BIFURCATION THEORY OF NONLINEAR BOUNDARY VALUE PROBLEMS

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William Finlay Langford

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ABSTRACT

The theory of bifurcation of solutions to two-point boundary value problems is developed for a system of nonlinear first order ordinary differential equations in which the bifurcation parameter is allowed to appear nonlinearly. An iteration method is used to establish necessary and sufficient conditions for bifurcation and to construct a unique bifurcated branch in a neighborhood of a bifurcation point which is a simple eigenvalue of the linearized problem. The problem of bifurcation at a degenerate eigenvalue of the linearized problem is reduced to that of solving a system of algebraic equations. Cases with no bifurcation and with multiple bifurcation at a degenerate eigenvalue are considered.

The iteration method employed is shown to generate approximate solutions which contain those obtained by formal perturbation theory. Thus the formal perturbation solutions are rigorously justified. A theory of continuation of a solution branch out of the neighborhood of its bifurcation point is presented. Several generalizations and extensions of the theory to other types of problems, such as systems of partial differential equations, are described.

The theory is applied to the problem of the axisymmetric buckling of thin spherical shells. Results are obtained which confirm recent numerical computations.

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CHAPTER I

INTRODUCTION

Bifurcation is a change in the number of solutions u of an equation

$$g(\lambda, u) = 0 \tag{1}$$

produced by a small change in the real parameter λ. Those values of λ at which bifurcation takes place are called the bifurcation points of the equation. Bifurcation theory deals with the existence and numerical values of the bifurcation points, and with the behavior of solutions in neighborhoods of the bifurcation points.

Bifurcation theory is of great practical importance in the analysis of nonlinear mathematical models of physical systems. Bifurcation in the model can correspond to such physical phenomena as buckling of engineering structures [41]*, ignition and extinction in reactors [13], change of phase of a solid, liquid or gas [50], and change of dynamical mode in mechanical systems [4] [27] [29].

Equation (1) can represent any type of mathematical equation; in this thesis we are primarily concerned with systems of nonlinear first order ordinary differential equations with two-point boundary conditions, as defined in Chapter III. Bifurcation in nonlinear integral equations has been studied by M.A. Krasnosel'skii [30],

*Numbers in square brackets refer to the list of references at the end of the thesis.

T. W. Laetsch [32] and G. H. Pimbley, Jr. [37], and in nonlinear partial differential equations by H. B. Keller [21] M. S. Berger [5] and others.

The systems of first order ordinary differential equations considered here contain a wide range of interesting problems. For example, systems of n^{th} order ordinary differential equations can easily be transformed to systems of first order equations and hence are included in our theory. Even nonlinearities involving the derivatives up to order $n-1$ are permitted. Systems of partial differential equations can also be treated by the methods presented here, as we indicate in § V. 3.

Our results are obtained by means of an iteration technique which is based on the work of H. B. Keller [21], here extended to include systems of equations and degenerate eigenvalues. This iteration method has the advantage over formal perturbation methods of giving mathematically rigorous results with little extra effort. We show in § V. 5 that our solutions contain those obtained by the perturbation method. Compared to other mathematically rigorous studies of bifurcation, which generally are either more abstract than our work [30] [37], or are limited to a single problem [4] [29], we are able to obtain more useful information about the bifurcation of solutions to a wider range of nonlinear boundary-value problems of practical interest.

We assume that $u \equiv 0$ is a "trivial" solution of the boundary-value problem, and that the problem can be linearized about this solution. (Of course we could as well consider a nontrivial solution, say w, and then rewrite the problem in terms of $v = u - w$ so that $v \equiv 0$ is again a trivial solution). We then investigate the possibility of nontrivial solutions branching away from $u \equiv 0$ at a bifurcation point λ_o, using the linearized problem as the starting point of our analysis. The problem of finding nontrivial solutions to the nonlinear problem in a neighborhood of a bifurcation point is reduced to that of solving a sequence of linear inhomogeneous problems and associated bifurcation equations.

We show that λ_o can be a bifurcation point of the nonlinear problem only if it is an eigenvalue of the linearized problem, and that if λ_o is a simple eigenvalue then it is always a bifurcation point. If λ_o is a degenerate eigenvalue of the linearized problem, then we can construct a distinct bifurcation branch for each simple root of an associated system of algebraic equations (usually quadratic) which we call the algebraic bifurcation equations. We show by an example that there may be no nontrivial solution at all bifurcating from a degenerate eigenvalue λ_o.

The contents of each chapter are adequately described in the Table of Contents and in the introduction at the beginning of each chapter. We only point out here that Chapter II is a review of well known linear theory and may be skipped at the reader's discretion, Chapters III, IV and V present the bifurcation theory, and Chapter VI applies the theory to a problem of current interest. The conventions followed in the use of symbols, and some important definitions, are listed in Appendix A. Proofs of the convergence of the sequences which arise in the iteration technique are greatly simplified by the use of the contracting mapping theorem, which is stated for ease of reference in Appendix B. Appendices C and D contain some results required for the application of the bifurcation theory to the problem of Chapter VI.

Numbering of equations and of theorems begins with 1 in each chapter. When a reference is made to an equation or theorem in a different chapter, the other chapter is named explicitly. Similarly symbols are uniquely defined within each chapter, but may have different meanings in different chapters.

CHAPTER II

LINEAR TWO-POINT BOUNDARY-VALUE PROBLEMS

II. 1. Introduction

This chapter contains a summary of well-known results from the theory of systems of first order ordinary differential equations with two-point boundary conditions, which are required for the developments of chapters III and IV. These results are stated as theorems in order to facilitate easy reference in the later chapters. Proofs of theorems are either outlined briefly or omitted entirely when they can be found in standard reference books on ordinary differential equations such as [12] and [17].

Self-adjointness is assumed nowhere in this chapter since it is not required for our bifurcation theory. This broadens the usefulness of the theory, since in the applications self-adjointness is less commonly a property of systems of differential equations than it is of scalar differential equations.

We consider linear problems of the form

$$y'(\xi) = [A(\xi) + \lambda J(\xi,\lambda)]\ y(\xi) \qquad \alpha \leq \xi \leq \beta \tag{1}$$

$$My(\alpha) + Ny(\beta) = 0, \tag{2}$$

where $y(\xi)$ is an n-component column vector, $A(\xi)$ and $J(\xi,\lambda)$ are $n \times n$ matrices continuous in $\xi \in [\alpha, \beta]$, λ is a parameter in some open interval $\mathcal{J}$ (possibly unbounded) of the real line, α and β are finite real constants with $\alpha < \beta$, and M and N are n x n constant matrices such that rank $[M, N] = n$. All scalars are assumed real.

II. 2. The Adjoint Problem

The adjoint problem corresponding to (1) and (2) is defined to be

$$z'(\xi) = -[A(\xi) + \lambda J(\xi, \lambda)]^* z(\xi) \qquad \alpha \leq \xi \leq \beta \tag{3}$$

$$P z(\alpha) + Q z(\beta) = 0. \tag{4}$$

Here P and Q are any two constant $n \times n$ matrices satisfying

$$\text{rank}\,[P, Q] = n \quad \text{and} \quad MP^* - NQ^* = 0. \tag{5}$$

This definition is justified by the following lemma:

Lemma 1

Matrices P and Q exist which satisfy (5), and the set of all pairs of vectors $z(\alpha)$ and $z(\beta)$ satisfying (4) is independent of the choice of such P and Q satisfying (5).

Furthermore $z(\xi)$ satisfies (4) if and only if

$$z^*(\beta)\,y(\beta) - z^*(\alpha)\,y(\alpha) = 0 \tag{6}$$

for all $y(\xi)$ satisfying (2); and conversely, $y(\xi)$ satisfies (2) if and only if (6) holds for all $z(\xi)$ satisfying (4).

For proof of this lemma, see [8] page 564, [17] page 407, and [12] chapter 11.

Define the operator L and its adjoint operator L^* by

$$Ly \equiv y' - [A(\xi) + \lambda J(\xi, \lambda)]\, y \tag{7}$$

$$L^*z \equiv z' + [A(\xi) + \lambda J(\xi, \lambda)]^*\, z. \tag{8}$$

The following theorem now follows by integration by parts and (6).

Theorem 1

For all $y(\xi)$ satisfying (2) and all $z(\xi)$ satisfying (4),

$$(z, Ly) = (L^*z, y) \quad . \tag{9}$$

See Appendix I for the definition of this inner product.

Let $Y(\xi)$ be the fundamental matrix solution of equation (1) with

$$Y(\alpha) = I, \tag{10}$$

and define the boundary matrix

$$B \equiv MY(\alpha) + NY(\beta). \tag{11}$$

Similarly let $Z(\xi)$ be the fundamental matrix solution of (3) with

$$Z(\alpha) = I, \tag{12}$$

and define

$$C \equiv PZ(\alpha) + QZ(\beta). \tag{13}$$

The existence and uniqueness of $Y(\xi)$ and $Z(\xi)$ on $[\alpha, \beta]$ are guaranteed by the elementary theory of ordinary differential systems [12]. We can now state the following well known results. For proofs see [8] and [17] page 62.

Theorem 2

$$Z(\xi)^* = Y(\xi)^{-1} \tag{14}$$

Theorem 3

If $\det B \neq 0$, problem (1) (2) has only the trivial solution

$y(\xi)$ 0, and similarly for C and problem (3) (4).

Theorem 4

If B has rank n-p, then problem (1) (2) has exactly p linearly independent solutions, and similarly for C and problem (3) (4).

Theorem 5

Rank(B) = Rank (C), and hence the problems (1) (2) and (3) (4) have the same number of independent solutions.

II. 3. The Basic Alternative Theorem

The inhomogeneous problem corresponding to (1) (2) is

$$u'(\xi) = [A(\xi) + \lambda J(\xi, \lambda)] u(\xi) + f(\xi) \qquad \alpha \leq \xi \leq \beta \tag{15}$$

$$M u(\alpha) + N u(\beta) = 0, \tag{16}$$

where $f(\xi) \in C_n[\alpha, \beta]$. Note that an inhomogeneity in the boundary condition (16) could be removed by a transformation which would simply alter $f(\xi)$; hence (15) (16) represents the general case.

All questions as to the solvability of this inhomogeneous problem are answered by the following standard theorem, known as the basic alternative theorem. [8] [14].

Theorem 6

Exactly one of the following two cases must hold with regard to problem (15) (16).

Case I: The inhomogeneous problem (15) (16) has a unique solution for all $f(\xi) \in Cn[\alpha, \beta]$. This is true if and only if the problem (1) (2) has only the trivial solution $y(\xi) \equiv 0$.

Case II: If (1) (2) has $p > 0$ linearly independent non-trivial solutions, then (15) (16) has solutions if and only if

$$(z^{(i)}, f) = 0 \qquad i = 1, \ldots, p \tag{17}$$

where $z^{(i)}(\xi)$, $i = 1, \ldots, p$ are the linearly independent solutions to the adjoint problem (3) (4). The most general solution, if (17) is satisfied, is

$$u(\xi) = v(\xi) + \sum_{i=1}^{p} \gamma_i \, y^{(i)}(\xi) \tag{18}$$

where $v(\xi)$ is a particular solution and $y^{(i)}(\xi)$ $i = 1, \ldots, p$ are the linearly independent solutions of (1) (2).

II. 4. Dependence on the Parameter λ

In general, all the solutions considered so far are functions of λ as well as of ξ. The matrices B and C and their rank also depend on λ. Our bifurcation theory will use Case II of the basic alternative theorem, in which problem (1) (2) has nontrivial solutions. Hence we are led by Theorem 3 to consider the equation

$$\det B(\lambda) = 0. \tag{19}$$

The roots $\lambda = \lambda_i$ of this equation are precisely the values of λ for which (1) (2) has nontrivial solutions. We call these λ_i the eigenvalues of problem (1) (2) and call the corresponding nontrivial solutions the eigensolutions.

Theorem 7

If the components of the matrix $J(\xi, \lambda)$ can be expanded in convergent power series in $\lambda \in \mathcal{I}$ for each fixed $\xi \in [\alpha, \beta]$, and if there exists a point $\mu \in \mathcal{I}$ such that the problem (1) (2) has only the trivial solution for $\lambda = \mu$, (i. e., if det $B(\mu) \neq 0$), then the eigenvalues in $\mathcal{I}$ of problem (1) (2) are isolated points, and are finite in number if $\mathcal{I}$ is bounded, or at most denumerably infinite in number if $\mathcal{I}$ is unbounded.

Proof:

The power series representation of $J_{ij}(\xi, \lambda)$ gives its analytic continuation into some neighborhood $\mathcal{N}$ of the real interval $\mathcal{I}$ in the complex λ-plane. Clearly the right-hand side of (1) is then an analytic function of λ and y. Hence by standard theory ([12] page 36), any solution $y(\xi, \lambda)$ of (1) satisfying an arbitrary initial condition is an analytic function of λ for $\lambda \in \mathcal{N}$ and for each $\xi \in [\alpha, \beta]$, and so the fundamental matrix $Y(\xi, \lambda)$ has this same analyticity property. It follows that det $B(\lambda)$ is an analytic function of λ for $\lambda \in \mathcal{N}$, and the conclusion follows from the well-known properties of the zeros of an analytic function.

Note that Theorem 7 does not guarantee the existence of eigenvalues in $\mathcal{I}$, nor does it guarantee that the eigenvalues are real. Neither of these is true in general. It does guarantee that any eigenvalues which do exist are isolated, and this is crucial to the development in Chapter III. Since we will be concerned with only one eigenvalue at a time, the "global" dependence on λ is of no importance to the bifurcation theory, and we can take the interval $\mathcal{J}$ to be just a neighborhood of an eigenvalue λ_o, say. On this small neighborhood it may be easier to check the analyticity of $J(\xi, \lambda)$ as a function of λ. Of course, in many cases of interest, $J(\xi, \lambda)$ will be analytic in λ for all λ, and if J is independent of λ the theorem is trivially satisfied.

Note also that if M and N are analytic functions of λ, the theorem still holds. In fact much less than analyticity of J, M, and N in λ is required for the result to hold, but we will not pursue this.

From now on, λ_o will represent an isolated eigenvalue of (1) (2) with $\lambda_o \in \mathcal{J}$. Let p be the number of linearly independent solutions to (1) (2) with $\lambda = \lambda_o$. By hypothesis $p > o$, and clearly $p \leq n$ since equation (1) has exactly n linearly independent solutions. We call p the multiplicity of the eigenvalue λ_o.

From Theorems 5 and 7 it follows that the adjoint problem (3) (4) has the same real eigenvalues with the same multiplicities as the problem (1) (2), and hence that the eigenvalues of the adjoint problem are also isolated points in $\mathcal{J}$ if the hypotheses of Theorem 7 are satisfied.

If problem (1) (2) is self-adjoint as defined in [2] or [8], then the eigenvalues are necessarily real.

II. 5. The Green's Matrix

The inverse of the operator L is given most conveniently in terms of the Green's matrix denoted $G(\xi, \tau)$. For Case I of the basic alternative theorem, in which problem (1) (2) has only the trivial solution, the existence and uniqueness of a Green's matrix is well known. (See [8] page 577, [17] page 408 and [14] page 393). It is usually defined by the following four properties:

$G(\xi, \tau)$ is an $n \times n$ matrix of functions of ξ and τ defined and continuously differentiable on the rectangle $\alpha \leq \xi \leq \beta$, $\alpha \leq \tau \leq \beta$, except on the line $\xi = \tau$, (20)

each column of $G(\xi, \tau)$ as a function of ξ is a solution of (1) (2) except at $\xi = \tau$, (21)

$$G(\tau^+, \tau) - G(\tau^-, \tau) = I, \tag{22}$$

$$MG(\alpha, \tau) + NG(\beta, \tau) = 0. \tag{23}$$

The following theorem is proven in [8].

Theorem 8

If $\operatorname{Det} B(\lambda) \neq 0$, a Green's matrix $G(\xi, \tau)$ exists satisfying (20) through (23) and it is unique. The solution to the inhomogeneous problem (15) (16) is given by

$$u(\xi) = \int_\alpha^\beta G(\xi, \tau) f(\tau) d\tau. \tag{24}$$

This Green's matrix for problem (1) (2) may be written explicitly as

$$G(\xi, \tau) = \begin{cases} -Y(\xi) B^{-1} N Y(\beta) Z^*(\tau) & \text{for } \alpha \leq \xi < \tau \leq \beta \\ Y(\xi) B^{-1} M Y(\alpha) Z^*(\tau) & \text{for } \alpha \leq \tau < \xi \leq \beta \end{cases} \tag{25}$$

or equivalently as

$$G(\xi,\tau) = \tfrac{1}{2}\, Y(\xi)\, [I \operatorname{sgn}(\xi-\tau) + B^{-1}D]\, Y^{-1}(\tau), \tag{26}$$

$$\text{where } \operatorname{sgn}(\xi-\tau) \equiv \begin{cases} 1 & \text{if } \xi > \tau \\ -1 & \text{if } \xi < \tau \end{cases} \tag{27}$$

$$\text{and} \quad D \equiv MY(\alpha) - NY(\beta). \tag{28}$$

Theorem 9

The unique Green's matrix $H(\xi,\tau)$ for the adjoint problem (3) (4) is given by

$$H_{ij}(\xi,\tau) = -G_{ji}(\tau,\xi) \qquad i,j = 1,\ldots, n. \tag{29}$$

Clearly formulae (25) and (26) are no longer valid in Case II of the basic alternative theorem, since then B^{-1} does not exist. Fortunately it is possible to define a generalized Green's matrix which plays the same role as the Green's matrix for Case II of the alternative. The following development of a generalized Green's matrix is based on the original paper by W. T. Reid [40]. See also [39], [9], and [33].

II. 6. The Generalized Green's Matrix

Assume that $\lambda = \lambda_o$ is an eigenvalue of problem (1) (2) with multiplicity p. That is we now assume that the problem

$$y'(\xi) = [A(\xi) + \lambda_o J(\xi,\lambda_o)]\, y(\xi) \tag{30}$$

$$My(\alpha) + Ny(\beta) = 0 \tag{31}$$

has exactly p linearly independent solutions $y^{(i)}(\xi)$ $i = 1, \ldots, p$, where $1 \leq p \leq n$. From any such linearly independent set we can construct an orthonormal set by the Gram-Schmidt process, say. Therefore, we assume that the set $\{y^{(i)}(\xi)\}_{i=1}^{p}$ is orthonormal. Now define the $n \times p$ matrix $R(\xi)$ with columns $R^{(i)}(\xi)$ $i = 1, \ldots, p$, by

$$R^{(i)}(\xi) = y^{(i)}(\xi) \quad , \quad i = 1, \ldots, p. \tag{32}$$

Similarly the adjoint problem with $\lambda = \lambda_o$,

$$z'(\xi) = -[A(\xi) + \lambda_o J(\xi, \lambda_o)]^* z(\xi) \tag{33}$$

$$P z(\alpha) + Q z(\beta) = 0 , \tag{34}$$

has exactly p linearly independent solutions $z^{(i)}(\xi)$ $i = 1, \ldots, p$, which, without loss of generality, we can assume to be orthonormal. Define the $n \times p$ matrix $S(\xi)$ by

$$S^{(i)}(\xi) = z^{(i)}(\xi) \qquad i = 1, \ldots, p. \tag{35}$$

The general solution of (30) (31) can now be written

$$y(\xi) = R(\xi)\, a, \qquad \text{where} \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix}, \tag{36}$$

and the orthogonality condition of the basic alternative theorem can be written

$$\int_{\alpha}^{\beta} S^*(\xi)\, f(\xi)\, d\xi = 0. \tag{37}$$

We seek an n x n matrix $G(\xi, \tau)$ which has properties (20), (21) with $\lambda = \lambda_o$, and (22), and is such that every solution of the inhomogeneous system

$$u'(\xi) = [A(\xi) + \lambda_o J(\xi, \lambda_o)]\, u(\xi) + f(\xi) \tag{38}$$

$$Mu(\alpha) + Nu(\beta) = 0 \tag{39}$$

may be written as

$$u(\xi) = \int_{\alpha}^{\beta} G(\xi, \tau)\, f(\tau)\, d\tau + R(\xi)\, a. \tag{40}$$

We define any such $G(\xi, \tau)$ to be a generalized Green's matrix for the problem (30) (31).

Let $\tilde{Y}(\xi)$ be an nxn fundamental matrix solution to (30) such that the first p columns of $\tilde{Y}(\xi)$ are just those of $R(\xi)$. Then

$$\tilde{Z}(\xi) \equiv \tilde{Y}^{*-1}(\xi) \tag{41}$$

is a fundamental matrix solution of the adjoint equation (33) but does not necessarily have the same first p columns as $S(\xi)$. Note also that $\tilde{Y}(\alpha)$ and $\tilde{Z}(\alpha)$ no longer equal the identity matrix in general. The matrix

$$\tilde{B} \equiv M\tilde{Y}(\alpha) + N\tilde{Y}(\beta) \tag{42}$$

has its first p columns identically zero, and has rank n-p. We may choose n-p rows of $\tilde{B}$, say the rows numbered $\ell_1, \ell_2, \ldots, \ell_{n-p}$, such that the (n-p) x (n-p) matrix F has rank n-p, where

$$F \equiv \begin{pmatrix} B_{\ell_1 p+1} & B_{\ell_1 p+2} & \cdots & B_{\ell_1 n} \\ \vdots & \vdots & & \vdots \\ B_{\ell_{n-p} p+1} & B_{\ell_{n-p} p+2} & \cdots & B_{\ell_{n-p} n} \end{pmatrix}. \qquad (43)$$

Matrix F has a unique (n-p) x(n-p) inverse, F^{-1}. Now define the n x n matrix $E = (E_{ij})$ by

$$E_{ij} = 0 \text{ if } i \leq p \text{ or } j \neq \ell_m, \; m = 1, \ldots, n-p.$$
$$E_{p+i, \ell_j} = (F^{-1})_{ij} \text{ for } i,j = 1, \ldots, n-p. \qquad (44)$$

Theorem 10

A generalized Green's matrix for (30) (31) exists and may be written as

$$G(\xi, \tau) = \tfrac{1}{2} \tilde{Y}(\xi) [I \operatorname{sgn}(\xi-\tau) + E\tilde{D}] \tilde{Y}^{-1}(\tau), \qquad (45)$$

where $\tilde{Y}(\xi)$ and E are defined above and $\tilde{D} \equiv M\tilde{Y}(\alpha) - N\tilde{Y}(\beta)$.

Proof: See [40].

Theorem 11

The generalized Green's matrix for (30) (31) is not unique. If $G_1(\xi, \tau)$ is one generalized Green's matrix, then every generalized Green's matrix is of the form

$$\mathbb{G}(\xi, \tau) = G_1(\xi, \tau) + R(\xi)U^*(\tau) + V(\xi)S^*(\tau), \qquad (46)$$

where $U(\tau)$ and $V(\xi)$ are $n \times p$ matrices continuously differentiable on $[\alpha, \beta]$. Furthermore, every $G(\xi, \tau)$ of the form (46) is a generalized Green's matrix for (30) (31).

Proof:

It is obvious that every $G(\xi, \tau)$ of the form (46) is a generalized Green's matrix. For the proof of the converse, see Reid [40].

II. 7. The Principal Generalized Green's Matrix

The generalized Green's matrix of Theorem 10 lacks three desirable features. First it is not unique; second it does not necessarily satisfy the boundary conditions (23) (although

$$u(\xi) \equiv \int_{\alpha}^{\beta} G(\xi, \tau) f(\xi) \, d\tau \tag{47}$$

does if $f(\tau)$ satisfies (37)), and third neither $G(\xi, \tau)$ nor $u(\xi)$ defined by (47) are necessarily orthogonal to the solutions $y^{(i)}(\xi)$, $i=1, \ldots, p$ of (30) (31). The third feature is especially important for our bifurcation theory. Fortunately it is easy to construct a generalized Green's matrix with these properties, through the use of the projection operators W and X defined by

$$W u \equiv R(\xi) \int_{\alpha}^{\beta} R^*(\tau) \, u(\tau) \, d\tau \tag{48}$$

and $$Xu \equiv S(\xi) \int_{\alpha}^{\beta} S^*(\tau) \, u(\tau) \, d\tau. \tag{49}$$

Here W projects the space $C_n^1[\alpha, \beta]$ onto the solution space of problem (30) (31), and X projects onto the solution space of the adjoint problem (33) (34). Following Reid, we call a generalized

Green's matrix with the above three properties the principal generalized Green's matrix, although the construction we use is due to Loud [33] [34]. For an alternate approach see [39] [40] and [9].

If $G(\xi, \tau)$ is any of the generalized Green's matrices of Theorems 10 and 11, define the matrix $G^{\dagger}(\xi, \tau)$ by

$$G^{\dagger}(\xi,\tau) \equiv G(\xi,\tau) - R(\xi)\int_{\alpha}^{\beta} R^*(\sigma)\, G(\sigma,\tau)\, d\sigma - \int_{\alpha}^{\beta} G(\xi,\sigma)\, S(\sigma)\, d\sigma\, S^*(\tau)$$
$$+ R(\xi)\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} R^*(\sigma) G(\sigma,\zeta)\, S(\zeta)\, d\sigma\, d\zeta\, S^*(\tau). \tag{50}$$

Theorem 12

The matrix $G^{\dagger}(\xi, \tau)$ defined by (50) is a generalized Green's matrix for problem (30) (31) and has the following properties:

$G^{\dagger}(\xi, \tau)$ is continuously differentiable in $[\alpha, \beta] \times [\alpha, \beta]$ except on $\xi = \tau$, (51)

$$\frac{d}{d\xi} G^{\dagger}(\xi,\tau) = [A(\xi) + \lambda_o J(\xi, \lambda_o)]\, G^{\dagger}(\xi,\tau) + I\delta(\xi-\tau) - S(\xi)\, S^*(\tau), \tag{52}$$

$$G^{\dagger}(\tau^+, \tau) - G^{\dagger}(\tau^-, \tau) = I, \tag{53}$$

$$M G^{\dagger}(\alpha, \tau) + N G^{\dagger}(\beta, \tau) = 0, \tag{54}$$

$$\int_{\alpha}^{\beta} R^*(\xi)\, G^{\dagger}(\xi, \tau)\, d\xi = 0, \tag{55}$$

Every solution to problem (38) (39) may be written in the form

$$u(\xi) = R(\xi)\, a + \int_{\alpha}^{\beta} G^{\dagger}(\xi,\tau)\, f(\tau)\, d\tau, \tag{56}$$

The matrix $G^{\dagger}(\xi, \tau)$ is independent of the choice of the matrix $G(\xi, \tau)$ in the definition (50). (57)

The matrix $G^{\dagger}(\xi, \tau)$ defined by (50) is the only matrix which satisfied properties (51) through (56).

We call $G^{\dagger}(\xi, \tau)$ the principal generalized Green's matrix of problem (30) (31).

Theorem 13

The principal generalized Green's matrix of the adjoint problem (33) (34) is

$$H^{\dagger}(\xi, \tau) = -G^{\dagger}(\tau, \xi)^{*}. \tag{58}$$

Proof: See [40].

It is convenient to summarize our results in the following:

Theorem 14

If $f(\xi)$ is any function in $C_n[\alpha, \beta]$ satisfying (37), then there is a unique solution $u(\xi)$ to problem (38) (39) which satisfies

$$\int_{\alpha}^{\beta} R(\xi)^{*} u(\xi)\, d\xi = 0, \tag{59}$$

and this $u(\xi)$ is given by

$$u(\xi) = \int_{\alpha}^{\beta} G^{\dagger}(\xi, \tau) f(\tau)\, d\tau. \tag{60}$$

Furthermore, there is a constant $\Phi > 0$ such that

$$\|u\| \leq \Phi \|f\| \tag{61}$$

for all such $f(\xi)$ and $u(\xi)$.

Proof: Take Φ $\|G^\dagger\|$, which, for example in the maximum norm, is given by

$$G^\dagger = \max_{\alpha \leq \xi \leq \beta} \quad \max_{1 \leq i \leq n} \int_\alpha^\beta \sum_{j=1}^{n} \left| G^\dagger_{ij} (\xi, \tau) \right| d\tau. \qquad (62)$$

Then $\|G^\dagger\|$ is guaranteed to exist by the continuity properties (51) and (53).

CHAPTER III

SIMPLE BIFURCATION

III. 1. Introduction

In this chapter we consider two-point boundary-value problems

$$u'(\xi) = A(\xi)\,u(\xi) + \lambda f(\xi,\lambda,u) \qquad \alpha \le \xi \le \beta\ , \tag{1}$$

$$M\,u(\alpha) + N\,u(\beta) = 0 \qquad . \tag{2}$$

Here λ is a parameter in some open real interval $\mathcal{J}$, $u(\xi) \in C^1_n[\alpha,\beta]$, $A(\xi)$ is an n×n matrix continuous on $[\alpha,\beta]$, and M and N are n×n constant matrices such that rank $[M,N] = n$. The n-vector function $f(\xi,\lambda,u)$ is assumed to be 2-times Fréchet differentiable in u and λ with its second Fréchet derivatives Lipschitz continuous in u and λ on the set S defined by (15), and with f and its derivatives continuous in ξ. (Fréchet derivatives are used mainly for notational convenience and are defined in Appendix A). We further assume that

$$f(\xi,\lambda,0) = 0 \tag{3}$$

for all $\xi \in [\alpha,\beta]$ and $\lambda \in \mathcal{J}$. Clearly then problem (1) (2) has the trivial solution $u(\xi) \equiv 0$ for all $\lambda \in \mathcal{J}$. Linearizing problem (1) (2) about this trivial solution gives the linearized problem or variational problem

$$y'(\xi) - [A(\xi) + \lambda\, J(\xi,\lambda)]\, y(\xi) = 0 \tag{4}$$

$$M\,y(\alpha) + N\,y(\beta) = 0 \qquad , \tag{5}$$

where $J(\xi,\lambda)$ is the n×n matrix defined by

$$J_{ij}(\xi, \lambda) \equiv \left. \frac{\partial f_i(\xi, \lambda, u)}{\partial u_j} \right|_{u=0} , \quad i, j = 1, \ldots, n . \tag{6}$$

The linearized problem (4) (5) is exactly the problem studied in Chapter II. Therefore, from Theorem 3 and §II. 4, λ is an eigenvalue of (4) (5) if and only if λ is a root of the equation

$$\det B(\lambda) = 0 \quad . \tag{7}$$

We assume $\lambda = \lambda_o$ is a real, isolated, nonzero eigenvalue of (4) (5), and throughout this chapter we make the additional assumption that λ_o is a simple eigenvalue. Under this assumption we are able to show that λ_o is a bifurcation point of problem (1) (2), and we construct a nontrivial solution branch in a neighborhood of λ_o.

