SOME APPROXIMATE SOLUTIONS OF DYNAMIC PROBLEMS IN THE LINEAR THEORY OF THIN ELASTIC SHELLS

Thesis by

Michael Philip Mortell

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1968

Submitted October 19, 1967

ACKNOWLEDGEMENTS

The author is pleased to acknowledge the contribution of his research adviser, Dr. J.K. Knowles. It was he who suggested the topic and the philosophy of the approach, and he gave freely of his time throughout.

This research was supported by a Graduate Teaching Assistantship from the California Institute of Technology, and also by funds from the Office of Naval Research and the National Science Foundation. The author expresses his appreciation of this support.

Finally, I wish to thank my wife, Pat, for typing the thesis and relieving me of many details.

ABSTRACT

Some aspects of wave propagation in thin elastic shells are considered. The governing equations are derived by a method which makes their relationship to the exact equations of linear elasticity quite clear. Finite wave propagation speeds are ensured by the inclusion of the appropriate physical effects.

The problem of a constant pressure front moving with constant velocity along a semi-infinite circular cylindrical shell is studied. The behavior of the solution immediately under the leading wave is found, as well as the short time solution behind the characteristic wavefronts. The main long time disturbance is found to travel with the velocity of very long longitudinal waves in a bar and an expression for this part of the solution is given.

When a constant moment is applied to the lip of an open spherical shell, there is an interesting effect due to the focusing of the waves. This phenomenon is studied and an expression is derived for the wavefront behavior for the first passage of the leading wave and its first reflection.

For the two problems mentioned, the method used involves reducing the governing partial differential equations to ordinary differential equations by means of a Laplace transform in time. The information sought is then extracted by doing the appropriate asymptotic expansion with the Laplace variable as parameter.

TABLE OF CONTENTS

If people bring so much courage to this world the world has to kill them to break them, so of course it kills them. The world breaks every one and afterwards many are strong at the broken places. But those that will not break it kills.

Hemingway

INTRODUCTION

The work presented here deals with approximations to the solutions of two transient dynamical problems, one involving a thin elastic cylindrical shell and the other a spherical shell.

The shells are described by approximate theories which include the effects of transverse shear deformation and rotatory inertia. This results in a system of hyperbolic equations, and thus disturbances in the shells are propagated with finite speed. The equations used for the cylindrical shell are essentially those derived by Naghdi and Cooper [1]. Prasad [2] gave a set of equations for a spherical shell, but some important terms had been neglected. These omissions are rectified here, and the resulting equations are found to agree exactly with those derived by Naghdi [3]. The latter author derived his equations in a general coordinate system using a variational principle due to Reissner [4]. The equations used here are obtained directly from the three-dimensional theory of elasticity.

The development of the subject has followed a definite pattern. Once the equations for spherical shells were derived, most of the work concentrated on the natural frequencies for the various theories. We make mention of Lamb [5], Silbiger [6], and Baker [7], who discussed the membrane theory. Kalnins [8] investigated the effect of bending on the frequency spectrum, and Wilkinson [9,10] extended the latter's analysis to include the effects of transverse shear deformation and rotatory inertia.

The representation of the transient response in terms of these modes was done principally by Prasad [2] and Wilkinson and Kalnins [11]. On the basis of the membrane theory, Huth and Cole [12] and Mann-Nachbar [13] studied the response of a spherical shell to an acoustic pressure wave using a modal approach. The method used in [13] did not lend itself to an analysis of the pressure distribution in the acoustic medium. Hayek took up this point and complemented the work in [14].

Investigations concerning the dynamical behavior of circular cylindrical shells proceeded along similar lines with initial efforts aimed at modeling more accurately the behavior at higher frequencies and shorter wavelengths. We mention the work of Herrmann and Mirsky [15], Lin and Morgan [16], and Naghdi and Cooper [1], all of whom compared the modes of the approximate theory with those predicted by the three dimensional theory of elasticity.

Payton [17, 18] treated transient propagation in the circumferential direction of a cylindrical shell by transform techniques. Berkowitz [19] studied the membrane theory of a longitudinal impact on a semi-infinite circular cylindrical shell. Using bending theory, Jones and Bhuta [20] examined the resonances involved in a ring load moving with constant velocity down such a shell. In a paper by Tang [21], the problem of the dynamic response of a cylindrical tube under internal moving pressure is studied. He obtains a steady-state solution which is very similar to certain of our results in Chapter II, and then analyzes the transient response by a numerical method. Finally, Keer, Fleming, and Hermann [22] extend Payton's work [17, 18] to include the bending effects, using a technique of Flugge and Zajac [23] for getting wave-front approximations and extending slightly their interval of validity.

The methods of attack on transient problems in the above references have been basically of two kinds. In one the solutions are represented in terms of an infinite series of modes, and in the other an integral transform is used. For the latter approach, inversion presents difficulties and various types of approximations on the inversion integral are introduced. The review [24] by Miklowitz gives an extensive account of the work of this type in the general area of elastic wave propagation.

The simplest concept of wave propagation is that of a wave progressing into a region of quiet in a stretched string. One is struck by the dearth of such representations in transient shell analysis.

As mentioned above, only a limited amount of information is gleaned even after the trouble of getting an exact representation of the Laplace transform of the solution. If, from the outset, we decide that only a certain limited type of information is wanted, e.g., wave-front behavior or long-time information, then it would seem more logical to work on the equations governing the motion and extract the information directly from them by the appropriate asymptotic procedure. By this means we would expect to bypass many extraneous details and in so doing keep the equations simple and capable of closed solutions in terms of convenient functions. The thesis presented here is a contribution in this direction.

Chapter I concerns itself with the derivation of a set of equations governing the motion of an elastic spherical shell. These equations are derived directly from the exact three-dimensional equations of the linear theory of elasticity for an homogeneous, isotropic body and include the effects of bending, transverse shear deformation, and rotatory inertia.

In Chapter II the problem of a pressure front moving with constant velocity down a semi-infinite circular cylindrical shell is treated.

The information sought is specifically confined to all wave-fronts in the problem. In particular, it is found that the main long time contribution travels with the velocity of very long longitudinal waves in a bar $\left(=\sqrt{\frac{E}{\rho}}\right)$ where E is Young's modulus and ρ is the density of the material of the shell. An expression for the behavior of the solution in this region is given.

We consider a constant moment applied to the lip of an incomplete spherical shell in Chapter III. Attention is confined to the leading wave-front only. Expressions are given for the wave-front behavior of the moment as it moves down the shell, into the pole $\theta = \pi$, and is then reflected.

The effects of bending in the problems treated can be traced through a parameter $\lambda_0 = \varepsilon^{-1}$.

CHAPTER I.

A set of equations governing the motion of a thin elastic spherical shell is derived from the equations of the three-dimensional linear theory of elasticity for an homogeneous isotropic body. The object is to find a theory which preserves the finiteness of the speed of propagation of disturbances. This is done by taking account of the effect of transverse shear deformation and rotatory inertia and results in a system of totally hyperbolic partial differential equations. The theory thus corresponds in certain qualitative respects to the Timoshenko theory for beams [25].

§1.1. Notation

Let (r, θ, φ) be spherical polar coordinates (Fig. 1) related to the (x,y,z) cartesian coordinate system by

$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

$$0 \le \theta \le \pi, \qquad 0 \le \varphi \le 2\pi, \qquad r \geq 0.$$

$\hat{r}$, $\hat{\theta}$, $\hat{\varphi}$ are unit vectors in the directions of increasing r, θ, φ respectively. X_r is the component of the vector $\underline{X}$ in the direction of $\hat{r}$. X_θ, X_φ are similarly defined.

For convenience we list here the notation to be employed in the derivation to follow.

$X_{,y}$ denotes $\frac{\partial X}{\partial y}$; $X_{,yz} \equiv \frac{\partial^2 X}{\partial y \partial z}$

$\underline{F} = (F_r, F_\theta, F_\varphi)$ is the body force vector per unit volume.

h is the thickness of the shell.

$r = R$ is the equation of the midsurface of the shell.

h and R are both constant.

ρ is the density of the material of the shell.

$\underline{u} = (u_r, u_\theta, u_\varphi)$ is the displacement vector.

$\underline{f} = (f_r, f_\theta, f_\varphi)$ is the acceleration vector, where

$$f_r = u_{r,tt} \quad , \quad f_\theta = u_{\theta,tt} \quad , \quad f_\varphi = u_{\varphi,tt} .$$

(u, v, w) is the displacement vector for the midsurface of the shell.

$\beta_\theta, \beta_\varphi$ are the rotations of the normal to the midsurface in the $\hat{\theta}$, $\hat{\varphi}$ directions respectively, during deformation.

λ, μ are the Lame constants for the material of the shell.

$\nu = \frac{\lambda}{2(\lambda+\mu)}$ is Poisson's ratio.

$e_{rr}, e_{r\theta}, e_{\theta\varphi}$ etc., are the strains.

$t_{rr}, t_{r\theta}, t_{\theta\varphi}$ etc., are the stresses.

$N_\theta, N_\varphi, N_{\theta\varphi}, Q_\theta, Q_\varphi$ are the stress resultants.

$M_\theta, M_\varphi, M_{\theta\varphi}$ are the stress couples.

$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is Young's modulus.

$D = \frac{Eh^3}{12(1-\nu^2)}$ is the bending rigidity of the shell.

$c_p = \sqrt{\frac{E}{\rho(1-\nu^2)}}$ is the speed of propagation of longitudinal waves in a plate.

$c_Q = \sqrt{\mu/\rho}$ is the speed of propagation of distortion waves in an unbounded medium.

$k_1 = 1 + \frac{1}{12}\left(\frac{h}{R}\right)^2$.

$k_2 = 1 + \frac{3}{20}\left(\frac{h}{R}\right)^2$.

Non-dimensional Quantities.

$$N_\theta' = \frac{1-\nu^2}{Eh} N_\theta \quad , \quad N_\varphi' = \frac{1-\nu^2}{Eh} N_\varphi \, ,$$

$$M_\theta' = \frac{R}{D} M_\theta \quad , \quad Q_\theta' = \frac{1-\nu^2}{Eh} Q_\theta \, ,$$

$$u' = \frac{u}{R} \quad , \quad w' = \frac{w}{R} \quad , \quad \beta_\theta = \beta$$

$$t' = \frac{t\, c_p}{R}$$

$$q' = \frac{1-\nu^2}{Eh} R q$$

where

$$q = \left[\left(\frac{r}{R}\right)^2 t_{rr}\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}$$

is the applied load per unit area in the $\vec{r}$ direction.

$\left[f(\xi)\right]_a^b$ means f(b) - f(a).

$$\varepsilon^2 = \frac{1}{12}\left(\frac{h}{R}\right)^2 \quad , \quad \lambda_0^2 = \varepsilon^{-2} .$$

§ 1.2. The Equations of the Classical Three-dimensional Linear Theory of Elasticity.

The momentum equations expressed in spherical polar coordinates are as follows [26].

$$t_{rr,r} + \frac{1}{r\sin\theta} t_{r\varphi,\varphi} + \frac{1}{r} t_{r\theta,\theta} + \frac{1}{r}\{2t_{rr} - t_{\varphi\varphi} - t_{\theta\theta} + t_{r\theta}\cot\theta\}$$

$$+ F_r = \rho f_r , \tag{1.1}$$

$$t_{r\theta,r} + \frac{1}{r\sin\theta} t_{\theta\varphi,\varphi} + \frac{1}{r} t_{\theta\theta,\theta} + \frac{1}{r}\{3t_{r\theta} + (t_{\theta\theta} - t_{\varphi\varphi})\cot\theta\}$$

$$+ F_\theta = \rho f_\theta , \tag{1.2}$$

$$t_{r\varphi,r} + \frac{1}{r\sin\theta} t_{\varphi\varphi,\varphi} + \frac{1}{r} t_{\theta\varphi,\theta} + \frac{1}{r}\{3t_{r\varphi} + 2t_{\theta\varphi}\cot\theta\}$$

$$+ F_\varphi = \rho f_\varphi . \tag{1.3}$$

The stress-strain relations are

$$\begin{aligned}
t_{rr} &= (\lambda + 2\mu) e_{rr} + \lambda(e_{\theta\theta} + e_{\varphi\varphi}) \\
t_{\theta\theta} &= (\lambda + 2\mu) e_{\theta\theta} + \lambda(e_{rr} + e_{\varphi\varphi}) \\
t_{\varphi\varphi} &= (\lambda + 2\mu) e_{\varphi\varphi} + \lambda(e_{rr} + e_{\theta\theta}) \\
t_{\theta\varphi} &= 2\mu e_{\theta\varphi} \\
t_{r\varphi} &= 2\mu e_{r\varphi} \\
t_{r\theta} &= 2\mu e_{r\theta} .
\end{aligned} \tag{1.4}$$

The strain-displacement equations are

$$e_{rr} = u_{r,r}$$

$$e_{\theta\theta} = \frac{1}{r} u_{\theta,\theta} + \frac{1}{r} u_r$$

$$e_{\varphi\varphi} = \frac{1}{r\sin\theta} u_{\varphi,\varphi} + \frac{1}{r} u_r + \frac{1}{r} \cot\theta\, u_\theta$$

$$e_{r\varphi} = \frac{1}{2}\left(\frac{1}{r\sin\theta} u_{r,\varphi} - \frac{1}{r} u_\varphi + u_{\varphi,r}\right) \tag{1.5}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r} u_{r,\theta} - \frac{1}{r} u_\theta + u_{\theta,r}\right)$$

$$e_{\varphi\theta} = \frac{1}{2}\left(\frac{1}{r} u_{\varphi,\theta} - \frac{1}{r}\cot\theta\, u_\varphi + \frac{1}{r\sin\theta} u_{\theta,\varphi}\right)$$

§1.3. Momentum Equations in Terms of Stress Resultants and Couples.

The midsurface of the spherical shell is defined by $r = R$, and its thickness is h. The stress resultants (representing the force per unit length along the midsurface) are defined by integrating the stresses through the thickness of the shell. Thus

$$N_\theta = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R} t_{\theta\theta}\, dr$$

$$N_\varphi = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R} t_{\varphi\varphi}\, dr \tag{1.6}$$

$$N_{\theta\varphi} = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R} t_{\theta\varphi}\, dr$$

$$Q_\theta = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R}\, t_{r\theta}\, dr$$

$$Q_\varphi = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R}\, t_{r\varphi}\, dr \quad . \tag{1.6}$$

Similarly, the stress couples are defined by

$$M_\theta = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R}(r-R)\, t_{\theta\theta}\, dr$$

$$M_\varphi = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R}(r-R)\, t_{\varphi\varphi}\, dr \tag{1.7}$$

$$M_{\theta\varphi} = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R}(r-R)\, t_{\theta\varphi}\, dr \quad .$$

The momentum equations (1.1), (1.2), (1.3) are integrated through the thickness of the shell and advantage is then taken of the definitions (1.6) and (1.7) to rewrite them in a form in which the r dependence has been removed. It is just this that we have in mind when we use the phrase "derived directly from the three-dimensional linear theory of elasticity."

As a sample case, (1.1) is multiplied through by r^2/R and then integrated with respect to r over $(R-\frac{1}{2}h,\ R+\frac{1}{2}h)$.

$$\int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R}\Big[t_{rr,r} + \frac{1}{r\sin\theta}\, t_{r\varphi,\varphi} + \frac{1}{r}\, t_{r\theta,\theta}$$

$$+\frac{1}{r}\left\{ 2t_{rr} - t_{\varphi\varphi} - t_{\theta\theta} + t_{r\theta}\cot\theta \right\}\Big]\, dr$$

$$+ \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R} F_r\, dr \quad = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho\frac{r^2}{R} f_r\, dr .$$

Using definitions (1.6), this equation is

$$\int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R} t_{rr,r}\, dr \;+ \operatorname{cosec}\theta\; Q_{\varphi,\varphi} \;+ Q_{\theta,\theta} \;+ Q_\theta \cot\theta$$

$$-(N_\theta + N_\varphi) \;+ \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} 2\frac{r}{R}\, t_{rr}\, dr \;+ \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R} F_r\, dr$$

$$= \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho\,\frac{r^2}{R}\, f_r\, dr.$$

Integration by parts gives

$$\int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \Big[\frac{r^2}{R} t_{rr,r} + 2\frac{r}{R}\, t_{rr}\Big] dr \;= \Big[\frac{r^2}{R}\, t_{rr}\Big]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} .$$

Then the equation is

$$\Big[\frac{r^2}{R}\, t_{rr}\Big]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} + Q_{\theta,\theta} + Q_{\varphi,\varphi}\operatorname{cosec}\theta + Q_\theta\cot\theta - (N_\theta + N_\varphi)$$

$$+\int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R} F_r\, dr \;= \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho\,\frac{r^2}{R}\, f_r\, dr \,. \tag{1.8}$$

In the same way, equations (1.2) and (1.3) lead to

$$\left[\frac{r^2}{R} t_{r\theta}\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} + N_{\theta\varphi,\varphi} \operatorname{cosec}\theta + N_{\theta,\theta} + Q_\theta + (N_\theta - N_\varphi)\cot\theta + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R} F_\theta \, dr = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho \frac{r^2}{R} f_\theta \, dr \tag{1.9}$$

and

$$\left[\frac{r^2}{R} t_{r\varphi}\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} + N_{\varphi,\varphi} \operatorname{cosec}\theta + N_{\theta\varphi,\theta} + Q_\varphi + 2N_{\theta\varphi}\cot\theta + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R} F_\varphi \, dr = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho \frac{r^2}{R} f_\varphi \, dr. \tag{1.10}$$

The stress couples, defined by (1.7), are used as follows: Equation (1.3) is multiplied through by $\frac{r^2}{R}(r-R)$ and integrated with respect to r over $(R-\frac{1}{2}h,\ R+\frac{1}{2}h)$.

$$\int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R}(r-R)\Big[t_{r\varphi,r} + \frac{1}{r\sin\theta} t_{\varphi\varphi,\varphi} + \frac{1}{r} t_{\theta\varphi,\theta} + \frac{1}{r}\left(3t_{r\varphi} + 2t_{\theta\varphi}\cot\theta\right)\Big] dr + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r^2}{R}(r-R) F_\varphi \, dr = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho \frac{r^2}{R}(r-R) f_\varphi \, dr.$$

Utilizing (1.7), this equation, after a little manipulation, may be re-written as

$$\left[t_{r\varphi}\ \frac{r^2}{R}(r-R)\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} - \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\left[3\,\frac{r}{R}(r-R)\,t_{r\varphi} + R\,\frac{r}{R}\,t_{r\varphi}\right]dr$$

$$+ M_{\varphi,\varphi}\,\operatorname{cosec}\theta + M_{\theta\varphi,\theta} + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} 3\,\frac{r}{R}(r-R)\,t_{r\varphi}\,dr$$

$$+ 2M_{\theta\varphi}\cot\theta + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\frac{r^2}{R}(r-R)\,F_\varphi\,dr$$

$$= \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\rho\,\frac{r^2}{R}(r-R)\,f_\varphi\,dr.$$

With the aid of (1.6), this reduces to

$$\left[t_{r\varphi}\ \frac{r^2}{R}(r-R)\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} - R\,Q_\varphi + M_{\varphi,\varphi}\,\operatorname{cosec}\theta + M_{\theta\varphi,\theta}$$

$$+ 2M_{\theta\varphi}\cot\theta + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\frac{r^2}{R}(r-R)\,F_\varphi\,dr \qquad (1.11)$$

$$= \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\rho\,\frac{r^2}{R}(r-R)\,f_\varphi\,dr.$$

Following the same procedure, we get from (1.2) that

$$\left[t_{r\theta}\ \frac{r^2}{R}(r-R)\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} - R\,Q_\theta + M_{\theta\varphi,\varphi}\,\operatorname{cosec}\theta + M_{\theta,\theta}$$

$$+ (M_\theta - M_\varphi)\cot\theta + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\frac{r^2}{R}(r-R)\,F_\theta\,dr \qquad (1.12)$$

$$= \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\rho\,\frac{r^2}{R}(r-R)\,f_\theta\,dr.$$

At this stage we point out that equations (1.8) to (1.12) are exact and no approximations have yet been made.

1.4. The Basic Approximations of the Theory. Expressions for Stress Resultants and Couples, and Inertia Terms.

We shall now derive a theory of shells based on three assumptions.

Assumption (a): $\frac{h}{R} \ll 1$.

Assumption (b): The Transverse Normal Stress, t_{rr}, on the curved surface of the shell is negligible when compared with $t_{\theta\theta}$ and $t_{\varphi\varphi}$ throughout the thickness of the shell.*

Assumption (c): $u_r \sim w(\theta, \varphi, t)$, independent of r .

$u_\theta \sim u(\theta,\varphi,t) + (r-R)\beta_\theta(\theta,\varphi,t)$, linear in r .

$u_\varphi \sim v(\theta,\varphi,t) + (r-R)\beta_\varphi(\theta,\varphi,t)$

Assumption (a) is just the definition of a thin shell, i.e., that the thickness of the shell is very small compared with the radius of its middle surface.

We give the background to assumption (b) by quoting firstly from Landau and Lifshitz [27], p.44. They are discussing a thin plate referred to cartesian coordinates and the z-axis is normal to the surface of the undeformed plate.

> Since the plate is thin, comparatively small forces on its surface are needed to bend it. These forces are always considerably less than the internal stresses

*Since t_{rr} on the surface of the shell is the applied pressure, it must be verified a posteriori that the applied pressure is negligible compared with $t_{\theta\theta}$, $t_{\varphi\varphi}$. That this is so in the very simple case of the equilibrium of a spherical shell under a uniform external pressure is shown in 7-7 of [28]. Reference may also be made to the results of §2.6.

> caused in the deformed plate by the extension and compression of its parts. ... Thus we must have on both surfaces of the plate $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. Since the plate is thin, however, these quantities must be small within the plate if they are zero on each surface.

Secondly, Fung[29], p. 456, says,

> A principal feature in straining a plate or shell is the relative smallness of tractions acting on surfaces parallel to the middle surface as compared with the maximum bending or stretching stresses in the body. ... When a plate is very thin, the smallness of tractions on the external faces implies the smallness of tractions on any surface parallel to the middle surface.

Further evidence in support of the reasonableness of assumption (b) is given by Sokolnikoff[26], p. 255, where he discusses the concept of plane stress in relation to a plate. Friedrichs and Dressler[30], when examining the equilibrium of a thin plate under normal pressure on the surfaces, exhibit explicitly the fact that t_{zz} throughout the plate is negligible compared with t_{xx} and t_{yy} (the midsurface of the plate is in the x-y plane).

Boundary layer phenomena which are typical for thin cylindrical or spherical shells are associated with the length $\sqrt{Rh}$. Furthermore, it is outside the domain of thin shell theory to include the effects of stresses which have rapid variation over a length of the order of the thickness of the shell[31]. Hence, the fact that t_{rr} is negligible on the curved surface of the shell implies that t_{rr} is negligible throughout the shell.

Assumption (c) incorporates the following statements.

(i) Plane sections remain plane after deformation and

(ii) normals to the midsurface before deformation are not necessarily normal to it after deformation.

It is this latter remark which allows us to include the effects of transverse shear and rotatory inertia.

By inverting the first three of equation (1.4), the strains e_{rr}, $e_{\theta\theta}$, and $e_{\varphi\varphi}$ may be written as linear functions of the stresses t_{rr}, $t_{\theta\theta}$, and $t_{\varphi\varphi}$. By assumption (b), t_{rr} is neglected in comparison with $t_{\theta\theta}$ and $t_{\varphi\varphi}$. We then have the three strain components e_{rr}, $e_{\theta\theta}$, and $e_{\varphi\varphi}$ in terms of the two stress components $t_{\theta\theta}$ and $t_{\varphi\varphi}$. Then e_{rr} may be eliminated in favor of $e_{\theta\theta}$ and $e_{\varphi\varphi}$. On inverting these final equations, we find

$$\begin{aligned} t_{\theta\theta} &= \frac{E}{1-\nu^2}\left(e_{\theta\theta} + \nu\, e_{\varphi\varphi} \right) \\ t_{\varphi\varphi} &= \frac{E}{1-\nu^2}\left(e_{\varphi\varphi} + \nu\, e_{\theta\theta} \right) \end{aligned} \tag{1.13}$$

Note, further, that

$$\begin{aligned} t_{\theta\varphi} &= 2\mu\, e_{\theta\varphi} = \frac{E}{1+\nu}\, e_{\theta\varphi} \\ t_{r\theta} &= \frac{E}{1+\nu}\, e_{r\theta} \\ t_{r\varphi} &= \frac{E}{1+\nu}\, e_{r\varphi} \quad . \end{aligned} \tag{1.14}$$

Assumption (c), together with (1.13) and (1.14), is now used to obtain expressions for the stress resultants and couples in terms of w, u, v, β_θ, β_φ, and their derivatives.

For example, using (1.6), (1.13), (1.5), and assumption (c), we find

$$N_\theta = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R}\, t_{\theta\theta}\, dr = \frac{E}{R(1-\nu^2)} \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} r\,(e_{\theta\theta} + \nu e_{\varphi\varphi})\, dr$$

$$= \frac{E}{R(1-\nu^2)} \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \Big[u_{,\theta} + (r-R)\beta_{\theta,\theta} + w + \nu \operatorname{cosec}\theta \{ v_{,\varphi} + (r-R)\beta_{\varphi,\varphi} \}$$

$$+ \nu w + \nu \cot\theta \{ u + (r-R)\beta_\theta \} \Big]\, dr .$$

Thus, since $(r-R)$ has a zero average through the thickness of the shell, we have

$$N_\theta = \frac{Eh}{R(1-\nu^2)} \left[u_{,\theta} + \nu u \cot\theta + (1+\nu) w + \nu \operatorname{cosec}\theta\, v_{,\varphi} \right]$$

A similar calculation gives

$$N_\varphi = \frac{Eh}{R(1-\nu^2)} \left[\nu u_{,\theta} + u \cot\theta + (1+\nu) w + \operatorname{cosec}\theta\, v_{,\varphi} \right]$$

$$N_{\theta\varphi} = \frac{Eh}{2(1+\nu)R} \left[u_{,\varphi} \operatorname{cosec}\theta + v_{,\theta} - v \cot\theta \right]$$

$$Q_\theta = \frac{Eh}{2(1+\nu)} \left[\beta_\theta + \frac{1}{R}(w_{,\theta} - u) \right]$$

$$Q_\varphi = \frac{Eh}{2(1+\nu)} \left[\beta_\varphi + \frac{1}{R}(w_{,\varphi} \operatorname{cosec}\theta - v) \right] .$$

Also

$$M_\theta = \frac{D}{R} \left[\beta_{\theta,\theta} + \nu \operatorname{cosec}\theta\, \beta_{\varphi,\varphi} + \nu \beta_\theta \cot\theta \right]$$

$$M_\varphi = \frac{D}{R} \left[\nu \beta_{\theta,\theta} + \beta_{\varphi,\varphi} \operatorname{cosec}\theta + \beta_\theta \cot\theta \right]$$

$$M_{\theta\varphi} = \frac{(1-\nu)D}{2R} \left[\beta_{\varphi,\theta} + \beta_{\theta,\varphi} \operatorname{cosec}\theta - \beta_\varphi \cot\theta \right] .$$

Many writers introduce an averaging coefficient, k_s , in the expressions for Q_θ and Q_φ . For example, Prasad [2] writes

$$Q_\theta = \frac{E h}{2(1+\nu) k_s} \left[\beta_\theta + \frac{1}{R} w_{,\theta} \right]$$

without further comment on k_s . Naghdi [3] takes $k_s = \frac{6}{5}$ which he says is consistent with the assumptions under which the equations were derived. Other authors, e.g., Herrmann and Mirsky [15] , following a precedent set by Mindlin [32], use this factor to make the frequency of very short waves in the lowest mode in the shell theory coincide with the corresponding frequency of the three-dimensional theory. This is equivalent to adjusting the shear wave speed to equal the Rayleigh wave speed. Lin and Morgan [16] take another viewpoint. They use the value $1/k_s = 8/9$ which they claim was experimentally determined by Filon, and quoted by Timoshenko [33], for the case of a bar of rectangular section. We have kept the factor k_s equal to unity for convenience, see equation (1.15), as it does not affect our results in any essential way.

