NON-LINEAR DISPERSIVE WAVES
WITH A SMALL DISSIPATION

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ABSTRACT

The general theory of Whitham for slowly-varying non-linear wavetrains is extended to the case where some of the defining partial differential equations cannot be put into conservation form. Typical examples are considered in plasma dynamics and water waves in which the lack of a conservation form is due to dissipation; an additional non-conservative element, the presence of an external force, is treated for the plasma dynamics example. Certain numerical solutions of the water waves problem (the Korteweg-de Vries equation with dissipation) are considered and compared with perturbation expansions about the linearized solution; it is found that the first correction term in the perturbation expansion is an excellent qualitative indicator of the deviation of the dissipative decay rate from linearity.

A method for deriving necessary and sufficient conditions for the existence of a general uniform wavetrain solution is presented and illustrated in the plasma dynamics problem. Peaking of the plasma wave is demonstrated, and it is shown that the necessary and sufficient existence conditions are essentially equivalent to the statement that no wave may have an amplitude larger than the peaked wave.

A new type of fully non-linear stability criterion is developed for the plasma uniform wavetrain. It is shown explicitly that this wavetrain is stable in the near-linear limit. The nature of this new type of stability is discussed.

Steady shock solutions are also considered. By a quite general method, it is demonstrated that the plasma equations studied here have no steady shock solutions whatsoever. A special type of steady shock is proposed, in which a uniform wavetrain joins across a jump discontinuity to a constant state. Such shocks may indeed exist for the Korteweg-de Vries equation, but are barred from the plasma problem because entropy would decrease across the shock front.

Finally, a way of including the Landau damping mechanism in the plasma equations is given. It involves putting in a dissipation term of convolution integral form, and parallels a similar approach of Whitham in water wave theory. An important application of this would be towards resolving long-standing difficulties about the "collisionless" shock.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

Appendices

TABLE OF CONTENTS (Cont'd)

Part I

Introduction

Non-linear dispersive waves are a well-known phenomenon in practically every branch of physics. The usual, and often the only, analytical approach which was made to such waves before the last decade was through various forms of perturbation theory beginning from the linearized solution. These small-amplitude analyses are certainly useful, and relevant, but the non-linear features of the wave phenomena were illuminated only dimly, at best.

In the past decade, new and powerful methods for dealing directly with non-linear dispersive waves have been developed. Among these are the averaging method, the averaged Lagrangian method, and the modified two-timing method of Whitham and his students(9,10,15,16,17,18). These methods are reviewed in Part II. As originally conceived, they apply to systems of partial differential equations in conservation form. One of the primary objectives of this thesis is to extend these methods to systems in non-conservation form, in particular dissipative systems. For this purpose, we consider two examples, one drawn from the theory of water waves (Part III) and the other from the theory of plasma waves (Part IV). The plasma waves example has the added interesting feature of being in non-conservative form even before dissipation is included.

The water wave problem considered in Part III is basically the Korteweg-de Vries equation, in which a small model dissipation term has been added. We shall begin our discussion with a brief review of linearized theory in general and as it applies to the Korteweg-de Vries equation in particular, and then move on to the steady progressing solutions, both with and without dissipation. These all are previously known results. Then, in §3, we shall apply the modified two-timing method to the Korteweg-de Vries equation with dissipation, and derive the resulting averaged equations in terms of elliptic integrals. For a special case of these averaged equations, numerical solutions are presented, whose qualitative features are predicted directly from a small-amplitude analysis as well.

The plasma equations studied in Part IV are the first three moments of the Boltzmann-Vlosov equation with no magnetic field. Their derivation is presented in §4, and possible terms for modeling the dissipation are considered. In particular, a new method of consistently including Landau damping is offered. Selected results from the linearized theory follow, including a study of the dispersion relation when a derivative dissipation term is included. Steady and unsteady shock solutions of the plasma equations are considered next, in §6 and §7.C. In particular, a new type of steady shock involving the joining of a uniform wavetrain and a constant state, found in §2 for the Korteweg-de Vries equation, is proved in §7.C to be impossible for the plasma wave case. It is further argued that no steady shock whatever exists. The existence of breaking solutions for properly rigged initial conditions is nevertheless demonstrated, leading one to suspect unsteady shock solutions. A method used to obtain some of the preceding results considers steady shocks to be solutions joining two singular points of a system of differential equations; while probably not new, this approach is useful in classifying the different kinds of steady shock.

A complete analysis of the uniform wavetrain solution is presented in §7.A and §7.B. Both the approach of §7.B and the results of §7.A and §7.B are new. In these sections, we shall obtain simple inequalities delimiting the region of parameter-space in which a uniform wavetrain solution may exist, which is important not only in and of itself but also for the analysis of stability (see §9). The phenomenon of peaking in the uniform wavetrain will also be uncovered, and examined in some detail.

§5, §6, and §7 form rather a unit, in that each considers an important aspect of the plasma equations, and each helps to illuminate the general structure of those equations. The rest of Part IV, with the exception of §11, is based upon the uniform wavetrain solution of §7. Using the modified two-timing method (see Part II) we are able to obtain the averaged equations for a slowly-varying wavetrain both without and with dissipation. In each case, we deal with equations in non-conservation form, and the extension of the averaging method to such equations in new. We shall then put the averaged equations in characteristic form, which leads directly into some very important stability considerations in §9. Finally, we shall demonstrate how the small-amplitude limit can be performed directly on the averaged equations, and analyze a few of the results so obtained.

The method of the averaged Lagrangian can be used in place of the Luke two-timing procedure in dissipationless problems. In §11, it is indicated how one might approach the plasma equations from this standpoint. The Lagrangian given there is somewhat unusual, and is an interesting result in its own right.

Part II

The Averaging Method and Two-Timing

Many systems of partial differential equations have steady propagating wave solutions in which each dependent variable is a function of just a single independent variable, θ, where

$$\theta = \underline{\kappa} \cdot \underline{x} - \omega t$$

$\underline{\kappa}$ is the vector wave number, and ω the frequency. Such waves are also variously called uniform wave trains, steady progressing waves, or just steady waves. The various terminologies will be used interchangeably in this thesis.

These solutions are the non-linear generalizations of the familiar $e^{i(\underline{\kappa}\cdot\underline{x}-\omega t)}$ ("plane wave") solutions of linearized theory. As the amplitude of the non-linear steady wave tends to zero, it will reduce to its corresponding linearized solution.

Uniform wave train solutions are of interest from the physical point of view because they can usually be excited and observed in the laboratory with relative ease; and also from the mathematical point of view, because they represent an important subclass of solutions of the system of partial differential equations under study. In fact, it is well-known in linear problems that any solution of the system whatsoever may be built up, via the Fourier integral, from members of this sub-class. This felicitous situation does not unfortunately persist into the non-linear regime, partly for lack of an appropriate non-linear generalization of linear superposition, and partly because of the greater variety of non-linear phenomena (most notably breaking and the formation of shocks). Nevertheless, steady wave solutions have been and will continue to be an important key to the understanding of non-linear phenomena.

In general, finding the steady wave solution of a system of partial differential equations reduces to the solution of a single ordinary differential equation[15]

$$\Phi_\theta^2 = F(\Phi, \kappa, \omega, \alpha_i) \tag{II.1}$$

where F is a rational function of Φ involving κ, ω, and the parameters (constants of integration) α_i. Only in the simplest cases, for example when F is a cubic or quartic polynomial in Φ, can (II.1) be solved in terms of known functions. Yet a great deal of information about the possible forms of solution of (II.1) can usually be extracted from (II.1), as we shall see in connection with the lukewarm plasma wave in §7.A and §7.B. In general, $\Phi(\theta)$ will be oscillatory, oscillating between two zeros of F.

We shall now consider the more general case of non-steady waves, but under the simplifying assumption that the amplitude (which is some unknown function of the α_i), the velocity $U = \frac{\omega}{\kappa}$, and other physically meaningful wave parameters all vary slowly in space and time. By "vary slowly," we mean that the relative change $\Delta C/C$ of a parameter C over one wave-length $\lambda = 2\pi/\kappa$ and over one period $\tau = 2\pi/\omega$ is small. We make the assumption of slow variation in preference to the more usual one of linearization, because we would like to retain the distinctly non-linear features of the problem.

The theory of slowly varying wavetrains has been developed in a series of papers by Whitham[15, 16, 17, 18] and he has called his method the averaging method after the Krylov-Bogoliubov method of the same name[23] for ordinary differential equations. Another approach, which rigorizes the averaging method in the sense of making it the first step in a perturbation procedure, was developed by Luke[9]. Luke's procedure we shall call two-timing, in analogy with a method of the same name in ordinary differential equations[6].

We shall present a brief resumé of the averaging and two-timing methods below, which will be sufficient to our purposes in this thesis. For a deeper discussion and some historical perspective, the reader is referred to the original references[9, 10, 15, 16, 17, 18].

Suppose we have a system of n partial differential equations (hereinafter abbreviated to p.d.e.'s). There are n dependent variables y_i (x,t) depending on a spatial coordinate x and the time t. (We consider one-dimensional problems exclusively, although in principle there is no difficulty in applying the method in more dimensions[15].) Suppose further that uniform wavetrain solutions $y_i(\theta)$ exist. These uniform wavetrain solutions will depend on n constants of integration A_i in addition to κ and ω, making for a total of n+2 parameters. One of these parameters, say A_n, is merely an additive constant to θ, from the integration of an equation like (II.1). It will drop out in the averaging method when integrals are taken over a full cycle of θ. In addition, a constraining relation will be found to hold among the parameters of the problem on account of specifying the as yet arbitrary period in θ.

This constraining relation is derived as follows. Suppose, in (II.1), that the roots Φ_1 and Φ_2 of $F(\Phi, \kappa, \omega, \alpha_i)$ form the limits of the oscillatory solution. Then assume that θ increases by Λ in one complete cycle. If (II.1) is written

$$G(\Phi) = \int_{\Phi_o}^{\Phi} \frac{d\Phi'}{\sqrt{F}} = \theta \qquad \text{(II.2)}$$

then it is clear that $G(\Phi)$ is a monotonic function of Φ, $G'(\Phi) > 0$. This tells us that as Φ goes from Φ_1 to Φ_2, half a cycle, θ will increase monotonically by an amount $\Lambda/2$,

$$G(\Phi_2) - G(\Phi_1) = \frac{\Lambda}{2} \tag{II.3}$$

which is equivalent to

$$\int_{\Phi_1}^{\Phi_2} \frac{d\Phi}{\sqrt{F}} = \frac{\Lambda}{2} \tag{II.4}$$

This is the promised relationship. It is implicit, since we cannot usually do the integral. (II.4) will be called the <u>non-linear dispersion relation</u> in this thesis, since in the small-amplitude limit it reduces to the relationship between ω and κ commonly called the dispersion relation. There is a non-linear dispersion relation for all non-linear dispersive wave problems.

Because of (II.4) and the disappearance of A_n, the uniform wavetrain solution will involve only n <u>independent</u> parameters. We shall obtain a set of n p.d.e.'s to describe the slow time and space variation of these n parameters.

Begin by extracting n p.d.e.'s in conservation form

$$\frac{\partial P_i}{\partial t} + \frac{\partial Q_i}{\partial x} = 0 \quad (i=1,\ldots,n) \tag{II.5}$$

from the original system. This often involves some ingenuity[15]. The P_i and Q_i will be related algebraically to the y_i. Now assume that each of the y_i is replaced by the uniform solution, $y_i(\theta)$. Average (II.5) over one cycle of θ,

$$\frac{\partial}{\partial t} \frac{1}{\Lambda} \int_0^{\Lambda} P_i \, d\theta + \frac{\partial}{\partial x} \frac{1}{\Lambda} \int_0^{\Lambda} Q_i \, d\theta = 0 \qquad \text{(II.6)}$$

which we write as

$$\frac{\partial}{\partial t} \langle P_i \rangle + \frac{\partial}{\partial x} \langle Q_i \rangle = 0 \qquad \text{(II.7)}$$

where the definition of $\langle\rangle$ is obvious. An approximation has been made here, which is that the averaging operation commutes with $\partial/\partial t$ and $\partial/\partial x$; this approximation is justified in[15]. But if we accept this approximation, we immediately have our desired system. For $\langle P_i \rangle$ and $\langle Q_i \rangle$ involve only κ, ω, α_i, and thus (II.7) is a system of equations for the variation of these parameters. These will be called the averaged equations.

The system (II.7) is unwieldy for even the simplest problems. Whitham was able to show, however, that tremendous simplifications could be made by introducing a master function W into the formalism.

A typical form for the master function would be

$$W = \oint \Phi_\theta \, d\Phi = \oint \sqrt{F(\Phi, \kappa, \omega, \alpha_i)} \, d\Phi$$

where the integration is over one complete cycle of Φ. It then proved possible, in the numerous physical examples considered, to express all the $\langle P_i \rangle$ and $\langle Q_i \rangle$ in terms of W and its partials with respect to the parameters of the problem. Furthermore, the averaged equations from widely disparate physical theories exhibited a remarkable unity when this was done.

The existence of this underlying unity suggested that the master function derived from some fundamental structure common to all physical problems. Such a structure is furnished by the Lagrangian formalism, and Whitham was able from these clues to show that W is none other than the <u>averaged Lagrangian</u> of the system in question(16). The Euler equations of this averaged Lagrangian with respect to variations in the α_i (i=1,...n-1) and in θ then could be reduced to (II.7). Or, (II.7) could be obtained directly by an application of Noether's Theorem(3) to the averaged Lagrangian. Either way, the averaged Lagrangian approach furnishes an esthetically satisfying approach to the derivation of (II.7), and at the same time simplifies the computations significantly.

We shall not actually use the averaged Lagrangian approach in this thesis. We shall point out the Lagrangian for the dissipationless plasma case in §11, but the master function for that case will be introduced in an *ad hoc* manner in §8. For the water waves case, §3, the master function was already known from the work of Whitham.

Instead we shall use the variant of the theory due to Luke[9], the two-timing method. This is because our primary concern in this thesis will be directed towards problems with dissipation, and it is well-known that such problems do not possess Lagrangians. On the other hand, as we shall see, the two-timing approach operates perfectly well when dissipation is present.

The two-timing method introduces an expansion of the form

$$y = y_0 (\theta, X, T) + \varepsilon y_1 (\theta, X, T) + \ldots \qquad \text{(II.8)}$$

for each variable y_i, where

$$X = \varepsilon x \qquad\qquad T = \varepsilon t$$

X and T are the "stretched" or "slow" space and time variables. The phase θ can no longer be written down explicitly as $\kappa x - \omega t$; instead we assume that

$$\left.\begin{aligned} \theta_t &= -\omega\,(X,T) \\ \theta_x &= \kappa\,(X,T) \end{aligned}\right\} \qquad \text{(II.9)}$$

There is a precedent for this treatment of the phase, θ, in the WKB method for ordinary differential equations. There it would be written[6]

$$\theta = \int^t \omega(T)\ dt$$

where ω(T) would be the slowly-varying frequency of some harmonic oscillator. Note that because of the definitions (II.9) the p.d.e.

$$\frac{\partial \kappa}{\partial T} + \frac{\partial \omega}{\partial X} = 0 \qquad \text{(II.10)}$$

holds automatically. Because it will be redundant with the n averaged equations, it must be implied by them. This always turns out to be the case.

The three variables θ, χ, and T are regarded as <u>independent</u> in the Luke method, much as the two time variables are regarded as independent in ordinary two-timing[6]. This is a key assumption in the application of the expansion (II.8). We illustrate this point by applying the Luke method to a conservation equation

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad \text{(II.11)}$$

With $\partial/\partial t$ and $\partial/\partial x$ written to reflect the independence of θ, X, and T, the expanded form of this conservation equation is

$$\left(-\omega \frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial T}\right) \left[P_o\ (\theta,X,T) + \varepsilon P_1\ (\theta,X,T) + \ldots\right] +$$

$$\left(\kappa \frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial X}\right) \left[Q_o\ (\theta,X,T) + \varepsilon Q_1\ (\theta,X,T) + \ldots\right] = 0$$

Equating the coefficient of each power of ε to zero gives

$$-\ \omega \frac{\partial P_o}{\partial\theta} + \kappa\frac{\partial Q_o}{\partial\theta} = 0$$

$$\omega \frac{\partial P_1}{\partial\theta} - \kappa\frac{\partial Q_1}{\partial\theta} = \frac{\partial P_o}{\partial T} + \frac{\partial Q_o}{\partial X}$$

$$\vdots$$

The first equation is the same one we would obtain if we were just looking for the uniform solution, only now when we integrate it

$$-\omega P_o + \kappa Q_o = A(X,T)$$

we see that the parameters of the uniform solution, like A, will depend on X and T. From the second equation, we deduce that $(\omega P_1 - \kappa Q_1)$ will be bounded as $\theta\to\infty$ if and only if

$$I = \int_o^\theta \left(\frac{\partial P_o}{\partial T} + \frac{\partial Q_o}{\partial X}\right) d\theta$$

is so bounded. Since P_o and Q_o, being the uniform solution, will be periodic of period Λ in θ, I will be bounded as $\theta \to \infty$ if and only if

$$\int_0^\Lambda \left(\frac{\partial P_o}{\partial T} + \frac{\partial Q_o}{\partial X}\right) d\theta = 0$$

or, because θ, X, and T are independent,

$$\frac{\partial}{\partial T} \int_0^\Lambda P_o \, d\theta + \frac{\partial}{\partial X} \int_0^\Lambda Q_o \, d\theta = 0$$

This is the same result we would have obtained by averaging.

Thus the averaged equations have the alternate interpretation as non-linear elimination-of-secularity conditions [for a discussion of secularity, see (6) or §1]. In other words, the averaged equations arise naturally upon enforcing boundedness on the $O(\varepsilon)$ term in a perturbation expansion whose $O(1)$ term is the uniform solution. Unfortunately, the two-timing method gives no hint of the simple way in which the averaged equations can be re-formulated, using the master function. Hence the averaged Lagrangian approach and the two-timing method complement each other, the former showing how the results may be simply formulated and the latter showing how we may consistently proceed to higher orders.

We have said nothing about the situation in which we cannot for one reason or another obtain all n p.d.e.'s in conservation form. This can happen because of dissipation, external forces, or other causes. We shall deal with such situations in §3, §8.A, and §8.B, and we reserve comment on this situation for those sections.

It may be noted that the averaged equations are only the first of a hierarchy of boundedness conditions. Suppose, in (II.11), that P and Q, unexpanded, are both functions of θ, X, and T. Then (II.11) becomes

$$\frac{\partial}{\partial\theta}(-\omega P + \kappa Q) + \varepsilon\left(\frac{\partial P}{\partial T} + \frac{\partial Q}{\partial X}\right) = 0$$

Integrating this from 0 to Λ, and assuming that P and Q are periodic in θ, we obtain

$$\int_0^\Lambda \left(\frac{\partial P}{\partial T} + \frac{\partial Q}{\partial X}\right) d\theta = 0$$

Now if we expand P and Q, we get the infinite set of boundedness conditions

$$\int_0^\Lambda \left(\frac{\partial P_i}{\partial T} + \frac{\partial Q_i}{\partial X}\right) d\theta = 0$$

(i=0,1,...).

Part III

Uniform and Slowly Varying Wavetrain Solutions of the Dissipative Korteweg-de Vries Equation

For the case of relatively long water waves, Korteweg and de Vries[7] derived the equation

$$\eta_t + \sqrt{gh_o}\,(1 + \frac{3\eta}{2h_o})\,\eta_x + \frac{1}{6}\,h_o^{\,2}\,\sqrt{gh_o}\,\eta_{xxx} = 0$$

for the elevation η of the water surface above the undisturbed depth h_o. By suitable re-definition of the variables, this equation may be transformed to

$$\eta_t + 6\eta\eta_x + \eta_{xxx} = 0 \; . \qquad \text{(III.1)}$$

We shall be interested primarily in a slightly modified form of this equation,

$$\eta_t + 6\eta\eta_x + \eta_{xxx} = \nu\eta_{xx} \; , \qquad \text{(III.2)}$$

in which a model dissipation is included, proportional to a damping coefficient ν. The Korteweg-de Vries equation is one of the simplest examples of a non-linear dispersive wave equation and, as such, has been considered by many authors.[11,15] Similarly, the modified form (3.2) which we shall consider contains, in a simple fashion, the three physical effects whose interplay is important in almost all fluid mechanics problems; namely, non-linearity ($\eta\eta_x$ term), dispersion (η_{xxx} term), and dissipation or diffusion ($\nu\eta_{xx}$ term). Thus, this equation has the virtue of being physically meaningful and yet not mathematically impenetrable. In the present work we shall try to illuminate the problem of uniform and slowly-varying wavetrain solutions of (III.2).

§1. Linearized Theory

The linearized theory of (III.2) is a typical and easy example of the application of perturbation methods to non-linear dispersive wave problems. These methods are used extensively, and form a backdrop and limiting case for the more general non-linear considerations to follow.

Steady solutions are those solutions which depend only on a single variable $\theta = \kappa x - \omega t$. The linearized steady solutions of equations like (III.1) and (III.2) fall into two categories:

(i) exponential solutions, which are only valid for a semi-infinite range of θ, including either $\theta = -\infty$ or $\theta = +\infty$ but not both;

(ii) sine and cosine solutions, which are valid for all θ.

Solutions of the first type can only be fragments of solutions which are essentially non-linear, e.g. the tails of shocks. We shall be concerned primarily with the second type of solution, and its non-linear generalization, in this thesis; that is, with uniform wavetrain solutions that are periodic in θ. We shall take the arbitrary period of the solution in θ to be 2π unless otherwise stated and, as we shall see, this fixes the dispersion relation uniquely.

Take (III.1) as an example. Steady linearized solutions of it must satisfy

$$\eta'' - \frac{\omega}{\kappa^3}\,\eta = \text{const.}$$

In order to have sine and cosine solutions of period 2π, we must require

$$\omega = -\kappa^3$$

which is the dispersion relation of the Korteweg-de Vries equation.

The linearized solution can be made the first term in a perturbation expansion of the form

$$\eta = \epsilon\eta_1(\theta) + \epsilon^2\eta_2(\theta) + \epsilon^3\eta_3(\theta) + \cdots \qquad (1.1)$$

but, if we actually substitute this expansion into (III.1) and carry out the calculations, we shall find secular terms ($\theta\sin\theta$, $\theta\cos\theta$) appearing in η_2 or η_3, which destroy the uniform validity of the expansion in θ. The remedy for this situation is well known, and consists in assuming ω to depend (analytically) on the amplitude ϵ as well as on the wavenumber κ. Hence

$$\omega = -\kappa^3 + \epsilon\omega_1(\kappa) + \epsilon^2\omega_2(\kappa) + \cdots$$

and, with the extra latitude afforded by ω_1, ω_2, etc., we are able to eliminate the secular terms. The sequence of equations generated is

$$\eta_1''' + \eta_1' = 0$$

$$\eta_2''' + \eta_2' = \frac{1}{\kappa^3}(\omega_1\eta_1' - 6\kappa\eta_1\eta_1')$$

$$\eta_3''' + \eta_3' = \frac{1}{\kappa^3}\left[\omega_2\eta_1' + \omega_1\eta_2' - 6\kappa(\eta_1\eta_2)'\right]$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Without loss of generality in the computation of ω_1, we may take

$$\eta_1 = A + B\sin\theta$$

so that the second equation of the sequence becomes

$$\eta_2'' + \eta_2 = \frac{1}{\kappa^3}\left[\omega_1(A + B\sin\theta) - 3\kappa(A + B\sin\theta)^2\right] + C$$

The coefficient of sin θ on the right-hand side must vanish to prevent secularity. This requires

$$\omega_1 = 6\kappa A ,$$

so that the first-order shift in the frequency is directly proportional to the mean level of η_1. Without extra loss of generality in the computation of ω_2, we may take

$$\eta_2 = D - \frac{B^2}{2\kappa^2}\cos 2\theta .$$

Putting this and η_1 into the equation for η_3, we require the coefficient of sin θ on the right-hand side to be zero to eliminate secular terms. This leads to

$$\omega_2 = 6\kappa D + \frac{3B^2}{2\kappa} .$$

The mean height D of η_2 enters just as did A in ω_1. The second term shows how the amplitude B of η_1 couples into the frequency. Continuing in this manner, we will obtain a uniformly valid small-amplitude expansion of η, and a concomitant expansion of ω. Later, in §10 , we shall see the method whereby the expansion of ω may be extracted directly from the results of the averaging method.

Let us now consider the dissipative form (III.2) of the Korteweg-de Vries eqaution. If we try an expansion of the form (1.1) in (III.2), we shall find the problem of secular terms no longer exists, for the first order solution η_1 is now of the form

$$e^{m\theta}$$

where

$$m = \frac{1}{2\kappa}(\nu \pm \sqrt{\nu^2 + 4U}) \qquad (U = \frac{\omega}{\kappa}) .$$

The former secular terms are now terms like $\theta e^{m\theta}$, which cause no trouble. Of course, the expansion will now be valid only in a semi-infinite range of θ (θ ⟶ + ∞ for −

sign in m, $\theta \longrightarrow -\infty$ for the + sign in m) and it describes only a fragment of a solution. Note that a dispersion relation is no longer engendered, because there are no periodicity in θ requirements, and that ω and κ now only appear in the combination $U = \omega/\kappa$.

More sophisticated techniques are necessary when the damping coefficient ν is the small parameter of the problem. An expansion of the form

$$\eta = \nu\eta_1(\theta) + \nu^2\eta_2(\theta) + \cdots$$

applied to (III.2) leads to secular terms in η_2. If we expand the frequency ω as before,

$$\omega = -\kappa^3 + \nu\omega_1(\kappa) + \nu^2\omega_2(\kappa) ,$$

we find that there is no way in which we can prescribe the ω_i to eliminate the secular terms. The additional latitude necessary to eliminate secularity can be obtained by introducing an additional independent variable $T = \nu t$ into the expansion[2,6]

$$\eta = \nu\eta_1(\theta, T) + \nu^2\eta_2(\theta, T) + \cdots \qquad (1.2)$$

and treating θ and T formally as distinct independent variables. In keeping with our original physical idea that the frequency ω is amplitude-dependent, it would seem reasonable to assume $\omega = \omega(T)$. It is not necessary to do this in first order (unlike the fully non-linear case!), however, for reasons which we shall presently see. It is sufficient to take $\omega = -\kappa^3$.

Putting the expansion (1.2) into (III.2), the following sequence of equations is generated:

$$\frac{\partial^3 \eta_1}{\partial \theta^3} + \frac{\partial \eta_1}{\partial \theta} = 0$$

$$\frac{\partial^3 \eta_2}{\partial \theta^3} + \frac{\partial \eta_2}{\partial \theta} = \frac{1}{\kappa^3}\left(\kappa^2 \frac{\partial^2 \eta_1}{\partial \theta^2} - 6\kappa \eta_1 \frac{\partial \eta_1}{\partial \theta} - \frac{\partial \eta_1}{\partial T}\right)$$

$$\frac{\partial^3 \eta_3}{\partial \theta^3} + \frac{\partial \eta_3}{\partial \theta} = \frac{1}{\kappa^3}\left(\kappa^2 \frac{\partial^2 \eta_2}{\partial \theta^2} - \frac{\partial \eta_2}{\partial T} - 6\kappa \frac{\partial (\eta_1 \eta_2)}{\partial \theta}\right)$$

$$\vdots$$

The solution for η_1 may be written

$$\eta_1 = A(T) + B(T) \sin\left[\theta + \phi(T)\right]$$

Putting this into the second equation,

$$\frac{\partial^3 \eta_2}{\partial \theta^3} + \frac{\partial \eta_2}{\partial \theta} = -\frac{1}{\kappa^3}\left[\frac{dA}{dT} + \left(\frac{dB}{dT} + \kappa^2 B\right) \sin(\theta + \phi) + B\left(\frac{d\phi}{dT} + 6\kappa A\right) \cos(\theta + \phi) - 3\kappa B^2 \sin 2(\theta + \phi)\right]$$

If we are to eliminate secular terms from η_2, it is clear that we must require

$$\left.\begin{aligned} \frac{dA}{dT} &= 0 \\ \frac{dB}{dT} + \kappa^2 B &= 0 \\ \frac{d\phi}{dT} + 6\kappa A &= 0 \end{aligned}\right\} \qquad (1.3)$$

These lead to

$$\left.\begin{aligned} A &= \text{const.} = A_o \\ B &= B_o e^{-\kappa^2 T} \\ \phi &= -6\kappa A_o T + \phi_o \end{aligned}\right\} \qquad (1.4)$$

We shall see the $e^{-\kappa^2 T}$ amplitude decay again in the linearized limit of the fully non-linear scheme (§3). We shall also find in §3 that A = const. reflects a fully non-linear result, which is that the mean level of η will not ever vary on the slow time scale T if there is no X-dependence. Finally, the result for ϕ explains why ω did not need to be expanded. For, by properly grouping terms,

$$\begin{aligned} \theta + \phi &= \kappa x + \kappa^3 t - 6\kappa A_o T + \phi_o \\ &= \kappa x - (-\kappa^3 + 6\kappa A_o \nu)t + \phi_o \end{aligned}$$

it becomes apparent that we have tacitly assumed an expansion of ω in taking $\phi = \phi(T)$. Unfortunately, to proceed to higher orders, it is necessary to re-introduce the expansion of ω in powers of ν.

We may solve for η_2

$$\eta_2 = A_2(T) + C(T) \sin(\theta + \phi) + D(T) \cos(\theta + \phi) - \frac{B^2(T)}{2\kappa^2} \cos 2(\theta + \phi)$$

and use this and the result for η_1 in the right-hand side of the η_3 - equation. The three conditions that η_3 have no secular terms may then be seen to be

$$\frac{dA_2}{dT} = 0$$

$$\frac{dC}{dT} + \kappa^2 C = 0$$

$$\frac{dD}{dT} + \kappa^2 D = (\omega_2 - 6\kappa A_2)B(T) - \frac{3}{2\kappa} B^3(T)$$

To ensure the uniform validity of the assumed expansion, we must bar coefficients like $Te^{-\kappa^2 T}$ (see (6)). This means

$$\omega_2 = 6\kappa A_2 ,$$

whereupon the solutions for C and D become

$$C = C_o e^{-\kappa^2 T}$$

$$D = D_o e^{-\kappa^2 T} + \frac{3B_o{}^3}{4\kappa^3} e^{-3\kappa^2 T} .$$

For the special case in which the first term η_1 acts as a driver for the higher terms, meaning all arbitrary constants that arise in η_2, η_3, etc. are set to zero, our expansion becomes

$$\eta = \nu A_o + \nu B_o e^{-\kappa^2 T} \sin\theta$$
$$+ \nu^2 \frac{3B_o^{\,3}}{4\kappa^3} e^{-3\kappa^2 T} \cos\theta - \nu^2 \frac{B_o^{\,2}}{2\kappa^2} e^{-2\kappa^2 T} \cos 2\theta$$
$$+ O(\nu^3)$$

where

$$\omega = -\kappa^3 + 6\kappa A_o \nu \ .$$

After a sufficient length of time, only the first two terms are significant. We see here the familiar phenomena of corrections to the fundamental wave and higher harmonics of the fundamental wave (third and fourth terms respectively).

This has all been, of course, the linearized limit of the Luke expansion procedure (Section II). The $X = \nu x$ - dependence could be introduced, at the expense of dealing with partial differential equations, but this is not necessary to eliminate secularity.

In the non-linear case, the procedure parallels that for the linear case. Because of the complexity, how- ever, we shall be content there to stop with the conditions analogous to (1.3), and so study only the first term of the expansion in η.

§ 2. Uniform Solutions

The uniform solution of the Korteweg-de Vries equation (III.1) satisfies

$$-\omega\eta' + 6\kappa\eta\eta' + \kappa^3\eta''' = 0$$

which integrates to

$$\begin{aligned}\kappa^3\ \eta'^2 &= 2\kappa(-\eta^3 + \frac{\omega}{2\kappa}\eta^2 + B\eta - A)\\ &= -\ 2\kappa(\eta - m_1)(\eta - m_2)(\eta - m_3)\end{aligned} \tag{2.1}$$

where A and B are constants of integration, and where the m_i are roots of the cubic which are constrained by

$$m_1 + m_2 + m_3 = \frac{\omega}{2\kappa} \tag{2.2}$$

We assume without loss of generality that $m_3 < m_1 < m_2$. The solution of (2.1) is then an elliptic function

$$\eta = m_1 + (m_2 - m_1)\ \mathrm{cn}^2(\beta\theta + \phi,\ k) \tag{2.3}$$

where ϕ is a constant of integration and

$$\beta = \frac{1}{\kappa}\sqrt{\frac{m_2 - m_3}{2}} \tag{2.4}$$

$$k = \sqrt{\frac{m_2 - m_1}{m_2 - m_3}} \qquad (0 \leq k \leq 1) \tag{2.5}$$

The notation for the elliptic function corresponds to that of Whittaker and Watson.[20] The solution (2.3) is oscillatory, oscillating between the pair of zeroes m_1, m_2 of the cubic and is the famous "cnoidal wave" solution.[7] It will be the fundamental solution for our investigations of §3.

The function $cn^2(u,k)$ is periodic of period $2K(k)$ in u, where

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

($K(k)$ is the complete elliptic integral of the first kind). Since we are regarding our periodic solutions as having period 2π in θ, this means from (2.3) that

$$2\pi\beta = 2K(k) \tag{2.6}$$

We call this the non-linear dispersion relation since it is derived just as in the linear regime from requiring the period in θ to be 2π. If we took the small-amplitude limit $(m_2 - m_1) \rightarrow 0$ we could recover the ordinary dispersion relation $\omega = -\kappa^3$ from (2.6).

K

Consider now the dissipative case (III.2). As per the comments of §1, it is sufficient to consider the solution to be of the form $\eta(X)$, where $X = x - Ut$. The equation then becomes

$$- U\eta' + 6\eta\eta' + \eta''' = \nu\eta'' .$$

By the transformation

$$\eta = \frac{U}{3}\hat{\eta} \qquad X = \frac{\xi}{\sqrt{U}}$$

this can be written in terms of a single parameter B,

$$\hat{\eta}_{\xi\xi} - B\hat{\eta}_{\xi} + \hat{\eta}^2 - \hat{\eta} = 0 ,$$

where $B = \nu/\sqrt{U}$. A trivial arbitrary change in level has been ignored. We re-write this as a system

$$\left.\begin{aligned} \frac{d\hat{\eta}}{d\xi} &= y \\ \frac{dy}{d\xi} &= By - \hat{\eta}^2 + \hat{\eta} \end{aligned}\right\} \qquad (2.7)$$

It turns out that this system has shock solutions, that is, solutions which approach different asymptotic values as $\xi \to +\infty$ and $\xi \to -\infty$. The shock asymptotes are singular points of the system; that is, points at which the right-hand sides vanish simultaneously. Clearly, these are

$$P_1: \; y = 0 \qquad \hat{\eta} = 0$$

$$P_2; \; y = 0 \qquad \hat{\eta} = 1$$

In the vicinity of P_1, the solutions are exponentials $e^{\lambda\xi}$, where

$$\lambda = \frac{1}{2}\left(B \pm \sqrt{B^2 + 4} \right)$$

In the language of differential equation theory,(5) P_1 is a saddle point and is an unstable equilibrium point of the system. All solutions in the neighborhood of P_1 diverge away from P_1 as $\xi \rightarrow \pm \infty$ except two special solutions, which approach P_1 as $\xi \rightarrow +\infty$ and as $\xi \rightarrow -\infty$ respectively. We shall single out the one which approaches P_1 for $\xi \rightarrow +\infty$ for study.

In the vicinity of P_2, the solutions are exponentials $e^{\bar{\lambda}\xi}$, where

$$\bar{\lambda} = \frac{1}{2}\left[B \pm \sqrt{B^2 - 4} \right]$$

For $B^2 > 4$, P_2 is thus a "nodal" point,(5) stable for $\xi \rightarrow -\infty$, in which limit all solutions in the neighborhood of P_2 approach P_2. For $\xi \rightarrow +\infty$, all solutions in the neighborhood of P_2 diverge from P_2, so P_2 may not be a shock asymptote for $\xi \rightarrow +\infty$. Thus, if we fix our solution by requiring it to enter the singular point P_2 as

$\xi \rightarrow -\infty$ and P_1 as $\xi \rightarrow +\infty$, it is uniquely defined, and has the general shape shown in Figure 1.

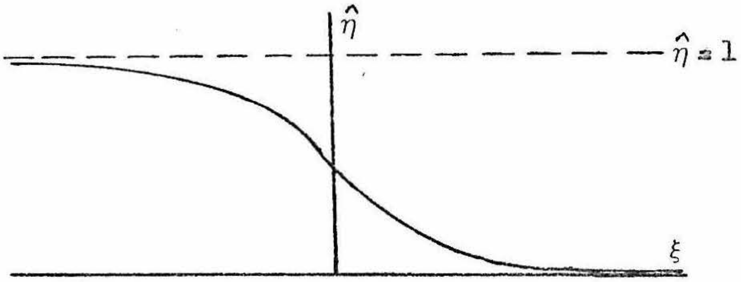

Figure 1. Smooth shock solution of Eq. (III.2).

A more interesting case is that in which $B^2 < 4$. Then the singular point P_2 is a "spiral" or "focal" point,[5] and the same shock solutions look as shown in Figure 2. Note that as the damping ν increases, the wiggles behind the shock front become less pronounced and the solution tends toward the smooth solution of Figure 1. In the other limit, as $\nu \rightarrow 0$, the solution tends toward a cnoidal wave (2.3) oscillating about $\hat{\eta} = 1$ connected across a discontinuity to the constant state $\hat{\eta} = 0$.

