NEARLY FREE MOLECULAR HEAT TRANSFER FROM A SPHERE

Thesis by

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ABSTRACT

Consider a sphere immersed in a rarefied monatomic gas with zero mean flow. The distribution function of the molecules at infinity is chosen to be a Maxwellian. The boundary condition at the body is diffuse reflection with perfect accommodation to the surface temperature. The microscopic flow of particles about the sphere is modeled kinetically by the Boltzmann equation with the Krook collision term. Appropriate normalizations in the near and far fields lead to a perturbation solution of the problem, expanded in terms of the ratio of body diameter to mean free path (inverse Knudsen number). The distribution function is found directly in each region, and intermediate matching is demonstrated. The heat transfer from the sphere is then calculated as an integral over this distribution function in the inner region. Final results indicate that the heat transfer may at first increase over its free flow value before falling to the continuum level.

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PART I

INTRODUCTION TO THE PROBLEM

Perturbation methods are of special interest to those who seek to analyze problems involving physically interesting phenomena in the field of rarefied gases. Such methods in general attempt to find a class of solutions dependent upon a set of parameters (usually just one) which take on certain limiting values. By making use of the size of these parameters, one may obtain a set of ordered equations to solve, which hopefully will be simpler than the original one. Many examples of such applications of perturbation theory appear in the literature, but they seem to have had limited use in solving problems in rarefied gas dynamics.

A rarefied gas is one in which the mean free path (λ), which is a measure of how far molecules travel between collisions, is large compared to other length scales in the particular problem being solved. Thus, λ is a natural parameter on which to base some sort of perturbation scheme.

It is virtually impossible to consider in detail the exact microscopic behavior of gases, but because of the large number of molecules usually present in the flow field one is led to a probabilistic formulation of the problem. This is the realm of kinetic theory.

The basic aim of kinetic theory is to find the probability distribution function of the gas. This function, usually denoted by the symbol f, is a six dimensional density measuring how many molecules lie in the cross product of a three dimensional increment of physical space $(\Delta x, \Delta y, \Delta z)$ with a three dimensional increment of velocity space $(\Delta \xi_x, \Delta \xi_y, \Delta \xi_z)$ where x, y, and z locate the molecules in a cartesian coordinate system and ξ_x, ξ_y, and ξ_z describe their velocities with respect to those cartesian axes. From this distribution function may be found the various physically interesting quantities such as density, temperature, and macroscopic velocity. These quantities are referred to as moments of f and are defined by the following integrals:

$$\rho = \int f \, d^3 \xi$$

$$\rho \vec{u} = \int \vec{\xi} f \, d^3 \xi$$

$$\rho RT = \int \tfrac{1}{3} (\vec{\xi} - \vec{u})^2 f \, d^3 \xi$$

To obtain an equation for f certain restrictions are imposed: 1) binary collisions between molecules, 2) molecular chaos, 3) a finite molecular cross section, and 4) the assumption that f varies slowly over a molecular dimension. [1, p.8] These limitations lead to the derivation of the famous Boltzman equation for the distribution function of the gas. This equation may be written symbolically

$$\frac{\partial f}{\partial t} + \xi_x \frac{\partial f}{\partial x} + \xi_y \frac{\partial f}{\partial y} + \xi_z \frac{\partial f}{\partial z} = \left.\frac{\delta f}{\delta t}\right|_{coll.}$$

in cartesian coordinates for no external force field. It is to be noted that this form of the equation holds only in cartesian coordinates, and as shown later it must be transformed carefully into other coordinate systems.

The term $\left.\frac{\partial f}{\partial t}\right|_{coll.}$ denotes the change in f along a molecular trajectory due to collisions between the particles. It may be expressed in terms of an integral involving f and thus the Boltzmann equation is actually an integro-differential one. The complexity of the collision integral makes exact solution all but impossible except in certain very special cases such as the two dimensional gas studied by Chahine. [2] The linearized, non-steady case has been considered extensively and Sirovich has found formal solutions for the initial-value problem. [3]

In order that solutions of the Boltzmann equation be found it has been proposed that the collision term be replaced by expressions which model its behavior to the extent that reasonable results may be expected. [4] To be a good model certain basic requirements must be met. [5, p. 66] First, mass, momentum, and energy must be conserved. In terms of the collision term this means that it must possess the same five collisional invariants as the full collision term,

$1, |\vec{\xi}|^2, \xi_x, \xi_y, \xi_z$. Symbolizing the invariant by ϕ and the collision term by δf these conditions are summarized by the equation

$$\int \phi \, \delta f \, d^3 \xi = 0$$

This guarantees that the same macroscopic equations of motion will be obtained from the model equation as from the full Boltzmann equation when the appropriate moments are taken. Second, we require that the distribution function should tend towards a Maxwellian in the equilibrium situation. Third, application of a Maxwell distribution to the model should produce results which agree as much as possible with the exact solution for the same case. Once these general requirements are satisfied, it may be necessary to apply further ones based on the particular problem being studied.

1.1 The Krook Model

Perhaps one of the most famous models of the Boltzmann equation is that proposed by Bhatnager, Gross , and Krook. This representation is most often referred to as the Krook Equation and it has found much use in the solution of gas dynamical problems. The Boltzmann collision term contains the exact intermolecular force field, which in practice is not known a priori, although one may think that it should be. In place of this exact field some sort of model is usually proposed which brings out the essential features of binary encounters and which duplicates to some reasonable extent the expected physical properties of the gas such as viscosity and heat conductivity. The assumption of the Krook model, however, is that no fine prescription of the force field is needed and that only a statistical model for the scattering process is required. [6, p. 1318]

Such a model is given by the expression

$$\left.\frac{\delta f}{\delta t}\right|_{\text{coll.}} = A\rho\,(f_m - f)$$

where ρ is the density, f_m is a Maxwellian evaluated at the local density, temperature, and velocity and A is a parameter dependent on the state of the gas. Narasimha presents an argument which leads to the conclusion that

$$A = \frac{\bar{c}}{\rho\lambda}$$

$$\bar{c} = \text{mean thermal velocity of the gas}$$

for hard sphere molecules. [5, pp. 84-85] Using this fact one arrives at the so-called single-relaxation time Krook equation,

$$\left.\frac{\delta f}{\delta t}\right|_{\text{coll.}} = \overline{\nu}\,(f_m - f)\ ,$$

by noting that $\bar{c}/\lambda$ is the collision frequency ν and presuming that some average value of ν, $\bar{\nu}$, may be substituted for ν. The name is derived from the fact that all moments of the distribution function f except the density, temperature, and velocity decay on a single time scale. Note also that relaxation to the equilibrium Maxwellian, prescribed by the Krook equation, satisfies one of the requirements of a good model. Of course, the assumption of being able to substitute an average value for ν would not be valid for any flow with wide variations in the state of the gas, such as those present in a shock wave. [6, p. 1318]

1.2 Methods of Solution

When a problem and a model equation have been chosen, the method of solution must be considered. The particular approach depends on many different factors. Generally, however, some sort of reasonable linearization is necessary to overcome the highly non-linear nature of the basic equation. In steady state problems, such as the one to be examined in this thesis, careful consideration must be given before proposing an expansion of the solution.

The sort of linearization involving the mean free path as a parameter often leads to a perturbation method for the solution. There are two distinctly important limits on λ. If a is some typical physical dimension in the problem being studied, such as body diameter or flat plate spacing, one usually defines the so-called Knudsen number

$$K = \lambda/a$$

or

$$\epsilon = 1/K = a/\lambda$$

We will confine our attention to ϵ. As $\epsilon \to 0$ the mean free path becomes quite large and the flow approaches what is termed free flow in the limit $\epsilon = 0$. Free flow is characterized by complete negligibility of inter-molecular collisions. All physical quantities such as density, temperature, and velocity may be calculated by strictly geometrical considerations. On the other end of the spectrum is the case $\epsilon \to \infty$, the continuum limit, where intermolecular collisions dominate. In the limit $\epsilon = \infty$ the need for a statistical model vanishes and the flow properties may be calculated strictly from the equations of motion such as the Navier-Stokes equations.

For large ϵ the famous Hilbert expansion is applied directly to the Boltzmann equation. After appropriate scaling of the physical space by λ one obtains

$$\frac{\partial f}{\partial t} + \vec{\xi} \cdot \nabla f = \epsilon \, \delta f$$

and Hilbert looks for solutions of the form

$$f = f_0 + \frac{1}{\epsilon} f_1 + \frac{1}{\epsilon^2} f_2 + \cdots$$

Inserting this expansion into the equation one finds that the successive terms are determined uniquely when the initial fluid conditions are specified.

Enskog also proposed a more general solution

$$f = f_0 + f_1 + f_2 + \cdots$$

requiring that

$$\sum_{i=0}^{n} f_i$$

be a solution for n = 0, 1, 2, ... of certain subdivisions of the collision operator. The division, of course, is not unique and is made in such a way that the resultant equations are soluble. It is in fact somewhat of an iteration scheme, where it may be shown that f_0 is a Maxwellian. [7, p. 109]

For small ϵ many procedures have been suggested. One of these is the moment method. First a representative form for the distribution function is chosen as a function of $\vec{\xi}$ and the basic moments: ρ, T, $\vec{u}$. Then this form of the solution is put into the model equation and the various moments are taken, arriving at a set of equations to solve for these unknown quantities. This method has been used notably by Lees [8] in rarefied gas heat transfer and his results agree remarkably with experiments performed by Takao. [9]

Alternatively one may take the approach suggested by Grad[10] and used by Rose [11] when the analysis involves steady flow about a body which is Maxwellian at infinity. The model equation uses Krook's collision term and is linearized about the distribution function and moments at infinity. Being essentially a far field analysis, the body is represented in the flow field by a source term in the equation whose strength may be calculated approximately by a knowledge of the zero order solution near the body (free flow). The Fourier transform of the linearized equation is taken resulting in an expression for the perturbation to f, which is then integrated to provide a set of equations for the moments. These moments depend directly on the source strength. This set of equations is then solved using the approximate source term. If the moments found are put back into the model equation, it can be solved directly for f by integrating along the characteristic curves determined by the geometry of the problem. A new source term may now be calculated to iterate the moment solutions previously found. A comment on the method as used by Rose appears at the end of this thesis.

In a well known paper by Baker and Charwat [12] the method of "first collisions" is used to obtain the first correction to the drag of a sphere in nearly free molecular flow over its value for free flow.

Their approach is a combination of geometrical considerations and elementary scattering theory which attempts to count certain dominant types of collisions, such as those between oncoming molecules and ones emitted from the surface of the sphere, which seem to contribute the majority of the first order drag perturbation. This seems to be a rather crude approach and the result is questioned by Willis. [13]

The Boltzmann equation is used by Liu, Pang, and Jew [14] to solve the problem posed above by Baker and Charwat. They apply the method of Knudsen iteration in which the exact collision integral is calculated using the free flow solution, allowing the equation to be solved for the first order perturbation. The non-uniform validity of the free flow solution limits the validity of this solution to a region near the body, and in fact only the first correction may be calculated in three dimensions by iterating from free flow. [1, p. 5]

Another method to solve the problem of sphere drag was presented by Rose in her Ph.D. thesis [15] but being an approach similar to that of Baker and Charwat leaves its validity open to the same questions.

1.3 Choice of the Problem

The original intent of this investigation was to solve the problem of high Mach number nearly free molecular drag of a sphere, chosen because of its interesting physical applicability. The aim was to apply a straightforward, mathematically sound procedure to some good model equation and to obtain a result which could be compared with known results, both experimental and theoretical. This problem at first proved intractable and it was decided to develop a successful method on a simpler problem which then could be used to solve the more difficult one. This thesis examines the heat transfer of a sphere immersed in a rarefied gas at rest macroscopically. The problem is posed and a method of solution is presented in the next part.

PART II

SPECIFICATION OF THE PROBLEM AND ITS SOLUTION

Consider a sphere immersed in a rarefied monatomic gas with zero mean flow. Let the molecules an infinite distance from the body be governed by a Maxwell distribution. Further, let the density and temperature at infinity be specified along with the body temperature. It is desired to calculate the heat transfer from or to the sphere.

Before proceeding with the problem it is necessary to chose a model equation, a method of solution, and a boundary condition to prescribe the re-emission of the molecules from the surface of the body.

2.1 Surface Condition

Two general types of surface conditions are usually considered: specular reflection and diffuse reflection. In the former the molecular velocity component parallel to the body is left unaffected by the surface interaction while that perpendicular is reversed in sign but keeps the same magnitude. Rose points out [15] that this approximation is valid when the deBroglie wave length of the incident particles is either equal to or greater than the surface irregularities or equal to the grating space of a scattering crystal. In diffuse reflection the incident molecules are "absorbed" by the surface and instantaneously re-emitted with a Maxwell distribution at some temperature between the incident temperature and that of the surface. It seems experimentally that the assumption of diffuse reflection is valid for many engineering surfaces and common gases [16, pp. 10-11]; thus, it will be assumed in this problem. In addition, the thermal accommodation coefficient, which indicates how much the incident molecules are "accommodated" to the surface temperature is very close to unity for most of the above surfaces, so it is reasonable to assume reflection at the surface temperature (perfect accommodation).

2.2 Model Equation

It is well recognized that the Krook equation satisfies the desirable properties of a good model of the Boltzmann equation. Further, it is particularly applicable when the flow properties do not vary too widely over the physical region in which a solution is desired. Finally, a quote from Narasimha is appropriate here.

"It may be concluded, therefore that the Krook model, in spite of its simplicity, describes the real behavior of gases fairly well both near the continuum limit and near the free molecule limit. Use of the model throughout the range between gasdynamics and gaskinetics may therefore be reasonably expected to be justified and worthwhile." [5, p. 85]

For the above reasons the Krook equation was chosen as the model for this problem. Constant collision frequency is assumed so we will be concerned with the single relaxation simulation.

2.3 Method of Solution

The usual method of solution of a problem of this type would be to apply some sort of far field or near field analysis to obtain the heat transfer. The former may require the specification of a source term such as Rose uses and an application of Fourier transforms, while in the latter case it could be appropriate to iterate the known free flow solution by means of a technique such as Knudsen iteration.

The inadequacy in these methods is that they only provide a first correction and one which is not uniformly valid in the flow field. One is led to conclude, therefore, that this is a good opportunity to make use of the inner-outer variable expansion procedure and the matching condition of Kaplun and Lagerstrom. [17]

Basically, the idea behind this perturbation theory is to choose (1) appropriate normalizations of the variables of the problem, valid respectively in the region close to the body (the inner region) and in the region far from the body (the outer region), along with (2) a parameter which may be used as the basis of an asymptotic expansion of the solution. The correct choice of the above will lead to two sets of equations, ordered in the parameter, one valid in each of the two regions defined. When solutions have been found, it is hoped that there will be some common region of validity, defined through a limit applied to the equations. Intuitively, this amounts to letting the inner solution expand to infinity while letting the outer one contract to the origin, to find the terms common to both expansions. More exactly it means defining an intermediate limit and an intermediate set of variables which remain fixed under this limit, so that the common terms may be evaluated. Very often this allows certain unknown functions in either region to be evaluated. A uniformly valid solution may then be constructed by adding together the outer and inner solutions and subtracting out the common part.