With assumption (c), the right hand sides of (1.8) to (1.12) become

$$\rho h R k_1 w_{,tt}$$
$$\rho h R k_1 u_{,tt} + \frac{1}{6} \rho h^3 \beta_{\theta,tt}$$
$$\rho h R k_1 v_{,tt} + \frac{1}{6} \rho h^3 \beta_{\varphi,tt}$$

$$\frac{1}{12}\,\rho h^3 R\left[\frac{2}{R}\,v_{,tt} + k_r\,\beta_{\varphi,tt}\right]$$

$$\frac{1}{12}\,\rho h^3 R\left[\frac{2}{R}\,u_{,tt} + k_r\,\beta_{\theta,tt}\right],$$

respectively.

§ 1.5. Summary of Equations.

The momentum equations are

$$\left[\frac{r^2}{R}\,t_{rr}\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} + Q_{\theta,\theta} + Q_{\varphi,\varphi}\operatorname{cosec}\theta + Q_\theta\cot\theta$$

$$- (N_\theta + N_\varphi) + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\frac{r^2}{R}F_r\,dr = \rho h R k_1 w_{,tt}\,, \tag{1.8}$$

$$\left[\frac{r^2}{R}\,t_{r\theta}\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} + N_{\theta\varphi,\varphi}\operatorname{cosec}\theta + N_{\theta,\theta} + Q_\theta + (N_\theta - N_\varphi)\cot\theta$$

$$+\int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\frac{r^2}{R}F_\theta\,dr = \rho h R k_1 u_{,tt} + \frac{1}{6}\rho h^3 \beta_{\theta,tt} \tag{1.9}$$

$$\left[t_{r\varphi}\,\frac{r^2}{R}\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} + N_{\varphi,\varphi}\operatorname{cosec}\theta + N_{\theta\varphi,\theta} + Q_\varphi + 2N_{\theta\varphi}\cot\theta$$

$$+\int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\frac{r^2}{R}F_\varphi\,dr = \rho h R k_1 v_{,tt} + \frac{1}{6}\rho h^3 \beta_{\varphi,tt} \tag{1.10}$$

$$\left[t_{r\varphi}\,\frac{r^2}{R}(r-R)\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} - R\,Q_\varphi + M_{\varphi,\varphi}\operatorname{cosec}\theta + M_{\theta\varphi,\theta}$$

$$+ 2M_{\theta\varphi}\cot\theta + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\frac{r^2}{R}(r-R)F_\varphi\,dr \tag{1.11}$$

$$= \frac{1}{12}\rho h^3 R\left[\frac{2}{R}\,v_{,tt} + k_r\,\beta_{\varphi,tt}\right]$$

and

$$\left[t_{r\theta}\,\frac{r^2}{R}(r-R)\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} - R\,Q_\theta + M_{\theta\varphi,\varphi}\operatorname{cosec}\theta + M_{\theta,\theta}$$

$$+(M_\theta - M_\varphi)\cot\theta + \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}\frac{r^2}{R}(r-R)\,F_\theta\,dr \tag{1.12}$$

$$= \frac{1}{12}\,\rho h^3 R\left[\frac{2}{R}u_{,tt} + k_r\,\beta_{\theta,tt}\right].$$

We note that the term $\left[\frac{r^2}{R}\,t_{rr}\right]_{R-\frac{1}{2}h}^{R+\frac{1}{2}h}$ in (1.8) will be replaced later by Rq, where q is the applied pressure. In almost all applications of shell theory $t_{r\theta}$ and $t_{r\varphi}$ are zero on the surfaces of the shell and so would not appear in (1.9) to (1.12). The expressions for the stress resultants and couples in terms of the displacements are

$$N_\theta = \frac{Eh}{R(1-\nu^2)}\left[u_{,\theta} + \nu u\cot\theta + (1+\nu)w + \nu\operatorname{cosec}\theta\, v_{,\varphi}\right]$$

$$N_\varphi = \frac{Eh}{R(1-\nu^2)}\left[\nu u_{,\theta} + u\cot\theta + (1+\nu)w + \operatorname{cosec}\theta\, v_{,\varphi}\right]$$

$$N_{\theta\varphi} = \frac{Eh}{2(1+\nu)R}\left[u_{,\varphi}\operatorname{cosec}\theta + v_{,\theta} - v\cot\theta\right]$$

$$Q_\theta = \frac{Eh}{2(1+\nu)}\left[\beta_\theta + \frac{1}{R}(w_{,\theta} - u)\right] \tag{1.15}$$

$$Q_\varphi = \frac{Eh}{2(1+\nu)}\left[\beta_\varphi + \frac{1}{R}(w_{,\varphi}\operatorname{cosec}\theta - v)\right]$$

$$M_\theta = \frac{D}{R}\left[\beta_{\theta,\theta} + \nu\operatorname{cosec}\theta\,\beta_{\varphi,\varphi} + \nu\beta_\theta\cot\theta\right]$$

$$M_\varphi = \frac{D}{R}\left[\nu \beta_{\theta,\theta} + \beta_{\varphi,\varphi} \operatorname{cosec}\theta + \beta_\theta \cot\theta\right]$$

$$M_{\theta\varphi} = \frac{(1-\nu)D}{2R}\left[\beta_{\varphi,\theta} + \beta_{\theta,\varphi} \operatorname{cosec}\theta - \beta_\varphi \cot\theta\right] . \tag{1.15}$$

Substitution of the quantities in (1.15) into the equations (1.8) to (1.12) would give a system of five simultaneous partial differential equations for u , v , w , β_θ , β_φ .

§1.6. Symmetry in φ, Scaling, and Final Equations.

We consider only problems which are symmetrical in φ , and in which there are no shearing tractions on the surface of the shell. The body force effects, which are mainly due to gravity, are also supposed negligible. We accordingly assume that

$$v = \beta_\varphi = 0$$

and that all physical quantities are independent of φ . Then the stress-strain relations (1.15) become

$$N_\theta = \frac{Eh}{R(1-\nu^2)}\left[u_{,\theta} + \nu \cot\theta\, u + (1+\nu) w\right]$$

$$N_\varphi = \frac{Eh}{R(1-\nu^2)}\left[\nu u_{,\theta} + u \cot\theta + (1+\nu) w\right]$$

$$N_{\theta\varphi} = 0 \tag{1.16}$$

$$Q_\theta = \frac{Eh}{2(1+\nu)}\left[\beta_\theta + \frac{1}{R}(w_{,\theta} - u)\right]$$

$$Q_\varphi = 0$$

$$M_\theta = \frac{D}{R}\left[\beta_{\theta,\theta} + \nu\beta_\theta \cot\theta\right]$$

$$M_\varphi = \frac{D}{R}\left[\nu\beta_{\theta,\theta} + \beta_\theta \cot\theta\right] \tag{1.16}$$

$$M_{\theta\varphi} = 0$$

The momentum equations, (1.8) to (1.12), reduce to the following three equations:

$$Q_{\theta,\theta} + Q_\theta \cot\theta - (N_\theta + N_\varphi) = \rho h R k_1 w_{,tt} - Rq$$

$$N_{\theta,\theta} + Q_\theta + (N_\theta - N_\varphi)\cot\theta = \rho h R k_1 u_{,tt} + \frac{1}{6}\rho h^3 \beta_{\theta,tt} \tag{1.17}$$

$$M_{\theta,\theta} + (M_\theta - M_\varphi)\cot\theta - RQ_\theta = \frac{1}{12}\rho h^3 R\left[\frac{2}{R}u_{,tt} + k_r \beta_{\theta,tt}\right].$$

The assumptions that $t_{r\theta} = t_{r\varphi} = 0$ on the surfaces of the shell, and that $F_r = F_\theta = F_\varphi = 0$ have been used to obtain equations (1.17).

We recall that

$$k_1 = 1 + \frac{1}{12}\left(\frac{h}{R}\right)^2$$

and

$$k_r = 1 + \frac{3}{20}\left(\frac{h}{R}\right)^2$$

When using equations (1.17), we shall henceforth set

$$k_1 = k_r = 1$$

since, by assumption (a), the neglected part is always small compared with 1. On the other hand, the term

$$\frac{1}{6}\rho h^3 \beta_{\theta,tt}$$

in the second equation of (1.17) is not neglected since we cannot assume, a priori, that β_θ and u are of the same order of magnitude. Furthermore, by hindsight we know that by retaining this term the characteristics for the set of equations (1.17) will all be distinct, and this fact will help to simplify subsequent analysis.

The basic length in the system is R, the radius of the mid-surface of the shell. The two speeds involved are c_p and c_Q and we choose c_p as the unit of speed. A dimensionless time, t', is then automatically

$$t' = \frac{t c_p}{R}$$

Of course, u and w are scaled by R to give dimensionless displacements. β_θ, being an angle of rotation, is dimensionless and so is left untouched.

N'_θ, N'_φ, Q'_θ are dimensionless stresses and M'_θ is a dimensionless moment, while q' is a dimensionless pressure.

With the scalings introduced in §1.1., we find that the dimensionless stress resultants and couples are given by

$$\begin{aligned}
N'_\theta &= u'_{,\theta} + \nu u' \cot\theta + (1+\nu) w' \\
N'_\varphi &= \nu u'_{,\theta} + u' \cot\theta + (1+\nu) w' \\
M'_\theta &= \beta_{,\theta} + \nu \beta \cot\theta
\end{aligned} \qquad (1.18)$$

$$M'_{\varphi} = \nu \beta_{,\theta} + \beta \cot\theta$$

$$Q'_{\theta} = \frac{1-\nu}{2}\left[\beta + w'_{,\theta} - u'\right] \tag{1.18}$$

For simplicity, we have now written β instead of β_{θ}. The momentum equations, after the appropriate scalings, become

$$N'_{\theta,\theta} + (N'_{\theta} - N'_{\varphi})\cot\theta + Q'_{\theta} = u'_{,t't'} + 2\varepsilon^2 \beta_{,t't'}$$

$$Q'_{\theta,\theta} + Q'_{\theta}\cot\theta - (N'_{\theta} + N'_{\varphi}) + q' = w'_{,t't'} \tag{1.19}$$

$$M'_{\theta,\theta} + (M'_{\theta} - M'_{\varphi})\cot\theta - \lambda_0^2 Q'_{\theta} = 2u'_{,t't'} + \beta_{,t't'} .$$

The substitution of (1.18) into (1.19) yields

$$\nabla^2 u - \left(\frac{1+\nu}{2} + \cot^2\theta\right) u + \frac{1-\nu}{2}\beta + \frac{3+\nu}{2} w_{,\theta} = u_{,tt} + 2\varepsilon^2 \beta_{,tt} \tag{1.20}$$

$$\nabla^2 w - \frac{4(1+\nu)}{1-\nu} w + \beta_{,\theta} + \beta\cot\theta - \frac{3+\nu}{1-\nu}(u_{,\theta} + u\cot\theta)$$
$$= \frac{2}{1-\nu}(w_{,tt} - q)$$

and

$$\nabla^2\beta - (\nu + \cot^2\theta)\beta - \frac{1-\nu}{2}\lambda_0^2(\beta + w_{,\theta} - u) = \beta_{,tt} + 2u_{,tt}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}, \qquad \lambda_0^2 = \varepsilon^{-2},$$

and, for convenience, all primes have been dropped. The investigation of (1.20) with suitable boundary and initial conditions will occupy us in

Chapter III.

By examining the highest order derivatives, it is readily seen that the structure of the system (1.23) for u , w , and β is that of a totally hyperbolic system [34] with characteristic speeds of

$$\pm \frac{1-\nu}{2} \quad , \quad \pm (1 \pm 2\varepsilon)^{1/2}$$

We note that if we set $\varepsilon = 0$ formally in (1.20), the third equation of that system then implies that

$$\beta + w_{,\theta} - u = 0$$

i.e.,

$$\beta = u - w_{,\theta} \tag{1.21}$$

Under this condition, assumption (c) on u_θ in §1.4. becomes (after due account is taken of the scaling)

$$u_\theta = u(\theta, t) + \left(\frac{r}{R} - 1\right)(u - w_{,\theta}) \tag{1.22}$$

Then, on referring to equations (1.5), using relation (1.22) for u_θ and $u_r = w$, it is found that

$$e_{r\theta} = 0 \, .$$

Equation (1.22) is thus recognised as one of the Love-Kirchoff hypotheses [35], viz., that points lying on the normal to the undeformed midsurface remain on the normal to the deformed midsurface. We also note that (1.21) implies that $Q_\theta = 0$.

If (1.21) is used to eliminate β from the first two equations of (1.20), the result is

$$\nabla^2 u - (\nu + \cot^2\theta)\, u + (1+\nu)\, w_{,\theta} = u_{,tt}$$

$$u_{,\theta} + u\cot\theta + 2w = -\frac{1}{1+\nu}\, w_{,tt} + \frac{1}{1+\nu}\, q \qquad (1.23)$$

These are the dynamical "membrane" equations for a spherical shell as used by Huth and Cole [12], except that in their equations w is positive radially inwards. We note that the speed $\frac{1}{2}(1-\nu)$ does not appear in (1.23). If w is regarded as known, then the first of (1.23) is a wave equation for u with speed of propagation equal to unity. If, on the other hand, u is regarded as known, then the structure of the second of (1.24) is that of an undamped linear oscillator with natural frequency $\omega = \sqrt{2(1+\nu)}$.

CHAPTER II.

We turn our attention to the problem of a constant pressure front moving with constant velocity, U, along a semi-infinite circular cylindrical shell. The equations governing the motion will be formulated to take advantage of the symmetry in the problem. Since the derivation of the governing equations is entirely analogous to the work of Chapter I, it will be given in considerably less detail.

The object of the analysis is to obtain information along all the wave-fronts in the problem. It is quite clear that one should examine the behavior of physical quantities along the characteristics. The appropriate asymptotic procedure yields this information. It becomes evident from this analysis that the dominant disturbance does not travel with either of the characteristic velocities. A procedure is given for determining the velocity of the dominant disturbance and the behavior of the various physical quantities in this region. The behavior of the solution immediately under the constant velocity pressure load is also found.

§2.1. Notation.

The notation introduced in Chapter I is still relevant here. We now only add whatever has not been previously mentioned.

Let (r, θ, x) form a circular cylindrical coordinate system as in Figure 3. The geometry of the shell is described by the inequalities

$$R - \tfrac{1}{2}h \leq r \leq R + \tfrac{1}{2}h \quad ,$$

$$0 \le \theta \le 2\pi, \qquad 0 \le x < \infty.$$

The resultant stresses and couples (shown in Figure 4) are N_x, N_θ, M_x, M_θ.

$q = Q\,H(t - \frac{x}{U})$ is the applied load per unit midsurface area, where $H(\xi)$ is the Heaviside unit step function.

$$Q_0 = \frac{R(1-\nu^2)}{Eh} Q$$

$c = \sqrt{\frac{1}{2}(1-\nu)}$ represents the speed of propagation of distortion (shear) waves in an unbounded medium when scaled by c_p.

$a = \sqrt{1-\nu^2}$ represents the speed of propagation of very long longitudinal waves in a bar when scaled by c_p.

$$G = \frac{Eh}{1-\nu^2}$$

w is the radial displacement of the midsurface, positive outwards.

u is the displacement of the midsurface in the x-direction.

β is the change of slope during deformation of the normal to the midsurface.

§ 2.2. The Equations of the Three-dimensional Linear Theory of Elasticity.

The momentum equations in cylindrical polar coordinates are [26]

$$
\begin{aligned}
t_{rr,r} + \frac{1}{r} t_{r\theta,\theta} + t_{rx,x} + \frac{1}{r}(t_{rr} - t_{\theta\theta}) &= \rho f_r \\
t_{r\theta,r} + \frac{1}{r} t_{\theta\theta,\theta} + t_{\theta x,x} + \frac{2}{r} t_{r\theta} &= \rho f_\theta \qquad (2.1) \\
t_{rx,r} + \frac{1}{r} t_{\theta x,\theta} + t_{xx,x} + \frac{1}{r} t_{rx} &= \rho f_x
\end{aligned}
$$

where t_{rr}, $t_{r\theta}$, t_{rx}, $t_{\theta\theta}$, $t_{\theta x}$, t_{xx} are the stresses. The body force terms have not been included as they shall be neglected in the subsequent analysis. The components of the acceleration vector are f_r, f_θ, f_x.

The stress-strain relations are

$$
\begin{aligned}
t_{rr} &= (\lambda+2\mu)\, e_{rr} + \lambda(e_{\theta\theta} + e_{xx}) \\
t_{\theta\theta} &= (\lambda+2\mu)\, e_{\theta\theta} + \lambda(e_{xx} + e_{rr}) \\
t_{xx} &= (\lambda+2\mu)\, e_{xx} + \lambda(e_{rr} + e_{\theta\theta}) \\
t_{r\theta} &= 2\mu\, e_{r\theta} \\
t_{rx} &= 2\mu\, e_{rx} \\
t_{x\theta} &= 2\mu\, e_{x\theta} .
\end{aligned}
\tag{2.2}
$$

Let the displacement vector have components u_r, u_θ, u_x. The problem treated in this chapter will be axisymmetric, and hence $u_\theta = 0$ and all physical quantities shall be independent of θ. With this in mind, the strain-displacement equations are

$$
\begin{aligned}
e_{rr} &= u_{r,r} \\
e_{\theta\theta} &= \frac{1}{r} u_r \\
e_{xx} &= u_{x,x}
\end{aligned}
\tag{2.3}
$$

$$e_{rx} = \tfrac{1}{2}(u_{x,r} + u_{r,x})$$

$$e_{r\theta} = e_{\theta x} = 0 \tag{2.3}$$

§1.3. Momentum Equations in Terms of Stress Resultants and Couples.

The definitions of the stress resultants are

$$N_x = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R} t_{xx}\, dr$$

$$N_\theta = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} t_{\theta\theta}\, dr \tag{2.4}$$

and

$$Q_x = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R} t_{rx}\, dr .$$

The midsurface of the cylindrical shell is defined by $r = R$ and its inner and outer surfaces are $r = R - \frac{1}{2}h$ and $r = R + \frac{1}{2}h$, respectively. The stress couples are defined by

$$M_\theta = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R}(r-R)\, t_{\theta\theta}\, dr$$

$$M_x = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \frac{r}{R}(r-R)\, t_{xx}\, dr . \tag{2.5}$$

We shall assume that there are no shearing tractions on either curved surface of the shell. Furthermore, the second equation of (2.1) is now vacuous due to the symmetry assumption. By the same procedure as was given in full detail in Chapter I, the momentum equations may be written as

$$N_{x,x} = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho \frac{r}{R} f_x \, dr$$

$$Q_{x,x} - \frac{1}{R} N_\theta = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho \frac{r}{R} f_r \, dr - q \tag{2.6}$$

$$M_{x,x} - Q_x = \int_{R-\frac{1}{2}h}^{R+\frac{1}{2}h} \rho \frac{r}{R} (r-R) f_x \, dr .$$

§ 2.4. The Basic Approximations and Final Equations.

Just as in Chapter I, it is assumed that

(a) $h/R \ll 1$.

(b) t_{rr} on the curved surface is negligible compared with t_{xx} throughout the thickness of the shell.

(c) $u_r(r, x, t) = w(R, x, t)$

$$u_x(r, x, t) = u(R, x, t) + (r-R)\beta(R, x, t) .$$

With conditions (a), (b), (c), the resultants (2.4) and (2.5) become

$$\frac{1}{G} N_x = u_{,x} + \frac{\nu}{R} w + R\varepsilon^2 \beta_{,x}$$

$$\frac{1}{G} N_\theta = \frac{1}{R} w + \nu u_{,x}$$

$$Q_x = \frac{Eh}{2(1+\nu)} (w_{,x} + \beta) \tag{2.7}$$

$$\frac{1}{D} M_x = \beta_{,x} + \frac{1}{R} u_{,x}$$

We define non-dimensional stress and couple resultants by

$$N_x' = \frac{1}{G} N_x \quad , \quad N_\theta' = \frac{1}{G} N_\theta$$

$$Q_x' = \frac{1-\nu^2}{Eh} Q_x \quad , \quad M_x' = \frac{R}{D} M_x$$

and

$$M_\theta' = \frac{R}{D} M_\theta$$

The dimensionless distance and displacements are

$$x' = \frac{x}{R} \quad , \quad u' = \frac{u}{R} \quad \text{and} \quad w' = \frac{w}{R}$$

and the dimensionless time is

$$t' = t c_p / R$$

The relations (2.7) may now be written

$$\begin{aligned} N_x' &= u'_{,x'} + \nu w' + \varepsilon^2 \beta_{,x'} \\ N_\theta' &= w' + \nu u'_{,x'} \\ Q_x' &= \varepsilon^2 (w'_{,x'} + \beta) \\ M_x' &= \frac{1}{R} (\beta_{,x'} + u'_{,x'}) \end{aligned} \qquad (2.8)$$

and the momentum equations (2.6) become

$$\begin{aligned} N'_{x,x'} &= u'_{,t't'} + \varepsilon^2 \beta_{,t't'} \\ Q'_{x,x'} - N_\theta' &= w'_{,t't'} - \frac{R(1-\nu^2)}{Eh} q \end{aligned} \qquad (2.9)$$

$$M'_{x,x'} - \frac{1}{R}\lambda_0^2 Q'_x = \frac{1}{R}\left(\beta_{,t't'} + u'_{,t't'}\right). \tag{2.9}$$

The substitution of (2.8) into (2.9) then gives

$$\begin{aligned}
&u_{,xx} + \nu w_{,x} + \varepsilon^2 \beta_{,xx} = u_{,tt} + \varepsilon^2 \beta_{,tt} \\
&c^2(w_{,xx} + \beta_{,x}) - (w + \nu u_{,x}) = w_{,tt} - \frac{1-\nu^2}{Eh} Rq \\
&\beta_{,xx} + u_{,xx} - \lambda_0^2 c^2 (w_{,x} + \beta) = \beta_{,tt} + u_{,tt} \, ,
\end{aligned} \tag{2.10}$$

where, for convenience, all the primes have been dropped. The equations (2.10) are those given by Naghdi and Cooper [1] except for a few differences. These authors, for example, do not include the term $\varepsilon^2\beta_{,tt}$ in the first equation, or $u_{,tt}$ in the third equation. At a later stage, we shall write a single sixth order equation for each of u , w , and β , and we will find that a more appropriate time to decide on whether to neglect certain terms or not.

The system (2.10) is a hyperbolic system of partial differential equations with single characteristics corresponding to speeds of $\pm c$, and double characteristics corresponding to speeds of ±1.

If we now set $\varepsilon = 0$ (or equivalently, $\lambda_0 = \infty$) formally in equations (2.10), they become

$$u_{,xx} + \nu w_{,x} = u_{,tt}$$

$$c^2(w_{,xx}+\beta_{,x})-(w+\nu u_{,x}) = w_{,tt}-\frac{1-\nu^2}{Eh}Rq$$

$$w_{,x}+\beta = 0$$

After eliminating β from these equations, we obtain

$$\begin{aligned} u_{,xx}+\nu w_{,x} &= u_{,tt} \\ w_{,tt}+w+\nu u_{,x} &= \frac{1-\nu^2}{Eh}Rq . \end{aligned} \tag{2.11}$$

Equations (2.11) are the dynamical "membrane" equations governing the axisymmetric deformation of a circular cylindrical shell[36].

The system (2.10), with suitable boundary and initial conditions, is the fundamental set of equations which shall be used to analyze the problem to be formulated in the next section.

§2.5. Statement of the Problem.

The circular cylindrical shell, $X \geqslant 0$, is initially at rest. Then at $t = 0$, a pressure front $q = Q\,H(t - \frac{x}{U})$, where Q is a constant force per unit area and U is the constant velocity at which the front travels, is incident at x = 0. At the end x = 0, the direct stress resultant, N_X, the shear stress resultant, Q_X, and the stress couple, M_X, are maintained equal to zero throughout the ensuing motion. We wish to describe the features of this motion which were mentioned in the introductory remarks to this chapter. The effects of bending can be traced through the parameter ε (or λ_0), in view of the reduction to membrane theory when ε = 0, as noted at the end of the previous section.

The mathematical statement of the problem is as follows.

The momentum equations are

$$
\begin{aligned}
&u_{,xx} + \nu w_{,x} + \varepsilon^2 \beta_{,xx} = u_{,tt} + \varepsilon^2 \beta_{,tt} \\
&c^2 (w_{,xx} + \beta_{,x}) - (w + \nu u_{,x}) = w_{,tt} - Q_0 H(t - \tfrac{x}{U})
\end{aligned}
\tag{2.12}
$$

and

$$\beta_{,xx} + u_{,xx} - \lambda_0^2 c^2 (w_{,x} + \beta) = \beta_{,tt} + u_{,tt}$$

with initial conditions, at t = 0

$$u = u_{,t} = w = w_{,t} = \beta = \beta_{,t} = 0 \qquad 0 \leq x < \infty . \tag{2.13}$$

The boundary conditions of zero direct and shear stress resultants (Q_x and N_x) and also zero couple stress (M_x) are, at x = 0, for $t \geqslant 0$

$$
\begin{aligned}
M_x &= \nu \beta_{,x} + u_{,x} = 0 \\
N_x &= u_{,x} + \nu w = 0 \\
Q_x &= c^2 (w_{,x} + \beta) = 0 .
\end{aligned}
\tag{2.14}
$$

We further impose a radiation condition that there are no waves coming from $x = +\infty$. (2.12), (2.13), (2.14) with the radiation condition constitute the complete statement of the problem.

§2.6. A Particular Integral.

