THE HYDRO-KINETIC THEORY OF LIQUID HELIUM II

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Abstract

In Part I the kinetic theory of excitations in flowing liquid He II is developed to a higher order than that carried out previously, by Landau and Khalatnikov, in order to demonstrate the existence of non-equilibrium terms of a new nature in the hydrodynamic equations. It is then shown that these terms can lead to spontaneous destabilization in counter currents when the relative velocity of the normal and super fluids exceeds a critical value that depends on the temperature, but not on geometry. There are no adjustable parameters in the theory. The critical velocities are estimated to be in the 14-20 m/sec range for $T \leq 2.0^{\circ}K$, but tend to zero as $T \rightarrow T_{\lambda}$. The possibility that these critical velocities may be related to the experimentally observed "intrinsic" critical velocities is discussed.

Part II consists of a semi-classical investigation of roton-quantized vortex line interactions. An essentially classical model is used for the collision and the behavior of the roton in the vortex field is investigated in detail. From this model it is possible to derive the HVBK mutual friction terms that appear in the phenomenological equations of motion for rotating liquid He II. Estimates of the Hall and Vinen B and B' coefficients are in good agreement with experiments. The claim is made that the theory does not contain any arbitrary adjustable parameters.

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I. Higher Order Kinetic Theory of Excitations in He II and Spontaneous Instability in a Counter Current*

1. Introduction.

Recently, considerable attention has been given to the subject of "intrinsic" critical velocities (i.e., those which do not depend on channel size) in the flow of superfluid He II [1-4, 11]. Notarys [4] and Kukich et al [11] have experimentally shown the existence of an intrinsic critical velocity at temperatures between 1.3°K and the λ point; while Langer and Fisher [2] have suggested a theory involving a mechanism for the spontaneous creation of quantized vortex rings.

The purpose of the present investigation is to examine the question of whether intrinsic critical velocities can be predicted on the basis of Landau's two fluid model of Helium II, in which the normal fluid is regarded as a gas of excitations moving in a superfluid background. A set of dissipative hydrodynamic equations has been derived by Khalatnikov [5-6], by using a Boltzmann equation for the excitations and general conservation laws. These equations reduce exactly to those of Landau [7], which were derived from purely continuum considerations, when the dissipative terms are put equal to zero. However, Khalatnikov's equations do not appear to predict an intrinsic critical velocity.

We shall show that the methods and ideas of Khalatnikov can be extended to derive a higher order version of the Landau-Khalatnikov

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equations which contain terms of a new nature. The terms are dissipative in the sense that they arise from a lack of thermodynamic equilibrium in the excitation gas, but nevertheless predict that some disturbances in a rapid counter current will grow in amplitude. Thus they provide a means of interpreting the phenomenon of critical velocities within the mathematical framework of a set of equations of motion derived in a consistent manner from the two fluid model. However, it will be seen that the critical velocities predicted by a rough calculation are larger than those observed, and there are differences in the dependence on temperature. The physical significance of our results is still therefore an open question.

In Section 2 we will derive the extended kinetic-hydrodynamic equations of motion. Section 3 contains an analysis of the dissipative terms which appear in the equations. In Sections 4 and 5 we analyze the effect on second sound waves in an unbounded medium in a counter current, and show that critical velocities arise at which second sound becomes spontaneously unstable. Section 6 will contain an analysis of the behavior of disturbances in flow through a two dimensional channel as described by a simplified version of the extended equations. Critical velocities found this way are similar to those given by the second sound instability. Finally, Section 7 will be devoted to a critical discussion of the extended equations, and a comparison of our results with experiments.

2. The equations of motion.

The Landau-Khalatnikov non-linear, dissipative equations of motion for Helium II can be written in the form,

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \underline{j} = 0, \tag{1}$$

$$\frac{\partial \underline{v}_s}{\partial t} = - \nabla (\mu + \tfrac{1}{2} v_s^2 + \phi) , \tag{2}$$

$$\frac{\partial \underline{j}}{\partial t} = - \nabla P - \operatorname{div}(\rho_n \underline{v}_n \underline{v}_n + \rho_s \underline{v}_s \underline{v}_s) + \operatorname{div} \underline{\underline{\pi}} , \tag{3}$$

$$\frac{\partial E}{\partial t} = -\operatorname{div} \left(\mu \underline{j} + TS \underline{v}_n + \tfrac{1}{2} v_s^2 \underline{j} + \underline{v}_n (\underline{v}_n \cdot \underline{j}) - \rho \underline{v}_n (\underline{v}_n \cdot \underline{v}_s) \right) + \nabla \cdot \underline{q} \tag{4}$$

Equation (1) expresses the conservation of mass, where ρ is the density and $\underline{j}$ the momentum density. In equation (2), $\underline{v}_s$ denotes the super fluid velocity, and is the velocity associated with the ground state or background in which the elementary excitations comprising the normal fluid move. Equation (2) is an equation of motion for the irrotational super fluid. The velocity of the normal fluid is denoted by $\underline{v}_n$. The momentum density is expressed in terms of a density ρ_n of the normal fluid by

$$\underline{j} = \rho \underline{v}_s + \rho_n (\underline{v}_n - \underline{v}_s) = \rho_s \underline{v}_s + \rho_n \underline{v}_n , \tag{5}$$

where $\rho_s = \rho - \rho_n$, and it is also convenient to write

$$\underline{J} = \rho_n (\underline{v}_n - \underline{v}_s) = \rho_n \underline{w} , \tag{6}$$

where $\underline{w} = \underline{v}_n - \underline{v}_s$ is the relative velocity of the normal and super fluids. Equation (3) expresses the conservation of momentum for the entire fluid and Equation (4) expresses the conservation of energy where E is the energy density (energy per unit volume). The quantities ϕ, $\underline{\underline{\pi}}$ and $\underline{q}$ are dissipative terms or fluxes which arise when departures from local thermodynamic equilibrium are not infinitesimal.

The symbols μ, P, S, T denote the chemical potential, pressure, entropy density, and temperature, and are related by

$$TdS = d(E - \underline{J} \cdot \underline{v}_s - \tfrac{1}{2}\rho v_s^2) - \underline{w} \cdot d\underline{J} - \mu d\rho \ , \tag{7}$$

$$P = ST + \underline{w} \cdot \underline{J} + \mu\rho - (E - \underline{J} \cdot \underline{v}_s - \tfrac{1}{2}\rho \underline{v}_s^2) \ , \tag{8}$$

when the departures from thermodynamic equilibrium are negligible. In this case, the equations can be derived from general considerations of Galilean invariance and the conservation of mass, momentum, energy and entropy [7].

The equations can also be derived from a model of Helium II in which the normal fluid is treated as a gas of excitations, each of which has momentum $\underline{p}$ and energy (in the laboratory frame)

$$e = \epsilon(p, \rho) + \underline{p} \cdot \underline{v}_s \quad (p = |\underline{p}|) \ . \tag{9}$$

$\epsilon(p, \rho)$ is given by the Landau spectrum. The density of excitations in phase space, $n(\underline{x}, \underline{p}, t)$ satisfies a Boltzmann equation

$$\mathcal{B}(n) \equiv \frac{\partial n}{\partial t} + \left(\frac{\partial \epsilon}{\partial p_i} + v_{si} \right) \frac{\partial n}{\partial x_i} - \left(\frac{\partial \epsilon}{\partial \rho} \frac{\partial \rho}{\partial x_i} + p_k \frac{\partial v_{sk}}{\partial x_i} \right) \frac{\partial n}{\partial p_i} = I(n) \quad (10)$$

where the right-hand side of Eq. (10) gives the rate of change of n due to collisions between the excitations. When the excitation gas is dilute, the equilibrium distribution (for Bose-Einstein statistics) is

$$n_0 = \left\{ \exp \left[(\epsilon - \underline{p} \cdot \underline{w})/k_B T \right] - 1 \right\}^{-1}, \quad (11)$$

and the thermodynamic properties of the excitation gas follow from standard methods of statistical mechanics. Using the fact that collisions between excitations conserve momentum and energy (but not necessarily number) and the assumption that the super fluid background has zero entropy and is irrotational, one can use Equation (10) to derive (1)-(8), without the dissipation terms, where n_0 is used for the distribution of the excitation gas.

In this model, the dissipative terms arise from deviations between n and the equilibrium value n_0, which occur when the flow is neither steady nor uniform. The energy and momentum of the excitation gas are still well defined quantities, and it is convenient to define the remaining thermodynamic variables of state (including $\underline{v}_n$) by the equilibrium functions of energy and momentum. One can then show [8], that the motion is given by Equations (1)-(8), and the dissipative terms are given by

$$\phi = - \int (n_0 - n) \frac{\partial \epsilon}{\partial \rho} \frac{d\underline{p}}{h^3}, \quad (12)$$

$$\pi_{ik} = \delta_{ik}\rho \int (n_0 - n) \frac{\partial \epsilon}{\partial \rho} \frac{d\underline{p}}{h^3} + \int (n_0 - n)\, p_i \frac{\partial \epsilon}{\partial p_k} \frac{d\underline{p}}{h^3} , \tag{13}$$

$$q_i = j_i \int (n_0 - n) \frac{\partial \epsilon}{\partial \rho} \frac{d\underline{p}}{h^3} + \int (n_0 - n)(\epsilon - \underline{p} \cdot \underline{w}) \frac{\partial \epsilon}{\partial p_i} \frac{d\underline{p}}{h^3} \tag{14}$$

In these expressions, the integral is over momentum space and h denotes Planck's constant. We shall refer to $\underline{\underline{\pi}}$ as the viscous stress tensor and $\underline{q}$ as the heat conduction vector. Further details of the derivation can be found in [8].

From equations (4) and (5), one obtains the equation for entropy change,

$$\frac{\partial S}{\partial t} + \operatorname{div}(S\underline{v}_n) = \frac{1}{T} \operatorname{div} \underline{q} - \frac{v_{ni}}{T} \frac{\partial \pi_{ij}}{\partial x_j} - \frac{\rho_s}{T} \underline{w} \cdot \nabla \phi \tag{15}$$

3. The dissipation terms.

The collision term in the Boltzmann equation for the excitations is not known with any certainty, and even if given, the Boltzmann equation is difficult to solve. We shall therefore proceed by assuming that departures from equilibrium are small and that the integrals (12), (13) and (14) can be evaluated with the standard approximation (see [8]).

$$
\begin{aligned}
n_0 - n &= \tau \mathcal{B}(n_0) \\
&= \frac{\tau n_0(1+n_0)}{k_B T} \left\{ \frac{(\epsilon - \underline{p}\cdot\underline{w})}{T} \left(\frac{\partial T}{\partial t} + \underline{v}_n \cdot \nabla T \right) \right. \\
&\quad + \frac{(\epsilon - \underline{p}\cdot\underline{w})}{T} \left(\frac{\partial \epsilon}{\partial \underline{p}} - \underline{w} \right) \cdot \nabla T + p_i \left(\frac{\partial w_i}{\partial t} + \underline{v}_n \cdot \nabla w_i \right) \\
&\quad \left. + p_i \left(\frac{\partial \epsilon}{\partial p_k} - w_k \right) \frac{\partial v_{ni}}{\partial x_k} - \frac{\partial \epsilon}{\partial \rho} \left(\frac{\partial \rho}{\partial t} + \underline{v}_n \cdot \nabla \rho \right) \right\} , \qquad (16)
\end{aligned}
$$

where τ is a relaxation time which depends on the details of the excitation collisions. For consistency, τ must be independent of $\underline{p}$, but can depend on density and/or pressure. Substitution of Equation (16) into the expressions for the dissipation terms then gives these terms as linear combinations of the gradients and time derivatives of T, $\underline{w}$, $\underline{v}_n$ and ρ, the coefficients being integrals of n_0, ϵ and $\underline{p}$ over momentum space.