Let $y_o(\xi)$ be a normalized solution of (4) (5) with $\lambda = \lambda_o$; then $y_o(\xi)$ satisfies

$$y_o'(\xi) - [A(\xi) + \lambda_o J(\xi, \lambda_o)] y_o(\xi) = 0 \tag{8}$$

$$M y_o(\alpha) + N y_o(\beta) = 0 \tag{9}$$

$$\int_\alpha^\beta y_o^*(\xi) y_o(\xi) d\xi = 1 \quad , \tag{10}$$

and $y_o(\xi)$ is unique within a sign. The corresponding adjoint problem is

$$z_o'(\xi) + [A(\xi) + \lambda_o J(\xi, \lambda_o)]^* z_o(\xi) = 0 \tag{11}$$

$$P z_o(\alpha) + Q z_o(\beta) = 0 \tag{12}$$

$$\int_\alpha^\beta z_o^*(\xi) z_o(\xi) d\xi = 1 \quad , \tag{13}$$

as defined in II.2. According to Theorem 5 of Chapter II, this problem has a solution $z_0(\xi)$ which is unique within a sign.

The matrix $J(\xi, \lambda_0)$ defined by (6) is assumed to be such that

$$\gamma \equiv \int_a^\beta z_0^*(\xi)\, J(\xi, \lambda_0)\, y_0(\xi) d\xi \neq 0 \quad . \tag{14}$$

Since the eigenvalue λ_0 is isolated, we can assume without loss of generality (rescale λ if necessary) that there is no other eigenvalue of problem (4)(5) in the interval $[\lambda_0 - 1, \lambda_0 + 1]$, and that this interval is contained in $\mathcal{J}$.

Let S be the n+2 dimensional domain

$$S \equiv \{(\xi, \lambda, u) \mid \xi \in [a, \beta],\ |\lambda - \lambda_0| \leq 1,\ u \in C_n[a, \beta], \|u\| \leq \|y_0\| + 1\} . \tag{15}$$

The norm $\|u\|$, $\|y_0\|$ and all other norms which we use, are defined in Appendix A. The function $f(\xi, \lambda, u)$ in equation (1) is assumed to be 2-times Fréchet differentiable in u and λ for each ξ in this set S, and its second Fréchet derivatives f_{uu} and $f_{u\lambda}$ are assumed Lipschitz continuous on S.

We impose one final restriction on f, which in effect says that its variation with λ must be mild. Assume that

$$\|f_{u\lambda}\| \leq \frac{|\gamma|}{2\,|\lambda_0|\, \|z_0\|_1\, \|y_0\|} \quad . \tag{16}$$

In § 2 we show that a necessary condition for λ to be a bifurcation point of (1)(2) is that λ be an eigenvalue of (4)(5). In § 3 we present an iteration scheme, which we claim generates a continuous branch of nontrivial solutions to problem (1)(2) in a (possibly one-sided) neighborhood of λ_0, and which intersects the trivial solution at $\lambda = \lambda_0$. All these claims are verified in § 4, by an application of the contracting mapping theorem. In § 5 we show that the solution branch constructed by the iteration scheme contains all possible nontrivial solutions in a neighborhood of the bifurcation point, and we remove the ambiguity in the choice of sign of y_0. Finally, in § 6 we calculate the asymptotic behavior of the solution branch near the bifurcation point.

III. 2. The Necessary Condition for Bifurcation

We prove that bifurcation of nontrivial solutions from the trivial solution of problem (1)(2) can occur only at an eigenvalue of the linearized problem (4)(5). This result has been established for similar problems by M. A. Krasnosel'skii [30], J. B. Keller [25], M. S. Berger [5], and others.

Theorem 1

If $\lambda \in [\lambda_0 - 1, \lambda_0 + 1]$ is not an eigenvalue of the linearized problem (4)(5), then (1)(2) can have no nontrivial solution in a sufficiently small neighborhood of the trivial solution $u(\xi) \equiv 0$.

Proof:

Rewrite (1)(2) in the form

$$u'(\xi) - [A(\xi) + \lambda\, J(\xi,\lambda)]\, u(\xi) = \lambda f(\xi,\lambda,u) - \lambda\, J(\xi,\lambda)\, u \qquad (17)$$

$$M\, u(\alpha) + N\, u(\beta) = 0 \quad . \qquad (18)$$

For $0 < \delta \leq \|\lambda_0\| + 1$, define the neighborhood η_δ by

$$\mathcal{N}_\delta \equiv \{ w \in C_n[\alpha,\beta] \;\big|\; \|w\| \le \delta \} \quad . \tag{19}$$

Consider the inhomogeneous problem, for $w \in \mathcal{N}_\delta$:

$$u'(\xi) - [A(\xi) + \lambda\, J(\xi,\lambda)]\, u(\xi) = \lambda\, f(\xi,\lambda,w) - \lambda\, J(\xi,\lambda) w \tag{20}$$

$$M\, u(\alpha) + N\, U(\beta) = 0 \tag{21}$$

From Theorems 6 and 8 of Chapter II it follows that this inhomogeneous problem has a unique solution $u(\xi)$ given in terms of the Green's matrix $G(\xi,\tau)$ by

$$u(\xi) = \lambda \int_\alpha^\beta G(\xi,\tau)\, [f(\tau,\lambda,w(\tau)) - J(\tau,\lambda) w(\tau)]\, d\tau \quad . \tag{22}$$

Let equation (22) define the mapping

$$T: \mathcal{N}_\delta \to C_n^1\, [\alpha,\beta] \tag{23}$$

where $Tw = u$.

Since $(\tau,\lambda,w) \in S$, we have

$$f(\tau,\lambda,w(\tau)) - J(\tau,\lambda) w(\tau) = \int_0^1 \int_0^1 f_{uu}(\tau,\lambda,\rho\,\sigma\, w) \rho\, d\sigma\, d\rho\; w^2(\tau) \quad . \tag{24}$$

Taking norms as defined in Appendix I, (22) and (24) give

$$\begin{aligned} \|u\| &\le |\lambda|\; \|G\|\; \tfrac{1}{2}\; \|f_{uu}\|_S\; \|w\|^2 \\ &\le \tfrac{1}{2}\,(|\lambda_0|+1)\, \|G\|\; \|f_{uu}\|_S\; \delta^2 \quad . \end{aligned} \tag{25}$$

Hence $\|u\| \le \delta$ if

$$\delta \le \frac{2}{(|\lambda_0|+1)\, \|G\|\; \|f_{uu}\|_S} \quad . \tag{26}$$

Similarly for $w^{(1)}$ and $w^{(2)}$ in $\mathcal{N}_\delta$, define $u^{(1)} \equiv T w^{(1)}$ and $u^{(2)} \equiv T w^{(2)}$. Then

$$\|u^{(1)} - u^{(2)}\| \le |\lambda| \, \|G\| \, \|f_{uu}\|_S \, \delta \, \|w^{(1)} - w^{(2)}\| \quad . \tag{27}$$

Now define, for any $0 < \theta < 1$,

$$\delta \equiv \min \left\{ \|y_0\| + 1, \; \frac{\theta}{(|\lambda_0|+1) \, \|G\| \, \|f_{uu}\|_S} \right\} , \tag{28}$$

and it follows that

$$\|u^{(1)} - u^{(2)}\| \le \theta \, \|w^{(1)} - w^{(2)}\| . \tag{29}$$

Thus, for δ defined by (28), T maps $\mathcal{N}_\delta$ into itself and is contracting on $\mathcal{N}_\delta$, so it follows from the contracting mapping theorem of Appendix II that T has a unique fixed point in $\mathcal{N}_\delta$.

But clearly the trivial solution $u(\xi) \equiv 0$ is already a fixed point of T in $\mathcal{N}_\delta$. Hence there cannot exist any nontrivial solution to (1)(2) in $\mathcal{N}_\delta$, since it would be a fixed point of T, violating the uniqueness result.

From Theorem 1 it clearly follows that the only possible candidate for a point of bifurcation of problem (1)(2) (from the trivial solution) in the interval $[\lambda_0 - 1, \lambda_0 + 1]$ is the eigenvalue λ_0 of the linearized problem (4)(5). If the differentiability assumptions on f hold for all λ in $\mathcal{J}$ instead of just in S, the same argument applies to any closed bounded subinterval of $\mathcal{J}$ by redefining S suitably. Then we conclude that bifurcation from the trivial solution in problem (1)(2) can occur only at the eigenvalues in $\mathcal{J}$ of problem (4)(5).

III. 3. The Iteration Scheme

We seek a continuous branch of nontrivial solutions to problem (1)(2) in the form

$$u(\xi, \epsilon) = \epsilon\, y_0(\xi) + \epsilon^2\, v(\xi, \epsilon) \tag{30}$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon\, \eta(\epsilon) \quad , \tag{31}$$

where ϵ is a small scalar which parameterizes the solution branch, and $v(\xi, \epsilon)$, $\eta(\epsilon)$ are functions to be determined. A possible source of non-uniqueness in this representation is removed by assuming that $v(\xi, \epsilon)$ is orthogonal to $y_0(\xi)$, that is

$$\int_a^\beta y_0^*(\xi)\, v(\xi, \epsilon)\, d\xi = 0 \quad . \tag{32}$$

We claim that such a solution branch exists and is given, for sufficiently small $|\epsilon|$, by the limit of the iteration scheme which we are about to define.

The ansatz (30)(31) is a solution of problem (1)(2) if and only if $v(\xi, \epsilon)$ and $\eta(\epsilon)$ satisfy

$$\begin{aligned} v'(\xi) - [A(\xi) + \lambda_0 J(\xi, \lambda_0)]\, v(\xi) = \frac{1}{\epsilon^2}(\lambda_0 + \epsilon\,\eta)\, [f(\xi, \lambda_0 + \epsilon\eta, \epsilon\, y_0 + \epsilon^2 v) \\ - J(\xi, \lambda_0)\,(\epsilon\, y_0 + \epsilon^2 v)] + \eta\, J(\xi, \lambda_0)(y_0 + \epsilon\, v) \end{aligned} \tag{33}$$

$$M\, v(a) + N\, v(\beta) = 0 \quad . \tag{34}$$

By Theorem 6 of Chapter II, problem (33)(34) has a solution only if the right-hand side of (33) is orthogonal to $z_0(\xi)$. This gives the condition

$$\eta \int_\alpha^\beta z_o^*(\xi) J(\xi, \lambda_o)(y_o(\xi)+\epsilon v(\xi))d\xi + \frac{1}{\epsilon^2}(\lambda_o+\epsilon\eta) \int_\alpha^\beta z_o^*(\xi)\Big[f(\xi, \lambda_o+\epsilon\eta, \epsilon y_o+\epsilon^2 v)$$

$$- J(\xi, \lambda_o)(\epsilon y_o + \epsilon^2 v)\Big]d\xi = 0 . \quad (35)$$

Hypothesis (3) implies that for $\lambda \in [\lambda_o - 1, \lambda_o + 1]$,

$$f_\lambda(\xi, \lambda, 0) = f_{\lambda\lambda}(\xi, \lambda, 0) = 0 . \quad (36)$$

Then the assumed differentiability and continuity properties of f give the Taylor expansion, for $(\xi, \lambda, u) \in S$,

$$f(\xi, \lambda, u) = f_u(\xi, \lambda_o, 0)u + \tfrac{1}{2} f_{uu}(\xi, \lambda_o, 0)u^2 + f_{u\lambda}(\xi, \lambda_o, 0)(\lambda-\lambda_o)u$$
$$+ E_1(\xi, u)u^2 + E_2(\xi, \lambda, u)(\lambda-\lambda_o)u , \quad (37)$$

where

$$E_1(\xi, u) \equiv \int_0^1 \int_0^1 [f_{uu}(\xi, \lambda_o, \rho\zeta u) - f_{uu}(\xi, \lambda_o, 0)]\zeta \, d\rho \, d\zeta , \quad (38)$$

and

$$E_2(\xi, \lambda, u) \equiv \int_0^1 \int_0^1 [f_{u\lambda}(\xi, \lambda_o+\rho(\lambda-\lambda_o), \zeta u) - f_{u\lambda}(\xi, \lambda_o, 0)]d\zeta d\rho. \quad (39)$$

Note that $E_1(\xi, 0) = E_2(\xi, \lambda_o, 0) = 0$. Furthermore, from the assumed Lipschitz continuity of f_{uu} and $f_{u\lambda}$ there exist constants Φ_1, Φ_2 and Φ_3 such that for $(\xi, \lambda^{(i)}, u^{(i)}) \in S$, $i = 1, 2$,

$$\|E_1(\xi, u^{(1)}) - E_1(\xi, u^{(2)})\| \le \Phi_1 \|u^{(1)} - u^{(2)}\| \quad (40)$$

and

$$\|E_2(\xi, \lambda^{(1)}, u^{(1)}) - E_2(\xi, \lambda^{(2)}, u^{(2)})\| \le \Phi_2 \|u^{(1)} - u^{(2)}\| + \Phi_3 |\lambda^{(1)} - \lambda^{(2)}| . \quad (41)$$

We use the expansion (37) for f to rewrite (35), grouping terms according to their order of magnitude in the small parameter ϵ:

$$\eta\left[\gamma + \lambda_o \int_\alpha^\beta z_o^*(\xi) f_{u\lambda}(\xi, \lambda_o, 0)\, y_o(\xi) d\xi\right]$$

$$+ \tfrac{1}{2}\lambda_o \int_\alpha^\beta z_o^*(\xi) f_{uu}(\xi, \lambda_o, 0)\, y_o^2(\xi) d\xi + \epsilon \int_\alpha^\beta z_o^*(\xi) h(\xi, \epsilon, \eta, v(\xi))\, d\xi$$

$$= 0 \quad , \tag{42}$$

where

$$h(\xi, \epsilon, \eta, v) \equiv \eta\,[f_u(\xi, \lambda_o, 0)v(\xi) + \tfrac{1}{2} f_{uu}(\xi, \lambda_o, 0)y_o^2(\xi) + \eta\, f_{u\lambda}(\xi, \lambda_o, 0)y_o(\xi)]$$

$$+ (\lambda_o + \epsilon\eta)\,[\tfrac{1}{2} f_{uu}(\xi, \lambda_o, 0)(2\, y_o v + \epsilon v^2) + \eta\, f_{u\lambda}(\xi, \lambda_o, 0)v(\xi)$$

$$+ \frac{1}{\epsilon} E_1(\xi, u)(y_o + \epsilon v)^2 + \frac{1}{\epsilon} E_2(\xi, \lambda, u)\eta(y_o + \epsilon v)] \; . \tag{43}$$

Similarly (33) becomes

$$v'(\xi) - [A(\xi) + \lambda_o J(\xi, \lambda_o)]v(\xi) = \eta[J(\xi, \lambda_o)y_o(\xi) + \lambda_o f_{u\lambda}(\xi, \lambda_o, 0)y_o(\xi)]$$

$$+ \tfrac{1}{2}\lambda_o f_{uu}(\xi, \lambda_o, 0)y_o^2(\xi) + \epsilon\, h(\xi, \epsilon, \eta, v). \tag{44}$$

Now we set up the iteration scheme for η and v according to the following rule: Wherever η or v appears in a term of (42) or (44) which is $O(\epsilon)$, we label it with the superscript "ℓ" to indicate that it is the old iterate, and when η or v appears in a O(1) term, we label it with the superscript "$(\ell+1)$" to make it the new iterate. This yields the sequence of linear problems for $\eta^{(\ell+1)}$ and $v^{(\ell+1)}(\xi)$, $\ell = 0, 1, 2, \ldots$, in equations (45) to (49). Note that the denominator in (46) is guaranteed nonzero by hypotheses (14) and (16).

$$\eta^{(0)} = 0, \quad v^{(0)} = 0 \quad , \tag{45}$$

$$\eta^{(\ell+1)} = - \frac{\frac{1}{2}\lambda_0 \int_\alpha^\beta z_0^*(\xi) f_{uu}(\xi, \lambda_0, 0) y_0^2(\xi) d\xi + \epsilon \int_\alpha^\beta z_0^*(\xi) h(\xi, \epsilon, \eta^{(\ell)}, v^{(\ell)}(\xi)) d\xi}{\gamma + \lambda_0 \int_\alpha^\beta z_0^*(\xi) f_{u\lambda}(\xi, \lambda_0, 0) y_0(\xi) d\xi} , \tag{46}$$

$$v^{(\ell+1)\prime}(\xi) - [A(\xi) + \lambda_0 J(\xi, \lambda_0)] v^{(\ell+1)}(\xi) = \eta^{(\ell+1)} [J(\xi, \lambda_0) y_0(\xi) + \lambda_0 f_{u\lambda}(\xi, \lambda_0, 0) y_0(\xi)]$$

$$+ \tfrac{1}{2} \lambda_0 f_{uu}(\xi, \lambda_0, 0) y_0^2(\xi) + \epsilon h(\xi, \epsilon, \eta^{(\ell)}, v^{(\ell)}(\xi))) , \tag{47}$$

$$M\, v^{(\ell+1)}(\alpha) + N\, v^{(\ell+1)}(\beta) = 0 \quad , \tag{48}$$

$$\int_\alpha^\beta y_0^*(\xi)\, v^{(\ell+1)}(\xi)\, d\xi = 0 \quad , \tag{49}$$

$$\ell = 0, 1, 2, \ldots \quad .$$

With $\eta^{(\ell+1)}$ defined by (46), the basic alternative theorem guarantees that the linear problem (47) (48) has a solution $v^{(\ell+1)}(\xi)$. The orthogonality condition (49) makes this solution unique. Thus the iteration scheme (45) to (49) uniquely defines the sequences of iterates

$$\{\eta^{(\ell)}\}, \qquad \{v^{(\ell)}(\xi)\}, \qquad \ell = 0, 1, \ldots, \tag{50}$$

provided only that $(\xi, \lambda_0 + \epsilon\eta^{(\ell)}, \ \epsilon y_0 + \epsilon^2 v^{(\ell)})$ remains in S for all ℓ, so that h remains defined. Assuming this to be so, the unique solution of problem (47) (48) (49) can be written in terms of the principal generalized Green's matrix $G^\dagger(\xi, \tau)$ of Chapter II §7 as

$$v^{(\ell+1)}(\xi) = \int_\alpha^\beta G^\dagger(\xi, \tau)\, [\eta^{(\ell+1)}(J(\tau, \lambda_0) y_0(\tau) + \lambda_0 f_{u\lambda}(\tau, \lambda_0, 0) y_0(\tau))$$

$$+ \tfrac{1}{2}\lambda_0 f_{uu}(\tau, \lambda_0, 0) y_0^2(\tau) + \epsilon h(\tau, \epsilon, \eta^{(\ell)}, v^{(\ell)}(\tau))] d\tau \quad ,$$

$$\ell = 0, 1, 2, \ldots \quad . \tag{51}$$

The iteration scheme defined by (45) to (49) is the optimum one in the following sense. If the ℓ^{th} iterates were allowed to appear in any of the O(1) terms, the convergence would be slower than the order ϵ convergence which our scheme gives. On the other hand, if $(\ell+1)^{\text{st}}$ iterates appeared in any $O(\epsilon)$ terms, we would have to impose additional restrictions on the range of ϵ in order to insure the solvability of (42) and (44) for the $(\ell+1)^{\text{st}}$ iterates.

III. 4. The Convergence Proof

We now verify the claims which have been made for the iteration scheme (45) to (49). Rather than work with the iterates directly, we consider the mapping $[\eta^{(\ell)}, v^{(\ell)}] \rightarrow [\eta^{(\ell+1)}, v^{(\ell+1)}]$. We show that this mapping is contracting on a domain which we define, and then all the desired properties follow from an application of the contracting mapping theorem.

Define bounds as follows. They are not the finest possible bounds, but have occasionally been chosen instead to simplify the calculations.

$$\Lambda \equiv \frac{2\|z_o\|_1}{|\gamma|}\left[\tfrac{1}{2}|\lambda_o|\,\|f_{uu}\|\,\|y_o\|^2 + 1\right] \tag{52}$$

$$\Omega \equiv \Lambda\,\|G^{\dagger}\|\left[\|f_u\|\,\|y_o\| + \frac{|\gamma|}{\|z_o\|_1}\right] \tag{53}$$

$$\Psi_1 \equiv \Lambda\left[\|f_u\|\Omega + \tfrac{1}{2}\|f_{uu}\|\,\|y_o\|^2 + \Lambda\,\|f_{u\lambda}\|\,\|y_o\|\right]$$

$$+ (|\lambda_o|+1)[\tfrac{1}{2}\|f_{uu}\|(2\|y_o\|+1)\Omega + \Lambda\,\|f_{u\lambda}\|\,\Omega$$

$$+ \Phi_1(\|y_o\|+1)^3 + \Phi_2(\|y_o\|+1)^2\Lambda + \Phi_3(\|y_o\|+1)\Lambda^2\,] \tag{54}$$

$$\Psi_2 \equiv \|f_u\|\,\Omega + \tfrac{1}{2}\|f_{uu}\|\,(\|y_0\|+1)^2 + 2\,\Lambda\,\|f_{u\lambda}\|\,(\|y_0\|+1) + \Phi_2(\|y_0\|+1)^2 + 2\,\Lambda\,\Phi_3(\|y_0\|+1) \tag{55}$$

$$\Psi_3 \equiv \Lambda\|f_u\| + (|\lambda_0|+1)\|f_{uu}\|\,(\|y_0\|+1) + \Lambda(|\lambda_0|+1)\,\|f_{u\lambda}\| + |\epsilon|\,[3\,\Phi_1(\|y_0\|+1)^2 + 2\,\Lambda\,\Phi_2(\|y_0\|+1) + \Lambda^2\,\Phi_3] \tag{56}$$

$$\Theta \equiv \frac{2\|z_0\|_1}{|\gamma|}\,(\Psi_2+\Psi_3)\,\max\left\{1,\ \frac{\Omega}{\Lambda}\right\} \tag{57}$$

$$\epsilon_0 \equiv \min\left\{1,\ \frac{1}{\Lambda},\ \frac{1}{\Omega},\ \frac{1}{\Psi},\ \frac{1}{2\Theta}\right\} \tag{58}$$

Define the Banach space $\{\mathcal{B}, \|\ \|\}$ by

$$\mathcal{B} \equiv \{[\eta, v] \mid \eta \in \mathcal{R},\ v \in C_n[\alpha,\beta],\ \int_\alpha^\beta y_0^*(\xi)v(\xi)d\xi = 0\}\ , \tag{59}$$

$$\|[\eta, v]\| \equiv \max\{|\eta|, \|v\|\}\,. \tag{60}$$

Define the closed subset $\mathcal{D} \subset \mathcal{B}$ by

$$\mathcal{D} \equiv \{[\eta, v] \in \mathcal{B} \mid \|v\| \le \Omega, |\eta| \le \Lambda\}\ . \tag{61}$$

Assume

$$|\epsilon| \le \epsilon_0\ , \tag{62}$$

and define the mapping $T_\epsilon : \mathcal{D} \to \mathcal{B}$ by

$$T_\epsilon[\eta, v] = [\tilde{\eta}, \tilde{v}]\ , \tag{63}$$

where

$$\tilde{\eta} = - \frac{\frac{1}{2}\lambda_0 \int_\alpha^\beta z_0^*(\xi) f_{uu}(\xi, \lambda_0, 0) y_0^2(\xi) d\xi + \epsilon \int_\alpha^\beta z_0^*(\xi) h(\xi, \epsilon, \eta, v(\xi)) d\xi}{\gamma + \lambda_0 \int_\alpha^\beta z_0^*(\xi) f_{u\lambda}(\xi, \lambda_0, 0) y_0(\xi) d\xi} , \tag{64}$$

$$\tilde{v}(\xi) \equiv \int_\alpha^\beta G^\dagger(\xi, \tau) [\tilde{\eta}(J(\tau, \lambda_0) y_0(\tau) + \lambda_0 f_{u\lambda}(\tau, \lambda_0, 0) y_0(\tau)) + \frac{1}{2}\lambda_0 (f_{uu}(\tau, \lambda_0, 0) y_0^2(\tau) + \epsilon h(\tau, \epsilon, \eta, v(\tau))] d\tau . \tag{65}$$

Now we show that for $|\epsilon| \le \epsilon_0$, T_ϵ maps $\mathcal{D}$ into itself. Note that $[\eta, v] \in \mathcal{D}$, $\xi \in [\alpha, \beta]$, and $|\epsilon| \le \epsilon_0$ together imply that

$$(\xi, \lambda_0 + \epsilon\, \eta,\ \epsilon\, y_0 + \epsilon^2 v) \in S \quad . \tag{66}$$

Hence $h(\xi, \epsilon, \eta, v)$ is still defined and continuous. From hypotheses (14) and (16), the denominator of (64) is $\ge \frac{|\gamma|}{2}$ in absolute value. Therefore,

$$|\tilde{\eta}| \le \frac{2}{|\gamma|} [\tfrac{1}{2} |\lambda_0| \, \|z_0\|_1 \, \|f_{uu}\| \, \|y_0\|^2 + |\epsilon| \, \|z_0\|_1 \, \|h\|]$$

But an easy calculation shows that for $|\epsilon| \le \epsilon_0$,

$$\|h\| \le \Psi_1 \quad , \qquad \text{(see (54))} \tag{67}$$

and since $\epsilon_0 \Psi_1 \le 1$, we have

$$|\tilde{\eta}| \le \frac{2\|z_0\|_1}{|\gamma|} [\tfrac{1}{2} |\lambda_0| \, \|f_{uu}\| \, \|y_0\|^2 + 1] = \Lambda \quad . \tag{68}$$

Similarly, for $|\epsilon| \le \epsilon_0$,

$$\|\tilde{v}\| \le \|G^{\dagger}\| [\Lambda(\|f_u\| \, \|y_0\| + |\lambda_0| \, \|f_{u\lambda}\| \, \|y_0\|) + \tfrac{1}{2}|\lambda_0| \, \|f_{uu}\| \, \|y_0\|^2 + |\epsilon| \, \|h\|]$$

$$\le \Lambda \|G^{\dagger}\| \left[\|f_u\| \, \|y_0\| + \frac{|\gamma|}{\|z_0\|_1} \right] \tag{69}$$

$$= \Omega \quad .$$

Thus $T_\epsilon : \mathcal{D} \to \mathcal{D}$.

Finally we show that T_ϵ is contracting on $\mathcal{D}$ for $|\epsilon| \le \epsilon_0$. It follows from (66) and the Lipschitz continuity hypotheses (40) and (41) that $h(\xi, \epsilon, \eta, v)$ is Lipschitz continuous in η and v for $[\eta, v] \in \mathcal{D}$. Take arbitrary $[\eta, v]$ and $[\mu, w] \in \mathcal{D}$. Then

$$\|h(\xi, \epsilon, \eta, v) - h(\xi, \epsilon, \mu, w)\| \le \Psi_2 \, |\eta - \mu| + \Psi_3 \, \|v - w\| \tag{70}$$

where Ψ_2 and Ψ_3 are defined by (55) and (56).

Therefore,

$$|\tilde{\eta} - \tilde{\mu}| \le |\epsilon| \, \frac{2\|z_0\|_1}{|\gamma|} \, [\Psi_2 \, |\eta - \mu| + \Psi_3 \, \|v - w\|], \tag{71}$$

and

$$\|\tilde{v} - \tilde{w}\| \le \|G^{\dagger}\| [(\|f_u\| \, \|y_0\| + |\lambda_0| \, \|f_{u\lambda}\| \, \|\lambda_0\|) \, |\tilde{\eta} - \tilde{\mu}| + |\epsilon| \, \|h(\xi, \epsilon, \eta, v) - h(\xi, \epsilon, \mu, w)\|], \tag{72}$$

which reduces to

$$\|\tilde{v} - \tilde{w}\| \le |\epsilon| \, \frac{2\|z_0\|_1}{|\gamma|} \, \frac{\Omega}{\Lambda} \, [\Psi_2 \, |\eta - \mu| + \Psi_3 \, \|v - w\|] \quad . \tag{73}$$

From (71), (73), (57), and $|\epsilon| \le \epsilon_0$, it follows that

$$\|[\tilde{\eta}, \tilde{v}] - [\tilde{\mu}, \tilde{w}]\| \le |\epsilon| \, \Theta \, \|[\eta, v] - [\mu, w]\| \quad . \tag{74}$$

But $\epsilon_0 \Theta \leq \frac{1}{2}$, so T_ϵ is contracting on $\mathcal{D}$ for $|\epsilon| \leq \epsilon_0$, and the contracting mapping theorem of Appendix II tells us that T_ϵ has a unique fixed point in $\mathcal{D}$. Note that our iterates $[\eta^{(\ell)}, v^{(\ell)}]$, $\ell = 0, 1, \ldots$ are precisely the elements of the sequence defined in the statement of the contracting mapping theorem. Also, a fixed point of T_ϵ is a solution of problem (32)(35) and vice versa. Thus we have established

Theorem 2.

The inhomogeneous problem (32)(33)(34)(35) has a unique solution $[\eta, v(\xi)]$ in D, for $|\epsilon| \leq \epsilon_0$. The iteration scheme of §3 defines a sequence $[\eta^{(\ell)}, v^{(\ell)}(\xi)]$ $\ell = 0, 1, \ldots,$ of elements of $\mathcal{D}$, which converges to a unique limit in $\mathcal{D}$, and this limit is the solution $[\eta, v(\xi)]$ of the inhomogeneous problem. Furthermore, the convergence of this sequence is given by

$$\|[\eta, v(\xi)] - [\eta^{(\ell)}, v^{(\ell)}(\xi)]\| \leq |\epsilon|^{\ell} \Theta^{\ell} \max(\Lambda, \Omega),$$

$$\ell = 0, 1, 2, \ldots .$$

Since $\epsilon_0 \Theta \leq \frac{1}{2}$, this convergence is uniform in ϵ for $|\epsilon| \leq \epsilon_0$. It is easily shown by induction that the iterates $\eta^{(\ell)} = \eta^{(\ell)}(\epsilon)$ and $v^{(\ell)} = v^{(\ell)}(\xi, \epsilon)$ are continuous in ϵ for $|\epsilon| \leq \epsilon_0$. Hence the limits $\eta(\epsilon)$ and $v(\xi, \epsilon)$ are also continuous in ϵ for $|\epsilon| \leq \epsilon_0$.

With $\eta(\epsilon)$ and $v(\xi, \epsilon)$ so determined, (30) (31) is a continuous branch of nontrivial solutions of problem (1) (2) for $|\epsilon| \leq \epsilon_0$ and $\epsilon \neq 0$, and this branch intersects the trivial solution $u(\xi) \equiv 0$ at the bifurcation point $\lambda = \lambda_0$.