The parameter to be used in the expansions is the ratio of mean free path to body diameter, ϵ, the inverse Knudsen number defined previously. Since the distribution of molecules far from the body tends towards a Maxwellian defined by the density and temperature at infinity, it may be thought adequate at first to linearize about this infinity Maxwellian, f_∞, in the outer region. It is found, however, that this is not a good choice due to the non-uniform approach to f_∞ in velocity space. This occurs apparently through the propagation of the discontinuity in the free flow solution into the outer region. There is, in fact, a finite contribution to the density and temperature perturbations which depends solely on this non-uniformity. In addition, it is assumed often that one may linearize about the free flow solution, with f_∞ governing the incoming distribution of particles, in the inner region. This usual assumption is verified mathematically in the present work by assuming an arbitrary incoming function and showing through intermediate matching that it indeed must be f_∞.

Physically one would require that the solution very close to the sphere be governed by the free flow equations, while that far away must tend toward the continuum solution given by the linearized equations of motion derived from the model equation. This is shown to be true in the analysis of the next part.

The equations for the outer region as well as those for the inner region are solved in a recursive manner by integrating along the characteristics determined by the strictly geometrical considerations of the problem. Note that no equation of state is presumed for the gas. Also, although the equations of motion for the continuum region are displayed, they are unnecessary for the solution of the problem, as shown after the matching is demonstrated.

Part III examines the foregoing considerations in detail and presents the complete solution to the problem. Because of the form of the solution for the distribution function, consisting of many complicated expressions left in integral form, both in the inner and outer regions, no uniformly valid solution for f is written down explicitly. Matching is demonstrated exactly, however and the heat transfer is found to order ϵ^2.

Part III

CALCULATION OF THE DISTRIBUTION FUNCTION

3.1 Transformation of the Basic Equation

This problem will be solved using the Krook model of the Boltzmann equation. Two basic forms of the equation will be considered here. These will be of use later. Written in spherical coordinates (see Appendix I), making use of the spherical symmetry of the problem in (r, θ, ϕ) space,

$$\xi_r \frac{\partial f}{\partial r} + \frac{\xi_\theta^2 + \xi_\phi^2}{r} \frac{\partial f}{\partial \xi_r} + \frac{\xi_\varphi^2 \cot\theta - \xi_r \xi_\theta}{r} \frac{\partial f}{\partial \xi_\theta}$$

$$- \xi_\phi \frac{\xi_r + \xi_\theta \cot\theta}{r} \frac{\partial f}{\partial \xi_\varphi} = h_i \quad , \qquad i = 1, 2$$

$$h_1 = h_1(\xi, r)$$

$$h_2 = h_2(\xi, r) - f \tag{3.1}$$

ξ is the vector quantity $(\xi_r, \xi_\theta, \xi_\phi)$. Making use of the symmetry expected in $(\xi_r, \xi_\theta, \xi_\phi)$ space write

$$f(\xi_r, \xi_\theta, \xi_\varphi; r) = F(r, \xi_r, \xi_T) \qquad \xi_T = (\xi_\theta^2 + \xi_\phi^2)^{1/2}$$

Now apply this transformation to the equation. The resulting expression is given by

$$\xi_r \frac{\partial F}{\partial r} + \frac{\xi_T^2}{r} \frac{\partial F}{\partial \xi_r} - \frac{\xi_r \xi_T}{r} \frac{\partial F}{\partial \xi_T} = h_i \tag{3.2}$$

This equation will be integrated along a characteristic curve, but to facilitate the calculations to follow, make the further transformation

$$F(r, \xi_r, \xi_T) = g(w, \xi_r, \xi_T)$$

$$w = - r\xi_r/\xi^2$$

Carrying out the transformation

$$- \frac{\partial g}{\partial w} - \frac{\xi_T^2 \xi_r}{\xi^2 w} \frac{\partial g}{\partial \xi_r} + \frac{\xi_r^2 \xi_T}{\xi^2 w} \frac{\partial g}{\partial \xi_T} = h_i \quad , \quad i = 1, 2$$

$$h_1 = H_1(\xi, w)$$

$$h_2 = H_2(\xi, w) - g \tag{3.3}$$

The characteristic equations for (3.3) are

$$\frac{dw}{-1} = \frac{-d\xi_r}{\xi_T^2 \xi_r} \xi^2 w = \frac{d\xi_T}{\xi_r^2 \xi_T} \xi^2 w = \frac{dg}{h_i}$$

Solving this system we obtain the following set of results

$$\xi_r^2 + \xi_T^2 = \bar{c}_1^2 \tag{3.4}$$

$$\frac{w\xi_T}{\xi_r} = \bar{c}_2 \tag{3.5}$$

$$g = - \int H_1(\xi, w)\, dw + k_1 \quad \text{for } i = 1 \tag{3.6}$$

$$g = -e^w \int H_2(\xi, w)\, e^{-w} dw + k_2 e^w \quad \text{for } i = 2 \tag{3.7}$$

Note

$$k_1 = k_1(\bar{c}_1, \bar{c}_2)$$
$$k_2 = k_2(\bar{c}_1, \bar{c}_2) \qquad (3.8)$$

The following section discusses division of the problem into two regions in the spirit of the perturbation methods used in many fluids problems. The small parameter in this case is a/λ, the ratio of body size to mean free path, which tends to zero as the gas becomes more and more rarefied.

3.2 The Inner Region

The region near the body where distances are measured in terms of the body diameter and are small compared to λ is referred to as the inner region. The appropriate normalization, linearization and resultant equations are discussed below.

Normalization,

$$\tilde{r} = r/a$$
$$\tilde{\xi} = \xi/c_\infty$$
$$\tilde{\rho} = \rho/\rho_\infty$$
$$\tilde{T} = (T/T_\infty)\tau = T/T_a$$
$$\tilde{f} = f/A \qquad A = \rho_\infty/(2\pi R T_\infty)^{3/2}$$

The Krook equation is written

$$\xi \cdot \nabla f = \nu(f_0 - f) \qquad \nu = c_\infty/\lambda$$
$$f_0 = \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{\xi^2}{2RT}\right)$$

Normalized as above, it becomes

$$\tilde{\xi} \cdot \tilde{\nabla} \tilde{f} = \epsilon(\tilde{f}_0 - \tilde{f})$$

$$\tilde{\xi} \cdot \tilde{\nabla} = \tilde{\xi}_r \frac{\partial}{\partial \tilde{r}} + \frac{\tilde{\xi}_T^2}{\tilde{r}} \frac{\partial}{\partial \tilde{\xi}_r} - \frac{\tilde{\xi}_r \tilde{\xi}_T}{\tilde{r}} \frac{\partial}{\partial \tilde{\xi}_T}$$

The linearization may be expressed in the form

$$\tilde{f} = \tilde{f}_1 + a_2(\epsilon)\tilde{f}_2 + a_3(\epsilon)\tilde{f}_3 + \ldots \quad (3.9)$$

$$\tilde{T} = \tilde{T}_1 + b_2(\epsilon)\tilde{T}_2 + b_3(\epsilon)\tilde{T}_3 + \ldots \quad (3.10)$$

$$\tilde{\rho} = \tilde{\rho}_1 + c_2(\epsilon)\tilde{\rho}_2 + c_3(\epsilon)\tilde{\rho}_3 + \ldots \quad (3.11)$$

$$\tilde{f}_0 = \tau^{3/2} \exp\left\{-\frac{3}{2}\tau\frac{\tilde{\xi}^2}{\tilde{T}_1}\right\}\left[\frac{\tilde{\rho}_1}{\tilde{T}_1^{3/2}} + c_2(\epsilon)\frac{\tilde{\rho}_2}{\tilde{T}_1^{3/2}} - \frac{3}{2}b_2(\epsilon)\frac{\tilde{\rho}_1\tilde{T}_2}{\tilde{T}_1^{5/2}} + b_2(\epsilon)\frac{3}{2}\tau\tilde{\xi}^2\frac{\tilde{\rho}_1\tilde{T}_2}{\tilde{T}_1^{7/2}}\right] + \ldots \quad (3.12)$$

To obtain the distinguished* limit for the system of equations resulting from this expansion of the system, we choose $a_2(\epsilon) = \epsilon$, and the non-dimensional ordered equations become

$$O(1): \quad \tilde{\xi} \cdot \tilde{\nabla} \tilde{f}_1 = 0 \quad (3.13)$$

$$O(\epsilon): \quad \tilde{\xi} \cdot \tilde{\nabla} \tilde{f}_2 = \tau^{3/2} \exp\left[-\frac{3}{2}\tau\frac{\tilde{\xi}^2}{\tilde{T}_1}\right] \cdot \frac{\tilde{\rho}_1}{\tilde{T}_1^{3/2}} - \tilde{f}_1 \quad (3.14)$$

$$O(\epsilon^2): \quad \tilde{\xi} \cdot \tilde{\nabla} \tilde{f}_3 = \tau^{3/2} \exp\left[-\frac{3}{2}\tau\frac{\tilde{\xi}^2}{\tilde{T}_1}\right]\left[\frac{\tilde{\rho}_2}{\tilde{T}_1^{3/2}} - \frac{3}{2}\frac{\tilde{\rho}_1\tilde{T}_2}{\tilde{T}_1^{5/2}} + \frac{3}{2}\tau\tilde{\xi}^2\frac{\tilde{\rho}_1\tilde{T}_2}{\tilde{T}_1^{7/2}}\right] - \tilde{f}_2 \quad (3.15)$$

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* Since a_2 must be of specific order and not just restricted to a certain class of orders.

where we have used the fact that $a_2 = b_2 = c_2 = \epsilon$, which follows from the moment definitions presented below, and $a_3(\epsilon) = \epsilon^2$ which is part of the distinguished limit.

3.3 Solution of the Free Flow Problem

The $O(1)$ equation is that of free flow

$$\tilde{\xi} \cdot \tilde{\nabla} \tilde{f}_1 = 0 \tag{3.16}$$

The boundary condition to be applied here is that $\tilde{f}_1 = f_{1_i}(\tilde{\xi}^2, \tilde{\mathcal{R}}\tilde{\xi}_T)$ on a sphere of radius $\tilde{\mathcal{R}}$ for $\tilde{\xi}_r < 0$. $\tilde{\mathcal{R}}$ has the following properties:

$$\begin{aligned} \tilde{\mathcal{R}} &\to \infty \quad \text{as} \quad \epsilon \to 0 \\ \epsilon\tilde{\mathcal{R}} &\to 0 \quad \text{as} \quad \epsilon \to 0 \end{aligned} \tag{3.17}$$

Solving (3.16) we obtain solutions of the form,

$$\tilde{f}_1 = f_1(\tilde{\xi}^2, \tilde{r}\, \tilde{\xi}_T)$$

The boundary condition above need only be stated for $\tilde{\xi}_r < 0$ since the free flow calculation gives us the outgoing flow at any point. To solve the $O(\epsilon)$ equation it is necessary to find $\tilde{\rho}_1$ and $\tilde{T}_1$, but they are completely determined in terms of f_{1_i} by the geometry of the body and by the boundary condition on its surface. In this particular case the usual cone, sphere geometry is used.[14, p. 790] Only diffuse reflection with complete accommodation is considered.

Before proceeding let us formulate the integral definitions of the moments needed in the calculation.

Inner Region

$$\rho = \int f \, d^3 \xi$$

$$\rho R T = \int \tfrac{1}{3} \xi^2 f \, d^3 \xi$$

$$\tilde{\rho}_1 = \left(\frac{3}{2\pi}\right)^{3/2} \int \tilde{f}_1 \, d^3 \tilde{\xi}$$

$$\tilde{\rho}_1 \tilde{T}_1 = \tau \left(\frac{3}{2\pi}\right)^{3/2} \int \tilde{f}_1 \tilde{\xi}^2 \, d^3 \tilde{\xi}$$

$$\tilde{\rho}_2 = \left(\frac{3}{2\pi}\right)^{3/2} \int \tilde{f}_2 \, d^3 \tilde{\xi}$$

$$\tilde{\rho}_2 \tilde{T}_1 + \tilde{\rho}_1 \tilde{T}_2 = \tau \left(\frac{3}{2\pi}\right)^{3/2} \int \tilde{f}_2 \tilde{\xi}^2 \, d^3 \tilde{\xi}$$

In a later section we will set down the corresponding formulas valid in the outer region.

Preliminary Considerations:

Once f_{1_i} is known we have $\tilde{f}_1$ for all $\tilde{r}$ since the flow is collisionless. The usual assumption is that $\tilde{f}_1$ is the free flow solution with $f_{1_i} = f^*_\infty$. This assumption will be justified at a later stage during the matching process to be defined.

Calculation of the free flow $\tilde{\rho}_1$ and $\tilde{T}_1$:

Let $\tilde{f}_r$ be the reflected distribution on the body, a Maxwellian at the body temperature. Let $\tilde{f}_i$ be the incoming distribution on the body. Applying the diffuse boundary condition along with the condition that the incoming and outgoing flux of particles must be the same, we obtain,

$$\tilde{f}_r = \tilde{N}_i \left(\frac{9}{2\pi}\right) \tau^2 \exp\left\{ -\frac{3}{2} \tau \tilde{\xi}^2 \right\}$$

$$\tilde{N}_i = \left| \iint_{-\infty}^{\infty} \int_{-\infty}^{0} \tilde{\xi}_r \, \tilde{f}_i \, d^3 \tilde{\xi} \right| \qquad \tilde{\xi}_r \in [-\infty, 0]$$

Since $\tilde{f}_i = f^*_\infty$ by assumption,

$$\tilde{N}_i = \iint_{-\infty}^{\infty} d\tilde{\xi}_\varphi \, d\tilde{\xi}_\theta \int_{-\infty}^{0} \tilde{\xi}_r \exp(-\frac{3}{2}\tilde{\xi}^2) \, d\tilde{\xi}_r$$

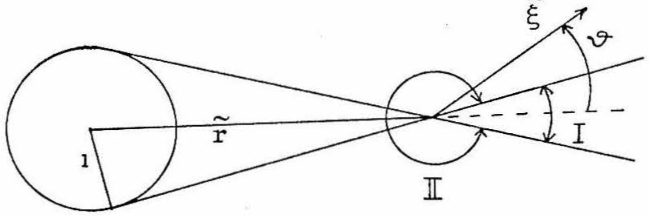
$$\tilde{N}_i = 2\pi/9$$

Hence,

$$\tilde{f}_r = \tau^2 \exp\{-\frac{3}{2}\tau\tilde{\xi}^2\}$$

$$\tau = T_\infty / T_a \qquad (3.18)$$

The density at any point may now be found by integrating $\tilde{f}_r$ and $\tilde{f}_i$ over the regions in velocity space in which they are valid.