By straightforward elimination the basic differential equations (2.12) may be reformulated as follows:

$$-\varepsilon^2 L_1 w = \left[\varepsilon^2 (1 - \varepsilon^2) \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right)^2 - c^2 \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \right] Q_0 H(t - \tfrac{x}{U}) \tag{2.15}$$

$$-\varepsilon^2 L_1 u = \left[-\varepsilon^2 \frac{1+\nu}{2} \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) + \nu c^2 \frac{\partial}{\partial x}\right] Q_0 H\left(t - \frac{x}{U}\right)$$

(2.15)

$$-\varepsilon^2 L_1 \beta = (c^2 + \nu\varepsilon^2) \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) Q_0 H\left(t - \frac{x}{U}\right)$$

where the operator $-\varepsilon^2 L_1$ is defined by

$$-\varepsilon^2 L_1 = \varepsilon^2(\varepsilon^2 - 1)\left(c^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)^2 + \varepsilon^2(\varepsilon^2 - 1)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)^2$$

$$-\left(c^2 \frac{\partial^2}{\partial t^2} + \nu\varepsilon^2 \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) - c^2\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) + \nu^2 c^2 \frac{\partial^2}{\partial x^2} .$$

It is now clear that terms involving ε^4 in the operator $-\varepsilon^2 L_1$ may be safely neglected. Thus, we are led to define the operator L by

$$L = \left(c^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)^2 + \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)^2$$

(2.16)

$$+\left(\lambda_0^2 c^2 \frac{\partial^2}{\partial t^2} + \nu \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) + \lambda_0^2 c^2\left(a^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) .$$

We may also neglect terms involving ε^4 on the right hand side of (2.15). The equations (2.15) for w , u , and β may now be rewritten as

$$L w = \left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)^2 + \lambda_0^2 c^2\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\right] Q_0 H\left(t - \frac{x}{U}\right),$$

(2.17)

$$L u = \left[\frac{1+\nu}{2} \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) - \nu \lambda_0^2 c^2 \frac{\partial}{\partial x}\right] Q_0 H\left(t - \frac{x}{U}\right),$$

$$L\beta = -(\lambda_0^2 c^2 + \nu)\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) Q_0 \, H\left(t - \frac{x}{U}\right) , \qquad (2.17)$$

where the opportunity has been taken to multiply both sides of (2.15) by $\varepsilon^{-2} = \lambda_0^2$.

Some pertinent features of the operator L will now be noted. Its characteristic speeds are ±1 and $\pm c$. When we recall that all the speeds have been scaled by c_p , we see that unity represents the velocity of longitudinal waves in an infinite plate, and c represents the velocity of distortional waves in an unbounded medium. We note, furthermore, that a speed "a" has appeared in the last term in L . This speed represents the velocity of propagation of very long longitudinal waves in a bar [37]. Its significance will emerge later in the analysis.

The time dependence in the problem will be suppressed by means of a Laplace transform. We define

$$\bar{w}(x, p) = \int_0^\infty e^{-pt} w(x,t)\, dt ,$$

where p is a complex variable, as the Laplace transform of $w(x, t)$. A bar over a dependent variable will always denote that the Laplace transform has been taken.

Taking note of the initial conditions (2.13), the Laplace transform of the equation for w in (2.17) is

$$\bar{L}\,\bar{w} = \left[\left(\frac{d^2}{dx^2} - p^2\right)^2 + \lambda_0^2 c^2\left(\frac{d^2}{dx^2} - p^2\right)\right]\frac{Q_0}{p}\, e^{-\frac{px}{U}} \qquad (2.18)$$

where

$$\bar{L} = \left(c^2 \frac{d^2}{dx^2} - p^2\right)\left(\frac{d^2}{dx^2} - p^2\right)^2 + \left(\frac{d^2}{dx^2} - p^2\right)^2$$

$$+ \left(\lambda_0^2 c^2 p^2 + \nu \frac{d^2}{dx^2}\right)\left(\frac{d^2}{dx^2} - p^2\right) + \lambda_0^2 c^2 \left(a^2 \frac{d^2}{dx^2} - p^2\right). \tag{2.19}$$

We may write, as a consequence of linearity,

$$\bar{w}(x, p) = \bar{w}_H(x, p) + \bar{w}_p(x, p) \tag{2.20}$$

where $\bar{w}_p(x, p)$ is a particular integral of (2.18) and $\bar{w}_H(x, p)$ will satisfy the homogeneous equation

$$\bar{L}\,\bar{w}_H = 0$$

with boundary conditions on $\bar{w}_H$ so chosen that the boundary conditions of $\bar{w}$ satisfy equations (2.14).

Let

$$\bar{w}_p = W(p)\, e^{-\frac{px}{U}} \tag{2.21}$$

and then, using (2.18) and (2.19), we find

$$W = \frac{Q_0}{p} \quad \frac{\left(\frac{1}{U^2} - 1\right) p^2 + \lambda_0^2 c^2}{p^4 \left(\frac{c^2}{U^2} - 1\right)\left(\frac{1}{U^2} - 1\right) + p^2\left(\frac{1}{U^2} - 1\right) + \left(\frac{\nu}{U^2} + \lambda_0^2 c^2\right) p^2 + \lambda_0^2 c^2 \frac{a^2 - U^2}{1 - U^2}} \tag{2.22}$$

The zeros of the denominator are determined by

$$p^2 = p_\pm^2 = \frac{-\left[\frac{1}{U^2}-1+\frac{\nu}{U^2}+\lambda_0^2 c^2 \pm \sqrt{\left(\frac{1}{U^2}-1+\frac{\nu}{U^2}+\lambda_0^2 c^2\right)^2 - 4\,\frac{c^2-U^2}{U^2}\cdot\frac{a^2-U^2}{U^2}\,\lambda_0^2 c^2}\right.}{2\left(\frac{c^2}{U^2}-1\right)\left(\frac{1}{U^2}-1\right)}$$

For a very thin shell, λ_0 is large compared with unity and we may approximate the zeros, $p_\pm$, by writing

$$p_+^2 = -\frac{a^2-U^2}{1-U^2} + O(1/\lambda_0^2)$$

and (2.23)

$$p_-^2 = -\frac{\lambda_0^2 c^2 U^4}{(c^2-U^2)(1-U^2)} + O(1)$$

We may now write

$$\bar{w}_p(x,p) = \frac{Q_0}{p}\,\frac{\left(\frac{1}{U^2}-1\right)p^2+\lambda_0^2c^2}{(p^2-p_+^2)(p^2-p_-^2)}\,e^{-\frac{px}{U}}$$

$$= \frac{Q_0\left(\frac{1}{U^2}-1\right)}{p_+^2-p_-^2}\left(\frac{p}{p^2-p_+^2}-\frac{p}{p^2-p_-^2}\right)e^{-\frac{px}{U}}$$

$$+\frac{\lambda_0^2c^2}{2p_-^2}\left[\frac{2}{p}-\frac{1}{p-p_-}-\frac{1}{p+p_-}\right]e^{-\frac{px}{U}} - \frac{\lambda_0^2c^2}{2p_+^2}\left[\frac{2}{p}-\frac{1}{p-p_+}-\frac{1}{p+p_+}\right]e^{-\frac{px}{U}}$$

It is noted that $p_+^2>0$ when $a<U<1$ and $p_+^2<0$ otherwise. Also $p_-^2>0$ when $c<U<1$, and $p_-^2<0$ if $U>1$ or $U<c$. The Laplace inversion of $\frac{p}{p^2-p_\pm^2}$ is $\frac{1}{p_\pm}\sinh p_\pm t$ and the inversion of $\frac{1}{p-p_\pm}+\frac{1}{p+p_\pm}$ gives $e^{p_\pm t}+e^{-p_\pm t}$. Thus hyperbolic functions are typical of the intersonic range, while trigonometric functions are typical of the subsonic and supersonic ranges. When the problem of a traveling force on a Timoshenko beam was examined by Florence [59], he also found this change in character of the solution. The exponential growth in the intersonic range does not cause any difficulty since account must still be taken of the solution of the homogeneous equations with the appropriate boundary conditions, and these latter are generated by the

functions just discussed. The final results presented are applicable for all values of U except $U=1$ and $U=c$.

Following the notation of (2.20) and the procedure similar to (2.21), it is found that

$$\bar{u}_p(x,p) = \left[-\frac{1+\nu}{2}\frac{Q_0}{U} + \frac{\nu U \lambda_0^2 c^2}{1-U^2}\frac{Q_0}{p^2}\right]\frac{e^{-\frac{px}{U}}}{(p^2-p_-^2)(p^2-p_+^2)} \tag{2.24}$$

and

$$\bar{\beta}_p(x,p) = \frac{Q_0}{U}(\lambda_0^2 c^2+\nu)\frac{e^{-\frac{px}{U}}}{(p^2-p_-^2)(p^2-p_+^2)} \tag{2.25}$$

We shall denote the stress and couple resultants corresponding to $\bar{w}_p$, $\bar{u}_p$, $\bar{\beta}_p$ by $\bar{N}_x^{(p)}$, $\bar{Q}_x^{(p)}$, and $\bar{M}_x^{(p)}$. Then it follows from (2.23), (2.24), (2.25), and the definitions (2.8) that

$$\bar{M}_x^{(p)} = \left[\frac{pQ_0}{U^2}(a^2-c^2-\nu\lambda_0^2c^2) - \frac{\nu\lambda_0^2c^2}{1-U^2}\frac{Q_0}{p}\right]\frac{e^{-\frac{px}{U}}}{(p^2-p_-^2)(p^2-p_+^2)}$$

$$\bar{N}_x^{(p)} = \left[\frac{pQ_0}{U^2}\left(\frac{3+\nu}{2}-U^2\right) + \frac{\nu Q_0\lambda_0^2c^2U^2}{p(1-U^2)}\right]\frac{e^{-\frac{px}{U}}}{(p^2-p_-^2)(p^2-p_+^2)}$$

(2.26)

$$\bar{Q}_x^{(p)} = -\frac{Q_0c^2}{U}\left[\frac{1-U^2}{U^2}p^2-\nu\right]\frac{e^{-\frac{px}{U}}}{(p^2-p_-^2)(p^2-p_+^2)}.$$

The remarks in §2.6. which led to the definition of L imply that the term $\varepsilon^2\beta_{,x}$ in N_x is negligible, as well as the terms $\varepsilon^2\beta_{,xx}$ and $\varepsilon^2\beta_{,tt}$ in the first of (2.10).

§2.7. Behavior along the Characteristics.

The resultant stresses and couples induced by the particular integrals of the last section at x = 0 are, from (2.6)

$$\overline{M}_x^{(p)}(0,p) = \left[\frac{pQ_0}{U^2}\left(a^2-c^2-\nu\lambda_0^2c^2\right) - \frac{\nu\lambda_0^2c^2}{1-U^2}\,\frac{Q_0}{p}\right]\frac{1}{(p^2-p_-^2)(p^2-p_+^2)},$$

$$\overline{N}_x^{(p)}(0,p) = \left[\frac{pQ_0}{U^2}\left(\frac{3+\nu}{2}-U^2\right) + \frac{\nu Q_0\lambda_0^2c^2U^2}{p(1-U^2)}\right]\frac{1}{(p^2-p_-^2)(p^2-p_+^2)}, \tag{2.27}$$

$$\overline{Q}_x^{(p)}(0,p) = -\frac{Q_0c^2}{U}\left[\frac{1-U^2}{U^2}p^2-\nu\right]\frac{1}{(p^2-p_-^2)(p^2-p_+^2)}.$$

We may decompose $\overline{M}_x$, $\overline{N}_x$, $\overline{Q}_x$ as follows:

$$\overline{M}_x(x,p) = \overline{M}_x^{(p)} + \overline{M}_x^{(H)},$$

$$\overline{N}_x(x,p) = \overline{N}_x^{(p)} + \overline{N}_x^{(H)},$$

$$\overline{Q}_x(x,p) = \overline{Q}_x^{(p)} + \overline{Q}_x^{(H)}.$$

Then, for example,

$$\overline{M}_x^{(H)} = \nu\,\overline{\beta}_{H,x} + \overline{u}_{H,x}$$

and $\overline{\beta}_H$, $\overline{u}_H$ are solutions of the homogeneous equations associated with (2.17):

$$\overline{L}\,\overline{\beta}_H = 0\ , \qquad \overline{L}\,\overline{u}_H = 0\ .$$

The boundary conditions at x = 0 then take the form

$$\overline{M}_x^{(H)}(0,p) = -\overline{M}_x^{(p)}(0,p),$$

$$\overline{N}_x^{(H)}(0,p) = -\overline{N}_x^{(p)}(0,p),$$

and

$$\overline{Q}_x^{(H)}(0,p) = -\overline{Q}_x^{(p)}(0,p).$$

The boundary value problem to be solved is

$$\begin{aligned}
&\bar{u}_{H,xx} + \nu\,\bar{w}_{H,x} - p^2\bar{u}_H = 0, \\
&c^2(\bar{w}_{H,xx} + \bar{\beta}_{H,x}) - (\bar{w}_H + \nu\,\bar{u}_{H,x}) - p^2\bar{w}_H = 0, \qquad (2.28) \\
&\bar{\beta}_{H,xx} + \bar{u}_{H,xx} - \lambda_0^2 c^2(\bar{w}_{H,x} + \bar{\beta}_H) - p^2(\bar{\beta}_H + \bar{u}_H) = 0
\end{aligned}$$

and at x = 0

$$\begin{aligned}
&\overline{M}_x^{(H)} = \nu\,\bar{\beta}_{H,x} + \bar{u}_{H,x} = -\overline{M}_x^{(p)}(0,p), \\
&\overline{N}_x^{(H)} = \bar{u}_{H,x} + \nu\,\bar{w}_H = -\overline{N}^{(p)}(0,p), \qquad (2.29) \\
&\overline{Q}_x^{(H)} = c^2(\bar{w}_{H,x} + \bar{\beta}_H) = -\overline{Q}_x^{(p)}(0,p).
\end{aligned}$$

The radiation condition is built in by accepting only negative exponentials in x as solutions.

No attempt will be made to solve the system (2.28), (2.29) exactly. Instead, the behavior of the solution near the characteristic wave-fronts will be found by doing an asymptotic expansion of the ordinary differential equations for large p . As well as being of interest in itself, it will serve as an introduction to the more difficult analysis involved in the problem for the sphere which will be treated in the next chapter.

If the resultants given in (2.27) are expanded in a power series in p^{-1}, it is found that

$$- \overline{M}_x^{(p)}(0, p) = \frac{1}{p^3} M_0 + \frac{1}{p^4} M_1 + \cdots$$

$$- \overline{N}_x^{(p)}(0, p) = \frac{1}{p^3} N_0 + \frac{1}{p^4} N_1 + \cdots$$

$$- \overline{Q}_x^{(p)}(0, p) = \frac{1}{p^2} Q_{x0} + \frac{1}{p^3} Q_{x1} + \cdots$$

where

$$M_0 = - \frac{Q_0}{U^2} \left(a^2 - c^2 - \nu \lambda_0^2 c^2\right),$$

$$N_0 = - \frac{Q_0}{U^2} \left(\frac{3+\nu}{2} - U^2 \right), \tag{2.30}$$

$$Q_{x0} = \frac{Q_0}{U^2} c^2 (1 - U^2) .$$

Correspondingly, we may assume expansions of the following form for $\overline{w}_H$, $\overline{u}_H$, and $\overline{\beta}_H$:

$$\overline{w}_H(\overline{x}, p) = \frac{1}{p^3} w_0(\overline{x}) + \frac{1}{p^4} w_1(\overline{x}) + \cdots$$

$$\overline{\beta}_H(\overline{x}, p) = \frac{1}{p^4} \beta_0(\overline{x}) + \frac{1}{p^5} \beta_1(\overline{x}) + \cdots$$

$$\overline{u}_H(\overline{x}, p) = \frac{1}{p^4} u_0(\overline{x}) + \frac{1}{p^5} u_1(\overline{x}) + \cdots$$

where

$$\bar{x} = p x .$$

In terms of the variable $\bar{x}$, after division by p^2 , equations (2.28) become

$$\begin{aligned}
&\bar{u}_{H,\bar{x}\bar{x}} - \bar{u}_H + \frac{1}{p} \nu \bar{w}_{H,\bar{x}} = 0 , \\
&c^2 \bar{w}_{H,\bar{x}\bar{x}} - \bar{w}_H + \frac{1}{p} c^2 \bar{\beta}_{H,\bar{x}} + \frac{1}{p} \nu \bar{u}_{H,\bar{x}} - \frac{1}{p^2} \bar{w}_H = 0 , \\
&\bar{\beta}_{H,\bar{x}\bar{x}} + \bar{u}_{H,\bar{x}\bar{x}} - \bar{\beta}_H - \bar{u}_H - \lambda_0^2 c^2 \left(\frac{1}{p} \bar{w}_{H,x} + \frac{1}{p^2} \bar{\beta}_H \right) = 0 .
\end{aligned} \tag{2.31}$$

The boundary value problem for the first approximation is

$$\begin{aligned}
&u_{0,\bar{x}\bar{x}} - u_0 + \nu w_{0,\bar{x}} = 0 , \\
&c^2 w_{0,\bar{x}\bar{x}} - w_0 = 0 , \\
&\beta_{0,\bar{x}\bar{x}} - \beta_0 + u_{0,\bar{x}\bar{x}} - u_0 - \lambda_0^2 c^2 w_{0,\bar{x}} = 0 ,
\end{aligned} \tag{2.32}$$

with

$$\begin{aligned}
\nu \beta_{0,\bar{x}} + u_{0,\bar{x}} &= M_0 , \\
u_{0,\bar{x}} + \nu w_0 &= N_0 , \\
c^2 w_{0,\bar{x}} &= Q_{x0} ,
\end{aligned} \tag{2.33}$$

at x = 0.

Bearing in mind the radiation condition, the solution to (2.32) and (2.33) is

$$w_0(\bar{x}) = -\frac{1}{c} Q_{x0} e^{-\bar{x}/c},$$

$$u_0(\bar{x}) = -\left(N_0 - \frac{\nu c}{1-c^2} Q_{x0}\right) e^{-\bar{x}} - \frac{\nu}{1-c^2} Q_{x0} e^{-\bar{x}/c},$$

$$\beta_0(\bar{x}) = -\left[\left\{\frac{c(1+\lambda_0^2)}{1-c^2} + \frac{\nu-1}{c(1-c^2)}\right\} Q_{x0} - \frac{1}{\nu}(N_0 - M_0)\right] e^{-\bar{x}}$$
$$+ \frac{Q_{x0}}{1-c^2}(\nu + \lambda_0^2 c^2) e^{-\bar{x}/c}.$$

Hence

$$\bar{Q}_x^{(H)}(x,p) = \frac{1}{p^2} Q_{x0} e^{-px/c} + O\left(\frac{1}{p^3}\right),$$

$$\bar{N}_x^{(H)}(x,p) = \frac{1}{p^3}\left[\left(N_0 - \frac{\nu c}{1-c^2} Q_{x0}\right) e^{-px} + \frac{\nu c}{1-c^2} Q_{x0} e^{-\frac{px}{c}}\right] + O\left(\frac{1}{p^4}\right), \tag{2.34}$$

$$\bar{M}_x^{(H)}(x,p) = \frac{1}{p^3}\left[M_0 + Q_{x0}\left\{\frac{\nu c \lambda_0^2}{1-c^2} + \frac{\nu(\nu-1)}{c(1-c^2)}\right\}\right] e^{-px}$$
$$- \frac{1}{p^3} Q_{x0}\left\{\frac{\nu c \lambda_0^2}{1-c^2} + \frac{\nu(\nu-1)}{c(1-c^2)}\right\} e^{-\frac{px}{c}} + O\left(\frac{1}{p^4}\right).$$

The inversion of (2.34) can be done immediately to give the following short-time approximations.

$$Q_x^{(H)}(x,t) = Q_{x0}\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right) + O\left(t - \frac{x}{c}\right)^2,$$

$$N_x^{(H)}(x,t) = \frac{1}{2}\left(N_0 - \frac{\nu c}{1-c^2} Q_{x0}\right)(t-x)^2 H(t-x) + O(t-x)^3$$
$$+ \frac{1}{2}\frac{\nu c}{1-c^2} Q_{x0}\left(t - \frac{x}{c}\right)^2 H\left(t - \frac{x}{c}\right) + O\left(t - \frac{x}{c}\right)^3, \tag{2.35}$$

$$M_x^{(H)}(x,t) = \frac{1}{2}\Big[M_0 + Q_{x0}\Big\{\frac{\nu c \lambda_0^2}{1-c^2} + \frac{\nu(\nu-1)}{c(1-c^2)}\Big\}\Big](t-x)^2 H(t-x)$$

$$-\frac{1}{2} Q_{x0}\Big\{\frac{\nu c \lambda_0^2}{1-c^2} + \frac{\nu(\nu-1)}{c(1-c^2)}\Big\}\Big(t-\frac{x}{c}\Big)^2 H\Big(t-\frac{x}{c}\Big) \tag{2.35}$$

$$+ O(t-x)^3 + O\Big(t-\frac{x}{c}\Big)^3 ,$$

where M_0 , N_0 , Q_{x0} are given by (2.30). We note that $Q_x^{(H)}$, $N_x^{(H)}$, and $M_x^{(H)}$ are continuous across the wave-fronts. Also, the presence of λ_0^2 in $M_x^{(H)}$ will give a sharp rise in the moment behind either wave-front.

If we continue the asymptotic procedure on equations (2.31), we find that $u_1(\bar{x})$ and $\beta_1(\bar{x})$ have terms involving $\bar{x}\, e^{-\bar{x}}$ and that $w_1(\bar{x})$ has a term involving $\bar{x}\, e^{-\bar{x}/c}$. Since $\bar{x} = px$, this means that the second terms in the expansions for $\bar{u}_H$, $\bar{w}_H$, and $\bar{\beta}_H$ are of the same order of magnitude as the first. We obviate this by introducing new variables $\tilde{x}_i$ and $\hat{x}$ where

$$\tilde{x}_i = \Big(1 + \frac{1}{p^2}\Omega_i\Big)\bar{x} \quad , \qquad (i = 1, 2, 3)$$

$$\hat{x} = \frac{1}{p}\bar{x} \ ,$$

and Ω_i are constants. We assume an expansion for $\bar{w}_H(x,p)$ of the form

$$\bar{w}_H(x, p) = \frac{1}{p^3} w_0(\hat{x}, \tilde{x}_3) + \frac{1}{p^4} w_1(\hat{x}, \tilde{x}_3) + \cdots$$

with similar expansions for $\bar{u}_H$ and $\bar{\beta}_H$. The two variable expansion procedure [38] is then used to find the terms of the expansion. The calculation of Ω_i is inherent in the procedure (and, in fact, is its essence). The first term in the expansion for $\bar{w}_H$ will be

$$\frac{1}{p^3}\tilde{w}_0(\hat{x},\tilde{x}_2) = -\frac{1}{p^3}\frac{Q_{x0}}{c}e^{-\frac{\tilde{x}}{c}\left(p+\frac{\Omega_3}{p}\right)}.$$

We note [55] that for $Re\,\nu > -1$, the inversion of $p^{-\nu-1}e^{\Omega/p}$ is $\Omega^{-\frac{1}{2}\nu}t^{\frac{1}{2}\nu}I_\nu(2\Omega^{1/2}t^{1/2})$ and the inversion of $p^{-\nu-1}e^{-\Omega/p}$ is $\Omega^{-\frac{1}{2}\nu}t^{\frac{1}{2}\nu}J_\nu(2\Omega^{1/2}t^{1/2})$. Thus, if $\Omega_3 > 0$ the expression for $\tilde{w}_H$ involves integrals of modified Bessel functions, and if $\Omega_3 < 0$ it involves integrals of Bessel functions. These replace the simple powers of $(t-x)$ in equation (2.35). A result of this kind was obtained for beams by Zajac [23], and for cylinders by Keer, Fleming, and Herrmann [22]. The results just cited were obtained from a representation of the exact solution in contrast to the method indicated here. We do not pursue the details of the calculation as nothing essential is gained. The interval of validity of the approximation is extended slightly, but a more precise statement is difficult. A calculation of a very similar nature is given in the next chapter when dealing with a spherical shell.

2.8. Preamble to the Long-Time Solution.

Some features of a paper by Whitham [39] are now reviewed as a prelude to examining the operator L defined in (2.16).

Suppose that the variable $\varphi(x,t)$ satisfies the equation

$$\left(\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial x}\right)\varphi+\lambda\left(\frac{\partial}{\partial t}+a_0\frac{\partial}{\partial x}\right)\varphi = 0. \qquad (2.36)$$

The characteristics are the lines $x - c_1 t$ = constant and $x - c_2 t$ = constant, which correspond to waves propagating with speeds c_1 and c_2, respectively.

If $\lambda = 0$, then the solution of (2.36) is

$$\varphi = g_1(x - c_1 t) + g_2(x - c_2 t). \tag{2.37}$$

On the other hand, if λ is large then it is very tempting to take

$$\left(\frac{\partial}{\partial t} + a_0 \frac{\partial}{\partial x}\right)\varphi = 0$$

as an approximation to (2.36), with the corresponding solution

$$\varphi = f(x - a_0 t) \tag{2.38}$$

For any λ between these two extremes $(\lambda = 0, \lambda = \infty)$ we would like to know how the lower order terms affect (2.37) and how the higher order terms modify (2.38).

A wave motion traveling with speed V satisfies the equation

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x}\right)\varphi = 0$$

We may interpret this observation as follows. For a wave motion traveling with speed V , the derivatives $\frac{\partial}{\partial t}$ and $-V\frac{\partial}{\partial x}$ are approximately equal. Let us now consider the motion described by (2.36) and we fix attention on the wave motion along the characteristic ray $x - c_1 t$ = constant. Along such a ray, $\frac{\partial}{\partial t}$ and $-c_1\frac{\partial}{\partial x}$ are equal. We may then rewrite (2.36) as

$$\left(\frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x}\right)(c_2 - c_1)\frac{\partial \varphi}{\partial x} + \lambda (a_0 - c_1)\frac{\partial \varphi}{\partial x} = 0 \tag{2.39}$$

and a solution of (2.39) is

$$\varphi_{c_1} = g\left(t - \frac{x}{c_1}\right) \exp\left[-\frac{\lambda}{c_1} \frac{c_1 - a_0}{c_1 - c_2} x\right]. \tag{2.40}$$

It is plausible to expect that (2.40) is a good approximation to the exact solution of (2.36) in the neighborhood of the ray $x - c_1 t$ = constant.

