A PERTURBATION PROCEDURE FOR NONLINEAR OSCILLATIONS
(THE DYNAMICS OF TWO OSCILLATORS WITH WEAK NONLINEAR COUPLING)

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ABSTRACT

This thesis considers in detail the dynamics of two oscillators with weak nonlinear coupling. There are three classes of such problems: non-resonant, where the Poincaré procedure is valid to the order considered; weakly resonant, where the Poincaré procedure breaks down because small divisors appear (but do not affect the O(1) term) and strongly resonant, where small divisors appear and lead to O(1) corrections. A perturbation method based on Cole's two-timing procedure is introduced. It avoids the small divisor problem in a straightforward manner, gives accurate answers which are valid for long times, and appears capable of handling all three types of problems with no change in the basic approach.

One example of each type is studied with the aid of this procedure: for the nonresonant case the answer is equivalent to the Poincaré result; for the weakly resonant case the analytic form of the answer is found to depend (smoothly) on the difference between the initial energies of the two oscillators; for the strongly resonant case we find that the amplitudes of the two oscillators vary slowly with time as elliptic functions of ϵt, where ϵ is the (small) coupling parameter.

Our results suggest that, as one might expect, the dynamical behavior of such systems varies smoothly with changes in the ratio of the fundamental frequencies of the two oscillators. Thus the pathological behavior of Whittaker's adelphic integrals as the frequency ratio is varied appears to be due to the fact that Whittaker ignored the small divisor problem. The energy sharing properties of these systems appear to depend strongly on the initial conditions, so that the systems

are not ergodic.

The perturbation procedure appears to be applicable to a wide variety of other problems in addition to those considered here.

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Introduction

Problems involving systems of oscillators with weak nonlinear (polynomial) coupling have long been of interest in connection with the ergodic problem of statistical mechanics (see Appendix A) and as simple models of nonlinear interactions. In the present work we will consider in detail problems of two such oscillators, with our objective being to determine the long-term behavior of the gross properties and the detailed dynamics of such systems.

One question of particular interest in connection with the ergodic problem is whether a particular coupled system permits significant energy to be exchanged among the degrees of freedom.* Since energy sharing is a necessary (but not sufficient) condition for ergodicity and presumably easier to establish than ergodicity, it can be used as a preliminary test - no energy sharing implies no ergodicity. Energy sharing is also of interest with respect to the question of equipartion - is a simple nonlinear coupling sufficient to insure that the time averaged energies of the degrees of freedom will be approximately the same?

Of course the direct relevance of these questions and indeed of these systems to statistical mechanics is describable only in terms of the thermodynamic limit, where the number of degrees of freedom of

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*We will use the term degrees of freedom with reference to coordinate systems in which the subsystems remain weakly coupled, e.g. particle coordinates, mode coordinates.

the system goes to infinity, but we suggest that it will be easier to understand large systems if we understand clearly the behavior of small ones. At the same time, we hope that the information obtained by our examination of small systems will shed some light on the behavior of large ones.

In addition to being model systems for statistical mechanical problems, the examples we will study here are directly relevant to understanding the long-time behavior of simple nonlinear systems. Perturbation methods have frequently been applied to such systems with varying degrees of success; however, a significant question has been left open, presumably because the answer is not obtainable by most usual approaches. This difficulty has been mentioned frequently in the literature, and is generally referred to as the small divisor problem.

The first chapter of this thesis describes some interesting results of the work of previous investigators of systems of coupled oscillators. Chapter II discusses the small divisor problem, points out certain of the other difficulties inherent in applying various standard perturbation procedures to coupled oscillator systems, and introduces and describes "N-timing," the approach which is to be used in the present work. In Chapter III we work a non-resonant problem which can be done equally well by other methods (e.g. Poincaré, Wigner-Brillouin), and in Chapter IV we work a problem which explicitly demonstrates the ability of N-timing to remove small divisors. Chapter V will present our solution of a "strongly resonant" problem, where the energy-sharing can be O(1). Chapter VI discusses our results and their relevance to some of the questions raised in the foregoing paragraphs.

Appendix A examines the importance of ergodicity in statistical mechanics, and Appendix B presents some algebraic details omitted from the text.

Chapter I

Discussion of Previous Literature on Coupled Oscillators.

E. T. Whittaker and Henri Poincaré were prominent among early workers contributing to the theory of nonlinear oscillations. Poincaré (1893)* devised a useful and ingenious perturbation procedure which we shall discuss in Chapter II. Whittaker (1916) studied in detail the problem of two oscillators with weak nonlinear coupling, and found that for such a system one can always, at least formally, construct a constant of the motion

$$\varphi\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}, \epsilon\right)$$

different from the total energy and analytic in the dynamical variables and ϵ.**

To construct these constants, which he called "adelphic integrals," Whittaker found it necessary to distinguish three cases depending on the frequencies ω_1 and ω_2 of the two oscillators and the form of the coupling:

Case 1) ω_1/ω_2 is irrational;

*A name followed by a date is used to refer to an entry in the list of references.

**Throughout this thesis, ϵ will generally be a small dimensionless parameter of the order of the coupling term in the Hamiltonian; i.e. - we are considering systems with

$$H = \sum_{k=1}^{n} \left[\left(\frac{dx_k}{dt}\right)^2 + \omega_k^2 x_k^2 \right] + \epsilon H'\left(\frac{d\vec{x}}{dt}, \vec{x}\right), \quad \epsilon << 1.$$

Case 2) ω_1/ω_2 is rational, and the system is "weakly resonant";

Case 3) ω_1/ω_2 is rational, and the system is "strongly resonant."

Here we have used our own terminology as to the types of resonance in order to avoid reference to and transformation to action angle variables of a specific problem. We will discuss weak and strong resonance in more detail later; for now it will be sufficient to state that, phenomenologically speaking in terms of applying the Poincaré procedure to the problem in question, by weak resonance we mean that there are no $O(1)^*$ additive corrections to the $O(1)$ term of the solution arising from higher iterations - by strong resonance we mean that there are such corrections. Each of the three cases distinguished above leads, in Whittaker's formalism, to a different analytic form for the adelphic integral.

Whittaker's work raises two questions: do the series defining the adelphic integrals converge, and if they do what is implied by the fact that the series change form drastically over arbitrarily small changes in the frequency ratio? With regard to the first question, we suspect that one can think of examples where Whittaker's formalism will lead to apparently divergent adelphic integrals. As to the second, we suggest that if Whittaker attempted to eliminate small divisors, the series for an irrational frequency ratio would resemble that for a neighboring rational ratio, since we expect a much smoother dependence on the frequency ratio than Whittaker's results appear to indicate. These

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*In this thesis $\varphi(\epsilon, t) = O\left(\psi(\epsilon)\right)$ will mean $\frac{\psi(\epsilon)}{\varphi(\epsilon,t)}$ and $\frac{\varphi(\epsilon,t)}{\psi(\epsilon)}$ both remain bounded as ϵ tends to 0 (t fixed) for "almost all" t (except possibly for a set of measure zero of t's).

questions are still open, but even if the adelphic integrals diverge it appears likely that one can find functions which are constant to some order, say ϵ^p, then variable only by an amount of order ϵ^{p+1}.

More recently, several authors have considered problems involving two or more oscillators with the aid of mechanical computation as well as analytic techniques, the former giving these authors, in a sense, experimental results to check their theoretical solutions. Among the significant contributions of this type are those of Fermi, Pasta and Ulam (1955), Ford and Waters (1963) and E. A. Jackson (1963a, 1963b).

FPU performed computer studies of a one dimensional chain of identical particles connected by identical, weakly nonlinear springs. The configuration of their system is represented in Fig. 1.

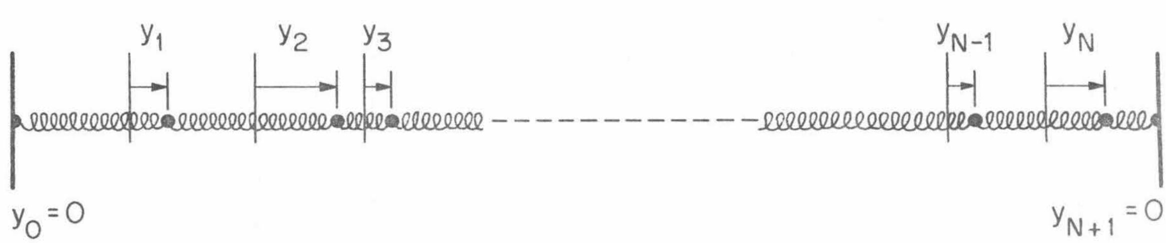

Fig. 1. Configuration of FPU System

y_k is the displacement of the kth particle from its equilibrium position. The potential energy of the spring connecting the kth particle to the k + 1 st is (in dimensionless coordinates)

$$V_{k,k+1} = \frac{1}{2}\left(y_{k+1} - y_k\right)^2 + \frac{\epsilon}{3}\left(y_{k+1} - y_k\right)^3$$

Thus the FPU Hamiltonian is (still in dimensionless coordinates)

$$H_{FPU} = \tfrac{1}{2} \sum_{k=1}^{N} \left(\frac{dy_k}{dt}\right)^2 + \sum_{k=0}^{N} \left\{ \tfrac{1}{2} \left(y_{k+1} - y_k\right)^2 + \frac{\epsilon}{3}\left(y_{k+1} - y_k\right)^3 \right\} \qquad (1)$$

Transforming to mode coordinates (based on the linear normal modes), the Hamiltonian becomes:

$$H_{FPU} = \tfrac{1}{2} \sum_{q=1}^{N} \left\{ \left(\frac{dx_q}{dt}\right)^2 + \omega_q^2 x_q^2 \right\} + \epsilon \sum_{k=1}^{N} \sum_{\ell=1}^{N} \sum_{m=1}^{N} A_{k\ell m} x_k x_\ell x_m \qquad (2)$$

with

$$\omega_q = 2 \sin \frac{q\pi}{2(N+1)}$$

We will refer to systems with Hamiltonians of the form (2) as "FPU-type systems."

Contrary to their expectations, FPU found that their system was not ergodic since energy was not shared uniformly among the modes - when the system was started with all the energy in the first mode only the first few modes became appreciably excited. This interesting result was the starting point for several subsequent studies of coupled oscillators, prominent among which were the work of Ford and Waters (1963) and of Jackson (1963a), (1963b).

Ford and Waters used the Wigner-Brillouin perturbation theory (which is similar to Poincaré's method) and mechanical computation to study conservative systems of from two to fifteen oscillators with weak cubic coupling in the Hamiltonian. Their theoretical work and supporting machine computations showed that such systems can exhibit significant (i.e., O(1)) energy sharing among all the modes only if the fundamental frequencies ω_k are respectively near certain integral multiples of some basic frequency. Furthermore, they found that even when the frequencies are in the right ratios there exist certain sets of initial conditions which do not permit significant energy exchange. This being the case, they concluded that nonlinear systems of the type they studied are not ergodic.

They also examined the behavior of a particular five-oscillator system which had the appropriate frequencies and initial conditions for energy sharing and found that in such a system, the amount of time a single oscillator (mode) spends with energy between E and $E + \delta E$ is roughly proportional to $\exp(-E/E_0)$, where E_0 is the total energy in the system divided by the number of oscillators. Thus, the remaining oscillators form a "heat bath" for the one under consideration and the canonical distribution appears without benefit of ergodicity.

Ford and Waters' analytic results for non-resonant systems were in agreement with their mechanical computations and the results of our N-timing procedure. However, the Wigner-Brillouin procedure, like the Poincaré technique, is inherently incapable of handling strongly resonant systems, since terms of O(1) will generally appear in an infinite number of iterations. As we shall see later, such behavior means we are putting the answer in the wrong analytic form; the slow variation of the solution is not necessarily a frequency shift, but possibly a different function of the slow time variables.

Jackson used a modified Wigner-Brillouin procedure to study FPU-type systems with moderate values of the coupling parameter (e.g.

$^1/_4$, $^1/_2$, $^3/_4$). He found that the sizes of the frequency shifts due to the nonlinear coupling were significant in determining the amount of energy exchanged among the modes and the recurrence time of the initial distribution of energy, and his theoretical results for these quantities agreed rather well with his machine calculations. Like Ford and Waters, Jackson pointed out that the energy sharing behavior of such systems depends strongly on initial conditions, and an FPU-type ensemble starting with all its energy in the low modes will tend to keep most of its energy in the low modes. Finally he pointed out that small changes in the initial conditions will not cause substantial changes in the gross properties of the solution.

Jackson's perturbation approach, while effective in predicting the gross behavior of the systems he studied, is somewhat less useful in giving explicit solutions to the dynamical problems. One reason is that his coupling constants are too large to permit the terms of his series to decrease in size rapidly, but more important, the same observations apply to his method as to Ford's method described above; the Wigner-Brillouin approach breaks down near resonances, and as an FPU system tends to a large number of degrees of freedom, the frequencies of the lower modes approach the resonant ratios. Thus Jackson's method is not useful for studying the detailed dynamics of large systems. Moreover, for certain classes of initial conditions higher order resonances are possible for systems of as few as two oscillators, even with incommensurable frequencies, and neither does Jackson's method apply to this problem.

Chapter II

The "N-timing" Perturbation Procedure

A. The Necessity for Uniform Validity and the Small Divisor Problem.

The problems to be studied in the present work are members of a class of problems involving small forces which are active for a long time. The basic systems we shall consider are derivable from Hamiltonians of the form (in dimensionless units):

$$H = \tfrac{1}{2}\sum_{k=1}^{N}\left\{\left(\frac{dx_k}{dt}\right)^2 + \omega_k^2 x_k^2\right\} - \epsilon g\left(x_1, x_2 \ldots, x_N\right)$$

with corresponding equations of motion

$$\frac{d^2 x_k}{dt^2} + \omega_k^2 x_k = \epsilon \frac{\partial}{\partial x_k} g\left(x_1, x_2, \ldots, x_k, \ldots, x_N\right) \qquad (1)$$

and

$$0 < |\epsilon| << 1, \quad x_k \leqslant O(1), \quad \frac{\partial g}{\partial x_k} \leqslant O(1)$$

The initial conditions to be satisfied are

$$x_k(0) = a_k \; ; \quad \left(\frac{dx_k}{dt}\right)_{|t=0} = b_k$$

g_k is a polynomial in the x_k's and at first we are thinking about the limit $\epsilon \to 0$, t fixed. The dynamic variables x_k and $\frac{dx_k}{dt}$ will usually correspond to the amplitudes and time rates of change of the amplitudes of the normal modes of oscillation of a system which is linear if $\epsilon \equiv 0$. We assume that we know by physical or other considerations that the dynamical variables x_k and $\frac{dx_k}{dt}$ are bounded.

The solutions of such systems are for times $t \leqslant O(1)$ equivalent to $O(1)$ to the solutions of the uncoupled equations; i.e., to the solutions of equations (1) with $\epsilon \equiv 0$. As time passes, however, the weak forces will cause the exact solutions of equations (1) to drift away or bifurcate from those of the uncoupled equations, and eventually the error engendered in using the initially valid solutions in place of the exact solutions will be O(1). Alternatively, if we look at the solution of system (1) in the time interval $\tau < t < \tau + \delta\tau$, where $\tau \geq O(1/\epsilon^2)$ and $\delta\tau = O(1)$, to O(1) the motion will look like the solution of the uncoupled equations with initial conditions different from equations (2) (but of course located on the same energy shell). Thus, to O(1) the solutions of equations (1) look like those of the uncoupled equations with slowly varying initial conditions.

A consequent difficulty often encountered in applying perturbation procedures to such problems is that the resulting solutions are only valid initially - after a certain initial time the magnitude of the higher correction terms becomes equivalent to or larger than that of the lowest order terms, and consequently an infinite number of terms is needed to adequately describe the answer. The classical procedure for eliminating this difficulty, and the one which has generally been applied to the kind of systems we are studying here, is the well-known Poincaré technique, which seeks to allow for the nonlinearity by introducing small shifts in the fundamental frequencies of the oscillators.

The Poincaré procedure works effectively if the frequencies are "sufficiently incommensurable," and in many cases for a certain delimited span of time when they are commensurable. However, trigonometric terms of various combination frequencies appear on the right hand sides of all iterations subsequent to the first, and if one of these combination frequencies is "sufficiently close" to the fundamental frequency of the equation in which it appears, the corresponding small resonance denominator can push the amplitude of the resulting term in the solution to a larger order than it was originally thought to be. Furthermore, once such a small divisor appears, it frequently appears in successively higher powers in subsequent interactions, raising a term from each of these iterations to the increased order of the term where the small divisor first appeared. The appearance of such terms in the solution thus tends to break down the uniform validity of the solution and raise questions about the convergence of the expansion.

A related situation is the case where the frequencies are commensurable but not in a ratio which will cause strong resonance (i.e., O(1) corrections to the solution due to small divisors). In this case one can obtain combination frequencies whose O(1) terms vanish identically. However, the small corrections to the combination frequencies usually do not vanish, so if an appropriately modified Poincaré scheme is used, this situation reduces to the usual small divisor problem. In the work that follows, we shall call cases where no small divisor appears to the order considered non-resonant, and where they do appear, weakly resonant. We suggest, however, that there is not necessarily a sharp dichotomy between the two cases - the small divisors in the latter case may just take longer to appear.*

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*It is not difficult to see how small divisors can appear at some point in the Poincaré expansion for any given pair of frequencies ω_1, ω_2. Given any pair of incommensurable numbers ω_1 and ω_2 and any $\delta > 0$, there exist infinitely many pairs of (positive) integers n, m such that $|n\omega_2 - m\omega_1| < \delta$. However, for polynomial coupling, the larger the values of n and m, the further out in the series the corresponding combination frequency will appear. It does not seem a priori obvious (without actually

In some cases, near particular frequency ratios which correspond to the form of the nonlinearity, one finds that small denominators occur in the initial iterations of the Poincaré scheme and raise the corresponding terms to $O(1)$, thus causing $O(1)$ corrections to the answer. In such cases, to which we will refer as "strongly resonant," the Poincaré procedure is useless (except for establishing that the strong resonance exists) and one usually attempts to establish the $O(1)$ solution by other means, most of which involve certain transformations of the original equations.

Several sophisticated techniques have been developed to deal with the difficulties mentioned in the foregoing paragraphs, including Krylov and Bogoliubov's method of averaging (Bogoliubov and Mitropolsky, 1961), Struble's general asymptotic method (Struble 1962) and Cole's two-timing procedure (Cole 1968). This chapter will introduce N-timing, which is an extension of two-timing.

In the work that will follow, we will find that the small divisor problem can be eliminated with the assistance of a sufficiently flexible perturbation procedure. N-timing appears to be such a procedure, and it will be used to study a non-resonant example which is also tractable by Poincaré's method, as well as weakly and strongly resonant examples

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(continued from preceding page)

calculating the expansion) how to tell whether or not a particular combination frequency $(n\omega_2 - m\omega_1)$ will appear but we can easily provide a bound for the order of the term in which a small divisor will first appear by the following procedure. Suppose H' (the perturbing Hamiltonian) is a polynomial of order k. Let the term where a small divisor first appears in the Poincaré expansion of the solution of the equations of motion resulting from the perturbed Hamiltonian $H_0 + \epsilon H'$ be $O(\epsilon^{\ell})$. Let (N, M) be the pair of integers satisfying $|n\omega_2 - m\omega_1| = O(\epsilon)$ and having the smallest value $|n+m|$ of all such pairs. Then $\ell \geqslant$ (smallest integer $\geqslant (N+M-1/k-2)-1$).

which are not.

B. The Method of N-timing.

The simplest perturbation procedure we can apply to a problem like (1) is a limit-process expansion wherein we let

$$x_k(t) = x_k^{(0)}(t) + \epsilon x_k^{(1)}(t) + \epsilon^2 x_k^{(2)}(t) + \dots \tag{2}$$

substitute (2) for x_k in (1), and set the coefficient of each ϵ^p in each of the N resulting equations separately equal to zero. This yields a system of N equations of each order in ϵ which in turn leads to a sequence of problems which can be solved in series (because of the limit process $\epsilon \to 0$). The equations of orders 1 and ϵ are:

$$O(1) \qquad \frac{d^2 x_k^{(0)}}{dt^2} + \omega_k^2 x_k^{(0)} = 0 \tag{3}$$

$$O(\epsilon) \qquad \frac{d^2 x_k^{(1)}}{dt^2} + \omega_k^2 x_k^{(1)} = \frac{\partial g}{\partial x_k}(x_1^{(0)}, \dots, x_k^{(0)}, \dots, x_N^{(0)}) \tag{4}$$

This system can be solved by first solving (3), then replacing the $x_k^{(0)}$'s in the right-hand side of (4) with the corresponding solutions of equations (3). However, one will generally find terms proportional to t appearing in the solutions for $x_k^{(1)}$ or $x_k^{(2)}$. Moreover, once such terms start to appear, one will find a higher power of t in each succeeding term, so that although these solutions may be valid to O(1) for $t = O(1)$, it is clear that the exact solutions cannot be represented accurately to O(1) by a finite number of terms for $t \geq O(1/\epsilon)$ or $O(1/\epsilon^2)$ (depending on whether the t first appears in the $O(\epsilon)$ or $O(\epsilon^2)$ term of the expression for x_k). This difficulty is clearly connected with the difficulty described in the introduction above, the fact that the solutions of (1) to O(1) resemble the solutions of the uncoupled equations with slowly varying initial conditions. The solutions obtained by this simple limit process expansion always have as an $O(1)$ term the solutions of the uncoupled equations with constant initial conditions. The divergent terms are telling us that we must allow for some change in the form of the $O(1)$ term due to the presence of the weak forces.

The classical procedure for eliminating this difficulty was developed by Poincaré (1893). He suggested that the effect of the weak forces is to cause a slight shift in the frequencies ω_k. Thus he proposed that one replace the ω_k's by

$$\Omega_k = \omega_k + \epsilon\omega_k^{(1)} + \epsilon^2\omega_k^{(2)} + \ldots \tag{5}$$

where the $\omega_k^{(i)}$'s and Ω_k are constants to be determined. Then equation (1) becomes

$$\frac{d^2 x_k}{dt^2} + \Omega_k^2 x_k = (\Omega_k^2 - \omega_k^2) x_k + \epsilon \frac{\partial}{\partial x_k} g(x_1, x_2, \ldots, x_k, \ldots, x_N) \tag{6}$$

where $\Omega_k^2 - \omega_k^2 = O(\epsilon)$. Substituting expansions of the form (2) for the x_k in equations (5) and setting the coefficients of each ϵ^p in each of the resulting equations equal to zero, we obtain a system of N equations of each order in ϵ. The equations of order 1 and ϵ are now:

$$O(1) \qquad \frac{d^2 x_k^{(0)}}{dt^2} + \Omega_k^2 x_k^{(0)} = 0 \tag{7}$$

$$O(\epsilon) \qquad \frac{d^2 x_k^{(1)}}{dt^2} + \Omega_k^2 x_k^{(1)} = 2\omega_k\omega_k^{(1)} x_k^{(0)} + \frac{\partial}{\partial x_k} g(x_1^{(0)}, x_2^{(0)}, \ldots, x_k^{(0)}, x_N^{(0)}) \tag{8}$$

Solving equation (7) we obtain:

$$x_k^{(0)} = \alpha_k^{(0)} \cos\{\Omega_k t - \varphi_k^{(0)}\} \tag{9}$$

Replacing each $x_k^{(0)}$ by its solution on the right hand side of (8) with its value (9), we obtain a set of linear oscillators with forcing terms whose frequencies are appropriate combinations of the Ω_k. The term $2\omega_k^{(0)}\omega_k^{(1)}x_k^{(0)}$ has a frequency identical to the fundamental, Ω_k. This and other of the driving terms which have frequency Ω_k (e.g., a term like $x_\ell^2 x_k$) are called secular terms and will lead to terms of the form $t\sin(\Omega_k t - \varphi_k^{(0)})$ in the expression for x_1, unless the sum of the coefficients of such driving terms vanishes. Since a useful perturbation theory requires that the terms be uniformly ordered - i.e., that the terms of $O(\epsilon)$ remain smaller than the terms of $O(1)$, etc. - and we are free to choose the $\omega_k^{(1)}$'s to suit our convenience, we choose the $\omega_k^{(1)}$'s such that the secular terms vanish. We then choose the $\omega_k^{(2)}$'s by repeating this procedure with the $O(\epsilon^2)$ equations and so forth.

The Poincaré procedure is quite useful for studying systems that reduce to a single oscillator. Unfortunately, it will generally break down in some order when $N \geqslant 2$. For example, suppose $N = 2$, $\omega_1 = 1$, $\omega_2 = 2$, and a term $x_1^2 x_2$ appears in the Hamiltonian. This leads to a term $\frac{1}{2}\alpha_1^{(0)2}\cos(2\Omega_1 t - 2\varphi_1^{(0)})$ in the right-hand side of the equation for $x_2^{(1)}$. This is not a secular term, so it leads to a term like

$$\frac{\alpha_1^{(0)2}}{2} \quad \frac{\cos\{2\Omega_1 t - 2\varphi_1^{(0)}\}}{\Omega_2^2 - 4\Omega_1^2}$$

in the expression for $x_2^{(1)}$. However, $\Omega_2 = 2 + \epsilon\,\omega_2^{(1)} + \ldots$ and $\Omega_1 = 1 + \epsilon\,\omega_1^{(1)} + \ldots$ so that $\Omega_2^2 - 4\Omega_1^2 = \epsilon\,(4\omega_2^{(1)} - 8\omega_1^{(1)}) + O(\epsilon^2)$, and thus one of the terms supposedly of $O(\epsilon)$ makes an $O(1)$ contribution to the expression for x_2. Furthermore, subsequent higher order terms may also make contributions in $O(1)$ and the Poincaré theory thus becomes useless for this case.