At this stage, Khalatnikov simplifies by linearizing in the velocities and spatial derivatives, and using the linearized 'inviscid' forms of Equations (1)-(4) to eliminate time derivatives. For our purposes it is necessary to work to a higher order of approximation. The use of Equation (16) gives expressions

$$\phi = \tau A^{(\phi)}\left(\frac{\partial T}{\partial t} + \underline{v}_n \cdot \nabla T\right) + \tau B_{ik}^{(\phi)} w_k \frac{\partial T}{\partial x_i} + \tau C_{ik}^{(\phi)} w_k\left(\frac{\partial w_i}{\partial t} + \underline{v}_n \cdot \nabla w_i\right)$$

$$+ \tau D_{ik}^{(\phi)} \frac{\partial v_{ni}}{\partial x_k} + \tau E^{(\phi)}\left(\frac{\partial \rho}{\partial t} + \underline{v}_n \cdot \nabla \rho\right) , \qquad (17)$$

$$\pi_{ik} = -\rho\phi\ \delta_{ik} + \tau A_{ik}^{(\pi)}\left(\frac{\partial T}{\partial t} + \underline{v}_n \cdot \nabla T\right) + \tau B_{ik\ell m}^{(\pi)} w_m \frac{\partial T}{\partial x_\ell}$$

$$+ \tau C_{ik\ell m}^{(\pi)} w_m\left(\frac{\partial w_\ell}{\partial t} + \underline{v}_n \cdot \nabla w_\ell\right) + \tau D_{ik\ell m}^{(\pi)} \frac{\partial v_{n\ell}}{\partial x_m}$$

$$+ \tau E_{ik}^{(\pi)}\left(\frac{\partial \rho}{\partial t} + \underline{v}_n \cdot \nabla \rho\right) , \qquad (18)$$

$$q_i = -\phi j_i + \tau A_{ik}^{(q)} w_k\left(\frac{\partial T}{\partial t} + \underline{v}_n \cdot \nabla T\right) + \tau B_{ik}^{(q)} \frac{\partial T}{\partial x_k}$$

$$+ \tau C_{ik}^{(q)}\left(\frac{\partial w_k}{\partial t} + \underline{v}_n \cdot \nabla w_k\right) + \tau D_{ik\ell m}^{(q)} w_m \frac{\partial v_{n\ell}}{\partial x_k}$$

$$+ \tau E_{ik}^{(q)} w_k\left(\frac{\partial \rho}{\partial t} + \underline{v}_n \cdot \nabla \rho\right) . \qquad (19)$$

The terms $A^{(\phi)}$, ..., $E_{ik}^{(q)}$ are integrals of n_0, ϵ and $\underline{p}$ and are therefore scalar or tensor functions of T, ρ and $\underline{w}$ alone. For example,

$$D_{ik\ell m}^{(\pi)} = \frac{1}{k_B T} \int n_0 (1+n_0)\ p_i p_m \frac{\partial \epsilon}{\partial p_k}\left(\frac{\partial \epsilon}{\partial p_\ell} - w_\ell\right) \frac{d\underline{p}}{h^3} , \qquad (20)$$

$$B_{ik}^{(q)} = \frac{1}{k_B T^2} \int n_0 (1+n_0)(\epsilon - \underline{p} \cdot \underline{w})^2 \frac{\partial \epsilon}{\partial p_i}\left(\frac{\partial \epsilon}{\partial p_k} - w_k\right) \frac{d\underline{p}}{h^3} , \qquad (21)$$

and so on. Symmetry requirements will reduce the number of

independent coefficients. The tensor dependence on $\underline{w}$ has been partially accounted for in writing Equations (17), (18), and (19), so that all the tensor coefficients are finite and non-zero as $\underline{w} \to 0$.

To a satisfactory approximation, the excitation spectrum can be approximated by two branches

$$\epsilon = c_1 p \qquad \text{for phonons,} \tag{22}$$

and

$$\epsilon = \Delta + \frac{(p - p_0)^2}{2\mu_0} \qquad \text{for rotons,} \tag{23}$$

where c_1 is the velocity of first sound. Typical values of the parameters in the roton branch are in c.g.s. units [9] :

$$\Delta \approx 1.2 \times 10^{-15}, \quad p_0 \approx 2.0 \times 10^{-19}, \quad \mu_0 \approx 1.1 \times 10^{-24}, \tag{24}$$

but there is a variation with density. (Neutron scattering experiments show a dependence on temperature. This is presumably due to interactions or 'dense gas effects'. The use of an excitation spectrum which depends on temperature is not consistent with the Boltzmann equation or the equilibrium distribution Equation (11)).

It is clear from the equilibrium distribution that the magnitude of the relative velocity $\underline{w}$ must be less than the minimum of ϵ/p (the Landau criterion). We suppose now that $w = |\underline{w}|$ is significantly less than the Landau critical velocity and that the coefficients $A^{(\phi)}, \ldots, E^{(q)}_{ik}$ may be replaced by their values for $\underline{w} = 0$. We then find after considerable labor that

$$\phi = \frac{\xi_1}{T}\left(\frac{\partial T}{\partial t} + \underline{v}_n \cdot \nabla T\right) + \frac{\xi_2}{T}\,\underline{w}\cdot\nabla T + \xi_3\,\underline{w}\cdot\left(\frac{\partial \underline{w}}{\partial t} + \underline{v}_n\cdot\nabla\underline{w}\right)$$

$$+ \xi_4\,\mathrm{div}\,\underline{v}_n + \frac{\xi_5}{\rho}\left(\frac{\partial \rho}{\partial t} + \underline{v}_n\cdot\nabla\rho\right), \qquad (25)$$

where the coefficients $\xi_1, \ldots, \xi_5$ are functions of ρ and T and independent of w^2. All coefficients have the dimensions of kinematic viscosity, except ξ_3 which is kinematic viscosity divided by velocity squared, i.e., time.

$$\xi_1 = -\frac{4\pi\tau}{k_B T}\int n'(1+n')\epsilon\,\frac{\partial\epsilon}{\partial\rho}\,p^2\,\frac{dp}{h^3},$$

$$\xi_2 = \frac{4\pi\tau}{k_B T}\int n'(1+n')\epsilon\,\frac{\partial\epsilon}{\partial\rho}\,p^2\,\frac{dp}{h^3} + \frac{4\pi\tau}{3k_B T}\int n'(1+n')\,\frac{\partial\epsilon}{\partial p}\,\frac{\partial\epsilon}{\partial\rho}\,p^3\,\frac{dp}{h^3}$$

$$- \frac{4\pi\tau}{3k_B^2 T^2}\int n'(1+n')(1+2n')\epsilon\,\frac{\partial\epsilon}{\partial p}\,\frac{\partial\epsilon}{\partial\rho}\,p^3\,\frac{dp}{h^3}, \qquad (26)$$

$$\xi_3 = -\frac{4\pi\tau}{3k_B^2 T^2}\int n'(1+n')(1+2n')\,\frac{\partial\epsilon}{\partial\rho}\,p^4\,\frac{dp}{h^3},$$

$$\xi_4 = -\frac{4\pi\tau}{k_B T}\int n'(1+n')\,\frac{\partial\epsilon}{\partial p}\,\frac{\partial\epsilon}{\partial\rho}\,p^3\,\frac{dp}{h^3},$$

$$\xi_5 = \rho\frac{4\pi\tau}{k_B T}\int n'(1+n')\left(\frac{\partial\epsilon}{\partial\rho}\right)^2 p^2\,\frac{dp}{h^3}$$

In these expressions, ϵ as a function of the scalar p can be approximated by Equations (22) and (23), and n' denotes the equilibrium distribution (Equation (11)) for $w = 0$. The dependence of roton parameters

and velocity of first sound on density is discussed by Wilks [9].

Similarly, we find that

$$\pi_{ik} + \rho\phi\,\delta_{ik} = \delta_{ik\ell m}\eta\,\frac{\partial v_{nm}}{\partial x_\ell} + \delta_{ik}\,\frac{\eta_1}{T}\left(\frac{\partial T}{\partial t} + \underline{v}_n\cdot\nabla T\right)$$

$$+ \delta_{ik\ell m}\,\frac{\eta_{21}}{T}\,w_\ell\,\frac{\partial T}{\partial x_m} - \delta_{ik}\,\frac{\eta_{22}}{T}\,\underline{w}\cdot\nabla T + \delta_{ik\ell m}\,\eta_3\,w_\ell\left(\frac{\partial w_m}{\partial t} + \underline{v}_n\cdot\nabla w_m\right)$$

$$+ \delta_{ik}\,\frac{\eta_5}{\rho}\left(\frac{\partial\rho}{\partial t} + \underline{v}_n\cdot\nabla\rho\right), \tag{27}$$

where

$$\delta_{ik\ell m} = \delta_{ik}\,\delta_{\ell m} + \delta_{i\ell}\,\delta_{km} + \delta_{im}\,\delta_{k\ell},$$

$$\eta = \frac{4\pi\tau}{15k_B T}\int n'(1+n')\left(\frac{\partial\epsilon}{\partial p}\right)^2 p^4\,\frac{dp}{h^3},$$

$$\eta_1 = \frac{4\pi\tau}{3k_B T}\int n'(1+n')\,\epsilon\,\frac{\partial\epsilon}{\partial p}\,p^3\,\frac{dp}{h^3},$$

$$\eta_{21} = \frac{4\pi\tau}{15k_B^2T^2}\int n'(1+n')(1+2n')\,\epsilon\left(\frac{\partial\epsilon}{\partial p}\right)^2 p^4\,\frac{dp}{h^3} - \frac{4\pi\tau}{15k_B T}\int n'(1+n')\left(\frac{\partial\epsilon}{\partial p}\right)^2 p^4\,\frac{dp}{h^3}, \tag{28}$$

$$\eta_{22} = \eta_1,$$

$$\eta_3 = \frac{4\pi\tau}{15k_B^2T^2}\int n'(1+n')(1+2n')\,\frac{\partial\epsilon}{\partial p}\,p^5\,\frac{dp}{h^3},$$

$$\eta_5 = \frac{4\pi\tau}{3k_B T}\,\rho\int n'(1+n')\,\frac{\partial\epsilon}{\partial p}\,\frac{\partial\epsilon}{\partial\rho}\,p^3\,\frac{dp}{h^3}.$$

All coefficients have dimensions of viscosity, except η_3 which is

viscosity/(velocity)^2.

