotic expansion:

$$Z_0(x) \sim \hat{Z}_0(x) + \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

$$\hat{Z}_0(0) = \begin{bmatrix} I & 0 & 0 \\ 0 & \frac{1}{\epsilon} I & 0 \\ -A_{33}^{L^{-1}} A_{31}^L & \mathcal{P}_5 & 0 \end{bmatrix} \tag{60'}$$

$$\hat{Z}_0(1) = \begin{bmatrix} Y_1(1) & -Y_1(1)A_{12}^{L}A_{22}^{L^{-1}} & A_{13}^{R}A_{33}^{R^{-1}} \\ -A_{22}^{R^{-1}}A_{21}^{R}Y_1(1) & A_{22}^{R^{-1}}A_{21}^{R}Y_1(1)A_{12}^{L}A_{22}^{L^{-1}} & \mathcal{P}_2 \\ 0 & 0 & \frac{1}{\epsilon}I \end{bmatrix} \qquad (60')$$

where:

$$\mathcal{P}_5 = \mathcal{P}_3(0) + A_{33}^{L^{-1}}A_{31}^{L}A_{12}^{L}A_{22}^{L^{-1}} \qquad \mathcal{P}_2 = \mathcal{P}_2(0)$$

Furthermore, we deduce:

$$Z_1(x) \sim \hat{Z}_1(x) + \theta(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

$$\hat{Z}_1(0) = \begin{bmatrix} I & A_{12}^{L}A_{22}^{L^{-1}} & 0 \\ 0 & \frac{1}{\epsilon}I & 0 \\ -A_{33}^{L^{-1}}A_{31}^{L} & \mathcal{P}_3(0) & 0 \end{bmatrix} \qquad (61)$$

$$\hat{Z}_1(1) = \begin{bmatrix} Y_1(1) & 0 & A_{13}^{R}A_{33}^{R^{-1}} \\ -A_{22}^{R^{-1}}A_{21}^{R}Y_1(1) & 0 & \mathcal{P}_2(0) \\ 0 & 0 & \frac{1}{\epsilon}I \end{bmatrix}$$

Proof: Let $Z_0^{(i)}(x)$ and e_i represent the i^{th} column of Z_0 and I respectively. We recognize $Z_0^{(i)}(x)$ satisfies the special problem:

$$\mathcal{L}\, Z_0^{(i)}(x) = 0 \qquad \mathcal{B}^* Z_0^{(i)} = e_i$$

This problem has, for all ϵ sufficiently small, a unique solution. Furthermore, from Corollary 1.50 we deduce $Z_0^{(i)}(x)$ admits an asymptotic expansion obtained from (30, 31). These asymptotic expansions lead directly to (60). By multiplying through by the matrix T_0, defined in (59), we also deduce the form given for Z_1. ##

Due to the improtant role the matrices Z_0, Z_1 play in the determination of the existence of solutions of the general problem

(1), we shall call $\mathcal{Z}_0, \mathcal{Z}_1$ the fundamental matrices associated with the special problem (6).

2.5 Existence of a Solution of the General Problem

A direct application of Theorems 1.28 and 2.33 is the following:

Theorem 2.62: Consider the general problem (1) subject to the continuity and eigenvalue conditions (3b, 4). Let $\mathcal{Z}_0, \mathcal{Z}_1$ be the fundamental matrices of the special problem (6). Then the general problem has a unique solution, for arbitrary $\underset{\sim}{f}, \underset{\sim}{g}$ and all ϵ sufficiently small, iff the matrix $\mathcal{B}\mathcal{Z}_0$ ($\mathcal{B}\mathcal{Z}_1$) is nonsingular. (62)

Proof: Under the continuity and eigenvalue conditions (3b, 4) we know, by Theorem 2.33, that the special problem has a unique solution for arbitrary $\underset{\sim}{f}, \underset{\sim}{g}$ and all sufficiently small ϵ. Since T_D, as defined in (59), is nonsingular we deduce the conclusion of this theorem by applying to Theorem 1.28. ##

Unfortunately, we do not have enough information to calculate either $\mathcal{B}\mathcal{Z}_0$ or $\mathcal{B}\mathcal{Z}_1$, so Theorem 2.62 is not immediately applicable. Before we apply Theorem 2.62 let us make the following:

Definition: The general problem (1) is said to be regular iff:

(a) the continuity and eigenvalue conditions (3a, 4) hold. (63')

(b) the matrix $B_0 = \lim_{\epsilon \to 0^+} \mathcal{B}\hat{Z}_0 \quad (B_1 = \lim_{\epsilon \to 0^+} \mathcal{B}\hat{Z}_1)$ exists and is nonsingular. (63')

With this definition we now prove the following:

Corollary 2.64: In the general problem (1) is regular then it has a unique solution for arbitrary $\underline{f}, \underline{g}$ for all sufficiently small ϵ. (64)

Proof: Since the general problem (1) is regular we know the matrix B_0 is nonsingular. Consider the identity:

$$\mathcal{B}Z_0 = B_0\{I + B_0^{-1}[\mathcal{B}Z_0 - B_0]\}$$

From the definitions of B_0, $\hat{Z}_0$ we find:

$$\lim_{\epsilon \to 0^+} B_0^{-1}[\mathcal{B}Z_0 - B_0] = 0$$

Therefore we infer from the Banach Lemma (1.20) that the matrix $I + B_0^{-1}[\mathcal{B}Z_0 - B_0]$ is nonsingular for all sufficiently small ϵ, i.e. the matrix $\mathcal{B}Z_0$ is nonsingular for all ϵ sufficiently small. Furthermore, since the matrices $L(\epsilon)$, $R(\epsilon)$ admit asymptotic expansions to $\mathcal{O}(\epsilon^2)$ we infer from (60) that:

$$\mathcal{B}Z_0 \sim B_0 + \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

##

If the general problem (1) is regular we deduce from the definition of $Z_1(x)$, the continuity of $\underline{g}(\epsilon)$, and the relations given in (30, 31) that $\underline{u}_0(0)$ may be determined as follows:

$$\underline{h} \equiv \underline{g}(0) + L(0)\begin{bmatrix} \underline{0} \\ \underline{0} \\ A_{33}^{L^{-1}}\underline{f}_3^L \end{bmatrix} + R(0)\begin{bmatrix} -\underline{F}_0 \\ A_{22}^{R^{-1}}\{A_{21}^R \underline{F}_0 + \underline{f}_2^R\} \\ \underline{0} \end{bmatrix} \tag{65'}$$

$$\underline{F}_0 \equiv \int_0^1 Y_1(1,\tau)\, \underline{F}(\tau)\, d\tau \qquad (65')$$

$$\begin{bmatrix} \underline{u}_0(0) \\ \epsilon\, \underline{v}(0) \\ \epsilon\, \underline{w}(1) \end{bmatrix} \sim B_1^{-1} \underline{h} + \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \qquad (66)$$

Therefore, from (30, 31) we may deduce an asymptotic expansion of the solution of the general problem. When we consider the nature of this solution we recognize in any closed subinterval of $(0, 1)$:

$$\underline{y}(x) \sim \underline{y}_0(x) + \mathcal{O}(\epsilon) \qquad \underline{y}_0(x) = \begin{bmatrix} \underline{u}_0(x) \\ \underline{v}_0(x) \\ \underline{w}_0(x) \end{bmatrix} \quad \text{as } \epsilon \to 0^+ \qquad (67)$$

Furthermore, we recognize from (17) that $\underline{y}_0(x)$ satisfies the initial-value problem:

$$\begin{aligned} \mathcal{L}^0 \underline{y}_0(x) &\equiv \Omega(0)\, D\underline{y}_0(x) - A(x,0)\, \underline{y}_0(x) = \underline{f}(x,0) \\ \Omega(0)\, \underline{y}_0(0) &= \Omega(0)\, B_1^{-1} \underline{h} \end{aligned} \qquad (68)$$

Since the solution of the general problem (1) reduces to the solution of (68) as $\epsilon \to 0^+$, we call (68) the reduced problem. From these statements we deduce the following:

Corollary 2.69: Suppose the general problem:

$$\mathcal{L}\underline{y}(x) = \underline{f}(x,\epsilon) \qquad \mathcal{B}\underline{y} = \underline{g}(\epsilon) \qquad (69)$$

is regular. Then the general problem has a unique solution for all sufficiently small ϵ. Furthermore, on every closed subinterval of $(0, 1)$ we find:

$$y(x) \sim y_0(x) + O(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{70}$$

where $y_0(x)$ is the solution of the reduced problem:

$$\begin{aligned} \mathcal{L}^0 y_0(x) &= f(x,0) \\ \Omega(0)\, y_0(0) &= \Omega(0)\, B_1^{-1} \underline{h} \end{aligned} \tag{71}$$

Let us consider several examples of regular problems. In each example we implicitly assume the continuity and eigenvalue conditions (3a, 4) hold.

Example 1: The special problem (6) is one example of a regular problem. In fact, the definition of a regular problem was based on the properties of this special problem. For the special problem we find $B_0 = I$.

Example 2: Suppose the boundary operator $\mathcal{B}$ has the following form:

$$L(\epsilon) = \begin{bmatrix} L^0 & -\epsilon L^0 A_{12}^L A_{22}^{L^{-1}} & 0 \\ 0 & \epsilon I & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R(\epsilon) = \begin{bmatrix} R^0 & 0 & -\epsilon R^0 A_{13}^R A_{33}^{R^{-1}} \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon I \end{bmatrix}$$

$L^0 + R^0 Y_1(1)$ ... nonsingular

Then the general problem is regular because the matrix:

$$B_1 = \begin{bmatrix} L^0 + R^0 Y_1(1) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

is nonsingular. Furthermore, the reduced problem may equivalently be written as:

$$\mathcal{L}^0 \underset{\sim}{y}_0(x) = \underline{f}(x,0)$$
$$L^0 \underline{u}_0(0) + R^0 \underline{u}_0(1) = g_1(0) \tag{72}$$

where $g_1(0)$ represents the first m rows of $g(0)$. The equivalence of the problems (71, 72) follows from the nonsingularity of the matrix $L^0 + R^0 Y_1(1)$.

Example 3: If the general problem is in diagonal form, that is the matrix $A(x,\epsilon)$ has the following form:

$$A(x,\epsilon) = \begin{bmatrix} A_{11}(x,\epsilon) & 0 & 0 \\ 0 & A_{22}(x,\epsilon) & 0 \\ 0 & 0 & A_{33}(x,\epsilon) \end{bmatrix} \tag{73}$$

then the boundary operator $\mathcal{B}$ with:

$$L(\epsilon) = \begin{bmatrix} L^0 & 0 & L^1 \\ L^2 & \epsilon I & L^3 \\ L^4 & 0 & L^5 \end{bmatrix} \qquad R(\epsilon) = \begin{bmatrix} R^0 & R^1 & 0 \\ R^2 & R^3 & 0 \\ R^4 & R^5 & \epsilon I \end{bmatrix}$$

$L^0 + R^0 Y_1(1)$ ... nonsingular

leads to a regular problem. In this case we find:

$$B_1 = \begin{bmatrix} L^0 + R^0 Y_1(1) & 0 & 0 \\ L^2 + R^2 Y_1(1) & I & 0 \\ L^4 + R^4 Y_1(1) & 0 & I \end{bmatrix}$$

is nonsingular. Again the reduced problem has the equivalent formulation given in (72). One simple case where we may transform the matrix A to the form (73) occurs when A does not depend on x.

We hasten to point out the following fact. The above examples constitute one representative of an entire class of equivalent regular problems, differing only in their boundary conditions. Let the problems:

$$\text{BV}(\nu): \quad \begin{aligned} \mathcal{L}\, y^{\nu}(x) &= f(x,\epsilon) \\ \mathcal{B}\, y^{\nu} &= g(\epsilon) \end{aligned} \qquad \nu = 0, 1$$

both be regular. Then we say the problems BV(ν) are equivalent iff the conditions under which $B_0^{(\nu)}$ ($\nu = 0, 1$) are nonsingular are equivalent.

Example 4: Let $L(\epsilon)$, $R(\epsilon)$ as given in example 2 be the boundary matrices for BV(0). Define

$$L^{(1)}(\epsilon) = L(\epsilon) + \epsilon^2 L_1 \qquad L^{(2)}(\epsilon) = \Lambda(\epsilon)\, L^{(1)}(\epsilon)$$

$$R^{1}(\epsilon) = R(\epsilon) + \epsilon^2 R_1 \qquad R^{(2)}(\epsilon) = \Lambda(\epsilon)\, R^{(1)}(\epsilon)$$

$\Lambda(\epsilon)$ ... a nonsingular matrix depending continuously on ϵ.

Then BV(ν) are equivalent regular problems because:

$$B_0^{(0)} = B_0^{(1)} = \Lambda^{-1}(0)\, B_0^{(2)}$$

We note, however, that examples 1-4 do not include the most basic singular perturbation problem described in (1.12). For this reason consider the following example:

Example 5: Consider the boundary-value problem:

$$\epsilon D^2 \underline{u}(x) - B(x) D\underline{u}(x) - C(x)\,\underline{u}(x) = \underline{f}(x)$$

$$\underline{u}(0) = \underline{\alpha} \qquad\qquad \underline{u}(1) = \underline{\beta} \tag{74}$$

We suppose B , C , $\underline{u}$, $\underline{f}$ have the following (compatible) partitioned forms:

$$B(x) = \begin{bmatrix} A_{22}(x) & 0 \\ 0 & A_{33}(x) \end{bmatrix}$$

A_{22} ... an $m_1 x m_1$ matrix
A_{33} ... an $m_2 x m_2$ matrix
$m = m_1 + m_2$

$$C(x) = \begin{bmatrix} A_{21}(x) \\ A_{31}(x) \end{bmatrix}$$

A_{21} ... an $m_1 x m$ matrix
A_{31} ... an $m_2 x m$ matrix

$$A_{12} = \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix}$$

A_{12} ... an $m x m_1$ matrix

$$A_{13} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$$

A_{13} ... an $m x m_2$ matrix

$$D\underline{u}(x) = \begin{bmatrix} \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} \qquad \underline{f}(x) = \begin{bmatrix} \underline{f}_2(x) \\ \underline{f}_3(x) \end{bmatrix}$$

$\underline{v}, \underline{f}_2$ ... m_1-vectors
$\underline{w}, \underline{f}_3$ ... m_2-vectors

Using these partitioned forms of B , C , $\underline{u}$, $\underline{f}$ we write (74) as the following equivalent system:

$$\Omega(\epsilon) D \begin{bmatrix} \underline{u}(x) \\ \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} - \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21}(x) & A_{22}(x) & 0 \\ A_{31}(x) & 0 & A_{33}(x) \end{bmatrix} \begin{bmatrix} \underline{u}(x) \\ \underline{v}(x) \\ \underline{w}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{f}_2(x) \\ \underline{f}_3(x) \end{bmatrix}$$

$$\begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}(0) \\ \underline{v}(0) \\ \underline{w}(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}(1) \\ \underline{v}(1) \\ \underline{w}(1) \end{bmatrix} = \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \end{bmatrix} \tag{75}$$

We assume the continuity and eigenvalue conditions (3a, 4) are satisfied by the system (75). Introduce the following partitions of $\underline{u}$, $\underline{\alpha}$, $\underline{\beta}$, $Y_1(1)$:

$$\underline{u} = \begin{bmatrix} \underline{u}^{I} \\ \underline{u}^{II} \end{bmatrix} \qquad \underline{\alpha} = \begin{bmatrix} \underline{\alpha}^{I} \\ \underline{\alpha}^{II} \end{bmatrix} \qquad \underline{\beta} = \begin{bmatrix} \underline{\beta}^{I} \\ \underline{\beta}^{II} \end{bmatrix}$$

$\underline{u}^{I}$, $\underline{\alpha}^{I}$, $\underline{\beta}^{I}$ ... m_1 -vectors

$\underline{u}^{II}$, $\underline{\alpha}^{II}$, $\underline{\beta}^{II}$ ... m_2 -vectors

$$Y_1(1) = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \qquad \begin{matrix} Y_{11} \ldots \text{ an } m_1 \times m_1 \text{ matrix} \\ Y_{12} \ldots \text{ an } m_2 \times m_2 \text{ matrix} \end{matrix}$$

Using these partitions we find the matrix B_1 has the following representation:

$$B_1 = \begin{bmatrix} I_{m_1} & 0 & A_{22}^{L^{-1}} & 0 \\ 0 & I_{m_2} & 0 & 0 \\ Y_{11} & Y_{12} & 0 & 0 \\ Y_{21} & Y_{22} & 0 & A_{33}^{R^{-1}} \end{bmatrix}$$

Since the conditions (3a, 4) are satisfied it follows that (75) is a regular problem iff the matrix Y_{11} is nonsingular. If Y_{11} is nonsingular we find the following representation for B_1^{-1} :

$$B_1^{-1} = \begin{bmatrix} 0 & -Y_{11}^{-1}Y_{12} & Y_{11}^{-1} & 0 \\ 0 & I_{m_2} & 0 & 0 \\ A_{22}^{L} & A_{22}^{L}Y_{11}^{-1}Y_{12} & -A_{22}^{L}Y_{11}^{-1} & 0 \\ 0 & -A_{33}^{R}\{Y_{22}-Y_{21}Y_{11}^{-1}Y_{12}\} & -A_{33}^{R}Y_{21}Y_{11}^{-1} & A_{33}^{R} \end{bmatrix}$$

Furthermore, if Y_{11} is singular we deduce B_1 is singular from the fact that:

$$\underline{\delta} = \begin{bmatrix} \underline{\gamma} \\ \underline{0} \\ -A_{22}^{L}\underline{\gamma} \\ -A_{33}^{R} Y_{21}\underline{\gamma} \end{bmatrix}$$

is a null-vector of B_1 whenever $\underline{\gamma}$ is a null-vector of Y_{11} .