Region I: Ω^* - particles coming from the body

Region II: $\Omega^{*}{}_1$ - particles missing the body or directed toward it

Figure I

The density is

$$\tilde{\rho}_1 = (\frac{3}{2\pi})^{3/2} \int \tilde{f}_1 \, d^3\tilde{\xi}$$

$$= (\frac{3}{2\pi})^{3/2} \left[\int_0^\infty \tilde{\xi}^2 d|\tilde{\xi}| \int_{\Omega^*} \tilde{f}_r \sin\vartheta \, d\vartheta \, d\varphi + \int_0^\infty \tilde{\xi}^2 d|\tilde{\xi}| \int_{\Omega^*{}_1} \tilde{f}_i \sin\vartheta \, d\vartheta \, d\varphi \right] \qquad (3.19)$$

(Region I)

$$\Omega^*: \vartheta \in \left[0, \sin^{-1}\frac{1}{\tilde{r}}\right]$$

$$\varphi \in [0, 2\pi]$$

(Region II)

$$\Omega^{*1}: \vartheta \in \left[\sin^{-1}\frac{1}{\tilde{r}}, \pi\right]$$

$$\varphi \in [0, 2\pi]$$

For the sake of convenience, the integral has been expressed in spherical coordinates for evaluation in velocity space. Carrying out the integration in (3.19):

$$\tilde{\rho}_1 = \tfrac{1}{2}(1+\tau^{1/2}) + \tfrac{1}{2}(1-\tau^{1/2})(1-\frac{1}{\tilde{r}^2})^{1/2} \tag{3.20}$$

Likewise,

$$\tilde{\rho}_1 \tilde{T}_1 = \tau \left(\frac{3}{2\pi}\right)^{3/2} \int \tilde{\xi}^2 \, \tilde{f}_1 \, d^3\tilde{\xi}$$

$$= \tau \left(\frac{3}{2\pi}\right)^{3/2} \left[\int_0^{\infty} \tilde{\xi}^4 \, d|\tilde{\xi}| \int_{\Omega^*} \tilde{f}_r \sin\vartheta \, d\vartheta \, d\varphi + \right.$$

$$\left. \int_0^{\infty} \tilde{\xi}^4 \, d|\tilde{\xi}| \int_{\Omega^{*1}} \tilde{f}_i \sin\vartheta \, d\vartheta \, d\varphi \right] \tag{3.21}$$

Integrating,

$$\tilde{\rho}_1 \tilde{T}_1 = \tfrac{1}{2}(\tau^{1/2}+\tau) + \tfrac{1}{2}(\tau-\tau^{1/2})(1-\frac{1}{\tilde{r}^2})^{1/2} \tag{3.22}$$

In the following sections asymptotic solutions for $\tilde{r} \to \infty$ will be found for $\tilde{f}_2$ and $\tilde{f}_3$ but first it is necessary to define the intermediate limit matching process under which these solutions are found. The exact solutions can be written as a series of quadratures, but they are useless for matching to the outer solution unless simplified.

3.4 The Matching Process Defined

Far from the body, where distances are measured in terms of λ we define the outer variable $r^* = r/\lambda$. Recalling $\tilde{r} = r/a$ we propose the intermediate limit:

$$\begin{aligned} & r_\eta = \eta(\epsilon)\tilde{r} \qquad \text{fixed as } \epsilon \to 0 \\ & 1 > \eta(\epsilon) > \epsilon \qquad \eta(\epsilon) \to 0 \end{aligned} \tag{3.23}$$

Then

$$\begin{aligned} & \tilde{r} = r_\eta/\eta \to \infty \\ & r^* = \epsilon\, r_\eta/\eta \to 0 \qquad \text{as } \epsilon \to 0 \end{aligned} \tag{3.24}$$

The matching condition for the inner and outer expansions is then

$$\lim_{r^* \to 0} [\text{Outer Exp.}] = \lim_{\tilde{r} \to \infty} [\text{Inner Exp.}] \tag{3.25}$$

under the condition r_η fixed. Note that this limit process is used explicitly or implied in all of the subsequent asymptotic treatment.

3.5 Asymptotic Solution for $\tilde{f}_2$ in the Inner Region

Exact formulation:

$$O(\epsilon): \tilde{\xi} \cdot \tilde{\nabla}\tilde{f}_2 = \tau^{3/2} \exp\left[-\frac{3}{2}\tau\frac{\tilde{\xi}^2}{\tilde{T}_1}\right] \cdot \frac{\tilde{\rho}_1}{\tilde{T}_1^{3/2}} - \tilde{f}_1 \tag{3.26}$$

$$\left.\begin{aligned} & \tilde{\rho}_1 = \tfrac{1}{2}\left\{(\tau^{1/2}+1) - (\tau^{1/2}-1)(1-\frac{1}{\tilde{r}^2})^{1/2}\right\} \\ & \tilde{\rho}_1\tilde{T}_1 = \frac{\tau^{1/2}}{2}\left\{(\tau^{1/2}+1) + (\tau^{1/2}-1)(1-\frac{1}{\tilde{r}^2})^{1/2}\right\} \end{aligned}\right\} \tag{3.27}$$

$\tilde{f}_1$ is the free flow solution and may be written:

$$\tilde{f}_1 = H(\tilde{\xi}_T^2 \tilde{r}^2 - \tilde{\xi}^2)\exp(-\tfrac{3}{2}\tilde{\xi}^2) + H(\tilde{\xi}^2 - \tilde{r}^2\tilde{\xi}_T^2)\tau^2 \cdot \exp(-\tfrac{3}{2}\tau\tilde{\xi}^2)H(\tilde{\xi}_r) + H(\tilde{\xi}^2 - \tilde{r}^2\xi_T^2)\exp(-\tfrac{3}{2}\tilde{\xi}^2)H(-\tilde{\xi}_r) \qquad (3.28)$$

H = the Heaviside function.

The solution $\tilde{f}_2$ is found by applying (3.6) and is written as a sum of two solutions: a particular one expressed as an integral and a solution to the homogeneous equation written as an arbitrary function of $\tilde{c}_1$ and $\tilde{c}_2$.

$$\tilde{c}_1^2 = \tilde{\xi}_r^2 + \tilde{\xi}_T^2$$

$$\tilde{c}_2 = -\tilde{r}\tilde{\xi}_T/\tilde{\xi}^2$$

$$\tilde{f}_{2P} = -\int \left[\tau^{3/2}\exp\left[-\frac{3}{2}\tau\frac{\tilde{\xi}^2}{\tilde{T}_1}\right]\cdot\frac{\tilde{\rho}_1}{\tilde{T}_1^{3/2}} - \tilde{f}_1\right]d\tilde{w} \qquad (3.29)$$

$$\tilde{f}_{2H} = \tilde{f}_{2H}(\tilde{c}_1^2, \tilde{c}_2) = \tilde{f}_{2H}(\tilde{\xi}^2, \tilde{r}\tilde{\xi}_T) \qquad (3.30)$$

here $\tilde{w} = -\tilde{r}\tilde{\xi}_r/\tilde{\xi}^2$

$$\tilde{r} = -\frac{\tilde{w}\tilde{\xi}^2}{\tilde{\xi}_r} = -\tilde{c}_1(\tilde{c}_2^2 + \tilde{w}^2)^{1/2}$$

Note that $\tilde{f}_1$ is constant along the characteristics of the equation since it is the solution to the homogeneous problem. Hence,

$$\int\tilde{f}_1\, d\tilde{w} = \tilde{f}_1\int d\tilde{w} = \tilde{f}_1\tilde{w} = -\tilde{f}_1\tilde{r}\tilde{\xi}_r/\tilde{\xi}^2$$

Since $\tilde{\rho}_1$ and $\tilde{T}_1$ are now known we may find $\tilde{f}_2$ by integrating the right hand side of (3.29). This integration will be carried out as

$\tilde{r} \to \infty$ with r_η fixed. The expression thus found will be used in matching.

$$\tilde{\rho}_1 \sim 1 - \frac{1}{4}\,\frac{(1-\tau^{1/2})}{\tilde{r}^2} \tag{3.31}$$

$$\tilde{\rho}_1 \tilde{T}_1 \sim \tau - \frac{1}{4}\,\frac{(\tau - \tau^{1/2})}{\tilde{r}^2} \qquad \text{as } \tilde{r} \to \infty \tag{3.32}$$

Introducing (3.31) and (3.32) into the right hand side of (3.29):

$$\tilde{\xi} \cdot \tilde{\nabla} \tilde{f}_2 = f^*_\infty \left\{1 + \frac{1}{\tilde{r}^2}\left[\overline{A} - \frac{3}{2}\overline{B}\,\tilde{\xi}^2\right]\right\} - \tilde{f}_1 + O(\eta^4)$$

$$\overline{A} = \frac{1}{8\sqrt{\tau}}\left[5\tau - 2\sqrt{\tau} - 3\right]$$

$$\overline{B} = \frac{1}{4\sqrt{\tau}}\left[\tau - 1\right]$$

Working out the integral:

$$\tilde{f}_{2P} = f^*_\infty\left[\overline{A} - \frac{3}{2}\overline{B}\,\tilde{\xi}^2\right]\frac{\cos^{-1}(\tilde{\xi}_T/|\tilde{\xi}|)}{\tilde{r}\,\tilde{\xi}_T} + (-\tilde{f}_1 + f^*_\infty)\frac{\tilde{r}\,\tilde{\xi}_r}{\tilde{\xi}^2} + O(\eta^3) \tag{3.33}$$

The total solution is written then

$$\tilde{f}_2 = f^*_\infty\left[\overline{A} - \frac{3}{2}\overline{B}\,\tilde{\xi}^2\right]\frac{\cos^{-1}(\tilde{\xi}_T/|\tilde{\xi}|)}{\tilde{r}\,\tilde{\xi}_T} + (f^*_\infty - \tilde{f}_1)\frac{\tilde{r}\,\tilde{\xi}_r}{\tilde{\xi}^2} + \tilde{f}_{2H}(\tilde{\xi}^2, \tilde{r}\tilde{\xi}_T) + O(\eta^3)$$

$$\text{as } \tilde{r} \to \infty \tag{3.34}$$

Matching determines $\tilde{f}_{2H}$.

3.6 Calculation of $\tilde{\rho}_2$ and $\tilde{T}_2$

The $O(\epsilon^2)$ equation for $\tilde{f}_3$, (3.15), cannot be solved until $\tilde{\rho}_2$ and $\tilde{T}_2$ are found. Again for matching we need only find $\tilde{f}_3$ as $\tilde{r} \to \infty$, hence equation (3.34) will be used for $\tilde{f}_2$ to calculate $\tilde{\rho}_2$ and $\tilde{T}_2$.

$$\tilde{\rho}_2 = \left(\frac{3}{2\pi}\right)^{3/2}\int \tilde{f}_2\, d^3\tilde{\xi}$$

define

$$\left(\frac{3}{2\pi}\right)^{3/2} \int \tilde{f}_{2H} d^3 \tilde{\xi} = \tilde{\rho}_{2H}$$

Consider

$$\int \frac{\cos^{-1}(\tilde{\xi}_T / |\tilde{\xi}|)}{\tilde{\xi}_T} f_\infty^* \left[\bar{A} - \frac{3}{2}\bar{B}\tilde{\xi}^2\right] d^3 \tilde{\xi}$$

Change the integral to polar coordinates,

$$\int_0^\infty \frac{\tilde{\xi}^2 f_\infty^* \left[\bar{A} - \frac{3}{2}\bar{B}\tilde{\xi}^2\right] d|\tilde{\xi}|}{|\tilde{\xi}|} \int_\Omega \frac{\cos^{-1}(\sin\vartheta)}{\sin\vartheta} \sin\vartheta \, d\vartheta \, d\varphi$$

$$\Omega : \vartheta \in [0, \pi] \qquad \varphi \in [0, 2\pi]$$

making the change of variable $\vartheta = \frac{\pi}{2} - x$ it is clear that the integral vanishes. Consider

$$\int (f_\infty^* - \tilde{f}_1) \frac{\tilde{r}\tilde{\xi}_r}{\tilde{\xi}^2} d^3 \tilde{\xi} =$$

$$\tilde{r} \int_0^\infty \frac{\tilde{\xi}^2 (f_\infty^* - \tilde{f}_1) |\tilde{\xi}| \, d|\tilde{\xi}|}{\tilde{\xi}^2} \int_\Omega \cos\vartheta \sin\vartheta \, d\vartheta \, d\varphi \tag{3.35}$$

$$\left.\begin{aligned} \tilde{f}_1 &= f_\infty^* \quad \text{in } \Omega^{*1} \\ \tilde{f}_1 &= \tau^2 \exp\left(-\frac{3}{2}\tau\tilde{\xi}^2\right) \quad \text{in } \Omega^* \end{aligned}\right\} \tag{3.36}$$

where Ω^*, Ω^{*1} are defined on p. 25. Using (3.36) in (3.35),

$$\tilde{r} \int_0^\infty \left[\exp\left(-\frac{3}{2}\tilde{\xi}^2\right) - \tau^2 \exp\left(-\frac{3}{2}\tau\tilde{\xi}^2\right)\right] |\tilde{\xi}| \, d|\tilde{\xi}| \int_{\Omega^*} \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi$$

Integrating,

$$\frac{\pi}{3\tilde{r}}(1-\tau)$$

hence,

$$\tilde{\rho}_2 = \tilde{\rho}_{2H} + \left(\frac{3}{2\pi}\right)^{3/2} \frac{\pi}{3\tilde{r}} (1-\tau) \tag{3.37}$$

Now for $\tilde{T}_2$ note from the moment definitions

$$\tilde{\rho}_2 \tilde{T}_1 + \tilde{\rho}_1 \tilde{T}_2 = \tau\left(\frac{3}{2\pi}\right)^{3/2} \int \tilde{f}_2 \tilde{\xi}^2 d^3 \tilde{\xi}$$

and the integrals may be evaluated as above. In this case two of them vanish leaving us with

$$\tilde{\rho}_2 \tilde{T}_1 + \tilde{\rho}_1 \tilde{T}_2 = \tilde{T}_{2H}$$

where,

$$\tilde{T}_{2H} \equiv \int \tau\left(\frac{3}{2\pi}\right)^{3/2} \tilde{f}_{2H} \tilde{\xi}^2 d^3 \tilde{\xi}$$

$$\tilde{T}_2 = \frac{\tilde{T}_{2H} - \tilde{\rho}_2 \tilde{T}_1}{\tilde{\rho}_1}$$

$$\tilde{T}_2 = \tilde{T}_{2H} - \tau\tilde{\rho}_{2H} - \frac{\pi\,\tau(1-\tau)}{3\tilde{r}} \left(\frac{3}{2\pi}\right)^{3/2} + O(\eta^2) \tag{3.38}$$

3.7 <u>Solution of the $O(\epsilon^2)$ Equation</u>.

Now that $\tilde{\rho}_2$ and $\tilde{T}_2$ are known (3.37), (3.38) we may solve the $O(\epsilon^2)$ equation for $\tilde{f}_3$ as $\tilde{r} \to \infty$. Introducing (3.31), (3.32), (3.37), (3.38) into the right hand side of (3.15):

$$\tilde{\xi} \cdot \tilde{\nabla} \tilde{f}_3 = f^*_\infty \left\{\tilde{\rho}_{2H} - \frac{3}{2}\left(\frac{T_{2H}}{\tau} - \tilde{\rho}_{2H}\right)(1-\tilde{\xi}^2) + \left(\frac{3}{2\pi}\right)^{3/2} \frac{\pi}{3\tilde{r}} [1-\tau]\left[\frac{5}{2} - \frac{3}{2}\tilde{\xi}^2\right]\right\} - \tilde{f}_2 + O(\eta^2) \quad \text{as} \quad \tilde{r} \to \infty$$

Again, the solution $\tilde{f}_3$ may be written as a sum of two solutions: a particular one expressed as an integral and a solution to the homogeneous equation written as an arbitrary function of $\tilde{c}_1$ and $\tilde{c}_2$.