For the heat flux vector, we have

$$q_i + \phi\, j_i = \kappa \frac{\partial T}{\partial x_i} + \frac{\zeta_1}{T} w_i \left(\frac{\partial T}{\partial t} + \underline{v}_n \cdot \nabla T\right) + \zeta_3 \left(\frac{\partial w_i}{\partial t} + \underline{v}_n \cdot \nabla w_i\right)$$

$$+ \zeta_{41}\, \delta_{ik\ell m}\, w_k \frac{\partial v_{nm}}{\partial x_\ell} - \zeta_{42} (\underline{w} \cdot \nabla)\, v_{ni}$$

$$+ \frac{\zeta_5}{\rho}\, w_i \left(\frac{\partial \rho}{\partial t} + \underline{v}_n \cdot \nabla \rho\right) , \qquad (29)$$

where

$$\kappa = \frac{4\pi\tau}{3k_B T^2} \int n'(1+n')\epsilon^2 \left(\frac{\partial \epsilon}{\partial p}\right)^2 p^2 \frac{dp}{h^3} ,$$

$$\zeta_1 = \frac{4\pi\tau}{3k_B^2 T^2} \int n'(1+n')(1+2n')\epsilon^2 \frac{\partial \epsilon}{\partial p} p^3 \frac{dp}{h^3} - 2\eta_1 ,$$

$$\zeta_3 = \eta_1$$

$$\zeta_{41} = \eta_{21} ,$$

$$\zeta_{42} = \eta_1 ,$$

$$\zeta_5 = \rho(\xi_1 + \xi_2) . \qquad (30)$$

The coefficients $\zeta_1, \ldots, \zeta_5$ all have the dimensions of viscosity.

The expressions (26), (28) and (30) depend specifically on the solution (16) of the Boltzmann equation. Also, the neglect of the w^2

dependence in the coefficients is a 10-20 % error when w is of order $k_B T/p_0 \sim 10\text{m/sec}$. However, the labor required to remove this latter error does not seem justified in view of the uncertainties inherent in the Boltzmann equation and its approximate solution. In fact, the purpose of the above calculation is to provide a basis for the hypothesis that the 'viscous' terms are of the form (25), (27), (29), where the coefficients $\xi_1, \ldots, \zeta_5$ are properties of the Helium and functions of ρ and T. In the temperature range where the use of a Boltzmann equation and the approximate solution (16) are valid (this is probably the case between 1.2° and 1.6°; above this temperature the excitations are dense and below this temperature phonon collisions, which may not be well described by (16), are important), we can use (26), (28) and (30) to estimate the coefficients. But outside this range, we make the hypothesis that the dissipation terms have the stated forms with coefficients that remain to be found.

When the non-linear terms and the terms involving $\underline{w}$ are neglected, the 'viscous' terms reduce essentially to the forms given by Khalatnikov [8]. This approximation seems most reasonable and estimates of the new coefficients (see Sections 4, 5, and 6) verify that the extra terms are small when velocities are small compared with the speed of second sound. However, the new terms contain qualitatively different properties. For instance, they can produce negative damping and thereby lead to destabilization of a laminar flow. Thus although they are small and apparently dominated by the familiar viscous and heat conduction damping terms, they can produce new effects. In the present work we shall confine the discussion to the propagation of second sound

in an unbounded fluid, and the behavior of incompressible disturbances in flow through a two-dimensional channel, and demonstrate the existence of critical values of a counter current above which the dissipation terms give negative damping and an exponential growth.

4. The effect of a counter current on the attenuation of second sound.

If we estimate the coefficients in (28) and (30) with the roton spectrum, and suppose that velocities are small compared with the speed of second sound, we find that the dominant terms in the irreversible momentum and energy transfer terms (27) and (29) are

$$\pi_{ik} + \rho\phi\,\delta_{ik} = \delta_{ik\ell m}\,\eta\,\frac{\partial v_{nm}}{\partial x_\ell} + \delta_{ik}\,\frac{\eta_1}{T}\,\frac{\partial T}{\partial t} + \delta_{ik\ell m}\,\frac{\eta_{21}}{T}\,w_\ell\,\frac{\partial T}{\partial x_m}, \tag{31}$$

where $$\frac{\eta_{21}}{\eta} \approx \frac{\Delta}{k_B T},$$

and $$q_i + \phi\, j_i = \kappa\,\frac{\partial T}{\partial x_i} + \delta_{ik\ell m}\,\eta_{21}\,w_k\,\frac{\partial v_{nm}}{\partial x_\ell}. \tag{32}$$

The substitution

$$\frac{\partial T}{\partial t} = -\frac{TS}{\rho C}\,\operatorname{div}\,\underline{v}_n,$$

(where C = specific heat) which follows from the linearized non-dissipative equations, convert the first two terms on the right-hand sides of (31) and (32) into the standard expressions given by Khalatnikov [8]. Also, the terms involving $\frac{\partial \epsilon}{\partial \rho}$ can be neglected to a first approximation so that ϕ can be taken as zero.

We now consider how the extra terms, which involve cross products of $\underline{w}$ with the gradients of temperature and normal velocity, will change the attenuation rate of second sound in a heat current. We neglect the non-linear terms in the convective terms; the approximation

is not completely self-consistent, but is sufficient for our purposes. One then finds that the temperature fluctuation T' (here and following, primed quantities denote perturbations) satisfies

$$\frac{\partial^2 T'}{\partial t^2} - u_{II}^2 \nabla^2 T' = \frac{1}{\rho C} \frac{\partial}{\partial t} \operatorname{div} \underline{q} - \frac{\rho_s TS}{\rho^2 \rho_n C} \frac{\partial^2 \pi_{ik}}{\partial x_i \partial x_k}, \tag{33}$$

where $u_{II}^2 = (S^2 T \rho_s / C\rho^2 \rho_n)$ is the speed of second sound. The right-hand side of (33) is evaluated by subsituting from (31) and (32).

For a wave of second sound in which all quantities vary like $e^{i(\underline{k} \cdot \underline{x} - \omega t)}$, we find after some algebra that the value of $\mathrm{Im}(\omega)$ is

$$\mathrm{Im}(\omega) = - \frac{3k^2 \eta_{21} u_{II} W_k}{ST} - \frac{\kappa k^2}{2\rho C} - \frac{3\eta k^2 \rho_s}{2\rho \rho_n} + \frac{\eta_1 S k^2 (\rho_n + 2\rho_s)}{2\rho \rho_n C}, \tag{34}$$

where W_k is the component of the undisturbed counter current in the direction of $\underline{k}$. The attenuation rate is proportional to $-\mathrm{Im}(\omega)$. Thus a wave propagating against the counter current decays more slowly. Further, if the counter current were increased sufficiently far, the attenuation could become negative so that spontaneously growing waves would appear. At 1.4° K, the expression (34) becomes zero when W_k is about 14 m/sec. However, the approximations leading to (34) are suspect for such large values, and in the next section we carry out a consistent approximation to find the critical value of the counter current for instability.

5. The instability of second sound in a counter current.

We consider small perturbations about a state of uniform motion. It is convenient to take a frame of reference moving with the normal fluid, so that $\underline{v}_s = \underline{W}$, say, in the undisturbed state. For the sake of simplicity, we shall only consider plane waves propagating parallel (or anti-parallel) to the counter current. Also, we shall neglect the coefficient of thermal expansion and the isothermal compressibility; this also includes the dependence of ρ on w^2. (These approximations appear to be good to within a few microdegrees of T_λ.) Then with ρ=constant, the equations of motion (1), (2), (3) and (15) give

$$\rho_n v_n' + \rho_s v_s' + W\rho_s' = 0, \tag{35}$$

$$\frac{\partial v_s'}{\partial t} + W\frac{\partial v_s'}{\partial x} = -\frac{1}{\rho}\frac{\partial P'}{\partial x} + s\frac{\partial T'}{\partial x} - \frac{\rho_n}{\rho} W\frac{\partial}{\partial x}(v_n' - v_s') - \frac{\partial \phi'}{\partial x}, \tag{36}$$

$$P' + W^2\rho_s' + 2\rho_s W v_s' = \pi_{11}', \tag{37}$$

$$\frac{\partial s'}{\partial t} + s\frac{\partial v_n'}{\partial x} = \frac{1}{\rho T}\frac{\partial q_1'}{\partial x} + \frac{\rho_s}{\rho T} W\frac{\partial \phi'}{\partial x}, \quad (s = \tfrac{S}{\rho}) \tag{38}$$

where the primed quantities are the departures from the uniform undisturbed values. In addition, we have from the differential equation of state

$$\rho_n' = -\rho_s' = \left(\frac{\partial \rho_n}{\partial T}\right)T' - 2W\left(\frac{\partial \rho_n}{\partial w^2}\right)\left(v_n' - v_s'\right), \tag{39}$$

$$s' = \left(\frac{\partial s}{\partial T}\right)T' - 2W\left(\frac{\partial s}{\partial w^2}\right)\left(v_n' - v_s'\right) . \tag{40}$$

From the identity

$$d\mu = \frac{dP}{\rho} - s\,dT - \frac{\rho_n}{2\rho}\,dw^2 ,$$

it follows that

$$\left(\frac{\partial s}{\partial w^2}\right) = \frac{1}{2\rho}\left(\frac{\partial \rho_n}{\partial T}\right) ,$$

since we are taking ρ to be constant.

When the dissipation terms are neglected, we find after some algebra that the velocity c of a wave propagating in the positive x-direction is given by the quadratic

$$c^2\left\{\ \rho\left[\rho_n + 2W^2\left(\frac{\partial \rho_n}{\partial w^2}\right)\right]\frac{\partial s}{\partial T} - W^2\left(\frac{\partial \rho_n}{\partial T}\right)^2\right\}$$

$$-2Wc\left\{\ \rho s\left(\frac{\partial \rho_n}{\partial T}\right) - \rho_n\left[\rho_s - 2W^2\left(\frac{\partial \rho_n}{\partial w^2}\right)\right]\frac{\partial s}{\partial T} - \frac{\rho_n W^2}{\rho}\left(\frac{\partial \rho_n}{\partial T}\right)^2\right\}$$

$$= \rho\left[\rho_s - 2W^2\left(\frac{\partial \rho_n}{\partial w^2}\right)\right]s^2 . \tag{41}$$

Moreover, the relative values of v_n', v_s' and T' are

$$v_n' \left\{ c \frac{\partial s}{\partial T} \left[\rho_s - 2W^2 \left(\frac{\partial \rho_n}{\partial w^2} \right) \right] + \frac{W^2 c}{\rho} \left(\frac{\partial \rho_n}{\partial T} \right)^2 \right\}^{-1}$$

$$= -v_s' \left\{ c \frac{\partial s}{\partial T} \left[\rho_n + 2W^2 \left(\frac{\partial \rho_n}{\partial w^2} \right) \right] - Ws \left(\frac{\partial \rho_n}{\partial T} \right) - \frac{W^2 c}{\rho} \left(\frac{\partial \rho_n}{\partial T} \right)^2 \right\}^{-1}$$

$$= T' \left\{ s \left[\rho_s - 2W^2 \left(\frac{\partial \rho_n}{\partial w^2} \right) \right] + Wc \left(\frac{\partial \rho_n}{\partial T} \right) \right\}^{-1} . \qquad (42)$$

In (41) and (42), all functions should be given their equilibrium values for given P, T, and W. For all reasonable flows, it can be shown that the quadratic (41) has real roots, so that second sound always exists in a counter current. Equation (41) was given by Khalatnikov [13] (see also [7]) but only up to O(W). We retain the W^2 terms for the sake of consistency.