In cases where such a breakdown does not occur in the $O(\epsilon)$ solution it may still occur later. For the case $N = 2$, it is easy to see how this can happen if ω_1 and ω_2 are commensurable. Of course, if they are incommensurable, it may also occur if we have $n\omega_1 - m\omega_2 = O(\epsilon)$ and one of the iterations, say $O(\epsilon^r)$ leads to a term with frequency $n\Omega_1 - m\Omega_2$. This implies a term contributing to $O(\epsilon^{r-1})$ and the further validity of the procedure becomes doubtful. The solution is probably valid to $O(\epsilon^{r-2})$ but questionable thereafter; thus we can follow the oscillator for a time $1/\epsilon^{r-2}$ but since we are not sure of the frequencies $\omega_k^{(r-1)}$, we lose track of it for larger times. This is the famous problem of small divisors.

The N-timing procedure which we shall now introduce and describe avoids the difficulties described above by anticipating that the slow changes with time of the solutions of equations (1) may be more complicated than simple frequency shifts.

To employ the N-timing procedure we assume that the slow variation of the solutions of equations (1) can be represented by formally considering the solutions to be functions of a sequence of related but (formally) independent variables; $t_0, t_1, \dots, t_k, \dots$, where the new variables are related to t by the relations $t_k = \epsilon^k t$.* We then use expansions for the x_k of the form:

$$x_k(t) = x_k^{(0)}(t_0, t_1, \dots, t_q, \dots) + \epsilon x_k^{(1)}(t_0, t_1, \dots) + \epsilon^2 x_k^{(2)}(t_0, t_1, \dots) \quad (10)$$

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*A more general plan would have $t_k = \varphi_k(\epsilon)\, t$, where the $\varphi_k(\epsilon)$ form an asymptotic sequence.

Derivatives with respect to t now become:

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \ldots = \sum_{k=0}^{\infty} \epsilon^k \frac{\partial}{\partial t_k} \tag{11}$$

so that

$$\frac{dx_k(t)}{dt} = \frac{\partial x_k^{(0)}}{\partial t_0} + \epsilon \left\{ \frac{\partial x_k^{(1)}}{\partial t_0} + \frac{\partial x_k^{(0)}}{\partial t_1} \right\} + \epsilon^2 \left\{ \frac{\partial x_k^{(2)}}{\partial t_0} + \frac{\partial x_k^{(1)}}{\partial t_1} + \frac{\partial x_k^{(0)}}{\partial t_2} \right\} + \ldots \tag{12}$$

and

$$\frac{d^2 x_k(t)}{dt^2} = \frac{\partial^2 x_k^{(0)}}{\partial t_0^2} + \epsilon \left\{ \frac{\partial^2 x_k^{(1)}}{\partial t_0^2} + 2 \frac{\partial^2 x_k^{(0)}}{\partial t_0 \, \partial t_1} \right\} + \epsilon^2 \left\{ \frac{\partial^2 x_k^{(2)}}{\partial t_0^2} + 2 \frac{\partial^2 x_k^{(1)}}{\partial t_0 \, \partial t_1} + \frac{\partial^2 x_k^{(0)}}{\partial t_1^2} + 2 \frac{\partial^2 x_k^{(0)}}{\partial t_0 \, \partial t_2} \right\}$$

We now replace $x_k(t)$ and $\frac{d^2 x_k}{dt^2}$ in equations (1) by their corresponding expansions, equations (10) and (12). Setting the coefficients of each power of ϵ in each equation separately equal to zero we obtain a system of N equations of each order in ϵ. The equations of $O(1)$ and $O(\epsilon)$ are respectively,

$$O(1) \quad \frac{\partial^2 x_k^{(0)}}{\partial t_0^2} + \omega_k^2 x_k^{(0)} = 0 \tag{13}$$

$$O(\epsilon) \quad \frac{\partial^2 x_k^{(1)}}{\partial t_0^2} + \omega_k^2 \, x_k^{(1)} = -2 \frac{\partial^2 x_k^{(0)}}{\partial t_0 \, \partial t_1} + \frac{\partial g}{\partial x_k} \left(x_1^{(0)}, \; x_2^{(0)}, \ldots, x_k^{(0)}, \ldots x_N^{(0)}\right) \tag{14}$$

The fundamental principle of expansion is that each term appearing in a solution of a particular order must be uniformly of that order. This excludes terms increasing like t, as well as terms with coefficients such as $1/\epsilon$ that raise a term to a larger* order. Such terms must be eliminated by choosing the proper dependence of larger order terms on the slow time variables, in the spirit of the procedure for removing

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*Throughout, to avoid ambiguity, larger order will mean larger magnitude - i.e., $O(1)$ is larger order than $O(\epsilon)$.

secular terms in the Poincaré technique.

The N-timing method is capable of handling all problems which can be done by the Poincaré technique, but more important, it provides a method for solving problems with resonant or near resonant frequencies where the Poincaré technique is manifestly inapplicable. N-timing also has the aesthetic feature that with no modification it is applicable to a variety of problems including both resonant and non-resonant systems like equations (1). In the present work we shall apply N-timing to three examples, one where no resonance appears to the order considered, and two others where resonances become significant early and lead to interesting consequences.

For a more detailed discussion and examples of the two-timing procedure, of which N-timing is an extension, refer to Cole (1968, Chapter III), and Kevorkian (1966). Unknown to the present author at the time this work was done similar expansions were proposed by Sandri (1966) previous to and Lick (1968) simultaneously with the present work. However, neither author applied the procedure to problems of the type we are considering. Lick applied it to singular problems and some partial differential equations from fluid mechanics and Sandri to some quantum mechanical examples.*

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*It should also be noted that Eckstein, Shi and Kevorkian (1964) studied an orbital mechanics problem which required the use of three time variables for solution. However, it appears that the third variable was used because the problem involved matching of solutions in two regions which required different slow variables. Therefore the approach used does not seem to be directly comparable with N-timing.

Chapter III

A Non-resonant Example

The first example we will consider is a non-resonant case, a two-oscillator FPU system, where to the order considered the Poincaré procedure will give the same result as N-timing.

The system of equations to be solved is derived from the Hamiltonian*:

$$\overline{H}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \tfrac{1}{2}\dot{z}_1^2 + \tfrac{1}{2}\dot{z}_2^2 + \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2 + \tfrac{1}{2}(z_2 - z_1)^2 + \overline{\epsilon}\left[z_1^3 + (z_2 - z_1)^3 - z_2^3\right] \quad (1)$$

For convenience, the equations will be solved in normal mode coordinates x, y where:

$$x = \frac{z_1 + z_2}{\sqrt{2}} \text{ is the amplitude of the symmetric mode and}$$

$$y = \frac{z_2 - z_1}{\sqrt{2}} \text{ is the amplitude of the antisymmetric mode.} \quad (2)$$

Applying transformation (2) to the Hamiltonian (1) and letting:

$$\epsilon = \frac{3}{\sqrt{2}}\,\overline{\epsilon} \quad (3)$$

we obtain:

$$H(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}\left(\frac{dy}{dt}\right)^2 + \tfrac{1}{2}x^2 + \tfrac{3}{2}y^2 + \epsilon\, y(y^2 - x^2). \quad (4)$$

The equations of motion are thus:

$$\frac{d^2x}{dt^2} + x = 2\epsilon xy \qquad x(0) = a \quad \left(\frac{dx}{dt}\right)\bigg|_{t=0} = b \quad (5a)$$

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* $\dot{f} \equiv \dfrac{df(t)}{dt}$

$$\frac{d^2y}{dt^2} + 3y = \epsilon(x^2 - 3y^2) \qquad y(0) = c \quad \left(\frac{dy}{dt}\right)\Big|_{t=0} = d \tag{5b}$$

A. <u>Solution of the Equations of Motion by N-timing.</u>

We shall apply the N-timing procedure to equations (5). Let

$$x(t) = \sum_{k=0}^{\infty} \epsilon^k \, x_k(t_0, t_1, \ldots) \tag{6a}$$

$$y(t) = \sum_{\ell=0}^{\infty} \epsilon^\ell \, y_\ell(t_0, t_1, \ldots) \tag{6b}$$

where

$$t_q = \epsilon^q t \tag{7}$$

The operator

$$\frac{d}{dt} = \sum_{k=0}^{\infty} \epsilon^k \frac{\partial}{\partial t_k} \quad ,$$

so that

$$\frac{d^2}{dt^2} = \sum_{k=0}^{\infty} \epsilon^k \left\{ \sum_{p=0}^{k} \frac{\partial^2}{\partial t_p \partial t_{k-p}} \right\} \tag{8}$$

Using expressions (6a) and (6b) for x and y and equation (8) for $\frac{d^2}{dt^2}$ in equations (5) and equating the coefficient of ϵ^k in each of the resulting equations to zero, we obtain a double sequence of equations of decreasing order in ϵ. The equations we shall need are:

$$O(1) \quad \left\{ \begin{array}{ll} \dfrac{\partial^2 x_0}{\partial t_0^2} + x_0 = 0 & \quad (9a) \\[2ex] \dfrac{\partial^2 y_0}{\partial t_0^2} + 3y_0 = 0 & \quad (9b) \end{array} \right.$$

$$O(\epsilon)\begin{cases}\dfrac{\partial^2 x_1}{\partial t_0^2} + x_1 = 2x_0\, y_0 - 2\dfrac{\partial^2 x_0}{\partial t_0\, \partial t_1} & (10a)\\[2ex] \dfrac{\partial^2 y_1}{\partial t_0^2} + 3y_1 = x_0^2 - 3y_0^2 - 2\dfrac{\partial^2 y_0}{\partial t_0\, \partial t_1} & (10b)\end{cases}$$

$$O(\epsilon^2)\begin{cases}\dfrac{\partial^2 x_2}{\partial t_0^2} + x_2 = 2x_0\, y_1 - 2y_0 x_1 - 2\dfrac{\partial^2 x_0}{\partial t_0\, \partial t_2} - \dfrac{\partial^2 x_0}{\partial t_1^2} - 2\dfrac{\partial^2 x_1}{\partial t_0\, \partial t_1} & (11a)\\[2ex] \dfrac{\partial^2 y_2}{\partial t_0^2} + 3y_2 = 2x_0\, x_1 - 6y_0\, y_1 - 2\dfrac{\partial^2 y_0}{\partial t_0\, \partial t_2} - \dfrac{\partial^2 y_0}{\partial t_1^2} - 2\dfrac{\partial^2 y_1}{\partial t_0\, \partial t_1} & (11b)\end{cases}$$

$$O(\epsilon^3)\begin{cases}\dfrac{\partial^2 x_3}{\partial t_0^2} + x_3 = 2x_0 y_2 + 2x_1 y_1 + 2x_2 y_0 - 2\dfrac{\partial^2 x_0}{\partial t_0\, \partial t_3} - 2\dfrac{\partial^2 x_0}{\partial t_1\, \partial t_2} - 2\dfrac{\partial^2 x_1}{\partial t_0 \partial t_2} - \dfrac{\partial^2 x_1}{\partial t_1^2} - 2\dfrac{\partial^2 x_2}{\partial t_0 \partial t_1} & (12a)\\[2ex] \dfrac{\partial^2 y_3}{\partial t_0^2} + 3y_3 = 2x_0 x_2 + x_1^2 - 6y_0 y_2 - 3y_1^2 - 2\dfrac{\partial^2 y_0}{\partial t_0 \partial t_3} - 2\dfrac{\partial^2 y_0}{\partial t_1 \partial t_2} - 2\dfrac{\partial^2 y_1}{\partial t_0 \partial t_2} - \dfrac{\partial^2 y_1}{\partial t_1^2} - 2\dfrac{\partial^2 y_2}{\partial t_0\, \partial t_1} & (12b)\end{cases}$$

In solving these equations we shall use the following notation for resonance denominators:

$$R_{m,n} = \frac{1}{1-(m\sqrt{3}+n)^2} \qquad\qquad S_{m,n} = \frac{1}{3-(m\sqrt{3}+n)^2} \tag{13}$$

The solutions of equations (9) are*:

$$x_0\,(t_0) = a_0\,(t_1)\cos(t_0 - \varphi_0\,(t_1)\,) \tag{14a}$$

$$y_0\,(t_0) = b_0\,(t_1)\cos(\sqrt{3}\, t_0 - \theta_0\,(t_1)\,) \tag{14b}$$

Substituting equations (13a) and (13b) in (10a) and (10b), we obtain:

$$\begin{aligned}\frac{\partial^2 x_1}{\partial t_0^2} + x_1 = {} & 2\,\frac{\partial a_0(t_1)}{\partial t_1}\sin(t_0 - \varphi_0(t_1)) - 2a_0(t_1)\,\frac{\partial \varphi_0(t_1)}{\partial t_1}\cos(t_0 - \varphi_0\,(t_1))\\ & + a_0(t_1)b_0(t_1)\cos\left\{(\sqrt{3}+1)t_0 - \theta_0(t_1) - \varphi_0(t_1)\right\}\\ & + a_0(t_1)b_0(t_1)\cos\left\{(\sqrt{3}-1)t_0 - \theta_0(t_1) + \varphi_0(t_1)\right\}\end{aligned} \tag{15a}$$

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*Throughout this thesis, we shall use the notation $f(t_k) = f(t_k, t_{k+1}, t_{k+2}, t_{k+3}, \dots)$

$$\frac{\partial^2 y_1}{\partial t_0^2} + 3y_1 = 2\sqrt{3}\ \frac{\partial b_0(t_1)}{\partial t_1} \sin\left(\sqrt{3}\, t_0 - \theta_0(t_1)\right) - 2\sqrt{3}\, b_0(t_1)\ \frac{\partial \theta_0(t_1)}{\partial t_1} \cos\left(\sqrt{3}\, t_0 - \theta_0(t_1)\right)$$

$$+ \frac{a_0^2(t_1)}{2} + \frac{a_0^2(t_1)}{2} \cos\left(2t_0 - 2\varphi_0(t_1)\right)$$

$$- \frac{3}{2} b_0^2(t_1) - \frac{3}{2} b_0^2(t_1) \cos\left(2\sqrt{3}\, t_0 - 2\theta_0(t_1)\right) \tag{15b}$$

The condition that the solutions of these equations be uniformly bounded requires the coefficients of $\sin[t_0 - \varphi_0(t_1)]$ and $\cos[t_0 - \varphi_0(t_1)]$ in the first equation and $\sin[\sqrt{3}\, t_0 - \theta_0(t_1)]$ and $\cos[\sqrt{3}\, t_0 - \theta_0(t_1)]$ in the second equation to vanish. Thus,

$$\frac{\partial a_0(t_1)}{\partial t_1} = 0 \qquad \frac{\partial \varphi_0(t_1)}{\partial t_1} = 0 \qquad a_0 = a_0(t_2) \qquad \varphi_0 = \varphi_0(t_2) \tag{16a}$$

$$\frac{\partial b_0(t_1)}{\partial t_1} = 0 \qquad \frac{\partial \theta_0(t_1)}{\partial t_1} = 0 \qquad b_0 = b_0(t_2) \qquad \theta_0 = \theta_0(t_2) \tag{16b}$$

The solutions of (15a) and (15b) are:

$$x_1 = a_1(t_1)\cos\left(t_0 - \varphi_1(t_1)\right) + a_0(t_2) b_0(t_2) R_{11} \cos\left\{(\sqrt{3}+1)t_0 - \theta_0(t_2) - \varphi_0(t_2)\right\}$$

$$+ a_0(t_2) b_0(t_2) R_{1-1} \cos\left\{(\sqrt{3}-1)t_0 - \theta_0(t_2) + \varphi_0(t_2)\right\} \tag{17a}$$

$$y_1 = b_1(t_1)\cos\left(\sqrt{3}\, t_0 - \theta_1(t_1)\right) + \left(\tfrac{1}{2} a_0^2(t_2) - \tfrac{3}{2} b_0^2(t_2)\right) S_{00}$$

$$+ \tfrac{1}{2} a_0^2(t_2) S_{02} \cos\left\{2t_0 - 2\varphi_0(t_2)\right\}$$

$$- \tfrac{3}{2} b_0^2(t_2) S_{20} \cos\left\{2\sqrt{3}\, t_0 - 2\theta_0(t_2)\right\} \tag{17b}$$

Using equations (14), (16) and (17) in equations (11) we obtain:

$$\frac{\partial^2 x_2}{\partial t_0^2} + x_2 = a_0(t_2)\left[-2\frac{\partial\varphi_0(t_2)}{\partial t_2} + a_0^2(S_{00}+\tfrac{1}{2}S_{02}) + b_0^2\{R_{11}+R_{1-1}-3S_{00}\}\right]\cos\left(t_0-\varphi_0(t_2)\right)$$

$$+2\frac{\partial a_0(t_2)}{\partial t_2}\sin\left(t_0-\varphi_0(t_2)\right) - 2a_1(t_1)\frac{\partial\varphi_1(t_1)}{\partial t_1}\cos\left(t_0-\varphi_1(t_1)\right) + 2\frac{\partial a_1(t_1)}{\partial t_1}\sin\left(t_0-\varphi_1(t_1)\right)$$

$$+\tfrac{1}{2}a_0^3(t_2)S_{02}\cos\left(3t_0-3\varphi_0(t_2)\right) + a_0(t_2)b_1(t_1)\cos\{(\sqrt{3}+1)t_0-\theta_1(t_1)-\varphi_0(t_2)\}$$

$$+a_1(t_1)b_0(t_2)\cos\{(\sqrt{3}+1)t_0-\theta_0(t_2)-\varphi_1(t_1)\} + a_0(t_2)b_1(t_1)\cos\{(\sqrt{3}-1)t_0-\theta_1(t_1)+\varphi_0(t_2)\}$$

$$+a_1(t_1)b_0(t_2)\cos\{(\sqrt{3}-1)t_0-\theta_0(t_2)+\varphi_1(t_1)\}$$

$$+a_0(t_2)b_0^2(t_2)\{R_{11}-\tfrac{3}{2}S_{20}\}\cos\{(2\sqrt{3}+1)t_0-2\theta_0(t_2)-\varphi_0(t_2)\}$$

$$+a_0(t_2)b_0^2(t_2)\{R_{1-1}-\tfrac{3}{2}S_{20}\}\cos\{(2\sqrt{3}-1)t_0-2\theta_0(t_2)+\varphi_0(t_2)\} \qquad (18a)$$

and

$$\frac{\partial^2 y_2}{\partial t_0^2} + 3y_2 = b_0(t_2)\left[a_0^2(t_2)\{R_{11}+R_{1-1}-3S_{00}\} + 9b_0^2(t_2)\{S_{00}+\tfrac{1}{2}S_{20}\} - 2\sqrt{3}\frac{\partial\theta_0(t_2)}{\partial t_2}\right]\cos\{\sqrt{3}t_0-\theta_0(t_2)\}$$

$$+2\sqrt{3}\frac{\partial b_0(t_2)}{\partial t_2}\sin\{\sqrt{3}t_0-\theta_0(t_2)\} + 2\sqrt{3}\frac{\partial b_1(t_1)}{\partial t_1}\sin\{\sqrt{3}t_0-\theta_1(t_1)\} - 2\sqrt{3}b_1(t_1)\frac{\partial\theta_1(t_1)}{\partial t_1}\cos\{\sqrt{3}t_0-\theta_1(t_1)\}$$

$$+a_0(t_2)a_1(t_1)\cos\{\varphi_1(t_1)-\varphi_0(t_2)\} - 3b_0(t_2)b_1(t_1)\cos\{\theta_1(t_1)-\theta_0(t_2)\}$$

$$+a_0(t_2)a_1(t_1)\cos\{2t_0-\varphi_0(t_2)-\varphi_1(t_1)\} - 3b_0(t_2)b_1(t_1)\cos\{2\sqrt{3}t_0-\theta_0(t_2)-\theta_1(t_1)\}$$

$$+a_0^2(t_2)b_0(t_2)\{R_{11}-\tfrac{3}{2}S_{02}\}\cos\{(\sqrt{3}+2)t_0-\theta_0(t_2)-2\varphi_0(t_2)\}$$

$$+a_0^2(t_2)b_0(t_2)\{R_{1-1}-\tfrac{3}{2}S_{02}\}\cos\{(\sqrt{3}-2)t_0-\theta_0(t_2)+2\varphi_0(t_2)\}$$

$$+\tfrac{9}{2}b_0^3(t_2)S_{20}\cos\{3\sqrt{3}t_0-3\theta_0(t_2)\} \qquad (18b)$$

Setting the secular terms equal to zero in each equation we obtain:

$$2\,\frac{\partial a_1(t_1)}{\partial t_1}\sin\{\varphi_1(t_1)-\varphi_0(t_2)\}+2a_1(t_1)\,\frac{\partial\varphi_1(t_1)}{\partial t_1}\cos\{\varphi_1(t_1)-\varphi_0(t_2)\}$$

$$=a_0(t_2)\left[-2\,\frac{\partial\varphi_0(t_2)}{\partial t_2}+a_0^2(t_2)(S_{00}+\tfrac{1}{2}S_{02})+b_0^2(t_2)\{R_{11}+R_{1-1}-3S_{00}\}\right]\qquad(19a)$$

$$2\,\frac{\partial a_1(t_1)}{\partial t_1}\cos\{\varphi_1(t_1)-\varphi_0(t_2)\}-2a_1(t_1)\,\frac{\partial\varphi_1(t_1)}{\partial t_1}\sin\{\varphi_1(t_1)-\varphi_0(t_2)\}=-2\,\frac{\partial a_0(t_2)}{\partial t_2}\qquad(19b)$$

$$2\sqrt{3}\,\frac{\partial b_1(t_1)}{\partial t_1}\sin\{\theta_1(t_1)-\theta_0(t_0)\}+2\sqrt{3}\,b_1(t_1)\,\frac{\partial\theta_1(t_1)}{\partial t_1}\cos\{\theta_1(t_1)-\theta_0(t_0)\}$$

$$=b_0(t_2)\left[-2\sqrt{3}\,\frac{\partial\theta_0(t_2)}{\partial t_2}+a_0^2(t_2)\{R_{11}+R_{1-1}-3S_{00}\}+9b_0^2(t_2)\{S_{00}+\tfrac{1}{2}S_{20}\}\right]\qquad(19c)$$

$$2\sqrt{3}\,\frac{\partial b_1(t_1)}{\partial t_1}\cos\{\theta_1(t_1)-\theta_0(t_2)\}-2\sqrt{3}\,b_1(t_1)\,\frac{\partial\theta_1(t_1)}{\partial t_1}\sin\{\theta_1(t_1)-\theta_0(t_2)\}$$

$$=-2\sqrt{3}\,\frac{\partial b_0(t_2)}{\partial t_2}\qquad(19d)$$

Let $A_{1C}(t_1)=a_1(t_1)\cos\{\varphi_1(t_1)-\varphi_0(t_2)\}$, $A_{1S}(t_1)=a_1(t_1)\sin\{\varphi_1(t_1)-\varphi_0(t_2)\}$ (20a)

$B_{1C}(t_1)=b_1(t_1)\cos\{\theta_1(t_1)-\theta_0(t_2)\}$, $B_{1S}(t_1)=b_1(t_1)\sin\{\theta_1(t_1)-\theta_0(t_2)\}$ (20b)

Equations (19) become:

$$2\,\frac{\partial A_{1S}(t_1)}{\partial t_1}=a_0(t_2)\left[-2\frac{\partial\varphi_0(t_2)}{\partial t_2}+a_0^2(t_2)\{S_{00}+\tfrac{1}{2}S_{02}\}+b_0^2(t_2)\{R_{11}+R_{1-1}-3S_{00}\}\right]\qquad(21a)$$

$$\frac{\partial A_{1C}(t_1)}{\partial t_1}=-\frac{\partial a_0(t_2)}{\partial t_2}\qquad(21b)$$

$$2\sqrt{3}\,\frac{\partial B_{1S}(t_1)}{\partial t_1}=b_0(t_2)\left[-2\sqrt{3}\,\frac{\partial\theta_0(t_2)}{\partial t_2}+a_0^2(t_2)\{R_{11}+R_{1-1}-3S_{00}\}+9b_0^2(t_2)\{S_{00}+\tfrac{1}{2}S_{20}\}\right]\qquad(21c)$$

$$\frac{\partial B_{1C}(t_1)}{\partial t_1}=-\frac{\partial b_0(t_2)}{\partial t_2}\qquad(21d)$$

The right hand side of each of these equations is independent of t_1, so to keep A_{1C}, A_{1S}, B_{1C} and B_{1S} bounded on the t_1 time scale we need:

$$2\,\frac{\partial\varphi_0(t_2)}{\partial t_2} = a_0^2(t_2)\left\{S_{00}+\tfrac{1}{2}S_{02}\right\} + b_0^2(t_2)\left\{R_{11}+R_{1-1}-3S_{00}\right\} \tag{22a}$$

$$\frac{\partial a_0(t_2)}{\partial t_2} = 0 \tag{22b}$$

$$2\sqrt{3}\,\frac{\partial\theta_0(t_2)}{\partial t_2} = a_0^2(t_2)\left\{R_{11}+R_{1-1}-3S_{00}\right\} + 9b_0^2(t_2)\left\{S_{00}+\tfrac{1}{2}S_{20}\right\} \tag{22c}$$

$$\frac{\partial b_0(t_2)}{\partial t_2} = 0 \tag{22d}$$

so that
$$\frac{\partial A_{1c}(t_1)}{\partial t_1} = \frac{\partial A_{1s}(t_1)}{\partial t_1} = \frac{\partial B_{1c}(t_1)}{\partial t_1} = \frac{\partial B_{1s}(t_1)}{\partial t_1} = 0 \tag{23}$$

Equation (23) together with equations (20) implies:

$$a_1 = a_1(t_2) \qquad \varphi_1 = \varphi_1(t_2) \qquad b_1 = b_1(t_2) \qquad \theta_1 = \theta_1(t_2) \tag{24}$$

