WAVE PROPAGATION IN AN INHOMOGENEOUS MEDIUM

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ABSTRACT

This paper is in two parts. In the first part we give a qualitative study of wave propagation in an inhomogeneous medium principally by geometrical optics and ray theory. The inhomogeneity is represented by a sound-speed profile which is dependent upon one coordinate, namely the depth; and we discuss the general characteristics of wave propagation which result from a source placed on the sound channel axis. We show that our mathematical model of the sound-speed in the ocean actually predicts some of the behavior of the observed physical phenomena in the underwater sound channel. Using ray theoretic techniques we investigate the implications of our profile on the following characteristics of SOFAR propagation: (i) the sound energy traveling further away from the axis takes less time to travel from source to receiver than sound energy traveling closer to the axis, (ii) the focusing of sound energy in the sound channel at certain ranges, (iii) the overall ray picture in the sound channel.

In the second part a more penetrating quantitative study is done by means of analytical techniques on the governing equations. We study the transient problem for the Epstein profile by employing a double transform to formally derive an integral representation for the acoustic pressure amplitude, and from this representation we obtain several alternative representations. We study the case where both source and receiver are on the channel axis and greatly separated. In particular we verify some of the earlier results derived by ray theory and obtain asymptotic results for the acoustic pressure in the far-field.

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Introduction

This paper is in two parts. In the first part we give a qualitative study of wave propagation in an inhomogeneous medium by geometrical optics and ray theory. In part two a more penetrating quantitative study is done by means of analytical techniques on the governing equations.

PART I

1. Introduction.

In the first part we give a qualitative study of acoustic wave propagation in an inhomogeneous medium principally by geometrical optics and ray theory. The inhomogeneity is represented by a sound-speed profile which is dependent upon one coordinate, namely the depth. The particular profile that will be analyzed in this paper can be used to describe several of the stratified media which occur in nature: for example, the ionesphere, the troposphere, and the deep ocean. We are primarily interested in this profile as a model for the speed of sound in the ocean.

The speed of sound in the ocean is principally a function of temperature, hydrostatic pressure, and salinity; and all of these quantities vary with depth. Several experimentally-determined sound-speed profiles for the Mid-Atlantic are illustrated in Figure 1-1. Above the level of minimum sound speed, temperature has the greatest influence; and since the temperature varies inversely with depth (the effect of solar radiation), the sound speed increases as the depth decreases. Below the sound speed minimum, the primary effect is due to the increase of hydrostatic pressure with depth; thus the sound speed increases with depth. With few exceptions, the salinity does not affect the character of the sound-speed. This type of profile leads to the formation of an acoustic waveguide (often called the underwater sound channel or SOFAR* sound channel), which allows sound to

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*Sound Fixing and Ranging

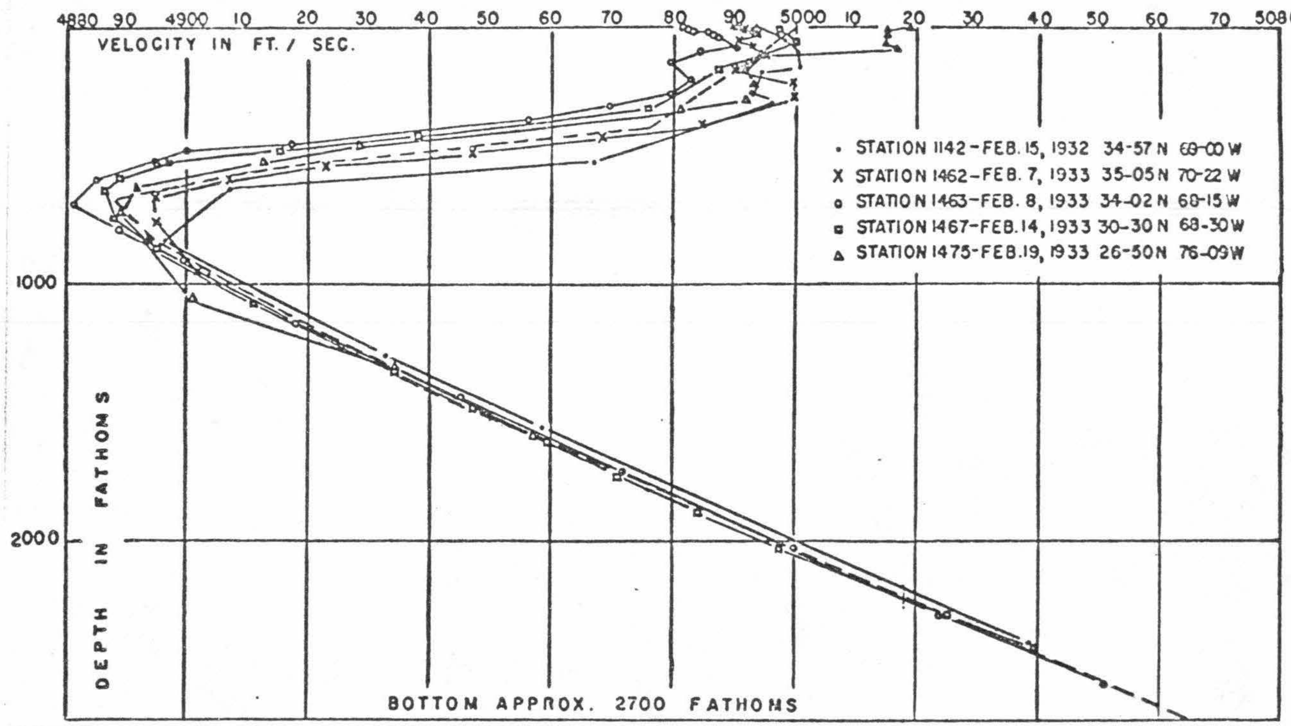

FIG. 1.1 EXPERIMENTALLY DETERMINED SOUND-SPEED PROFILES (REPRODUCED FROM [9])

propagate considerable distances with relatively little dissipation.

Consider the wave propagation resulting from a point source of spherical waves on the channel axis. Then a receiver placed on the axis detects the following characteristics of SOFAR or long-range wave propagation: (i) the transmission of sounds over extremely large distances (in the neighborhood of 10,000 miles for small explosions), (ii) a signal which grows in amplitude after its initial detection and terminates abruptly, (iii) the duration of the SOFAR signal, depending upon the distance in such a way that the distance from source to receiver can be measured very accurately, (iv) the sound energy traveling further away from the axis takes less time to travel from source to receiver than sound energy traveling closer to the axis, (v) the focusing of sound energy in the sound channel at certain ranges.

In this first part we will apply ray theory to illustrate the above properties of SOFAR wave propagation. In Section 2 we describe the mathematical model used to represent the SOFAR sound channel. In each of the succeeding sections we use ray theoretic techniques to investigate the implications of our profile on the above-mentioned properties (ii) - (v).

2. The Mathematical Model.

We first discuss several general characteristics of SOFAR propagation in the ocean. Rays emanating from a source located at the depth of minimum sound speed (called the channel axis) are refracted toward the axis as illustrated in Figure 2-1. Those rays that leave the source at small angles (we shall be more precise later) undergo total refraction. Those rays leaving at large angles are reflected from surface or bottom boundaries, and in the case of a source and receiver

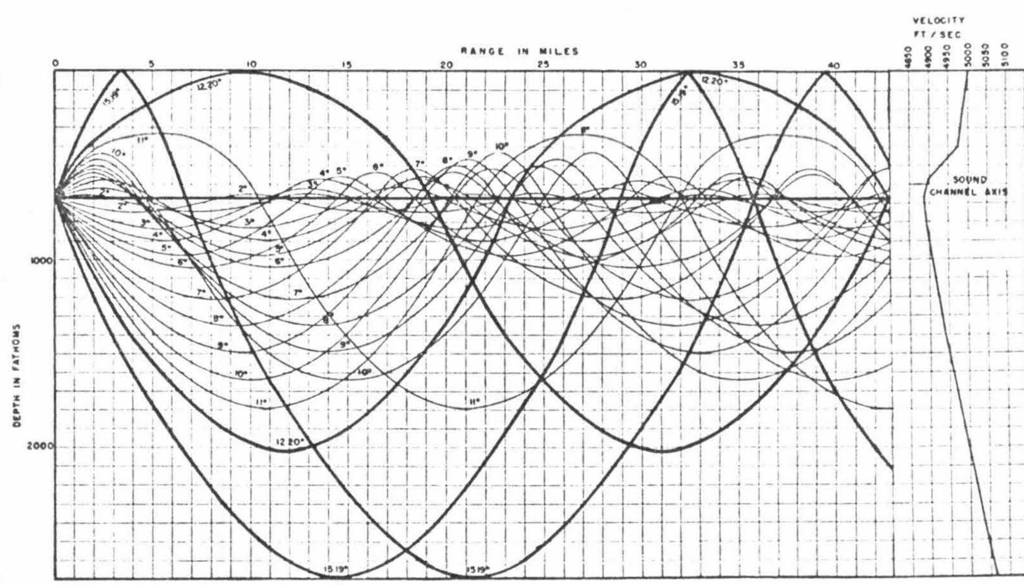

FIG. 2.1 RAY DIAGRAM FOR A TYPICAL ATLANTIC OCEAN SOUND CHANNEL (REPRODUCED FROM [9])

both on the axis, these rays are unimportant at long ranges because of high attenuation at the boundaries. Therefore it is justifiable to assume that the main part of the SOFAR signal is composed of contributions from small-angle rays which oscillate about the channel axis. The path of these rays is determined only by the structure of the sound-speed profile in the region of the minimum.

When the source (SOFAR bomb) and receiver are both on the channel axis, it is known that the last of the SOFAR arrivals travels along the axis (see [5], [9], [21]), and that ray tubes which deviate most from the axis arrive first. A typical SOFAR record is sketched in Figure 2-2.

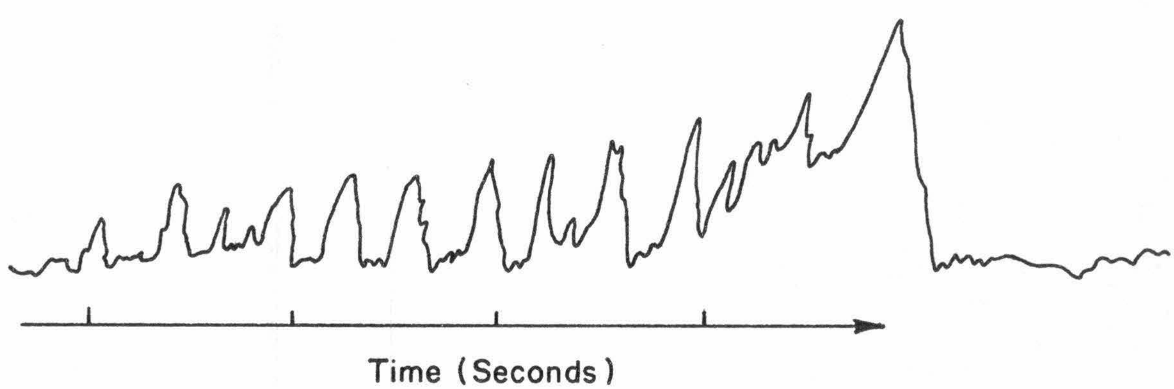

FIG. 2.2

We shall study the situation in an inhomogeneous medium in which the sound speed $c(z)$ is given by

$$(2.1) \qquad c(z) = c_\infty \left[1 + \frac{M}{\cosh^2(\frac{1}{2}mz)}\right]^{-\frac{1}{2}}, \quad M > 0,$$

where c_∞, m, and M are constants (see Figure 2-3).

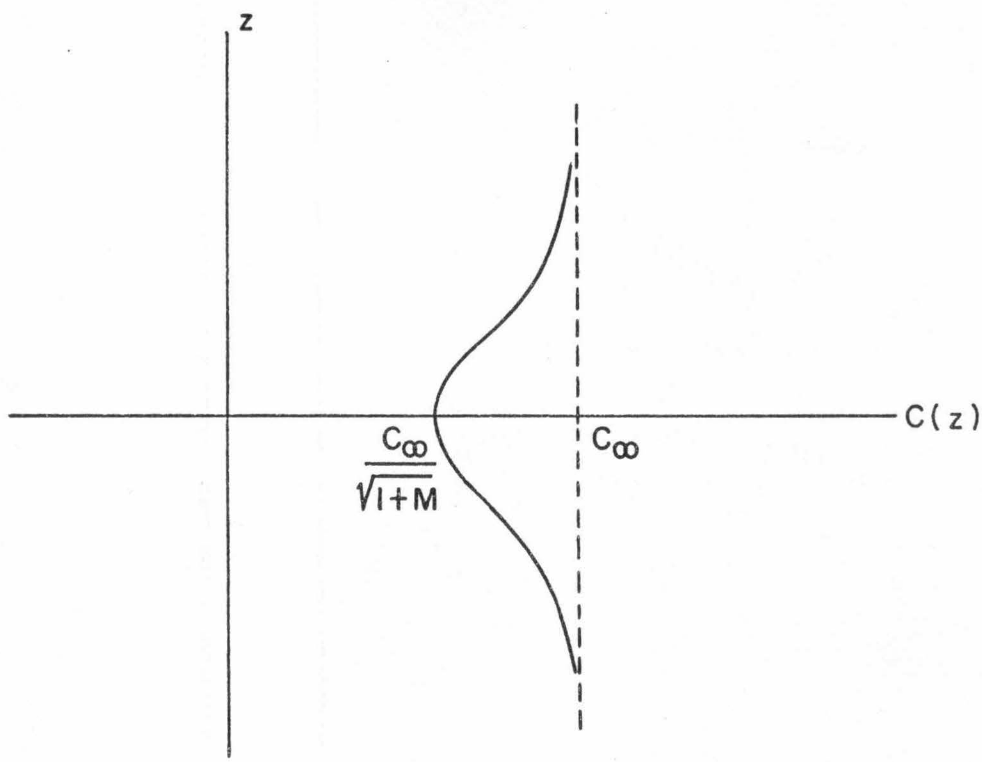

FIG. 2.3

Thus we have an axisymmetric situation in which the sound speed changes only with depth. The sound speed given by (2.1) is known as the Epstein symmetric profile. It was first used by P. Epstein [8] in 1930, and has been studied periodically since then in various wave propagation problems in several fields; see [1], [2], [4], for example.

Another sound speed profile which has been frequently used (E. T. Kornhauser [12], P. Hirsch [6]) is given by

$$c(z) = c_0(1 - \alpha^2 z^2)^{-1/2} \tag{2.2}$$

(see Figure 2-4). However, this profile leads to results which contradict the physical situation. Not only does the sound speed become infinite at $z = \pm \frac{1}{\alpha}$, but also this profile leads to rays where the axial arrival precedes the off-axis arrivals (see P. Hirsch and A. Carter [5]).

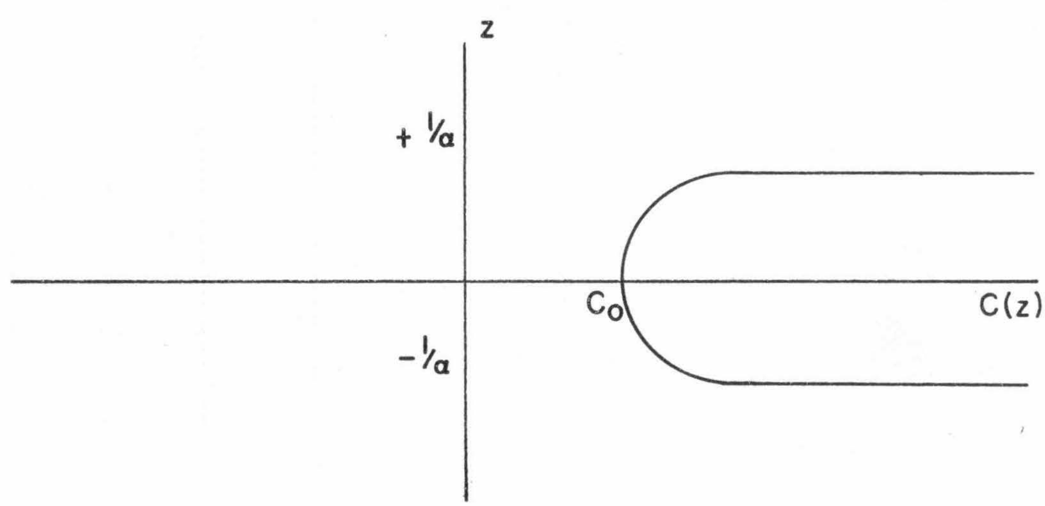

FIG. 2.4

3. Determination of Ray Arrival Times.

We will show that, for the Epstein profile (2.1), the off-axis arrivals precede the axial arrivals. Also we will show that, for small initial angles, the rays which deviate most from the axis arrive first. It should be pointed out that this property is not shared by all profiles having a pronounced minimum. For example, the profiles treated by P. Hirsch and A. Carter [5], and used by P. Hirsch [6] and E. T. Kornhauser [12], qualitatively represent the Epstein profile around the channel axis. However, as shown by P. Hirsch and A. Carter [5], the time arrivals are not correct for SOFAR wave propagation, and thus it is not clear a priori that their quantitative results are correct. Later in Part II we give another reason to question the use of their profile.

Consider a ray which is emitted from a point source on the channel axis and is observed at a point $P(r,z)$ (see Figure 3-1). Each of these rays is characterized either by the angle that it makes with the channel axis or by the number of times that it crosses the axis.

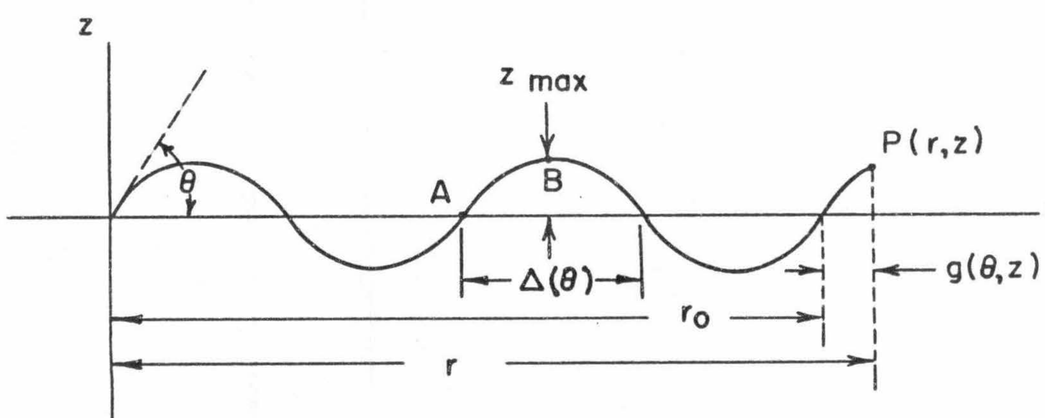

FIG. 3.1

In order to form a general expression for r as a function of θ and z, we define the following quantities which are illustrated in Figure 3-1:

θ = initial angle between the ray and the channel axis,

$\Delta(\theta)$ = distance between successive intersections with the channel axis,

$g(\theta, z)$ = projection on $z = 0$ of a ray from point $P(r,z)$ to the nearest intersection of the ray and the axis,

N = number of intersections of the ray with the axis after leaving the source,

z_{max} = turning point of the ray.

If the point $P(r,z)$ is on the rising portion of the ray and $z > 0$, or on the descending portion of the ray and $z < 0$, then we can write:

$$r = N\Delta(\theta) + g(\theta, z), \quad N = 0, 1, 2, \ldots \quad . \tag{3.1}$$

On the other hand if the point $P(r,z)$ is on the descending portion of the ray and $z > 0$, or on the rising part of the ray and $z < 0$, then

$$r = (N+1)\Delta(\theta) - g(\theta, z), \quad N = 0, 1, 2, \ldots \quad . \tag{3.2}$$

To calculate $g(\theta, z)$ we consider an infinitesimal element of a ray as illustrated in Figure 3-2.