Solutions such as those in Figure 2 have been observed experimentally in water waves.[11] However, the model dissipation term of Eq. (III.2), which represents an "eddy viscosity," falls short of being able to give the correct dissipation by a factor of 10 or so.[19] Hence, in spite of the experimental evidence, not much physical significance can be attached to the model (III.2).

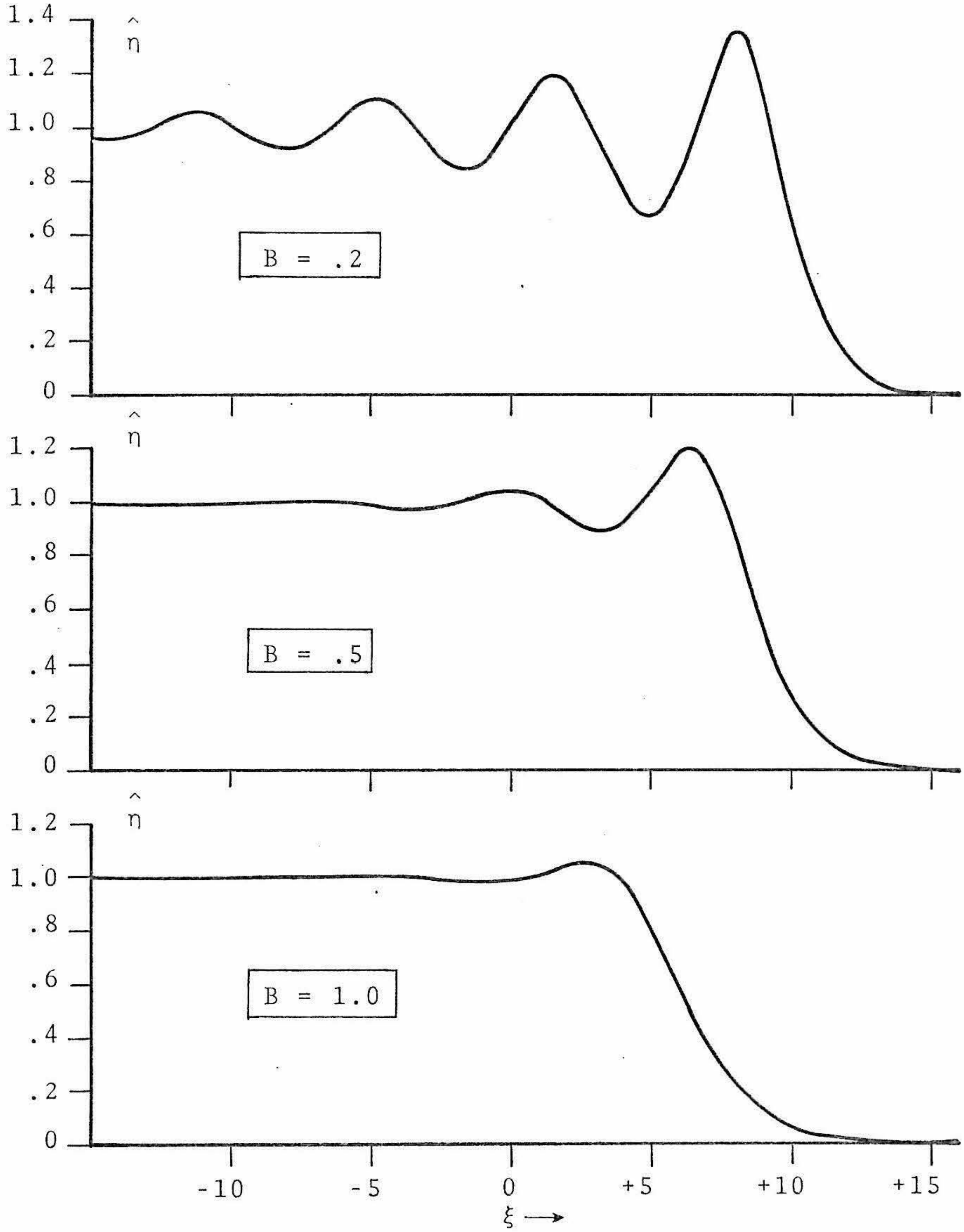

Figure 2. Oscillatory Shock Solutions of Equation (III.2).

A final comment needs to be made regarding our earlier discussion of singular points. One might ask, why did we only consider solutions joining the two singular points P_1 and P_2? The answer is, because these are the only solutions which are uniformly bounded for $-\infty < \xi < \infty$. Rather than prove this, however, we merely appeal to physical considerations for justification. Because the system (2.7) contains damping, we expect that everything at $\xi = -\infty$ and $\xi = +\infty$ will have settled down to a constant state. In particular, derivatives will be zero. Thus, the left-hand sides of (2.7) will be zero at $\xi = \pm\infty$ and so the shock asymptotes must indeed be singular points of the system.

§3. Slowly-Varying Wavetrain Solutions

We shall now apply the averaging method (in the guise of the Luke two-timing expansion) to slowly-varying wave solutions of the Korteweg-de Vries equation with dissipation, Eq. (III.2). The dissipationless case, Eq. (III.1), has been studied by Whitham[15], and various of his results will be used without derivation to simplify the presentation.

Expand η according to

$$\eta = \eta_o(\theta,X,T) + \nu\eta_1(\theta,X,T) + \ldots$$

where

$$\theta_t = -\omega(X,T)$$

$$\theta_x = \kappa(X,T)$$

$$X = \nu x$$

$$T = \nu t$$

The expansion is for small damping, $\nu \to 0$.

The zero-order solution η_o will just be the uniform solution (2.3), where now the various parameters which occur there (m_1, m_2, m_3, κ, Φ) will be functions of X and T. To derive the secularity conditions that the first-order solution η_1 be uniformly bounded in θ, we turn to modified forms of two of the conservation equations derived by Whitham for the Korteweg-de Vries equation. They are

$$(\eta)_t + (3\eta^2+\eta_{xx})_x = \nu\eta_{xx} \tag{3.1}$$

$$(\tfrac{1}{2}\eta^2)_t + (2\eta^3+\eta\eta_{xx}-\tfrac{1}{2}\eta_x^2)_x = \nu(\eta\eta_x)_x - \nu\eta_x^2 \tag{3.2}$$

and may be derived from (III.2). We write (3.1) and (3.2) symbolically as

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = \nu\,\eta_{xx}$$

$$\frac{\partial \bar{P}}{\partial t} + \frac{\partial \bar{Q}}{\partial x} = \nu(\tfrac{1}{2}\eta^2)_{xx} - \nu\eta_x^2$$

The expansions of these equations take the form [à la Eq. (II.12)]

$$(-\omega\frac{\partial}{\partial\theta} + \nu\frac{\partial}{\partial T})(P_o + \nu P_1 + \ldots)$$
$$+ (\kappa\frac{\partial}{\partial\theta} + \nu\frac{\partial}{\partial X})(Q_o + \nu Q_1 + \ldots) = \nu(\kappa\frac{\partial}{\partial\theta} + \nu\frac{\partial}{\partial X})^2(\eta_o + \nu\eta_1 + \ldots)$$

$$(-\omega\frac{\partial}{\partial\theta} + \nu\frac{\partial}{\partial T})(\bar{P}_o + \nu\bar{P}_1 + \ldots)$$
$$+ (\kappa\frac{\partial}{\partial\theta} + \nu\frac{\partial}{\partial X})(\bar{Q}_o + \nu\bar{Q}_1 + \ldots) = \frac{\nu}{2}(\kappa\frac{\partial}{\partial\theta} + \nu\frac{\partial}{\partial X})^2(\eta_o + \nu\eta_1 + \ldots)^2$$
$$-\nu\left\{(\kappa\frac{\partial}{\partial\theta} + \nu\frac{\partial}{\partial X})(\eta_o + \nu\eta_1 + \ldots)\right\}^2$$

The two $O(\nu)$ equations out of the previous set are

$$\frac{\partial}{\partial\theta}\left(\kappa Q_1 - \omega P_1\right) = \kappa^2 \frac{\partial^2 \eta_o}{\partial\theta^2} - \frac{\partial P_o}{\partial T} - \frac{\partial Q_o}{\partial X}$$

$$\frac{\partial}{\partial\theta}\left(\kappa \bar{Q}_1 - \omega \bar{P}_1\right) = \frac{\kappa^2}{2} \frac{\partial^2}{\partial\theta^2}\left(\eta_o^{\,2}\right) - \kappa^2\left(\frac{\partial\eta_o}{\partial\theta}\right)^2 - \frac{\partial \bar{P}_o}{\partial T} - \frac{\partial \bar{Q}_o}{\partial X}$$

As explained in Section II, these lead to the two boundedness conditions

$$\int_o^{2\pi}\left(\kappa^2 \frac{\partial^2\eta_o}{\partial\theta^2} - \frac{\partial P_o}{\partial T} - \frac{\partial Q_o}{\partial X}\right) d\theta = 0$$

$$\int_o^{2\pi}\left[\frac{\kappa^2}{2} \frac{\partial^2}{\partial\theta^2}\left(\eta_o^{\,2}\right) - \kappa^2\left(\frac{\partial\eta_o}{\partial\theta}\right)^2 - \frac{\partial \bar{P}_o}{\partial T} - \frac{\partial \bar{Q}_o}{\partial X}\right] d\theta = 0$$

Because η_o and $\partial\eta_o/\partial\theta$ are periodic in θ, the first term in each condition drops out, leaving

$$\frac{\partial}{\partial T}\int_o^{2\pi} P_o \, d\theta + \frac{\partial}{\partial X}\int_o^{2\pi} Q_o \, d\theta = 0 \qquad (3.3)$$

$$\frac{\partial}{\partial T}\int_0^{2\pi}\overline{P}_o\,d\theta+\frac{\partial}{\partial X}\int_0^{2\pi}\overline{Q}_o\,d\theta+\kappa^2\int_0^{2\pi}\left(\frac{\partial\eta_o}{\partial\theta}\right)^2 d\theta=0 \qquad (3.4)$$

We now go back to (3.1) and (3.2) to obtain the specific values for the P's and Q's:

$$P_o = \eta$$

$$Q_o = 3\eta^2 + \kappa^2\frac{\partial^2\eta}{\partial\theta^2} = U\eta + B$$

$$\overline{P}_o = \frac{1}{2}\eta^2$$

$$\overline{Q}_o = 2\eta^3 + \kappa^2\eta\frac{\partial^2\eta}{\partial\theta^2} - \frac{\kappa^2}{2}\left(\frac{\partial\eta}{\partial\theta}\right)^2 \qquad (3.5)$$

$$= \frac{U}{2}\eta^2 + A$$

The subscript "0" is dropped here and hereafter. In these formulas, (2.1) has been used to substitute for the derivatives of η. The equations (3.3) and (3.4) then become

$$\frac{\partial}{\partial T}\int_0^{2\pi}\eta\,d\theta+\frac{\partial}{\partial X}\int_0^{2\pi}(U\eta+B)\,d\theta=0 \qquad (3.6)$$

$$\frac{\partial}{\partial T}\int_0^{2\pi}\frac{1}{2}\eta^2\,d\theta+\frac{\partial}{\partial X}\int_0^{2\pi}(\frac{U}{2}\eta^2+A)\,d\theta+\kappa^2\int_0^{2\pi}(\frac{\partial\eta}{\partial\theta})^2\,d\theta=0 \quad (3.7)$$

At this point, following Whitham[15], we define

$$W=-\frac{\kappa}{2\pi}\oint\frac{\partial\eta}{\partial\theta}\,d\eta=-\frac{\kappa}{2\pi}\int_0^{2\pi}(\frac{\partial\eta}{\partial\theta})^2\,d\theta$$

$$=-\frac{\sqrt{2}}{2\pi}\oint\sqrt{-\eta^3+\frac{1}{2}U\eta^2+B\eta-A}\;d\eta$$

It then proves possible to write (3.6) and (3.7) as

$$\frac{\partial}{\partial T}(\kappa W_B)+\frac{\partial}{\partial X}(\kappa UW_B-B)=0 \quad (3.8)$$

$$\frac{\partial}{\partial T}(\kappa W_U)+\frac{\partial}{\partial X}(\kappa UW_U-A)+\kappa W=0 \quad (3.9)$$

These are similar to the equations for the dissipationless case, the only difference being the extra term κW. It was fortuitous that the dissipation term could be represented in terms of W. In general, this will not be so.

There are four variables A,B,U, and κ to determine. In addition to (3.8) and (3.9) we need two more relations among these variables. The non-linear dispersion relation

(2.6) furnishes one. The other may be taken as the conservation of waves

$$\frac{\partial \kappa}{\partial T} + \frac{\partial}{\partial X}(\kappa U) = 0$$

which follows from the definitions of κ and ω.

Because W may be written as an elliptic integral, it is possible to make some analytical headway into the present equations. The natural variables to use, however are m_1, m_2, and m_3, in terms of which η is written explicitly in (2.3), rather than A, B, and U. Thus the forms (3.8) and (3.9) are of little use for calculation. We return instead to (3.6) and (3.7) and put the expression (2.3) for η directly into the integrals. We eliminate all explicit dependence on A, B, U by using the first forms of Q_o and $\bar{Q}_o$ in (3.5) rather than the second forms. The rest is a matter of using the proper elliptic function identities, and the details may be found in Appendix G. What we obtain, after using (2.6) to eliminate one of the variables, is a set of three quite complicated partial differential equations:

$$\frac{\partial M_1}{\partial T} + \frac{\partial M_2}{\partial X} = 0 \qquad (3.11)$$

$$\frac{\partial M_2}{\partial T} + \frac{\partial M_3}{\partial X} + M_4 = 0 \qquad (3.12)$$

$$\frac{\partial \kappa}{\partial T} + \frac{\partial \omega}{\partial X} = 0 \qquad (3.13)$$

where

$$M_1 = \pi m_1 + \frac{2}{\pi} \kappa^2 K(k) \left\{ E(k) - (1-k^2) K(k) \right\}$$

$$M_2 = 3\pi m_1{}^2 + \frac{4}{\pi} \kappa^2 K(k) E(k) \left\{ 3m_1 + \frac{2}{\pi^2} \kappa^2 (2k^2-1) K^2(k) \right\}$$

$$+ \frac{2}{\pi} \kappa^2 (1-k^2) K^2(k) \left\{ -6m_1 + \frac{2}{\pi^2} \kappa^2 (2-3k^2) K^2(k) \right\}$$

$$M_3 = 12 \left[\frac{2}{\pi} \kappa^2 K(k) E(k) \left\{ 3m_1{}^2 + \frac{4}{\pi^2} m_1 \kappa^2 (2k^2-1) K^2(k) \right.\right.$$

$$\left. + \frac{4}{15\pi^4} \kappa^4 (23k^4 - 23k^2 + 8) K^4(k) \right\}$$

$$+ \frac{2}{\pi} \kappa^2 K^2(k) (1-k^2) \left\{ -3m_1{}^2 + \frac{2}{\pi^2} m_1 \kappa^2 (2-3k^2) K^2(k) \right.$$

$$\left. - \frac{4}{15\pi^4} \kappa^4 (15k^4 - 19k^2 + 8) K^4(k) \right\}$$

$$\left. + \pi m_1{}^3 \right] - \frac{3}{2} M_4$$

$$M_4 = \frac{32}{5\pi^5} \kappa^6 K^5(k) \left[2(k^4 - k^2 + 1) E(k) - (1-k^2)(2-k^2) K(k) \right]$$

$$\omega = 2\kappa \left[3m_1 + \frac{2}{\pi^2} \kappa^2 (2k^2-1) K^2(k) \right]$$

Eq. (3.11) comes from (3.6), (3.12) from (3.7), and (3.13) from (3.10). The three variables to be determined by these three equations are κ, m_1, and k, where k was defined in (2.5). The fourth unknown has been eliminated, as noted, by using the nonlinear dispersion relation (2.6). The expression for ω in terms of κ, m_1, and k follows from (2.2) after some manipulation.

We do not propose to solve the system (3.11-13). It has been written out in detail to give some indication of the difficulty of the averaged equations, even for this simple case. In general it is not even possible to write the averaged equations in terms of known functions, as we have done here. This is because the master function is a hyper-elliptic integral in almost all cases of physical interest[15], and so are its derivatives, which are what enter the averaged equations. Nevertheless, the picture as regards numerical solution is quite bright. We shall **illustrate** in the present problem using the special case of no X-dependence in (3.11-13). Physically, this means we are more interested in the damping than in the propagation, for the damping will be primarily a temporal effect.

For $\frac{\partial}{\partial X} = 0$, it follows immediately that

$$\kappa = \text{const.}$$

$$M_1 = \text{const.}$$

so that we need only consider a single differential equation

$$\frac{dM_2}{dT} + M_4 = 0 \tag{3.14}$$

Furthermore, from Eq. (3.6), the physical interpretation of M_1 is as the mean value of η, which we may without loss of generality take to be zero. $M_1 = 0$ may then be solved for m_1:

$$m_1 = \frac{2}{\pi^2} \kappa^2 K(k) \left[(1-k^2) K(k) - E(k)\right] \tag{3.15}$$

This may be used to replace m_1 in the expression for M_2, whereupon (3.14) becomes a differential equation involving only one dependent variable, k. After some manipulation, and the use of the formulae[20]

$$\frac{d}{dk} E(k) = \frac{E(k) - K(k)}{k}$$

$$\frac{d}{dk} K(k) = \frac{E(k) - (1-k^2) K(k)}{k(1-k^2)}$$

this differential equation may be brought into the form

$$\frac{dk}{dT} = \frac{8\kappa^2}{5\pi^2}\;\frac{k(1-k^2)\;K^4(k)\;\left[(2-k^2)(1-k^2)\;K(k)\;-\;2(1-k^2+k^4)\;E(k)\right]}{6E(k)\;\left[K(k)-E(k)\right]\;\left[E(k)-(1-k^2)\;K(k)\right]} \tag{3.16}$$

Before proceeding to the numerics, let us examine the small-amplitude limit of this differential equation. Since $a = m_2 - m_1$ is the amplitude, and k goes like $a^{\frac{1}{2}}$ by (2.5), the appropriate limit is $k \to 0$. Using the expansions[22]

$$K(k) = \frac{\pi}{2}\;(1+\tfrac{1}{4}k^2+\tfrac{9}{64}k^4+\tfrac{25}{256}k^6+\ldots) \tag{3.17}$$

$$E(k) = \frac{\pi}{2}\;(1-\tfrac{1}{4}k^2-\tfrac{3}{64}k^4-\tfrac{5}{256}\;k^6-\ldots) \tag{3.18}$$

we find for $k \to 0$

$$\frac{dk}{dT} = -\;\frac{\kappa^2}{2}\;k\;+\;\frac{\kappa^2}{4}\;k^3\;+\;O(k^5)$$

The solution of this differential equation is

$$k = k_o e^{-\frac{\kappa^2}{2}T} - \frac{1}{4}\;k_o^{\,3} e^{-\frac{3\kappa^2}{2}T} \;+\; O(k_o^{\,5} e^{-\frac{5\kappa^2}{2}\;T}) \tag{3.19}$$

where either k_o is small or T is large. We see that the correction term from the non-linear effects tends to make dk/dT less negative, so that k decays less rapidly than the linearized solution.

No direct physical interpretation may be placed on k. The quantities which are of more interest physically are $U = \omega/\kappa$, the velocity; a, the amplitude; and m_1, the level of the wave troughs. Eq. (3.15) gives us m_1 in terms of k; hence from (3.19) and the expansions (3.17) and (3.18) we deduce the small-amplitude expansion of m_1,

$$m_1 = \frac{\kappa^2}{4}\left[-k_o^{\,2}e^{-\kappa^2 T} + \frac{1}{8}k_o^{\,4}e^{-2\kappa^2 T} + O(k_o^{\,6}e^{-3\kappa^2 T})\right] \quad (3.20)$$

From the relation $\pi\beta = K(k)$ and the definitions (2.4) and (2.5) of β and k, one may obtain the amplitude a as a function of k,

$$a = m_2 - m_1 = 2\left\{\frac{\kappa}{\pi}\,k\,K(k)\right\}^2 \quad (3.21)$$

Its small-amplitude expansion is

$$a = \frac{\kappa^2}{2}\left[k_o^{\,2}e^{-\kappa^2 T} + O(k_o^{\,6}e^{-3\kappa^2 T})\right] \quad (3.22)$$

Note that the first-order correction term vanishes and hence that the deviation from linearity will be felt only through higher-order terms. Finally, from the relations following (3.13),

$$U = \frac{\omega}{\kappa} = 2\left[3m_1 + \frac{2}{\pi^2}K^2(2k^2-1)\ K^2(k)\right]$$
$$= \frac{4\kappa^2}{\pi^2} K(k) \quad \left[(2-k^2)\ K(k) - 3E(k)\right] \tag{3.23}$$

which in the small-amplitude limit becomes

$$U = -\ \kappa^2 \left[1-\frac{3}{32}\ k_o{}^4 e^{-2\kappa^2 T} - \frac{15}{256}\ k_o{}^6 e^{-3\kappa^2 T} + O(k_o{}^8 e^{-4\kappa^2 T})\right] \tag{3.24}$$

The negative velocity as $T \to \infty$ is only because of the normalization of η used here. If we took the mean height $\bar{\eta}$ to be non-zero, then $\bar{\eta}$ would be added to the right hand side above and U could be made positive.

What about the other limit in which K(k) and E(k) can be simply approximated, that in which $k \to 1$? It turns out this is not a meaningful limit for our problem without the qualification that $\kappa \to 0$ at the same time. K(k) has a logarithmic singularity at k = 1, which means that the amplitude, by (3.21), blows up there. In reality, we know that the Korteweg-de Vries equation has a uniform wavetrain solution of maximum amplitude, the solitary wave, which is formed when the roots m_1 and m_3 coalesce[11]. But by the definition of k, (2.5), $m_1 = m_3$ implies k = 1.

So why does the amplitude come out infinite instead of finite at $k = 1$? The reason is, that the solitary wave has infinite wavelength ($\kappa \to 0$) and we have tacitly assumed κ fixed and finite. It would be perfectly fine to study the $k \to 1$ limit if we required $\kappa \to 0$ at the same time. But we cannot let $k \to 1$ for fixed κ. Since this limit is rather complex, we shall not study it here.

Let us now look at a few numerical solutions of Eq. (3.16). In Figure 3 are plotted three solutions of (3.16) corresponding to initial values $k(0)$ of .5, .7, and .9. The effect of the non-linearity is more pronounced the larger the starting value of k, as we might have expected. We see that the deviations from the linearized approximations follow the pattern predicted by Eq. (3.19), that is, the values of k fall below the linearized values.

In Figure 4 we have the variation of the trough level m_1 with T. This is computed from the solutions for k using (3.15). Again, we see that the deviations from linearity are of the sign predicted by (3.20).

Finally, in Figure 5, the amplitude a has been plotted against T for solutions II and III of Figure 3. It is computed from Eq. (3.21). No prediction was made as to the sign of the deviation of a from linearity because the first-order correction term vanished (see (3.22)). However, it

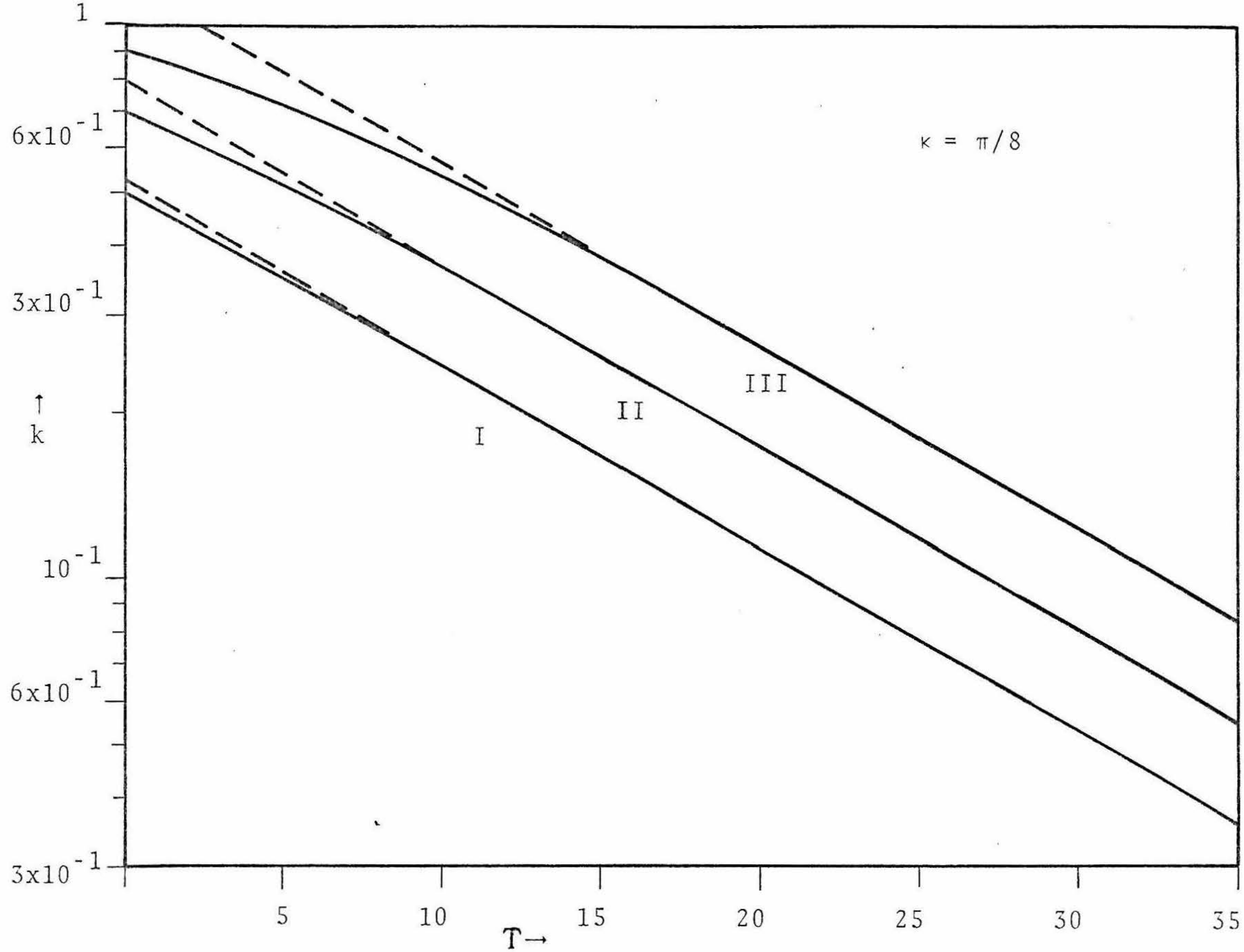

Figure 3. Several Solutions of Eq. (3.16). The dashed straight lines are the linearized approximations into which the true solutions finally merge.

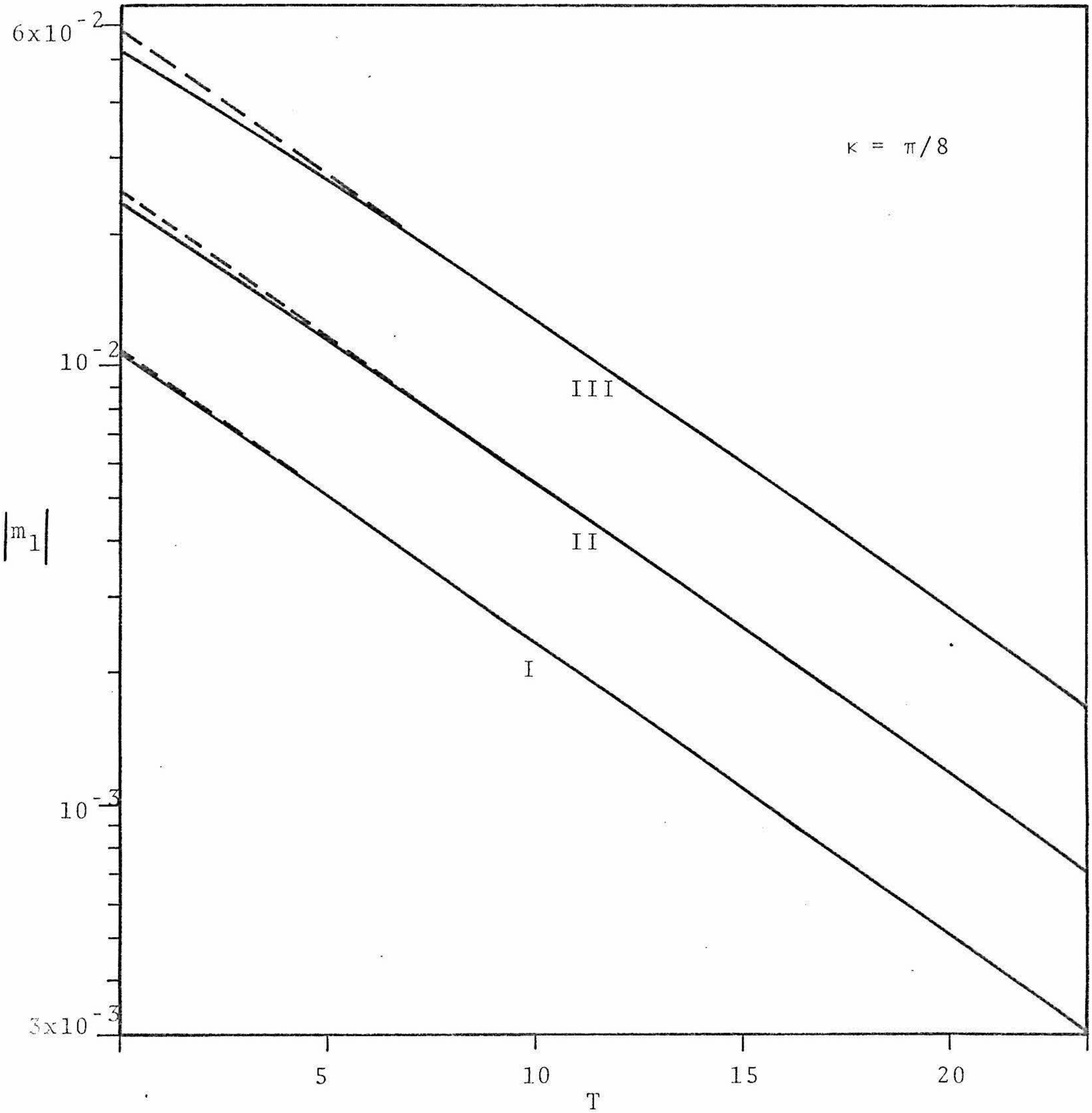

Figure 4. The decay of the trough level m_1 with T for the three solutions of Figure 3. The dashed straight lines are the linearized approximations into which the true solutions merge.

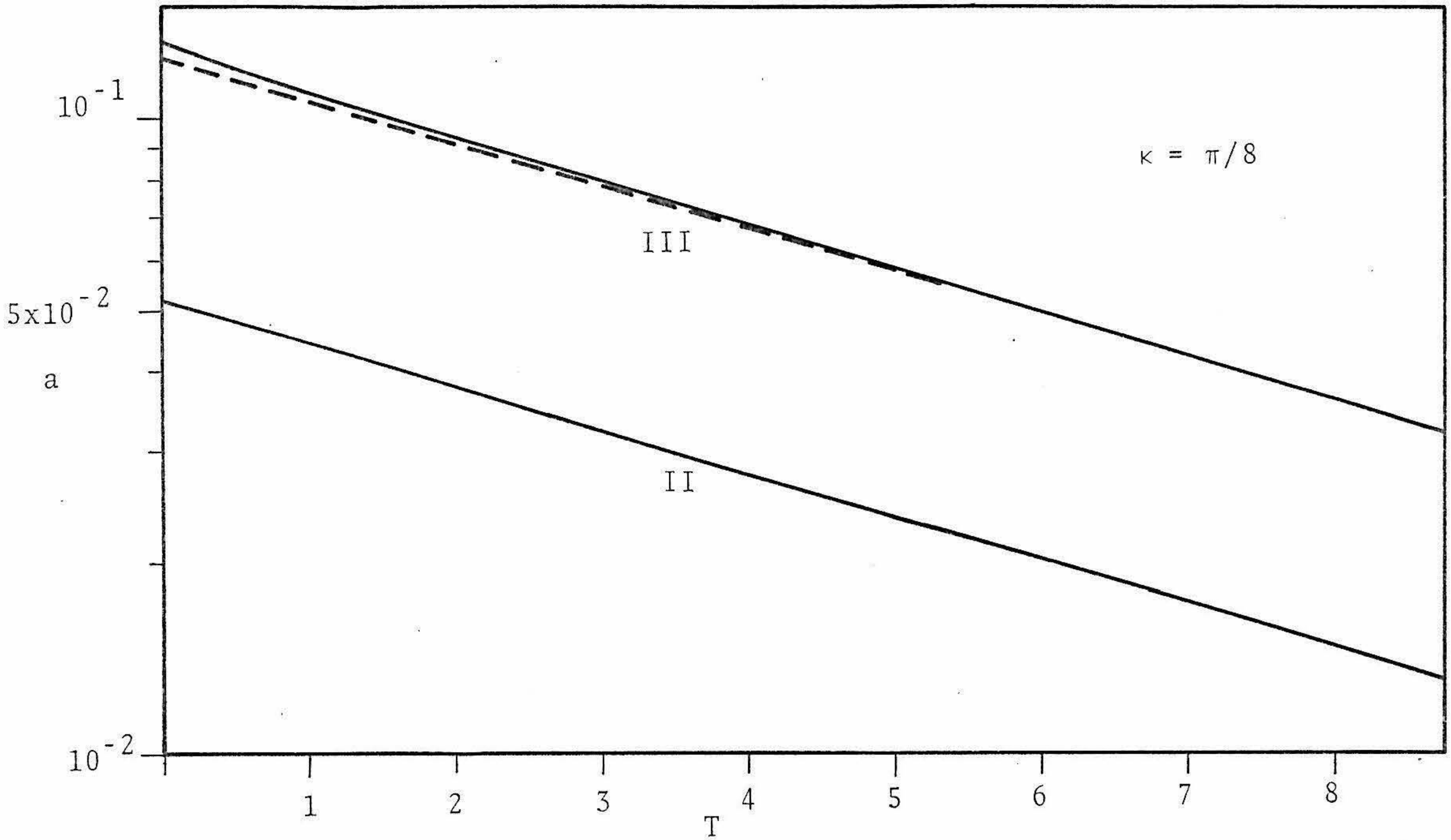

Figure 5. The decay of the amplitude a with T for solutions II and III of Figure 3. Solution I agrees exactly with the linearized approximation, as does solution II, and so is not shown. The dashed line is the linearized approximation with which solution III merges.

is clear that the deviation of a is considerably smaller than that of k or m_1. This is in keeping with the deviation being a second-order effect. It is interesting to note that the true amplitude is larger than the linearized version, a circumstance we find also in the plasma wave case (cf. Appendix C and Figure B-2, Appendix B).

Even though the solutions plotted do not exhibit a high degree of non-linearity (the largest starting amplitude was .13) the qualititative behaviours observed here persist throughout the non-linear regime. Thus small-amplitude expansions such as (3.19) are seen to be useful tools even in highly non-linear problems.

The problem of slowly-varying wavetrain solutions of the Korteweg-de Vries equation with dissipation is certainly a problem of interest in its own right. However it also serves as a model, a paradigm, of how the calculations should go for the more complicated plasma problem to be considered next. Such analogies are often possible in non-linear dispersive wave problems because of the underlying unity of their mathematical formulation, as brought out by the averaged Lagrangian method (see Part II).

Part IV

Uniform and Slowly-Varying Wavetrain Solutions of the Lukewarm Plasma Equations

There are many different approximate sets of equations which are used to study plasmas[4]. All of these have applicability in one limit or another of real plasmas, e.g. as the temperature goes to zero. In the present part we shall examine in some detail one such set of equations, which is derived in §4 from the Boltzmann-Vlasov equation. We shall be interested in this set of equations perhaps more from the mathematical point of view than the physical. It affords a reasonably difficult test of the method developed in §3 for treating small dissipation, and at the same time brings out new features of the averaging method, in particular the handling of equations in non-conservation form (§8). It also has a distinctly non-trivial uniform wavetrain solution, which nevertheless can be completely analyzed by the methods of §7, which are quite general. This in turn allows one to render judgment on the possibility of steady shocks, whose existence is suggested by the results of §6 but finally barred by the arguments of §7.C. The treatment in §7.C parallels that in §2, and, again, may be expected to apply to a wide variety of problems. Thus the plasma equations we shall consider are somewhat of a backdrop for the mathematical methods to be considered here; the ultimate goal is to illustrate new methods of attack upon systems of partial differential equations.

The later sections of this part, §8.C, §9, and §10, are devoted to the properties of the averaged equations derived in §8.A and §8.B; of particular note is the discussion of non-linear stability in §9. We conclude in §11 with a derivation of the Lagrangian and averaged Lagrangian for the plasma equations, which would be an alternate starting point for the derivation of the averaged equations.

§4. Derivation of the Lukewarm Plasma Equations; Possible Choices for a Damping Term

In sub-section A we shall derive the system of plasma equations which we shall study in the ensuing seven sections. This system shall be an appropriately truncated system of moment equations of the Boltzmann-Vlasov equation. In sub-section B, we shall comment on an ad hoc procedure for including Landau damping in the equations.