Equations (22b) and (22d) imply:

$$a_0 = a_0(t_3), \quad b_0 = b_0(t_3) \tag{25}$$

so that, integrating (22a) and (22c),

$$\varphi_0(t_2) = \left[\tfrac{1}{2}\left\{S_{00}+\tfrac{1}{2}S_{02}\right\} a_0^2(t_3) + \tfrac{1}{2}\left\{R_{11}+R_{1-1}-3S_{00}\right\} b_0^2(t_3)\right] t_2 + \varphi_0^{(3)}(t_3) \tag{26a}$$

and

$$\theta_0(t_2) = \left[\frac{\sqrt{3}}{6}\left\{R_{11}+R_{1-1}-3S_{00}\right\} a_0^2(t_3) + \frac{3\sqrt{3}}{2}\left\{S_{00}+\tfrac{1}{2}S_{20}\right\} b_0^2(t_3)\right] t_2 + \theta_0^{(3)}(t_3). \tag{26b}$$

Integrating equations (18) we obtain:

$$\begin{aligned} x_2 = {} & a_2(t_1)\cos\left(t_0-\varphi_2(t_1)\right) \\ & +\tfrac{1}{2}a_0^3(t_3)R_{03}S_{02}\cos\left\{3t_0-3\varphi_0(t_2)\right\} + a_0(t_3)b_1(t_2)R_{11}\cos\left\{(\sqrt{3}+1)t_0-\theta_1(t_2)-\varphi_0(t_2)\right\} \\ & +a_1(t_2)b_0(t_3)R_{11}\cos\left\{(\sqrt{3}+1)t_0-\theta_0(t_2)-\varphi_1(t_2)\right\} + a_0(t_3)b_1(t_2)R_{1-1}\cos\left\{(\sqrt{3}-1)t_0-\theta_1(t_2)+\varphi_0(t_2)\right\} \\ & +a_1(t_2)b_0(t_3)R_{1-1}\cos\left\{(\sqrt{3}-1)t_0-\theta_0(t_2)+\varphi_1(t_2)\right\} \end{aligned}$$

$$+ a_0(t_3)b_0^2(t_3)R_{21}\left\{R_{11} - \frac{3}{2}S_{20}\right\} \cos\left\{(2\sqrt{3}+1)t_0 - 2\theta_0(t_2) - \varphi_0(t_2)\right\}$$

$$+ a_0(t_3)b_0^2(t_3)R_{2-1}\left\{R_{1-1} - \frac{3}{2}S_{20}\right\} \cos\left\{(2\sqrt{3}-1)t_0 - 2\theta_0(t_2) + \varphi_0(t_2)\right\} \qquad (27a)$$

and

$$y_2 = b_2 \cos\left\{\sqrt{3}\, t_0 - \theta_2(t_1)\right\}$$

$$+ a_0(t_3)a_1(t_2)S_{00}\cos\left\{\varphi_1(t_1) - \varphi_0(t_2)\right\} - 3b_0(t_3)b_1(t_2)S_{00}\cos\left\{\theta_1(t_2) - \theta_0(t_2)\right\}$$

$$+ a_0(t_3)a_1(t_2)S_{02}\cos\left\{2t_0 - \varphi_1(t_2) - \varphi_0(t_2)\right\} - 3b_0(t_3)b_1(t_2)S_{20}\cos\left\{2\sqrt{3}t_0 - \theta_1(t_2) - \theta_0(t_2)\right\}$$

$$+ a_0^2(t_3)b_0(t_3)S_{12}\left\{R_{11} - \frac{3}{2}S_{02}\right\}\cos\left\{(\sqrt{3}+2)t_0 - \theta_0(t_2) - 2\varphi_0(t_2)\right\}$$

$$+ a_0^2(t_3)b_0(t_3)S_{1-2}\left\{R_{1-1} - \frac{3}{2}S_{02}\right\}\cos\left\{(\sqrt{3}-2)t_0 - \theta_0(t_2) + 2\varphi_0(t_2)\right\}$$

$$+ \frac{9}{2} b_0^3(t_3)S_{30}S_{20}\cos\left\{3\sqrt{3}\, t_0 - 3\theta_0(t_2)\right\} \qquad (27b)$$

Substituting equations (14), (17) and (27) in equations (12) we obtain the equations for x_3 and y_3. The algebra involved in writing these equations is rather tedious but completely straightforward. The secular terms are:

$$2\frac{\partial a_2(t_1)}{\partial t_1}\sin\left\{t_0 - \varphi_2(t_1)\right\} - 2a_2(t_1)\frac{\partial \varphi_2(t_1)}{\partial t_1}\cos\left\{t_0 - \varphi_2(t_1)\right\} + 2\frac{\partial a_1(t_2)}{\partial t_2}\sin\left\{t_0 - \varphi_1(t_2)\right\}$$

$$- 2a_1(t_2)\frac{\partial \varphi_1(t_2)}{\partial t_2}\cos\left\{t_0 - \varphi_1(t_2)\right\} + 2\frac{\partial a_0}{\partial t_3}\sin\left\{t_0 - \varphi_0(t_2)\right\} - 2a_0\frac{\partial \varphi_0(t_2)}{\partial t_3}\cos\left\{t_0 - \varphi_0(t_2)\right\}$$

$$+ a_1(t_2)\left[a_0^2(t_3)\left\{S_{02} + S_{00}\right\} + b_0^2(t_3)\left\{R_{11} + R_{1-1} - 3S_{00}\right\}\right]\cos\left\{t_0 - \varphi_1(t_2)\right\}$$

$$+ \left[2a_0^2(t_3)a_1(t_2)\cos\left\{\varphi_1(t_2) - \varphi_0(t_2)\right\} - 6a_0(t_3)b_0(t_3)b_1(t_2)\cos\left\{\theta_1(t_2) - \theta_0(t_2)\right\}\right]S_{00}\cos\left\{t_0 - \varphi_0(t_2)\right\}$$

$$+ \frac{1}{2} a_0^2(t_3)a_1(t_2)S_{02}\cos\left\{t_0 - 2\varphi_0(t_2) + \varphi_1(t_2)\right\}$$

$$+ a_0(t_3)b_0(t_3)b_1(t_2)\left[R_{11} + R_{1-1}\right]\cos\left\{t_0 - \varphi_0(t_2) + \theta_1(t_2) - \theta_0(t_2)\right\} = 0 \qquad (28a)$$

and

$$2\sqrt{3}\,\frac{\partial b_2(t_1)}{\partial t_1}\sin\{\sqrt{3}\,t_0-\theta_2(t_1)\} - 2\sqrt{3}\,b_2(t_1)\frac{\partial\theta_2(t_1)}{\partial t_1}\cos\{\sqrt{3}\,t_0-\theta_2(t_1)\}$$

$$+\,2\sqrt{3}\,\frac{\partial b_1(t_2)}{\partial t_2}\sin\{\sqrt{3}\,t_0-\theta_1(t_2)\} - 2\sqrt{3}b_1(t_2)\frac{\partial\theta_1(t_2)}{\partial t_2}\cos\{\sqrt{3}\,t_0-\theta_1(t_2)\}$$

$$+\,2\sqrt{3}\,\frac{\partial b_0(t_3)}{\partial t_3}\sin\{\sqrt{3}t_0-\theta_0(t_2)\} - 2\sqrt{3}\,b_0(t_3)\frac{\partial\theta_0(t_2)}{\partial t_3}\cos\{\sqrt{3}t_0-\theta_0(t_2)\}$$

$$+\,b_1(t_2)\Big[a_0^2(t_3)\{R_{11}+R_{1-1}-3S_{00}\}+9b_0^2(t_3)\{S_{00}+S_{20}\}\Big]\cos\{\sqrt{3}t_0-\theta_1(t_2)\}$$

$$+\Big[18b_0^2b_1\cos\{\theta_1(t_2)-\theta_0(t_2)\}-6a_1(t_2)a_0(t_3)b_0(t_3)\cos\{\varphi_1(t_2)-\varphi_0(t_2)\}\Big]S_{00}\cos\{\sqrt{3}t_0-\theta_0(t_2)\}$$

$$+a_1(t_2)a_0(t_3)b_0(t_3)\Big[R_{11}+R_{1-1}\Big]\cos\{\sqrt{3}\,t_0-\theta_0(t_2)+\varphi_1(t_2)-\varphi_0(t_2)\}$$

$$+a_1(t_2)a_0(t_3)b_0(t_3)\Big[R_{11}+R_{1-1}\Big]\cos\{\sqrt{3}t_0-\theta_0(t_2)+\varphi_0(t_2)-\varphi_1(t_2)\}$$

$$+\,\frac{9}{2}S_{20}\,b_0^2(t_3)b_1(t_2)\cos\{\sqrt{3}\,t_0-2\theta_0(t_2)+\theta_1(t_2)\} = 0 \tag{28b}$$

Setting the coefficients of $\cos\{t_0-\varphi_0(t_2)\}$ and $\sin\{t_0-\varphi_0(t_2)\}$ in (28a) and of $\cos\{\sqrt{3}\,t_0-\theta_0(t_2)\}$ and $\sin\{\sqrt{3}\,t_0-\theta_0(t_2)\}$ in (28b) separately equal to zero, we obtain:

$$2\,\frac{\partial a_2(t_1)}{\partial t_1}\sin\{\varphi_2(t_1)-\varphi_0(t_2)\} + 2a_2(t_1)\frac{\partial\varphi_2(t_1)}{\partial t_1}\cos\{\varphi_2(t_1)-\varphi_0(t_1)\}$$

$$= -\,2\,\frac{\partial a_1(t_2)}{\partial t_2}\sin\{\varphi_1(t_2)-\varphi_0(t_2)\}-2a_1(t_2)\frac{\partial\varphi_1(t_2)}{\partial t_2}\cos\{\varphi_1(t_2)-\varphi_0(t_2)\}-2a_0(t_3)\frac{\partial\varphi_0(t_2)}{\partial t_3}$$

$$+a_1(t_2)\Big[3a_0^2(t_3)\{S_{00}+\tfrac{1}{2}S_{02}\}+b_0^2(t_3)\{R_{11}+R_{1-1}-3S_{00}\}\Big]\cos\{\varphi_1(t_2)-\varphi_0(t_2)\}$$

$$+\,2a_0(t_3)b_0(t_3)b_1(t_2)\Big[R_{11}+R_{1-1}-3S_{00}\Big]\cos\{\theta_1(t_2)-\theta_0(t_2)\} \tag{29a}$$

$$-2\,\frac{\partial a_2(t_1)}{\partial t_1}\cos\{\varphi_2(t_1)-\varphi_0(t_2)\}+2a_2(t_1)\frac{\partial\varphi_2(t_1)}{\partial t_1}\sin\{\varphi_2(t_1)-\varphi_0(t_1)\}$$

$$=2\,\frac{\partial a_1(t_2)}{\partial t_2}\cos\{\varphi_1(t_2)-\varphi_0(t_2)\}-2a_1(t_2)\frac{\partial\varphi_1(t_2)}{\partial t_2}\sin\{\varphi_1(t_2)-\varphi_0(t_2)\}+2\,\frac{\partial a_0(t_3)}{\partial t_3}$$

$$+a_1(t_2)\Big[a_0^2(t_3)\,[\,S_{00}+\tfrac{1}{2}S_{02}\,]+b_0^2(t_3)\{R_{11}+R_{1-1}-3S_{00}\}\Big]\sin\{\varphi_1(t_2)-\varphi_0(t_2)\} \qquad (29b)$$

$$2\sqrt{3}\,\frac{\partial b_2(t_1)}{\partial t_1}\sin\{\theta_2(t_1)-\theta_0(t_2)\}+2\sqrt{3}\,b_2(t_1)\frac{\partial\theta_2(t_1)}{\partial t_1}\cos\{\theta_2(t_1)-\theta_0(t_2)\}$$

$$=-2\sqrt{3}\,\frac{\partial b_1(t_2)}{\partial t_2}\sin\{\theta_1(t_2)-\theta_0(t_2)\}-2\sqrt{3}\,b_1(t_2)\frac{\partial\theta_1(t_2)}{\partial t_2}\cos\{\theta_1(t_2)-\theta_0(t_2)\}$$

$$-2\sqrt{3}\,b_0(t_3)\frac{\partial\theta_0(t_2)}{\partial t_3}+b_1(t_2)\Big[a_0^2(t_3)\{R_{11}+R_{1-1}-3S_{00}\}+27b_0^2(t_3)\{S_{00}+\tfrac{1}{2}S_{20}\}\Big]\cos\{\theta_1(t_2)-\theta_0(t_2)\}$$

$$+2a_1(t_2)a_0(t_3)b_0(t_3)\Big[R_{11}+R_{1-1}-3S_{00}\Big]\cos\{\varphi_1(t_2)-\varphi_0(t_2)\} \qquad (29c)$$

$$-2\sqrt{3}\,\frac{\partial b_2(t_1)}{\partial t_1}\cos\{\theta_2(t_1)-\theta_0(t_2)\}+2\sqrt{3}\,b_2(t_1)\frac{\partial\theta_2(t_1)}{\partial t_1}\sin\{\theta_2(t_1)-\theta_0(t_2)\}$$

$$=+2\sqrt{3}\,\frac{\partial b_1(t_2)}{\partial t_2}\cos\{\theta_1(t_2)-\theta_0(t_2)\}-2\sqrt{3}\,b_1(t_2)\frac{\partial\theta_1(t_2)}{\partial t_2}\sin\{\theta_1(t_2)-\theta_0(t_2)\}$$

$$+2\sqrt{3}\,\frac{\partial b_0(t_3)}{\partial t_3}+b_1(t_2)\Big[a_0^2(t_3)\{R_{11}+R_{1-1}-3S_{00}\}+9b_0^2(t_3)\{S_{00}+\tfrac{1}{2}S_{20}\}\Big]\sin\{\theta_1(t_2)-\theta_0(t_2)\} \qquad (29d)$$

The right hand sides of equations (29) are all independent of t_1. Let

$$A_{2C}(t_1)=a_2(t_1)\cos\{\varphi_2(t_1)-\varphi_0(t_2)\}$$
$$A_{2S}(t_1)=a_2(t_1)\sin\{\varphi_2(t_1)-\varphi_0(t_2)\} \qquad (30a)$$

$$B_{2C}(t_1)=b_2(t_1)\cos\{\theta_2(t_1)-\theta_0(t_2)\}$$
$$B_{2S}(t_1)=b_2(t_1)\sin\{\theta_2(t_1)-\theta_0(t_2)\} \qquad (30b)$$

Then to keep A_{2C}, A_{2S}, B_{2C}, B_{2S} bounded on the t_1 time scale we must have for each of equations (29)

Right-hand side = Left-hand side = 0

$$\text{Then} \qquad \frac{\partial A_{2C}(t_1)}{\partial t_1} = \frac{\partial A_{2S}(t_1)}{\partial t_1} = \frac{\partial B_{2C}(t_1)}{\partial t_1} = \frac{\partial B_{2S}(t_1)}{\partial t_1} = 0$$

$$\text{So that } a_2 = a_2(t_2);\ b_2 = b_2(t_2);\ \varphi_2 = \varphi_2(t_2);\ \theta_2 = \theta_2(t_2) \qquad (31)$$

and equations (29) become:

$$2\frac{\partial A_{1S}(t_2)}{\partial t_2} + A_{1C}(t_2)\left[\left\{S_{00}+\tfrac{1}{2}S_{02}\right\}a_0^2(t_3)+\left\{R_{11}+R_{1-1}-3S_{00}\right\}b_0^2(t_3)\right]$$

$$= A_{1C}(t_2)\left[3\left\{S_{00}+\tfrac{1}{2}S_{02}\right\}a_0^2(t_3)+\left\{R_{11}+R_{1-1}-3S_{00}\right\}b_0^2(t_3)\right]$$

$$+ 2a_0(t_3)b_0(t_3)\left[R_{11}+R_{1-1}-3S_{00}\right]B_{1C}(t_2) - 2a_0(t_3)\frac{\partial\varphi_0(t_2)}{\partial t_3} \qquad (32a)$$

$$\frac{\partial A_{1C}(t_2)}{\partial t_2} = -\frac{\partial a_0(t_3)}{\partial t_3} \qquad (32b)$$

$$2\sqrt{3}\frac{\partial B_{1S}(t_2)}{\partial t_2} + B_{1C}(t_2)\left[\left\{R_{11}+R_{1-1}-3S_{00}\right\}a_0^2(t_3)+9\left\{S_{00}+\tfrac{1}{2}S_{20}\right\}b_0^2(t_3)\right]$$

$$= B_{1C}(t_2)\left[\left\{R_{11}+R_{1-1}-3S_{00}\right\}a_0^2(t_3)+27\left\{S_{00}+\tfrac{1}{2}S_{20}\right\}b_0^2(t_3)\right]$$

$$+ 2a_0(t_3)b_0(t_3)\left[R_{11}+R_{1-1}-3S_{00}\right]A_{1C}(t_2) - 2\sqrt{3}b_0(t_3)\frac{\partial\theta_0(t_2)}{\partial t_3} \qquad (32c)$$

$$\frac{\partial B_{1C}(t_2)}{\partial t_2} = -\frac{\partial b_0(t_3)}{\partial t_3} \qquad (32d)$$

To keep A_{1C} and B_{1C} bounded on the t_2 time scale we must have for equations (32b) and (32d), Left-hand side = Right-hand side = zero

Thus,

$$A_{1C} = A_{1C}(t_3); \quad B_{1C} = B_{1C}(t_3); \quad a_0 = a_0(t_4); \quad b_0 = b_0(t_4) \tag{33}$$

and equations (32a) and (32c) become:

$$\frac{\partial A_{1S}(t_2)}{\partial t_2} = \left\{S_{00} + \tfrac{1}{2} S_{02}\right\} a_0^2(t_4) A_{1C}(t_3) + \left\{R_{11} + R_{1-1} - 3S_{00}\right\} a_0(t_4) b_0(t_4) B_{1C}(t_3) - a_0(t_4) \frac{\partial \varphi_0^{(3)}(t_3)}{\partial t_3} \tag{34a}$$

$$\frac{\partial B_{1S}(t_2)}{\partial t_2} = \frac{\sqrt{3}}{3} a_0(t_4) b_0(t_4) \left[R_{11} + R_{1-1} - 3S_{00}\right] A_{1C}(t_3) + 3\sqrt{3}\, b_0^2(t_4) \left\{S_{00} + \tfrac{1}{2} S_{20}\right\} B_{1C}(t_3) - b_0(t_4) \frac{\partial \theta_0^{(3)}(t_3)}{\partial t_3} \tag{34b}$$

The right hand sides of (34a) and (34b) are independent of t_2, so to keep A_{1S} and B_{1S} bounded on the t_2 time scale, we must have:

$$\frac{\partial A_{1S}(t_2)}{\partial t_2} = \frac{\partial B_{1S}(t_2)}{\partial t_2} = 0 \tag{35}$$

so that

$$A_{1S} = A_{1S}(t_3), \qquad B_{1S} = B_{1S}(t_3)$$

and

$$\frac{\partial \varphi_0^{(3)}(t_3)}{\partial t_3} = \left\{S_{00} + \tfrac{1}{2} S_{02}\right\} a_0(t_4) A_{1C}(t_3) + \left\{R_{11} + R_{1-1} - 3S_{00}\right\} b_0(t_4) B_{1C}(t_3) \tag{36a}$$

$$\frac{\partial \theta_0^{(3)}(t_3)}{\partial t_3} = \frac{\sqrt{3}}{3} \left\{R_{11} + R_{1-1} - 3S_{00}\right\} a_0(t_4) A_{1C}(t_3) + 3\sqrt{3}\left\{S_{00} + \tfrac{1}{2} S_{20}\right\} b_0(t_4) B_{1C}(t_3) \tag{36b}$$

It is possible, although tedious, to show $\frac{\partial A_{1c}(t_3)}{\partial t_3} = \frac{\partial B_{1c}(t_3)}{\partial t_3} = 0$ by considering the $O(\epsilon^4)$ equations. Assuming $A_{1c}=A_{1c}(t_4)$, $B_{1c}=B_{1c}(t_4)$ we have

$$\varphi_0^{(3)}(t_3) = \left[\left\{S_{00}+\tfrac{1}{2}S_{02}\right\}a_0(t_4)A_{1c}(t_4)+\left\{R_{11}+R_{1-1}-3S_{00}\right\}b_0(t_4)B_{1c}(t_4)\right]t_3+\varphi_0^{(4)}(t_4) \qquad (37a)$$

$$\theta_0^{(3)}(t_3) = \left[\frac{\sqrt{3}}{3}\left\{R_{11}+R_{1-1}-3S_{00}\right\}a_0(t_4)A_{1c}(t_4)+3\sqrt{3}\left\{S_{00}+\tfrac{1}{2}S_{20}\right\}b_0(t_4)B_{1c}(t_4)\right]t_3+\theta_0^{(4)}(t_4) \qquad (37b)$$

Collecting all the results, letting $t_4 = 0$, assuming the frequency shifts in φ_1 and φ_2 are the same as those in φ_0,* and the a_k's and b_k's are constants, we have to $O(\epsilon^2)$:

$$\begin{aligned}
x(t) = {} & a_0(0)\cos\{t_0-\varphi_0(t_2)\} \\
& +\epsilon\Big[a_1(0)\cos\{t_0-\varphi_0(t_2)+\varphi_0(0)-\varphi_1(0)\} \\
& \quad + a_0(0)b_0(0)R_{11}\cos\{(\sqrt{3}+1)t_0-\theta_0(t_2)-\varphi_0(t_2)\} \\
& \quad + a_0(0)b_0(0)R_{1-1}\cos\{(\sqrt{3}-1)t_0-\theta_0(t_2)+\varphi_0(t_2)\}\Big] \\
& +\epsilon^2\Big[a_2(0)\cos\{t_0-\varphi_0(t_2)+\varphi_0(0)-\varphi_2(0)\}+\tfrac{1}{2}a_0^3(0)R_{03}S_{02}\cos\{3t_0-3\varphi_0(t_2)\} \\
& \quad + a_0(0)b_1(0)R_{11}\cos\{(\sqrt{3}+1)t_0-\theta_0(t_2)-\varphi_0(t_2)+\theta_0(0)-\theta_1(0)\} \\
& \quad + a_1(0)b_0(0)R_{11}\cos\{(\sqrt{3}+1)t_0-\theta_0(t_2)-\varphi_0(t_2)+\varphi_0(0)-\varphi_1(0)\} \\
& \quad + a_0(0)b_1(0)R_{1-1}\cos\{(\sqrt{3}-1)t_0-\theta_0(t_2)+\varphi_0(t_2)+\theta_0(0)-\theta_1(0)\} \\
& \quad + a_1(0)b_0(0)R_{1-1}\cos\{(\sqrt{3}-1)t_0-\theta_0(t_2)+\varphi_0(t_2)-\varphi_0(0)+\varphi_1(0)\} \\
& \quad + a_0(0)b_0^2(0)R_{21}\left\{R_{11}-\tfrac{3}{2}S_{20}\right\}\cos\{(2\sqrt{3}+1)t_0-2\theta_0(t_2)-\varphi_0(t_2)\} \\
& \quad + a_0(0)b_0^2(0)R_{2-1}\left\{R_{1-1}-\tfrac{3}{2}S_{20}\right\}\cos\{(2\sqrt{3}-1)t_0-2\theta_0(t_2)+\varphi_0(t_2)\}\Big]
\end{aligned} \qquad (38a)$$

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*One can demonstrate this by computing further terms in the approximation.

and:

$$y(t)=b_0(0)\cos\{\sqrt{3}t_0-\theta_0(t_2)\}$$

$$+\epsilon\Big[b_1(0)\cos\{\sqrt{3}t_0-\theta_0(t_2)+\theta_0(0)-\theta_1(0)\}+\tfrac{1}{2}\{a_0^2(0)-3b_0^2(0)\}S_{00}$$

$$+\tfrac{1}{2}a_0^2(0)S_{02}\cos\{2t_0-2\varphi_0(t_2)\}-\tfrac{3}{2}b_0^2(0)S_{20}\cos\{2\sqrt{3}t_0-2\theta_0(t_2)\}\Big]$$

$$+\epsilon^2\Big[b_2(0)\cos\{\sqrt{3}t_0-\theta_0(t_2)+\theta_0(0)-\theta_2(0)\}$$

$$+a_0(0)a_1(0)S_{00}\cos\{\varphi_1(0)-\varphi_0(0)\}-3b_0(0)b_1(0)S_{00}\cos\{\theta_1(0)-\theta_0(0)\}$$

$$+a_0(0)a_1(0)S_{02}\cos\{2t_0-2\varphi_0(t_2)+\varphi_0(0)-\varphi_1(0)\}-3b_0(0)b_1(0)S_{20}\cos\{2\sqrt{3}t_0-2\theta_0(t_2)$$

$$+\theta_0(0)-\theta_1(0)\}$$

$$+a_0^2(0)b_0(0)S_{12}\{R_{11}-\tfrac{3}{2}S_{02}\}\cos\{(\sqrt{3}+2)t_0-\theta_0(t_2)-2\varphi_0(t_2)\}$$

$$+a_0^2(0)b_0(0)S_{1-2}\{R_{1-1}-\tfrac{3}{2}S_{02}\}\cos\{(\sqrt{3}-2)t_0-\theta_0(t_2)+2\varphi_0(t_2)\}$$

$$+\tfrac{9}{2}b_0^3(0)S_{30}S_{20}\cos\{3\sqrt{3}t_0-3\theta_0(t_2)\}\Big] \qquad (38b)$$

with

$$\varphi_0(t_2)=\tfrac{1}{2}\Big[\{S_{00}+\tfrac{1}{2}S_{02}\}a_0^2(0)+\{R_{11}+R_{1-1}-3S_{00}\}b_0^2(0)\Big]\epsilon^2 t$$

$$+\Big[\{S_{00}+\tfrac{1}{2}S_{02}\}a_0(0)a_1(0)\cos\{\varphi_1(0)-\varphi_0(0)\}+\{R_{11}+R_{1-1}-3S_{00}\}b_0(0)b_1(0)\cos\{\theta_1(0)$$

$$-\theta_0(0)\}\Big]\epsilon^3 t+\varphi_0(0) \qquad (39a)$$

$$\theta_0(t_2)=\frac{\sqrt{3}}{6}\Big[\{R_{11}+R_{1-1}-3S_{00}\}a_0^2(0)+9\{S_{00}+\tfrac{1}{2}S_{20}\}b_0^2(0)\Big]\epsilon^2 t$$

$$+\frac{\sqrt{3}}{3}\Big[\{R_{11}+R_{1-1}-3S_{00}\}a_0(0)a_1(0)\cos\{\varphi_1(0)-\varphi_0(0)\}+9\{S_{00}+$$

$$+\tfrac{1}{2}S_{20}\}b_0(0)b_1(0)\cos\{\theta_1(0)-\theta_0(0)\}\Big]\epsilon^3 t+\theta_0(0) \qquad (39b)$$

These expressions should be uniformly valid to $O(\epsilon^2)$ for times $t=O(\frac{1}{\epsilon})$, to $O(\epsilon)$ for $t=O(\frac{1}{\epsilon^2})$ and to $O(1)$ for $t=O(\frac{1}{\epsilon^3})$.