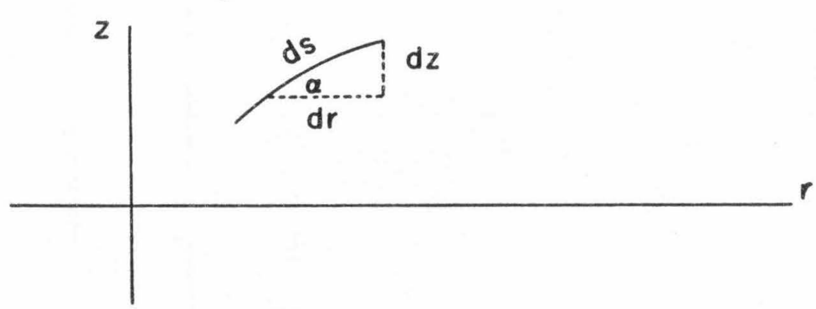

FIG. 3.2

We can write

$$g(\theta,z) = \int_{r_0}^{r} dr ,$$

and since $dr = dz/\tan\alpha$

(3.3) $$g(\theta,z) = \int_0^z \frac{dz}{\tan\alpha} .$$

We can express $\tan\alpha$ in terms of θ and z by noting Snell's Law

(3.4) $$\frac{\cos\alpha}{c(z)} = \frac{\cos\theta}{c_0} ,$$

where $c_0 = c(0)$. Therefore, we obtain

(3.5) $$g(\theta,z) = \cos\theta \int_0^z \frac{dz}{\sqrt{\frac{c_0^2}{c^2(z)} - \cos^2\theta}} .$$

Obviously

$$\Delta(\theta) = 2g(\theta, z_{max}) .$$

Applying the above expressions to the case where both source and receiver are on the axis, we obtain for the distance, R, between source and receiver

$$R = N\Delta(\theta) ,$$

or

(3.6) $$R = 2N\cos\theta \int_0^{z_{max}} \frac{dz}{\sqrt{\frac{c_0^2}{c^2(z)} - \cos^2\theta}} .$$

To determine z_{max} for a given ray leaving the source at an angle θ, we use Snell's Law, noting the fact that at $z = z_{max}$ (a turning point of the ray), $\alpha = 0$. Thus

$$c(z_{max}) = \frac{c_0}{\cos\theta} \quad . \tag{3.7}$$

Now, we determine $R = R(N,\theta)$ for our profile (2.1):

$$R = 2N\cos\theta \int_0^{z_{max}} \frac{dz}{\left[\left\{\frac{1}{1+M}\right\}\left\{1+\frac{M}{\cosh^2(\frac{1}{2}mz)}\right\} - \cos^2\theta\right]^{1/2}} \ , \tag{3.8}$$

where

$$z_{max} = \frac{2}{m}\cosh^{-1}\left[\left\{\frac{M}{(1+M)\cos^2\theta - 1}\right\}^{1/2}\right] .$$

Upon letting

$$y = \sinh\left(\frac{mz}{2}\right) , \qquad r^2(\theta) = \frac{(M+1)(1-\cos^2\theta)}{(M+1)\cos^2\theta - 1} ,$$

(3.8) becomes

$$R = \frac{4N(1+M)^{1/2}\cos\theta}{m[(1+M)\cos^2\theta - 1]^{1/2}} \int_0^{\sinh(\frac{mz_{max}}{2})} \frac{dy}{[r^2(\theta) - y^2]^{1/2}} = \frac{2\pi N\cos\theta}{m[\cos^2\theta - \frac{1}{1+M}]^{1/2}} . \tag{3.9}$$

Next, we calculate the time that it takes for a ray to travel from the source to receiver (again the source and receiver are both on the axis and separated by a distance R). This travel time can be expressed by

$$T = 2N\tau , \tag{3.10}$$

where τ is the time that it takes for a ray to go from point A to point B in Figure 3-1. Noting Figure 3-2, we have

$$(3.11)\qquad \tau = \int_0^{z_{max}} \frac{dS}{c(z)} = \int_0^{z_{max}} \frac{dz}{c(z)\sin\alpha}\ ,$$

and upon using Snell's Law, (3.11) becomes

$$(3.12)\qquad \tau = c_0 \int_0^{z_{max}} \frac{dz}{c^2(z)\sqrt{\frac{c_0^2}{c^2(z)} - \cos^2\theta}}\ .$$

For our profile, given by (2.1),

$$(3.13)\qquad T = \frac{2N}{c_\infty\sqrt{1+M}} \int_0^{z_{max}} \frac{\left[1 + \frac{M}{\cosh^2(\frac{mz}{2})}\right] dz}{\left[\left(\frac{1}{1+M}\right)\left(1 + \frac{M}{\cosh^2(\frac{mz}{2})}\right) - \cos^2\theta\right]^{1/2}}\ .$$

Now we note that the first term in the above integrand is identical to the integrand of (3.8). Thus, we can write

$$(3.14)\qquad T = \frac{2N\pi}{c_\infty m[(M+1)\cos^2\theta - 1]^{1/2}} + \frac{2NM}{c_\infty\sqrt{1+M}} \int_0^{z_{max}} \frac{dz}{\cosh^2(\frac{mz}{2})\left[\left(\frac{1}{1+M}\right)\left(1 + \frac{M}{\cosh^2(\frac{mz}{2})}\right) - \cos^2\theta\right]^{1/2}}\ .$$

By setting

$$x = \cosh^2\left(\frac{mz}{2}\right),\quad s^2(\theta) = \frac{M}{(1+M)\cos^2\theta - 1}$$

the above integral becomes

$$(3.15) \qquad I = \frac{1}{m\left[\cos^2\theta - \frac{1}{1+M}\right]^{1/2}} \int_1^{\cosh^2\left(\frac{mz_{max}}{2}\right)} \frac{dx}{x\left[-x^2+\left(s^2(\theta)+1\right)x - s^2(\theta)\right]^{1/2}} = \frac{\pi}{m}\left(\frac{1+M}{M}\right)^{1/2} .$$

Hence, we obtain finally that

$$(3.16) \qquad T = \frac{2NM}{mc_\infty}\left[M^{1/2} + \frac{1}{[(M+1)\cos^2\theta - 1]^{1/2}} \right] .$$

By eliminating N from the expressions (3.9) and (3.16) we obtain T as a function of R and θ:

$$(3.17) \qquad T = \frac{R}{c_0}\left\{ \frac{[(M+1)\cos^2\theta-1]^{1/2} M^{1/2} + 1}{(1+M)\cos\theta} \right\} .$$

Since we are primarily interested in the small-angle rays we expand (3.17) for small θ:

$$(3.18) \qquad T = \frac{R}{c_0}\left\{1 - \frac{1}{8M}\theta^4 + O(\theta^6)\right\} .$$

From obvious considerations, the travel time for the "direct arrival" (the ray traveling exclusively along the axis) is R/c_0. Clearly, (3.18) implies that it takes less time for a ray to travel from source to receiver when the ray has a non-zero initial angle θ, and furthermore, the larger the initial angle of the ray the faster the time of travel, with the direct arrival being the last ray to arrive at the receiver. Thus we conclude that for any values of m and $M > 0$ the sound-speed profile (2.1) adequately accounts for this aspect of SOFAR wave propagation.

We can also calculate the duration of a signal received on the axis. Solving (3.9) for $\cos\theta$,

$$\cos\theta = \frac{1}{\left[(M+1)\left(1 - \frac{4\pi^2 N^2}{m^2 R^2}\right)\right]^{1/2}} , \tag{3.19}$$

and upon substituting this into (3.17) we obtain

$$T_N = \frac{R}{c_\infty}\left\{\frac{2\pi N M^{1/2}}{mR} + \left(1 - \frac{4\pi^2 N^2}{m^2 R^2}\right)^{1/2}\right\} \tag{3.20}$$

for the travel time of a ray going through N cycles from source to receiver. Since we are considering only those rays with initial angles such that

$$\frac{1}{(M+1)^{1/2}} < \cos\theta < 1 ,$$

we find that

$$0 < N < N_{max} ,$$

where $N_{max} \equiv$ greatest integer less than $\left(\frac{mR}{2\pi}\sqrt{\frac{M}{M+1}}\right)$. Thus we see that there are a finite number of rays going from source to receiver, and that number depends on the parameters of the profile (2.1) and the distance between the source and receiver. Now the duration of the signal is the time interval between the arrival of the ray going through one cycle $(N=1)$ and the direct arrival, thus

$$T_d = \frac{R}{c_0} - T_1 = R\left\{\frac{1}{c_0} - \frac{1}{c_\infty}\left[\left(1 - \frac{4\pi^2}{m^2 R^2}\right)^{1/2} - \frac{2\pi M^{1/2}}{mR}\right]\right\} . \tag{3.21}$$

In the far-field (as $R \to \infty$), we see that

$$T_d = R\left\{\left(\frac{1}{c_0} - \frac{1}{c_\infty}\right) + \frac{2\pi M^{1/2}}{m c_\infty R} + O\left(\frac{1}{R^2}\right)\right\} ;$$

thus the duration of the signal is proportional to the distance between source and receiver.

4. Focusing of Rays. In this section the phenomenon of focusing is illustrated using ray theory. More precisely, we will show that paraxial (i.e., small angle) rays converge periodically along the channel axis. Later in Part II we will show, using normal mode theory, that we obtain a resonance effect along the axis at exactly the same positions.

We recall that the path of a ray with initial angle θ is given by

$$\frac{\cos\theta}{c_0} = \frac{\cos\alpha}{c(z)} .$$

Noting from Figure 3-2 that

$$\cos\alpha = \frac{1}{\sqrt{1+(\frac{dz}{dr})^2}} ,$$

we find on using our profile (2.1) that this becomes

$$\frac{dz}{dr} = \frac{1}{A}\left[1 - A^2 + \frac{M}{\cosh^2(\frac{1}{2}mz)}\right]^{1/2} , \tag{4.1}$$

where $A^2 = (1+M)\cos^2\theta$. Since the ray emanates from $r = 0$, $z = 0$, then

$$r = \frac{1}{T}\int_0^{\frac{1}{2}mz} \frac{\cosh\zeta\, d\zeta}{[M+B\cosh^2\zeta]^{1/2}} , \tag{4.2}$$

where

$$T = \frac{m}{2A} , \quad B = 1-A^2.$$

Now upon setting $\eta = \cosh^2\zeta + \frac{M-B}{2B}$, (4.2) becomes

$$r = \frac{1}{2T\sqrt{B}}\int_D^{\eta_1(z)} \frac{d\eta}{[\eta^2-D^2]^{1/2}} , \tag{4.3}$$

where

$$D = \frac{M+B}{2B} \quad , \qquad \eta_1(z) = \cosh^2(\tfrac{1}{2}mz) + \frac{M-B}{2B} \ .$$

After evaluating the integral in (4.3) and some algebraic manipulation, we obtain

$$(4.4) \qquad \cosh^2(\tfrac{1}{2}mz) = \frac{(M+B)\cos(rm\sqrt{\frac{B}{B-1}}) - (M-B)}{2B} \ .$$

In general the equation for ray paths is specified by

$$(4.5) \qquad z = z(r;\theta) \ ;$$

therefore focusing is expected at those values of r such that

$$(4.6) \qquad \frac{\partial z}{\partial \theta} = 0$$

(where the envelope of the family of rays crosses the axis). Since we are primarily interested in the axial focusing of paraxial rays (small angle rays), we differentiate expression (4.4) with respect to θ, assume $\theta \ll 1$, and solve for those values of r making (4.6) valid. Upon differentiating (4.4) and setting $\frac{dz}{d\theta} = 0$, we obtain

$$(4.7) \ B\left[\cos(rm\sqrt{\tfrac{B}{B-1}}) + (M+B)\sin(rm\sqrt{\tfrac{B}{B-1}}) + 1\right] - (M+B)\cos(rm\sqrt{\tfrac{B}{B-1}}) + (M-B) = 0;$$

and upon letting $\theta \to 0$, $B \to -M$, (4.7) becomes

$$\cos(rm\sqrt{\frac{M}{1+M}}) = 1,$$

or

$$(4.8) \qquad r = \frac{2\ell\pi}{m}\sqrt{\frac{1+M}{M}} \ , \quad \ell = 0, 1, 2, \cdots \ .$$

Thus we have shown that the paraxial rays exhibit focusing at equally spaced points on the channel axis.

It is interesting to note that P. Hirsch [6], and E. T. Kornhauser [12] have shown focusing for the parabolic profile (2.2). Observing the two profiles (2.1) and (2.2), if we set $c_{\infty} = \sqrt{1+M}\, c_0$ and equate the second derivitives at $z = 0$, then focusing occurs at the same points on the axis for both profiles. Also if we assume $M >> 1$, then our result agrees with E. T. Kornhauser and A. D. Yaghjian [3].

5. Complete Ray Picture.

In order to obtain a better qualitative understanding of our sound speed profile, we will now look at the complete ray picture. In particular we will show that there are no shadow zones for our profile. In contrast, we will analyze the ray picture resulting from the bilinear profile. In this case we will show that there are shadow zones. Finally, we will show what class of profiles leads to wave propagation where there are no shadow zones.

Consider the Epstein profile

$$c(z) = c_\infty \left[1 + \frac{M}{\cosh^2(\frac{1}{2}mz)}\right]^{-\frac{1}{2}} . \tag{5.1}$$

From Section 3 we note that z_{max}, the coordinate at which a ray turns back toward the axis, is given by

$$c(z_{max}) = \frac{c_0}{\cos\theta} . \tag{5.2}$$

Now since

$$c_0 = \frac{c_\infty}{\sqrt{1+M}} \leq c(z) < c_\infty ,$$

we observe that those rays which have turning points are such that their initial angles, θ, obey the following relation:

$$\frac{1}{\sqrt{1+M}} < \cos\theta < 1 , \tag{5.3}$$

and the limiting ray (the ray with initial angle θ^ℓ such that all rays with initial angles $\theta > \theta^\ell$ have no turning points) has an initial angle

$$\theta^\ell = \cos^{-1}\left(\frac{1}{\sqrt{M+1}}\right) . \tag{5.4}$$

Since the profile is symmetric, we confine our attention to those rays having an initial angle between 0 and $\pi/2$. From Section 3 we note that

(5.5) $$z_{max} = \frac{2}{m} \cosh^{-1}\left\{\left[\frac{M}{(M+1)\cos^2\theta - 1}\right]^{1/2}\right\},$$

and for the limiting ray

$$z^{\ell}_{max} = \infty \quad .$$

Now we can determine r_{max}, the distance from the source (along the channel axis) at which a ray goes through its first turning point:

(5.6) $$r_{max} = \int_0^{z_{max}} \frac{dz}{\tan\alpha} \quad .$$

For the above profile

(5.7) $$r_{max} = \frac{\pi \cos\theta}{m\left[\cos^2\theta - \frac{1}{M+1}\right]^{1/2}} \quad ;$$

and for the limiting ray (as $\theta \rightarrow \theta^{\ell}$), $z_{max} \rightarrow \infty$, and $r_{max} \rightarrow \infty$.

The result of the above analysis is illustrated in Figure 5-1, and we can see that there is no zone of geometrical shadow.

Now consider the following sound-speed profile which is illustrated in Figure 5-2:

(5.8) $$c(z) = \begin{cases} c_1 & z > z_1 \\ \beta z + c_0 & 0 \leq z \leq z_1 \\ -\beta z + c_0 & -z_1 \leq z \leq 0 \\ c_1 & z < -z_1 \end{cases} \quad ,$$

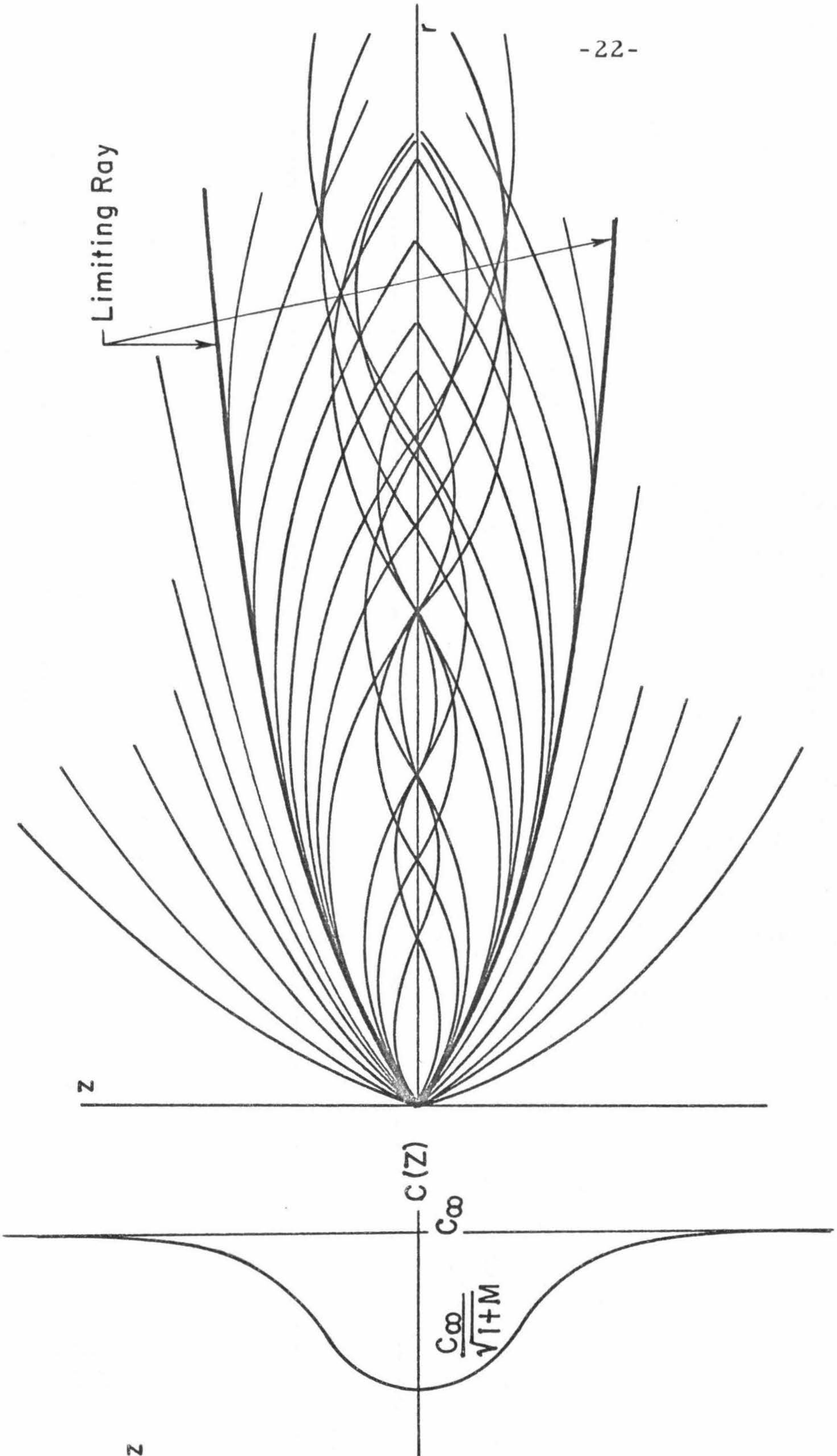

FIG. 5.1

where $\beta = \frac{c_1 - c_0}{z_1}$. For this profile

(5.9) $$c_0 \leqslant c(z) \leqslant c_1 ;$$

therefore the rays which have turning points are such that their initial angles are in the range

(5.10) $$\frac{c_0}{c_1} \leqslant \cos\theta < 1.$$

The limiting ray has an initial angle

(5.11) $$\theta^{\ell} = \cos^{-1}\frac{c_0}{c_1} .$$

We can easily solve for z_{max} by substituting (5.8) into (5.2). Thus

(5.12) $$z_{max} = \frac{c_0}{\beta}\left(\frac{1}{\cos\theta} - 1\right) ,$$

and for the limiting ray,

(5.13) $$z^{\ell}_{max} = z_1 .$$

For the above profile,

(5.14) $$r_{max} = \frac{c_0 z_1 \tan\theta}{c_1 - c_0} ,$$

and for the limiting ray

(5.15) $$r^{\ell}_{max} = z_1 \sqrt{\frac{c_1 + c_0}{c_1 - c_0}} .$$

The actual ray paths are given by Brekhovskikh [4]:

$$\left(r - \frac{c_0}{\beta}\tan\theta\right)^2 + \left(z + \frac{c_0}{\beta}\right)^2 = \frac{c_0}{\beta^2\cos^2\theta} .$$

These rays, which are produced by the above profile, are illustrated in

Figure 5-2; and we see that there is a geometrical shadow for this case.