We now include the dissipation terms. Since these are small, they can be evaluated by substituting the ratios (42) and relating time and space derivatives by $\frac{\partial}{\partial t} = -c \frac{\partial}{\partial x}$, where c is a root of (41). It is clear that these terms will make c complex. For W small, the dissipation terms will reduce to the familiar viscosity and heat conduction terms, which will produce an attenuation or damping of second sound. But as W increases, the possibility exists that the imaginary part of c will change sign, which implies a spontaneous amplification. We now find the equation for W_c, the critical value at which the imaginary part of c changes sign.

Eliminating p' and writing $\frac{\partial}{\partial t} = -c \frac{\partial}{\partial x}$, we can write (35)-(38) as

$$v_n'\left[\rho_n + 2W^2\left(\frac{\partial\rho_n}{\partial w^2}\right)\right] + v_s'\left[\rho_s - 2W^2\left(\frac{\partial\rho_n}{\partial w^2}\right)\right] - T'\left[W\frac{\partial\rho_n}{\partial T}\right] = 0,$$

$$v_n'\left[s + \frac{Wc}{\rho}\left(\frac{\partial\rho_n}{\partial T}\right)\right] - v_s'\left[\frac{Wc}{\rho}\left(\frac{\partial\rho_n}{\partial T}\right)\right] - T'\left[c\frac{\partial s}{\partial T}\right] = \frac{1}{\rho T}q_1' + \frac{\rho_s}{\rho T}W\phi',$$

$$-v_n'\left[W\left(\rho_n - 2W^2\frac{\partial\rho_n}{\partial w^2}\right)\right] + v_s'\left[\rho c + \left(\rho_s - 2W^2\frac{\partial\rho_n}{\partial w^2}\right)W\right] + T'\left[s\rho - W^2\frac{\partial\rho_n}{\partial T}\right] = \pi_{11}' + \rho\phi'. \quad (43)$$

The determinant formed by the terms in square brackets is the quadratic (41). The condition for $\mathrm{Im}(c) = 0$ is that the three equations in (43) are linearly dependent when the determinant vanishes. This gives

$$\left\{\frac{1}{\rho T}q_1' + \frac{\rho_s W}{\rho T}\phi'\right\}\left\{\rho\rho_n c + 2W^2\rho c\left(\frac{\partial\rho_n}{\partial w^2}\right) + 2W\rho_s\rho_n - 4W^3\rho_n\left(\frac{\partial\rho_n}{\partial w^2}\right)\right\}$$

$$+\left\{\pi_{11}' + \rho\phi'\right\}\left\{s\rho_s - 2sW^2\left(\frac{\partial\rho_n}{\partial w^2}\right) + Wc\frac{\partial\rho_n}{\partial T}\right\} = 0. \quad (44)$$

For this equation, we have from (29) that

$$q_1' + \rho_s W\phi' = \frac{\partial}{\partial x}\left[\kappa T' + \frac{cW}{T}\zeta_1 T' - c\zeta_3\left(v_n' - v_s'\right) - \left(3\zeta_{41} - \zeta_{42}\right)Wv_n'\right], \quad (45)$$

and from (27)

$$\pi_{11}' + \rho\phi' = \frac{\partial}{\partial x}\left[3\eta v_n' - \frac{c\eta_1 T'}{T} - 3\eta_{21}\frac{W}{T}T' + \frac{\eta_{22}}{T}WT'\right]. \quad (46)$$

The ratios of v_n', v_s' and T' are given by (42), and c is a root of (41). For given values of the dissipation coefficients, equation (44) gives the critical values of W. If we accept our approximate solution of the Boltzmann equation, the coefficients are given by the integrals (28) and

(30). Note that only the ratios of the coefficients enter, so the results are independent of the relaxation time τ.

These equations were too complicated to examine analytically and were therefore studied on the IBM 360/75 computer over a temperature range between $1.4^{\circ}K$ and T_{λ}. The results are shown by the full curve in Fig. 1.

Some of the assumptions used in the numerical analysis should be mentioned explicitly. The viscosity coefficients were evaluated using a relaxation time τ that was assumed independent of the excitation momenta. Under this assumption it was found that the roton contributions dominated the numerical evaluation of the viscosity integrals.

It was not possible to find formulas or experimental data giving the dependence of the various thermodynamic quantities on the relative velocity $\underline{w}$ for temperatures above about $1.4^{\circ}K$, although Khalatnikov [8] has derived formulas valid at lower temperatures. Fig.1 was computed using the experimental data of Donnelly [10] up to $2.0^{\circ}K$, and that of Clow and Reppy [14] near T_{λ}. These data are essentially that for $w = 0$. It was felt that this approximation, although crude, was the most consistent one we could make for all temperatures. W_c was also computed at 1.40° and 1.80° using the Khalatnikov formulas and good agreement was found with Fig.1 at these temperatures. It should be noted that both the experimental approximation and the Khalatnikov formulas are extremely rough near $2.0^{\circ}K$ where $W_c \simeq 20$ m/sec.

Since no experimental data were available for the quantity $\partial\rho_n/\partial w^2$ we used the Khalatnikov formula up to $2.0^{\circ}K$. Although this is a dubious approximation at such high temperatures, this term was found to have only a small effect on W_c anyway, so the error involved here is

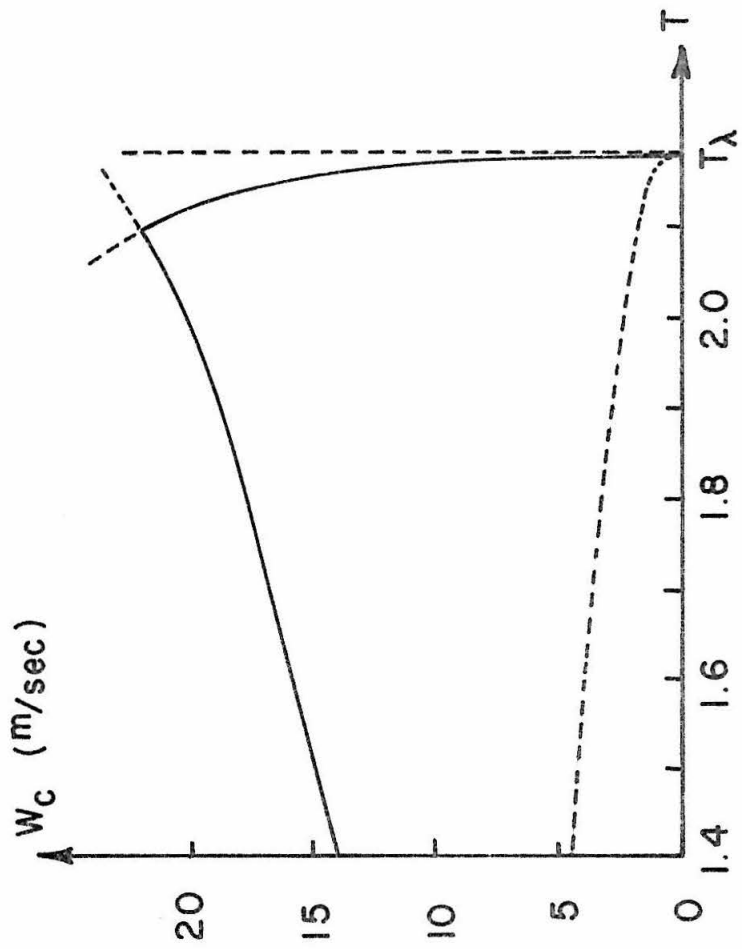

FIG. 1 THE CRITICAL RELATIVE VELOCITY W_c

probably not very large. It was felt that this term should decay to zero rapidly near T_λ and was simply dropped above 2.0°K. A small discontinuity in W_c at this temperature has been smoothed out in Fig. 1.

The behavior of W_c above 2.10°K is due to the fact that the first coefficient in Equation (41) can change sign for relatively small values of W since $\frac{\partial \rho_n}{\partial T} \to \infty$ as $T \to T_\lambda$ [14]. This results in a discontinuity which changes the sign of the net damping effect, as is most easily seen from an analysis similar to that of Section 4. The decay of W_c near T_λ is roughly

$$W_c \sim 10^4 (1-T/T_\lambda)^{\frac{1}{3}} \quad \text{as} \quad T \to T_\lambda. \tag{47}$$

As an independent check, the full set of hydrodynamic Equations (1), (2), (3), and (15) were also analyzed on the computer. These equations were perturbed about the same equilibrium flow described at the beginning of this section, but without the simplifying assumptions that led to Equations (35) to (38) by enabling the second sound modes to be uncoupled from those of first sound. The critical velocities obtained from this analysis agreed closely with those shown in Fig. 1 and seem to verify the assumptions used in this section.

It should be mentioned that there do not seem to be any instabilities in first sound below the Landau critical velocity.

6. Stability of flow through a channel.

An analysis of the complete set of viscous thermohydrodynamic equations for flows in more than one dimension with boundary conditions is extremely involved. In this section we will give a stability analysis for a very simplified form of the equations in a two-dimensional channel.

We simplify the viscous terms by retaining only the shear viscosity η and the destabilizing viscosity η_{21} in the total momentum equation. All other viscous terms will be dropped.

We further reduce the equations with the assumptions that s = const. and that both the normal and super fluids are incompressible (see Landau and Lifshitz, [7]).

More explicitly, under the above approximations, the equations of motion become:

$$\rho_n = \text{const}, \quad \rho_s = \text{const}, \quad \rho = \text{const}, \quad s = \text{const}, \quad \operatorname{div} \underline{v}_n = \operatorname{div} \underline{v}_s = 0,$$

$$\rho_n \frac{\partial \underline{v}_n}{\partial t} + \rho_n \underline{v}_n \cdot \nabla \underline{v}_n + \rho_s \frac{\partial \underline{v}_s}{\partial t} + \rho_s \underline{v}_s \cdot \nabla \underline{v}_s = -\nabla P + \eta \nabla^2 \underline{v}_n + \frac{\eta_{21}}{T} \operatorname{div}(\underline{w} \nabla T),$$

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{\nabla P}{\rho} + s \nabla T + \frac{\rho_n}{2\rho} \nabla w^2 . \qquad (48)$$

We consider a channel in two dimensions with walls at $y = \pm 1$; the x-axis being taken as the center line. (Using ± 1 results in no loss of generality since the final result will be independent of the channel size.)