We assume the problem (75) is regular, i.e. (3a, 4) hold and the matrix Y_{11} is nonsingular. Define:

$$\underline{F}_0 = \begin{bmatrix} \underline{F}_0^{I} \\ \underline{F}_0^{II} \end{bmatrix} = \int_0^1 Y_1(x,\tau)\,\underline{F}(\tau)\,d\tau$$

$\underline{F}_0^{I}$ ... an m_1 - vector

$\underline{F}_0^{II}$ ... an m_2 -vector

From (65, 66) we infer the initial condition satisfied by the reduced problem corresponding to (75) is:

$$\underline{u}_0(0) = \begin{bmatrix} Y_{11}^{-1}\{-Y_{12}\underline{\alpha}^{II} + \underline{\beta}^{I} - \underline{F}_0^{I}\} \\ \underline{\alpha}^{II} \end{bmatrix}$$

Since Y_{11} is nonsingular we find $\underline{u}_0(0)$ also is the unique solution of the following linear system:

$$\left\{\begin{bmatrix} 0 & 0 \\ 0 & I_{m_2} \end{bmatrix} + \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}\right\}\underline{u}_0(0) = \begin{bmatrix} \underline{\beta}^{I} \\ \underline{\alpha}^{II} \end{bmatrix} - \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}\underline{F}_0$$

From this result we infer the reduced problem corresponding to (75) is :

$$-A(x)\,D\underline{u}_0(x) - B(x)\,\underline{u}_0(x) = \underline{f}(x)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & I_{m_2} \end{bmatrix} \underline{u}_0(0) + \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \underline{u}_0(1) = \begin{bmatrix} \underline{\beta}^{I} \\ \underline{\alpha}^{II} \end{bmatrix} \qquad (76)$$

Furthermore, from the definition of $Y_1(x)$ and Theorem 1.28 it follows that (76) has a unique solution for all $\underline{f}$, $\underline{\beta}^{I}$, $\underline{\alpha}^{II}$ iff the matrix Y_{11} is nonsingular.

Combining these results we find under the continuity and eigenvalue conditions (3a, 4) that the original problem (75) is regular iff the reduced problem (76) has a unique solution for every $\underline{f}$, $\underline{\beta}^{I}$, $\underline{\alpha}^{II}$.

3. DIFFERENCE APPROXIMATION OF THE GENERAL PROBLEM

3.0 Introduction

Consider the general problem:

$$\begin{aligned} \mathcal{L}\, \underset{\sim}{y}(x) &= \underset{\sim}{f}(x,\epsilon) \\ \mathcal{B}\, \underset{\sim}{y} \equiv L(\epsilon)\, \underset{\sim}{y}(0) + R(\epsilon)\, \underset{\sim}{y}(1) &= \underset{\sim}{g}(\epsilon) \end{aligned} \qquad x \in I \tag{1}$$

described in detail in (2.1, 2.2). Associated with the general problem (1) is the special problem:

$$\begin{aligned} \mathcal{L}\, \underset{\sim}{y}(x) &= \underset{\sim}{f}(x,\epsilon) \\ \mathcal{B}^{*} \underset{\sim}{y} \equiv L^{*}(\epsilon)\, \underset{\sim}{y}(0) + R^{*}(\epsilon)\, \underset{\sim}{y}(1) &= \underset{\sim}{g}^{*}(\epsilon) \end{aligned} \qquad x \in I \tag{2}$$

described in detail in (2.6, 2.7). We will assume:

(a) The general problem (1) is regular.
(b) The boundary operator $\mathcal{B}$ does not involve $\underset{\sim}{v}(1)$ or $\underset{\sim}{w}(0)$. (3)

As a consequence of (3a) we find:

(a) The special problem (2) is regular.
(b) The fundamental matrix Z_0 defined in (2.57) exists and admits the asymptotic expansion (2.60). (4)
(c) The matrix B_0 defined in (2.63) is nonsingular.

We make the following definitions:

$$h = \frac{1}{J} \qquad x_\alpha = \alpha h$$

$$A^h_{ij}(\ell) = A_{ij}(x_{\ell+\frac{1}{2}}, \epsilon) \qquad \underline{f}^h(j) = \underline{f}(x_{j+\frac{1}{2}}, \epsilon)$$

$$A_h(j) = \begin{bmatrix} A^h_{11}(j)E & A^h_{12}(j)T & A^h_{13}(j) \\ A^h_{21}(j)E & A^h_{22}(j)T & \epsilon A^h_{23}(j) \\ A^h_{31}(j)E & \epsilon A^h_{23}(j)T & A^h_{33}(j) \end{bmatrix} \tag{5}$$

T ... shift operator $\qquad T\,\underline{F}(j) = \underline{F}(j+1)$

I ... identity operator $\qquad I\,\underline{F}(j) = \underline{F}(j)$

$E = \frac{1}{2}\{T+I\}$ ... averaging operator

$D^h = \frac{1}{h}\{T-I\}$ ... forward difference operator

Using these definitions the numerical scheme used to solve (1) can be written as follows:

$$\begin{aligned} \mathcal{L}_h \underline{y}^h(j) &\equiv \Omega(\epsilon) D^h \underline{y}^h(j) - A_h \underline{y}^h(j) = \underline{f}^h(j) \\ \mathcal{B}_h \underline{y}^h &\equiv L(\epsilon)\underline{y}^h(0) + R(\epsilon)\underline{y}^h(J) = \underline{g}(\epsilon) \end{aligned} \quad 0 \le j \le J-1 \tag{6}$$

Associated with the general difference problem (6) is the special difference problem:

$$\begin{aligned} \mathcal{L}_h \underline{y}^h(j) &= \underline{f}^h(j) \\ \mathcal{B}^*_h \underline{y}^h &\equiv L^*(\epsilon)\underline{y}^h(0) + R(\epsilon)\underline{y}^h(J) = \underline{g}^*(\epsilon) \end{aligned} \quad 0 \le j \le J-1 \tag{7}$$

We note (6) and (7) may differ only in the boundary conditions.

The principal result of this chapter is given in Corollary 3.72.

3.1 Definitions and Useful Identities

Analogous to the definitions make in (2.9) we define:

$$
\begin{aligned}
a^h(j) &\equiv A^h_{11}(j) - A^h_{12}(j)\, A^{h^{-1}}_{22}(j)\, A^h_{21}(j) - A^h_{13}(j)\, A^{h^{-1}}_{33}(j)\, A^h_{31}(j) \\
\underline{F}^h(j) &\equiv \underline{f}^h_1(j) - A^h_{12}(j)\, A^{h^{-1}}_{22}(j)\, \underline{f}^h_2(j) - A^h_{13}(j)\, A^{h^{-1}}_{33}(j)\, \underline{f}^h_3(j) \\
A^{hL}_{ij} &\equiv A_{ij}(0) \qquad\qquad A^R_{ij} \equiv A_{ij}(J-1) \\
a^{hL} &\equiv a^h(0) \qquad\qquad a^{hR} \equiv a^h(J-1) \\
\underline{f}^{hL}_j &\equiv \underline{f}^h_j(0) \qquad\qquad \underline{f}^{hR}_j \equiv \underline{f}^h_j(J-1) \\
C^L_1 &\equiv [I - \tfrac{h}{2} a^{hL}]^{-1} A^{hL}_{12} A^{hL^{-1}}_{22} \qquad C^L_2 \equiv [I + \tfrac{h}{2} a^{hL}]^{-1} A^{hL}_{12} A^{hL^{-1}}_{22} \\
C^R_3 &\equiv [I + \tfrac{h}{2} a^{hR}]^{-1} A^{hR}_{13} A^{hR^{-1}}_{33} \\
A^h_{ij}\, \underline{Z}(\ell) &\equiv A^h_{ij}(\ell)\, \underline{Z}(\ell) \qquad a^h \underline{Z}(\ell) = a^h(\ell)\, \underline{Z}(\ell)
\end{aligned}
\tag{8'}
$$

Analogous to the operators defined in (2.35) we define:

$$
\begin{aligned}
K^h_0\, \underline{Z}(j) &\equiv h \sum_{k=0}^{j-1} \underline{Z}(k) \\
K^h_1\, \underline{Z}(j) &\equiv h \sum_{k=0}^{j-1} Y^h_1(j,k) [I + \tfrac{h}{2} a^h(k)]^{-1} \underline{Z}(k) \\
K^h_2\, \underline{Z}(j) &\equiv h \sum_{k=0}^{j-1} Y^h_2(j,k)\, \underline{Z}(k) \\
K^h_3\, \underline{Z}(j) &\equiv -h \sum_{k=j}^{J-1} Y^h_3(j,k+1)\, \underline{Z}(k)
\end{aligned}
\tag{8'}
$$

In the definition of the operators K_i^h we have used the discrete analogs Y_i^h of the fundamental solution matrices Y_i defined in (2.32). The matrices Y_i^h are defined to be the solutions of the following difference equations:

$$
\begin{aligned}
D_j^h Y_1(j,k) &= a^h(j)\, E_j Y_1^h(j,k) \qquad Y_1^h(k,k) = I \qquad Y_1^h(j) \equiv Y_1^h(j,0) \\
\epsilon D_j^h Y_2(j,k) &= A_{22}^h(j)\, T\, Y_2^h(j,k) \qquad Y_2^h(k,k) = I \qquad Y_2^h(j) \equiv Y_2^h(j,0) \qquad (8') \\
\epsilon D_j^h Y_3(j,k) &= A_{33}^h(j)\, Y_3^h(j,k) \qquad Y_3^h(k,k) = I \qquad Y_3^h(j) \equiv Y_3^h(j,J)
\end{aligned}
$$

Using the matrices Y_ℓ^h we define the following operators:

$$Y_\ell^h \underline{Z}(j) \equiv Y_\ell^h(j)\, \underline{Z}(j) \qquad \ell = 1,2,3 \qquad (8')$$

Analogous to the norm $\|\cdot\|_1$ defined on $\mathcal{C}^N[0,1]$ we define:

$$\|\underline{Z}\|_1 = h \sum_{k=0}^{J-1} |\underline{Z}(k)| \qquad \underline{Z} \varepsilon\, \mathcal{D}^N[0,J-1] \qquad (8')$$

Although not explicitly shown as an argument we note the matrices $A_{ij}^h, a^h, Y_\ell^h, C_i^{L,R}$ and the vectors $\underline{F}^h, \underline{f}_\ell^h$ depend on ϵ.

From the eigenvalue condition (2.4) satisfied by the matrices A_{22}, A_{33} we deduce the matrices Y_2^h, Y_3^h are well defined for all positive h and ϵ. In chapter four we prove the following:

Theorem 3.9: (Exponential Dichotomy) Let $A_{22}(x,\epsilon), A_{33}(x,\epsilon)$ be continuous functions of x and ϵ, for $(x,\epsilon) \varepsilon I \times E_1$, which satisfy the eigenvalue condition (2.4). Let $A_{22}^h, A_{33}^h, Y_2^h, Y_3^h$

be as defined in (4) and (8). Then there exist positive constants C_0, Δ, ϵ_2 such that for all $0<\epsilon\le\epsilon_2$ and $0<h\le 1$:

$$\| Y_2(j,k) \| \le C_0/\{1+\Delta\tfrac{h}{\epsilon}\}^{j-k} \qquad 0\le k\le j\le J$$
$$\| Y_3(j,k) \| \le C_0/\{1+\Delta\tfrac{h}{\epsilon}\}^{k-j} \qquad 0\le j\le k\le J \tag{9}$$

From the difference equation satisfied by Y_1^h we deduce:

$$Y_1^h(j+1,j) = [I-\tfrac{h}{2}a^h(j)]^{-1}[I+\tfrac{h}{2}a^h(j)] \tag{10}$$

Therefore, for:

$$0<h\le h_0\le 1 \qquad \text{where} \qquad h_0\|a\|_\infty\le 1 \tag{11}$$

we may use the Banach Lemma (1.20) to prove Y_1^h is a well defined nonsingular matrix. Through the use of (10), the difference equation satisfied by Y_1^h given in (8), and the estimates:

$$\frac{1+x}{1-x}\le 1+4x \qquad 0\le x\le\tfrac{1}{2}$$

$$1+4x\le \exp\{4x\} \qquad 0\le x$$

we deduce:

$$\| Y_1^h(j,k) \| \le \exp\{2\|a\|_\infty\} \qquad \begin{matrix} 0\le j,k\le J \\ 0<h\le h_0 \end{matrix} \tag{12}$$

Through the use of the identity:

$$\frac{h}{\epsilon}\sum_{k=1}^{\infty}\frac{1}{\{1+\Delta\frac{h}{\epsilon}\}^k} = \frac{1}{\Delta} \tag{13}$$

and the bounds given in (9, 12) we deduce K_0^h, K_1^h, K_2^h, K_3^h are bounded linear operators on $\mathcal{D}_\infty^N[0,J]$. These bounds can be chosen to be independent of ϵ.

Analogous to the rules governing the differentiation of a product of matrices and the integration by parts formula we find:

$$D^h[B(k)C(k)] = \begin{cases} D^h B(k)\cdot C(k+1) + B(k)\cdot D^h C(k) \\ B(k+1)\cdot D^h C(k) + D^h B(k)\cdot C(k) \end{cases}$$

$$h\sum_{k}{}_a^b D^h B(k)\cdot C(k) = B(b+1)C(b) - B(a)C(a) - h\sum_{k}{}_a^{b-1} B(k+1)\cdot D^h C(k) \tag{14}$$

$$h\sum_{k}{}_a^b B(k)\cdot D^h C(k) = B(b)C(b+1) - B(a)C(a) - h\sum_{k}{}_a^{b-1} D^h B(k)\cdot C(k+1)$$

From the difference equation satisfied by Y_2^h and Y_3^h, the identities given in (8, 14), and the identity:

$$Y_\ell^h(k,j)\,Y_\ell^h(j,k) = I \qquad \ell = 2,3 \qquad 0 \le j,k \le J$$

we deduce:

$$\begin{aligned} \epsilon D_j^h Y_2^h(k,j) &= -Y_2^h(k,j)\,A_{22}^h(j) \\ \epsilon D_j^h Y_3^h(k,j) &= -Y_3^h(k,j+1)\,A_{33}^h(j) \end{aligned} \qquad 0 \le j,k \le J \tag{15}$$

Consider the following set of initial-value problems:

$$\begin{aligned} D^h \underline{u}^h(j) &= \mathcal{A}^h(j)\,E\,\underline{u}^h(j) + \underline{f}(j) && \underline{u}^h(0)\ \ldots \text{ given} \\ \epsilon D^h \underline{v}^h(j) &= A_{22}^h(j)\,T\,\underline{v}^h(j) + \underline{f}(j) && \underline{v}^h(0)\ \ldots \text{ given} \\ \epsilon D^h \underline{w}^h(j) &= A_{33}^h(j)\,\underline{w}^h(j) + \underline{f}(j) && \underline{w}^h(J)\ \ldots \text{ given} \end{aligned} \tag{16}$$

Analogous to the variation of parameters formula (1.25) we find the solutions of (16) admit the representations:

$$
\begin{aligned}
\underline{u}^h(j) &= Y_1^h(j)\,\underline{u}^h(0) + h\sum_{k=0}^{j-1} Y_1^h(j,k)\,[I+\tfrac{h}{2}a^h(k)]^{-1}\,\underline{f}(j) \\
\underline{v}^h(j) &= Y_2^h(j)\,\underline{v}^h(0) + h\sum_{k=0}^{j-1} Y_2^h(j,k)\,\underline{f}(j) \\
\underline{w}^h(j) &= Y_3^h(J)\,\underline{w}^h(J) - h\sum_{k=j}^{J-1} Y_3^h(j,k+1)\,\underline{f}(j)
\end{aligned}
\tag{17}
$$

The representation given in (17) for $\underline{u}^h$ can be obtained as follows. Suppose $\hat{\underline{u}}^h$ satisfies the following relationship:

$$\underline{u}^h(j) = Y_1^h(j)\,\hat{\underline{u}}^h(j)$$

Substituting the above form for $\underline{u}^h$ into the difference equation satisfied by $\underline{u}^h$ and using (14) we arrive at the relation:

$$D^h Y_1^h(j)\cdot\hat{\underline{u}}^h(j+1) + Y_1^h(j)\cdot D^h\hat{\underline{u}}^h(j) = a^h(j)\,E Y_1^h\hat{\underline{u}}^h(j) + \underline{f}(j)$$

From the difference equation satisfied by Y_1^h this relation upon rearrangement becomes:

$$[I+\tfrac{h}{2}a^h(j)]\,Y_1^h(j)\cdot D^h\hat{\underline{u}}^h(j) = \underline{f}(j)$$

Using the identity :

$$\hat{\underline{u}}^h(j) = \hat{\underline{u}}^h(0) + h\sum_{k=0}^{j-1} D^h\hat{\underline{u}}^h(k)$$

we find:

$$\hat{\underline{u}}^h(j) = \hat{\underline{u}}^h(0) + h\sum_{k=0}^{j-1} Y_1^{h^{-1}}(k)\,[I+\tfrac{h}{2}a^h(k)]^{-1}\,\underline{f}(k)$$

From the above expression for $\hat{\underline{u}}^h$ and the relationship between $\underline{u}^h$ and $\hat{\underline{u}}^h$ we deduce the representation given in (17) for $\underline{u}^h$.