Similarly, an approximation of the form

$$\varphi_{c_2} = g_2\left(t - \frac{x}{c_2}\right) \exp\left[-\frac{\lambda}{c_2}\ \frac{a_0 - c_2}{c_1 - c_2} x\right] \tag{2.41}$$

should be useful near the ray $x - c_2 t$ = constant. The sub-characteristic $x - a_0 t$ = constant is relevant for the lower order terms in (2.36), and replacing $\frac{\partial}{\partial t}$ by $-a_0 \frac{\partial}{\partial x}$ in the higher order terms results in

$$\frac{(c_1 - a_0)(a_0 - c_2)}{\lambda}\ \varphi_{,xx} = \left(\frac{\partial}{\partial t} + a_0 \frac{\partial}{\partial x}\right)\varphi . \tag{2.42}$$

The transformation

$$\xi = a_0 t - x \quad , \quad \eta = t$$

puts (2.42) into the form

$$K\, \varphi_{,\xi\xi} - \varphi_{,\eta} = 0$$

where K = $\frac{1}{\lambda}(c_1 - a_0)(a_0 - c_2)$. It is thus seen that (2.42) represents diffusion about $x - a_0 t$ = constant. In fact, the solution of (2.42) is

$$\varphi_{a_0} = \frac{1}{\sqrt{4\pi \frac{Kx}{a_0^3}}}\ \exp\left[-\frac{(t - x/a_0)^2}{4Kx/a_0^3}\right] , \tag{2.43}$$

which we expect to be a good approximation to the solution of (2.36) near to $x - a_0 t$ = constant.

If $\lambda > 0$ and $c_1 < a_0 < c_2$, then the approximations given by (2.40) and (2.41) are exponentially damped along the rays $x - c_1 t$ = constant and $x - c_2 t$ = constant. In contrast to this, φ_{a_0} as given by (2.43), has diffusion about $x - a_0 t$ = constant and the damping along this ray is proportional to $x^{-1/2}$.

Clearly, it now makes sense to say that the dominant disturbance described by (2.39) is the one which travels with speed a_0. We rephrase this statement to say that the dominant disturbance in (2.39) travels with the speed of the lowest order term. The higher order terms then produce a diffusion of this wave.

A precise meaning is now given to the above somewhat vague remarks. The solution $\varphi(x,t)$ to (2.36) may be represented by

$$\varphi(x,t) = \frac{1}{2\pi i} \int_{Br} \bar{\varphi}(p, x)\, e^{pt}\, dp \tag{2.44}$$

where Br is the familiar Bromwich contour. Substitution of (2.44) into the equation (2.36) shows that $\bar{\varphi}(p, x)$ has the form

$$e^{P_1(p)x} \qquad \text{and} \qquad e^{P_2(p)x}$$

where $P_1(p)$, $P_2(p)$ are solutions of the equation

$$c_1 c_2 P^2 + \{(c_1 + c_2)p + \lambda a_0\} P + p(p + \lambda) = 0 \tag{2.45}$$

and

$$P_1 \sim -p/c_1 \quad , \quad P_2 \sim -p/c_2$$

for large p with λ fixed.

The form of $P_1(p)$ for large p is

$$P_1(p) = -p/c_1 - \frac{\lambda}{c_1}\,\frac{c_1 - a_0}{c_1 - c_2} + \cdots$$

and so near the wave-front $x = c_1 t$ the behavior of the solution is

$$\varphi = e^{-\frac{\lambda}{c_1}\frac{c_1 - a_0}{c_1 - c_2}}\ \varphi_0\left(t - \frac{x}{c_1}\right) + \cdots$$

This agrees exactly with φ_{c_1} of equation (2.40). The same procedure using $P_2(p)$ will give the solution as in (2.41).

Thus the ruse of making $\frac{\partial}{\partial t}$ and $-V\frac{\partial}{\partial x}$ approximately equal yields the wave-front behavior of the exact solution in the case when V is a characteristic speed.

Equation (2.44) is now rewritten as

$$\varphi(x,t) = \frac{1}{2\pi i}\int_{Br} e^{t\{p + \frac{x}{t}P_1(p)\}}\,dp \tag{2.46}$$

where $P_1(p)$ is determined by (2.45). We examine the asymptotic form of the right hand side of (2.46) as $t \to \infty$ along the rays $\frac{x}{t}$ = constant. The method of steepest descents [40] is used to this end, and by keeping $\frac{x}{t}$ = constant we are able to deduce wave properties.

The saddle point is at $p = p_1$ where

$$1 + \xi\,\frac{dP_1}{dp}(p_1) = 0 \tag{2.47}$$

and $\xi = \frac{x}{t}$.

By solving (2.47) for p_1, one gets

$$p_1 = p_1(\xi)$$

i.e., the saddle point is a function of the ray one is considering. The first term in the asymptotic expansion of (2.46) then is [40]

$$\frac{\exp\left[t\left\{p_1(\xi) + \xi P_1\left(p_1(\xi)\right)\right\}\right]}{\sqrt{2\pi x \quad \frac{d^2 P_1}{dp^2}\left(p_1(\xi)\right)}} \tag{2.48}$$

We would now like to find the ray on which, for fixed t, the exponential in (2.48) takes its maximum and also its value thereon. This value of ξ is given by

$$\frac{dp_1}{d\xi}\left\{1 + \xi \frac{dP_1}{dp}\left(p_1(\xi)\right)\right\} + P_1\left(p_1(\xi)\right) = 0 .$$

Hence, by (2.47),

$$P_1\left(p_1(\xi)\right) = 0$$

If we now set $P = 0$ in (2.45), we are left with

$$p(p+\lambda) = 0$$

Thus $p = 0$ or $p = -\lambda$. It may be checked that $p = -\lambda$ corresponds to $P_2\left(p(\xi)\right) = 0$, so we disregard this value of p. ξ is now calculated from (2.47) to give

$$\frac{x}{t} = \xi(p_1 = 0) = -\frac{1}{\frac{dP_1}{dp}(p_1 = 0)} .$$

We differentiate (2.45) with respect to p and then set $P = p = 0$ to give $\frac{dP_1}{dp}(p_1 = 0) = -\frac{1}{a_0}$. Hence $\xi = a_0$ for $P = 0$, which is just the speed in the lowest order terms of equation (2.36). Thus, the exponential in (2.48) has its maximum on the ray $x - a_0 t = 0$. Along this ray, the expression (2.48) has a decay proportional to $x^{-1/2}$. Exactly this behavior is exhibited by φ_{a_0} given in (2.43).

At this stage we can say that the approximate solution found by equating $\frac{\partial}{\partial t}$ and $-V\frac{\partial}{\partial x}$ when V is the lowest order speed in the equation is equivalent to maximizing over all the rays the first term of the asymptotic expansion for large t of the exact solution. The value of the heuristic approach is that one can tell at a glance which is the most important group of terms in the equation under consideration.

§2.9. Examination of the Operator L.

In the previous section we examined a simple operator and showed the equivalence of two approaches. The prime lesson learned was that the maximum disturbance moves with the speed of the lowest order terms. Attention is now turned to the more complicated operator L defined in equation (2.16).

We note immediately that the speed in the lowest order term is a, and so it is to be expected that the dominant disturbance described by L will travel with speed a. This conjecture will be verified first. The Laplace transform of $L w$ will be written as $\bar{L}\bar{w}$. On noting the initial conditions (2.13),

$$\bar{L}\bar{w} = \left(c^2 \frac{d^2}{dx^2} - p^2\right)\left(\frac{d^2}{dx^2} - p^2\right)^2 \bar{w} + \left(\frac{d^2}{dx^2} - p^2\right)^2 \bar{w}$$

$$+ \left(\lambda_0^2 c^2 p^2 + \nu \frac{d^2}{dx^2}\right)\left(\frac{d^2}{dx^2} - p^2\right)\bar{w} + \lambda_0^2 c^2 \left(a^2 \frac{d^2}{dx^2} - p^2\right)\bar{w}.$$

We wish to solve $\bar{L}\bar{w} = 0$. Let $\bar{w} = e^{P(p)x}$, then $P(p)$ satisfies

$$\begin{aligned}(c^2 P^2 - p^2)(P^2 - p^2)^2 + (P^2 - p^2)^2 \\ + (\lambda_0^2 c^2 p^2 + \nu P^2)(P^2 - p^2) + \lambda_0^2 c^2 (a^2 P^2 - p^2) = 0.\end{aligned} \tag{2.49}$$

Following the procedure laid out in §2.8., we set $P = 0$ to find the corresponding values of p from (2.49). The equation satisfied by p is

$$p^6 - p^4(1 - \lambda_0^2 c^2) + \lambda_0^2 c^2 p^2 = 0 \tag{2.50}$$

Thus

$$p = 0 \quad \text{or} \quad p^2 = \tfrac{1}{2}\left[(1 - \lambda_0^2 c^2) \pm \sqrt{(1 - \lambda_0^2 c^2)^2 - 4\lambda_0^2 c^2}\,\right] \tag{2.51}$$

Equation (2.50) for p^2 yields complex values of p. Now the method used involves taking a maximum and so it is tacitly assumed that all quantities are real. Hence we accept only the root $p = 0$ from equation (2.50). We differentiate (2.49) with respect to p and set $P = p = 0$. This yields

$$\frac{dP}{dp}(p = 0) = \pm 1/a \tag{2.52}$$

and then equation (2.47) gives

$$\xi(p = 0) = a.$$

Thus, as anticipated, the speed of the dominant disturbance is a.

Further differentiations give that

$$\frac{d^2P}{dp^2}(p = 0) = 0 \tag{2.53}$$

and

$$\frac{d^3P}{dp^3}(p = 0) = \frac{3(1 - a^2)}{a^3}\left[1 + \mu_0\right] \tag{2.54}$$

where

$$\mu_0 = \frac{a}{\lambda_0^2 c^2 (1-a^2)} \left[\frac{1+\nu}{a^3} + a - \frac{2+\nu}{a} \right] . \tag{2.55}$$

The bending terms involved in the operator L are represented here by the term μ_0 . For a thin shell $\mu_0 << 1$, as can be seen by the presence of λ_0^2 in (2.55). Thus, as far as the calculations here are concerned, the membrane equations are adequate. The bending terms enter through the boundary conditions (2.29).

The equation $Lw = 0$, where L is given in (2.16), is a linear dispersive equation. Consequently, it is usual to look for solutions of the form

$$w = e^{i(kx - \omega t)}$$

where k is the wave number and ω is the frequency. The equation $Lw = 0$ then gives ω as a function of k . This relationship, $\omega = \omega(k)$, is called the dispersion relation. The phase velocity, c_{ph} , is given by

$$\frac{x}{t} = c_{ph} = \frac{\omega}{k} , \tag{2.56}$$

and the group velocity, c_g , is defined by

$$c_g(k) = \frac{d\omega}{dk}$$

The group velocity is the velocity at which the energy in the system is transmitted[41]. From (2.56), it is seen that

$$\omega = k c_{ph} \tag{2.57}$$

and thus

$$c_g = \frac{d\omega}{dk} = c_{ph} + k \frac{dc_{ph}}{dk}$$

Therefore, $c_g = c_{ph}$ if $\frac{d}{dk} c_{ph} = 0$. Now, the phase velocity a is the speed of very long longitudinal waves in a bar. Since the wavelength is very large, the wave number, k , equals zero. Equation (2.57) with $c_{ph} = a$ and $k = 0$ implies that $\omega = 0$. Thus, the phase velocity, a, with very long wavelength corresponds to $\omega = 0,\ k = 0$.

On substituting $w = e^{i(kx-\omega t)}$ into the equation $Lw = 0$, we obtain

$$(\omega^2 - c^2k^2)(\omega^2 - k^2)^2 + (\omega^2 - k^2)^2 - (\lambda_0^2 c^2 \omega^2 + \nu k^2)(\omega^2 - k^2)$$

$$+ \lambda_0^2 c^2 (\omega^2 - a^2 k^2) = 0 .$$

Then, for $\omega = k = 0$,

$$c_g = \frac{d\omega}{dk} = a ,$$

$$\frac{d^2\omega}{dk^2} = 0 ,$$

and

$$\frac{d^3\omega}{dk^3} < 0 .$$

Thus, along the ray $x - at = 0$, the group velocity equals the phase velocity and the group velocity has its maximum value.

The disturbance associated with a stationary value of the group velocity has been called the "Airy phase" by Pekeris [42]. The motion is characterized by its regular period, namely, that corresponding to the stationary group velocity.

Naghdi and Cooper [1] have examined the natural vibrations of a shell described by (2.10). They say that for very long waves the characteristic equation yields for the lowest mode of motion a speed $\sqrt{1-\nu^2}$ — which is our speed a. Figure 1(b) in their paper is a plot of phase velocity versus $(\text{wavelength})^{-1}$. It can be seen that for the lowest mode as the wavelength tends to infinity, the phase speed tends to a and the curve has zero slope at this point, i.e., $\frac{d}{dk} c_{ph} = 0$. This again confirms that

$$c_{ph} = c_g = a.$$

They do not give a corresponding plot involving the group velocity.

§2.10. The Behavior Near $x = at$.

We know that when ξ is near a, p is near zero. Thus, by Taylor's theorem with (2.52), (2.53), and (2.54),

$$P(p_1(\xi)) = -\frac{1}{a} p_1(\xi) + \frac{1-a^2}{2a^3}(1+\mu_0) p_1^3(\xi) + O(p_1^4), \tag{2.58}$$

when ξ is near a. The saddle point equation, (2.47), determines p_1 as a function of ξ. With the aid of (2.58), equation (2.47) becomes

$$1 + \xi\left[-\frac{1}{a} + \frac{3}{2}\frac{1-a^2}{a^3}(1+\mu_0) p_1^2(\xi) + \cdots\right] = 0$$

Therefore,

$$p_1(\xi) = \pm\sqrt{\frac{2}{3}}\,\frac{a^{3/2}}{(1-a^2)(1+\mu_0)}\left(\frac{1}{a} - \frac{1}{\xi}\right)^{1/2}, \tag{2.59}$$

and

$$p_1(\xi) + \xi P(p_1(\xi)) = \left(\frac{2}{3}\right)^{3/2}\frac{a}{\sqrt{(1-a^2)(1+\mu_0)}}\left(1 - \frac{x}{at}\right)\left(1 - \frac{at}{x}\right)^{1/2}. \tag{2.60}$$

The exponential part of (2.48) then is constant for $x = at$, decaying for

$x > at$, i.e., in front of the wavefront, and oscillatory for $x < at$, i.e., behind the wavefront.

When $\xi \neq a$, the first term in the asymptotic evaluation of (2.46) is given by the formula (2.48) in which the amplitude decays like $x^{-1/2}$ along the ray. As $\xi \to a$, two saddle points coalesce and (2.48) is no longer applicable. The first term of the asymptotic formula for this case is given by Jones [43] on p. 445. Without giving the exact expression, we note that the decay factor is $x^{-1/3}$. Thus there is a non-uniformity in the asymptotic expansion of (2.46) as ξ tends to a. Chester et. al. [44] have examined just such a situation and have given a method of obtaining an asymptotic expansion which is uniformly valid as $\xi \to a$. We will give here the mechanics of the calculation, but we do not attempt a full exposition. It is noted that the form of the solution presented here makes the application of the technique very simple.

With the aid of (2.58), the right hand side of (2.46) may be written as

$$\frac{1}{2\pi i} \int_{Br} e^{t[(1-\xi/a)p_1(\xi) + A\xi p_1^3(\xi) + \cdots]} \, dp_1 \tag{2.61}$$

where

$$A = \frac{1-a^2}{2a^3}(1+\mu_0).$$

We define

$$Q(p_1, \xi) = (1-\xi/a)p_1(\xi) + A\xi p_1^3(\xi). \tag{2.62}$$

For a very thin shell, $\mu_0 \ll 1$ and we are essentially dealing with the membrane equations. The Bromwich contour in this case is given in Fig. 5.

There are no singularities in the right half plane. The points $\pm i$ and $\pm ia$ are branch points of the integrand and the appropriate branch cuts are shown. Equation (2.61) may now be written as

$$-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{t\,Q(p_1,\,\xi)}\,dp_1 \tag{2.63}$$

and we have shown that the major contribution to this integral comes from $p_1 = 0$, $\xi = a$.

Following the method in [44], the variable v is introduced by the relation

$$Q\,(p_1,\xi) = \frac{1}{3}v^3 - \zeta(\xi)\,v + \nu(\xi). \tag{2.64}$$

If this is to be a regular (1,1) transformation we must have $\frac{dp_1}{dv} \neq 0$ or ∞, where

$$\frac{\partial Q}{\partial p_1}\,\frac{dp_1}{dv} = v^2 - \zeta(\xi). \tag{2.65}$$

Now $\frac{\partial Q}{\partial p_1}(p_1,\xi)$ vanishes at the two saddle points $p_+(\xi)$, $p_-(\xi)$, while $v^2 - \zeta(\xi)$ vanishes at $v = \pm\,\zeta^{1/2}(\xi)$. If the transformation is to be regular, these points must correspond, and so we have from (2.64) the equations

$$Q\,(p_+,\xi) = -\frac{2}{3}\,\zeta^{3/2}(\xi) + \nu(\xi)$$

and

$$Q\,(p_-,\xi) = \frac{2}{3}\,\zeta^{3/2}(\xi) + \nu(\xi)$$

to determine $\zeta(\xi)$ and $\nu(\xi)$. Solving for ζ and ν yields

$$2\,\nu(\xi) = (1-\xi/a)(p_+ + p_-) + A\xi(p_+^3 + p_-^3)$$

and

$$\frac{4}{3}\,\zeta^{3/2}(\xi) = (1-\xi/a)(p_- - p_+) + A\xi(p_-^3 - p_+^3).$$

Since, by (2.59), $p_- = -p_+$, we obtain

$$\nu(\xi) = 0$$

and (2.66)

$$\zeta^{3/2}(\xi) = \frac{3}{2}\,p_-\left[(1-\xi/a) + A\,\xi\,p_-^2\right].$$

Since the variable of integration in (2.63) is changed to v, we write

$$\frac{dp_1}{dv} = \sum_{m=0}^{\infty} p_m(\xi)(v^2-\zeta)^m + \sum_{m=0}^{\infty} q_m(\xi)\,v\,(v^2-\zeta)^m \tag{2.67}$$

and we now proceed to calculate p_0 and q_0. On putting $v = \pm\zeta^{1/2}(\xi)$ and $p_1 = p_\pm(\xi)$ we find from (2.67) that

$$\left(\frac{dp_1}{dv}\right)_\pm = p_0(\xi) \pm \zeta^{1/2}(\xi)\,q_0(\xi) \tag{2.68}$$

and the left hand side is known when $\frac{dp_1}{dv}$ is known at $v = \pm\zeta^{1/2}(\xi)$.

But, from (2.65),

$$\frac{\partial^2 Q}{\partial p_1^2}\left(\frac{dp_1}{dv}\right)^2 + \frac{\partial Q}{\partial p_1}\,\frac{d^2 p_1}{dv^2} = 2v ,$$

and then

$$\left(\frac{\partial^2 Q}{\partial p_1^2}\right)_\pm \left(\frac{dp_1}{dv}\right)_\pm^2 = \pm\,2\,\zeta^{1/2}$$

since

$$\frac{\partial Q}{\partial p_1} = 0 \quad at \quad p_1 = p_\pm .$$

By (2.62)

$$\frac{\partial^2 Q}{\partial p_1^2} = 6 A \xi p_1 (\xi)$$

and then

$$\left(\frac{d p_1}{d v}\right)_+^2 = \frac{2 \zeta^{1/2} (\xi)}{6 A \xi p_+} \tag{2.69}$$

and

$$\left(\frac{d p_1}{d v}\right)_-^2 = \frac{- 2 \zeta^{1/2}}{6 A \xi p_-} . \tag{2.69}$$

Substitution of equations (2.69) into equations (2.68) yields

$$p_0 (\xi) = \frac{\zeta^{1/2}}{3 A \xi p_+} \quad \text{and} \quad q_0 (\xi) = 0 . \tag{2.70}$$

We now gather up the fragments. Equation (2.63) is transformed into

$$- \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{t\left[\frac{1}{3} v^3 - \zeta(\xi) v + \nu(\xi)\right]} \frac{dp_1}{dv} \, dv$$

by the transformation (2.64). The functions $\zeta(\xi)$ and $\nu(\xi)$ are given by (2.66). $\frac{dp_1}{dv}$ is approximated by the first term in the expansion (2.67), i.e., we write

$$\frac{dp_1}{dv} = p_0 (\xi) + v \, q_0 (\xi)$$

and p_0 , q_0 are given by (2.70).

Using formula (2.3) of [44], equation (2.63) may now be approximated by

$$-\frac{1}{2\pi i}\, p_0(\xi) \int_{\infty e^{-\frac{1}{3}\pi i}}^{\infty e^{\frac{1}{3}\pi i}} \exp\left[t\left(\tfrac{1}{3}v^3 - \zeta v\right)\right] dv$$

$$= -\frac{p_0(\xi)}{t^{1/3}}\, Ai\left[t^{2/3}\,\zeta(\xi)\right], \tag{2.71}$$

since

$$Ai(z) = \frac{1}{2\pi i}\int_{\infty e^{-\frac{1}{3}\pi i}}^{\infty e^{\frac{1}{3}\pi i}} \exp\left[\tfrac{1}{3}u^3 - zu\right] du.$$

$Ai(z)$ is the Airy function and is discussed in [41]. The final result may be stated now as

$$\frac{1}{2\pi i}\int_{Br} e^{t[p + \xi P(p)]}\, dp \sim -t^{-1/3} p_0(\xi)\, Ai\left[t^{2/3}\,\zeta(\xi)\right] \tag{2.72}$$

as $t \to \infty$ and ξ is in the neighborhood of a. With $A = \frac{1-a^2}{2a^3}(1+\mu_0)$, we find on using (2.66) and (2.70) that

$$\zeta(\xi) = \frac{1}{(3A\xi)^{1/3}}\left(\xi/a - 1\right)$$

and

$$p_0(\xi) = (3A\xi)^{-1/6}$$

Thus, for $P(p)$ given by equation (2.49)

$$\frac{1}{2\pi i}\int_{Br} e^{t[p+\xi P(p)]}$$

$$\sim \frac{1}{t^{1/3}(3A\xi)^{1/6}}\, Ai\left[t^{2/3}\, \frac{\xi/a - 1}{(3A\xi)^{1/2}} \right]$$

as $t \to \infty$ and ξ is near a. This expression is uniformly valid as $\xi \to a$.

The situation described here near the ray $x = at$ is entirely analagous to Kelvin's ship-wave pattern as expounded by Ursell [45]. We quote the relevant part of his introduction.

> When a concentrated pressure travels with constant velocity over the free surface of water, it carries with it a familiar pattern of ship-waves. Let viscosity and surface tension be neglected, let the free-surface condition be linearized, let the depth of the water be assumed infinite, and let initial transient effects be ignored. Then, the wave motion everywhere can be found by standard methods in the form of a double integral. The wave pattern at a great distance behind the disturbance can be found by an application of the ordinary method of stationary phase, which shows that the wave amplitude is considerable inside an angle bounded by the two horizontal rays $\theta = \pm\theta_c$ from the disturbance, where $\theta_c = \sin^{-1}\frac{1}{3} \doteq 19\frac{1}{2}°$. But the method fails near the critical lines $\theta = \pm\theta_c$ . ... Near the critical lines the surface elevation at a greater distance behind the pressure point can be expressed in terms of Airy functions, and this expression goes over into the known wave pattern inside the critical angle. It is shown that near the critical lines the crest length increases as the cube root of the distance, and the separation between crests remains constant.

The last statement here refers to the regular period of the Airy phase which is due to the stationary group velocity. This point is discussed by Pekeris [42] and Newlands [46].

§ 2.11. The Long-Time Solution.

Since we are dealing with linear equations, $\overline{M}_x^{(H)}(x, p)$ satisfies the equation

$$\bar{L}\ \bar{M}_x^{(H)}(x, p) = 0$$

and

$$\bar{M}_x^{(H)}(0, p) = -\bar{M}_x^{(p)}(0, p)$$

where $\bar{M}_x^{(p)}(0,p)$ is given by (2.27). We may thus write

$$M_x^{(H)}(x,t) \sim -\frac{1}{\left(\frac{x}{a}\right)^{1/3}(3A\xi)^{1/6}} \int_0^t M_x^{(p)}(0, \tau)\, Ai\left[\frac{x/a-1}{(3A\xi)^{1/2}}(t-\tau)^{2/3}\right] d\tau \tag{2.73}$$

for $t \to \infty$ in the neighborhood of the ray $x - at$ = constant, where, by inversion of (2.27)

$$M_x^{(p)}(0,t) = \frac{Q_0(a^2 - c^2 - \nu\lambda_0^2 c^2)}{U^2(p_+^2 - p_-^2)}\left[\cosh p_+ t - \cosh p_- t\right]$$

$$-\frac{\nu\lambda_0^2 c^2 Q_0}{1-U^2}\left[\frac{1}{2p_-^2}\left\{2 - e^{p_- t} - e^{-p_- t}\right\} - \frac{1}{2p_+^2}\left\{2 - e^{p_+ t} - e^{-p_+ t}\right\}\right],$$

and p_+, p_- are given by (2.23). We note that the bending effects, represented by λ_0^2, enter in a fairly innocuous fashion. (It had been hoped at the outset that λ_0 would enter in such a way that (2.73) would be valid for $\lambda_0 t \gg 1$ and then we would not have been restricted to very large t).

In a similar fashion, the expressions for $N_x(x,t)$ and $Q_x(x,t)$ are

$$N_x^{(H)}(x,t) \sim -\frac{1}{\left(\frac{x}{a}\right)^{1/3}(3A\xi)^{1/6}} \int_0^t N_x^{(p)}(0, \tau)\, Ai\left[\frac{x/a-1}{(3A\xi)^{1/2}}(t-\tau)^{2/3}\right] d\tau, \tag{2.74}$$

where

$$N_x^{(\phi)}(0,t) = \frac{Q_0}{U}\left(\frac{3+\nu}{2} - U^2\right)\frac{1}{p_+^2 - p_-^2}\left[\cosh p_+ t - \cosh p_- t\right]$$

$$+ \frac{\nu Q_0 \lambda_0^2 c^2 U^2}{1-U^2}\left[\frac{1}{2p_-^2}\left\{2 - e^{p_- t} - e^{-p_- t}\right\} - \frac{1}{2p_+^2}\left\{2 - e^{p_+ t} - e^{-p_+ t}\right\}\right],$$

and

$$Q_x^{(H)}(x,t) \sim \frac{-\nu Q_0 c^2}{U p_-^2 p_+^2} \frac{1}{\left(\frac{x}{a}\right)^{1/3}(3A\xi)^{1/6}} Ai\left[\frac{\xi/a - 1}{(3A\xi)^{1/3}} t^{2/3}\right]. \qquad (2.75)$$

(2.74) and (2.75) are valid for large t in the neighborhood of the rays

$x - at$ = constant.

We examine $Q_x(x,t)$ to get a better feel for the formula in (2.75). The others are similar.