We assume the following undisturbed state

$$\underline{v}_{n_0} = 0, \qquad \underline{v}_{s_0} = U\underline{e}_x$$

$$T = T_0 = \text{const}, \quad P = P_0 = \text{const} . \tag{49}$$

For boundary conditions at the wall, we use the insulating wall conditions at $y = \pm 1$:

$$v_{sy} = 0,$$

$$v_{nx} = v_{ny} = 0 . \tag{50}$$

Assuming that the super fluid is irrotational allows us to use the super fluid potential φ_s defined by

$$\underline{v}_s = \nabla\varphi_s + U\underline{e}_x . \tag{51}$$

We try the following forms for the fluctuations

$$\begin{aligned}
\varphi_s &= \Phi(y)e^{i\alpha(x-ct)} , \\
P' &= P(y)e^{i\alpha(x-ct)} , \\
T' &= T(y)e^{i\alpha(x-ct)} , \\
v_{nx} &= D\psi(y)e^{i\alpha(x-ct)} , \\
v_{ny} &= -i\alpha\,\psi(y)e^{i\alpha(x-ct)} ,
\end{aligned} \tag{52}$$

where

$$D = \frac{d}{dy} .$$

Note that Equations (48) have the steady state inviscid solution

$$P' + \tfrac{1}{2}\rho_n v_n^2 + \tfrac{1}{2}\rho_s v_s^2 = \text{const on streamlines,} \tag{53}$$

$$-\frac{P'}{\rho} + sT' + \frac{\rho_n}{2\rho} W^2 - \tfrac{1}{2} v_s^2 = \text{const everywhere.} \tag{54}$$

Substitution of (52) into the super fluid equation in (48) gives

$$\left(-i\alpha c + i\alpha U - i\alpha \frac{\rho_n U}{\rho}\right)\Phi = -\frac{P}{\rho} + sT - \frac{\rho_n U}{\rho} D\psi . \tag{55}$$

Substituting (52) into the total momentum equations in (48) results in

$$\left[-i\alpha\rho_n c - \eta(D^2-\alpha^2)\right] D\psi = -i\alpha P + \alpha^2 \rho_s (U-c)\Phi - \frac{U\eta_{21}}{T_0}(D^2-3\alpha^2)T , \tag{56}$$

and

$$-i\alpha\left[-i\alpha\rho_n c - \eta(D^2-\alpha^2)\right]\psi = -DP - i\alpha\rho_s(U-c)D\Phi - 2i\alpha\frac{U\eta_{21}}{T_0} DT . \tag{57}$$

Using Eqs. (48), (50)-(57) we eliminate Φ, P, and T to obtain:

$$\left(\frac{s\rho}{\alpha^2} - \frac{2U\eta_{21}}{iT\alpha}\right)\left[\rho_n(U+c) + \frac{\eta}{i\alpha}(D^2-\alpha^2)\right] D^2\psi = \left[s\rho + \frac{U\eta_{21}}{i\alpha T}(D^2-3\alpha^2)\right]\left[\rho_n c + \frac{\rho_n U D^2}{\alpha^2} + \eta\frac{(D^2-\alpha^2)}{i\alpha}\right]\psi \tag{58}$$

Using $\psi \propto e^{\pm\alpha\lambda y}$, substituting into Eq. (58), and cancelling common factors gives

$$\rho\rho_n sc = \frac{s\rho\eta\alpha}{i}(1-\lambda^2) + \frac{U\eta_{21}\alpha\rho_n}{iT_0}(3c+\lambda^2 U) \quad , \quad \text{for } \lambda \neq 1. \tag{59}$$

Note that for $\lambda = 1$, Eq. (58) is satisfied identically.

If we assume the explicit form

$$\psi = A \sinh \alpha y + B \sinh \alpha \lambda y \tag{60}$$

for ψ then Eqs. (50, (52), and (60) applied at the wall produces the determining equation for λ :

$$\lambda \tanh \alpha = \tanh \alpha\lambda \ . \tag{61}$$

Eq. (61) has the real roots $\lambda = 0, 1$ and the complex roots $\lambda = i\mu_n$, where $\mu_n = \frac{n\pi}{\alpha} + \epsilon_n$, $n = 1, 2, \ldots$, (The exact values of ϵ_n are immaterial here.)

The flow will be unstable when $\mathrm{Im}(\alpha c) > 0$ (see (52)). From (59) the condition for neutral stability is

$$U^2 = \frac{s\rho T_0}{\rho_n} \frac{\eta}{\eta_{21}} \ . \tag{62}$$

For U larger than U_c given by (62), the fluctuations (52) are unbounded in time.

The critical velocity found for this problem is very similar to the one in Section 5 for second sound in a counter current. In the temperature range between 1.4°K and 2.0°K both velocities increase

roughly in proportion to T, are independent of any geometric length scale, and are of the order of about 10-20 m/sec.

It should be noted that the thermodynamic assumptions used here cannot possibly produce the wave modes experimentally observed for He II in very small channels (i.e., fourth sound and the fifth, or "no sound", wave mode [12]). The equations of motion were also analyzed for narrow channel instabilities without the thermodynamic and incompressibility assumptions made at the beginning of this section. Only the η and η_{21} viscosity terms were retained, but all quadratic inertia terms were dropped. The analysis was extremely involved and it was only possible to solve the problem for the two lowest modes. These proved to be stable for U less than the Landau critical velocity. As yet, it is impossible to say anything about the higher modes.

7. Discussion.

The critical velocities found in the preceding sections do not seem to correspond to any of the critical velocities known to exist in He II, although there are some similarities with the intrinsic critical velocities observed by Clow and Reppy [1], Kukich et al. [11], Notarys [4], and Jinchvelashvili et al [3]. All of these experimental critical velocities are of the order of a few meters per second and decay to zero at T_λ. (See dotted curve in Fig.1). The observed decay near T_λ is, in [1], [11], and [4], somewhat faster than the calculated values shown in Fig.1. The experiments described in [4] and [11] consider temperatures well below T_λ, but they observe a decrease in the critical velocity as T increases, and not the increase shown in Fig.1. It should be noted, however, that the first three experiments use finely porous flows and so differ very considerably from the flows analyzed in Sections 5 and 6. Jinchvelashvili et al. used narrow, rotating annuli and found a wide variety of results that give both faster and slower decays than shown in Fig.1.

The only other theory of the intrinsic critical velocity, that of Langer and Fisher [2], tries to associate the critical velocity with the spontaneous nucleation of quantized vortex rings by thermal fluctuations. Notarys, in a private communication, has pointed out a serious inconsistency in this theory. It can be shown that these rings would have to be larger than the channel size and it is not clear how they could fit in.

In light of this we feel that we have either predicted a new

critical velocity for the breakdown of superfluidity, or have found an old one theoretically, but have obtained poor numerical correlations with experiments because of the crudeness of some of our approximations or because the flows analyzed in Sections 5 and 6 are so different from the experiments. If we have found a new critical velocity, it is of limited importance since it occurs at high velocities and is masked by the earlier occurrence of other critical velocities.

Accepting the basic structure of the equations, which depends on the approximations made on the collision terms in the Boltzmann equation in Section 3, we see that different choices of values for the viscosity coefficients could produce instabilities at critical velocities that would agree well with experiments. That is, if we regard the viscosity coefficients as adjustable parameters, they could be chosen so that the calculated W_c would fit any experimental data. At the high temperatures considered here there is no question that the use of the integrals (26) to (30) is only a rough approximation. Even accepting these integrals, it may be necessary to evaluate them using more sophisticated methods. In particular, the $\tau \neq \tau(p)$ assumption might be dropped in some consistent manner (see Khalatnikov [5], [6], [8]) and reevaluation of these viscosity integrals might give a very different $W_c(T)$ curve.

There does not seem to be a simple physical picture for our instability. All we can say is that it appears that for $W > W_c$ the kinetic energy of the relative motion feeds energy into a small disturbance or fluctuation faster than the dissipative viscosities can drain it out. Of course, the disturbance cannot grow indefinitely,

but we are not able to predict what would happen once the flow becomes unstable.

Lastly, we would like to point out the unusual qualitative nature of the "cross" terms involving η_{21} and ζ_{41} in Eqs. (27) and (29) respectively. These "vorticity" terms have no counterparts in the theory of classical fluids. Their existence appears to depend on the peculiar nature of liquid He II, and they can be destabilizing. As yet, we have not been able to analyze their effects mathematically in three dimensions or interpret them physically.

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II. A Semi-Classical Model of the Roton-Quantized Vortex Line Interaction with an Application to Rotating Liquid He II.

1. Introduction.

It is a well established experimental fact [1] that, on a macroscopic scale, the superfluid equilibrium motion in uniformly rotating liquid He II closely resembles the solid body rotation exhibited by classical liquids. Other experiments also seem to indicate the presence of vorticity in the superfluid.

These experimental observations have been explained by Onsager and Feynman [2] in a manner that allows us to retain the irrotationality condition

$$\nabla \times \underline{v}_s = 0 \tag{1}$$

almost everywhere. Their model introduces circulation into the superfluid by postulating that the helium is threaded by a distribution of vortex lines of microscopic diameter which behave very much like ideal classical vortex lines except that their circulation is quantized in units of Planck's constant h, i.e.

$$\Gamma = \oint \underline{v}_s \cdot d\underline{\ell} = n \frac{h}{m_{He}} \quad , \quad n = 1, 2, 3, \ldots \quad . \tag{2}$$

These quantized vortex lines (QVL) are considered to be part of the superfluid.

In the rotating vessel experiment these lines are thought to hang parallel to the axis of rotation and be stationary in a frame rotating with the container. The distribution of the vortices is chosen so that the circulation around any macroscopic closed curve in the liquid helium is the same as that around an identical curve in a liquid moving with a solid body rotation. This results in a line density per unit area perpendicular to the axis of rotation given by

$$N = \frac{2\omega_0}{\Gamma_0} , \tag{3}$$

where ω_0 = angular velocity of the bucket, and Γ_0 is given by Eq. (2) with n=1. (Energy arguments show that the n=1 case is much more likely than any of the larger quanta of circulation.) It can then be argued that the superfluid velocity fields of these lines imitate solid body rotation on a macroscopic scale (see [1] and [2] for complete details).

If this model of rotating helium is correct, then it would be reasonable to suppose that these QVL would act as scattering centers for the thermal excitations that constitute the normal fluid. The velocity field $\underline{v}_s$ of a QVL would influence the motion of a nearby excitation through the $\underline{p} \cdot \underline{v}_s$ interaction term in the excitation Hamiltonian. When the excitations have a mean drift velocity relative to the vortex lines, we would expect a significant momentum transfer between the excitations (normal fluid) and the vortex lines (superfluid). This momentum exchange between the two fluids is commonly called mutual friction. The line's "massiveness" and

tension should keep it fairly rigid during a collision with an excitation, and it would be expected that there would be no (or a very small) component of mutual friction parallel to the vortex line (axis of rotation) since $\underline{v}_s$ would have no component in this direction.