III. 5. Uniqueness

The uniqueness part of the contracting mapping theorem, as used in Theorem 2, implies that there is only one nontrivial solution branch of problem (1)(2) which has the form of (30)(31) with given $y_o(\xi)$ and λ_o. It remains to show that there are no nontrivial solution branches of any other form bifurcating from λ_o. One obvious candidate is the solution branch obtained by choosing $-y_o(\xi)$ instead of $y_o(\xi)$ for the normalized solution of the linearized problem, and then proceeding as in the previous two sections. This does indeed yield a branch of nontrivial solutions; but it coincides identically with the branch (30)(31). Similarly, any different normalization of $y_o(\xi)$ just gives the same branch with a new parameterization. The only other possibility is a solution which is orthogonal to $y_o(\xi)$. Such a solution could not be obtained from our iteration scheme. We now show that nontrivial solutions orthogonal to $y_o(\xi)$ cannot exist in a sufficiently small neighborhood of the bifurcation point. Define the neighborhood $\mathcal{M}_{\delta_1}$ for $0 < \delta_1 \leq 1$ by

$$\mathcal{M}_{\delta_1} \equiv \{ [\lambda, u] \mid \lambda \in \mathcal{J},\ u \in C_n[\alpha, \beta],\ |\lambda - \lambda_o| \leq \delta_1,\ \|u\| \leq \delta_1 \} \quad . \tag{76}$$

Then

$$[\alpha, \beta] \times \mathcal{M}_{\delta_1} \subset S \quad . \tag{77}$$

Let $u(\xi)$ be any nontrivial solution of (1)(2) for some value of λ such that

$$[\lambda, u] \in \mathcal{M}_{\delta_1} \quad . \tag{78}$$

Define

$$\sigma \equiv \int_{\alpha}^{\beta} y_0^*(\xi)\, u(\xi) d\xi \quad . \tag{79}$$

Then $u(\xi)$ can be written

$$u(\xi) = \sigma\, y_0(\xi) + w(\xi) \tag{80}$$

where $w(\xi)$ satisfies

$$\int_{\alpha}^{\beta} y_0^*(\xi)\, w(\xi) d\xi = 0 \quad . \tag{81}$$

Here we are allowing the possibility $\sigma = 0$.

<u>Lemma</u>

If $u(\xi)$ is a nontrivial solution of problem (1)(2) such that (78) holds with δ_1 defined by (88), then $u(\xi)$ cannot be orthogonal to $y_0(\xi)$.

Proof:

From the hypotheses that $u(\xi)$ satisfies (1)(2) and $y_0(\xi)$ satisfies (4)(5), it follows that $w(\xi)$ must satisfy

$$w'(\xi) - [A(\xi) + \lambda_0 J(\xi, \lambda_0)]w(\xi) = \lambda f(\xi, \lambda, u) - \lambda_0 J(\xi, \lambda_0)u \tag{82}$$

$$M\, w(\alpha) + N\, w(\beta) = 0 \quad . \tag{83}$$

And so, from the basic alternative theorem,

$$\lambda \int_{\alpha}^{\beta} z_0^*(\xi)\, [\lambda f(\xi, \lambda, \sigma\, y_0(\xi)+w(\xi)) - \lambda_0 J(\xi, \lambda_0)(\sigma y_0(\xi)+w(\xi))] d\xi = 0 \tag{84}$$

Clearly $\sigma = 0$, $w(\xi) \equiv 0$ gives a solution of (84). If (84) has no other solution, then (1)(2) has no nontrivial solution for this value of λ, contradicting our hypothesis. Therefore we assume (84) is satisfied for some $w(\xi) \neq 0$.

Then (81)(82)(83) together are equivalent to the integral equation

$$w(\xi) = \int_a^\beta G^\dagger(\xi,\tau)[\lambda f(\tau,\lambda,u) - \lambda_0 J(\tau,\lambda_0)u]\,d\tau \quad , \tag{85}$$

where $G^\dagger(\xi,\tau)$ is the principal generalized Green's Matrix of §II. 7.

By virtue of (77),

$$w(\xi) = \int_a^\beta G^\dagger(\xi,\tau)\,[\lambda \int_0^1\int_0^1 f_{u\lambda}(\tau,\lambda_0+\rho(\lambda-\lambda_0),\zeta u)\,d\rho\,d\lambda\,(\lambda-\lambda_0)$$

$$+ (\lambda-\lambda_0)\int_0^1 f_u(\tau,\lambda_0,\zeta u)d\zeta + \lambda_0 \int_0^1\int_0^1 f_{uu}(\tau,\lambda_0,\rho\zeta u)\zeta\,d\rho\,d\zeta\,u]$$

$$(\sigma y_0(\tau) + w(\tau))\,d\tau \tag{86}$$

Taking norms as in Appendix I and using (78) gives

$$\|w\| \le \|G^\dagger\|\Big[(|\lambda_0|+1)\|f_{u\lambda}\|_S + \|f_u\|_S + |\lambda_0|\tfrac{1}{2}\|f_{uu}\|_S\Big]\,\delta_1\,\|\sigma y+w\|. \tag{87}$$

Now for any $0 < \theta < 1$, define δ_1 by

$$\delta_1 \equiv \frac{\theta}{\|G^\dagger\|\Big[(|\lambda_0|+1)\|f_{u\lambda}\|_S + \|f_u\|_S + \tfrac{1}{2}|\lambda_0|\,\|f_{uu}\|_S\Big]} \quad , \tag{88}$$

and we have

$$\|w\| \le \theta\,\|\sigma y + w\| \quad . \tag{89}$$

But it follows from (89) that $\sigma = 0$ implies $w(\xi) \equiv 0$ and so $u(\xi)$ is orthogonal to $y_0(\xi)$ only if $u(\xi) \equiv 0$, which proves the lemma.

On the strength of this lemma, we can make the following definitions

$$\epsilon \equiv \sigma = \int_{\alpha}^{\beta} y_0^*(\xi)\, u(\xi) d\xi \tag{90}$$

$$\bar{v}(\xi, \epsilon) \equiv \frac{1}{\epsilon^2} (u(\xi) - \epsilon\, y_0(\xi)) \tag{91}$$

$$\bar{\eta}(\epsilon) \equiv \frac{1}{\epsilon} (\lambda - \lambda_0) \quad . \tag{92}$$

It is clear that $[\bar{\eta}, \bar{v}]$ must satisfy the same equations as $[\eta, v]$ in Theorem 2, and so if $[\bar{\eta}, \bar{v}] \in \mathfrak{D}$, the two must coincide by the uniqueness part of the contracting mapping theorem.

Now we consider the ambiguity in the choice of normalization of $y_0(\xi)$. Suppose instead of $y_0(\xi)$ we had chosen the eigenfunction

$$x_0(\xi) = \omega\, y_0(\xi) \tag{93}$$

for any real $\omega \neq 0$.

Then, for sufficiently small $\sigma \neq 0$, we could proceed as in §3 and §4 to construct the solution branch

$$u(\xi) = \sigma\, x_0(\xi) + \sigma^2\, w(\xi, \sigma) \tag{94}$$

$$\lambda = \lambda_0 + \sigma\, u(\sigma) \quad . \tag{95}$$

By inspection, the equations (32)(33)(34) and (35) which define v and η in §3 are unchanged by the substitution

$$
\begin{aligned}
y_0(\xi) &\rightarrow \omega\, y_0(\xi)\\
\epsilon &\rightarrow \frac{1}{\omega}\,\epsilon\\
\eta &\rightarrow \omega\,\eta\\
v &\rightarrow \omega^2\, v \quad .
\end{aligned}
\tag{96}
$$

Hence the solution branch (94)(95) coincides with (30)(31) in a neighborhood of the bifurcation point with the equivalence

$$
\begin{aligned}
y_0 &= \frac{1}{\omega}\, x_0\\
\epsilon &= \omega\,\sigma\\
\eta &= \frac{1}{\omega}\,\mu\\
v &= \frac{1}{\omega^2}\, w \quad .
\end{aligned}
\tag{97}
$$

Thus (94)(95) yields no new solutions, but is just a new parameterization of the unique solution branch of Theorem 2.

We summarize these results in

<u>Theorem 3</u>

Problem (1)(2) has no small nontrivial solutions in a sufficiently small neighborhood of λ_0, other than those on the branch (30)(31) given by Theorem 2. This branch is unique at least for $\epsilon \leq \epsilon^*$, where

$$
\epsilon^* \equiv \min\left\{\epsilon_0,\ \frac{\delta_1}{\Lambda}\ ,\ \frac{\delta_1}{\|y_0\| + 1}\right\}.
$$

Here δ_1 is defined by (88).

III. 6 Asymptotic Behavior

It is useful for the applications to have an asymptotic expansion of solution branch (30) (31) in the neighborhood of its bifurcation point. Such asymptotic expansions are often computed formally, without rigorous justification. On the basis of Theorem 2, we are able to obtain the first terms of an asymptotic expansion with very little effort, to estimate the error, and to confirm that the expansion is indeed asymptotic. For a much deeper treatment of this subject, see §V. 5. For the definition of the order symbol O, see Appendix A.

From (30) and Theorem 2 we have immediately that

$$\|u(\xi,\varepsilon) - \varepsilon y_o(\xi)\| = O(\varepsilon^2), \qquad \varepsilon \to 0. \tag{100}$$

since

$$\|v\| \leq \Omega \quad \text{for} \quad |\varepsilon| \leq \varepsilon_o . \tag{101}$$

From (42) we have

$$\eta(\varepsilon) = - \frac{\frac{1}{2}\lambda_0 \int_a^\beta z_o^*(\xi) f_{uu}(\xi,\lambda_o,0) y_o^2(\xi) d\xi + \varepsilon \int_a^\beta z_o^*(\xi) h(\xi,\varepsilon,\eta,v(\xi)) d\xi}{\gamma + \lambda_o \int_a^\beta z_o^*(\xi)\, f_{u\lambda}(\xi,\lambda_o,0) y_o(\xi) d\xi} \tag{102}$$

Define λ_1 by

$$\lambda_1 = - \frac{\frac{1}{2}\lambda_o \int_a^\beta z_o^*(\xi) f_{uu}(\xi,\lambda_o,0) y_o(\xi) y_o(\xi) d\xi}{\gamma + \lambda_o \int_a^\beta z_o^*(\xi) f_{u\lambda}(\xi,\lambda_o,0) y_o(\xi) d\xi} . \tag{103}$$

Hypotheses (14) and (16) guarantee that the denominator of (102) and (103) is $\geq \frac{|\gamma|}{2}$. Therefore

$$|\eta(\varepsilon) - \lambda_1| \leq \frac{2}{|\gamma|} \, |\varepsilon| \, \|z_o\|_1 \, \|h\|$$

$$\leq |\varepsilon| \frac{2}{|\gamma|} \|z_o\|_1 \Psi_1 \tag{104}$$

from (67) and (54), assuming $|\varepsilon| \leq \varepsilon_o$. Thus we have proven

$$\eta(\varepsilon) = \lambda_1 + O(\varepsilon), \quad \varepsilon \to 0, \tag{105}$$

which we substitute in (31) and get

$$\lambda(\varepsilon) = \lambda_o + \varepsilon\lambda_1 + O(\varepsilon^2), \quad \varepsilon \to 0. \tag{106}$$

Then (105) and (106) give the asymptotic behavior of the nontrivial solution branch up to $O(\varepsilon^2)$.

In order to calculate higher order terms in the expansion, in general we need to know more about the behavior of $v(\xi, \varepsilon)$. However, in the special case when $f_{uu}(\xi, \lambda_o, 0) = 0$ and higher derivatives of f exist, we can easily get higher order terms in (31) without knowing $v(\xi, \varepsilon)$. In fact, let us assume that f is m-times Fréchet differentiable, and in the notation of Appendix A,

$$f_{u^k}(\xi, \lambda_o, 0) = 0 \qquad \text{for } k = 2, \ldots, m-1, \tag{107}$$

$$f_{u^m}(\xi, \lambda_o, 0) \neq 0, \tag{108}$$

where $m \geq 3$. Furthermore let us assume that f_{u^m} is Lipschitz continuous in u, as defined in Appendix A, for $(\xi, \lambda_o, u) \in S$.

Now (107) implies that

$$\lambda_1 = 0 \tag{109}$$

so (102) becomes

$$\eta(\varepsilon) = \frac{-\varepsilon \int_a^\beta z_o^*(\xi) h(\xi, \varepsilon, \eta, v(\xi) d\xi}{\gamma + \lambda_o \int_a^\beta z_o^*(\xi) f_{u\lambda}(\xi, \lambda_o, 0) y_o(\xi) d\xi} \; . \tag{110}$$

Now by inspection of (43), we can rewrite h as

$$h(\xi, \varepsilon, \eta, v) = \eta [f_u(\xi, \lambda_o, 0)v + f_{u\lambda}(\xi, \lambda_o, 0)(\eta y_o + (\lambda_o + \varepsilon\eta)v) \tag{111}$$

$$+ \frac{\lambda_o + \varepsilon\eta}{\varepsilon} E_2(\xi, \lambda, u)(y_o + \varepsilon v)] + \frac{\lambda_o + \varepsilon\eta}{\varepsilon} E_1(\xi, u)(y_o + \varepsilon v)^2 \; .$$

But $E_1(\xi, u)$ defined by (38) is now equal to

$$E_1(\xi, u) = \frac{1}{m!} f_{u^m}(\xi, \lambda_o, 0)\, u^{m-2} + E^{(m)}(\xi, u) u^{m-2} \tag{112}$$

where

$$E^{(m)}(\xi, u) \equiv \int_o^1 \int_o^1 \cdots \int_o^1 [f_{u^m}(\xi, \lambda_o, \rho_1 \rho_2 \cdots \rho_m u) \tag{113}$$

$$- f_{u^m}(\xi, \lambda_o, 0)] \; \rho_2 \rho_3^2 \cdots \rho_m^{m-1} \, d\rho_1 \cdots d\rho_m \; .$$

Define s and t to represent terms which appear in h as follows

$$s(\xi) = f_u(\lambda_o, 0)v + f_{u\lambda}(\xi, \lambda_o, 0)(\eta y_o + \lambda_o v) + \frac{\lambda_o + \varepsilon\eta}{\varepsilon} E_2(\xi, \lambda, u) y_o \tag{114}$$

$$t(\xi) = \eta f_{u\lambda}(\xi, \lambda_o, 0)v + (\lambda_o + \varepsilon\eta) E_2(\xi, \lambda, u)v \; , \tag{115}$$

so now

$$h(\xi, \varepsilon, \eta, v) = \eta s(\xi) + \varepsilon\eta t(\xi) + (\lambda_o + \varepsilon\eta)\frac{\varepsilon^{m-3}}{m!} f_{u^m}(\xi, \lambda_o, 0)(y_o + \varepsilon v)^m$$

$$+ (\lambda_o + \varepsilon\eta)\varepsilon^{m-3} E^{(m)}(\xi, u)(y_o + \varepsilon v)^m, \tag{116}$$

and (110) can be written

$$\eta(\varepsilon) = \frac{-(\lambda_o+\varepsilon\eta)\varepsilon^{m-2}\int_a^\beta z_o^*(\xi)\left[\frac{1}{m!} f_{u^m}(\xi, \lambda_o, 0) + E^{(m)}(\xi, u)\right](y_o+\varepsilon v)^m d\xi}{\gamma + \int_a^\beta z_o^*(\xi)\left[\lambda_o f_{u\lambda}(\xi, \lambda_o, 0) y_o + \varepsilon s(\xi) + \varepsilon^2 t(\xi)\right] d\xi} \tag{117}$$

provided the denominator is nonzero. But this is assured if $|\varepsilon| \leq \tilde{\varepsilon}_o$ where

$$\tilde{\varepsilon}_o \equiv \min\left\{\varepsilon_o, \frac{1}{\|z_o\|_1}\left[\Omega(\|f_u\| + \lambda_o\|f_{u\lambda}\|) + \|f_{u\lambda}\|(\|y_o\|+1)\Lambda\right.\right.$$

$$\left.\left. + \Phi_2(\|y_o\|+1)^2 + \Lambda\Phi_3(\|y_o\|+1)\right]^{-1}\right\} . \tag{118}$$

Define

$$\lambda_{m-1} \equiv \frac{-\lambda_o \frac{1}{m!}\int_a^\beta z_o^*(\xi) f_{u^m}(\xi, \lambda_o, 0) y_o^m(\xi) d\xi}{\gamma + \lambda_o \int_a^\beta z_o^*(\xi) f_{u\lambda}(\xi, \lambda_o, 0) y_o(\xi) d\xi} . \tag{119}$$

Then we can write

$$\eta(\varepsilon) = \varepsilon^{m-2}\lambda_{m-1} + \varepsilon^{m-1}\tilde{\eta}(\varepsilon), \tag{120}$$

where

$$\tilde{\eta}(\varepsilon) \equiv \frac{-\int_a^\beta z_o^*(\xi)\left[\frac{\eta}{m!} f_{u^m}(\xi, \lambda_o, 0) + \frac{(\lambda_o+\varepsilon\eta)}{\varepsilon} E^{(m)}(\xi, u)\right](y_o+\varepsilon v)^m d\xi + \int_a^\beta z_o^*(\xi)}{\gamma + \int_a^\beta z_o^*(\xi)\left[\lambda_o f_{u\lambda}(\xi, \lambda_o, 0)y_o + \varepsilon s(\xi) + \varepsilon^2 t(\xi)\right] d\xi}$$

$$\frac{\left[f_{u^m}(\xi, \lambda_o, 0)\left(\sum_{j=1}^m \frac{\varepsilon^{j-1} y_o^{m-j} v^j}{j!\,(m-j)!}\right) + \lambda_{m-1}\,(s(\xi)+\varepsilon t(\xi))\right] d\xi}{} \,. \quad (121)$$

It is clear that $\tilde{\eta}(\varepsilon)$ is bounded for $|\varepsilon| \leqslant \tilde{\varepsilon}_o$, because the Lipschitz continuity of f_{u^m} implies

$$\frac{1}{\varepsilon} E^m(\xi, u) \leqslant \Phi^{(m)} \left(\|y_o\| + 1\right) \quad (122)$$

for $(\xi, \lambda, u) \in S$.

Hence we have proven that

$$\eta(\varepsilon) = \varepsilon^{m-2} \lambda_{m-1} + O(\varepsilon^{m-1}), \qquad \varepsilon \to 0 \quad (123)$$

or that

$$\lambda(\varepsilon) = \lambda_o + \varepsilon^{m-1} \lambda_{m-1} + O(\varepsilon^m), \qquad \varepsilon \to 0\,. \quad (124)$$

The qualitative behavior of the solution is now clear from (124) and (100). Let Λ_m be a bound on $|\tilde{\eta}(\varepsilon)|$ for $|\varepsilon| \leqslant \varepsilon_o$, and define

$$\tilde{\varepsilon}_o \equiv \min\left\{\varepsilon_o, \frac{|\lambda_{m-1}|}{\Lambda_m}\right\} \quad .$$

Then the following two types of behavior can arise in a neighborhood of λ_o. See Figure 1 for illustrations.

Case (i): m is even.

Then a nontrivial solution u exists for all λ in an open interval containing λ_o as an interior point, except for the point λ_o itself (where of course $u \equiv 0$). In a sufficiently small interval containing λ_o as an interior point, the "amplitude" ε of the solution is monotone increasing or decreasing, depending on whether λ_{m-1} is positive or negative respectively.

Case (ii) m is odd.

Then if λ_{m-1} is positive (negative) there is no small nontrivial solution for λ in some interval below (above) λ_o and containing λ_o as its upper (lower) end point. There are exactly two small nontrivial solutions for each λ in the open interval $(\lambda_o, \lambda(\tilde{\varepsilon}_o))$ if λ_{m-1} is positive, or in $(\lambda(\tilde{\varepsilon}_o), \lambda_o)$ if λ_{m-1} is negative.

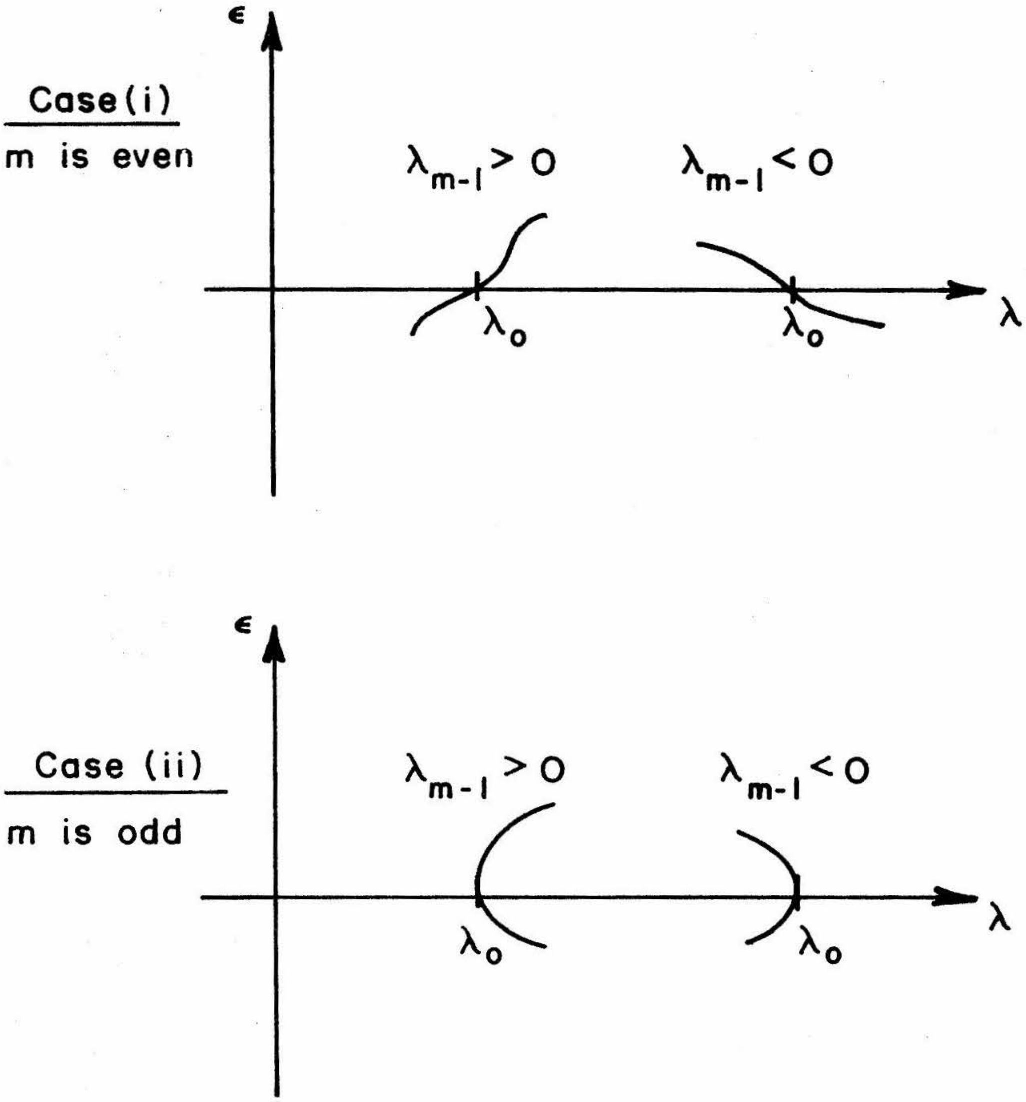

FIG. 1 ASYMPTOTIC BEHAVIOR NEAR A BIFURCATION POINT

CHAPTER IV

DEGENERATE BIFURCATION THEORY

IV. 1. Introduction

Throughout Chapter III, λ_o was assumed to be a simple eigenvalue of the linearized problem. Now we remove this restriction and assume that λ_o is an eigenvalue with any multiplicity p. From the linear theory in Chapter II we know that p is finite, in fact $p \leq n$. When $p > 1$, the eigenvalue λ_o is said to be degenerate; we extend this terminology and define "degenerate bifurcation theory" to be the theory of bifurcation at eigenvalues of the linearized problem which have multiplicity p, where $1 < p \leq n$. The simple case $p = 1$ is of course included in the theory of this chapter, but the theory of Chapter III gives stronger results for this case so we ignore it here.

The problem considered in this chapter is identical in form to that of Chapter III:

$$u'(\xi) = A(\xi)\,u(\xi) + \lambda f(\xi, \lambda, u(\xi)) \qquad \alpha \leq \xi \leq \beta \tag{1}$$

$$Mu(\alpha) + N\,u(\beta) = 0. \tag{2}$$

The matrices A, M and N, the vectors u and f and the scalars α, β, ξ, and λ are all defined at the beginning of Chapter III. We seek nontrivial solutions $u(\xi)$ to problem (1) (2) for λ in a neighborhood of an isolated degenerate eigenvalue λ_o of the linearized problem

$$y'(\xi)-[A(\xi) + \lambda_o J(\xi,\lambda_o)]\, y(\xi) = 0 \qquad \alpha \le \xi \le \beta \tag{3}$$

$$My(\alpha) + Ny(\beta) = 0. \tag{4}$$

Since λ_o now has multiplicity p where $1 < p \le n$, there exists a set of p linearly independent eigenfunctions of (3) (4) which we can choose to be orthonormal, and which we designate

$$y^{(j)}(\xi) \quad , \quad j = 1, \ldots, p. \tag{5}$$

Any solution to (3) (4) may be expressed as a linear combination of these $y^{(j)}(\xi)$.

From Chapter II, Theorem 5, the problem adjoint to (3) (4) also has exactly p orthonormal solutions

$$z^{(i)}(\xi) \, , \quad i = 1, \ldots, p, \tag{6}$$

where

$$z^{(i)'}(\xi) + [A^*(\xi) + \lambda_o J^*(\xi,\lambda_o)]\, z^{(i)}(\xi) = 0 \qquad \alpha \le \xi \le \beta \tag{7}$$

$$\begin{aligned} P z^{(i)}(\alpha) + Q z^{(i)}(\beta) = 0, \\ i = 1, \ldots, p. \end{aligned} \tag{8}$$

The hypothesis (14) of Chapter III is replaced by the following generalization: define the $p \times p$ matrix C by

$$C_{ij} = \int_\alpha^\beta z^{(i)*}(\xi)\, J(\xi,\lambda_o)\, y^{(j)}(\xi)\, d\xi \tag{9}$$

and assume

$$\det C \neq 0. \tag{10}$$

A bound on $\|f_{u\lambda}\|$, analogous to (16) of Chapter III, is assumed in section 5.

Theorem 1 of Chapter III still applies to problem (1) (2), so bifurcation cannot occur for values of λ other than the eigenvalues of (3) (4). In §IV. 2 we show that sometimes bifurcation does not occur even at an eigenvalue, if this eigenvalue is degenerate. A simple example is given to demonstrate this possibility. Then in §IV. 3 we define the algebraic bifurcation equations, and present an iteration scheme which generates a nontrivial solution to the problem (1) (2), given a simple root of the algebraic bifurcation equations. Section IV. 4 contains the proofs of the statements made in §IV. 3, and in §IV. 5 we present conditions under which the algebraic bifurcation equations are solvable. Finally in §IV. 6 we indicate an extension of the theory of the preceding three sections to problems with nonlinearities for which the lowest order term has degree higher than two.

IV. 2. Non-existence Example

Unlike the case of a simple eigenvalue, for which we proved in Chapter III that bifurcation always occurs, bifurcation does not always occur at a degenerate eigenvalue. The following example, based on one by Berger [5], demonstrates this point. Let

$$u_1'(\xi) = \lambda[u_1(\xi) + u_2^3(\xi)]$$
$$u_2'(\xi) = -\lambda[u_2(\xi) - u_1^3(\xi)] \qquad (11)$$

for $0 \leq \xi \leq 1$ and λ real, and

$$\begin{aligned} e\,u_1(0) &= u_1(1) \\ u_2(0) &= e\,u_2(1). \end{aligned} \tag{12}$$

Clearly this has the form of problem (1) (2). The linearized problem has the eigenvalue $\lambda_o = 1$ of multiplicity 2 and corresponding linearly independent eigensolutions

$$y^{(1)}(\xi) = \begin{pmatrix} e^{\xi} \\ 0 \end{pmatrix}, \quad y^{(2)}(\xi) = \begin{pmatrix} 0 \\ e^{-\xi} \end{pmatrix}. \tag{13}$$

Now if we multiply the two equations in (11) by u_2 and u_1, respectively, integrate by parts, use (12) and add, we get

$$\lambda \int_0^1 [u_2^4(\xi) + u_1^4(\xi)]\, d\xi = 0. \tag{14}$$

But (14) implies that $u(\xi) \equiv 0$ for $\lambda \neq 0$, and so problem (11) (12) has no nontrivial solution for λ near the eigenvalue $\lambda_o = 1$, (or in fact for any real λ). Thus bifurcation does not occur at λ_o.

IV. 3. The Iteration Scheme and the Algebraic Bifurcation Equations

By analogy with Chapter III, we seek a nontrivial solution branch of small norm in a neighborhood of λ_o, of the form

$$u(\xi, \epsilon) = \epsilon \sum_{j=1}^{p} q_j(\epsilon)\, y^{(j)}(\xi) + \epsilon^2 v(\xi, \epsilon) \tag{15}$$

$$\lambda(\epsilon) = \lambda_o + \epsilon\,\eta(\epsilon). \tag{16}$$

Here $\sum_{j=1}^{p} q_j(\epsilon)\, y^{(j)}(\xi)$ is some element (to be determined) of the eigensolution space of problem (3) (4), normalized such that

$$\sum_{j=1}^{p} q_j^2(\epsilon) = 1. \tag{17}$$

The summation convention will frequently be used to abbreviate this term

$$q_j y^{(j)} \equiv \sum_{j=1}^{p} q_j(\epsilon)\, y^{(j)}(\xi). \tag{18}$$

As before, ϵ is a small parameter, $v(\xi, \epsilon)$ and $\eta(\epsilon)$ are functions to be determined, and v is made unique by imposing

$$\int_{\alpha}^{\beta} y^{(j)*}(\xi)\, v(\xi, \epsilon)\, d\xi = 0 \quad , \quad i = 1, \ldots, p. \tag{19}$$

Substituting (15) (16) into (1) (2) and using (3) (4) gives the following boundary value problem which q, η and v must satisfy:

$$v'(\xi) - [A(\xi) + \lambda_o J(\xi, \lambda_o)]\, v(\xi) = \frac{\lambda_o}{\epsilon^2}\, [f(\xi, \lambda_o + \epsilon\eta, \epsilon q_j y^{(j)} + \epsilon^2 v)$$

$$- J(\xi, \lambda_o)(\epsilon q_i y^{(j)} + \epsilon^2 v)] + \frac{1}{\epsilon}\, \eta\, f(\xi, \lambda_o + \epsilon\eta, \epsilon q_j y^{(j)} + \epsilon^2 v)$$

$$\alpha \leq \xi \leq \beta \tag{20}$$

$$M v(\alpha) + N v(\beta) = 0. \tag{21}$$

From the basic alternative theorem of Chapter II, problem (20) (21) has a solution only if the right hand side of (20) is orthogonal to $z^{(i)}(\xi)$, $i = 1, \ldots, p$, that is,

$$\begin{aligned} &\frac{\lambda_o}{\epsilon^2} \int_\alpha^\beta z^{(i)*}(\xi)\,[f(\xi, \lambda_o + \epsilon\eta, \epsilon q_j y^{(j)} + \epsilon^2 v) \\ &\qquad\qquad - J(\xi, \lambda_o)(\epsilon q_j y^{(j)} + \epsilon^2 v)]\, d\xi \\ &+ \frac{1}{\epsilon}\eta \int_\alpha^\beta z^{(i)*}(\xi)\, f(\xi, \lambda_o + \epsilon q_j y^{(j)} + \epsilon^2 v)\, d\xi = 0 \\ &\qquad i = 1, \ldots, p. \end{aligned} \tag{22}$$

The $n+2$ dimensional domain S of Chapter III, on which f is assumed Fréchet differentiable with Lipschitz continuous second derivatives, is now defined by

$$\begin{aligned} S \equiv \{ (\xi, \lambda, u) \,\big|\, &\xi \in [\alpha, \beta],\ |\lambda - \lambda_o| < 1,\ u \in C_n[\alpha, \beta], \\ &\|u\| \leq \Phi + 1 \} \end{aligned} \tag{23}$$

where

$$\Phi \equiv \sum_{j=1}^{p} \|y^{(j)}\|.$$

Therefore f has the Taylor expansion with remainder for $(\xi, \lambda, u) \in S$, just as in Chapter III:

$$f(\xi, \lambda, u) = f_u(\xi, \lambda_o, 0)\, u + \tfrac{1}{2} f_{uu}(\xi, \lambda_o, 0)\, u^2 + f_{u\lambda}(\xi, \lambda_o, 0)(\lambda - \lambda_o)\, u$$

$$+ E_1(\xi, u)\, u^2 + E_2\, (\xi, \lambda, u)\, (\lambda - \lambda_o) u, \tag{24}$$

where E_1 and E_2 are defined by (38) and (39) of Chapter III and have the Lipschitz continuity properties (40) and (41) of Chapter III, for $(\xi, \lambda, u) \in S$ as defined above.