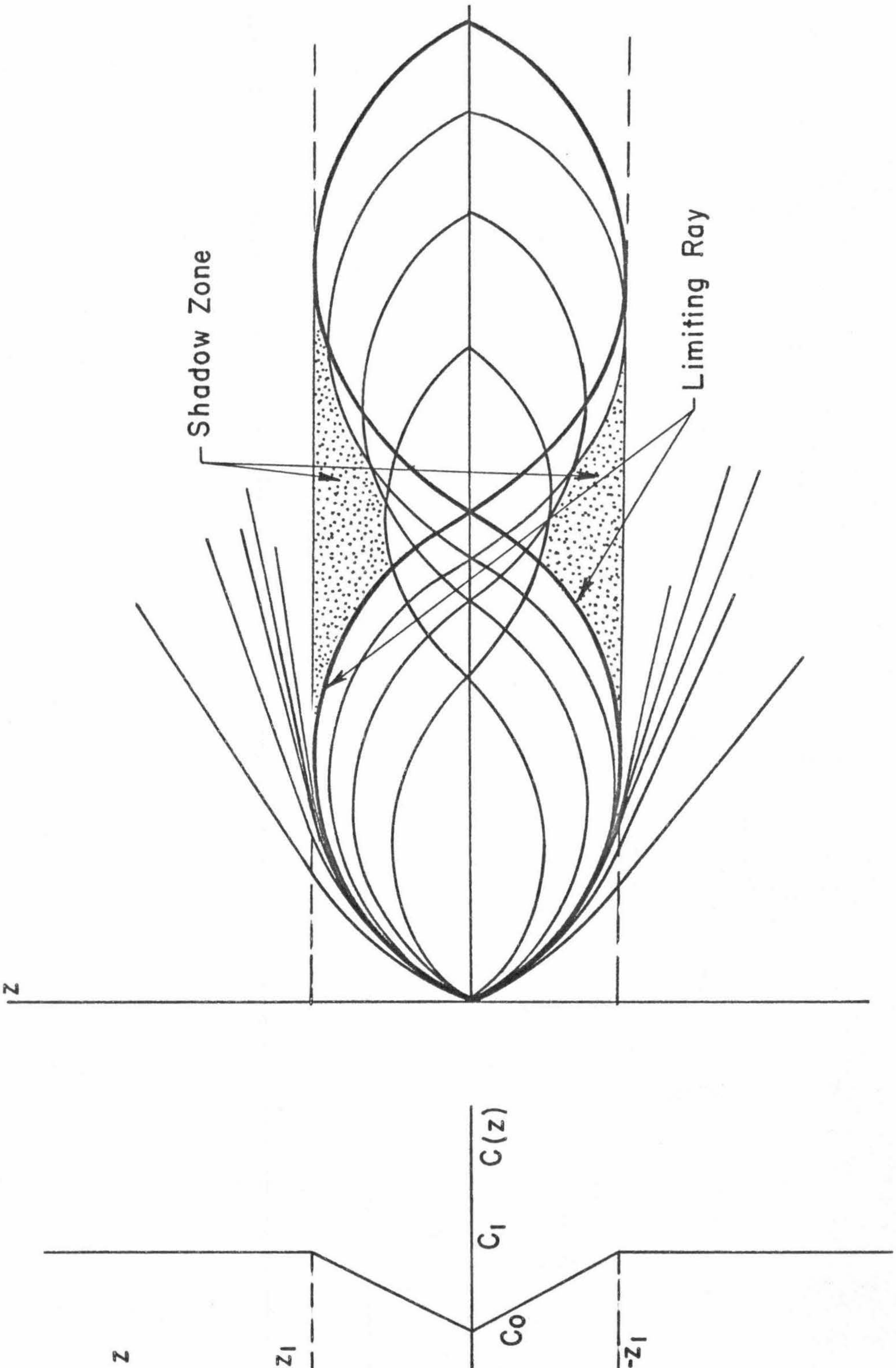

FIG. 5.2

From the previous analysis we can see that there is no shadow zone if as $\theta \to \theta^{\ell}$ then $r_{max} \to \infty$; and obviously, if $\theta \to \theta^{\ell}$, and r_{max} approaches a finite number, we would expect a shadow zone. We will apply this observation quantitatively to obtain a condition on the sound-speed profile which dictates whether or not there is a shadow zone.

We analyze the following integral:

$$(5.15) \qquad r^{\ell}_{max} = \int_0^{z^{\ell}_{max}} \frac{dz}{\tan\alpha} \quad .$$

Using Snell's Law, we can write the above integral as

$$(5.16) \qquad r^{\ell}_{max} = \int_0^{z^{\ell}_{max}} \frac{c(z)}{\sqrt{f(z)}} \quad ,$$

where

$$f(z) = c^2(z^{\ell}_{max}) - c^2(z) \; .$$

Expanding $f(z)$ about $z = z^{\ell}_{max}$ gives

$$(5.17) \quad f(z) = -2\left[c(z)c'(z)\right]\Bigg|_{z=z^{\ell}_{max}}(z-z^{\ell}_{max}) - \left[c'^2(z)+ c(z)c''(z)\right]\Bigg|_{z=z^{\ell}_{max}}(z-z^{\ell}_{max})^2 +$$

$$O(z-z^{\ell}_{max})^3 \; .$$

From this expression we see that a sufficient condition for r^{ℓ}_{max} to be infinite is

$$\left.\frac{dc}{dz}\right|_{z=z^{\ell}_{max}} = 0.$$

Thus we can conclude that if $c'(z^{\ell}_{max}) = 0$, then there are no shadow zones present in the sound channel duct; and if $c'(z^{\ell}_{max}) \neq 0$, then we would expect shadow zones.

PART II

1. Introduction.

In this part a more penetrating quantitative study is done by means of analytical techniques on the governing equations. Several people have derived expressions for the acoustic field in the underwater sound channel from a CW (continuous wave) source. E. T. Kornhauser [12] has used the parabolic profile in expression (2.2), Part I, as a model for the sound speed. R. L. Deavenport [1], Iu L. Gazarian [2], and E. T. Kornhauser and A. D. Yaghjian [3] have used either the Epstein profile or similar profiles to derive expressions for the acoustic field.

These expressions for the acoustic field from a CW source are too cumbersome to give much insight into the physical phenomena which result in the ocean. Thus we will study the transient problem which is more complex analytically; however, the results are much simpler to interpret and are much more relevant to the acoustic field produced by an actual explosion in the ocean. P. Hirsch [6] has investigated the problem for the parabolic duct; but as we have already pointed out in Part I, there are several reasons to doubt the validity of this model for acoustic wave propagation in the ocean. We shall study here the transient problem for the more realistic Epstein profile.

In Section 2 we employ a double transform to formally derive an integral representation for the acoustic pressure amplitude, and from this representation we obtain several alternative representations. The case where both source and receiver are on the channel axis and greatly separated is studied in Sections 3 and 4. In particular, we verify some of the earlier results derived by ray theory in Part I, this time, of course, giving them by means of a more quantitative formulation. Furthermore, complete asymptotic results are obtained for the far-field.

Hence our problem is to determine the acoustic pressure field $p = p(r, z, t)$ satisfying

$$\Delta p - \frac{1}{c^2(z)} p_{tt} = \frac{-\delta(r)\,\delta(z)\,f(t)}{2\pi r}, \quad r \geq 0, \ -\infty < z < \infty, t > 0, \tag{1.1}$$

$$p(r, z, 0) = 0, r \geq 0, \quad -\infty < z < \infty, \tag{1.2}$$

$$p_t(r, z, 0) = 0, \ r \geq 0, \ -\infty < z < \infty, \tag{1.3}$$

where

$$c^2(z) = c_\infty^2 \left[1 + \frac{M}{\cosh^2(\frac{1}{2} m\dot{z})} \right]^{-1} . \tag{1.4}$$

Later in our analysis we shall explicitly use the conditions that $p(r, z, t) \rightarrow 0$ as $r \rightarrow \infty$, $z \rightarrow \pm \infty$, for any fixed t, and $p(r, z, t)$ is regular at $r = 0$ for all z and t; of course, these conditions are implicit in the formulation (1.1), (1.2), (1.3).

2. Formal Solution.

We now derive several representations of the solution of problem (1.1), (1.2), (1.3). For the time-harmonic case where $f(t) = e^{-i\omega t}$ problems have been solved by Iu L. Gazarian [2], L. M. Brekhovskikh [4], R. L. Deavenport [1], and E. T. Kornhauser and A. D. Yaghjian [3]. In principle the solution for a general time dependence is achieved merely by taking a Fourier transform of these previously derived solutions. Such an integral representation, however, does not yield even a superficial understanding of the properties of the solution, and in fact the forms generally given for the time-harmonic solution are incomplete in the sense that not enough is known for a study of a Fourier integral of the representation.

We present a complete derivation of the solution, carefully stressing certain pertinent details needed in our subsequent analysis. For the time harmonic case our results agree with those of [1] - [4].

Our first representation is obtained by applying first a Fourier transform in t and then a Hankel transform in r. Define

$$(2.1) \qquad P(r,z,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(r,z,t)e^{i\omega t}dt, \quad p(r,z,t) = \int_{-\infty+i\bar{\sigma}}^{\infty+i\bar{\sigma}} P(r,z,\omega)e^{-i\omega t}\,d\omega,$$

where $\bar{\sigma} > 0$ and $p(r,z,t) \equiv 0$ for $t < 0$. In the standard way we find that the boundary value problem (1.1), (1.2), (1.3) transforms to

$$(2.2) \qquad \nabla^2 P + \frac{\omega^2}{c^2(z)} P = \frac{-\delta(r)\,\delta(z)\,F(\omega)}{2\pi r}, \quad r \geq 0, \ -\infty < z < \infty,$$

where

$$(2.3) \qquad F(\omega) = \frac{1}{2\pi}\int_0^{\infty} e^{+i\omega t} f(t)\,dt,$$

and where, as usual, we append to (2.2) the boundary conditions that P satisfy either the Sommerfeld radiation condition or appropriate decay conditions at infinity. Now, define

$$(2.4)\quad \phi(z,u,\omega) = \int_0^\infty rP(r,z,\omega)\, J_0(ru)\, dr, \qquad P(r,z,\omega) = \int_0^\infty u\phi(z,u,\omega) J_0(ru)du,$$

and apply these in the usual way to find that ϕ satisfies

$$(2.5)\qquad \frac{d^2\phi}{dz^2} + \left[k^2(z) - u^2\right]\phi = \frac{-\delta(z)\,F(\omega)}{2\pi}, \quad -\infty < z < \infty,$$

$$(2.6)\qquad \lim_{z\to\pm\infty} [\phi] = 0,$$

where $k(z) = \frac{\omega}{c(z)}$.

Now, let

$$(2.7)\quad \zeta = \frac{mz}{2}, \quad k_\infty = \frac{\omega}{c_\infty}, \quad \nu(\nu-1) = \left(\frac{2k_\infty}{m}\right)^2 M, \quad \mu^2 = \left(\frac{2}{m}\right)^2 (u^2 - k_\infty^2).$$

Then, (2.5), (2.6) become

$$(2.8)\qquad \frac{d^2\phi}{d\zeta^2} + \left[\frac{\nu(\nu-1)}{\cosh^2\zeta} - \mu^2\right]\phi = \frac{-\delta(\zeta)\,F(\omega)}{m\pi}, \quad -\infty < \zeta < \infty,$$

$$(2.9)\qquad \lim_{\zeta\to\pm\infty} [\phi] = 0.$$

It is shown in Appendix A that two linearly independent solutions ϕ_1 and ϕ_2 of the homogeneous equation corresponding to (2.8) are given by

$$(2.10)\qquad \phi_1 = e^{\mu\zeta}\left(1+e^{2\zeta}\right)^{\nu} F\left(\mu+\nu, \nu; \mu+1; -e^{2\zeta}\right),$$

$$(2.11)\qquad \phi_2 = e^{-\mu\zeta}\left(1+e^{-2\zeta}\right)^{\nu} F\left(\mu+\nu, \nu; \mu+1; -e^{-2\zeta}\right),$$

where $F(a,b;c;z)$ is the hypergeometric function. With the choice that $\mathrm{Re}\{\mu\} > 0$, we have $\phi_1 \to 0$ as $\zeta \to -\infty$ and $\phi_2 \to 0$ as $\zeta \to \infty$. Thus, by standard techniques we find that the solution of (2.8), (2.9) is

$$\phi(\zeta) = \frac{-F(\omega)\phi_1(\zeta_<)\phi_2(\zeta_>)}{m\pi\; W(\phi_1,\phi_2)}\,, \tag{2.12}$$

where $\zeta_<$ and $\zeta_>$ are respectively the lesser and greater of 0 and ζ. Here $W(\phi_1,\phi_2)$ is the Wronskian of ϕ_1 and ϕ_2 which, as we show in Appendix B, is given by

$$W(\phi_1,\phi_2) = \frac{-2\Gamma^2(\mu+1)}{\Gamma(\mu+1-\nu)\,\Gamma(\mu+\nu)}\,, \tag{2.13}$$

where $\Gamma(z)$ is the Gamma function. Thus, we have finally that the solution of (2.5), (2.6) is given by

$$\phi(z,u,\omega) = \frac{F(\omega)\,\Gamma(\mu+1-\nu)\,\Gamma(\mu+\nu)e^{-\frac{m\mu}{2}|z|}(1+e^{-m|z|})^{\nu}F(\mu+\nu,\nu;\mu+1;e^{-m|z|})}{2^{\mu+1}\sqrt{\pi}\,m\,\Gamma(\mu+1)\,\Gamma(1+\frac{\mu-\nu}{2})\,\Gamma(\frac{1+\mu+\nu}{2})}. \tag{2.14}$$

Therefore, using (2.14) and inverting the Hankel and Fourier transforms, we conclude that the solution of our boundary value problem (1.1), (1.2), (1.3) is given by

$$p(r,z,t) = \tfrac{1}{2}\int_{-\infty+i\bar{\sigma}}^{\infty+i\bar{\sigma}} \left[\int_{-\infty}^{\infty} u\phi(z,u,\omega)\,H_0^{(1)}(ru)\,du\right] e^{-i\omega t}\,d\omega, \tag{2.15}$$

where $H_0^{(1)}(x)$ is the Hankel function of the first kind. In inverting the Hankel transform we have used the fact that

$$P(r,z,\omega) = \int_0^\infty u\phi(z,u,\omega)\,J_0(ru)\,du = \tfrac{1}{2}\int_{-\infty}^{\infty} u\phi(z,u,\omega)\,H_0^{(1)}(ru)\,du,$$

which easily follows by using the properties that (i) $\phi(z,u,\omega)$ and $J_0(ru)$ are even functions of u, (ii) $J_0(z) = \frac{1}{2}H_0^{(1)}(z) + \frac{1}{2}H_0^{(2)}(z)$, and (iii) $H_0^{(2)}(z) = -H_0^{(1)}(ze^{\pi i})$.

While (2.15) does provide an exact answer to the problem, it does not lend itself, in its present form, to any sort of analysis for the extraction of properties of the solution. Hence, we now derive alternative representations. First, it will be convenient to make the following changes of variables: Let

$$\alpha = \frac{mc_\infty}{4\sqrt{M}}, \quad \gamma = \frac{\alpha}{c_\infty}, \quad \omega = \alpha\xi , \quad u = \gamma v , \quad \tau = \alpha t, \quad \beta = \frac{\bar{\sigma}}{\alpha} .$$

Then, (2.15) becomes

$$(2.16) \qquad p(r,z,\tau) = \frac{\gamma^2\alpha}{2}\int_{-\infty+i\beta}^{\infty+i\beta} e^{-i\xi\tau}\int_{-\infty}^{\infty} vH_0^{(1)}(\gamma r v)\phi(z,\gamma v,\alpha\xi)\,dv\,d\xi,$$

where, in the new variables,

$$(2.17) \qquad \nu = \tfrac{1}{2}+\tfrac{1}{2}(1+\xi^2)^{1/2}, \qquad \mu = \frac{1}{2\sqrt{M}}(v^2-\xi^2)^{1/2} .$$

Complex contour integration will now be used to evaluate the integrals in (2.16). First, it is advantageous to formally interchange orders of integration, and thus we first consider the integral Q given by

$$(2.18) \qquad Q = \int_{-\infty+i\beta}^{\infty+i\beta} \phi(z,\gamma v,\alpha\xi)\,e^{-i\xi\tau}\,d\xi .$$

Initially, we assume v is a real variable. Let $\delta = (1+\xi^2)^{1/2} = (\xi-i)^{1/2}(\xi+i)^{1/2}$, and define the square roots so that $-\frac{\pi}{2} < \arg(\xi-i), \arg(\xi+i) < \frac{3\pi}{2}$. This together with our previously imposed requirement that $\mathrm{Re}\{\mu\} > 0$ implies that we take branch cuts as illustrated in Figure 2-1. From (2.14) we see that poles of the integrand in (2.18) can result only from singularities of $\Gamma(\mu+1-\nu)$, $\Gamma(\mu+\nu)$, and $F(\alpha\xi)$, the Fourier transform of f(t). For all reasonable functions f(t), the poles of $F(\alpha\xi)$ lie in the half-plane

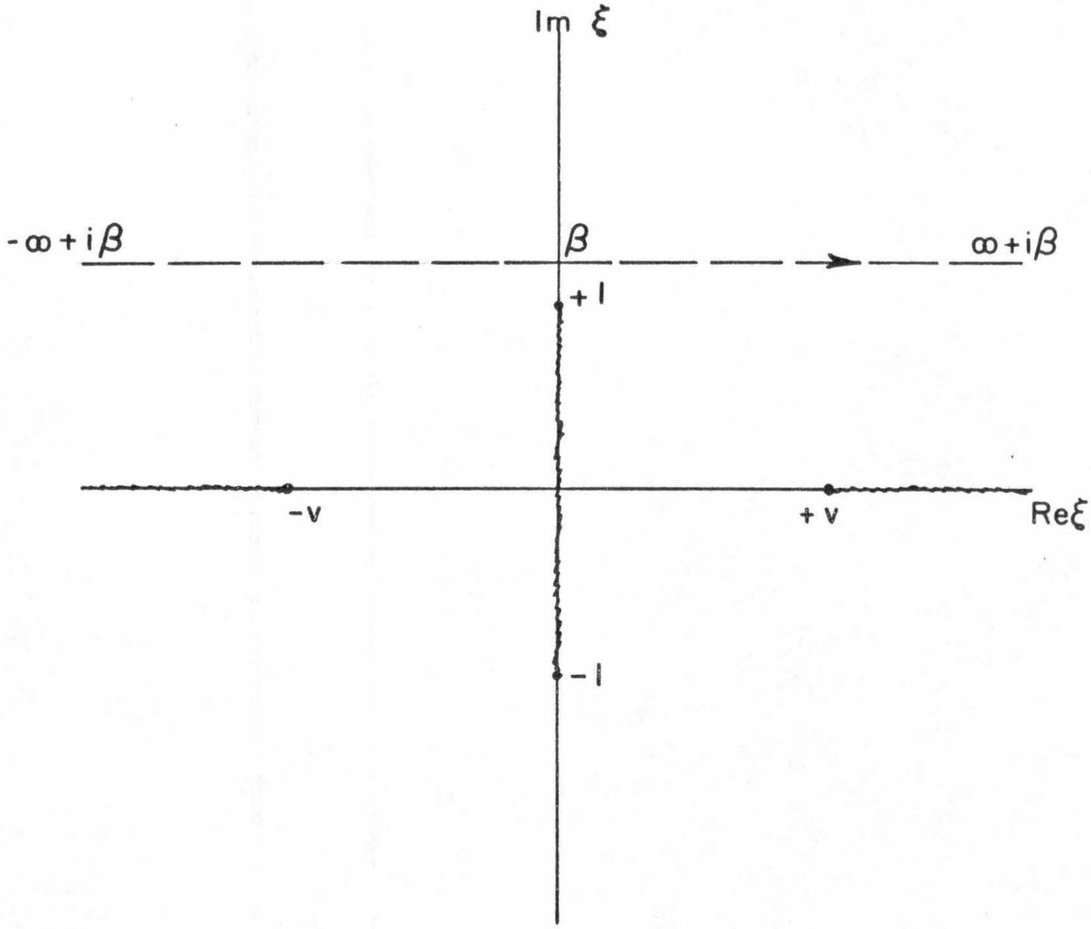

FIG. 2.1

$Im\{\xi\} \leq 0$; and as shown in Appendix C, the poles from the Gamma functions lie on the real axis in the ξ-plane between $-v$ and v and in fact are given by the solutions ξ_n of

$$\mu + \tfrac{1}{2} + \tfrac{1}{2}\delta = -t \quad (t = 0,1,2, \ldots) \quad \text{for} \quad -v \leq Re\{\xi\} < 0,$$

(2.19)

$$\mu + \tfrac{1}{2} - \tfrac{1}{2}\delta = -s \quad (s = 0,1,2,\ldots) \quad \text{for} \quad 0 < Re\{\xi\} \leq v.$$

Thus, we can write (2.18) as

(2.20) $$Q = \int_{-\infty+i\beta}^{\infty+i\beta} = \int_{p_{1L}} + \int_{p_{2L}} + \int_{p_{2R}} + \int_{p_{1R}} .$$

The meaning of the integrals on the right in (2.20) is clear from Figure 2-2. In the usual way, the integrals on the contours p_{3L}, p_{3R}, p_ρ vanish in the appropriate limits (see Appendix D).