For $\xi = a$, i.e., along the ray $x - at = 0$,

$$Q_x^{(H)}(x,t) \sim -\frac{\nu Q_0 c^2}{U p_-^2 p_+^2} \frac{Ai(0)}{(3Aa)^{1/6}} t^{-1/3},$$

and so the amplitude decays like $t^{-1/3}$. Also the amplitude is proportional to λ_0^{-2} due to the presence of p_-^2 . There is an exponential decay in front of $x - at = 0$ and an oscillatory behavior behind $x - at = 0$ as remarked subsequent to equation (2.60).

CHAPTER III

Attention is now fixed on a spherical shell which is described by the equations derived in detail in Chapter I. The inclusion of the effects of transverse shear and rotatory inertia in this description led to a system of simultaneous equations for the displacements which is totally hyperbolic. Thus it makes sense to examine the behavior of a disturbance in the shell as it moves into a region of quiet. We are interested in the focusing effect which results from the geometry of the shell, and especially in the influence of bending on this effect. To this end, the problem of a constant moment $M_\theta = M_0$ applied at the lip $\theta = \theta_0 > 0$ of a spherical shell is studied. From the outset, the objective is limited to the leading wavefront behavior of M_θ as it makes its first traversal from $\theta = \theta_0$ to $\theta = \pi$, and its first reflection therefrom. This objective is achieved by the appropriate asymptotic expansion without recourse to any representation of the exact solution. See Fig. 6 for a picture of the situation.

§3.1. Notation.

The notation of Chapters I and II is again relevant here. In addition we have the following.

$$\bar{\theta} = p(\theta - \theta_0) \quad , \quad p \text{ is the Laplace transform variable.}$$

$$\tilde{\theta} = p(\pi - \theta) \quad .$$

$$\theta^+ = \left(1 + \frac{\Omega}{p^2}\right)\bar{\theta} \quad , \quad \Omega \text{ is a constant.}$$

$$\hat{\theta} = \beta^{-1}\bar{\theta} \quad .$$

$$q_1 = (1+2\varepsilon)^{1/2} \quad ; \quad q_2 = (1-2\varepsilon)^{1/2} \quad .$$

$$a_1 = \pi - \theta_0 \quad ; \quad b_1 = \pi - \theta \quad .$$

$$\alpha = a_1 - b_1 \quad ; \quad \gamma = a_1 + b_1 \quad .$$

$m = \dfrac{t-\alpha}{\gamma-\alpha}$, i.e., m traces the wavefront.

$$u(\theta, p) = \sin^{1/2}\theta \; \bar{u}(\theta, p) \quad .$$

$M_\theta(wf)$ is the value of $M_\theta(\theta, t)$ at the leading wavefront.

§ 3.2. Statement of the Problem.

The momentum equations derived in Chapter I are as follows.

$$\nabla^2 u - \left(\frac{1+\nu}{2} + \cot^2\theta\right)u + c^2\beta + \frac{3+\nu}{2} w_{,\theta} = u_{,tt} + 2\varepsilon^2 \beta_{,tt}$$

$$\nabla^2 w - \frac{2(1+\nu)}{c^2} w + \beta_{,\theta} + \beta\cot\theta - \frac{3+\nu}{2c^2}(u_{,\theta} + u\cot\theta) = \frac{1}{c^2} w_{,tt} \tag{3.1}$$

$$\nabla^2\beta - (\nu + \cot^2\theta)\beta - (\beta + w_{,\theta} - u)\, c^2 \lambda_0^2 = \beta_{,tt} + 2u_{,tt} \, .$$

Equations (3.1) are valid for $0 < \theta_0 \leq \theta \leq \pi$ and for $t > 0$. The pressure term q in (1.20) has been set equal to zero and it has been assumed that the motion is independent of φ . Again, we note that

$$\nabla^2 = \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta}$$

We assume that the sphere is initially at rest in its undeformed state corresponding to the initial conditions

$$t = 0 : \quad u = u_{,t} = \beta = \beta_{,t} = w = w_{,t} = 0 \quad , \quad \theta_0 \leq \theta \leq \pi . \tag{3.2}$$

The sphere is to be loaded by a suddenly applied uniform bending moment at the edge $\theta = \theta_0$:

$$\theta = \theta_0 : \qquad M_\theta = \beta_{,\theta} + \nu\beta \cot\theta = M_0 H(t)$$

$$N_\theta = u_{,\theta} + \nu u \cot\theta + (1+\nu) w = 0 \tag{3.3}$$

$$Q_\theta = c^2 (\beta + w_{,\theta} - u) = 0 .$$

Furthermore, the radiation condition that the leading wave be moving into an undisturbed region on its first traversal from $\theta = \theta_0$ to $\theta = \pi$ is enforced.

The problem is to be attacked by reducing (3.1) - (3.3) to a set of simultaneous ordinary differential equations on the interval $\theta_0 \leq \theta \leq \pi$ by means of a Laplace transform in time. We define

$$\bar{u}(\theta, p) = \int_0^\infty u(\theta,t)\, e^{-pt}\, dt,$$

with corresponding definitions for $\bar{w}(\theta, p)$ and $\bar{\beta}(\theta, p)$.

The boundary value problem after the Laplace transform is taken is

$$\nabla^2 \bar{u} - \left(\frac{1+\nu}{2} + \cot^2\theta \right) \bar{u} + c^2 \bar{\beta} + \frac{3+\nu}{2} \bar{w}_{,\theta} - p^2 \bar{u} - 2\varepsilon^2 p^2 \bar{\beta} = 0 \tag{3.4}$$

$$\nabla^2 \bar{w} - \frac{2(1+c)}{c^2}\bar{w} + \bar{\beta}_{,\theta} + \bar{\beta}\cot\theta - \frac{3+\nu}{2c^2}(\bar{u}_{,\theta} + \bar{u}\cot\theta) - \frac{p^2}{c^2}\bar{w} = 0$$

$$\nabla^2 \bar{\beta} - (\nu + \cot^2\theta)\bar{\beta} - (\bar{\beta} + \bar{w}_{,\theta} - \bar{u})c^2\lambda_0^2 - p^2\bar{\beta} - 2p^2\bar{u} = 0 \qquad (3.4)$$

for $\theta_0 \le \theta \le \pi$, where we have now written

$$\nabla^2 = \frac{d^2}{d\theta^2} + \cot\theta \, \frac{d}{d\theta} \quad ;$$

the boundary conditions are, at $\theta = \theta_0$,

$$\bar{\beta}_{,\theta} + \nu\bar{\beta}\cot\theta = \frac{1}{p}M_0$$

$$\bar{u}_{,\theta} + \nu\bar{u}\cot\theta + (1+\nu)\bar{w} = 0 \qquad (3.5)$$

$$\bar{\beta} + \bar{w}_{,\theta} - \bar{u} = 0 \; .$$

The radiation condition is now put in the form that for $t < \pi - \theta_0$, the Laplace transform of any variable should have the form of a decaying exponential in $\theta - \theta_0$ for large values of the Laplace variable p .

We further impose the condition that all solutions $\bar{u}(\theta, p)$, $\bar{\beta}(\theta, p)$, and $\bar{w}(\theta, p)$ and their derivatives with respect to θ are bounded functions of θ for $\theta_0 \le \theta \le \pi$ where p is just considered as a parameter. In other words, solutions of the ordinary differential equations for $\bar{u}(\theta, p)$, $\bar{\beta}(\theta, p)$, and $\bar{w}(\theta, p)$ which have a singularity in the range $\theta_0 \le \theta \le \pi$ are not acceptable.

The problem is now fully specified by equations (3.4) and (3.5) and the subsequent remarks.

It is not proposed to find the exact solution of the problem just posed. Instead we shall consider the asymptotic structure of the solution for large p . Before proceeding to the approximate solution, a few remarks on asymptotic expansions may not be out of place.

§ 3.3. On Asymptotic Expansions.

It is the intention here to remark on what may be termed the hierarchy of sophistication in regard to asymptotic expansions of ordinary differential equations involving a large real parameter. In particular, we are interested in an approximation which is uniformly valid on the range on which the differential equation holds.

Consider the equation

$$\frac{d^2y}{dx^2} + \lambda^2 f(x)\, y = 0 \tag{3.6}$$

over the range $x_1 \leq x \leq x_2$, where λ is large and real.

If $f(x) > 0$ in $[x_1, x_2]$, then [47]

$$y \sim c_1 \{f(x)\}^{-1/4} \cos\{\lambda \int [f(x)]^{1/2} dx\} + c_2 \{f(x)\}^{-1/4} \sin\{\lambda \int [f(x)]^{1/2} dx\}. \tag{3.7}$$

If $f(x) < 0$ in $[x_1, x_2]$, then

$$y \sim \{f(x)\}^{-1/4} \left[c_3 \exp\{\lambda \int [-f(x)]^{1/2} dx\} + c_4 \exp\{-\lambda \int [-f(x)]^{1/2} dx\}\right. \tag{3.8}$$

If now $f(x_0) = 0$ for $x_0 \in (x_1, x_2)$ with $f(x) > 0$ for $x \in (x_0, x_2]$ and $f(x) < 0$ for $x \in [x_1, x_0)$, then (3.7) is an asymptotic approximation in any closed interval to the right of x_0 , while (3.8) is an asymptotic approximation in any closed interval to the left of x_0 . The problem of finding a uniform asymptotic expansion over $[x_1, x_2]$, or, in other words, to find the relation connecting c_1, c_2 to c_3, c_4 was solved by Jeffreys [48] among others. An asymptotic formula which is valid over $[x_1, x_2]$ may be written in terms of Airy functions, see Erdelyi [47]. The basis for future progress was laid by Langer in a series of papers beginning in 1931 [49, 50] when he introduced his method of the "related equation." The rough idea is to construct a related equation which approaches the given equation asymptotically, with the idea of being able to solve the related equation explicitly. Then, under certain circumstances, a solution of the related equation is asymptotically equivalent to a solution of the given equation.

If $f(x)$ in (3.6) is now allowed to have a regular singularity, the problem of the asymptotic behavior is more complex. The uniform asymptotic approximation to the exact solution can be written in terms of Bessel functions, as shown by Thorne [51].

An account of the historical development of the subject is given by Pike [52, 53] as well as a proposed new existence theorem. Since the differential equations (3.4) have a regular singularity at $\theta = \pi$, and since our objective is to obtain an asymptotic approximation which is uniformly valid in $\theta_0 \le \theta \le \pi$, we may expect to become involved with Bessel functions.

§3.4. The First Order Approximation Uniformly Valid in $\theta_0 \le \theta \le \pi$.

The variable $u(\theta, p)$ is introduced by the relations

$$\bar{u}(\theta, p) = u(\theta, p) \exp\left[-\tfrac{1}{2} \int \cot\theta \, d\theta\right]$$

i.e.,

$$u(\theta, p) = \sin^{1/2}\theta \;\; \bar{u}(\theta, p) \qquad (3.9)$$

Similarly we define

$$w(\theta, p) = \sin^{1/2}\theta \; \bar{w}(\theta, p),$$

and (3.9)

$$\beta(\theta, p) = \sin^{1/2}\theta \; \bar{\beta}(\theta, p).$$

It is now noted that

$$\bar{u}_{,\theta\theta} + \cot\theta \; \bar{u}_{,\theta} = \frac{1}{\sin^{1/2}\theta}\left[u_{,\theta\theta} - \tfrac{1}{4} u \cot^2\theta - \tfrac{1}{2} u \frac{d}{d\theta}(\cot\theta)\right] \qquad (3.10)$$

Thus the transformation in (3.9) eliminates the first derivative terms in the operator ∇^2. Equations (3.4) and (3.5), with the aid of (3.9) and (3.10), may now be written as

$$u_{,\theta\theta} - \left[\tfrac{3}{4}\cot^2\theta + \tfrac{\nu}{2} + p^2\right]u + \beta(c^2 - 2\varepsilon^2 p^2) + \tfrac{3+\nu}{2}\left(w_{,\theta} - \tfrac{1}{2} w \cot\theta\right) = 0$$

$$w_{,\theta\theta} + \left[\tfrac{1}{4}\cot^2\theta + \tfrac{1}{2} - \tfrac{2a^2}{c^4} - \tfrac{1}{c^2}p^2\right]w + \beta_{,\theta} + \tfrac{1}{2}\beta\cot\theta$$

$$- \frac{3+\nu}{2c^2}\left(u_{,\theta} + \tfrac{1}{2} u \cot\theta\right) = 0 \qquad (3.11)$$

$$\beta_{,\theta\theta} - \left[\tfrac{3}{4}\cot^2\theta - \tfrac{1}{2} + \nu + c^2\lambda_0^2 + p^2\right]\beta - \lambda_0^2 c^2\left[w_{,\theta} - \tfrac{1}{2} w \cot\theta - u\right]$$

$$- 2p^2 u = 0$$

with the boundary conditions at $\theta = \theta_0$

$$\frac{1}{p} M_0 \sin^{1/2}\theta_0 = \beta_{,\theta} - \left(\tfrac{1}{2} - \nu\right)\beta\cot\theta$$

(3.12)

$$0 = u_{,\theta} - \left(\tfrac{1}{2} - \nu\right)u\cot\theta + (1+\nu)w$$

$$0 = \beta - u + w_{,\theta} - \frac{1}{\alpha} w \cot\theta \tag{3.12}$$

We consider p to be very large and ε (hence λ_0) fixed. The approximation to the solution of (3.11) and (3.12) which is valid near $\theta = \theta_0$ is first obtained. It will be found that this approximation is not valid at $\theta = \pi$.

We define

$$\bar{\theta} = p(\theta - \theta_0)$$

and then assume the following expansions for u , w , and β in terms of the variable $\bar{\theta}$

$$u(\theta, p) = u_0(\bar{\theta}) + \frac{1}{p} u_1(\bar{\theta}) + \cdots$$

$$w(\theta, p) = w_0(\bar{\theta}) + \frac{1}{p} w_1(\bar{\theta}) + \cdots$$

$$\beta(\theta, p) = \beta_0(\bar{\theta}) + \frac{1}{p} \beta_1(\bar{\theta}) + \cdots$$

The substitution of these expansions into equations (3.11) and (3.12) yields the following boundary value problem for the zeroth order approximations u_0 , w_0 , and β_0 .

$$\begin{aligned} u_{0,\bar{\theta}\bar{\theta}} - u_0 - 2\varepsilon^2 \beta_0 &= 0 \\ w_{0,\bar{\theta}\bar{\theta}} - \frac{1}{c^2} w_0 &= 0 \\ \beta_{0,\bar{\theta}\bar{\theta}} - \beta_0 - 2u_0 &= 0 \end{aligned} \tag{3.13}$$

with the boundary conditions at $\bar{\theta} = 0$

$$\beta_{0,\bar{\theta}} = \frac{1}{p^2} M_0 \sin^{1/2}\theta_0$$

$$u_{0,\bar{\theta}} = 0 \tag{3.14}$$

$$w_{0,\bar{\theta}} = 0$$

Solutions of (3.13) are

$$w_0 = A_0 e^{-\frac{1}{c}\bar{\theta}}$$

$$\beta_0 = B_1 e^{-q_1\bar{\theta}} + B_2 e^{-q_2\bar{\theta}}$$

$$u_0 = \frac{1}{2}(q_1^2-1) B_1 e^{-q_1\bar{\theta}} + \frac{1}{2}(q_2^2-1) e^{-q_2\bar{\theta}}$$

where $q_1^2 = 1+2\varepsilon$ and $q_2^2 = 1-2\varepsilon$. The exponentials which increase with $\bar{\theta}$ have been rejected since they do not represent waves going into the region $\theta > \theta_0$.

The boundary conditions (3.14) determine the coefficients A_0 , B_1 , and B_2 and the final result is that

$$w_0 = 0$$

$$\beta_0 = \frac{M_0}{p^2} \frac{q_2^2-1}{q_1(q_1^2-q_2^2)} \sin^{1/2}\theta_0 \, e^{-q_1\bar{\theta}} - \frac{M_0}{p^2} \frac{q_1^2-1}{q_2(q_1^2-q_2^2)} \sin^{1/2}\theta_0 \, e^{-q_2\bar{\theta}} \tag{3.15}$$

$$u_0 = \frac{M_0}{p^2} \frac{(q_1^2-1)(q_2^2-1)}{2q_1(q_1^2-q_2^2)} \sin^{1/2}\theta_0 \, e^{-q_1\bar{\theta}} - \frac{M_0}{p^2} \frac{(q_1^2-1)(q_2^2-1)}{2q_2(q_1^2-q_2^2)} \sin^{1/2}\theta_0 \, e^{-q_2\bar{\theta}} .$$

If we now revert to the dependent variables $\tilde{w}$, $\tilde{u}$, $\tilde{\beta}$, we see, for example, that the first approximation to $\tilde{\beta}(\theta, p)$ is $\sin^{-1/2}\theta \ \beta_0(\bar{\theta})$, and this has a singularity at $\theta = \pi$. We could have foreseen this non-uniformity by noticing the terms involving $\cot^2\theta$ in equation (3.11). These terms were neglected in comparison with p^2 , for p large, but this is valid only when θ is bounded away from π .

The variable

$$\tilde{\theta} = p(\pi - \theta)$$

is introduced to rectify this difficulty, and expansions of the following form are assumed for u , β , and w :

$$w(\theta, p) = \tilde{w}_0(\tilde{\theta}) + \frac{1}{p} \tilde{w}_1(\tilde{\theta}) + \cdots$$

$$u(\theta, p) = \tilde{u}_0(\tilde{\theta}) + \frac{1}{p} \tilde{u}_1(\tilde{\theta}) + \cdots$$

$$\beta(\theta, p) = \tilde{\beta}_0(\tilde{\theta}) + \frac{1}{p} \tilde{\beta}_1(\tilde{\theta}) + \cdots$$

The ordinary differential equations for $\tilde{w}_0$, $\tilde{u}_0$, and $\tilde{\beta}_0$ then are

$$\begin{aligned}
&\tilde{u}_{0,\tilde{\theta}\tilde{\theta}} - \left(\frac{3}{4\tilde{\theta}^2} + 1\right)\tilde{u}_0 - 2\varepsilon^2\tilde{\beta}_0 = 0 \\
&\tilde{w}_{0,\tilde{\theta}\tilde{\theta}} + \left(\frac{1}{4\tilde{\theta}^2} - \frac{1}{c^2}\right)\tilde{w}_0 = 0 \\
&\tilde{\beta}_{0,\tilde{\theta}\tilde{\theta}} - \left(\frac{3}{4\tilde{\theta}^2} + 1\right)\tilde{\beta}_0 - 2\tilde{u}_0 = 0
\end{aligned} \tag{3.16}$$

We note that the main difference between equations (3.13) and (3.16) is that the latter have a term involving $1/\tilde{\theta}^2$ which is there to represent the regular singularity at $\theta = \pi$ in the original equations (3.11).

Solutions of (3.16) are

$$\tilde{w}_0 = \tilde{A}_0 \, \tilde{\theta}^{1/2} \, I_0(\tfrac{1}{c}\tilde{\theta})$$

$$\tilde{\beta}_0 = \tilde{B}_1 \, \tilde{\theta}^{1/2} \, I_1(q_1\tilde{\theta}) + \tilde{B}_2 \, \tilde{\theta}^{1/2} \, I_1(q_2\tilde{\theta}) \tag{3.17}$$

$$\tilde{u}_0 = \tfrac{1}{2}(q_1^2-1)\,\tilde{B}_1 \, \tilde{\theta}^{1/2} \, I_1(q_1\tilde{\theta}) + \tfrac{1}{2}(q_2^2-1)\,\tilde{B}_2 \, \tilde{\theta}^{1/2} \, I_1(q_2\tilde{\theta})$$

where I_0, I_1 are modified Bessel functions. The functions K_0, K_1 are rejected as solutions of (3.17) as they are singular at $\tilde{\theta} = 0$.

The approximation (3.17) is valid at least in a neighborhood of $\tilde{\theta} = 0$, i.e., near $\theta = \pi$, while the approximation (3.15) is valid on any closed interval which excludes the point $\theta = \pi$. Thus the two approximations have a common interval of validity. By matching equations (3.17) to equations (3.15) on this common interval of validity the coefficients $\tilde{A}_0$, $\tilde{B}_1$, and $\tilde{B}_2$ are determined.

We fix $\theta \neq \pi$ and let $p \gg 1$, then

$$I_\nu(q\tilde{\theta}) = I_\nu(qp(\pi-\theta)) \sim \frac{e^{qp(\pi-\theta)}}{\sqrt{2\pi q p(\pi-\theta)}}\left[1 + O(\tfrac{1}{p})\right]$$

Thus, for $\theta \neq \pi$

$$\tilde{\beta}_0 \sim \tilde{B}_1 \frac{1}{\sqrt{2\pi q_1}} e^{q_1 p(\pi-\theta)} + \tilde{B}_2 \frac{1}{\sqrt{2\pi q_2}} e^{q_2 p(\pi-\theta)} .$$

This expression for $\tilde{\beta}_0$ may be written as

$$\tilde{\beta}_0 \sim \tilde{B}_1 \frac{1}{\sqrt{2\pi q_1}} e^{q_1 p(\pi-\theta_0)} e^{-q_1 p(\theta-\theta_0)}$$

$$+ \tilde{B}_2 \frac{1}{\sqrt{2\pi q_2}} e^{q_2 p(\pi-\theta_0)} e^{-q_2 p(\theta-\theta_0)}$$

and on comparison with (3.13) it is evident that

$$\tilde{B}_1 = \frac{M_0}{p^2} \left(\frac{2\pi}{q_1}\right)^{1/2} \frac{q_2^2-1}{q_1^2-q_2^2} \sin^{1/2}\theta_0 \; e^{-q_1 p(\pi-\theta_0)}$$

and

$$\tilde{B}_2 = -\frac{M_0}{p^2} \left(\frac{2\pi}{q_2}\right)^{1/2} \frac{q_1^2-1}{q_1^2-q_2^2} \sin^{1/2}\theta_0 \; e^{-q_2 p(\pi-\theta_0)} .$$

By a similar procedure

$$\tilde{A}_0 = 0$$

Thus, the first order approximation which is uniformly valid in θ in the range $\theta_0 \le \theta \le \pi$ as $p \to \infty$ is given by

$$\bar{w}_0(\theta, p) = 0$$

$$\bar{\beta}_0(\theta,p) = \frac{M_0}{p^{3/2}} \frac{q_2^2-1}{q_1^2-q_2^2} (2\pi \sin\theta_0)^{1/2} \left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} e^{-q_1 p(\pi-\theta_0)} I_1[q_1 p(\pi-\theta)]$$

$$- \frac{M_0}{p^{3/2}} \frac{q_1^2-1}{q_1^2-q_2^2} (2\pi \sin\theta_0)^{1/2} \left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} e^{-q_2 p(\pi-\theta_0)} I_1[q_2 p(\pi-\theta)] \qquad (3.18)$$

$$\bar{u}_0(\theta,p) = \frac{M_0}{p^{3/2}} \frac{(q_1^2-1)(q_2^2-1)}{q_1^2-q_2^2} \left(\frac{\pi}{2} \sin\theta_0\right)^{1/2} \left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} e^{-q_1 p(\pi-\theta_0)} I_1[q_1 p(\pi-\theta)]$$

$$- \frac{M_0}{p^{3/2}} \frac{(q_1^2-1)(q_2^2-1)}{q_1^2-q_2^2} \left(\frac{\pi}{2} \sin\theta_0\right)^{1/2} \left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} e^{-q_2 p(\pi-\theta_0)} I_1[q_2 p(\pi-\theta)] .$$

We have written $\bar{\beta}_0 = \sin^{-1/2}\theta \; \tilde{\beta}_0$ and similarly for $\bar{u}_0$ and $\bar{w}_0$.

The essence of an asymptotic expansion is that the succeeding terms are diminishing so that each term is a small correction to the preceding ones. If we now proceed to calculate the second terms $u_1(\bar{\theta})$ and $\beta_1(\bar{\theta})$ of the expansion valid near $\theta = \theta_0$, we will find, for example, that $\beta_1(\bar{\theta})$ will have a term of the form $\bar{\theta}\, e^{-q_1\bar{\theta}}$. Due to the definition of $\bar{\theta}$,

$$\frac{1}{p}\bar{\theta}\, e^{-q_1\bar{\theta}} = (\theta - \theta_0)\, e^{-q_1\bar{\theta}}$$

and so the second term in the expansion has the same order of magnitude as the first. We could term this a "resonance" phenomenon. This defect may be obviated, and at the same time the effect of higher order coupling terms in the equations is obtained, by the two-variable expansion technique originated by Cole and Kevorkian and described by Kevorkian [38].

3.5. Uniformity by the Two-Variable Expansion Method.

We consider a linear oscillator with a driving force, viz.,

$$y_{,tt} + (1+\varepsilon)y = \sin t$$

where ε is a small number. The frequency of the driving force is very close to the natural frequency of the oscillator. Since $\varepsilon << 1$, we may approximate the given equation by

$$y_{,tt} + y = \sin t.$$

Solutions of this equation are unbounded as $t \to \infty$ due to resonance, whereas solutions to the original equation are bounded. If we followed a naive asymptotic procedure in powers of ε, such as the one in the previous section, then the first approximation would lead to the resonance phenomenon. The resonance thus obtained is not inherent in the original equation, but appears as a result of poor mathematics. It is a difficulty such as this which arises when we try to construct the second terms in the expansions proposed in the previous section. The source of the difficulty is made patent by the simple example cited. A device must be found to correct the argument of the solution of the homogeneous equation given by the first approximation and then resonance does not occur.