Now rewrite the orthogonality conditions (22), grouping terms according to their order of magnitude in the small parameter ϵ. This gives

$$g_i(q, \eta) + \epsilon\, h_i(\epsilon, q, \eta, v) = 0 \tag{25}$$

$$i = 1, \ldots, p,$$

where

$$g_i(q, \eta) \equiv \eta \left[\int_\alpha^\beta z^{(i)*}(\xi)\, J(\xi, \lambda_o) q_j y^{(j)}(\xi) d\xi + \lambda_o \int_\alpha^\beta z^{(i)*}(\xi)\, f_{u\lambda}(\xi, \lambda_o, 0) \right.$$

$$\left. q_j y^{(j)}(\xi) d\xi \right] + \lambda_o \tfrac{1}{2} \int_\alpha^\beta z^{(i)*}(\xi)\, f_{uu}(\xi, \lambda_o, 0)\, q_j y^{(j)}(\xi)\, q_k y^{(k)}(\xi)\, d\xi, \tag{26}$$

and

$$h_i(\epsilon, q, \eta, v) \equiv \eta \int_\alpha^\beta z^{(i)*}(\xi)\, [f_u(\xi, \lambda_o, 0) v(\xi) + \tfrac{1}{2} f_{uu}(\xi, \lambda_o, 0) q_j y^{(j)}(\xi) q_k y^{(k)}(\xi)$$

$$+ \eta\, f_{u\lambda}(\xi, \lambda_o, 0) q_j y^{(j)}(\xi)\,]\, d\xi$$

$$+(\lambda_o + \epsilon\eta) \int_\alpha^\beta z^{(i)*}(\xi) \left[\tfrac{1}{2} f_{uu}(\xi, \lambda_o 0) (2 q_j y^{(j)}(\xi) + \epsilon v^2(\xi)) \right.$$

$$+ \eta f_{u\lambda}(\xi, \lambda_o, 0) v(\xi) + \frac{1}{\epsilon} E_1(\xi, \epsilon q_j y^j + \epsilon^2 v) (q_j y^{(j)} + \epsilon v)^2$$

$$\left. + \frac{1}{\epsilon} E_2(\xi, \lambda_o + \epsilon\eta, \epsilon q_j y^j + \epsilon^2 v) \eta (q_j y^{(j)} + \epsilon v) \right] d\xi,$$

$$i = 1, \ldots, p. \qquad (27)$$

Setting $\epsilon = 0$ in (25) gives us the "algebraic bifurcation equations", defined by

$$g_i(q, \eta) = 0 \quad , \quad i = 1, \ldots, p,$$
$$\sum_{i=1}^{p} q_i^2 = 1. \qquad (28)$$

Define the following arrays which appear in (26)

$$C_{ij} \equiv \int_\alpha^\beta z^{(i)*}(\xi) J(\xi, \lambda_o) y^{(j)}(\xi) d\xi \qquad (29)$$

$$D_{ij} \equiv \int_\alpha^\beta z^{(i)*}(\xi) f_{u\lambda}(\xi, \lambda_o 0) y^{(j)}(\xi) d\xi \qquad (30)$$

$$F_{ijk} \equiv \tfrac{1}{2} \int_\alpha^\beta z^{(i)*}(\xi) f_{uu}(\xi, \lambda_o, 0) y^{(j)}(\xi) y^{(k)}(\xi) d\xi \qquad (31)$$

$$i, j, k = 1, \ldots, p.$$

Then the algebraic bifurcation equations (28) can be written

$$\omega \sum_{j=1}^{p} [C_{ij} + \lambda_o D_{ij}] \; x_j + \lambda_o \sum_{j,k=1}^{p} F_{ijk} \, x_j x_k = 0, \; i = 1, \ldots, p,$$

$$\sum_{j=1}^{p} x_j^2 = 1. \tag{32}$$

Here (x, ω) denotes a simple root of (32). Note that the algebraic bifurcation equations are independent of v and ϵ. They are a system of p+1 quadratic equations in the p+1 unknowns $x_1, \ldots, x_p$, and ω. We assume throughout this section that a simple root (x, ω) of (32) has been found, and show how to construct a nontrivial solution branch to (1) (2), given this root.

This iteration scheme, which we now present, is really a double iteration scheme, consisting of "inner" and "outer" iterations. The outer iterations correspond roughly to the iteration scheme of Chapter III, and the inner iterations generate the new value of q and η at each step of the outer scheme.

The outer iteration scheme is defined by the following equations, which come directly from (17) (19) (20) (21) and (22).

$$v^{(0)} = 0, \tag{33}$$

$$\frac{\lambda_o}{\epsilon^2} \int_\alpha^\beta z^{(i)*}(\xi)\Big[f(\xi, \lambda_o + \epsilon\eta^{(\ell+1)}, \epsilon q_j^{(\ell+1)} y^{(j)} + \epsilon^2 v^{(\ell)}) - J(\xi, \lambda_o)$$

$$(\epsilon q_j^{(\ell+1)} h^{(j)} + \epsilon^2 v^{(\ell)})\Big] d\xi \tag{34}$$

$$+ \frac{1}{\epsilon}\eta^{(\ell+1)} \int_\alpha^\beta z^{(i)*}(\xi) f(\xi, \lambda_o + \epsilon\eta^{(\ell+1)}, \epsilon q_j^{(\ell+1)} y^{(j)} + \epsilon^2 v^{(\ell)}) d\xi = 0, \; i = 1, \ldots, p,$$

$$\sum_{j=1}^{p} q_j^{(\ell+1)\,2} = 1 \tag{35}$$

$$v^{(\ell+1)'}(\xi) - [A(\xi)+\lambda_o J(\xi,\lambda_o)]\, v^{(\ell+1)}(\xi) = \frac{\lambda_o}{\epsilon^2}\Big[f(\xi,\lambda_o+\epsilon\eta^{(\ell+1)}, \epsilon q_j^{(\ell+1)} y^{(j)} + \epsilon^2 v^{(\ell)})$$

$$-J(\xi,\lambda_o)(\epsilon q_j^{(\ell+1)} y^{(j)} + \epsilon^2 v^{(\ell)})\Big] \tag{36}$$

$$+ \frac{1}{\epsilon}\eta^{(\ell+1)} f(\xi,\lambda_o+\epsilon\eta^{(\ell+1)}, \epsilon q_j^{(\ell+1)} y^{(j)} + \epsilon^2 v^{(\ell)}),$$

$$M\, v^{(\ell+1)}(\alpha) + N v^{(\ell+1)}(\beta) = 0, \tag{37}$$

$$\int_\alpha^\beta y^{(i)*}(\xi)\, v^{(\ell+1)}(\xi)\, d\xi = 0, \qquad i = 1,\ldots,p, \tag{38}$$

where $\ell = 0, 1, 2, \ldots$.

At each step of the iterations, (34) represents p transcendental equations in the $p+1$ unknowns $q_1^{(\ell+1)} \ldots q_p^{(\ell+1)}$ and $\eta^{(\ell+1)}$. We call (34) together with (35) the transcendental bifurcation equations. They are solved using the inner iteration scheme, which we define shortly, using the assumed root of the algebraic bifurcation equations as a starting point. When (34) is satisfied, the basic alternative theorem guarantees that (37) (37) has a solution $v^{(\ell+1)}(\xi)$, and (38) makes this solution unique.

This unique solution of (36) (37) (38) can be written in terms of the generalized Green's matrix of § II. 7 as

$$v^{(\ell+1)}(\xi) = \int_\alpha^\beta G^\dagger(\xi,\tau)\Bigg[\frac{\lambda_o}{\epsilon^2}[f(\tau,\lambda_o+\epsilon\eta^{(\ell+1)},\ \epsilon q_j y^{(j)} + \epsilon^2 v^{(\ell)})$$

$$-J(\tau,\lambda_o)\ (\epsilon q_j^{(\ell+1)} y^{(j)} + \epsilon^2 v^{(\ell)})\,] \qquad (39)$$

$$+\frac{1}{\epsilon}\eta^{(\ell+1)} f(\tau,\lambda_o+\epsilon\eta^{(\ell+1)},\ \epsilon q_j^{(\ell+1)} y^{(j)} + \epsilon^2 v^{(\ell)})\Bigg] d\tau.$$

Now we define the inner iteration scheme. In order to simplify the notation, we define the augmented vectors $\bar{x}$ and $\bar{q}$ to be the p+1 dimensional vectors with first p components the same as x and q respectively, and $(p+1)^{st}$ components ω and η respectively. Similarly we define the augmented vector functions $\bar{g}$ and $\bar{h}$ to be the p+1 dimensional vectors with first p components the same as g and h respectively, and $(p+1)^{st}$ components defined by

$$\bar{g}_{p+1}(\bar{q}) \equiv \sum_{j=1}^{p} q_j^2 - 1, \qquad (40)$$

$$\bar{h}_{p+1}(\epsilon, \bar{q}, v) \equiv 0. \qquad (41)$$

Now the algebraic bifurcation equations are equivalent to

$$\bar{g}(\bar{x}) = 0, \qquad (42)$$

and the transcendental bifurcation equations are equivalent to

$$\bar{g}(\bar{q}^{(\ell+1)}) + \epsilon\,\bar{h}(\epsilon, \bar{q}^{(\ell+1)}, v^{(\ell)}) = 0, \qquad (43)$$

as can be seen from the equivalence of (22) and (25).

The inner iteration scheme consists of solving (43) by the chord method, using $\bar{x}$ as an initial guess. We use subscripts in parentheses to number the iterates in the inner iteration scheme, and in the following discussion we suppress the superscripts of the outer iteration scheme, for convenience, since they do not change anyway. Define $\bar{a}$ by

$$\bar{q} = \bar{x} + \bar{a}. \tag{44}$$

Then the inner iteration scheme is defined by

$$\bar{a}_{(o)} = 0 \tag{45}$$

$$\bar{a}_{(m+1)} = \bar{a}_{(m)} - K\,[\,\bar{g}\,(\bar{x}+\bar{a}_{(m)}) + \epsilon\,\bar{h}\,(\epsilon,\, \bar{x} + \bar{a}_{(m)}, v)\,], \qquad m = 0, 1, \ldots \tag{46}$$

where

$$K \equiv [\,\bar{g}_{\bar{x}}\,(\bar{x})\,]^{-1}, \tag{47}$$

and K exists by the hypothesis that $\bar{x}$ is a simple root. (By "simple" root we mean that the Jacobian $\bar{g}_{\bar{x}}$ is nonsingular at the root $\bar{x}$).

In the following section we prove that the inner iterates defined by (45) (46) converge to a limit $\bar{a}$ for sufficiently small ϵ, and this gives the root $\bar{q} \equiv \bar{x} + \bar{a}$ of the transcendental bifurcation equations at each step ℓ of the outer iteration scheme. Then the outer iterates defined by (33) to (38) converge to a solution of the boundary value problem (17) (19) (20) (21) for sufficiently small $|\epsilon|$, and hence give a nontrivial solution to problem (1) (2).

IV. Convergence Proofs

Define the following bounds.

$$\Lambda \equiv |\omega| + 1 \tag{48}$$

$$\Omega \equiv \|G\| \left[\tfrac{1}{2} |\lambda_o| \, \|f_{uu}\|_S \, (\Phi + 1) + \Lambda \|f_{u\lambda}\|_S + \Lambda \|f_u\|_S \right] (\Phi + 1) \tag{49}$$

$$\Delta_1 \equiv \|G\| \left[|\lambda_o| \, \|f_{uu}\|_S \, (\Phi + 1) + \Lambda \|f_{u\lambda}\|_S + \Lambda \|f_u\|_S \right] \tag{50}$$

$$\Delta_2 \equiv \|G\| \left[|\lambda_o| \, \|f_{uu}\|_S \, (\Phi + 1) + \Lambda \|f_{u\lambda}\|_S + \|f_u\|_S \right] \tag{51}$$

$$\Delta_3 \equiv \|G\| \left[2 \|f_{u\lambda}\|_S + \|f_u\| \right] \tag{52}$$

$$\epsilon_1 \equiv \min \left\{ 1, \frac{1}{\Lambda}, \frac{1}{\Omega} \right\}. \tag{53}$$

Define the Banach space $\{\mathcal{B}, \| \ \|\}$ by

$$\mathcal{B} \equiv \left\{ v(\xi) \in C_n[\alpha, \beta] \;\middle|\; \int_\alpha^\beta y^{(i)*}(\xi) \, v(\xi) \, d\xi = 0, \ i = 1, \ldots, p \right\}, \tag{54}$$

and let $\| \ \|$ be the maximum norm. Let

$$\mathcal{S} \equiv \{ v(\xi) \in \mathcal{B} \mid \|v\| \leq \Omega \}. \tag{55}$$

First we show that the inner iteration scheme defined by (45) (46) converges for any $v \in \mathscr{S}$ and sufficiently small ϵ. It is clear from (32) that $\bar{g}_{\bar{x}\bar{x}}$ exists and is a (p+1) x (p+1) x (p+1) array of real constants. Define the norms $\|\bar{g}_{\bar{x}\bar{x}}\|$ and $\|k\|$ according to the conventions in Appendix A. Then let

$$\rho \equiv \min \left\{ 1, \ \frac{1}{3\|k\| \ \|\bar{g}_{\bar{x}\bar{x}}\|} \right\} . \tag{56}$$

Define the ρ - neighborhood of $\bar{x}$ by

$$\mathcal{N}_{\rho}(\bar{x}) = \left\{ \bar{r} \in \mathcal{R}^{p+1} \ \middle| \ \|\bar{r} - \bar{x}\| \leq \rho \right\} . \tag{57}$$

Just as in Chapter III, the assumed differentiability and Lipschitz continuity of f implies that $\bar{h}(\epsilon, \bar{q}, v)$ is bounded for $|\epsilon| \leq \epsilon_1$, $\bar{q} \in \mathcal{N}_{\rho}(\bar{x})$ and $v \in \mathscr{S}$, and is Lipschitz continuous in $\bar{q}$ and v there. Therefore there exist positive constants Δ_4 and Δ_5 such that

$$\|\bar{h}(\epsilon, \bar{q}, v) - \bar{h}(\epsilon, \bar{r}, w)\| \leq \Delta_4 \|\bar{q} - \bar{r}\| + \Delta_5 \|v - w\| \tag{58}$$

for $|\epsilon| \leq \epsilon_1$, $\bar{q}, \bar{r}, \in \mathcal{N}_{\rho}(\bar{x})$, and $v, w \in \mathscr{S}$.

Define

$$\mathcal{M}_{\rho} \equiv \{ \bar{a} \in \mathcal{R}^{p+1} \ | \ \bar{x} + \bar{a} \in \mathcal{N}_{\rho}(\bar{x}) \} . \tag{59}$$

Now for $\bar{a} \in \mathcal{M}_{\rho}$ and a fixed $v \in \mathscr{S}$, define the mapping U which generates the inner iteration scheme (46) by

$$U(\bar{a}) \quad \bar{a} - K[\bar{g}(\bar{x}+\bar{a}) + \epsilon h(\epsilon, \bar{x}+\bar{a}, v)]. \tag{60}$$

Clearly

$$\begin{aligned} \|U(0)\| &= \|0 - K[0 + \epsilon\bar{h}(\epsilon, \bar{x}, v)]\| \\ &\leq |\epsilon| \, \|K\| \, \|\bar{h}\| \end{aligned} \tag{61}$$

If $\bar{a}$ and $\bar{b}$ are any two points in $\mathcal{M}_\rho$, then

$$\begin{aligned} \|U(\bar{a}) - U(\bar{b})\| &\leq \| \bar{a} - \bar{b} - K[\bar{g}(\bar{x}+\bar{a}) - \bar{g}(\bar{x}+\bar{b}) \\ &\qquad + \epsilon\bar{h}(\epsilon, \bar{x}+\bar{a}, v) - \epsilon h(\epsilon, \bar{x}+\bar{b}, v)]\| \\ &\leq \|I - K\int_0^1 \bar{g}_{\bar{x}}(\bar{x} + \sigma\bar{a} + (1-\sigma)\bar{b})d\sigma\| \, \|\bar{a} - \bar{b}\| \\ &\qquad + |\epsilon| \, \|K\| \, \|\bar{h}(\epsilon, \bar{x}+\bar{a}, v) - \bar{h}(\epsilon, \bar{x}+\bar{b}, v)\| \\ &\leq \|K\| \, \|\bar{g}_{\bar{x}\bar{x}}\| \, \rho \, \|a - b\| + |\epsilon| \, \|K\| \, \Delta_4 \, \|\bar{a} - \bar{b}\| \\ &= \|K\| \, [\|\bar{g}_{\bar{x}\bar{x}}\| \, \rho + |\epsilon| \, \Delta_4] \, \|\bar{a} - \bar{b}\| \end{aligned} \tag{62}$$

With ρ defined by (56), define

$$\epsilon_2 \equiv \min \left\{ \epsilon_1, \quad \frac{1}{3\|K\|\Delta_4} \quad , \quad \frac{\rho}{3\|K\| \, \|h\|} \right\} \tag{63}$$

and it follows from (61) and (62) that for $|\epsilon| \leq \epsilon_2$,

$$\|U(0)\| \leq \tfrac{1}{3}\rho, \tag{64}$$

and

$$\|U(\bar{a}) - U(\bar{b})\| \leq \text{[illegible]} \; \|\bar{a} - \bar{b}\| .$$

Hence the contracting mapping theorem of Appendix B applies to U, and U has a unique fixed point $\bar{a}^{\#}$ in $\mathfrak{M}_\rho$. But a fixed point of U clearly is equivalent to a solution

$$\bar{q}^{\#} = \bar{x} + \bar{a}^{\#} \tag{65}$$

of the transcendental bifurcation equations. Thus we have proven:

<u>Theorem 1:</u>

If the algebraic bifurcation equations have a simple root $\bar{x}$, and if $|\epsilon| \leq \epsilon_2$, then the transcendental bifurcation equations with a given $v \in \mathcal{S}$ have a unique root (65) in $\mathcal{N}_\rho(\bar{x})$, where $\bar{a}^{\#}$ is the limit of the inner iteration scheme (44) (45).

Now we see how this root $\bar{q}^{\#}$ depends on the choice of $v \in \mathcal{S}$. Let $\bar{q} = \bar{x} + \bar{a}$ and $\bar{r} = \bar{x} + \bar{b}$ be solutions in $\mathcal{N}_\rho(\bar{x})$ of the transcendental bifurcation equations, corresponding to v and w respectively in $\mathcal{S}$. That is,

$$\bar{g}(\bar{q}) + \epsilon \bar{h}(\epsilon, \bar{q}, v) = 0 \tag{66}$$

$$\bar{g}(\bar{r}) + \epsilon \bar{h}(\epsilon, \bar{r}, w) = 0 . \tag{67}$$

Subtract (67) from (66) and manipulate as in (62) to get

$$[I + K \int_0^1 \int_0^1 \bar{g}_{\bar{x}\bar{x}} (\bar{x} + \sigma \zeta \bar{a} + \sigma(1-\zeta)\bar{b}) (\zeta \bar{a} + (1-\zeta)\bar{b}) \, d\sigma d\zeta] (\bar{q} - \bar{r})$$

$$+ \epsilon K [\bar{h}(\epsilon, \bar{q}, v) - \bar{h}(\epsilon, \bar{r}, w)] = 0 \tag{68}$$

From (56) it follows that the matrix coefficient of $(\bar{q} - \bar{r})$ in (68) is invertible. Define

$$B \equiv [I + K \int_0^1 \int_0^1 \bar{g}_{\bar{x}\bar{x}} (\bar{x} + \sigma \zeta \bar{a} + \sigma(1-\zeta)\bar{b}) (\zeta \bar{a} + (1-\zeta)\bar{b}) \, d\sigma d\zeta]^{-1}. \tag{69}$$

Therefore from (68)

$$\begin{aligned} \|\bar{q} - \bar{r}\| &\leq |\epsilon| \, \|B\| \, \|K\| \, \|\bar{h}(\epsilon, \bar{q}, v) - \bar{h}(\epsilon, \bar{r}, w)\| \\ &\leq |\epsilon| \, \|B\| \, \|K\| \, [\Delta_4 \|\bar{q} - \bar{r}\| + \Delta_5 \|v - w\|]. \end{aligned} \tag{70}$$

Define

$$\epsilon_3 \equiv \min \left\{ \epsilon_2 , \quad \frac{1}{2\|B\| \, \|K\| \, \Delta_4} \right\} \tag{71}$$

and the following lemma is obvious.

Lemma:

If $|\epsilon| \leq \epsilon_3$, then the roots of the transcendental bifurcation equations considered as functions of $v \in \mathcal{B}$, satisfy

$$\|\bar{q} - \bar{r}\| \leq |\epsilon| \, 2 \, \|B\| \, \|K\| \, \Delta_5 \, \|v - w\| \tag{72}$$

where $\bar{q}$ and $\bar{r}$ are the roots in $\mathcal{N}_\rho(\bar{x})$ corresponding to v and w respectively in $\mathcal{B}$.

Finally we turn our attention to the outer iteration scheme. Define the mapping

$$T : \mathcal{B} \to \mathcal{B} \tag{73}$$

by

$$T\,v \equiv \int_\alpha^\beta G^\dagger(\xi,\tau)\Big[\frac{\lambda_o}{\epsilon^2}\,[f(\tau,\lambda_o+\epsilon\eta,\ \epsilon q_j y^{(j)}+\epsilon^2 v) - J(\tau,\lambda_o)(\epsilon q_j y^{(j)}+\epsilon^2 v)]$$

$$+\frac{1}{\epsilon}\,\eta f(\tau,\lambda_o+\epsilon\eta,\ \epsilon q_j y^{(j)}+\epsilon^2 v)\Big]\,d\tau, \tag{74}$$

where $(q,\eta) \equiv \bar{q}$ is the corresponding root of the transcendental bifurcation equations given for each $v \in \mathcal{B}$ by Theorem 1. That is, q and η satisfy, for the given v,

$$\frac{\lambda_o}{\epsilon^2}\int_\alpha^\beta z^{(i)*}(\xi)\Big[f(\xi,\lambda_o+\epsilon\eta,\ \epsilon q_j y^{(j)}+\epsilon^2 v) - J(\xi,\lambda_o)(\epsilon q_j\ y^{(j)}+\epsilon^2 v)\Big]\,d\xi$$

$$+\frac{1}{\epsilon}\,\eta\int_\alpha^\beta z^{(i)*}(\xi)\ f(\xi,\lambda_o+\epsilon\eta,\ \epsilon q_j y^{(j)}+\epsilon^2 v)\,d\xi = 0 \tag{75}$$

$$\sum_{j=1}^{p} q_j^2 = 1.$$

Then, using definitions (48) to (53), it is easy to show that T satisfies

$$\|T\,v\| \leq \Omega \tag{76}$$

$$\|T\,v - T\,w\| \leq |\epsilon|\,\Delta_1\,\|v-w\| + \Delta_2\|q-r\| + \Delta_3|\eta-\mu| \tag{77}$$

where v and w are in $\mathscr{B}$ and (q,η) (r,η) are the corresponding roots of (75), assuming $|\epsilon| \leq \epsilon_3$. Combining (77) with (72) gives

$$\|Tv - Tw\| \leq |\epsilon| \, [\Delta_1 + 2 \|B\| \, \|K\| \, \Delta_5(\Delta_2 + \Delta_3)] \, \|v - w\|. \tag{78}$$

Hence $T : \mathscr{B} \to \mathscr{B}$ and is contracting on $\mathscr{B}$ for

$$|\epsilon| \leq \epsilon_o \equiv \min\left\{\epsilon_3, \, [\Delta_1 + 2\|B\| \, \|K\| \Delta_5 (\Delta_2 + \Delta_3)]^{-1}\right\}. \tag{79}$$

An application of the contracting mapping theorem now yields

Theorem 2.

Corresponding to each simple root of the algebraic bifurcation equations (32), there is a nontrivial solution branch of the form (15) (16) for $|\epsilon| \leq \epsilon_o$, satisfying the nonlinear boundary-value problem (1) (2) near $\lambda = \lambda_o$. This solution branch is the limit of the sequences defined by the iteration schemes of §IV. 3.

Continuity of this solution branch in ϵ follows just as in Theorem 2 of Chapter III.

Distinct roots of the algebraic bifurcation equations lead to distinct solution branches, at least in a small neighborhood of λ_o, since we may choose ρ as small as we please and thus make $\eta_\rho(\bar{x}^{(1)})$ and $\eta_\rho(\bar{x}^{(2)})$ disjoint sets where $x^{(1)}$ and $x^{(2)}$ are distinct simple roots of the algebraic bifurcation equations.

IV. 5. Roots of the Algebraic Bifurcation Equations

The problem of solving the algebraic bifurcation equations is not a trivial one, although it is much easier than solving the original nonlinear boundary value problem. Without going into the computational aspects of the problem, we present sets of sufficient conditions which guarantee existence of a root. If these conditions are not met, of course it is still possible for the algebraic bifurcation equations to have a root. Recall that the equations are

$$\lambda_o \sum_{j,k=1}^{p} F_{ijk} x_j x_k + \omega \sum_{j=1}^{p} (C_{ij} + \lambda_o D_{ij}) x_j = 0 \qquad i = 1, \ldots, p, \tag{80}$$

$$\sum_{j=1}^{p} x_j^2 = 1. \tag{81}$$

The arrays F_{ijk} C_{ij} and D_{ij} are defined by (29) (30) and (31) in § IV. 3. Recall that the matrix C is nonsingular, by (10). The matrix D is identically zero if $f(\xi, \lambda, u)$ is independent of λ; we make the assumption that f varies slowly with λ, specifically that

$$\|f_{u\lambda}\| < \frac{1}{|\lambda_o| \|C^{-1}\| \left(\max_{1 \le i \le p} \|z^{(i)}\|_1\right) \left(\sum_{j=1}^{p} \|y^{(j)}\|\right)} \tag{82}$$

This condition allows us to rewrite (80) as

$$T(x) = \omega x, \tag{83}$$

where T is the homogeneous quadratic operator defined by

$$T(x) = -\lambda_o [I + \lambda_o C^{-1} D]^{-1} C^{-1} F\, xx\,. \tag{84}$$

The problem (81) (83) looks remarkably like an eigenvalue problem. Problems of this type have been studied by Birkhoff and Kellogg [7] and Berger and Berger [6] under the name of invariant direction problems, and we adopt this terminology. The problem is now to find a unit vector x whose direction remains unchanged under the mapping T. The scalar ω just gives the length of the image vector T(x) (within a sign). It is traditional not to include a solution with $\omega = 0$ as an invariant direction, since then clearly T(x) has no direction. Therefore we first dispose of this case.

Case (i)

If (82) holds and there exists a unit vector $x^{\#}$ such that $T(x^{\#}) = 0$, then the algebraic bifurcation equations (80) (81) have the solution $x = x^{\#}$, $\omega = 0$.

If p is even, then (83) need not have a solution, for example take $p = 2$ and T(x) a pure rotation. However, for odd p we have the important Birkhoff-Kellogg invariant direction theorem:

Case (ii)

If (82) holds, p is odd, and $T(x) \neq 0$ for all x on the unit sphere, then there exists a unit vector $x^{\#}$ such that

$$T(x^{\#}) = \omega x^{\#} \tag{85}$$

with $\omega \neq 0$, and these $x^{\#}$ and ω solve (80) (81).

Another interesting case arises when T(x) is a gradient system, that is when there exists a scalar "potential function" $\varphi(x)$ such that

$$T_i(x) = \frac{\partial}{\partial x_i} \varphi(x) \qquad i, 1, \ldots, p. \tag{86}$$

Then we have the following result of Berger [6] (page 63).

Case (iii)

If (82) holds, $T(x) \neq 0$ for all unit vectors x, and T(x) is a gradient system, then T(x) has at least two invariant directions and (80) (81) correspondingly has two solutions.

The proof of this is simply that $\varphi(x)$ being a continuous function on a closed bounded set must have a maximum and a minimum there.

There is an important type of bifurcation problem which always gives rise to a mapping T(x) which is a gradient system. This is the case of a scalar self-adjoint boundary value problem, for example the elliptic problems studied in [13], [15], [22] and [45]. We assume that f is independent of λ so that $D \equiv 0$. Then the matrix C can be made equal to the identity and the array F becomes

$$F_{ijk} \equiv \tfrac{1}{2} \int f_{uu}(\xi, 0)\, \varphi^{(i)}(\xi)\, \varphi^{(j)}(\xi)\, \varphi^{(k)}(\xi)\, d\xi \tag{87}$$

where $\varphi^{(i)}$ $i = 1, \ldots, p$ are the orthonormalized scalar eigenfunctions of the linearized problem. Then clearly F_{ijk} is symmetric in its indices ijk, i.e., is unchanged by any permutation of them. Whenever this is the case, we have

$$T(x) = -\tfrac{1}{3}\lambda_o \operatorname{grad}\left(\sum_{ijk=1}^{p} F_{ijk}\, x_i\, x_j\, x_k\right) \tag{88}$$

so $T(x)$ is a gradient system.

Another invariant direction theorem, which we will apply in the next section, is also due to Berger [6] (page 85).

Case (iv)

If T is a continuously differentiable gradient system defined and nonzero on the unit sphere, and

$$T(-x) = -T(x) \tag{89}$$

then T has at least $2p$ distinct invariant directions.

Clearly (89) is never satisfied by $T(x)$ defined by (84). However, the higher degree algebraic bifurcation equations of the next section can give rise to such mappings.

IV. 6. Higher Degree Nonlinearities

If the array F_{ijk} defined by (31) is identically zero, then none of the theory of §IV.3 and §IV.4 is applicable. This case is analogous to the situation discussed in §III.6 when $f_{uu}(\xi, \lambda_o, 0) \equiv 0$ at a simple eigenvalue. There we were able to calculate the next higher degree term in the expansion of $\lambda(\epsilon)$

assuming only the existence of higher derivatives of f. We can proceed along the same lines in the degenerate case and obtain a higher degree algebraic bifurcation equation.