We end this section by presenting the following:

Lemma 3.18: Suppose ϵ, Δ, x are positive constants and $0 < h \le \frac{1}{L}$. Then:

$$\max_{j \ge 0} \frac{jh}{\{1+\Delta\frac{h}{\epsilon}\}^j} \le \frac{\epsilon}{\Delta}$$
$$\frac{1}{\{1+\Delta\frac{h}{\epsilon}\}^{x/h}} \le \left\{\frac{L\epsilon}{\Delta}\right\}^{Lx} \tag{18}$$

Proof: For $h \ge 0$ we find the function $g(h)$ defined by:

$$g(h) = \frac{1}{h}\log\{1+\Delta\frac{h}{\epsilon}\}$$

is a decreasing function of h. We therefore find for $0 < h \le \frac{1}{L}$:

$$\begin{aligned}\frac{1}{\{1+\Delta\frac{h}{\epsilon}\}^{x/h}} &= \exp\{-xg(h)\} \\ &\le \exp\{-xg(\tfrac{1}{L})\} \\ &\le \left(\frac{L\epsilon}{\Delta}\right)^{Lx}\end{aligned}$$

Next consider the function $f(j)$ defined by:

$$f(j) = \frac{jh}{\{1+\Delta\frac{h}{\epsilon}\}^j} = jh\exp\{-j\log[1+\Delta\tfrac{h}{\epsilon}]\}$$

For $y \ge 0$ the function $f(y)$ has a maximum at $y = y_0$ where:

$$y_0 = \frac{1}{\log[1+\Delta\frac{h}{\epsilon}]}$$

Since j assumes only integer values we find:

$$\max_{j \geq 0} f(j) \leq \begin{cases} f(1) & y_0 \leq 1 \\ f(y_0) & y_0 \geq 1 \end{cases}$$

We note:

$$y_0 \geq 1 \iff \log\{1 + \Delta \tfrac{h}{\epsilon}\} \leq 1 \iff h \leq \frac{e-1}{\Delta}\epsilon$$

Therefore, for $y_0 \geq 1$:

$$f(y_0) = e^{-1}/g(h) \leq e^{-1}/g(\tfrac{e-1}{\Delta}\epsilon) = \frac{1-e^{-1}}{\Delta}\epsilon \leq \frac{\epsilon}{\Delta}$$

We also estimate:

$$f(1) = \frac{h}{1 + \Delta \frac{h}{\epsilon}} \leq \frac{\epsilon}{\Delta}$$

Therefore (18) has been established. ##

3.2 Existence of a Solution of the Special Difference Problem

In this section we prove the special difference problem (7) has a unique solution for all sufficiently small h and ϵ . Furthermore, this solution satisfies the a priori bound given in (19).

Theorem 3.19: Suppose the special problem (2) is regular. Then for all sufficiently small ϵ and h the special difference problem (7) has a unique solution. Furthermore, this unique solution satisfies the bound:

$$\|\underline{u}^h\|_\infty \le C_1\{\|\underline{u}^h(0)\| + \epsilon\|\underline{v}^h(0)\| + \epsilon\|\underline{w}^h(J)\| + \|\underline{f}_1^h\|_1 + \|\underline{f}_2^h\|_\infty + \|\underline{f}_3^h\|_\infty\}$$

$$\|\underline{v}^h\|_\infty \le C_1\{\|\underline{u}^h(0)\| + \|\underline{v}^h(0)\| + \epsilon\|\underline{w}^h(J)\| + \|\underline{f}_1^h\|_1 + \|\underline{f}_2^h\|_\infty + \|\underline{f}_3^h\|_\infty\} \quad (19)$$

$$\|\underline{w}^h\|_\infty \le C_1\{\|\underline{u}^h(0)\| + \epsilon\|\underline{v}^h(0)\| + \|\underline{w}^h(J)\| + \|\underline{f}_1^h\|_1 + \|\underline{f}_2^h\|_\infty + \|\underline{f}_3^h\|_\infty\}$$

Here C_1 is a positive constant independent of ϵ .

Proof: This proof is similar to that given in section 2.2 , the similarity will be made clear through the use of the same notation. The operators we consider belong to the space $\mathcal{D}_\infty^N[0,J]$ and when such an operator is bounded we will always imply the bound can be chosen to be independent of ϵ .

In the special difference scheme (7), sum the equation for $\underline{u}^h$ and apply the variation of parameters formula (17) to the equations for $\underline{v}^h$ and $\underline{w}^h$. The result may be written, through the use of (8), as follows:

$$\theta^h \underline{y}^h = \underline{H}^h$$

$$\theta^h = \theta_1^h - \epsilon K_4^h = \begin{bmatrix} I_m - K_0^h A_{11}^h E & -K_0^h A_{12}^h T & -K_0^h A_{13}^h \\ -K_2^h A_{21}^h E & I_{m_1} & 0 \\ -K_3^h A_{31}^h E & 0 & I_{m_2} \end{bmatrix} - \epsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K_2^h A_{23}^h \\ 0 & K_3^h A_{32}^h T & 0 \end{bmatrix}$$

$$\underline{y}^h = \begin{bmatrix} \underline{u}^h \\ \underline{v}^h \\ \underline{w}^h \end{bmatrix} \qquad \underline{f}^h = \begin{bmatrix} \underline{f}_1^h \\ \underline{f}_2^h \\ \underline{f}_3^h \end{bmatrix} \qquad \underline{H}^h = \begin{bmatrix} \underline{u}^h(0) + K_0^h \underline{f}_1^h \\ Y_2^h \cdot \underline{v}^h(0) + K_2^h \underline{f}_2^h \\ Y_3^h \cdot \underline{w}^h(J) + K_3^h \underline{f}_3^h \end{bmatrix} \quad (20)$$

I_N ... the identity operator in $\mathcal{D}^N[0,J]$

As in (2.36, 2.37) we find:

$$\theta_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{m_1} & 0 \\ 0 & 0 & I_{m_2} \end{bmatrix} + \begin{bmatrix} I_m \\ K_2^h A_{21}^h E \\ K_3^h A_{31}^h E \end{bmatrix} Q^{h^{-1}} [I_m \quad K_0^h A_{12}^h T \quad K_0^h A_{13}^h] \tag{21}$$

$$Q^h = I_m - K_0^h A_{11}^h E - K_0^h A_{12}^h T K_2^h A_{21}^h E - K_0^h A_{13}^h K_3^h A_{31}^h E$$

provided $Q^{h^{-1}}$ exists. We will prove for some ϵ_3 in $(0, \epsilon_2]$, where ϵ_2 is the constant used in (9):

$$Q^{h^{-1}} = [I + \epsilon K_8^h][I + K_1^h a^h E] \qquad 0 < \epsilon \le \epsilon_3 \tag{22}$$

Here K_8^h is a bounded linear operator. Once (22) has been established it will follow that $\theta_1^{h^{-1}}$ exists as a bounded linear operator. To establish (22) we first prove:

$$\begin{aligned} K_0^h A_{12}^h T K_2^h A_{21}^h &= -K_0^h A_{12}^h A_{22}^{h^{-1}} A_{21}^h + \epsilon K_5^h \\ K_0^h A_{13}^h K_3^h A_{31}^h &= -K_0^h A_{13}^h A_{33}^{h^{-1}} A_{31}^h + \epsilon K_6^h \end{aligned} \qquad 0 < \epsilon \le \epsilon_3 \tag{23}$$

Here K_5^h, K_6^h are bounded linear operators. To establish the first identity given in (23) we note:

$$K_0^h A_{12}^h T K_2^h A_{21}^h \underline{Z}(j) \qquad h \sum_{k=0}^{j-1} A_{12}^h(k) \cdot \frac{h}{\epsilon} \sum_{\ell=0}^{k} Y_2^h(k+1, \ell) A_{21}^h(\ell) Z(\ell)$$

... change the order of summation.

$$h \sum_{\ell=0}^{j-1} \left\{ \frac{h}{\epsilon} \sum_{k=\ell}^{j-1} A_{12}^h(k) Y_2^h(k+1, \ell) \right\} A_{21}^h(\ell) \underline{Z}(\ell)$$

$$\ldots \quad \frac{1}{\epsilon} Y_2^h(k+1, \ell) = A_{22}^{h^{-1}}(k) D_k^h Y_2^h(k, \ell)$$

sum by parts using (14).

$$= -h\sum_{\ell=0}^{j-1} A_{12}^h(\ell) A_{22}^{h^{-1}}(\ell) A_{21}^h(\ell) \underline{Z}(\ell) + \epsilon K_5^h \underline{Z}(j)$$

$$= -K_0^h A_{12}^h A_{22}^{h^{-1}} A_{21}^h \underline{Z}(j) + \epsilon K_5^h \underline{Z}(j)$$

In the above derivation the operator K_5^h is defined to be:

$$K_5^h \underline{Z}(j) = \frac{h}{\epsilon}\sum_{\ell=0}^{j-1}\Big\{ A_{12}^h(j-1) A_{22}^{h^{-1}}(j-1) Y_2^h(j,\ell) -$$

$$h\sum_{k=\ell}^{j-2} D_k^h\left[A_{12}^h(k) A_{22}^{h^{-1}}(k)\right] Y_2^h(k+1,\ell)\Big\} A_{21}^h(\ell)\, \underline{Z}(\ell)$$

For $0<\epsilon\le\epsilon_2$ the exponential dichotomy (9) and the identity (13) allow us to estimate:

$$\| K_5^h \underline{Z}(j) \| \sim \mathcal{O}\Big(\frac{h}{\epsilon}\sum_{\ell=0}^{j-1}\Big\{\frac{1}{(1+\Delta\frac{h}{\epsilon})^{j-\ell}} + h\sum_{k=\ell}^{j-2}\frac{1}{(1+\Delta\frac{h}{\epsilon})^{k+1-\ell}}\Big\}\Big)\|\underline{Z}\|_\infty$$

$$\sim \mathcal{O}\Big(\frac{h}{\epsilon}\Big\{\frac{\epsilon}{\Delta h} + h\sum_{\ell=0}^{j-1}\frac{\epsilon}{\Delta h}\Big\}\Big)\|\underline{Z}\|_\infty \qquad \text{as } \epsilon\to 0^+$$

$$\sim \mathcal{O}\Big(\frac{1}{\Delta}\Big)\|\underline{Z}\|_\infty$$

We therefore conclude K_5^h is a bounded linear operator. To establish the second identity given in (23) we note:

$$K_0^h A_{13}^h K_3^h A_{31}^h \underline{Z}(j) = -h\sum_{k=0}^{j-1} A_{13}^h(k)\cdot\frac{h}{\epsilon}\sum_{\ell=k}^{J-1} Y_3^h(k,\ell+1) A_{31}^h(\ell)\, \underline{Z}(\ell)$$

... split the ℓ sum.

$$= -h\sum_{k=0}^{j-1} A_{13}^h(k)\cdot\frac{h}{\epsilon}\sum_{\ell=k}^{j-1} Y_3^h(k,\ell+1) A_{31}^h(\ell)\, \underline{Z}(\ell) + \epsilon\bar{K}_6^h \underline{Z}(j)$$

... change the order of summation.

$$= -h\sum_{\ell=0}^{j-1}\Big\{\frac{h}{\epsilon}\sum_{k=0}^{\ell} A_{13}^h(k) Y_3^h(k,\ell+1)\Big\} A_{31}^h(\ell)\, \underline{Z}(\ell) + \epsilon\bar{K}_6^h \underline{Z}(j)$$

$$\ldots \quad \tfrac{1}{\epsilon} Y_3^h(k,\ell+1) = A_{33}^{h^{-1}}(k) D_k^h Y_3^h(k,\ell+1),$$

sum by parts using (14).

$$= -h \sum_{\ell=0}^{j-1} A_{13}^h(\ell) A_{33}^{h^{-1}}(\ell) A_{31}^h(\ell) \underline{Z}(\ell) + \epsilon K_6^h \underline{Z}(j)$$

$$= -K_0^h A_{13}^h A_{33}^{h^{-1}} A_{31}^h \underline{Z}(j) + \epsilon K_6^h \underline{Z}(j)$$

In the above derivation the operators $\bar{K}_6^h$, K_6^h are defined to be:

$$\bar{K}_6^h \underline{Z}(j) = -\frac{h}{\epsilon} \sum_{k=0}^{j-1} A_{13}^h(k) \cdot \frac{h}{\epsilon} \sum_{\ell=j}^{J-1} Y_3^h(k,\ell+1) A_{31}^h(\ell) \underline{Z}(\ell)$$

$$K_6^h \underline{Z}(j) = \bar{K}_6^h \underline{Z}(j) + \frac{h}{\epsilon} \sum_{\ell=0}^{j-1} \Big\{ A_{13}^h(0) A_{33}^{h^{-1}}(0) Y_3^h(0,\ell+1) +$$

$$h \sum_{k=0}^{\ell-1} D_k^h \left[A_{13}^h(k) A_{33}^{h^{-1}}(k) \right] Y_3^h(k+1,\ell+1) \Big\} A_{31}^h(\ell) \underline{Z}(\ell)$$

For $0 < \epsilon \leq \epsilon_2$ we may use (9, 13) to estimate:

$$\| \bar{K}_6^h \underline{Z}(j) \| \sim \mathcal{O}\Big(\frac{h}{\epsilon} \sum_{k=0}^{j-1} \frac{h}{\epsilon} \sum_{\ell=j}^{J-1} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{\ell+1-k}} \Big) \| \underline{Z} \|_\infty$$

$$\sim \mathcal{O}\Big(\frac{h}{\epsilon} \sum_{k=0}^{j-1} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{j-k}} \cdot \frac{h}{\epsilon} \sum_{\ell=j}^{J-1} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{\ell+1-j}} \Big) \| \underline{Z} \|_\infty$$

$$\sim \mathcal{O}\Big(\frac{1}{\Delta^2} \Big) \| \underline{Z} \|_\infty \qquad \text{as } \epsilon \to 0^+$$

$$\| K_6^h \underline{Z}(j) - \bar{K}_6^h \underline{Z}(j) \| \sim \mathcal{O}\Big(\frac{h}{\epsilon} \sum_{\ell=0}^{j-1} \Big\{ \frac{1}{(1+\Delta\frac{h}{\epsilon})^{\ell+1}} + h \sum_{k=0}^{\ell-1} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{\ell-k}} \Big\} \Big) \| \underline{Z} \|_\infty$$

$$\sim \mathcal{O}\Big(\frac{1}{\Delta} + \frac{h}{\epsilon} \sum_{\ell=0}^{j-1} \frac{\epsilon}{\Delta} \Big) \| \underline{Z} \|_\infty$$

$$\sim \mathcal{O}\Big(\frac{1}{\Delta} \Big) \| \underline{Z} \|_\infty$$

We therefore conclude K_6^h is a bounded linear operator. From the definition of $\mathcal{Q}^h$ and the identities (23) we find:

$$\mathcal{Q}^h = I - K_0^h a^h E - \epsilon K_7^h$$
$$K_7^h = K_5^h E + K_6^h E \qquad (24)$$

Suppose $\underline{G}$ and $\underline{F}$ are related through the equation:

$$\underline{G} = [I + K_1^h a^h E]\, \underline{F}$$

$$\underline{G}(j) = \underline{F}(j) + h \sum_{k=0}^{j-1} Y_1^h(j,k) [I - \tfrac{h}{2} a^h(k)]^{-1} a^h(k) E\, \underline{F}(k)$$

We note the sum involved in the above expression satisfies the difference equation given for $\underline{\nu}^h$ in (16). From this fact we find upon differencing the above equation:

$$D^h \underline{G}(j) = D^h \underline{F}(j) + [a^h(j) E \{\underline{G}(j) - \underline{F}(j)\} + a^h(j) E\, \underline{F}(j)$$
$$= D^h \underline{F}(j) + a^h(j) E\, \underline{G}(j)$$

By summing this difference equation for $\underline{G}$ we find:

$$\underline{G}(j) - \underline{G}(0) = \underline{F}(j) - \underline{F}(0) + h \sum_{k=0}^{j-1} a^h(k) E\, \underline{G}(k)$$
$$\underline{G}(j) = \underline{F}(j) + K_0^h a^h E\, \underline{G}(j)$$
$$\underline{F}(j) = \underline{G}(j) - K_0^h a^h E\, \underline{G}(j)$$
$$\underline{F} = [I - K_0^h a^h E]\, \underline{G}$$

From this last identity and the original relationship between $\underline{F}$ and $\underline{G}$ we conclude:

$$[I - K_0^h a^h E]^{-1} = I + K_1^h a^h E \tag{25}$$

By the same arguments used in the derivation of (2.43-2.46) we find for some ϵ_3 in $(0, \epsilon_2]$:

$$\begin{aligned} [I - \epsilon \theta_1^{h-1} K_4^h]^{-1} &= I + \epsilon K_9^h \\ [I - \epsilon \{I + K_1^h a^h E\} K_7^h]^{-1} &= I + \epsilon K_8^h \\ \theta^{h-1} &= [I + \epsilon K_9^h] \theta_1^{h-1} \qquad 0 < \epsilon \le \epsilon_3 \\ Q^{h-1} &= [I + \epsilon K_8^h][I + K_1^h a^h E] \end{aligned} \tag{26}$$

Here K_8^h and K_9^h are bounded linear operators. From the identity:

$$[I - \epsilon \theta_1^{h-1} K_4^h]^{-1} = I + \epsilon [I - \epsilon \theta_1^{h-1} K_4^h]^{-1} \theta_1^{h-1} K_4^h$$

which is valid for $0 < \epsilon \le \epsilon_3$ we deduce:

$$\theta^{h-1} = [I + \epsilon \{I + \epsilon K_9^h\} \theta_1^{h-1} K_4^h] \theta_1^{h-1} \tag{27}$$

Following steps similar to those used to derive (25) we find:

$$\begin{aligned} [I + K_1^h a^h E] K_0^h &= K_1^h \\ [I + K_1^h a^h E] Y_{1(j)}^h &= Y_{1(j)}^h \end{aligned} \tag{28}$$

Collecting the results contained in (21, 26, 28) we find:

$$\Theta^{h-1} = \begin{cases} [I + \epsilon K_9^h]\,\theta_1^{h-1} \\ [I + \epsilon\,\theta_1^{h-1} K_4^h + \epsilon^2 K_9^h\,\theta_1^{h-1} K_4^h]\,\theta_1^{h-1} \end{cases}$$

$$\theta_1^{h-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} I \\ K_2^h A_{21}^h E \\ K_3^h A_{31}^h E \end{bmatrix} Q^{h-1} [I \quad K_0^h A_{12}^h T \quad K_0^h A_{13}^h]$$

$$Q^{h-1} = [I + \epsilon K_8^h][I + K_1^h a^h E]$$

$$K_4^h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K_2^h A_{23}^h \\ 0 & K_3^h A_{32}^h T & 0 \end{bmatrix} \tag{29}$$

$$[I + K_1^h a^h E]\, K_0^h = K_1^h$$

$$[I + K_1^h a^h E]\, Y_{1(j)}^h = Y_{1(j)}^h$$

Through the use of the identities given in (29) we calculate:

$$Q^{h-1}[I \quad K_0^h A_{12}^h T \quad K_0^h A_{13}^h]\,\underline{H}^h = [I + \epsilon K_8^h][\,Y_1^h \cdot \underline{u}^h(0) + K_1^h \underline{f}_1^h + K_1^h A_{12}^h T\{Y_2^h \cdot \underline{v}^h(0) + K_2^h \underline{f}_2^h\} + K_1^h A_{13}^h \{Y_3^h \cdot \underline{w}^h(J) + K_3^h \underline{f}_3^h\}\,] \tag{30}$$

For some positive constant C_2 and $0 < \epsilon \le \epsilon_3$ we estimate through the use of (9, 13, 30):

$$\| Q^{h-1}[I \quad K_0^h A_{12}^h T \quad K_0^h A_{13}^h]\,\underline{H}\|_\infty \le C_2 [\,\| \underline{u}^h(0) \| + \epsilon \| \underline{v}^h(0) \| + \epsilon \| \underline{w}^h(J) \| + \| \underline{f}_1^h \|_1 + \| \underline{f}_2^h \|_\infty + \| \underline{f}_3^h \|_\infty\,] \tag{31}$$

Through the use of the results and definitions given in (20, 29, 31) we deduce the a priori bound given in (19).