With these remarks in mind, we introduce the new independent variables,

$$\theta_i^+ = \left(1 + \frac{\Omega_i}{p^2}\right)\bar{\theta} \quad , \qquad i = 1, 2, 3$$

where Ω_i is constant and, as before, $\bar{\theta} = p\,(\theta - \theta_0)$. Further, let

$$\hat{\theta} = \frac{1}{p}\,\bar{\theta}$$

If $\alpha = \alpha(\bar{\theta})$ and we assume an expansion for α of the form

$$\alpha(\bar{\theta}) = \alpha_0(\hat{\theta}, \theta^+) + \frac{1}{p}\,\alpha_1(\hat{\theta}, \theta^+) + \cdots$$

then

$$\frac{d\alpha}{d\bar{\theta}} = \alpha_{0,\theta^+} + \frac{1}{p}\left(\alpha_{0,\hat{\theta}} + \alpha_{1,\theta^+}\right) + \frac{1}{p^2}\left(\Omega\,\alpha_{0,\theta^+} + \alpha_{1,\hat{\theta}} + \alpha_{2,\theta^+}\right)$$
$$+ \cdots$$

and

$$\frac{d^2\alpha}{d\bar{\theta}^2} = \alpha_{0,\theta^+\theta^+} + \frac{1}{p}\left(2\alpha_{0,\theta^+\hat{\theta}} + \alpha_{1,\theta^+\theta^+}\right)$$

$$+ \frac{1}{p^2}\left(2\Omega\,\alpha_{0,\theta^+\theta^+} + \alpha_{0,\hat{\theta}\hat{\theta}} + 2\alpha_{1,\theta^+\hat{\theta}} + \alpha_{2,\theta^+\theta^+}\right) + \cdots$$

Equations (3.11) are rewritten using $\bar{\theta}$ as independent variable. They become

$$p^2 u_{,\bar\theta\bar\theta} - \left[\tfrac{3}{4}\cot^2\left(\theta_0+\tfrac{\bar\theta}{p}\right) + \tfrac{\nu}{2} + p^2\right] u + \beta\left(c^2 - 2\varepsilon^2 p^2\right)$$
$$+ \tfrac{3+\nu}{2}\left[p\, w_{,\bar\theta} - \tfrac{1}{2} w \cot\left(\theta_0 + \tfrac{\bar\theta}{p}\right)\right] = 0$$

$$p^2 w_{,\bar\theta\bar\theta} + \left[\tfrac{1}{4}\cot^2\left(\theta_0+\tfrac{\bar\theta}{p}\right) + \tfrac{1}{2} - \tfrac{2a^2}{c^4} - \tfrac{1}{c^2}p^2\right] w + p\,\beta_{,\bar\theta}$$
$$+ \tfrac{1}{2}\beta\cot\left(\theta_0+\tfrac{\bar\theta}{p}\right) - \tfrac{3+\nu}{2c^2}\left[p\, u_{,\bar\theta} + \tfrac{1}{2} u \cot\left(\theta_0+\tfrac{\bar\theta}{p}\right)\right] = 0 \tag{3.19}$$

$$p^2\beta_{,\bar\theta\bar\theta} - \left[\tfrac{3}{4}\cot^2\left(\theta_0+\tfrac{\bar\theta}{p}\right) - \tfrac{1}{2} + \nu + c^2\lambda_0^2 + p^2\right]\beta$$
$$-\lambda_0^2 c^2\left[p\, w_{,\bar\theta} - \tfrac{1}{2} w\cot\left(\theta_0+\tfrac{\bar\theta}{p}\right) - u\right] - 2p^2 u = 0$$

The boundary conditions (3.12) become: At $\bar\theta = 0$

$$\tfrac{1}{p} M_0 \sin^{1/2}\theta_0 = p\,\beta_{,\bar\theta} - \left(\tfrac{1}{2}-\nu\right)\beta\cot\left(\theta_0+\tfrac{\bar\theta}{p}\right)$$

$$0 = p\, u_{,\bar\theta} - \left(\tfrac{1}{2}-\nu\right) u\cot\left(\theta_0+\tfrac{\bar\theta}{p}\right) + (1+\nu) w \tag{3.20}$$

$$0 = \beta - u + p\, w_{,\bar\theta} - \tfrac{1}{2} w \cot\left(\theta_0 + \tfrac{\bar\theta}{p}\right)$$

We now assume that the following expansions are valid.

$$\beta(\bar\theta, p) = \beta_0(\hat\theta, \theta_1^+) + \tfrac{1}{p}\beta_1(\hat\theta, \theta_1^+) + \cdots$$

$$u(\bar\theta, p) = u_0(\hat\theta, \theta_2^+) + \tfrac{1}{p} u_1(\hat\theta, \theta_2^+) + \cdots$$

$$w(\bar\theta, p) = w_0(\hat\theta, \theta_3^+) + \tfrac{1}{p} w_1(\hat\theta, \theta_3^+) + \cdots$$

Then the first approximation to the system given by (3.19) and (3.20) is

$$u_{0,\theta^+\theta^+} \quad - u_0 \quad - 2\varepsilon^2 \beta_0 \qquad = 0$$

$$w_{0,\theta^+\theta^+} \quad - \frac{1}{c^2} w_0 \qquad = 0$$

$$\beta_{0,\theta^+\theta^+} \quad - \beta_0 \quad - 2u_0 \qquad = 0$$

with

$$\frac{1}{\beta^2} \quad M_0 \quad \sin^{1/2}\theta_0 \qquad = \beta_{0,\theta^+}$$

$$0 \qquad = u_{0,\theta^+}$$

$$0 \qquad = w_{0,\theta^+}$$

at $\theta^+ = 0$.

The general solution is

$$w_0 \quad = A_{10}(\hat{\theta}) \quad e^{-\frac{1}{c}\theta_3^+}$$

$$\beta_0 \quad = B_{10}(\hat{\theta})\ e^{-q_1\theta_1^+} + B_{20}(\hat{\theta})\ e^{-q_2\theta_1^+}$$

$$u_0 \quad = \frac{1}{2}(q_1^2-1)\,B_{10}(\vec{\theta})\ e^{-q_1\theta_2^+} + \frac{1}{2}(q_2^2-1)\,B_{20}(\vec{\theta})\ e^{-q_2\theta_2^+} \tag{3.21}$$

and the boundary conditions require

$$A_{10}(0) \qquad = 0$$

$$B_{10}(0) = \frac{1}{\beta^2}\,\frac{M_0}{q_1}\,\frac{q_2^2-1}{q_2^2-q_1^2}\ \sin^{1/2}\theta_0 \tag{3.22}$$

$$B_{20}(0) = -\frac{1}{\beta^2} \frac{M_0}{q_2} \frac{q_1^2-1}{q_2^2-q_1^2} \sin^{1/2}\theta_0 \tag{3.22}$$

where, for example, $A_{10}(0)$ means $A_{10}(\hat{\theta})$ evaluated at $\hat{\theta} = 0$. We note that the first order approximation given by (3.21) and(3.22) differs from (3.15) in some simple, but yet essential, aspects. The expressions for w_0, β_0, and u_0 in (3.15) are fully determined. In contrast to this, the solutions given by (3.21) and(3.22) have a $\hat{\theta}$ dependence which has not yet been fully determined and θ_1^+, θ_2^+, θ_3^+ are as yet unknown since the Ω associated with each must be found. We propose to use this indeterminancy to impose the condition that there be no resonance in the second term of the asymptotic expansion. The remainder of this section concerns itself with the details of this calculation.

The boundary value problem for u_1, β_1, and w_1 is

$$\begin{aligned}
u_{1,\theta^+\theta^+} - u_1 - 2\epsilon^2\beta_1 + 2u_{0,\theta^+\hat{\theta}} + \frac{3+\nu}{2} w_{0,\theta^+} &= 0 \\
w_{1,\theta^+\theta^+} - \frac{1}{c^2} w_1 + \beta_{0,\theta^+} - \frac{3+\nu}{2c^2} u_{0,\theta^+} + 2 w_{0,\theta^+\hat{\theta}} &= 0 \\
\beta_{1,\theta^+\theta^+} - \beta_1 - 2u_1 + 2\beta_{0,\theta^+\hat{\theta}} - \lambda_0^2 c^2 w_{0,\theta^+} &= 0
\end{aligned} \tag{3.23}$$

with

$$\begin{aligned}
\beta_{1,\theta_1^+} &= (\tfrac{1}{2}-\nu)\beta_0 \cot\theta_0 \\
u_{1,\theta_2^+} &= (\tfrac{1}{2}-\nu) u_0 \cot\theta_0 - (1+\nu) w_0
\end{aligned} \tag{3.24}$$

$$w_{1,\theta_3^+} = u_0 - \beta_0 + \frac{1}{\alpha} w_0 \cot \theta_0 \tag{3.24}$$

at $\hat{\theta} = \theta^+ = 0$. On using the forms of w_0, β_0, u_0 given in (3.21), (3.23) may be written as

$$u_{1,\theta_2^+\theta_2^+} - u_1 - 2\varepsilon^2 \beta_1 = \frac{3+\nu}{2c} A_{10} e^{-\frac{1}{c}\theta_3^+} + q_1 (q_1^2-1) B_{10}' e^{-q_1\theta_2^+} + q_2 (q_2^2-1) B_{20}' e^{-q_2\theta_2^+}$$

$$w_{1,\theta_3^+\theta_3^+} - \frac{1}{c^2} w_1 = \frac{2}{c} A_{10}' e^{-\frac{1}{c}\theta_3^+} + q_1 B_{10} e^{-q_1\theta_1^+} + q_2 B_{20} e^{-q_2\theta_1^+} - \frac{3+\nu}{2c^2}\left[\frac{1}{\alpha} q_1 (q_1^2-1) B_{10} e^{-q_1\theta_2^+} + \frac{1}{\alpha} q_2 (q_2^2-1) B_{20} e^{-q_2\theta_2^+}\right] \tag{3.25}$$

$$\beta_{1,\theta_1^+\theta_1^+} - \beta_1 - 2u_1 = 2q_1 B_{10}' e^{-q_1\theta_1^+} + 2q_2 B_{20}' e^{-q_2\theta_1^+} - \lambda_0^2 c A_{10} e^{-\frac{1}{c}\theta_3^+}$$

where a prime denotes differentiation with respect to $\hat{\theta}$.

The right hand side of the equation for w_1 in (3.25) has a term $\frac{2}{c} A_{10}' e^{-\frac{1}{c}\theta_3^+}$ which will give a resonance on integration. This difficulty may be avoided by setting

$$A_{10}'(\hat{\theta}) = 0 ,$$

i.e.,

$$A_{10}(\vec{\theta}) = \text{constant} = A_{10}(0).$$

For similar reasons, we set

$$B'_{10} = B'_{20} = 0$$

and hence

$$B_{10}(\vec{\theta}) = B_{10}(0) \quad , \qquad B_{20}(\vec{\theta}) = B_{20}(0).$$

Thus, the coefficients in the first approximation have been determined, and the undesirable terms in the equation for the second approximation have been eliminated at one stroke. This is the first aspect of the flexibility allowed us by the two-variable technique.

At this stage, we have

$$\begin{aligned} A_{10} &\equiv A_{10}(\vec{\theta}) = 0 \\ B_{10} &\equiv B_{10}(\vec{\theta}) = \frac{1}{\beta^2}\frac{M_0}{q_1}\frac{q_2^2-1}{q_2^2-q_1^2}\sin^{1/2}\theta_0 \\ B_{20} &\equiv B_{20}(\vec{\theta}) = -\frac{1}{\beta^2}\frac{M_0}{q_2}\frac{q_1^2-1}{q_2^2-q_1^2}\sin^{1/2}\theta_0 \end{aligned} \qquad (3.26)$$

To complete the first approximation we must yet find Ω_1, Ω_2, Ω_3.

Equation (3.25) now becomes

$$u_{1,\theta_2^+\theta_2^+} - u_1 - 2\varepsilon^2\beta_1 = 0$$

$$w_{1,\theta_3^+\theta_3^+} - \frac{1}{c^2}w_1 = q_1 B_{10} e^{-q_1\theta_1^+} + q_2 B_{20} e^{-q_2\theta_1^+}$$

$$- \frac{3+\nu}{2c^2}\left[\tfrac{1}{2}q_1(q_1^2-1)B_{10}e^{-q_1\theta_2^+} + \tfrac{1}{2}q_2(q_2^2-1)B_{20}e^{-q_2\theta_2^+}\right]$$

$$\beta_{1,\theta_1^+\theta_1^+} \quad - \beta_1 \quad - 2u_1 \qquad = 0$$

Solutions to these are

$$\beta_1 = B_{11}(\vec{\theta})\, e^{-q_1\theta_1^+} + B_{21}(\vec{\theta})\, e^{-q_2\theta_1^+}$$

$$u_1 = \tfrac{1}{2}(q_1^2-1)\, B_{11}\, e^{-q_1\theta_2^+} + \tfrac{1}{2}(q_2^2-1)\, B_{21}\, e^{-q_2\theta_2^+}$$

$$w_1 = A_{11}(\vec{\theta})\, e^{-\frac{1}{c}\theta_3^+} + \frac{q_1}{q_1^2 - 1/c^2}\, B_{10}\, e^{-q_1\theta_1^+} + \frac{q_2}{q_2^2 - 1/c^2}\, B_{20}\, e^{-q_2\theta_1^+} \tag{3.27}$$

$$- \frac{3+\nu}{2c^2}\left[\frac{q_1(q_1^2-1)}{2(q_1^2-1/c^2)}\, B_{10}\, e^{-q_1\theta_2^+} + \frac{q_2(q_2^2-1)}{2(q_2^2-1/c^2)}\, B_{20}\, e^{-q_2\theta_2^+}\right]$$

and the boundary conditions (3.24), together with the result $w_0 = 0$, imply

$$-q_1\, B_{1\phi}(0) - q_2\, B_{21}(0) = (\tfrac{1}{2}-\nu)\cot\theta_0\,(B_{10} + B_{20})$$

$$-\tfrac{1}{2}q_1(q_1^2-1)\, B_{11}(0) - \tfrac{1}{2}q_2(q_2^2-1)\, B_{21}(0) = (\tfrac{1}{2}-\nu)\cot\theta_0\left[\tfrac{1}{2}(q_1^2-1)\, B_{10} + \tfrac{1}{2}(q_2^2-1)\, B_{20}\right]$$

$$-\tfrac{1}{c}A_{11}(0) + a_1 B_{10} + a_2 B_{20} = \tfrac{1}{2}(q_1^2-3)\, B_{10} + \tfrac{1}{2}(q_2^2-3)\, B_{20}$$

where

$$a_1 = \frac{-q_1^2}{q_1^2 - \frac{1}{c^2}} + \frac{3+\nu}{2c^2}\;\frac{q_1^2(q_1^2-1)}{2(q_1^2-1/c^2)}$$

and

$$a_2 = \frac{-q_2^2}{q_2^2 - 1/c^2} + \frac{3+\nu}{2c^2} \frac{q_2^2(q_2^2-1)}{2(q_2^2 - 1/c^2)}$$

In particular

$$A_{11}(0) = c\left[\{a_1 - \tfrac{1}{2}(q_1^2-3)\}B_{10} + \{a_2 - \tfrac{1}{2}(q_2^2-3)\}B_{20}\right] \tag{3.28}$$

When cognizance is taken of the fact that u_0 and β_0 are independent of $\hat{\theta}$ and $w_0 = 0$, the equations for β_2, u_2, and w_2 are

$$2\Omega_2 u_{0,\theta_2^+\theta_2^+} + 2u_{1,\theta_2^+\hat{\theta}} + u_{2,\theta_2^+\theta_2^+} - (\tfrac{3}{4}\cot^2\theta_0 + \tfrac{\nu}{2})u_0$$
$$- u_2 + c^2\beta_0 - 2\varepsilon^2\beta_2 + \frac{3+\nu}{2} w_{1,\theta_3^+} = 0$$

$$2w_{1,\theta_3^+\hat{\theta}} + w_{2,\theta_3^+\theta_3^+} - \frac{1}{c^2}w_2 + \beta_{1,\theta_1^+} + \tfrac{1}{2}\beta_0\cot\theta_0$$
$$- \frac{3+\nu}{2c^2}\left(u_{1,\theta_2^+} + \tfrac{1}{2}u_0\cot\theta_0\right) = 0 \tag{3.29}$$

$$2\Omega_1\beta_{0,\theta_1^+\theta_1^+} + 2\beta_{1,\theta_1^+\hat{\theta}} + \beta_{2,\theta_1^+\theta_1^+} - \left[\tfrac{3}{4}\cot^2\theta_0 - \tfrac{1}{2} + \nu + \lambda_0^2c^2\right]\beta_0$$
$$- \beta_2 - \lambda_0^2c^2\left(w_{1,\theta_3^+} - u_0\right) - 2u_2 = 0$$

The second of these equations is

$$w_{2,\theta_3^+\theta_3^+} - \frac{1}{c^2} w_2 = -2 w_{1,\theta_3^+\hat{\theta}} - \beta_{1,\theta_1^+} - \frac{1}{2}\beta_0 \cot\theta_0 + \frac{3+\nu}{2c^2}\left(u_{1,\theta_2^+} + \frac{1}{2} u_0 \cot\theta_0\right)$$

The term $e^{-\frac{1}{c}\theta_3^+}$ does not occur in β_0 , β_1 , u_0 , or u_1 , and so a resonance will not arise from these terms. On the other hand, w_1 does have such a term. As before, we may eliminate the resonance by demanding that $A_{11}(\vec{\theta})$ be independent of $\hat{\theta}$, or equivalently, that $A_{11}(\vec{\theta})$ constant = $A_{11}(0)$. Thus, w_1 is given by equation (3.27) where $A_{11}(\vec{\theta})$ is replaced by $A_{11}(0)$ and $A_{11}(0)$ is given by (3.28).

If we consider the first and third equations of (3.29), there are resonant terms present. These may be grouped as follows on the right hand side.

$$e^{-q_1\theta_2^+}\left[-q_1(q_1^2-1)B_{11}' + a_{11}\Omega_2 + a_{12}\right] + e^{-q_2\theta_2^+}\left[-q_2(q_2^2-1)B_{21}' + a_{13}\Omega_2 + a_{14}\right] + a_{15}e^{-q_1\theta_1^+} + a_{16}e^{-q_2\theta_1^+}$$

and

$$e^{-q_1\theta_1^+}\left[-2q_1 B_{11}' + b_{11}\Omega_1 + b_{12}\right] + e^{-q_2\theta_1^+}\left[-2q_2 B_{21}' + b_{13}\Omega_1 + b_{14}\right. + b_{15}e^{-q_1\theta_2^+} + b_{16}e^{-q_2\theta_2^+}$$

where

$$a_{11} = q_1^2(q_1^2-1)B_{10}$$

$$a_{12} = -\left(\frac{3}{4}\cot^2\theta_0 + \frac{\nu}{2}\right)\frac{1}{2}(q_1^2-1)B_{10} + \frac{(3+\nu)^2}{4c^2}\frac{q_1^2(q_1^2-1)}{2(q_1^2-1/c^2)}B_{10}$$

$$a_{13} = q_2^2(q_2^2-1)B_{20}$$

$$a_{14} = -\left(\frac{3}{4}\cot^2\theta_0 + \frac{\nu}{2}\right)\frac{1}{2}(q_2^2-1)B_{20} + \frac{(3+\nu)^2}{4c^2}\frac{q_2^2(q_2^2-1)}{2(q_2^2-1/c^2)}B_{20}$$

$$a_{15} = c^2 B_{10} - \frac{3+\nu}{2}\frac{q_1^2}{q_1^2-1/c^2}B_{10}$$

$$a_{16} = c^2 B_{20} - \frac{3+\nu}{2}\frac{q_2^2}{q_2^2-1/c^2}B_{20}$$

$$b_{11} = 2q_1^2 B_{11}$$

$$b_{12} = -\left(\frac{3}{4}\cot^2\theta_0 - \frac{1}{2}+\nu+\lambda_0^2c^2\right)B_{10} + \frac{\lambda_0^2c^2q_1^2}{q_1^2-1/c^2}B_{10}$$

$$b_{13} = 2q_2^2 B_{21}$$

$$b_{14} = -\left(\frac{3}{4}\cot^2\theta_0 - \frac{1}{2}+\nu+\lambda_0^2c^2\right)B_{20} + \frac{\lambda_0^2c^2q_2^2}{q_2^2-1/c^2}B_{20}$$

$$b_{15} = \frac{3+\nu}{2c^2}\frac{q_1^2(q_1^2-1)}{2(q_1^2-1/c^2)}B_{10} + \lambda_0^2c^2\frac{1}{2}(q_1^2-1)B_{10}$$

$$b_{16} = \frac{3+\nu}{2c^2}\frac{q_2^2(q_2^2-1)}{2(q_2^2-1/c^2)}B_{20} + \lambda_0^2c^2\frac{1}{2}(q_2^2-1)B_{20}.$$

As before, some of these terms may be eliminated by setting

$$B_{11}' = B_{21}' = 0$$

Thus,

$$B_{11}(\vec{\theta}) = B_{11}(0)$$

and (3.30)

$$B_{21}(\vec{\theta}) = B_{21}(0) .$$

At this stage, the equations under consideration may be written as

$$u_{,\theta\theta} - u - 2\epsilon^2\beta = e^{-q_1\theta}\left[a_{11}\Omega_1 + d_{11}\right] + e^{-q_2\theta}\left[a_{13}\Omega_2 + d_{12}\right]$$

(3.31)

$$\beta_{,\theta\theta} - \beta - 2u = e^{-q_1\theta}\left[b_{11}\Omega_1 + d_{21}\right] + e^{-q_2\theta}\left[b_{13}\Omega_1 + d_{22}\right]$$

where

$$d_{11} = a_{12} + a_{15}$$

$$d_{12} = a_{14} + a_{16}$$

$$d_{21} = b_{12} + b_{15}$$

$$d_{22} = b_{14} + b_{16} .$$

All the coefficients of the right hand side of (3.31) are constants.

If we can now choose Ω_1 and Ω_2 so that the only solutions to the non-homogeneous equations (3.31) are the solutions to the homogeneous equations, we will have achieved our purpose of eliminating the terms giving resonance.

If we eliminate u, say, from (3.31) and write a non-homogeneous fourth order equation for β, then the non-homogeneity may be removed by choosing Ω_1 and Ω_2 as solutions of the algebraic equations

$$\tfrac{1}{2}(q_1^2-1)\left[b_{11}\Omega_1 + d_{21}\right] + \left[a_{11}\Omega_2 + d_{11}\right] = 0$$

$$\tfrac{1}{2}(q_2^2-1)\left[b_{13}\Omega_1 + d_{22}\right] + \left[a_{13}\Omega_2 + d_{12}\right] = 0$$

Then

$$\Omega_1 = \mathcal{D}^{-1}\left[a_{11}\left\{\tfrac{1}{2}(q_2^2-1)d_{22} + d_{12}\right\} - \left\{\tfrac{1}{2}(q_1^2-1)d_{21} + d_{11}\right\}a_{13}\right]$$

and (3.32)

$$\Omega_2 = \mathcal{D}^{-1}\left[\left\{\tfrac{1}{2}(q_1^2-1)d_{21} + d_{11}\right\}\tfrac{1}{2}(q_2^2-1)b_{13} - \left\{\tfrac{1}{2}(q_2^2-1)d_{22} + d_{12}\right\}\tfrac{1}{2}(q_1^2-1)b_{11}\right]$$

where

$$\mathcal{D} = \tfrac{1}{2}(q_1^2-1)\,b_{11}\,a_{13} - \tfrac{1}{2}(q_2^2-1)\,a_{11}\,b_{13} .$$

The determination of Ω_3 follows in the same way at the next order of the approximation.

We now see that the zeroth order approximation is fully determined and the first correction to it does not have any undesirable growth.

To sum up in general terms, the two-variable expansion technique gives an indeterminancy in the coefficient multiplying the solution function and also in the argument of the solution function at the zeroth order of approximation. The rule that resonance cannot be tolerated determines the coefficient at the first order, and the argument at the second order. Then, the correct expansion has been found.

The expansion for $\bar{\beta}(\vartheta, p)$ now is

$$\bar{\beta}(\theta,p) = \bar{\beta}_0(\theta,p) + \frac{1}{p}\bar{\beta}_1(\vartheta,p) + \cdots$$

$$= \sin^{-\frac{1}{2}}\theta \left[\beta_0(\hat{\theta},\vartheta_1^+) + \frac{1}{p}\beta_1(\hat{\theta},\vartheta_1^+) + \cdots \right]$$

$$= \sin^{-\frac{1}{2}}\theta \left[\frac{1}{p^2}\frac{M_0}{q_1}\frac{q_2^2-1}{q_2^2-q_1^2}\sin^{1/2}\theta_0\, e^{-q_1\vartheta_1^+} \right.$$

$$\left. -\frac{1}{p^2}\frac{M_0}{q_2}\frac{q_1^2-1}{q_2^2-q_1^2}\sin^{1/2}\theta_0\, e^{-q_2\vartheta_1^+} + \cdots \right] \tag{3.33}$$

where $\vartheta_1^+ = (1+\frac{\Omega_1}{p^2})p(\theta-\theta_0)$ and Ω_1 is given in (3.32). The expansion given in (3.33) is not uniformly valid in θ for $\theta_0 \le \theta \le \pi$, but this is easily rectified. The argument of the modified Bessel functions in equations (3.18) is changed from $p(\pi-\theta)$ to $(1+\frac{\Omega}{p^2})p(\pi-\theta)$. Then, for example,

$$\bar{\beta}_0(\theta,p) = \frac{M_0}{p^{3/2}}\frac{q_2^2-1}{q_1^2-q_2^2}(2\pi\sin\theta_0)^{1/2}\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2}$$
$$\times\, e^{-q_1 p(\pi-\theta_0)(1+\frac{\Omega_1}{p^2})} I_1[q_1 p(\pi-\theta)(1+\frac{\Omega_1}{p^2})]$$

$$-\frac{M_0}{p^{3/2}}\frac{q_1^2-1}{q_1^2-q_2^2}(2\pi\sin\theta_0)^{1/2}\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2}$$
$$\times\, e^{-q_2 p(\pi-\theta_0)(1+\frac{\Omega_1}{p^2})} I_1[q_1 p(\pi-\theta)(1+\frac{\Omega_1}{p^2})] \tag{3.34}$$

and

$$\frac{\bar{\beta}_1(\theta,p)}{\bar{\beta}_0(\theta,p)} = O\left(\frac{1}{p}\right) \quad \text{as } p \to \infty, \quad \theta_0 \le \theta \le \pi, \tag{3.35}$$

where $\bar{\beta}_0$ is given by (3.34) and $\bar{\beta}_1$ has a similar structure. A statement such as (3.35) could not be made about $\bar{\beta}_0$ as given by (3.18) when $\bar{\beta}_1$ has a similar form.

If we had not made the transformation (3.9), then the procedure of the preceding section would not have yielded a solution uniformly valid in $\theta_0 \le \theta \le \pi$. In the present section, when the transformation (3.9) was made, the procedure gave an ordinary differential equation for $A_{10}(\vec{\theta})$, $B_{10}(\vec{\theta})$, and $B_{20}(\vec{\theta})$ which resulted in all three quantities being constant. If (3.9) had not been used, then this latter procedure would again have given an ordinary differential equation for $B_{10}(\vec{\theta})$, say. But this time the solution of the equation would have given the factor $\sin^{-1/2}\theta$.

We now examine whether we may expect to get terms in the solution (3.34) which represent disturbances reflected from $\theta = \pi$. Firstly, a simple example from [34], p.527, is given just to fix ideas.