Without going into any details or proofs, we state that if

$$f_{u^k}(\xi, \lambda_o, 0) = 0, \qquad k = 2, \ldots, m-1 \tag{90}$$

and

$$f_{u^m}(\xi, \lambda_o, 0) \neq 0 \tag{91}$$

then the relevant algebraic bifurcation equation is

$$\lambda_o F_{ij_1 \cdots j_m} x_{j_1} \cdots x_{j_m} + \omega (C_{ij} + \lambda_o D_{ij}) x_j = 0 \tag{92}$$

(summation convention understood) where C_{ij} and D_{ij} are as before, and

$$F_{ij_1 \cdots j_m} \equiv \frac{1}{m!} \int_\alpha^\beta z^{(i)*}(\xi) f_{u^m}(\xi, \lambda_o, 0) y^{(j_1)}(\xi) \ldots y^{(j_m)}(\xi) d\xi. \tag{93}$$

In particular if $m = 3$ and the homogeneous cubic mapping T defined analogously to (84) is a gradient system nonzero on the unit sphere, then Case (iv) of the previous section applies and (92) has 2p distinct solutions.

CHAPTER V

GENERALIZATIONS AND EXTENSIONS

V. 1. Introduction

The bifurcation theory of Chapters III and IV can be generalized in a number of ways. In the following section we present several rather trivial generalizations of the theory. In § V. 3 we show how the techniques of this thesis can be applied to systems of nonlinear partial differential equations. The important problem of the extension of a solution branch out of the neighborhood of its bifurcation point is dealt with by the continuation theory in § V. 4. Finally in § V. 5 we compare the approximate solutions from our iteration scheme with the asymptotic solutions obtained from formal perturbation theory, and show that the former contain the latter.

V. 2. Generalizations

The generalization to include complex coefficients is straightforward. It involves only redefining the norms, inner products, and adjoints in the obvious way. For example, given a matrix A with complex components, A^* would represent the complex conjugate transpose instead of just the transpose.

As indicated after Theorem 7 of Chapter II, the eigenvalues of the linearized problem may be complex even when all the coefficients are real. Bifurcation at these complex eigenvalues may be studied with no additional difficulty, except of course that the two-dimensional graphs of Figure 1 are no longer valid.

The assumption that all the coefficients appearing in the nonlinear boundary-value problem be continuous functions of $\xi \in [\alpha, \beta]$ can be weakened considerably. The existence and all the properties of the Green's matrix and principal generalized Green's matrix presented in Chapter II have been shown to hold for the case of Lebesgue integrable coefficients by W. M. Whyburn [51] and W. T. Reid [40]. We need only relax our definition of a "solution" of a boundary value problem to mean an absolutely continuous function (see page 90 in [43]) which satisfies the differential equations "almost everywhere." Our bifurcation theory then remains valid if equations are understood to hold "almost everywhere" where necessary, and our norms are replaced by the $\mathcal{L}_\infty$ and $\mathcal{L}_1$ norms, where appropriate. This generalization to include Lebesgue integrable coefficients contains several subcases of practical importance, such as piecewise continuous coefficients and certain mild singularities.

Problems in which the matrices $A(\xi)$ and $J(\xi, \lambda)$ are analytic in $\xi \in [\alpha, \beta]$ except for simple poles at α or β or both, lead us to the theory of regular singular end points as discussed in [12] and [17]. Then a fundamental solution matrix exists for the linearized differential equations, and is analytic for $\alpha < \xi < \beta$, but it either has poles or is non-invertible at α and β. Thus the choice of boundary conditions is severely restricted. However, for suitable boundary conditions it is often possible to construct a Green's matrix and a generalized Green's matrix for the linearized problem, and to apply the bifurcation theory of Chapters III and IV. The theory of generalized Green's matrices for systems with singular end points is not well developed, but in [11] and Chapter 10 of [12], a Green's function has been constructed for scalar problems consisting of a singular n^{th} order self-adjoint differential equation and suitable boundary conditions. An example of a problem with regular singular end-points, for which both the Green's matrix and the generalized Green's matrix exist, and to which the bifurcation theory can be applied, is given in Chapter VI.

V. 3. Systems of Elliptic Partial Differential Equations

Often the techniques used to study ordinary differential equations cannot be extended to partial differential equations. Our iteration method does not have this limitation. We now discuss a special class of systems of partial differential equations to which the method of Chapter IV is particularly applicable. Our approach is very similar to that in [21]. For another approach, see [5].

Let $\mathcal{D}$ be a closed bounded domain in $\mathbb{R}^m$ with smooth boundary. Let $\varphi(x) \in C^2[\mathcal{D}]$. Define the uniformly elliptic self-adjoint second order partial differential operator L by

$$L\,\varphi(x) \equiv -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\,\frac{\partial\varphi(x)}{\partial x_j}\right] + a_o(x)\,\varphi(x) \tag{1}$$

where

$$\sum_{i,j=1}^{m} a_{ij}(x)\,q_i q_j \geq \alpha \sum_{i=1}^{m} q_i^2 \quad , \qquad \alpha > 0 \quad , \tag{2}$$

for all $x \in \mathcal{D}$ and $q \in \mathbb{R}^m$,

$$a_{ij}(x) = a_{ji}(x) \in C^1[\mathcal{D}] \tag{3}$$

and $$a_o(x) \geq 0 \; , \; a_o(x) \in C\,[\mathcal{D}] \quad . \tag{4}$$

Let the function $g(x, \lambda, \varphi)$, together with its derivatives $g_\varphi, g_\lambda, g_{\varphi\varphi}, g_{\varphi\lambda}$ be defined and continuous on some set S_o defined by

$$S_o \equiv \{(x, \lambda, \varphi) \mid x \in \mathcal{A},\ \lambda \in \mathcal{J},\ \varphi \in C[\mathcal{A}],\ \|\varphi\| \leq \Phi\} \ , \tag{5}$$

and assume

$$g(x, \lambda, 0) = 0 \ , \qquad g_\varphi(x, \lambda, 0) > 0 \ , \tag{6}$$

for $x \in \mathcal{A}$ and $\lambda \in \mathcal{J}$.

Then we consider the problem

$$L\, \varphi(x) = \lambda\, g(x, \lambda, \varphi) \qquad x \in \mathcal{A} \tag{7}$$

$$\varphi(x) = 0 \ . \qquad x \in \partial \mathcal{A} \ . \tag{8}$$

This problem is one of a class studied in [21] and shown there to give rise to a nontrivial bifurcation branch at each simple eigenvalue of the linearized problem

$$L\, \psi(x) = \lambda\, g_\varphi(x, \lambda, 0)\, \psi(x) \qquad x \in \mathcal{A} \tag{9}$$

$$\psi(x) = 0 \qquad x \in \partial \mathcal{A} \ . \tag{10}$$

Now consider the following generalization of problem (7)(8). Let T be the $n \times n$ diagonal matrix with all diagonal elements equal to L. Let $u(x) \in C_n^2[\mathcal{A}]$, and let $f(x, \lambda, u)$ be an n-dimensional vector function satisfying the same hypotheses as $f(\xi, \lambda, u)$ in Chapter IV with $[\alpha, \beta]$ replaced by $\mathcal{A}$. Then we have the vector problem

$$T\,u(x) = \lambda\, f(x, \lambda, u) \qquad x \in \mathcal{a} \qquad (11)$$

$$u(x) = 0 \qquad x \in \partial \mathcal{a} \qquad (12)$$

Now assume that the n×n matrix $f_u(x, \lambda, 0)$ is independent of x, and write it as $J(\lambda)$. Then the linearization of (11)(12) is

$$T\,y(x) = \lambda\, J(\lambda)\, \lambda(x) \qquad x \in \mathcal{a} \qquad (13)$$

$$y(x) = 0 \qquad x \in \partial \mathcal{a} \qquad (14)$$

Assume further that $J(\lambda)$ is diagonalizable, so that there exists a non-singular matrix S, in general depending on λ, such that

$$S^{-1} J(\lambda) S = \begin{pmatrix} \mu_1(\lambda) & & & O \\ & \mu_2(\lambda) & & \\ & & \ddots & \\ O & & & \mu_n(\lambda) \end{pmatrix} . \qquad (15)$$

Then (13)(14) can be separated into n scalar problems similar to (9)(10), i.e.

$$[L - \lambda\, \mu_i(\lambda)]\psi_i(x) = 0 \qquad x \in \mathcal{a} \qquad (16)$$

$$\psi_i(x) = 0 \qquad x \in \partial \mathcal{a} \qquad (17)$$

$$i = 1, 2, \ldots, n ,$$

where $w(x) \equiv \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} = S^{-1}\, y(x)$. (18)

Therefore the linearized problem (13)(14) has a nontrivial solution if and only if $\lambda\, \mu_i(\lambda)$ is an eigenvalue of L for some $\mu_i(\lambda)$ $i = 1, \ldots, n$ which is an eigenvalue of the matrix $J(\lambda)$.

Degeneracy may arise in two ways. Either a given $\lambda\,\mu_i(\lambda)$ may be a degenerate eigenvalue of L with linearly independent normalized eigenfunctions

$$\psi_i^{(1)}(x),\ \psi_i^{(2)}(x),\ldots,\ \psi_i^{(k_i)}(x)\ , \tag{19}$$

or more than one of the $\lambda\,\mu_i(\lambda)$ may be eigenvalues of L for the same value of λ but different values of i, say $i_1, i_2, \ldots, i_\ell$.

Now assume that $\lambda = \lambda_0$ is such that the total number of such eigenfunction solutions to the problems (16)(17) is $p > 0$. Define the linearly independent n-vector functions $w_i^{(j)}(x)$, $i = i_1, i_2, \ldots, i_\ell$, $j = 1, \ldots, k_i$, by

$$w_i^{(j)}(x) \quad = \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi_i^{(j)}(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{20}$$

where $\psi_i^{(j)}(x)$ occupies the i^{th} position in the n-vector indicated. Then a set of linearly independent solutions of problem (13)(14) is

$$\begin{aligned} y^{(1)}(x) &= S(\lambda_0)\, w_{i_1}^{(1)}(x) \\ y^{(2)}(x) &= S(\lambda_0)\, w_{i_1}^{(2)}(x) \\ &\vdots \\ y^{(k_{i_1})}(x) &= S(\lambda_0)\, w_{i_1}^{(k_{i_1})}(x) \\ &\vdots \\ y^{(p)}(x) &= S(\lambda_0)\, w_{i_\ell}^{(k_{i_\ell})}(x) \end{aligned} \tag{21}$$

We call λ_0 an eigenvalue of degeneracy p of the linearized problem

(13) (14).

The basic alternative theorem for the inhomogeneous form of (13)(14) is the same as for ordinary differential equations. The adjoint problem to (13)(14) is

$$[T - \lambda J(\lambda)^{*}] z(x) = 0 \qquad x \in \mathcal{A} \tag{22}$$

$$z(x) = 0 \qquad x \in \partial\mathcal{A} \tag{23}$$

Clearly for $\lambda = \lambda_o$, problem (22)(23) also has p linearly independent solutions

$$z^{(i)}(x) \qquad i = 1, \ldots, p . \tag{24}$$

Now we assume that the domain $\mathcal{A}$ is such that we can construct Green's function and generalized Green's function for L satisfying the boundary condition (8). We do this only to preserve the analogy with Chapter IV; it is not really necessary to construct these Green's functions. All that is necessary is that L have a bounded inverse, which is true quite generally. See [21] and [31].

Let $G^{(i)}(x, t)$, $i = 1, \ldots, n$ be the Green's functions and generalized Green's functions determined by the problems

$$[L-\lambda_o \mu_i(\lambda_o)] G^{(i)}(x, t) = \delta(x-t) - \sum_{j=1}^{k_i} \psi_i^{(j)}(x) \psi_i^{(j)}(t), x, t \in \mathcal{A}, \tag{25}$$

$$G^{(i)}(x, t) = 0, \qquad x \in \partial\mathcal{A}, \tag{26}$$

$$i = 1, \ldots, n.$$

Here $\psi_i^{(j)}(x) \equiv 0$ if $\lambda_o \mu_i(\lambda_o)$ is not an eigenvalue of L. Then a generalized Green's matrix for (13)(14) is given by

$$G^{\dagger}(x,t) = S \begin{pmatrix} G^{(1)}(x,t) & & & O \\ & G^{(2)}(x,t) & & \\ & & \ddots & \\ O & & & G^{(n)}(x,t) \end{pmatrix} S^{-1} . \tag{27}$$

Return now to consider the nonlinear problem (11)(12). It can be rewritten as

$$[T-\lambda_0 J(\lambda_0)]\, u(x) = \lambda f(x,\lambda,u) - \lambda_0 J(\lambda_0) u(x) \qquad x \in \mathcal{a} \tag{28}$$

$$u(x) = 0 \qquad x \in \partial \mathcal{a} . \tag{29}$$

Try a solution of the form considered in Chapter IV, namely

$$u(x,\epsilon) = \epsilon \sum_{j=1}^{p} q_j(\epsilon) y^{(j)}(x) + \epsilon^2 v(x,\epsilon) \tag{30}$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon\, \eta(\epsilon) , \tag{31}$$

where

$$\sum_{j=1}^{p} q_j^2 = 1 . \tag{32}$$

Now we can proceed as in Chapter IV: Set up an iteration scheme and prove its convergence, for sufficiently small $|\epsilon|$, using $G^{\dagger}(x,t)$ and the contracting mapping theorem, to a solution (q, η, v) with q and η in a neighborhood of an isolated root of the appropriate algebraic bifurcation equations.

The algebraic bifurcation equations for this problem are

$$\omega \sum_{j=1}^{p} [C_{ij} + \lambda_0 D_{ij}] r_{ij} + \lambda_0 \sum_{j,k=1}^{p} F_{ijk}\, r_j r_k = 0 , \quad i=1,\ldots,p, \tag{33}$$

$$\sum_{j=1}^{p} r_j^2 = 1 \tag{34}$$

where C_{ij}, D_{ij}, and F_{ijk} are defined by

$$C_{ij} \equiv \int_{\mathcal{a}} z^{(i)*}(x)\, J(\lambda_o) y^{(j)}(x)dx \tag{35}$$

$$D_{ij} \equiv \int_{\mathcal{a}} z^{(i)*}(x)\, f_{u\lambda}(x, \lambda_o, 0) y^{(j)}(x)dx \tag{36}$$

$$F_{ijk} \equiv \int_{\mathcal{a}} z^{(i)*}(x)\, f_{uu}(x, \lambda_o, 0) y^{(j)}(x)\, y^{(k)}(x)dx \tag{37}$$

$$i, j, k = 1, \ldots, p \ .$$

Thus the class of partial differential equation problems defined here can be treated by the same method as the ordinary differential equation problems of Chapter IV.

V. 4. Continuation of Solution Branches and Secondary Bifurcation

In Chapters III and IV we constructed a nontrivial solution branch (or branches) in a small neighborhood of a bifurcation point. We now show how such a branch can be extended out of this small neighborhood.

Any process which extends the domain of a function beyond its original domain of definition, while preserving certain characterizing properties, is called a continuation. In our case, the continued function must be a solution of the nonlinear boundary-value problem, be continuous in λ, and coincide with the solution of Chapter III or IV in its neighborhood of definition. The question of continuation of solutions of bifurcation problems has been studied by many authors, including Hildebrandt and Graves [18], Simpson and Cohen [45], Pimbley [37], and H. B. Keller [22].

It may happen that a nontrivial solution branch itself splits into two or more new branches at some value of $\lambda \neq \lambda_0$. We refer to this as secondary bifurcation. Clearly the bifurcation theory of Chapters III and IV applies equally well to secondary bifurcation if we linearize the nonlinear boundary-value problem about this nontrivial solution instead of about the trivial solution. Each new branch can often be continued by the methods of this section. Thus, by repeated applications of the bifurcation and continuation theorems, we can in many cases obtain a global solution consisting of many branches, all of which are ultimately connected to the trivial solution. Of course, this process fails to yield any solution branches which are not connected to the trivial solution.

The nonlinear boundary-value problem is the same as that considered in Chapters III and IV, namely

$$u'(\xi) = A(\xi)\, u(\xi) + \lambda\, f(\xi, \lambda, u(\xi)) \;, \qquad \alpha \le \xi \le \beta \;, \tag{41}$$

$$M\, u(\alpha) + N\, u(\beta) = 0 \;. \tag{42}$$

All the hypotheses of Chapters III or IV are assumed true here, and we will later extend the domain of λ and u values on which $f(\xi, \lambda, u)$ is assumed to be defined.

We assume that a branch of nontrivial solutions to (41)(42) has been found by the methods of Chapter III or IV in a neighborhood of a bifurcation point. This solution branch is represented parametrically by

$$u(\xi, \epsilon) = \epsilon \sum_{j=1}^{p} q_j(\epsilon)\, y^{(j)}(\xi) + \epsilon^2 v(\xi, \epsilon) \tag{43}$$

$$\lambda(\epsilon) \;=\; \lambda_0 + \epsilon\, \eta(\epsilon) \quad . \tag{44}$$

Here we assume $|\epsilon| \le \epsilon_0$ where ϵ_0 is as defined in Chapter III or IV respectively, and

$$\sum_{j=1}^{p} q_j^2 \;=\; 1 \quad , \tag{45}$$

where p is the multiplicity of the eigenvalue λ_0 of the linearized problem (49)(50) below. If p = 1, then define $q_1(\epsilon) \equiv 1$ and $y^{(1)}(\xi) \equiv y_0(\xi)$ and we have the solution branch of Chapter III. Otherwise, (43)(44) represents a solution branch as constructed in Chapter IV.

We further assume that the parameter ϵ can be eliminated from the equations (43)(44) of the nontrivial branch, that is that we can solve (44) for ϵ to obtain the single valued function

$$\epsilon \;=\; \epsilon(\lambda) \tag{46}$$

for each λ in some open interval, say $\mathcal{J}_0$, where

$$\mathcal{J}_0 \;\subset \{\lambda \,|\, \lambda = \lambda(\epsilon) \text{ and } |\epsilon| \le \epsilon_0 \} \quad . \tag{47}$$

Then we substitute (46) into (43) to obtain a single-valued function defined on $[\alpha, \beta] \times \mathcal{J}_0$, which we write as $u(\xi, \epsilon(\lambda))$, or simply as

$$u \;=\; u(\xi, \lambda) \quad . \tag{48}$$

This will be true in a neighborhood $\mathcal{J}_0$ of any ϵ for which $\frac{d\lambda(\epsilon)}{d\epsilon}$ exists and is nonzero. In particular, if $\eta(0) \neq 0$, then (48) is valid in a neighborhood of λ_o.

The linear problem obtained by linearizing (41)(42) about the trivial solution is

$$y'(\xi) - [A(\xi) + \lambda_o J(\xi, \lambda_o)]y(\xi) = 0 \quad (49)$$

$$My(\alpha) + Ny(\beta) = 0 \quad . \quad (50)$$

The linear problem obtained by linearizing (41)(42) about the nontrivial solution plays a more important role in the continuation theory. Using (48), this problem can be written

$$w'(\xi) - [A(\xi) + \lambda f_u(\xi, \lambda, u(\xi, \lambda))] w(\xi) = 0 \quad (51)$$

$$M w(\alpha) + N w(\beta) = 0 \quad . \quad (52)$$

Any value of λ for which this problem has a nontrivial solution $w(\xi)$ will be called an eigenvalue of (51)(52).

The fundamental continuation theorem which we shall prove (Theorem 3) is valid only at those values of λ which are not eigenvalues of problem (51)(52). In this regard we have the following theorem. (The set S of this theorem is defined by (15) of Chapter III or by (23) of Chapter IV depending on whether $p = 1$ or $p > 1$ respectively.)

Theorem 1

If $f(\xi, \lambda, u)$ is analytic in λ and u for each ξ with $(\xi, \lambda, u) \in S$, if (46) is valid for $\lambda \in \mathcal{J}_o$, and if $u(\xi, \epsilon(\lambda))$ is a solution of (41)(42) for $\lambda \in \mathcal{J}_o$, then the linearized problem (51)(52) satisfies the following dichotomy: either a nontrivial solution of (51)(52) exists for all $\lambda \in \mathcal{J}_o$, or at most a finite set of isolated eigenvalues exists in $\mathcal{J}_0$.

Proof:

From the contracting mapping theorem, the sequences $\{v^{(\ell)}, \eta^{(\ell)}\}_{\ell=0}^{\infty}$ of Chapter III and $\{v^{(\ell)}, q^{(\ell)}, \eta^{(\ell)}\}_{\ell=0}^{\infty}$ of Chapter IV converge uniformly in ε for $|\varepsilon| \le \varepsilon_0$. But an easy inductive argument on the respective iteration schemes shows that $v^{(\ell)}, q^{(\ell)}$, and $\eta^{(\ell)}$ are analytic in ε for each ℓ and for each $\xi \in [\alpha, \beta]$. Therefore the limits of these sequences, v, q, and η are analytic in ε for each $\xi \in [\alpha, \beta]$. From the implicit function theorem and hypothesis (46), $u(\xi, \lambda)$ defined by (48) is an analytic function of $\lambda \in \mathcal{J}_0$ for each $\xi \in [\alpha, \beta]$, and so $f(\xi, \lambda, u(\xi, \lambda))$ has this same property. We can now proceed as in the proof of Theorem 7 of Chapter II, using the fact that $\mathcal{J}_0$ is finite, to reach the stated conclusion.

For continuation to be possible, the solution branch (43)(44) (or (48)) must fall in the second case of the dichotomy.

The following result is now obvious.

<u>Corollary:</u>

If $f(\xi, \lambda, u)$ is analytic in λ and u for each ξ with $(\xi, \lambda, u) \in S$, and if there exists a point $(\tilde{u}, \tilde{\lambda})$ on the solution branch (43)(44) such that $\lambda'(\varepsilon) \neq 0$ at $\lambda = \tilde{\lambda}$ and $\tilde{\lambda}$ is not an eigenvalue of (11)(12), then there exists an open λ-interval containing $\tilde{\lambda}$ throughout which problem (51)(52) has no nontrivial solutions.

In case linearized problem (51)(52) can have only simple eigenvalues, for example if (51)(52) is equivalent to a second-order self-adjoint Sturm-Lionville problem, then we are in an especially fortunate position. The uniqueness theorem of § III. 5 can always be applied to the solution branch (43)(44) and yields the result that no secondary bifurcation occurs on (43)(44) for $|\epsilon| \leq \epsilon^*$. Thus, since simple eigenvalues always result in bifurcation, the linearized problem (51)(52) can have no simple eigenvalues on the branch (43)(44) with $0 < |\epsilon| \leq \epsilon^*$. Because of its importance, we state this as a theorem. First we define the set

$$\mathcal{J}_1 \equiv \{\lambda \mid \lambda = \lambda(\epsilon), \quad 0 < |\epsilon| < \epsilon^* \} \tag{53}$$

where $\lambda(\epsilon)$ is defined by (44) and ϵ^* is defined in § III. 5.

Theorem 2:

If the linearized problem (51)(52) is defined and can have only simple eigenvalues on $\mathcal{J}_1$, then it has no eigenvalues at all there.

For any point $(\widetilde{u}(\xi), \widetilde{\lambda}) \in C_n[\alpha, \beta] \times \mathcal{R}$, define the following sets:

$$\widetilde{S} \equiv \{(\xi, \lambda, u) \mid \xi \in [\alpha, \beta], |\lambda - \widetilde{\lambda}| \leq 1, \; u \in C_n[\alpha, \beta], \|u - \widetilde{u}\| \leq 1\} \tag{54}$$

$$\mathcal{U}_{\delta_1}(\widetilde{u}) \equiv \{u(\xi) \in C_n[\alpha, \beta] \mid \; \|u - \widetilde{u}\| \leq \delta_1\} \tag{55}$$

$$\mathcal{W}_{\delta_1} \equiv \{w(\xi) \in C_n[\alpha, \beta] \mid \; |w| \leq \delta_1 \} \tag{56}$$

$$\eta_\delta(\widetilde{\lambda}) \equiv \{\lambda \in \mathcal{R} \mid \; |\lambda - \widetilde{\lambda}| \leq \delta \} \; . \tag{57}$$

We now state and prove the basic continuation theorem.

Theorem 3:

If $f(\xi, \lambda, u)$ is defined in $\mathfrak{F}$ and f_{uu} and $f_{u\lambda}$ exist and are continuous there; if $\tilde{u}(\xi)$ is a nontrivial solution of the nonlinear boundary-value problem (41)(42) for $\lambda = \tilde{\lambda}$; and if the linear problem (51)(52) has no nontrivial solution for $\lambda = \tilde{\lambda}$ and $u = \tilde{u}$; then there exist constants δ and δ_1 in $(0, 1]$, such that the nonlinear boundary-value problem (41)(42) has a unique solution $u(\xi, \lambda) \in \mathcal{N}_{\delta_1}(\tilde{u})$ for each $\lambda \in \mathcal{N}_\delta(\tilde{\lambda})$. This solution $u(\xi, \lambda)$ has the following properties:

$$u(\xi, \tilde{\lambda}) = \tilde{u}(\xi) , \tag{58}$$

$u(\xi, \lambda)$ is continuous in λ for $\lambda \in \mathcal{N}_\delta(\tilde{\lambda})$ and for each $\xi \in [\alpha, \beta]$, (59)

and $u(\xi, \lambda)$ is the limit of the sequence $\{u(\xi, \lambda)^{(\ell)}\}_{\ell=0}^{\infty}$ defined by the iteration scheme

$$u(\xi, \lambda)^{(0)} = \tilde{u}(\xi) \tag{60}$$

$$u^{(\ell+1)'} - [A(\xi) + \tilde{\lambda} f_u(\xi, \tilde{\lambda}, \tilde{u})]u^{(\ell+1)} = \lambda f(\xi, \lambda, u^{(\ell)}) - \tilde{\lambda} f_u(\xi, \tilde{\lambda}, \tilde{u})u^{(\ell)} \tag{61}$$

$$Mu(\alpha, \lambda)^{(\ell+1)} + N u(\beta, \lambda)^{(\ell+1)} = 0, \quad \ell = 0, 1, 2, \dots \quad . \tag{62}$$

Proof:

Choose some $u(\xi) \in \mathcal{N}_{\delta_1}(\tilde{u})$ and define

$$w(\xi) \equiv u(\xi) - \tilde{u}(\xi) \quad . \tag{63}$$

Then $w \in \mathcal{W}_{\delta_1}$, and u is a solution of (1)(2) iff w satisfies

$$w'(\xi) - [A(\xi) + \tilde{\lambda} f_u(\xi, \tilde{\lambda}, \tilde{u})]w(\xi) = \lambda f(\xi, \lambda, u) - \tilde{\lambda} f(\xi, \tilde{\lambda}, \tilde{u}) - \tilde{\lambda} f_u(\xi, \tilde{\lambda}, \tilde{u})w \tag{64}$$

$$M\, w(\alpha) + N w(\beta) = 0 \quad . \tag{65}$$

By hypothesis, the linear problem obtained from (64)(65) by setting the right-hand side of (64) equal to zero has no nontrivial solutions, so a Green's matrix $\tilde{G}(\xi, \tau)$ exists by which (64)(65) may be transformed to the equivalent integral equation:

$$w(\xi) = \int_a^\beta \tilde{G}(\xi, \tau)[\lambda f(\tau, \lambda, \tilde{u}+w) - \tilde{\lambda} f(\tau, \tilde{\lambda}, \tilde{u}) - \tilde{\lambda} f_u(\tau, \tilde{\lambda}, \tilde{u}) w(\tau)]d\tau \quad . \tag{66}$$

Let the operator on the right side of (66) be represented by $T_\lambda w$. Then

$$T_\lambda : \mathscr{W}_{\delta_1} \to C_n[a, \beta] \quad \text{for each } \lambda \in \mathscr{N}_\delta(\tilde{\lambda}) \quad , \tag{67}$$

and w is a solution of (66) iff w is a fixed point of T_λ.

Defining norms in the usual way with respect to the set $\tilde{S}$, we have, for w, $w^{(1)}$ and $w^{(2)}$ in $\mathscr{W}_{\delta_1}$ and $\lambda \in \mathscr{N}_\delta(\tilde{\lambda})$,

$$\|T_\lambda 0\| \le \|\tilde{G}\| \{\|f\| + |\tilde{\lambda}| \|f_\lambda\|\} |\lambda - \tilde{\lambda}|$$

$$\le (1-\theta)\delta_1 \quad , \tag{68}$$

if

$$\delta \le \frac{(1-\theta)\delta_1}{\|\tilde{G}\| \{\|f\| + |\tilde{\lambda}| \|f_u\|\}} \quad . \tag{69}$$

Also,

$$\|T_\lambda w^{(1)} - T_\lambda w^{(2)}\| \le \|\tilde{G}\| \{\|\lambda f(\tau, \lambda, \tilde{u}+w^{(1)}) - \lambda f(\tau, \lambda, u+w^{(2)}) - \lambda f_u(\tau, \tilde{\lambda}, \tilde{u})(w^{(1)} - w^{(2)})\|\}$$

$$\le \|\tilde{G}\| \{\|f_u\|\delta + |\tilde{\lambda}| \|f_{u\lambda}\|\delta + |\tilde{\lambda}| \|f_{uu}\|\delta_1\} \|w^{(1)} - w^{(2)}\| .$$

So,

$$\|T_\lambda w^{(1)} - T_\lambda w^{(1)}\| \le \theta \|w^{(1)} - w^{(2)}\| \quad \text{for any } 0 < \theta < 1 \; , \tag{70}$$

if

$$\delta \leq \frac{\theta}{2\,\|\tilde{G}\|\,\{\|f_u\| + |\tilde{\lambda}|\,\|f_{u\lambda}\|\}}$$

and

$$\delta_1 \leq \frac{\theta}{2\,\|\tilde{G}\|\,|\tilde{\lambda}|\,\|f_{uu}\|} .$$

Therefore we define, for any $0 < \theta < 1$,

$$\delta \equiv \min\left\{1, \frac{\theta}{2\|\tilde{G}\|\{\|f_u\|+|\tilde{\lambda}|\,\|f_{u\lambda}\|\}}, \frac{\theta\,(1-\theta)}{2\,\|\tilde{G}\|^2\,|\tilde{\lambda}|\,\|f_{uu}\|\{\|f\|+|\tilde{\lambda}|\,\|f_u\|\}}\right\} \quad (71)$$

and

$$\delta_1 \equiv \min\left\{ \frac{\theta}{2\,\|\tilde{G}\|\,|\tilde{\lambda}|\,\|f_{uu}\|}, 1\right\} . \quad (72)$$

With these definitions the contracting mapping theorem applies and we conclude that T_λ has a unique fixed point in $\mathcal{U}_{\delta_1}$ for each $\lambda \in \mathcal{N}_\delta(\tilde{\lambda})$. This fixed point w is a unique solution of (66) in $\mathcal{U}_{\delta_1}$ and is continuously differentiable and so is the unique solution of (64)(65) in $\mathcal{U}_{\delta_1}$. Therefore $u(\xi, \lambda) \equiv \tilde{u}+w$ is a unique solution of (41)(42) in $\mathcal{U}_{\delta_1}$, for $\lambda \in \mathcal{N}_\delta$.

Furthermore, it follows from the contracting mapping theorem that $u(\xi, \lambda)$ is the limit of the sequence of iterates (60)(61)(62), and the convergence of this sequence is uniform in $\lambda \in \mathcal{N}_\delta(\tilde{\lambda})$. The continuity of $u(\xi, \lambda)$ in λ follows from the continuity of these iterates and their uniform convergence. Also, from (69), $\lambda = \tilde{\lambda}$ when $\delta_1 \equiv 0$, thus verifying (58), and the theorem is proved.