##

3.3 Asymptotic Solution of the Special Difference Problem

We will use the results contained in the derivation of Theorem 3.19 to derive and asymptotic expansion of the solution of the special difference problem. This expansion is given in (32) and is the discrete analog of the expansion given in (2.31).

Corollary 3.32: Suppose the special problem (2) is regular. Then for all sufficiently small ϵ and h the unique solution of the special difference problem (7) admits the asymptotic expansion:

$$\underline{y}^h(j) \sim \hat{y}^h(j) + \theta(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

where:

$$\hat{\underline{u}}^h(j) = \begin{cases} \underline{u}_0^0 & j=0 \\ \underline{u}_0^h(j) + C_2^L Y_2^h(j)\,\underline{v}_{-1}^0 + C_3^R Y_3^h(j)\,\underline{w}_{-1}^1 & 1 \leq j \leq J \end{cases}$$

$$\hat{\underline{v}}^h(j) = Y_2^h(j)\left\{\frac{1}{\epsilon}\underline{v}_{-1}^0 + \underline{v}_0^0\right\} + K_2^h\left\{A_{21}^h E\,\hat{\underline{u}}^h + \underline{f}_2^h + A_{23}^h Y_3^h\,\underline{w}_{-1}^1\right\}(j) \tag{32}$$

$$\hat{\underline{w}}^h(j) = Y_3^h(j)\left\{\frac{1}{\epsilon}\underline{w}_{-1}^1 + \underline{w}_0^1\right\} + K_3^h\left\{A_{31}^h E\,\hat{\underline{u}}^h + \underline{f}_3^h + A_{32}^h T Y_2^h\,\underline{v}_{-1}^0\right\}(j)$$

$$\hat{\underline{u}}_0^h(j) = Y_1^h(j)\left\{\underline{u}_0^0 - C_1^L \underline{v}_{-1}^0\right\} + K_1^h \underline{F}^h(j)$$

Furthermore, if $D^h\underline{Z} \in \mathcal{D}_\infty^N[0, J-1]$ we find:

$$K_2^h \underline{Z}(j) \sim Y_2^h(j)\,A_{22}^{h^{L^{-1}}}\underline{Z}(0) - A_{22}^{h^{-1}}(j-1)\,\underline{Z}(j-1) + \theta(\epsilon) \qquad 1 \leq j \leq J$$

$$\text{as } \epsilon \to 0^+$$

$$K_3^h \underline{Z}(j) \sim Y_3^h(j)\,A_{33}^{h^{R^{-1}}}\underline{Z}(J) - A_{33}^{h^{-1}}(j)\,\underline{Z}(j) + \theta(\epsilon) \qquad 0 \leq j \leq J-1$$

Proof: We recall from (7) that the boundary-values for the special problem are:

$$\begin{bmatrix} \underline{u}^h(0) \\ \underline{v}^h(0) \\ \underline{w}^h(J) \end{bmatrix} \sim \frac{1}{\epsilon}\begin{bmatrix} 0 \\ \underline{v}^0_{-1} \\ \underline{w}^1_{-1} \end{bmatrix} + \begin{bmatrix} \underline{u}^0_0 \\ \underline{v}^0_0 \\ \underline{w}^1_0 \end{bmatrix} + \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{33}$$

From the definitions (20) and identities (29) we find:

$$\underline{y}^h = \left[I + \epsilon\, \Theta_1^{h^{-1}} K_4^h + \epsilon^2 K_9^h \Theta_1^{h^{-1}} K_4^h \right] \Theta_1^{h^{-1}} \underline{H}^h \tag{34}$$

Since K_9^h, $\Theta_1^{h^{-1}}$, K_4^h are bounded linear operators and:

$$\| \underline{H}^h \|_\infty \sim \mathcal{O}\left(\frac{1}{\epsilon}\right) \qquad \text{as } \epsilon \to 0^+$$

we estimate:

$$\| \epsilon^2 K_9^h \Theta_1^{h^{-1}} K_4^h \Theta_1^{h^{-1}} \underline{H}^h \|_\infty \sim \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{35}$$

By calculations similar to those used to derive (23) we find:

$$\left.\begin{aligned} K_0^h A_{12}^h T K_2^h \underline{f}_2^h &\sim -K_0^h A_{12}^h A_{22}^{h^{-1}} \underline{f}_2^h + \mathcal{O}(\epsilon) \\ K_0^h A_{13}^h K_3^h \underline{f}_3^h &\sim -K_0^h A_{13}^h A_{33}^{h^{-1}} \underline{f}_3^h + \mathcal{O}(\epsilon) \end{aligned}\right\} \text{ as } \epsilon \to 0^+ \tag{36}$$

We note for $1 \le j \le J$:

$$\left.\begin{aligned} K_1^h A_{12}^h T Y_2^h(j) &\sim \epsilon C_2^L Y_2^h(j) - \epsilon Y_1^h(j) C_1^L + \mathcal{O}(\epsilon^2) \\ K_1^h A_{13}^h Y_3^h(j) &\sim \epsilon C_3^R Y_3^h(j) + \mathcal{O}(\epsilon^2) \end{aligned}\right\} \text{ as } \epsilon \to 0^+ \tag{37}$$

To derive the expansions given in (37) we argue as follows:

$$K_1^h A_{12}^h T Y_2^h(j) = h \sum_{k=0}^{j-1} Y_1^h(j,k) [I + \tfrac{h}{2} a^h(k)]^{-1} A_{12}^h(k) Y_2^h(k+1)$$

... $Y_2^h(k+1) = \epsilon A_{22}^h(k) D^h Y_2^h(k)$, for $j \geq 1$ sum by parts using (14), estimate the remaining sum as $\mathcal{O}(\epsilon^2)$.

$$\sim \epsilon [I - \tfrac{h}{2} a^h(j-1)]^{-1} A_{12}^h(j-1) A_{22}^{h^{-1}}(j-1) Y_2^h(j) - \epsilon Y_1^h(j) [I + \tfrac{h}{2} a^h(0)]^{-1} A_{12}^h(0) A_{22}^{h^{-1}}(0) + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0^+$$

... use the Lipschitz continuity of a, A_{ij} with respect to x and (18) to estimate the error in replacing $j-1$ by 0 as $\mathcal{O}(\epsilon^2)$.

$$\sim \epsilon C_1^L Y_2^h(j) - \epsilon Y_1^h(j) C_2^L + \mathcal{O}(\epsilon^2) \qquad \text{as } \epsilon \to 0^+$$

$$K_1^h A_{13}^h Y_3^h(j) = h \sum_{k=0}^{j-1} Y_1^h(j,k) [I + \tfrac{h}{2} a^h(k)]^{-1} A_{13}^h(k) Y_3^h(k)$$

... $Y_3^h(k) = \epsilon A_{33}^{h^{-1}}(k) D^h Y_3^h(k)$, for $j \geq 1$ sum by parts using (14), estimate the remaining sum as $\mathcal{O}(\epsilon^2)$.

$$\sim \epsilon [I - \tfrac{h}{2} a^h(j-1)]^{-1} A_{13}^h(j-1) A_{33}^{h^{-1}}(j-1) Y_3^h(j) - \epsilon Y_1^h(j) [I + \tfrac{h}{2} a^h(0)]^{-1} A_{13}^h(0) A_{33}^{h^{-1}}(0) Y_3^h(0) + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0^+$$

... for $h \leq 1$ use (18) to estimate $Y_3^h(0)$ as $\mathcal{O}(\epsilon^2)$, use the

Lipschitz continuity of a, A_{ij} with respect to x and (18) to estimate the error in replacing $j-1$ by $J-1$ as $\theta(\epsilon^2)$.

$$\sim \epsilon C_3^R Y_3^h(j) + \theta(\epsilon^2) \qquad \text{as } \epsilon \to 0^+$$

Combining the results given in (30, 33, 36, 37) we find:

$$Q^{h^{-1}}[I \ \ K_0^h A_{12}^h T \ \ K_0^h A_{13}^h] \underline{H}^h \sim \underline{\hat{u}}^h(j) + \theta(\epsilon^2) \qquad \text{as } \epsilon \to 0^+ \tag{38}$$

where:

$$\begin{aligned} \underline{\hat{u}}^h(j) &= \begin{cases} \underline{u}_0^0 \\ \underline{u}_0^h(j) + C_2^L Y_2^h(j) \underline{v}_{-1}^0 + C_3^R Y_3^h(j) \underline{w}_{-1}^1 \end{cases} \\ \underline{u}_0^h(j) &= Y_1^h(j) \{ \underline{u}_0^0 - C_1^L \underline{v}_{-1}^0 \} + K_1^h \underline{F}^h(j) \end{aligned} \tag{39}$$

From (29, 33, 38) we find $\theta_1^{h^{-1}} \underline{H}^h$, the first term in (34), admits the asymptotic expansion:

$$\theta_1^{h^{-1}} \underline{H}^h \sim \frac{1}{\epsilon} \begin{bmatrix} \underline{0} \\ Y_2^h \cdot \underline{v}_{-1}^0 \\ Y_3^h \cdot \underline{w}_{-1}^1 \end{bmatrix} + \begin{bmatrix} \underline{\hat{u}}^h \\ Y_2^h \cdot \underline{v}_0^0 + K_2^h \{ A_{21}^h E \underline{\hat{u}}^h + \underline{f}_{-2}^h \} \\ Y_3^h \cdot \underline{w}_0^0 + K_3^h \{ A_{31}^h E \underline{\hat{u}}^h + \underline{f}_{-3}^h \} \end{bmatrix} + \theta(\epsilon^2) \tag{40}$$

From the definition of K_4^h given in (20) and the estimate given in (40) we find:

$$\epsilon K_4^h \theta_1^{h^{-1}} \underline{H}^h \sim \begin{bmatrix} \underline{0} \\ K_2^h A_{23}^h Y_3^h \cdot \underline{w}_{-1}^1 \\ K_3^h A_{32}^h T Y_2^h \cdot \underline{v}_{-1}^0 \end{bmatrix} + \theta(\epsilon) \tag{41}$$

Using the identities given in (29) and the estimate (41) we find:

$$\epsilon Q^{h^{-1}} [I \;\; K_0^h A_{12}^h T \;\;\; K_0^h A_{13}^h] K_4^h \theta_1^{h^{-1}} \underline{H}^h \sim \begin{bmatrix} \underline{0} \\ K^h A_{12}^h T K_2^h A_{23}^h Y_3^h \underline{w}_{-1}^1 \\ K_1^h A_{13}^h K_3^h A_{31}^h T Y_2^h \underline{v}_{-1}^0 \end{bmatrix} + O(\epsilon^2) \qquad (42)$$

Through the use of (9, 13) we estimate the terms in the above expression as follows:

$$\| K_1^h A_{12}^h T K_2^h A_{23}^h Y_3^h(j) \| \sim O\left(h \sum_{k=0}^{j-1} \frac{h}{\epsilon} \sum_{\ell=0}^{k} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{k+1-\ell}} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{j-\ell}} \right)$$

$$\sim O\left(h \sum_{k=0}^{j-1} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{j-k}} \cdot \frac{h}{\epsilon} \sum_{\ell=0}^{k} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{k+1-\ell}} \right)$$

$$\sim O(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

$$\| K_1^h A_{13}^h K_3^h A_{32}^h T Y_2^h(j) \| \sim O\left(h \sum_{k=0}^{j-1} \frac{h}{\epsilon} \sum_{\ell=k}^{J-1} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{\ell+1-k}} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{\ell+1}} \right)$$

$$\sim O\left(h \sum_{k=0}^{j-1} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{k+1}} \cdot \frac{h}{\epsilon} \sum_{\ell=k}^{J-1} \frac{1}{(1+\Delta\frac{h}{\epsilon})^{\ell+1-k}} \right)$$

$$\sim O(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

The above bounds allow us to estimate (42) as follows:

$$\epsilon Q^{h^{-1}} [I \;\; K_0^h A_{12}^h T \;\;\; K_0^h A_{13}^h] K_4^h \theta_1^{h^{-1}} \underline{H}^h \sim O(\epsilon) \qquad \text{as } \epsilon \to 0^+ \qquad (43)$$

Combining (29, 41, 43) we find $\epsilon \theta_1^{h^{-1}} K_4^h \theta_1^{h^{-1}} \underline{H}^h$, the second term in (34), admits the asymptotic expansion:

$$\epsilon \theta_1^{h^{-1}} K_4^h \theta_1^{h^{-1}} \underline{H}^h \sim \begin{bmatrix} \underline{0} \\ K_2^h A_{23}^h Y_3^h \cdot \underline{w}_{-1}^1 \\ K_3^h A_{32}^h T Y_2^h \cdot \underline{v}_{-1}^0 \end{bmatrix} + O(\epsilon) \qquad \text{as } \epsilon \to 0^+ \qquad (44)$$

By combining the estimates given in (35, 40, 44) we obtain from (34) the asymptotic expansion given in (32). The estimates of $K_2^h \underline{Z}$ and $K_3^h \underline{Z}$ given in (32) are obtained by using (15), preforming a summation by parts using (14), and estimating the remaining sum by using (9) and (13). ##

3.4 Fundamental Matrix for the Special Difference Problem

Analogous to the definition of the fundamental matrix Z_0 given in (2.60) we make the following:

Definition: Let $Z_0^h(j)$ be that matrix which satisfies the boundary-value problem:

$$\mathcal{L}_h Z_0^h(j) = 0 \qquad \mathcal{B}^* Z_0^h = I \tag{45}$$

Due to the important role the matrix Z_0^h plays in the general difference problem we call Z_0^h the fundamental matrix associated with the special difference problem. We derive an asymptotic expansion for the boundary-values assumed by Z_0^h in the following:

Corollary 3.46: Suppose the special problem (2) is regular. Then for all sufficiently small ϵ and h the matrix $Z_0^h(j)$ defined by (45) exists, is unique, and admits the asymptotic expansion:

$$Z_0^h(j) \sim \hat{Z}_0^h(j) + \theta(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{46'}$$

where:

$$\hat{Z}_0^h(0) = \begin{bmatrix} I & 0 & 0 \\ 0 & \frac{1}{\epsilon} I & 0 \\ -A_{33}^{hL^{-1}} A_{31}^{hL} E Y_1^h(0) & \mathcal{P}_5^h & 0 \end{bmatrix}$$

$$\hat{Z}_0^h(J) = \begin{bmatrix} Y_1^h(J) & -Y_1^h(J) C_1^L & C_3^R \\ -A_{22}^{hR^{-1}} A_{23}^{hR} E Y_1^h(J-1) & A_{22}^{hR^{-1}} A_{21}^{hR} E Y_1^h(J-1) C_1^L & \mathcal{P}_2^h \\ 0 & 0 & \frac{1}{\epsilon} I \end{bmatrix} \qquad (46')$$

$$\mathcal{P}_2^h = K_2^h \{ A_{21}^h E C_3^R Y_3^h + A_{23}^h Y_3^h \}(J)$$

$$\mathcal{P}_5^h = K_3^h \{ A_{31}^h E C_2^L Y_2^h + A_{32}^h T Y_2^h \}(0) + A_{33}^{hL^{-1}} A_{31}^{hL} E Y_1^h(0) C_1^L + \mathcal{P}_6^h$$

$$\mathcal{P}_6^h = \frac{1}{2} \frac{h}{\epsilon} Y_3^h(0,1) A_{31}^{hL} (C_2^L - C_1^L)$$

Proof: We derive below the asymptotic expansion of the second block column of Z_0^h. The asymptotic expansions of the remaining block columns of Z_0^h are determined in a similar manner.

Since each column of Z_0^h satisfies the special difference problem, Theorem 3.19 establishes the existence and uniqueness of Z_0^h for all sufficiently small ϵ and h. We obtain the desired asymptotic estimates of Z_0^h through the use of Corollary 3.32.