We can differentiate these expressions with respect to t and obtain:

$$\frac{dx(t)}{dt} = -a_0(0)\sin\{t_0-\varphi_0(t_2)\}$$

$$+\epsilon\Big[-a_1(0)\sin\{t_0-\varphi_0(t_2)+\varphi_0(0)-\varphi_1(0)\}$$

$$-(\sqrt{3}+1)a_0(0)b_0(0)R_{11}\sin\{(\sqrt{3}+1)t_0-\theta_0(t_2)-\varphi_0(t_2)\}$$

$$-(\sqrt{3}-1)a_0(0)b_0(0)R_{1-1}\sin\{(\sqrt{3}-1)t_0-\theta_0(t_2)+\varphi_0(t_2)\}\Big]$$

$$+\epsilon^2\Big[\tfrac{1}{2}a_0(0)\{(S_{00}+\tfrac{1}{2}S_{02})a_0^2(0)+(R_{11}+R_{1-1}-3S_{00})b_0^2(0)\}\sin\{t_0-\varphi_0(t_2)\}$$

$$-a_2(0)\sin\{t_0-\varphi_0(t_2)+\varphi_0(0)-\varphi_2(0)\}-\frac{3}{2}a_0^3(0)R_{03}S_{02}\sin\{3t_0-3\varphi_0(t_2)\}$$

$$-(\sqrt{3}+1)a_0(0)b_1(0)R_{11}\sin\{(\sqrt{3}+1)t_0-\theta_0(t_2)-\varphi_0(t_2)+\theta_0(0)-\theta_1(0)\}$$

$$-(\sqrt{3}+1)a_1(0)b_0(0)R_{11}\sin\{(\sqrt{3}+1)t_0-\theta_0(t_2)-\varphi_0(t_2)+\varphi_0(0)-\varphi_1(0)\}$$

$$-(\sqrt{3}-1)a_0(0)b_1(0)R_{1-1}\sin\{(\sqrt{3}-1)t_0-\theta_0(t_2)+\varphi_0(t_2)+\theta_0(0)-\theta_1(0)\}$$

$$-(\sqrt{3}-1)a_1(0)b_0(0)R_{1-1}\sin\{(\sqrt{3}-1)t_0-\theta_0(t_2)+\varphi_0(t_2)-\varphi_0(0)+\varphi_1(0)\}$$

$$-(2\sqrt{3}+1)a_0(0)b_0^2(0)R_{21}\{R_{11}-\frac{3}{2}S_{20}\}\sin\{(2\sqrt{3}+1)t_0-2\theta_0(t_2)-\varphi_0(t_2)\}$$

$$-(2\sqrt{3}-1)a_0(0)b_0^2(0)R_{2-1}\{R_{1-1}-\frac{3}{2}S_{20}\}\sin\{(2\sqrt{3}-1)t_0-2\theta_0(t_2)+\varphi_0(t_2)\}\Big] \quad (40a)$$

and

$$
\begin{aligned}
\frac{dy(t)}{dt} = & -\sqrt{3}\, b_0(0) \sin\{\sqrt{3}\, t_0 - \theta_0(t_2)\} \\
& + \epsilon \Big[-\sqrt{3}\, b_1(0) \sin\{\sqrt{3} t_0 - \theta_0(t_2) + \theta_0(0) - \theta_1(0)\} \\
& \quad - a_0^2(0) S_{02} \sin\{2t_0 - 2\varphi_0(t_2)\} + 3\sqrt{3}\, b_0^2(0) S_{20} \sin\{2\sqrt{3}\, t_0 - 2\theta_0(t_2)\} \Big] \\
& + \epsilon^2 \Big[\frac{\sqrt{3}}{6} b_0(0) \{(R_{11} + R_{1-1} - 3S_{00}) a_0^2(0) + 9(S_{00} + \tfrac{1}{2} S_{20}) b_0^2(0)\} \sin\{\sqrt{3} t_0 - \theta_0(t_2)\} \\
& \quad - \sqrt{3}\, b_2(0) \sin\{\sqrt{3}\, t_0 - \theta_0(t_2) + \theta_0(0) - \theta_2(0)\} \\
& \quad - 2a_0(0) a_1(0) S_{02} \sin\{2t_0 - 2\varphi_0(t_2) + \varphi_0(0) - \varphi_2(0)\} \\
& \quad + 6\sqrt{3}\, b_0(0) b_1(0) S_{20} \sin\{2\sqrt{3}\, t_0 - 2\theta_0(t_2) + \theta_0(0) - \theta_1(0)\} \\
& \quad - (\sqrt{3}+2) a_0^2(0) b_0(0) S_{12} \{R_{11} - \tfrac{3}{2} S_{02}\} \sin\{(\sqrt{3}+2) t_0 - \theta_0(t_2) - 2\varphi_0(t_2)\} \\
& \quad - (\sqrt{3}-2) a_0^2(0) b_0(0) S_{1-2} \{R_{1-1} - \tfrac{3}{2} S_{02}\} \sin\{(\sqrt{3}-2) t_0 - \theta_0(t_2) + 2\varphi_0(t_2)\} \\
& \quad - \frac{27\sqrt{3}}{2} b_0^3(0) S_{30} S_{20} \sin\{3\sqrt{3}\, t_0 - 3\theta_0(t_2)\} \Big]
\end{aligned}
\qquad (40b)
$$

B. Comparison with Computer Experiments.

Equations (5) were solved numerically on the CIT IBM 7094 computer in order to compare the numerical results with the predictions of the theory. Several sets of parameters (i.e., ϵ and the initial conditions) were studied and for relatively small t $(t \sim \frac{1}{\epsilon^2})$ were found to yield results agreeing with the theory. One case was then selected as an example to study the longer-time behavior of the system. The particular set of parameters chosen was:

$$x(0) = 0.00\ldots \qquad \left(\frac{dx(t)}{dt}\right)_{t=0} = 1.00\ldots$$

$$y(0) = 0.00\ldots \qquad \left(\frac{dy(t)}{dt}\right)_{t=0} = 1.00\ldots$$

$$\epsilon = 0.10\ldots$$

System (5) with these parameters was integrated numerically for $0 \leq t \leq 2946.40$, corresponding to about 470 cycles of the $\omega = 1$ oscillator and 800 cycles of the $\omega = \sqrt{3}$ oscillator. The step size taken was $\Delta t = 0.10$ and the maximum error per step was 10^{-12}, so that the total error accumulated was negligible and the numerical results can be considered an exact solution. To illustrate the accuracy of the theory some numerical and theoretical results for this case are presented in Tables 1 and 2.

Table 1 compares theoretical predictions of the values of the dynamical variables $x(t)$, $\frac{dx(t)}{dt}$, $y(t)$, $\frac{dy(t)}{dt}$ at various representative times t with "exact" values of the same quantities. The theoretical results used for the comparison, equations (38) and (40), include only terms through $O(\epsilon^3)$ in time variation and $O(\epsilon^2)$ in magnitude. Thus the theoretical solution can be expected to be accurate through $O(\epsilon^2)$ for $t \leqslant O(1/\epsilon)$. However, when t becomes $O(1/\epsilon^2)$, the neglected frequency shift of $O(\epsilon^4)$ in the $O(1)$ term becomes $O(\epsilon^2)$, thus causing phase shifts and errors of $O(\epsilon^2)$ in the various quantities. Similarly, for $t = O(1/\epsilon^4)$, error $= O(\epsilon)$ and when $t =)(1/\epsilon^4)$, the error becomes $O(1)$ and we have completely lost track of the motion. Table one shows that the agreement between theoretical and exact values is well within these allowable errors.

Table 2 compares theoretical predictions of the time average energy "in each mode" and the energy of interaction with exact values of the same quantities. The quantities considered are:

$$E_1 = \text{energy "of symmetric mode"} = \tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}x^2 .$$

$$E_2 = \text{energy "of antisymmetric mode"} = \tfrac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^2 .$$

$$E_I = \text{energy "of interaction"} = \epsilon\, y\,(y^2 - x^2) .$$

The theoretical values of the barred quantities in Table 2 are obtained by computing the quantities under the bars to $O(\epsilon^2)$ from equations (38) and (39), then averaging these quantities for a time $= O(\frac{1}{\epsilon^3})$. If we assume that the next few time variations of each term are just additional frequency shifts, E_1, E_2 and E_I then consist of sums of constant terms and sinusoidal terms. The time averages of the sinusoidal terms are $\leqslant O(1/\epsilon^3)$, so we are just left with the constant terms. We have:

$$\overline{E}_1 = \tfrac{1}{2} a_0^2(0) + \epsilon\, a_0(0) a_1(0) \cos\{\varphi_1(0) - \varphi_0(0)\}$$

$$+\epsilon^2 \Big[a_2(0) a_0(0) \cos\{\varphi_2(0) - \varphi_0(0)\} + \tfrac{1}{2} a_1^2(0) + \tfrac{1}{2} a_0^2(0) b_0^2(0) (\tfrac{5}{2} + \sqrt{3}) R_{11}^2$$

$$+ \tfrac{1}{2} a_0^2(0)\, b_0^2(0)\, (\tfrac{5}{2} - \sqrt{3}) R_{1-1}^2 - \tfrac{1}{4} (S_{00} + \tfrac{1}{2} S_{02}) a_0^4(0)$$

$$- \tfrac{1}{4} (R_{11} + R_{1-1} - 3S_{00}) a_0^2(0) b_0^2(0) \Big] + O(\epsilon^3)$$

$$\overline{E}_2 = \tfrac{3}{2} b_0^2(0) + 3\epsilon\, b_0(0) b_1(0) \cos\{\theta_1(0) - \theta_0(0)\}$$

$$+\epsilon^2 \Big[3b_2(0) b_0(0) \cos\{\theta_2(0) - \theta_0(0)\} + \tfrac{3}{2} b_1^2(0) + \tfrac{3}{8}\{a_0^2(0) - 3b_0^2(0)\}^2 S_{00}^2$$

$$+ \tfrac{7}{16} a_0^4(0) S_{02}^2 + \tfrac{135}{16} b_0^4(0) S_{20}^2 - \tfrac{1}{4} a_0^2(0) b_0^2(0) (R_{11} + R_{1-1} - 3S_{00})$$

$$- \tfrac{9}{4} b_0^4(0) (S_{00} + \tfrac{1}{2} S_{20}) \Big] + O(\epsilon^3)$$

$$\overline{E}_3 = \epsilon^2 \Big[-\tfrac{1}{4} S_{00} \{a_0^2(0) - 3b_0^2(0)\}^2 - \tfrac{1}{8} S_{02}\, a_0^4(0) - \tfrac{1}{2}(R_{11} + R_{1-1})\, a_0^2(0) b_0^2(0) - \tfrac{9}{8} S_{20}\, b_0^4(0) \Big] + O(\epsilon^3)$$

The "exact" values of the time-averaged quantities are not really terribly exact, since the time averages have not yet settled down to precisely constant values in the duration of the machine computation. However, they are constant to $O(1/\epsilon^3)$ by the time the numerical integration is completed, and that is sufficient for our purposes.

Table 1. Comparison of Theory and Exact Values for Dynamical Variables at Selected Times t

t	1.00	10.00	100.00	1000.00	2945.00
x_{theory}	.8541	-.3675	-.8001	.0578	1.080
x_{exact}	.8538	-.3685	-.7973	.0656	1.0718
$x_{theory} - x_{exact}$	.0003	.0010	-.0028	-.0078	.008
$\frac{dx}{dt}$ theory	.582	-1.026	.7703	1.124	-.3141
$\frac{dx}{dt}$ exact	.582	-1.022	.772	1.120	-.335
$\frac{dx}{dt}$ theory $- \frac{dx}{dt}$ exact	.000	-.004	-.002	.004	.021
y_{theory}	.5615	-.5210	.1131	-.4631	.4468
y_{exact}	.5616	-.5194	.1131	-.4618	.4463
$y_{theory} - y_{exact}$	.0001	-.0016	.0000	-.0013	.0005
$\frac{dy}{dt}$ theory	-.1805	-.1631	-.8625	.3433	-.4700
$\frac{dy}{dt}$ exact	-.181	-.159	-.863	.347	-.476
$\frac{dy}{dt}$ theory $- \frac{dy}{dt}$ exact	.000	-.004	.000	-.004	.006
Allowable Error	0(.001)	0(.001)	0(.01)	0(.1)	0(.1)

Table 2. Comparison of Theory and Exact Values for Time Averages of Mode Energies and Interaction Energy*

	$\overline{E}_1$	$\overline{E}_2$	$\overline{E}_I$
Theoretical	.5210	.4809	-.0019
Exact	.5206	.4813	-.0019
Error	.0004	-.0004	.0000
Allowable Error	O(.001)	O(.001)	O(.0001)

*"Exact" values are based on mechanical calculations of averages for about 470 cycles of slower oscillator.

Chapter IV

A "Weakly Resonant" Example

A simple model example of a "weakly" resonant case is the system with Hamiltonian

$$H\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}\right) = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}x^2 + \frac{9}{2}y^2 + \epsilon\frac{x^2y^2}{2} \tag{1}$$

The equations of motion for this system are:

$$\frac{d^2x}{dt^2} + x = -\epsilon xy^2 \qquad x(0) = a, \quad \left.\frac{dx(t)}{dt}\right|_{t=0} = b \tag{2a}$$

$$\frac{d^2y}{dt^2} + 9y = -\epsilon x^2 y \qquad y(0) = c \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = d \tag{2b}$$

Let

$$x(t) = x_0(t_0, t_1, t_2, \ldots) + \epsilon x_1(t_0, t_1, t_2, \ldots) + \epsilon^2 x_2(t_0, t_1, t_2, \ldots) + O(\epsilon^3)$$

$$y(t) = y_0(t_0, t_1, t_2, \ldots) + \epsilon y_1(t_0, t_1, t_2, \ldots) + \epsilon^2 y_2(t_0, t_1, t_2, \ldots) + O(\epsilon^3)$$

Substituting these expressions in equations (2) and setting the coefficients of ϵ^k separately equal to zero we obtain:

$$O(1) \quad \frac{\partial^2 x_0}{\partial t_0^2} + x_0 = 0 \tag{3a}$$

$$\frac{\partial^2 y_0}{\partial t_0^2} + 9y_0 = 0 \tag{3b}$$

$$O(\epsilon) \quad \frac{\partial^2 x_1}{\partial t_0^2} + x_1 = -2\frac{\partial^2 x_0}{\partial t_0 \partial t_1} - x_0 y_0^2 \tag{4a}$$

$$\frac{\partial^2 y_1}{\partial t_0^2} + 9y_1 = -2\frac{\partial^2 y_0}{\partial t_0 \partial t_1} - x_0^2 y_0 \tag{4b}$$

$$O(\epsilon^2) \quad \frac{\partial^2 x_2}{\partial t_0^2} + x_2 = -2\frac{\partial^2 x_0}{\partial t_0 \partial t_2} - \frac{\partial^2 x_0}{\partial t_1^2} - 2\frac{\partial^2 x_1}{\partial t_0 \partial t_1} - x_1 y_0^2 - 2x_0 y_0 y_1 \tag{5a}$$

$$\frac{\partial^2 y_2}{\partial t_0^2} + 9y_2 = -2\frac{\partial^2 y_0}{\partial t_0 \partial t_2} - \frac{\partial^2 y_0}{\partial t_1^2} - 2\frac{\partial^2 y_1}{\partial t_0 \partial t_1} - 2x_0 y_0 x_1 - x_0^2 y_1 \tag{5b}$$

$$O(\epsilon^3) \quad \frac{\partial^2 x_3}{\partial t_0{}^2} + x_3 = -2\frac{\partial^2 x_0}{\partial t_0 \partial t_3} - 2\frac{\partial^2 x_0}{\partial t_1 \partial t_2} - 2\frac{\partial^2 x_1}{\partial t_0 \partial t_2} - \frac{\partial^2 x_1}{\partial t_1{}^2} - 2\frac{\partial^2 x_2}{\partial t_0 \partial t_1} - x_2 y_0^2 - 2x_1 y_0 y_1$$

$$-2x_0 y_0 y_2 - x_0 y_1^2 \tag{6a}$$

$$\frac{\partial^2 y_3}{\partial t_0^2} + 9y_3 = -2\frac{\partial^2 y_0}{\partial t_0 \partial t_3} - 2\frac{\partial^2 y_0}{\partial t_1 \partial t_2} - 2\frac{\partial^2 y_1}{\partial t_0 \partial t_2} - \frac{\partial^2 y_1}{\partial t_1{}^2} - 2\frac{\partial^2 y_2}{\partial t_0 \partial t_1} - 2x_0 x_2 y_0$$

$$-x_1^2 y_0 - 2x_0 x_1 y_1 - x_0^2 y_2 \tag{6b}$$

The solution of equations (3) is

$$x_0 = a_0(t_1)\cos\{t_0 - \varphi_0(t_1)\} \tag{7a}$$

$$y_0 = b_0(t_1)\cos\{3t_0 - \theta_0(t_1)\} \tag{7b}$$

Substituting equations (7) into equations (4) we obtain:

$$\frac{\partial^2 x_1}{\partial t_0^2} + x_1 = \underline{2\frac{\partial a_0}{\partial t_1}\sin\{t_0 - \varphi_0(t_1)\}} \underline{-2a_0\frac{\partial \varphi_0(t_1)}{\partial t_1}\cos\{t_0 - \varphi_0(t_1)\}}$$

$$\underline{-\frac{a_0 b_0^2}{2}\cos\{t_0 - \varphi_0(t_1)\}} - \frac{a_0 b_0^2}{4}\cos\{5t_0 - 2\theta_0(t_1) + \varphi_0(t_1)\}$$

$$-\frac{a_0 b_0^2}{4}\cos\{7t_0 - 2\theta_0(t_1) - \varphi_0(t_1)\} \tag{8a}$$

and

$$\frac{\partial^2 y_1}{\partial t_0{}^2} + 9y_1 = \underline{6\frac{\partial b_0}{\partial t_1}\sin\{3t_0 - \theta_0(t_1)\}} \underline{-6b_0\frac{\partial \theta_0}{\partial t_1}\cos\{3t_0 - \theta_0(t_1)\}}$$

$$\underline{-\frac{a_0^2 b_0}{2}\cos\{3t_0 - \theta_0(t_1)\}} - \frac{a_0^2 b_0}{4}\cos\{t_0 - \theta_0(t_1) + 2\varphi_0(t_1)\}$$

$$-\frac{a_0^2 b_0}{4}\cos\{5t_0 - \theta_0(t_1) - 2\varphi_0(t_1)\} \tag{8b}$$

For a uniformly bounded solution, the underlined terms in each equation must vanish. This requires

$$\frac{\partial a_0}{\partial t_1} = 0 \qquad \frac{\partial \varphi_0(t_1)}{\partial t_1} = -\frac{b_0^2}{4}$$

$$\frac{\partial b_0}{\partial t_1} = 0 \qquad \frac{\partial \theta_0(t_1)}{\partial t_1} = -\frac{a_0^2}{12}$$

so that:

$$a_0(t_1, t_2, \dots) = a_0(t_2, \dots) \qquad \varphi_0(t_1) = -\frac{b_0^2 t_1}{4} + \varphi_0^{(2)}(t_2) \tag{9a}$$

$$b_0(t_1, t_2, \dots) = b_0(t_2, \dots) \qquad \theta_0(t_1) = -\frac{a_0^2 t_1}{12} + \theta_0^{(2)}(t_2) \tag{9b}$$

Actually, if a_0 and b_0 turned out to depend upon t_2 or some higher order time scale, we would have some difficulties with the present procedure. If, for example, a_0 depended on t_k, we would have $\frac{\partial \theta_0}{\partial t_k} = -\frac{a_0}{6}\frac{\partial a_0}{\partial t_k} t_1$. Since our procedure of writing a hierarchy of equations ordered in ϵ and solving the equations sequentially depends upon our ability to separate the variables in the equations of various order, and terms such as $t_1 \frac{\partial a_k}{\partial t_k}$ would make the equations inseparable, we would be unable to uniformize the expansion. However, we shall find that a_0 and b_0 are independent of time for time scales up to t_3 which is as far as we shall carry the present calculation. Solving equations (8) with the underlined terms eliminated, we obtain:

$$x_1(t_0, \dots) = a_1(t_1)\cos\{t_0 - \varphi_1(t_1)\} + \frac{a_0 b_0^2}{96}\cos\{5t_0 - 2\theta_0(t_1) + \varphi_0(t_1)\}$$
$$+ \frac{a_0 b_0^2}{192}\cos\{7t_0 - 2\theta_0(t_1) - \varphi_0(t_1)\} \tag{10a}$$

$$y_1(t_0, \ldots) = b_1(t_1)\cos\{3t_0-\theta_1(t_1)\} - \frac{a_0^2 b_0}{32}\cos\{t_0-\theta_0(t_1)+2\varphi_0(t_1)\}$$

$$+ \frac{a_0^2 b_0}{64}\cos\{5t_0-\theta_0(t_1)-2\varphi_0(t_1)\} \tag{10b}$$

Substituting equations (7), (9) and (10) in equations (5), we obtain

$$\frac{\partial^2 x_2}{\partial t_0^2} + x_2 = \underline{2\frac{\partial a_0}{\partial t_2}\sin\{t_0-\varphi_0(t_1)\}} \underline{-2a_0\frac{\partial \varphi_0}{\partial t_2}\cos\{t_0-\varphi_0(t_1)\}} + \underline{\frac{a_0 b_0^4}{16}\cos\{t_0-\varphi_0(t_1)\}}$$

$$\underline{+2\frac{\partial a_1}{\partial t_1}\sin\{t_0-\varphi_1(t_1)\}} \underline{-2a_1\frac{\partial \varphi_1}{\partial t_1}\cos\{t_0-\varphi_1(t_1)\}}$$

$$+\frac{5a_0 b_0^2}{48}\left\{\frac{a_0^2}{6}-\frac{b_0^2}{4}\right\}\cos\{5t_0-2\theta_0(t_1)+\varphi_0(t_1)\} + \frac{7a_0 b_0^2}{96}\left\{\frac{a_0^2}{6}+\frac{b_0^2}{4}\right\}\cos\{7t_0$$

$$-2\theta_0(t_1) - \varphi_0(t_1)\}$$

$$\underline{-\frac{b_0^2}{2}a_1(t_1)\cos\{t_0-\varphi_1(t_1)\}} - \frac{a_0 b_0^4}{192}\cos\{5t_0-2\theta_0(t_1)+\varphi_0(t_1)\}$$

$$-\frac{a_0 b_0^4}{384}\cos\{7t_0-2\theta_0(t_1)-\varphi_0(t_1)\} - \frac{a_1 b_0^2}{4}\cos\{5t_0-2\theta_0(t_1)+\varphi_1(t_1)\}$$

$$-\frac{a_1 b_0^2}{4}\cos\{7t_0-2\theta_0(t_1)-\varphi_1(t_1)\} \underline{-\frac{a_0 b_0^4}{384}\cos\{t_0-\varphi_0(t_1)\}} \underline{-\frac{a_0 b_0^4}{768}\cos\{t_0-\varphi_0(t_1)\}}$$

$$-\frac{a_0 b_0^4}{768}\cos\{13t_0-4\theta_0(t_1)-\varphi_0(t_1)\} - \frac{a_0 b_0^4}{384}\cos\{11t_0-4\theta_0(t_1)+\varphi_0(t_1)\} -$$

$$\underline{-\frac{a_0 b_0 b_1}{2}\cos\{t_0+\theta_1(t_1)-\theta_0(t_1)-\varphi_0(t_1)\}}$$

$$-\frac{a_0 b_0 b_1}{2}\cos\{7t_0-\theta_1(t_1)-\theta_0(t_1)-\varphi_0(t_1)\} \underline{-\frac{a_0 b_0 b_1}{2}\cos\{t_0-\theta_1(t_1)+\theta_0(t_1)-\varphi_0(t_1)\}}$$

$$-\frac{a_0 b_0 b_1}{2}\cos\{5t_0-\theta_1(t_1)-\theta_0(t_1)+\varphi_0(t_1)\} + \frac{a_0^3 b_0^2}{64}\cos\{3t_0-3\varphi_0(t_1)\}$$

$$+\frac{a_0^3 b_0^2}{64}\cos\{5t_0-2\theta_0(t_1)+\varphi_0(t_1)\}+\frac{a_0^3 b_0^2}{64}\cos\{3t_0-2\theta_0(t_1)+3\varphi_0(t_1)\}$$

$$+\underline{\frac{a_0^3 b_0^2}{64}\cos\{t_0-\varphi_0(t_1)\}}-\underline{\frac{a_0^3 b_0^2}{128}\cos\{t_0-\varphi_0(t_1)\}}-\frac{a_0^3 b_0^2}{128}\cos\{9t_0-2\theta_0(t_1)-3\varphi_0(t_1)\}$$