It is now clear how to proceed with the continuation of a non-trivial solution branch away from its bifurcation point. We start with the branch (43)(44) given in a neighborhood of the bifurcation point $(0, \lambda_0)$

by the theory of Chapter III or IV, and, if possible, find a point $(\tilde{u}, \tilde{\lambda})$ on that branch to which Theorem 3 applies. Then the iteration scheme (60)(61)(62) generates a solution $u(\xi, \lambda)$ for λ in the neighborhood $\eta_\delta(\tilde{\lambda})$, and by the uniqueness, it must coincide with the original branch where there domains of definition overlap. Call $\tilde{\lambda} \equiv \lambda^{(1)}$ and $\eta_\delta(\tilde{\lambda}) \equiv \eta^{(1)}$. Then we can pick a point $\lambda^{(2)}$ which is further from the bifurcation point λ_0 than $\lambda^{(1)}$ was, and if it is not an eigenvalue of (11)(12), apply Theorem 3 again to extend the definition of $u(\xi, \lambda)$ into the new neighborhood $\eta^{(2)} \equiv \eta_{\delta^{(2)}}(\lambda^{(2)})$. This process may be repeated indefinitely, provided no eigenvalue of the linearized problem (51)(52) is encountered. In this regard we are helped by Theorems 1 and 2.

However, even if problem (51)(52) has no eigenvalues, we have not yet shown that this process will take us anywhere. That is, suppose that, for some $n \geq 1$,

$$\eta^{(m)} \subseteq \eta^{(n)} \quad \text{for all} \quad m \geq n \quad . \tag{73}$$

In such a case, the points $\lambda^{(k)}$ will still be a monotone sequence moving away from λ_0, but they will be bounded above, and the neighborhoods $\eta^{(k)}$ will shrink in such a way that the domain of definition of $u(\xi, \lambda)$ is not extended at all. We prove in the next theorem that such a frustrating situation can occur only for very good reasons.

We note in passing that the iteration scheme (60)(61)(62) may not be the best way to compute the continuation in practice. The method of Poincaré continuation, described by H. B. Keller [23], pages 146-149, is more practical.

In the following theorem we assume, for convenience, that the solution branch is being continued to the right; that is in the direction of increasing λ, from λ_0. The case of decreasing λ is of course equivalent.

Theorem 4

Assume $f(\xi, \lambda, u)$ is continuous in (ξ, λ, u) and f_u, f_{uu}, and $f_{u\lambda}$ are defined and continuous in (ξ, λ, u), for each $\xi \in [\alpha, \beta]$, for all $\lambda > \lambda_0$ and for all $u \in C_n[\alpha, \beta]$. Suppose the continuation process of Theorem 3 is carried out on a monotone increasing sequence of $\lambda^{(k)}$, $k = 1, 2, \ldots$, thus defining the continued solution branch $u(\xi, \lambda)$. Let λ^* be the least upper bound of all possible such $\lambda^{(k)}$. Then exactly one of the following three possibilities must occur:

(a) $\lambda^* = \infty$, (that is, $u(\xi, \lambda)$ exists for all finite λ);

(b) $\lambda^* < \infty$, and $\|u(\xi, \lambda)\| \to \infty$ as $\lambda \to \lambda^*$;

(c) $\lambda^* < \infty$, $u(\xi, \lambda^*) \equiv \lim_{\lambda \to \lambda^*} u(\xi, \lambda)$ exists, and λ^* is an eigenvalue of the linearized problem (11)(12).

Proof:

Clearly (a), (b) and (c) are mutually exclusive.

Suppose neither (a) nor (b) occur, and λ^* is not an eigenvalue of problem (51)(52). Then λ^* is finite, and $u(\xi, \lambda^*) \equiv \lim_{\lambda \to \lambda^*} u$ exists. From Theorem 3, $u(\xi, \lambda)$ is continuous in λ for $\lambda^{(1)} \le \lambda < \lambda^*$, so this definition makes $u(\xi, \lambda)$ continuous in λ for $\lambda^{(1)} \le \lambda \le \lambda^*$.

Choose one sequence out of the many possible monotone increasing sequences $\{\lambda^{(k)}\}_{k=1}^{\infty}$ with $\lambda^{(k)} \to \lambda^*$ and $\lambda^{(k)} \neq \lambda^*$, $k = 1, 2, \ldots$.

Then clearly $u(\xi, \lambda^{(k)})$ is continuous in $\xi \in [\alpha, \beta]$ for $k = 1, 2, \ldots$. The continuity of $u(\xi, \lambda^*)$ in $\xi \in [\alpha, \beta]$ then follows from the equi-continuity of the sequence $\{u(\xi, \lambda^{(k)})\}_{k=1}^{\infty}$ which is proven in the Lemma below.

By the hypothesis that λ^* is not an eigenvalue of (51)(52), a Green's matrix $G(\xi, \tau, \lambda)$ exists for (51)(52), and it follows from the continuity of $u(\xi, \lambda)$ that $G(\xi, \tau, \lambda)$ is continuous in λ for $\lambda_1 \leq \lambda \leq \lambda^*$. Consider the following integral equation which is equivalent to (41)(42) for $\lambda_1 \leq \lambda < \lambda^*$:

$$u(\xi, \lambda) = \lambda \int_\alpha^\beta G(\xi, \tau, \lambda)[f(\tau, \lambda, u(\tau, \lambda)) - f_u(\tau, \lambda, u(\tau, \lambda))u(\tau, \lambda)]d\tau \quad . \tag{74}$$

Both sides of (74) are continuous in λ and uniformly continuous in ξ for $\lambda \rightarrow \lambda^*$, so (74) remains valid if we let $\lambda = \lambda^*$. Hence $u(\xi, \lambda^*)$ is the unique solution of (74) with $\lambda = \lambda^*$, and by Theorem 8 of Chapter II, is the unique continuously differentiable solution to problem (41)(42), where by unique we now mean unique in a sufficiently small neighborhood of the solution branch under consideration.

But now Theorem 3 is applicable at λ^* and so the solution branch can be continued beyond λ^* into an open neighborhood of λ^*, which contradicts the hypothesis that λ^* is the least upper bound. Therefore at least one of (a), (b), or (c) must occur, but since they are mutually exclusive, exactly one occurs.

Lemma

Suppose that the hypotheses of Theorem 4 are satisfied, and that λ^* is finite and $u(\xi, \lambda^*)$ exists as defined in Theorem 4. Let $\{\lambda^{(k)}\}_{k=1}^{\infty}$

be the monotone increasing sequence of Theorem 4 with $\lambda^{(k)} \to \lambda^*$ and $\lambda^{(k)} \neq \lambda^*$, $k = 1, 2, \ldots$. Then the sequence of functions $u^{(k)}(\xi) \equiv u(\xi, \lambda^{(k)})$, $k = 1, 2, \ldots$, is equicontinuous in ξ for $\xi \in [a, \beta]$.

Proof:

Choose any $\tilde{\lambda}$ such that $\lambda^{(1)} \leq \tilde{\lambda} < \lambda^*$; this $\tilde{\lambda}$ will remain fixed throughout the proof. Define $u(\xi) \equiv u(\xi, \tilde{\lambda})$. Clearly $\tilde{\lambda}$ is not an eigenvalue of (51)(52).

For each $k = 1, 2, \ldots$, $u^{(k)}(\xi)$ is the solution of the problem

$$u^{(k)}(\xi)' - [A(\xi)+\tilde{\lambda} f_u(\xi, \tilde{\lambda}, \tilde{u})]u^{(k)}(\xi) = \lambda^{(k)} f(\xi, \lambda^{(k)}, u^{(k)}) - \tilde{\lambda} f_u(\xi, \tilde{\lambda}, \tilde{u}) u^{(k)}(\xi) \tag{75}$$

$$M\, u^{(k)}(a) + N\, u^{(k)}(\beta) = 0 \quad , \tag{76}$$

which is equivalent to

$$u^{(k)}(\xi) = \int_a^\beta \tilde{G}(\xi, \tau)[\lambda^{(k)} f(\tau, \lambda^{(k)}, u^{(k)}) - \tilde{\lambda} f_u(\tau, \tilde{\lambda}, \tilde{u}) u^{(k)}(\tau)]d\tau \tag{77}$$

where the Green's matrix $\tilde{G}(\xi, \tau)$ is defined by

$$\tilde{G}(\xi, \tau) \equiv G(\xi, \tau, \tilde{\lambda}) \tag{78}$$

which exists. Now,

$$\|u^{(k)}(\xi^{(1)}) - u^{(k)}(\xi^{(2)})\| \leq \int_a^\beta \|\tilde{G}(\xi^{(1)}, \tau) - \tilde{G}(\xi^{(2)}, \tau)\| d\tau \|\lambda^{(k)} f(\tau, \lambda^{(k)}, u^{(k)}) - \tilde{\lambda} f_u(\tau, \tilde{\lambda}, \tilde{u}) u^{(k)}\| \tag{79}$$

Since $\lambda^{(k)}$ and $\|u^{(k)}\|$ are bounded for all k, and for $k \to \infty$, there exists a constant Φ, such that

$$\|\lambda^{(k)} f(\tau, \lambda^{(k)}, u^{(k)}) - \tilde{\lambda} f_u(\tau, \tilde{\lambda}, \tilde{u}) u^{(i)}\| \leq \Phi \quad \text{for all } k \tag{80}$$

Use a tilde to denote that the quantities so marked are to be evaluated at $\lambda = \tilde{\lambda}$. Then, from (26) of Chapter II,

$$\int_a^\beta \|\tilde{G}(\xi^{(1)},\tau)-\tilde{G}(\xi^{(1)},\tau)\|d\tau \le \tfrac{1}{2}\int_a^\beta \| \tilde{Y}(\xi^{(1)})[I\,\mathrm{sgn}(\xi^{(1)}-\tau)+\tilde{B}^{-1}\tilde{D}]-Y(\xi^{(2)})$$

$$[I\,\mathrm{sgn}(\xi^{(2)}-\tau)+\tilde{B}^{-1}\tilde{D}]\|\|\tilde{Y}^{-1}(\tau)\|d\tau$$

$$\le \tfrac{1}{2}\|\tilde{Y}(\xi^{(1)})-\tilde{Y}(\xi^{(2)})\| \int_a^\beta \|I\,\mathrm{sgn}(\xi^{(1)}-\tau)+\tilde{B}^{-1}\tilde{D}\|\;\|\tilde{Y}^{-1}(\tau)\|d\tau$$

$$+\tfrac{1}{2}\|\tilde{Y}(\xi^{(2)})\|\cdot 2\,|\xi^{(1)}-\xi^{(2)}|\;\|Y^{-1}\| \quad .$$

Since $\tilde{Y}$ and $\tilde{Y}^{-1}$ are continuously differentiable in ξ (or τ), there exists a constant Φ_2 such that

$$\int_a^\beta \|\tilde{G}(\xi^{(1)},\tau)-\tilde{G}(\xi^{(2)},\tau)\|d\tau \le \Phi_2\,|\xi^{(1)}-\xi^{(2)}| \quad .$$

Let $\Phi = \Phi_1\,\Phi_2$, and we have from (39), (40) and (41) that

$$\|u^{(k)}(\xi^{(1)})-u^{(k)}(\xi^{(2)})\| \le \Phi\,|\xi^{(1)}-\xi^{(2)}| \quad ,$$

with Φ independent of $k = 1, 2, \ldots$.

Thus the sequence $\{u^{(k)}(\xi)\}_{k=1}^{\infty}$ is equicontinuous in ξ for $\xi \in [a,\beta]$.

V.5. Comparison with the Perturbation Method

Formal perturbation theory is often used to obtain very useful approximations to solutions of bifurcation problems. This method originated in the work of Lindstedt and Poincaré [38] on periodic motions in celestial mechanics. It has recently been applied by J. B. Keller and others [26], [27], [36] to a number of physically important nonlinear boundary-value problems, which arise in such diverse areas as nonlinear optics, heat conduction, vibrations, and superconductivity.

In this section we will show how the iteration method which we have presented can be used to justify rigorously the approximate solutions obtained formally by perturbation theory, and furthermore we prove that the n^{th} iterate obtained by means of our iteration scheme contains the first (n+1) terms of the perturbation expansion.

First we define what we mean by "the formal perturbation method" for a class of bifurcation problems. This definition is essentially that presented in [27]. Let L be a linear differential operator of one of the types which has been discussed in this thesis; that is

$$Lu \equiv u' + A(\xi)u \quad \text{as in Chapter II; or} \tag{83}$$

$$Lu \equiv -\sum_{ij=1}^{m} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial u(\xi)}{\partial \xi_j}\right) + a_o(\xi)\, u(\xi) \tag{84}$$

as in §V.3.

Let B represent the appropriate boundary operator, viz

$$Bu \equiv Mu(\alpha) + Nu(\beta) = 0 \qquad \text{or} \tag{85}$$

$$Bu \equiv u(\xi) = 0, \tag{86}$$

respectively. Let $f(\xi, \lambda, u)$ be the appropriate nonlinearity as previously defined for each of the above operators. Then we can represent any of the above problems as

$$Lu = \lambda f(\xi, \lambda, u) \qquad \xi \in \mathcal{D} \tag{87}$$

$$B u = 0 \quad . \qquad \xi \in \partial \mathcal{D} \quad . \tag{88}$$

The formal perturbation method for problem (87)(88) proceeds as follows. Clearly $u \equiv 0$ is a trivial solution of (87)(88) for all λ. A one-parameter family of nontrivial solutions of (87)(88) is sought, which splits off of this trivial solution at some value of λ, say $\lambda = \lambda_0$. Assume that such a one-parameter family exists and that it can be expressed in the form

$$u(\xi, \epsilon) = \epsilon\, x_1(\xi) + \epsilon^2\, x_2(\xi) + \epsilon^3 x_3(\xi) + \ldots , \tag{89}$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon\, \lambda_1 + \epsilon^2\, \lambda_2 + \ldots . \tag{90}$$

These power series in ϵ are not assumed to be convergent, but are generally assumed to be asymptotically valid, uniformly in ξ. That is, it is assumed that for each $n = 1, 2, \ldots, m$ (m may be finite or infinite), the following is true uniformly in ξ :

$$\| u(\xi, \epsilon) - (\epsilon\, x_1(\xi) + \ldots + \epsilon^n x_n(\xi)) \| = O(\epsilon^{n+1}) \tag{91}$$

$$| \lambda(\epsilon) - (\lambda_0 + \epsilon\lambda_1 + \ldots + \epsilon^n \lambda_n) | = O(\epsilon^{n+1}) \tag{92}$$

as $\epsilon \to 0$. (The order symbol $O(\epsilon^n)$ is defined in Appendix A.) Substitute (89)(90) into (87)(88), differentiate repeatedly with respect to ϵ, and set $\epsilon = 0$. Then the following sequence of linear problems is obtained, assuming of course that the indicated derivatives of f exist:

$$[L - \lambda_0 f_u(\xi, \lambda_0, 0)] x_1 = 0 \quad , \tag{93}$$

$$[L - \lambda_0 f_u(\xi, \lambda_0, 0)]x_2 = \lambda_1 [f_u(\xi, \lambda_0, 0)x_1 + \lambda_0 f_{u\lambda}(\xi, \lambda_0, 0)x_1]$$

$$+ \tfrac{1}{2} \lambda_0 f_{uu}(\xi, \lambda_0, 0)x_1 \; x_1 \quad , \tag{94}$$

etc., with boundary conditions

$$B x_i = 0 \qquad i = 1, 2, \ldots \; . \tag{95}$$

The first of these is just the familiar linearized problem and so λ_0 must be an eigenvalue and x_1 the corresponding eigenfunction $y_0(\xi)$ of (13)(14). We assume throughout this section that λ_0 is a simple eigenvalue. The subsequent linear problems are inhomogeneous, and so λ_i is determined by applying the basic alternative theorem to the $(i+1)^{th}$ problem, which is then solved for $x_{i+1}(\xi)$. The solution $x_{i+1}(\xi)$ is made unique by the condition $\int y_0(\xi)x_{i+1}(\xi)d\xi = 0$, $i = 1, 2, \ldots$. This method fails when the derivatives of f fail to exist or when the coefficient of λ_i in the $(i+1)^{th}$ equation is zero. For example, (94) yields

$$\lambda_1 = \frac{\lambda_0 \int z_0^*(\xi)\, f_{uu}(\xi, \lambda_0, 0)x_1^2(\xi)d\xi}{\int z_0^*(\xi)f_u(\xi, \lambda_0, 0)x_1(\xi)d\xi + \lambda_0 \int z_0^*(\xi)f_{u\lambda}(\xi, \lambda_0, 0)x_1(\xi)d\xi} \, . \tag{96}$$

We now compare the asymptotic forms of the approximate solutions obtained by the iteration scheme presented earlier and the perturbation method just described. Note that the contracting mapping theorem as used in Chapters III and IV to prove that the iteration schemes converge, also tells us that this convergence is geometric in ϵ. That is:

$$\| v(\xi, \epsilon) - v^{(j)}(\xi, \epsilon) \| = O(\epsilon^j), \quad \text{and} \tag{97}$$

$$| \eta(\epsilon) - \eta^{(j)}(\epsilon) | = O(\epsilon^j) \quad , \tag{98}$$

for $j = 1, 2, \ldots$, as $\epsilon \to 0$. Here $v(\xi, \epsilon)$ and $\eta(\epsilon)$ are the exact solution and $v^{(i)}(\xi, \epsilon), \eta^{(j)}(\epsilon)$ are the j^{th} iterates. If we define

$$u^{(j)}(\xi, \epsilon) \equiv \epsilon\, y_0 + \epsilon^2\, v^{(j)}(\xi, \epsilon) \tag{99}$$

$$\lambda^{(j)}(\epsilon) = \lambda_0 + \epsilon\, \eta^{(j)}(\epsilon) \tag{100}$$

for $j = 1, 2, \ldots$, and $|\epsilon| \leq \epsilon_0$, then we have from (97)(98) that

$$\|u(\xi, \epsilon) - u^{(j)}(\xi, \epsilon)\| = O(\epsilon^{j+2}), \tag{101}$$

$$|\lambda(\epsilon) - \lambda^{(j)}(\epsilon)| = O(\epsilon^{j+1}), \quad \text{as } \epsilon \to 0 . \tag{102}$$

Combining (101)(102) with (91) (92) we have the following theorem.

Theorem 5

If the perturbation method is valid, that is if it generates expansions of the form (89)(90) which have properties (91)(92), then the perturbation expansions (89)(90) are related to the iterates (99)(100) by

$$\|u^{(j)}(\xi, \epsilon) - (\epsilon x_1 + \epsilon^2 x_2 + \ldots + \epsilon^{j+1} x_{j+1})\| = O(\epsilon^{j+2}), \tag{103}$$

$$|\lambda^{(j)}(\epsilon) - (\lambda_0 + \epsilon \lambda_1 + \ldots + \epsilon^j x_j)| = O(\epsilon^{j+1}), \tag{104}$$

for each $j = 1, \ldots, m-1$.

This leaves open the question of whether or not the formal perturbation method is "valid" for a given problem. In many cases the validity can be proven by a simple extension of the iteration method. In fact, in § III. 6 we have already done this for (104) with $j = 1$, or greater in some special cases. More generally, we could take as our ansatz

$$u(\xi, \epsilon) = \epsilon\, x_1(\xi) + \epsilon^2\, x_2(\xi) + \ldots + \epsilon^n x_n(\xi) + \epsilon^{n+1}\tilde{v}(\xi, \epsilon) \tag{105}$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon\,\lambda_1 + \ldots + \epsilon^{n-1}\lambda_{n-1} + \epsilon^n\,\tilde{\eta}(\xi)\ , \tag{106}$$

where $x_i(\xi)$ and λ_i are determined by the perturbation method, and with sufficiently differentiable $f(\xi, \lambda, u)$, we could find the equations which $\tilde{v}$ and $\tilde{\eta}$ must satisfy and then attempt to prove as we did with v and η, via the contracting mapping theorem, that $\tilde{v}$ and $\tilde{\eta}$ exist and are unique and bounded, which would verify (91) and (92). However, this approach can become very tedious and we do not attempt it here.

An entirely different approach is to prove (103)(104) by induction, and then use (97)(98) to verify (91)(92). We now outline this inductive argument. For convenience in the notation, we assume that f and u are scalar functions and f is independent of λ. Assume that $f(\xi, u)$ has continuous derivatives with respect to u up to order $m \geq 3$, uniformly in ξ. Then define the functions

$$\begin{aligned} g^{(j)}(\xi) \equiv\ & f_u(\xi, 0)x_j(\xi) + \tfrac{1}{2} f_{uu}(\xi, 0)(x_1\, x_{j-2} + x_2 x_{j-2} + \ldots + x_{j-1} x_1) \\ & + \frac{1}{6} f_{uuu}(\xi, 0)(x_1^2 x_{j-2} + x_1\, x_2\, x_{j-3} + \ldots + x_{j-2}\, x_1^2) \\ & + \ldots \\ & + \frac{1}{j\,!}\ f_{u^j}(\xi, 0)\ x_1^j \end{aligned} \tag{107}$$

for $j = 1, \ldots, m$.

We also define

$$\tilde{g}^{(j)}(\xi) \equiv g^{(j)}(\xi) - f_u(\xi, 0)\, x_j(\xi) \qquad j = 1, \ldots, m\ . \tag{108}$$

Then the equations which determine the coefficients in the perturbation expansion are:

$$[L - \lambda_0 f_u(\xi, 0)]x_1 = 0 \tag{109}$$

$$[L - \lambda_0 f_u(\xi, 0)]x_2 = \lambda_0 \tilde{g}^{(2)}(\xi) + \lambda_1 g^{(1)}(\xi)$$

$$[L - \lambda_0 f_u(\xi, 0)]x_3 = \lambda_0 \tilde{g}^{(3)} + \lambda_1 g^{(2)} + \lambda_2 g^{(1)} \tag{110}$$

or in general ,

$$[L-\lambda_0 f_u(\xi, 0)]x_{j+1} = \lambda_0 \tilde{g}^{(j+1)} + \lambda_1 g^{(j)} + \ldots + \lambda_{j-1} g^{(2)} + \lambda_j g^{(1)}, \ j=0, \ldots, m-1. \tag{111}$$

The boundary conditions in each case are

$$B x_{j+1} = 0 , \qquad j = 0, 1, \ldots m-1 , \tag{112}$$

and for uniqueness we require

$$\int y_0(\xi) x_{j+1}(\xi) d\xi = 0 , \qquad j = 1, 2, \ldots . \tag{113}$$

The orthogonality condition applied to (32)(33) determines

$$\lambda_j = \frac{-\int z_0(\xi) \, \{\lambda_0 \tilde{g}^{(j+1)}(\xi) + \lambda_1 g^{(j)}(\xi) + \ldots + \lambda_{j-1} g^{(2)}(\xi)\} d\xi}{\int z_0(\xi) \, g^{(1)}(\xi) d\xi} \tag{114}$$

for $j = 1, 2, \ldots, m-1$. Here $z_0(\xi)$ is the eigenfunction of the adjoint problem corresponding to (104) (112) .

Now we consider the iteration method. By hypothesis, $f(\xi, u)$ satisfies the identity

$$f(\xi, u) = 0+f_u(\xi, 0)u + \tfrac{1}{2} f_{uu}(\xi, 0)\, u^2 + \ldots + \frac{1}{(m-1)!} f_{u^{m-1}}(\xi, 0)u^{m-1}$$
$$+ \int_0^1 \int_0^1 \cdots \int_0^1 f_{u^m}(\xi, \rho_m\rho_{m-1}\cdots\rho_1 u)\rho_{m-1}\rho_{m-2}^2\cdots\rho_1^{m-1}$$
$$d\rho_n \ldots d\rho_1\, u^m \quad ,$$

so

$$f(\xi, u) = f_u(\xi, 0)u + \tfrac{1}{2} f_{uu}(\xi, 0)u^2 + \ldots + \frac{1}{(m-1)!} f_{u^{m-1}}(\xi, 0)u^{m-1} + O(u^m) \tag{115}$$

as $\|u\| \to 0$, uniformly in ξ. The iterates $v^{(j+1)}$ and $\eta^{(j+1)}$ are determined by the equations

$$[L-\lambda_0 f_u(\xi, 0)]v^{(j+1)} = \frac{1}{\epsilon^2}(\lambda_0+\epsilon\,\eta^{(j)})[f(\xi, \epsilon y_0+\epsilon^2 v^{(j)})-f_u(\xi,0)(\epsilon y_0+\epsilon^2 v^{(j)})]$$
$$+ \eta^{(j+1)} f_u(\xi, 0)\cdot y_0 + \epsilon\eta^{(j)} f_u(\xi, 0)v^{(j)} \quad , \tag{116}$$

$$\int y_0\,(\xi)v^{(j+1)}(\xi)d\xi = 0, \quad j = 0, 1, \ldots , \tag{117}$$

$$B\, v^{(j+1)} = 0 \quad , \tag{118}$$

$$\eta^{(j+1)} = \frac{-1}{\int z_0(\xi)f_u(\xi_0)y_0(\xi)d\xi} \Big\{\frac{1}{\epsilon^2}(\lambda_0+\epsilon\,\eta^{(j)}) \int z_0[f(\xi,\epsilon y_0+\epsilon^2 v^{(j)})$$
$$-f_u(\xi, 0)(\epsilon\, y_0+\epsilon^2 v^{(j)})]d\xi$$
$$+ \epsilon\eta^{(j)} \int z_0(\xi)f_u(\xi, 0)v^{(j)}(\xi)d\xi\Big\}, \tag{119}$$

for j = 0, 1, . . . , where the integrals are taken over the appropriate set $\xi \in [\alpha, \beta]$ or $\xi \in \mathfrak{D}$, say, and where

$$v^{(o)} = \eta^{(o)} \equiv 0. \tag{120}$$

Now we take for our induction hypothesis the following equations:

$$v^{(j)}(\xi, \epsilon) = x_2(\xi) + \epsilon\, x_3(\xi) + \ldots + \epsilon^{j-1} x_{j+1}(\xi) + O(\epsilon^j) \qquad (121)$$

$$\eta^{(j)}(\epsilon) = \lambda_1 + \epsilon\, \lambda_2 + \ldots + \epsilon^{j-1} \lambda_j + O(\epsilon^j) \quad . \qquad (122)$$

It is easily seen that (121) (122) is valid for $j = 1$. We assume (121) (122) valid for $j = 1, \ldots,$ n and prove the validity for $j = n+1$, where $n \leq m-3$. Substitute (115) and (121)(122) into (119) with $j = n$. Then

$$\eta^{(j+1)} = \frac{-1}{\int z_o f_u y_o d\xi} \Big\{(\lambda_o + \epsilon\lambda_1 + \epsilon^2 \lambda_2 + \ldots + \epsilon^j \lambda_j + O(\epsilon^{j+1}))\int z_o(\xi)\big[\tfrac{1}{2} f_{uu}(\xi, 0) \times (y_o + \epsilon v^{(j)})^2$$

$$+ \frac{\epsilon}{6} f_{uuu}(\xi, 0)(y_o + \epsilon v^{(j)})^3 + \ldots + \frac{\epsilon^{m-3}}{(m-1)!} f_{u^{m-1}}(\xi, 0) \times (y_o + \epsilon v^{(j)})^{m-1}\big]d\xi$$

$$+ O(\epsilon^{m-2}) + (\epsilon\, \lambda_1 + \epsilon^2 \lambda_2 + \ldots + \epsilon^j \lambda_j + O(\epsilon^{j+1}))\int z_o(\xi) f_u(\xi, 0) v^{(j)}(\xi) d\xi\Big\}$$

$$= \frac{-1}{\int z_o f_u y_o d\xi} \Big\{\lambda_o \int z_o \tilde{g}^{(2)} d\xi + \epsilon[\lambda_o \int z_o \tilde{g}^{(3)} d\xi + \lambda_1 \int z_o g^{(2)} d\xi]$$

$$+ \epsilon^2 [\lambda_o \int z_o \tilde{g}^{(4)} d\xi + \lambda_1 \int z_o g^{(3)} d\xi + \lambda_2 \int z_o g^{(2)} d\xi]$$

$$+ \ldots$$

$$+ \epsilon^j [\lambda_o \int z_o \tilde{g}^{(j+2)} d\xi + \lambda_1 \int z_o g^{(j+1)} d\xi + \ldots + \lambda_j \int z_o g^{(2)} d\xi]$$

$$+ O(\epsilon^{j+1}) \Big\}$$

Hence

$$\eta^{(j+1)} = \lambda_1 + \epsilon\, \lambda_2 + \ldots + \epsilon^j \lambda_{j+1} + O(\epsilon^{j+1}) \quad . \qquad (123)$$

Using this in (116) we get

$$[L-\lambda_0 f_u(\xi,0)]v^{(j+1)}(\xi) = (\lambda_0+\epsilon\lambda_1+\ldots+\epsilon^j\lambda_j+O(\epsilon^{j+1}))[\tfrac{1}{2}f_{uu}(\xi,0)(y_0+\epsilon v^{(j)})^2$$
$$+\frac{\epsilon}{6}f_{uuu}(\xi,0)(y_0+\epsilon v^{(j)})^3$$
$$+\frac{\epsilon^{m-3}}{(m-1)!}f_{u^{m-1}}(\xi,0)(y_0+\epsilon v^{(j)})^{m-1}+O(\epsilon^{m-2})]$$
$$+(\lambda_1+\epsilon\lambda_2+\ldots+\epsilon^{j-1}\lambda_j+O(\epsilon^j))f_u(\xi,0)(y_0+\epsilon v^{(j)})$$
$$+\epsilon^j\lambda_{j+1}f_u(\xi,0)y_0$$
$$=[\lambda_0\tilde{g}^{(2)}+\lambda_1 g^{(1)}]+\epsilon[\lambda_0\tilde{g}^{(3)}+\lambda_1 g^{(2)}+\lambda_2 g^{(1)}]$$
$$+\epsilon^2[\lambda_0\tilde{g}^{(4)}+\lambda_1 g^{(3)}+\lambda_2 g^{(2)}+\lambda_3 g^{(1)}] \tag{124}$$
$$+\ldots$$
$$+\epsilon^j[\lambda_0\tilde{g}^{(j+2)}+\lambda_1 g^{(j+1)}+\lambda_2 g^{(j)}+\ldots+\lambda_j g^{(2)}+\lambda_{j+1}g^{(1)}]$$
$$+O(\epsilon^{j+1}) \quad .$$

Since this is true for all sufficiently small ϵ, we get from (111) that

$$[L-\lambda_0 f_u(\xi,0)](v^{(j+1)}-x_2-\epsilon x_3-\ldots-\epsilon^j x_{j+2}) = O(\epsilon^{j+1}) . \tag{125}$$

But now, since $v^{(j+1)}$ and the x_i are orthogonal to y_0 and satisfy the same boundary conditions, a trivial argument involving the principal generalized Green's function gives:

$$v^{(j+1)}(\xi,\epsilon) = x_2(\xi)+\epsilon x_3(\xi)+\ldots+\epsilon^j x_{j+2}(\xi)+O(\epsilon^{j+1}) , \tag{126}$$

which, with (123), is what we set out to prove.