Define:

$$\tilde{\rho}_{2H} - \frac{3}{2}\left(\frac{\tilde{T}_{2H}}{\tau} - \tilde{\rho}_{2H}\right)\left(1-\tilde{\xi}^2\right) = \tilde{f}_H$$

$$\bar{\sigma} = \left(\frac{3}{2\pi}\right)^{3/2} \frac{\pi(1-\tau)}{3}$$

Then,

$$\tilde{f}_{3P} = -\int\left\{ f_\infty^*\left[\tilde{f}_H + \frac{\bar{\sigma}}{\tilde{r}}\left(\frac{5}{2} - \frac{3}{2}\tilde{\xi}^2\right)\right] - \tilde{f}_2\right\} d\tilde{w} \qquad (3.39)$$

$$\tilde{f}_{3H} = \tilde{f}_{3H}(\tilde{c}_1^2, \tilde{c}_2) = \tilde{f}_{3H}(\tilde{\xi}^2, \tilde{r}\tilde{\xi}_T) \qquad (3.40)$$

Rewriting (3.4), (3.5), (3.31) in terms of the intermediate variable, we obtain

$$\left.\begin{aligned} w_\eta &= -r_\eta\, \tilde{\xi}_r / \tilde{\xi}^2 \\ r_\eta &= -w_\eta\, \tilde{\xi}^2/\tilde{\xi}_r = -\tilde{c}_1(c_{2\eta}^2 + w_\eta^2)^{1/2} \\ c_{2\eta} &= w_\eta\, \tilde{\xi}_T / \tilde{\xi}_r \end{aligned}\right\} \qquad (3.41)$$

Noting (3.41), the integrals in (3.39) may be readily evaluated.

Define

$$-\int f_\infty^*\, \tilde{f}_H\, d\tilde{w} = \tilde{g}_H$$

Note

1. $$\int \frac{d\tilde{w}}{\tilde{r}} = \int \frac{-dw_\eta}{\tilde{c}_1(c_{2\eta}^2 + w_\eta^2)^{1/2}}$$

$$= -\frac{1}{\tilde{c}_1}\log\left[\frac{r_\eta(\tilde{c}_1 + \tilde{\xi}_r)}{\tilde{\xi}^2}\right]$$

2. $$\int \tilde{f}_{2H}\, d\tilde{w} = \tilde{w}\tilde{f}_{2H}$$

3. $$-\int \frac{(f_\infty^* - \tilde{f}_1)\,\tilde{r}\tilde{\xi}_r}{\tilde{\xi}^2}\, d\tilde{w} = \int (f_\infty^* - \tilde{f}_1)\tilde{w}\, d\tilde{w} = (f_\infty^* - \tilde{f}_1)\,\tilde{w}^2/2$$

Observe

$$\tilde{\xi}_T = \frac{\tilde{c}_1 \tilde{c}_2}{\sqrt{\tilde{c}_2^2 + w^2}}$$

Then

$$\tilde{r}\,\tilde{\xi}_T = -\tilde{c}_1^2\,\tilde{c}_2 \,, \quad \text{also a constant.}$$

Hence,

$$\int \frac{\cos^{-1}(\tilde{\xi}_T/|\tilde{\xi}|)}{\tilde{r}\,\tilde{\xi}_T}\, d\tilde{w} = -\frac{1}{\tilde{c}_1^2\,\tilde{c}_2}\int \tan^{-1}(\tilde{\xi}_r/\tilde{\xi}_T)\, d\tilde{w}$$

$$= -\frac{1}{\tilde{c}_1^2\,\tilde{c}_2}\int \tan^{-1}\left(\frac{\tilde{w}}{\tilde{c}_2}\right) d\tilde{w}$$

$$= -\frac{1}{\tilde{\xi}^2}\left[\frac{\tilde{\xi}_r}{\tilde{\xi}_T}\tan^{-1}\left(\frac{\tilde{\xi}_r}{\tilde{\xi}_T}\right) - \tfrac{1}{2}\log\left[\frac{\tilde{\xi}^2}{\tilde{\xi}_T^2}\right]\right]$$

Summarizing the integrals,

$$\tilde{f}_{3P} \sim \tilde{g}_H + \bar{\sigma}\left(\frac{5}{2} - \frac{3}{2}\tilde{\xi}^2\right)\frac{f_\infty^*}{|\tilde{\xi}|}\log\left[\frac{r_\eta(\tilde{c}_1 + \tilde{\xi}_r)}{\tilde{\xi}^2}\right] - (f_\infty^* - \tilde{f}_1)\frac{\tilde{w}^2}{2} +$$

$$+ \tilde{w}\tilde{f}_{2H} - f_\infty^*\left[\bar{A} - \frac{3}{2}\bar{B}\,\tilde{\xi}^2\right]\cdot\frac{1}{\tilde{\xi}^2}\left[\frac{\tilde{\xi}_r}{\tilde{\xi}_T}\tan^{-1}\left(\frac{\tilde{\xi}_r}{\tilde{\xi}_T}\right) +\right.$$

$$\left. + \log\left(\frac{\tilde{\xi}_T}{|\tilde{\xi}|}\right)\right] + O(\eta)$$

$$\text{as} \quad \tilde{r} \to \infty$$

$$\tilde{f}_3 = \tilde{f}_{3P} + \tilde{f}_{3H}$$

3.8 The Outer Region

In the region of physical space which is far from the body, that is, many mean free paths away, we must choose a normalization and linearization which are different from those in the inner region where the free flow solution is the dominant term. We expect the solution to tend in some way to the continuum behavior at infinity.

The presumption that this occurs uniformly in velocity space has led previous calculations based on linearization about f^*_∞ to error. In fact, the velocity space must again be divided into Ω^* and Ω^{*1} as noted below.

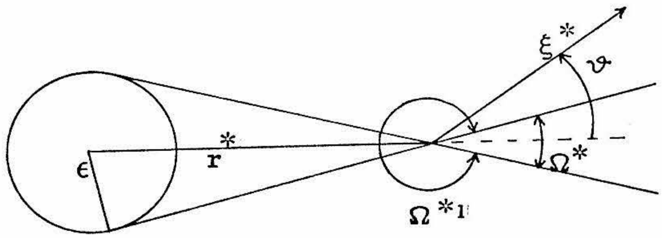


Figure II

$$\Omega^*: \vartheta \in [0, \sin^{-1} \epsilon/r^*]$$
$$\varphi \in [0, 2\pi]$$

$$\Omega^{*1}: \vartheta \in [\sin^{-1} \epsilon/r^*, \pi]$$
$$\varphi \in [0, 2\pi]$$

$$\Omega = \Omega^* + \Omega^{*1}$$

As shown in the following section, the linearization about f^*_∞ is valid in Ω^{*1}, while that in Ω^* must be about another function yet to be determined.

Normalization,
$$r^* = r/\lambda$$
$$\xi^* = \xi/c_\infty$$
$$\rho^* = \rho/\rho_\infty$$
$$T^* = T/T_\infty$$
$$f^* = f/A$$

Linearization,

$$f^* = g_1 + \alpha_1(\epsilon) g_2 + \dots$$
$$T^* = 1 + \beta_1(\epsilon) T_1^* + \dots$$
$$\rho^* = 1 + \gamma_1(\epsilon) \rho_1^* + \dots$$
$$f_0^* = f_\infty^* \{1 + \gamma_1(\epsilon)\rho_1^* - \frac{3}{2}\beta_1(\epsilon) T_1^*(1-\xi^{*2})\}, \quad f_\infty^* = \exp(-\frac{3}{2}\xi^{*2})$$

Note that T^* and ρ^* may be linearized about "one" since the contributions to them from Ω^* are second order as will be shown. Again, following from the moment definitions we must have

$$\alpha_1(\epsilon) = \beta_1(\epsilon) = \gamma_1(\epsilon)$$

and the non-dimensional ordered set of equations to be solved becomes

$$O(1): \ \xi^* \cdot \nabla^* g_1 = f_\infty^* - g_1 \tag{3.42}$$

$$O(\alpha(\epsilon)): \ \xi^* \cdot \nabla^* g_2 = f_\infty^* \left[\rho_1^* - \frac{3}{2} T_1^*(1-\xi^{*2})\right] - g_2$$
$$\xi^* \cdot \nabla^* = \xi_r^* \frac{\partial}{\partial r^*} + \frac{\xi_T^{*2}}{r^*}\frac{\partial}{\partial \xi_r^*} - \frac{\xi_r^* \xi_T^*}{r^*}\frac{\partial}{\partial \xi_T^*} \tag{3.43}$$

The boundary condition to be applied to g_1 is that it approach f_∞^* as $r^* \to \infty$. That imposed on g_2 requires its vanishing at infinity. Solution of the O (1) equation presents no particular problem. Letting $h = g_1 - f_\infty^*$,

$$\xi^* \cdot \nabla^* h = -h$$

which from p. 19 has the solution

$$h = K(c_1^*, c_2^*) \exp\left(-\frac{r^* \xi_r^*}{\xi^{*2}}\right) \qquad \text{or}$$

$$g_1 = f^*_\infty + K(c_1^*, c_2^*) \exp\left(-\frac{r^* \xi_r^*}{\xi^{*2}}\right)$$

$$c_1^{*2} = \xi_r^{*2} + \xi_T^{*2}$$

$$c_2^* = -r^* \xi_T^* / \xi^{*2}$$

where $K(c_1^*, c_2^*)$ is to be determined by matching. Applying the intermediate limit just to the first terms of $\tilde{f}$ and f^* tells us that,

$$f^*_\infty + K(c_1^*, 0) = \tilde{f}_1$$

where $\tilde{f}_1$ is the free flow solution previously found. Hence

$$K(c_1^*, 0) = \tilde{f}_1 - f^*_\infty$$

and we may write

$$g_1 = \begin{cases} f^*_\infty & \text{in } \Omega^{*1} \\ f^*_\infty + (\tilde{f}_1 - f^*_\infty) \exp\left(-\frac{r^* \xi_r^*}{\xi^{*2}}\right) & \text{in } \Omega^* \end{cases}$$

$$\tilde{f}_1 = \tau^2 \exp\left(-\frac{3}{2} \tau \xi^{*2}\right) \quad \text{in } \Omega^*$$

noting that there are no problems with the exponential since $\xi_r^* > 0$ in Ω^*.

K has only been found as a function of c_1^*. Expanding K in the form

$$K(c_1^*, c_2^*) = K(c_1^*, 0) + c_2^* K'(c_1^*, 0) + \ldots$$

and using this expansion in the final matching process shows that it is reasonable to assume K is a function of c_1^* only and hence is determined completely by the above expression found by the first order matching.

3.9 The Equations of Motion

Before solving (3.43) we must have ρ_1^* and T_1^* explicitly. The contributions to these moments from Ω^{*1}, in which the linearization is about the continuum solution, are governed by the continuum equations which will be derived below from the Krook equation. The part of ρ_1^* and T_1^* coming from Ω^* is determined by integrals over g_1 in that region. Also we expect the equation of state to hold in Ω^{*1}. Subsequent matching will demonstrate that it is not necessary to consider either the equations of motion or the equation of state in order to find the distribution function. This section is included in order to verify that the resultant solution does indeed approach that required by the linearized equations of motion derived from the Krook model. Another interesting derivation of the equations of motion where there is a macroscopic velocity is presented in Appendix II.

Define

$$\rho_1^* = \rho_{11}^* + \rho_{12}^* \quad , \qquad T_1^* = T_{11}^* + T_{12}^*$$

$$\rho_{11}^*,\ T_{11}^* = \text{"contribution from } \Omega^{*1}\text{"}$$

$$\rho_{12}^*,\ T_{12}^* = \text{"contribution from } \Omega^*\text{"}$$

We must have

$$P_{11}^* = \rho_{11}^* + T_{11}^* ,$$

the linearized equation of state. The following form of the Krook equation will be used in deriving the equations of motion:

$$\xi_r^* \frac{\partial f}{\partial r^*} + \frac{\xi_T^{*}}{r^*} \frac{\partial f}{\partial \xi_r^*} - \frac{\xi_r^* \xi_T^*}{r^*} \frac{\partial f}{\partial \xi_T^*} = \frac{\delta f}{\delta t} \tag{3.44}$$

Multiplying (3.44) by one and integrating over all ξ space in cylindrical coordinates

$$2\pi \frac{\partial}{\partial r}\int \xi_r^* \xi_T^* f d\xi_r^* d\xi_T^* + \frac{2\pi}{r}\int \frac{\partial}{\partial \xi_r^*}(\xi_T^{*3} f) d\xi_r^* d\xi_T^* - \frac{2\pi}{r}\int \xi_T^{*2}\frac{\partial}{\partial \xi_T}(f\xi_r^*) d\xi_r^* d\xi_T^* = 0$$

Integrating by parts

$$2\pi \frac{\partial}{\partial r}\int \xi_T^* \xi_r^* f d\xi_r^* d\xi_T^* + \frac{2\pi}{r}\int 2\xi_T^* \xi_r^* f d\xi_r^* d\xi_T^* = 0$$

$$\left(2\pi \frac{\partial}{\partial r} + \frac{4\pi}{r}\right)\int \xi_T^* \xi_r^* f\, d\xi_r^* d\xi_T^* = 0 \qquad (3.45)$$

further,

$$u_r^* = u_T^* \equiv 0 \quad \text{[no mean flow]}$$

$$u_r^* = 2\pi \int \xi_T^* \xi_r^* f\, d\xi_r d\xi_T$$

hence (3.45) is satisfied identically. Similar reasoning tells us that the momentum equation is satisfied for $p = \text{constant}$. To obtain the energy equation, multiply (3.44) by ξ^{*2} and integrate.

Define

$$q_r^* = \int \xi_r^* \xi^{*2} f^* d^3\xi^* \qquad \text{Heat flux}$$

The moment equation obtained is

$$\left(\frac{\partial}{\partial r^*} + \frac{2}{r^*}\right) q_r^* = 0 \qquad \text{or}$$

$$\frac{1}{r^{*2}}\frac{\partial}{\partial r}\left(r^{*2} q_r^*\right) = 0$$

We expect $q_{r_1}^*$, the contribution to the heat flux from the region Ω^{*1}, to satisfy this equation. Hence,

$$q_{r_1}^* = \frac{\text{constant}}{r^{*2}}$$

Using the further relation

$$q^*_{r_1} = -k \frac{dT^*_{11}}{dr^*}$$

we find

$$T^*_{11} = K_1/r^* \tag{3.46}$$

where T^*_{11} satisfies the Navier-Stokes equation. Utilizing the fact that the pressure is constant and that T^*_{11} and ρ^*_{11} obey the linearized equation of state, we find

$$\begin{aligned} \rho^*_{11} &= P^*_{11} - T^*_{11} \\ &= P^*_{11} - K_1/r^* \end{aligned} \tag{3.47}$$

Subsequent matching will tell us that we must choose $\alpha_1(\epsilon) = \epsilon^2$. To simplify the next calculation let us assume this subject to verification.