At this point it is convenient to confine ourselves to the case where $z = 0$ (i.e., both source and receiver on the channel axis). Thus, (2.20) and (2.16) imply

(2.21) $$p(r,0,\tau) = \frac{\gamma^2\alpha}{2}\int_{F_R} G(r,\xi)\, e^{-i\xi\tau}\, d\xi,$$

where

(2.22) $$G(r,\xi) = \int_{-\infty}^{\infty} v\, H_0^{(1)}(\gamma r v)\, \phi(0, \gamma v, \alpha\xi)\, dv,$$

and where $F_R \equiv p_{1L} + p_{2L} + p_{2R} + p_{1R}$. We now evaluate $G(r,\xi)$ as given by (2.22); clearly, this must be done in each of the four cases corresponding to ξ on p_{1L}, p_{2L}, p_{2R}, or p_{1R}. In all cases there is a branch point at $v = 0$ due to the Hankel function, and in all cases we

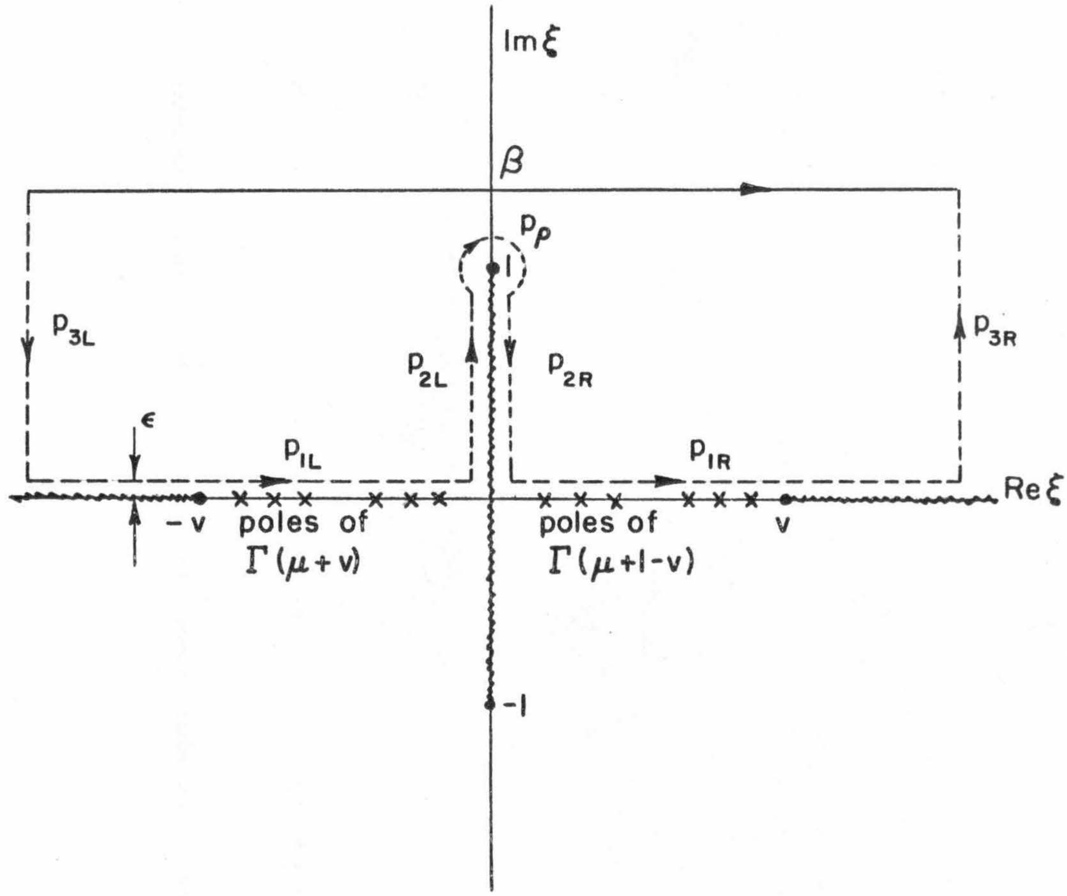
Im ξ
β
p_ρ
1
p_{3L}
p_{2L}
p_{2R}
p_{3R}
ε
p_{1L}
p_{1R}
Re ξ
−v
poles of
Γ(μ+v)
poles of
Γ(μ+1−v)
v
−1

FIG. 2.2

choose the branch cut in the v-plane as $\frac{-\pi}{2} < \arg(v) < \frac{3\pi}{2}$. There will also be branch cuts from the requirement that $\mathrm{Re}\{\mu\} > 0$, where μ is defined in (2.17), and clearly, here the position of the cuts depends on the values of ξ.

For ξ on p_{1L} the requirement that $\mathrm{Re}\{\mu\} > 0$ implies that we take branch cuts as illustrated in Figure 2-3. In addition, for ξ on p_{1L}, the only poles in the v-plane are at those v for which $\Gamma(\mu+\nu)$ is singular. For fixed ξ on p_{1L}, these consist of a finite number of simple poles given by those v_t for which $\mu + \frac{1}{2} + \frac{\delta}{2} = -t$, $t = 0,1,2,\ldots,M(\xi)$, where $M(\xi)$ is the greatest integer less than $\mathrm{Re}\{-\frac{1}{2}\delta - \frac{1}{2}\}$. Thus, by deforming our contour into the upper half of the complex v-plane as shown in Figure 2-3, we proceed in the usual way to obtain

$$G(r,\xi) = \int_{b_1+b_2} v H_0^{(1)}(\gamma r v)\phi \, dv + 2\pi i \sum_{t=0}^{M(\xi)} (\text{residue at } v = v_t). \tag{2.23}$$

We now confine ourselves to the far field. It is well known (see [4], pp.466) that in the limit as $r \to \infty$ the contribution of the integrals in (2.23) is of a smaller order of magnitude than that from the series. (As $r \to \infty$ the integral is $O(\frac{1}{r})$ while the series is $O(\frac{1}{r^{1/2}})$). Thus, we have that, for ξ on p_{1L}, as $r \to \infty$,

$$G(r,\xi) \sim \frac{8M\pi i}{m} \sum_{t=0}^{M(\xi)} \frac{(-1)^t 2^{2t-1-a+bi}(a-1-2t-bi)\,\Gamma(a-t-bi) H_0^{(1)}(\gamma r v_t) F(\alpha\xi)}{\Gamma(1+t)\,\Gamma^2(\frac{1-t}{2})\,\Gamma^2(\frac{a+1-t-bi}{2})}, \tag{2.24}$$

where $a = -\mathrm{Re}\{\delta\}$ and $b = \mathrm{Im}\{\delta\}$. A similar treatment can be given for ξ on p_{1R}. This contour integration is illustrated in Figure 2-4 (see Appendix E). We now let $\epsilon \to 0$ (see Figure 2-2) so that p_{1L}, p_{1R}

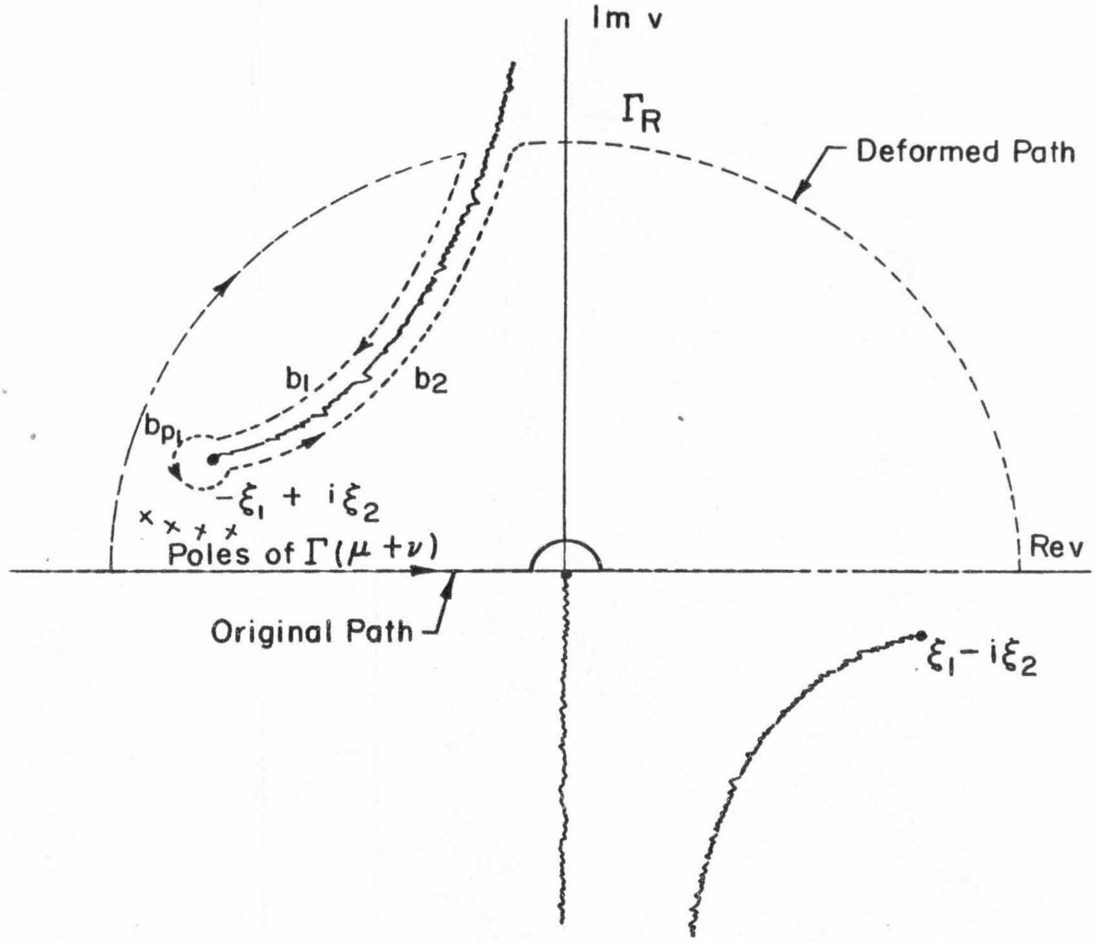
Im v
Γ_R
Deformed Path
b_1
b_2
b_{p_1}
$-\xi_1 + i\xi_2$
Poles of $\Gamma(\mu+\nu)$
Rev
Original Path
$\xi_1 - i\xi_2$

FIG. 2.3

approach the real axis, and <u>henceforth we assume ξ is real.</u> In summary, we obtain (as $r \to \infty$)

$$(2.25) \qquad G(r,z) = \sum_{s=0}^{M(\xi)} g_s(\xi)\, H_0^{(1)}(\pm r\, |u_s|) + o\left(\frac{1}{r^{1/2}}\right),$$

where the plus or minus sign is taken for ξ on the positive or negative real axis, respectively (see Appendix F), and where

$$g_s(\xi) = \frac{8M\pi i}{m} \; \frac{(-1)^s\, 2^{2s-1-a}(a-1-2s)\,\Gamma(a-s)\,F(\alpha\xi)}{\Gamma(1+s)\Gamma^2\left(\frac{1-s}{2}\right)\Gamma^2\left(\frac{a+1-s}{2}\right)},$$

$$(2.26) \qquad a = \sqrt{1+\xi^2},$$

$$|u_s| = \frac{m}{2}\left|\left(\frac{\xi^2}{4M} + \left[\frac{a-1}{2} - s\right]^2\right)\right|,$$

and $M(\xi)$ is the greatest integer less than $\left(\frac{a-1}{2}\right)$. It is shown in Appendix G that for ξ on p_{2L} or p_{2R} we have $G(r,\xi) = o\left(\frac{1}{r^{1/2}}\right)$ as $r \to \infty$. Therefore, (2.22) provides the desired evaluation of $G(r,\xi)$ asymptotically as $r \to \infty$.

Now, we use the following well-known properties of the Hankel function:

$$H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\; e^{i(z-\frac{\pi}{4})} + o\left(\frac{1}{z^{1/2}}\right) \quad \text{as } z \to \infty,$$

$$H_0^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\; e^{-i(z-\frac{\pi}{4})} + o\left(\frac{1}{z^{1/2}}\right) \text{ as } z \to \infty,$$

$$H_0^{(1)}(-r\,|u_s|) = -H_0^{(2)}(r\,|u_s|).$$

Then, as $r \to \infty$,

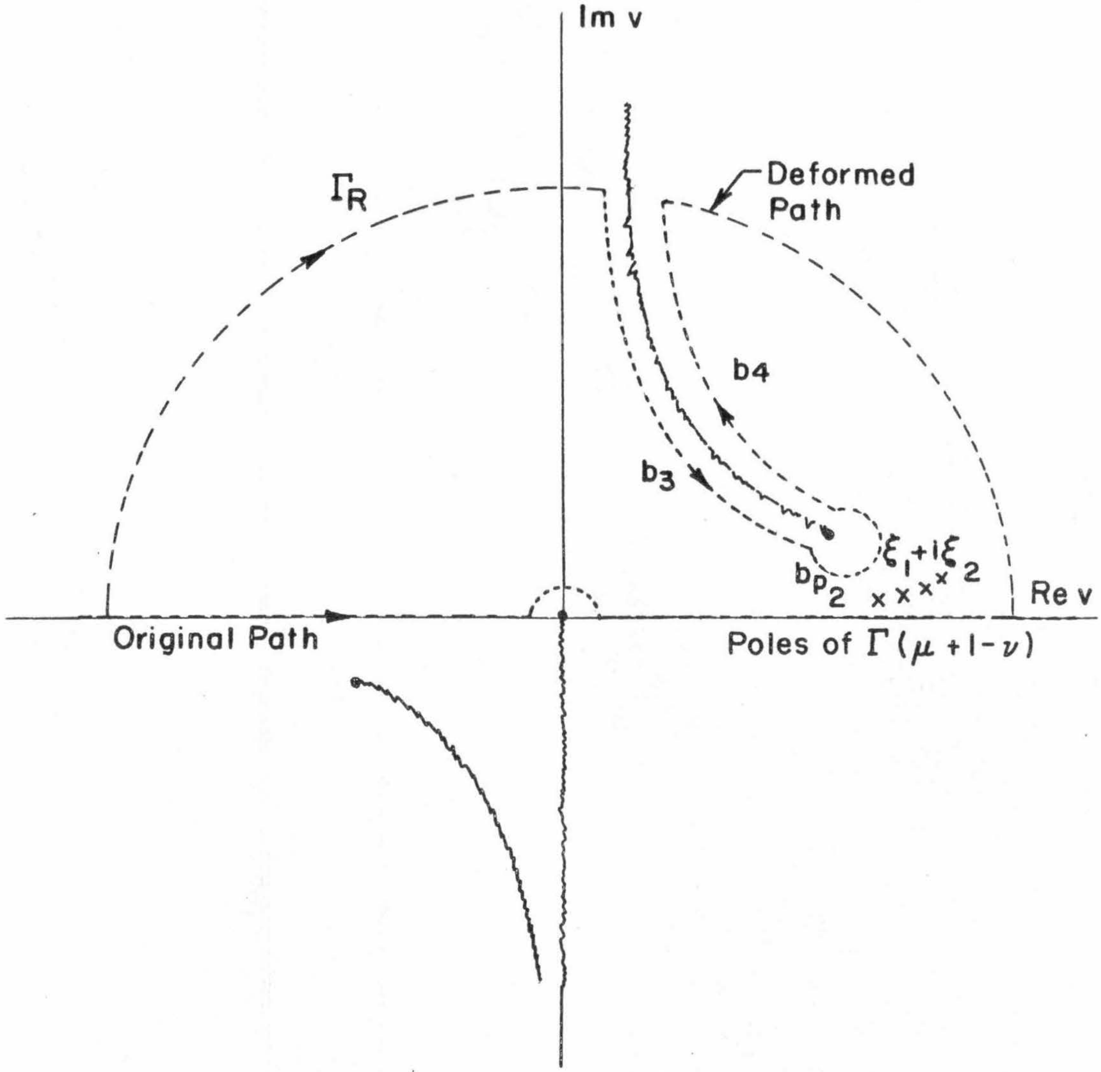
Im v
Γ_R
Deformed Path
b4
b3
$\xi_1 + i\xi_2$
b_{p_2}
Re v
Original Path
Poles of $\Gamma(\mu+1-\nu)$

FIG. 2.4

$$G(r,\xi) = \sum_{s=0}^{M(\xi)} h_s(\xi)\, F(\alpha\xi)\, e^{\pm i(r|u_s| - \frac{\pi}{4})} + o(\frac{1}{r^{1/2}}) , \tag{2.27}$$

where the plus or minus sign is taken for ξ on the positive or negative real axis, respectively, and where

$$h_s(\xi) = \sqrt{\frac{2}{\pi r |u_s|}}\ \frac{g_s(\xi)}{F(\alpha\xi)} . \tag{2.28}$$

We now substitute (2.27) into (2.21) which we write as

$$p(r,0,\tau) = \frac{\gamma^2\alpha}{2} \int_{P_{1L}} G(r,\xi) e^{-i\xi\tau} d\xi + \frac{\gamma^2\alpha}{2} \int_{P_{1R}} G(r,\xi) e^{-i\xi\tau} d\xi . \tag{2.29}$$

Finally, we make the change of variable $\xi = -x$ in the first integral of (2.29) and use the facts that $M(\xi) = M(-\xi)$, $h_s(\xi) = h_s(-\xi), F(-\alpha\xi) = F^*(\alpha\xi)$, and $\Gamma(\frac{1-s}{2})$ is singular for $s = 1, 3, 5, \ldots$, to obtain that as $r \to \infty$,

$$p(r,0,\tau) \sim -\alpha m \sqrt{\frac{\pi}{2r}}\ \mathrm{Im} \left\{ \int_0^\infty \sum_{s=0}^{N(\xi)} \frac{2^{4s-1-a} (a-1-4s)\Gamma(a-2s)F(\alpha\xi) e^{i(r|u_s| - \xi\tau - \frac{\pi}{4})}}{\Gamma(1+2s)\Gamma^2(\frac{1}{2}-s)\Gamma^2(\frac{a+1-2s}{2})\sqrt{|u_{2s}|}} d\xi \right\} , \tag{2.30}$$

where $N(\xi)$ is the greatest integer less than $(\frac{a-1}{4})$.