$$-\frac{a_0^3 b_0^2}{128}\cos\{3t_0-3\varphi_0(t_1)\}-\frac{a_0^3 b_0^2}{128}\cos\{7t_0-2\theta_0(t_1)-\varphi_0(t_1)\} \qquad (11a)$$

and

$$\frac{\partial^2 y_0}{\partial t_0^2}+9y_2=\underline{6\frac{\partial b_0}{\partial t_2}\sin\{3t_0-\theta_0(t_2)\}}\ \underline{-6b_0\frac{\partial\theta_0}{\partial t_2}\cos\{3t_0-\theta_0(t_1)\}}+\underline{\frac{a_0^4 b_0}{144}\cos\{3t_0-\theta_0(t_1)\}}$$

$$+\underline{6\frac{\partial b_1}{\partial t_1}\sin\{3t_0-\theta_1(t_1)\}}\ \underline{-6b_1\frac{\partial\theta_1}{\partial t_1}\cos\{3t_0-\theta_1(t_1)\}}$$

$$-\frac{a_0^2 b_0}{16}\left\{\frac{a_0^2}{12}-\frac{b_0^2}{2}\right\}\cos\{t_0-\theta_0(t_1)+2\varphi_0(t_1)\}+\frac{5a_0^2 b_0}{32}\left\{\frac{a_0^2}{12}+\frac{b_0^2}{2}\right\}\cos\{5t_0-\theta_0(t_1)$$

$$-2\varphi_0(t_1)\}$$

$$-\frac{a_0 b_0 a_1}{2}\cos\{5t_0-\theta_0(t_1)-\varphi_0(t_1)-\varphi_1(t_1)\}-\underline{\frac{a_0 b_0 a_1}{2}\cos\{3t_0-\theta_0(t_1)-\varphi_0(t_1)+\varphi_1(t_1)\}}$$

$$-\underline{\frac{a_0 b_0 a_1}{2}\cos\{3t_0-\theta_0(t_1)+\varphi_0(t_1)-\varphi_1(t_1)\}}-\frac{a_0 b_0 a_1}{2}\cos\{t_0-\theta_0(t_1)+\varphi_0(t_1)+\varphi_1(t_1)\}$$

$$-\frac{a_0^2 b_0^3}{192}\cos\{t_0-\theta_0(t_1)+2\varphi_0(t_1)\}-\frac{a_0^2 b_0^3}{192}\cos\{9t_0-3\theta_0(t_1)\}$$

$$-\underline{\frac{a_0^2 b_0^3}{192}\cos\{3t_0-\theta_0(t_1)\}}-\frac{a_0^2 b_0^3}{192}\cos\{7t_0-3\theta_0(t_1)+2\varphi_0(t_1)\}$$

$$-\frac{a_0^2 b_0^3}{384}\cos\{5t_0-\theta_0(t_1)-2\varphi_0(t_1)\}-\frac{a_0^2 b_0^3}{384}\cos\{9t_0-3\theta_0(t_1)\}$$

$$-\underline{\frac{a_0^2 b_0^3}{384}\cos\{3t_0-\theta_0(t_1)\}}-\frac{a_0^2 b_0^3}{384}\cos\{11t_0-3\theta_0(t_1)-2\varphi_0(t_1)\}$$

$$-\underline{\frac{a_0^2}{2}b_1(t_1)\cos\{3t_0-\theta_1(t_1)\}}+\frac{a_0^4b_0}{64}\cos\{t_0-\theta_0(t_1)+2\varphi_0(t_1)\}$$

$$-\frac{a_0^4b_0}{128}\cos\{5t_0-\theta_0(t_1)-2\varphi_0(t_1)\}-\frac{a_0^2b_1}{4}\cos\{5t_0-\theta_1(t_1)-2\varphi_0(t_1)\}$$

$$-\frac{a_0^2b_1}{4}\cos\{t_0-\theta_1(t_1)+2\varphi_0(t_1)\}+\underline{\frac{a_0^4b_0}{128}\cos\{3t_0-\theta_0(t_1)\}}$$

$$+\frac{a_0^4b_0}{128}\cos\{t_0+\theta_0(t_1)-4\varphi_0(t_1)\}-\underline{\frac{a_0^4b_0}{256}\cos\{3t_0-\theta_0(t_1)\}}$$

$$-\frac{a_0^4b_0}{256}\cos\{7t_0-\theta_0(t_1)-4\varphi_0(t_1)\} \qquad (11b)$$

The requirement that the solutions be uniformly bounded implies that we must set the terms of frequencies 1 and 3 respectively in the two equations equal to zero. The coefficients of $\sin(t_0-\varphi_0)$, $\cos(t_0-\varphi_0)$, $\sin(3t_0-\theta_0)$ and $\cos(3t_0-\theta_0)$ are respectively:

$$2\frac{\partial a_0}{\partial t_2}+2\frac{\partial a_1}{\partial t_1}\cos(\varphi_1-\varphi_0)-2a_1\frac{\partial\varphi_1}{\partial t_1}\sin(\varphi_1-\varphi_0)-\frac{a_1b_0^2}{2}\sin(\varphi_1-\varphi_0)=0 \qquad (12a)$$

$$-2a_0\frac{\partial\varphi_0}{\partial t_2}+\frac{a_0b_0^4}{16}-2\frac{\partial a_1}{\partial t_1}\sin(\varphi_1-\varphi_0)-2a_1\frac{\partial\varphi_1}{\partial t_1}\cos(\varphi_1-\varphi_0)-\frac{b_0^2}{2}a_1\cos(\varphi_1-\varphi_0)-\frac{a_0b_0^4}{256}$$
$$-a_0b_0b_1\cos(\theta_1-\theta_0)+\frac{a_0^3b_0^2}{128}=0 \qquad (12b)$$

$$6\frac{\partial b_0}{\partial t_2}+6\frac{\partial b_1}{\partial t_1}\cos(\theta_1-\theta_0)-6b_1\frac{\partial\theta_1}{\partial t_1}\sin(\theta_1-\theta_0)-\frac{a_0^2b_1}{2}\sin(\theta_1-\theta_0)=0 \qquad (12c)$$

$$-6b_0\frac{\partial\theta_0}{\partial t_2}+\frac{a_0^4b_0}{144}-6\frac{\partial b_1}{\partial t_1}\sin(\theta_1-\theta_0)-6b_1\frac{\partial\theta_1}{\partial t_1}\cos(\theta_1-\theta_0)-a_0b_0a_1\cos(\varphi_1-\varphi_0)-\frac{a_0^2b_0^3}{128}$$
$$-\frac{a_0^2}{2}b_1\cos(\theta_1-\theta_0)+\frac{a_0^4b_0}{256}=0 \qquad (12d)$$

To simplify equations (12),

let

$$A_{1c} = a_1(t_1)\cos\{\varphi_1(t_1)-\varphi_0(t_1)\} \qquad A_{1s} = a_1(t_1)\sin\{\varphi_1(t_1)-\varphi_0(t_1)\}$$

$$B_{1c} = b_1(t_1)\cos\{\theta_1(t_1)-\theta_0(t_1)\} \qquad B_{1s} = b_1(t_1)\sin\{\theta_1(t_1)-\theta_0(t_1)\}$$

The equations (12) then become respectively

$$\frac{\partial A_{1c}(t_1)}{\partial t_1} + \frac{\partial a_0(t_2)}{\partial t_2} = 0 \tag{13a}$$

$$\frac{\partial A_{1s}(t_1)}{\partial t_1} + a_0\frac{\partial \varphi_0(t_1)}{\partial t_2} = \frac{a_0 b_0^2}{2}\left\{\frac{15}{256}b_0^2 + \frac{1}{128}a_0^2\right\} - \tfrac{1}{2}a_0 b_0 B_{1c}(t_1) \tag{13b}$$

$$\frac{\partial B_{1c}(t_1)}{\partial t_1} + \frac{\partial b_0(t_2)}{\partial t_2} = 0 \tag{13c}$$

$$\frac{\partial B_{1s}(t_1)}{\partial t_1} + b_0\frac{\partial \theta_0(t_1)}{\partial t_2} = \frac{a_0^2 b_0}{6}\left\{\frac{25}{2304}a_0^2 - \frac{1}{128}b_0^2\right\} - \frac{1}{6}a_0 b_0 A_{1c}(t_1) \tag{13d}$$

Consider equation (13a). We have from (9a) that a_0 is independent of t_1. Then, integrating

$$A_{1c}(t_1) = t_1\frac{\partial a_0(t_2)}{\partial t_2} + A_{1c}^{(2)}(t_2)\,.$$

But this would prevent our expansion from being uniform unless

$$\frac{\partial a_0(t_2)}{\partial t_2} = 0 \quad \text{so that} \quad \frac{\partial A_{1c}(t_1)}{\partial t_1} = 0 \tag{14a}$$

Similarly, equation (13c) yields

$$\frac{\partial b_0(t_2)}{\partial t_2} = 0 \qquad \frac{\partial B_{1c}(t_1)}{\partial t_1} = 0 \tag{14b}$$

Thus only the first terms on the left-hand sides of equations (13b) and (13d) depend on t_1, so to keep A_{1s} and B_{1s} uniformly bounded we must have

$$\frac{\partial A_{1s}(t_1)}{\partial t_1} = 0 \qquad \frac{\partial B_{1s}(t_1)}{\partial t_1} = 0$$

and

$$\frac{\partial \varphi_0^{(2)}(t_2)}{\partial t_2} = \frac{b_0^2}{2}\left\{\frac{15}{256}\ b_0^2 + \frac{1}{128}\ a_0^2\right\} - \tfrac{1}{2}\ b_0\ B_{1c}(t_2) \tag{15a}$$

$$\frac{\partial \theta_0^{(2)}(t_2)}{\partial t_2} = \frac{a_0^2}{6}\left\{\frac{25}{2304}\ a_0^2 - \frac{1}{128}\ b_0^2\right\} - \frac{1}{6}\ a_0\ A_{1c}(t_2) \tag{15b}$$

From equations (11) we obtain

$$\begin{aligned}
x_2 = {} & a_2(t_1)\cos\{t_0 - \varphi_2(t_1)\} - \frac{1}{24}\left\{\frac{19}{576} a_0^3 b_0^2 - \frac{1}{32} a_0 b_0^4\right\}\cos\{5t_0 - 2\theta_0(t_1) + \varphi_0(t_1)\} \\
& - \frac{1}{48}\left\{+\frac{5}{1152} a_0^3 b_0^2 + \frac{a_0 b_0^4}{64}\right\}\cos\{7t_0 - 2\theta_0(t_1) - \varphi_0(t_1)\} \\
& + \frac{a_0^3 b_0^2}{80\cdot 128}\cos(9t_0 - 2\theta_0 - 3\varphi_0) + \frac{a_0 b_0^4}{120\cdot 384}\cos\{11t_0 - 4\theta_0(t_1) + \varphi_0(t_1)\} \\
& + \frac{a_0\ b_0^4}{168\cdot 768}\cos\{13t_0 - 4\theta_0(t_1) - \varphi_0(t_1)\} \\
& - \frac{a_0^3\ b_0^2}{1024}\cos\{3t_0 - 3\varphi_0(t_1)\} - \frac{a_0^3 b_0^2}{512}\cos\{3t_0 - 2\theta_0(t_1) + 3\varphi_0(t_1)\} \\
& + \frac{a_1 b_0^2}{96}\cos\{5t_0 - 2\theta_0(t_1) + \varphi_1(t_1)\} + \frac{a_0 b_0 b_1}{48}\cos\{5t_0 - \theta_0(t_1) - \theta_1(t_1) + \varphi_0(t_1)\} \\
& + \frac{a_1 b_0^2}{192}\cos\{7t_0 - 2\theta_0(t_1) - \varphi_1(t_1)\} + \frac{a_0 b_0 b_1}{96}\cos\{7t_0 - \theta_0(t_1) - \theta_1(t_1) - \varphi_0(t_1)\}
\end{aligned} \tag{16a}$$

$$\begin{aligned}
y_2 = {} & b_2(t_1)\cos\{3t_0 - \theta_2(t_1)\} + \left\{+\frac{a_0^4 b_0}{768} + \frac{5a_0^2 b_0^3}{1536}\right\}\cos\{t_0 - \theta_0(t_1) + 2\varphi_0(t_1)\} \\
& + \frac{a_0^4 b_0}{1024}\cos\{t_0 + \theta_0(t_1) - 4\varphi_0(t_1)\} - \frac{1}{16}\left\{\frac{1}{192} a_0^4\ b_0 + \frac{29}{384} a_0^2 b_0^3\right\}\cos\{5t_0 - \theta_0(t_1) - 2\varphi_0(t_1)\} \\
& + \frac{a_0^2 b_0^3}{40\cdot 192}\cos\{7t_0 - 3\theta_0(t_1) + 2\varphi_0(t_1)\} + \frac{a_0^4\ b_0}{40\cdot 256}\cos\{7t_0 - \theta_0(t_1) - 4\varphi_0(t_1)\} \\
& - \frac{a_0 b_0 a_1}{16}\cos\{t_0 - \theta_0(t_1) + \varphi_0(t_1) + \varphi_1(t_1)\} - \frac{a_0^2 b_1}{32}\cos\{t_0 - \theta_1(t_1) + 2\varphi_0(t_1)\} \\
& + \frac{a_0 b_0 a_1}{32}\cos\{5t_0 - \theta_0(t_1) - \varphi_0(t_1) - \varphi_1(t_1)\} + \frac{a_0^2 b_1}{64}\cos\{5t_0 - \theta_1(t_1) - 2\varphi_0(t_1)\} \\
& + \frac{a_0^2\ b_0^3}{72\cdot 128}\cos\{9t_0 - 3\theta_0(t_1)\} + \frac{a_0^2\ b_0^3}{112\cdot 384}\cos\{11t_0 - 3\theta_0(t_1) - 2\varphi_0(t_1)\}
\end{aligned} \tag{16b}$$

However, we do not yet know the t_2 dependence of A_{1c} and B_{1c}, so we cannot integrate equations (15) without looking at equations (6). Substituting the appropriate expressions for $x_0, x_1, x_2, y_0, y_1, y_2$ and their derivatives in the right-hand sides of equations (6) and setting the coefficients of $\sin(t_0-\varphi_0)$ on the right-hand side of equation (6a) and of $\sin(3t_0-\theta_0)$ on the right-hand side of (6b) equal to zero, we obtain respectively:

$$\frac{\partial A_{2c}(t_1)}{\partial t_1}+\frac{\partial A_{1c}(t_2)}{\partial t_2}+\frac{\partial a_0}{\partial t_3}=-\frac{3}{8192}a_0^5 b_0^2 \sin\left\{\frac{1}{6}(a_0^2-9b_0^2)t_1+6\varphi_0^{(2)}(t_2)-2\theta_0^{(2)}(t_2)\right\} \quad (17a)$$

$$\frac{\partial B_{2c}(t_1)}{\partial t_1}+\frac{\partial B_{1c}(t_2)}{\partial t_2}+\frac{\partial b_0}{\partial t_3}=\frac{1}{24576}a_0^6 b_0 \sin\left\{\frac{1}{6}(a_0^2-9b_0^2)t_1+6\varphi_0^{(2)}(t_2)-2\theta_0^{(2)}(t_2)\right\} \quad (17b)$$

The arguments of the sin on the right-hand sides of equations (17) comes from secular terms like $\cos\{t_0-2\theta_0+5\varphi_0\}$ on the right-hand side of (6a); the sin on the right-hand side of (17b) comes from a term like $\cos(3t_0+\theta_0-6\varphi_0)$ on the right-hand side of (6b). These are basically terms whose combination frequencies are to O(1) identical to the fundamental frequencies of the equations in which they appear - those which give rise to "vanishing" or small divisors in the Poincaré procedure, or contribute to the nonuniformity of the expansion in Jackson's modified Wigner-Brillouin theory. (For details, see Appendix B.)

Thus we have reached the point where the standard Poincaré procedure breaks down. It would continue to work only if the right-hand sides of equations (17) vanished identically. Equations (17) are also interesting because we can begin to see the relation between the strength of the resonance and the relative amplitudes of the two oscillators. There are basically two ways we must now proceed, depending upon the value of $(a_0^2-9b_0^2)$, the difference between the O(1) energies of the two oscillators.

<u>Case A.</u> If $(a_0^2 - 9b_0^2) = O(1)$, the right-hand sides of (17a) and (17b) depend upon t_1 and we have, applying the uniformity requirement;

$$\frac{\partial A_{2c}(t_1)}{\partial t_1} = -\frac{3}{8192} a_0^5 b_0^2 \sin\left\{\frac{1}{6}(a_0^2 - 9b_0^2)t_1 + 6\varphi_0^{(2)}(t_2) - 2\theta_0^{(2)}(t_2)\right\} \tag{18a}$$

$$\frac{\partial B_{2c}(t_1)}{\partial t_1} = \frac{1}{24576} a_0^6 b_0 \sin\left\{\frac{1}{6}(a_0^2 - 9b_0^2)t_1 + 6\varphi_0^{(2)}(t_2) - 2\theta_0^{(2)}(t_2)\right\} \tag{18b}$$

$$\frac{\partial A_{1c}(t_2)}{\partial t_2} = \frac{\partial B_{1c}(t_2)}{\partial t_2} = 0 \tag{19}$$

and $$\frac{\partial a_0}{\partial t_3} = \frac{\partial b_0}{\partial t_3} = 0$$

From equations (18) we obtain:

$$A_{2c} = \frac{9}{4096} \frac{a_0^5 b_0^2}{a_0^2 - 9b_0^2} \cos\left\{\frac{1}{6}(a_0^2 - 9b_0^2)t_1 + 6\varphi_0^{(2)}(t_2) - 2\theta_0^{(2)}(t_2)\right\} + A_{2c}^{(3)}(t_3) \tag{20a}$$

$$B_{2c} = -\frac{1}{4096} \frac{a_0^6 b_0}{a_0^2 - 9b_0^2} \cos\left\{\frac{1}{6}(a_0^2 - 9b_0^2)t_1 + 6\varphi_0^{(2)}(t_2) - 2\theta_0^{(2)}(t_2)\right\} + B_{2c}^{(3)}(t_3) \tag{20b}$$

and since equations (19) imply A_{1c} and B_{1c} are constant on the t_2 time scale, we have, integrating equations (15),

$$\varphi_0^{(2)}(t_2) = \left[\frac{b_0^2}{2}\left\{\frac{15}{256} b_0^2 + \frac{1}{128} a_0^2\right\} - \frac{1}{2} b_0 B_{1c}\right] t_2 + \varphi_0^{(3)}(t_3) \tag{21a}$$

$$\theta_0^{(2)}(t_2) = \left[\frac{a_0^2}{6}\left\{\frac{25}{2304} a_0^2 - \frac{1}{128} b_0^2\right\} - \frac{1}{6} a_0 A_{1c}\right] t_2 + \theta_0^{(3)}(t_3) \tag{21b}$$

Since these solutions correspond to additional sinusoidal terms, they could probably be obtained by Jackson's modified Wigner-Brillouin method. However, we notice that if $(a_0^2 - 9b_0^2)$ were $O(\epsilon)$, A_{2c} and B_{2c} would be $O(\frac{1}{\epsilon})$ which would imply that x_2 and y_2 were $O(\epsilon)$ instead of $O(\epsilon^2)$, and the expansion would no longer be uniform. Thus a different separation of equations (17) must be used if $a_0^2 - 9b_0^2$ is $O(\epsilon)$, which brings us to:

Case B. $(a_0^2 - 9b_0^2) = O(\epsilon) = \lambda\epsilon \qquad \lambda = O(1)$

In this case the argument of the sin on the right-hand sides of equations (17) becomes:

$$\sin\left\{\frac{\lambda\epsilon}{6} t_1 + 6\varphi_0^{(2)}(t_2) - 2\theta_0^{(2)}(t_2)\right\}.$$

To avoid an ϵ in the denominator and thus keep the expansion uniform we must use the fact that the time variables are related and write:

$$\epsilon t_1 = t_2 \; .$$

Now the right-hand sides of equations (17) are independent of t_1 and we must have:

$$\frac{\partial A_{2c}}{\partial t_1} = \frac{\partial B_{2c}}{\partial t_1} = 0 \tag{22}$$

$$\frac{\partial a_0}{\partial t_3} = \frac{\partial b_0}{\partial t_3} = 0$$

and

$$\frac{\partial A_{1c}(t_2)}{\partial t_2} = -\frac{3}{8192} a_0^5 b_0^2 \sin\left\{\frac{\lambda t_2}{6} + 6\varphi_0^{(2)}(t_2) - 2\theta_0^{(2)}(t_2)\right\} \tag{23a}$$

$$\frac{\partial B_{1c}(t_2)}{\partial t_2} = \frac{1}{24576} a_0^6 b_0 \sin\left\{\frac{\lambda t_2}{6} + 6\varphi_0^{(2)}(t_2) - 2\theta_0^{(2)}(t_2)\right\} \tag{23b}$$

These equations cannot be integrated as they stand, but must be solved simultaneously with equations (15)

$$\frac{\partial\varphi_0^{(2)}(t_2)}{\partial t_2} = \frac{b_0^2}{2}\left\{\frac{15}{256} b_0^2 + \frac{1}{128} a_0^2\right\} - \tfrac{1}{2} b_0 B_{1c}(t_2) \tag{15a}$$

$$\frac{\partial\theta_0^{(2)}(t_2)}{\partial t_2} = \frac{a_0^2}{6}\left\{\frac{25}{2304} a_0^2 - \frac{1}{128} b_0^2\right\} - \frac{1}{6} a_0 A_{1c}(t_2) \tag{15b}$$

Taking $\frac{\partial}{\partial t_2}$ of equations (15) and using (23) we obtain respectively (dropping arguments of the unknown functions except where necessary):

$$\frac{\partial^2 \varphi_0^{(2)}}{\partial t_2^2} = -\tfrac{1}{2}b_0 \frac{\partial B_{1c}}{\partial t_2} = -\frac{1}{49152} a_0^6 b_0^2 \sin\left\{\frac{\lambda t_2}{6} + 6\varphi_0^{(2)} - 2\theta_0^{(2)}\right\} \tag{24a}$$

$$\frac{\partial^2 \theta_0^{(2)}}{\partial t_2^2} = -\frac{1}{6} a_0 \frac{\partial A_{1c}}{\partial t_2} = \frac{1}{16384} a_0^6 b_0^2 \sin\left\{\frac{\lambda t_2}{6} + 6\varphi_0^{(2)} - 2\theta_0^{(2)}\right\} \tag{24b}$$

Subtracting two times (24b) from six times (24a) and letting

$$u = u(t_2) = 6\varphi_0^{(2)} - 2\theta_0^{(2)} \tag{25}$$

we obtain:

$$\frac{\partial^2 u}{\partial t_2^2} = -\frac{a_0^6 b_0^2}{4096} \sin\left\{u + \frac{\lambda t_2}{6}\right\} \tag{26}$$

Setting

$$z = u + \frac{\lambda t_2}{6} \quad \text{and} \quad g = \frac{a_0^6 b_0^2}{4096} \tag{27}$$

we obtain finally

$$\frac{\partial^2 z}{\partial t_2^2} + g \sin z = 0 , \tag{28}$$

the equation of motion of a simple pendulum!

Multiply by $\frac{\partial z}{\partial t_2}$ and integrate, obtaining:

$$\left(\frac{\partial z}{\partial t_2}\right)^2 = 2g \cos z + c \tag{29}$$

Suppose that when z is at the lowest point of the circle (i.e. $z = 0$),

$$\frac{\dot{z}^2}{2g} = h \; * \tag{30}$$

Then $c = 2g(h-1)$, and (29) becomes:

$$\dot{z}^2 = 2g\left[h-(1-\cos z)\right] = 2g\left[h - 2\sin^2 z/2\right] \tag{31}$$

---

*For convenience in the remainder of this discussion, we shall write $\frac{\partial z}{\partial t_2} \equiv \dot{z}$, $\frac{\partial q}{\partial t_2} = \dot{q}$.

Making the substitution

$$q = \sin z/2 \tag{32}$$

we obtain, finally,

$$\dot{q}^2 = q\left(\frac{h}{2} - q^2\right)(1-q^2) \tag{33}$$

The pendulum (see e.g.,Whittaker, 1937) has two basic motions: "oscillating" and "circulatory" - in the first the "gravitational" attraction dominates the initial angular momentum and the pendulum oscillates back and forth about the bottom point of the circle (refer to **Fig.** 2), $z = 0$; in the second the energy is sufficient to carry it over the highest point of the circle and it rotates about its center, always in same sense. These features can be seen easily from the phase diagram (Fig. 3). A third possible situation - where the pendulum has just enough energy to reach $z = \pi$ with zero momentum - is usually referred to as the separatrix.

Let us now examine the solutions of our system for these three cases:

<u>Oscillatory Case</u>

Equation (33) will have an oscillatory-type solution if $\frac{h}{2} < 1$. Let $h = 2k^2$ where k^2 is less than 1. The resulting solution of equation (33) is

$$q = k\,\mathrm{sn}\left\{\sqrt{g}\,(t_2 - \tau), k\right\} = \sin z/2 \tag{34}$$

where k and τ are arbitrary constants which must be determined from the initial conditions.