Moment Definitions in the Outer Region

$$\rho = \int f \, d^3\xi$$

$$\rho^* = \left(\frac{3}{2\pi}\right)^{3/2} \int f^* \, d^3\xi^*$$

$$1 + \epsilon^2 \rho^*_1 = \left(\frac{3}{2\pi}\right)^{3/2} \left[\int g_1 \, d^3\xi^* + \int \epsilon^2 g_2 \, d^3\xi^* \right]$$

$$\therefore \quad \rho^*_1 = \left(\frac{3}{2\pi}\right)^{3/2} \int g_2 \, d^3\xi^* + \left[\left(\frac{3}{2\pi}\right)^{3/2} \int g_1 \, d^3\xi^* - 1 \right] \cdot \frac{1}{\epsilon^2}$$

$$\rho^*_{11} + \rho^*_{12} = \left(\frac{3}{2\pi}\right)^{3/2} \int_{\Omega} g_2 \, d^3\xi^* + \frac{1}{\epsilon^2} \left(\frac{3}{2\pi}\right)^{3/2} \int_{\Omega^*} (g_1 - f^*_\infty) \, d^3\xi^*$$

and we identify

$$\rho^*_{11} = \left(\frac{3}{2\pi}\right)^{3/2} \int_{\Omega} g_2 \, d^3\xi^* \tag{3.48}$$

$$\rho^*_{12} = \frac{1}{\epsilon^2}\left(\frac{3}{2\pi}\right)^{3/2} \int_{\Omega^*} (g_1 - f^*_\infty)\, d^3\xi^* \tag{3.49}$$

$$\tilde{f}_1 = \tau^2 \exp\left(-\frac{3}{2}\tau\xi^{*2}\right) \quad \text{in } \Omega^* \tag{3.50}$$

$$g_1 - f^*_\infty = (\tilde{f}_1 - f^*_\infty)\exp\left(-\frac{r^*\xi^*_r}{\xi^{*2}}\right) \quad \text{in } \Omega^* \tag{3.51}$$

Proceeding along the same line with T^*_1:

$$\rho^* T^* = \left(\frac{3}{2\pi}\right)^{3/2} \int \xi^{*2} f^* d^3\xi^*$$

$$(1 + \rho^*_1\epsilon^2)(1 + T^*_1\epsilon^2) = \left(\frac{3}{2\pi}\right)^{3/2} \int \xi^{*2}\left[g_1 + \epsilon^2 g_2\right] d^3\xi^*$$

$$\rho^*_1 + T^*_1 = \left(\frac{3}{2\pi}\right)^{3/2} \int \xi^{*2} g_2\, d^3\xi^* + \frac{1}{\epsilon^2}\left(\frac{3}{2\pi}\right)^{3/2} \int \xi^{*2}(g_1 - f^*_\infty)\, d^3\xi^*$$

and comparing both sides of the equation,

$$T^*_{11} = \left(\frac{3}{2\pi}\right)^{3/2} \int \xi^{*2} g_2\, d^3\xi^* - \rho^*_{11} \qquad \text{[linearized eqn. of state]} \tag{3.52}$$

$$T^*_{12} = \frac{1}{\epsilon^2}\left(\frac{3}{2\pi}\right)^{3/2} \int_{\Omega^*} \xi^{*2}(g_1 - f^*_\infty)\, d^3\xi^* - \rho^*_{12} \tag{3.53}$$

$$g_1 - f^*_\infty = \left(\tau^2\exp\left(-\frac{3}{2}\tau\xi^{*2}\right) - f^*_\infty\right)\exp\left(-\frac{r^*\xi^*_r}{\xi^{*2}}\right) \tag{3.54}$$

both T^*_{11} and ρ^*_{11} are known from p. 38. It remains to calculate T^*_{12} and ρ^*_{12} following the same routine as in previous integrals.

3.10 Evaluation of T^*_{12} and ρ^*_{12}

For the purposes of matching we will expand $\exp\left(-\frac{r^*\xi^*_r}{\xi^{*2}}\right)$ for small r^* and retain only those terms of appropriate order. The exact expression for these moments will be displayed later. The results follow.

$$\rho^*_{12} = -\frac{1}{4}\frac{(1-\tau^{1/2})}{r^{*2}} + \frac{\pi}{3r^*}(1-\tau)\left(\frac{3}{2\pi}\right)^{3/2} + \frac{3}{8}(\tau^{3/2}-1) + O\left(\frac{\epsilon}{\eta}\right) \quad (3.55)$$

$$T^*_{12} = \frac{1}{4r^{*2}}\quad\frac{1-\tau}{\tau^{1/2}} + \frac{\tau^{1/2}-1}{8} - \frac{3}{8}(\tau^{3/2}-1) - \frac{\pi}{3r^*}(1-\tau)\left(\frac{3}{2\pi}\right)^{3/2} + O\left(\frac{\epsilon}{\eta}\right) \quad (3.56)$$

The exact calculation reduces to a single quadrature.

Consider

$$\int_0^\infty \xi^{*2}(\tilde{f}_1 - f^*_\infty)\int_{\Omega^*}\exp\left(-\frac{r^*\xi^*_r}{\xi^{*2}}\right) d|\xi^*|\, d\Omega^*$$

$$\int_0^\infty \xi^{*2}(\tilde{f}_1 - f^*_\infty)\int_0^{\sin^{-1}\frac{\epsilon}{r^*}}\exp\left(-\frac{r^*\xi^*_r}{\xi^{*2}}\right)\sin\vartheta\, d\vartheta\, d\varphi\, d|\xi^*|$$

Noting $\qquad \xi^*_r = |\xi^*|\cos\vartheta,$

$$\int_0^\infty \xi^{*2}(\tilde{f}_1 - f^*_\infty)\frac{2\pi|\xi^*|}{r^*}\exp\left(-\frac{r^*\cos\vartheta}{|\xi^*|}\right)\Bigg|_0^{\sin^{-1}\frac{\epsilon}{r^*}} d|\xi^*|$$

$$\frac{2\pi}{r^*}\int_0^\infty |\xi^*|^3\,[\tilde{f}_1 - f^*_\infty]\left[\exp\left(\frac{-r^*}{|\xi^*|}\left(1-\frac{\epsilon^2}{r^{*2}}\right)^{1/2}\right) - \exp\left(\frac{-r^*}{|\xi^*|}\right)\right] d|\xi^*|$$

Multiplying by $\left(\frac{3}{2\pi}\right)^{3/2}\cdot\frac{1}{\epsilon^2}$ and noting that ϵ^2/r^{*2} is small compared to one, we obtain

$$\rho^*_{12} = \frac{2\pi}{r^{*2}}\int_0^\infty |\xi^*|^2\left(\frac{\tilde{f}_1 - f^*_\infty}{2}\right)\exp\left(\frac{-r^*}{|\xi^*|}\right) d|\xi^*| \quad (3.57)$$

$$\left.\begin{aligned}\tilde{f}_1 &= \tau^2\exp\left(-\frac{3}{2}\tau\xi^{*2}\right)\\ f^*_\infty &= \exp\left(-\frac{3}{2}\xi^{*2}\right)\end{aligned}\right\} \quad (3.58)$$

T^*_{12} is the same as ρ^*_{12} with ξ^{*2} replaced by ξ^{*4} in the integral.

3.11 Solution of the $O(\epsilon^2)$ Equation

Repeating here equation (3.43)

$$O\left(\alpha(\epsilon)\right) : \xi^* \cdot \nabla^* g_2 = f_\infty^* \left[\rho_1^* - \frac{3}{2} T_1^*(1-\xi^{*2})\right] - g_2 \tag{3.59}$$

and using (3.46), (3.47), (3.55), (3.56)

$$\rho_1^* = P_{11}^* - K_1/r^* - \frac{1}{4}\frac{(1-\tau^{1/2})}{r^{*2}} + \frac{\pi}{3r^*}(1-\tau)\left(\frac{3}{2\pi}\right)^{3/2} + \frac{3}{8}(\tau^{3/2}-1) + O\left(\frac{\epsilon}{\eta}\right) \tag{3.60}$$

$$T_1^* = K_1/r^* + \frac{1}{4r^{*2}}\frac{1}{\tau^{1/2}}(1-\tau) + \frac{\tau^{1/2}-1}{8} - \frac{3}{8}\frac{\tau^{3/2}-1}{1} - \frac{\pi}{3r^*}(1-\tau)\left(\frac{3}{2\pi}\right)^{3/2} + O\left(\frac{\epsilon}{\eta}\right) \tag{3.61}$$

Following p. 19

$$g_2 = -e^{w^*}\int f_\infty^* \cdot \left[\rho_1^* - \frac{3}{2} T_1^*(1-\xi^{*2})\right] e^{-w^*} dw^* + K_2 e^{w^*} \tag{3.62}$$

$$w^* = -r^* \xi_r^* \Big/ \xi^{*2} \tag{3.63}$$

We set $K_2 = 0$ because this term has the wrong behavior for $r^* \to \infty$ when $\xi_r^* < 0$. Examine now the terms in the integrand. There are three basic forms of integrals involved.

1. $$\int C e^{-w^*} dw^* \quad , \quad C = \text{Constant} \tag{3.64}$$

2. $$\int \frac{C}{r^*} e^{-w^*} dw^* \tag{3.65}$$

3. $$\int \frac{C}{r^{*2}} e^{-w^*} dw^* \tag{3.66}$$

Following (3.41) we define

$$\left.\begin{aligned} w_\eta &= -r_\eta \xi_r^* \Big/ \xi^{*2} \\ r_\eta &= -w_\eta \xi^{*2} \Big/ \xi_r^* = -c_1^*\left(c_{2\eta}^2 + w_\eta^2\right)^{1/2} \\ c_{2\eta} &= w_\eta \, \xi_T^* / \xi_r^* \end{aligned}\right\} \tag{3.67}$$

The first integral is of course easily evaluated.

1. $$\int C e^{-w^*} dw^* = - C e^{-w^*}$$

Expressing the second in terms of the intermediate variable,

2. $$\int \frac{C}{r^*} e^{-w^*} dw^* = C \int \frac{e^{-\frac{\epsilon}{\eta} w_\eta} dw_\eta}{- c_1^* (c_{2\eta}^2 + w_\eta^2)^{1/2}}$$

$\frac{\epsilon}{\eta} \to 0$ so expanding

$$= - \frac{C}{c_1^*} \left[\int \frac{d w_\eta}{(c_{2\eta}^2 + w_\eta^2)^{1/2}} + \int \frac{- \frac{\epsilon}{\eta} w_\eta dw_\eta}{(c_{2\eta}^2 + w_\eta^2)^{1/2}} + O\left(\frac{\epsilon^2}{\eta^2}\right) \right]$$

$$= - \frac{C}{c_1^*} \log \left[\frac{r_\eta (c_1^* + \xi_r^*)}{\xi^{*2}} \right] + O\left(\frac{\epsilon}{\eta}\right)$$

3. $$\int \frac{C}{r^{*2}} e^{-w^*} dw^* = C \int \frac{e^{-\frac{\epsilon}{\eta} w_\eta} dw_\eta}{c_1^{*2} (c_{2\eta}^2 + w_\eta^2)} \cdot \left(\frac{\eta}{\epsilon}\right)$$

Expanding the integrand again,

$$= \left(\frac{\eta}{\epsilon}\right) \frac{C}{c_1^{*2}} \left[\int \frac{d w_\eta}{c_{2\eta}^2 + w_\eta^2} + \int \frac{- \frac{\epsilon}{\eta} w_\eta d w_\eta}{c_{2\eta}^2 + w_\eta^2} + O\left(\frac{\epsilon^2}{\eta^2}\right) \right]$$

$$= \left(\frac{\eta}{\epsilon}\right) \frac{C}{c_1^{*2}} \left[\frac{1}{c_{2\eta}} \tan^{-1} \left(\frac{\xi_r^*}{\xi_T^*}\right) + \frac{\epsilon}{\eta} \log \left(\frac{\xi_T^*}{|\xi^*|}\right) + O\left(\frac{\epsilon^2}{\eta^2}\right) \right]$$

Putting 1-3 into (3.62) and collecting together all the constants, we have the following result in which only terms of the indicated order have been retained.

$$g_2 = \frac{f^*_\infty}{16}\left[\left(15\tau^{3/2} - 12 - 3\tau^{1/2}\right) + \xi^{*2}\left(3\tau^{1/2} - 9\tau^{3/2} + 6\right) + 16P^*_{11}\right]$$

$$+ \frac{K_1}{|\xi^*|} f^*_\infty \left(-\frac{5}{2} + \frac{3}{2}\xi^{*2}\right) \log\left[\frac{r_\eta(\tilde{c}_1 + \xi^*_r)}{\xi^{*2}}\right] - f^*_\infty\left[\overline{A} - \frac{3}{2}\overline{B}\xi^{*2}\right] \cdot$$

$$\cdot \frac{1}{\xi^{*2}}\left[\frac{\xi^*_r}{\xi^*_T}\tan^{-1}\left(\frac{\xi^*_r}{\xi^*_T}\right) + \log\left(\frac{\xi^*_T}{|\xi^*|}\right)\right] + f^*_\infty\left[\overline{A} - \frac{3}{2}\overline{B}\xi^{*2}\right] \cdot$$

$$\cdot \frac{\eta}{\epsilon}\,\frac{\cos^{-1}(\xi^*_T/|\xi^*|)}{\xi^*_T\, r_\eta} + \overline{\sigma}\left(\frac{5}{2} - \frac{3}{2}\xi^{*2}\right)\frac{f^*_\infty}{|\xi^*|}\log\left[\frac{r_\eta(\tilde{c}_1 + \xi^*_r)}{\xi^{*2}}\right] + O\left(\frac{\epsilon}{\eta}\right) \quad (3.68)$$

$$\left.\begin{aligned} \overline{A} &= \frac{1}{8\tau^{1/2}}\left[5\tau - 2\tau^{1/2} - 3\right] \\ \overline{B} &= \frac{1}{4\tau^{1/2}}\left[\tau - 1\right] \\ \overline{\sigma} &= \left(\frac{3}{2\pi}\right)^{3/2}\frac{\pi(1-\tau)}{3} \end{aligned}\right\} \quad (3.69)$$

$$\tilde{c}_1 = c^*_1$$

3.12 Intermediate Matching

The inner and outer solutions are now known and it is possible to show that they match under the limit defined in (3.25). Note that the intermediate limit has been applied in both regions to obtain explicit solutions necessary for matching. The exact forms cannot be expressed in terms of elementary functions.

Before proceeding with verification of (3.25), we summarize the results found thus far.