Hall and Vinen [3] performed experiments on the attenuation of second sound in a uniformly rotating sample of liquid helium, and found an extra attenuation due to rotation which could be qualitatively explained by this theory of mutual friction. Quantitative agreement between their experiments and the Landau hydrodynamic equations could be obtained by adding an ad hoc force of the form:

$$\underline{F} = - B \frac{\rho_s \rho_n}{\rho} \frac{\underline{\omega}_0 \times [\underline{\omega}_0 \times (\bar{\underline{v}}_n - \bar{\underline{v}}_s)]}{\omega_0} - B' \frac{\rho_s \rho_n}{\rho} \left[\underline{\omega}_0 \times (\bar{\underline{v}}_n - \bar{\underline{v}}_s)\right] \qquad (4)$$

to the superfluid equation. In (4), $\bar{\underline{v}}_n$ and $\bar{\underline{v}}_s$ are values of the normal and superfluid velocities that have been averaged over regions whose dimensions are large compared to the distance between vortex lines, but small compared to the size of the experimental apparatus (hydrodynamic velocity fields). B and B' are chosen to fit the experimental data and are functions of the temperature T.

If this $\underline{F}$ is due to excitation - QVL collisions, then Eq. (4) should be derivable from microscopic considerations of such a scattering and B and B' should be numerically predicted by this derivation. This is what we have tried to do here.

We shall first review earlier work on this problem. Then in Section 2, we will consider an individual encounter between a thermal excitation and a QVL using an essentially classical model for the

interaction. In Section 3, we further describe the mechanism responsible for mutual friction and give a derivation of Eq. (4) which produces expressions for B and B' that do not contain any arbitrary adjustable parameters. Finally, Section 4 contains a critical discussion of our theory that includes comparisons with experiment and previous work on the problem.

The experiment by Hall and Vinen [3] mentioned earlier only measures B quantitatively. Snyder and Linekin [4] have measured B' from mode splitting experiments on second sound in a rotating cavity. Tsakadze [5] has verified the strongly anisotropic character of mutual friction from an oscillation experiment that shows that any force component parallel to the axis of rotation must be at least two orders of magnitude smaller than the components normal to the axis. Hall and Vinen also observed this, but were not able to make any measurements.

By using general conservation laws and the assumption that the internal energy is increased in a manner proportional to the averaged local superfluid vorticity, Bekarevich and Khalatnikov [6] were able to derive hydrodynamic equations which include the force given by Eq. (4). However, theirs is a purely formal continuum derivation that neither considers the microscopic nature of mutual friction nor does it predict B or B'.

Hall and Vinen [7] analyze the excitation-QVL interaction using a Born approximation and derive expressions for $\underline{F}$, B, and B' from classical kinetic-hydrodynamic arguments. Lifshitz and Pitaevskii [8] use the same classical derivation as [7], but use a quasi-

classical approximation to calculate the excitation-QVL cross section. It can be shown [9] that the quasi-classical method is appropriate to this interaction, whereas the Born approximation is not. Iordansky (see [4]) uses a quantum kinetic analysis applied to a dilute weakly interacting Bose gas to find B and B'. None of these analyses have been very successful in predicting B'. In addition, there are several aspects about the work in [7] and [8] that appear to be unsatisfactory. In particular, both of these papers use cross sections that contain arbitrary adjustable parameters which are very important to their calculations. These points will be discussed further in Section 4.

2. The Roton-Vortex Line Interaction.

All the experiments on mutual friction have been carried out with $T > 1.1^{\circ}K$. At such temperatures the roton contribution to ρ_n is at least two orders of magnitude greater than the phonon contribution [10], and it appears safe to simplify our analysis by only considering roton-QVL encounters, as was done in [7] and [8]. Also, as will be shown, phonons would not be capable of undergoing the strong and long range scattering characteristic of rotons.

In treating the interaction between a single roton and a QVL, we shall regard the vortex as a fixed center of force, and the roton as a point particle with a free particle energy given by the Landau spectrum [11]

$$\epsilon(p) = \Delta + \frac{(p-p_0)^2}{2\mu_0} \quad , \tag{5}$$

where $\underline{p}$ = roton momentum and $p = |\underline{p}|$. The parameters in (5) will be taken as constants with the following values in c.g.s. units [12]:

$$\begin{aligned} \Delta &\simeq (1.2)\, 10^{-15} \\ p_0 &\simeq 2 \cdot 10^{-19} \\ \mu_0 &\simeq 1.1 \cdot 10^{-24}. \end{aligned} \tag{6}$$

The motion of the roton will be influenced by the superfluid velocity field produced by the vortex. In the presence of such a $\underline{v}_s$ field the roton Hamiltonian is given by [10]:

$$H = \epsilon(p) + \underline{p} \cdot \underline{v}_s \ , \tag{7}$$

as seen in a frame in which the line is stationary. The QVL will be assumed to be perfectly straight and its velocity field taken as the planar counterclockwise field of a classical vortex line in an ideal fluid [13]:

$$(v_{sx},\ v_{sy}) = \left(- \frac{\Gamma_0 y}{2\pi r^2} \ , \ \frac{\Gamma_0 x}{2\pi r^2} \right) , \tag{8}$$

where $r^2 = x^2 + y^2$, and the line has been taken to lie along the z axis. In c.g.s. units $\Gamma_0 \simeq 10^{-3}$ (see Eq. (2)).

The vortex is assumed to have a core of radius $a_0 \simeq 3\mathring{A}$, inside of which there is no superfluid. This choice of a_0 is made on the assumption that the superfluid velocity at the edge of the core is equal to the Landau critical velocity [14] for the destruction of superfluidity, i.e. a_0 is the solution of the equation

$$\frac{\Gamma_0}{2\pi a_0} \simeq \frac{\Delta}{p_0} = 60 \text{ m/sec.}$$

This choice of a_0 is consistent with all known estimates of the core radius.

If the line is moving, it is assumed to do so with a velocity $\overline{\underline{v}}_s$, defined following Eq. (4). The roton-line collision will always be analyzed in a frame in which the line is stationary, i.e. a frame moving with $\overline{\underline{v}}_s$. In such a frame the superfluid velocity is given by

Eq. (8).

We will assume that all the momentum transfer takes place between the rotons and the core, i.e. that there is no direct transfer of momentum from the rotons to the superfluid. The momentum may actually be considered as going into the impulse of the entire vortex system, and thus eventually ends up in the superfluid considered as a whole. What cannot happen is that the roton loses momentum directly to the superfluid in its immediate neighborhood. This would produce a rotational flow in the superfluid.

Other points, such as a possible Magnus effect, will be considered later.

The approach described so far must be considered inadequate for at least one reason. In our model we are only considering momentum transfer due to the line's field acting on the roton. The roton, however, is not a point particle, and the presence of the "finite" roton must alter the velocity field produced by the vortex and therefore have some kind of effect on the line. For example, the line might be bent so that its velocity field would no longer be strictly two dimensional. Unfortunately, our knowledge of the structures of both the roton and the vortex core is much too incomplete to take such effects into account. However, from what we do know about rotons and QVL's, we do not expect this to be a serious problem as far as this investigation is concerned. The scattering is probably very similar to that of a light particle by a rigid structure. To a certain degree, this analysis may be regarded as a test of this assumption.

One may also feel somewhat uncomfortable about treating a QVL as a classical vortex line (except, of course, for its quantized circulation and microscopic core size). Here again, our ignorance of the line structure precludes any better approximation. However, this model has proven itself extremely successful in the past, and it is felt that it should be adequate for the purposes of this investigation. Such blending of quantum and classical mechanics has become so characteristic of the whole subject of liquid helium that it is difficult to imagine understanding helium phenomena without such a mixing of ideas.

We shall now consider the scattering of a roton from an initial momentum state $\underline{p}$ to a final state $\underline{p}_f$. Our analysis will depart from standard scattering treatments in two important ways. First, we will follow the detailed motion of the roton as it passes through the vortex "interaction region" (say $r \leq L$, with the possibility of $L \to \infty$); and, secondly, we shall assume this motion to be governed by the classical Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \qquad \dot{p}_i = - \frac{\partial H}{\partial q_i} \quad , \tag{9}$$

where

$$\{ q_i \} = \{ x, y, z \} \quad \text{and} \quad \{ p_i \} = \{ p_x, p_y, p_z \} \ .$$

Before trying to justify the use of Eq. (9), we must first consider what happens near the core. Absolutely nothing is known about what goes on when a roton gets near the core, so the best we

can do is guess. We will take the attitude of trying to do the simplest reasonable thing we can think of. Rotons passing near the core will also be assumed to be governed by classical mechanics and rotons that actually hit the core will be assumed to be absorbed, i.e. they give up their entire momentum to the vortex. We don't expect this model to pass for rigorous quantum mechanical "truth"; the "actual" interaction might involve some sort of weird bound state. What we are saying is that we expect some very strong interaction to take place near the core, and we guess that it should be something like an absorption. We hope that this model will be a reasonably good approximation as far as a phenomenological description of He II is concerned. This investigation can be considered to be a test of this, and, in fact, is a test to see just how far we can go in using the simple, semi-classical, Landau-Feynman two fluid model to account for macroscopic hydrodynamic phenomena.

Given our model of a roton and the core, it is relatively easy to make a rough test of the applicability of classical mechanics to our scattering situation. For an arbitrary potential, it can be shown* (see [17]) that a classical approximation is reasonable when

$$\frac{\hbar}{2b\Delta p} << 1 ,$$

where b is the impact parameter (the distance at which the roton would pass the QVL if there were no interaction between them), and Δp is the momentum transferred during the collision as calculated

*This derivation includes a detailed consideration of Uncertainty Principle effects.

from classical mechanics. Our analysis will produce typical values for the left hand side of the above relation that are between .02 and .10 for all b. Lifshitz and Hall [9] have carried out a somewhat more detailed calculation and found essentially the same result. So the use of a classical model seems reasonably justified. This result is not really that surprising since rotons essentially obey Maxwell-Boltzmann statistics.

Substitution of (5), (7), and (8) into (9) produces:

$$\frac{dx}{dt} = \frac{(p - p_0)}{\mu_0} \frac{p_x}{p} - \frac{\Gamma_0}{2\pi} \frac{y}{r^2} \quad , \tag{10}$$

$$\frac{dy}{dt} = \frac{(p - p_0)}{\mu_0} \frac{p_y}{p} + \frac{\Gamma_0}{2\pi} \frac{x}{r^2} \quad , \tag{11}$$

$$\frac{dp_x}{dt} = - p_x \frac{\Gamma_0}{\pi} \frac{xy}{r^4} - p_y \frac{\Gamma_0}{2\pi r^2} + p_y \frac{\Gamma_0}{2\pi} \frac{x^2}{r^4} \quad , \tag{12}$$

$$\frac{dp_y}{dt} = p_x \frac{\Gamma_0}{2\pi} \frac{1}{r^2} - p_x \frac{\Gamma_0}{\pi} \frac{y^2}{r^4} + p_y \frac{\Gamma_0}{\pi} \frac{xy}{r^4} \quad . \tag{13}$$

The z equations show that p_z = constant.