Then

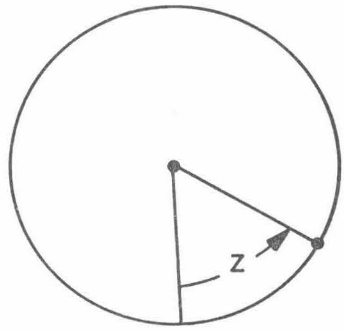

Fig. 2. Coordinate System for Simple Pendulum

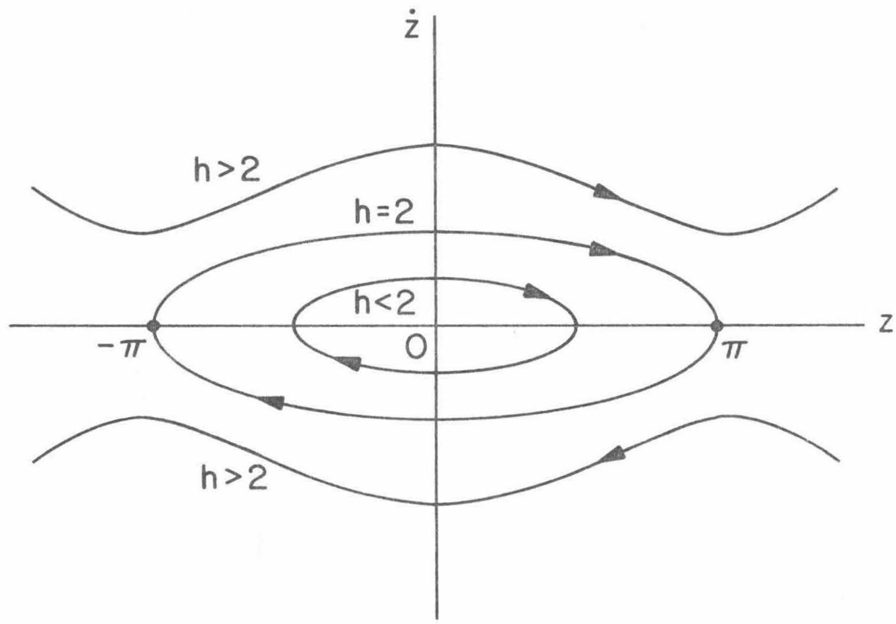

Fig. 3. Phase Plane Diagram for Simple Pendulum - Equation (28)

$$\cos z/2 = \sqrt{1-k^2 \sin^2\{\sqrt{g}\,(t_2-\tau),k\}} = \mathrm{dn}\{\sqrt{g}(t_2-\tau),k\} \tag{35}$$

Equations (23) thus become respectively:

$$\frac{\partial A_{1C}}{\partial t_2} = -\frac{3}{8192}\, a_0^5\, b_0^2 \sin\left\{\frac{\lambda t_2}{6} + 6\varphi_0^{(2)}(t_2) - 2\theta_0^{(2)}(t_2)\right\}$$

$$= -\frac{3}{8192}\, a_0^5\, b_0^2 \sin z$$

$$= -\frac{3}{4096}\, a_0^5\, b_0^2 \sin z/2 \cos z/2$$

$$= -\frac{3k}{4096}\, a_0^5\, b_0^2\, \mathrm{sn}\{\sqrt{g}(t_2-\tau),k\}\, \mathrm{dn}\{\sqrt{g}(t_2-\tau),k\} \tag{36a}$$

and

$$\frac{\partial B_{1C}}{\partial t_2} = +\frac{k}{12288}\, a_0^6\, b_0\, \mathrm{sn}\{\sqrt{g}(t_2-\tau),k\}\, \mathrm{dn}\{\sqrt{g}(t_2-\tau),k\} \tag{36b}$$

Performing the quadratures, we obtain:

$$A_{1C}(t_2) = \frac{3k}{4096\sqrt{g}}\, a_0^5\, b_0^2\, \mathrm{cn}\{\sqrt{g}(t_2-\tau),k\} + \mu_1 \tag{37a}$$

$$B_{1C}(t_2) = -\frac{k}{12288\sqrt{g}}\, a_0^6\, b_0\, \mathrm{cn}\{\sqrt{g}(t_2-\tau),k\} + \mu_2 \tag{37b}$$

μ_1, μ_2 are constants.

Using equations (37) in equations (15) and integrating we obtain (using equation (27) for g):

$$\varphi_0^{(2)}(t_2) = \left\{\frac{b_0^2}{2}\left(\frac{1}{128}\, a_0^2 + \frac{15}{256}\, b_0^2\right) - \tfrac{1}{2} b_0 \mu_2\right\} t_2 + \frac{1}{6}\cos^{-1}\left[\mathrm{dn}\{\sqrt{g}(t_2-\tau),k\}\right] + \nu_1 \tag{38a}$$

$$\theta_0^{(2)}(t_2) = \left\{\frac{a_0^2}{6}\left(\frac{25}{2304}\, a_0^2 - \frac{1}{128}\, b_0^2\right) - \frac{1}{6} a_0 \mu_1\right\} t_2 - \tfrac{1}{2}\cos^{-1}\left[\mathrm{dn}\{\sqrt{g}(t_2-\tau),k\}\right] + \nu_2 \tag{38b}$$

We originally had four first order differential equations so we are entitled to four constants - however we have six, so we must eliminate two. Also, it would be useful to express our constants in terms of the values of A_{1C},

B_{1C}, $\varphi_0^{(2)}$ and $\theta_0^{(2)}$ at $t_2 = 0$, which in turn are obtainable from the initial conditions.

From equation (34) we have at $t = 0$:

$$3\varphi_0^{(2)}(0) - \theta_0^{(2)}(0) = \sin^{-1}\left[k\,\mathrm{sn}\{-\sqrt{g}\ \tau, k\}\right] \tag{39a}$$

However from equations (38), since $\cos^{-1}\left[\mathrm{dn}\{\ \}\right] = \pm\sin^{-1}\left[k\,\mathrm{sn}\{\ \}\right]$

$$3\varphi_0^{(2)}(0) - \theta_0^{(2)}(0) = \sin^{-1}\left[k\,\mathrm{sn}\{-\sqrt{g}\ \tau, k\}\right] + 3\nu_1 - \nu_2 \tag{39b}$$

Since both of the expressions for $3\varphi_0^{(2)}(t_2) - \theta_0^{(2)}(t_2)$ must have the same dependence on t_2, the first terms on the right hand sides of (39a) and (39b) must be identical. Thus we have:

$$3\nu_1 - \nu_2 = 0$$

and therefore:

$$\nu_2 = 3\nu_1$$

If at $t_2 = 0$ we have

$$3\varphi_0^{(2)} + \theta_0^{(2)} = \Omega \tag{40}$$

then

$$\nu_1 = \frac{\Omega}{6}\ ,\quad \nu_2 = \frac{\Omega}{2}$$

so that

$$\varphi_0^{(2)}(0) = +\frac{1}{6}\cos^{-1}\left[\mathrm{dn}\{-\sqrt{g}\ \tau, k\}\right] + \frac{\Omega}{6}$$

$$\theta_0^{(2)}(0) = -\frac{1}{2}\cos^{-1}\left[\mathrm{dn}\{-\sqrt{g}\ \tau, k\}\right] + \frac{\Omega}{2}$$

Similarly, by again comparing $3\varphi_0^{(2)}(t_2) - \theta_0^{(2)}(t_2)$ from equations (34) and (38), we find:

$$\frac{3}{2}b_0^2\left(\frac{1}{128}a_0^2+\frac{15}{256}b_0^2\right)-\frac{3}{2}b_0\mu_2-\frac{a_0^2}{6}\left(\frac{25}{2304}a_0^2-\frac{1}{128}b_0^2\right)+\frac{1}{6}a_0\mu_1=-\frac{\lambda}{12}$$

So that

$$a_0\mu_1=-\frac{\lambda}{2}+\frac{25}{2304}a_0^4-\frac{5}{64}a_0^2b_0^2-\frac{135}{256}b_0^4+9b_0\mu_2 \tag{41}$$

This eliminates the two redundant constants. Now we have to solve for μ_1, Ω, τ and k in terms of $A_{1c}(0)=\alpha$, $B_{1c}(0)=\beta$, $\varphi_0^{(2)}(0)=\varphi_{00}$, $\theta_0^{(2)}(0)=\theta_{00}$. We have from (40)

$$\Omega = 3\varphi_{00}+\theta_{00} \tag{42}$$

from (39a) $$\pm k\,\mathrm{sn}\{-\sqrt{g}\,\tau,k\}=\sin\{3\varphi_{00}-\theta_{00}\} \tag{43}$$

from (37)
$$a_0\alpha+9b_0\beta=-\frac{\lambda}{2}+\frac{25}{2304}a_0^4-\frac{5}{64}a_0^2b_0^2-\frac{135}{256}b_0^4+18b_0\mu_2 \tag{44}$$

and

$$a_0\alpha-9b_0\beta=\pm\frac{3k}{2048\sqrt{g}}a_0^6b_0^2\,\mathrm{cn}\{-\sqrt{g}\,\tau,k\}-\frac{\lambda}{2}+\frac{25}{2304}a_0^4-\frac{5}{64}a_0^2b_0^2-\frac{135}{256}b_0^4$$

$$\Longrightarrow \pm\frac{3k}{2048\sqrt{g}}a_0^6b_0^2\,\mathrm{cn}\{-\sqrt{g}\,\tau,k\}=a_0\alpha-9b_0\beta+\frac{\lambda}{2}-\frac{25}{2304}a_0^4+\frac{5}{64}a_0^2b_0^2+\frac{135}{256}b_0^4$$

$$\sqrt{g}=\frac{a_0^3b_0}{64}$$

$$\Longrightarrow \pm k\,\mathrm{cn}\{-\sqrt{g}\,\tau,k\}=\frac{1}{6\sqrt{g}}\left\{a_0\alpha-9b_0\beta+\frac{\lambda}{2}-\frac{25}{2304}a_0^4+\frac{5}{64}a_0^2b_0^2+\frac{135}{256}b_0^4\right\} \tag{45}$$

Adding the squares of equations (43) and (45), we obtain:

$$k^2=\sin^2(3\varphi_{00}-\theta_{00})+\frac{1024}{9a_0^6b_0^2}\left\{a_0\alpha-9b_0\beta+\frac{\lambda}{2}-\frac{25}{2304}a_0^4+\frac{5}{64}a_0^2b_0^2+\frac{135}{256}b_0^4\right\}^2<1 \tag{46}$$

Taking the positive sign for k, τ must then be chosen appropriately so that the phases come out right in (43) and (45).

Circulatory Case.

Equation (33) will have a circulatory solution if $h > 2$: Let

$$hk^2 = 2 \tag{47}$$

so that $k^2 < 1$. The solution of (33) in this case is:

$$q = \mathrm{sn}\left\{\frac{\sqrt{g}}{k}(t_2 - \tau), k\right\} = \sin z/2 \tag{48}$$

where k and τ are again constants which must be determined from the initial conditions. We have

$$\cos z/2 = \mathrm{cn}\left\{\frac{\sqrt{g}}{k}(t_2 - \tau), k\right\} \tag{49}$$

Substituting (48) and (49) in (23) and performing the quadratures we have:

$$A_{1C} = \frac{3a_0^5 b_0^2}{4096\sqrt{g}\,k}\,\mathrm{dn}\left\{\frac{\sqrt{g}}{k}(t_2 - \tau), k\right\} + \mu_1 \tag{50a}$$

$$B_{1C} = -\frac{a_0^6 b_0}{12288\sqrt{g}\,k}\,\mathrm{dn}\left\{\frac{\sqrt{g}}{k}(t_2 - \tau), k\right\} + \mu_2 \tag{50b}$$

Substituting (50) in (23) and integrating:

$$\varphi_0^{(2)}(t_2) = \left\{\frac{b_0^2}{2}\left(\frac{a_0^2}{128} + \frac{15b_0^2}{256}\right) - \tfrac{1}{2}b_0\,\mu_2\right\}t_2 + \frac{1}{6}\sin^{-1}\left[\mathrm{sn}\left\{\frac{\sqrt{g}}{k}(t_2 - \tau), k\right\}\right] + \nu_1 \tag{51a}$$

$$\theta_0^{(2)}(t_2) = \left\{\frac{a_0^2}{6}\left(\frac{25}{2304}a_0^2 - \frac{1}{128}b_0^2\right) - \frac{1}{6}a_0\,\mu_1\right\}t_2 - \tfrac{1}{2}\sin^{-1}\left[\mathrm{sn}\left\{\frac{\sqrt{g}}{k}(t_2 - \tau), k\right\}\right] + \nu_2 \tag{51b}$$

Using the same procedures as in the previous case to eliminate the extraneous constants and put the remaining constants in terms of the initial conditions given, we obtain (using similar notation):

$$a_0\mu_1 = \tfrac{1}{2}\left\{-\frac{\lambda}{2}+\frac{25}{2304}a_0^4-\frac{5}{64}a_0^2 b_0^2-\frac{135}{256}b_0^4+a_0\alpha+9b_0\beta\right\} \tag{52a}$$

$$9b_0\mu_2 = \tfrac{1}{2}\left\{\frac{\lambda}{2}-\frac{25}{2304}a_0^4+\frac{5}{64}a_0^2 b_0^2+\frac{135}{256}b_0^4+a_0\alpha+9b_0\beta\right\} \tag{52b}$$

$$\nu_1 = \frac{1}{6}(3\varphi_{00}+\theta_{00}) \tag{53a}$$

$$\nu_2 = \tfrac{1}{2}(3\varphi_{00}+\theta_{00}) \tag{53b}$$

$$k^2 = \left\{\sin^2(3\varphi_{00}-\theta_{00})+\frac{1024}{9a_0^6 b_0^2}\left[a_0\alpha-9b_0\beta+\frac{\lambda}{2}-\frac{25}{2304}a_0^4+\frac{5}{64}a_0^2 b_0^2+\frac{135}{256}b_0^4\right]^2\right\}^{-1} \tag{54}$$

$$\left\{\begin{aligned} &\mathrm{sn}\left\{-\frac{\tau\sqrt{g}}{k},k\right\} = \sin(3\varphi_{00}-\theta_{00}) \\ &\mathrm{dn}\left\{-\frac{\tau\sqrt{g}}{k},k\right\} = \frac{32}{3}\frac{k}{a_0^3 b_0}\left(a_0\alpha-9b_0\beta+\frac{\lambda}{2}-\frac{25}{2304}a_0^4+\frac{5}{64}a_0^2 b_0^2+\frac{135}{256}b_0^4\right)\end{aligned}\right. \tag{55}$$

$$k^2 < 1$$

As a simple check, we have for both the circulatory and oscillatory case

$$h = 2\left\{\sin^2(3\varphi_{00}-\theta_{00})+\frac{1024}{9a_0^6 b_0^2}\left[a_0\alpha-9b_0\beta+\frac{\lambda}{2}-\frac{25}{2304}a_0^4+\frac{5}{64}a_0^2 b_0^2+\frac{135}{256}b_0^4\right]^2\right\} \tag{56}$$

<u>Separatrix</u>

If $h=2$, equation (33) becomes:

$$\dot{q}^2 = g(1-q^2)^2 \tag{57}$$

which has the solutions:

$$q = \sin z/2 = \tanh\left\{\sqrt{g}\,(t_2-\tau)\right\} \tag{58}$$

Then

$$A_{1c} = \pm\frac{3}{4096}\,\frac{a_0^5 b_0^2}{\sqrt{g}\cosh\left|\sqrt{g}\,(t_2-\tau)\right|}+\mu_1 \tag{59a}$$

$$B_{1c} = \mp \frac{1}{12288} \frac{a_0^6 b_0}{\sqrt{g}\cosh\left[\sqrt{g}\,(t_2-\tau)\right]} + \mu_2 \tag{59b}$$

$$\varphi_0^{(2)} = \left[\frac{b_0^2}{2}\left\{\frac{15}{256} b_0^2 + \frac{1}{128} a_0^2\right\} - \tfrac{1}{2} b_0 \mu_2\right] t_2 \pm \frac{1}{6}\tan^{-1}\sinh\left[\sqrt{g}(t_2-\tau)\right] + \frac{\Omega}{6} \tag{60a}$$

$$\theta_0^{(2)} = \left[\frac{a_0^2}{6}\left\{\frac{25}{2304} a_0^2 - \frac{1}{128} b_0^2\right\} - \frac{1}{6} a_0 \mu_1\right] t_2 \mp \tfrac{1}{2}\tan^{-1}\sinh\left[\sqrt{g}(t_2-\tau)\right] + \frac{\Omega}{2} \tag{60b}$$

The initial conditions $\mu_1, \mu_2, \nu_1, \nu_2$ are the same as those for the previous case, equations (52) and (53); τ is determined by

$$\tanh\left\{-\tau\sqrt{g}\right\} = \sin(3\varphi_{00} - \theta_{00}) \tag{61}$$

and since $h = 2$ we have from (56) as the condition for the separatrix case

$$\cos^2(3\varphi_{00} - \theta_{00}) = \frac{1024}{9a_0^6 b_0^2}\left\{a_0\alpha - 9b_0\beta + \frac{\lambda}{2} - \frac{25}{2304} a_0^4 + \frac{5}{64} a_0^2 b_0^2 + \frac{135}{256} b_0^4\right\}^2 \tag{62}$$

Thus, to recapitulate, we have four possible situations; writing solutions valid to O(1) for a time $O(\frac{1}{\epsilon^2})$, we have:

A. $a_0 - 9b_0^2 = O(1) \quad \Rightarrow x_0 = a_0 \cos\left\{t\left(1 + \frac{\epsilon b_0^2}{4} - \epsilon^2\left[\frac{b_0^2}{2}\left\{\frac{15}{256} b_0^2 + \frac{1}{128} a_0^2\right\} - \tfrac{1}{2} b_0 B_{1c}\right]\right) + \varphi_0^{(3)}(0)\right\}$

B. $a_0 - 9b_0^2 = \lambda\epsilon,\ \lambda = O(1)$

a. Circulatory Case: $h > 2$

$$x_0 = a_0\cos\left\{t\left(1 + \frac{\epsilon b_0^2}{4} - \epsilon^2\left[\frac{b_0^2}{2}\left\{\frac{15}{256} b_0^2 + \frac{1}{128} a_0^2\right\} - \tfrac{1}{2} b_0 \mu_2\right]\right) - \frac{1}{6}\sin^{-1}\left[\mathrm{sn}\{\sqrt{g}\,(t_2-\tau), k\}\right] - \nu_1\right\}$$

b. Oscillatory Case: $h < 2$

$$x_0 = a_0\cos\left\{t\left(1 + \frac{\epsilon b_0^2}{4} - \epsilon^2\left[\frac{b_0^2}{2}\left\{\frac{15}{256} b_0^2 + \frac{1}{128} a_0^2\right\} - \tfrac{1}{2} b_0 \mu_2\right]\right) - \frac{1}{6}\cos^{-1}\left[\mathrm{dn}\{\sqrt{g}(t_2-\tau), k\}\right] - \nu_1\right\}$$

c. Separatrix: $h = 2$

$$x_0 = a_0 \cos\left\{t\left(1+\frac{\epsilon b_0^2}{4} - \epsilon^2\left[\frac{b_0^2}{2}\left\{\frac{15}{256}b_0^2 + \frac{1}{128}a_0^2\right\} - \frac{1}{2}b_0\,\mu_2\right]\right) - \frac{1}{6}\tan^{-1}\left[\sinh\sqrt{g}(t_2-\tau)\right] - \frac{\Omega}{6}\right\}$$

and the N-timing procedure and its requirement of uniformity leads to different analytic forms for the solutions of the same pair of equations, depending on the initial amplitudes of the two oscillators.

Chapter V

A Strongly Resonant Example

In the two cases we have studied so far, we have found that the phases of the O(1) solutions can change with time, but their amplitudes remain constant. For certain frequency ratios depending upon the form of the nonlinearity, it has been found (e.g., Ford and Waters, 1963) that successive iterations of the Poincaré procedure lead to O(1) corrections to the motion, since the small divisors appear immediately. Thus the amplitudes of the O(1) terms are no longer constant, but vary slowly with time. These problems, which we have called "strongly resonant," also appear to be amenable to our procedure, but due to the early appearance of complicated functions we have been unable to carry the calculations very far. However, for one such problem we have determined the first two time variations of the O(1) terms and will present our results here. It should be noted that our use of only two time scales means that for this example our procedure is identical to Cole's two-timing method.

We will consider a system of equations which are similar to those of the non-resonant case studied in Chapter III, equations III-(5) but we will let $\omega_2 = 2$ instead of $\sqrt{3}$. Then we have:

$$\ddot{x} + x = 2\epsilon xy \qquad x(0) = a \qquad \left(\frac{dx}{dt}\right)\Big|_{t=0} = b \qquad \text{(1a)}$$

$$\ddot{y} + 4y = \epsilon(x^2 - 3y^2) \qquad y(0) = c \qquad \left(\frac{dy}{dt}\right)\Big|_{t=0} = d \qquad \text{(1b)}$$

Ford and Waters examined this system using the Wigner-Brillouin approach and found that there would be one term of O(1) from each iteration, so that little information could be obtained about the dynamics of the system, except that with appropriate initial conditions there could be O(1) energy sharing.

Applying the N-timing procedure to equations (1), we obtain

$$O(1) \quad \begin{cases} \dfrac{\partial^2 x_0}{\partial t_0^2} + x_0 = 0 \\[2ex] \dfrac{\partial^2 y_0}{\partial t_0^2} + 4y_0 = 0 \end{cases}$$

so that

$$x_0 = a_0 \cos(t_0 - \varphi_0) \tag{2a}$$

$$y_0 = b_0 \cos(2t_0 - \theta_0) \tag{2b}$$

where a_0, b_0, φ_0 and θ_0 are thus far undetermined functions of $t_1, t_2, \ldots$ The $O(\epsilon)$ equations are:

$$\frac{\partial^2 x_1}{\partial t_0^2} + x_1 = -2\frac{\partial^2 x_0}{\partial t_0\, \partial t_1} + 2x_0 y_0$$

$$\frac{\partial^2 y_1}{\partial t_0^2} + 4y_1 = -2\frac{\partial^2 y_0}{\partial t_0\, \partial t_1} + x_0^2 - 3y_0^2$$

Substituting equations (2) for x_0 and y_0 on the right-hand sides of these equations, eliminating the secular terms and solving the resulting equations, we obtain:

$$x_1 = a_1 \cos(t_0 - \varphi_1) - \frac{a_0 b_0}{8} \cos(3t_0 - \theta_0 - \varphi_0) \tag{3a}$$

$$y_1 = b_1 \cos(2t_0 - \theta_1) + \frac{a_0^2 - 3b_0^2}{8} + \frac{b_0^2}{8} \cos(4t_0 - 2\theta_0) \tag{3b}$$

and from the secular terms we obtain:

$$2\frac{\partial a_0}{\partial t_1} = a_0 b_0 \sin(2\varphi_0 - \theta_0) \tag{4a}$$

$$2a_0 \frac{\partial \varphi_0}{\partial t_1} = a_0 b_0 \cos(2\varphi_0 - \theta_0) \tag{4b}$$

$$4 \frac{\partial b_0}{\partial t_1} = \frac{a_0^2}{2} \sin(\theta_0 - 2\varphi_0) \tag{4c}$$

$$4b_0 \frac{\partial \theta_0}{\partial t_1} = \frac{a_0^2}{2} \cos(\theta_0 - 2\varphi_0) \tag{4d}$$

To determine the O(1) behavior of the solutions of equations (1) on the t_1 time scale, we need to solve system (4). Multiplying (4a) by $a_0/2$, (4c) by b_0 and adding the resulting equations, we obtain:

$$a_0 \frac{\partial a_0}{\partial t_1} + 4b_0 \frac{\partial b_0}{\partial t_1} = 0$$

$$\Rightarrow \tfrac{1}{2}(a_0^2 + 4b_0^2) = E_0(t_2) \text{ - constant on } t_1 \text{ time scale.} \tag{5}$$

Comparing with equations (2) we see that

$\frac{1}{2} a_0^2 (t_1)$ is to O(1) the energy in the $\omega = 1$ mode

and $\frac{1}{2}[4b_0^2 (t_1)]$ is to O(1) the energy in the $\omega = 2$ mode.

Squaring equations (4c) and (4d), adding the resulting equations and using equation (5), we obtain:

$$16 \left[\left(\frac{\partial b_0}{\partial t_1}\right)^2 + b_0^2 \left(\frac{\partial \theta_0}{\partial t_1}\right)^2 \right] = \frac{a_0^4}{4} = (E_0 - 2b_0^2)^2 \tag{6}$$

If we can obtain another expression for $\frac{\partial \theta_0}{\partial t_1}$ in terms of b_0, we can eliminate $\frac{\partial \theta_0}{\partial t_1}$ and be left with an equation for $b_0(t_1)$. From equations (4b) and (4d) we have:

$$\theta_0' = \frac{a_0^2}{4b_0^2} \varphi_0' \;{}^{*} \tag{7}$$

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*For the remainder of this section we will use the notation $(\ )' = \frac{\partial(\ )}{\partial t_1}$

Using this relation and those equations, we obtain

$$2\varphi_0'' = \frac{2\varphi_0' b_0'}{b_0} - (2\varphi_0' - \frac{a_0^2}{4b_0^2}\varphi_0') \cdot \frac{2a_0'}{a_0}$$

so that

$$\frac{\varphi_0''}{\varphi_0'} = \frac{b_0'}{b_0} - \frac{2a_0'}{a_0} + \frac{a_0 a_0'}{4b_0^2}$$

But from equation (5),

$$a_0 a_0' = -4b_0 b_0'$$

so that

$$\frac{\varphi_0''}{\varphi_0'} = -\frac{2a_0'}{a_0}$$

which implies that

$$\varphi_0' = \frac{\lambda_0}{a_0^2} \qquad \lambda_0 = \lambda_0(t_2) \tag{8a}$$

so that
$$\theta_0' = \frac{\lambda_0}{4b_0^2} \tag{8b}$$

which is the desired relation.* Substituting (8b) in (6), we have

$$16\, b_0^2\, b_0'^2 + \lambda_0^2 = b_0^2\, (E_0 - 2b_0^2)^2$$

This equation can be simplified slightly be setting $E_2^{(0)} = 2b_0^2 = O(1)$ energy in second mode. Then $E_2^{(0)'} = 4b_0\, b_0'$ and we have:

$$E_2^{(0)'2} = \tfrac{1}{2} E_2^{(0)} (E_0 - E_2^{(0)})^2 - \lambda_0^2 \tag{9}$$

The solutions $E_2^{(0)}$ of this equation are Weierstrass elliptic functions but

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* Note that λ_0 corresponds to the first term of Whittaker's adelphic integral for this case.

for our purposes we can obtain sufficient information by studying their behavior in the $(E_2^{(0)}, E_2^{(0)\prime})$ phase plane.