A. Derivation of the equations

The Boltzmann-Vlasov equation, or "collisionless" Boltzmann equation, is, in one dimension[4]

$$\frac{\partial f'}{\partial t'} + v' \frac{\partial f'}{\partial x'} - \frac{e}{m} E' \frac{\partial f'}{\partial v'} = 0$$

$$\frac{\partial E'}{\partial x'} = 4\pi e \left(n_o - \int f' \, dv'\right)$$

where $f'(x', v', t')$ is the distribution function and E' is the electric field. It may be non-dimensionalized by the following transformations:

$$x' = \lambda_D \, x = \sqrt{\frac{kT_o}{4\pi n_o e^2}} \, x$$

$$t' = \frac{t}{\omega_p} = \sqrt{\frac{m}{4\pi n_o e^2}} \, t$$

$$v' = \sqrt{\frac{kT_o}{m}} \, v$$

$$E' = \sqrt{4\pi n_o k T_o}\ E$$

$$f' = n_o \sqrt{\frac{m}{kT_o}}\ f$$

leading to

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} - E\,\frac{\partial f}{\partial v} = 0 \qquad (4.1)$$

$$\frac{\partial E}{\partial x} = 1 - \int f\, dv \qquad (4.2)$$

In the above formulas, λ_D is the Debye length, ω_p is the longitudinal plasma frequency, e and m are the electronic charge and mass, T_o is the electronic temperature, and n_o is the number density of ions. The ions are assumed fixed in place in this model, providing a uniform background of positive charge in which the electrons move. This is a reasonable approximation at frequencies of the order of the plasma frequency, for the ions cannot begin to follow such rapid oscillations.

The type of plasma being considered is classical (non-quantum and non-relativistic), does not interact with the radiation field, and is described at equilibrium by a Maxwellian velocity distribution. The induced B-field (from the changing E-field) is neglected.

The word "collisionless" attached to the Boltzmann equation in (4.1) is somewhat of a misnomer. The collisions that are being neglected are the binary ones, the short and violent impacts which in a plasma are much less important than the continuously-acting and long-range Coulombic interactions. Each electron is constantly "in collision" with all the other electrons inside its Debye sphere; each electron follows a trajectory determined by the external fields plus the smoothed-out fields produced by all the other electrons in the plasma. This is of course an approximation (called the <u>self-consistent field</u> approximation because the charge density in Poisson's equation (4.2) is determined by f) but it is not a "collisionless" approximation.

We shall work with only one velocity component here, that is, the model will be truly one-dimensional. This simplifies the mathematics a little. At the end of the derivation the minor modifications necessary to incorporate the full three dimensions in velocity space will be indicated.

We begin by defining the moments of f in the usual manner:

$$n = \int f \, dv$$

$$nu = \int vf \, dv$$

$$p = \int (v-u)^2 \; f \; dv$$

$$q = \int (v-u)^3 \; f \; dv$$

n is the number density of electrons, u is their mean velocity, p is the pressure, and q is the heat flux. The last two may be re-written using the first two:

$$\left.\begin{aligned} p + nu^2 &= \int v^2 \; f \; dv \\ q + 3pu + nu^3 &= \int v^3 \; f \; dv \end{aligned}\right\} \qquad (4.3)$$

Using (4.3), the first three moments of (4.1) may be written down immediately:

$$n_t + (nu)_x = 0 \qquad (4.4)$$

$$(nu)_t + (nu^2 + p)_x + nE = 0 \qquad (4.5)$$

$$(nu^2 + p)_t + (nu^3 + 3pu + q)_x + 2nuE = 0 \qquad (4.6)$$

The terms involving E come from integrations by parts. These three equation describe the conservation of mass (or charge), momentum, and energy, respectively.

Using $n = 1 - E_x$ in the first term of (4.4) and integrating through, we, obtain

$$E_t = nu \qquad (4.7)$$

An arbitrary function of t has been set to zero because that is its value at equilibrium ($E=u=0$). Eq. (4.7) will be called the current equation. It is basically just one of Maxwell's equations.

Replacing nu by E_t in the last term of (4.6),

$$(nu^2+p+E^2)_t + (nu^3+3pu+q)_x = 0 \qquad (4.8)$$

This is the conservation form of the energy equation. It is not possible to put the momentum equation, (4.5), in conservation form. This is because it requires an external force to hold the ions fixed in place, and through the medium of the ion-electron field E, this external force communicates itself to the electrons. Momentum cannot be conserved in the presence of an external force. On the other hand, it is still perfectly reasonable that we have a conservation form energy equation (4.8), for the ions are immobile and hence the external force does no work on the system.

We are now faced with the usual problem of closing our system of moment equations. There are 5 unknowns n,u,p,E, and q, and only four equations (4.2) and (4.4-6). It is standard[4] to assume that the heat flow q is zero. This is justified if the plasma is not too hot, and we make this assumption here. The resulting closed system of p.d.e.'s we shall call the lukewarm plasma equations, or LPE's for short.

We can use the continuity equation (4.4) to simplify the momentum equation (4.5), and the momentum and continuity equations together to simplify the energy equation (4.6). The result is:

$$n_t + (nu)_x = 0$$

$$u_t + uu_x + \frac{p_x}{n} + E = 0 \qquad (4.9)$$

$$p_t + up_x + 3pu_x = 0 \qquad (4.10)$$

$$E_x = 1 - n$$

This is almost the simplest form in which the LPE's can be written. One further simplification is possible. We note that if $E = 0$ these equations become essentially the Euler equations of fluid mechanics (with $\gamma = 3$ rather than $\gamma = 5/3$). In the Euler equations it is convenient to introduce the entropy S as a monotonic function (usually the logarithm) of p/ρ^{γ}. For our problem we shall simply take

$$S = \frac{p}{n^3} \qquad (4.11)$$

It then proves possible, using the continuity equation, to reduce (4.10) to

$$S_t + uS_x = 0 \qquad (4.12)$$

By adding n times (4.11) to S times (4.4), we obtain the equation for conservation of entropy

$$(nS)_t + (nuS)_x = 0 \qquad (4.13)$$

Either (4.12) or (4.13) may be taken in place of (4.10).

What about using a three-dimensional velocity space? The Boltzmann equation (4.1) is unaltered in form, only now v means v_x and f is a function of v_y and v_z as well as v_x. Poisson's equation (4.2) is unaltered. The moments become

$$n = \int f \, d^3v$$

$$nu_\alpha = \int v_\alpha f \, d^3v$$

$$P_{\alpha\beta} = \int (v_\alpha - u_\alpha)\,(v_\beta - u_\beta)\, f\, d^3v$$

$$q_\alpha = \int (v_\alpha - u_\alpha)\,[(v_x-u_x)^2 + (v_y-u_y)^2 + (v_z-u_z)^2]\, f\, d^3v$$

Multiplying (4.1) through by 1, v_x, v_y, v_z, and $(v_x^2+v_y^2+v_z^2)$ and integrating over velocity space, and assuming in accordance with one-dimensionality that $u_y = u_z = 0$,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu_x) = 0 \qquad (4.13)$$

$$\frac{\partial}{\partial t}(nu_x) + \frac{\partial}{\partial x}(nu_x^2 + p_{xx}) + nE = 0 \qquad (4.14)$$

$$\frac{\partial p_{xy}}{\partial x} = 0 \qquad (4.15)$$

$$\frac{\partial p_{xz}}{\partial x} = 0 \qquad (4.16)$$

$$\frac{\partial}{\partial t}(nu_x^2 + p_{xx} + p_{yy} + p_{zz}) + \frac{\partial}{\partial x}\left\{nu_x^3 + u_x(3p_{xx} + p_{yy} + p_{zz}) + q_x\right\} + 2nu_x E = 0 \qquad (4.17)$$

The pressure tensor $p_{\alpha\beta}$ will be diagonal, as suggested by (4.15) and (4.16), because of the spatial one-dimensionality. The diagonal elements p_{xx}, p_{yy}, and p_{zz} are what we usually think of as pressures, that is, normal forces across an area perpendicular to the x-, y-, and z-directions respectively. Unless there is some preferred direction in the problem, caused say by an impressed magnetic field, there is no reason to suppose the pressure will be anisotropic. Hence

$$p_{xx} = p_{yy} = p_{zz}$$

which makes (4.17) read

$$\frac{\partial}{\partial t}(nu_x{}^2 + 3p_{xx}) + \frac{\partial}{\partial x}(nu_x{}^3 + 5up_{xx} + q_x) + 2nu_xE = 0 \quad (4.18)$$

When this is reduced to a form analogous to (4.10), it becomes (with q = 0 and dropping the xx-subscript)

$$p_t + up_x + \frac{5}{3} pu_x = 0$$

Thus the only change in going to three dimensions is the replacement of 3 by 5/3 in (4.10). (Dropping the x-subscripts in (4.13) and (4.14) makes them identical to (4.4) and (4.5).) This will mean , of course, that the entropy will be $p/n^{5/3}$, and it is to avoid dealing with such fractional powers that we restrict ourselves to the one-dimensional case.

B. Possible choices for a damping term

If dissipation is to enter the LPE's, it must be through the term q we have dropped, for otherwise the LPE's are exact consequences of the Boltzmann-Vlasov equation. To make a sound choice for this term, we must go back to the underlying microscopic description (4.1).

Landau[8] linearized (4.1-2) by assuming

$$f = f_o(v) + \tilde{f}(x,v,t)$$

where $\tilde{f} << f_o$ and where $f_o(v)$ is a Maxwellian. In the present units, $f_o(v)$ is

$$f_o(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}$$

The linearization leads to

$$\frac{\partial\tilde{f}}{\partial t} + v\frac{\partial\tilde{f}}{\partial x} - E\frac{df_o}{dv} = 0$$

$$\frac{\partial E}{\partial x} = -\int\tilde{f}\,dv$$

Then by essentially studying fundamental solutions of the form

$$\tilde{f} = g(v)\, e^{i(\kappa x - \omega t)}$$

Landau was led to his famous damping term in the dispersion relation. For $\kappa \to o$, this dispersion relation assumes the asymptotic form

$$\omega^2 = 1 + 3\kappa^2 - \frac{ia}{\kappa^3} \exp\left(-\frac{1}{2\kappa^2}\right) \tag{4.19}$$

where

$$a = \sqrt{\frac{\pi}{2}}\, e^{-3/2}$$

The real part of ω^2 has been expanded to two terms here to reproduce the dispersion relation of the linearized LPE's (see §5). It would actually be legitimate to expand the real part to any number of terms, while keeping only the one term for the damping, because

$$e^{-\frac{1}{2\kappa^2}} = o(\kappa^n) \tag{4.20}$$

as $\kappa \to 0$ for any $n > 0$.

The last is an interesting point. Each time we enlarge our our system of moment equations by one, by truncating at the next higher level, we gain an extra term of accuracy in the real part of the dispersion relation, as $\kappa \to 0$. But because of (4.20), the procedure which we shall give below for including the damping will be useful for arbitrarily large moment systems.

If the form of q which we picked to reproduce (4.19) consisted only of n, u, p, E and their partial derivatives, the best we could hope for would be a damping term of $O(\kappa^n)$ as $\kappa \to 0$. This is because the dispersion relation for any system of p.d.e.'s, on account of the replacement

$$\frac{\partial}{\partial t} \to -i\omega$$

$$\frac{\partial}{\partial x} \to i\kappa ,$$

will always be simply a polynomial in ω and in κ. And it can be shown that the roots ω of any such polynomial always behave as a power of κ as $\kappa \to 0$.

Thus no system of partial differential equations could have (4.19) for its dispersion relation. One must, therefore range more widely in choosing a form for q. A similar situation prevails in water wave theory, where the exact dispersion relation is transcendental and cannot be correctly given by the various approximate theories (shallow water, Boussinesq, Korteweg-de Vries, etc.) There Whitham has suggested[18] the use of an integral term of convolution form in the approximate theory, which is constructed to give the correct dispersion relation. A similar approach can be used here.

Naturally the construction of the convolution is to a large extent arbitrary, since it must only reproduce the correct linearized limit. We give only the simplest form which could be chosen:

$$\frac{\partial q}{\partial x} = \int_{-\infty}^{\infty} K\,(x-\xi)\; u\,(\xi,t)\; d\xi \qquad (4.21)$$

where

$$K\,(x) = \frac{a}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\kappa^4}\, e^{-\frac{1}{2\kappa^2} - i\kappa x}\, d\kappa \qquad (4.22)$$

The LPE's become a set of integro-p.d.e.'s with this definition of q, and of course no analytical solution is possible. We convolute with u rather than n,p, or E primarily because the dispersion relation (4.19) then drops directly out of the equations with no further approximations.

Integrals of the form (4.22) occur throughout the literature[24,25,26] and have been tabulated[22]. They always occur in connection with a Maxwellian velocity distribution [the damping term in (4.19) comes from putting $v = \frac{\omega}{\kappa}$ in $e^{-v^2/2}$ and expanding ω]. The standard form is obtained from making the change of variables $\kappa = \frac{1}{u}$:

$$K(x) = \frac{a}{2\pi}\left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\} \frac{1}{\kappa^4} \exp\left(-\frac{1}{2\kappa^2} - i\kappa x\right) d\kappa$$

$$= \frac{a}{2\pi}\left\{\int_0^\infty u^2 \exp\left(-\tfrac{1}{2}u^2 + \frac{ix}{u}\right) du + \int_0^\infty u^2 \exp\left(-\tfrac{1}{2}u^2 - \frac{ix}{u}\right) du\right\}$$

From these forms one may obtain the asymptotic expansions[26]

$$K(x) \sim \begin{cases} \frac{1}{2}e^{-3/2}\left(1 - \frac{1}{2}x^2 + \sqrt{\frac{\pi}{2}}\, x^3\right) + O(x^4) & x \to 0 \\ \frac{1}{\sqrt{3}}e^{-3/2}\, x^{2/3}\, e^{-\frac{3}{4}x^{2/3}} \cos\left(\frac{3\sqrt{3}}{4}x^{2/3} - \frac{\pi}{3}\right) & x \to \infty \end{cases}$$

A reasonable approximation might be to replace K(x) by its asymptotic formula for $x \to \infty$, since it is a rule-of-thumb that $x \to \infty$ corresponds to $\kappa \to 0$ under Fourier transformation, and the damping term is correct for $\kappa \to 0$. The whole question of approximating kernels for complicated integral equations is in its infancy, however[1], and there does not seem to be any rigorous justification for such a procedure at the moment.

With the integral term in the LPE's, it is no longer possible to do even some of the simple things that we can do with p.d.e.'s. For example, we can no longer find characteristic velocities, do geometrical optics expansions, or obtain steady solutions. A numerical approach is the best we can hope for at this time, and even this is complicated by the non-local nature of integro-p.d.e.'s. Still, we judge that this approach of including an integral term merits attention because it holds promise of elucidating the effects of Landau damping on non-linear plasma phenomena, most notably the "collisionless" shock wave.

We close with some brief remarks on another possible type of damping term. Suppose a mock collision term of the Krook type[27] is put into (4.1),

$$f_t + vf_x - Ef_v = \nu(f_o - f) \qquad (4.23)$$

where ν is the collision frequency, f_o (v; n, u, T) is a local Maxwellian and T is the temperature. Such an equation has been considered by Weitzner[14]. Without going into the details, we merely state that if one applies the Chapman-Enskog procedure to (4.23) and carries it out to the Navier-Stokes level, one may compute an approximation for q

$$q = -\beta \, (p/n)_x \tag{4.24}$$

in which the electric field drops out entirely. The coefficient β depends on the temperature, but to a good approximation it may be treated as a constant. The validity of (4.24) is formally in the limit $\nu \to \infty$, thus binary collisions are assumed to be dominant. This is not the limit we wish to consider, as explained earlier, but in spite of that the form (4.24) of the damping is useful. It is simple, unlike (4.21), and it is more amenable to analytical calculations. In addition, it may be expected to hold for moderate ν's. Finally, any true description of a plasma will have to contain some of both kinds of damping terms, the collisional (4.24) and the Landau (4.21).

§5. Linearized Theory

In the next three sections (§5, §6, §7) we shall study different aspects of the LPE's which throw light on their general structure. In the present section we shall study the linearized theory of uniform wavetrain solutions (sub-section A) and the dispersion relation when the model dissipation (4.24) is included (sub-section B). The first is valuable because any linear and near-linear solution can be constructed as a superposition of uniform wavetrains. The second will show the inherent limitations in any derivative-like dissipation. We will go on to consider breaking and shock solutions in §6, and again in §7.C. These discussions are important, not only because shocks are an important sub-class of non-linear solutions, but also because the methods for getting at them are important. We shall conclude that there are no steady shock solutions of the LPE's, but that unsteady shocks are possible. And finally, in §7.A and §7.B, we shall completely analyze the uniform wavetrain solution of the LPE's. Again, the methods used are worthy of note (in particular the Sturm sequence method of §7.B). Each of the three different kinds of solution, linearized, shock, and uniform wavetrain, adds a piece to the total picture, and each is a kind of solution which we are accustomed to observing experimentally.

We may linearize the LPE's by assuming

$$n = 1 + \tilde{n} \qquad u = \tilde{u} \qquad p = S_o + \tilde{p} \qquad E = \tilde{E}$$

where each of the tilda-ed quantities is small compared to unity. The mean level of p is of course arbitrary. The linearized forms of (4.4), (4.9), (4.10), and (4.2) become

$$\tilde{n}_t + \tilde{u}_x = 0$$

$$\tilde{u}_t + \tilde{p}_x + \tilde{E} = 0$$

$$\tilde{p}_t + 3S_o\tilde{u}_x = 0$$

$$\tilde{E}_x + \tilde{n} = 0$$

which can be reduced to a single equation for, say, $\tilde{p}$:

$$\tilde{p}_{tt} - 3S_o\tilde{p}_{xx} + \tilde{p} = 0 \qquad (5.1)$$

This is a Klein-Gordon, or telegraph, equation, and its solutions have been well-studied.[3] Its dispersion relation is

$$\omega_o = \sqrt{1 + 3S_o\kappa^2} \qquad (5.2)$$

It is interesting to note that the entrance of E into the equations leads to the term $\tilde{p}$ in (5.1) and hence to dispersion (5.2). Thus the plasma case is qualitatively different from the gas-dynamic case $E = 0$, in which sound waves propagate without dispersion.

A. Small-amplitude uniform wavetrains

When all variables n, u, S, E in the LPE's depend only on $\theta = \kappa x - \omega t$, we have from the entropy equation (4.12) and from the current (4.7) and Poisson (4.2) equations that

$$\frac{dS}{d\theta} = 0 \tag{5.3}$$

$$u = U(1 - 1/n) \tag{5.4}$$

where $U = \omega/\kappa$. These can be used to reduce the number of variables in the momentum equation (4.9):

$$\kappa(3S_o n - \frac{U^2}{n^3}) \frac{dn}{d\theta} + E = 0 \tag{5.5}$$

This, together with Poisson's equation

$$\kappa \frac{dE}{d\theta} = 1 - n \ , \tag{5.6}$$

forms a set of two equations for n and E.

We expand n and E as in §1,

$$n = 1 + \epsilon n_1(\theta) + \epsilon^2 n_2(\theta) + \cdots$$

$$E = \epsilon E_1(\theta) + \epsilon^2 E_2(\theta) + \ldots$$

We also expand ω as in §1

$$\omega = \omega_o(\kappa) + \epsilon\omega_1(\kappa) + \epsilon^2\omega_2(\kappa) + \ldots$$

to allow for the elimination of secular terms. Putting all these expansions into (5.5) and (5.6), we have in $O(\epsilon)$

$$n_1(\theta) = A \cos \theta + B \sin \theta$$

$$E_1(\theta) = -A/K \sin \theta + B/K \cos \theta$$

It is sufficient for our purposes to take A = 0. Then, in $O(\epsilon^2)$, the solution of

$$\frac{d^2 n_2}{d\theta^2} + n_2 = 2\omega_o \omega_1 B \sin \theta + 3(1 + 4S_o K^2) B^2 \cos 2\theta$$

gives n_2. If n_2 is to have no secular terms, we must take

$$\omega_1 = 0$$

whereupon

$$n_2 = A_2 \sin \theta + B_2 \cos \theta - (1 + 4S_o K^2) B^2 \cos 2\theta$$

With a similar solution for E_2, one then obtains the following equation for n_3:

$$\frac{d^2 n_3}{d\theta^2} + n_3 = \left[\frac{2\omega_o \omega_2}{B^2} + \frac{3}{2} \omega_o^2 - \frac{3}{2}(1 + 4S_o K^2)^2 \right] B^3 \sin \theta$$

$$+ (\quad) \sin 2\theta + (\quad) \sin 3\theta$$

The choice

$$\omega_2 = \frac{3K^2 B^2}{4\omega_o}(16S_o^2 K^2 + 5S_o) \tag{5.7}$$

makes the coefficient of sin θ vanish, and so eliminates secular terms from n_3. Later we shall derive this same result from a small-amplitude expansion of the non-linear dispersion relation (§10).

B. With dissipation

Let us now consider the linearized form of the LPE's with a small derivative dissipation

$$q = -\epsilon p_x \tag{5.8}$$

analogous to (4.24). The properties of the linearized solution are not particularly sensitive to the choice of q in terms of derivatives of n, p, and u, so we use the simpler form (5.8) rather than (4.24).

With this dissipation, the linearized LPE's can be reduced to a single equation for, say, $\tilde{p}$:

$$\tilde{p}_{ttt} - 3\tilde{p}_{xxt} + \tilde{p}_t = \epsilon(\tilde{p}_{xx} + \tilde{p}_{xxtt}) \tag{5.9}$$

S_o has been set to one for simplicity. We begin by investigating steady-profile solutions

$$\tilde{p} = \tilde{p}(x - Ut)$$

For such solutions, (5.9) becomes an ordinary differential equation satisfied by exponentials

$$e^{m(x - Ut)} \tag{5.10}$$

The equation for m is

$$\epsilon U^2 m^3 + U(U^2 - 3)m^2 + \epsilon m + U = 0$$

Assuming $U^2 > 3$, we can write the expansions of the three roots for m as $\epsilon \longrightarrow 0$,

$$m_1 = -\frac{U^2 - 3}{U\epsilon} + O(\epsilon)$$

$$m_2, m_3 = \pm \frac{i}{\sqrt{U^2 - 3}} + \frac{3\epsilon}{2U(U^2 - 3)^2} + O(\epsilon^2)$$

The root m_1 gives the proper decay of (5.10) as $(x - Ut) \longrightarrow +\infty$, and the roots m_2, m_3 give the proper decay as $(x - Ut) \longrightarrow -\infty$. As mentioned earlier, (§1), these exponentials can only be fragments of non-linear solutions. Thus it is possible to envision a steady-profile solution described in its forward tail by (5.10) with $m = m_1$ and in its rear tail by (5.10) with $m = m_2$ or $m = m_3$. It would look somewhat as in Figure 6.

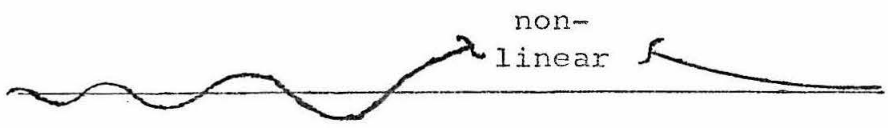

Figure 6. A possible steady profile solution.

It can be shown that m_1 is strictly real for $\epsilon \longrightarrow 0$, so the tail in front is strictly a decay. m_2 and m_3 have imaginary parts, however, and so lead to decay with oscillation at the other end.

In the gas-dynamic "limit" $E \longrightarrow 0$, the equation (5.9) simplifies to

$$\tilde{p}_{tt} - 3\tilde{p}_{xx} = \epsilon\tilde{p}_{xxt}$$

steady-profile solutions of this have but a single decay constant

$$m_o = -\frac{U^2 - 3}{U\epsilon}$$

Since this matches m_1 for $\epsilon \longrightarrow 0$, it is reasonable to say that the forward tail of the LPE solution is "gas-dynamics dominated," that is, that the effect of the electric field is negligible there.

We now consider the dispersion relation of (5.9). In the usual manner, it may be found to be

$$\omega^3 + i\epsilon\kappa^2\omega^2 - (3\kappa^2 + 1)\omega - i\epsilon\kappa^2 = 0 \qquad (5.11)$$

The expansions of the three roots for $\epsilon \longrightarrow 0$ are

$$\omega_1 = -i\epsilon\frac{\kappa^2}{3\kappa^2 + 1} + O(\epsilon^3)$$

$$\omega_2, \omega_3 = \pm\sqrt{3\kappa^2 + 1} - i\epsilon\frac{3\kappa^4}{2(3\kappa^2 + 1)} + O(\epsilon^2)$$

All the imaginary parts here of the correct signs to produce damping in $e^{i(\kappa x - \omega t)}$.

The introduction of damping has thus produced a new, purely diffusive mode ω_1, and added damping to the propagating modes ω_2 and ω_3. This is similar to what happens when we introduce the Navier-Stokes damping terms into the Euler equations of fluid mechanics.[28] The root locus of (5.11) in the complex ω-plane, as κ increases, is also similar to the Navier-Stokes case, and is shown in Figure 7. The arrows indicate the directions of increasing κ.

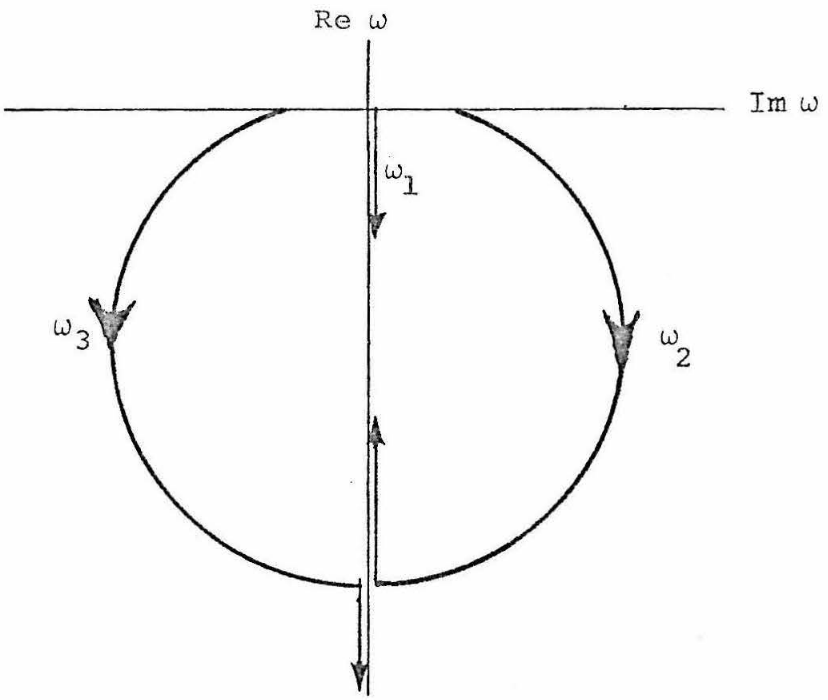

Figure 7. Root locus of dispersion relation (5.11) as κ increases (arrows indicate direction of increasing κ).

The roots ω_2 and ω_3 travel in almost exact semi-circles until they both simultaneously hit the imaginary axis at $K = K_o$. Then ω_2 heads back up the imaginary axis, approaching the limiting value

$$- i\left[\frac{3}{2\epsilon} + \sqrt{\left(\frac{3}{2\epsilon}\right)^2 - 1}\right] \tag{5.12}$$

as $K \longrightarrow \infty$. ω_3 heads down the imaginary axis, having the asymptotic behaviour

$$\omega_3 \sim - i\epsilon K^2$$

as $K \longrightarrow \infty$. ω_1 travels strictly down the imaginary axis, approaching the limiting value

$$- i\left[\frac{3}{2\epsilon} - \sqrt{\left(\frac{3}{2\epsilon}\right)^2 - 1}\right]$$

as $K \longrightarrow \infty$, which is above the limiting value (5.12) for ω_2.

As ϵ increases from small values, the radii of the semi-circles in Figure 7 decrease approximately as $1/\epsilon$. By the time we reach $\epsilon = 3/2$, the semi-circles have shrunk to vanishingly small radii. If we increase ϵ any farther, the whole qualitative picture of Figure 7 is altered. This might be guessed from (5.12), since the square root term becomes imaginary for $\epsilon > 3/2$. We shall not go into the case $\epsilon > 3/2$ here, since our object is to consider moderately small damping.

The behavior of the value κ_o as a function of damping ϵ can be deduced in the limit $\epsilon \longrightarrow 0$ from the discriminant of (5.11). When this discriminant vanishes, $\omega_2 = \omega_3$. Hence we set the discriminant to zero and find κ as a function of ϵ for $\epsilon \longrightarrow 0$. This leads to

$$\kappa_o \sim \frac{\sqrt{12}}{\epsilon}$$

as $\epsilon \longrightarrow 0$, a formula which turns out to be quite accurate for $0 < \epsilon < 3/2$ (as substantiated by numerical calculations).

Propagation can take place (on the branches ω_2, ω_3) only for $0 < \kappa < \kappa_o$. The damping $\text{Im}(\omega)$ increases with κ for $0 < \kappa < \kappa_o$. These facts agree qualitatively with experiment (shorter waves experience a larger damping) but of course no sharp cut-off is observed. The sharp cut-off at $\kappa = \kappa_o$ lies in the nature of the model. Similar cut-offs are observed for sound wave propagation when more detailed kinetic theory models are made (cf. 12, 28).

The general features noted above, in particular, the existence of a cut-off wavenumber, can be expected in any derivative dissipation model. This would be difficult to prove rigorously, but enough examples have accrued to make a pretty strong case for it. The meaningfulness of all such models after cut-off, that is, for $K > K_0$, is extremely dubious. Thus they will suffer from inaccuracy in regions where the solutions vary too rapidly, either spatially or temporally. In particular, derivative dissipation models cannot be expected to be useful for calculating detailed shock structure.

§6. Breaking and Shocks

We shall demonstrate two properties of the LPE's in this section:

(1) there is no possibility of a shock solution which is just a simple jump in level;

(2) certain solutions of the LPE's break.

The second property shows that shock solutions of the LPE's may be possible. The first shows that if shock solutions do exist, they will not be as simple as in gas dynamics.

For the first, we shall show that there is only a single constant-state solution of the LPE's. Putting $\partial/\partial t$ and $\partial/\partial x$ to zero everywhere, we obtain from the momentum equation (4.9) that E=0. This in turn leads, through Poisson's equation (4.2), to n=1, and through the current equation, (4.7), to u=0. Thus if there is a constant state both before and behind the proposed shock front, it must be the same constant state in both places. This means neither n, nor u, nor E experiences a jump across the shock front. Thus, there is no shock front, and shocks of the simple jump discontinuity type do not exist.

For completeness, it should be mentioned that a jump in the pressure is allowed by the equations. It is, however, difficult to imagine a jump discontinuity in pressure propagating through an otherwise quiescent ($n=1$, $u=E=0$) medium.

The reason a step solution does not exist is its inherent instability. If we set it up at time $t=0$, it would decay almost instantly, because the charge imbalance caused by the deviation of n from unity would result in a large restoring electric field, which would always act to pull n back to one. Hence, oscillations are possible, but not extended regions in which $n>1$ or $n<1$ permanently. Plasma oscillations and steady progressing steps are incompatible phenomena.

In spite of the non-existence of steady progressing step shocks, we still suspect the existence of breaking solutions of the LPE's, simply because they are a hyperbolic system. Hyperbolic systems are well-known to have breaking solutions whenever neighboring characteristics belonging to the same family cross[3]. The characteristic form of the LPE's may be found to be

$$(u_t + c_+ u_x) + \frac{1}{\sqrt{3pn}} (p_t + c_+ p_x) + E = 0 \qquad (6.1)$$

$$(u_t+c_-u_x) - \frac{1}{\sqrt{3pn}}(p_t+c_-p_x) + E = 0 \tag{6.2}$$

$$(n_t+un_x) - \frac{n}{3p}(p_t+up_x) = 0 \tag{6.3}$$

$$\left.\begin{array}{c} E_t - nu = 0 \\ \text{or} \\ E_t + uE_x - u = 0 \end{array}\right\} \tag{6.4}$$

where

$$c_\pm = u \pm \sqrt{3p/n}$$

The families of characteristics are thus the "particle paths"

$$\frac{dx}{dt} = u$$

and the sound-wave characteristics

$$\frac{dx}{dt} = u \pm \sqrt{3p/n} \tag{6.5}$$

so-called because $\sqrt{3p/n}$ is the soundspeed for $\gamma = 3$. The characteristic form (6.4) indicates a characteristic velocity of either u or zero. Whichever we pick, it amounts to somewhat of a degeneracy, since there are only 3 non-zero characteristic velocities for 4 equations.

Eq. (6.3) is equivalent to

$$S_t + uS_x = 0$$

which says that S is constant along particle paths, as in gas dynamics.

While we can prove nothing about the general case, as regards crossing of characteristics and breaking, it is possible to demonstrate breaking for a special class of solutions. This class consists of solutions which are initially weakly discontinuous; that is, have initial discontinuities in slope. We study such solutions in the neighborhood of the wavefront

$$\theta(x,t) = 0$$

by use of the following geometrical optics expansions:

$$n = \begin{cases} 1 + n_1(t)\,\theta + n_2(t)\,\frac{\theta^2}{2!} + \ldots & \theta > 0 \\ 1 & \theta < 0 \end{cases}$$

$$u = \begin{cases} u_1(t)\,\theta + u_2(t)\,\frac{\theta^2}{2!} + \ldots & \theta > 0 \\ 0 & \theta < 0 \end{cases}$$

$$p = \begin{cases} 1 + p_1(t)\ \Theta + p_2(t)\ \frac{\Theta^2}{2!} + \dots & \Theta > 0 \\ 1 & \Theta < 0 \end{cases}$$

$$E = \begin{cases} E_2(t)\ \frac{\Theta^2}{2!} + \dots & \Theta > 0 \\ 0 & \Theta < 0 \end{cases}$$

The initial discontinuities in slope are seen from these expansions to be

$$\left.\begin{aligned} [n_x]_{\Theta=0-}^{\Theta=0+} &= n_s = n_1(0)\ \Theta_x\ (x,0) \\ [u_x]_{\Theta=0-}^{\Theta=0+} &= u_s = u_1(0)\ \Theta_x(x,0) \\ [p_x]_{\Theta=0-}^{\Theta=0+} &= p_s = p_1(0)\ \Theta_x(x,0) \end{aligned}\right\} \quad (6.6)$$

Note that the initial discontinuity in E has been assumed to occur in its <u>second</u> derivatiive, not its first. This agrees with Eq.(4.2), which says that $E_{xx} = -n_x$, so that a discontinuity in n_x is matched by a discontinuity in E_{xx}. E_x behaves like n, and so is continuous.

We put the geometrical optics expansions into the LPE's of §4, and group terms according to the power of Θ that they multiply. In this procedure, we assume Θ_t and Θ_x are $O(1)$ as $\Theta \to 0$. This assumption is later validated by our choice of Θ. We then set the coefficient of each power of Θ to zero.

The procedure is illustrated for the continuity equation (4.4). For $\Theta > 0$,

$$0 = n_t + (nu)_x$$

$$= n_1'(t)\Theta + n_1(t)\ \Theta_t + n_2(t)\Theta\Theta_t$$

$$+ u_1(t)\ \Theta_x + [u_2(t)+2n_1(t)u_1(t)]\Theta\ \Theta_x + O(\Theta^2)$$

$$= n_1\ \Theta_t + u_1\ \Theta_x + [n_1'+n_2\Theta_t+(u_2+2n_1u_1)\Theta_x]\Theta + O(\Theta^2)$$

Hence

$$n_1\Theta_t + u_1\Theta_x = 0 \tag{6.7}$$

$$n_1'(t) + n_2(t)\Theta_t + (2n_1u_1 + u_2)\Theta_x = 0 \tag{6.8}$$

$$\vdots$$

Eq. (6.7) is one of the set of three "eikonal" equations for the LPE's (there is no eikonal equation resulting from $E_x = 1 - n$). They may be manipulated to yield

$$p_1(t) = 3n_1(t) \tag{6.9}$$

$$u_1(t) = \pm \sqrt{3}\, n_1(t) \tag{6.10}$$

$$\Theta_t \pm \sqrt{3}\, \Theta_x = 0 \tag{6.11}$$

The $\pm$ sign corresponds to the possibility of the waves traveling either to the right or to the left. We pick the plus sign for definiteness, corresponding to rightward motion. Then the general solution of (6.11) will be

$$\Theta(x,t) = \psi(x - \sqrt{3}\, t)$$

If we assume the wavefront started at x=0 at time t=0, and if we assume Θ can be written

$$\Theta(x,t) = t - W(x)$$

then Θ is uniquely determined as

$$\Theta(x,t) = t - \frac{x}{\sqrt{3}} \tag{6.12}$$

Using (6.12), the three remaining equations of the type (6.8) are

$$u_1'(t) + 2n_1u_1 + u_2 - \frac{2}{\sqrt{3}} u_1^2 - \frac{1}{\sqrt{3}} p_1 = 0 \tag{6.13}$$

$$p_1'(t) - \frac{4}{\sqrt{3}} u_1p_1 - \sqrt{3}\, u_2 + p_2 = 0 \tag{6.14}$$

$$E_2 - \sqrt{3}\, n_1 = 0 \tag{6.15}$$

Multiplying (6.13) by $\sqrt{3}$ and adding it to (6.14), the variables u_2 and p_2 drop out, leaving an equation in n_1, u_1, and p_1 only. Then u_1 and p_1 may be written in terms of n_1 using (6.9) and (6.10). The resultant equation for n_1 is

$$n_1'(t) = 2n_1^2$$

whose solution is

$$n_1(t) = -\frac{1}{2t+c}$$

where c is a constant. c may be evaluated from (6.6) and (6.12) as

$$c = \frac{1}{\sqrt{3}\, n_s} \tag{6.16}$$

It is clear that if c is negative, $n_1(t)$ will become infinite at time

$$t_B = -\frac{c}{2}$$

while if c is positive $n_1(t)$ will decay to zero smoothly as $t \to \infty$. Since $n_1(t)$ is proportional to the jump in derivative n_x across $\Theta = 0$ at time t, this means that n_x becomes infinite at $t = t_B$ when $c < 0$, which is what we call breaking. The solution becomes a shock front at this point, and the LPE's must be supplemented by shock conditions if we wish to continue the solution beyond $t = t_B$. We shall have more to say about shock solutions in §7.C.