Finally, (123) and (126) can be combined with (99) and (100) to prove (103)(104), which together with (101)(122) verifies (91)(92). We have thus proven:

Theorem 6

If $f(\xi, u)$ has continuous derivatives up to order m with respect to u, uniformly in ξ, for some $m \geq 3$, then the two approximate solutions obtained by the iteration method and the formal perturbation method are related by the equations

$$u^{(j)}(\xi, \epsilon) = \epsilon\, x_1(\xi) + \epsilon^2 x_2(\xi) + \ldots + \epsilon^{j+1} x_{j+1}(\xi) + O(\epsilon^{j+2}) \qquad (127)$$

$$\lambda^{(j)}(\epsilon) = \lambda_0 + \epsilon\, \lambda_1 + \ldots + \epsilon^j \lambda_j + O(\epsilon^{j+1}) \qquad (128)$$

as $\epsilon \to 0$, for each $j = 1, 2, \ldots, m-2$. Furthermore, the perturbation method is "valid" in the asymptotic sense previously defined.

CHAPTER VI

AXISYMMETRIC BUCKLING OF THIN SPHERICAL SHELLS

VI. 1. Introduction

One of the outstanding problems of applied mechanics is that of finding a mathematical theory for reliably predicting the buckling behavior of thin structures. By buckling we mean, roughly speaking, a large change in the displacement of some part of the structure caused by a small change in the magnitude of the applied load, occurring while all parts of the structure are well below the elastic limit of the material.

An elementary example is the buckling of a slender elastic column due to axial compression. This problem was first analyzed theoretically by Bernoulli and Euler. Experimentally it is well known that as the axial load is increased from zero, the column at first shortens and thickens slightly, but remains straight. However a critical load is soon reached at which the slender column begins to buckle; that is it bends into a curved state. If the load is increased further, the column bends more sharply until the elastic limit is passed at some point in the material, and the column eventually breaks. However, if the load is removed before this occurs, that is while the column is still behaving elastically, then the column returns to its original straight condition. This elastic transition between the straight and curved states caused by a small change in the axial load is what we call buckling. The mathematical theory of column buckling is well developed; see for example [19] and [41].

A nontrivial example of buckling in which there is much current interest is the buckling of spherical shells subject to a uniform external pressure. T. von Karman and H.S. Tsien [48], having observed experimentally that the buckling deformation is usually a "dimple" confined to a small region of the sphere, initiated the theoretical investigations of a clamped spherical cap, which would correspond to the region of the sphere in which the dimple occurs. This approach was developed in [10] [24] and elsewhere. In recent years interest has returned to the theory of buckling of complete spherical shells, see [3], [28], [42], [46] and [49]. The agreement between theoretically predicted and experimentally measured buckling behavior is still relatively poor. In most cases the discrepancies are attributed to unavoidable imperfections in the real shells used in the experiments, and some attempts have been made to include the imperfections in the theory.

Without attempting to solve the outstanding problems in the theory of shell buckling, we show in this chapter how the bifurcation theory of the previous chapters can be applied to a mathematical model of the buckling of a spherical shell under a uniform external pressure. We do not exploit the full potential of the bifurcation theory for predicting buckling behavior, and in fact the theory is capable of providing much more information than is presented here. However, such data are abundantly available in [3]. The theory is admittedly unphysical in that we make the following two assumptions:

first that the shell is free of imperfections, and second that only axisymmetric deformations occur.

The mathematical model (1) (2) (3) is that derived by E. Reiss in [3]. The nonlinearities come from the strain-displacement relations and thus are geometric in origin. It is assumed that the linear stress-strain relations are valid, that is that Hooke's Law is valid. Other customary assumptions of shell theory made here are: the shell is thin, normals to the midsurface remain normal to the deformed midsurface, the normal stress in the radial direction is negligible compared to the other normal stresses, and the strains are small compared to 1. The derivation in [3] is based on the variations of the energy integral. Equivalently, equations (1) (2) (3) can be obtained from the equations of equilibrium of forces and moments as is done for the linearized model by Timoshenko [47].

The resulting mathematical model of sphere buckling is the nonlinear boundary value problem

$$(1-\xi^2)\frac{d^2x_1(\xi)}{d\xi^2} - 2\xi\frac{dx_1(\xi)}{d\xi} + \left(\nu - \frac{\xi^2}{1-\xi^2}\right)x_1 = -x_2(\xi) - \frac{1}{2}\frac{\xi}{\sqrt{1-\xi^2}}x_2^2(\xi) \tag{1}$$

$$(1-\xi^2)\frac{d^2x_2(\xi)}{d\xi^2} - 2\xi\frac{dx_2(\xi)}{d\xi} + \left(\frac{1-\nu^2}{\varkappa}\rho - \nu - \frac{\xi^2}{1-\xi^2}\right)x_2 = \left(\frac{1-\nu^2}{\varkappa}\right)\left(x_1(\xi) + \frac{\xi}{\sqrt{1+\xi^2}}x_1(\xi)x_2(\xi)\right) \tag{2}$$

$$-1 < \xi < 1$$

$$x_1(-1) = x_1(1) = 0$$
$$x_2(-1) = x_2(1) = 0 \qquad (3)$$

The symbols in these equations have the following definitions and physical interpretations:

$\xi \equiv \cos\theta$, where θ is the polar angle measured from the north pole, $0 \leq \theta \leq \pi$,

$x_1(\xi)$ is proportional to the shear strain,

$x_2(\xi)$ is proportional to the rotation of a tangent to a meridian,

$\nu \equiv$ Poisson's ratio,

$\varkappa \equiv \frac{1}{3}\left(\frac{h}{R}\right)^2$ is the dimensionless thickness parameter, where h = thickness of shell and R = radius of sphere,

$\rho \equiv \frac{1}{4}\frac{PR}{Eh}$ is the dimensionless load parameter, where E = Young's modulus and P is the uniform external pressure.

Our $x_1(\xi)$ and $x_2(\xi)$ are identical to q and v respectively in [3].

Clearly $x(\xi) \equiv 0$ is a solution of (1) (2) (3) for all values of ρ. We call this the trivial solution; physically it corresponds to a uniform radial contraction of the sphere due to the load ρ. Experimentally it is well known that for a sufficiently large load the shell buckles into a non-spherical shape. In the mathematical model (1) (2) (3) there are values of the load parameter ρ at which bifurcation occurs, that is nontrivial solutions appear. These nontrivial solutions describe possible buckled states of the sphere. Whether or not a physical sphere actually buckles onto one of these states depends on its relative energy; in general a structure buckles into states of lower energy. We do not go into such energy considerations here; see [3].

The radial displacement of the uniformly contracted unbuckled sphere is $(1-\nu)\rho$. Superimposed on this is the buckling displacement of a point on the sphere at polar angle θ given by the 2-component vector $u(\theta)$ where

$u_1(\theta) \equiv$ tangential displacement in θ direction,

$u_2(\theta) \equiv$ radial displacement toward center.

We may neglect azimuthal displacement by the axisymmetry assumption. The displacement components are given in terms of x_1 and x_2 by

$$u_1(\theta) = -(1+\nu)\, x_1(\theta) - \frac{\sin\theta}{2} \int_0^\theta \frac{x_2^2(\cos\varphi)}{\sin\varphi}\, d\varphi, \tag{4}$$

$$u_2(\theta) = -x_1(\cos\theta)\cot\theta - \frac{dx_1(\cos\theta)}{d\theta} - \frac{\cos\theta}{2} \int_0^\theta \frac{x_2^2(\cos\varphi)}{\sin\varphi}\, d\varphi. \tag{5}$$

The theory of the preceding chapters cannot be applied directly to the sphere buckling problem (1) (2) (3). These equations can easily be reformulated as a system of first order equations which have the form of the problem of Chapters III and IV except that the coefficients are singular at the end points $\xi = 1$ and $\xi = -1$. Therefore none of the linear theory of Chapter II is applicable, since there we required that all coefficients be continuous (or at least integrable -- see Chapter V), and so it is necessary to verify that the basic alternative theorem still holds and the generalized Green's matrix exists for this problem. This is done in Appendices C and D.

Since the linearizations of equations (1) and (2) are closely related to Legendre's differential equation, it is convenient to keep the problem in the form (1) (2), rather than in the form of a first-order system, so that the well-known properties of Legendre's equation may be utilized. Once the necessary linear theory is verified, the nonlinear bifurcation theory follows in the same general manner as it did in the previous chapters. This demonstrates the power and generality of the method.

For convenience, we rewrite the problem (1) (2) (3) as follows. Define

$$L \equiv -(1-\xi^2)\frac{d^2}{d\xi^2} + 2\xi\frac{d}{d\xi} + \frac{\xi^2}{1-\xi^2}$$

$$= -\frac{d}{d\xi}\left[(1-\xi^2)\frac{d}{d\xi}\right] + \frac{\xi^2}{1-\xi^2}\,, \tag{6}$$

$$T \equiv \begin{pmatrix} L & O \\ O & L \end{pmatrix}, \tag{7}$$

$$x(\xi) \equiv \begin{pmatrix} x_1(\xi) \\ x_2(\xi) \end{pmatrix}, \tag{8}$$

$$A(\rho) \equiv \begin{pmatrix} \nu & 1 \\ -\left(\frac{1-\nu^2}{\varkappa}\right) & \rho\left(\frac{1-\nu^2}{\varkappa}\right) - \nu \end{pmatrix}, \tag{9}$$

$$f(\xi, x(\xi)) \equiv \begin{pmatrix} \frac{1}{2} \frac{\xi}{\sqrt{1-\xi^2}} \; x_2(\xi)^2 \\ -\left(\frac{1-\nu^2}{\varkappa}\right) \frac{\xi}{\sqrt{1-\xi^2}} \; x_1(\xi)\, x_2(\xi) \end{pmatrix}. \tag{10}$$

Then (1) (2) (3) is

$$T\,x(\xi) = A(\rho)\, x(\xi) + f(\xi, x(\xi)), \; -1 < \xi < 1, \tag{11}$$

$$x(-1) = x(1) = 0. \tag{12}$$

VI. 2. The Linearized Problem

The linearized problem corresponding to (11) (12) is clearly

$$T\,y(\xi) = A(\rho)\, y(\xi) \tag{13}$$

$$y(-1) = y(1) = 0, \tag{14}$$

where $y(\xi) \equiv \begin{pmatrix} y_1(\xi) \\ y_2(\xi) \end{pmatrix}$.

We seek values of the load parameter ρ for which this problem has nontrivial solutions. When suitably normalized, we call such solutions eigensolutions, and the corresponding values of ρ the eigenpressures.

Legendre's differential equation for a scalar function $\varphi(\xi)$ is

$$(1-\xi^2)\,\varphi''(\xi) - 2\xi\,\varphi'(\xi) + \left[n(n+1) - \frac{\sigma^2}{1-\xi^2}\right] \varphi(\xi) = 0, \tag{15}$$

and has linearly independent solutions called the associated Legendre functions and designated

$$P_n^{(\sigma)}(\xi) \quad , \quad Q_n^{(\sigma)}(\xi).$$

Setting $\sigma = 1$, (15) becomes

$$L\,\varphi(\xi) = [n(n+1) - 1]\,\varphi(\xi), \tag{16}$$

and (16) has solutions $P_n^{(1)}(\xi)$ and $Q_n^{(1)}(\xi)$. If n is an integer, $P_n^{(1)}(\xi)$ satisfies the boundary conditions

$$P_n^{(1)}(-1) = P_n^{(1)}(1) = 0, \tag{17}$$

and $Q_n^{(1)}(\xi)$ blows up at $\xi = \pm 1$. If n is not an integer, neither

$P_n^{(1)}(\xi)$ nor $Q_n^{(1)}(\xi)$ satisfies the boundary conditions (17). Furthermore $P_o^{(1)}(\xi) = 0$ and

$$P_{-n-1}^{(1)}(\xi) = P_n^{(1)}(\xi). \tag{18}$$

Hence the scalar problem consisting of differential equation (16) with boundary conditions of the form (17) has the eigenfunctions

$$P_n^{(1)}(\xi) \quad , \qquad n = 1, 2, \ldots$$

and corresponding eigenvalues

$$\lambda_n \equiv n(n+1) - 1 \ , \ n = 1, 2, \ldots \ . \tag{19}$$

Now consider the matrix $A(\rho)$ defined by (9). It can have two eigenvalues, say $\lambda^+(\rho)$ and $\lambda^-(\rho)$ which are functions of ρ and are given by the roots of the characteristic equation

$$\lambda^2 - \rho\left(\frac{1-\nu^2}{\varkappa}\right) \lambda + \frac{1-\nu^2}{\varkappa} + \rho\left(\frac{1-\nu^2}{\varkappa}\right)\nu - \nu^2 = 0. \tag{20}$$

Assume for the moment that (20) had unequal roots. Then $A(\rho)$ can be diagonalized, that is there exists a non-singular matrix S such that

$$S^{-1}A(\rho)S = \begin{pmatrix} \lambda^+(\rho) & 0 \\ 0 & \lambda^-(\rho) \end{pmatrix} . \tag{21}$$

Define

$$w(\xi) = S^{-1} y(\xi). \tag{22}$$

Then problem (13) (14) is equivalent to

$$\begin{pmatrix} L & O \\ O & L \end{pmatrix} \begin{pmatrix} w_1(\xi) \\ w_2(\xi) \end{pmatrix} = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} \begin{pmatrix} w_1(\xi) \\ w_2(\xi) \end{pmatrix} \tag{23}$$

$$w(-1) = w(1) = 0, \tag{24}$$

which is just two independent scalar problems which have a nontrivial solution if and only if one of the eigenvalues λ^+ or λ^- of $A(\rho)$ is equal to an eigenvalue λ_n of L defined by (19). If for some value of ρ, exactly one of $\lambda^+(\rho)$ and $\lambda^-(\rho)$ is an eigenvalue of L, say $\lambda^+ = \lambda_n$ and $\lambda^- \neq \lambda_m$ for all m, then (13) (14) has exactly one eigensolution given by

$$y(\xi) = S \begin{pmatrix} P_n^{(1)}(\xi) \\ 0 \end{pmatrix} \tag{25}$$

In this case we write $\rho = \rho_n$ and we call ρ_n a simple eigenpressure of (13) (14). If both $\lambda^+(\rho)$ and $\lambda^-(\rho)$ are eigenvalues of L for the same ρ, say $\lambda^+ = \lambda_n$ and $\lambda^- = \lambda_m$, then (13) (14) has a two-dimensional space of eigensolutions spanned by

$$S\begin{pmatrix} P_n^{(1)}(\xi) \\ 0 \end{pmatrix} \quad \text{and} \quad S\begin{pmatrix} 0 \\ P_m^{(1)}(\xi) \end{pmatrix}, \tag{26}$$

and we call this value of ρ a degenerate eigenpressure.

The simple eigenpressure ρ_n is easily determined by substituting $\lambda = \lambda_n$ in (20), which gives

$$\rho_n = \left(\frac{\varkappa}{1-\nu^2}\right) (\lambda_n + \nu) + \frac{1}{(\lambda_n - \nu)} \quad n = 1, 2, \ldots \quad . \tag{27}$$

Then the condition for a degenerate eigenpressure to exist is that $\rho_n = \rho_m$ for some n and m with $n \neq m$, which yields

$$(\lambda_n - \nu)(\lambda_m - \nu) \left(\frac{\varkappa}{1-\nu^2}\right) = 1. \tag{28}$$

In most of what follows we assume that ρ_n is a simple eigenpressure (i.e., that (28) is false for all m) and that $A(\rho_n)$ has distinct eigenvalues.

Now suppose that (20) has a double root, so that $A(\rho)$ has two equal eigenvalues. Then we can easily show that $A(\rho)$ cannot be diagonalized and that problem (13) (14) can have at most one independent eigensolution, which exists only if this double root equals some λ_n, $n = 1, 2, \ldots$. Thus a double root does not lead to a degenerate eigenpressure.

Define the matrices

$$A_n \equiv A(\rho_n) \tag{29}$$

for each simple eigenpressure ρ_n. Then the eigenvalues of A_n, from (20) are

$$\lambda^+ = \lambda_n = n(n+1) - 1 \tag{30}$$

$$\lambda^{-} = \lambda_{\mu} \quad \nu + - \frac{1 - \nu^2}{n(\lambda_n - \nu)} . \tag{31}$$

Here μ is defined to be the positive root of

$$\mu(\mu+1) - 1 = \lambda_{\mu} , \tag{32}$$

and μ is not an integer when ρ_n is simple.

The normalized eigensolution of (13) (14) corresponding to a simple eigenpressure ρ_n is

$$y(\xi) = y^{(n)}(\xi) \equiv \begin{pmatrix} a_1 \\ \\ a_2 \end{pmatrix} P_n^{(1)}(\xi) \tag{33}$$

where
$$a_1 \equiv \left[\frac{n + \frac{1}{2}}{n(n+1)\,(1 + (\lambda_n - \nu)^2)} \right]^{\frac{1}{2}} \tag{34}$$

$$a_2 \equiv (\lambda_n - \nu)\, a_1 . \tag{35}$$

Here we have normalized $y^{(n)}(\xi)$ by

$$\int_{-1}^{1} (y_1^{(n)}(\xi)^2 + y_2^{(n)}(\xi)^2)\, d\xi = 1 \tag{36}$$

using

$$\int_{-1}^{1} P_n^{(1)}(\xi)^2\, dz = \frac{n(n+1)}{n + \frac{1}{2}} . \tag{37}$$

A matrix S_n which diagonalizes A_n is

$$S_n \equiv \begin{pmatrix} 1 & \left(\frac{\varkappa}{1-\nu^2}\right) & (\lambda_n - \nu) \\ \lambda_n - \nu & & 1 \end{pmatrix}, \tag{38}$$

and its inverse is

$$S_n^{-1} \equiv \begin{pmatrix} 1 & -\left(\frac{\varkappa}{1-\nu^2}\right) & (\lambda_n - \nu) \\ -(\lambda_n - \nu) & & 1 \end{pmatrix} \frac{1}{1-\left(\frac{\varkappa}{1-\nu^2}\right)(\lambda_n - \nu)^2}. \tag{39}$$

Corresponding to the eigenvalue $\lambda\mu$ of A_n is the function $P_\mu^{(1)}(\xi)$ which is a solution to the equation (16) with $n = \mu$, but does not satisfy the boundary conditions. We use $P_\mu^{(1)}(\xi)$ in constructing the Green's matrix in Appendix D.

The problem adjoint to (13) (14) is

$$T\ z(\xi) = A^*(\rho)\ z(\xi) \tag{40}$$

$$z(-1) = z(1) = 0. \tag{41}$$

This problem has the same eigenpressures as (13) (14) and the normalized eigensolutions

$$z(\xi) = z^{(n)}(\xi) \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} P_n^{(1)}(\xi) \tag{42}$$

where $$b_1 = \left[\frac{n+\frac{1}{2}}{n(n+1)\left(1+\left(\frac{\varkappa}{1-\nu^2}\right)^2(\lambda_n-\nu)^2\right)}\right]^{\frac{1}{2}} \tag{43}$$

$$b_2 = -\left(\frac{\varkappa}{1-\nu^2}\right)(\lambda_n-\nu)\ b_1 . \tag{44}$$

The properties of Legendre's functions which are used throughout this chapter, and many useful formulae, may be found in [1], [16] and [20].

VI. 3. Bifurcation at a Simple Eigenpressure

In this section we indicate very briefly how a slight modification of the proof of Theorem 2 in Chapter III enables us to prove that bifurcation actually occurs in the nonlinear problem (11) (12) at the simple eigenpressures ρ_n of the linearized problem (13) (14). We seek a nontrivial solution branch of the form

$$x(\xi,\epsilon) = \epsilon y(\xi) + \epsilon^2\sqrt{1-\xi^2}\ v(\xi,\epsilon) \tag{45}$$

$$\rho(\epsilon) = \rho_n + \epsilon\,\eta(\epsilon) . \tag{46}$$

Here $y(\xi)$ is the normalized eigensolution of (13) (14) corresponding to the simple eigenpressure ρ_n. Note that we have written an explicit factor of $\sqrt{1-\xi^2}$ in the second term of x. This stratagem knocks out the singularities which otherwise appear in f. The function v is no longer required to satisfy the boundary conditions but only to be continuous and hence bounded in $[-1, 1]$. Since the eigensolutions behave like $\sqrt{1-\xi^2}$ at ± 1, it is not unreasonable to hope that x does too.

Define A_n as in (29) and $B_n(\epsilon)$ by

$$B_n(\epsilon) \equiv \frac{1}{\epsilon} (A(\rho) - A_n) = \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{1-\nu^2}{\varkappa}\right) \eta(\epsilon) \end{pmatrix} \tag{47}$$

Substitute (45) (46) into (11) (12) and use (13) (14) to obtain the equation which v must satisfy:

$$\begin{aligned} [T - A_n] \sqrt{1-\xi^2} \; v(\xi, \epsilon) &= B_n (y + \epsilon \sqrt{1-\xi^2} \; v) \\ &+ \frac{1}{\epsilon^2} f(\xi, \epsilon y + \epsilon^2 \sqrt{1-\xi^2} \; v) \end{aligned} \tag{48}$$

Writing f out explicity, (48) becomes

$$\begin{aligned} [T - A_n] \sqrt{1-\xi^2} \; v &= B_n (y + \epsilon \sqrt{1-\xi^2} \; v) \\ &+ \frac{\xi P_n^{(1)}(\xi)^2}{\sqrt{1-\xi^2}} \begin{pmatrix} \frac{1}{2} a_2^2 \\ -\left(\frac{1-\nu^2}{\varkappa}\right) a_1 a_2 \end{pmatrix} \end{aligned}$$

$$+ \epsilon \xi P_n^{(1)}(\xi) \begin{pmatrix} a_2 v_2 \\ -\left(\frac{1-\nu^2}{\kappa}\right) (a_1 v_2 + a_2 v_1) \end{pmatrix}$$

$$+ \epsilon^2 \xi \sqrt{1-\xi^2} \begin{pmatrix} \frac{1}{2} v_2^2 \\ -\left(\frac{1-\nu^2}{\kappa}\right) v_1 v_2 \end{pmatrix} . \tag{49}$$

Note that every term on the right side of (49) behaves like $\sqrt{1-\xi^2}$ at ± 1 if v is bounded. From the basic alternative theorem in Appendix C, (49) can have a solution only if the right hand side is orthogonal to $z(\xi)$. Therefore $\eta(\epsilon)$ must satisfy:

$$\left(\frac{1-\nu^2}{\kappa}\right) \eta(\epsilon) \int_{-1}^{1} z_2(\xi) \left[y_2 + \epsilon \sqrt{1-\xi^2}\; v_2 \right] d\xi$$

$$+ \frac{1}{\epsilon^2} \int_{-1}^{1} z^*(\xi) f(\xi, \epsilon y + \epsilon^2 \sqrt{1-\xi^2}\, v)\, d\xi = 0. \tag{50}$$

Define

$$\gamma \equiv \int_{-1}^{1} z_2(\xi)\, y_2(\xi)\, d\xi \left(\frac{1-\nu^2}{\kappa}\right)$$

$$= - \frac{(\lambda_n - \nu)^2}{(1+(\lambda_n-\nu)^2)^{\frac{1}{2}}(1+(\frac{\kappa}{1-\nu^2})(\lambda_n-\nu)^2)^{\frac{1}{2}}} . \tag{51}$$

Now define a mapping

$$M : [\eta , v] \rightarrow [\tilde{\eta} , \tilde{v}] \tag{52}$$

by

$$\tilde{\eta} = -\frac{1}{\gamma}\left[\frac{1}{\epsilon^2} \int_{-1}^{1} z^* f(\xi, \epsilon y + \epsilon^2 \sqrt{1-\xi^2}\, v)\, d\xi \right.$$
$$\left. + \epsilon\left(\frac{1-\nu^2}{\varkappa}\right) \eta \int_{-1}^{1} z_2 v_2\, d\xi \right] \tag{53}$$

$$\tilde{v}(\xi) = \frac{1}{\sqrt{1-\xi^2}} \int_{-1}^{1} G^{\dagger}(\xi,\tau)\left[\left(\frac{1-\nu^2}{\varkappa}\right) [\tilde{\eta}\begin{pmatrix}0\\ y_2\end{pmatrix} + \epsilon\, \eta \begin{pmatrix}0\\ v_2\end{pmatrix} \sqrt{1-\xi^2}\,] \right.$$
$$\left. + \frac{1}{\epsilon^2} f(\tau, \epsilon y + \epsilon^2 \sqrt{1-\xi^2}\, v) \right] d\tau, \tag{54}$$

where $G^{\dagger}(\xi,\tau)$ is the generalized Green's matrix from Appendix D. Now the iteration scheme

$$\eta^{(o)} = 0 \quad , \quad v^{(o)} = 0 \tag{55}$$

$$[\eta^{(\ell+1)}, v^{(\ell+1)}] = M[\eta^{(\ell)}, v^{(\ell)}], \quad \ell = 0, 1, \ldots, \tag{56}$$

is formally the same as that defined in Chapter III. Define vector functions $r(\xi, \epsilon, \tilde{\mu}, \eta, v)$ and $s(\xi, \epsilon, \eta, v)$ by rewriting (53) and (54) as

$$\tilde{\eta} = \frac{-1}{\gamma} \int_{-1}^{1} z^*(\xi)\, s(\xi, \epsilon, \eta, v)\, d\xi \tag{57}$$

$$\tilde{v} = \frac{1}{\sqrt{1-\xi^2}} \int_{-1}^{1} G^{\dagger}(\xi,\tau)\, r(\tau, \epsilon, \tilde{\eta}, v)\, d\tau. \tag{58}$$

Then a careful inspection verifies that s is just the right hand side of (49) with the term $B_n y$ removed, and r is the right hand side of (49) with the term $B_n y$ replaced by $\tilde{B}_n y$, where $\tilde{B}_n$ indicates the matrix B_n defined by (47) with η replaced by $\tilde{\eta}$. Then r and s have the following properties. They are continuous in η, v, and ξ and behave like $\sqrt{1-\xi^2}$ as $\xi \to \pm 1$ provided $v(\xi)$ is bounded, and they satisfy the Lipschitz conditions:

$$\| s(\xi, \epsilon, \eta, v) - s(\xi, \epsilon, \zeta, w) \| \leq |\epsilon| \Phi_1 | \eta - \zeta | + |\epsilon| \Phi_2 \| v - w \| \tag{59}$$

$$\begin{aligned} &\| r(\xi, \epsilon, \tilde{\eta}, \eta, v) - r(\xi, \epsilon, \tilde{\zeta}, \zeta, w) \| \\ &\quad \leq \Phi_3 | \tilde{\eta} - \tilde{\zeta} | + |\epsilon| \Phi_4 | \eta - \zeta | + |\epsilon| \Phi_5 \| v - w \| . \end{aligned} \tag{60}$$

Finally we note that the Green's matrix is so well behaved at ± 1 that the norm

$$\left\| \frac{1}{\sqrt{1-\xi^2}} G^{\dagger}(\xi, \tau) \right\| \tag{61}$$

exists. Thus the procedure used in Chapter III to prove T_ϵ contracting works also for M, with $\| G^{\dagger} \|$ replaced by (61). It follows that M has a bounded fixed point and hence (11) (12) has a solution branch of the form (45) (46).

The asymptotic form of this solution branch is easily obtained for $\epsilon \to 0$.

$$x(\xi,\epsilon) = \epsilon \left[\frac{n+\frac{1}{2}}{n(n+1)(1+(\lambda_n-\nu)^2)}\right]^{\frac{1}{2}} \begin{pmatrix} 1 \\ \lambda_n-\nu \end{pmatrix} P_n^{(1)}(\xi) + O(\epsilon^2) \tag{62}$$

$$\rho(\epsilon) = \rho_n + \epsilon\frac{3}{2}\left(\frac{n+\frac{1}{2}}{n(n+1)}\right)^{3/2} \frac{1}{\sqrt{1+(\lambda_n-\nu)^2}} \int_{-1}^{1} \frac{\xi}{\sqrt{1-\xi^2}} P_n^{(1)}(\xi)^3 d\xi$$

$$+ O(\epsilon^2). \tag{63}$$

When n is odd, the integral in (63) is zero, but we are able to calculate the next term in the expansion for $\rho(\epsilon)$,

$$\rho(\epsilon) = \rho_n + \tfrac{1}{2}\,\epsilon^2\,\rho''(o) + O(\epsilon^3), \tag{64}$$

where

$$\rho''(0) = -\frac{(n+\frac{1}{2})^2(\lambda_n-\nu)^2}{n^2(n+1)^2\left[\left(\frac{\varkappa}{1-\nu^2}\right)(\lambda_n-\nu)^2-1\right]\left[1+(\lambda_n-\nu)^2\right]} \tag{65}$$

$$\cdot\int_{-1}^{1}\frac{\xi}{\sqrt{1-\xi^2}}P_n^{(1)}(\xi)^2\left\{\int_{-1}^{\xi}\frac{\zeta}{\sqrt{1-\zeta^2}}P_n^{(1)}(\zeta)^2\left[\frac{3}{n(n+1)}Q_n^{(1)}(\xi)P_n^{(1)}(\zeta)\right.\right.$$

$$\left.+\frac{\pi\left[\frac{1}{2}\left(\frac{\varkappa}{1-\nu^2}\right)(\lambda_n-\nu)^2+1\right]}{\mu(\mu+1)\sin\pi\mu}P_\mu^{(1)}(\xi)\,P_\mu^{(1)}(-\zeta)\right]d\zeta$$

$$+\int_{\xi}^{1}\frac{\zeta}{\sqrt{1-\zeta^2}}P_n^{(1)}(\zeta)^2\left[\frac{3}{n(n+1)}P_n^{(1)}(\xi)\,Q_n^{(1)}(\zeta)\right.$$

$$+ \frac{\pi [\frac{1}{2} (\frac{\varkappa}{1-\nu^2}) (\lambda_n - \nu)^2 + 1]}{\mu(\mu+1) \sin \pi\mu} \; P_\nu^{(1)}(-\xi) \, P_\nu^{(1)}(\zeta) \Big] \, d\zeta \Bigg\} \, d\xi . \tag{65}$$

Note that $\rho''(o)$ blows up when

$$\left(\frac{\varkappa}{1-\nu^2}\right) (\lambda_n - \nu)^2 = 1, \tag{66}$$

which is the condition that A_n have equal eigenvalues, (see 28)). The numerical solutions in [3] appear to confirm this singular behavior.

VI. 4. Degenerate Eigenpressures

As was pointed out in §VI. 2, the maximum possible degeneracy in this problem is 2, and this occurs when $\rho_n = \rho_m$ for some $n \neq m$, which is equivalent to saying that n and m satisfy (28) with $n \neq m$. We now assume this to be the case.