In order to interchange orders of summation and integration in (2.30) we observe that

$$\sum_{s=0}^{N(\xi)} A_s = \sum_{s=0}^{\infty} A_s H\left[N(\xi) - s\right] ,$$

where $H[x]$ is the Heaviside function defined as 1 for $x \geq 0$ and 0 for $x < 0$. Then, (2.30) becomes finally

$$p(r,0,\tau) \sim -\alpha m \sqrt{\frac{\pi}{2r}}\ \mathrm{Im}\left\{ e^{-\frac{i\pi}{4}} \sum_{s=0}^{\infty} \frac{2^{4s}}{\Gamma(1+2s)\Gamma^2(\frac{1}{2}-s)} I_s(r,t) \right\} , \tag{2.31}$$

where

$$I_s(r,t) = \int_0^{\infty} \frac{2^{-a}(\frac{a-1}{2}-2s)\,\Gamma(a-2s)\,F(\alpha\xi)\,H[N(\xi)-s]\,e^{i(r|u_s|-\xi\tau)}}{\Gamma^2(\frac{a+1}{2}-s)\sqrt{|u_{2s}|}}\,d\xi . \tag{2.32}$$

3. Asymptotic Evaluation of The $I_s(r,t)$.

In order to determine properties of the far field we shall evaluate the integrals $I_s(r,t)$, given by (2.32), asymptotically. From the definition of $N(\xi)$ in (2.30) we see that $N(\xi) = n$ when $n < \frac{1}{4}\left(\sqrt{1+\xi^2}-1\right) \leq n+1$. Thus, the integrand of I_s is identically zero for $0 < \xi \leq \sqrt{(4s+1)^2}-1$ because $N(\xi) \geq s$ for $\xi > \sqrt{(4s+1)^2}-1$. Therefore, I_s can be written as

$$I_s(r,t) = \int_{\sqrt{(4s+1)^2}-1}^{\infty} q_s(\xi)\, e^{irf_s(\xi)}\, d\xi, \tag{3.1}$$

where

$$q_s(\xi) = \frac{2^{-a}\left(\frac{a-1}{2}-2s\right)\Gamma(a-2s)\,F(\alpha\xi)}{\Gamma^2\left(\frac{a+1}{2}-s\right)\sqrt{|u_{2s}|}}\,, \quad f_s(\xi) = |u_{2s}| - \frac{\xi\tau}{r}\,. \tag{3.2}$$

In the present section we shall be primarily interested in the time history of the acoustic pressure $p(r,0,t)$ at any fixed point in the far field. Since $p(r,0,t)$ undergoes significant changes in both its quantitative and qualitative behavior, we shall need different asymptotic expansions for several different time intervals. Furthermore, <u>in the remainder of this section we assume that $M >> 1$</u> for the purpose of simplifying many of the necessary algebraic manipulations. This assumption implies that the sound velocity profile possesses a very pronounced "well" near the channel axis (see Figure 2-3, Part I).

First, we apply the principle of stationary phase for fixed $\frac{\tau}{r}$ as $r \to \infty$. The stationary points are the zeroes of $f_s'(\xi)$. Now, (2.26) and (3.2) imply that

$$f_s(\xi) = \frac{m}{4}\left|\left[\xi^2(1+\frac{1}{M}) - 2a_{2s}\sqrt{1+\xi^2}+1+a_{2s}^2\right]\right| - \frac{\xi\tau}{r}. \tag{3.3}$$

Then, $M>>1$ and $f_s'(\bar{\xi})=0$ imply that the stationary point $\bar{\xi}$ is given by

$$\bar{\xi} = \frac{1}{\sqrt{M\chi-1}} \quad \text{where} \quad \chi = \frac{r^2}{c_\infty^2 t^2}. \tag{3.4}$$

If this stationary point $\bar{\xi}$ lies within the range of integration in (3.1), then, as is well known, $I_s = O(\frac{1}{r^{1/2}})$; if $\bar{\xi}$ does not fall within the range of integration, then $I_s = O(\frac{1}{r})$. Since the integration range is $\xi > \sqrt{(4s+1)^2}-1$, the stationary point does lie within this range if $\bar{\xi} > \sqrt{(4s+1^2}-1$, that is,if

$$s < \frac{1}{4}\left[\sqrt{\frac{M\chi}{M\chi-1}}-1\right] = \frac{1}{4}\left[\sqrt{\frac{1}{1-\frac{c_\infty^2 t^2}{Mr^2}}}-1\right] \equiv Q(r,t,M). \tag{3.5}$$

Therefore, for fixed r,t the modes (i.e., terms of the series in (2.31)) corresponding to $s=0,1,2,\ldots, Q(r,t,M)$ dominate the rest of the modes. Thus, using the stationary phase formula, we find that as $r\to\infty$ for fixed $\frac{r}{\tau}$,

$$I_s(r,t) \sim \sqrt{\frac{8\pi}{rm}}\;\frac{\eta^{3/2}}{2^\eta}\;\frac{\Gamma(\eta-2s)(\eta-a_{2s})F\left(\frac{mc_\infty\eta}{4M\sqrt{\chi}}\right)}{\Gamma^2\left(\frac{1}{2}+\frac{\eta}{2}-s\right)}\,e^{i[\frac{m}{4}(\frac{1}{\eta}-a_{2s})r+\frac{\pi}{4}]}, \tag{3.6}$$

so that (2.31) yields

$$p(r,0,t) \sim \frac{\Phi\eta^{3/2}}{r2^\eta}\sum_{s=0}^{\mathcal{J}(\eta)}\frac{2^{4s}\,\Gamma(\eta-2s)(\eta-a_{2s})}{\Gamma(1+2s)\Gamma^2(\frac{1}{2}-s)\Gamma^2(\frac{1+\eta}{2}-s)}\,\mathrm{Im}\left\{F\left(\frac{mc_\infty\eta}{4M\sqrt{\chi}}\right)e^{i[\frac{m}{4}(\frac{1}{\eta}-a_{2s})r+\pi]}\right\}, \tag{3.7}$$

where

$$\Phi = \frac{m^{1/2}c_\infty\pi}{2M^{1/2}}, \quad a_{2s} = 1+4s, \quad \eta = \sqrt{\frac{M\chi}{M\chi-1}}, \quad \chi = \frac{r^2}{c_\infty^2 t^2},$$

and $\mathcal{J}(\eta)$ is the greatest integer less than $Q(r,t,M)$.

Several observations must now be stated. First, we recall that the method of stationary phase is not valid near those points for which $f_s''(\overline{\xi}) = 0$. In the present problem this occurs for $\eta = 0$ (i.e., for $M\chi=1$). Hence, we conclude that (3.6) and (3.7) are not valid near $t = \frac{\sqrt{M}r}{c_\infty} \equiv T$. (Note that for $M >> 1$, T is the arrival time for a ray traveling along the channel axis; see Part I of this paper.)

Next, we note that for $t << \frac{\sqrt{M}r}{c_\infty}$ (i.e., $M\chi >> 1$), (3.4) implies $\overline{\xi} = \frac{1}{\sqrt{M\chi}}\left[1+O(\frac{1}{M\chi})\right]$, so that $Q(r,t,M) = \frac{1}{8M\chi}\left[1+O(\frac{1}{M\chi})\right]$. Thus, in the time range $t << T$, the only mode present in the series in (3.7) is the zeroth mode (in fact, this mode is always present and yields the familiar precusor effect associated with the integral transform - stationary phase type of treatment). As time goes on (still satisfying $t < T$), more and more modes of the series in (3.7) come into play. The time t_k at which the k^{th} mode comes into play is clearly given by $Q(r,t_k,M) = k$, which yields, upon solving for t_k, that

$$t_k = \frac{\sqrt{M}\,r}{c_\infty}\sqrt{\frac{(4k+1)^2-1}{4k+1}}\;.$$

We have found that for $t < T$ the far-field is adequately described by (3.7). (An interesting alternative derivation is given in Appendix H.) It now remains to study $I_s(r,t)$ for t in the range $t \approx T$.

In order to discuss the behavior of $I_s(r,t)$ in the neighborhood of $t = T$, we use a method similar to that used by C. L. Pekeris [9] in studying Airy waves. This analysis arose in the study of the propagation of a pulse in a liquid layer, and is used where the method of stationary phase becomes invalid. Again we study $I_s(r,t)$ for r large and fixed.

Actually from purely physical reasoning one would expect some kind of change in the behavior of $I_s(r,t)$ for t near T, because a wave tail should begin at a time corresponding to the direct arrival of energy along the channel axis.

We will show that the dominant behavior of $I_s(r,t)$ is dependent on the behavior of its integrand for high frequencies and is critically dependent upon the high frequency behavior of the Fourier transform of the forcing function $f(t)$. This is analogous to Pekeris' result where the dominant behavior is dependent on the frequency ξ_0 such that $f''(\xi_0)=0$.

To begin the analysis we note that the phase function in the integrand of (3.1) can be written as

$$r f_s(\xi) = \phi_s(\xi) = k(\xi)\, r - \xi \bar{c}_0\, t, \tag{3.8}$$

where

$$k(\xi) = \frac{m}{4}\left(\sqrt{1+\xi^2} - a_{2s}\right), \quad \bar{c}_0 = \frac{m}{4} c_0 . \tag{3.9}$$

Recalling the definition of group velocity, $U = \frac{d\xi}{dk}$, we obtain

$$U(\xi) = \frac{4}{m} \frac{\sqrt{1+\xi^2}}{\xi}, \tag{3.10}$$

and we observe the group velocity as a function of frequency from Figure 3-1. Thus $U(\xi)$ has a minimum at $\xi_m = \infty$ $(k''(\xi_m) = 0)$. Also we note that the point of stationary phase $\bar{\xi} \to \infty$ as $t \to T$.

In order to evaluate $I_s(r,t)$ about $t = T$ (where $f''(\xi_m) = 0, \xi_m$ infinite) it will be instructive to recall the analysis done by Pekeris [9] and summarized by Brekhovskikh [4]. For this case, the group velocity as a function of frequency is illustrated in Figure 3-2. We observe that,

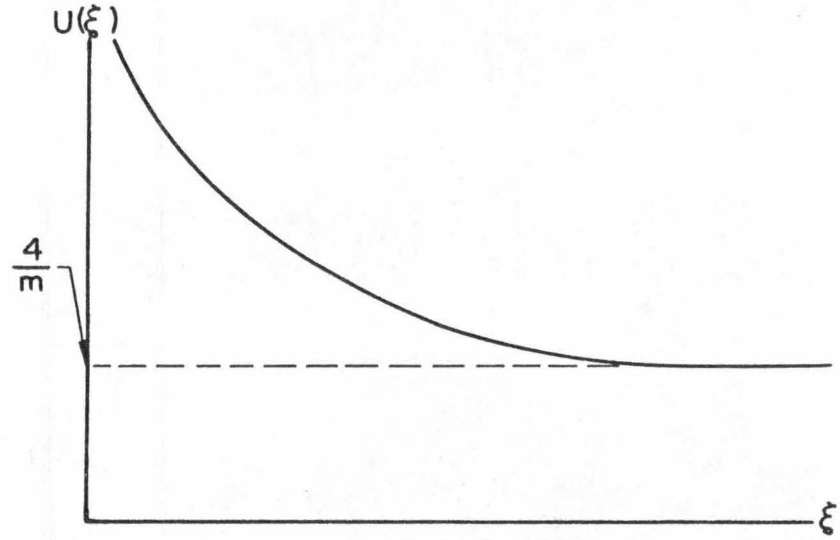

FIG. 3.1

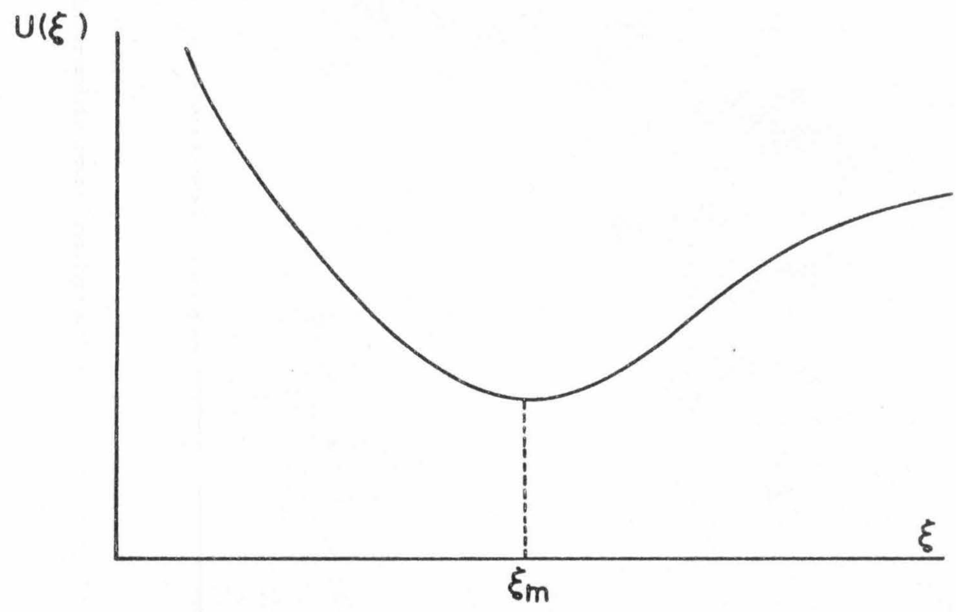

FIG. 3.2

for r large and fixed, the method of stationary phase breaks down at that $t=t_m$ (where the point of stationary phase $\overline{\xi}=\xi_m$). Thus we observe that the main contribution to the integral arises from the point ξ_m (finite) for $t \sim t_m$, and we expand the integrand of $I_s(r,t)$ about $\xi=\xi_m$. Thus setting $\xi=\xi_m+u$, we obtain

$$(3.11)\qquad \begin{cases} q_s(\xi) \sim q_s(\xi_m) + O(u), \\ \\ \phi_s(\xi) = \phi_s(\xi_m) + [\,t - k'(\xi_m)\,r\,]\,u - \frac{r}{6}\,k'''(\xi_m)u^3 + O(u^4)\ . \end{cases}$$

We note that the derivatives of $k(\xi)$ evaluated at ξ_m are independent of r and t, whereas using stationary phase they would be dependent upon r and t. Therefore, if we substitute (3.11) into (3.1), we find that, for sufficiently large r and for t sufficiently close to t_m, $I_s(r,t)$ is asymptotic to an Airy function. These are the so-called Airy waves. In principle we should be able to do the same type of analysis for ξ_m infinite.

Now we evaluate $I_s(r,t)$ in the far-field for $t \sim T = r/c_0$. We recall from expressions (3.4) that the point of stationary phase goes to infinity as $t \to T$; thus we can write

$$(3.12)\qquad I_s(r,t) = \int_{\sqrt{(4s+1)^2-1}}^{N} + \int_{N}^{\infty}\ ,$$

where $N >> 1$ and is such that $N < \overline{\xi} < \infty$. Therefore, in the far-field

$$(3.13)\qquad I_s(r,t) = \int_N^{\infty} Q_s(\xi)\,F(\alpha\xi)\,e^{i\phi_s(\xi)}\,d\xi + O(\tfrac{1}{r}).$$

Assuming that $Q_s(\xi)$ is slowly varying as $\xi \to \infty$ (we have shown that

$\lim_{\xi \to \infty} Q_s(\xi) = A_s$), $F(\omega) \sim g(\omega)$ as $\omega \to \infty$, and noting that

$$\phi_s(\xi) = \frac{m}{4}\left[\xi + \frac{1}{2\xi} + O(\tfrac{1}{\xi^3})\right] r - \bar{c}_0 \xi t, \quad \text{as} \quad \xi \to \infty, \tag{3.14}$$

then

$$I_s(r,t) \sim \frac{Q_s(\infty)}{\alpha} e^{-ir\frac{m}{4}a_{2s}} \int_{\alpha N}^{\infty} g(\omega) e^{i\omega(\frac{r}{c_0} - t)} e^{i\frac{\alpha^2 r}{2c_0\omega}} d\omega . \tag{3.15}$$

Since as $r \to \infty$, for $\frac{r}{t}$ fixed, there is no point of stationary phase in the above integrand in the interval $0 \le \xi \le \alpha N$, then to within $O(\frac{1}{r})$

$$I_s(r,t) \sim \Theta \int_0^{\infty} g(\omega)\, e^{-i\omega\bar{\tau}}\, e^{i B/\omega}\, d\omega, \tag{3.16}$$

where

$$\Theta = \frac{Q_s(\infty) e^{-ir\frac{m}{4}a_{2s}}}{\alpha}, \quad \bar{\tau} = t - \frac{r}{c_0}, \quad B = \frac{\alpha^2 r}{2c_0} .$$

Assuming that $g(\omega) = \frac{1}{\omega^{\lambda+1}}$, $0 < \lambda < 1$, $\bar{\tau} > 0$, and letting

$$\gamma = \sqrt{B/\bar{\tau}}, \quad \chi = 2\sqrt{B\bar{\tau}}, \quad \omega = \gamma\eta,$$

we obtain

$$I_s(r,t) \sim \frac{\Theta}{\gamma^{\lambda}} \int_0^{\infty} \frac{1}{\eta^{\lambda+1}} e^{-i\frac{\chi}{2}[\eta - \frac{1}{\eta}]} d\eta . \tag{3.17}$$

From G. N. Watson [22], we recognize the above integral as the integral representation of the Bessel function $K_\nu(\chi)$, thus

$$I_s(r,t) \sim \frac{Q_s(\infty)\, e^{-i(r\frac{m}{4}a_{2s} - \frac{\lambda}{2}\pi)}}{\alpha\, 2^{\lambda-1} b^{\lambda} r^{\lambda}} \chi^{\lambda} K_{\lambda}(\chi) , \tag{3.18}$$

where

$$b = \frac{m^2 c_0}{32}, \quad \alpha = \frac{mc_0}{4}, \quad \chi = 2\sqrt{br(t - \frac{r}{c_0})}, \quad 0 < \lambda < 1.$$

In summary, we have evaluated $I_s(r,t)$ in the far-field for $t \sim T = r/c_0$ for any forcing function $f(t)$ such that its Fourier transform, $F(\omega) = O\left(\frac{1}{|\omega|^\beta}\right)$, as $\omega \to \infty$ $1 < \beta < 2$. One reasonable class of such functions is

$$f(t) = \frac{1}{\Gamma(\beta)} t^{\beta - 1} e^{-\beta t} \tag{3.19}$$

We wish to point out explicitly that our result depends critically upon the assumption that our forcing function $f(t)$ has a Fourier transform $F(\omega)$ which behaves like

$$F(\omega) = O\left(\frac{1}{|\omega|^\beta}\right), \tag{3.20}$$

as $\omega \to \infty$. It is well known that the high frequency components of any function play a very important role in the Fourier synthesis of that function, and thus, since our analysis in the range $t \sim T$ corresponds to the point of stationary phase moving off to infinity in the frequency domain, it is reasonable that we must make an assumption such as (3.20).

For an interesting calculation of $I_s(r,t)$ as $r \to \infty$ for $t = T$, see Appendix I.

4. Focusing from Normal Mode Theory.

Since this convergence phenomenon is time-independent, it will be illustrated using the time-independent solution to our problem. Therefore, we seek the solution to (1.1) with $f(t) = e^{-i\omega t}$. Assuming that

$$p(r,z,t) = P(r,z,\omega)\, e^{-i\omega t} ,$$

then $P(r,z,\omega)$ is determined from (2.2) with $F(\omega) = 1$. Since we are interested in focusing along the channel axis we want $P(r,0,\omega)$, and this has been calculated previously in expression (2.25):

$$(4.1) \qquad P(r,0,\omega) = \sum_{s=0}^{M(\xi)} g_s(\xi)\, H_0^{(1)}(r|u_s|) + o\left(\frac{1}{(|u_s|r)^{1/2}}\right), \quad \text{as } |u_s|r \to \infty .$$

Since the quantity $(|u_s|r)$ is large we can use the asymptotic form of the Hankel function; therefore for $|u_s|r \to \infty$,

$$(4.2) \qquad P(r,0,\omega) \sim \sum_{s=0}^{N(\xi)} h_{2s}(\xi)\, e^{i(r|u_{2s}| - \frac{\pi}{4})} ,$$

where

$$h_{2s}(\xi) = \sqrt{\frac{2}{\pi r |u_{2s}|}}\; g_{2s}(\xi) ,$$

and $N(\xi)$ is the greatest integer less than $(\frac{a-1}{4})$. In obtaining (4.2) from (4.1), we have also used the fact that $\Gamma(\frac{1-s}{2}) = \infty$, $s = 1,3,5,\cdots$.

We now examine the range-dependent part of the above expression and focusing will be exhibited for some values of ξ*. The range-dependent part can be written as

$$(4.3) \qquad e^{ir|u_{2s}|} = e^{ir\frac{m}{4}\left[\xi^2\left(\frac{M+1}{M}\right) - 2a_{2s}\sqrt{1+\xi^2} + 1 + a_{2s}^2\right]^{1/2}} .$$

*The variable ξ is the normalized frequency. To get a "real" frequency range, we must consider the real frequency ω.