In terms of the initial discontinuities, the breaking condition $c < 0$ becomes

$$n_s < 0$$

$$u_s < 0$$

$$p_s < 0$$

from (6.6) and (6.9-10). Thus when

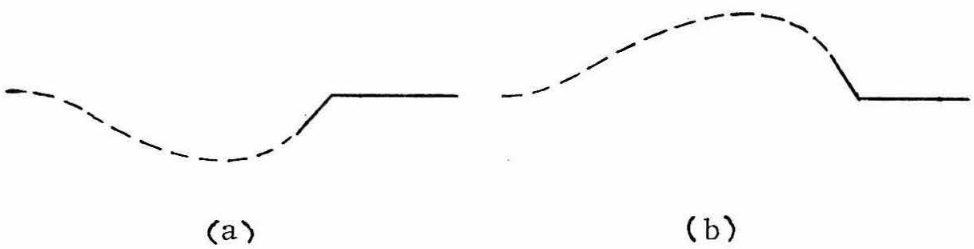

Figure 8. Form of breaking and non-breaking disturbances in the LPE's.

n,u, and p have initial disturbances of the form of Figure 8(a), the rightward-propagating disturbance will break at time $t = t_B$. When they have initial disturbances of the form of Figure 8(b), the break in derivative will smooth out like 1/t as it propagates rightward.

It might be remarked that the above results do not depend in any essential way on E. Thus the breaking is purely a fluid mechanical phenomenon. The form of shocks, etc., will of course depend on E.

§7. Uniform Wavetrain Solutions

In this section we shall study the steady progressing wave or uniform wavetrain solution of the LPE's (see Part II for definitions). In sub-section A, we shall formulate the LPE's in a way which facilitates the study of uniform wavetrain solutions. Then we shall explore the solutions themselves, noting in particular the limiting cases of the solitary wave and the peaked wave. In sub-section B necessary and sufficient conditions for the existence of such waves will be formulated, using the method of Sturm sequences. And finally, in sub-section C, we shall show the impossibility of joining a steady wave solution across a jump discontinuity to a constant state, a rather unusual type of shock suggested by the analogous solution in the Korteweg-de Vries problem (cf. §2).

A. Equations for the uniform wavetrain

It is desirable, when finding steady-wave solutions of a system of partial differential equations, to express as many of the equations in conservation form

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (7.1)$$

as possible. For when all variables are functions of $\theta = \kappa x - \omega t$, this becomes

$$-\omega \frac{d}{d\theta} P + K \frac{d}{d\theta} Q = 0$$

and the integration is immediate:

$$KQ - \omega P = \text{const.} \tag{7.2}$$

Added motivation for seeking equations in conservation form comes from the fact that they are the most natural framework within which to study slowly-varying wavetrains, as we shall do in the next section. Also, the constants occurring in the integrated forms (7.2) turn out to be useful parameters for describing the steady-wave solution.

We have already obtained three of the LPE's in conservation form. These were the continuity equation (4.4), the energy equation (4.8), and the entropy equation (4.13). Hence these equations integrate immediately to

$$Knu - \omega n = \text{const.} \tag{7.3}$$

$$K(nu^3 + 3pu) - \omega(nu^2 + p + E^2) = \text{const.} \tag{7.4}$$

$$KnuS - \omega nS = \text{const.} \tag{7.5}$$

Using (7.3) in (7.5), it follows that

$$S = \text{const.}$$

We will therefore take S to be one of the parameters of the problem.

For a steady wave, the current (4.7) and Poisson (4.2) equations become

$$-\omega \frac{dE}{d\theta} = nu \qquad (7.6)$$

$$\kappa \frac{dE}{d\theta} = 1-n \qquad (7.7)$$

Eliminating $dE/d\theta$ between these, we obtain

$$\kappa nu - \omega n = -\omega$$

Thus the constant in (7.3) is $-\omega$, and so we have finally

$$n(u - U) = - U \qquad (7.8)$$

where $U = \omega/\kappa$.

Before passing on, we note that the physical restriction of a positive number density, $n > 0$, requires that

$$u < U \ .$$

This follows from (7.8) when we, without loss of generality, assume $U > 0$. Since the pressure p is likewise positive, (4.11) gives the restriction $S > 0$.

We remarked in §4 that the momentum equation could not be put in conservation form. It proves possible, nevertheless, to obtain a "pseudo-conservation" form which, for our present purposes, has the same properties as a conservation form. Beginning from Eq. (4.5), we replace n in the last term by $(1 - E_x)$ to obtain

$$(nu)_t + (nu^2 + Sn^3 - 1/2\ E^2)_x + E = 0 \tag{7.9}$$

Combining Eqs. (4.9) and (4.11), we obtain

$$u_t + (1/2\ u^2)_x + 1/n(Sn^3)_x + E = 0$$

Subtracting the last two equations,

$$\{(n-1)u\}_t + \{(nu^2 - 1/2\ u^2 + Sn^3 - 1/2\ E^2\}_x - 1/n\ (Sn^3)_x = 0$$

It would be desirable to have the last term in this equation merely involve S_x, since we know S is constant for the steady wave. And indeed, this can be achieved by absorbing part of the term under the x-derivative:

$$\{(n-1)u\}_t + \{nu^2 - 1/2\ u^2 + Sn^3 - 1/2\ E^2 - 3/2\ Sn^2\}_x + 1/2\ n^2 S_x = 0 \tag{7.10}$$

Now, for the steady wave, this equation is effectively in conservation form (this is what we meant by "pseudo-conservation form"). It integrates immediately to

$$-\omega(n-1)u + \kappa(nu^2 - 1/2\ u^2 + Sn^3 - 1/2\ E^2 - 3/2\ Sn^2) = -\kappa A \tag{7.11}$$

where A is a constant.

By substituting for u in terms of n from Eq. (7.8), we may solve for E^2 in Eq. (7.11) entirely in terms of n,

$$E^2 = 2A + S(2n^3 - 3n^2) - U^2 \frac{(n-1)^2}{n^2} \tag{7.12}$$

Reversing the process, we may obtain E^2 entirely in terms of u,

$$E^2 = 2A - u^2 - \frac{2SU^3}{(u-U)^3} - \frac{3SU^2}{(u-U)^2} \tag{7.13}$$

With E given by either of the above formulas, Poisson's equation

$$\kappa E'(\theta) = 1-n = \frac{u}{u-U} \tag{7.14}$$

leads to a single differential equation in either n or u. In terms of u, it is

$$\kappa u_\theta = \frac{-(U-u)^3}{3SU^2 - (U-u)^4} \sqrt{2A - u^2 + \frac{2SU^3}{(U-u)^3} - \frac{3SU^2}{(U-u)^2}}$$

$$= \frac{-(U-u)^{3/2}}{3SU^2 - (U-u)^4} \sqrt{Q(u)} \equiv R(u) \tag{7.15}$$

where Q(u) is a fifth-degree polynomial in <u>u</u> which we shall use in the various forms

$$Q(u) = (U-u)^3 (2A - u^2) - S(U^3 - 3U^2u) \tag{7.16}$$

$$= (U-u)^3 (a^2 - u^2) - S(u^3 - 3Uu^2) \tag{7.17}$$

$$= - w^5 + 2Uw^4 + (2A - U^2)w^3 - 3SU^2w + 2SU^3 \tag{7.18}$$

and where

$$w = U - u > 0$$

$$a^2 = 2A - S > 0$$

We identify a as a measure of the amplitude of the oscillatory solutions to be expected from Eq. (7.15). This is an extrapolation from the linearized theory, where we find a to be proportional to the amplitude.

The linearized theory is obtained by first noting that u = 0 is the equilibrium point. We expand the expression under the radical in the first form of Eq. (7.15) to $O(u^2)$ as $u \rightarrow 0$, and approximate the expression in front of the radical by its value at u = 0:

$$\kappa u_\theta = \frac{U}{U^2 - 3S} \sqrt{(2A - S) - \frac{U^2 - 3S}{U^2} u^2} \qquad (7.19)$$

If $2A-S = a^2 > 0$ and $U^2 > 3S$, the solutions of this equation are seen to be sines and cosines with amplitudes proportional to a. This is what one should obtain, as may be verified by a direct linearization of the lukewarm plasma equations. No other senses of these inequalities will lead to the correct linearized solution and assuming that the non-linear solution develops continuously from the linearized one by increasing the quantity a, we deduce that these inequalities will hold in general.

Since $\sqrt{3Sn^2}$ is the sound-speed for this problem, $U^2 > 3S$ is the familiar statement that the wavespeed is larger than the linearized sound-speed.

The statement was made above that the solutions of (7.15) will be oscillatory. This is because the slope, which equals $R(u)$, vanishes at certain well-defined values of $\underline{u}$; namely, the real roots of $Q(u)$ ($u=U$ is not a possibility on the physical grounds that $\underline{n}$ is infinite there).

Approximating the differential equation in the neighborhood of a simple root u_o of $Q(u)$, $u_\theta \propto \sqrt{u_o - u}$, we see that the local behavior of the solution is parabolic tangency to u_o, as indicated in Figure 9.

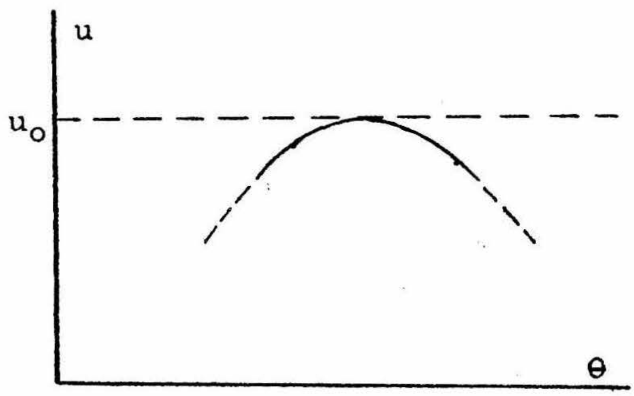

Figure 9. Behavior of the steady wave solution near a simple root u_o of $Q(u)$.

Hence the solution is "turned around" whenever it hits a root of $Q(u)$, and so must travel back and forth between two adjacent roots of $Q(u)$ endlessly. The only restrictions are that $u < U$ everywhere and that $Q(u)$ be positive between the two adjacent roots of interest (this since we are taking its square root in the differential equation).

These last two restrictions are sufficient to eliminate all but one of the possible solutions of (7.15). To show this, consider the following deductions from Descartes' Rule of Signs applied to the forms (7.16) and (7.18) of Q:

1. Q has 3 or 1 real roots less than U.
2. Q has one real root less than zero.

In case Q(u) has only one root less than U, the solution would approach $u = U$ as $\theta \rightarrow \infty$, which we disallow. In case Q has 3 roots less than U, 2 of them will be positive because of statement (2). We designate the 3 roots as u_1, u_2, u_3, and assume the following ordering:

$$u_1 < 0 < u_2 < u_3 < U$$

(the case $u_2 = u_3$ represents a peaked wave, which is treated in Appendix B). Then since

$$Q(0) = U^3 a^2 > 0$$

we have

$$Q(u) > 0 \quad \text{for} \quad u_1 < u < u_2$$

The only possible solution is thus that oscillating between u_1 and u_2, for $Q(u) < 0$ in $u < u_1$ and in $u_2 < u < u_3$, so that these regions are disbarred. Any solution in $u > u_3$ must eventually approach $u = U$, which we again disallow.

The above analysis has tacitly assumed that the relevant root of the denominator of R(u) in Eq. (7.15),

$$u_\infty = U - (3SU^2)^{1/4}$$

does not lie in the range of oscillation $[u_1, u_2]$. Since

$$u_\infty > 0$$

because $U^2 > 3S$, u_∞ cannot coincide with u_1; let us assume that it also does not coincide with u_2, but lies in $(0, u_2)$. Obviously, $u_\theta = \infty$ for $u = u_\infty$. Let us inquire how u_θ approaches infinity as $u \to u_\infty$. Near u_∞ Eq. (7.15) becomes

$$\kappa u_\theta \approx - \frac{\sqrt{Q(u_\infty)}}{4(3SU^2)^{3/8}} \left[\frac{1}{u-u_\infty}\right]$$

which integrates to

$$u-u_\infty \approx \left\{ \frac{Q^{1/2}(u_\infty)}{2\kappa(3SU^2)^{3/8}} \right\}^{1/2} \sqrt{\theta_o - \theta}$$

We note that this sol$^{\underline{n}}$ may not continue past $\theta = \theta_o$ (θ_o an arbitrary constant of integration). Since it may not just "stop dead" in its tracks, its only recourse is to become multivalued as in Figure 10, which is allowed because $\sqrt{\theta_o - \theta}$ may have either sign. But multi-valued-ness must be rejected on physical grounds. Hence u_∞ must not penetrate the region $(0, u_2)$.

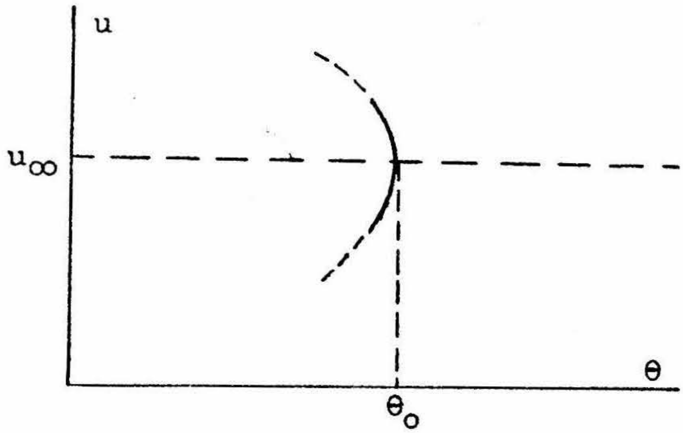

Figure 10. Behavior of a steady wave solution which crosses $u = u_\infty$.

Up to this point we have left open the question of whether $Q(u)$ does indeed have three real roots $u < U$. In sub-section B, a necessary and sufficient condition for this assumption to be true is given. It is

$$\frac{a^2}{U^2} = \frac{2A - S}{U^2} < \frac{1}{3}(1-\beta)^3\ (3+\beta) \tag{7.20}$$

where $\beta = (3S/U^2)^{1/4}$. This condition, as a byproduct, helps us to locate the position of u_∞ relative to u_2 and u_3. For, from Eqs. (7.17) and (7.20), we find

$$\begin{aligned} Q(u_\infty) &= Q(U - \beta U) \\ &= (\beta U)^3 \left\{a^2 - U^2(1-\beta)^2\right\} + SU^3(1-\beta)^2(2+\beta) \\ &= (\beta U)^3 \left[a^2 - \frac{U^2}{3}(1-\beta)^3(3+\beta)\right] < 0 \end{aligned}$$

Since $u_\infty > 0$, the only way $Q(u_\infty)$ can be negative is if

$$u_2 < u_\infty < u_3$$

so that, when an oscillatory solution exists, the problems connected with $u_\infty \in (0, u_2)$ never arise.

Numerous additional inequalities concerning the roots of Q(u) can be found, requiring varying levels of ingenuity. We mention only two simple ones which will be of use in sub-section B:

$$\left.\begin{aligned} a &< |u_1| \\ a &< u_2 \end{aligned}\right\} \tag{7.21}$$

They are obtained by noting, from the form (7.17) of Q, that $Q > 0$ for $-a \leq u \leq a$. It follows that the roots u_1, u_2 must lie outside of $[-a, a]$.

We have said nothing of the special cases $u_2 = u_3$ and $u_2 = u_\infty$. These are the cases of the solitary wave (wave of infinite wavelength, with only one crest or one trough) and the cusped peaked wave, respectively, as may be verified by an analysis similar to that leading to Figures 9 and 10. These cases are treated in Appendices A and B. The conclusion reached there is that neither case may occur, independently, so that the coalescence of any two of u_2, u_3, u_∞ entails the coalescence of all three. The type of wave produced when this happens is as illustrated in Figure 11, that is, a peaked wave with

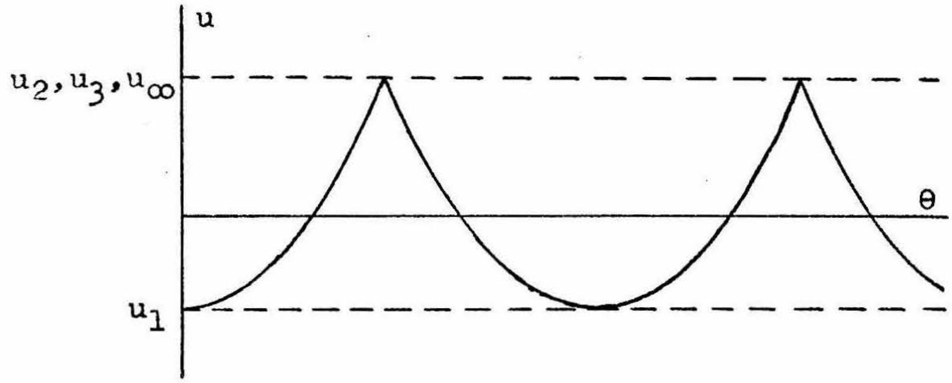

Figure 11. The steady wave solution of maximum amplitude.

a finite peak angle. This peaked wave is also the solution of maximum amplitude. For, from Appendix B, the parameters of the peaked wave solution are constrained by

$$\frac{a^2}{U^2} = \frac{1}{3}(1-\beta)^3(3+\beta)$$

and taking note of inequality (7.20), it follows immediately that for fixed β and U, the peaked wave is the solution of maximum <u>a</u>, that is, of maximum amplitude.

A picture of the development of the non-linear solution from the linearized one may now be formed. We consider the entropy S and speed U of the wave to be fixed, and observe the movement of u_1, u_2, and u_3 as <u>a</u> increases from small values (where $u_2 \approx - u_1 \approx a$). The results are indicated schematically in Figure 12. They are borne out by the numerical experiments of Appendix C.

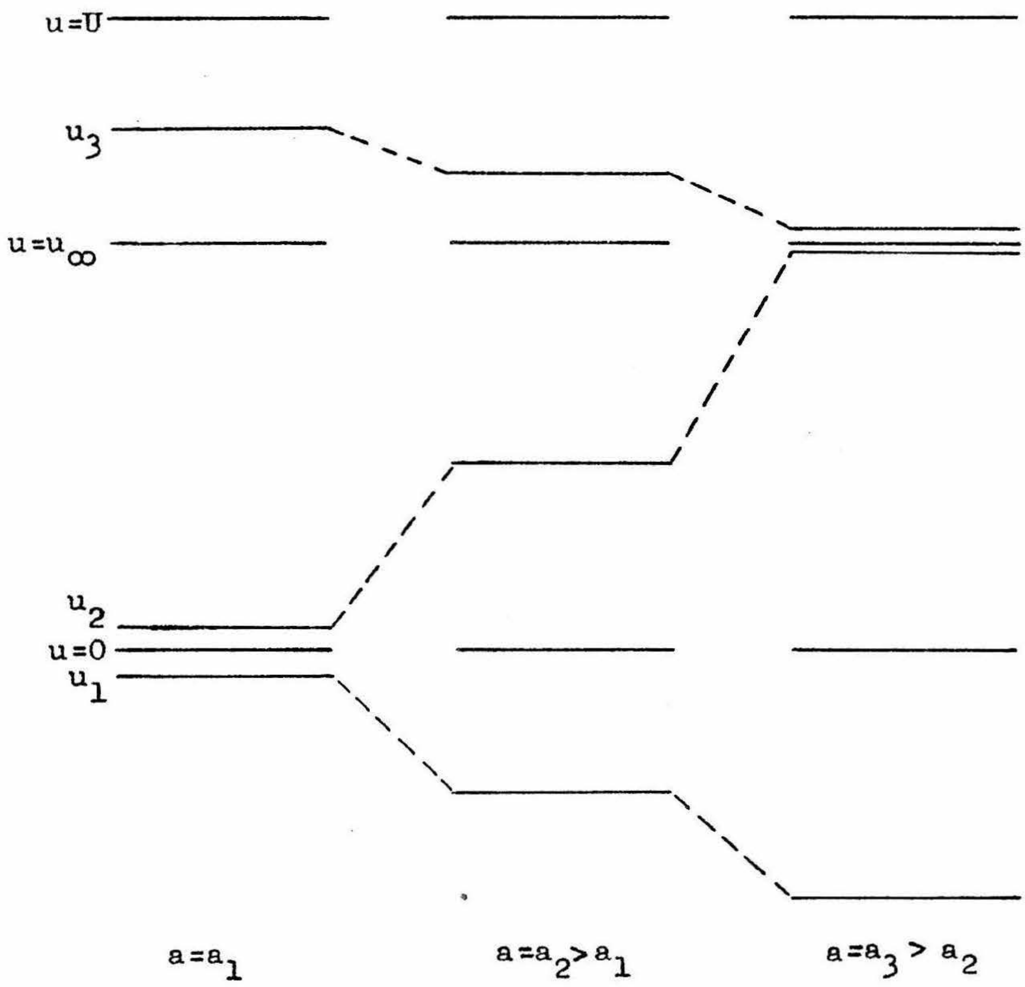

Figure 12. Movement of the roots u_1, u_2, u_3 of $Q(u)$ as the amplitude $\underline{a}$ increases from small values.

We close this sub-section with a brief derivation of the average values of n, nu, and E for a steady wave. To begin, we note from Eq. (7.14) that when u is oscillatory, so also is the electric field E. Hence, when we integrate Poisson's equation over one wavelength λ of the steady wave

$$\int_0^\lambda \kappa E'(\theta)d\theta = \int_0^\lambda \left[1-n(\theta)\right]d\theta$$

the left-hand side vanishes by periodicity, and we are left with

$$1/\lambda \int_0^\lambda n(\theta)d\theta = 1 \qquad (7.22)$$

This says that the average value of n over one cycle is unity. A similar procedure applied to the current equation, (7.5), yields

$$\int_0^\lambda nu\ d\theta = 0 \qquad (7.23)$$

which says that there is no mean mass flow in the wave.

To obtain the average of E, we go back to the momentum equation in the form (4.9),

$$n(u_t + uu_x) + p_x + nE = 0$$

For steady waves, this becomes

$$(\kappa p - \omega u)_\theta + nE = 0$$

Integrating this over one cycle gives

$$\int_0^\lambda nE\ d\theta = 0 \qquad (7.24)$$

Replacing n via Poisson's equation, this reduces to

$$\int_{0}^{\lambda} E \, d\theta = 0 \qquad (7.25)$$

which says that the average value of E over one cycle is zero.

B. Necessary and sufficient conditions for the existence of a steady wave

In sub-section A, we assumed that Q(u) had exactly two roots, u_2 and u_3, on (0, U). Upon this assumption hinged the existence of a steady wave. Now we shall develop a necessary and sufficient condition for this to be true.

We shall employ for this purpose the method of Sturm sequences, which is a device for finding the number of zeroes of a polynomial on the interval (a, b) of the real axis. For a typical n^{th} degree polynomial P(x), the Sturm sequence assumes the form[21]

$$\left\{P(x),\ P'(x),\ P_3(x),\ \ldots,\ P_n(x),\ P_{n+1}\right\}$$

where P_{n+1} is a constant. Then if $V(x_o)$ denotes the number of sign variations in the Sturm sequence when it is evaluated at $x = x_o$, we have the result[21]

No. of roots of P(x) on (a,b) = V(a) - V(b).

This method is often referred to in numerical analysis literature as a way to obtain starting iterates for rapidly-converging polynomial root-finders. It can also be a powerful analytical tool, however, whenever one is concerned with finding conditions for a parameter-dependent algebraic equation $f(x,a_1,\ldots,a_k) = 0$ to have a specified number of roots on a specified interval of the real x-axis. (The extension of the method to algebraic equations is possible because of the possibility of obtaining excellent low-order polynomial approximations to most transcendental functions.) It makes possible analytical, as opposed to trial-and-error numerical, parameter studies.

The construction of the sequence proceeds somewhat along the lines of the Euclidean algorithm[13] for finding the greatest common divisor of two integers. Taking the first two members of the sequence to be $P_1(x) = P(x)$, $P_2(x) = P'(x)$, the rest are given by[21]

$$\frac{P_k(x)}{P_{k+1}(x)} = q_k(x) - \frac{P_{k+2}(x)}{P_{k+1}(x)} \quad (k = 1,2,\ldots,n-1) \quad (7.26)$$

where $q_k(x)$ is the quotient and P_{k+2} is the remainder. In other words, $P_{k+2}(x)$ is the negative of the remainder obtained upon dividing $P_k(x)$ by $P_{k+1}(x)$, and is a polynomial of lower degree than $P_{k+1}(x)$. Sturm sequences are of course not unique (e.g. any member may be multiplied by a positive constant) but the sequence generated by Eq. (7.26) proves convenient, and has the virtue of relative simplicity.

In case the iteration process (7.26) terminates, in the sense that $P_{m-1}(x) \not\equiv 0$ and $P_m(x) \equiv 0$ for $m \leq n+1$, then one or more multiple roots of P(x) are indicated. In fact, if the last j members of a Sturm sequence vanish identically, then we can say that P(x) will have N multiple roots of multiplicities m_i $(i=1,\ldots,N)$ such that

$$\sum_{i=1}^{N} (m_i - 1) = j$$

Thus, for a given j, there are as many different possibilities for the set $\{m_i\}$ as there are combinations of non-zero integers which sum to j. For example, for j=2, the possibilities are one triple root ($N=1$, $m_1=3$) or two double roots ($N=2$, $m_1 = m_2 = 2$) of P(x). For j=1, there is no ambivalence, however, and the statement there is: P(x) has one double root and no other multiple roots if and only if

$$P_{n+1} = 0 \quad \text{and} \quad P_n(x) \not\equiv 0$$

The above result might seem to have some merit in the study of the solitary wave, where Q(u) has a double root on (0,U). The situation is actually the reverse in practice, however; a direct investigation of the double-root case (Appendix A) without benefit of Sturm sequences leads to a condition much simpler than $P_{n+1} = 0$. This simple condition can then be used to go back and factor P_{n+1} in a non-obvious way. This brings us to an unfortunate fact of life for the method of Sturm sequences, at least in the present case, but probably in general: it produces overly complicated expressions, which require considerable skill and ingenuity to simplify. The advantage of the method is that it unfailingly yields necessary and sufficient conditions in various questions of root existence, complicated as those conditions may be.

For the present case, the method was useful because it provided, after some lengthy manipulations, a single relatively simple necessary and sufficient condition for Q(u) to have two zeroes between u = 0 and u = U. That condition is derived in Appendix D, and is

$$\alpha < 2\beta^2 - 8/3 \;\; \beta + 1 \tag{7.27}$$

where

$$\alpha = \frac{2A}{U^2} \qquad\qquad \beta = \left(\frac{3S}{U^2}\right)^{1/4}$$

It may be transformed into an inequality on the amplitude

$\underline{a}$

$$\frac{a^2}{U^2} = \alpha - \frac{1}{3}\beta^4$$

$$< -\frac{1}{3}\beta^4 + 2\beta^2 - \frac{8}{3}\beta + 1 = \frac{1}{3}(1-\beta)^3(3+\beta)$$

which is Eq. (7.20) of sub-section A. The assumptions used in deriving (7.27) were that $u < U$ and that

$$S > 0 \qquad (7.28)$$

$$U^2 > 3S \qquad (7.29)$$

$$\alpha - \frac{1}{3}\beta^4 > 0 \qquad (7.30)$$

which were all deduced in sub-section A. The last is merely a restatement of $a^2 > 0$ in terms of α and β.

The inequalities (7.27) through (7.30), taken together, delimit a region in the $\alpha\beta$-plane, which is shown hatched in Figure 13. Because of (7.28) and (7.29) it is only necessary to consider $0 < \beta < 1$, and (7.27) and (7.30) then give the upper and lower bounding curves of the region, respectively. It is only for α and β within the region so delineated that a steady wave may exist.

It is interesting to note from Figure 13 that, as $\beta \rightarrow 1$, both the range of possible amplitudes and the possible amplitudes themselves become exceedingly small. Since $\beta \rightarrow 1$ corresponds to $U \rightarrow \sqrt{3S}$, this has the physical

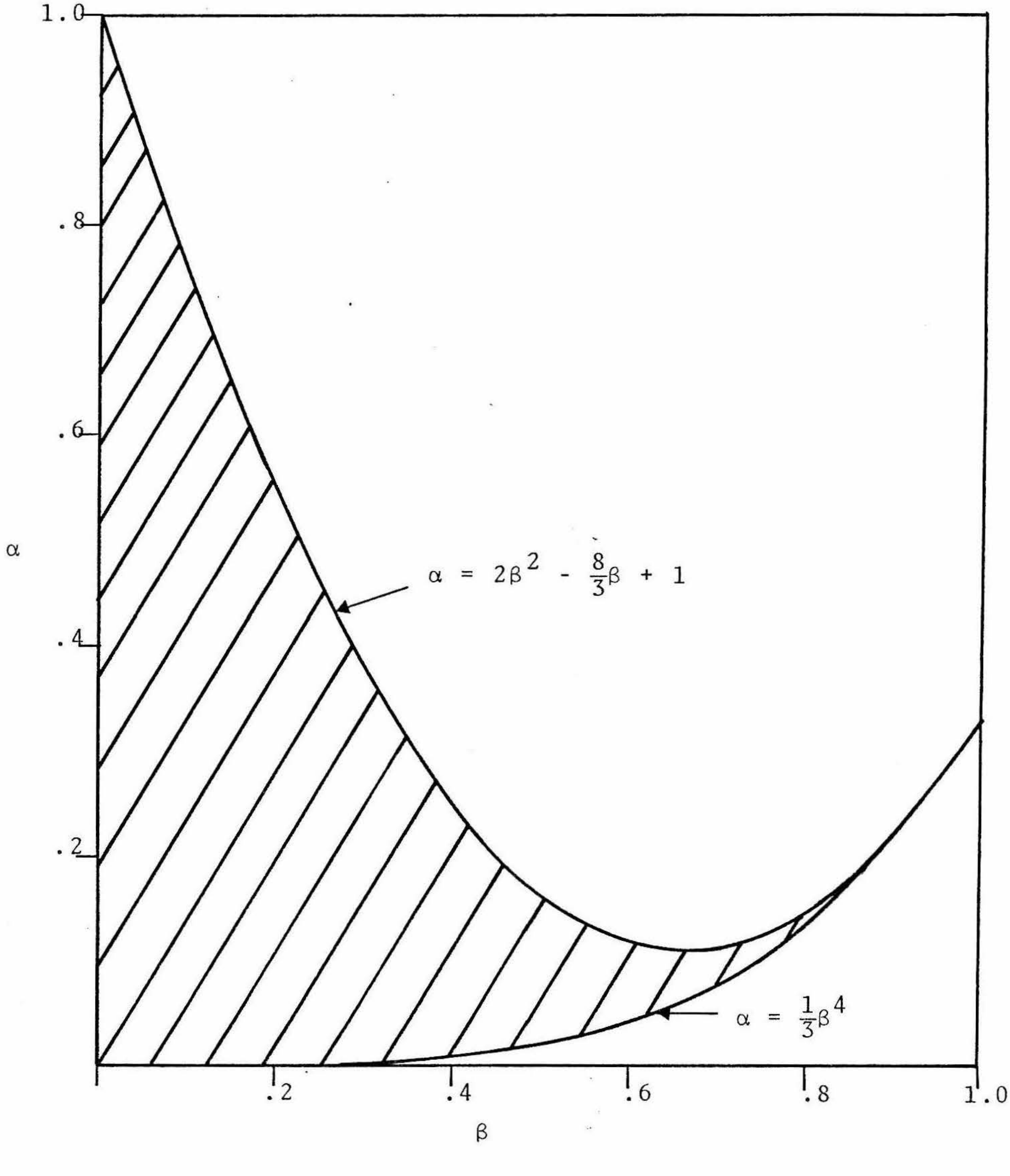

Figure 13. The region defined by inequalities (7.27) through (7.30). A steady wave solution is only possible for α and β within this region.

interpretation that waves travelling near the soundspeed must have very small amplitudes, as we would expect in the sound wave limit. Even for $U = 2\sqrt{3S}$, the maximum amplitude (actually amplitude-velocity ratio, a/U) is only about 0.17. At the other extreme, as $\beta \to 0$ and hence $U \to \infty$, the maximum possible amplitude

$$(a/U)_{max} \longrightarrow 1 \quad \text{as } \beta \to 0,$$

and we have

$$a_{max} = O(U)$$

as $U \to \infty$. Of course, in reality, excessively high amplitudes and velocities would not occur because the assumptions used in deriving the LPE's would break down. In particular, dissipative mechanisms, represented by the neglected term q, would become active because of large temperature gradients; and furthermore, because of the large associated electric fields, it would no longer be reasonable to assume the ions to be immobile. Thus the region shown in Figure 13 should be even further delimited by model breakdown, but we have not attempted to estimate this effect.

C. A Shock-like solution compounded of a steady wave and a constant state

Earlier, in §6, certain solutions of the LPE's were shown to break, namely, those with an initial discontinuity in slope of the proper sign. At the same time, the impossibility of a steady shock joining two constant states was demonstrated. It is of course possible that breaking solutions of the LPE's produce only unsteady shocks; but before reaching this conclusion we shall examine the one other possible type of steady shock, that in which a steady wave solution is joined onto a constant state.

We receive some motivation in this direction from a possible steady solution of the Korteweg-de Vries equation with damping (Eq. (III.2)). This solution is shown in Figure 2(§2), and is a true bore joining two different water levels. The solution only exists for a limited range of the damping coefficient ν (for fixed velocity U and water depth h_o). What is of interest for us is that as $\nu \rightarrow 0$ the solution becomes a steady wave at the upper level joined across a jump discontinuity to a constant state at the lower level. It is for such solutions that we shall search in the LPE case.

The type of solution we are seeking is shown schematically in Figure 14. It is the only remaining possibility for a steady shock, since the only steady solutions of the LPE's are the constant state

$$n_o = 1 \qquad u_o = 0 \qquad S=S_o \qquad E_o=0$$

and the steady wave of sub-section A. The change of level will not be as pronounced as in the Korteweg-de Vries case, as we have tried to indicate in Figure 14. This is because, taking u as a typical example, the steady wave solution oscillates between levels u_2 and u_1 above and below the constant state value of u=0. The oscillation is not symmetric about u=0, but its mean level does not differ much from u=0 either.

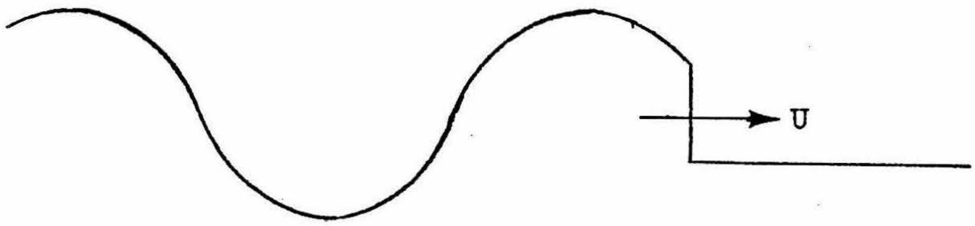

Figure 14. Steady wave joined to a constant state

For solutions such as shown in Figure 14 to exist, it is necessary that mass, momentum, and energy be conserved across the shock front. It would also be reassuring if entropy increased across the shock front. We shall therefore formulate these conditions for the proposed solution, beginning from the LPE's in the form

$$n_t + (nu)_x = 0 \tag{7.31}$$

$$(nu)_t + (nu^2 + Sn^3)_x + nE = 0 \tag{7.32}$$

$$(nu^2 + Sn^3 + E^2)_t + (nu^3 + 3Sn^3u)_x = 0 \tag{7.33}$$

$$E_x = 1-n \tag{7.34}$$

and then see if all these conditions can be satisfied simultaneously.

There is a standard body of theory concerning the derivation of shock conditions from equations in conservation form (see, for example, Ref. 3). For an equation

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

there is a corresponding shock condition

$$[Q] = U[P]$$

connecting the discontinuities $[P]$ and $[Q]$ across the shock with the shock velocity, U. For Eqs. (7.31) and (7.33) these formulas become

$$[nu] = U[n]$$

$$[nu^2 + 3Sn^3u] = U[nu^2 + Sn^3]$$

or, using the subscript 's' to refer to quantities on the steady-wave side of the shock front,

$$n_s u_s - n_o u_o = U(n_s - n_o)$$

$$n_s u_s = U(n_s - 1) \qquad (7.35)$$

and

$$n_s u_s^3 + 3S_s n_s^3 u_s = U(n_s u_s^2 + S_s n_s^3 + E_s^2 - S_o) \qquad (7.36)$$

These account for the conservation of mass and energy, respectively.