Inner Region:

$$\tilde{\rho} \sim 1 - \tfrac{1}{4}\, \frac{(1-\tau^{1/2})}{\tilde{r}^2} + \epsilon \left\{ \tilde{\rho}_{2H} + \Gamma \frac{\pi}{3\tilde{r}} (1-\tau) \right\} + O(\eta^4) \qquad (3.70)$$

$$\tilde{T} \sim \tau + \tfrac{1}{4}\, \frac{(1-\tau)\,\tau^{1/2}}{\tilde{r}^2} + \epsilon \left\{ \tilde{T}_{2H} - \tau\, \tilde{\rho}_{2H} - \Gamma \frac{\pi}{3\tilde{r}}\, \tau(1-\tau) \right\} + O(\eta^4) \qquad (3.71)$$

$$\tilde{f} \sim \tilde{f}_1 + \epsilon \left\{ f^*_\infty \left[\overline{A} - \frac{3}{2}\overline{B}\,\tilde{\xi}^2\right] \frac{\cos^{-1}(\tilde{\xi}_T/|\tilde{\xi}|)}{\tilde{r}\,\tilde{\xi}_T} + \right.$$

$$\left. + (f^*_\infty - \tilde{f}_1)\, \frac{\tilde{r}\,\tilde{\xi}_r}{\tilde{\xi}^2} + \tilde{f}_{2H} \right\} + \epsilon^2 \left\{ \tilde{g}_H + \overline{\sigma} \left(\frac{5}{2} - \frac{3}{2}\tilde{\xi}^2 \right) . \right.$$

$$\cdot \frac{f^*_\infty}{|\tilde{\xi}|} \log \left[\frac{r_\eta(\tilde{c}_1 + \tilde{\xi}_r)}{\tilde{\xi}^2} \right] - \left(f^*_\infty - \tilde{f}_1\right) \frac{\tilde{w}^2}{2} + \tilde{w}\,\tilde{f}_{2H} - f^*_\infty \left[\overline{A} - \frac{3}{2}\overline{B}\tilde{\xi}^2\right] .$$

$$\left. \cdot \frac{1}{\tilde{\xi}^2} \left[\frac{\tilde{\xi}_r}{\tilde{\xi}_T} \tan^{-1}\!\left(\frac{\tilde{\xi}_r}{\tilde{\xi}_T}\right) + \tfrac{1}{2} \log \left(\frac{\tilde{\xi}_T}{|\tilde{\xi}|}\right)^2 \right] \right\} + O(\eta^3\epsilon) + \epsilon^2 \tilde{f}_{3H} \qquad (3.72)$$

where

$$\Gamma = \left(\frac{3}{2\pi}\right)^{3/2}$$

$\tilde{f}_1$ = free flow with f^*_∞ incoming

Outer Region:

$$\rho^* \sim 1 + \gamma_1(\epsilon) \left\{ P^*_{11} - K_1/r^* - \frac{1}{4} \frac{(1-\tau^{1/2})}{r^{*2}} + \Gamma \frac{\pi}{3r^*} (1-\tau) + \right.$$

$$\left. \frac{3}{8} (\tau^{3/2} - 1) \right\} + O\left(\frac{\epsilon}{\eta} \gamma_1(\epsilon)\right) \qquad (3.73)$$

$$T^* \sim 1 + \beta_1(\epsilon) \left\{ \frac{K_1}{r^*} + \frac{1}{4r^{*2}} \frac{1}{\tau^{1/2}} (1-\tau) + \frac{\tau^{1/2}-1}{8} + \right.$$

$$\left. - \frac{3}{8} \frac{\tau^{3/2}-1}{1} - \Gamma \frac{\pi}{3r^*} (1-\tau) \right\} + O\left(\frac{\epsilon}{\eta} \beta_1(\epsilon)\right) \qquad (3.74)$$

$$f^* \sim \frac{f^*_\infty}{16}\Bigg\{ \Bigg[\left(15\,\tau^{3/2} - 12 - 3\tau^{1/2}\right) + \xi^{*2}\left(3\tau^{1/2} - 9\tau^{3/2} + 6\right) + 16\,P^*_{11}\Bigg] +$$

$$+ \frac{K_1}{|\xi^*|} f^*_\infty \left(-\frac{5}{2} + \frac{3}{2}\xi^{*2}\right) \log\left[\frac{r_\eta(\tilde{c}_1 + \xi^*_r)}{\xi^{*2}}\right] - f^*_\infty\left[\overline{A} - \frac{3}{2}\overline{B}\xi^{*2}\right] \cdot$$

$$\cdot \frac{1}{\xi^{*2}}\left[\frac{\xi^*_r}{\xi^*_T}\tan^{-1}\left(\frac{\xi^*_r}{\xi^*_T}\right) + \log\left(\frac{\xi^*_T}{|\xi^*|}\right)\right] + f^*_\infty\left[\overline{A} - \frac{3}{2}\overline{B}\xi^{*2}\right] \cdot$$

$$\cdot \frac{\eta}{\epsilon}\,\frac{\cos^{-1}(\xi^*_T/|\xi^*|)}{\xi^*_T r_\eta} + \overline{\sigma}\left(\frac{5}{2} - \frac{3}{2}\xi^{*2}\right)\frac{f^*_\infty}{|\xi^*|}\log\left[\frac{r_\eta(\tilde{c}_1 + \xi^*_r)}{\xi^{*2}}\right]\Bigg\}\alpha_1(\epsilon)$$

$$+ f^*_\infty + (\tilde{f}_1 - f^*_\infty)\exp\left(\frac{-r^*\xi^*_r}{\xi^{*2}}\right) + o(\epsilon^2) \qquad (3.75)$$

In each term of the inner expansion there occurs a term of the form $\epsilon^n \frac{\tilde{w}^n}{n!}(f^*_\infty - \tilde{f}_1)$. The number of terms of this type to be included in the asymptotic expansion is determined by the class of intermediate limits, since written in the intermediate variables we have

$$\frac{w_\eta}{n!}(f^*_\infty - \tilde{f}_1)\left(\frac{\epsilon}{\eta}\right)^n \quad \text{and} \quad \left(\frac{\epsilon}{\eta}\right)^n < \epsilon^2 \qquad (3.76)$$

for the expansion to be valid to ϵ^2. It is also clear from (3.70), (3.71), (3.72) that

(A) $\quad \epsilon > \eta^4$

(B) $\quad \epsilon^2 > \eta^3\epsilon$

The terms of form (3.76) are matched by the term

$$(\tilde{f}_1 - f^*_\infty)\exp\left(\frac{-r^*\xi^*_r}{\xi^{*2}}\right) = \sum_{n=0}^{\infty} \frac{w_\eta}{n!}(f^*_\infty - \tilde{f}_1)\left(\frac{\epsilon}{\eta}\right)^n$$

in the outer region. Term by term matching of the density yields the following results:

(A) $\gamma_1(\epsilon) = \epsilon^2$

(B) $K_1 = 0$

(C) $\rho_{2H} = T_{2H} = 0$

The physically reasonable conditions that the perturbations vanish for $\tau \to 1$ and $r^* \to \infty$ requires,

(D) $P_{11}^* = 0$

The density now matches to $O(\epsilon)$. There is still a term of $O(\epsilon^2)$, $\frac{3}{8}(\tau^{3/2}-1)$, left unmatched but it will be shown to come from $\tilde{f}_{3H}$, a term in the distribution function yet to be found.

Term by term matching of the temperature yields

(A) $\beta_1(\epsilon) = \epsilon^2$

(B) $K_1 = 0$

agreeing with the assumption $\gamma_1(\epsilon) = \beta_1(\epsilon)$. The rest of the unknown functions, $\tilde{g}_H$, $\tilde{f}_{2H}$, $\tilde{f}_{3H}$, etc., are found by comparing (3.72), (3.75) which give the intermediate limits on f in the inner and outer regions, respectively.

It is concluded,

(A) $\tilde{f}_1 = f_\infty^*$ in Ω^{*1} verifying the previous assumption.

(B) $\alpha_1(\epsilon) = \epsilon^2$ again verified

(C) $K_1 = 0$

(D) $\tilde{f}_{2H} = 0$, $\tilde{g}_H = 0$

(E) $\tilde{f}_{3H} = \frac{f_\infty^*}{16}\left[(15\tau^{3/2} - 12 - 3\tau^{1/2}) + \xi^{*2}(3\tau^{1/2} - 9\tau^{3/2} + 6)\right]$

Note that the density and temperature computed from $\tilde{f}_{3H}$, ρ_{3H}^* and T_{3H}^*, are (use (3.48), (3.52))

$$\rho_{3H}^* = \frac{3}{8}(\tau^{3/2} - 1)$$

$$T^*_{3H} = \frac{1}{8}(\tau^{1/2}-1) - \frac{3}{8}(\tau^{3/2}-1)$$

which account for the presence of these terms in the outer region. To find the density and temperature to $O(\epsilon^2)$ it is necessary to integrate the distribution function found above to $O(\epsilon^2)$.

3.13 Comments on the Boundary Conditions Applied to $\tilde{f}_2$ and $\tilde{f}_3$

The boundary condition applied to the functions $\tilde{f}_2$ and $\tilde{f}_3$ in the inner region for particles in Ω^{*1} is given on a sphere of radius $\mathcal{R}$ with the following properties:

$$\tilde{\mathcal{R}} = \mathcal{R}/a \rightarrow \infty \qquad \text{as} \qquad \epsilon \rightarrow 0$$
$$\mathcal{R}^* = \mathcal{R}/\lambda \rightarrow 0 \qquad \text{as} \qquad \epsilon \rightarrow 0$$

i.e.

$$\tilde{\mathcal{R}} \rightarrow \infty$$
$$\mathcal{R}^* \quad \epsilon\tilde{\mathcal{R}} \rightarrow 0$$

on the sphere

$$\epsilon \tilde{f}_2 = F_2(\tilde{\xi}, \tilde{\mathcal{R}})$$
$$\epsilon^2 \tilde{f}_3 = F_3(\tilde{\xi}, \tilde{\mathcal{R}})$$

and we require

$$F_2(\tilde{\xi}, \tilde{\mathcal{R}}) \rightarrow 0 \qquad \text{as} \qquad \epsilon \rightarrow 0$$
$$F_3(\tilde{\xi}, \tilde{\mathcal{R}}) \rightarrow 0 \qquad \text{as} \qquad \epsilon \rightarrow 0$$

A check of the inner solutions found previously verifies this boundary condition.

The boundary condition applied to $\tilde{f}_2$ and $\tilde{f}_3$ in Ω^* is that of diffuse reflection at the body. This condition along with the equality of incoming and outgoing flux of particles tells us

$$\tilde{f}_r = \tilde{N}_i \left(\frac{9}{2\pi}\right) \tau^2 \exp\left(-\frac{3}{2}\tau \tilde{\xi}^2\right)$$

$$\tilde{N}_i = \left| \int_{-\infty}^{\infty}\int \int_{-\infty}^{0} \tilde{\xi}_r \tilde{f}_i \, d^3\tilde{\xi} \right| = -\int_{-\infty}^{\infty}\int \int_{-\infty}^{0} \tilde{\xi}_r \tilde{f}_i \, d^3\tilde{\xi}$$

$\tilde{f}_r \equiv$ reflected distribution

$\tilde{N}_i \equiv$ incident flux

$\tilde{\xi}_r \in [-\infty, 0]$ in $\tilde{N}_i$

Thus, such quantities as heat transfer to (or from) the body may easily be evaluated by solving the inner equations near the surface to obtain the incoming distribution and then applying the above conditions to find $\tilde{f}_r$, finally taking the appropriate moments desired. Part IV examines the heat transfer in this manner.

PART IV

HEAT TRANSFER

4.1 The Inner Solution on the Body ($\tilde{r} = 1$)

In order to find the heat flow from (or to) the body, it is necessary to know $\tilde{f}$ near $\tilde{r} = 1$. Following the same procedure as on pp. 26-32 we simplify the right hand side of equations (3.14), (3.15) under the limit $\tilde{r} \sim 1$ and then evaluate the characteristic integrals.

O(1): $\tilde{f}_1$ is again the free flow solution and no further simplification is necessary.

O(ϵ): Using (3.20), (3.22)

$$\tilde{\rho}_1 \sim \tfrac{1}{2}(1 + \tau^{1/2})$$
$$\tilde{T}_1 \sim \tau^{1/2} \qquad \text{as} \quad \tilde{r} \to 1$$

(3.29) yields

$$f_{2P} \sim -\int \left[\tau^{3/2} \exp(-\tfrac{3}{2}\tau^{1/2}\tilde{\xi}^2)\, \frac{\frac{1}{2}(1+\tau^{1/2})}{\tau^{3/4}} - \tilde{f}_1\right] d\tilde{w}$$

$$f_{2P} \sim \left[\tilde{f}_1 - \frac{\tau^{3/4}}{2} \exp(-\tfrac{3}{2}\tau^{1/2}\tilde{\xi}^2)\,(1+\tau^{1/2})\right]\tilde{w} \tag{4.1}$$

To find $\tilde{f}_{3P}$, $\tilde{\rho}_2$ and $\tilde{T}_2$ are needed. They are evaluated directly from $\tilde{f}_{2P}$, remembering that $\tilde{f}_{2H}$ was found to be zero. The density is given by,

$$\tilde{\rho}_2 = \Gamma \left[\tilde{r} \int_0^\infty \frac{\tau^{3/4}}{2} \exp(-\frac{3}{2}\tau^{1/2}\tilde{\xi}^2)(1+\tau^{1/2}) |\tilde{\xi}| d|\tilde{\xi}| \int_\Omega \sin\vartheta\cos\vartheta \, d\vartheta \, d\varphi \right]$$

$$-\Gamma\tilde{r} \left[\int_0^\infty \tau^2 \exp(-\tau\frac{3}{2}\tilde{\xi}^2) |\tilde{\xi}| d|\tilde{\xi}| \int_{\Omega^*} \sin\vartheta\cos\vartheta \, d\vartheta d\varphi \, + \right.$$

$$\left. + \int_0^\infty \exp(-\frac{3}{2}\tilde{\xi}^2) \, |\tilde{\xi}| \, d|\tilde{\xi}| \int_{\Omega^{*1}} \sin\vartheta\cos\vartheta \, d\vartheta d\varphi \right]$$

$$\tilde{\rho}_2 \sim \frac{\Gamma\pi}{3\tilde{r}}(1-\tau) \qquad \text{as} \quad \tilde{r} \sim 1$$

Finding $\tilde{T}_2$ in the same manner:

$$\tilde{\rho}_2 \tilde{T}_1 + \tilde{\rho}_1 \tilde{T}_2 = \tau\Gamma \int \tilde{f}_2 \tilde{\xi}^2 d^3 \tilde{\xi}$$

we find the integral to be zero, hence

$$\tilde{T}_2 = \frac{-\tilde{\rho}_2 \tilde{T}_1}{\tilde{\rho}_1}$$

$$\tilde{T}_2 \sim \frac{\tau^{1/2}(\tau-1)}{1+\tau^{1/2}} \frac{2\pi\Gamma}{3\tilde{r}} \qquad \text{as} \quad \tilde{r} \sim 1$$

$$f_{3P} \sim -\int \left[(\frac{5}{2} - \frac{3}{2}\tau^{1/2}\tilde{\xi}^2) \frac{\Gamma\pi(1-\tau)}{3\tilde{r}} \tau^{3/4} \exp(-\frac{3}{2}\tau^{1/2}\tilde{\xi}^2) - \tilde{f}_{2P} \right] d\tilde{w}$$

From previous calculations (p.31)

$$\int \frac{d\tilde{w}}{\tilde{r}} = -\frac{1}{|\tilde{\xi}|} \log\left[\frac{\tilde{r}(|\tilde{\xi}|+\tilde{\xi}_r)}{\tilde{\xi}^2}\right]$$

$$\tilde{f}_{3P} \sim \left(\frac{5}{2}-\frac{3}{2}\tau^{1/2}\tilde{\xi}^2\right)\frac{\Gamma\pi(1-\tau)}{3|\tilde{\xi}|}\tau^{3/4}\exp\left(-\frac{3}{2}\tau^{1/2}\tilde{\xi}^2\right)\cdot$$

$$\cdot \log\left[\frac{\tilde{r}(|\tilde{\xi}|+\tilde{\xi}_r)}{\tilde{\xi}^2}\right] + \frac{\tilde{r}^2\tilde{\xi}_r^2}{2\tilde{\xi}^4}\left[\tilde{f}_1 - \frac{\tau^{3/4}}{2}\exp\left(-\frac{3}{2}\tau^{1/2}\tilde{\xi}^2\right)(1+\tau^{1/2})\right]$$

$$\text{as} \quad \tilde{r} \sim 1 \tag{4.2}$$

$$\tilde{f}_{3H} = \frac{f_\infty^*}{16}\left[(15\tau^{3/2}-12-3\tau^{1/2})+\tilde{\xi}^2(3\tau^{1/2}-9\tau^{3/2}+6)\right] \tag{4.3}$$

$$\tilde{f}_3 = \tilde{f}_{3P} + \tilde{f}_{3H}$$

The next sections outline the calculation of the heat transfer. First, the free flow heat transfer is found. Then, as observed previously, it is necessary to find the incoming flux from $\tilde{f}_1$, $\tilde{f}_2$, and $\tilde{f}_3$ which in turn allows the reflected distribution to be displayed. Finally, the heat transfer is calculated from the incident and reflected distributions on the body.