Eqs. (10) to (13) can be made dimensionless with the variables

$$\underline{x}' = \frac{\underline{x}}{a_0} \quad , \quad \underline{p}' = \frac{\underline{p}}{p_0} \quad , \quad t' = \frac{\Gamma_0}{a_0^2} t \quad .$$

Substitution of these variables into (10) to (13) yields:

$$
\begin{aligned}
\frac{dx'}{dt'} &= A(p'-1)\,\frac{p'_x}{p'} - \frac{1}{2\pi}\,\frac{y'}{r'^2}\,, \\
\frac{dy'}{dt'} &= A(p'-1)\,\frac{p'y}{p'} + \frac{1}{2\pi}\,\frac{x'}{r'^2}\,, \\
\frac{dp'_x}{dt'} &= \frac{1}{\pi}\left[-p'_x\,\frac{x'y'}{r'^4} - p'_y\,\frac{1}{2r'^2} + p'_y\,\frac{x'^2}{r'^4}\right], \\
\frac{dp'_y}{dt'} &= \frac{1}{\pi}\left[p'_x\,\frac{1}{2r'^2} - p'_x\,\frac{y'^2}{r'^4} + p'_y\,\frac{x'y'}{r'^4}\right],
\end{aligned}
\tag{14}
$$

where $A = \dfrac{a_0 p_0}{\Gamma_0 \mu_0} \simeq 5.4$.

The scattering of a roton by a QVL takes place as follows. The roton starts out at infinity in an initial state $\underline{p}$ and with an impact parameter b. The subsequent motion of the roton is governed by Eqs. (14). The roton passes the QVL, has its state altered, and has its final state $\underline{p}_f$ recorded when it returns to infinity. The momentum transferred to the core is simply $\underline{p} - \underline{p}_f$. If a roton hits the core, its entire momentum is assumed to be absorbed by the line. We do not have to worry about conserving the number of rotons. This total absorption and the large loss of momentum suffered by rotons passing very close to the core is similar to the "hard" cross section guessed at in [8]. This will be discussed again in Section 4.

Although too nonlinear and strongly coupled to be treated analytically, Eqs. (14) could easily be solved numerically on a high speed computer. This was what was actually done. A large number of numerically accurate solutions for different $\underline{p}$ and b were

necessary to compute the cross sections to be described in Section 3. It was also possible to solve highly simplified forms of these equations analytically.

It was found that all the rotons were scattered as small deflections, hits, or "snap-backs" as shown in Fig. 1 (in two dimensions). The particular course taken by a roton depended on its initial state $\underline{p}$ and its impact parameter b. The scattering was found to be unsymmetric about the forward direction. Note that a roton can approach the QVL even if $\underline{p}$ is pointed away from the core if $p < p_0$ (see Eq. (10)). The $p > p_0$ and $p < p_0$ cases have different asymmetries about the forward direction, and there is almost no scattering through angles near 90°.

The snap-back scattering is peculiar to rotons. This behavior is a consequence of the strange dispersion relation (5) and the form of the Hamiltonian (7). This can be explained qualitatively as follows. Since H is not explicitly time dependent, (9) implies that

$$H = \Delta + \frac{(p-p_0)^2}{2\mu_0} + \underline{p} \cdot \underline{v}_s = \text{const.} \qquad (15)$$

At infinity $\underline{v}_s = 0$, so that $\underline{p} \cdot \underline{v}_s = 0$. As roton b_3 (as shown in Fig. 1) moves toward the core, $|\underline{v}_s|$ increases, we find $\underline{p} \cdot \underline{v}_s > 0$, and this term becomes more positive as the roton moves closer to the core. The only way to keep H = const is to have p decrease, i.e. the roton moves toward (p_0, Δ) on the dispersion curve as shown in Fig. 2. Just how far the roton moves along this curve is a function of $\underline{p}$ and

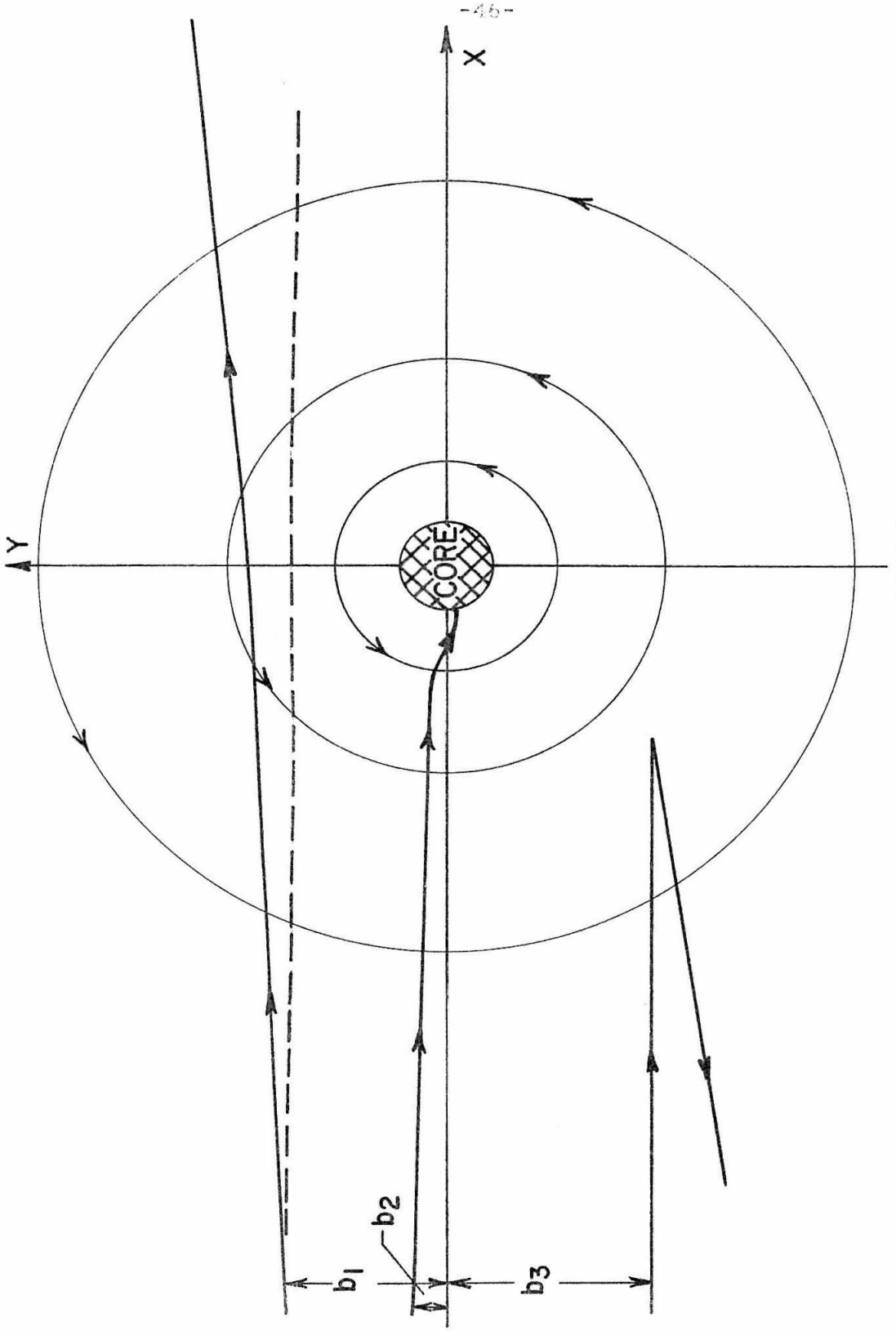

FIG. 1

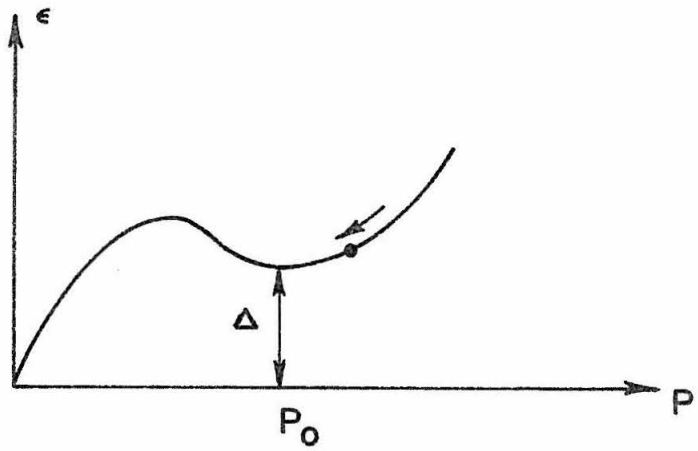

FIG. 2

b. Some can be moved across the $p = p_0$ point. From (10)* we see that these rotons come to a stop and start backing up. From Eqs. (14) it can be shown that such a roton continues to move back in the direction it came from. Not all rotons, of course, do this. Some have large enough p or b so that they can pass the QVL before they move across $p = p_0$ on the dispersion curve. These rotons then start to move back up the curve and eventually end up with $p_f = p$.

Note that $\underline{p}$ need not change direction for one of these turn arounds to occur. All we need is a relatively small force to just barely push the roton across $p = p_0$ on the dispersion curve before it passes the QVL. But the amount of momentum lost by the roton in this process is sizeable. In this way we can have a fairly strong interaction at large distances (on an atomic scale) from the core. For example, a fairly strong roton, with an impact parameter of 60 Å, can transfer 10% of its total momentum to the core.

---

*The second term in Eq. (10) is generally much smaller than the first.

In the next section we will use this model of the roton-QVL interaction to quantitatively calculate the force of mutual friction.

3. The Force of Mutual Friction.

If the two fluids are moving together, i.e. $\underline{u} \triangleq \underline{\bar{v}}_n - \underline{\bar{v}}_s = 0$, then the distribution function for the excitations that collide with a QVL can be taken as the Bose-Einstein equilibrium distribution function [14]:

$$n_0(p) = \left[e^{\frac{\epsilon(p)}{k_B T}} - 1 \right]^{-1} \simeq e^{-\frac{\epsilon(p)}{k_B T}}, \tag{16}$$

where k_B = Boltzmann's constant and T = absolute temperature. Eq.(16) has no directional preference so a roton with momentum p is just as likely to collide with a QVL from any direction, and there would be no net force on a line because the collision intensity is the same for all angles.

When $\underline{u} = \underline{\bar{v}}_n - \underline{\bar{v}}_s \neq 0$, the excitation distribution function becomes [14]:

$$n(p, \underline{p} \cdot \underline{u}) = \left[\exp\left(\frac{\epsilon - \underline{p} \cdot \underline{u}}{k_B T} \right) - 1 \right]^{-1} \simeq e^{-\frac{1}{k_B T}(\epsilon - \underline{p} \cdot \underline{u})} \tag{17}$$

For a given p, it follows from Eqs.(16) and (17) that:

$$n(p, \underline{p} \cdot \underline{u}) > n_0(p) \quad \text{for} \quad \underline{p} \cdot \underline{u} > 0$$

and

$$n(p, \underline{p} \cdot \underline{u}) < n_0(p) \quad \text{for} \quad \underline{p} \cdot \underline{u} < 0 .$$

The intensity of excitations colliding with a vortex now varies with direction and we expect this to produce a net force on the line. The force of mutual friction will be identified with the total force on all the line elements in a unit volume of the liquid.