We consider, for the interval $0 \le x \le l$, the problem of solving the partial differential equation

$$\varphi_{,tt} - \varphi_{,xx} = 0$$

with the initial and boundary conditions

$$\varphi(x,0) = \varphi_{,t}(x,0) = 0$$

$$\varphi(0,t) = 1 \quad , \quad \varphi_{,x}(l,t) = 0$$

Let $\bar{\varphi}(x,p) = \int_0^\infty \varphi(x,t)\, e^{-pt}\, dt$, then

$$\bar{\varphi}(x,p) = \frac{1}{p}\,\frac{\cosh p(\ell - x)}{\cosh p\ell} = \frac{1}{p}\,\frac{e^{-px} + e^{-p(2\ell - x)}}{1 + e^{-2p\ell}} \tag{3.36}$$

$$= \frac{1}{p} e^{-px} + \frac{1}{p}\sum_{\nu=1}^{\infty}(-1)^{\nu+1}\left[e^{-p(2\nu\ell - x)} - e^{-p(x+2\nu\ell)}\right]$$

On performing the inversion of this, we find

$$\varphi(x,t) = H(t-x) + \sum_{\nu=1}^{\infty}(-1)^{\nu+1}\left[H(t+x-2\nu\ell) - H(t-x-2\nu\ell)\right].$$

We rewrite the right hand side, and then

$$\varphi(x,t) = H(t-x) + H(t-(2\ell-x)) - H(t-(x+2\ell)) + \cdots$$

Now the first term represents a wave going from $x = 0$ to $x = \ell$; the second term is the reflection of this wave from $x = \ell$ and so on. Thus the negative exponentials in (3.36) correspond to the various reflections of the wave at $x = 0$ and $x = \ell$.

The type of function that has been met as solutions to the problem defined by (3.4) and (3.5) is

$$f(\theta)\, e^{-qp(\pi-\theta_0)}\, I_1\left[qp(\pi-\theta)\right]. \tag{3.37}$$

The question is whether the modified Bessel function is rich enough to account for reflections of the wave from $\theta = \pi$ and $\theta = \theta_0$. Watson [54], p. 203, gives the asymptotic formula for $I_\nu(z)$ when z is large and $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$,

$$I_\nu(z) \sim \frac{e^{z}}{\sqrt{2\pi z}}\left[1+O\left(\tfrac{1}{z}\right)\right] + \frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}}\left[1+O\left(\tfrac{1}{z}\right)\right].$$

Even though the second term in the asymptotic formula given is exponentially small compared with the first, it is crucial for our results.

We may write formula (3.37) for p large and $\theta \neq \pi$ as

$$f(\theta)\, e^{-qp(\pi-\theta_0)}\, I_1[qp(\pi-\theta)]$$

$$\sim \frac{f(\theta)}{\sqrt{2\pi q p(\pi-\theta)}}\left[e^{-qp(\theta-\theta_0)} + e^{-qp(2\pi-\theta-\theta_0)+\frac{3}{2}\pi i}\right].$$

The first term is recognized as the representation of a wave going from $\theta = \theta_0$ to $\theta = \pi$; the second term represents its reflection from $\theta = \pi$.

§ 3.6. $M_\theta(\theta,t)$ at the Wavefront.

We let $\varepsilon \to 0$ and then q_1 and q_2 both tend to unity. Then we get from (3.34) that

$$\bar{\beta}_0(\theta,p) = -\frac{M_0}{p^{3/2}}(2\pi \sin\theta_0)^{1/2}\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} e^{-p(\pi-\theta_0)(1+\frac{\Omega_1}{p})} I_1\left[p(\pi-\theta)(1+\tfrac{\Omega_1}{p})\right] \tag{3.38}$$

Let

$$\bar{M}_\theta(\theta,p) = \int_0^\infty e^{-pt} M_\theta(\theta,t)\,dt$$

and

$$\bar{M}_\theta(\theta, p) = \bar{M}_{0\theta}(\theta, p) + \frac{1}{p}\bar{M}_{1\theta}(\theta, p) + \cdots$$

Then

$$\bar{M}_{0\theta}(\theta, p) = \bar{\beta}_{0,\theta} + \nu\bar{\beta}_0 \cot\theta$$

For the remainder of the work, Ω_1 is set equal to zero as nothing essential is lost and the details are less tedious.

Note that

$$\frac{d}{dz} I_1(z) = I_0(z) - \frac{1}{z} I_1(z)$$

so that

$$\bar{M}_{0\theta}(\theta, p) = M_0(2\pi \sin\theta_0)^{1/2}\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} e^{-p(\pi-\theta_0)} p^{-1/2} I_0[p(\pi-\theta)]$$

$$+ M_0\left(\frac{\pi}{2}\sin\theta_0\right)^{1/2}\left(\frac{1}{\pi-\theta} + \cot\theta\right)\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} e^{-p(\pi-\theta_0)} p^{-3/2} I_1[p(\pi-\theta)]$$

$$- \nu M_0(2\pi\sin\theta_0)^{1/2}\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2}\cot\theta\, e^{-p(\pi-\theta_0)} p^{-3/2} I_1[p(\pi-\theta)] \qquad (3.39)$$

$$- M_0(2\pi\sin\theta_0)^{1/2}\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} e^{-p(\pi-\theta_0)} \frac{1}{p^{3/2}(\pi-\theta)} I_1[p(\pi-\theta)] .$$

We examine the terms in (3.39) to see which give the major contribution. Let p be fixed and let θ tend to π .

In the first term:

$$\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} \quad \text{is bounded as} \quad \theta \to \pi$$

and

$$p^{-1/2} I_0[p(\pi-\theta)] = p^{-1/2} + O(\pi-\theta) \qquad \text{as } \theta \to \pi. \qquad \text{(i)}$$

In the second term:

$$\left(\frac{1}{\pi-\theta} + \cot\theta\right)\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} \qquad \text{is bounded as} \quad \theta \to \pi$$

and

$$p^{-3/2} I_1[p(\pi-\theta)] = \frac{1}{2} p^{-1/2}(\pi-\theta) + O(\pi-\theta)^2 \qquad \text{as } \theta \to \pi. \qquad \text{(ii)}$$

In the third term:

$$\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} \quad \text{is bounded as} \quad \theta \to \pi$$

and

$$\cot\theta\, p^{-3/2} I_1[p(\pi-\theta)] = \cot\theta\,\frac{1}{2} p^{-1/2}(\pi-\theta) + O(\pi-\theta) \qquad \text{as } \theta \to \pi. \qquad \text{(iii)}$$

In the fourth term:

$$\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2} \quad \text{is bounded as} \quad \theta \to \pi$$

and

$$\frac{1}{p^{3/2}(\pi-\theta)} I_1[p(\pi-\theta)] = \frac{1}{2} p^{-1/2} + O(\pi-\theta) \qquad \text{as } \theta \to \pi. \qquad \text{(iv)}$$

The inverse Laplace transform of $p^{-1/2}$ is $\frac{1}{\sqrt{\pi}} t^{-1/2}$. We recall that the speed has been scaled to unity and that $t = 0$ is at the wavefront. Then for $\theta = \pi$, at the arrival time of the first wavefront

(i) has a square root singularity.

(ii) is equal to zero.

(iii) has a square root singularity.

(iv) has a square root singularity.

Thus in getting the wavefront contribution to $M_\theta(\theta, t)$ we can neglect the second term.

Some formulae for the inversion of

$$e^{-p(\pi-\theta_0)} I_0[p(\pi-\theta)]$$

and

$$e^{-p(\pi-\theta_0)} I_1[p(\pi-\theta)]$$

are needed. These are supplied by Erdelyi et. al. [55], p. 276.

For $b_1 > a_1 \geq 0$

$$e^{-\frac{1}{2}(a_1+b_1)p} \; I_n[\tfrac{1}{2}(b_1-a_1)p]$$

$$\sim \begin{cases} 0 & 0 < t < a_1 \\ \dfrac{\cos\left(n \cos^{-1} \frac{2t - a_1 - b_1}{b_1 - a_1}\right)}{\pi (t-a_1)^{1/2} (b_1 - t)^{1/2}} & a_1 < t < b_1 \\ 0 & t > b_1 \end{cases}$$

Thus

$$e^{-a_1 p} I_o(b_1 p) \sim \begin{cases} \dfrac{1}{\pi (t-\alpha)^{1/2} (\gamma - t)^{1/2}} & , \quad \alpha < t < \gamma \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

$$e^{-a_1 p} I_1(b_1 p) \sim \begin{cases} \dfrac{t - a}{b_1 \pi (t-\alpha)^{1/2} (\gamma - t)^{1/2}} & , \quad \alpha < t < \gamma \\ 0 & , \quad \text{otherwise} \end{cases}$$

where

$$\alpha = a_1 - b_1 = \theta - \theta_o ,$$

and

$$\gamma = a_1 + b_1 = 2\pi - \theta - \theta_o ,$$

with

$$a_1 = \pi - \theta_o \quad ; \quad b_1 = \pi - \theta .$$

Aside: Had we kept Ω_1 in (3.38) then the formula in p. 133 of Erdelyi *et. al.*[55] would have been used, viz.,

$$p^{-2\nu-1} g\left(p + \frac{\Omega}{p}\right) \sim \int_0^t \left(\frac{t-u}{\Omega u}\right)^{\nu} J_{2\nu}\left[2\left(\Omega u t - \Omega u^2\right)^{1/2}\right] du$$

where

$$g(p) \sim f(t).$$

It seems superfluous to remark that

$$g(p) \sim f(t)$$

means "the inverse Laplace transform of $g(p)$ is $f(t)$."

Thus

$$p^{-1/2} e^{-a_1 p} I_0(b_1 p) \sim \frac{1}{\pi^{3/2}} \int_\alpha^t \frac{du}{\sqrt{(t-u)(u-\alpha)(\gamma-u)}} \tag{3.40}$$

and

$$p^{-3/2} e^{-a_1 p} I_1(b_1 p) \sim \frac{2^{1/2}}{\pi^{3/2} b_1} \int_\alpha^t \frac{(t-u)^{1/2}(u-a_1)}{\sqrt{(u-\alpha)(\gamma-u)}}\, du \tag{3.41}$$

Let $w = \dfrac{u-\alpha}{t-\alpha}$,

then

$$\frac{1}{\pi^{3/2}} \int_\alpha^t \frac{du}{\sqrt{(t-u)(u-\alpha)(\gamma-u)}} = \frac{1}{\pi^{3/2}(\gamma-\alpha)^{1/2}} \int_0^1 \frac{dw}{\sqrt{w(1-w)(1-mw)}}$$

where

$$m = \frac{t - \alpha}{\gamma - \alpha} .$$

Now let $w = \sin^2 \varphi$ and

$$\frac{1}{\pi^{3/2}} \int_{\alpha}^{t} \frac{du}{\sqrt{(t-u)(u-\alpha)(\gamma-u)}} = \frac{2}{\pi^{3/2}(\gamma-\alpha)^{1/2}} \int_{0}^{\pi/2} \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}} .$$

Thus

$$p^{-1/2} e^{-a_1 p} I_0(b_1 p) \sim \frac{2}{\pi^{3/2}(\gamma - \alpha)^{1/2}} K(m) \tag{3.42}$$

where $K(m)$ is the complete elliptic integral of the first kind ([56], p. 590). Also

$$\int_{\alpha}^{t} \frac{(t-u)^{1/2}(u-a_1)}{\sqrt{(u-\alpha)(\gamma-u)}} \, du$$

$$= \frac{t-\alpha}{(\gamma-\alpha)^{1/2}} \int_{0}^{1} \frac{\sqrt{1-w}\,\{\alpha - a_1 + (t-\alpha) w\}}{\sqrt{w(1-mw)}} \, dw .$$

Near the wavefront

$$t - \alpha \ll 1$$

and hence we neglect the $(t-\alpha)$ term in the numerator compared with $\alpha - a_1$. Then

$$p^{-3/2} e^{-a_1 p} I_1(b_1 p) \sim \frac{2^{3/2}}{\pi^{3/2} b_1} \frac{(t-\alpha)(\alpha - a_1)}{(\gamma-\alpha)^{1/2}} \int_{0}^{\pi/2} \frac{\cos^2\varphi}{\sqrt{1 - m \sin^2 \varphi}} \, d\varphi . \tag{3.43}$$

We now denote the wavefront contribution to $M_\theta(\theta,t)$ by $M_\theta(Wf)$. Then, from (3.39) with (3.42) and (3.43) in mind,

$$M_\theta(Wf) = M_o(2\pi \sin\theta_o)^{1/2}\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2}\left[\frac{2}{\pi^{3/2}(\gamma-\alpha)^{1/2}}K(m)\right.$$

$$\left.-\left(\nu\cot\theta+\frac{1}{\pi-\theta}\right)\frac{2^{3/2}}{\pi^{3/2}b_1}\frac{(t-\alpha)(\alpha-a_1)}{(\gamma-\alpha)^{1/2}}\int_0^{\pi/2}\frac{\cos^2\varphi}{\sqrt{1-m\sin^2\varphi}}\,d\varphi\right]. \tag{3.44}$$

Note $\gamma-\alpha=2(\pi-\theta)$; $\alpha-a_1=-(\pi-\theta)$, and for $\theta_o\le\theta<\pi$, $t=\alpha$ is the wavefront. Then, for $\theta_o\le\theta<\pi$, at $t=\theta-\theta_o$, $M_\theta(Wf)$ reduces to

$$M_o(2\pi\sin\theta_o)^{1/2}\left(\frac{\pi-\theta}{\sin\theta}\right)^{1/2}\frac{2}{\pi^{3/2}2^{1/2}(\pi-\theta)^{1/2}}K(0)$$

or

$$M_\theta(Wf) = M_o\left(\frac{\sin\theta_o}{\sin\theta}\right)^{1/2} \tag{3.45}$$

i.e., a growing discontinuity in M_θ is propagated with the wavefront. For $\theta=\pi$ when $t=\pi-\theta_o$

$$M_\theta(Wf) = M_o(2\pi\sin\theta_o)^{1/2}\left[\frac{2}{\pi^{3/2}2^{1/2}(\pi-\theta)^{1/2}}\cdot\frac{\pi}{2}\right.$$

$$\left.-\left(\nu\cot\theta+\frac{1}{\pi-\theta}\right)\frac{2^{3/2}}{\pi^{3/2}(\pi-\theta)}\frac{\{t-(\pi-\theta_o)\}\{-(\pi-\theta)\}}{2^{1/2}(\pi-\theta)^{1/2}}\cdot\frac{\pi}{4}\right]$$

$$= M_o \frac{\sin^{1/2}\theta_o}{(\pi-\theta)^{1/2}} + \frac{1}{\sqrt{2}} M_o \sin^{1/2}\theta_o \left(\nu \cot\theta + \frac{1}{\pi-\theta}\right) \frac{t-(\pi-\theta_o)}{(\pi-\theta)^{1/2}} .$$

$$\lim_{\theta \to \pi} M_\theta (Wf) = K (\pi-\theta)^{-1/2} \tag{3.46}$$

where

$$K = M_o \sin^{1/2}\theta_o \left(\frac{1+\sqrt{2}}{\sqrt{2}} - \frac{\nu}{\sqrt{2}} \right) .$$

By (3.46), the effect of the focusing on the applied moment is to give a square root singularity at $\theta = \pi$ at the arrival time of the first wave.

It now remains only to get the reflected wave. So far

$$m = \frac{t-\alpha}{\gamma-\alpha}$$

and since $t = \alpha$ is the wavefront, $m = 0$ on the wavefront for $\theta_o \leq \theta \leq \pi$. Now

$$m = \frac{t-\alpha}{\gamma-\alpha} = \frac{t-\alpha}{2\pi-\theta-\theta_o-\alpha}$$

Let

$$\theta = \pi - \theta' , \qquad \theta_o \leq \theta' < \pi .$$

The time for a wave to leave $\theta = \theta_o$, go to $\theta = \pi$, and be reflected back to $\theta = \pi - \theta'$ is

$$\pi - \theta_o + \theta' = 2\pi - \theta_o - \theta$$

since the speed is unity.

Thus it is reasonable to interpret $2\pi - \theta - \vartheta_0$ as the time for a reflected wavefront and so in this case, we set $m = 1$ as a determination of a once reflected wavefront.

Then, by (3.44),

$$M_\theta(Wf) = M_0 (2\pi \sin\theta_0)^{1/2} \left(\frac{\pi - \theta}{\sin\theta}\right)^{1/2} \frac{2}{\pi^{3/2}\, 2^{1/2} (\pi - \theta)^{1/2}} K(1)$$

Now from [56], p. 591,

$$K(m) = -\tfrac{1}{2} \log(1-m) + O(1) \qquad \text{as } m \uparrow 1.$$

Thus, for $\pi < \theta < 2\pi - \theta_0$,

$$M_\theta(Wf) = -\frac{M_0}{\pi} \left(\frac{\sin\theta_0}{\sin\theta}\right)^{1/2} \log(1-m), \quad m \uparrow 1, \tag{3.47}$$

i.e., there is a logarithmic singularity on the reflected wavefront.

(3.45), (3.46), and (3.47) have all emerged from (3.35), and the function which synthesizes these is $K(m)$.

G.N. Ward [57] has considered the internal supersonic flow past a tube of nearly constant radius. The motion is governed by the wave equation. If $\varphi(r, \theta, t)$ is the scalar potential suitably scaled, then the Prandtl-Glauert equation for φ is

$$\varphi_{,rr} + \frac{1}{r}\varphi_{,r} + \frac{1}{r^2}\varphi_{,\theta\theta} = \varphi_{,tt}.$$

When $\varphi = \varphi_{,t} = 0$ for $t \leq 0$, the Laplace transform of this equation is

$$\bar{\varphi}_{,rr} + \frac{1}{r}\bar{\varphi}_{,r} + \frac{1}{r^2}\bar{\varphi}_{,\theta\theta} = p^2\bar{\varphi}$$

The solution for $\bar{\varphi}$ which is independent of θ is

$$\bar{\varphi}(r,p) = A\,I_0(pr) + C\,K_0(pr)$$

where I_0 and K_0 are modified Bessel functions. The solution, $\bar{\varphi}(r,p)$, appropriate for internal flow, i.e., the axis $r=0$ lies in the region of interest, at zero incidence is

$$\bar{\varphi}(r,p) = A\,I_0(pr)$$

Ward goes on to examine the derivatives of the potential given by

$$\overline{\left(\frac{\partial\varphi}{\partial t}\right)} = \eta\,\frac{I_0(pr)}{I_1(p)}$$

and

$$\frac{\partial\bar{\varphi}}{\partial r} = \eta\,\frac{I_1(pr)}{I_1(p)}$$

where η is a constant. On taking the inverse Laplace transform of these two quantities, he finds

$$\frac{\partial\varphi}{\partial t} = 2\eta\left\{t + \sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n J_1'(\lambda_n)}\sin\lambda_n t\right\}$$

and

$$\frac{\partial\varphi}{\partial r} = 2\eta\left\{\frac{1}{2}r + \sum_{n=1}^{\infty}\frac{J_1(\lambda_n r)}{\lambda_n J_1'(\lambda_n)}\cos\lambda_n t\right\}$$

where λ_n is the n^{th} zero of $J_1(\lambda)=0$, $\lambda_0 = 0$.

After a somewhat involved analysis, Ward finds that on the first wavefront, both $\varphi_{,t}$ and $\varphi_{,r}$ carry a discontinuity of magnitude $\eta/\sqrt{r}$. Then for the wave reflected from $r = 0$, $\varphi_{,t}$ has the behavior

$-\frac{\eta}{\pi\sqrt{r}}$ log $|\zeta_{-} - 1|$ and $\varphi_{,r}$ has the behavior $\frac{\eta}{\pi\sqrt{r}}$ log $|\zeta_{-} - 1|$ where

$\zeta_{-} = 1$ at the wavefront of the reflected wave. These results are found in Table 3 of Ward's paper. In Table 4, we find the square root singularity at the point $r = 0$ at the arrival time of the wave. These results confirm ours as given by equations (3.45), (3.46), and (3.47).

Appendix A.

In connection with assumption (c) of § 1.4., it is reasonable to ask whether the inclusion of a term linear in r in the assumed form of u_r would significantly alter the resulting equations. To examine this we trace the effect of such an additional term and show that it is indeed negligible to the order considered.

For simplicity, the analysis is restricted to the axisymmetric case and assumption (c) is revised so that the basic assumptions now are as follows:

(a') $\quad h/R << 1$

(b') t_{rr} is negligible compared with $t_{\theta\theta}$ and $t_{\varphi\varphi}$.

(c') $\quad u_r \sim w(\theta,t) + (r-R)\, w_1(\theta,t)$

$$u_\theta \sim u(\theta,t) + (r-R)\, \beta(\theta,t) .$$

Assumption (b') implies

$$t_{\theta\theta} = \frac{E}{1-\nu^2}\left(e_{\theta\theta} + \nu\, e_{\varphi\varphi}\right)$$

$$t_{\varphi\varphi} = \frac{E}{1-\nu^2}\left(e_{\varphi\varphi} + \nu\, e_{\theta\theta}\right)$$

$$e_{rr} = -\frac{\nu}{1-\nu}\left(e_{\theta\theta} + e_{\varphi\varphi}\right)$$

Since $\quad e_{rr} = u_{r,r}$,

$$w_1 = -\frac{\nu}{1-\nu}\left(e_{\theta\theta} + e_{\varphi\varphi}\right)$$

Assumptions (a') and (c') together with definitions (1.6) and (1.7) yield

$$
\begin{aligned}
N_\theta &= \frac{Eh}{R(1-\nu^2)}\left[u_{,\theta} + \nu u \cot\theta + (1+\nu) w\right] \\
N_\varphi &= \frac{Eh}{R(1-\nu^2)}\left[\nu u_{,\theta} + u \cot\theta + (1+\nu) w\right] \\
Q_\theta &= \frac{Eh}{2(1+\nu)}\left[\beta + \frac{1}{R}(w_{,\theta} - u)\right] \\
M_\theta &= \frac{Eh^3}{12(1-\nu^2)R}\left[\beta_{,\theta} + \nu\beta\cot\theta + (1+\nu) w_1\right] \\
M_\varphi &= \frac{Eh^3}{12(1-\nu^2)R}\left[\nu\beta_{,\theta} + \beta\cot\theta + (1+\nu) w_1\right]
\end{aligned}
\tag{A.3}
$$

These differ from equations (1.16) only in the additional term involving w_1 in M_θ and M_φ .

The momentum equations now become

$$
\begin{aligned}
Q_{\theta,\theta} + Q_\theta\cot\theta - (N_\theta + N_\varphi) + Rq &= \rho h R\, w_{,tt} + \tfrac{1}{6}\rho h^3 w_{1,tt} \\
N_{\theta,\theta} + Q_\theta + (N_\theta - N_\varphi)\cot\theta &= \rho h R\, u_{,tt} + \tfrac{1}{6}\rho h^3 \beta_{,tt} \\
M_{\theta,\theta} + (M_\theta - M_\varphi)\cot\theta - R\,Q_\theta &= \tfrac{1}{12}\rho h^3 R\left[\tfrac{2}{R} u_{,tt} + \beta_{,tt}\right]
\end{aligned}
$$

An order of magnitude analysis will now be made on these equations to decide which terms are comparable.

For the problem of the static equilibrium of a spherical shell under an external pressure P_0 , it is known that $N_\theta = O(RP_0)$, (see § 7-7 of [28]). The static analysis of a uniform moment $M_\theta = M_0$ applied to the edge of an incomplete spherical shell yields $M_\theta = O(M_0)$ (see [58], p. 547). These observations are the motivation for the introduction of suitable dimensionless variables in equations (A.4). The definitions of N_θ and M_θ show that $M_\theta = O(hN_\theta)$, so that, for example, in the case of a sphere under external pressure, we shall have

$N_\theta = O(RP_0)$ and $M_\theta = O(hRP_0)$. Analogous results hold for the case of an applied edge moment. Since we do not consider boundary layer phenomena and since it is assumed that the wavelengths involved are large compared with the thickness of the shell, derivatives with respect to angle or time will have the same order of magnitude as the undifferentiated quantities.

Equations (A.1) and the definitions of N_θ and N_φ give that

$$w_1 = -\frac{\nu}{Eh}\left(N_\theta + N_\varphi\right) \tag{A.5}$$

We shall make use of a "typical" stress resultant magnitude, N, in doing the subsequent scalings. For the pressure case $N = RP_0$, and for the edge moment case $N = M_0/h$. Non-dimensional quantities will be denoted by primed variables. Using (A.3), (A.4), (A.5), we introduce the dimensionless quantities by

$$N_\theta = \frac{1}{1-\nu^2} N N_\theta'$$

$$N_\varphi = \frac{1}{1-\nu^2} N N_\varphi'$$

$$Q_\theta = \frac{1}{1-\nu^2} N Q_\theta'$$

$$M_\theta = \frac{1}{12(1-\nu^2)} h N M_\theta'$$

$$w_1 = \frac{N}{Eh} w_1'$$

$$\beta = \frac{RN}{Eh^2} \beta'$$

$$u = \frac{RN}{Eh} u'$$

$$w = \frac{RN}{Eh} w'$$

$$t = T t' .$$

When written in terms of the primed quantities, the momentum equations are

$$Q'_{\theta,\theta} + Q'_{\theta} \cot\theta - (N'_{\theta} + N'_{\varphi}) + \frac{(1-\nu^2)Rq}{N} = \frac{R^2}{T^2} \frac{\rho(1-\nu^2)}{E} \left[w'_{,t't'} + \frac{1}{6}\left(\frac{h}{R}\right)^2 w'_{1,t't'} \right]$$

$$N'_{\theta,\theta} + Q'_{\theta} + (N'_{\theta} - N'_{\varphi})\cot\theta = \frac{R^2}{T^2} \frac{\rho(1-\nu^2)}{E} \left[u'_{,t't'} + \frac{1}{6} \frac{h}{R} \beta'_{,t't'} \right]$$

$$M'_{\theta,\theta} + (M'_{\theta} - M'_{\varphi})\cot\theta - \frac{12R}{h} Q'_{\theta} = \frac{R}{T^2} \frac{\rho(1-\nu^2)}{E} \left[\beta'_{,t't'} + 2\frac{h}{R} u'_{,t't'} \right]$$

where

$$N'_{\theta} = u'_{,\theta} + \nu u' \cot\theta + (1+\nu) w'$$

$$N'_{\varphi} = \nu u'_{,\theta} + u' \cot\theta + (1+\nu) w'$$

$$Q'_{\theta} = \frac{1-\nu}{2}\left(\frac{R}{h}\beta' + w'_{,\theta} - u' \right)$$

$$M'_{\theta} = \beta'_{,\theta} + \nu\beta' \cot\theta + (1+\nu)\frac{h}{R} w'_1$$

$$M'_{\varphi} = \nu\beta'_{,\theta} + \beta' \cot\theta + (1+\nu)\frac{h}{R} w'_1$$

The momentum equations in terms of the displacements are

$$\varepsilon \nabla^2 w' + (\beta'_{,\theta} + \beta' \cot\theta) - \frac{3+\nu}{1-\nu} \varepsilon (u'_{,\theta} + u' \cot\theta)$$

$$- \frac{4(1+\nu)}{1-\nu} \varepsilon w' + 2\varepsilon \frac{(1+\nu)Rq}{N} = \frac{2(1+\nu)\rho}{E} \frac{R^2}{T^2} \left(\varepsilon w'_{,t't'} + \frac{1}{6}\varepsilon^3 w'_{1,t't'} \right) \qquad (A.6)$$

$$\varepsilon \nabla^2 u' - \varepsilon \left(\frac{1+\nu}{2} + \cot^2\theta \right) u' + \frac{3+\nu}{2} \varepsilon w'_{,\theta} + \frac{1-\nu}{2} \beta'$$

$$= \frac{(1-\nu^2)\rho}{E} \frac{R^2}{T^2} \left(\varepsilon u'_{,t't'} + \frac{1}{6} \varepsilon^2 \beta'_{,t't'} \right) \tag{A.6}$$

$$\varepsilon^2 \nabla^2 \beta' - \varepsilon^2 (\nu + \cot^2\theta) \beta' - 6(1-\nu) \left(\beta' + \varepsilon w'_{,\theta} - \varepsilon u' \right)$$

$$= \frac{(1-\nu^2)\rho}{E} \frac{R^2}{T^2} \left(\varepsilon^2 \beta'_{,t't'} + 2 \varepsilon^3 u'_{,t't'} \right)$$

where $\varepsilon = h/R$. If only the terms of order ε^0 , ε , and ε^2 are retained, these equations are replaced by

$$\varepsilon \nabla^2 w' + \left(\beta'_{,\theta} + \beta' \cot\theta \right) - \frac{3+\nu}{1-\nu} \varepsilon \left(u'_{,\theta} + u' \cot\theta \right) - \frac{4(1+\nu)}{1-\nu} \varepsilon w'$$

$$+ 2 \varepsilon \frac{(1+\nu) R q}{N} = \frac{2(1+\nu)\rho}{E} \frac{R^2}{T^2} \varepsilon w'_{,t't'}$$

$$\varepsilon \nabla^2 u' - \varepsilon \left(\frac{1+\nu}{2} + \cot^2\theta \right) u' + \frac{3+\nu}{2} \varepsilon w'_{,\theta} + \frac{1-\nu}{2} \beta'$$

$$= \frac{(1-\nu^2)\rho}{E} \frac{R^2}{T^2} \left(\varepsilon . u'_{,t't'} + \frac{1}{6} \varepsilon^2 \beta'_{,t't'} \right) \tag{A.7}$$

$$\varepsilon^2 \nabla^2 \beta' - \varepsilon^2 (\nu + \cot^2\theta) \beta' - 6(1-\nu) \left(\beta' + \varepsilon w'_{,\theta} - \varepsilon u' \right)$$

$$= \frac{(1-\nu^2)\rho}{E} \frac{R^2}{T^2} \varepsilon^2 \beta'_{,t't'} .$$

The equations (A.7) are bereft of w_1 . Thus, w_1 occurs only in the boundary conditions and even there it is of smaller order. The effect of w_1 as contained in the boundary conditions can be traced through in the problems considered and can be shown to be inconsequential in so far as first approximations are concerned.