Physical solution curves must have $E_0 \geqslant E_2 \geqslant 0$. The cubic form in $E_2^{(0)}$ on the right-hand side has:

(i) two positive real roots $\leqslant E_0$ and one real root $> E_0$ if $(0<)\lambda_0^2 < \frac{2}{27} E_0^3$

(ii) one double root at $E_2^{(0)} = E_0/3$ and one real root $> E_0$ if $\lambda_0^2 = \frac{2}{27} E_0^3$

(iii) two complex conjugate and one real root $> E_0$ if $\lambda_0^2 > \frac{2}{27} E_0^3$.

Case (iii) clearly corresponds to a non-physical situation, since $(E_2^{(0)\prime})^2$ must have a positive real root $\leqslant E_0$ if E_2 is to remain bounded. Case (ii) corresponds to a singular solution at $E_2^{(0)} = \frac{1}{3} E_0$. Case (i) corresponds to a family of simple closed curves in the $(E_2^{(0)\prime}, E_2^{(0)})$ plane symmetric about the $E_2^{(0)}$ axis (with the curve corresponding to $\lambda_0^2 = 0$ forming an envelope for the family) and having zeroes of $E_2^{(0)\prime}$ at $E_2^{(0)} = E_0$ and $E_2^{(0)} = 0$. Thus the family of physical solutions looks like Fig. 4.

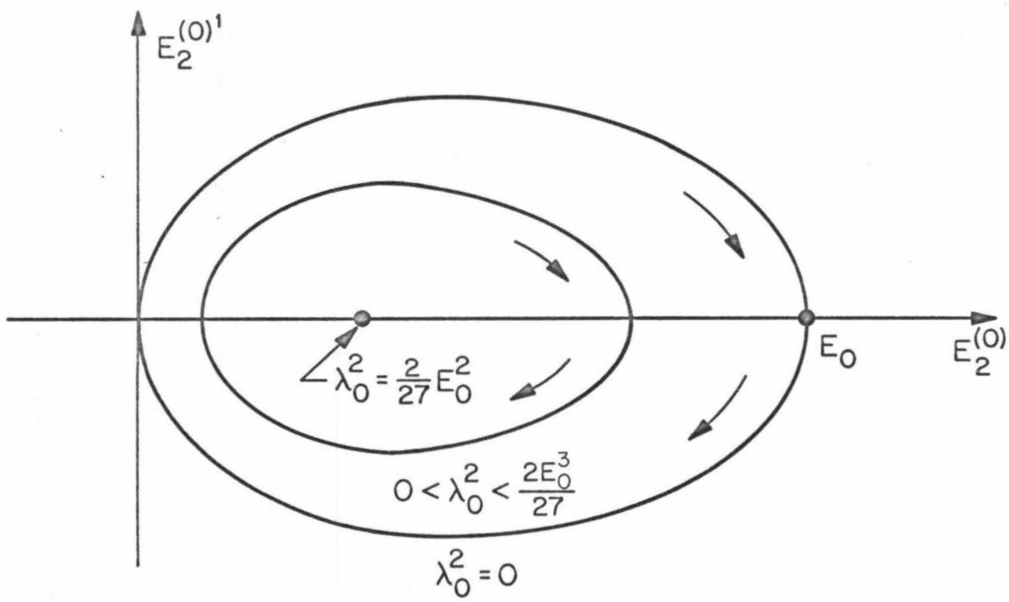

Fig. 4. $(E_2^{(0)}, E_2^{(0)\prime})$ Phase Plane Diagram for Equation (9)

The range of $E_2^{(0)}$ for the various values of λ_0^2 should correspond to O(1) to the actual range of the energy in the second mode, at least for a time $O(1/\epsilon)$ and possibly for all time. However, the range of E_2' cannot be determined without considering the $O(\epsilon)$ terms in the equations for E_2, since

$$\frac{dE_2}{dt} = \epsilon\left[\frac{\partial E_2^{(0)}}{\partial t_1} + \frac{\partial E_2^{(1)}}{\partial t_0}\right] + O(\epsilon^2)$$

Unfortunately, we can go no further without looking at the secular terms in the $O(\epsilon^2)$ equations. These become quite involved, so instead we will check the predictions we have made so far by considering the computer solutions of equations (1a) and (1b) for two special cases. One will be the case where $\lambda_0 = 0$. In this case our theory predicts complete energy sharing so that E_2 should have a range from zero to E_0. We can start anywhere on the curve described by $\lambda_0^2 = 0$. For conveneience we choose the initial conditions $a = 0$, $b = 1$, $c = 0$, $d = 1$, so that $E_0 = 1$. This leads to the (E_2, E_2') phase plane diagram shown in Fig. 5 which resembles a martini glass embedded in a fishbowl.*

The other interesting limiting case is where $\lambda_0^2 = \frac{2}{27} E_0^3$. This case requires

$$\sin\left(2\varphi_0(0) - \theta_0(0)\right) = 0 \tag{10a}$$

and if we want $E = E_0 = 1$ we need

$$a_0^2 \cos^2 \varphi_0 = b_0^2 \cos^2 \theta_0 \tag{10b}$$

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* Patent pending.

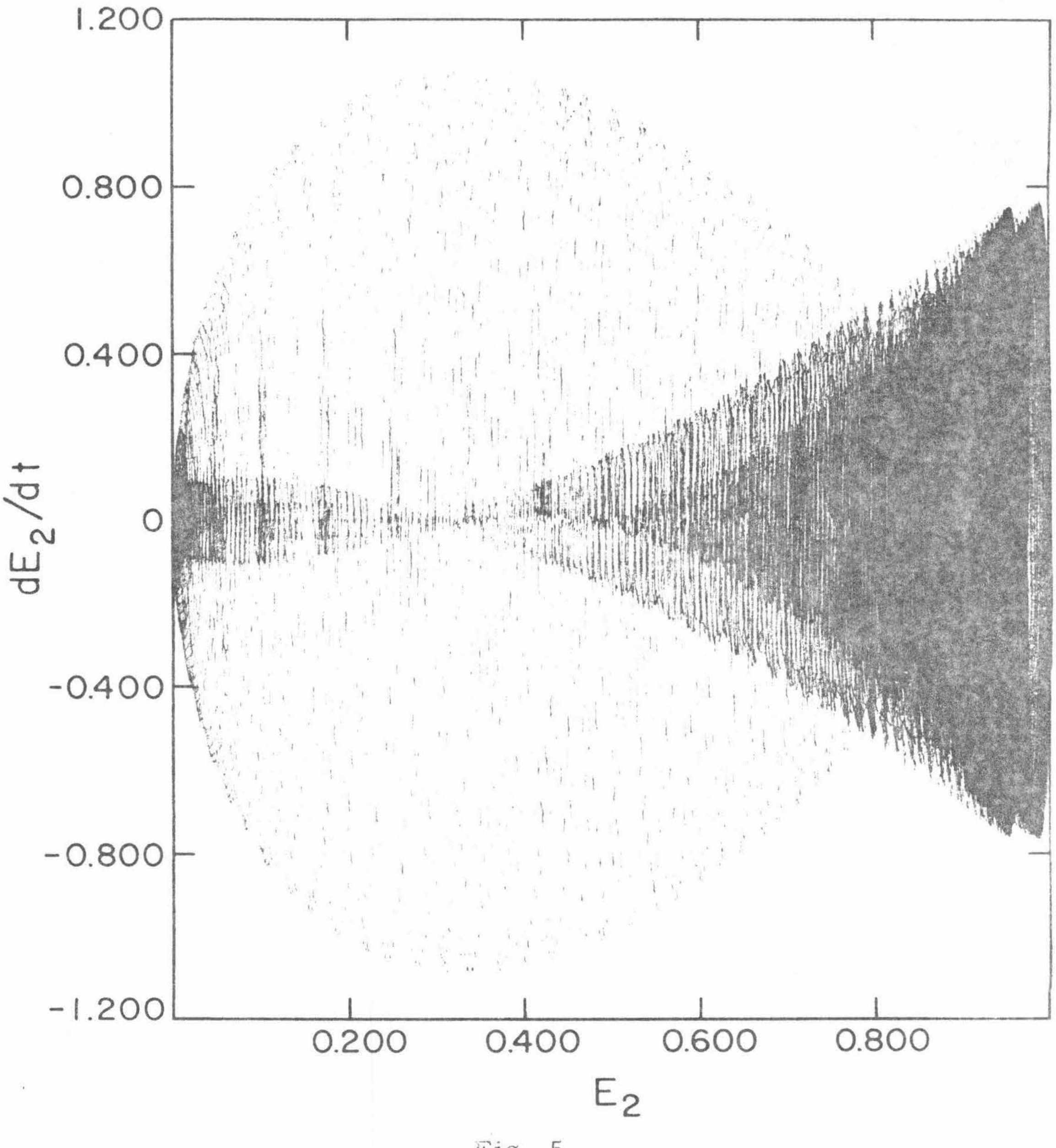

Fig. 5

Region of $\left(E_2, \frac{dE_2}{dt}\right)$ Phase Plane

Covered by Computer Solution of Equations (1)

With Initial Conditions a = 0, b = 1, c = 0, d = 1

Since $E_2^{(0)} = \frac{1}{3}$ we need $a_0^2 = \frac{4}{3}$ and $b_0^2 = \frac{1}{6}$. Substituting these values in (10b) we obtain the system

$$\sin(2\varphi_0(0) - \theta_0(0) = 0 \tag{11a}$$

$$\cos^2\varphi_0 = \frac{1}{8}\cos^2\theta_0 \tag{11b}$$

one solution of which occurs when:

$$a = 0.3382 \tag{12a}$$

$$b = 1.104 \tag{12b}$$

$$c = -0.3382 \tag{12c}$$

$$d = 0.457 \tag{12d}$$

These initial conditions should lead to $\frac{1}{3} - O(\epsilon) < E_2 < \frac{1}{3} + O(\epsilon)$ for at least a time $O(1/\epsilon)$ and perhaps for all time.

We performed a computer experiment, solving equations (1) with initial conditions (12) and $\epsilon = 0.1$ for a time $O(1/\epsilon^2)$ and the energy sharing was less than 0.1).260 $\leq E_2 \leq$ .380). The region of the E_2-E_2' phase plane covered by the motion is shown in Figure 6 .* We conclude that the approximations obtained by integrating equations (4) appear to lead to an estimate of the motion valid for times $O(1/\epsilon)$.

Another computer experiment was run, comparing the solutions for identical initial conditions but different ϵ's. For one case $\epsilon = 0.1$ and for the other $\epsilon = 0.01$. We find by comparing the resulting plots of E_2 against ϵt for the two cases that the two results are nearly identical. These curves are presented superimposed in Fig. 7. Thus we see that the slow time variation of the energy in a given mode

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* The reader can decide for himself what this one looks like.

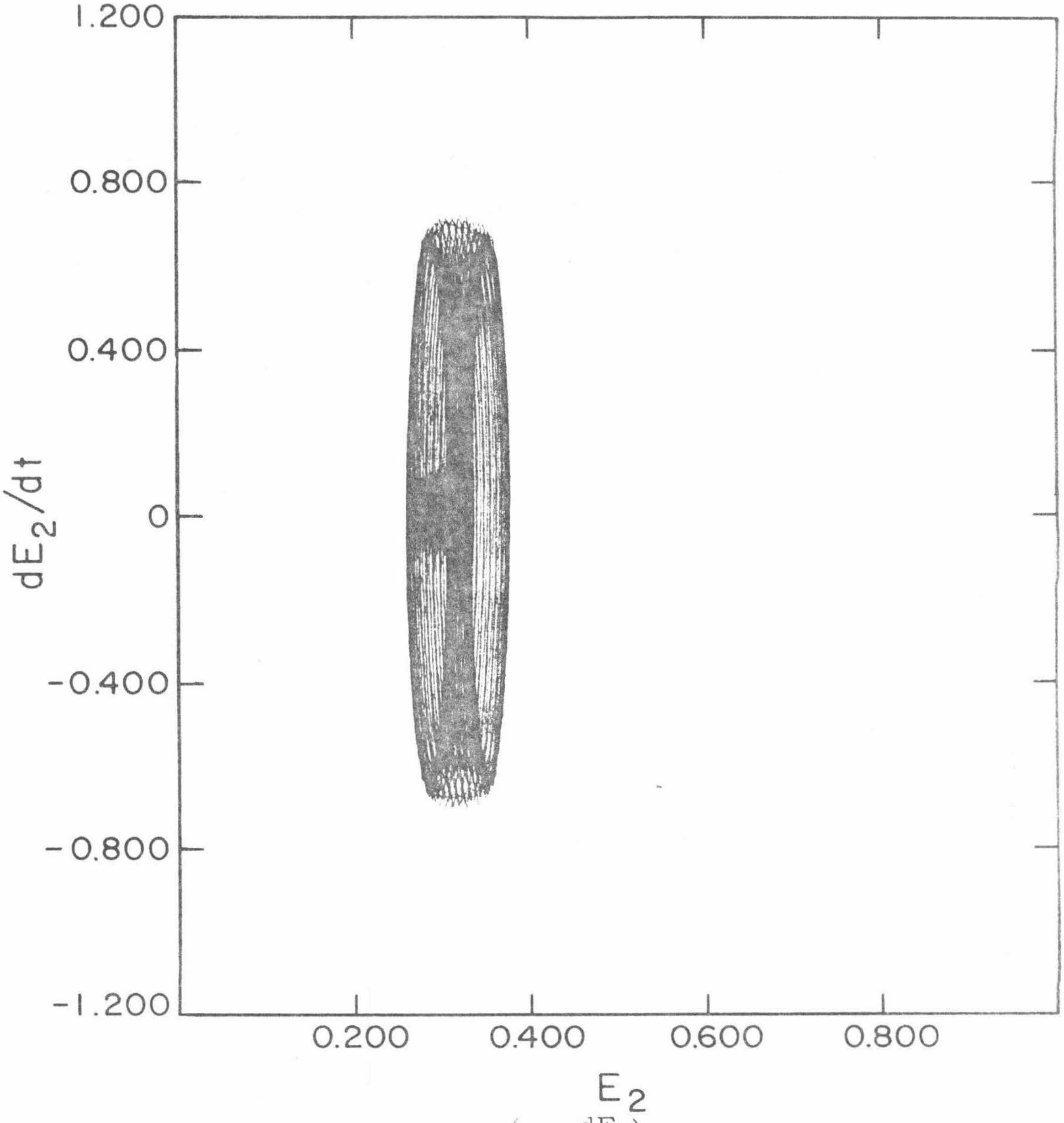

Fig. 6. Region of $\left(E_2, \frac{dE_2}{dt}\right)$ Phase Plane Covered by Computer Solution of Equations (1) with Initial Conditions Given by Equations (12)

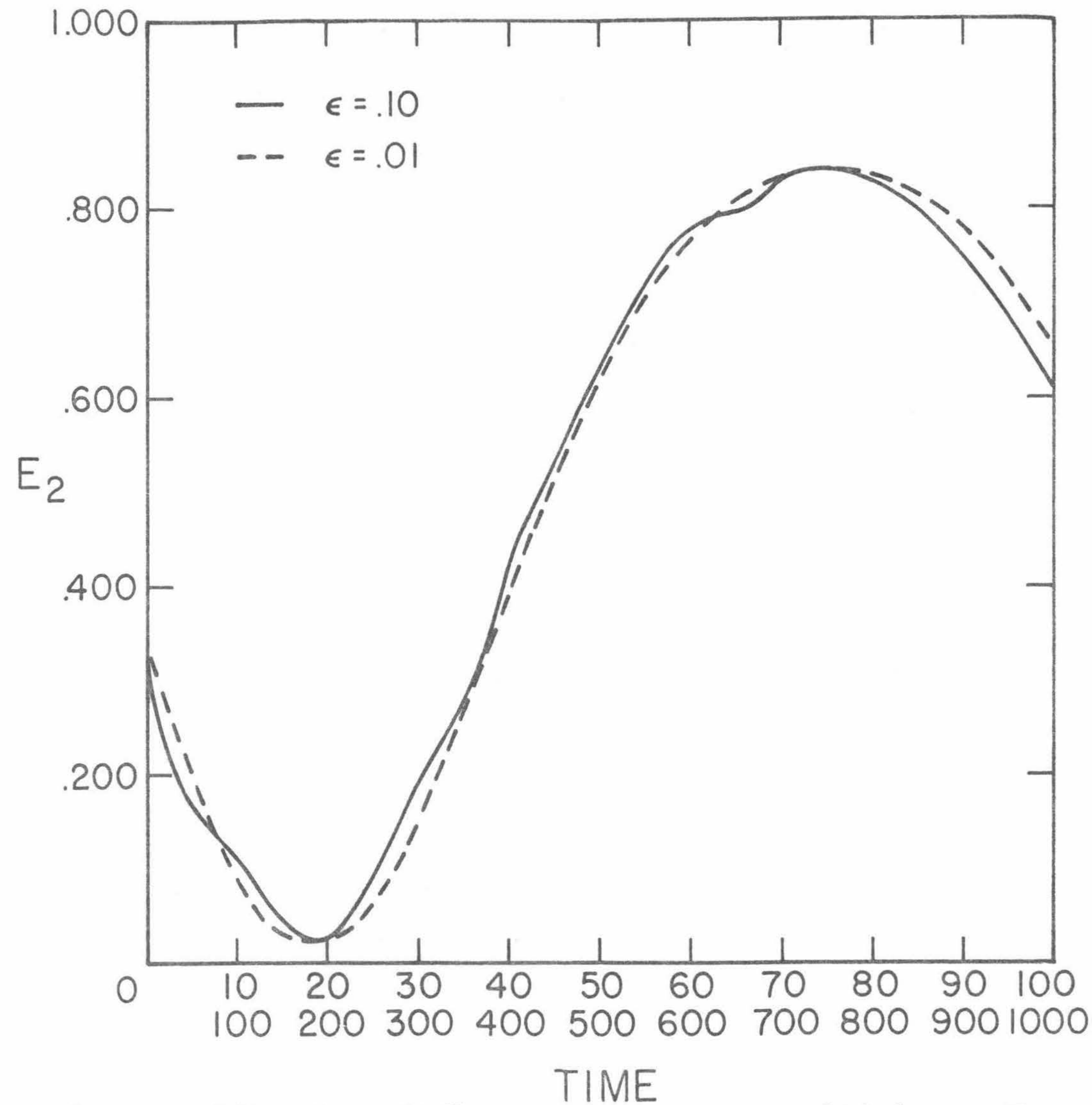

Fig. 7. Computer Solution of Equations (1) Showing Energy in Second Mode as a Function of Time for Two Cases with Identical Initial Conditions but Different Values of ϵ

is indeed a function of ϵt. One would expect these curves to drift slowly apart for times sufficiently large that the $\epsilon^2 t$ dependence becomes important.

Chapter VI

Summary and Conclusions

This chapter will summarize our results, attempt to answer the questions raised earlier in this thesis, mention some of the successes and difficulties of the N-timing procedure and consider some of the possible directions that this work suggests for future research.

N-timing appears to be a procedure which is applicable to each of the three types of conservative systems we have considered without modification of the general approach. For nonresonant problems the slow time variation of the solution due to the nonlinearity can be described by simple frequency shifts. The accuracy of our representation in predicting the long term "exact" dynamics of nonresonant systems is demonstrated by the magnitude of the error in the approximation given by our theory for the problem considered in Chapter III. After about 800 cycles of the faster oscillator the average error in the dynamical variables is about 0.1, which corresponds to an error in the shifted frequency of about one part in 5×10^4, and three terms in the energy series predict the average energy of each mode with error of less than one part in 10^3.

The weakly and strongly resonant cases studied in Chapters IV and V respectively were not motivated by consideration of simple spring-mass systems, but were chosen because they were relatively simple examples of the kinds of problems we wished to study. The weakly resonant problem yields what we believe is a new and interesting result by explicitly demonstrating the dependence of the analytic form of the solution upon the relative amplitudes of the two oscillators. This example shows at the same time that the small divisor problem presents no special conceptual difficulty within the framework of the N-timing formalism.

The strongly resonant case considered in Chapter V was studied primarily to demonstrate that that problem can also be discussed in terms of a multiple time scale expansion. It is unfortunate that the functions describing the ϵt variation of the O(1) solutions are sufficiently complicated that we have thus far been unable to carry out calculations of the $\epsilon^2 t$ variations of the O(1) terms. However, the validity of our approach is confirmed by two computer experiments. The first of these shows that our theory is capable of predicting within the allowable error initial conditions which will lead to maximum and minimum energy sharing - the second, that for a time $O(1/\epsilon)$ the energy of each mode indeed varies as a function of ϵt as our theory predicts. The invariants of the motion we found for this case are in agreement with those obtained by other authors using various methods (e.g., Whittaker 1916: Ford and Waters 1963: Jackson 1963b: Kronauer and Musa 1966); however, most of these authors handled this problem in different ways from the ways in which they handled nonresonant problems and to our knowledge none actually solved for the dynamics of the system.

We have found that two oscillators with weak nonlinear coupling share energy significantly within a "reasonable" length of time only if the ratio of the frequencies of the uncoupled system is near certain special values and the initial conditions belong to a special class. However, it should be noted that constructive procedures such as

N-timing have certain limitations with regard to answering the question of whether the oscillators will "ever" share energy. N-timing is intended to solve problems where t_{max} and the allowable error are specified in advance; that is, it can at best be expected to provide solutions valid to a certain specified order for a certain specified time. Therefore, none of our calculations can be considered proofs of the non-existence of energy sharing even where our results indicate that energy does not appear to be shared. The strongest statement we can make is that the nonresonant and weakly resonant examples considered in Chapters III and IV respectively will not exhibit O(1) energy sharing within any length of time where a uniform expansion* of the form we have chosen is a valid representation of their solutions.

To see this, consider for example the solution of the non-resonant problem discussed in Chapter III. The energy of the first mode is, to O(1), $\frac{1}{2}a_0^2$, and of the second mode, $3/2\ b_0^2$. Thus there can be O(1) energy sharing only if a_0 and b_0 vary with time. Suppose a_0 and b_0 vary on some t_k time scale; i.e., $a_0 = a_0(\epsilon^k t)$, $b_0 = b_0(\epsilon^k t)$. Recall to O(1) we have:

$$x(t) = a_0 \cos\left[\, t_0 - \varphi_0\,(t_2\,, \ldots)\right]$$

$$y(t) = b_0 \cos\left[\sqrt{3}\, t_0 - \theta_0(t_2\,, \ldots)\right]$$

Recalling also equations (39) for $\varphi_0(t_2)$ and $\theta_0(t_2)$, we see that the expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ involve terms like:

*A uniform expansion, for our purposes, will mean an expansion having the property that the term of each order remains of that order and never becomes larger.

$$\text{const} \quad \epsilon^{k+2} \, a_0^2 \, \frac{\partial a_0}{\partial t_k} \, t \sin\left[\, t - \varphi_0(t_2, \ldots)\right]$$

and

$$\text{const} \quad \epsilon^{k+2} \, a_0^2 \, \frac{\partial b_0}{\partial t_k} \, t \sin\left[\sqrt{3}\, t - \theta_0(t_2, \ldots)\right]$$

respectively. Unless such terms add up to zero, which appears unlikely, the $O(\epsilon^{k+2})$ terms of the expansions for $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ would not be uniform. Thus the condition of uniformity requires that a_0, b_0 and thus to $O(1)$ the energy in each mode be constant on all relevant time scales.

In Chapter I we raised the question of whether the change in the analytic form of the adelphic integrals as the frequency ratio varies from rational to irrational values is significant in describing the motion. By studying systems similar to those considered here but with slightly shifted frequency ratios (e.g., $\frac{\omega_2}{\omega_1} \rightarrow \frac{\omega_2}{\omega_1} + O(\epsilon)$), one finds that the analytic form of the motion depends rather smoothly on the frequencies. For instance, in the strongly resonant system studied in Chapter I, if one shifts ω_2 from 2 to $2 + O(\epsilon)$, significant energy sharing is still obtained, and the amount of energy sharing decreases smoothly as ω_2 is varied away from 2. This behavior was also exhibited in a series of computer calculations done by Ford and Waters (1963). The existence of a band width for resonance and the generally smooth dependence of the analytic form of the motion upon the frequency ratio suggest that the pathological variation of the analytic form of Whittakers' adelphic integrals is probably due to the fact that Whittaker ignored the small divisor problem and did not require his expansions to be uniform.

The N-timing procedure has turned out to be effective for attacking the various types of conservative weakly nonlinear problems considered in this thesis and appears to eliminate the problem of small divisors in a direct and apparently meaningful way. It predicts accurately the exact dynamics of such systems for very long times. Perhaps most important, it provides a conceptually straight-forward framework within which one can seek bounded solutions of a class of problems containing small parameters which is probably much broader than that considered in this thesis and includes damped, conservative and singular systems of ordinary and in some cases partial differential equations.

To pay for these advantages we find that for the present we are required to accept certain technical difficulties of the method. The most obvious of these is that the number of terms grows rapidly as we carry out calculations to smaller and smaller order. Part of this difficulty can be eliminated by leaving out the homogeneous solutions of the equations of lower order, but we have found it useful to include them for two reasons: first, they enable one to write down the initial conditions directly from those given with the problem without requiring the solution of possibly complicated algebraic equations, and second, slow variations of the smaller order homogeneous terms may differ in analytic form from one another, as in the example in Chapter IV.

The second and more significant difficulty is that essentially no proofs are presently available of the validity of solutions obtained by multiple variable procedures (except the empirical evidence that such solutions appear to be correct) or the ability of the procedure to yield uniform approximations, in principle, to arbitrary order. This difficulty is not just a mathematical fine point desirable for logical completeness, but can present real and as yet unresolved difficulties, such as the one mentioned at the end of Appendix B in carrying the weakly resonant problem beyond a certain order. The resolution of such difficulties would be an interesting and useful direction for further study.