Assuming that $\xi >> 1$, then

$$\sqrt{1+\xi^2} = \xi\left[1+\frac{1}{2\xi^2}+\cdots\right];$$

and upon assuming that

$$\xi >> \frac{1+a_{2s}^2}{2a_{2s}},$$

we find that

$$e^{ir|u_{2s}|} \sim e^{ir\frac{m}{4}\xi\sqrt{\frac{M+1}{M}}\left[1-\frac{2a_{2s}M}{\xi(M+1)}\right]^{1/2}}. \tag{4.4}$$

Finally if we assume that $\xi >> \frac{2a_{2s}M}{M+1}$, the range dependence becomes asymptotic to the following

$$e^{ir|u_{2s}|} \sim e^{ir\frac{m}{4}\xi\sqrt{\frac{M+1}{M}}\left[1-\frac{a_{2s}M}{\xi(M+1)}+\cdots\right]}, \tag{4.5}$$

and the time independent solution can be written:

$$P(r,0,\omega) \sim K(r;\xi)\sum_{s=0}^{N(\xi)} h_{2s}(\xi)\, e^{-imr\sqrt{\frac{M}{M+1}}\,s}, \tag{4.6}$$

where

$$K(r;\xi) = e^{i\left[\frac{mr}{4}\left(\xi\sqrt{\frac{M+1}{M}}-\sqrt{\frac{M}{M+1}}\right)-\frac{\pi}{4}\right]}. \tag{4.7}$$

Observing (4.6), we see that if $e^{-imr\sqrt{\frac{M}{M+1}}\,s} = +1$ for all s, we would have constructive interference from all terms in the series, and thus would have focusing. This is indeed the case when

$$mr\sqrt{\frac{M}{M+1}} = 2\ell\pi,\quad \ell = 1,2,\cdots, \tag{4.8}$$

or

$$r = \frac{2\ell\pi}{m}\sqrt{\frac{M+1}{M}},\quad \ell = 1,2,\cdots. \tag{4.9}$$

These positions of convergence along the channel axis are the same as those which were predicted using ray theory in Section 4, Part I.

It must be noted that for a given frequency of the forcing function, $e^{-i\omega t}$, we have demonstrated focusing only for the modes s, such that

$$(4.10)\qquad \begin{aligned} &\text{(i)} \qquad \omega >> \frac{m\,c_{\infty}}{4\sqrt{M}}\ \frac{(1+4s+8s^2)}{(1+4s)} \ , \\ &\text{(ii)} \qquad \omega >> \frac{m\,c_{\infty}\sqrt{M}\,(1+4s)}{2(M+1)} \ . \end{aligned}$$

From the above conditions we see that for fixed m and c_{∞}, the larger the parameter M, the greater the number of modes that are known to produce focusing.

Appendix A

We will find two linearly independent solutions of

$$\frac{d^2\phi}{d\zeta^2} + \left[\frac{\nu(\nu-1)}{\cosh^2\zeta} - \mu^2\right]\phi = 0, \qquad -\infty < \zeta < \infty \tag{A.1}$$

(P. Epstein [8]).

Let

$$\eta = e^{2\zeta}, \tag{A.2}$$

and note that

$$\frac{d}{d\zeta} = 2\eta \frac{d}{d\eta}, \qquad \frac{d^2}{d\zeta^2} = 4\left[\eta^2 \frac{d^2}{d\eta^2} + \eta \frac{d}{d\eta}\right].$$

Then (A.1) becomes

$$\frac{d^2\phi}{d\eta^2} + \frac{1}{\eta}\frac{d\phi}{d\eta} + \left[\frac{-\mu^2}{4\eta^2} + \frac{\nu(\nu-1)}{\eta(\eta+1)^2}\right]\phi = 0. \tag{A.3}$$

Now let $\phi(\eta)$ be of the form

$$\phi(\eta) = (1+\eta)^{\alpha}\,\eta^{\beta}\,H(\eta) \tag{A.4}$$

where α and β are arbitrary constants. Upon substituting (A.4) into (A.3) and letting $\alpha = \frac{\mu}{2}$, $\beta = \nu$, we find that $H(\eta)$ satisfies the hypergeometric equation

$$v(1-v)\,H''(\eta) + \left[C-(A+B+1)v\right]H'(\eta) - A\,B\,H(\eta) = 0 \quad , \tag{A.5}$$

where

$$v = -\eta, \quad A = \mu+\nu, \quad B = \nu, \quad C = \mu+1.$$

Thus one solution of (A.1) is

$$\phi_1 = e^{\mu\zeta}\,(1+e^{\;2\zeta})^{\nu}\;F\,(\mu+\nu\,,\;\nu;\,\mu+1\,;\;-e^{2\zeta}).$$

If we assume that $\mathrm{Re}\{\mu\} > 0$, we see that $\lim\limits_{\zeta \to -\infty} [\,\phi_1\,] = 0.$

Similarly, if we let

$$\eta = e^{-2\zeta}\,,$$

we find that

$$\phi_2 = e^{-\mu\zeta}\,(1+e^{-2\zeta})^{\nu}\;F\,(\mu+\nu,\,\nu\,;\;\mu+1;\;-e^{-2\zeta})$$

is a solution of (A.1); this solution is such that $\lim\limits_{\zeta \to \infty} [\phi_2] = 0.$

Appendix B

To evaluate the Wronskian $W(\phi_1, \phi_2)$ we adopt the following notation:

$$
(\text{B.1}) \quad \begin{cases}
F_+ = F(\mu+\nu,\ \nu;\ \mu+1;\ -e^{-2\zeta}), \\
F_- = F(\mu+\nu,\ \nu;\ \mu+1;\ -e^{2\zeta}), \\
G_+ = F(\mu+\nu+1,\ \nu+1;\ \mu+2;\ -e^{-2\zeta}), \\
G_- = F(\mu+\nu+1,\ \nu+1;\ \mu+2;\ -e^{2\zeta}), \\
f = \dfrac{2(\mu+\nu)\nu}{\mu+1}, \\
g = \mu+2\nu .
\end{cases}
$$

Then in terms of the above notation:

$$\phi_1 = e^{\mu\zeta}(1+e^{2\zeta})^{\nu}\ F_- ,$$

$$\phi_2 = e^{-\mu\zeta}(1+e^{-2\zeta})^{\nu}\ F_+ ,$$

$$\phi_1' = e^{\mu\zeta}(1+e^{2\zeta})^{\nu-1}[\mu+g\,e^{2\zeta}]\,F_- - f e^{(2+\mu)\zeta}(1+e^{2\zeta})^{\nu}\ G_- ,$$

$$\phi_2' = -e^{-\mu\zeta}(1+e^{-2\zeta})^{\nu-1}[\mu+g\,e^{-2\zeta}]\,F_+ + f e^{-(2+\mu)\zeta}(1+e^{-2\zeta})^{\nu}\ G_+ ;$$

and $W(\phi_1, \phi_2)$ can be written as

$$(\text{B.2}) \quad W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = 2^{\nu}(1+\cosh 2\zeta)^{\nu}\left\{f[e^{-2\zeta}G_+F_- + e^{2\zeta}F_+G_-] - (\mu+g)F_+F_-\right\}.$$

Since the system is self-adjoint $W(\phi_1, \phi_2) =$ constant, and we will evaluate it as $\zeta \to +\infty$.

From the series definition of the hypergeometric function $F(a, b;c;z)$,

$$F_+ = 1 + O(e^{-2\zeta}), \quad G_+ = 1 + O(e^{-2\zeta})$$

as $\zeta \to +\infty$. Using the analytic continuation of the hypergeometric function (see [11]), we can express F_- and G_- in terms of hypergeometric functions with argument $e^{-2\zeta}$:

$$\text{(B.3)} \qquad \left\{ \begin{aligned} F_- &= B_1 e^{-2(\mu+\nu)\zeta} F_+ + B_2 e^{-2\nu\zeta} K_+ , \\ G_- &= e^{-2\zeta}\{B_3 e^{-2(\mu+\nu)\zeta} H_+ + B_4 e^{-2\nu\zeta} I_+\} , \end{aligned} \right.$$

where

$$K_+ = F(\nu, \nu-\mu; 1-\mu; -e^{-2\zeta}) ,$$

$$H_+ = F(\mu+\nu+1, \nu; \mu+1; -e^{-2\zeta}) ,$$

$$I_+ = F(\nu+1, \nu-\mu; 1-\mu; -e^{-2\zeta}),$$

and B_1, B_2, B_3, B_4 are functions of μ and ν. Substituting (B.3) into (B.2) we obtain for $W(\phi_1, \phi_2)$,

$$\text{(B.4)} \qquad W(\phi_1, \phi_2) = 2^\nu \left[\tfrac{1}{2} + e^{-2\zeta} + \frac{e^{-4\zeta}}{2}\right]^\nu \Big[f\{e^{-2\zeta}(B_1 e^{-2\mu\zeta} F_+ G_+ + B_2 K_+ G_+) + (B_3 e^{-2\mu\zeta} H_+ F_+ + B_4 I_+ F_+)\} - (\mu+g)(B_1 e^{-2\mu\zeta} F_+^2 + B_2 K_+ F_+)\Big] ;$$

and noting that

$$K_+ = 1 + O(e^{-2\zeta}), \quad H_+ = 1 + O(e^{-2\zeta}), \quad I_+ = 1 + O(e^{-2\zeta})$$

as $\zeta \to \infty$, we find that

$$W(\phi_1, \phi_2) = \lim_{\zeta\to\infty} \Big[W(\phi_1, \phi_2) \Big] = \frac{-2\Gamma^2(\mu+1)}{\Gamma(\mu+1-\nu)\,\Gamma(\mu+\nu)} .$$

Appendix C

It will be shown that the poles of the gamma functions $\Gamma(\mu+1-\nu)$ and $\Gamma(\mu+\nu)$ lie on the real axis of the ξ-plane in the range $-v \leq \mathrm{Re}\,\xi \leq v$. In order to carry out this analysis we must know where in the ξ-plane the real and imaginary parts of μ and δ assume positive and negative values. This is illustrated in Figure C-1.

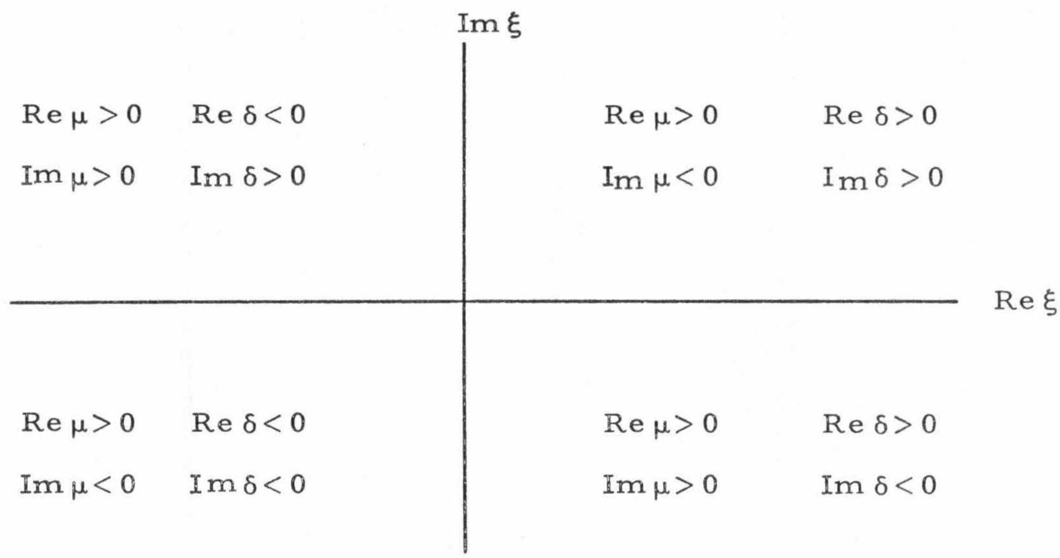

Figure C.1

First we will show that there are no poles in the first quadrant ($\mathrm{Re}\,\xi > 0$, $\mathrm{Im}\,\xi > 0$). Since, in this quadrant, $\mathrm{Re}\,\mu > 0$ and $\mathrm{Re}\,\delta > 0$, the only possible poles are at those ξ such that $\mu + \frac{1}{2} - \frac{\delta}{2} = -s$, $s = 0, 1, 2, \ldots$; however, since $\mathrm{Im}\,\mu < 0$ and $\mathrm{Im}\,\delta > 0$, $\mathrm{Im}\,\mu \neq \mathrm{Im}\,\frac{\delta}{2}$. Therefore, $\mu + \frac{1}{2} - \frac{\delta}{2} \neq -s$, $s = 0, 1, 2, \ldots$ in the first quadrant and there are no poles. In the second quadrant $\mathrm{Re}\,\mu > 0$ and $\mathrm{Re}\,\delta < 0$; therefore, the only possible poles are at those ξ such that $\mu + \frac{1}{2} + \frac{\delta}{2} = -t$, $t = 0, 1, 2, \ldots$. But here $\mathrm{Im}\,\mu < 0$ and $\mathrm{Im}\,\delta > 0$; thus $\mathrm{Im}\,\mu \neq -\mathrm{Im}\,\frac{\delta}{2}$; again we observe that there are no poles. Similarly, it can be shown that there are no poles of the gamma functions in the third or fourth quadrants. Now consider the portion of the imaginary axis $\text{Im}\,\xi > 1$. Here $\text{Re}\,\delta = 0$, and $\text{Re}\,\mu > 0$; therefore, (2.19) cannot be satisfied for any ξ. Thus there are no poles. For the portion of the imaginary axis $0 < \text{Im}\,\xi \leq 1$ we have two cases to consider. First we approach the axis from the right. For this case $\text{Im}\,\mu = \text{Im}\,\delta = 0$, $\text{Re}\,\mu > 0$, $0 \leq \delta \leq 1$; therefore, (2.19) cannot be satisfied. Similarly, as the axis is approached from the left, $\text{Im}\,\mu = \text{Im}\,\delta = 0$, $\text{Re}\,\mu > 0$, $-1 \leq \delta \leq 0$; again (2.19) is not satisfied. On the portion of the real axis $\text{Re}\,\xi > v$, $\text{Re}\,\mu = 0$ and $\text{Re}\,\delta > 0$; therefore, poles may exist at those ξ such that $\mu + \frac{1}{2} - \frac{\delta}{2} = -s$, $s = 0, 1, 2, \ldots$. However, $\text{Im}\,\delta = 0$, and $\text{Im}\,\mu < 0$ or $\text{Im} > 0$, depending upon the direction from which the axis is approached. Again there are no poles. Similarly it can be shown that there are no poles on the portion of the real axis $\text{Re}\,\xi < -v$. Finally, upon applying the above analysis for those ξ on the remaining part of the real axis,we find that there are poles of $\Gamma(\mu+\nu)$ on $-v \leq \text{Re}\,\xi < 0$ and poles of $\Gamma(\mu+1-\nu)$ on $0 < \text{Re}\,\xi \leq v$.

Appendix D

We will show that if $F(\alpha\xi) \sim O(|\xi|^{-\epsilon})$ as $|\xi| \to \infty$ where $\epsilon > 1$, then

$$I_R = \int_{-R+i\beta}^{-R} \phi(0, \gamma v, \alpha\xi) e^{i\xi\tau} d\xi \to 0, \quad \text{as} \quad R \to \infty \quad . \tag{D.1}$$

The same analysis can be used to show that the integrals over p_{3R} and p_ρ vanish in the appropriate limits.

Noting that

$$F(\mu+\nu, \nu; \mu+1; -1) = \frac{2^{-(\mu+\nu)}\sqrt{\pi}\,\Gamma(\mu+1)}{\Gamma(1+\frac{\mu-\nu}{2})\,\Gamma(\frac{1+\mu+\nu}{2})} \quad , \tag{D.2}$$

upon setting $z = 0$ in (2.14) we obtain

$$\phi(0, \gamma v, \alpha\,\xi) = \frac{F(\alpha\xi)\,\Gamma(\mu+1-\nu)\,\Gamma(\mu+\nu)}{m2^{2\mu+1}\,\Gamma^2(1+\frac{\mu-\nu}{2})\,\Gamma^2(\frac{1+\mu+\nu}{2})} \quad . \tag{D.3}$$

Substituting (D.3) into (D.1) and letting $\xi = -R+i\eta$, we obtain

$$I_R = \frac{1}{2mi}\int_0^\beta \frac{F[\alpha(-R+i\eta)]\,\Gamma(\mu+1-\nu)\Gamma(\mu+\nu)\,e^{iR\tau}\,e^{\eta\tau}}{2^{2\mu}\,\Gamma^2(1+\frac{\mu-\nu}{2})\,\Gamma^2(\frac{1+\mu+\nu}{2})}\, d\eta \quad , \tag{D.4}$$

where δ and μ are functions of $(-R+i\eta)$. It is obvious that $\delta = (1+\xi^2)^{1/2} \sim \xi$ as $|\xi| \to \infty$, thus $\delta \sim -R+i\eta$ as $R \to \infty$. To evaluate μ as $|\xi| \to \infty$ on p_{3L}, refer to Figure D-1.

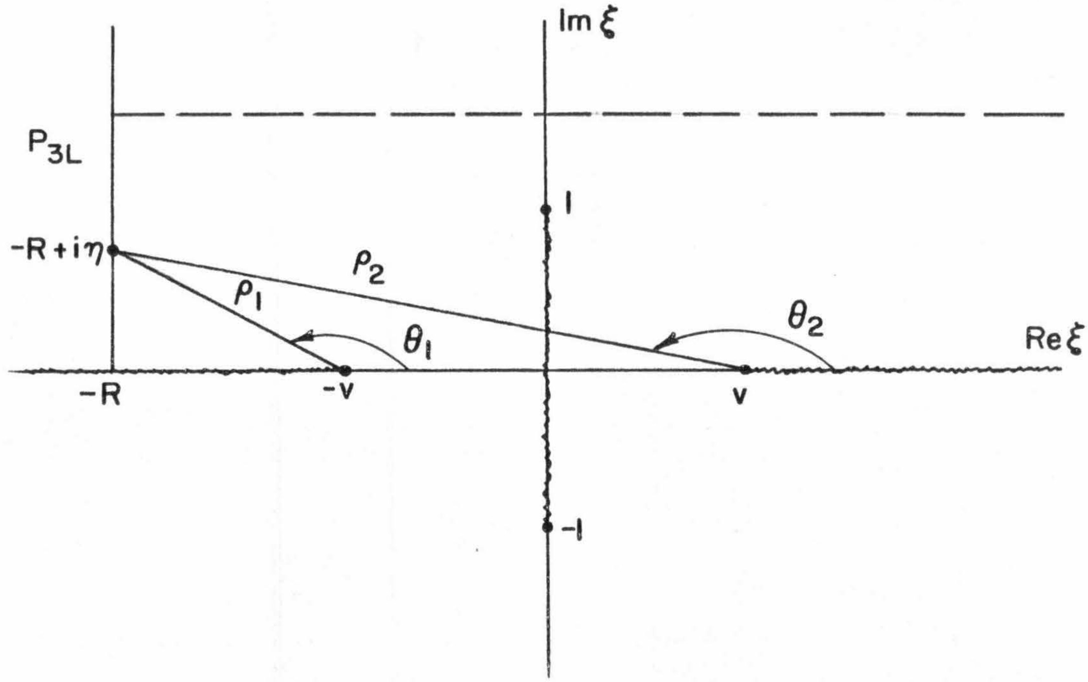

FIG. D.1

Therefore,

$$\text{(D.5)} \qquad \mu = \frac{1}{2\sqrt{M}}\,(\xi+v)^{1/2}(\xi-v)^{1/2} = \frac{\sqrt{\rho_1\rho_2}}{2\sqrt{M}}\left[\sin\left(\frac{\theta_1+\theta_2}{2}\right)-i\,\cos\left(\frac{\theta_1+\theta_2}{2}\right)\right],$$

where

$$\rho_1 = \sqrt{\eta^2+(R-v)^2}\,, \qquad \rho_2 = \sqrt{\eta^2+(R+v)^2} \quad .$$

Expressing μ in terms of the angle $\theta_1+\theta_2$, we find that

$$\text{(D.5)} \qquad \mu = \sqrt{\frac{\rho_1\rho_2}{8M}}\left[\sqrt{1-\cos(\theta_1+\theta_2)}+i\,\sqrt{1+\cos(\theta_1+\theta_2)}\right];$$

and noting that

$$\cos(\theta_1+\theta_2) = \frac{R^2-v^2-\eta^2}{\rho_1\rho_2} \quad ,$$

then

$$\text{(D.6)} \qquad \mu = \frac{1}{\sqrt{8M}} \left[\sqrt{\rho_1\rho_2 - R^2 + v^2 + \eta^2} + i\sqrt{\rho_1\rho_2 + R^2 - v^2 - \eta^2} \right].$$

Letting $R \to \infty$,

$$\rho_1\rho_2 = R^2 + \eta^2 - v^2 + o(1) ,$$

and

$$\mu \sim \frac{1}{2\sqrt{M}} [\eta + iR] .$$

Thus

$$\text{(D.7)} \qquad \left\{ \begin{array}{lll} \mu + 1 - \nu & \sim & \frac{R}{2}\left[1 + \frac{i}{\sqrt{M}}\right] \\ \mu + \nu & \sim & \frac{R}{2}\left[\frac{i}{\sqrt{M}} - 1\right] \\ 1 + \frac{\mu+\nu}{2} & \sim & \frac{R}{4}\left[1 + \frac{i}{\sqrt{M}}\right] \\ \frac{1+\mu+\nu}{2} & \sim & \frac{R}{4}\left[\frac{i}{\sqrt{M}} - 1\right] \end{array} \right.$$

as $R \to \infty$. Using the asymptotic expansion of the gamma function:

$$\Gamma(z) \sim \sqrt{2\pi}\; e^{-z} z^{z-\frac{1}{2}} \quad \text{as } z \to \infty \text{ in } |\arg z| < \pi ,$$

we obtain as $R \to \infty$

$$\frac{\Gamma(\mu+1-\nu)\,\Gamma(\mu+\nu)}{\Gamma^2(1+\frac{\mu-\nu}{2})\,\Gamma^2(\frac{1+\mu+\nu}{2})} \sim \frac{i}{8\pi}\sqrt{\frac{1+M}{M}}\; R\, 2^{\frac{2Ri}{\sqrt{M}}} .$$

Finally

$$I_R \sim \frac{1}{16\pi m}\sqrt{\frac{M+1}{M}} \int_0^\beta F\left[\alpha(-R+i\eta)\right] 2^{-\frac{1}{\sqrt{M}}\eta}\, e^{\eta\tau}\, 2^{\frac{iR}{\sqrt{M}}}\, e^{iR\tau}\, R\, d\eta$$

as $R \to \infty$; therefore, if $F\left[\alpha(-R+i\eta)\right] \sim O(R^{-\epsilon})$ as $R \to \infty$ $(\epsilon > 1)$, then

$$I_R \to 0.$$

Appendix E

We will calculate the residues from the contour integration

$$\oint vH_0^{(1)}(\gamma r v)\,\phi(0,\gamma v,\alpha\xi)\,dv \tag{E.1}$$

for two separate cases: $\mathrm{Re}\,\xi < 0$ and $\mathrm{Re}\,\xi > 0$.