The final equations (A.7) have a double characteristic $c_p = \sqrt{\frac{E}{\rho(1-\nu^2)}}$.

In carrying out the calculations, we include the term $2\epsilon^3 u'_{,t't'}$ of the third equation of (A.6) simply as a calculational aid, since this has the effect of splitting the double characteristic. The final results are obtained by allowing the characteristics to coalesce once again.

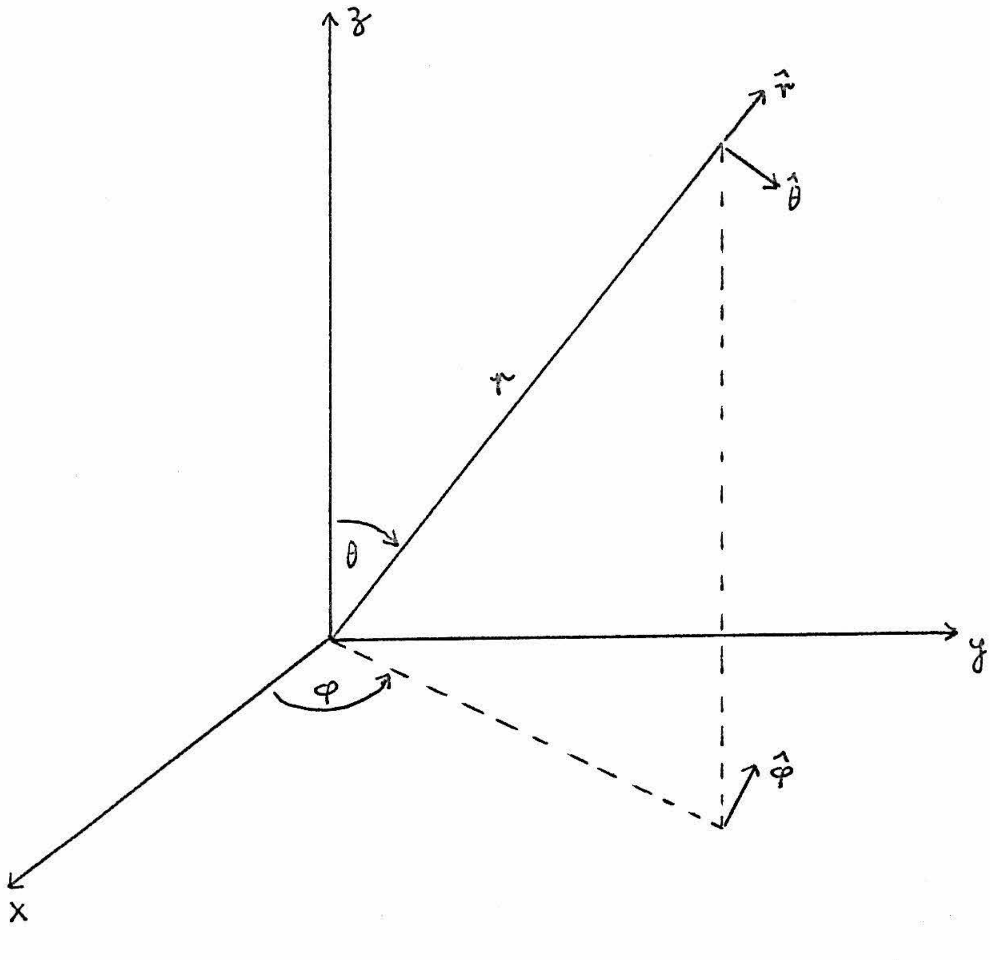

Figure 1

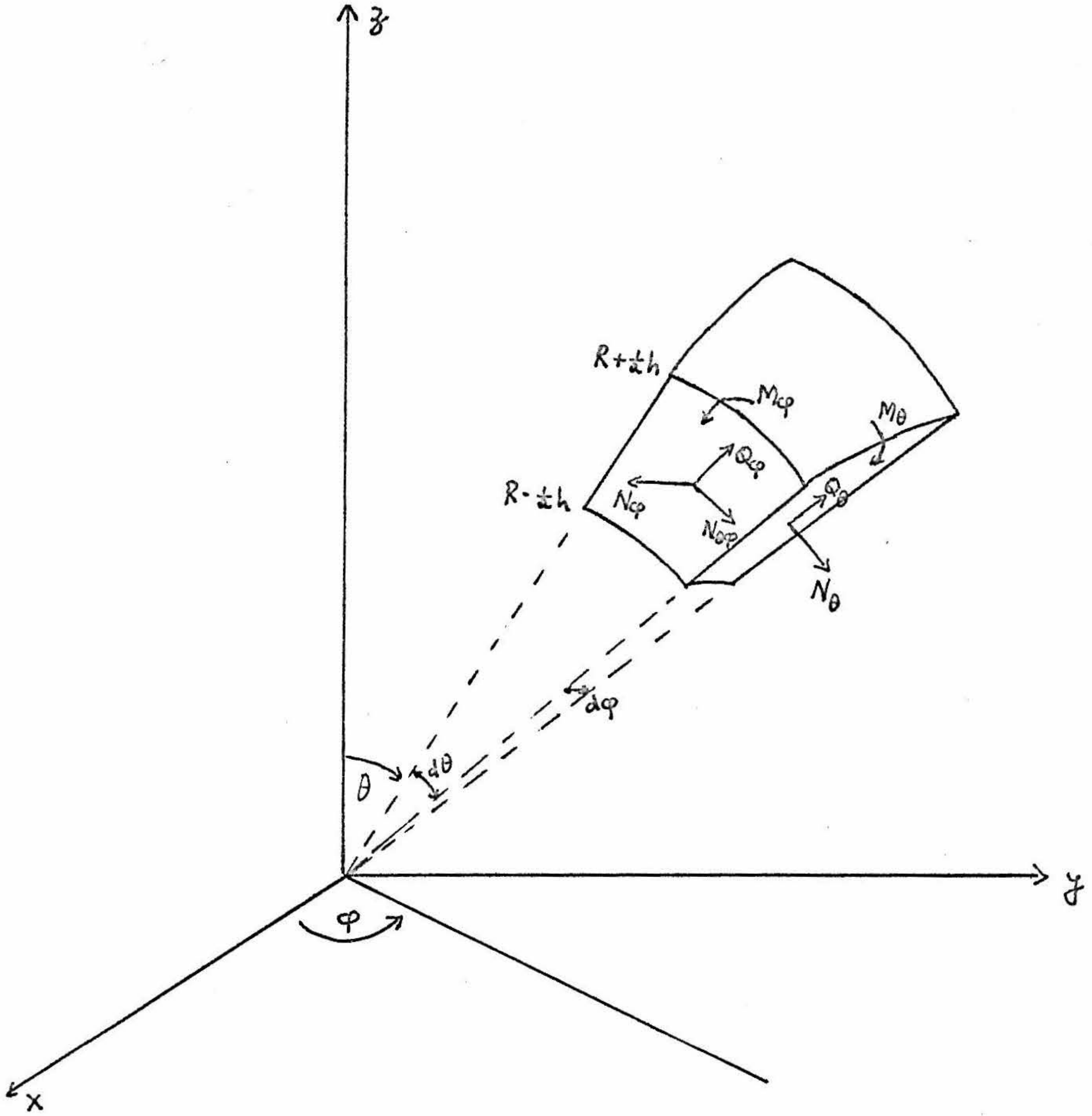

Figure 2

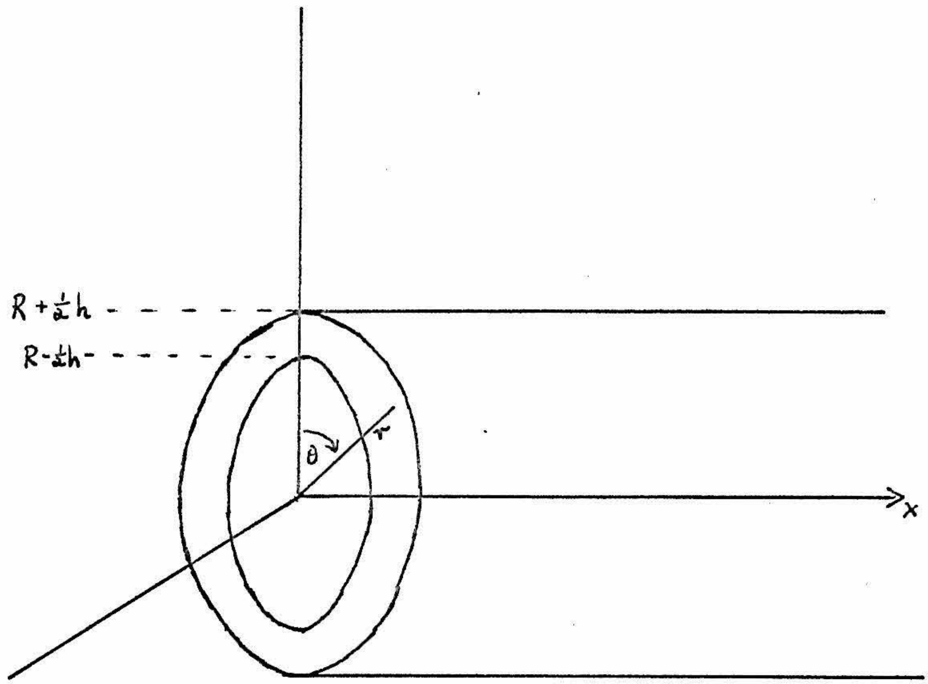

Figure 3

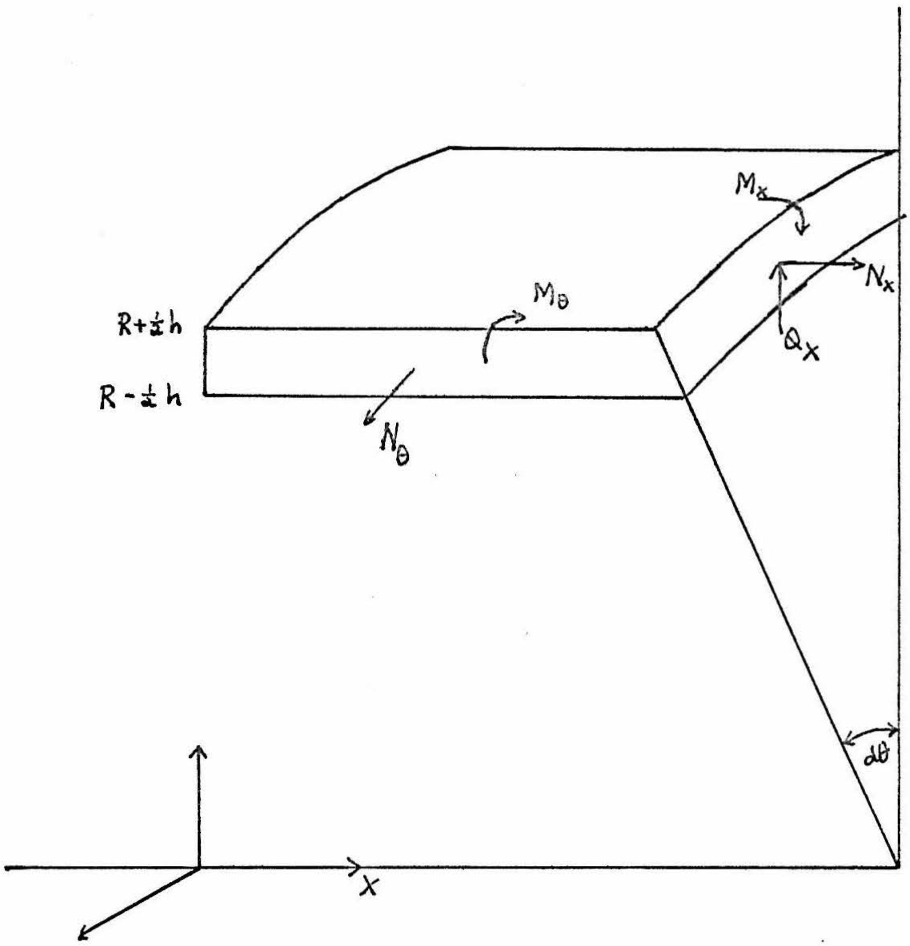

Figure 4

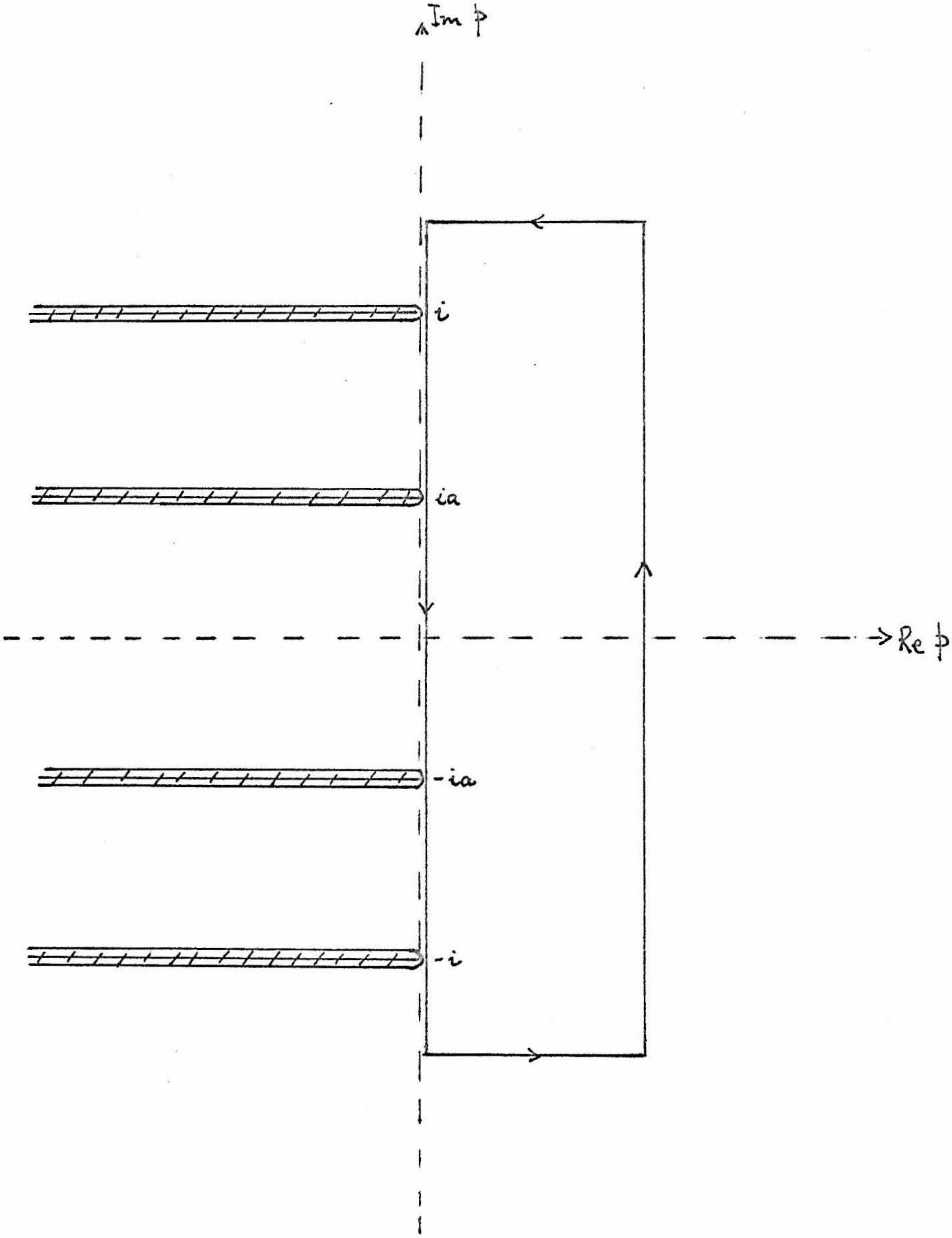

Figure 5

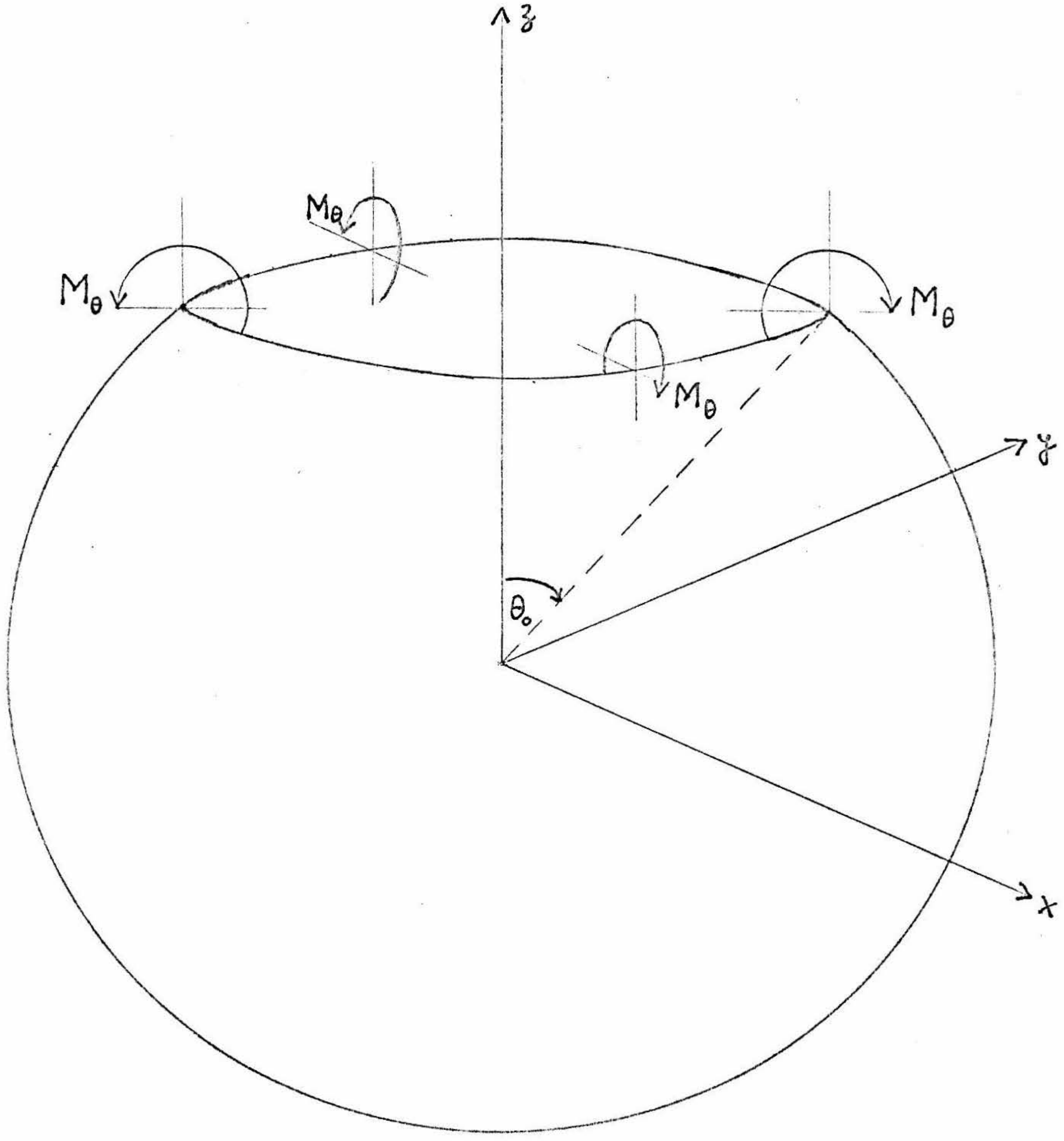

Figure 6

REFERENCES

1. J. Ac. Soc. Amer., 28, #1, 1956, p. 56.
2. J. Ac. Soc. Amer., 36, 1964, p. 489.
3. Quart. Appl. Math., 14, 1957, p. 369.
4. J. Math. and Phys., 29, 1950, p. 90.
5. Proc. London Math. Soc., 14, 1882, p. 52.
6. J. Ac. Soc. Amer., 34, 1962, p. 862.
7. J. Ac. Soc. Amer., 33, 1961, p. 1749.
8. J. Ac. Soc. Amer., 36, 1964, p. 74.
9. J. Ac. Soc. Amer., 40, 1966, p. 801.
10. J. Ac. Soc. Amer., 38, 1965, p. 367.
11. J. Appl. Mech., 3, 1965, p. 525.
12. J. Appl. Mech., (A.S.M.E., 22), 1955, p. 473.
13. Quart. Appl. Math., 15, 1957, p. 83.
14. J. Ac. Soc. Amer., 39, 1966, p. 556.
15. J. Appl. Mech., (A.S.M.E., 23), 1956, p. 563.
16. J. Appl. Mech., (A.S.M.E., 23), 1956, p. 255.
17. J. Appl. Mech., 1961, p. 417.
18. J. Ac. Soc. Amer., 32, 1960, p. 722.
19. J. Appl. Mech., 1963, p. 347.
20. J. Appl. Mech., 1964, p. 105.
21. Am. Soc. Civil Eng. Proc., Em. 5, Oct. 1965, p. 97.
22. J. Ac. Soc. Amer., 41, 1967, p. 358.
23. Ing.-Archiv., 28, 1959, p. 59.
24. Appl. Mech. Surveys, Spartan Books, p. 809.

25. Proc. Roy. Soc., Ser. A, 93, 1917, p. 223.

26. I.S. Sokolnikoff, Mathematical Theory of Elasticity, McGraw-Hill, 1956.

27. Landau and Lifshitz, Theory of Elasticity, Pergamon, 1964.

28. Housner and Vreeland, The Analysis of Stress and Deformation, MacMillan Co., 1966.

29. Y.C. Fung, Foundations of Solid Mechanics, Prentice-Hall, Inc., 1965.

30. Comm. Pure and Appl. Math., Vol. XIV, 1961, p. 1.

31. Jour. Math. and Phys., 37, 1958, p. 371.

32. Jour. Appl. Mech., (A.S.M.E., 73), 1951, p. 31.

33. Phil. Mag., Ser. 6, 43, 1922, p. 125.

34. Courant and Hilbert, Methods of Mathematical Physics, Vol. II, Interscience.

35. Theory of Thin Elastic Shells, Ed. Koiter, North-Holland Publishing Co., Delft, 1959, p. 12.

36. Jour. Aero. Sci., 29, 1962, p. 648.

37. Kolsky, Stress Waves in Solids, Dover Publications.

38. A.M.S. Lectures in Appl. Math., 7, p. 206.

39. Comm. Pure and Appl. Math., Vol. XII, 1959, p. 113.

40. Copson, Asymptotic Expansions, Cambridge, 1965.

41. Jeffreys and Jeffreys, Methods of Mathematical Physics, 2nd Ed., Cambridge.

42. Geol. Soc. Amer., Mem. #27, 1948.

43. Jones, The Theory of Electromagnetism, Pergamon Press, 1964.

44. Camb. Phil. Soc. Proc., 53, 1957, p. 599.

45. Jour. Fluid Mech., 8, 1960, p. 418.

46. Roy. Soc. London, Phil. Trans. A, 245, 1952-53, p. 213.

47. Erdelyi, Asymptotic Expansions, Dover Publications.

48. Proc. London Math. Soc., (2), 23, p. 428.

49. Trans. Am. Math. Soc., 33, 1931, p. 23.

50. Phys. Rev., 51, 1937, p. 669.

51. Jour. Austr. Math. Soc., 1, p. 439.

52. Quart. Jour. Mech. and Appl. Math., Vol. XVII, Pt. 1, 1964, p. 105.

53. Quart. Jour. Mech. and Appl. Math., Vol. XVII, Pt. 3, 1964, p. 369.

54. Watson, Theory of Bessel Functions, 2nd Ed., Cambridge, 1944.

55. Bateman Manuscript Project, Table of Integral Transforms, Vol. 1.

56. Handbook of Mathematical Functions, A.M.S., 55, (N.B.S.).

57. Quart. Jour. Mech. and Appl. Math., Vol. I, Pt. 2, 1948, p. 225.

58. Timoshenko and Woinowsky-Krieger, "Theory of Plates and Shells," 2nd Edition, McGraw-Hill.

59. Jour. Appl. Mech., 32, 1965, p. 351.