Let the second block column of Z_0^h be represented by:

$$Z_{02}^h(j) = \begin{bmatrix} U^h(j) \\ V^h(j) \\ W^h(j) \end{bmatrix}$$

From the definition of Z_0^h given in (45) we deduce Z_{02}^h satisfies the special difference problem:

$$\mathcal{L}_h Z_{02}^h = 0 \qquad\qquad \mathcal{B}^* Z_{02}^h = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}$$

Using Corollary 3.32 we therefore find:

$$\begin{aligned} U^h_{(j)} &\sim \left\{ \begin{matrix} 0 & j=0 \\ -Y^h_{1(j)}\, C_1^L + C_2^L\, Y^h_{2(j)} & 1 \le j \le J \end{matrix} \right\} + \mathcal{O}(\epsilon) \\ V^h_{(j)} &\sim \tfrac{1}{\epsilon} Y^h_{3(j)} + K_2^h A^h_{21} E\, U^h_{(j)} + \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \\ W^h_{(j)} &\sim K_3^h \{ A^h_{31} E\, U^h + A^h_{32} T\, Y_2^h \}_{(j)} + \mathcal{O}(\epsilon) \end{aligned} \tag{47}$$

From (9, 13, 18) and the last estimates given in (32) we find:

$$\begin{aligned} K_2^h A^h_{21} E\, U^h_{(J)} &\sim -K_2^h A^h_{21} E\, Y^h_{1(J)}\, C_1^L + \mathcal{O}(\epsilon) \\ &\sim A_{22}^{h^{R^{-1}}} A_{21}^{h^R} E\, Y^h_{1(J-1)}\, C_1^L + \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \\ K_3^h A^h_{31} E\, U^h_{(0)} &= K_3^h A^h_{31} E \{ C_2^L Y_2^h - C_1^L Y_1^h \}_{(0)} + \mathcal{P}_6^h \\ &\sim K_3^h A^h_{31} E\, C_2^L Y^h_{2(0)} + A_{33}^{h^{L^{-1}}} A_{31}^{h^L} E\, C_1^L Y^h_{1(0)} + \mathcal{P}_6^h + \mathcal{O}(\epsilon) \end{aligned} \tag{48}$$

The matrix $\mathcal{P}_6^h$ is defined to be:

$$\mathcal{P}_6^h = \frac{1}{2}\, \frac{h}{\epsilon}\, Y^h_{3(0,1)}\, A_{31}^{h^L} \{ C_2^L - C_1^L \} \tag{49}$$

From the definitions of C_2^L, C_1^L we find $\mathcal{P}_6$ is $\mathcal{O}(h)$.

Combining (47, 48, 49) we obtain the asymptotic expansions of Z^h_{02} given in (46). ##

The most significant difference between the expansions given in (2.60) and (46) occurs in the terms $\mathcal{P}_2$, $\mathcal{P}_5$ and $\mathcal{P}_2^h$, $\mathcal{P}_5^h$ respectively. For $h \gg \epsilon$ detailed calculations of the differences $\mathcal{P}_2 - \mathcal{P}_2^h$, $\mathcal{P}_5 - \mathcal{P}_5^h$ show them to be of order one. This difference explains why we have made assumption (3b), for when (3b) is true the errors in $\underline{v}^h(J), \underline{w}^h(0)$ cannot affect the determination of the boundary-values assumed by the solution of the general problem. However for $h \ll \epsilon$ we find the differences $\mathcal{P}_2 - \mathcal{P}_2^h$, $\mathcal{P}_5 - \mathcal{P}_5^h$ are of order $\frac{h}{\epsilon}$. This fact may be proved by appealing to the stability and consistency of the scheme as $h \to 0^+$. We should also note the dependence of the matrices $A^h_{i\ell}(j)$ upon j and ϵ found in $\mathcal{P}_2^h$, $\mathcal{P}_5^h$ does not occur in their analogs $\mathcal{P}_2$, $\mathcal{P}_5$. Through the use of (9, 13, 18) we estimate the error incurred by replacing $A^h_{i\ell}(j)$ by $A_{i\ell}(x_{J-\frac{1}{2}}, 0)$ in $\mathcal{P}_2^h$ is of order ϵ . In a similar manner we estimate the error incurred by replacing $A^h_{i\ell}(j)$ by $A_{i\ell}(x_{\frac{1}{2}}, 0)$ in $\mathcal{P}_5^h$ is of order ϵ . Therefore we conclude the dependence of $A^h_{i\ell}$ upon j and ϵ in $\mathcal{P}_2^h$, $\mathcal{P}_5^h$ is inessential.

3.5 Properties of the Solution of the General Difference Problem

The main results of this section are given in Lemma 3.57 and Corollary 3.72 . To prove these results we need the following lemmas.

Lemma 3.50: Suppose the special problem (2) is regular. Let $\underline{u}_0(x)$, $\underline{u}_0^h(j)$ be the solutions of the following initial-value problems:

$$D\underline{u}_0(x) = a(x)\,\underline{u}_0(x) + \underline{F}(x) \qquad 0 \le x \le 1 \qquad \underline{u}_0(0) = \underline{\alpha}$$

$$D^h\underline{u}_0^h(j) = a^h(j)\,E\underline{u}_0^h(j) + \underline{F}^h(j) \qquad 0 \le j \le J-1 \qquad \underline{u}_0^h(0) = \underline{\alpha}$$

Then for sufficiently small h there exist continuous functions $\{\underline{u}_m(x)\}_1^\infty$ independent of h and ϵ such that for every N :

$$\underline{u}_0^h(j) \sim \underline{u}_0(x_j) + \sum_{\ell=1}^{N-1} h^{2\ell}\,\underline{u}_m(x_j) + O(h^{2N}+\epsilon) \quad \text{as } \epsilon, h \to 0^+ \tag{50}$$

Proof: The regularity of the special problem (2) implies the matrices $A_{ij}(x,\epsilon)$ and vectors $\underline{f}_i(x,\epsilon)$ are infinitely differentiable functions of x and ϵ for $(x,\epsilon) \varepsilon I \times E_1$. Recalling the definitions (8) and (2.9) these continuity properties imply:

$$\left.\begin{aligned} a^h(j) &\sim a(x_{j+\frac{1}{2}}) + O(\epsilon) \\ \underline{F}^h(j) &\sim \underline{F}(x_{j+\frac{1}{2}}) + O(\epsilon) \end{aligned}\right\} \qquad 0 \le j \le J-1 \qquad \text{as } \epsilon \to 0^+$$

These estimates are independent of h . From the stability of the difference scheme used to determine $\underline{u}_0^h$, see Keller [8], we find:

$$\underline{u}_0^h(j) \sim \hat{\underline{u}}_0^h(j) + O(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{51}$$

where $\hat{\underline{u}}_0^h$ is the solution of the initial-value problem:

$$D\hat{\underline{u}}_0^h(j) = \mathcal{Q}(x_{j+\frac{1}{2}})\,E\,\hat{\underline{u}}_0^h(j) + \underline{F}(x_{j+\frac{1}{2}}) \qquad \hat{\underline{u}}^h(0) = \underline{\alpha}$$

Using the continuity properties of $\mathcal{Q}(x)$ and $\underline{F}(x)$ we obtain, as shown in Keller [8], the asymptotic expansion:

$$\hat{\underline{u}}_0^h(j) \sim \underline{u}_0(x_j) + \sum_{\ell=1}^{N-1} h^{2\ell}\,\underline{u}_\ell(x_j) + \mathcal{O}(h^{2N}) \qquad \text{as } h \to 0^+ \tag{52}$$

The functions $\{\underline{u}_m(x)\}_1^\infty$ are continuous and independent of h while N is any fixed positive integer. Combining the estimates (51, 52) we obtain the asymptotic expansion given in (50). ##

We note the general and special difference schemes (6, 7) can be equivalently formulated as the following linear systems:

$$\begin{aligned} \mathbb{G}_h \mathbb{Y}^h &= \mathbb{F}_h && \text{... the general difference scheme} \\ \mathbb{S}_h \mathbb{Y}^h &= \mathbb{F}_h^* && \text{... the special difference scheme} \end{aligned} \tag{53}$$

The matrices $\mathbb{G}_h$, $\mathbb{S}_h$ and vectors $\mathbb{Y}^h, \mathbb{F}_h, \mathbb{F}_h^*$ are defined to be:

$$\mathbb{Y}^h = \begin{bmatrix} \underline{y}^h(0) \\ \cdot \\ \cdot \\ \cdot \\ \underline{y}^h(J) \end{bmatrix} \qquad \mathbb{F}_h = \begin{bmatrix} \underline{g}(\epsilon) \\ \underline{f}^h(0) \\ \cdot \\ \cdot \\ \cdot \\ \underline{f}^h(J-1) \end{bmatrix} \qquad \mathbb{F}_h^* = \begin{bmatrix} \underline{g}^*(\epsilon) \\ \underline{f}^h(0) \\ \cdot \\ \cdot \\ \cdot \\ \underline{f}^h(J-1) \end{bmatrix} \tag{54'}$$

$$\mathbb{B}_h = [\,I_m, 0, \ldots, 0\,]^T \qquad \text{... an } m(J+1)\times m \text{ matrix}$$

$$\mathbb{C}_h = [\,L(\epsilon) - L^*(\epsilon), 0, \ldots, 0, R(\epsilon) - R^*(\epsilon)\,] \quad \text{... an } m\times m(J+1) \text{ matrix}$$

$$\mathbb{S}_h = \begin{bmatrix} L^*(\epsilon) & & & & R^*(\epsilon) \\ G_0 & H_0 & & 0 & \\ & \ddots & \ddots & & \\ & & & & \\ 0 & & & G_{J-1} & H_{J-1} \end{bmatrix}$$

$$\mathbb{G}_h = \begin{bmatrix} L(\epsilon) & & & & R(\epsilon) \\ G_0 & H_0 & & 0 & \\ & \ddots & \ddots & & \\ & & & & \\ 0 & & & G_{J-1} & H_{J-1} \end{bmatrix} \tag{54'}$$

$$G_j = -\frac{1}{h}\Omega(\epsilon) - \begin{bmatrix} \frac{1}{2}A^h_{11}(j) & 0 & A^h_{13}(j) \\ \frac{1}{2}A^h_{21}(j) & 0 & \epsilon A^h_{23}(j) \\ \frac{1}{2}A^h_{31}(j) & 0 & A^h_{33}(j) \end{bmatrix}$$

$$0 \leq j \leq J-1$$

$$H_j = \frac{1}{h}\Omega(\epsilon) - \begin{bmatrix} \frac{1}{2}A^h_{11}(j) & A^h_{12}(j) & 0 \\ \frac{1}{2}A^h_{21}(j) & A^h_{22}(j) & 0 \\ \frac{1}{2}A^h_{31}(j) & \epsilon A^h_{33}(j) & 0 \end{bmatrix}$$

The following lemma will be used to relate the nonsingularity of the matrices $\mathbb{S}_h$ and $\mathbb{G}_h$.

<u>Lemma 3.55:</u> (Woodbury Formula) Suppose the matrices A, B, C, D have the dimensions:

$A \ldots N \times N \qquad D \ldots m \times m$

$B \ldots N \times m \qquad C \ldots m \times N$

Let the matrices A, D be nonsingular. Then the matrix:

$$H = A + BDC$$

is nonsingular iff the matrix $D^{-1} + CA^{-1}B$ is nonsingular. Furthermore, if H is nonsingular then:

$$H^{-1} = A^{-1} - A^{-1}B[D^{-1} + CA^{-1}B]^{-1}CA^{-1} \qquad (55)$$

Proof: Suppose the matrix $D^{-1} + CA^{-1}B$ is nonsingular. From the result :

$$\{A^{-1} - A^{-1}B[D^{-1} + CA^{-1}B]^{-1}CA^{-1}\}\{A + BDC\} = I$$

we deduce the matrix H is nonsingular and H^{-1} has the representation given in (55).

Suppose the matrix H is nonsingular. Let $\underset{\sim}{y}$ satisfy:

$$[D^{-1} + CA^{-1}B]\,\underset{\sim}{y} = \underline{0} \qquad (56)$$

Multiplying (56) by BD and rearranging terms we find:

$$[A + BDC]A^{-1}B\underset{\sim}{y} = \underline{0}$$

Since H was assumed to be nonsingular we conclude $A^{-1}B\underset{\sim}{y} = \underline{0}$. Using this fact in (56) we find $\underset{\sim}{y} = \underline{0}$. Therefore the nonsingularity of H implies $D^{-1} + CA^{-1}B$ is nonsingular. ##

The Woodbury formula (55) is the basis of the method of rank annihilation , a method which is used to calculate the inverse of a matrix. See Noble [9], section 5.7, for other uses of (55).

We now consider the relationship between the solutions of the special problems (2) and (7).

Lemma 3.57: Suppose the special problem (2) is regular. Let $\underset{\sim}{y}_0(x)$ be the solution of the reduced problem corresponding to (2) and $\underset{\sim}{y}^h(j)$ the solution of the special difference problem (7). Then there exist continuous functions $\{\underset{\sim}{y}_m(x)\}_1^\infty$ independent of h and ϵ such that for $\delta > 0$ and $\delta \leq x_j \leq 1-\delta$:

$$\underset{\sim}{y}^h(j) \sim \underset{\sim}{y}_0(x_j) + \sum_{\ell=1}^{N-1} h^\ell \underset{\sim}{y}_\ell(x_j) + \mathcal{O}(h^N + \epsilon) \quad \text{as } \epsilon, h \to 0^+ \tag{57}$$

Here N is any fixed positive integer. Furthermore, if the boundary conditions for (2) and (7) have $\underset{\sim}{g}^*(\epsilon) = \underline{0}$ then as $\epsilon, h \to 0^+$:

$$\begin{aligned} \underline{u}^h(j) &\sim \underline{u}_0(x_j) + \sum_{\ell=1}^{N-1} h^\ell \underline{u}_\ell(x_j) + \mathcal{O}(h^N + \epsilon) && 0 \leq x_j \leq 1 \\ \underline{v}^h(j) &\sim \underline{v}_0(x_j) + \sum_{\ell=1}^{N-1} h^\ell \underline{v}_\ell(x_j) + \mathcal{O}(h^N + \epsilon) && \delta \leq x_j \leq 1 \\ \underline{w}^h(j) &\sim \underline{w}_0(x_j) + \sum_{\ell=1}^{N-1} h^\ell \underline{w}_\ell(x_j) + \mathcal{O}(h^N + \epsilon) && 0 \leq x_j \leq 1-\delta \end{aligned} \tag{58}$$

Here N is any fixed positive integer and the vectors $\{\underset{\sim}{y}_m(x)\}_1^\infty$ which occur in (57) have been partitioned into the form $\{[\underline{u}_m^T(x), \underline{v}_m^T(x), \underline{w}_m^T(x)]^T\}_1^\infty$.

Proof: Let $\underset{\sim}{y}^*(x)$, $\underset{\sim}{y}^{*h}(j)$ be the solutions of the following boundary-value problems:

$$\begin{aligned} \mathcal{L}\, \underset{\sim}{y}^*(x) &= \underline{f}(x,\epsilon) & \mathcal{B}^* \underset{\sim}{y}^* &= \underline{0} \\ \mathcal{L}_h \underset{\sim}{y}^{*h}(j) &= \underline{f}^h(j) & \mathcal{B}^*_h \underset{\sim}{y}^{*h} &= \underline{0} \end{aligned} \tag{59}$$

Introducing the partition $\underline{y}^* = [\underline{u}^{*T}, \underline{v}^{*T}, \underline{w}^{*T}]^T$ we find from (2.31):

$$\begin{aligned}
\underline{u}^*(x) &\sim \underline{u}_0^*(x) + O(\epsilon) && 0 \le x \le 1 \\
\underline{v}^*(x) &\sim -A_{22}^{-1}(x)\{A_{21}(x)\,\underline{u}_0^*(x) + \underline{f}_2(x)\} + O(\epsilon) && \delta \le x \le 1 \\
& && \text{as } \epsilon \to 0^+ \qquad (60) \\
\underline{w}^*(x) &\sim -A_{33}^{-1}(x)\{A_{31}(x)\,\underline{u}_0^*(x) + \underline{f}_3(x)\} + O(\epsilon) && 0 \le x \le 1-\delta \\
\underline{u}_0^*(x) &= \int_0^x Y_1(x,\tau)\,\underline{F}(\tau)\,d\tau
\end{aligned}$$

Introducing the partition $\underline{y}^{*h} = [\underline{u}^{*h^T}, \underline{v}^{*h^T}, \underline{w}^{*h^T}]^T$ we find from (32):

$$\begin{aligned}
\underline{u}^{*h}(j) &\sim \underline{u}_0^{*h}(j) + O(\epsilon) && 0 \le x_j \le 1 \\
\underline{v}^{*h}(j) &\sim -A_{22}^{h^{-1}}(j-1)\{A_{21}^h(j-1)\,E\,\underline{u}_0^{*h}(j-1) + \underline{f}_2^h(j-1)\} + O(\epsilon) && \delta \le x_j \le 1 \\
& && \text{as } \epsilon, h \to 0^+ \qquad (61) \\
\underline{w}^{*h}(j) &\sim -A_{33}^{h^{-1}}(j)\{A_{31}^h(j)\,E\,\underline{u}_0^{*h}(j) + \underline{f}_3^h(j)\} + O(\epsilon) && 0 \le x_j \le 1-\delta \\
\underline{u}_0^{*h}(j) &= h\sum_{k=0}^{j-1} Y_1^h(j,k)\,[I + \tfrac{h}{2}\,a^h(k)]^{-1}\,\underline{F}^h(k)
\end{aligned}$$

We note $\underline{u}_0^*$ and $\underline{u}_0^{*h}$ are the solutions of the initial-value problems:

$$D\,\underline{u}_0^*(x) = a(x)\,\underline{u}_0^*(x) + \underline{F}(x) \qquad \underline{u}_0^*(0) = \underline{0}$$

$$D\,\underline{u}_0^{*h}(j) = a^h(j)\,E\,\underline{u}_0^{*h}(j) + \underline{F}^h(j) \qquad \underline{u}_0^{*h}(0) = \underline{0}$$

By Lemma 3.50 there exist continuous functions $\{\underline{u}_m^*(x)\}_1^\infty$ independent of ϵ and h such that:

$$\underline{u}_0^{*h}(j) \sim \underline{u}_0^*(x_j) + \sum_{\ell=1}^{N-1} h^\ell \underline{u}_m^*(x_j) + \mathcal{O}(h^N+\epsilon) \quad \text{as } \epsilon, h \to 0^+ \tag{62}$$

Here N is any fixed positive integer. From the continuity properties satisfied by A_{ij} and $\underline{f}_i$ we find:

$$\left.\begin{aligned} A_{i\ell}^h(j) &\sim A_{i\ell}(x_j+\tfrac{1}{2}h) + \mathcal{O}(\epsilon) \\ \underline{f}_i^h(j) &\sim \underline{f}_i(x_j+\tfrac{1}{2}h) + \mathcal{O}(\epsilon) \end{aligned}\right\} \quad \text{as } \epsilon \to 0^+ \tag{63}$$

Substituting (62, 63) into (61) and preforming further expansions in powers of h we obtain the asymptotic expansions given in (58).