We consider the case $\mathrm{Re}\,\xi > 0$, which is illustrated in Figure 2-3. It has been noted that the poles of the above integrand are at

$$\mu(v_t) + \tfrac{1}{2} + \frac{\delta}{2} = -t\,, \quad t = 0,1,2,\ldots, M(\xi)\,, \tag{E.2}$$

or

$$v_t = \left[\, \xi^2 + M(\delta+1+2t)^2 \,\right]^{1/2} . \tag{E.3}$$

Since the poles of the gamma function are simple poles, we obtain for the residue, R_t, at the pole $v = v_t$:

$$R_t = \frac{\Gamma(-\delta-t)\, K(v_t;\xi)}{\left.\dfrac{d}{dv}\left(\dfrac{1}{\Gamma(\mu+\frac{1}{2}+\frac{\delta}{2})}\right)\right|_{v=v_t}} , \tag{E.4}$$

where

$$K(v;\xi) = \frac{vH_0^{(1)}(\gamma r v)\,\phi(0,\gamma v,\alpha\xi)}{\Gamma\left(\mu+\frac{1}{2}+\frac{\delta}{2}\right)\Gamma(\mu+\frac{1}{2}-\frac{\delta}{2})} .$$

To evaluate the denominator of (E.4), we let $\eta = \frac{1}{2} - \frac{\delta}{2} - \mu$ and note the identity

$$\Gamma(1-\eta) = \frac{\pi}{\Gamma(\eta)\sin(\pi\eta)} .$$

Then

$$\Gamma(\mu+\tfrac{1}{2}+\tfrac{\delta}{2}) = \frac{1}{\Gamma(\eta)\sin(\pi\eta)} \quad ,$$

and

$$\text{(E.5)} \qquad \frac{d}{dv}\left(\frac{1}{\Gamma(\mu+\frac{1}{2}+\frac{\delta}{2})}\right)\Bigg|_{v=v_t} = \frac{d}{d\eta}\Big(\Gamma(\eta)\sin(\pi\eta)\Big)\Bigg|_{\eta=1+t}\left(\frac{d\eta}{dv}\right)\Bigg|_{v=v_t} \quad ,$$

since at $v = v_t$, $\eta = 1+t$. Using the expressions:

$$\mu^2 = \left(\tfrac{1}{2}-\tfrac{\delta}{2}-\eta\right)^2 \quad ,$$

$$\frac{d(\mu^2)}{dv}\Bigg|_{v=v_t} = -2\mu(v_t)\left(\frac{d\eta}{dv}\right)\Bigg|_{v=v_t} \quad ,$$

$$\frac{d(\mu^2)}{dv}\Bigg|_{v=v_t} = \frac{v_t}{2M} \quad ,$$

we obtain

$$\text{(E.6)} \qquad \frac{d}{dv}\left(\frac{1}{\Gamma(\mu+\frac{1}{2}+\frac{\delta}{2})}\right)\Bigg|_{v=v_t} = \frac{(-1)^t\,\Gamma(1+t)\,v_t}{4M\,\mu(v_t)} \quad .$$

Finally upon substituting (E.6) into (E.4), we obtain

$$\text{(E.7)} \qquad R_t = \frac{4M}{m}\,\frac{(-1)^t\,2^{2t-1-a+bi}(a-1-2t-bi)\,\Gamma(a-t-bi)\,H_0^{(1)}(\gamma r v_t)F(\alpha\xi)}{\Gamma(1+t)\,\Gamma^2(\frac{1-t}{2})\,\Gamma^2\left(\frac{a+1-t-bi}{2}\right)} , t=0,1,2,\ldots,M(\xi).$$

For the case $\operatorname{Re}\xi > 0$, which is illustrated in Figure 2-4, we use the same analysis; however, now the poles of the integrand in (E.1) are at those $v = v_s$ such that

(E.8) $$\mu(v_s) + \tfrac{1}{2} - \frac{\delta}{2} = -s, \quad s = 0, 1, 2, \ldots, M(\xi),$$

or

$$v_s = \left[\xi^2 + M(\delta - 1 - 2s)^2\right]^{1/2} .$$

In this case we obtain for the residues,

$$R_s = \frac{4M}{m} \frac{(-1)^s \; 2^{2s-1-a-bi}(a-1-2s+bi)\,\Gamma(a-s+bi)H_0^{(1)}(\gamma r v_s)F(\alpha\xi)}{\Gamma(1+s)\,\Gamma^2(\frac{1-s}{2})\,\Gamma^2(\frac{a+1-s+bi}{2})}, s=0,1,2,\ldots,M(\xi).$$

Appendix F

We will show that the poles of $\Gamma(\mu+\nu)$ lie in the second quadrant of the v-plane for those ξ on the path p_{1L} in the ξ-plane; and as $\epsilon \rightarrow 0$, the poles approach the negative real axis. For those ξ on the path p_{1R}, we will show that the poles of $\Gamma(\mu+1-\nu)$ lie in the first quadrant of the v-plane; and as $\epsilon \rightarrow 0$, these poles approach the positive real axis.

We recall from Appendix E that the poles of $\Gamma(\mu+\nu)$ are

$$\text{(F.1)} \qquad v_t = \left[\xi^2 + M(\delta+1+2t)^2\right]^{1/2}, \quad \text{for } \operatorname{Re}\xi < 0 \ (\xi \text{ on } p_{1L}),$$

and the poles of $\Gamma(\mu+1-\nu)$ are

$$\text{(F.2)} \qquad v_s = \left[\xi^2 + M(\delta-1-2s)^2\right]^{1/2}, \quad \text{for } \operatorname{Re}\xi > 0 \ (\xi \text{ on } p_{1R}).$$

Letting

$$a_t = 1 + 2t, \quad B = 1 + \frac{1}{M} \quad ,$$

we can write (F.1) and (F.2) as

$$\text{(F.3)} \qquad v_t = M^{1/2}(Y_t)^{1/2}, \quad \text{where } Y_t = B\xi^2 + 1 + a_t^2 - 2a_t(1+\xi^2)^{1/2},$$

and

$$\text{(F.4)} \qquad v_s = M^{1/2}(Z_s)^{1/2}, \quad \text{where } Z_s = B\xi^2 + 1 + a_s^2 + 2a_s(1+\xi^2)^{1/2}.$$

On path p_{1R}, $\xi = \xi_1 + i\xi_2$ $(\xi_1 > 0, \xi_2 > 0)$; and from Figure F-1, we note that $(1+\xi^2)^{1/2} = \sqrt{\rho_1 \rho_2}\ e^{i\eta}$. Assuming that $\eta << 1$, we can show that

$$\text{(F.5)} \qquad \rho_1\rho_2 = (1+\xi_1^2)\left[1 + O(\eta^2)\right], \quad \xi_2 = \frac{(1+\xi_1^2)}{\xi_1}\,\eta\left[1 + O(\eta)\right].$$

On path p_{1L}, $\xi = -\xi_1 + i\xi_2$, $(1+\xi^2)^{1/2} = -\sqrt{\rho_1\rho_2}\, e^{-i\eta}$; and for $\eta \ll 1$, $\rho_1\rho_2$ and ξ_2 are given by (F.5).

Noting that for $\eta << 1$,

$$
\text{(F.6)} \qquad
\begin{aligned}
(1+\xi^2)^{1/2} &\sim \sqrt{1+\xi^2}\,(1+i\eta)\,, && (\xi \text{ on } p_{1R})\,,\\
(1+\xi^2)^{1/2} &\sim \sqrt{1+\xi^2}\,(-1+i\eta)\,, && (\xi \text{ on } p_{1L})\,,
\end{aligned}
$$

we obtain

$$
\text{(F.7)} \quad Y_t = B\xi_1^2 + 1 - 2a_t\sqrt{1+\xi_1^2} + a_t^2 - B\frac{(1+\xi_1^2)^2}{\xi_1^2}\eta - 2\sqrt{1+\xi_1^2}\,\eta(B\sqrt{1+\xi_1^2} - a_t)\,i\,,
$$

$$
\text{(F.8)} \quad Z_s = B\xi_1^2 + 1 - 2a_s\sqrt{1+\xi_1^2} + a_s^2 - B\frac{(1+\xi_1^2)^2}{\xi_1^2}\eta + 2\sqrt{1+\xi_1^2}\,\eta(B\sqrt{1+\xi_1^2} - a_s)\,i\,.
$$

Now

$$
B\xi_1^2 + 1 - 2a_t\sqrt{1+\xi_1^2} + a_t^2 > \left(\sqrt{1+\xi_1^2} - a_t\right)^2 > 0,
$$

$$
B\sqrt{1+\xi_1^2} - a_t > \sqrt{1+\xi_1^2} - a_t \geq 0\,;
$$

therefore, for $\epsilon << 1$ ($\eta << 1$), Y_t and Z_s can be represented by the complex numbers:

$$
Y_t = \alpha - i\beta\,, \quad Z_s = \alpha + i\beta \quad (\alpha > 0,\ \beta > 0)\,.
$$

Thus the poles of $\Gamma(\mu+\nu)$ lie in the second quadrant and the poles of $\Gamma(\mu+1-\nu)$ lie in the first quadrant; and in the limit as $\epsilon \to 0$ ($\eta \to 0$),

$$
\begin{aligned}
\nu_t &\to -\sqrt{(1+M)\xi_1^2 + 1 - 2a_t\sqrt{1+\xi_1^2} + a_t^2}\\
\nu_s &\to +\sqrt{(1+M)\xi_1^2 + 1 - 2a_s\sqrt{1+\xi_1^2} + a_s^2}\,.
\end{aligned}
$$

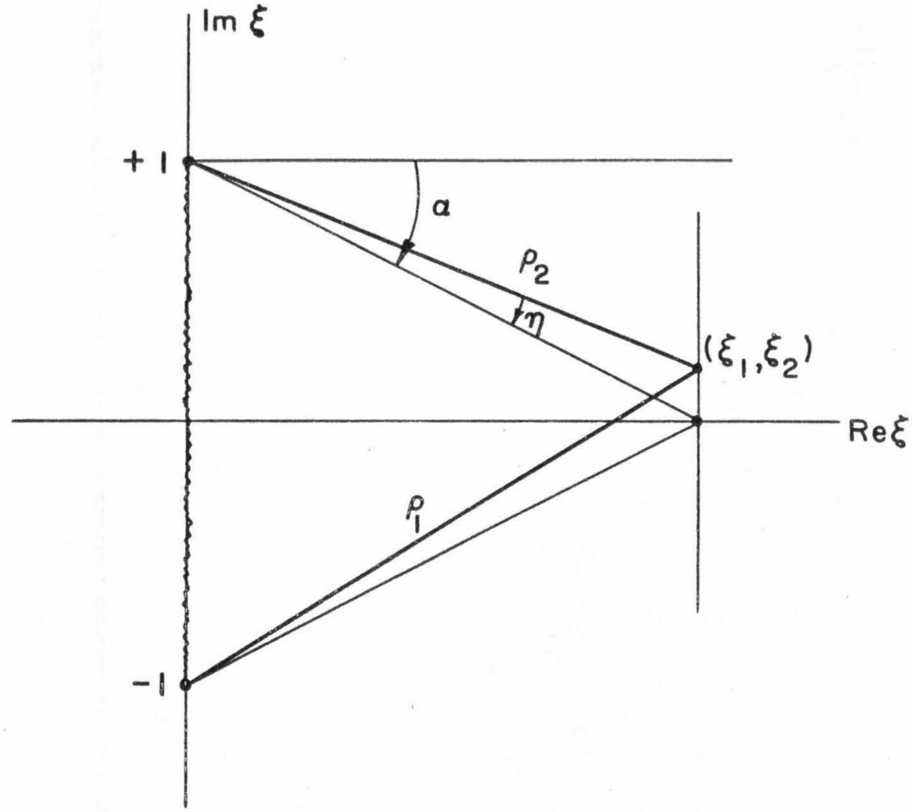

FIG. F.1

Appendix G

We will show that, for ξ on the path p_{2L}, then

$$G(r,\xi) = \int_{-\infty}^{\infty} vH_0^{(1)}(\gamma rv)\phi(0,\gamma v,\alpha\xi)dv = o\left(\frac{1}{r^{1/2}}\right) \tag{G.1}$$

as $r \to \infty$. On p_{2L}, $\xi = i\xi_2\,(0<\xi_2<1)$, and $\delta = -\sqrt{1-\xi_2^2}$. As before, we will evaluate $G(r,\xi)$ by contour integration in the v-plane. From the requirement that $\text{Re}\{\mu\} > 0$, we take the branch cuts as illustrated in Figure G-1.

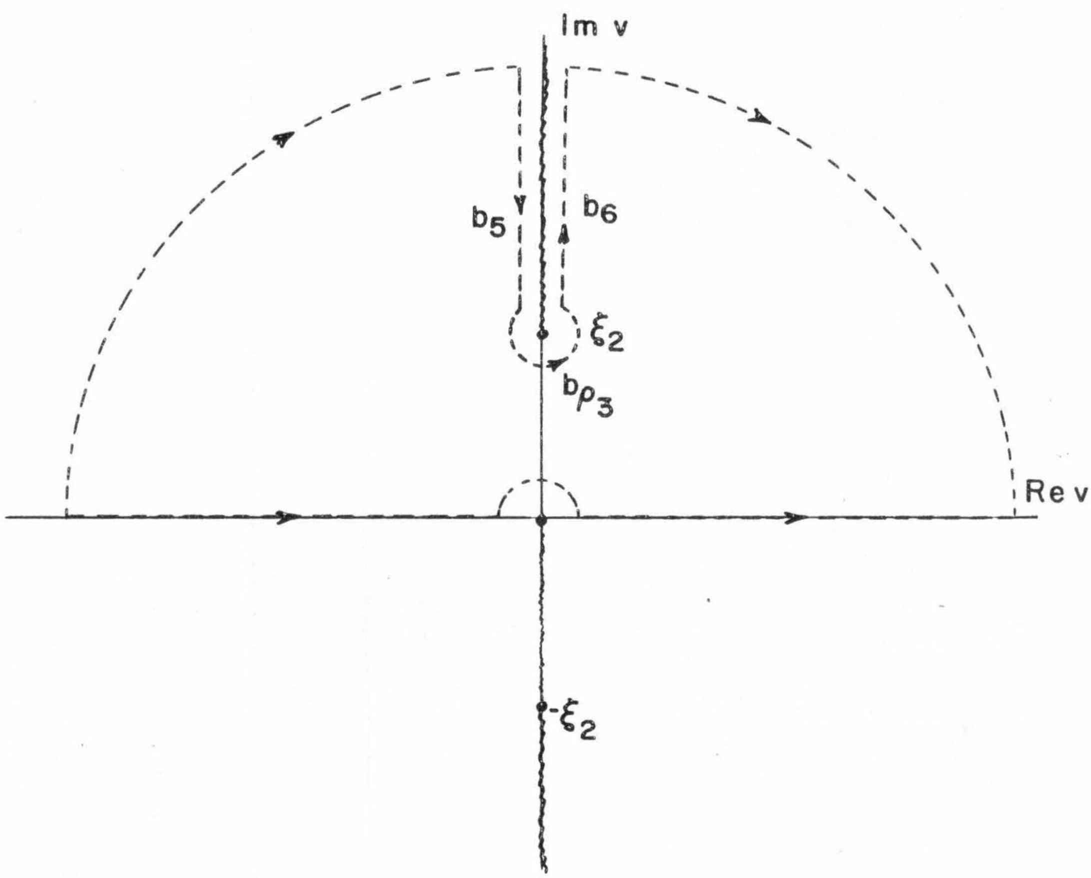

FIG. G.1

Since $-1 < \delta < 0$, the integrand of (G.1) does not have any poles in the complex v-plane; therefore we proceed in the usual way to obtain

(G.2) $$G(r,\xi) = \int_{b_5 + b_6} vH_0^{(1)}(\gamma rv)\,\phi\,dv\ .$$

Now if we let $v = e^{\frac{\pi i}{2}}\nu$, and note that

$$\mu = \frac{-i}{2\sqrt{M}}\sqrt{\nu^2 - \xi_2^2} \quad \text{on } b_5\ ,$$

and

$$\mu = \frac{i}{2\sqrt{M}}\sqrt{\nu^2 - \xi_2^2} \quad \text{on } b_6\ ,$$

then $G(r,\xi)$ can be written as

(G.3) $$G(r,\xi) = 2F(\alpha\xi)\,i\int_{\xi_2}^{\infty} \nu\, H_0^{(1)}(\gamma r e^{\frac{i\pi}{2}}\nu)\,\theta\, d\nu,$$

where

$$\theta = \mathrm{Im}\left[\frac{\phi}{F(\alpha\xi)}\right].$$

Confining ourselves to the far field we obtain

(G.4) $$G(r,\xi) \sim \sqrt{\frac{8}{\pi\gamma r}}\,F(\alpha\xi)\int_{\xi_2}^{\infty} \nu^{1/2}\, e^{-\gamma r\nu}\,\theta\, d\nu;$$

and since

$$\int_{\xi_2}^{\infty} \nu^{1/2}\, e^{-\gamma r \nu}\,\theta\, d\nu \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty,$$

then

$$G(r,\xi) = o\left(\frac{1}{r^{1/2}}\right).$$

By the same analysis we can show that as $r \rightarrow \infty$,

$$G(r,\xi) \sim o\left(\frac{1}{r^{1/2}}\right),$$

for ξ on the path p_{2R}.