4.2 Free Flow Heat Transfer and Corrections to $O(\epsilon^2)$

The heat flux (heat per unit time per unit area) at any given place in the flow field is defined as

$$q_r = \frac{1}{2}\int \xi_r\, \xi^2 f\, d^3\xi$$

Note only the radial component is considered since the flow is spherically symmetric. Defining, further, the dimensionless quantity

$$\tilde{q}_r = \int \tilde{\xi}_r \tilde{\xi}^2 \tilde{f} \, d^3 \tilde{\xi} \qquad \text{we have}$$

$$q_r = \frac{\Gamma}{2} \rho_\infty (3 \, RT_\infty)^{3/2} \quad \tilde{q}_r$$

The free flow heat transfer is calculated in the same way as $\tilde{\rho}_1$ and $\tilde{T}_1$ (pp. 24-25)

$$\tilde{q}_{r_1} \equiv \int_0^\infty |\tilde{\xi}|^5 \tau^2 \exp(-\frac{3}{2}\tau \tilde{\xi}^2) \, d|\tilde{\xi}| \int_{\Omega^*} \sin\vartheta \cos\vartheta \, d\vartheta d\varphi +$$

$$\int_0^\infty |\tilde{\xi}|^5 \exp(-\frac{3}{2}\tilde{\xi}^2) \, d|\tilde{\xi}| \int_{\Omega^*_1} \sin\vartheta \cos\vartheta \, d\vartheta d\varphi$$

$$\tilde{q}_{r_1} = \frac{8\pi}{27\tilde{r}^2} \left[\frac{1}{\tau} - 1\right] \tag{4.4}$$

The incoming flux is defined as

$$\tilde{N}_i = \left| \int_{-\infty}^{\infty}\int d\tilde{\xi}_\theta \, d\tilde{\xi}_\varphi \int_{-\infty}^{0} \tilde{\xi}_r \tilde{f}_i \, d\tilde{\xi}_r \right| \tag{4.5}$$

$$\tilde{f}_i = \tilde{f}_1 + \epsilon \tilde{f}_2 + \epsilon^2 \tilde{f}_3 \qquad \text{at} \quad \tilde{r} = 1 \tag{4.6}$$

Three separate integrals must be done. Using (4.1), (4.2), (4.3)

$$\tilde{f}_i = f^*_\infty + \epsilon \left[\frac{\tau^{3/4}(1+\tau^{1/2})}{2} \exp(-\frac{3}{2}\tau^{1/2}\tilde{\xi}^2) - f^*_\infty \right] \frac{\tilde{\xi}_r}{\tilde{\xi}^2}$$

$$+ \epsilon^2 \left[\frac{f^*_\infty}{16} \left[(15\tau^{3/2} - 12 - 3\tau^{1/2}) + \tilde{\xi}^2 (3\tau^{1/2} - 9\tau^{3/2} + 6) \right] + \right.$$

$$(\frac{5}{2} - \frac{3}{2}\tau^{1/2}\tilde{\xi}^2) \frac{\Gamma n(1-\tau)}{3|\tilde{\xi}|} \tau^{3/4} \exp(-\frac{3}{2}\tau^{1/2}\tilde{\xi}^2) \log\left[\frac{|\tilde{\xi}| + \tilde{\xi}_r}{\tilde{\xi}^2}\right] +$$

$$\left. + \frac{\tilde{\xi}_r^2}{2\tilde{\xi}^4} \left[f^*_\infty - \frac{\tau^{3/4}}{2} \exp(-\frac{3}{2}\tau^{1/2}\tilde{\xi}^2)(1+\tau^{1/2}) \right] \right], \quad \tilde{r} = 1 \tag{4.7}$$

To facilitate integration, we write

$$\tilde{N}_i = \left| \int_0^\infty |\tilde{\xi}|^3 \, d|\tilde{\xi}| \int_{\Omega^*} \tilde{f}_i \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi \right| \tag{4.8}$$

After laborious integration, one obtains

$$\tilde{N}_i = \frac{2\pi}{9} + \frac{\epsilon}{12\Gamma}(1-\tau^{1/2}) - \epsilon^2 \left[\frac{\pi}{72}(4-3\tau^{3/2} - \tau^{1/2}) - \pi(1-\tau) \cdot \left(\frac{1}{24}\log\frac{3}{2} + \frac{1}{48}\log\tau + \frac{1}{\sqrt{\pi}}\left(\frac{n}{3} - \frac{5}{6}m\right) - \frac{3}{24} \right) + \frac{\pi}{24}(\tau^{3/4} + \tau^{1/4} - 2) \right]$$

$$m = \int_0^\infty x^2 \, e^{-x^2} \log x \, dx$$

$$n = \int_0^\infty x^4 \, e^{-x^2} \log x \, dx \tag{4.9}$$

and the reflected distribution on the surface is given by

$$\tilde{f}_r = \tilde{N}_i \frac{9}{2\pi} \tau^2 \exp\left(-\frac{3}{2}\tau \tilde{\xi}^2\right) \tag{4.10}$$

Since the free flow heat transfer is known, it is only necessary to calculate that due to $\tilde{f}_2$, $\tilde{f}_3$, and $\tilde{f}_r$. Observe

$$\tilde{q}_r = \int_0^\infty |\tilde{\xi}|^5 \, d|\tilde{\xi}| \int_\Omega f \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi \tag{4.11}$$

where f is the distribution function on the surface of the sphere.

$$\tilde{q}_r = \int_0^\infty |\tilde{\xi}|^5 \, d|\tilde{\xi}| \int_{\Omega^*_1} \tilde{f}_i \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi + \int_0^\infty |\tilde{\xi}|^5 \, d|\tilde{\xi}| \int_{\Omega^*} \tilde{f}_r \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi \tag{4.12}$$

Define

$$\tilde{N}_i = \tilde{N}_{i1} + \epsilon \tilde{N}_{i2} + \epsilon^2 \tilde{N}_{i3}$$

$$\tilde{f}_{rj} = \tilde{N}_{ij} \frac{9}{2\pi} \tau^2 \exp(-\frac{3}{2} \tau \tilde{\xi}^2)$$

$$\tilde{f}_i = \tilde{f}_{i1} + \epsilon \tilde{f}_{i2} + \epsilon^2 \tilde{f}_{i3}$$

Then

$$\tilde{q}_r = \tilde{q}_{r1} + \epsilon \tilde{q}_{r2} + \epsilon^2 \tilde{q}_{r3}$$

$$\tilde{q}_{r1} = \text{free flow heat transfer}$$

$$\tilde{q}_{rj} = \int_0^\infty |\tilde{\xi}|^5 d|\tilde{\xi}| \int_{\Omega^{*}_1} \tilde{f}_{ij} \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi + \int_0^\infty |\tilde{\xi}|^5 d|\tilde{\xi}| \int_{\Omega^{*}} \tilde{f}_{rj} \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi$$

Evaluate the above integrals, using the fact that the expressions m and n may be found in terms of the Euler psi function, which is tabulated, along with several identities for ψ. [18, p.197 (3.723)], [19, pp.255-258]

$$m = \frac{\sqrt{\pi}}{2} \left(2 + \psi(\tfrac{1}{2})\right)$$

$$n = \frac{3}{4} \sqrt{\pi} \left(\frac{2}{3} + 2 + \psi(\tfrac{1}{2})\right)$$

$$\psi(\tfrac{1}{2}) \doteq -1.96$$

$$\tilde{q}_r = \frac{8\pi}{27}\left[\frac{1}{\tau} - 1\right] + \frac{\epsilon}{36\Gamma\tau}\left[4 - 3\tau - \tau^{1/2}\right] + \epsilon^2 \left[\frac{\pi}{54\tau} \cdot (3\tau^{5/2} + \tau^{1/2} - 4) + \frac{\pi(1-\tau)}{\tau}\left(.069 + \frac{1}{36}\log\tau - \frac{\tau^{1/2}}{6}\right) + \frac{\pi}{36\tau}\left(3\tau^{3/4} + \tau^{5/4} + 2\tau^{1/4} - 2\tau - 4\right)\right] \quad (4.13)$$

To obtain the total heat transfer to the sphere, use

$$Q = \tilde{q}_r \cdot \text{area of sphere} = \tilde{q}_r \cdot 4\pi \tag{4.14}$$

In order to compare the above result with that found by other investigators, make the further assumption that

$$\left.\begin{aligned} \tau &= 1 - \Delta \\ \Delta &<< 1 \end{aligned}\right\} \tag{4.15}$$

That is to say that the body is only slightly hot compared to the free stream. Introducing (4.15) into (4.13), retaining only the linear terms in Δ yields the following expression for $\tilde{q}_r$

$$\tilde{q}_r = \frac{8\pi}{27}\Delta + \frac{7\Delta\epsilon}{72\Gamma} - .228\,\pi\,\Delta\,\epsilon^2 \tag{4.16}$$

Now normalize by $\frac{8\pi}{27}$ so that this may be equivalent to that obtained by Lees in his paper on the moment method.[8]

$$\frac{27}{8\pi}\tilde{q}_r = \Delta\{1 + .316\epsilon - .770\epsilon^2\} \tag{4.17}$$

whereas, Lees finds

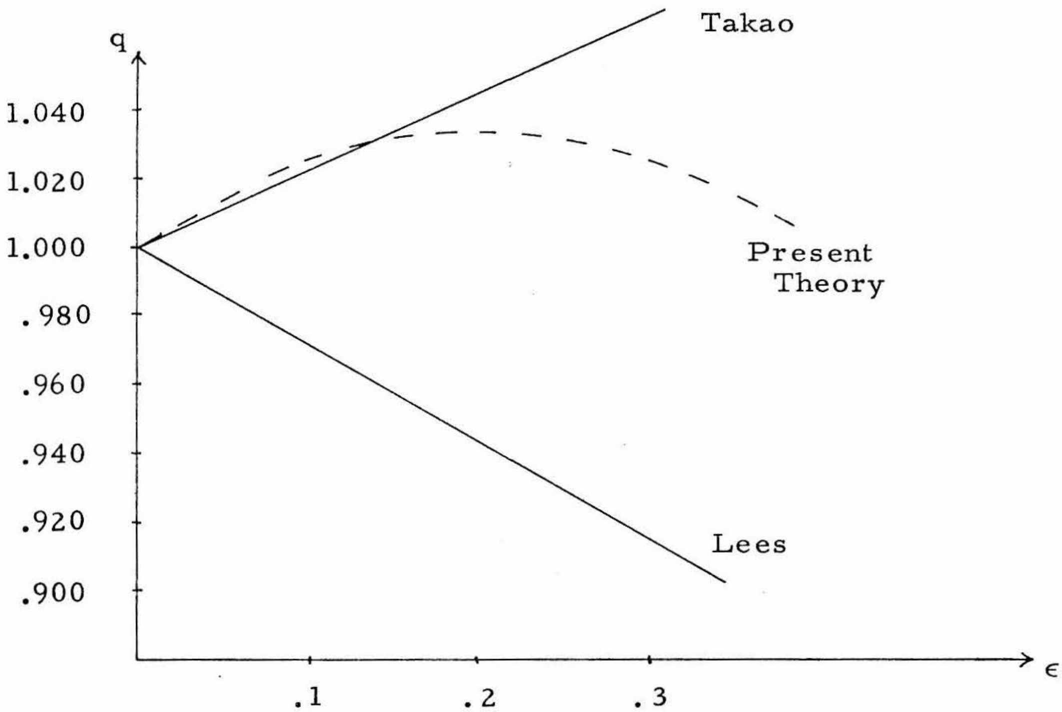
$$\frac{27}{8\pi}\tilde{q}_r = \frac{\Delta}{1+\frac{4}{15}\epsilon} \doteq \Delta\,\{1 - .267\epsilon\} \tag{4.18}$$

It is at first discouraging to note the difference in sign found in the $O(\epsilon)$ corrections. However, the moment method is of questionable validity for $\epsilon << 1$. In a slightly more complicated fashion, Takao [9] obtains a result which may be expressed as those above, after extensive investigation into his notation. According to a comparison of his theoretical calculation with experimental values he obtained, it appears that there is good agreement, even for small ϵ. In our notation, Takeo's result becomes

$$\frac{27}{8\pi}\,\tilde{q}_r = \Delta\left\{1 + .216\,\epsilon\right\} \tag{4.19}$$

Note here that the sign agrees with that in (4.17).

For the sake of comparison, the following figure illustrates (4.17), (4.18), (4.19). The experimental points are omitted since they are for a diatomic gas.



$$q = \frac{27}{8\pi}\,\tilde{q}_r \cdot \frac{1}{\Delta}$$

Figure III

In Appendix III it is shown that in the continuum limit q may be written as

$$q = \frac{f}{\epsilon}$$

where f is a numerical factor close to unity. The next figure shows the behavior of q from the very near free flow region, where the present theory is valid, through the near free flow and transition regions, where the theory of Lees and Takao holds, to the continuum, where the heat transfer is found by a simple physical calculation.

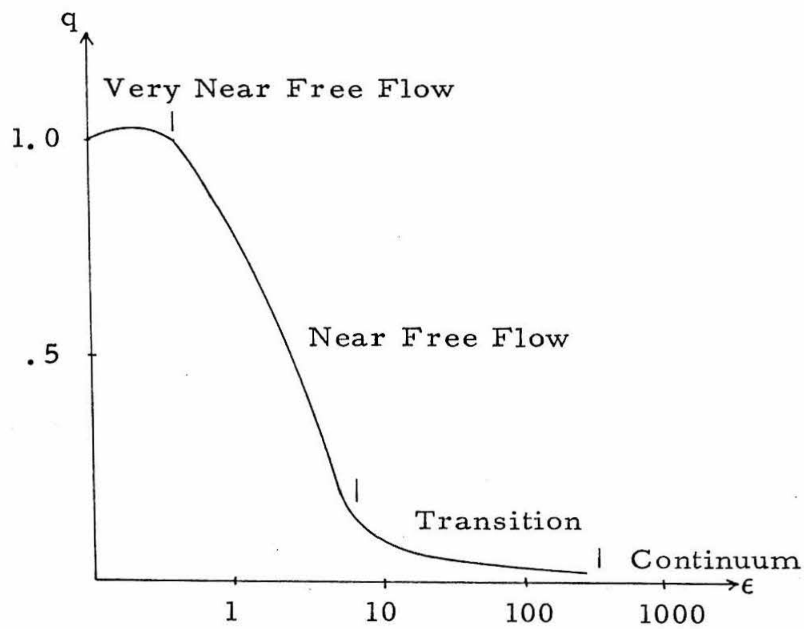


Figure IV

PART V

CONCLUSIONS AND FURTHER COMMENTS

From a comparison of the present theory with existing ones, reasonable results have been obtained for the heat transfer from the sphere. This confirms the apparent increase of q over its free flow value for a small range of ϵ, approximately (0, .4), found also by Takao. It does not seem that there is any simple physical argument which can support this behavior, such as that which explains the eventual decrease in heat transfer caused by collisions (the shielding effect of molecules colliding in front of the body). In fact there are no experimental data at present which can either confirm or disprove the initial increase.

There is an interesting comparison between the sign of the first correction to q and the sign of the first correction to the hypersonic drag of a cold blunt body in nearly free flow. The author has recently examined a most interesting paper by Willis [13] dealing with this. The problem considered is nearly free flow drag of a cold blunt body in a hypersonic stream, and, in particular, sphere drag is analyzed. It is shown that the sign of the drag correction is different as obtained by the two methods considered: 1) the B-G-K model solved by the transform method of Rose and the first collision method of Baker and Charwat and 2) the B-G-K model solved by Knudsen iteration. Several reasons are given for the anomaly, and it is concluded that either Knudsen iteration is not valid for the problem or the B-G-K model fails in highly non-linear or non-equilibrium flows.