Now suppose $\underline{g}^*(\epsilon) \neq \underline{0}$. We note the solutions of (2) and (7) may be written as:

$$\begin{aligned} \underline{y}(x) &= \underline{y}^*(x) + \tilde{\underline{y}}(x) \\ \underline{y}^h(j) &= \underline{y}^{*h}(j) + \tilde{\underline{y}}^h(j) \end{aligned} \tag{64}$$

Here $\underline{y}^*(x)$, $\underline{y}^{*h}(j)$ are the solutions of (59) and $\tilde{\underline{y}}(x), \tilde{\underline{y}}^h(j)$ are the solutions of the boundary-value problems:

$$\begin{aligned} \mathcal{L}\,\tilde{\underline{y}}(x) &= \underline{0} & \qquad \mathcal{B}^*\tilde{\underline{y}} &= \underline{g}^*(\epsilon) \\ \mathcal{L}_h\,\tilde{\underline{y}}^h(j) &= \underline{0} & \qquad \mathcal{B}^*\tilde{\underline{y}}^h &= \underline{g}^*(\epsilon) \end{aligned} \tag{65}$$

From (9) and (18) we find for $0 < h \leq \frac{1}{2\delta}$:

$$\left.\begin{aligned} \| Y_2^h(j) \| &\sim \mathcal{O}(\epsilon^2) & \qquad \delta &\leq x_j \leq 1 \\ \| Y_3^h(j) \| &\sim \mathcal{O}(\epsilon^2) & \qquad 0 &\leq x_j \leq 1-\delta \end{aligned}\right\} \quad \text{as } \epsilon \to 0^+ \tag{66}$$

Introducing the partition $\tilde{\underline{y}} = [\tilde{\underline{u}}^T, \tilde{\underline{v}}^T, \tilde{\underline{w}}^T]^T$ we find from (2.31):

$$\left.\begin{aligned} \tilde{\underline{u}}(x) &\sim \tilde{\underline{u}}_0(x) + \theta(\epsilon) \\ \tilde{\underline{v}}(x) &\sim -A_{22}^{-1}(x)\, A_{21}(x)\, \tilde{\underline{u}}_0(x) + \theta(\epsilon) \\ \tilde{\underline{w}}(x) &\sim -A_{33}^{-1}(x)\, A_{31}(x)\, \tilde{\underline{u}}_0(x) + \theta(\epsilon) \\ \tilde{\underline{u}}_0(x) &= Y_1(x)\,\{\underline{u}_0^0 - A_{12}^L A_{22}^L \underline{v}_{-1}^0\} \end{aligned}\right\} \quad \begin{aligned} &\delta \le x \le 1-\delta \\ &\text{as } \epsilon \to 0^+ \end{aligned} \qquad (67)$$

Introducing the partition $\tilde{\underline{y}}^h = [\tilde{\underline{u}}^{hT}, \tilde{\underline{v}}^{hT}, \tilde{\underline{w}}^{hT}]^T$ we find from (32) and the estimates (66) :

$$\left.\begin{aligned} \tilde{\underline{u}}^h(j) &\sim \tilde{\underline{u}}_0^h(j) + \theta(\epsilon) \\ \tilde{\underline{v}}^h(j) &\sim -A_{22}^{h-1}(j-1)\, A_{21}^h(j-1)\, E\,\tilde{\underline{u}}_0^h(j) + \theta(\epsilon) \\ \tilde{\underline{w}}^h(j) &\sim -A_{33}^{h-1}(j)\, A_{31}^h(j)\, E\,\tilde{\underline{u}}_0^h(j) + \theta(\epsilon) \\ \tilde{\underline{u}}_0^h(j) &= Y_1^h(j)\,\{\underline{u}_0^0 - C_1^L \underline{v}_{-1}^0\} \end{aligned}\right\} \quad \begin{aligned} &\delta \le x_j \le 1-\delta \\ &\text{as } \epsilon \to 0^+ \end{aligned} \qquad (68)$$

Recall $Y_1(x), Y_1^h(j)$ satisfy the initial-value problems:

$$D\,Y_1(x) = a(x)\,Y_1(x) \qquad Y_1(0) = I$$

$$D^h\,Y_1^h(j) = a^h(j)\,E\,Y_1^h(j) \qquad Y_1^h(0) = I$$

By an application of Lemma 3.50 to each column of $Y_1(x)$, $Y_1^h(x)$ we infer the existence of continuous matrix-valued functions $\{W_m(x)\}_1^\infty$ independent of ϵ and h such that:

$$Y_1^h(j) \sim Y_1(x_j) + \sum_{\ell=1}^{N-1} h^{2\ell} W_\ell(x_j) + \theta(h^{2N}+\epsilon) \quad \text{as } \epsilon, h \to 0^+ \tag{69}$$

Here N is any positive integer.

From the definition of C_1^L in (8) and the asymptotic expansions given in (63, 69) we infer the existence of continuous functions $\{\underline{\tilde{u}}_m(x)\}_1^\infty$ independent of ϵ and h such that:

$$\underline{\tilde{u}}_0^h(j) \sim \underline{\tilde{u}}_0(x_j) + \sum_{\ell=1}^{N-1} h^\ell \underline{\tilde{u}}_\ell(x_j) + \theta(h^N+\epsilon) \quad \text{as } \epsilon, h \to 0^+ \tag{70}$$

Here N is any fixed positive integer.

Substituting (63, 70) into (68) and preforming further expansions in powers of h we infer the existence of continuous functions $\{\underline{\tilde{y}}_m(x)\}_1^\infty$ independent of ϵ and h such that for $\delta \le x_j \le 1-\delta$:

$$\underline{\tilde{y}}^h(j) \sim \underline{\tilde{y}}_0(x_j) + \sum_{\ell=1}^{N-1} h^\ell \underline{\tilde{y}}_\ell(x) + \theta(h^N+\epsilon) \quad \text{as } \epsilon, h \to 0^+ \tag{71}$$

Combining the asymptotic expansions given in (58, 64, 71) we obtain the asymptotic expansion given in (57). ##

Through the use of the preceding lemma we prove the following:

<u>Corollary 3.72:</u> Suppose the general problem (1) satisfies the conditions given in (3). Then for all ϵ and h sufficiently small, the general difference problem (6) has a unique solution. Furthermore, for $\delta > 0$ and $\delta \le x_j \le 1-\delta$ this solution admits the asymptotic expansion:

$$y^h(j) \sim y_0^h(x_j) + \sum_{\ell=1}^{N-1} h^\ell y_\ell(x_j) + \mathcal{O}(h^N + \epsilon) \quad \text{as } \epsilon, h \to 0^+ \tag{72}$$

Here N is any positive integer, $y_0(x)$ is the solution of the reduced problem corresponding to (1), and $\{y_n(x)\}_1^\infty$ are continuous functions independent of ϵ and h.

Proof: The general difference problem (6) has a unique solution iff the matrix $\mathbb{G}_h$ defined in (53, 54) in nonsingular. From the definitions of $\mathbb{S}_h$, $\mathbb{G}_h$, $\mathbb{B}_h$, $\mathbb{C}_h$ given in (54) we deduce the identity:

$$\mathbb{G}_h = \mathbb{S}_h + \mathbb{B}_h \mathbb{C}_h$$

From Theorem 3.19 we infer the matrix $\mathbb{S}_h$ is nonsingular for all ϵ and h sufficiently small. Therefore Lemma 3.55 implies $\mathbb{G}_h$ is nonsingular iff the following matrix is nonsingular:

$$\mathbb{B}_h \mathbb{Z}_0^h = \mathbb{I} + \mathbb{C}_h \mathbb{S}_h^{-1} \mathbb{B}_h \tag{73}$$

We are justified in calling the matrix on the right of the equality in (73) $\mathbb{B}_h \mathbb{Z}_0^h$ because:

$$\mathbb{I} + \mathbb{C}_h \mathbb{S}_h^{-1} \mathbb{B}_h = L(\epsilon)\, \mathbb{Z}_0^h(0) + R(\epsilon)\, \mathbb{Z}_0^h(J) \tag{74}$$

To obtain (74) we note the definition of $\mathbb{Z}_0^h$ given in (45) implies:

$$L^*(\epsilon)\, \mathbb{Z}_0^h(0) + R^*(\epsilon)\, \mathbb{Z}_0^h(J) = \mathbb{I} \tag{75'}$$

$$\mathbb{S}_h^{-1}\mathbb{B}_h = \begin{bmatrix} Z_0^h(0) \\ \cdot \\ \cdot \\ \cdot \\ Z_0^h(J) \end{bmatrix} \tag{75'}$$

From (75) we find:

$$\begin{aligned} I + \mathbb{C}_h \mathbb{S}_h^{-1}\mathbb{B}_h &= I + [L(\epsilon) - L^*(\epsilon)]\, Z_0^h(0) + [R(\epsilon) - R^*(\epsilon)]\, Z_0^h(J) \\ &= L(\epsilon)\, Z_0^h(0) + R(\epsilon)\, Z_0^h(J) \\ &= \mathbb{B} Z_0^h \end{aligned}$$

This last equality justifies the relationship given in (73). By combining (3b, 46, 63, 69) and performing asymptotic expansions in terms of ϵ, h we infer the existence of matrices $\{\tilde{B}_m\}_1^\infty$ independent of ϵ and h such that:

$$\mathbb{B}_h Z_0^h \sim B_0 + \sum_{\ell=1}^{N-1} h^\ell \tilde{B}_\ell + \mathcal{O}(h^N + \epsilon) \quad \text{as } \epsilon, h \to 0^+ \tag{76}$$

In (76) we have used the matrix B_0 defined in (2.63). By (4c) we know B_0 is nonsingular. Applying Theorem 1.21 we conclude for all sufficiently small ϵ and h the matrix $\mathbb{B}_h Z_0^h$ is nonsingular, i.e. for all sufficiently small ϵ and h the general difference problem (6) has a unique solution.

From (76) we infer the existence of matrices $\{B_m\}_1^\infty$ independent of ϵ and h such that:

$$[\mathcal{B}_h Z_0^h]^{-1} \sim B_0^{-1} + \sum_{\ell=1}^{N-1} h^\ell B_\ell + \mathcal{O}(h^N + \epsilon) \qquad \text{as } \epsilon, h \to 0^+ \tag{77}$$

Define $\tilde{y}(x)$, $\tilde{y}^h(j)$ to be the solutions of the special boundary-value problems:

$$\begin{aligned} \mathcal{L}\,\tilde{y}(x) &= f(x,\epsilon) & \mathcal{B}^*\tilde{y} &= 0 \\ \mathcal{L}_h\,\tilde{y}^h(j) &= f^h(j) & \mathcal{B}_h^*\tilde{y}^h &= 0 \end{aligned} \tag{78}$$

The solutions $y(x)$, $y^h(j)$ of the general boundary-value problems (1) and (6) then admit the representations:

$$\begin{aligned} y(x) &= \tilde{y}(x) + Z_0(x)\,\alpha_0 \\ y^h(j) &= \tilde{y}^h(j) + Z_0^h(j)\,\alpha^h \end{aligned} \tag{79}$$

Here α_0, α^h are the unique solutions of the linear systems:

$$\begin{aligned} [\mathcal{B} Z_0]\,\alpha_0 &= g(\epsilon) - \mathcal{B}\tilde{y} \\ [\mathcal{B}^h Z_0^h]\,\alpha^h &= g(\epsilon) - \mathcal{B}^h\tilde{y}^h \end{aligned} \tag{80}$$

From (78) and the representations (79) we find y, y^h also satisfy the boundary-value problems:

$$\begin{aligned} \mathcal{L}\,y(x) &= f(x,\epsilon) & \mathcal{B}^* y &= \alpha_0 \\ \mathcal{L}_h\,y^h(j) &= f^h(j) & \mathcal{B}_h^* y^h &= \alpha^h \end{aligned} \tag{81}$$

From the definitions of $\tilde{y}$, $\tilde{y}^h$ given in (78) and the asymptotic expansion of these solutions given in (58) we infer the existence of vectors $\{\underset{\sim}{\beta}_m\}_1^\infty$ independent of ϵ and h such that:

$$\mathcal{B}_h \tilde{\underset{\sim}{y}}^h \sim \mathcal{B}\tilde{\underset{\sim}{y}} + \sum_{\ell=1}^{N-1} h^\ell \underset{\sim}{\beta}_\ell + \mathcal{O}(h^N+\epsilon) \qquad \text{as } \epsilon, h \to 0^+ \tag{82}$$

From the proof of Corollary 2.64 we recall:

$$\mathcal{B}\underset{\sim}{z}_0 \sim \underset{\sim}{B}_0 + \mathcal{O}(\epsilon) \qquad \text{as } \epsilon \to 0^+ \tag{83}$$

By combining (77, 80, 82, 83) we deduce the existence of vectors $\{\underset{\sim}{\alpha}_m\}_1^\infty$ independent of ϵ and h such that:

$$\underset{\sim}{\alpha}^h \sim \underset{\sim}{\alpha}_0 + \sum_{\ell=1}^{N-1} h^\ell \underset{\sim}{\alpha}_\ell + \mathcal{O}(h^N+\epsilon) \qquad \text{as } \epsilon, h \to 0^+ \tag{84}$$

From the asymptotic expansion (57) given in Corollary 3.57 and (81, 84) we obtain the expansion given in (72).

##

One application of Corollary 3.72 arises when we use the general difference scheme (6) to numerically solve the boundary-value problem described in (2.75). From the asymptotic expansion (76) we infer the numerical solution of the general difference problem accurately represents the solution of the reduced problem corresponding to (1) on the interval $[\delta, 1-\delta]$. Furthermore, the expansion (76) also shows Richardson extrapolation may be used to further increase the accuracy of the numerical solution, the limiting accuracy obtainable being $\mathcal{O}(\epsilon)$.

4. APPENDIX

4.1 A Matrix Transformation

In this section we will prove the following:

Theorem 4.1: Let K be a compact subset of $\mathbb{R}^M$ which is star-shaped about the origin. Let $A(\underline{x})$ be an $N \times N$ complex-valued matrix which depends continuously on $\underline{x}$ for $\underline{x} \varepsilon K$. If for each $\underline{x} \varepsilon K$ no eigenvalue of $A(\underline{x})$ has its real part equal to zero then:

(a) There exist positive constants μ, A_∞ such that for each $\underline{x} \varepsilon K$ and any eigenvalue $\lambda(\underline{x})$ of $A(\underline{x})$:

$$|Re\, \lambda(\underline{x})| \geqslant \mu \qquad \| A(\underline{x}) \|_\infty \leqslant A_\infty \tag{1}$$

(b) The number of eigenvalues of $A(\underline{x})$, counting multiplicities, whose real part is positive (negative) is independent of $\underline{x}$.

(c) There exists a nonsingular matrix $U(\underline{x})$, with the same continuity properties as $A(\underline{x})$, which "block diagonalizes" $A(\underline{x})$ as follows:

$$U^{-1}(\underline{x}) A(\underline{x}) U(\underline{x}) = \begin{pmatrix} A_+(\underline{x}) & 0 \\ 0 & A_-(x) \end{pmatrix} \quad \underline{x} \varepsilon K \tag{2}$$

Here, every eigenvalue of $A_+(\underline{x})$ ($A_-(\underline{x})$) has its real part positive (negative).

In the proof of this theroem we will use the following lemmas:

Lemma 4.3: Let $\{\lambda_j\}_1^m$ be the m distinct eigenvalues of the $N \times N$ matrix A, and m_j the multiplicity of the eigenvalue λ_j. Then for all sufficiently (3) small positive ϵ there is a positive number δ such that if $\| B-A \| < \delta$ then the matrix B has exactly m_j eigenvalues in the disk of radius ϵ about λ_j .

Proof: See Franklin [3] , page 191. ##

Lemma 4.4: (Spectral Projection) Let A be as described in Lemma 4.3, and $\mathcal{C}$ a contour which encloses the eigenvalues $\{\lambda_j\}_1^s$. Then the matrix:

$$P = \frac{1}{2\pi i} \oint_{\mathcal{C}} (zI-A)^{-1} dz \tag{4}$$

has the following properties:

(a) The matrix P is idempotent, commutes with A , and has rank $p_s = \sum_{j=1}^{s} m_j$.