The condition (7.35) is really no restriction at all on the location of the shock front, for it is identical to Eq. (7.8) and thus holds everywhere in the steady wave.

From relation (7.11) for the steady wave, we obtain

$$U(n_s-1)u_s - n_s u_s^2 + \frac{1}{2} u_s^2 - S_s n_s^3 + \frac{1}{2} \quad E_s^2 + \frac{3}{2} S_s n_s^2 = A$$

The first two terms cancel by (7.35), leaving

$$u_s^2 - 2S_s n_s^3 + 3S_s n_s^2 + E_s^2 = 2A \tag{7.37}$$

From Eq. (7.36), again using Eq. (7.35),

$$u_s^2 + S_s n_s^3 (1 - \frac{3u_s}{U}) + E_s^2 = S_o$$

Substracting the last two equations,

$$2A - S_o = S_s n_s^2 (-2n_s + 3 - n_s + \frac{3}{U} n_s u_s)$$

$$= \frac{3}{U} S_s n_s^2 \left[n_s u_s + U(1-n_s) \right]$$

Using Eq. (7.35) once more, this reduces to

$$2A - S_o = 0$$

But

$$a^2 = 2A - S_s > 0$$

from sub-section A, so that

$$S_o > S_s$$

This says that the entropy must decrease across the shock front, a result which is contrary to thermodynamics. Hence we reject shock solutions of the proposed form.

To add to the weight of the argument against steady shocks, we shall show that it is not possible to obtain a smooth shock solution by adding the dissipation (4.24) to the LPE's. The proof will be patterned after the discussion of §2; we shall obtain a system of ordinary differential equations and then show that it has but one singular point.

With the dissipation term (4.24) for q in the LPE's, we look for solutions which are functions of $\xi = x - Ut$. The integrated form of the continuity equation, (7.8), is unaltered:

$$n(u-U) = -U$$

as are Poisson's equation

$$E' = 1-n$$

and the momentum equation (4.5)

$$(-Unu + nu^2 + p)' + nE = 0$$

The integrated form of the energy equation (4.8) contains the contribution from q:

$$-U(nu^2+p+E^2) + nu^3 + 3pu - \beta(p/n)' = -B$$

where B is a constant.

By defining

$$Q = p/n$$

it is possible to reduce the preceding four equations to a system of three ordinary differential equations:

$$\left.\begin{aligned} \beta Q' &= F(Q,u,E) \\ E' &= \frac{u}{u-U} \\ u' &= \frac{u-U}{Q-(u-U)^2}\left[E + \frac{1}{\beta}F(Q,u,E)\right] \end{aligned}\right\} \qquad (7.38)$$

where

$$F(Q,u,E) = B-U(u^2+E^2)-UQ\,\frac{3u-U}{u-U}$$

The singular points of (7.38) are where the three right-hand sides vanish simultaneously. One may quickly find that

$$u = 0, \qquad E = 0, \qquad Q = B/U$$

is the only singular point. Since a shock solution must have two singular points, one to start from and one to arrive at, the system (7.38) has no shock solutions.

The non-existence of steady shocks indicates that any shock solutions of the LPE's must be unsteady. We found evidence of breaking in §6, so unsteady shocks are a distinct possibility. There might even be an unsteady shock of the type shown in Figure 14, only with the wavetrain behind the shock moving relative to the shock front, carrying away momentum and energy to preserve the shock conditions. We shall not pursue such questions here, however.

§8. Slowly-Varying Wavetrain Solutions; The Averaged Equations

The last section was devoted to the uniform wavetrain solution. The present section will consider the slowly-varying wavetrain, using the two-timing method of Part II. We shall derive the averaged equations in sub-section A. Then we shall show how an arbitrary but small dissipation can be taken into account in the averaged equations, in sub-section B. Finally, in sub-section C, we shall put the averaged equations into characteristic form and study their type (elliptic, hyperbolic, parabolic). This will lead naturally into a discussion of non-linear stability in §9.

There are numerous reasons for considering slowly-varying wavetrain solutions. Foremost of these is the likelihood that, for a suitably restricted class of initial distrubances, the solutions of the LPE's develop asymptotically (as $x,t \rightarrow \infty$) into slowly varying wavetrains. This is a reasonable extrapolation from the result in linearized LPE theory (and indeed in any linear dispersive wave theory) that <u>all</u> localized initial disturbances disperse into slowly varying wavetrains as $x,t \rightarrow \infty$, x/t fixed (proved by the method of stationary phase). The entrance of non-linear distortion brings with it the possibility of breaking into a shock, however. Thus our restricted set of initial disturbances would be those for which the dispersion would dominate the non-linear distortion. We have already seen an example of the trade-off of these two effects in our study, (§6), of solutions of the LPE's with an initial discontinuity in slope. For one sign of this discontinuity, non-linear distortion dominated and the solution steepened and broke; for the other sign, the solution smoothed out with time. The latter solutions are of the type that develop into slowly-varying wavetrains.

Another important application of the slowly-varying wavetrain theory is in deducing *stability criteria* for the uniform wavetrain. We shall have more to say on this matter in §9.

A. Deduction of the averaged equations

Proceeding via the Luke expansion or two-timing method of Part II, let us expand each and every variable in the lukewarm plasma equations according to the following pattern:

$$S = S_o(\theta,X,T) + \epsilon S_1(\theta,X,T) + \ldots \tag{8.1}$$

The zero-order quantities in each expansion will be the uniform wavetrain solutions of §7, only now the various parameters A, U, etc. will depend on $X = \epsilon X$ and $T = \epsilon t$.

We shall apply the Luke expansions (8.1) to the LPE's in the form:

$$\frac{\partial}{\partial t}\Big[(1-n)u\Big] + \frac{\partial}{\partial x}\Big[nu^2 - \frac{1}{2}\, u^2 + Sn^3 - \frac{3}{2}\, Sn^2 - \frac{1}{2}\, E^2\Big] = -\frac{1}{2}\, n^2\, \frac{\partial S}{\partial x} \qquad (8.2)$$

$$\frac{\partial}{\partial t}(nu^2 + Sn^3 + E^2) + \frac{\partial}{\partial x}(nu^3 + 3Sn^3u) = 0 \qquad (8.3)$$

$$\frac{\partial}{\partial t}(nS) + \frac{\partial}{\partial x}(nuS) = 0 \qquad (8.4)$$

$$E_x = 1-n \qquad (8.5)$$

Equation (8.2) is the modified form of the momentum equation derived in the previous chapter. It is useful because its right-hand side is $O(\epsilon)$, so that in $O(1)$ it is a conservation equation. Equation (8.3) expresses the conservation of energy; it may be derived directly from the Boltzmann equation (§4). The last two equations are the entropy and Poisson equations, respectively, just as they were used in the previous chapter.

The absence of the continuity equation from this set may be noted. This is because its averaged form

$$\frac{\partial}{\partial t}\int_0^{2\pi} n\, d\theta + \frac{\partial}{\partial x}\int_0^{2\pi} nu\, d\theta = 0$$

becomes

$$\frac{\partial}{\partial t}(2\pi) + \frac{\partial}{\partial x}(0) = 0$$

or simply a tautology, using the results (7.22) and (7.23) of the last chapter.

Any choice for the lukewarm equations would lead, with patience, to the results we shall obtain here. In these types of problems, however, a wise initial choice of equations can often save one a good deal of calculation. This is partly a matter of experience, but there are certain guidelines that one may follow, most important of which is that equations in conservation form are desirable. That is why as many of the LPE's as possible have been written in this form (including the unnatural pseudo-conservation form (8.2) of the momentum equation).

For equations in conservation form

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

we showed in Part II that the condition that the solution be bounded in first order was

$$\frac{\partial}{\partial T} \int_0^{2\pi} P_o \, d\theta + \frac{\partial}{\partial X} \int_0^{2\pi} Q_o \, d\theta = 0 \qquad (8.6)$$

where P_o and Q_o are the zero-order terms in the expansions (8.1) of P and Q. Equivalently, (8.6) may be obtained by the averaging method (Part II). In any case, from the conservation equations (8.3) and (8.4), we have the averaged equations

$$\frac{\partial}{\partial T}\int_0^{2\pi}(n_o u_o^{\,2}+ S_o n_o^{\,3}+ E_o^{\,2})d\theta + \frac{\partial}{\partial X}\int_0^{2\pi}(n_o u_o^{\,3}+ 3S_o n_o^{\,3} u_o)d\theta = 0 \tag{8.7}$$

$$\frac{\partial}{\partial T}\int_0^{2\pi} n_o S_o d\theta + \frac{\partial}{\partial X}\int_0^{2\pi} n_o u_o S_o d\theta = 0 \tag{8.8}$$

To simplify Eq. (8.8), we must note that S_o, being the zero-order solution for the entropy S, is independent of θ according to the preceding section, so that it depends only on X and T. Then (8.8) becomes

$$\frac{\partial}{\partial T} S_o \int_0^{2\pi} n_o d\theta + \frac{\partial}{\partial X} S_o \int_0^{2\pi} n_o u_o d\theta = 0$$

From the results (7.22) and (7.23) for $< n_o >$ and $< n_o u_o >$, this reduces to

$$\frac{\partial S_o}{\partial T} = 0 \tag{8.9}$$

Thus $S_o = S_o(X)$, that is, S_o retains its initial distribution for all time.

For dealing with the momentum equation, an expression for $\partial S_1/\partial\theta$ in terms of zero-order quantities will be needed. We therefore expand the particle-path form (4.12) of the entropy equation

$$(-\omega\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial T})(S_o+ \epsilon S_1 + \dots)$$
$$+ (u_o+ \epsilon u_1 + \dots)(\kappa\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial X})(S_o+ \epsilon S_1 + \dots) = 0$$

and take the $O(\epsilon)$ equation, using (8.9) to simplify:

$$(\kappa u_o - \omega)\frac{\partial S_1}{\partial \theta} = -u_o \frac{\partial S_o}{\partial X}$$

Multiplying this through by n_o and using (7.8) of the last section, we arrive at

$$\omega \frac{\partial S_1}{\partial \theta} = n_o u_o \frac{\partial S_o}{\partial X} \qquad (8.10)$$

The modified momentum equation (8.2) can be written

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} + \frac{1}{2} n^2 \frac{\partial S}{\partial x} = 0 \qquad (8.11)$$

which is expanded to become

$$\left(-\omega\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial T}\right)(P_o + \epsilon P_1 + \dots)$$
$$+ \left(\kappa\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial X}\right)(Q_o + \epsilon Q_1 + \dots)$$
$$+ \frac{1}{2}(n_o{}^2 + 2\epsilon n_o n_1 + \dots)\left(\kappa\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial X}\right)\left[S_o(X) + \epsilon S_1 + \dots\right] = 0$$

Setting the $O(1)$ term to zero gives

$$-\omega\frac{\partial P_o}{\partial\theta} + \kappa\frac{\partial Q_o}{\partial\theta} = 0$$

which we integrate as in the last section to obtain

$$Q_o - UP_o = A(X,T)$$

Notice that the dangling S_x term does not contribute in $O(1)$, but only in $O(\epsilon)$, as we mentioned earlier. The $O(\epsilon)$

equation is

$$-\omega\frac{\partial P_1}{\partial\theta}+K\frac{\partial Q_1}{\partial\theta}+\frac{\partial P_o}{\partial T}+\frac{\partial Q_o}{\partial X}+\frac{1}{2}n_o^{\,2}(K\frac{\partial S_1}{\partial\theta}+\frac{\partial S_o}{\partial X})=0 \qquad (8.11.1)$$

Replacing $\partial S_1/\partial\theta$ using (8.10),

$$\frac{\partial}{\partial\theta}(\kappa Q_1-\omega P_1)+\frac{\partial P_o}{\partial T}+\frac{\partial Q_o}{\partial X}+\frac{1}{2}n_o^{\,2}(\frac{n_o u_o}{U}+1)\frac{\partial S_o}{\partial X}=0$$

With $u_o = U - (U/n_o)$, this reduces to

$$\frac{\partial}{\partial\theta}(\kappa Q_1-\omega P_1)=-\frac{\partial P_o}{\partial T}-\frac{\partial Q_o}{\partial X}-\frac{1}{2}n_o^{\,3}\frac{\partial S_o}{\partial X}$$

By the same argument as before, the condition that $(\kappa Q_1-\omega P_1)$ be bounded as $\theta\rightarrow\infty$ is

$$\frac{\partial}{\partial T}\int_o^{2\pi}P_o d\theta+\frac{\partial}{\partial X}\int_o^{2\pi}Q_o\,d\theta+\frac{\partial S_o}{\partial X}\int_o^{2\pi}1/2\;n_o^{\,3}\,d\theta=0 \qquad (8.12)$$

where, from (8.2),

$$P_o=(n_o-1)u_o$$

$$Q_o=n_o u_o^{\,2}-\frac{1}{2}u_o^{\,2}+S_o n_o^{\,3}-\frac{3}{2}S_o n_o^{\,2}-\frac{1}{2}E_o^{\,2}$$

It is interesting to note that this is similar to what we would obtain from averaging (8.11) directly, the crucial difference being the extra factor of <u>n</u> we obtain in the third term. This in turn comes from the $\partial S_1/\partial\theta$ term, which would be missed in a naive application of averaging to (8.11).

Supplementing the three boundedness conditions (8.7), (8.9), and (8.12) will be the non-linear dispersion relation derived in Part II. The equation analogous to (II.1) for the present problem is (7.15). However, it is preferable to work with n_o in the present calculation, so we re-phrase (7.15) in terms of n_o. Transcribing Eqs. (7.12) and (7.14),

$$E_o^2 = 2A + S_o(2n_o^3 - 3n_o^2) - U^2\left(\frac{n_o-1}{n_o}\right)^2 \equiv F(n_o) \quad (8.12.1)$$

$$K\frac{\partial E_o}{\partial \theta} = 1-n_o \quad (8.12.2)$$

it is clear that the equation for n_o is

$$K\frac{\partial n_o}{\partial \theta} = \frac{2(1-n_o)\sqrt{F(n_o)}}{F'(n_o)} \quad (8.13)$$

Now, dropping zero subscripts from here on, the non-linear dispersion relation for (8.13) is

$$K\int_{n_1}^{n_2} \frac{F'(n)}{2(1-n)\sqrt{F(n)}}\,dn = \pi \quad (8.14)$$

which is (II.4) with $\Lambda = 2\pi$; the limits n_1 and n_2 are roots of $F(n)$, since $n=1$ can be shown to be an unsatisfactory limit of oscillation [e.g., using (7.22)], and n_1 and n_2 must be roots of the right-hand side of (8.13).

Equation (8.14) together with the three aforementioned averaged equations (8.7), (8.9), (8.12), furnish a sufficient number of equations, in principle, to determine A, S, U, and κ as functions of X and T.

Before proceeding, let us take a moment to discuss the reformulation of integrals like (8.14) as complex loop integrals. The subject is well treated in some of the older analysis texts (e.g. (29)) under the title of hyper-elliptic integrals. We begin by taking n to be a complex variable. Then, to make $\sqrt{F(n)}$ single-valued in the n-plane, we put cuts from n_1 to n_2 and from the other roots of F(n) to ∞ (Fig. 15). For definiteness, take $\sqrt{F(n)}$ positive on the top side of the branch cut. Then (8.14) may be written

$$\kappa \int_{\Gamma_1} \frac{F'(n)}{2(1-n)\sqrt{F(n)}} \, dn = \pi \tag{8.15}$$

where Γ_1 goes from n_1 to n_2 along the top side of the branch cut (Fig. 15(a)). But $\sqrt{F(n)}$ takes exactly the same values just below the branch cut as it does just above, only with the sign reversed. Hence we also have

$$\kappa \int_{\Gamma_2} \frac{F'(n)}{2(1-n)\sqrt{F(n)}} \, dn = \pi \tag{8.16}$$

where Γ_2 runs from n_2 to n_1 just below the cut (Fig. 15(a)). Adding (8.15) and (8.16),

$$K \oint_{\Gamma} \frac{F'(n)}{2(1-n)\sqrt{F(n)}}\, dn = 2\pi \qquad (8.17)$$

where Γ is a closed contour hugging the branch cut (Fig. 15(b)). We have omitted the simple proof that the contributions from the vanishingly small circles around n_1 and n_2 vanish.

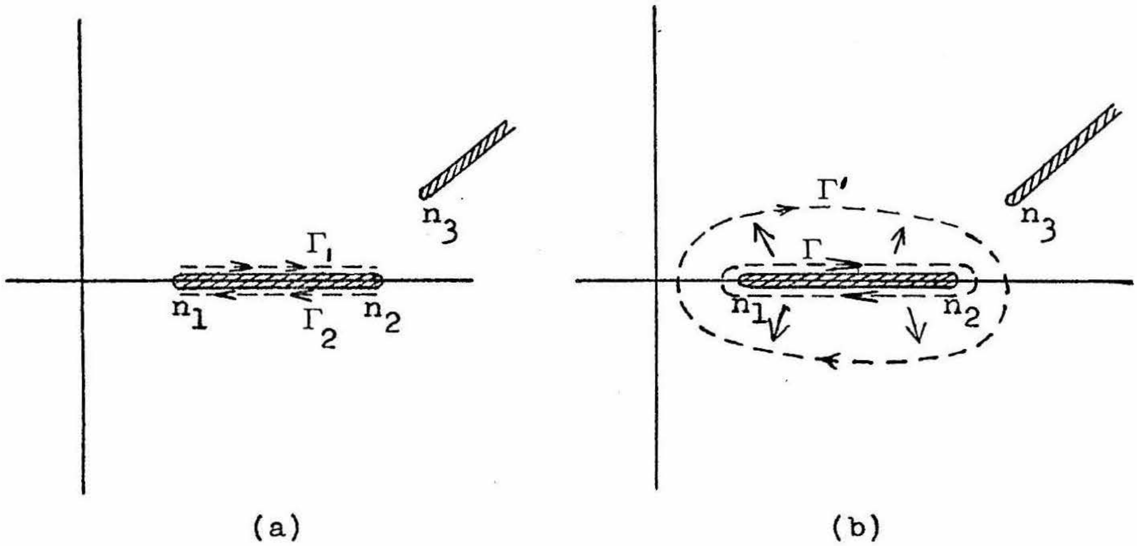

Figure 15. The complex n-plane; branch cuts and contours used in converting certain hyper-elliptic integrals along the real axis to complex loop integrals.

But once the integrals are in the form (8.17) there is no longer any need to tie the loop contour down to the branch cut. It may be any closed contour surrounding the cut, by Cauchy's Theorem, such as the contour Γ' of Fig. (15(b)).

Thus from now on we shall write loop integrals without reference to any particular contour.

There are several advantages to using these complex loop integrals, among which are:

(1) numerous techniques of complex analysis (e.g. Laurent's Theorem;see§10) become available to us;

(2) certain integrations by parts, illegal along the real axis, may be performed;

(3) derivatives with respect to a parameter, say A, commute with the loop integral, but do not with the real integral because n_1 and n_2 depend on A.

As an illustration of (2), and to set the equation up for further work, let us integrate (8.17) by parts:

$$-K\oint \frac{\sqrt{F(n)}}{(1-n)^2}\, dn = 2\pi \qquad (8.18)$$

The apparent singularity in (8.17) at n=1 was removable; now it is genuine, and (8.18) would be nonconvergent along the real axis, because n=1 (the equilibrium value) will always lie between n_1 and n_2.

We will now derive the master function W for this problem from (8.18). Our procedure will be essentially ad hoc, but could be justified by an appeal to the averaged Lagrangian (§11) which is equivalent to W. The derivation consists basically in noting that the canonical form found by Whitham for the non-linear dispersion relation is[15]

$$\kappa W_A = 1 \qquad (8.19)$$

where A is the integration constant from the conservative form of the problem's momentum equation. In our problem A was only the integration constant from the pseudo-conservative form of the momentum equation (7.11), but it turns out the trick works anyway. Identifying from (8.18)

$$W_A = -\frac{1}{2\pi} \oint \frac{\sqrt{F(n)}}{(1-n)^2}\, dn \qquad (8.20)$$

and noting from (8.12.1) the A-dependence of F(n), we deduce

$$W = -\frac{1}{6\pi} \oint \frac{F^{3/2}(n)}{(1-n)^2}\, dn \qquad (8.21)$$

With this form of W, it is possible to write the averaged equations (8.7) and (8.12) in terms of W and its partials W_A, W_U, W_S. First, however, let us manipulate the definition of an averaged quantity into a form which is convenient for us. We begin with a general quantity P(n,u,E) such as occurs in (8.7) or (8.12), and substitute for u and E according to

$$u = U - U/n \qquad (8.22)$$

$$E = \sqrt{F(n)} \qquad (8.23)$$

to produce a quantity entirely in terms of n, say $\hat{P}(n)$.

Then

$$< P(n,u,E) > = \frac{1}{2\pi}\int_0^{2\pi} P \, d\theta = \frac{1}{2\pi}\oint \hat{P}(n) \frac{dE}{E_\theta}$$

$$= \frac{K}{2\pi}\oint \frac{\hat{P}(n)}{1-n} dE$$

$$= -\frac{K}{2\pi}\oint E(n)\frac{d}{dn}\left(\frac{\hat{P}}{1-n}\right) dn$$

$$= -\frac{K}{2\pi}\oint \sqrt{F(n)} \frac{\hat{P}(n)+(1-n)\hat{P}'(n)}{(1-n)^2} dn \quad (8.24)$$

To get from the first line to the second, we converted to a loop integral in the complex plane and used Poisson's equation. Then we did an integration by parts, and finally used (8.23).

The computations of the averages $< P(n,u,E) >$ for the specific P's of (8.7) and (8.12) are relegated to Appendix E. It is also shown there how these averages may be written in terms of W, W_U, etc. The results are

$$\frac{\partial}{\partial T}(\kappa W_U) + \frac{\partial}{\partial X}(\kappa U W_U - A) - \kappa W_S \frac{\partial S}{\partial X} = 0 \quad (8.25)$$

$$\frac{\partial}{\partial T}\left[\kappa(A W_A + U W_U - W)\right] + \frac{\partial}{\partial X}\left[\kappa U(U W_U - W)\right] = 0 \quad (8.26)$$

Equation (8.25) comes from (8.12), and (8.26) comes from (8.7). Coupled with

$$\frac{\partial S}{\partial T} = 0$$

$$\kappa W_A = 1$$

these form a completely determinate system for A, S, κ, and U.

Since the identity (II.10)

$$\frac{\partial \kappa}{\partial T} + \frac{\partial \omega}{\partial X} = \frac{\partial \kappa}{\partial T} + \frac{\partial (\kappa U)}{\partial X} = 0 \tag{8.27}$$

holds from the definitions of ω and κ, it must follow from the four preceding equations. This can be demonstrated by expanding (8.26) and using the other three equations to simplify the result. One must note in so doing that

$$\frac{\partial W}{\partial T} = W_A \frac{\partial A}{\partial T} + W_U \frac{\partial U}{\partial T} + W_S \frac{\partial S}{\partial T}$$

with a similar expression for $\partial W/\partial X$.

The conservation law (8.27) is called the conservation of waves. It says that the number of wave crests $\kappa/2\pi \, \Delta X$ in a length ΔX of the wave may only change due to the net flux $1/2\pi\left[(\kappa U)_{X+\Delta X} - (\kappa U)_X\right]$ of wave crests across the boundaries of this interval. This explains why the slowly-varying theory will generally hold for late times, $t \rightarrow \infty$, if it holds at all. For (8.27) assumes that no waves are being created or destroyed; if they were, (8.27) would need a source term on the right-hand side. Thus (8.27) cannot be valid near t=0 for an initial value problem, for the initial disturbance usually requires a period of time to resolve itself into its component wave trains. Similarly, the slowly-varying theory would only be valid sufficiently far from a boundary at which waves were being created or absorbed.

Using (8.27), (8.25) may be simplified to

$$\kappa \frac{\partial W_U}{\partial T} + \kappa U \frac{\partial W_U}{\partial X} - \frac{\partial A}{\partial X} - \kappa W_S \frac{\partial S}{\partial X} = 0 \qquad (8.28)$$

We take this together with

$$\frac{\partial \kappa}{\partial T} + \frac{\partial(\kappa U)}{\partial X} = 0$$

$$\frac{\partial S}{\partial T} = 0 \qquad (8.29)$$

$$\kappa W_A = 1 \qquad (8.30)$$

as our simplest set of averaged equations. If desired, κ may be eliminated from this system entirely by using (8.30). The result is

$$\left.\begin{aligned} \frac{\partial W_U}{\partial T} + U \frac{\partial W_U}{\partial X} - W_A \frac{\partial A}{\partial X} - W_S \frac{\partial S}{\partial X} &= 0 \\ \frac{\partial W_A}{\partial T} + U \frac{\partial W_A}{\partial X} - W_A \frac{\partial U}{\partial X} &= 0 \\ \frac{\partial S}{\partial T} &= 0 \end{aligned}\right\} \qquad (8.31)$$

Even with S=constant, this system is inaccessible to our present-day analytical tools. We shall deduce some facts about its characteristic velocities in §8.C, about its stability in §9, and about its small-amplitude limit in §10, but we shall not attempt to solve it in this thesis.

One remarkable result about the averaged equations is that it is possible to put the first of (8.31), which comes from a non-conservative momentum equation, into conservation form:

$$\frac{\partial W_U}{\partial T} + \frac{\partial (UW_U)}{\partial X} - W_A \frac{\partial A}{\partial X} - W_S \frac{\partial S}{\partial X} - W_U \frac{\partial U}{\partial X}$$

$$= \frac{\partial W_U}{\partial T} + \frac{\partial}{\partial X}(UW_U - W) = 0$$

We close with some remarks about the physical interpretation of the partials of W. Starting from the formula (8.24) for a general average, we deduce:

$$<n> = -\frac{\kappa}{2\pi}\oint \frac{\sqrt{F(n)}}{(1-n)^2}\,dn = \kappa W_A$$

$$<u> = <U - \frac{U}{n}> = -\frac{\kappa U}{2\pi}\oint \frac{\sqrt{F(n)}}{n^2}\,dn = -\kappa W_U$$

$$<p> = <Sn^3> = \frac{\kappa S}{2\pi}\oint (2n^3 - 3n^2)\frac{\sqrt{F(n)}}{(n-1)^2}\,dn$$

$$= -2\kappa S W_S$$

$$<E> = <\sqrt{F(n)}> = -\frac{\kappa}{2\pi}\oint \frac{F(n) + \frac{1}{2}(1-n)F'(n)}{(1-n)^2}\,dn$$

$$= 0$$

The last result follows from Cauchy's Theorem since the integrand is analytic. Thus the three partials of W have physical interpretations as the average values of n, u, and p. (We had already deduced the average of E was zero, Eq. (7.25), but the present derivation is more straightforward.)

B. The inclusion of dissipation

When there is dissipation in a system, it is not possible to write down a Lagrangian for that system, and the method of the averaged Lagrangian is no longer strictly applicable. Provided that we consider the dissipation to be small, of $O(\epsilon)$, however, the Luke expansion method can still be applied. The results can no longer be written strictly in terms of a single quantity W and its partials, as in the first section; but the equations are substantially the same, with only the addition of an extra term to the entropy and momentum equations. This term takes the form of a weighted average of the dissipation term.

The application of the Luke method to the Korteweg-de Vries equation with a small dissipation term has been considered as an example in §3.

The reason we must consider the dissipation to be small is that, if it were significant, it would destroy the possibility of a steady-profile wave over an extended region of space and time. Any such wave would decay too rapidly for us to use it as a basic solution and consider slowly-varying perturbations of it. The whole idea of slowly-varying perturbations becomes meaningless in such a situation. When the dissipation is $O(\epsilon)$, however, a steady wave may persist for many cycles, and its amplitude will only decay slowly, so that this decay fits into the category of a slowly-varying perturbation.

From the mathematical standpoint, the presence of dissipation generally alters the form of the conservation equations to

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial X} + R = 0 \qquad (8.32)$$

The "dangling" term R describes the dissipation, and causes the quantity P not to be conserved. If R is 0(1), the averaged form of Eq. (8.32) is

$$\frac{\partial \langle P \rangle}{\partial t} + \frac{\partial \langle Q \rangle}{\partial x} + \langle R \rangle = \epsilon \frac{\partial \langle P \rangle}{\partial T} + \epsilon \frac{\partial \langle Q \rangle}{\partial X} + \langle R \rangle = 0$$

This is clearly inconsistent, since all terms are not of the same order, unless we require

$$\langle R \rangle = 0$$

which means that the dissipation has no net effect on the wave; such a dissipation would be unphysical. Thus, to use the averaging method, we need to have

$$\langle R \rangle = O(\epsilon)$$

For the present, we shall achieve $\langle R \rangle = O(\epsilon)$ by taking R to be explicitly proportional to ϵ. This need not necessarily be the case, however. In the dissipationless case, Eq. (8.2) was of the form of Eq. (8.32) with

$$R = 1/2\ n^2 \frac{\partial s}{\partial x}$$

Because of the special circumstance that S_o was independent of θ, the averaged form of this term was $O(\epsilon)$. Thus we may make the general observation that for systems of p.d.e.'s in non-conservation form like Eq. (8.32), the averaging method is still applicable, provided only that the averages of the dangling terms are $O(\epsilon)$. This holds whether or not the dangling terms represent dissipation, and whether or not they are explicitly proportional to ϵ.

In our derivation of the LPE's from the "collisionless" Boltzmann equation, we noted that the energy equation was

$$(nu^2 + p + E^2)_t + (nu^3 + 3pu + q)_x = 0 \qquad (8.33)$$

where

$$q = \int (v-u)^3 f\, dv$$

and $f = f(x,v,t)$ was the distribution function. For the dissipationless LPE's, we took $q=0$. Now we assume

$$q = \epsilon r \qquad (8.34)$$

where r depends on n,u,p,E and is to model the dissipation in the system. Two physically reasonable forms of q were discussed in §4, but the success of the method to follow is independent of the nature of q.

The only two equations among the LPE's which are altered by taking a non-zero q are the energy equation (8.33) and the entropy equation, which becomes

$$S_t + uS_x + \frac{\epsilon r_x}{n^3} = 0 \qquad (8.35)$$

We now reconsider the Luke expansion of each of the LPE's which are affected by dissipation.

Writing the modified energy equation as

$$\hat{P}_t + (\hat{Q} + \epsilon r)_x = 0$$

we may apply the Luke expansion just as before

$$(-\omega\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial T})(\hat{P}_o + \epsilon\hat{P}_1 + \ldots)$$
$$+ (\kappa\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial X})(\hat{Q}_o + \epsilon\hat{Q}_1 + \epsilon r_o + \ldots) = 0$$

The O(1) result is clearly unaltered by dissipation. The $O(\epsilon)$ result is

$$\frac{\partial}{\partial\theta}(\omega\hat{P}_1 - \kappa\hat{Q}_1) = \frac{\partial\hat{P}_o}{\partial T} + \frac{\partial\hat{Q}_o}{\partial X} + \kappa\frac{\partial r_o}{\partial\theta}$$

To prevent $(\omega\hat{P}_1 - \kappa\hat{Q}_1)$ from being unbounded as $\theta \to \infty$, we must require

$$\int_0^{2\pi}(\frac{\partial\hat{P}_o}{\partial T} + \frac{\partial\hat{Q}_o}{\partial X} + \kappa\frac{\partial r_o}{\partial\theta})\, d\theta = 0$$

The term involving r_o clearly integrates to zero, and we are left with the same result (Eq. (8.7)) as in the dissipationless case. This may at first glance seem surprising, but it is perfectly reasonable, for the energy that is lost from the wave motion due to dissipation is gained by the random thermal motion, which is measured by p. No energy disappears from the system, with or without dissipation.

Omitting the details, we note that in O(1) the entropy equation, (8.35), reproduces the dissipationless result

$$\frac{\partial s_o}{\partial \theta} = 0 \tag{8.36.1}$$

and in $O(\epsilon)$ yields

$$(\omega - \kappa u_o) \frac{\partial s_1}{\partial \theta} = \frac{\partial s_o}{\partial T} + u_o \frac{\partial s_o}{\partial X} + \frac{\kappa}{n_o^3} \frac{\partial r_o}{\partial \theta}$$

Multiplying through by n_o and using Eq. (7.8)

$$n_o(u_o - U) = - U \tag{7.8}$$

leads to

$$\omega \frac{\partial s_1}{\partial \theta} = n_o \frac{\partial s_o}{\partial T} + n_o u_o \frac{\partial s_o}{\partial X} + \frac{\kappa}{n_o^2} \frac{\partial r_o}{\partial \theta} \tag{8.36.2}$$

The condition that s_1 be bounded as $\theta \to \infty$ is then

$$\int_o^{2\pi} \left(n_o \frac{\partial s_o}{\partial T} + n_o u_o \frac{\partial s_o}{\partial X} + \frac{\kappa}{n_o^2} \frac{\partial r_o}{\partial \theta} \right) d\theta = 0$$

By virtue of Eq. (8.36.1) and results (7.22) and (7.23) for $\langle n_o \rangle$ and $\langle n_o u_o \rangle$, this reduces to

$$\frac{\partial s_o}{\partial T} = D \tag{8.37}$$

where

$$D = -\frac{\kappa}{2\pi}\int_o^{2\pi}\frac{1}{n_o^2}\frac{\partial r_o}{\partial\theta}\,d\theta$$

$$= -\frac{\kappa}{2\pi}\int_o^{2\pi}\frac{2r_o}{n_o^3}\frac{\partial n_o}{\partial\theta}\,d\theta = -\kappa\left\langle\frac{2r_o}{n_o^3}\frac{\partial n_o}{\partial\theta}\right\rangle \quad (8.38)$$

The second form of D follows from an integration by parts. Putting (8.37) back into (8.36.2) and using (7.8), we obtain

$$\omega\frac{\partial s_1}{\partial\theta} = n_o D + U(n_o-1)\frac{\partial s_o}{\partial X} + \frac{\kappa}{n_o^2}\frac{\partial r_o}{\partial\theta} \quad (8.39)$$

Since the momentum equation is unchanged in form, the O(1) and $O(\epsilon)$ results (Eqs. (7.11) and (8.11.1)) from the previous sections still apply. Now, however, we must replace $\partial s_1/\partial\theta$ in Eq. (8.11.1) according to Eq. (8.39), leading to

$$\frac{\partial}{\partial\theta}(\omega P_1 - \kappa Q_1) = \frac{\partial P_o}{\partial T} + \frac{\partial Q_o}{\partial X} + \frac{1}{2}n_o^2\left[\frac{n_o D}{U} + n_o\frac{\partial s_o}{\partial X} + \frac{\kappa}{Un_o^2}\frac{\partial r_o}{\partial\theta}\right]$$

The boundedness condition associated with this equation is

$$\int_o^{2\pi}\left\{\frac{\partial P_o}{\partial T} + \frac{\partial Q_o}{\partial X} + \frac{1}{2}n_o^3\left(\frac{D}{U} + \frac{\partial s_o}{\partial X}\right) + \frac{\kappa}{2U}\frac{\partial r_o}{\partial\theta}\right\}d\theta = 0$$

The term involving r_o integrates to zero. Putting in the expressions for $\langle P_o\rangle$, $\langle Q_o\rangle$, and $\langle\frac{1}{2}n_o^3\rangle$ from Appendix E just as in the dissipationless calculation, we arrive finally at

$$\frac{\partial}{\partial T}(\kappa W_U) + \frac{\partial}{\partial X}(\kappa U W_U - A) - \kappa W_s\left(\frac{\partial S}{\partial X} + \frac{D}{U}\right) = 0$$

The complete set of averaged equations for the LPE's with dissipation are thus

$$\frac{\partial}{\partial T}\left[\kappa(UW_U - W)+A\right] + \frac{\partial}{\partial X}\left[\kappa U(UW_U - W)\right] = 0 \tag{8.40}$$

$$\frac{\partial S}{\partial T} = D \tag{8.41}$$

$$\frac{\partial}{\partial T}(\kappa W_U) + \frac{\partial}{\partial X}(\kappa UW_U - A) - \kappa W_S\left(\frac{\partial S}{\partial X} + \frac{D}{U}\right) = 0 \tag{8.42}$$

$$\kappa W_A = 1 \tag{8.43}$$

Equation (8.40), unchanged from §8.A, comes from the energy equation. Equations (8.41) and (8.42) were derived above. Equation (8.43) carries over unchanged from the dissipationless case. If we expand Eq. (8.40) and simplify it using Eqs. (8.41), (8.42), and (8.43), we recover the conservation of waves equation

$$\frac{\partial \kappa}{\partial T} + \frac{\partial}{\partial X}(\kappa U) = 0 \tag{8.44}$$

exactly as in the non-dissipative case. This may be used to simplify Eq. (8.42) to

$$\frac{\partial W_U}{\partial T} + U\frac{\partial W_U}{\partial X} - \frac{1}{\kappa}\frac{\partial A}{\partial X} - W_S\left(\frac{\partial S}{\partial X} + \frac{D}{U}\right) = 0 \tag{8.45}$$

Equations (8.41), (8.43), (8.44), and (8.45) then form a simpler set to investigate than the original four equations.