Appendix H

The asymptotic expansion obtained in Section 3 for $t < T$ breaks down as $t \to T$. We will develop an expansion which can be used for those t approaching T but not equal to T. Again we consider the integral for I_s:

$$\text{(H.1)} \qquad I_s = \int_{C_s}^{\infty} Q_s(\xi)\, F(\alpha\xi)\, e^{irf_s(\xi)}\, d\xi \,,$$

where

$$Q_s(\xi) = \frac{q_s(\xi)}{F(\alpha\xi)}\,, \quad C_s = \sqrt{(4s+1)^2-1}\,, \quad f_s(\xi) = \frac{m}{4}\left[\sqrt{1+\xi^2} - a_{2s} - b\xi\right], \; b = \frac{t}{T} \,.$$

We have previously found that the stationary phase point of the above integral is $\xi = \dfrac{b}{\sqrt{1-b^2}}$, and our immediate interest is to determine how large r must be to obtain an asymptotic expansion of I_s for b near 1 and fixed. We will now manipulate the above integral (H.1) to make the point of stationary phase independent of b. Let

$$\xi = \sinh\rho, \qquad \frac{b}{\sqrt{1-b^2}} = \cosh\zeta, \qquad \rho_s = \sinh^{-1}(C_s) \;;$$

therefore, (H.1) becomes

$$\text{(H.2)} \quad I_s = e^{-\frac{im}{4} ra_{2s}} \int_{\rho_s}^{\infty} Q_s[\sinh\rho]\, F[\alpha\sinh\rho]\, e^{\frac{im}{4} r\sqrt{1-b^2}\cosh(\rho-\zeta)} \cosh\rho\, d\rho \,.$$

Now upon letting

$$I_s = e^{-\frac{im}{4} ra_{2s}}\, J_s, \quad \varphi = \rho - \zeta \,,$$

we obtain

$$\text{(H.3)}\quad J_s = \int_{\rho_s - \zeta}^{\infty} Q_s[\sinh(\varphi+\zeta)]\, F[\alpha \sinh(\varphi+\zeta)]\, e^{ir\sqrt{1-b^2}\frac{m}{4}\cosh\varphi} \cosh(\varphi+\zeta)d\varphi .$$

Now we apply stationary phase as the new variable $r\sqrt{1-b^2} \to \infty$, obtaining

$$J_s \sim \sqrt{\frac{8\pi}{mr\sqrt{1-b^2}}}\; Q_s\left[\frac{b}{\sqrt{1-b^2}}\right] F\left[\frac{\alpha b}{\sqrt{1-b^2}}\right] \frac{1}{\sqrt{1-b^2}}\, e^{i\left(\frac{m}{4}r\sqrt{1-b^2}+\frac{\pi}{4}\right)} .$$

This expansion is valid if $F(\omega) \sim O\left(\frac{1}{\omega^\alpha}\right)$, $\alpha \geq 1$ as $\omega \to \infty$, since $Q_s[\chi] \to$ constant as $\chi \to \infty$.

In summary, the above analysis shows how large r must be in order to obtain an asymptotic expansion for I_s for t near T.

Appendix I

We will evaluate expression (3.1) as $r \to \infty$ for $t = T$, and we will show that the series in (2.31) converges for all r (large) except possibly at a denumerable number of points.

We recall that for $t = T$

$$I_s = e^{-ir\frac{m}{4}a_{2s}} \int_{C_s}^{\infty} Q_s(\xi) F(\alpha\xi) e^{ir\frac{m}{4}h(\xi)} d\xi , \tag{I.1}$$

where $Q_s(\xi)$ and C_s have been defined in Appendix H, and $h(\xi)=\sqrt{1+\xi^2}-\xi$. Also let

$$J = \sum_{s=0}^{\infty} A_s I_s , \tag{I.2}$$

where

$$A_s = \frac{2^{4s}}{\Gamma(1+2s)\,\Gamma^2(\frac{1}{2}-s)} .$$

Observing the behavior of the phase function $h(\xi)$ in Figure I-1, we see that (i) $h'(\xi) < 0$ for all finite ξ, (ii) $\lim_{\xi\to\infty} h'(\xi) = \lim_{\xi\to\infty} h(\xi) = 0$, (iii) $h(C_s) = a_{2s} - \sqrt{a_{2s}^2 - 1}$.

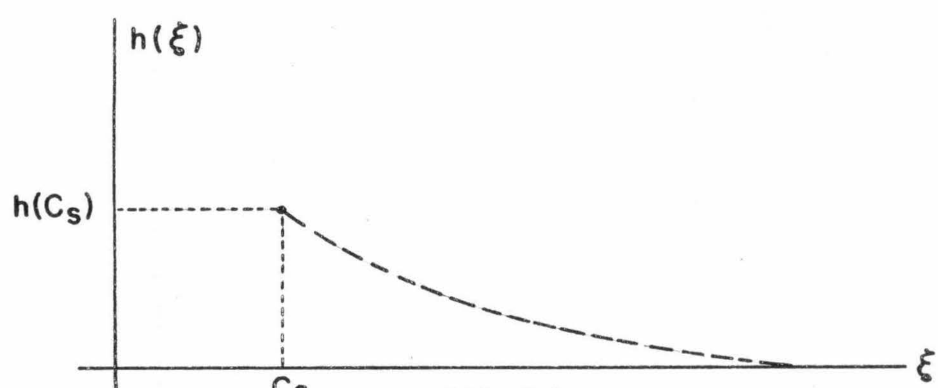

FIG. I.1

Letting

$$\eta = h(\xi), \ \eta_s = h(C_s), \ \tilde{r} = \frac{mr}{4}, \ I_s = e^{ir\frac{m}{4}a_{2s}} J_s ,$$

and noting that

$$\xi = \frac{1-\eta^2}{2\eta} , \qquad d\xi = -\frac{(1+\eta^2)\,d\eta}{2\eta^2} ,$$

then

$$\text{(I.3)} \qquad J_s = \frac{1}{2}\int_0^{\eta_s} Q_s\left[\frac{1-\eta^2}{2\eta}\right] F\left[\alpha\left(\frac{1-\eta^2}{2\eta}\right)\right] \frac{(1+\eta^2)}{\eta^2}\, e^{i\tilde{r}\eta}\, d\eta .$$

Now if we assume that as $\eta \to 0$,

$$\text{(I.4)} \qquad F\left[\alpha\left(\frac{1-\eta^2}{2\eta}\right)\right] \sim \eta^{\chi}\, \mathcal{J}(\eta)$$

for $1 < \chi < 2$, and where $\lim_{\eta \to 0} \mathcal{J}(\eta) = \text{const} \neq 0$, then J_s becomes

$$\text{(I.5)} \qquad J_s = \int_0^{\eta_s} \Phi_s(\eta)\, \eta^{\lambda-1}\, e^{i\tilde{r}\eta}\, d\eta ,$$

where

$$\Phi_s(\eta) = \frac{Q_s\left[\frac{1-\eta^2}{2\eta}\right](1+\eta^2)\,\mathcal{J}(\eta)}{2} ,$$

and

$$0 < \lambda < 1.$$

The above integral is merely a Fourier integral with a singularity at one end and can be evaluated asymptotically as $\tilde{r} \to \infty$ (Erdélyi [15]):

$$J_s \sim \frac{\Gamma(\lambda)\, e^{\frac{\pi\lambda}{2}i}\, \Phi_s(0)}{\tilde{r}^{\lambda}}$$

or

$$J_s \sim \frac{\Gamma(\lambda)\left(\frac{4}{m}\right)^{\lambda} e^{\frac{\pi\lambda}{2}i} Q_s(\infty)\, \mathcal{J}(0)}{2r^{\lambda}} \qquad (I.6)$$

Now using the expansions as $z \to \infty$:

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}} \sqrt{2\pi} \quad , \qquad (I.7)$$

$$(z+\alpha)^{z+\beta} \sim z^{z+\beta} e^{\alpha} , \qquad (I.8)$$

we find that

$$\lim_{\xi \to \infty \ (\text{fixed } s)} Q(\xi) = \frac{2^{-2s}}{\sqrt{2\pi m}} \quad . \qquad (I.9)$$

Finally we can write for the first term in the asymptotic expansion for I_s as $r \to \infty$:

$$I_s \sim \Theta(r)\, 2^{-2s} e^{-irms} \quad , \qquad (I.10)$$

where

$$\Theta(r) = \frac{e^{\frac{i}{2}(\pi\lambda - \frac{mr}{2})}\, \Gamma(\lambda)\, \mathcal{J}(0)}{2^{3/2-2\lambda} m^{1/2+\lambda} \pi^{1/2} r^{\lambda}} \quad . \qquad (I.11)$$

Now that I_s has been determined asymptotically we return to the original series in expression (I. 2), and investigate the convergence of this series:

$$J \sim \Theta(r) \sum_{s=0}^{\infty} A_s 2^{-2s} e^{-irms} \quad . \qquad (I.12)$$

Define

$$\hat{J} = \sum_{s=0}^{\infty} A_s\, 2^{-2s} e^{-irms} \quad , \qquad (I.13)$$

$$\hat{J} = \sum_{s=0}^{\infty} A_s 2^{-2s} \cos(rms) - i \sum_{s=0}^{\infty} A_s 2^{-2s} \sin(rms) \ , \tag{I.14}$$

and note the following test for convergence:

Let $u_0 + u_1 + \cdots + u_n + \cdots$ be a series which converges or is indeterminate $(\sum_{s=0}^{N} u_s < A,$ independent of N). Let $\epsilon_0, \epsilon_1, \cdots, \epsilon_n, \cdots$ be a monotonically decreasing sequence of positive numbers which approach zero as n becomes infinite. Then the series

$$\sum_{s=0}^{\infty} \epsilon_s u_s$$

converges.

We will apply this test to both series in (I.14). First we will show that the quantity

$$B_s = A_s 2^{-2s} = \frac{2^{2s}}{\Gamma(1+2s)\,\Gamma^2(\frac{1}{2}-s)} \tag{I.15}$$

forms a decreasing sequence of numbers. Using the identity

$$\Gamma(1-\eta) = \frac{\pi}{\Gamma(\eta)\sin(\pi\eta)} \ ,$$

we obtain

$$B_s = \frac{2^{2s}\,\Gamma^2(s+\frac{1}{2})}{\pi^2\,\Gamma(1+s)} \ ; \tag{I.16}$$

and upon examining the ratio $\frac{B_{s+1}}{B_s}$, we find that

$$\frac{B_{s+1}}{B_s} = \frac{s+\frac{1}{2}}{s+1} < 1 \tag{I.17}$$

for all s. Thus the B_s form a decreasing sequence of numbers and as $s \to \infty$,

$$B_s \sim \frac{1}{\pi^{3/2}\, s^{1/2}} . \tag{I.18}$$

Now we must show that

$$\sum_{s=0}^{N} \cos(rms) < A^c \quad * \tag{I.19}$$

and

$$\sum_{s=0}^{N} \sin(rms) < A^s . \tag{I.20}$$

Consider the geometric series $H = \sum_{s=0}^{N} e^{i\theta s}$, with sum

$$H = \frac{1-e^{i\theta(N+1)}}{1-e^{i\theta}} . \tag{I.21}$$

Manipulating (I.21), we find that:

$$\sum_{s=0}^{N} \sin(\theta s) = \operatorname{Im} H = \frac{\sin\theta - \sin[(N+1)\theta] + \sin(N\theta)}{2\,(1-\cos\theta)} , \tag{I.22}$$

$$\sum_{s=0}^{N} \cos(\theta s) = \operatorname{Re} H = \frac{1-\cos\theta - \cos[(N+1)\theta] + \cos(N\theta)}{2\,(1-\cos\theta)} . \tag{I.23}$$

Therefore, upon using the identities:

(i) $1-\cos\theta = 2\sin^2\frac{\theta}{2}$

(ii) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$,

(iii) $\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$,

(iv) $\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$,

---

*Obviously for $mr = 2k\pi, k=0,1,2,\ldots,$ $\sum_{s=0}^{N} \cos(rms) = N+1$; thus the sum is not independent of N and we cannot prove convergence. In fact, for this case the first series in (I.14) diverges and the second one is zero.

we can easily show that

$$\sum_{s=0}^{N} \sin(\theta s) = \frac{\sin\frac{N\theta}{2}\sin\left[\frac{(N+1)\theta}{2}\right]}{\sin\frac{\theta}{2}} ,$$

and thus

$$\left|\sum_{s=0}^{N} \sin(\theta s)\right| \leqslant \frac{1}{\left|\sin\frac{\theta}{2}\right|} = A^{s} \text{ (independent of N) .}$$

Similarly, a bound for the cosine series can be determined which is independent of N.

In summary, we have taken the series in expression (I.2) and evaluated it asymptotically as $r \to \infty$ for $t = T$. We determined that

$$J \sim \theta(r) \sum_{s=0}^{\infty} A_s 2^{-2s} e^{-irms} ,$$

which converges for all r except for $r = 2k\pi$, $k = 0, 1, \ldots$. This result is valid for all forcing functions $f(t)$ such that $F(\alpha\xi) = F\left[\frac{\alpha(1-\eta^2)}{2\eta}\right] \sim \eta^{\chi}$ as $\eta \to 0$, where $1 < \chi < 2$.

APPENDIX J

We will show that the wave equation is a good approximation of the equation which is satisfied by the acoustic pressure in the ocean (see P. Bergmann [22]).

We consider the following equations for a perfect fluid:

(J.1) $$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\overline{u}) = 0$$ Continuity Equation

(J.2) $$\rho\left[\frac{\partial\overline{u}}{\partial t} + (\overline{u}\cdot\nabla)\overline{u}\right] + \nabla p + \rho\overline{g} = 0$$ Equations of Motion

(J.3) $$p = p(\rho, S)$$ Equation of State

where the density ρ, pressure p, velocity $\overline{u}$, and entropy S are functions of position $\overline{x}$ and time t; and $\overline{g}$ is the acceleration due to gravity.

Now, if we assume a reference state $p = p_0(\overline{x})$, $\rho = \rho_0(\overline{x})$, $\overline{u} = 0$, $S = S_0(\overline{x})$, we observe that the equation of continuity is identically satisfied, and the equations of motion and the equation of state become

(J.4) $$\nabla p_0 + \rho_0\overline{g} = 0,$$

(J.5) $$p_0 = p_0(\rho_0, S_0)\ .$$

Assuming constant entropy we can show that

(J.6) $$\frac{dp}{dt} = c^2(\rho)\frac{d\rho}{dt}\ ,$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\overline{u}\cdot\nabla)\ , \quad c^2(\rho) = \left(\frac{\partial p}{\partial\rho}\right)_S\ .$$

Thus the continuity equation becomes

$$\frac{dp}{dt} + \rho\, c^2(\rho)\, \mathrm{div}\,\overline{u} = 0. \tag{J.7}$$

We now linearize

$$\left\{\begin{aligned} &\frac{\partial p}{\partial t} + (\overline{u}\cdot\nabla)p + \rho\, c^2(\rho)\,\mathrm{div}\,\overline{u} = 0, \\ &\frac{\partial p}{\partial t} + (\overline{u}\cdot\nabla)p - c^2(\rho)\left[\frac{\partial \rho}{\partial t} + (\overline{u}\cdot\nabla)\rho\right] = 0, \\ &\rho\left[\frac{\partial \overline{u}}{\partial t} + (\overline{u}\cdot\nabla)\overline{u}\right] + \nabla p + \rho\,\overline{g} = 0 \end{aligned}\right. \tag{J.8}$$

about the above static solution. Therefore let

$$\left\{\begin{aligned} &p(\overline{x},t) = p_0(\overline{x}) + p_1(\overline{x},t), \\ &\rho(\overline{x},t) = \rho_0(\overline{x}) + \rho_1(\overline{x},t), \\ &\overline{u}(\overline{x},t) = \overline{u}_1(\overline{x},t), \end{aligned}\right. \tag{J.9}$$

where

$$\frac{|\overline{u}_1|}{c_0}, \quad \frac{p_1}{p_0}, \quad \frac{\rho_1}{\rho_0} << 1.$$

Upon substituting (J.9) into (J.8), and noting that $c^2(\rho) = c^2(\rho_0) + O(\rho_1)$, the terms of highest order are:

$$\left\{\begin{aligned} &\frac{\partial p_1}{\partial t} + (\overline{u}_1\cdot\nabla)p_0 + \rho_0\, c_0^2\,\mathrm{div}\,\overline{u}_1 = 0, \\ &\rho_0\frac{\partial \overline{u}_1}{\partial t} + \nabla p_1 + \rho_1\,\overline{g} = 0, \\ &\frac{\partial p_1}{\partial t} + \overline{u}_1\cdot\nabla p_0 - c_0^2\left[\frac{\partial \rho_1}{\partial t} + \overline{u}_1\cdot\nabla\rho_0\right] = 0, \end{aligned}\right. \tag{J.10}$$

where $c_0 \equiv c(\rho_0)$. These five equations determine the acoustic quantities $p_1(\overline{x},t)$, $\rho_1(\overline{x},t)$, and $u_1(\overline{x},t)$.

We now study the time harmonic solution of the system of equations (J.10). Let

$$(J.11)\qquad \begin{cases} \bar{u}_1(\bar{x},t) = \bar{U}_1(\bar{x})e^{i\omega t}, \\ p_1(\bar{x},t) = P_1(\bar{x})e^{i\omega t}, \\ \rho_1(\bar{x},t) = \Gamma_1(\bar{x})e^{i\omega t}. \end{cases}$$

Then the system (J.10) becomes:

$$(J.12)\qquad i\omega P_1 + \bar{U}_1 \cdot \nabla p_0 + \rho_0 c_0^2 \operatorname{div}\bar{U}_1 = 0,$$

$$(J.13)\qquad i\omega\rho_0 \bar{U}_1 + \nabla P_1 + \rho_1 \bar{g} = 0,$$

$$(J.14)\qquad i\omega P_1 + \bar{U}_1 \cdot \nabla p_0 - c_0^2 (i\omega\Gamma_1 + \bar{U}_1 \cdot \nabla\rho_0) = 0.$$

Solving (J.13) for $\bar{U}_1$, we obtain

$$(J.15)\qquad \bar{U}_1 = \frac{i}{\omega\rho_0}(\nabla P_1 + \Gamma_1 \bar{g});$$

and upon substituting this into (J.14), we determine Γ_1 as a function of P_1:

$$(J.16)\qquad \Gamma_1 = \frac{1}{c_0^2}\left[\frac{P_1 - \frac{c_0^2}{\omega^2}\bar{G}\cdot\nabla P_1}{1 - \frac{\bar{G}\cdot\nabla p_0}{\omega^2\rho_0}}\right],$$

where

$$\bar{G} = \frac{1}{\rho_0}\left(\nabla\rho_0 - \frac{\nabla p_0}{c_0^2}\right), \quad \bar{g} = -\frac{\nabla p_0}{\rho_0}.$$

After substituting (J.15) into (J.12) and (J.14) and taking a linear combination of the results, we obtain

(J.17) $$\nabla^2 P_1 + \omega^2 \Gamma_1 - \frac{\nabla p_0}{\rho_0} \nabla \Gamma_1 = 0.$$

Finally, upon substituting (J.16) into (J.17) we obtain

(J.18) $$\nabla^2 P_1 + \frac{\omega^2}{c_0^2}\left[\frac{P_1 - \frac{c_0^2}{\omega^2}\overline{G}\cdot\nabla P_1}{1 - \frac{\overline{G}\cdot\nabla p_0}{\omega^2\,\rho_0}}\right] - \frac{\nabla p_0}{\rho_0}\nabla\left[\frac{1}{c_0^2}\left\{\frac{P_1 - \frac{c_0^2}{\omega^2}\overline{G}\cdot\nabla P_1}{1 - \frac{\overline{G}\cdot\nabla p_0}{\omega^2\,\rho_0}}\right\}\right] = 0.$$

Now we assume that the frequency ω is such that the ratio $\frac{|\overline{G}|}{\omega^2}$ is small. (This can be done by either considering very high frequencies or assuming that the layer of fluid is in adiabatic equilibrium). Then the effect of gravity waves is negligible and (J.18) becomes

$$\nabla^2 P_1 + \frac{\omega^2}{c_0^2} P_1 - \frac{\nabla p_0}{\rho_0}\nabla P_1 + 2\,\frac{\nabla p_0}{\rho_0}\,\frac{\nabla c_0}{c_0} P_1 = 0.$$

Since $\frac{\nabla p_0}{\rho_0}$, $\frac{\nabla c_0}{c_0} << 1$, we can neglect the last two terms; therefore, we finally obtain that

$$\nabla^2 P_1 + \frac{\omega^2}{c_0^2} P_1 = 0$$

We have been considering relatively high frequency propagation, and thus for our problem $\frac{|\overline{G}|}{\omega^2}$ is negligible on the basis of a high frequency assumption. Hence for a time dependent forcing function f(t) <u>composed of comparatively high-frequency components</u> we expect by the usual Fourier synthesis that the time dependent pressure field $p_1(\underline{x}, t)$ satisfies

$$\nabla^2 p_1 - \frac{1}{c_0^2}\frac{\partial^2 p_1}{\partial t^2} = F(\overline{x}, t),$$

where $F(\overline{x}, t)$ is proportional to f(t).

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