(b) Let U be any nonsingular matrix which diagonalizes P as follows:

$$PU = U \begin{pmatrix} I_{p_s} & 0 \\ 0 & 0 \end{pmatrix} \tag{5}$$

Then the matrix U diagonalizes A as follows:

$$AU = U \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \tag{6a}$$

where:

$$\begin{aligned} A_+ &\ldots \text{ a square matrix whose eigenvalues are } \{\lambda_j\}_1^s \text{ with multiplicities } m_j . \\ A_- &\ldots \text{ a square matrix whose eigenvalues are } \{\lambda_j\}_{s+1}^m \text{ with multiplicities } m_j . \end{aligned} \tag{6b}$$

Proof: The proof of (a) may be found in Lancaster [4] , chapters four and five. Since P is idempotent we know it is simple. Furthermore, since P has rank p_s we know

P has 1 as an eigenvalue p_s times and 0 as an eigenvalue $N-p_s$ times. Define:

$$Q = I - P$$

$$U = [\underline{u}_1, \ldots, \underline{u}_N]$$

$$U^{-1} = \begin{bmatrix} \underline{v}_1^T \\ \vdots \\ \underline{v}_N^T \end{bmatrix}$$

From (5) we deduce:

$$\begin{aligned} P\underline{u}_j &= \begin{cases} \underline{u}_j & 1 \le j \le p_s \\ \underline{0} & p_s+1 \le j \le N \end{cases} \\ Q\underline{u}_j &= \begin{cases} \underline{0} & 1 \le j \le p_s \\ \underline{u}_j & p_s+1 \le j \le N \end{cases} \end{aligned} \tag{7}$$

Furthermore:

$$UU^{-1} = I \quad \Rightarrow \quad I = \sum_{j=1}^{N} \underline{u}_j \underline{v}_j^T$$

and so from (7):

$$\begin{aligned} P &= P \cdot I = \sum_{j=1}^{p_s} \underline{u}_j \underline{v}_j^T \\ Q &= Q \cdot I = \sum_{j=p_s+1}^{N} \underline{u}_j \underline{v}_j^T \end{aligned} \tag{8}$$

Since P, Q commute with A we deduce from (8):

$$A\underline{u}_\ell = AP\underline{u}_\ell = PA\underline{u}_\ell = \sum_{j=1}^{p_s} (\underline{v}_j^T A \underline{u}_\ell)\, \underline{u}_j \qquad 1 \le \ell \le p_s \tag{9'}$$

$$A\underline{u}_\ell = AQ\underline{u}_\ell = QA\underline{u}_\ell = \sum_{j\,p_s+1}^{N} (\underline{v}_j^T A \underline{u}_\ell)\,\underline{u}_j \qquad p_s+1 \le j \le N \tag{9'}$$

Therefore, for some matrices A_+, A_- :

$$AU = U \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$$

Suppose λ were an eigenvalue of A_+ that lies outside the contour $\mathcal{C}$. Then for some $\underline{w} \neq \underline{0}$:

$$A_+ \underline{w} = \lambda \underline{w}$$

Consider the vector:

$$\hat{\underline{w}} = U \begin{pmatrix} \underline{w} \\ \underline{0} \end{pmatrix} \neq \underline{0}$$

From (5, 6) we conclude:

$$\begin{aligned} P\hat{\underline{w}} &= \hat{\underline{w}} \\ A\hat{\underline{w}} &= \lambda\hat{\underline{w}} \end{aligned} \tag{10}$$

Thus $\hat{\underline{w}}$ is an eigenvector of A belonging to the eigenvalue λ. Since λ lies outside $\mathcal{C}$ we deduce from (4) that:

$$P\hat{\underline{w}} = \frac{1}{2\pi i}\oint_{\mathcal{C}} (zI - A)^{-1}\hat{\underline{w}}\,dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} (z-\lambda)^{-1}\hat{\underline{w}}\,dz = \underline{0}$$

which contradicts (10). Therefore, every eigenvalue of A_+ lies within $\mathcal{C}$. In an analogous manner we deduce every eigenvalue of A_- lies outside $\mathcal{C}$. Since the totality of all eigenvalues of A_+ and A_- are the eigenvalues of A, we conclude (6) is true. ##

Proof of Theorem 4.1:

Since $\|A(\underline{x})\|$ is a continuous function of $\underline{x}$ on the compact set K, it is bounded. Therefore, A_∞ exists.

For each $\underline{x} \in K$ let $\{\lambda_j(\underline{x}) : 1 \le j \le m(\underline{x})\}$ be the distinct eigenvalues of $A(\underline{x})$, each of multiplicity $m_j(\underline{x})$. Define:

$$\mu(\underline{x}) = \tfrac{1}{2} \min\{ |\mathrm{Re}\, \lambda_j(\underline{x})| : 1 \le j \le m(\underline{x})\}$$

$$d(\underline{x}) = \tfrac{1}{3} \min\{ |\lambda_j(\underline{x}) - \lambda_i(\underline{x})| : 1 \le i \ne j \le m(\underline{x})\}$$

$$\epsilon(\underline{x}) = \min\{\mu(\underline{x}), d(\underline{x})\}$$

$$B(\underline{x}, \delta) = \{ \underline{y} : \|\underline{y} - \underline{x}\| < \delta\}$$

Since every eigenvalue of $A(\underline{x})$ has non-zero real part it follows that $\mu(\underline{x}) > 0$. Choose $\epsilon \in (0, \epsilon(\underline{x}))$ and $\delta > 0$ such that Lemma 4.3 holds. By the continuity of A at $\underline{x}$ we may choose $\delta(\underline{x}) > 0$ such that:

$$\|A(\underline{y}) - A(\underline{x})\| < \delta \qquad \text{if} \qquad \underline{y} \in B(\underline{x}, \delta(\underline{x})) \cap K$$

From Lemma 4.3 we then conclude that for $\underline{y} \in B(\underline{x}, \delta(\underline{x})) \cap K$ the matrix $A(\underline{y})$ has exactly m_j eigenvalues, counting multiplicities, in the disk of radius $\epsilon(\underline{x})$ about $\lambda_j(\underline{x})$.

Since $\{B(\underline{x}, \delta(\underline{x})) : \underline{x} \in K\}$ is an open cover of the compact set K there must exist a finite subcover $\{B_\ell = B_\ell(\underline{x}, \delta(\underline{x})) : 1 \le \ell \le L\}$ of K. We may then choose:

$$\mu = \min\{\mu(\underline{x}_\ell) : 1 \le \ell \le L\}$$

Therefore, (1a) has been established.

To establish (1b), define $p(\underline{x})$ to be the number of eigenvalues of $A(\underline{x})$, counting multiplicities, with positive real part. We consider p to be a mapping of K into $\mathbb{R}$. From the above arguments we deduce p is integer-valued and constant on the sets $B_\ell \cap K$. By considering sequential limits we find $p(\underline{x})$ is a continuous function. Since K is star-shaped we know it is connected, therefore $p(K)$ is connected since p is continuous. We recall the only connected subsets of $\mathbb{R}$ are the intervals. Since $p(K)$ is connected and yet consists solely of integer values we deduce $p(K)$ consists of a single integer, that is p is constant on K. Therefore, (1b) has been established.

Let us recall $\|\cdot\|$ is the infinity vector or matrix norm. From Gerschgorin's Theorem, see Franklin [3], we deduce for every $\underline{x} \in K$ and $1 \le j \le m(\underline{x})$:

$$|\lambda_j(\underline{x})| \le A_\infty \qquad\qquad |\operatorname{Re} \lambda_j(\underline{x})| \geq \mu$$

Therefore, as shown in Figure 4.1, $\lambda_j(\underline{x})$ must lie in either Σ_+ or Σ_-. From (1b) we also know the number of eigenvalues of $A(\underline{x})$, counting multiplicities, in Σ_+ (Σ_-) is independent of $\underline{x}$. Let $\mathcal{C}_+$ be the contour enclosing Σ_+ shown in Figure 4.1. Define:

$$P(\underline{x}) = \frac{1}{2\pi i} \oint_{\mathcal{C}_+} (zI - A(\underline{x}))^{-1}\, dz \tag{11a}$$

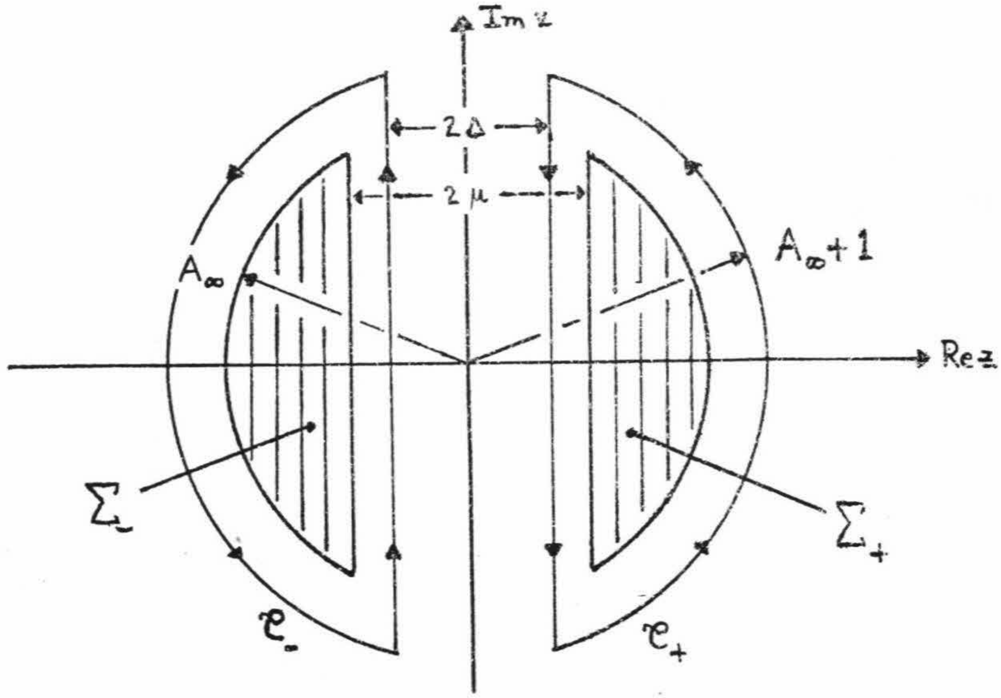

Figure 4.1

By considering sequential limits we deduce $\|(zI-A(\underline{x}))^{-1}\|$ is a continuous function of $(z, \underline{x})$ on the compact set $\mathcal{C}_+ \times K$. Therefore, for some $K_0 > 0$:

$$\|(zI-A(\underline{x}))^{-1}\| \leq K_0 \qquad (z,\underline{x}) \in \mathcal{C}_+ \times K \tag{11b}$$

We know $P(\underline{x})$ is idempotent by Lemma 4.4, and from the remarks above we also deduce the rank of $P(\underline{x})$ is independent of $\underline{x}$ for $\underline{x} \in K$. For $\underline{x}, \underline{y} \in K$ and $z \in \mathcal{C}_+$ we find:

$$(zI-A(\underline{x}))^{-1} - (zI-A(\underline{y}))^{-1} = (zI-A(\underline{x}))^{-1}(A(\underline{x})-A(\underline{y}))(zI-A(\underline{y}))^{-1}$$

$$\| P(\underline{x}) - P(\underline{y}) \| \leq \frac{1}{2\pi} K_0^2 \oint_{\mathcal{C}_+} |dz| \cdot \| A(\underline{x}) - A(\underline{y}) \|$$

Hence, we deduce $P(\underline{x})$ is a continuous function of $\underline{x}$ on

K . For $\underline{x}, \underline{y} \varepsilon K$ define:

$$S(\underline{x}, \underline{y}) = I + [P(\underline{x}) - P(\underline{y})][2P(\underline{y}) - I]$$

Since $\| 2P(\underline{y}) - I \|$ is a continuous function of $\underline{y}$ on the compact set K we know for some constant $K_1 > 0$:

$$\| 2P(\underline{y}) - I \| \leq K_1 \qquad \underline{y} \varepsilon K$$

From the uniform continuity of $P(\underline{x})$ on the compact set K we deduce for some $h > 0$:

$$\| P(\underline{x}) - P(\underline{y}) \| \leq 1/2K_1 \qquad \underline{y} \varepsilon B(\underline{x}, h)$$

An application of the Banach Lemma (1.20) then tells us $S(\underline{x}, \underline{y})$ is a nonsingular matrix for $\underline{y} \varepsilon B(\underline{x}, h)$. We also note, for $\underline{x}, \underline{y} \varepsilon K$:

$$P(\underline{x}) S(\underline{x}, \underline{y}) = S(\underline{x}, \underline{y}) P(\underline{y}) \tag{12}$$

Since K is a compact set there exists an integer N such that:

$$\| \underline{x} \| < Nh \qquad \underline{x} \varepsilon K$$

Define:

$$\alpha_m = m/N \qquad m = 0, 1, \ldots, N$$

From the fact that K is star-shaped with respect to the origin and the estimate:

$$\| \alpha_{j+1} \underline{x} - \alpha_j \underline{x} \| < h \qquad \underline{x} \varepsilon K \quad j = 0, 1, \ldots, N-1$$

we deduce the matrix:

$$\begin{aligned} T(\underline{x}) &= S(\alpha_0\underline{x}, \alpha_1\underline{x}) \ \ldots \ S(\alpha_{N-1}\underline{x}, \alpha_N\underline{x}) \\ &= S(\underline{0}, \alpha_1\underline{x}) \ \ldots \ S(\alpha_{N-1}\underline{x}, \underline{x}) \end{aligned} \tag{13}$$

is a well-defined, continuous, nonsingular matrix for all $\underline{x} \in K$. Furthermore, from (12) we find:

$$P(\underline{0})\, T(\underline{x}) = T(\underline{x})\, P(\underline{x}) \qquad \underline{x} \in K \tag{14}$$

Let $U(\underline{0})$ be any nonsingular matrix which diagonalizes $P(\underline{0})$ as in (5). Define:

$$U(\underline{x}) = T^{-1}(\underline{x})\, U(\underline{0}) \qquad \underline{x} \in K \tag{15}$$

Then $U(\underline{x})$ is a continuous nonsingular matrix for all $\underline{x} \in K$. Furthermore, from (5, 14, 15) we find:

$$P(\underline{x})\, U(\underline{x}) = U(\underline{x}) \begin{pmatrix} I_{ps} & 0 \\ 0 & 0 \end{pmatrix} \qquad \underline{x} \in K$$

Therefore, by applying Lemma 4.4 we deduce:

$$A(\underline{x})\, U(\underline{x}) = U(\underline{x}) \begin{pmatrix} A_+(\underline{x}) & 0 \\ 0 & A_-(\underline{x}) \end{pmatrix} \qquad \underline{x} \in K$$

$A_+(\underline{x})$ ... a square matrix, depending continuously on $\underline{x}$ for $\underline{x} \in K$, whose eigenvalues lie in Σ_+.

$A_-(\underline{x})$ ... a square matrix depending continuously on $\underline{x}$ for $\underline{x} \in K$, whose eigenvalues lie in Σ_-.

By considering (11) we deduce $A(\underline{x})$ and $P(\underline{x})$ share the same continuity properties (differentiate under the integral sign). Therefore, since K is star-shaped with respect to the origin the matrices $S(\alpha_j \underline{x}, \alpha_{j+1} \underline{x})$, $T(\underline{x})$, $U(\underline{x})$ share the continuity properties of $A(\underline{x})$. Therefore (2) has been established.

##

We recall the existence of the matrix $S(\underline{x}, \underline{y})$ allowed us to construct the matrix $U(\underline{x})$. Other interesting uses of the matrix $S(\underline{x}, \underline{y})$ may be found in Coppel [5, 6].

To prove Theorem 1.5 we note the correspondence:

$$\underline{x} \to (x, \epsilon) \qquad K \to I \times E_0$$

Therefore, Theorem 1.5 follows as a corollary of Theorem 4.1.

4.2 Exponential Dichotomy

Let us prove the exponential dichotomy mentioned in 1.29, 2.5, 2.10, 3.9 exists. First, make the following:

Definition: Let $K(A_\infty, \mu)$, for $\mu, A_\infty > 0$, be the set of all $N \times N$ complex-valued matrices A satisfying:

$$\| A \| \leq A_\infty \qquad \mathrm{Re}\, \lambda(A) \leq -\mu \tag{16}$$

Here, $\lambda(A)$ is any eigenvalue of A.

With the definition of $K(A_\infty, \mu)$ we now prove the following:

Lemma 4.17: The set $K(A_\infty, \mu)$ is a compact subset of $\mathbb{C}^{N^2}$. (17)

Proof: The compact subsets of $\mathbb{C}^{N^2}$ are those which are closed and bounded. Clearly $K(A_\infty,\mu)$ is a bounded set. To show $K(A_\infty,\mu)$ is closed let $\{A_m\}_1^\infty$ be any Cauchy sequence in $K(A_\infty,\mu)$. Since $\mathbb{C}^{N^2}$ is complete $A = \lim_{m\to\infty} A_m$ exists. From the continuity of the norm $\|\cdot\|$ we deduce:

$$\|A\| = \lim_{m\to\infty} \|A_m\| \leq A_\infty$$

Furthermore, if some eigenvalue $\lambda(A)$ of A satisfies the condition:

$$Re\,\lambda(A) > -\mu$$

we conclude from Lemma 4.3 that some eigenvalue $\lambda(A_m)$ of A_m, where m is sufficiently large, also satisfies the condition:

$$Re\,\lambda(A_m) > -\mu$$

But this contradicts the fact that $A_m \varepsilon K(A_\infty,\mu)$. Therefore:

$$Re\,\lambda(A) \leq -\mu$$

for all eigenvalues $\lambda(A)$ of A. Thus $A \varepsilon K(A_\infty,\mu)$ and hence $K(A_\infty,\mu)$ is a closed subset of $\mathbb{C}^{N^2}$. ##

Since $\|\cdot\|$ is the infinity norm we deduce from Gerschforin's Theorem (see Franklin [3]) that the eigenvalues of any matrix $A \varepsilon K(A_\infty,\mu)$ lie in the region Σ_- illustrated in Figure 4.1 From this fact we have the following:

Lemma 4.18: There exists a constant $K_1 > 0$ such that for any $A \varepsilon K(A_\infty, \mu)$:

$$\frac{1}{2\pi} \oint_{\mathcal{C}_-} \| (zI - A)^{-1} \| \, |dz| \leq K_1 \tag{18}$$

Proof: Since $\|(zI-A)^{-1}\|$ is a continuous function of (z, A) on the compact set $\mathcal{C}_- \times K(A_\infty, \mu)$ we know it is bounded. Therefore, since $\mathcal{C}_-$ has finite length, (18) immediately follows. Note the constant K_1 depends only on Δ, A_∞, μ. ##

Using these lemmas we now prove the following:

Theorem 4.19: Let $\epsilon_0 > 0$, $I = [0, 1]$, $E_0 = (0, \epsilon_0)$. Suppose $A(x,\epsilon)$ is a square matrix, depending continuously on x and ϵ for $(x, \epsilon) \varepsilon I \times E_0$, with the property that for each $(x, \epsilon) \varepsilon I \times E_0$ every eigenvalue of $A(x,\epsilon)$ has its real part negative. Then for some positive constants K_0, Δ_1, ϵ_1 and all $\epsilon \varepsilon (0, \epsilon_1]$ the F.S.M. $Y(x,\tau)$ for $\frac{1}{\epsilon} A(x,\epsilon)$ satisfies the bound:

$$\| Y(x,\tau) \| \leq K_0 \exp\left\{-\frac{\Delta_1}{\epsilon}(x-\tau)\right\} \qquad 0 \leq \tau \leq x \leq 1 \tag{19}$$