For the special case of no X-dependence, the system simplifies to

$$\kappa(UW_U - W) + A = \text{const.} \tag{8.46}$$

$$\frac{dS}{dT} = D \tag{8.47}$$

$$U\frac{dW_U}{dT} - DW_S = 0 \tag{8.48}$$

$$\kappa W_A = 1 \tag{8.49}$$

$$\kappa = \text{const.}$$

Since κ is constant (by Eq. (8.44)), there are only three variables, A, S, and U, to solve for. Thus, as in the Korteweg-de Vries problem, (§3), we would only need to solve a single differential equation (probably Eq. (8.47), since it is simplest). The remaining variables would then be determined by algebraic relations, Eqs. (8.46) and (8.49). While analytic solution eludes us even in this special case, it is no problem in principle to solve (8.46 - 49) numerically as we did in the Korteweg-de Vries case, §3.

It should be noted that the dissipation term could be an integral, as suggested in §4, Eq. (4.21). Some extra work in computing D would result, but it seems tractable enough numerically. It would be of great interest to see how this integral term controlled the damping.

C. Characteristic form and group velocities

We shall obtain the characteristic form for the dissipationless averaged equations (8.31), which we transcribe here for convenience:

$$\frac{\partial S}{\partial T} = 0$$

$$\frac{\partial W_A}{\partial T} + U\frac{\partial W_A}{\partial X} - W_A\frac{\partial U}{\partial X} = 0$$

$$\frac{\partial W_U}{\partial T} + U\frac{\partial W_U}{\partial X} - W_A\frac{\partial A}{\partial X} - W_S\frac{\partial S}{\partial X} = 0$$

The first equation is already in characteristic form. It has a characteristic velocity of zero. We expand the derivative terms in the second and third equations according to the example

$$\frac{\partial W_A}{\partial X} = W_{AA}\frac{\partial A}{\partial X} + W_{AU}\frac{\partial U}{\partial X} + W_{AS}\frac{\partial S}{\partial X}$$

and add λ times the second equation to the third one to obtain

$$(W_{UU}+\lambda W_{AU})\left\{\frac{\partial U}{\partial T} + (U - \frac{\lambda W_A}{W_{UU}+\lambda W_{AU}}\frac{\partial U}{\partial X}\right\} \tag{8.50}$$

$$+ (W_{AU}+\lambda W_{AA})\left\{\frac{\partial A}{\partial T} + (U - \frac{W_A}{W_{AU}+\lambda W_{AA}})\frac{\partial A}{\partial X}\right\}$$

$$+ (UW_{US} - W_S + \lambda UW_{AS})\frac{dS}{dX} = 0$$

To make this the characteristic form, we must choose λ such that

$$C \equiv U - \frac{\lambda W_A}{W_{UU} + \lambda W_{AU}} = U - \frac{W_A}{W_{AU} + \lambda W_{AA}} \tag{8.51}$$

which becomes a quadratic in λ

$$\lambda^2 W_{AA} - W_{UU} = 0$$

with roots

$$\lambda_{\pm} = \pm\sqrt{\frac{W_{UU}}{W_{AA}}}$$

Using these values of λ, we find two characteristic velocities from (8.51):

$$\left.\begin{aligned} C_+ &= U - \frac{W_A}{W_{AU} + \sqrt{W_{AA}W_{UU}}} \\ C_- &= U - \frac{W_A}{W_{AU} - \sqrt{W_{AA}W_{UU}}} \end{aligned}\right\} \tag{8.52}$$

The corresponding characteristic forms are, from (8.50),

$$\frac{dA}{dT} + \sqrt{\frac{W_{UU}}{W_{AA}}}\frac{dU}{dT} + f_+ \frac{dS}{dX} = 0 \tag{8.53}$$

on $\frac{dX}{dT} = C_+$, where $\frac{d}{dT} = \frac{\partial}{\partial T} + C_+ \frac{\partial}{\partial X}$, and

$$\frac{dA}{dT} - \sqrt{\frac{W_{UU}}{W_{AA}}}\frac{dU}{dT} + f_- \frac{dS}{dX} = 0 \tag{8.54}$$

on $\frac{dX}{dT} = C_-$, where $\frac{d}{dT} = \frac{\partial}{\partial T} + C_- \frac{\partial}{\partial X}$, and where

$$f_{\pm} = \frac{UW_{US} - W_S + \lambda_{\pm} UW_{AS}}{W_{AU} + \lambda_{\pm} W_{AA}}$$

The various quantities involved in this calculation can be found directly from W to be

$$W_{AA} = -1/2\pi \oint \frac{dn}{(n-1)^2 \sqrt{F}}$$

$$W_{AU} = U/2\pi \oint \frac{dn}{n^2 \sqrt{F}}$$

$$W_{UU} = -U^2/2\pi \oint \frac{(n-1)^2}{n^4 \sqrt{F}} dn$$

$$W_A = -1/2\pi \oint \frac{\sqrt{F}}{(n-1)^2} dn$$

$$= -1/2\pi \oint (3S - \frac{U^2}{n^3}) \frac{dn}{\sqrt{F}}$$

$$W_{AS} = -1/4\pi \oint \frac{2n^3 - 3n^2}{(n-1)^2 \sqrt{F}} dn$$

$$W_{US} = U/4\pi \oint \frac{2n-3}{\sqrt{F}} dn$$

$$W_S = -1/4\pi \oint \frac{2n^3 - 3n^2}{(n-1)^2} \sqrt{F}\ dn$$

By inspection, all but W_{AA} and W_{AS} can be re-expressed as integrals along the real axis simply by shrinking the contour of integration back down around the branch cut (Fig. 15). For W_A and W_S, a preliminary integration by parts is necessary, for n=1 will most assuredly lie on the branch cut.

These partial integrations eliminate the double poles at $n=1$ at the expense of introducing $\sqrt{F}$ into the denominator of the integrand, which means square root singularities at the ends of the branch cut. This is all right, because such singularities are integrable. But for W_{AA} and W_{AS}, the same integration by parts leads to an $F^{3/2}$ in the denominator, and the singularities at the ends of the branch cut are no longer integrable.

It is especially important to be able to calculate W_{AA}, as we shall see in § 9. Therefore Appendix H gives a method for so doing, which would be especially suited for numerical computations. We shall proceed here to investigate the signs of the other derivatives of W needed in the calculation of $C_{\pm}$.

The sign of W_A is the same as the sign of κ, from (8.30). Since we may without loss of generality take $\kappa > 0$, it follows that

$$W_A > 0$$

The quantities W_{AU} and W_{UU} may be re-expressed as integrals over θ by the following ploy:

$$W_{AU} = U/2\pi \int_0^{2\pi} \frac{n_\theta}{n^2\sqrt{F(n)}} \, d\theta$$

$$W_{UU} = - U^2/2\pi \int_0^{2\pi} \frac{(n-1)^2}{n^4\sqrt{F(n)}} \, n_\theta \, d\theta$$

Then if we take $\partial n/\partial\theta$ from (8.13), with the actual expression for F'(n) inserted, these reduce to

$$W_{AU} = \frac{1}{2\pi\omega}\int_0^{2\pi} \frac{n}{1-(\beta n)^4}\, d\theta$$

$$W_{UU} = -\frac{1}{2\pi\kappa}\int_0^{2\pi} \frac{(n-1)^2}{n\left[1-(\beta n)^4\right]}\, d\theta$$

where $\beta = (3S/U^2)^{1/4}$ as in § 7. The denominators in the integrands would vanish if $n = 1/\beta$, which by (7.8) translates to

$$u = U - \beta U = u_\infty$$

But in § 7 we showed that $u < u_\infty$ for the steady wave, which by (7.8) becomes $\beta n < 1$. Thus the denominators of the integrands are positive. So are the numerators, so that with $\kappa > 0$ and $\omega = \kappa U > 0$ we have immediately that

$$\left.\begin{array}{l} W_{AU} > 0 \\ W_{UU} < 0 \end{array}\right\} \qquad (8.55)$$

It follows from (8.55) that the characteristic velocities, or non-linear group velocities[15], $C_\pm$, will only be real if

$$W_{AA} < 0 \qquad (8.56)$$

If this is true, then we can say something about the relationship of the group velocities to the phase velocity U. If $W_{AU} < (W_{AA}W_{UU})^{1/2}$ the group velocities flank the phase velocity

$$C_+ < U < C_-$$

while if $W_{AU} > (W_{AA}W_{UU})^{1/2}$ they are both less than the phase velocity

$$C_- < C_+ < U$$

The latter situation parallels the linearized case, in which

$$\frac{d\omega_o}{d\kappa} = \frac{3S\kappa}{\sqrt{1+3S\kappa^2}} < \frac{\sqrt{1+3S\kappa^2}}{\kappa} = \frac{\omega_o}{\kappa}$$

and hence that would be the situation we would expect to hold in the small-amplitude limit.

The type of the averaged equations becomes parabolic exactly at $W_{AU} = (W_{AA}W_{UU})^{1/2}$. Then C_- is infinite and the averaged quantities would diffuse and tend to smear out as they propagated at the single velocity C_+. We would thus not expect shocks to form in the averaged quantities (see (15)) at or even near this parabolic limit.

It is also of note that the parabolic case does not mark a transition between the elliptic and hyperbolic cases. For both ${W_{AU}}^2 > W_{AA}W_{UU}$ and ${W_{AU}}^2 < W_{AA}W_{UU}$ the averaged equations are hyperbolic. On the other hand, the actual transitional case between elliptic and hyperbolic regimes, $W_{AA} = 0$, is not parabolic. It is a degenerate hyperbolic case in which $C_+ = C_-$, that is, both propagation speeds are equal.

§9. Stability of the Plasma Wave

An extremely important application of the preceding work (§8) is to give truly non-linear stability criteria for plasma waves. The familiar stability considerations of linearized theory center around determining the sign of $\mathrm{Im}(\omega)$ in $e^{i(\kappa x-\omega t)}$- type solutions. These considerations are useful as far as they go, but it was not until the work of Whitham(15,16,17,18) that anyone formulated stability conditions, or even defined stability rigorously, for fully non-linear wavetrains, Now we have such a definition, and it is simply:

> Unfform wavetrains are unstable if the type of the averaged equations is elliptic.

Thus, from (8.56), the condition that the uniform wavetrain in the present problem be stable is

$$W_{AA} < 0 \tag{9.1}$$

To understand the above definition of stability, consider for a moment the hyperbolic case. In a hyperbolic system, small-amplitude disturbances, sound waves, are well known to travel out along the characteristics. They may be represented by

$$e^{i\mu(x-ct)} \tag{9.2}$$

where c is one of the (real) characteristic velocities. Now consider what happens when the system becomes elliptic. The characteristic velocities c become complex conjugate pairs, and small disturbances like (9.2) will grow with time. This is the sense in which the uniform wavetrain is unstable when the averaged equations are elliptic. If we have a uniform wavetrain propagating along, and all of a sudden it is subjected to some minute slowly-varying perturbation, such as entering a medium whose properties vary slowly, the wavetrain will be unstable to this perturbation. It will most likely break up and dissipate its organized motion into turbulence.

The above instability was demonstrated by Whitham[17] for Stokes waves in deep water, a hitherto unexpected result. It agreed with an inability to manufacture these waves in the laboratory which had long frustrated experimenters[19]. It appears likely that similar dividends may be reaped by such stability analyses of uniform wavetrains in other areas of physics. Plasma physics, which is particularly fraught with instabilities of all kinds, seems an especially fertile hunting ground.

We cannot, unfortunately, push the stability condition (9.1) very far analytically. The difficulties in even writing W_{AA} as an integral along the real axis were explained in §8.C, and there we referred to Appendix H for a technique for so doing. The result obtained in Appendix H is, however, quite formidable, and seemingly only suited for numerical evaluation. It is possible, nevertheless, to make a determination of stability in the small-amplitude limit. In §10 we shall see explicitly that the characteristic velocities are real when we take the small-amplitude limit of the averaged equations. In the meantime, we may note that result will be only a special case of a general theorem noted by Whitham[18], which is that:

> Small-amplitude stability holds if and only if $\omega_o''(\kappa)\ \omega_2(\kappa) > 0$, where ω_o and ω_2 come from the expansion of the frequency as
>
> $$\omega = \omega_o(\kappa) + a^2\ \omega_1(\kappa) + \ldots$$
>
> as a power series in the amplitude <u>a</u> in the near-linear limit $a \rightarrow 0$.

The details of the computation of ω_o, ω_2, etc. are given in §1 and §5.A. In particular, in §5.A, we found

$$\omega_o\ (\kappa) = \sqrt{1+3S\kappa^2}$$

$$\omega_2\ (\kappa) = \frac{3\kappa^2 B^2}{4\omega_o}\ (16S^2\kappa^2 + 5S)$$

from which it may be easily verified that the stability criterion $\omega_o''\,\omega_2 > 0$ is satisfied.

Thus, at least in the limit of small-amplitude slowly-varying perturbations, the uniform wavetrain solution of the LPE's is stable.

In general, when we formulate stability criteria for non-linear wavetrains with parameters α_i, one of two situations will hold: (1) it will be immediately obvious that the stability criterion is or is not satisfied from an examination of the integrands, for example as in §8.C when we proved that $W_{UU} < 0$; (2) it will not be obvious, and the stability criterion may or may not be satisfied depending on the α_i. In the latter situation, which pertains to the plasma case, an analysis of the sort of §7.B is essential. Such an analysis determines the region R in parameter-space (α_i-space) for which a uniform wavetrain exists. Armed with this information, one may then test the stability criterion just over R, analytically if the inequalities defining R are simple enough, otherwise numerically. This will delimit the sub-region of R in which waves may be expected to the stable.

§10. Small Amplitude Expansions

In order to do small amplitude expansions of the non-linear dispersion relation (sub-section A), the averaged equations (sub-section B), and the averaged equations with a model dissipation (sub-section C), it is necessary to expand the hyper-elliptic integrals represented by W, W_A, etc. for small amplitudes. The method which we have found useful for these expansions is given in Appendix F. It will be used without further reference below.

A. The non-linear dispersion relation

The non-linear dispersion relation for the plasma waves, written out in detail, is Eq. (8.18)

$$1 = -\frac{K}{2\pi}\oint \frac{1}{(n-1)^2}\sqrt{2A + S n^2(2n-3) - U^2\left(\frac{n-1}{n}\right)^2}\, dn$$

We begin by changing to a variable centered on the branch cut between n_1 and n_2 (Fig. 15). Since $\langle n \rangle = 1$ from (7.22), it is certain that n=1 lies on the branch cut, so we take

$$n = 1 + \xi$$

which leads to

$$1 = -\frac{K}{2\pi}\oint \frac{1}{\xi^2}\sqrt{a^2 - b^2\xi^2 + \xi^3\left(2S + U^2\frac{2+\xi}{(1+\xi)^2}\right)}\, d\xi \quad (10.1)$$

where

$$a^2 = 2A - S$$

$$b^2 = U^2 - 3S$$

That 2A-S > 0 was discussed in connection with the linearized solution (§7.A). It was also noted there that $U^2-3S > 0$ and that a is a measure of amplitude. We will want to expand, then, for small a.

We regard the contour of integration, for the moment, to be wrapped fairly closely around the branch cut; this ensures that ξ is small, that is, $\xi = O(a)$ as $a \to 0$. One can readily convince oneself that $b = O(1)$ as $a \to 0$ (otherwise there is no linearized solution). This means that the first two terms under the root in (10.1) are $O(a^2)$, while the third is $O(a^3)$. Thus we are led to expand the root in terms of the third term divided by the sum of the first two terms. To implement this idea, we transform

$$\xi = \frac{a}{b}\eta$$

$$1 = -\frac{\kappa b}{2\pi}\oint \frac{1}{\eta^2}\sqrt{1-\eta^2 + a\left(\frac{\eta}{b}\right)^3\left(2S + U^2\,\frac{2 \pm \frac{a}{b}\eta}{(1+\frac{a}{b}\eta)^2}\right)}\,d\eta$$

Now expand the contour of integration away from the branch cut so that $\eta = \pm 1$ do not lie on or near the contour. Then we may take a factor $(1-\eta^2)$ out of the root and expand for $a \to 0$:

$$1 = -\frac{\kappa b}{2\pi}\oint \frac{1}{\eta^2}\sqrt{1-\eta^2}\left[1 + \frac{a}{2}\,\frac{(\eta/b)^3}{1-\eta^2}\left(2S + U^2\,\frac{2+\frac{a}{b}\eta}{(1+\frac{a}{b}\eta)^2}\right) + \dots\right]d\eta$$

Each individual term of the expanded radical must also be expanded for a $\to 0$, which yields

$$1 = -\frac{Kb}{2\pi}\oint \frac{1}{\eta^2}\sqrt{1-\eta^2}\left[1+a\frac{c}{b^3}\frac{\eta^3}{1-\eta^2} - a^2\frac{3U^2}{2b^4}\frac{\eta^4}{1-\eta^2} - a^2\frac{c^2}{2b^6}\frac{\eta^6}{(1-\eta^2)^2} + O(a^3)\right]d\eta \qquad (10.2)$$

where

$$c = S + U^2$$

Since an asymptotic expansion may be integrated term-by-term, we may integrate term-by-term in (10.2). The individual η-integrals are then done by the Laurent method outlined in Appendix F, yielding

$$1 = Kb\left\{1 - \frac{3a^2}{4b^6}(S^2 + 5SU^2) + O(a^4)\right\}$$

If, in this equation, we substitute the expansion

$$\omega = \omega_o(K) + a\omega_1(K) + a^2\omega_2(K) + \ldots \qquad (10.3)$$

of the frequency $\omega(K,a)$, which now depends on a because this is a non-linear problem, we may solve for the ω_i. In particular,

$$\omega_o = \sqrt{1 + 3S\kappa^2}$$

$$\omega_1 = 0$$

$$\omega_2 = \frac{3K^4}{4\omega_o}(16S^2\kappa^2 + 5S) \qquad (10.4)$$

Upon comparing (10.4) with the corresponding result (5.7) obtained by a direct linearization and elimination of secular terms, we see that it is necessary to identify $\epsilon B = \kappa a$ to make the two expansions of ω agree. This means that the assumed form of expansion of n there,

$$n = 1 + \epsilon B \sin\theta + \ldots$$

becomes here

$$n = 1 + \kappa a \sin\theta + \ldots$$

in terms of the amplitude a.

B. The averaged equations

In order to obtain the small-amplitude expansion of the averaged equations, we shall find it convenient to replace the variable A by a, where a was defined earlier as $\sqrt{2A-S}$. The set of equations we shall work with then becomes (see (8.27-29))

$$\frac{\partial W_U}{\partial T} + U\frac{\partial W_U}{\partial X} - \frac{1}{2\kappa}\frac{\partial a^2}{\partial X} - (W_S + \frac{1}{2\kappa})\frac{\partial S}{\partial X} = 0 \qquad (10.5)$$

$$\frac{\partial \kappa}{\partial T} + \frac{\partial}{\partial X}(\kappa U) = 0 \qquad (10.6)$$

$$\frac{\partial S}{\partial T} = 0 \qquad (10.7)$$

There are three equations for three unknowns, κ, a, and S. We are going to regard U as completely determined in terms of the other variables, via its expansion in powers of a^2 from sub-section A.

The conservation form of the momentum equation

$$\frac{\partial W_U}{\partial T} + \frac{\partial}{\partial X}(UW_U - W) = 0 \qquad (10.8)$$

which follows (8.31), turns out to be more convenient for calculation than (10.5). This advantage is seemingly offset by the necessity of expanding W, however, for W does not appear in (10.5). There is a trick, however, by which such extra effort can be avoided. If we look at W as a function of A, U, and S, we find that (see Eq. (8.21) for definition of W)

$$W(\lambda^2 A,\ \lambda^2 S, \lambda U) = \lambda^3 W(A,S,U)$$

If we differentiate this result with respect to λ, and set $\lambda=1$, we arrive at

$$2AW_A + 2SW_S + UW_U = 3W$$

or

$$a^2 W_A + S(W_A + 2W_S) + UW_U = 3W \qquad (10.9)$$

This is merely a modification of the device used by Euler to study homogeneous functions. With (10.9), we are able to get the $a \rightarrow 0$ expansion of W from those for W_A, W_U, W_S, with comparatively little effort.

Using either (10.5) or (10.8), we shall need a pair of the following small-amplitude expansions:

$$W = \frac{1}{2K} a^2 + \frac{3K^3}{16}(16\ K^2 S^2 + 5S)a^4 + O(a^6)$$

$$W_U = \frac{\omega_o}{2} a^2 + \frac{3SK^4}{16\omega_o}(15 + 95\ SK^2 + 144S^2K^4)a^4 + O(a^6)$$

$$W_S = -\frac{1}{2K} - \frac{3}{4} Ka^2 + O(a^4)$$

These were obtained by the method of sub-section A, or Appendix F. To carry these expansions to more terms would require expanding the dispersion relation out to $O(a^4)$; that is, ω_4 would be required. The dispersion relation to $O(a^2)$ has been used to replace U everywhere in these expansions, so that only κ, S, and a^2 are involved.

Keeping only $O(a^2)$ quantities, and assuming that

$$\frac{\partial a^2}{\partial T}, \frac{\partial a^2}{\partial X} = O(a^2)$$

the averaged equations become

$$\left.\begin{aligned} &\frac{\partial}{\partial T}\left\{\frac{1}{2}\ \omega_o(K)a^2\right\} + \frac{\partial}{\partial X}\left\{\frac{3}{2}\ SKa^2\right\} = 0 \\ &\frac{\partial \kappa}{\partial T} + \frac{\partial}{\partial X}\left\{\omega_o(K) + a^2\omega_2(K)\right\} = 0 \\ &\qquad \frac{\partial S}{\partial T} = 0 \end{aligned}\right\} \qquad (10.10)$$

to $O(a^2)$. The third equation is already in characteristic form; the other two equations may be put in characteristic form:

$$\frac{d\kappa}{dT} \pm \frac{\omega_o}{a}\sqrt{\frac{\omega_o\omega_2}{3S}}\frac{da^2}{dT} + \frac{\partial \omega_o}{\partial S} S'(X) = 0 \qquad (10.11)$$

on

$$\frac{dX}{dT} = C_{\pm} = \frac{\partial \omega_o}{\partial K} \pm a \sqrt{\frac{3S\omega_2}{\omega_o^{\ 3}}} + O(a^2)$$

where

$$\frac{d}{dT} = \frac{\partial}{\partial T} + C_{\pm} \frac{\partial}{\partial X}$$

Only quantities consistent with the order of the approximation have been retained. In particular, $C_{\pm}$ is only correct to two terms, and has been so expanded from the form in which it may be obtained from Eqs. (10.10). Likewise Eq. (10.11) is not the exact characteristic form obtainable from Eqs. (10.10), but rather an expansion of that exact form consistent with the order of the approximation. Note that because $\omega_o > 0$ and $\omega_2 > 0$, it is explicitly verified here that the characteristic velocities $C_{\pm}$ are real as $a \rightarrow 0$ (see §9).

In the special case S(X) = const., we can find Riemann invariants for our problem. Replacing ω_o and ω_2 by their expressions in terms of κ and S, Eq. (10.11) becomes

$$\frac{d\kappa}{dT} \pm \kappa^2 \sqrt{(1+3S\kappa^2)(5+16S\kappa^2)} \quad \frac{da}{dT} = 0$$

which may be integrated

$$a \pm F(\kappa) = \text{const.}$$

where

$$F(K) = \int \frac{dK}{K^2\sqrt{(1+3SK^2)(5+16SK^2)}}$$

F(K) is an elliptic integral of the third kind. By use of the transformation which reduces elliptic integrals to standard form, which in this case is

$$SK^2 = \frac{5}{16}\,\text{ctn}^2\theta$$

we arrive at

$$F(K) = -\frac{4\sqrt{S}}{5}\int^{\tan^{-1}\sqrt{\frac{1}{16SK^2}}} \frac{\tan^2\theta}{\sqrt{1-\frac{1}{16}\cos^2\theta}}\,d\theta$$

A quite good approximation to F(K) is possible if one expands the square root and integrates term-by-term:

$$F(K) = -\frac{4\sqrt{S}}{5}\int \tan^2\theta\left(1 + \frac{1}{32}\cos^2\theta + \ldots\right) d\theta$$

$$\cong \frac{63\sqrt{S}}{80}\tan^{-1}\sqrt{\frac{5}{16SK^2}} - \frac{1}{\sqrt{5}\,K}\left(1 - \frac{\frac{1}{4}SK^2}{5+16SK^2}\right)$$

The terms omitted are of the form

$$a_n\int \sin^2\theta\,\cos^n\theta\,d\theta \qquad (10.12)$$

where $n = 2,4,6,\ldots$ and a_n is negligibly small, the largest being a_2 which is approximately 1/700. Since in general

$$\int \sin^2\theta\,\cos^n\theta\,d\theta = \sum_j \beta_j \sin^{n_j}\theta\,\cos^{m_j}\theta$$

and since

$$\cos\theta = \left(\frac{16SK^2}{5 + 16SK^2}\right)^{1/2}$$

we see that the integrals in (10.12) are $O(1)$ for all K, so that the omitted terms (10.12) are indeed uniformly small for all K.

Knowing explicit expressions for the Riemann invariants, it is possible to study simple wave solutions, etc., just as in gas dynamics. Because a is small, both sets of characteristics will be forward-leaning. This leads to some interesting situations, including the prediction of shocks for many sets of initial conditions. This is in sharp contrast to our usual experience with small-amplitude theories, for we are accustomed to regarding shocks as a finite-amplitude phenomenon. It is important to remember here, however, that we are considering not the LPE's themselves but a derived set of equations which assume slow variations in the relevant wave parameters of a non-linear wave. In the case of shock formation, it is necessary to ask whether the assumption of slow variation is still justified. The answer is no, because then a, U, etc. would experience large changes and/or large gradients over small distances. Yet, in the Euler equations of fluid mechanics, a similar situation was dealt with quite success-fully by putting in jump discontinuities and shock conditions. From their derivation from the Boltzmann equation, it is clear that the Euler equations are only valid for slow variations of ρ, $\underline{u}$, T with respect to a mean free path and mean free time, and gasdynamic shocks violate these restrictions. Because of the success of this method in gas dynamics, Whitham proposed it for the averaged equations as well.[15] The experimental evidence is not yet in, however, and it might well be that the tendency to shock formation in a, U, etc. would be unstable, in the sense that it would tend to destroy the underlying uniform wavetrain.

We have only considered the averaged equations to $O(a^2)$. To proceed to higher orders would involve a marked increase in the algebra with no corresponding increase in understanding. For finite amplitudes, it is probably better to do a numerical solution anyway, which would then naturally cover the small-amplitude case.

C. The averaged equations with dissipation

We consider the averaged equations with dissipation only in the case of no X-dependence

$$U \frac{dW_U}{dT} - W_S D = 0 \tag{10.13}$$

$$\frac{dS}{dT} = D \tag{10.14}$$

$$A + \kappa(UW_U - W) = \text{const.} \tag{10.15}$$

κ is constant in this case, and

$$D = -\frac{\kappa}{2\pi}\int_0^{2\pi} \frac{2r_o n_{o\theta}}{n_o^3}\, d\theta$$

$$= -\frac{\kappa}{2\pi}\oint \frac{2r_o}{n_o^3}\, dn_o$$

For definiteness, we pick a model dissipation term of the form

$$r = -\sigma\frac{\partial n}{\partial x}$$

so that

$$r_o = -\kappa\sigma\frac{\partial n_o}{\partial \theta}$$

Putting this into D, and dropping the zero subscript from now on, we have

$$D = \frac{\sigma\kappa^2}{\pi}\oint \frac{n_\theta}{n^3}\, dn$$

$$= \frac{\sigma\kappa}{\pi}\oint \frac{F^{1/2}(n)}{U^2 - 3Sn^4}\, dn$$

The small-amplitude expansion of D is, by the method of sub-section A,

$$D = \sigma\kappa^4 a^2 + D_2 a^4 + O(a^6)$$

where

$$D_2 = \frac{3}{8}\sigma\kappa^6 (2 + 37S\kappa^2 + 176S^2\kappa^4)$$

The small-amplitude expansion of the energy equation, (10.15), is

$$S + (1+3SK^2)a^2 + \frac{3SK^4}{8}(20+111SK^2+ 144S^2K^4)a^4 = \Sigma + O(a^6) \quad (10.16)$$

where Σ is a constant. It is clear from (10.16) that S is given correctly to $O(a^2)$ by

$$S = \Sigma - (1+3\Sigma K^2)a^2 \qquad (10.17)$$

If we use this expansion for S in the second and third terms on the left-hand side of (10.16), we obtain S correctly to $O(a^4)$

$$S = \Sigma - \Sigma_2 a^2 + \Sigma_4 a^4 \qquad (10.18)$$

where

$$\Sigma_2 = 1 + 3\Sigma K^2$$

$$\Sigma_4 = \frac{3K^2}{8}\left(8+4\Sigma K^2 - 111(\Sigma K^2)^2 - 144(\Sigma K^2)^3\right)$$

Inserting only the two-term expansion (10.17) of the entropy into the entropy equation (10.14),

$$-\Sigma_2 \frac{da^2}{dT} = \sigma K^4 a^2 + O(a^4)$$

whose solution is

$$a^2 = a_o^2 e^{-\gamma T} \quad (\gamma = \sigma K^4/\Sigma_2) \qquad (10.19)$$

This result is consistent with the assumptions used in its derivation if $a_o << 1$ or T is large. In particular, then

$$\frac{da^4}{dT} = O(a^4)$$

In this order of approximation, the entropy is

$$S = \Sigma - (1+3\Sigma K^2)a_o^2 e^{-\gamma T}$$

which increases monotonically to its limiting value Σ as $T \rightarrow \infty$.

By taking the three-term expansion (10.18) for S and the two-term expansion for D and putting them into the entropy equation, we have

$$\Sigma_2 \frac{da^2}{dT} - 2\Sigma_4 a^2 \frac{da^2}{dT} = -\sigma K^4 a^2 - D_2 a^4 + O(a^6)$$

The solution of this, to the order of approximation that we are considering, is

$$a^2 = a_o^2 e^{-\gamma T} + \alpha a_o^4 e^{-2\gamma T} \tag{10.20}$$

where

$$\alpha = \frac{3K^2}{8\Sigma_2}\left[18 + 51\Sigma K^2 + 65(\Sigma K^2)^2 + 240(\Sigma K^2)^3\right]$$

The fact that the correction is strictly positive indicates that the non-linear solution decays less rapidly than the linearized one. To this same order of approximation, the entropy is

$$S = \Sigma - \Sigma_2 a_o^2 e^{-\gamma T} - (\Sigma_2\alpha - \Sigma_4)a_o^4 e^{-2\gamma T}$$

Since

$$\Sigma_2\alpha - \Sigma_4 = \frac{3K^2}{8}\left[10+47\Sigma K^2+ 176(\Sigma K^2)^2+ 384(\Sigma K^2)^3\right]$$

which is strictly positive, we see that the entropy also decays more slowly to its asymptotic value as a result of the non-linearity.

Qualitative arguments concerning the effect of the non-linearity, such as those presented above, were shown to be quite useful in §3, where explicit numerical solutions were available for comparison. They may be expected to be equally useful for the plasma case. Of especial interest would be the effect of the Landau damping term (4.1) on the non-linear decay rates.

The behavior of the damping coefficient

$$\gamma = \frac{\sigma \kappa^4}{1+3\sum \kappa^2}$$

with κ is of some interest, for it turns out that this behavior is the same for all reasonable forms of the damping term, e.g. (4.24). For short wavelengths, $\kappa \rightarrow \infty$, the damping becomes infinite like κ^2, while for long wavelengths, $\kappa \rightarrow 0$, it goes to zero like κ^4. Thus in the long-wavelength limit the damping is quite small compared to the frequency, although the disparity is nowhere near as large as that found by Landau (Eq. (4.19)).

It is also found that the general effect of the non-linearity is independent of the particular form of derivative dissipation chosen. That is why we used $r = -\sigma n_x$ rather than (4.24). In all cases it is found that the non-linearity decreases the decay rate from its linearized value.

§ 11. Variational Formulation of the Lukewarm Plasma Equations

The whole subject of variational principles in continuum mechanics has received a thorough treatment by Seliger and Whitham in Ref. 10. Of particular note is their general conclusion that, for any system of partial differential equations, about half should be identically satisfied by defining appropriate potentials, while the other half should follow from the variational principle.

In the present problem a rather unusual situation prevails, in that the electric field itself acts as a "potential." Looking at the problem of introducing a potential to satisfy the continuity equation

$$n_t + (nu)_x = 0 \ ,$$

we see that this may be done by defining

$$n = -\Psi_x \qquad nu = \Psi_t \ .$$

But, from the current equation $E_t = nu$, it is clear that Ψ will equal E to within a function of x. By the trivial redefinition

$$n = 1 - \Psi_x$$

it is clear that the continuity equation, Poisson equation

$E_x = 1 - n$, and current equation $E_t = nu$ are satisfied by the choice $\psi = E$. Thus we shall formulate our variational principle as in Ref. 10 with n and u assumed to be replaced everywhere according to

$$n = 1 - E_x \tag{11.1}$$

$$u = \frac{E_t}{1 - E_x} \tag{11.2}$$

We shall have to obtain two additional equations out of the variational principle, the equation of motion

$$n(u_t + uu_x) + (Sn^3)_x + nE = 0 \tag{11.3}$$

and the equation of entropy conservation

$$(nS)_t + (nuS)_x = 0 \;. \tag{11.4}$$

The latter we shall take as a side condition on the variational principle, by adding it to the Lagrangian with a Lagrange multiplier η. The basic Lagrangian itself we take to be the difference of the kinetic energy and the sum of the potential energy in the electric field and the random thermal energy measured by p :

$$L = \frac{1}{2}\, nu^2 - \frac{1}{2}\, Sn^3 - \frac{1}{2}\, E^2 + \eta\left[(nS)_t + (nuS)_x\right] \tag{11.5}$$

where n and u are assumed replaced according to (11.1) and (11.2). The variational principle is then

$$\delta \iint_R L \, dx \, dt = 0 \tag{11.6}$$

where R is some region of x - t space.

Variations $\delta\eta$ with respect to η lead to the entropy conservation equation (11.4), as expected. Variations δS in S lead to the Euler equation

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial S_t} + \frac{\partial}{\partial x} \frac{\partial L}{\partial S_x} - \frac{\partial L}{\partial S} = 0 ,$$

which becomes an equation for η

$$\eta_t + u\eta_x + \frac{1}{2} n^2 = 0 . \tag{11.7}$$

For variations δE in E, the Euler equation is

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial E_t} + \frac{\partial}{\partial x} \frac{\partial L}{\partial E_x} - \frac{\partial L}{\partial E} = 0 \tag{11.8}$$

Before dealing with this equation, let us write the Lagrangian in the partially integrated form

$$L = \frac{1}{2} nu^2 - \frac{1}{2} Sn^3 - \frac{1}{2} E^2 - Sn(\eta_t + u\eta_x) \tag{11.9}$$

which is suggested by integration of (11.5) by parts in (11.6). We note in passing that this eliminates double derivatives of E and that the new Lagrangian still gives

(11.4) and (11.7) upon varying η and S, respectively. In terms of E, this new Lagrangian is

$$L = \frac{1}{2}\frac{{E_t}^2}{1 - E_x} - \frac{1}{2} S(1 - E_x)^3 - \frac{1}{2} E^2 - S\eta_t(1 - E_x) - S\eta_x E_t$$

Then (11.8) becomes

$$\frac{\partial}{\partial t}\left(\frac{E_t}{1 - E_x} - S\eta_x\right) + \frac{\partial}{\partial X}\left(\frac{1}{2}\frac{{E_t}^2}{(1 - E_x)^2} + \frac{3}{2} S(1 - E_x)^2 + S\eta_t\right) + E = 0$$

Translating back to n and u and cancelling a pair of terms,

$$\frac{\partial u}{\partial t} - S_t\eta_x + \frac{\partial}{\partial x}\left(\frac{1}{2} u^2 + \frac{3}{2} Sn^2\right) + S_x\eta_t + E = 0$$

Replacing η_t according to its Euler equation (11.7) and using the entropy equation,

$$u_t + uu_x + \frac{3}{2}(Sn^2)_x - \frac{1}{2}n^2 S_x + E = 0$$

The latter is the equation of motion, (11.3), in an expanded form.

Note that if we formally insert the Euler equation (11.7) for η back into the Lagrangian (11.9), it reduces to

$$L = \frac{1}{2} nu^2 - \frac{1}{2} E^2 \qquad (11.10)$$

which is the traditional form of the Lagrangian as the difference of the kinetic and potential energies. This is a purely formal manipulation, however, for (11.10) gives neither the entropy equation nor the equation of motion of the system.