There are several other classes of problems which this thesis suggests for further work. One obvious extension would be to continue solutions beyond the point where complicated functions begin to arise (e.g., to solve for the $\epsilon^2 t$ variation of the $O(1)$ solution of the strongly resonant case). Another would be to attempt to analyze systems of three or more oscillators where the frequencies are such as to lead to strong resonance. A third and probably most interesting would be to apply N-timing to physical situations such as, for example, classical formulations of nonlinear optics and orbital mechanics where nonlinear problems with small parameters tend to arise.

Appendix A

The Significance of Ergodicity in Statistical Mechanics

The fundamental problem of statistical mechanics is to predict the macroscopic behavior of dynamical systems which have so many degrees of freedom that their equations of motion cannot be solved. Statistical techniques are useful for this problem in that they appear to give correct answers when we are able to do the mathematics, but their applicability is based on an assumption which has not yet been shown to be valid for any realistic system. The present section will discuss briefly the significance of this assumption, the so-called ergodic hypothesis.

For purposes of illustration, let us focus attention on a system which is typical of those to which statistical mechanics is usually applied. Consider a classical-mechanical system S consisting of a large number N of particles, each particle having $2m$ degrees of freedom, contained in a box of volume V. Suppose we know how the particles interact with the walls and with one another, and would like to predict the functional dependence of certain macroscopic quantities describing the system upon other such quantities.

In most treatments of statistical mechanics, it is assumed either explicitly or implicitly that the system S is interacting in some way with another system. Therefore, let us suppose S is interacting "weakly" with some much larger system S' in such a way that the following conditions hold:

a) The total energy

$$H_T\left[\vec{p}_S, \vec{q}_S, \vec{p}_{S'}, \vec{q}_{S'}\right] = H_S(\vec{p}_S, \vec{q}_S) + H_{S'}(\vec{p}_{S'}, \vec{q}_{S'}) + H_{int}(\vec{p}_S, \vec{q}_S, \vec{p}_{S'}, \vec{q}_{S'})$$
$$= \text{constant} = E_T \tag{1}$$

b) H_{int} is sufficiently small that each of the systems can be thought of as having instantaneously its own "private" energy, depending only on the values of its own dynamical variables.

c) S and S' are free to exchange energy but not particles.

d) S' is supposed to be so much "larger" than S that its macroscopic variables are not significantly affected by its interaction with S.

If S' has a total of 2ℓ degrees of freedom we can visualize a $2\ell + 2m$ dimensional phase space whose coordinates correspond to the degrees of freedom of S and S'. Each state $(\vec{q}_T, \vec{p}_T)$ of the combined systems $S_T = S + S'$ corresponds to a single point in this phase space, and as the dynamical processes associated with H_T take place, the points corresponding to the successive states through which S_T passes trace out a curve $s_T(\vec{q}_T(t), \vec{p}_T(t))$ on the $2\ell + 2mN-1$ dimensional "surface" corresponding to H_T = constant.

Let us now consider the curve $s_S(\vec{q}_S(t), \vec{p}_S(t))$, which we define to be the projection of s_T on the $2Nm$ dimensional reduced manifold whose coordinates correspond to the degrees of freedom of S. A complete solution of the equations of motion of S_T would give us the functional forms of the $\vec{q}_S(t)$ and $\vec{p}_S(t)$ and we could then compute the time averages of functions of the $\vec{q}_S$ and $\vec{p}_S$. However, for the systems in which we are interested, let us suppose this is impossible, and in order to get any answers it is necessary to make another assumption.

This is where the ergodic hypothesis comes in.

The ergodic hypothesis amounts essentially to the assumption that for "almost all" trajectories*, the proportion of the time s_T spends in a region of the surface $E_T = H_T$ is directly proportional to the "area" of that region. That is:

$$\frac{dt_{inA}}{t} = \frac{dA}{A}$$

If the ergodic hypothesis applies to S_T, we must have:

$$\overline{f(\vec{q}_S, \vec{p}_S)} = \lim_{t\to\infty} \frac{1}{t} \int_0^t f\left[\vec{p}_S(t), \vec{q}_S(t)\right] dt = \frac{\int f(\vec{p}_S, \vec{q}_S) d\vec{p}_S d\vec{q}_S d\vec{p}_{S'} d\vec{q}_{S'} \delta(E_T - H_T)}{\int d\vec{p}_S d\vec{q}_S d\vec{p}_{S'} d\vec{q}_{S'} \delta(E_T - H_T)} \tag{2}$$

Furthermore, one can also show that if S' is a perfect gas $\left(H_{S'} = \frac{1}{2}\sum_{i=1}^{\ell/3} \frac{(\vec{p}_i)^2}{2m_i}\right)$ with a very large number of particles,

$$\lim_{\ell\to\infty} \int d\vec{p}_{S'} d\vec{q}_{S'} \delta(E - H_T) = \exp\{-\beta H_S(\vec{p}_S, \vec{q}_S)\} \tag{3}$$

where $1/\beta$ is the average energy per particle of S'. Thus we must have:

$$\overline{f(\vec{p}_S, \vec{q}_S)} = \frac{\int f(\vec{p}_S, \vec{q}_S) \exp\{-\beta H_S(\vec{p}_S, \vec{q}_S)\} d\vec{p}_S d\vec{q}_S}{\int \exp\{-\beta H_S(\vec{p}_S, \vec{q}_S)\} d\vec{p}_S d\vec{q}_S} \tag{4}$$

The distribution function $\exp\{-\beta H_S(\vec{p}_S, \vec{q}_S)\}$ is called the canonical distribution, and the integral in the denominator of the right hand side of equation (4) is known as the canonical partition function.

As we have seen from the foregoing discussion, equation (4) can be applied only if the ergodic hypothesis is valid for the system S_T.

*Except possibly for a set of measure zero.

There are in principle at least two methods by which one can decide whether it is valid for a particular system S_T. One way would be to solve completely the equations of motion of S_T, but that would bring us back to the problem we had hoped to avoid through the use of statistical techniques. The other possible approach is to show that S_T satisfies the hypotheses of the ergodic theorems of Birkhoff.

Birkhoff's theorem applied to S_T states essentially that on the energy shell $E_T = H_T(\vec{p}_T, \vec{q}_T)$, time averages of dynamical quantities exist and are equal to phase averages for "almost all" trajectories if and only if the energy shell $E_T = H_T$ is metrically indecomposable. Metrical indecomposability means essentially that the surface $E_T = H_T$ cannot be divided into two or more regions, each of non-zero measure, such that trajectories starting in one region cannot enter the other. Physically, metrical indecomposability implies that there is no function $g(\vec{p}_T, \vec{q}_T)$ = constant except for H_T (Khinchin, pp. 55-6). As one can guess, it is an extremely difficult problem to show that any realistic system has metrically indecomposable energy surfaces (although it has been demonstrated by Sinai (1963) for a system of rigid spheres).

The intractability of realistic systems to application of Birkhoff's theorem has led various workers (Fermi, Pasta and Ulam 1955: Jackson 1963a, 1963b: Ford and Waters 1963: Northcote and Potts 1964) to study systems which are sufficiently simple that one can in some sense solve their equations of motion. For such a purpose, it is sufficient to study "closed" systems since, referring to our example, it is really the closed system S_T which is required to be ergodic.

Most of these workers have studied one dimensional chains of coupled oscillators, since these are relatively simple systems to analyze. Work with polynomial coupling and identical springs (FPU systems) has led to the conclusion that such systems are not ergodic; most of the energy remains in the low modes if the system starts with the energy in the low modes. However, Northcote and Potts performed a computer experiment, studying a model consisting of hard rods of finite diameter connected by linear springs, and found that energy was indeed shared equally among the modes. This model is more realistic than polynomial coupling for gases at reasonably high temperatures where the interactions look like collisions; however,one would expect the polynomial model to correspond more or less to solids at relatively low temperatures. Certainly the results of Northcote and Potts imply that the form taken for the coupling is an important determinant of ergodicity. In addition, we should consider the possibility that the nature of the model is important - oscillator models do not allow the particles to migrate from one lattice site to another in the polynomial model - the particles are drawn back to their original equilibrium positions even if they have exchanged positions by passing through one another.

Appendix B

Algebraic Details of and Comments on Chapter IV

Terms on the right hand sides of equations IV-(6) which can cause non-uniformities in the solutions are terms from (6a) whose frequencies to O(1) are 1 and terms from (6b) whose frequencies to O(1) are 3. To keep our solutions uniform we must therefore set the sum of these terms from each equation separately equal to zero. Thus:

$$
\begin{aligned}
& 2\frac{\partial a_0}{\partial t_3}\sin(t_0-\varphi_0)-2a_0\frac{\partial\varphi_0}{\partial t_3}\cos(t_0-\varphi_0)+2\frac{\partial\varphi_0}{\partial t_1}\frac{\partial\varphi_0}{\partial t_2}a_0\cos(t_0-\varphi_0) \\
& +2\frac{\partial a_1}{\partial t_2}\sin(t_0-\varphi_1)-2a_1\frac{\partial\varphi_1}{\partial t_2}\cos(t_0-\varphi_1)+\left(\frac{\partial\varphi_1}{\partial t_1}\right)^2 a_1\cos(t_0-\varphi_1) \\
& +2\frac{\partial a_2}{\partial t_1}\sin(t_0-\varphi_2)-2a_2\frac{\partial\varphi_2}{\partial t_1}\cos(t_0-\varphi_2)-\frac{b_0^2}{2}a_2\cos(t_0-\varphi_2) \\
& +\frac{1}{96}a_0 b_0^4\left\{\frac{19}{576}a_0^2-\frac{1}{32}b_0^2\right\}\cos(t_0-\varphi_0)+\frac{1}{192}a_0 b_0^4\left\{\frac{5}{1152}a_0^2+\frac{1}{64}b_0^2\right\}\cos(t_0-\varphi_0) \\
& -\frac{a_1 b_0^4}{384}\cos(t_0-\varphi_1)-\frac{a_0 b_0^3 b_1}{192}\cos(t_0-\varphi_0+\theta_1-\theta_0)-\frac{a_1 b_0^4}{768}\cos(t_0-\varphi_1) \\
& -\frac{a_0 b_0^3 b_1}{384}\cos(t_0-\varphi_0+\theta_0-\theta_1)-a_1 b_0 b_1\cos(t_0-\varphi_1)\cos(\theta_1-\theta_0) \\
& -\frac{a_0 b_0^3 b_1}{192}\cos(t_0-\varphi_0+\theta_0-\theta_1)-\frac{a_0 b_0^3 b_1}{384}\cos(t_0-\varphi_0+\theta_1-\theta_0) \\
& +\frac{a_0^3 b_0^4}{64\cdot 96}\cos(t_0-\varphi_0)+\frac{a_0^2 b_0^2 a_1}{64}\cos(t_0-\varphi_0+\varphi_1-\varphi_0)-\frac{a_0^3 b_0^4}{128\cdot 192}\cos(t_0-\varphi_0) \\
& -\frac{a_1 a_0^2 b_0^2}{128}\cos(t_0-\varphi_0+\varphi_1-\varphi_0)-\frac{a_0 b_1^2}{2}\cos(t_0-\varphi_0)-\frac{a_0^5 b_0^2}{2048}\cos(t_0-\varphi_0) \\
& \underline{-\frac{a_0^5 b_0^2}{4096}\cos(t_0-2\theta_0+5\varphi_0)}-\frac{a_0^5 b_0^2}{8192}\cos(t_0-\varphi_0)+\frac{a_0^3 b_0 b_1}{64}\cos(t_0-\varphi_0+\theta_0-\theta_1) \\
& -\frac{a_0^3 b_0 b_1}{128}\cos(t_0-\varphi_0+\theta_1-\theta_0)-a_0 b_0 b_2\cos(t_0-\varphi_0)\cos(\theta_2-\theta_0) \\
& -\left\{\frac{a_0^5 b_0^2}{1536}+\frac{5a_0^3 b_0^4}{3072}\right\}\cos(t_0-\varphi_0)\underline{-\frac{a_0^5 b_0^2}{2048}\cos(t_0-2\theta_0+5\varphi_0)} \\
& +\frac{1}{32}\left\{\frac{1}{192}a_0^4 b_0+\frac{29}{384}a_0^2 b_0^3\right\}a_0 b_0\cos(t_0-\varphi_0)+\frac{a_0^2 b_0^2 a_1}{32}\cos(t_0-\varphi_1) \\
& +\frac{a_0^3 b_0 b_1}{64}\cos(t_0-\varphi_0+\theta_1-\theta_0)-\frac{a_0^2 b_0^2 a_1}{64}\cos(t_0-\varphi_1)-\frac{a_0^3 b_0 b_1}{128}\cos(t_0-\varphi_0+\theta_0-\theta_1) \\
& =0 \qquad\qquad (1a)
\end{aligned}
$$

$$6\,\frac{\partial b_0}{\partial t_3}\sin(3t_0-\theta_0) - 6b_0\,\frac{\partial \theta_0}{\partial t_3}\cos(3t_0-\theta_0) + 6\,\frac{\partial b_1}{\partial t_2}\sin(3t_0-\theta_1)$$

$$-6b_1\,\frac{\partial \theta_1}{\partial t_2}\cos(3t_0-\theta_1) + \frac{a_0^4}{144}b_1\cos(3t_0-\theta_1) + 6\,\frac{\partial b_2}{\partial t_1}\sin(3t_0-\theta_2)$$

$$-6b_2\,\frac{\partial \theta_2}{\partial t_1}\cos(3t_0-\theta_2) - \frac{a_0^2 b_0}{36}\left\{\frac{25}{2304}a_0^4 - \frac{1}{128}a_0^2 b_0^2 - a_0 A_{1c}\right\}\cos(3t_0-\theta_0)$$

$$-a_0 b_0 a_2\cos(3t_0-\theta_0)\cos(\varphi_2-\varphi_0) + \frac{1}{48}a_0 b_0\left\{\frac{19}{576}a_0^3 b_0^2 - \frac{1}{32}a_0 b_0^4\right\}\cos(3t_0-\theta_0)$$

$$+\frac{1}{96}a_0 b_0\left\{\frac{5}{1152}a_0^3 b_0^2 + \frac{1}{64}a_0 b_0^4\right\}\cos(3t_0-\theta_0) - \frac{a_1 a_0 b_0^3}{192}\cos(3t_0-\theta_0+\varphi_1-\varphi_0)$$

$$-\frac{a_0^2 b_0^2 b_1}{96}\cos(3t_0-\theta_1) - \frac{a_1 a_0 b_0^3}{38}\cos(3t_0-\theta_0+\varphi_0-\varphi_1) - \frac{a_0^2 b_0^2 b_1}{192}\cos(3t_0-\theta_1)$$

$$-\frac{a_1^2 b_0}{2}\cos(3t_0-\theta_0) - \frac{a_0^2 b_0^5}{96\cdot 192}\cos(3t_0-\theta_0) - \frac{a_0^2 b_0^5}{192\cdot 384}\cos(3t_0-\theta_0)$$

$$-\frac{a_1 a_0 b_0^3}{384}\cos(3t_0-\theta_0+\varphi_1-\varphi_0) - \frac{a_1 a_0 b_0^3}{192}\cos(3t_0-\theta_0+\varphi_0-\varphi_1)$$

$$-a_0 a_1 b_1\cos(3t_0-\theta_1)\cos(\varphi_1-\varphi_0) + \frac{a_1 a_0^3 b_0}{64}\cos(3t_0-\theta_0+\varphi_0-\varphi_1)$$

$$-\frac{a_1 a_0^3 b_0}{128}\cos(3t_2-\theta_0+\varphi_1-\varphi_0) - \frac{a_0^2 b_0^2 b_1}{192}\cos(3t_0-2\theta_0+\theta_1) - \frac{a_0^4 b_0^3}{6144}\cos(3t_0-\theta_0)$$

$$-\frac{a_0^2 b_0^2 b_1}{384}\cos(3t_0-2\theta_0+\theta_1) - \frac{a_0^4 b_0^3}{64\cdot 384}\cos(3t_0-\theta_0) - \frac{a_0^2 b_2}{2}\cos(3t_0-\theta_2)$$

$$-\frac{a_0^4 b_0}{3072}\left\{a_0^2 + \frac{5}{2}b_0^2\right\}\cos(3t_0-\theta_0) - \underline{\frac{a_0^6 b_0}{4096}\cos(3t_0+\theta_0-6\varphi_0)}$$

$$+\frac{a_0^4 b_0}{64}\left\{\frac{a_0^2}{192} + \frac{29}{384}b_0^2\right\}\cos(3t_0-\theta_0) + \frac{a_0^3 b_0 a_1}{64}\cos(3t_0-\theta_0+\varphi_1-\varphi_0) + \frac{a_0^4 b_1}{128}\cos(3t_0-\theta_1)$$

$$-\frac{a_0^3 b_0 a_1}{128}\cos(3t_0-\theta_0+\varphi_0-\varphi_1) - \frac{a_0^4 b_1}{256}\cos(3t_0-\theta_1) = 0 \qquad (1b)$$

In equation (6a) set the coefficients of $\sin(t_0-\varphi_0)$ and $\cos(t_0-\varphi_0)$ separately equal to zero and in (6b) set the coefficients of $\sin(3t_0-\theta_0)$ and $\cos(3t_0-\theta_0)$ separately equal to zero.

For convenience, let

$$\begin{aligned} A_{1C} &= a_1\cos(\varphi_1-\varphi_0) & A_{1S} &= a_1\sin(\varphi_1-\varphi_0) \\ A_{2C} &= a_1\cos(\varphi_2-\varphi_0) & A_{2S} &= a_2\sin(\varphi_2-\varphi_0) \\ B_{1C} &= b_1\cos(\theta_1-\theta_0) & B_{1S} &= b_1\sin(\theta_1-\theta_0) \\ B_{2C} &= b_2\cos(\theta_2-\theta_0) & B_{2S} &= b_2\sin(\theta_2-\theta_0) \end{aligned} \tag{2}$$

We then obtain from the coefficients of:

$$\sin(t_0-\varphi_0)=0 \Longrightarrow \frac{\partial A_{2C}}{\partial t_1}+\frac{\partial A_{1C}}{\partial t_2}+\frac{\partial a_0}{\partial t_3} = -\frac{3}{8192}a_0^5 b_0^2\sin\{6\varphi_0-2\theta_0\} \tag{3a}$$

$$\begin{aligned}\cos(t_0-\varphi_0)=0 \Longrightarrow \frac{\partial A_{2S}}{\partial t_1}+\frac{\partial A_{1S}}{\partial t_2}+a_0\frac{\partial \varphi_0}{\partial t_3} &= \frac{1}{128}a_0 b_0\{a_0^2+15b_0^2\}B_{1C}+\frac{1}{128}a_0^2 b_0^2 A_{1C} \\ &\quad -\tfrac{1}{4}a_0 B_{1C}^2-\tfrac{1}{4}a_0 B_{1S}^2-\tfrac{1}{2}a_0 b_0 B_{2C} \\ &\quad -\frac{3}{8192}a_0^5 b_0^2\cos(6\varphi_0-2\theta_0) \\ &\quad -\frac{a_0 b_0^2}{8192}\{61b_0^4+3a_0^2 b_0^2+\tfrac{9}{2}a_0^4\}\end{aligned} \tag{3b}$$

$$\sin(3t_0-\theta_0)=0 \Longrightarrow \frac{\partial B_{2C}}{\partial t_1}+\frac{\partial B_{1C}}{\partial t_2}+\frac{\partial b_0}{\partial t_3} = \frac{a_0^6 b_0}{24576}\sin(6\varphi_0-2\theta_0) \tag{3c}$$

$$\begin{aligned}\cos(3t_0-\theta_0)=0 \Longrightarrow \frac{\partial B_{2S}}{\partial t_1}+\frac{\partial B_{1S}}{\partial t_2}+b_0\frac{\partial \theta_0}{\partial t_3} &= -\frac{1}{384}a_0^2 b_0^2 B_{1C}+\frac{a_0 b_0}{3456}\{25a_0^2-9b_0^2\}A_{1C} \\ &\quad -\frac{1}{12}b_0 A_{1C}^2-\frac{1}{12}b_0 A_{1S}^2-\frac{1}{6}a_0 b_0 A_{2C} \\ &\quad -\frac{181}{486\cdot 4096}a_0^6 b_0+\frac{17}{6(192)^2}a_0^4 b_0^3-\frac{41}{36\cdot 2048}a_0^2 b_0^5 \\ &\quad -\frac{a_0^6 b_0}{24576}\cos(6\varphi_0-2\theta_0)\end{aligned} \tag{3d}$$

Note that $6\varphi_0-2\theta_0 = \frac{1}{6}(a_0^2-9b_0^2)t_1+6\varphi_0^{(2)}t_2)-2\theta_0^{(2)}(t_2)$ so that equations (3a) and (3c) are identical with equations IV-(17). Notice that equations (3b) and (3d) can be solved for the t_1 dependence of A_{2s} and B_{2s}, but not for the t_2 dependence of A_{1s} and B_{1s} without knowing the t_2 dependence of $A_{2c}^{(2)}$ and $B_{2c}^{(2)}$, which comes from the $O(\epsilon^4)$ equations.

For convenience, suppose we have case A, so that $(6\varphi_0 - 2\theta_0)$ depends on t_1. The condition that B_{2c} and B_{2s} be uniform requires that the terms not depending on t_1 add up to zero. Thus, rewriting (3b) and (3d), leaving out terms depending on t_1 we obtain:

$$\frac{\partial A_{1s}}{\partial t_2} + a_0 \frac{\partial \varphi_0}{\partial t_3} = \frac{1}{128} a_0 b_0 (a_0^2 + 15 b_0^2) B_{1c} + \frac{1}{128} a_0^2 b_0^2 A_{1c} - \frac{1}{4} a_0 B_{1c}^2 - \frac{1}{4} a_0 B_{1s}^2$$

$$- \frac{1}{2} a_0 b_0 B_{2c}^{(2)} - \frac{a_0^2 b_0^2}{8192} \left\{ 61 b_0^4 + 3 a_0^2 b_0^2 + \frac{9}{2} a_0^4 \right\} \qquad (4a)$$

$$\frac{\partial B_{1s}}{\partial t_2} + b_0 \frac{\partial \theta_0}{\partial t_3} = - \frac{1}{384} a_0^2 b_0^2 B_{1c} + \frac{a_0 b_0}{3456} \left\{ 25 a_0^2 - 9 b_0^2 \right\} A_{1c} - \frac{1}{12} b_0 A_{1c}^2$$

$$- \frac{1}{12} b_0 A_{1s}^2 - \frac{1}{6} a_0 b_0 A_{2c}^{(2)} - \frac{181}{486 \cdot 4096} a_0^6 b_0 + \frac{17}{6 \cdot (492)^2} a_0^4 b_0^3$$

$$- \frac{41}{36 \cdot 2048} a_0^2 b_0^5 \qquad (4b)$$

The following quantities are known not to depend on t_2:

$$a_0 , \; b_0 , \; A_{1c}, \; B_{1c}$$

Our experience suggests that $A_{2c}^{(2)}$ and $B_{2c}^{(2)}$ will also be independent of t_2, and this will be assumed to avoid writing out the terms of the next higher order. Then the only terms depending on t_2 in equations (4) are:

$$\frac{\partial A_{1s}}{\partial t_2} , \; B_{1s}^2 , \; \frac{\partial B_{1s}}{\partial t_2} , \; A_{1s}^2$$

Lumping all other terms in equations (4a) and (4b) into $c_1(t_3) = c_1$ and $c_2(t_3) = c_2$ respectively, we obtain:

$$\frac{dA_{1s}}{dt_2} = c_1 - \frac{1}{4} a_0 B_{1s}^2 \tag{5a}$$

$$\frac{dB_{1s}}{dt_2} = c_2 - \frac{1}{12} b_0 A_{1s}^2 \tag{5b}$$

a pair of coupled nonlinear first order differential equations.

It would be obvious how to proceed if these equations had a single stable solution and the condition for stability of A_{1s} and B_{1s}* were describable in terms of a pair of algebraic equations involving c_1 and c_2. However, life is not that simple, as we can see by dividing (5b) by (5a).

We obtain:

$$\frac{dB_{1s}}{dA_{1s}} = \frac{c_2 - \frac{1}{12} b_0 A_{1s}^2}{c_1 - \frac{1}{4} a_0 B_{1s}^2} \tag{6}$$

Examining this equation in the (A_{1s}, B_{1s}) phase plane, we find that for $c_2 > \frac{1}{12} b_0 A_{1s}^2$ and $c_1 > \frac{1}{4} a_0 B_{1s}^2$ there is a continuum of solutions which are apparently stable centered about:

$$A_{1s} = \pm \sqrt{\frac{12c_2}{b_0}}$$

$$B_{1s} = \mp \sqrt{\frac{4c_1}{a_0}}$$

This means that there is a range of values of c_1 and c_2 which will yield stable solutions, and we have two difficulties. The first is that it is not clear what values of c_1 and c_2 to choose, so that $\varphi_0^{(3)}$ and $\theta_0^{(3)}$

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*Needed to maintain uniformity of the $O(\epsilon)$ terms in the expansion of the solutions of the original equations.

are not uniquely determined, and the second is that even if we can decide on some values for c_1 and c_2, the resulting functions describing the t_2 variation of A_{1s} and B_{1s} will be difficult to represent analytically.

One obvious procedure for choosing particular special values for c_1 and c_2 is to assume $\frac{dA_{1s}}{dt_2} = \frac{dB_{1s}}{dt_2} = 0$. This choice is necessary when a similar difficulty arises in the case of a single nonlinear oscillator and seems like the most reasonable thing to try in the present example. A detailed consideration of this difficulty will, however, have to await further study, since the algebra involved in the analysis of further terms is extremely complicated.

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