Then the algebraic bifurcation equations for this problem are:

$$\sum_{jk=1}^{2} F_{ijk} \, q_j \, q_k + \omega \sum_{j=1}^{2} C_{ij} \, q_j = 0$$
$$q_1^2 + q_2^2 = 1, \tag{67}$$

where the coefficients are defined by

$$C_{ij} = \left(\frac{1-\nu^2}{\varkappa}\right) \int_{-1}^{1} z_2^{(n_i)}(\xi) \, y_2^{(n_j)}(\xi) \, d\xi \tag{68}$$

$$F_{ijk}=\int_{-1}^{1} \frac{\xi}{\sqrt{1-\xi^2}} \left[\tfrac{1}{2} z_1^{(n_i)}(\xi) y_2^{(n_j)}(\xi) y_2^{(n_k)}(\xi) - \left(\frac{1-\nu^2}{\varkappa}\right) - \left(\frac{1-\nu^2}{\varkappa}\right) z_2^{(n_i)}(\xi) y_1^{(n_j)}(\xi) y_2^{(n_k)}(\xi) \right] d\xi \quad i,j,k = 1, \tag{69}$$

and we define $n_1 \equiv n$, $n_2 \equiv m$.

Now C is a diagonal matrix, so we can divide through and get the equivalent equations

$$A_{111}\, q_1^2 + (A_{112} + A_{121})\, q_1 q_2 + A_{122}\, q_2^2 + \omega\, q_1 = 0 \tag{70}$$

$$A_{211}\, q_1^2 + (A_{212} + A_{221})\, q_1 q_2 + A_{222}\, q_2^2 + \omega\, q_2 = 0 \tag{71}$$

$$q_1^2 + q_2^2 = 1, \tag{72}$$

Define constants Γ_n Φ_n and $\Theta(n,m)$ by

$$\Gamma_n \equiv \sqrt{\frac{n+\frac{1}{2}}{n(n+1)}} \quad , \tag{73}$$

$$\Phi_n \equiv \frac{1}{\sqrt{1+(\lambda_n-\nu)^2}} \tag{74}$$

$$\Theta(n,m) \equiv \int_{-1}^{1} \frac{\xi}{\sqrt{1-\xi^2}}\, P_n^{(1)}(\xi)^2\, P_m^{(1)}(\xi)\, d\xi\,. \tag{75}$$

Then the A_{ijk} can be written

$$A_{111} = \frac{3}{2}\,\Phi_n \Gamma_n^3\,\Theta(n,n)$$

$$A_{112} = \frac{3}{2}\,\Phi_m \left(\frac{\lambda_m - \nu}{\lambda_n - \nu}\right) \Gamma_n^2\,\Gamma_m\,\Theta(n,m)$$

$$A_{121} = \Phi_m \frac{\frac{1}{2}(\lambda_m - \nu) + (\lambda_n - \nu)}{\lambda_n - \nu}\,\Gamma_n^2\,\Gamma_m\,\Theta(n,m)$$

$$A_{122} = \frac{\Phi_m^2}{\Phi_n}\,\frac{(\lambda_m - \nu)\,[\frac{1}{2}(\lambda_m - \nu) + (\lambda_n - \nu)]}{(\lambda_n - \nu)^2}\,\Gamma_n\Gamma_m^2\,\Theta(m,n)$$

$$A_{211} = \frac{\Phi_n}{\Phi_m^2}\,\frac{(\lambda_n - \nu)\,[\frac{1}{2}(\lambda_n - \nu) + (\lambda_m - \nu)]}{(\lambda_m - \nu)^2}\,\Gamma_n^2\,\Gamma_m\,\Theta(n,m) \tag{76}$$

$$A_{212} = \Phi_n \frac{\frac{1}{2}(\lambda_n - \nu) + (\lambda_m - \nu)}{(\lambda_m - \nu)}\,\Gamma_n\,\Gamma_m^2\,\Theta(m,n)$$

$$A_{221} = \frac{3}{2}\,\Phi_n \left(\frac{\lambda_n - \nu}{\lambda_m - \nu}\right) \Gamma_m^2\,\Gamma_n\,\Theta(m,n)$$

$$A_{222} = \frac{3}{2}\,\Phi_m\,\Gamma_m^3\,\Theta(m,m)$$

If n and m are both odd, the coefficients A_{ijk} all vanish, which implies $\omega = 0$. If n and m are both even, the coefficients are all nonzero and (70) (71) (72) does not simplify. However if n is odd and m is even, we have

$$A_{111} = A_{122} = A_{212} = A_{221} = 0 \tag{77}$$

so the problem reduces to

$$(A_{112} + A_{121})\, q_1 q_2 + \omega q_1 = 0 \tag{78}$$

$$A_{211}\, q_1^2 + A_{222}\, q_2^2 + \omega q_2 = 0 \tag{79}$$

$$q_1^2 + q_2^2 = 1. \tag{80}$$

One solution to this is

$$q_1 = 0, \quad q_2 = 1 \quad , \omega = -A_{222}, \tag{81}$$

and a second solution is

$$q_1^2 = \frac{A_{112} + A_{121} - A_{222}}{A_{112} + A_{121} + A_{211} - A_{222}}$$

$$q_2^2 = \frac{A_{211}}{A_{112} + A_{121} + A_{211} - A_{222}} \tag{82}$$

$$\omega = -(A_{112} + A_{121})\, \mathrm{sgn}\,(q_2) \sqrt{\frac{A_{211}}{A_{112} + A_{121} + A_{211} - A_{222}}}$$

So the algebraic bifurcation equations have in general two distinct roots when n is odd and m is even.

Of course, the analogous situation occurs when n is even and m is odd.

VI. 5. Numerical Results

We take $\nu = 0.3$ and consider $\kappa = 10^{-3}$ (thick shell) and $\kappa = 10^{-5}$ (thin shell). Then the eigenpressures ρ_n are easily evaulated from (27) and (19). The numerical values of ρ_n are given in Table I of [3], where the symbol P_n is used instead of our ρ_n.

From § VI. 3 we see that the simple ρ_n are the bifurcation points of (11) (12).

The asymptotic formula of § VI. 3 gives us the slopes of the nontrivial branches at the bifurcation points. Use $\rho_n'(0)$ to designate $\frac{d}{d\epsilon}\rho(\epsilon)\Big|_{\epsilon=0}$ for each n. Then

$$\rho_n'(0) = \frac{3}{2}\left[\frac{n+\frac{1}{2}}{n(n+1)}\right]^{3/2} \frac{1}{\sqrt{1+(\lambda_n-\nu)^2}} \int_{-1}^{1} \frac{\xi}{\sqrt{1-\xi^2}} P_n^{(1)\,3}(\xi)\, d\xi. \quad (83)$$

Physically, it is of more interest to know how the radial displacement $u_2(\theta)$ defined by (5) varies with the pressure near a bifurcation point. This is easily calculated from

$$\left.\frac{d\|u_2\|_2}{d\rho}\right|_{\rho=\rho_n} = \frac{\left.\frac{d\|u_2\|_2}{d\epsilon}\right|_{\epsilon=0}}{\rho_n'(0)} \quad (84)$$

where from (5) and (62), using formulae in [16],

$$\left.\frac{d\|u_2\|_2}{d\epsilon}\right|_{\epsilon=0} = \left[\frac{\pi\, n(n+1)\,(n+\frac{1}{2})}{(1+(\lambda_n-\nu)^2)\; 2^{4n+1}} \sum_{k=0}^{n} \frac{(2k)!^2(2n-2k)!^2}{k!^4(n-k)!^4}\right]^{\frac{1}{2}} \tag{85}$$

Here $\| \quad \|_2$ designates the norm defined by

$$\|u_2\|_2 \equiv \left[\tfrac{1}{2}\int_0^{\pi} u_2(\theta)^2\, d\theta\right]^{\frac{1}{2}} \tag{86}$$

which is used in [3]. The slopes (84) have also been evaluated from the numerical solutions presented in [3], which were computed using a "shooting" method. We tabulate the values obtained by a direct evaluation of (84) using (83) and (85), along with the values from the numerical solutions in [3] for comparison. Note that $\rho_n'(0)$ is zero for all odd n, so odd n are not tabulated. Note also that formulae (83) (84) (85) are all independent of the thickness parameter $\varkappa$.

n	$\rho_n'(0)$	$\left.\frac{d\|u_2\|_2}{d\epsilon}\right\|_{\epsilon=0}$	$\left.\frac{d\|u_2\|_2}{d\rho}\right\|_{\rho=\rho_n}$ from (84)	Slope from Numerical solutions in [3]
2	.2590716	.5922635	2.28611	2.286
4	.0922344	.2965424	3.21510	3.214
6	.0514012	.2051203	3.99058	3.984
8	.0339498	.1585912	4.67135	4.67
10	.0245807	.1299478	5.28658	5.27
12	.01886128	.110395	5.85300	5.84
14	.01506605	.0961394	6.38119	6.38
16	.01239492	.0852572	6.87840	6.88
18	.01043059	.0766628	7.34981	7.35
20	.008936037	.0696946	7.79928	7.81

APPENDIX A

Notation Conventions and Definitions

Except for one or two cases where standard usage dictates otherwise, the following notation conventions have been adopted:

Greek capital letters represent positive real bounds.

Greek small letters represent real numbers.

Roman capital letters represent matrices and operators.

Script Roman capital letters represent sets and spaces.

Roman small letters a to h represent known or constant column vectors.

Roman small letters i to p represent integers.

Roman small letters q to z represent unknown or variable column vectors.

I is the $n \times n$ identity matrix.

* denotes the transpose for real matrices and vectors, and the adjoint for operators.

' denotes differentiation with respect to ξ.

$\equiv$ means "is identically equal to" or "is defined by".

R is the set of all real numbers.

R^n is the n-dimensional real vector space.

$C_n^i[\alpha, \beta]$ is the set of real n-dimensional vector functions with components i times continuously differentiable on $[\alpha, \beta]$.

The scalar product of two real vectors in R^n is

$$z^* y \equiv \sum_{i=1}^{n} z_i y_i \,.$$

The inner product of $z(\xi)$ and $y(\xi)$ in $C_n[\alpha, \beta]$ is

$$(z, y) \equiv \int_{\alpha}^{\beta} z^*(\xi)\, y(\xi)\, d\xi.$$

The order symbols O and o are defined as follows:

$$\eta(\epsilon) = O(\epsilon) \quad \text{as} \quad \epsilon \to 0$$

iff there exist positive constants Φ and δ such that

$$|\eta(\epsilon)| \leq \Phi\, |\epsilon| \qquad \text{for all} \quad |\epsilon| \leq \delta;$$

$$\eta(\epsilon) = o(\epsilon) \quad \text{as} \quad \epsilon \to 0$$

$$\text{iff } \frac{\eta(\epsilon)}{\epsilon} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

If M and N are $n \times n$ matrices, then $[M, N]$ represents the $n \times 2n$ matrix consisting of the n columns of M followed by the n columns of N.

We define

$$f(\tau^+) \equiv \lim_{\substack{\xi \to \tau \\ \xi > \tau}} f(\xi)$$

and

$$f(\tau-) \equiv \lim_{\substack{\xi \to \tau \\ \xi < \tau}} f(\xi)$$

Fréchet Differentials and Derivatives

The Fréchet derivative (or strong derivative) is defined as follows. For a thorough and rigorous treatment, see chapter VI in [35].

Let $\mathcal{L}$ and $\mathcal{L}^1$ be normed linear spaces, x and h be elements of $\mathcal{L}$, and g be a mapping of $\mathcal{L}$ into $\mathcal{L}^1$. Then g is said to be Fréchet differentiable at $x_o \in \mathcal{L}$ iff there exists a linear operator $G: \mathcal{L} \to \mathcal{L}^1$, which depends in general on x_o, such that

$$g(x_o+h) - g(x_o) = Gh + a(x_o, h)$$

where

$$\|a(x_o, h)\| = o(\|h\|) \text{ as } \|h\| \to 0.$$

Then Gh is called the Fréchet differential of g at the point x_o for the increment h, and is designated by $Dg(x_o, h)$. The linear operator G is called the Fréchet derivative of g at the point x_o and is denoted $g_x(x_o)$. Thus

$$Dg(x_o, h) = g_x(x_o)h \equiv Gh.$$

If $\mathcal{L}$ and $\mathcal{L}^1$ are n-dimensional linear spaces and we have the representations

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \quad g(x) = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix},$$

then $g_x(x_o)$ has the matrix representation $J(x_o)$, where

$$J_{ij}(x_o) \equiv \frac{\partial g_i(x_o)}{\partial x_j} \qquad i, j = 1, \ldots, n,$$

and $J(x_o)h$ is defined by the usual matrix multiplication.

Before we can define higher order differentials and derivatives, we must define homogeneous forms and polynomials in h, as in [35]. If $h_1, h_2, \ldots, h_m$ are in $\mathfrak{L}$, then a function

$$a_m(h_1, h_2, \ldots, h_m)$$

taking values in $\mathfrak{L}^1$, is called a m-termed linear form if it is linear and homogeneous in each of its arguments h_i, $i = 1, \ldots, m$. It is called symmetric if

$$a_m(h_1, h_2, \ldots, h_m) = a_m(h_{i_1}, h_{i_2}, \ldots, h_{i_m})$$

where $i_1, \ldots, i_m$ is an arbitrary permutation of the indices $1, 2, \ldots, m$. The norm of $a_m = a_m(h_1, h_2, \ldots, h_m)$ is defined by

$$\|a_m\| \equiv \sup \frac{\|a_n(h_1, h_2, \ldots, h_m)\|}{\|h_1\| \; \|h_2\| \ldots, \|h_m\|} \; .$$

Clearly the totality of m-termed linear forms a_m is a normed linear space.

The form $a_m(h, h, \ldots, h)$ obtained from a symmetric form $a_m(h_1, h_2, \ldots, h_m)$ by setting $h_1 = h_2 = \ldots = h_m = h$, is called a homogeneous form of degree m. It is generally abbreviated

$$a_m(h, h, \ldots, h) = a_m h^m.$$

Clearly $a_m(\xi h)^m = \xi^m a_m h^m$, and

$$\| a_m h^m \| \leq \| a_m \| \ \| h \|^m .$$

A sum

$$P_m(h) \equiv \sum_{k=1}^{m} a_k h^k ,$$

of homogeneous forms, all of which take values in $\mathcal{L}^1$, is called a polynomial in h of degree m.

Now we can define higher order differentials and derivatives. Let x, h, and g be as before. Suppose there exists a polynomial in h, $P_m(h)$, and a function $r_m(h) : \mathcal{L} \to \mathcal{L}^1$ such that

$$g(x_o + h) - g(x_o) = P_m(h) + r_m(h)$$

where $\| r_m(h) \| = o(\| h \|^m)$ as $\| h \| \to 0$.

Then g is said to be <u>n times Fréchet differentiable at x_o.</u> The polynomial $P_n(h)$ is called <u>Taylor's sum of degree m</u> for $g(x_o + h)$ and the m^{th} term multiplied by m! is called the <u>m^{th} Fréchet differential</u> of g at the point x_o, and is designated

$$D^m g(x_o, h) \equiv n! \ a_m h^m.$$

The corresponding symmetric m-termed linear form is $D^m g(x_o, h_1, h_2, \ldots, h_m) = m!\; a_m(h_1, h_2, \ldots, h_m)$. This linear form $m!\, a_m$ is called the <u>m^{th} Fréchet derivative</u> of g at x_o and is designated by $g_{x^m}(x_o)$. Thus

$$D^m g(x_o, h) = g_{x^m}(x_o)\, h^m = m!\; a_m h^m.$$

The Taylor sum for $g(x_o+h)$ gives therefore

$$g(x_o+h) - g(x_o) = g_x(x_o)\, h + \tfrac{1}{2} g_{xx}(x_o)\, h^2 + \ldots$$

$$+ \frac{1}{m!} g_{x^m}(x_o)\, h^m + r_m(h).$$

If g is a function with more than one argument, the definitions are extended in the obvious way; see §43 in [35].

If $\mathcal{L}$ and $\mathcal{L}^I$ are n-dimensional and x, h, and g have vector representations as before, then $y = D^m g(x_o, h)$ is an n-vector with components

$$y_i = \sum_{j_1, j_2, \ldots j_m = 1}^{n} \left(\frac{\partial g_i(x_o)}{\partial x_{j_1}\, \partial x_{j_2} \ldots \partial x_{j_m}} \right) h_{j_1} h_{j_2} \ldots h_{j_m} .$$

The norm of the m^{th} Fréchet derivative is defined by

$$\| g_{x^m}(x_o) \| \equiv \sup \frac{\| g_{x^m}(x_o) h^m \|}{\|h\|^m} .$$

In the n dimensional case, using maximum norms, this gives

$$\| g_{x^m}(x_o) \| = \max_{1 \le i \le n} \sum_{j_1, j_2 \ldots j_m = 1}^{n} \left| \frac{\partial^m g_i(x_o)}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_m}} \right| .$$

The Maximum Norms

Throughout this thesis we use the maximum norm and its related norms. They are defined as follows:

For $x, z \in R^n$,

$$\|x\| \equiv \max_{1 \le i \le n} |x_i|,$$

$$\|z\|_1 \equiv \sum_{i=1}^{n} |z_i| .$$

Then

$$z^* x \le \|z\|_1 \; \|x\|.$$

For $x, z \in C_n [\alpha, \beta]$,

$$\|x\| \equiv \max_{1 \le i \le n} \; \max_{\xi \in [\alpha, \beta]} |x_i(\xi)|$$

$$\|z\|_1 \equiv \sum_{i=1}^{n} \int_{\alpha}^{\beta} |z_i(\xi)| \; d\xi.$$

Then

$$(z, x) \leq \|z\|_1 \; \|x\|.$$

For a constant matrix M,

$$\|M\| \equiv \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left| M_{ij} \right| .$$

For a Green's matrix $G(\xi, \tau)$,

$$\|G\| \equiv \max_{1 \leq i \leq n} \; \max_{\xi \in [\alpha, \beta]} \; \sum_{j=1}^{n} \int_{\alpha}^{\beta} \left| G_{ij}(\xi, \tau) \right| d\tau.$$

The norms of the nonlinearity $f(\xi, \lambda, u)$ in the boundary-value problem of chapters III and IV, and its derivatives, are evaluated at $\lambda = \lambda_o$ and $u = 0$, and defined as follows:

$$\|f\| \equiv \max_{1 \leq i \leq n} \; \max_{\xi \in [\alpha, \beta]} \left| f_i \, (\xi, \lambda_o, 0) \right|$$

$$\|f_u\| \equiv \max_{1 \leq i \leq n} \; \max_{\xi \in [\alpha, \beta]} \sum_{j=1}^{n} \left| \frac{\partial f_i}{\partial u_i} (\xi, \lambda_o, 0) \right|$$

$$\|f_{uu}\| \equiv \max_{1 \leq i \leq n} \; \max_{\xi \in [\alpha, \beta]} \sum_{j=1}^{n} \sum_{k=1}^{n} \left| \frac{\partial^2 f_i}{\partial u_j \partial u_k} (\xi, \lambda_o 0) \right|$$

$$\|f_{u\lambda}\| \equiv \max_{1 \leq i \leq n} \; \max_{\xi \in [\alpha, \beta]} \sum_{j=1}^{n} \left| \frac{\partial^2 f_i}{\partial u_j \partial y} (\xi, \lambda_o, 0) \right| .$$

Sometimes we need bounds on f and its derivatives when the arguments are allowed to range over a closed bounded set S .

Therefore we define

$$\|f\|_S \equiv \max_{1 \leq i \leq n} \max_{(\xi, \lambda, u) \in S} \left| f_i(\xi, \lambda, u) \right|$$

$$\|f_u\|_S \equiv \max_{1 \leq i \leq n} \max_{(\xi, \lambda, u) \in S} \sum_{j=1}^{n} \left| \frac{\partial f_i}{\partial u_j} (\xi, \lambda, u) \right|$$

$$\|f_{uu}\|_S \equiv \max_{1 \leq i \leq n} \max_{(\xi, \lambda, u) \in S} \sum_{j=1}^{n} \sum_{k=1}^{n} \left| \frac{\partial^2 f_i}{\partial u_j \partial u_k} (\xi, \lambda, u) \right|$$

$$\|f_{u\lambda}\|_S \equiv \max_{1 \leq i \leq n} \max_{(\xi, \lambda, u) \in S} \sum_{j=1}^{n} \left| \frac{\partial^2 f_i}{\partial u_j \partial \lambda} (\xi, \lambda, u) \right|$$

These bounds exist if the corresponding derivatives are defined and continuous on S. They satisfy the characteristic property of operator norms, e.g.:

$$\|f_{uu}(\xi, \lambda, u)\, u^2\| \leq \|f_{uu}\|_S \, \|u\|^2 \quad \text{for } (\xi, \lambda, u) \in S.$$

The m^{th} Frechet derivative $g_{x^m}(x)$ is said to be <u>Lipschitz continuous</u> in x on some set $\mathcal{D}$ iff

$$\|g_{x^m}(x^{(1)}) - g_{x^m}(x^{(2)})\| \leq \Phi \, \| x^{(1)} - x^{(2)} \|$$

for all $x^{(1)}$ and $x^{(2)} \in \mathcal{D}$.

APPENDIX B

The Contracting Mapping Theorem

Let $\mathcal{B}$ be a Banach space (complete normed linear space) and $\mathcal{N}$ be the neighborhood of the origin 0 of $\mathcal{B}$ defined by

$$\mathcal{N} = \left\{ x \in \mathcal{B} \quad \|x\| \leq \Phi \right\} . \tag{1}$$

Let T be a mapping of $\mathcal{N}$ into $\mathcal{B}$. Suppose that for some constant $0 < \alpha < 1$, T satisfies the Lipschitz condition

$$\| Tx - Ty \| \leq \alpha \|x-y\| \quad \forall \quad x,y \in \mathcal{N}, \tag{2}$$

and also satisfies

$$\| T \cdot 0 \| \leq (1-\alpha) \Phi. \tag{3}$$

Then T has exactly one fixed point, say $\tilde{x}$, in $\mathcal{N}$, and furthermore $\tilde{x}$ is the limit of the sequence defined by

$$x^{(o)} = 0$$

$$x^{(n+1)} = T x^{(n)} \qquad n = 0, 1, 2, \ldots .$$

The convergence of this sequence is given by

$$\|x^{(n)} - \tilde{x}\| \leq \alpha^n \Phi .$$

For proof of this theorem, see page 30 in [23] or page 27 in [35].

Note that if T is known to map $\mathcal{N}$ into itself, that is if

$$Tx \in \mathcal{N} \text{ for all } x \in \mathcal{N}, \tag{4}$$

then condition (3) is no longer needed.

APPENDIX C

BASIC ALTERNATIVE THEOREM FOR THE BUCKLING SPHERE PROBLEM

Let T and A_n be as defined in Chapter VI, and take $g(\xi) \in C_2[\alpha, \beta]$. Consider the inhomogeneous problem

$$[T - A_n]\ x(\xi) = g(\xi) \tag{1}$$

$$x(-1) = x(1) = 0, \tag{2}$$

and the homogeneous adjoint problem

$$[T - A_n^*]\ z(\xi) = 0 \tag{3}$$

$$z(-1) = z(1) = 0. \tag{4}$$

Define inner products as in Appendix A. The following result is used in Chapter VI.

Theorem:

Problem (1) (2) has a solution if and only if

$$(z, g) = 0 \tag{5}$$

for all solutions $z(\xi)$ of problem (1) (2).

Proof:

The proof of necessity is trivial. Assume x is a solution to (1) (2) and z is a solution to (3) (4). Then

$$(z, g) = (z, [T - A_n]x)$$

$$= ([T - A_n^*] z, x) \qquad (6)$$

$$= 0.$$

To prove sufficiency, diagonalize A_n using the nonsingular matrix S_n defined in § VI. 2, and consider the scalar problems

$$[L - \lambda_n] w_1(\xi) = h_1(\xi) \qquad (7)$$

$$w_1(-1) = w_1(1) = 0, \qquad (8)$$

$$[L - \lambda_\mu] w_2(\xi) = h_2(\xi) \qquad (9)$$

$$w_2(-1) = w_2(1) = 0. \qquad (10)$$

Here λ_n is an eigenvalue of L and λ_μ is not and $h = S_n^{-1} g$. Let $\varphi(\xi)$ be an eigensolution of the self-adjoint problem

$$[L - \lambda_n] \varphi(\xi) = 0 \qquad (11)$$

$$\varphi(-1) = \varphi(1) = 0. \qquad (12)$$

Then $\varphi(\xi) = P_n^{(1)}(\xi)$.

Proceeding formally by the method of variation of parameters, we get as a candidate for the solution of (7) (8)

$$w_1(\xi) = \frac{Q_n^{(1)}(\xi)}{J(P_n^{(1)}, Q_n^{(1)})} \int_{-1}^{\xi} h_1(\xi)\, P_n^{(1)}(\xi)\, d\xi$$

$$- \frac{P_n^{(1)}(\xi)}{J(P_n^{(1)}, Q_n^{(1)})} \int_{-1}^{\xi} h_1(\xi)\, Q_n^{(1)}(\xi)\, d\xi, \tag{13}$$

where

$$J(P_n^{(1)}, Q_n^{(1)}) \equiv -(1-\xi^2)\,(P_n^{(1)}\, Q_n^{(1)'} - Q_n^{(1)}\, P_n^{(1)'})$$

$$= -n(n+1) \tag{14}$$

is the conjunct of $P_n^{(1)}$ and $Q_n^{(1)}$.

A consideration of the asymptotic properties of $P_n^{(1)}(\xi)$ and $Q_n^{(1)}(\xi)$ as $\xi \to \pm 1$, as given in [16], verifies that both terms in (13) are bounded on $[-1, 1]$. In fact the second term vanishes at ± 1 without any special conditions on $h_1(\xi)$ other than continuity. Similarly the first term vanishes as $\xi \to -1$, using only the asymptotic properties of $Q_n^{(1)}(\xi)$ and $P_n^{(1)}(\xi)$. However, we use the orthogonality condition

$$\int_{-1}^{1} h_1(\xi)\, P_n^{(1)}(\xi)\, d\xi = 0 \tag{15}$$

to show that the first term in (13) vanishes as $\xi \to +1$:

$$\lim_{\xi\to 1}\left\{Q_n^{(1)}(\xi)\int_{-1}^{\xi} h_1(\sigma)\, P_n^{(1)}(\sigma)\, d\sigma\right\}$$

$$= \lim_{\xi\to 1}\left\{0 - Q_n^{(1)}(\xi)\int_{\xi}^{1} h_1(\sigma)\, P_n^{(1)}(\sigma)\, d\sigma\right\}$$

$$= \lim_{\epsilon\to 0} - \left[-\frac{1}{\sqrt{2\epsilon}} + O(1)\right]\int_0^{\epsilon}\left[h_1(1) + o(1)\right]\left[-n(n+1)\sqrt{\frac{\delta}{2}} + O(\delta)\right] d\delta \tag{16}$$

$$= \lim_{\epsilon\to 0} - \left[\frac{1}{3} h_1(1)\, n(n+1)\, \epsilon + o(\epsilon)\right]$$

$$= 0 .$$

A straightforward substitution shows that (13) satisfies (7). Hence (13) satisfies (7) (8) provided that the orthogonality condition (15) holds.

Similarly we can show that (9) (10) has a solution $w_2(\xi)$ for all $h_2(\xi) \epsilon\, C[-1, 1]$, assuming λ_μ is not an eigenvalue.

Now transform back to the original problem (1) (2). Then a solution of (1) (2) is

$$x(\xi) = S_n\, w(\xi)\,, \tag{17}$$

and the orthogonality condition (15) becomes

$$\begin{aligned} 0 &= (p, h) \\ &= (p, S_n^{-1} g) \\ &= (S_n^{-1*} p, g) \end{aligned} \tag{18}$$

where we have defined $p(\xi) = \begin{pmatrix} P_n^{(1)}(\xi) \\ 0 \end{pmatrix}$.

But $S_n^{-1*}\, p$ is just an eigensolution of (3) (4), so the theorem is proved.

APPENDIX D

GENERALIZED GREEN'S MATRIX FOR THE BUCKLING SPHERE PROBLEM

We calculate a generalized Green's matrix for problem (13) (14) of Chapter VI, with ρ equal to a simple eigenpressure ρ_n. The procedure is to first diagonalize (13) (14), then find the appropriate Green's functions for the two scalar problems in (23) (24), and then transform back to the original problem.

$$T\, y(\xi) = A(\rho_n)\, y(\xi) \tag{1}$$

$$y(-1) = y(1) = 0 . \tag{2}$$

Let $y = S_n w$, then

$$Lw_1 = \lambda_n w_1 , \quad w_1(-1) = w_1(1) = 0 \tag{3}$$

$$Lw_2 = \lambda_\mu w_2 , \quad w_2(-1) = w_2(1) = 0 . \tag{4}$$

The generalized Green's function for (3) is

$$\begin{aligned} G^{(1)}(\xi,\tau) = {} & - \begin{cases} \dfrac{P_n^{(1)}(\tau)Q_n^{(1)}(\xi)}{n(n+1)} & -1 \leq \tau \leq \xi \leq 1 \\[2ex] \dfrac{P_n^{(1)}(\xi)Q_n^{(1)}(\tau)}{n(n+1)} & -1 \leq \xi \leq \tau \leq 1 \end{cases} \\ & + \frac{n+\frac{1}{2}}{n^2(n+1)^2} P_n^{(1)}(\tau) \left\{ Q_n^{(1)}(\xi) \int_{-1}^{\xi} P_n^{(1)}(\sigma)^2 d\sigma + P_n^{(1)}(\xi) \int_{\xi}^{1} P_n^{(1)}(\sigma) Q_n^{(1)}(\sigma) d\sigma \right\} \\ & + \frac{n+\frac{1}{2}}{n^2(n+1)^2} P_n^{(1)}(\xi) \left\{ Q_n^{(1)}(\tau) \int_{-1}^{\tau} P_n^{(1)}(\sigma)^2 d\sigma + P_n^{(1)}(\tau) \int_{\tau}^{1} P_n^{(1)}(\sigma) Q_n^{(1)}(\sigma) d\sigma \right\} \end{aligned} \tag{5}$$

$$+2P_n^{(1)}(\xi)\, P_n^{(1)}(\tau)\, \frac{(n+\frac{1}{2})^2}{n^3(n+1)^3} \int_{-1}^{1}\int_{\sigma}^{1} P_n^{(1)}(\sigma)^2 P_n^{(1)}(\zeta) Q_n^{(1)}(\zeta)\, d\zeta\, d\sigma \ .$$

The Green's function for (4) is

$$G^{(2)}(\xi,\tau) = \begin{cases} \dfrac{\pi P_\mu^{(1)}(\xi) P_\mu^{(1)}(-\tau)}{2\mu(\mu+1)\sin\pi\mu} & -1 \leqslant \tau \leqslant \xi \leqslant 1 \\[2ex] \dfrac{\pi P_\mu^{(1)}(-\xi) P_\mu^{(1)}(\tau)}{2\mu(\mu+1)\sin\pi\mu} & -1 \leqslant \xi \leqslant \tau \leqslant 1 \end{cases} \qquad (6)$$

Then the generalized Green's matrix for (1) (2) is

$$G^{\dagger}(\xi,\tau) \equiv S_n \begin{pmatrix} G^{(1)}(\xi,\tau) & 0 \\ 0 & G^{(2)}(\xi,\tau) \end{pmatrix} S_n^{-1} \qquad (7)$$

where S_n is defined by (38) of Chapter VI.

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