Referring critically to Rose's paper [11, p. 1267], however, can cast some doubt as to whether the anomaly really exists or not. In order to do her calculation of the drag, she obtains the moments ρ, T, and $\vec{u}$ through the transform method and then uses them to go back and find the distribution function. The latter calculation relies heavily on an integration along the characteristics of the Krook equation and cannot conceivably be correct since she has written down the wrong equation. In transforming the Boltzmann equation to spherical coordinates, the derivatives with respect to the velocity coordinates have been omitted. There may be justifiable grounds for leaving these terms out in some sort of far field analysis, yet many of her crucial steps, such as finding the source function, rely on being close to the body. It seems, then, that Knudsen iteration may have given the correct result.

Equations of Motion and State

It is to be noted in the final analysis that neither the equations of motion nor the equation of state were necessary to solve the problem. The equation of state, in a linearized form, follows from the moment definitions and the linearization of the distribution function (see 3.52). Matching shows that the dependence of the solutions on the equations of motion (through K_1) is non-existent, since $K_1 = 0$.

In addition, the existence of the logarithmic singularity in the present solution for f should be no mystery and is supported by Yu [20, p. 2473]. Although Yu has considered the Boltzmann equation and not the Krook model, he obtains terms of the form r^n and $r^n \log r$ in agreement with those found above. He points out that the fluid behaves like a macroscopic fluid for large r, concurring with the present theory which indicates a r^{-1} decay in the density and temperature perturbations.

Extension of the Method

The perturbation method used in this thesis should be applicable to many similar problems in rarefied gasdynamics. In particular, it could be used to solve the hypersonic drag problems considered by

Rose above. Admittedly, the simplicity of the present work, due to the complete spherical symmetry involved, would not always be present; however, the case of sphere drag in a uniform free stream possesses cylindrical symmetry which makes the equations a bit easier to solve.

APPENDIX I

The Boltzmann Equation in Various Coordinate Systems

Expressed in cartesian coordinates, the Boltzmann equation is written

$$\xi \cdot \nabla f = \frac{\delta f}{\delta t} ,$$

where $\delta f/\delta t$ denotes the change in f due to collisions between particles. We are interested in the form of the left hand side of the equation when the coordinate system is not cartesian, but still an orthogonal curvilinear system. At first glance one might make the naive assumption that it is only necessary to express "∇" in the various coordinate systems. However, on further consideration, it is noted that in both cylindrical and spherical systems, the six independent variables on which f depends are not independent of each other and, thus, derivatives with respect to the velocity coordinates will now appear even in the absence of an external force field.

To give an example of the necessary calculations, the Boltzmann equation will now be transformed to cylindrical coordinates.

$$f = f(x, y, z; \xi_x, \xi_y, \xi_z) = f(r, \phi, z; \xi_r, \xi_\phi, \xi_z) .$$

In transforming the derivatives, we use

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial q_j} \frac{\partial q_j}{\partial x_i}$$

$$q_j = (r, \phi, z; \xi_r, \xi_\phi, \xi_z)$$

$$r = (x^2 + y^2)^{\frac{1}{2}}$$

$$\sin\phi = y/r$$

$$\cos\phi = x/r$$

$$\phi = \tan^{-1} y/x$$
$$\xi_x = \xi_r \cos\phi - \xi_\phi \sin\phi$$
$$\xi_y = \xi_r \sin\phi + \xi_\phi \cos\phi$$
$$\xi_z = \xi_z;$$
$$\xi_r = \xi_x \cos\phi + \xi_y \sin\phi$$
$$\xi_\phi = -\xi_x \sin\phi + \xi_y \cos\phi$$
$$\xi_z = \xi_z$$

Calculating all the appropriate derivatives,

$$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi} + \frac{\sin\phi}{r} \xi_r \frac{\partial}{\partial \xi_\phi} - \frac{\sin\phi}{r} \xi_\phi \frac{\partial}{\partial \xi_r}$$
$$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial r} + \frac{\cos\phi}{r} \frac{\partial}{\partial \phi} - \frac{\cos\phi}{r} \xi_r \frac{\partial}{\partial \xi_\phi} + \frac{\cos\phi}{r} \xi_\phi \frac{\partial}{\partial \xi_r} .$$

Hence, the equation becomes

$$\frac{\delta f}{\delta t} = \xi_z \frac{\partial f}{\partial z} + \xi_r \frac{\partial f}{\partial r} + \frac{\xi_\phi}{r} \frac{\partial f}{\partial \phi} + \frac{\xi_\phi^2}{r} \frac{\partial f}{\partial \xi_r} - \frac{\xi_\phi \xi_r}{r} \frac{\partial f}{\partial \xi_\phi} .$$

We may arrive at the same result in a different manner by using the Lagrangian formulated Boltzmann equation developed by Pao [1, pp. 7-9]. Briefly, he derives the Boltzmann equation in general orthogonal curvilinear coordinates as follows:

Let $\overline{q} = (x_1, x_2, x_3; \xi_1, \xi_2, \xi_3)$ where $\xi_i = g_i \dot{x}_i$ (no sum) describe the dynamical state of a particle. If $f(x_i;\xi_i)$ is the distribution function of the flow, then the change of f along the particle's path is due to intermolecular collisions. We may write, then

$$\frac{df}{dt} = \frac{\delta f}{\delta t} .$$

Following f along the path of a particle, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\xi_i}{g_i} \frac{\partial f}{\partial x_i} + \dot{\xi}_i \frac{\partial f}{\partial \xi_i} \right) ,$$

where it is now necessary to evaluate $\dot{\xi}_i$ in terms of x_i and ξ_i. This may be done using Lagrange's equations. For no external force,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$L = T - V$

T = kinetic energy of the particle

V = potential energy of the particle

$$L = \left(\sum_{k=1}^{3} \frac{m}{2} \xi_k^2 \right) - V$$

m = mass of the particle.

Using Lagrange's equations, it follows that

$$\dot{\xi}_i = \frac{1}{g_i} \left[\sum_{k=1}^{3} \frac{\xi_k}{g_k} \left(\xi_k \frac{\partial g_k}{\partial x_i} - \xi_i \frac{\partial g_i}{\partial x_k} \right) - \frac{1}{m} \frac{\partial V}{\partial x_i} \right] ,$$

and finally,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \frac{\xi_i}{g_i} \frac{\partial f}{\partial x_i} + \sum_{i=1}^{3} \frac{1}{g_i} \left[\sum_{k=1}^{3} \frac{\xi_k}{g_k} \left(\xi_k \frac{\partial g_k}{\partial x_i} - \xi_i \frac{\partial g_i}{\partial x_k} \right) - \frac{1}{m} \frac{\partial V}{\partial x_i} \right] \frac{\partial f}{\partial \xi_i} = \frac{\delta f}{\delta t} .$$

In the cylindrical coordinate system,

$$(x_1, x_2, x_3) = (r, \phi, z)$$

$$(g_1, g_2, g_3) = (1, r, 1) ,$$

and exactly the same equation is found as on page 63.

In the spherical coordinate system,

$$(x_1, x_2, x_3) = (\rho, \theta, \phi)$$

$$(g_1, g_2, g_3) = (1, \rho, \rho\sin\theta) ,$$

and the Boltzmann equation becomes

$$\xi_\rho \frac{\partial f}{\partial \rho} + \frac{\xi_\theta}{\rho} \frac{\partial f}{\partial \theta} + \frac{\xi_\phi}{\rho \sin\theta} \frac{\partial f}{\partial \phi} + \frac{\xi_\theta^2 + \xi_\phi^2}{\rho} \frac{\partial f}{\partial \xi_\rho} + \frac{\xi_\phi^2 \cot\theta - \xi_\rho \xi_\theta}{\rho} \frac{\partial f}{\partial \xi_\theta} -$$

$$- \xi_\phi \frac{\xi_\rho + \xi_\theta \cot\theta}{\rho} \frac{\partial f}{\partial \xi_\phi} = \frac{\delta f}{\delta t} .$$

The above result may be verified by a simple but somewhat tedious calculation similar to that on the previous pages.

APPENDIX II

Derivation of the Equations of Motion from the Linearized Krook Equation

The purpose of this appendix is to examine the equations of motion derived from the linearized Krook equation. The linearizations used are those of the outer region and the distribution function is expanded about f_∞^*. Normalization also follows the pattern in the outer region. In this case, the fluid is presumed to have a non-zero macroscopic velocity $\vec{u}$. The moments are defined as follows:

density

$$\rho = \int f d^3\xi$$

velocity

$$\rho\vec{u} = \int \xi f d^3\xi$$

pressure tensor

$$\mathbb{P} = \int (\xi-\vec{u})_i(\xi-\vec{u})_j f d^3\xi$$

pressure

$$p = \frac{1}{3}\,\mathrm{trace}\ \mathbb{P}$$

heat flux vector

$$q = \frac{1}{2}\int (\xi-\vec{u})^2(\xi-\vec{u}) f d^3\xi \ .$$

The pressure tensor and heat flux vector may be linearized in the same way as the previous thermodynamic quantities.

$$\mathbb{P} = p_\infty\delta_{ij} + \mathbb{P}_1$$

$$\mathbb{P}_1 = \int c_i c_j g_2 d^3 c$$

$$f = f_\infty + g_2$$

$$c = \xi - \vec{u}$$

$$\rho = \rho_\infty + \rho_1$$

$$p = p_\infty + p_1$$

$$T = T_\infty + T_1$$

$$q = \frac{1}{2}(Q - 3\vec{u}_1 p + 2\vec{u}_1 \cdot \mathbb{P})$$

$$\vec{u} = \vec{u}_\infty + \vec{u}_1$$

$$Q = \frac{1}{3}\int c_i^2 c_j g_2 d^3 \xi$$

The linearized Krook equation is written

$$\xi \cdot \nabla g_2 = \frac{\delta g_2}{\delta t} ,$$

and to obtain the equations of motion, it is multiplied by 1 , c , and c^2 , and then integrated over ξ . After the first integration

$$\nabla \cdot (\vec{u}_1 \rho_\infty + \vec{u}_\infty \rho_1) = 0 ,$$

which is the linearized continuity equation, where the expansion is about the infinity variables. The second integration yields

$$\nabla \cdot (\rho_\infty \vec{u}_\infty \vec{u}_1 + \mathbb{P}_1) = 0 .$$

This equation is obtained from the momentum equation by linearization and application of the continuity equation. When the Krook equation is multiplied by c^2 and integrated over ξ , the equation of motion resulting from this moment is

$$\nabla \cdot \{ Q + \vec{u}_{\infty} p_1 \} = 0 .$$

After appropriate linearization and use of some vector identities, the Navier-Stokes energy equation reduces to the following form

$$\nabla \cdot \left\{ Q + \frac{2}{3} p_{\infty} \vec{u}_1 \left(\frac{1}{\gamma - 1} - \frac{3}{2} \right) + \frac{2\vec{u}_{\infty} p_1}{3(\gamma - 1)} \right\} = 0 .$$

To be equivalent to the above derived equation of motion, we must choose $\gamma = 5/3$, the value appropriate to a monatomic gas.

APPENDIX III

Heat Transfer in the Continuum Limit

The equation for the heat flux is

$$q_r = -K \frac{dT}{dr}$$

Integrating the equation under the restrictions,

$$T = T_a \qquad \text{at } r = a$$

$$T = T_\infty \qquad \text{at } r = \infty$$

$$q_r = Q/4\pi r^2 \qquad \text{(spherical symmetry)},$$

we obtain

$$Q = 4\pi a K[T_a - T_\infty] .$$

Normalizing and making the substitution

$$\frac{T_\infty}{T_a} = 1 - \Delta = \tau$$

$$\Delta << 1$$

the heat transfer may be written at the surface of the body

$$\tilde{q}_r = \frac{\Delta}{\Pr \mathrm{Re} \frac{3\Gamma}{2}} \left(\frac{\gamma}{\gamma - 1} \right)$$

where use has been made of the expressions

$$\Pr = c_p \mu/K$$

$$\mathrm{Re} = c_\infty a/\nu .$$

From kinetic theory

$$K = d_1 \mu c_v$$

$$\nu = d_2 \cdot \tfrac{1}{2}\bar{c}\lambda$$

$$\bar{c} = c_\infty/1.09$$

d_1, d_2 are numerical factors close to unity.

Finally, q, as plotted in Figure IV, can be written

$$\frac{27}{8\pi} \cdot \frac{1}{\Delta} \tilde{q}_r = q \sim \frac{1}{\epsilon} .$$

LIST OF SYMBOLS

General:

a	sphere radius
λ	mean free path
ϵ	a/λ (inverse Knudsen number)
ρ	density
T	temperature
q_r	radial heat transfer
T_a	body temperature
T_∞	temperature at infinity
τ	T_∞/T_a
R	gas constant
c_∞	$(3RT_\infty)^{3/2}$ mean square speed at infinity
ρ_∞	density at infinity
f_∞	$[\rho_\infty/(2\pi RT_\infty)^{3/2}]\exp(-\xi_2/2\ \ T_\infty)$
r	spherical radius
ξ_r	radial velocity component
ξ_T	tangential velocity component
w	$-r\xi_r/\xi^2$ characteristic coordinate
$\eta(\epsilon)$	intermediate expansion parameter
$\Omega^{*}{}', \Omega^{*}$	divisions of the velocity space
A	$\rho_\infty/(2\pi RT_\infty)^{3/2}$

Numerical Constants:

$\overline{A}$	$\frac{1}{8\tau^{\frac{1}{2}}}[5\tau - 2\tau^{\frac{1}{2}} - 3]$

$\overline{B}$	$\frac{1}{4\tau^{\frac{1}{2}}}[\tau-1]$
$\overline{\sigma}$	$\Gamma\frac{\pi}{3}[1-\tau]$
Γ	$(\frac{3}{2\pi})^{3/2}$

Inner Normalization:

$\tilde{r}$	r/a
$\tilde{\xi}$	ξ/c_{∞}
$\tilde{\rho}$	ρ/ρ_{∞}
$\tilde{T}$	T/T_a
$\tilde{f}$	f/A
$\tilde{w}$	$-\tilde{r}\tilde{\xi}_r/\tilde{\xi}^2$
$\tilde{c}_1$	$\lvert\tilde{\xi}\rvert$
$\tilde{c}_2$	$-\tilde{r}\tilde{\xi}_T/\tilde{\xi}^2$
$\tilde{q}_r$	$2q_r/\Gamma\rho_{\infty}c_{\infty}^3$

Outer Normalization:

r^*	r/λ
ξ^*	ξ/c_{∞}
ρ^*	ρ/ρ_{∞}
T^*	T/T_{∞}
f^*	f/A
w^*	$-r^*\xi_r^*/\xi^{*2}$
c_1^*	$\lvert\xi^*\rvert$
c_2^*	$-r^*\xi_T^*/\xi^{*2}$
P_{11}^*	outer pressure perturbation

Intermediate Variables and Constants:

w_η	$-r_\eta \tilde{\xi}_r / \dot{\tilde{\xi}}^2 = -r_\eta \xi^*_r / \xi^{*2}$
r_η	$r/\eta(\epsilon)$
$c_{2\eta}$	$w_\eta \tilde{\xi}_T / \tilde{\xi}_r = w_\eta \xi^*_T / \xi^*_r$

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