We will now see how the master function W arises from the <u>averaged</u> Lagrangian

$$\langle L \rangle = \frac{1}{2\pi}\int_0^{2\pi} L \, d\theta$$

when all the variables in L are assumed to take their uniform wavetrain forms n(θ), u(θ), etc. We repeat here for convenience certain relations among the uniform wavetrain solutions which we shall need:

$$u = U\left(\frac{n-1}{n}\right) \qquad (11.11)$$

$$E^2 = F(n) = 2A + 2Sn^3 - 3Sn^2 - U^2\left(\frac{n-1}{n}\right)^2 \qquad (11.12)$$

Using the form (11.9) of L, it then follows that

$$\langle L \rangle = \frac{1}{2}\langle nu^2 - Sn^3 - E^2 \qquad - 2Sn\eta_\theta(\kappa u - \omega)\rangle$$

$$= \frac{1}{2}\langle U^2 n\left(\frac{n-1}{n}\right)^2 - Sn^3 - F(n) + 2\omega S\eta_\theta\rangle$$

The last term vanishes, since

$$\langle \eta_\theta \rangle = \frac{1}{2\pi}\int_0^{2\pi} \eta_\theta \, d\theta$$

$$= \frac{1}{2\pi}[\eta(2\pi) - \eta(0)]$$

and this vanishes by the periodicity of η. Then $\langle L \rangle$

reduces to

$$<L> = -A + \frac{1}{2}<U^2(n+1)\left(\frac{n-1}{n}\right)^2 - 3Sn^2(n-1)>. \quad (11.13)$$

We now invoke the definition of an average developed in (8.24), which reads

$$<P(n)> = -\frac{K}{2\pi}\oint \sqrt{F}\,\frac{P(n) + (1-n)P'(n)}{(1-n)^2}\,dn$$

Using this on (11.13), we find

$$<L> = -A - \frac{K}{2\pi}\oint \sqrt{F}\,\left(3Sn - \frac{U^2}{n^3}\right)dn \quad (11.14)$$

It may be verified that

$$3Sn - \frac{U^2}{n^3} = \frac{1}{2}\,\frac{F'(n)}{n-1}$$

The last result, put into (11.14), and followed by an integration by parts, leads to

$$<L> = -A - \frac{K}{6\pi}\oint \frac{F^{3/2}(n)}{(1-n)^2}\,dn \quad (11.15)$$

From the definition of W, (8.21), we see that what we have obtained is essentially W:

$$<L> = -A + KW$$

From this the various averaged equations may be derived as Euler equations. The details of the method are supplied by Whitham.[16]

Part V
Conclusion

Our attention in the preceding pages has been primarily focussed on specific physical problems, first in the theory of water waves and next in plasma waves. The volume of detail in these investigations has perhaps tended to obscure the fundamental methods and ideas which were to be illustrated. It is, therefore, the function of the present section to establish a certain perspective on the results which we have obtained.

Unquestionably the most important result of this thesis is that the averaging method of Whitham can, in the guise of the Luke two-timing procedure, be extended to systems of partial differential equations which are not expressible in conservation form. This extension must, however, be qualified by the proviso that the "dangling terms" (see §8.B) are, or can be made to be, $O(\varepsilon)$, so that in $O(1)$ all equations appear to be in conservation form. It is incidental whether or not these dangling terms represent dissipation, as far as the method is concerned, but of course as far as physical applications are concerned dissipation is of exceeding interest. We discussed the effect of certain model dissipation terms on both the water waves and the plasma waves. For the former, numerical solutions for the case of no X-dependence were found and compared with results from the small-amplitude limit, which took the form

$$(\ )e^{-\gamma T} + (\ )e^{-2\gamma T} + \ldots \qquad \text{(V.1)}$$

We discovered that the second term of the small-amplitude expansion provided a very reliable qualitative description of the deviation from linearity, even in highly non-linear situations. It would be reasonable to expect this to be the case for most wave problems with dissipation.

Non-linear corrections to the linearized damping, à la Eq. (V.1), were also derived for the plasma waves, and in both the plasma waves and the water waves the effect of these non-linear corrections was to decrease the decay rate from the linearized value. That is, all the quantities of interest decayed more slowly than the linearized solution (the one exception was the amplitude of the water wave, which, however, had the small-amplitude behaviour

$$(\ )e^{-\gamma T} + (\ )e^{-3\gamma T} + \ldots$$

so that the increase in its decay rate was the result of higher-order terms). Again, we may take these results as indicators that the general effect of non-linearity will be to decrease decay rates.

In connection with dissipation, a method was proposed for consistently including Landau damping in the plasma equations. What it amounted to was setting the heat flux q to be an integral

$$\frac{\partial q}{\partial x} = \int_{-\infty}^{\infty} K(x - \xi)u\ (\xi,t)\ d\xi \qquad \text{(V.2)}$$

where u(x,t) was the velocity and K(x) was a kernel which could be expressed in terms of certain tabulated functions. The equations become unmanageable analytically with this form of q, so that a numerical solution is called for; one was not attempted here, however.

A second important area considered in connection with the plasma problem was that of the stability of the uniform wavetrain. The method of deducing fully non-linear stability criteria was illustrated, and it was shown that for the plasma case a single inequality

$$W_{AA} < 0 \qquad \text{(V.3)}$$

guaranteed stability, where W was the master function introduced in §8. While (V.3) proved intractable analytically (see Appendix H), it was possible to deduce that at least in the small-amplitude limit the plasma waves are stable.

Tied in closely with the stability question is the problem of deciding what inequalities must be satisfied by the parameters α_i of the uniform wavetrain solution, in order that that solution may exist. It would be senseless to test the stability criteria in regions where these inequalities are not satisfied. In fact, one could be led to erroneous conclusions of instability if one tested the criteria with no knowledge of the permitted region of α_i-space. Thus the delimitation of α_i-space is vital. It was possible to obtain a completely satisfactory answer to this problem in the plasma case, using the method of Sturm sequences. The answer was that: (1) the wave velocity U must be greater than the sound-wave velocity $\sqrt{3S}$; (2) the "linearized" wave amplitude a must be positive; (3) the actual wave amplitude must be less than or equal to the amplitude of the peaked wave. The method used in obtaining these results is recommended highly for all similar investigations. The results themselves are also of great importance because they are so simple and so seemingly general. Perhaps the set (1) - (3) would be sufficient to delimit α_i-space for a wide variety of physical problems.

The extension of the averaging method to non-conservative systems, the studies of dissipation in the non-linear regime, the stability considerations, and the α_i-space results form the essential elements of the present thesis. Certain less major elements are nevertheless also of interest. Foremost of these is the discussion of steady shock solutions where a uniform wavetrain is joined to a constant state (see §2 and §7.C), a situation which can occur in the water waves case but is disallowed for plasmas because the entropy would decrease across such a shock. The analysis of the peaked plasma wave (see Appendix B) has some special twists to it which make it rather unique, among which is the possibility of all peak angles, including $\theta_{peak} \to 0$ in the limit of large velocity U. The small-amplitude expansion of the hyper-elliptic integrals in the averaged equations (see Appendix F and §10) is also a technique worthy of note, for it often furnishes the only analytical hold we can get on the averaged equations. And the discussion of the plasma dispersion relation in the presence of dissipation (§5.B), and of the Lagrangian for the plasma (§11), are both interesting sidelights.

It would be of great interest to obtain many of the same results we have found for a derivative dissipation using the integral dissipation of (V.2). In particular, to shed some light on the long-standing problem of collisionless shocks, we would like to be able to demonstrate or disallow breaking in the presence of (V.2). We would also like to know how expansions such as (V.1) are affected and whether the steady wavetrain is still stable or not.

The answers to questions such as these await new techniques in the theory of integro-differential equations.

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APPENDIX A: The Solitary Wave Solution of the LPE's

The mathematical characterization of the solitary wave solution of an equation like (7.15) is that the quantity under the radical, here Q(u), must have a double root, call it u_o. Then it may be shown that in the neighborhood of $u = u_o$ the solution approaches $u = u_o$ exponentially, so that it never turns back, as it would if $u = u_o$ were a simple root. Thus the wave consists of a single crest or trough with exponential tails. This presumes, of course, the existence of a second root of Q(u), $u = u_1$, which forms the other bound of the "oscillation."

The condition that u_o be a double root of Q(u) may be expressed by the two statements

$$Q(u_o) = 0$$
$$Q'(u_o) = 0$$

which in terms of $w_o = U - u_o$ (see Eq. (7.18)) become

$$-w_o^5 + 2Uw_o^4 + (2A-U^2)w_o^3 - 3SU^2w_o + 2SU^3 = 0$$

$$5w_o^4 - 8Uw_o^3 - 3(2A-U^2)w_o^2 + 3SU^2 = 0$$

The second of these may be solved for $(2A-U^2)$,

$$2A-U^2 = \frac{5w_o^4 - 8Uw_o^3 + 3SU^2}{3w_o^2}$$

and the result put into the first equation to yield

$$w_o^5 - Uw_o^4 - 3SU^2w_o + 3SU^3 = 0$$

This may be factored

$$(w_o-U)(w_o^4-3SU^2) = 0$$

The candidates for w_o are thus

$$w_o = U, \pm (3SU^2)^{\frac{1}{4}}$$

or, in terms of u_o,

$$u_o = 0, U(1\pm\beta)$$

where $\beta = (3S/U^2)^{\frac{1}{4}}$:

We reject the root $U(1+\beta)$ because it is larger than U (the reason for only considering roots smaller than U is given in §7.A). The root $U(1-\beta)$ is none other than u_∞, and the case when u_∞ is a double root is treated in Appendix B (the solution then is not a solitary wave). So we are left with $u_o = 0$ as the only possibility. When we set $Q(0) = 0$, we find

$$a^2 = 2A - S = 0$$

so that the quantity we thought of as a measure of amplitude is here zero. Accepting this, we put $a^2 = 0$ back into Q(u) to obtain

$$Q(u) = u^2 P(w)$$

where

$$P(w) = -w^3 + Sw + 2SU$$

The other "limit of oscillation" $u = u_1$ must be a root of $P(w) = 0$.

Q(u) must be positive in the interval between $u = 0$ and $u = u_1$ for a solution to exist. This means

$$P(w)\Big|_{u=0} = - U^3 + 3SU > 0$$

or $U^2 < 3S(\beta>1)$. Thus the conditions $a^2 > 0$ and $U^2 > 3S$ are both violated, and if a solitary wave solution exists, it cannot be developed continuously from a linearized solution (see Eq. (7.19)).

From two applications of Descartes' Rule of Signs,[21] once to P(w) as given and once to P(w) as a function of u, using $\beta > 1$ we are able to show that P(w) has only one root in $u < 0$ and no roots in $0 < u < U$. Thus the root in $u < 0$ must be u_1.

This presents a problem for, because $\beta > 1$, we also have $u_\infty = U(1-\beta) < 0$, and as we explained in §7.A, u_∞ may not lie within the range of oscillation of the solution. Thus a solution will exist only if $u_\infty \leq u_1 \leq 0$. It is a simple matter to check the location of u_∞ relative to u_1, for we know that

$$\text{(A.1)} \qquad P(w) \begin{cases} > 0 & u_1 < u < 0 \\ < 0 & u < u_1 \end{cases}$$

from the last paragraph. Thus the sign of P(w) evaluated at $u = u_\infty$ will determine the location of u_∞. We find

$$P(w_\infty) = -(\beta U)^3 + S\beta U + 2SU$$

or, writing S in terms of U and β from the definition of β,

$$P(w_\infty) = \frac{1}{3}(\beta U)^3 (\beta+3)(\beta-1)$$

Because $\beta > 1$,

$$P(w_\infty) > 0$$

which places u_∞ between $u = u_1$ and $u = 0$ according to (A.1). Thus no solitary wave solution is possible.

APPENDIX B: The Peaked Wave Solution of the LPE's

The peaked wave solution comes about when the upper limit of oscillation, u_2, coincides with u_∞. If this were the whole story, then it could be demonstrated from Eq.(7.15) that the solution would behave near $u = u_2 = u_\infty$ as indicated in Figure B-1; that is, it would have a cusped peak formed from two parabolic arcs (a "square-root cusp"). It turns out, however, that $u_2 = u_\infty$ necessarily implies $u_2 = u_3$, so that the solutions are peaked waves with finite peak angles.

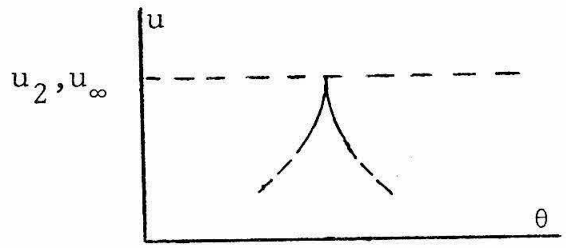


Figure B-1. Cusp Peak

If we demand that $u_\infty = u_2$, then u_∞ is a root of Q(u), and the equation

$$Q(u_\infty) = Q\{U(1-\beta)\} = 0$$

can be reduced to

(B.1)
$$\alpha = 2\beta^2 - \frac{8}{3}\beta + 1$$

where

$$\alpha = \frac{2A}{U^2} \qquad \beta = \left(\frac{3S}{U^2}\right)^{\frac{1}{4}}$$

If we now factor Q(u),

$$Q(u) = (u-u_\infty)P_4(u)$$

then we shall discover that $P_4(u)$ contains an additional factor of $(u-u_\infty)$, giving finally

$$Q(u) = (u-u_\infty)^2 P_3(w)$$

where, with $w = U-u$,

$$\text{(B.2)} \qquad P_3(w) = -w^3 + 2(1-\beta)Uw^2 + \beta(\tfrac{4}{3} - \beta)U^2 w + \tfrac{2}{3}\beta^2 U^3$$

and where of course A has been replaced according to (B.1). Thus if u_∞ is a single root, it must be a double root, and since u_3 is the only other available positive root of Q(u), it must be that

$$u_2 = u_\infty = u_3.$$

Because $U^2 > 3S$, we have $0 < \beta < 1$, and for this range of β it may be shown that the discriminant of $P_3(w)$ is positive. This means that $P_3(w)$ has only one real root, w_1, and so of necessity $w_1 = U - u_1$, where u_1 is the root referred to in §7.A. We may write down an expression for w_1 from the cubic formula

$$\text{(B.3)} \qquad w_1 = \frac{U}{3}\left[(F+G)^{1/3} + (F-G)^{1/3} - 2(\beta-1)\right]$$

where

$$F = \beta^3 + 12\beta^2 - 12\beta + 8$$

$$G = 2\beta\sqrt{9\beta^3 + 15\beta^2 - 28\beta + 24}$$

The discriminant is proportional to the quantity under the square root in G.

A quantity of some interest for this special case, primarily because we have explicit formulas for its calculation, is the amplitude $u_2 - u_1 = u_\infty - u_1$ of the wave. In the limits $\beta \to 0$ and $\beta \to 1$, it is found to be

$$u_2 - u_1 = 2U - \frac{7}{3}U\beta + O(\beta^2) \qquad \text{as} \quad \beta \to 0$$

$$= \frac{3}{2}u_\infty + O\{(1-\beta)^2\} \qquad \text{as} \quad \beta \to 1$$

where $u_\infty = U(1-\beta)$. The limit $\beta \to 0$ corresponds to $U \to \infty$, and is the large-amplitude limit. In that limit, u_1 and u_∞ are about equally spaced on either side of $u = 0$. The small-amplitude limit corresponds to $\beta \to 1$ $(U \to \sqrt{3S})$, and in that limit u_∞ is approximately twice as far from $u = 0$ as u_1 is,

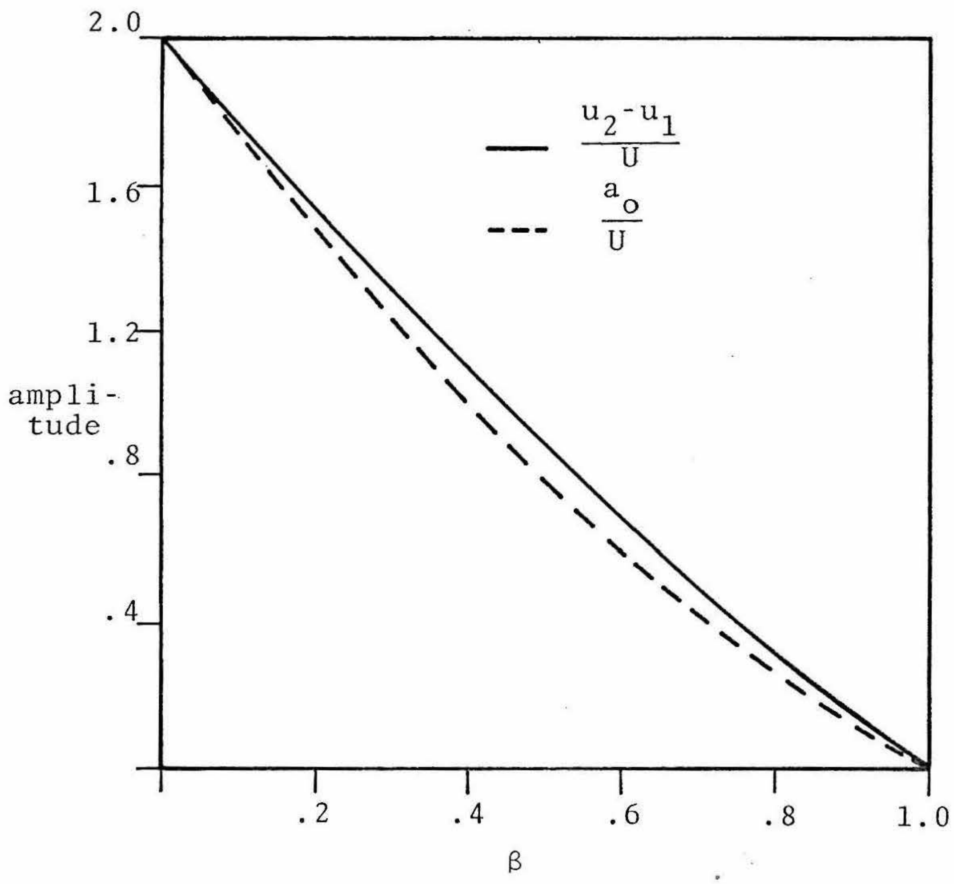


Figure B-2. The true amplitude $(u_2 - u_1)$ and the linearized amplitude a_o of the peaked solution plotted against $\beta = (3S/U^2)^{1/4}$.

making for a somewhat top-heavy wave. The exact amplitude for all β, scaled by U, may be computed using Eq. (B.3) and $u_\infty = U(1-\beta)$; the results are plotted in Figure B-2. The approximate linearity of the curve should be noted.

On the same graph, we have plotted the amplitude which would be deduced from linearized theory. From the linearized form (7.19) of the steady wave equation, this amplitude can be seen to be

$$a_o = 2\sqrt{\frac{U^2(2A-S)}{U^2-3S}} = 2\sqrt{\frac{2A-S}{1-\beta^4}} = 2U\sqrt{\frac{\alpha-\frac{1}{3}\beta^4}{1-\beta^4}}$$

When α is replaced according to Eq. (B.1) and the resultant polynomial in β factored, this reduces to

$$a_o = 2U(1-\beta)\sqrt{\frac{3+\beta}{3(1+\beta)(1+\beta^2)}}$$

Again, the scaled amplitude a_o/U is plotted in Figure B-2. The two curves are in reasonably good agreement, showing that the amplitude from linear theory is a good approximation to the fully nonlinear wave amplitude (this holds generally, although we have only demonstrated it for the peaked case).

The form of the peak may be deduced from the governing differential equation (7.15). Approximating to this equation in the neighborhood of $u = u_\infty$,

$$\kappa u_\theta \approx \frac{\pm w_\infty^{3/2}}{4w_\infty^3\ (u-u_\infty)}\ (u-u_\infty)\ \sqrt{P_3(w_\infty)} \tag{B.4}$$

$$\approx \pm \frac{\sqrt{P_3(w_\infty)}}{4w_\infty^{3/2}}$$

The ± sign comes about because $\sqrt{(u-u_\infty)^2}$ may be of either sign. And at $u = u_\infty$, a solution may in fact switch branches continuously, from the branch with the + sign to the branch with the - sign or vice versa. This is indeed how the peak is formed.

From Eq. (B.4), the peak is clearly made up of two straight line segments of slopes,

$$\pm \frac{\sqrt{P_3(w_\infty)}}{4w_\infty^{3/2}} = \pm \frac{1}{2}\sqrt{\frac{1-\beta}{\beta}} \tag{B.5}$$

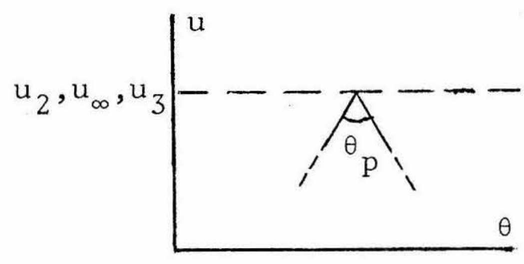
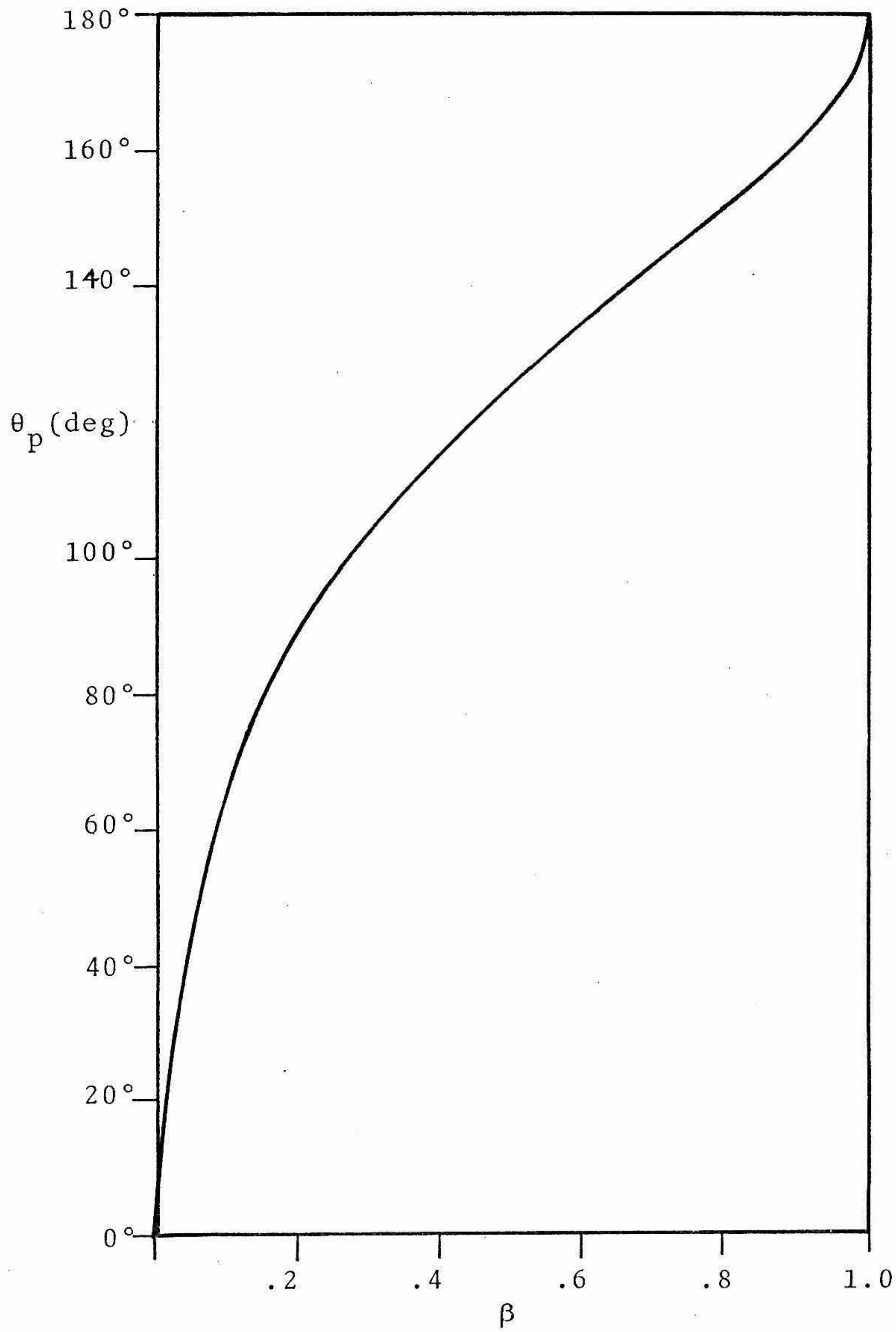


Figure B-3. Local behavior of the peaked solution

(the second expression being obtained by using the explicit form of P_3). These segments meet and form a finite peak angle θ_p, as indicated in Figure B-3. The formula for θ_p is easily found from (B.5) to be

$$\tan \frac{\theta_p}{2} = 2\sqrt{\frac{\beta}{1-\beta}}$$

Figure B-4. The peak angle θ_p as a function of $\beta = (3S/U^2)^{1/4}$.



which is plotted in Figure B-4. Notice that the peaks are very flat ($\theta_p \to \pi$) in the small amplitude limit $\beta \to 1$, and very accentuated ($\theta_p \to 0$) in the large amplitude limit $\beta \to 0$.

Since small amplitudes are certainly allowed in the present case, the question might well be asked, why are peaked solutions not found in the linearized theory? The answer is, the linearized theory as derived from (7.19) or directly from the linearized LPE's ignores the existence of u_∞. That is, it regards β as fixed, and u_∞ as fixed, and then lets the amplitude go to zero, so that we have $u_2 < u_\infty$ and in fact $u_2 << u_\infty$. The failure of the linearized theory to come up with peaked solutions thus lies in the nature of the limiting processes implicit in linearization.

APPENDIX C: Numerical Calculations of Root Structure of Q(u)

By defining

$$\bar{u} = \frac{u}{U}$$

Q(u) becomes

$$Q(u) = U^5\left[(1-\bar{u})^3\left(\frac{a^2}{U^2} - \bar{u}^2\right) - \frac{1}{3}\beta^4\bar{u}^2(\bar{u}-3)\right]$$

where $\beta = (3S/U^2)^{\frac{1}{4}}$. Hence U functions primarily as a scale factor for u and for the amplitude a, and so for our present purposes we may without loss of generality take U = 1. Then for three values of β, $\beta = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, we watch the development of the roots u_1, u_2, u_3 of Q(u) as a increases from small values. The results are presented in Table C-1.

In all cases the roots u_1 and u_2, and hence the actual wave height u_2-u_1, increase in approximate proportionality to a. This bears out the assumption of §7.A that a is a good measure of amplitude. Actually, u_2 increases somewhat more rapidly than the first power of a as it nears u_∞, because the wave is becoming more peaked and yet must still satisfy the area conservation equations (7.22) and (7.23),

$$\frac{1}{2\pi}\int_0^{2\pi} n(\theta)d\theta = 1$$

$$\int_0^{2\pi} n(\theta)u(\theta)d\theta = 0$$

independently of a. While these do not directly imply anything about the area under $u(\theta)$, the data seem to indicate that that area varies little with a, so that the peaks, which enclose less area, must rise more rapidly than the troughs in order to keep the balance.

Table C-1 also provides quantitative substantiation for various facts about the roots deduced in §7. For example, it is clear that $u_2 < u_\infty < u_3$, that $u_2 \to u_\infty$ from below and that $u_3 \to u_\infty$ from above as a increases, and that when a increases beyond the point where $u_2 = u_3 = u_\infty$ there is no longer a solution, so that $u_2 = u_3 = u_\infty$ represents the solution of maximum amplitude.

	$\beta = \frac{1}{4}, u_\infty = \frac{3}{4}$			$\beta = \frac{1}{2}, u_\infty = \frac{1}{2}$			$\beta = \frac{3}{4}, u_\infty = \frac{1}{4}$		
a	u_1	u_2	u_3	u_1	u_2	u_3	u_1	u_2	u_3
.025	-.0250	.0250	.8592	-.0258	.0259	.6331	-.0297	.0308	.3446
.050	-.0501	.0501	.8591	-.0514	.0519	.6325	-.0586	.0633	.3406
.075	-.0751	.0752	.8589	-.0770	.0781	.6315	-.0868	.0983	.3334
.100	-.1002	.1003	.8586	-.1025	.1045	.6300	-.1145	.1374	.3211
.125	-.1252	.1253	.8582	-.1279	.1311	.6281	-.1418	.1866	.2981
.150	-.1502	.1505	.8578	-.1533	.1580	.6256	-.1687	-	-
.20	-.2002	.2007	.8566	-.2039	.2131	.6188			
.25	-.2503	.2511	.8549	-.2544	.2706	.6087			
.30	-.3003	.3016	.8528	-.3048	.3327	.5931			
.35	-.3503	.3522	.8501	-.3550	.4064	.5650			
.40	-.4003	.4032	.8466	-.4052	-	-			
.50	-.5003	.5069	.8363						
.60	-.6003	.6173	.8165						
.675	-.6753	.7389	.7604						
.70	-.7003	-	-						

Table C-1. A numerical study of the real roots of $Q(u) = 0$ for $U = 1$. The hyphens indicate that u_2 and u_3 have become complex, and remain so for all larger values of <u>a</u>. The root u_1 is not tabulated after this happens.

APPENDIX D: The Sturm Sequence Calculation for Q(u)

We shall not only obtain the direct results of the Sturm sequence calculation here, but also demonstrate how these results may be tremendously simplified using various results of §7.A and Appendix A.

We shall work with a scaled version of Q(u)

$$P(V) = -V^5 + 2V^4 + (\alpha-1)V^3 - \beta^4 V + \frac{2}{3}\beta^4$$

where

$$\alpha = \frac{2A}{U^2} \qquad \beta = \left(\frac{3S}{U^2}\right)^{\frac{1}{4}}$$

and which is related to Q(u) by

$$P(V) = \frac{1}{U^5} Q(U-UV)$$

This is essentially the form (7.18) of Q with w replaced by UV to eliminate U as an independent third parameter.

The inequality $a^2 = 2A-S > 0$ becomes in the present notation

(D.1) $$\frac{a^2}{U^2} = \alpha - \frac{1}{3}\beta^4 > 0$$

which proves to be extremely useful in what follows.

By proceeding according to the algorithm of §7.B, we derive the members of the Sturm sequence as:

$$P_2(V) = P'(V) = -5V^4 + 8V^3 + 3(\alpha-1)V^2 - \beta^4$$

$$P_3(V) = -(5\alpha+3)V^3 - 3(\alpha-1)V^2 + 10\beta^4 V - \frac{22}{3}\beta^4$$

$$P_4(V) = 3\left\{(10\alpha+6)\beta^4 - 3\alpha(\alpha-1)^2\right\}V^2 - 8\beta^4(11\alpha+3)V + \beta^4(3\alpha^2+52\alpha+9)$$

$$P_5 = CV + D$$

$$P_6 = 1296\beta^8 - 8(81\alpha^2+414\alpha+17)\beta^4 + 81(\alpha-1)^4$$

where

$$C = -360\beta^8 + 2(99\alpha^2+496\alpha-3)\beta^4 - 9\alpha(\alpha-1)^2(3\alpha-11)$$

$$D = 264\beta^8 - 2(111\alpha^2+328\alpha+9)\beta^4 + 27\alpha(\alpha-1)^3$$

In these computations, use has been made of the fact that a positive factor may be dropped from any member of the sequence without loss of generality. In particular, two positive factors have been dropped from P_6. One of them is $(\alpha-\frac{1}{3}\beta^4)$, which is positive by Eq. (D.1). The other is

$$(D.2) \qquad \left\{(10\alpha+6)\beta^4 - 3\alpha(\alpha-1)^2\right\}^2$$

which we may note is the square of the leading coefficient of $P_4(V)$. It is quite natural that the leading coefficient of P_4 should be a factor of P_6, for if it vanishes, then P_4 will be linear in V, P_5 will be a constant, and P_6 will vanish. However, this type of reasoning cannot be pushed back another step, for there are no factors of $(5\alpha+3)$, the leading coefficient of P_3, in P_6. There is, however, a double factor of $(5\alpha+3)$ in P_5 (in both C and D) which has been dropped. Thus the general rule seems to be, that the leading coefficient of P_n is a factor in P_{n+2}.

It is also natural that $(\alpha-\frac{1}{3}\beta^4)$ is a factor in P_6, in light of the interpretation placed on P_6 in §7.B. There it was stated that the vanishing of P_6 signaled a double root of Q(u). Going back to the form (7.17) of Q(u), we see that when a = 0, u = 0 is a double root of Q(u). Thus P_6 has to vanish when $\alpha = \frac{1}{3}\beta^4$ (a = 0).

The reasoning in the last paragraph can be extended to obtain two more factors of P_6. In Appendix A we found that, in addition to u = 0, Q(u) might have the double roots $u_I = U(1-\beta)$ and $u_{II} = U(1+\beta)$. Each of these situations leads to a constraining relation between α and β, which is conveniently expressed by

$$Q(u_I) = U^5 P(\beta) = \beta^3 U^5 (\alpha - 2\beta^2 + \tfrac{8}{3}\beta - 1) = 0$$

$$Q(u_{II}) = U^5 P(-\beta) = -\beta^3 U^5 (\alpha - 2\beta^2 - \tfrac{8}{3}\beta - 1) = 0$$

Thus it is logical to seek factors $(\alpha - 2\beta^2 + \frac{8}{3}\beta - 1)$ and $(\alpha - 2\beta^2 - \frac{8}{3}\beta - 1)$ in P_6. Such factors are indeed found, and the resultant factored form of P_6 is

$$P_6 = 9(\alpha - 2\beta^2 + \tfrac{8}{3}\beta - 1)(\alpha - 2\beta^2 - \tfrac{8}{3}\beta - 1)\left\{9(1 - \alpha - 2\beta^2)^2 + 64\beta^2\right\}$$

Since the last factor is strictly positive, we drop it, leaving P_6 in the final form

$$P_6 = (\alpha - 2\beta^2 + \tfrac{8}{3}\beta - 1)(\alpha - 2\beta^2 - \tfrac{8}{3}\beta - 1) \tag{D.3}$$

We may now proceed to the evaluation of the variation-of-sign functions $v(0)$ and $v(1)$, from which we will obtain the number of roots of $P(V)$ between $V = 0$ and $V = 1$ (which means the number of zeroes of $Q(u)$ between $u = 0$ and $u = U$). Evaluating the Sturm sequence at $V = 0$ gives

$$P(0) = \tfrac{2}{3}\beta^4 > 0 \qquad P_4(0) = \beta^4(3\alpha^2 + 52\alpha + 9) > 0$$

$$P_2(0) = -\beta^4 < 0 \qquad P_5(0) = D$$

$$P_3(0) = -\tfrac{44}{75}\beta^4 < 0 \qquad P_6(0) = P_6$$

while for V = 1:

$$P(1) = \alpha - \frac{1}{3}\beta^4 > 0$$

$$P_2(1) = 3(\alpha - \frac{1}{3}\beta^4) > 0$$

$$P_3(1) = -\frac{16}{25}(\alpha - \frac{1}{3}\beta^4) < 0$$

$$P_4(1) = -9(\alpha-1)^2(\alpha - \frac{1}{3}\beta^4) < 0$$

$$P_5(1) = 72(\alpha - \frac{1}{3}\beta^4)\left\{4\beta^4 + (\alpha-1)^2\right\} > 0$$

$$P_6(1) = P_6$$

$P_5(1)$ is the only one of the quantities given (other than P_6 and D) which required factoring, and in view of the four results immediately above it, it was not too surprising to find it had the factor $(\alpha - \frac{1}{3}\beta^4)$.

The variation-of-sign functions obtained from the above data are

$$v(0) = \begin{cases} 2 & \text{if } D > 0 \text{ and } P_6 > 0 \\ 3 & \text{if } P_6 < 0 \\ 4 & \text{if } D < 0 \text{ and } P_6 > 0 \end{cases}$$

$$v(1) = \begin{cases} 2 & \text{if } P_6 > 0 \\ 3 & \text{if } P_6 < 0 \end{cases}$$

The number of roots between V = 0 and V = 1, which is given by

$$\text{No. roots} = \nu(0) - \nu(1)$$

will equal two if and only if the inequalities

$$D < 0$$

$$P_6 > 0$$

are fulfilled.

We may simplify the inequality $P_6 > 0$, and then show that the inequality $D < 0$ is redundant. Beginning with $P_6 > 0$ and P_6 in the form (D.3), it is clear that either both factors in P_6 must be positive or both negative. Because $\beta > 0$, one of the factors is always greater than the other

$$\alpha - 2\beta^2 + \frac{8}{3}\beta - 1 > \alpha - 2\beta^2 - \frac{8}{3}\beta - 1$$

and so the case where they are both positive reduces to

$$\alpha > 2\beta^2 + \frac{8}{3}\beta + 1 \qquad \text{(D.4)}$$

while the case where they are both negative reduces to

$$\alpha < 2\beta^2 - \frac{8}{3}\beta + 1 \qquad \text{(D.5)}$$

The inequality (D.4) may be rewritten

$$\frac{a^2}{U^2} = \alpha - \frac{1}{3}\beta^4 > -\frac{1}{3}\beta^4 + 2\beta^2 + \frac{8}{3}\beta+1$$

$$= \frac{1}{3}(3-\beta)(1+\beta)^3 \equiv F(\beta)$$

Now since $U^2 > 3S$, $\beta < 1$, and over the interval $0 \leq \beta \leq 1$ the function $F(\beta)$ takes an absolute minimum of unity (at $\beta = 0$). Hence

$$\frac{a^2}{U^2} > 1 \tag{D.6}$$

But in §7.A it was remarked that $u_2 > a$, so that we are forced by (D.6) to conclude that

$$u_2 > U$$

which is not allowed. Thus the case (D.4) is physically unrealizable, and $P_6 > 0$ reduces to (D.5).

The inequality $D < 0$ may be put in the form

$$G(\alpha,\beta) = 27\alpha(1-\alpha)^3 + 2(111\alpha^2+328\alpha+9)\beta^4 - 264\beta^8 > 0$$

Using $\alpha > \frac{1}{3}\beta^4$ in the first term of G leads to

$$G(\alpha,\beta) > \beta^4\left[9(1-\alpha)^3 + 2(111\alpha^2+328\alpha+9) - 264\beta^4\right]$$

Using $-\beta^4 > -3\alpha$ in the last term,

$$G(\alpha,\beta) > \beta^4\left[9(1-\alpha)^3 + 2(111\alpha^2+328\alpha+9) - 792\alpha\right]$$

$$= \beta^4(-9\alpha^3+249\alpha^2-163\alpha+27)$$

$$= \beta^4(3\alpha-1)^2(27-\alpha)$$

Hence $G > 0$ automatically provided that $\alpha < 27$. But in fact $\alpha < 1$, as can be verified from (D.5), for over the range $0 \leq \beta \leq 1$ the function $(2\beta^2-\frac{8}{3}\beta+1)$ takes an absolute maximum of unity. Hence the inequality $D < 0$ is implied by (D.5), and may be dispensed with entirely.