Proof: Let μ, A_∞ be as given in Theorem 4.1 and $\mathcal{C}_-$ the path illustrated in Figure 4.1. We know:

$$\epsilon D_x Y(x,\tau) = A(x,\epsilon) Y(x,\tau) \qquad Y(\tau,\tau) = I$$

Let $x_0 \varepsilon I$, then we may write the above differential equation as:

$$\epsilon D_x Y(x,\tau) = A(x_0,\epsilon) Y(x,\tau) + \{A(x,\epsilon) - A(x_0,\epsilon)\} Y(x,\tau)$$

By the V.O.P. formula (1.25) we therefore deduce:

$$Y(x,\tau) = \exp\left\{\frac{x-\tau}{\epsilon} A(x_0,\epsilon)\right\} + \frac{1}{\epsilon}\int_{\tau}^{x} \exp\left\{\frac{x-s}{\epsilon} A(x_0,\epsilon)\right\}\left\{A(s,\epsilon) - A(x_0,\epsilon)\right\} Y(s,\tau)\, ds \qquad (20)$$

We know:

$$\exp\left\{\frac{x-\tau}{\epsilon} A(x_0,\epsilon)\right\} = \frac{1}{2\pi i}\oint_{\underline{C}} (zI - A(x_0,\epsilon))^{-1} \exp\left\{\frac{x-\tau}{\epsilon} z\right\} dz$$

and so from Lemma 4.18 we deduce:

$$\left\| \exp\left\{\frac{x-\tau}{\epsilon} A(x_0,\epsilon)\right\}\right\| \le K_1 \exp\left\{-\frac{\Delta}{\epsilon}(x-\tau)\right\} \qquad x-\tau \ge 0 \qquad (21)$$

Since $A(x,\epsilon)$ is a continuous function on $I \times E_0$ its modulus of continuity exists. Therefore, there is a continuous increasing function $\omega(\delta)$ for $\delta \in I$ such that:

$$\begin{aligned} &\| A(x,\epsilon) - A(y,\epsilon) \| \le \omega(|x-y|) \qquad x,y \in I \\ &\lim_{\delta \to 0^+} \omega(\delta) = 0 \end{aligned} \qquad (22')$$

We extend the definition of ω as follows:

$$\omega(\delta) = \omega(1) \qquad \delta > 1 \qquad (22')$$

Choose $\alpha \in (0,1)$ and set $\beta = 1-\alpha$. Define:

$$E(x,\tau) = \| Y(x,\tau) \| \exp\left\{\frac{\alpha\Delta}{\epsilon}(x-\tau)\right\} \qquad (23)$$

If we take $x_0 = x$ in (20) and consider only $0 \le \tau \le x \le 1$ then from (20, 21, 22, 23) it follows that:

$$E(x,\tau) \le K_1 + \frac{1}{\epsilon} K_1 \int_{\tau}^{x} \omega(x-s) \exp\left\{-\frac{\beta\Delta}{\epsilon}(x-s)\right\} E(s,\tau)\, ds \qquad (24)$$

Define the constants E_∞, ϵ_1 as follows:

$$E_\infty = \sup\left\{ E(x,\tau) : 0 \le \tau \le x \le 1 \right\}$$

$$\frac{K_1}{\beta\Delta}\left[w(\sqrt{\epsilon_1}) + w(1)\exp\left\{-\frac{\beta\Delta}{\sqrt{\epsilon_1}}\right\}\right] = \frac{1}{2}$$

Then for $0 \leq \tau \leq x \leq 1$ and $\epsilon \varepsilon (0, \epsilon_1]$ we estimate:

$$\begin{aligned}
\frac{K_1}{\epsilon}\int_\tau^x w(x-s)\exp\left\{-\frac{\beta\Delta}{\epsilon}(x-s)\right\} ds &\leq \frac{K_1}{\epsilon}\int_0^\infty w(s)\exp\left\{-\frac{\beta\Delta}{\epsilon}s\right\} ds \\
&\leq \frac{K_1}{\epsilon}\left[\int_0^{\sqrt{\epsilon}} + \int_{\sqrt{\epsilon}}^\infty\right] w(s)\exp\left\{-\frac{\beta\Delta}{\epsilon}s\right\} ds \\
&\leq \frac{K_1}{\epsilon}\left[\frac{\epsilon}{\beta\Delta} w(\sqrt{\epsilon}) + \frac{\epsilon}{\beta\Delta} w(1)\exp\left\{-\frac{\beta\Delta}{\sqrt{\epsilon}}\right\}\right] \\
&\leq \frac{1}{2}
\end{aligned} \qquad (25)$$

Hence for $0 \leq \tau \leq x \leq 1$ and $\epsilon \varepsilon (0, \epsilon_1]$ we find (24, 25) imply:

$$E(x,\tau) \leq K_1 + \frac{1}{2}E_\infty \quad \Rightarrow \quad E_\infty \leq 2K_1$$

Therefore using (23) we deduce for $\epsilon \varepsilon (0, \epsilon_1]$:

$$\| Y(x,\tau) \| \leq 2K_1 \exp\left\{-\frac{\alpha\Delta}{\epsilon}(x-\tau)\right\} \qquad 0 \leq \tau \leq x \leq 1$$

Choosing $\Delta_1 = \alpha\Delta$ and $K_0 = 2K_1$ we recognize (19) holds. This proof differs from that given in Flatto and Levinson [7] only in the fact that Δ_1 may be chosen to lie anywhere in the interval $(0, \mu)$. ##

Suppose $B(x,\epsilon)$ were a square matrix, depending continuously on x and ϵ for $(x, \epsilon) \varepsilon I \times E_0$, with the property that for each $(x, \epsilon) \varepsilon I \times E_0$ every eigenvalue of $B(x,\epsilon)$ has positive real part. Define $W(x,\tau)$ to be the F.S.M. for $\frac{1}{\epsilon}B(x,\epsilon)$, that is:

$$\epsilon D_x W(x,\tau) = B(x,\epsilon) W(x,\tau) \qquad W(\tau,\tau) = I$$

If we define:

$$y = 1 - x \qquad s = 1 - \tau$$

$$A(y,\epsilon) = B(1-y,\epsilon)$$

$$Y(y,s) = W(1-y,\, 1-s)$$

then we conclude:

$$\epsilon D_y Y(y,s) = -A(y,\epsilon)\, Y(y,s) \qquad Y(s,s) = I$$

By applying Theorem 4.19 we conclude for some positive constants K_0, Δ, ϵ_1 and all $\epsilon \varepsilon (0,\epsilon_1]$:

$$\| Y(y,s) \| \leq K_0 \exp\{-\tfrac{\Delta}{\epsilon}(y-s)\} \qquad 0 \leq s \leq y \leq 1$$

or in terms of $W(x,\tau)$:

$$\| W(x,\tau) \| \leq K_0 \exp\{-\tfrac{\Delta}{\epsilon}(\tau - x)\} \qquad 0 \leq x \leq \tau \leq 1 \tag{26}$$

Combining (19, 26) we obtain the continuous versions of the exponential dichotomy used in 1.29, 2.5, 2.10. The derivation of the discrete version of the exponential dichotomy 3.9 is only slightly more complicated.

Theorem 4.27: Let $A(x,\epsilon)$ be as described in Theorem 4.16. Then for some positive constants K_0, Δ_1, ϵ_1 and all $\epsilon \varepsilon (0,\epsilon_1]$ the matrix $Y(j,k)$ defined by:

$$\epsilon D_j Y(j,k) = A([j+\gamma]h,\epsilon)\, Y(j+1,k) \qquad Y(k,k) = I$$

satisfies, for all $0 \leq \gamma \leq 1$ and $h = 1/J$, the bound:

$$\| Y(j,k) \| \leq K_0/[1+\Delta \tfrac{h}{\epsilon}]^{j-k} \qquad 0 \leq k \leq j \leq J \tag{27}$$

Proof: Let μ , A_∞ be as given in Theorem 4.1 and $\mathcal{C}_-$ the path illustrated in Figure 4.1. Since the proof below works for any $\gamma \varepsilon [0,1]$ we choose to carry it out for $\gamma = 0$. We note:

$$[I - \tfrac{h}{\epsilon} A(jh,\epsilon)]^{-1} = \frac{1}{2\pi i} \oint_{\mathcal{C}} [zI - A(jh,\epsilon)]^{-1} [1 - \tfrac{h}{\epsilon} z]^{-1} dz$$

and so from Lemma 4.18:

$$\| [I - \tfrac{h}{\epsilon} A(jh,\epsilon)]^{-1} \| \leq K_1/[1+\Delta \tfrac{h}{\epsilon}] \tag{28}$$

We note:

$$Y(j,k) = \prod_{i=k}^{j-1} [I - \tfrac{h}{\epsilon} A(ih,\epsilon)]^{-1} \qquad 0 \leq k \leq j \leq J \tag{29}$$

Therefore, for $K_1 \leq 1$ the bound (27) easily follows from (28, 29). If $K_1 > 1$ we proceed as follows. Define:

$$C_0 = \frac{2}{\Delta}[K_1 - 1] \qquad \Delta_2 = \frac{\Delta}{2K_1}$$

and note:

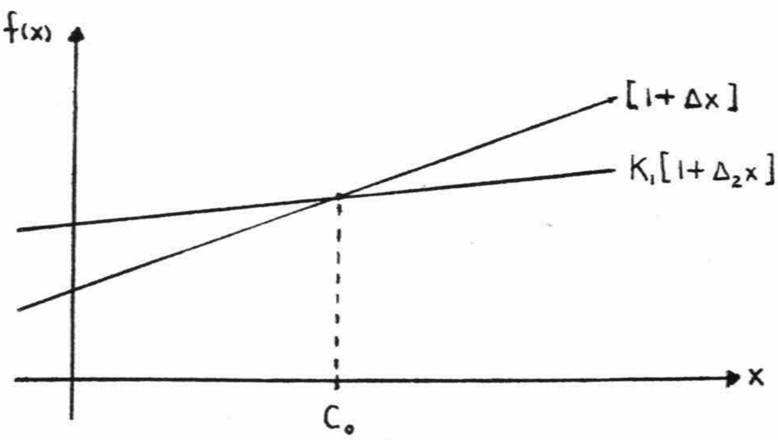

Figure 4.2

Therefore, for $\frac{h}{\epsilon} \geq C_0$:

$$K_1 (1 + \Delta_2 \tfrac{h}{\epsilon}) \leq (1 + \Delta \tfrac{h}{\epsilon})$$

$$\Rightarrow \quad K_1 / [1 + \Delta \tfrac{h}{\epsilon}] \leq 1 / [1 + \Delta_2 \tfrac{h}{\epsilon}]$$

and from (29) we conclude:

$$\| Y(j,k) \| \leq 1/[1+\Delta_2 \tfrac{h}{\epsilon}]^{j-k} \qquad \tfrac{h}{\epsilon} \geq C_0 \quad 0 \leq k \leq j \leq J \tag{30}$$

For $\frac{h}{\epsilon} \leq C_0$ we carry out the discrete version of the proof given in Theorem 4.19. Write the difference equation for $Y(j,k)$ as follows:

$$\epsilon D_j Y(j,k) = A(j_0 h, \epsilon) Y(j,k) + \{ A(jh,\epsilon) - A(j_0 h, \epsilon) \} Y(j,k)$$

By the discrete version of the V.O.P. formula we have:

$$\begin{aligned} Y(j,k) &= W(j,k) + \tfrac{1}{\epsilon} h \sum_{\ell = k}^{j-1} W(j,\ell) \{ A(\ell h,\epsilon) - A(j_0 h,\epsilon) \} Y(\ell,k) \\ W(j,k) &= [I - \tfrac{h}{\epsilon} A(j_0 h, \epsilon)]^{-(j-k)} \\ &= \frac{1}{2\pi i} \oint_{\mathcal{C}_-} [zI - A(j_0 h,\epsilon)]^{-1} [1 - \tfrac{h}{\epsilon} z]^{-(j-k)} dz \end{aligned} \tag{31}$$

By Lemma 4.18 we deduce:

$$\| W(j,k) \| \leq K_1 / [1 + \Delta \tfrac{h}{\epsilon}]^{j-k} \tag{32}$$

Choose $\alpha \in (0,1)$ and define:

$$\begin{aligned} \beta &= \frac{1-\alpha}{1+\alpha C_0 \Delta} \\ E(j,k) &= \| Y(j,k) \| [1 + \alpha \Delta \tfrac{h}{\epsilon}]^{j-k} \end{aligned} \tag{33}$$

We note for $\frac{h}{\epsilon} \le C_0$:

$$\frac{1+\kappa\Delta\frac{h}{\epsilon}}{1+\Delta\frac{h}{\epsilon}} \le \frac{1}{1+\beta\Delta\frac{h}{\epsilon}} \qquad \frac{h}{\epsilon} \le C_0 \tag{34}$$

Let ω be as described in the proof of Theorem 4.19 and choose $j_0 = j$ in (31). Combining (31, 32, 33, 34) we deduce:

$$E(j,k) \le K_1 + K_1\frac{h}{\epsilon}\sum_{\ell}^{j-1}\sum_{k}\omega([j-\ell]h)\,E(\ell,k)/[1+\beta\Delta\tfrac{h}{\epsilon}]^{j-\ell} \tag{35}$$

If we note the function $f(h)$ defined by:

$$f(h) = 1/[1+\beta\Delta\tfrac{h}{\epsilon}]^{\sqrt{\epsilon}/h} \qquad \beta, \Delta, h, \epsilon > 0$$

is an increasing function of h then:

$$f(h) \le 1/[1+\beta\Delta C_0]^{1/C_0\sqrt{\epsilon}} \qquad \frac{h}{\epsilon} \le C_0$$

As $\epsilon \to 0^+$ this upper bound on $f(h)$ decreases. This means we may choose $\epsilon_1 > 0$ such that:

$$\frac{K_1}{\beta\Delta}\left[\omega(\sqrt{\epsilon}) + \omega(1)/[1+\beta\Delta\tfrac{h}{\epsilon}]^{\sqrt{\epsilon}/h}\right] \le \frac{1}{2} \qquad \frac{h}{\epsilon} \le C_0,\ \epsilon \in (0,\epsilon_1] \tag{36}$$

For $\frac{h}{\epsilon} \le C_0$ and $\epsilon \in (0,\epsilon_1]$ we estimate:

$$\begin{aligned} K_1\frac{h}{\epsilon}\sum_{\ell}^{j-1}\sum_{k}\omega([j-\ell]h)/[1+\beta\Delta\tfrac{h}{\epsilon}]^{j-\ell} &\le K_1\frac{h}{\epsilon}\sum_{\ell=1}^{\infty}\omega(\ell h)/[1+\beta\Delta\tfrac{h}{\epsilon}]^{\ell} \\ &\le K_1\frac{h}{\epsilon}\left[\sum_{\ell=1}^{\sqrt{\epsilon}/h} + \sum_{\ell=\sqrt{\epsilon}/h+1}^{\infty}\right]\omega(\ell h)/[1+\beta\Delta\tfrac{h}{\epsilon}]^{\ell} \\ &\le \frac{K_1}{\beta\Delta}\left[\omega(\sqrt{\epsilon}) + \omega(1)/[1+\beta\Delta\tfrac{h}{\epsilon}]^{\sqrt{\epsilon}/h}\right] \\ &\le \frac{1}{2} \end{aligned} \tag{37}$$

If we define:

$$E_\infty = \max\{ E(j,k) : 0 \le k \le j \le J \} \tag{38}$$

then we deduce from (35, 37) for $\frac{h}{\epsilon} \le C_0$ and $\epsilon \varepsilon (0, \epsilon_1]$:

$$E(j,k) \le K_1 + \tfrac{1}{2} E_\infty$$

$$\Rightarrow \quad E_\infty \le 2K_1$$

Recalling (33, 38) we therefore have for all $\epsilon \varepsilon (0, \epsilon_1]$:

$$\| Y(j,k) \| \le 2K_1/[1+\alpha\Delta \tfrac{h}{\epsilon}]^{j-k} \qquad 0 \le k \le j \le J, \ \tfrac{h}{\epsilon} \le C_0 \tag{39}$$

If we then define:

$$K_0 = 2K_1 \qquad\qquad \Delta_1 = \min\{ \Delta_2, \alpha\Delta \}$$

we deduce from (30, 39) that for all $h = \frac{1}{J}$ and $\epsilon \varepsilon (0, \epsilon_1]$:

$$\| Y(j,k) \| \le K_0/[1+\Delta_1 \tfrac{h}{\epsilon}]^{j-k} \qquad 0 \le k \le j \le J$$

This establishes (27). ##

Suppose $B(x,\epsilon)$ were a square matrix, depending continuously on x and ϵ for $(x, \epsilon) \varepsilon I \times E_0$, with the property that for each $(x, \epsilon) \varepsilon I \times E_0$ every eigenvalue of $B(x,\epsilon)$ has positive real part. Let the matrix $W(j,k)$ satisfy:

$$\epsilon D_j W(j,k) = B([j+\gamma]h, \epsilon)\, W(j,k) \qquad W(k,k) = I$$

where $0 \le \gamma \le 1$. By following steps analogous to those used to derive (26) we conclude for some positive constants K_0, Δ_1, ϵ_1 and all $\epsilon \in (0, \epsilon_1]$, $h = \frac{1}{J}$:

$$\| W(j,k) \| \leq K_0 / [1 + \Delta_1 \frac{h}{\epsilon}]^{k-j} \qquad 0 \leq j \leq k \leq J \tag{40}$$

Of course, in the proof of (40) we appeal to Theorem 4.27.

This completes the derivation of the exponential dichotomies used in the previous chapters.

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