Therefore the single necessary and sufficient condition for $Q(u)$ to have two zeroes on $(0,U)$ is

$$\alpha < 2\beta^2 - \frac{8}{3}\beta+1$$

Nothing has been assumed in this derivation beyond the positivity of (U^2-3S), $(\alpha-\frac{1}{3}\beta^4)$, S, and $U-u$.

APPENDIX E: Computation of the Averaged Quantities for the Lukewarm Plasma Equations

The values of the following averaged quantities are needed in terms of **W** and its derivatives:

$$A_1 = \langle nu^2 + Sn^3 + E^2 \rangle$$

$$A_2 = \langle nu^3 + 3Sn^3 u \rangle$$

$$A_3 = \langle (n-1)u \rangle$$

$$A_4 = \langle nu^2 - \frac{1}{2} u^2 + Sn^3 - \frac{3}{2} Sn^2 - \frac{1}{2} E^2 \rangle$$

$$A_5 = \langle \frac{1}{2} n^3 \rangle$$

A_1 and A_2 are needed in the averaged energy equation (8.7); A_3, A_4, and A_5 are needed in the averaged momentum equation (8.12).

The last three averages are simpler, and so we shall do them first. Eq. (7.11) of §7.A, which is

$$nu^2 - \frac{1}{2} u^2 + Sn^3 - \frac{3}{2} Sn^2 - \frac{1}{2} E^2 - U(n-1)\, u = -A$$

allows us to simplify A_4:

$$\begin{aligned} A_4 &= U \langle (n-1)u \rangle - A \\ &= UA_3 - A \end{aligned} \qquad \text{(E.1)}$$

Then with Eq. (7.8), which may be solved for u,

$$u = U\left(\frac{n-1}{n}\right) \tag{E.2}$$

we have

$$A_3 = U \left< \frac{(n-1)^2}{n} \right>$$

The form (8.24) of the averaging operation

$$\left< P(n) \right> = -\frac{\kappa}{2\pi} \oint \sqrt{F(n)}\, \frac{P + (1-n)P'}{(1-n)^2}\, dn \tag{E.3}$$

produces the following results for A_3 and A_5:

$$A_3 = \frac{\kappa U}{2\pi} \oint \frac{\sqrt{F(n)}}{n^2}\, dn$$

$$A_5 = \frac{\kappa}{4\pi} \oint \frac{2n^3 - 3n^2}{(n-1)^2} \sqrt{F(n)}\, dn$$

The U- and S-derivatives of W may be computed from (8.21) as

$$W_U = \frac{U}{2\pi} \oint \frac{\sqrt{F(n)}}{n^2}\, dn$$

$$W_S = -\frac{1}{4\pi} \oint \frac{2n^3 - 3n^2}{(1-n)^2} \sqrt{F(n)}\, dn$$

whereupon we may identify immediately that

$$A_3 = \kappa W_U$$

$$A_5 = -\kappa W_S$$

From (E.1) we then have

$$A_4 = \kappa U W_U - A$$

The energy equation averages A_1 and A_2 are harder. We begin by reducing them to "n-form" using (E.2) and Eq. (8.12.1) to replace u and E in favor of n:

$$A_1 = 2A + U^2 \left\langle \frac{(n-1)^3}{n^2} \right\rangle + 3S \langle n^3 - n^2 \rangle$$

$$A_2 = U^3 \left\langle \frac{(n-1)^3}{n^2} \right\rangle + 3SU \langle n^3 - n^2 \rangle$$

These are patently related,

$$A_2 = U(A_1 - 2A) \tag{E.4}$$

Using (E.3), the last expression for A_1 assumes the loop integral form

$$A_1 = 2A + \frac{\kappa}{\pi} \oint \left(3Sn + U^2 \frac{n-1}{n^3}\right) \sqrt{F}\, dn$$

From the formula for W_U above, this simplifies slightly:

$$A_1 = 2A + 2\kappa U W_U + \frac{\kappa}{\pi} \oint \left(3Sn - \frac{U^2}{n^3}\right) \sqrt{F}\, dn$$

We may now identify the last term of A_1 as a modified form of W. A straightforward integration by parts applied to the original form (8.21) of W yields

$$W = \frac{1}{4\pi} \oint \frac{F'(n)}{1-n} \sqrt{F} \; dn$$

$$= \frac{1}{2\pi} \oint \left(\frac{U^2}{n^3} - 3Sn\right)\sqrt{F} \; dn$$

This result may be substituted into (E.5) to yield

$$A_1 = 2A + 2\kappa U W_U - 2\kappa W$$

Then from (E.4) we have finally

$$A_2 = U(2\kappa U W_U - 2\kappa W)$$

Collecting all of our results,

$$A_1 = 2(A + \kappa U W_U - \kappa W)$$

$$A_2 = 2\kappa U \; (U W_U - W)$$

$$A_3 = \kappa W_U$$

$$A_4 = \kappa U W_U - A$$

$$A_5 = -\kappa W_S$$

Appendix F: Manipulation of Loop Integrals in the Complex Plane

We consider here integrals of the general form

$$\int_a^b \frac{P(x)}{\sqrt{R(x)}}\, dx \tag{F.1}$$

where P(x) and R(x) are polynomials of arbitrary degree and a, b are simple real roots of R(x).

To illustrate the techniques involved, it will be sufficient to consider the special cases in which P(x) = 1 and R(x) is a quadratic or a cubic. The methods employed for these two special cases will then be readily extensible to the more complicated integrals of the type (F.1).

The first example we shall consider is

$$I_1 = \int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$$

By the transformation

$$x = \frac{a+b}{2} + \frac{b-a}{2}\, z$$

this becomes

$$I_1 = \int_{-1}^{1} \frac{dz}{\sqrt{1-z^2}}$$

If we cut the complex z-plane from -1 to +1, and define

$\sqrt{1-z^2}$ to be positive on the top side of the cut, then I_1 may be viewed as an integral along the top side of the cut. Since $\sqrt{1-z^2}$ is then negative on the bottom side of the cut, I_1 is also the negative of the integral along the bottom side of the cut. This suggests writing I_1 as

$$I_1 = \frac{1}{2} \oint_{\Gamma_1} \frac{dz}{\sqrt{1-z^2}}$$

where Γ_1 is the closed contour shown

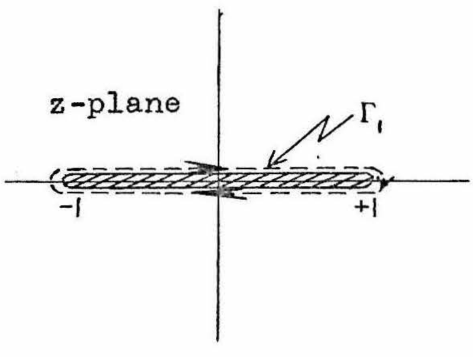


Figure F-1.

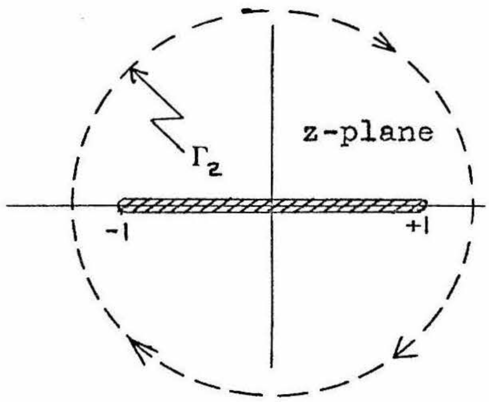


Figure F-2.

in Figure F-1. A rigorous argument, using the usual vanishingly small loops around the branch points, shows this to be correct.

But now, since there are no other singularities of the integrand in the finite part of the plane, we may write I_1 as

$$I_1 = \frac{1}{2} \oint_{\Gamma_2} \frac{dz}{\sqrt{1-z^2}}$$

where Γ_2 is any closed contour enclosing both branch points; this follows from Cauchy's Theorem. By taking Γ_2 to be a contour outside $|z| = 1$, as shown in Figure F-2, we may take advantage of Laurent's Theorem, which holds in the annulus $1 < |z| < \infty$ where $1/\sqrt{1-z^2}$ is analytic. Laurent's Theorem guarantees that $1/\sqrt{1-z^2}$ has an expansion of the form

$$\frac{1}{\sqrt{1-z^2}} = \sum_{n=-\infty}^{\infty} a_n z^n \tag{F.2}$$

in $1< |z| <\infty$, and that furthermore this expansion is uniformly convergent (in any closed sub-annulus). This latter property allows us to integrate the expansion term-by-term

$$\oint_{\Gamma_2} \frac{dz}{\sqrt{1-z^2}} = \sum_{n=-\infty}^{\infty} a_n \oint_{\Gamma_2} z^n \, dz = 2\pi i a_{-1}$$

The expansion of $1/\sqrt{1-z^2}$ is readily found for $|z| > 1$:

$$\frac{1}{\sqrt{1-z^2}} = \frac{1}{iz} \left(1-\frac{1}{z^2}\right)^{-\frac{1}{2}} = \frac{1}{iz} + \frac{1}{2iz^3} + \ldots$$

Since the Laurent expansion is unique, this must be identical to (F.2). We identify $a_{-1} = \frac{1}{i}$, whereupon

$$I_1 = \pi i a_{-1} = \pi$$

Now consider the integral

$$I_2 = \int_a^b \frac{dx}{\sqrt{(x-a)(b-x)(c-x)}} \tag{F.3}$$

where c>b>a. By the same transformation as before,

$$I_2 = A\int_{-1}^{1} \frac{dz}{\sqrt{(1-z^2)(1-Bz)}}$$

where

$$A = \frac{1}{\sqrt{c-\frac{a+b}{2}}} \qquad B = \frac{b-a}{2c-(a+b)}$$

To move into the complex plane this time, we need an additional branch cut from $z = \frac{1}{B}$ (we can easily show that $\frac{1}{B} > 1$) to $z = \infty$. Thus the annulus in which we apply Laurent's Theorem is now finite, $1 < |z| < \frac{1}{B}$. Otherwise, everything goes through the same as before, and we may

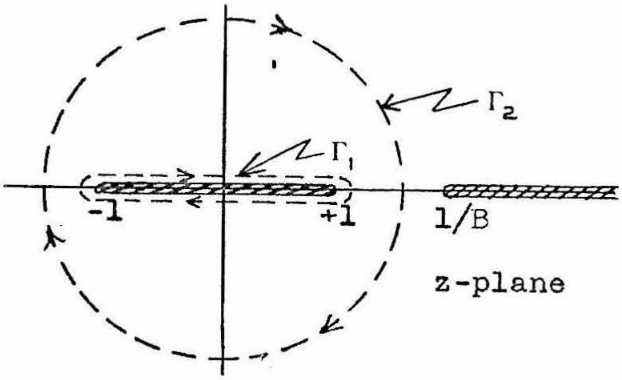


Figure F-3.

calculate as follows:

$$I_2 = \frac{1}{2} A \oint_{\Gamma_1} \frac{dz}{\sqrt{(1-z^2)(1-Bz)}}$$

$$= \frac{1}{2} A \oint_{\Gamma_2} \frac{dz}{\sqrt{(1-z^2)(1-Bz)}}$$

$$= \frac{1}{2} A \oint_{\Gamma_2} \frac{1}{iz} \sum_{n=o}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})} \frac{1}{z^{2n}} \sum_{m=o}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{m!\ \Gamma(\frac{1}{2})} (Bz)^m$$

$$= \frac{A}{2i} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\pi n!\ m!} B^m \oint_{\Gamma_2} z^{m-2n-1}\,dz$$

$$= \frac{A}{2i} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\pi n!\ m!} B^m\, 2\pi i \delta_{m,2n}$$

$$= A \sum_{n=o}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(2n+\frac{1}{2})}{n!\ (2n)!} B^{2n} \qquad (F.4)$$

where the contours Γ_1 and Γ_2 are illustrated in Figure F-3. The coefficients in the series (F.4) are $O(1/n)$ for $n\to\infty$, so it will be slowly converging unless B is fairly small, i.e. not near one. But the case $B \ll 1$ is equivalent to the case b-a << 1, which is the small-amplitude case discussed in the main text. So what we have really obtained in Eq. (F.4) is an expansion of I_2 in powers of the amplitude (b-a).

It is not necessary to know the roots of the denominator of the integrand in order to obtain an expansion like (F.4); this is fortunate, for in all but the simplest cases we do not know these roots explicitly. Let us reconsider I_2, written in the form

$$I_2 = \int_a^b \frac{dx}{\sqrt{R+Sx+Tx^2+x^3}}$$

and show how we can obtain an expansion like (F.4) assuming no knowledge of the roots.

We first center the variable of integration somewhere between $\underline{a}$ and $\underline{b}$ (the equilibrium value $\underline{x_o}$ of the variable $\underline{x}$ always lies between $\underline{a}$ and $\underline{b}$, so we use that):

$$x = x_o + \xi$$

$$I_2 = \int_{-(x_o-a)}^{b-x_o} \frac{d\xi}{\sqrt{R'+S'\xi+T'\xi^2+\xi^3}}$$

The small-amplitude limit now amounts to the assumption that the limits of integration in this integral are $O(\varepsilon)$ as $\varepsilon \to 0$ (where ε is an order-of-smallness parameter). Then

$$R' = O(\epsilon^2)$$

$$T' = O(1)$$

because R' is the product of the roots of the cubic, two of which are $(b-x_o)$ and $-(x_o-a)$, and T' is the sum of the roots, one of which is O(1) by assumption. S', which is the sum of the pairwise products of the roots, can be shown to be $O(\epsilon^2)$ as long as

$$x_o - \frac{a+b}{2} = O(\epsilon^2)$$

which we certainly expect to be true, since x_o is the equilibrium and this is the linearized limit.

Using these results, we have immediately that

$$R',\ T'\xi^2 = O(\epsilon^2)$$

$$S'\xi,\ \xi^3 = O(\epsilon^3)$$

since the variable of integration ξ is clearly $O(\epsilon)$. Hence the proper small-amplitude expansion of the integrand would clearly seem to consist in extracting $(R'+T'\xi^2)$ from the cubic and expanding the integrand in terms of the $O(\epsilon)$ ratio

$$f(\xi) = \frac{S'\xi+\xi^3}{R'+T'\xi^2}$$

as follows:

$$I_2 = \int_{-(x_o-a)}^{(b-x_o)} \frac{d\xi}{(R'+T'\xi^2)^{\frac{1}{2}}\ (1+f(\xi))^{\frac{1}{2}}}$$

$$= \int_{-(x_o-a)}^{(b-x_o)} \frac{1}{(R'+T'\xi^2)^{1/2}} \left(1 - \frac{1}{2}\ \frac{S'\xi+T'\xi^3}{R'+T'\xi^2} + \ldots\right) d\xi$$

If we attempt to integrate this expansion term-by-term, however, we will find in general that all terms after the first will involve non-convergent integrals. This is due to the fact that the roots of the factor $(R'+T'\xi^2)$ will generally lie on the path of integration $[a-x_o,\ b-x_o]$, which means physically that the linearized amplitude will be less than the true amplitude (this is demonstrated explicitly for the LPE's by Table C-1 of Appendix C, where a is the linearized amplitude and u_2 the true amplitude).

It may also be noted that the fact that $(R'+T'\xi^2)$ has real roots comes from a consideration of the linearized limit, in which it is established that $R' > 0$ and $T' < o$.

The difficulty which we have encountered above may be eliminated by, as before, allowing I_2 to become a loop integral in the complex plane. We first transform

$$\xi = (-\frac{R'}{T'})^{\frac{1}{2}}\ n$$

and then make I_2 into a loop integral. The result is

$$I_2 = \frac{1}{2\sqrt{-T'}} \oint \frac{dn}{\sqrt{1-n^2+\alpha(n-\frac{R'}{S'T'}n^3)}}$$

where

$$\alpha = \frac{S'}{(-R'T')^{1/2}} = O(\varepsilon)$$

Since n is now an O(1) variable, and (R'/S'T') = O(1), the term multiplying α is O(1), and so we may expand in powers of α:

$$I_2 = \frac{1}{2\sqrt{-T'}} \oint \frac{1}{\sqrt{1-n^2}} \left\{ 1 - \frac{\alpha}{2} \frac{n-(R'/S'T')n^3}{1-n^2} + \dots \right\} dn \qquad (F.5)$$

Now we move the loop contour out beyond $|n| = 1$, so that $n = \pm 1$ do not lie on the contour. Then, if α is small enough, the series in curly brackets in (F.5) converges uniformly on the contour, and we may integrate term-by-term

$$I_2 = \frac{1}{2\sqrt{-T'}} \oint \frac{dn}{\sqrt{1-n^2}} - \frac{\alpha}{4\sqrt{-T'}} \oint \frac{n-(R'/S'T')n^3}{(1-n^2)^{3/2}} dn + \dots \qquad (F.6)$$

to obtain the small-amplitude expansion of I_2.

The individual loop integrals in (F.6) are done by Laurent's Theorem, just as before. For example, the second integral is done as follows:

$$\oint \frac{n-(R'/S'T')n^3}{(1-n^2)^{3/2}}\,dn = \oint \frac{n-(R'/S'T')n^3}{i^3 n^3 (1-\frac{1}{n^2})^{3/2}}\,dn$$

$$= \oint i(\frac{1}{n^2} - \frac{R'}{S'T'})(1+\frac{3/2}{n^2}+ \ldots)\,dn$$

$$= i \oint (-\frac{R'}{S'T'} + (1-\frac{3R'}{2S'T'})\frac{1}{n^2} + \ldots)\,dn$$

$$= 0$$

The procedures given above extend in a straightforward way to integrals of the more general form (F.1).

Appendix G: The Details of the Slowly-Varying Wavetrain Calculation for the Korteweg-de Vries Equation With Dissipation

We shall be concerned here with the expression of five integrals from §3 in terms of the fundamental integrals

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

and

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 x}\; dx$$

(which are the complete elliptic integrals of the first and second kind, respectively). The five integrals are those occurring in Eqs. (3.16) and (3.17):

$$I_1 = \int_0^{2\pi} P_o \, d\theta = \int_0^{2\pi} \eta \, d\theta$$

$$I_2 = \int_0^{2\pi} Q_o \, d\theta = \int_0^{2\pi} (3\eta^2 + \kappa^2 \frac{\partial^2 \eta}{\partial \theta^2}) \, d\theta = \int_0^{2\pi} 3\eta^2 d\theta$$

$$I_3 = \int_0^{2\pi} \bar{P}_o \, d\theta = \int_0^{2\pi} \frac{1}{2} \eta^2 \, d\theta$$

$$I_4 = \int_0^{2\pi} \kappa^2 (\frac{d\eta}{d\theta})^2 \, d\theta$$

$$I_5 = \int_0^{2\pi} \bar{Q}_o d\theta = \int_0^{2\pi} \left[2\eta^3 + \kappa^2 \eta \frac{\partial^2 \eta}{\partial \theta^2} - \frac{\kappa^2}{2} (\frac{\partial \eta}{\partial \theta})^2 \right] d\theta$$

$$= \int_0^{2\pi} \left[2\eta^3 - \frac{3}{2} \kappa^2 (\frac{\partial \eta}{\partial \theta})^2 \right] d\theta$$

$$= \int_0^{2\pi} 2\eta^3 d\theta - \frac{3}{2} I_4$$

Integration by parts and the periodicity of η were employed to simplify I_5.

We re-write Eq. (3.9) here for convenient reference:

$$\eta = m_1 + a \, cn^2 (\beta\theta + \Phi, k) \tag{H.1}$$

where

$$\left.\begin{aligned} a &= m_2 - m_1 \\ b &= m_2 - m_3 \\ \beta &= \frac{1}{\kappa}\sqrt{\frac{b}{2}} \\ k &= \sqrt{\frac{a}{b}} \end{aligned}\right\} \tag{H.2}$$

It is possible to write I_1 through I_5 as linear combinations of integrals of the form

$$K_n = \int_0^{2\pi} cn^{2n} (\beta\theta+\Phi, k)\ d\theta$$

This follows immediately for all except I_4 upon substituting the form (H.2) of η:

$$I_1 = 2\pi m_1 + aK_1$$

$$I_2 = 3(2\pi m_1^2+2m_1 aK_1+a^2K_2)$$

$$I_3 = \frac{1}{6} I_2$$

$$I_5 = 2(2\pi m_1^3 + 3m_1^2\ a\ K_1 + 3m_1\ a^2\ K_2 + a^3\ K_3) - \frac{3}{2} I_4$$

For I_4, we need to refer back to Eq. (3.7), which is

$$\kappa^2 (\frac{\partial\eta}{\partial\theta})^2 = 2(\eta-m_1)(m_2-\eta)(\eta-m_3)$$

Using this, I_4 becomes

$$I_4 = 2 \int_0^{2\pi} \left[a\ cn^2(\beta\theta+\Phi)\right] \left[a-a\ cn^2(\beta\theta+\Phi)\right] \left[m_1-m_3+a\ cn^2(\beta\theta+\Phi)\right] d\theta$$

$$= 2a^2 \left[(m_1-m_3)\ K_1 - (m_1-m_3-a)\ K_2-aK_3\right]$$

It merely remains to find K_1, K_2, K_3 in terms of $K(k)$ and $E(k)$. We first reduce K_n to standard form, as follows:

$$K_n = \int_0^{2\pi} cn^{2n}\ (\beta\theta + \Phi, k)\ d\theta$$

$$= \frac{1}{\beta} \int_{\Phi}^{2\pi\beta+\Phi} cn^{2n}(u,k)\ du$$

$$= \frac{1}{\beta} \int_0^{2K(k)} cn^{2n}(u,k)\ du$$

The non-linear dispersion relation (3.12) has been used to replace $\pi\beta$ by $K(k)$ in the upper limit, and the fact that $2K(k)$ is a period of $cn^{2n}(u,k)$ has been used in setting $\Phi=0$ in the limits.

The fundamental formulae we shall need in our attack on K_1, K_2, and K_3 are:

$$E(k) = \frac{1}{2}\int_{0}^{2K(k)} dn^2(u,k)\ du \qquad (H.4)$$

$$dn^2(u,k) = 1 - k^2 + k^2 cn^2\ (u,k) \qquad (H.5)$$

$$\frac{1}{2}\frac{d^2}{du^2} cn^2(u,k) = 1 - k^2 + 2(2k^2-1) cn^2(u,k) - 3k^2\ cn^4(u,k) \qquad (H.6)$$

$$\frac{1}{4}\frac{d^2}{du^2} cn^4\ (u,k) = 3(1-k^2)\ cn^2\ (u,k) + 4(2k^2-1)\ cn^4(u,k) - 5k^2\ cn^6(u,k) \qquad (H.7)$$

(H.4) is a modification of Whittaker and Watson's definition [section 22.73] of the elliptic integral E(k). (H.5) is the definition of the elliptic function dn(u,k). (H.6) and (H.7) were suggested by Example 4, Section 22.72, of Whittaker and Watson, but may be verified directly by differentiation if desired. We make use of (H.6) and (H.7) by integrating each from 0 to 2K(k). The left hand sides integrate immediately and drop out by the periodicity properties of elliptic functions. We are left with

$$0 = 2(1-k^2)\ K(k) + 2(2k^2-1)\ \beta K_1 - 3k^2\ \beta K_2$$

$$0 = 3(1-k^2)\beta K_1 + 4(2k^2-1)\ \beta K_2 - 5k^2 \beta K_3$$

From these it is clear that we may express K_2 and K_3 in terms of K_1:

$$\beta K_2 = \frac{2}{3k^2} \left[(1-k^2)\ K(k) + (2k^2-1)\ \beta K_1\right]$$

$$\beta K_3 = \frac{1}{15k^4} \left[8\ (2k^2-1)\ (1-k^2)\ K(k) + (23k^4-23k^2+8)\beta K_1\right]$$

Hence we only need to know K_1 in terms of elliptic integrals. But from (H.4) and (H.5), this is easy. The steps are

$$E(k) = \frac{1}{2}\int_0^{2K(k)} \left\{1-k^2+k^2 cn^2(u,k)\right\} du$$

$$= (1-k^2)\ K(k) + \frac{1}{2}\ k^2 \beta K_1$$

$$\beta K_1 = \frac{2}{k^2} \left[E(k) - (1-k^2)\ K(k)\right]$$

Collecting the preceding results, we may write the sought-after integrals, I_1 through I_5, as

$$I_1 = 2\pi m_1 + \frac{2a}{k^2\beta} \left[E(k) - (1-k^2)\ K(k)\right]$$

$$= 2\pi m_1 + 2K\sqrt{2b} \left[E(k) - (1-k^2)\ K(k)\right]$$

$$I_2 = 2\left[3\pi m_1^2 + 2\kappa\sqrt{2b}\left\{3m_1 + b(2k^2-1)\right\} E(k) + \kappa\sqrt{2b}\,(1-k^2)\left\{-6m_1 + b(2-3k^2)\right\} K(k)\right]$$

$$I_3 = \frac{1}{6} I_2$$

$$I_4 = \frac{4\sqrt{2}}{15}\kappa b^{5/2}\left[2(k^4-k^2+1)K(k) - (1-k^2)(2-k^2)\; K(k)\right]$$

$$I_5 = 4\left[\pi m_1^3 + \kappa\sqrt{2b}\left\{3m_1^2 + 2m_1 b(2k^2-1) + \frac{b^2}{15}(23k^4-23k^2+8)\right\} E(k) + \kappa\sqrt{2b}\,(1-k^2)\left\{-3m_1^2 + m_1 b(2-3k^2) - \frac{b^2}{15}(15k^4-19k^2+8)\right\} K(k)\right] - \frac{3}{2} I_4$$

Some simplifications have been wrought by use of the identities (H.2). The natural set of variables which arise in this calculation are m_1, b, k, κ rather than m_1, m_2, m_3, κ.

One would probably want to avail oneself of the information in the nonlinear dispersion relation

$$\pi\beta = K(k)$$

which can be solved for b

$$b = \frac{2}{\pi^2} k^2 K^2(k)$$

to eliminate b from I_1 through I_5. This is presumed to have been done when these formuae are introduced in §3.

The integrals I_i are used in (3.16) and (3.17) as follows:

$$\frac{\partial I_1}{\partial T} + \frac{\partial I_2}{\partial X} = 0$$

$$\frac{\partial I_3}{\partial T} + \frac{\partial I_5}{\partial X} + I_4 = 0$$

APPENDIX H: Reducing W_{AA} and Similar Integrals to Real Form.

W_{AA} was computed in §8.C to be

$$W_{AA} = -\frac{1}{2\pi}\oint \frac{dn}{(n-1)^2\sqrt{F}}$$

It was noted that in neither this form nor in the integrated-by-parts version

$$W_{AA} = \frac{1}{2\pi}\oint \left(3Sn-\frac{U^2}{n^3}\right)\frac{dn}{F^{3/2}}$$

could the loop contour be shrunk back down around the branch cut, between $\underline{n_1}$ and $\underline{n_2}$ say, on the real axis. For the first form, this was precluded by the double pole at n = 1, and for the second form, by the non-integrable singularities of the integrand at $\underline{n_1}$ and $\underline{n_2}$ (which are roots of F(n)).

Let us, therefore, go back to W_A and see what went wrong. W_A is given by

$$W_A = -\frac{1}{2\pi}\oint \left(3Sn-\frac{U^2}{n^3}\right)\frac{dn}{\sqrt{F}}$$

which <u>can</u> be shrunk back to an integral along the branch cut,

$$W_A = -\frac{1}{\pi}\int_{n_1}^{n_2} \left(3Sn-\frac{U^2}{n^3}\right)\frac{dn}{\sqrt{F}} \qquad (H.1)$$

where $\underline{n_2}$ and $\underline{n_1}$ will naturally depend on $\underline{A}$. If we now try and take the A-derivative of this integral by the usual rule, that is

$$\frac{\partial}{\partial A}\int_{n_1}^{n_2} Q(n)\, dn = Q(n_2)\,\frac{\partial n_2}{\partial A} - Q(n_1)\,\frac{\partial n_1}{\partial A} + \int_{n_1}^{n_2} \frac{\partial Q}{\partial A}\, dn$$

we see that the first two terms are infinite and the third is a non-convergent integral. We must, therefore, seek to transform the integral in some way before we take its A-derivative.

One way to do this is to fix the limits by transforming

$$n = \frac{n_1 + n_2}{2} + \frac{n_2 - n_1}{2}\,\xi \qquad \text{(H.2)}$$

To keep the calculations simple, we illustrate how this works on the integral

$$I = \int_{n_1}^{n_2} \frac{dn}{\sqrt{a + bn + cn^2 + n^3}}$$

$$= \int_{n_1}^{n_2} \frac{dn}{\sqrt{(n-n_1)\,(n_2-n)\,(n_3-n)}}$$

$$= \int_{-1}^{+1} \frac{d\xi}{\sqrt{(1-\xi^2)\,\left(n_3-\frac{n_1+n_2}{2} - \frac{n_2-n_1}{2}\xi\right)}}$$

Since A is one of the coefficients of F(n), it will be analogous in the present problem to consider $\partial I/\partial a$:

$$\frac{\partial I}{\partial a} = -\frac{1}{2}\int_{-1}^{+1} \frac{\frac{\partial}{\partial a}\left(n_3 - \frac{n_1+n_2}{2}\right) - \frac{\partial}{\partial a}\left(\frac{n_2-n_1}{2}\right)\xi}{\sqrt{1-\xi^2}\left[n_3 - \frac{n_1+n_2}{2} - \frac{n_2-n_1}{2}\xi\right]^{3/2}}\, d\xi \tag{H.3}$$

The roots n_1, n_2, n_3 can always be found very accurately on a computer; numerical differentiation tends to be quite inaccurate, however, and so we show how the $\partial n_i/\partial a$ can be expressed in terms of the n_i themselves.

The coefficients of the cubic in I in terms of its roots are

$$a = -n_1 n_2 n_3$$

$$b = n_1 n_2 + n_1 n_3 + n_2 n_3$$

$$c = -n_1 - n_2 - n_3$$

Taking the partial of each equation with respect to a,

$$-n_2 n_3 \frac{\partial n_1}{\partial a} - n_1 n_3 \frac{\partial n_2}{\partial a} - n_1 n_2 \frac{\partial n_3}{\partial a} = 1$$

$$(n_2+n_3)\frac{\partial n_1}{\partial a} + (n_1+n_3)\frac{\partial n_2}{\partial a} + (n_1+n_2)\frac{\partial n_3}{\partial a} = 0 \tag{H.4}$$

$$\frac{\partial n_1}{\partial a} + \frac{\partial n_2}{\partial a} + \frac{\partial n_3}{\partial a} = 0$$

This system can be solved for the $\partial n_i/\partial a$:

$$\frac{\partial n_1}{\partial a} = \frac{1}{(n_3-n_1)\ (n_1-n_2)}$$

with the other solutions given by cyclic permutation of the indices. Putting these back into (H.3), we have a convergent integral for $\partial I/\partial a$ involving only the $\underline{n_i}$.

We can proceed in the same way for $\partial W_A/\partial A$. In the form (H.1) of W_A, we write

$$F = \frac{1}{n^2} P(n)$$

where

$$P(n) = 2Sn^5 - 3Sn^4 + (2A-U^2)n^2 + 2U^2n - U^2$$

$$= 2S\ (n-n_1)\ (n_2-n)\ (n_3-n)\ (n-n_4)\ (n-n_5)$$

By expanding the factored form of P(n), we find

$$\frac{3}{2} = \sum_{i=1}^{5} n_i$$

$$0 = \sum n_i n_j \qquad \text{(H.5)}$$

$$\frac{U^2-2A}{2S} = \sum n_i n_j n_k$$

$$\frac{U^2}{S} = \sum n_i n_j n_k n_l$$

$$\frac{U^2}{2S} = n_1 n_2 n_3 n_4 n_5$$

where the sums after the first are over all possible combinations of $\{1, 2, 3, 4, 5\}$ with no two indices the same. For example,

$$\sum n_i n_j n_k n_l = n_1 n_2 n_3 n_4 + n_1 n_2 n_3 n_5 + n_1 n_2 n_4 n_5 + n_1 n_3 n_4 n_5 + n_2 n_3 n_4 n_5$$

Taking the partial derivative of each equation in the set (H.5) with respect to A, we obtain a system of linear equations similar to those in (H.4) for the $\partial n_i / \partial a$. After considerable manipulation, we find that

$$\frac{\partial n_1}{\partial A} = -\frac{1}{S}\frac{n_1^2}{(n_1 - n_2)\ (n_1 - n_3)\ (n_1 - n_4)\ (n_1 - n_5)} \tag{H.6}$$

where the other solutions may be obtained by four successive cyclic permutations of the indices. The computations are considerably simplified in this case by performing column operations on the determinants involved to obtain these determinants in factored form. We may illustrate this point using the determinant of the coefficients in (H.4):

$$D = \begin{vmatrix} -n_2n_3 & -n_1n_3 & -n_1n_2 \\ n_2+n_3 & n_1+n_3 & n_1+n_2 \\ 1 & 1 & 1 \end{vmatrix}$$

Subtract the first column from each of the other two,

$$D = \begin{vmatrix} -n_2n_3 & n_3(n_2-n_1) & n_2(n_3-n_1) \\ n_2+n_3 & n_1-n_2 & n_1-n_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} n_3(n_2-n_1) & n_2(n_3-n_1) \\ n_1-n_2 & n_1-n_3 \end{vmatrix}$$

$$= (n_1-n_2)\ (n_1-n_3) \begin{vmatrix} -n_3 & -n_2 \\ 1 & 1 \end{vmatrix}$$

$$= (n_1-n_2)\ (n_1-n_3)\ (n_2-n_3)$$

The manipulations follow the same pattern for the larger system (H.5).

Replacing F(n) in W_A by the equivalent expression

$$F(n) = \frac{1}{n^2} (n-n_1)\ (n_2-n)\ Q(n)$$

where Q(n) is a cubic with roots n_3, n_4, n_5

$$Q(n) = 2S(n_3-n)\ (n-n_4)\ (n-n_5)$$

we obtain

$$W_A = -\frac{1}{\pi} \int_{n_1}^{n_2} (3Sn^2 - \frac{U^2}{n^2}) \frac{dn}{\sqrt{(n-n_1)\ (n_2-n)\ Q(n)}}$$

Transforming this according to (H.2),

$$n = \alpha + \beta\xi$$

$$\left(\alpha = \frac{n_1 + n_2}{2},\ \beta = \frac{n_2 - n_1}{2}\right)$$

$$W_A = -\frac{1}{\pi} \int_{-1}^{1} [3S(\alpha+\beta\xi)^2 - \frac{U^2}{(\alpha+\beta\xi)^2}] \frac{d\xi}{\sqrt{(1-\xi^2)\ Q(\alpha+\beta\xi)}}$$

It is now an easy matter to take the A-derivative, following which we transform back to the original variable n. The result is

$$W_{AA} = \frac{-1}{\pi}\int_{n_1}^{n_2} \Big\{ \left(I(n) - J(n)\frac{Q'(n)}{Q(n)}\right)\left(\frac{\partial\alpha}{\partial A} + \frac{\partial\beta}{\partial A} + \frac{n-\alpha}{\beta}\right) +$$

$$J(n)\left(\frac{\partial n_3/\partial A}{n-n_3} + \frac{\partial n_4/\partial A}{n-n_4} + \frac{\partial n_5/\partial A}{n-n_5}\right)\Big\} \frac{dn}{\sqrt{(n-n_1)\ (n_2-n)\ Q(n)}}$$

where $$I(n) = 6Sn + \frac{2U^2}{n^3}$$

$$J(n) = \frac{1}{2}\left(3Sn^2 - \frac{U^2}{n^2}\right)$$

By substituting the values of α, β in terms of n_1, n_2, this reduces to

$$W_{AA} = -\frac{1}{\pi\ (n_2-n_1)}\int_{n_1}^{n_2}\left[I(n) - J(n)\frac{Q'(n)}{Q(n)}\right]\frac{1}{\sqrt{Q(n)}}$$

$$\times\left[\sqrt{\frac{n_2-n}{n-n_1}}\ \frac{\partial n_1}{\partial A} \pm \sqrt{\frac{n-n_1}{n_2-n}}\ \frac{\partial n_2}{\partial A}\right] dn$$

$$-\frac{1}{\pi}\int_{n_1}^{n_2} J(N)\left(\frac{\partial n_3/\partial A}{n-n_3} + \frac{\partial n_4/\partial A}{n-n_4} + \frac{\partial n_5/\partial A}{n-n_5}\right)\frac{dn}{\sqrt{(n-n_1)(n_2-n)Q(n)}}$$

where the $\partial n_i/\partial A$ are assumed to be replaced according to (H.6).

The numerical evaluation of the integrals in W_{AA} presents no great difficulty, once we decide which of the roots n_i to take for n_1 and n_2; and that question is answered in our study of the steady-profile solution.

There are excellent Gaussian quadrature formulas for integrals with square-root singularities at the end-points[22] which may be used.

It might be noted that the sign of W_{AA} is of great interest because it determines the stability ($W_{AA} < 0$) or instability ($W_{AA} > 0$) of the fundamental wavetrain to slowly varying perturbations.