Our method of calculating the force of mutual friction is as follows. We assume high enough temperatures, so that phonon effects can be neglected, but these temperatures should be low enough so that the roton gas may still be considered dilute. This roughly restricts us to the range $1.1^{\circ}K \leq T < 1.6^{\circ}K$. The interaction between the rotons and a QVL will be considered in a frame moving with the line, i.e. moving with $\underline{\bar{v}}_s$. Momentum transferred by a roton to the line during a scattering will be calculated as described in Section 2. Eq. (17) will be assumed to govern the distribution of incoming rotons. The effects of these collisions will then be integrated over all p, b, and incoming directions. In this way, we shall calculate the force per unit length on a single line. The force per unit volume (the force of mutual friction) can then be calculated from a knowledge of the length of vortex line per unit volume (see Eq. (3)). Physically, this method has the advantages of being very direct and picturesque.

It can be argued [7] that the momentum transfer mechanism described here would lead to a Magnus force on the line, so that the vortex would move with a velocity $\underline{v}_L$ which is slightly different from $\underline{\bar{v}}_s$. However, such a Magnus force correction can be shown to be entirely negligible in the temperature region where our theory is most likely to be valid, i.e. for $1.1^{\circ} \leq T < 1.6^{\circ}$. Hall and Vinen [7]

found that the Magnus effect is only important near T_λ.

We will also assume that the rotons do not interact with each other, although we will take some account of this later. This assumption seems reasonable for the temperature range considered here, since the roton density is low.

We assume $\underline{u}$ to lie in a plane perpendicular to the QVL (as it does in most experiments). The generalization to arbitrary $\underline{u}$ is not difficult [7]. It is convenient to divide the calculation of the force per unit length, $\underline{f}$, on the vortex into x and y components, parallel and perpendicular to $\underline{u}$ respectively (see Fig. 3). The QVL lies along the z axis. We show the derivation of f_x for $p > p_0$ in complete detail. The other force components can be calculated in exactly the same way.

If a single roton, with an impact parameter b, undergoes a scattering from the state $\underline{p}$ to the state $\underline{p}_f$, then the momentum transferred to the core in the x-direction is given by:

$$\Delta p_x(\underline{p}, b) = (\underline{p}-\underline{p}_f) \cdot \underline{i}_x , \qquad (p > p_0)$$

which can be rewritten in the convenient form:

$$\Delta p_x(\underline{p}, b) = \left[(\underline{p} - \underline{p}_f) \cdot \underline{i}_B\right] \underline{i}_B \cdot \underline{i}_x$$

$$+ \left[(\underline{p} - \underline{p}_f) \cdot \underline{i}_{B\perp}\right] \underline{i}_{B\perp} \cdot \underline{i}_x .$$

(see Fig. 3).

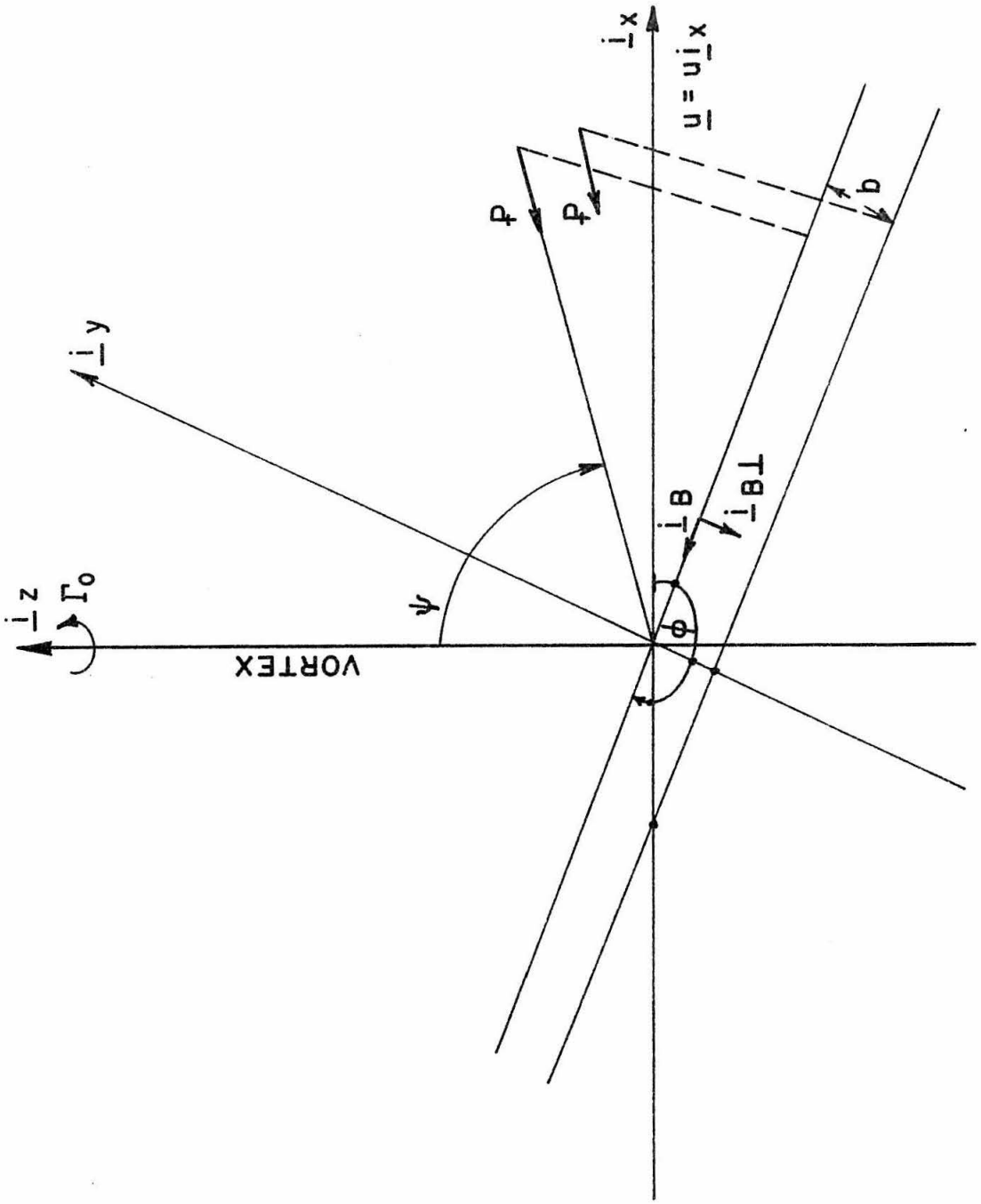
$\underline{i}_x$
$\underline{u} = u\underline{i}_x$
P
P
b
$\underline{i}_y$
$\underline{i}_{BT}$
$\underline{i}_B$
$\underline{i}_z$
Γ_o
ψ
ϕ
VORTEX

FIG. 3

Using Fig. 3, we can rewrite the last equation as follows

$$\Delta p_x(\underline{p}, b) = \left(\frac{\underline{p}\cdot\underline{i}_B - \underline{p}_f\cdot\underline{i}_B}{p}\right) p\cos\varphi$$

$$+ \left(\frac{\underline{p}\cdot\underline{i}_{B\perp} - \underline{p}_f\cdot\underline{i}_{B\perp}}{p}\right) p\sin\varphi$$

$$\triangleq \overset{+}{W}_{\parallel} p\cos\varphi + \overset{+}{W}_{\perp} p\sin\varphi \,. \tag{18}$$

Eq. (18) defines $\overset{+}{W}_{\parallel}(\underline{p}, b)$ and $\overset{+}{W}_{\perp}(\underline{p}, b)$. The + superscript denotes the case $p > p_0$, and a - superscript will denote the case $p < p_0$. The W's are the quantities that are calculated on the computer. They are independent of φ.

By an elementary kinetic argument (see, for example, [18]) the number of rotons in unit z depth with $\underline{p} \in [\underline{p}, \underline{p} + d\underline{p}]$ and $b \in [b, b + db]$ that will interact with the core per unit time is given by

$$n(\underline{p}, \underline{p}\cdot\underline{u}) \frac{d\underline{p}}{h^3} \, db \, v_G \sin\psi \,, \tag{19}$$

where h = Planck's const., and $v_G = \frac{p - p_0}{\mu_0}$ is the group (particle) velocity of the rotons. The distribution function n is given by Eq. (17). From (18) and (19) we find the force per unit length on a vortex due to all rotons with $\underline{p} \in [\underline{p}, \underline{p} + d\underline{p}]$:

$$n(\underline{p}, \underline{p}\cdot\underline{u}) \frac{d\underline{p}}{h^3} \, v_G \int db \, p \sin\psi \left[\overset{+}{W}_{\parallel} \cos\varphi + \overset{+}{W}_{\perp} \sin\varphi\right].$$

So that for all $\underline{p}$ such that $p > p_0$, we have

$$f_x(p>p_0) = \iiint_{p>p_0} \frac{dp}{h^3}\, n(\underline{p},\underline{p}\cdot\underline{u})\, v_G\, p \sin\psi \left[\overset{+}{\sigma}_{\parallel} \cos\varphi + \overset{+}{\sigma}_{\perp} \sin\varphi \right] \quad , \tag{20}$$

where

$$\left.\begin{aligned} \overset{+}{\sigma}_{\parallel}(p) &= \int db\, \overset{+}{W}_{\parallel} \quad , \\ \text{and} \qquad \overset{+}{\sigma}_{\perp}(p) &= \int db\, \overset{+}{W}_{\perp} \quad . \end{aligned}\right\} \tag{21}$$

It can be shown analytically (see Appendix A), using a small angle scattering approximation [15], that $\overset{+}{\sigma}_{\perp}$ and $\overset{-}{\sigma}_{\perp}$ diverge logarithmically as $b \to \infty$. This makes it necessary to use a cut-off or screening radius on the $\underline{v}_s$ field produced by the vortex. We choose the following cut-off potential to replace Eq. (8):

$$\left.\begin{aligned} (v_{sx},\ v_{sy}) &= \left(-\frac{\Gamma_0 y}{2\pi r^2} ,\ \frac{\Gamma_0 x}{2\pi r^2} \right) , \quad r \leq L \\ &= 0, \quad r > L \ , \end{aligned}\right\} \tag{22}$$

where L = a typical roton-roton mean free path in the T range where we expect our theory to be most valid. This choice is based on the assumption that the weak QVL potential is not felt beyond L because it is screened by strong roton-roton collisions. We take L = 150 Å, which corresponds to the roton-roton mean free path for 1.2°K or 1.3°K as calculated from the Khalatnikov theory [14]. Our final results are insensitive to the exact choice of L as long as

$L \simeq 0(10^2\ \mathring{A})$, since the roton-QVL interaction is very weak at these distances. (We suspect that the dependence on L is logarithmic).

The cross sections (21) make the explicit evaluation of (20) extremely difficult and complicated. However, the presence of the factors n and v_G results in the fact that only a very small range of p contributes to the integrals (20). The σ's vary slowly over this range, and we will make the major computational simplification of factoring out average values of these quantities from Eq.(20). Our method of averaging will be to evaluate Eqs.(21) for $p = \bar{p}$, where $\bar{p}$ is the momentum corresponding to the average group velocity, $(\bar{v}_G^2)^{1/2}$, of a roton in thermal equilibrium at a given temperature. More explicitly, $\bar{p}$ is a solution of: (see Ref. [16])

$$\bar{v}_G^2 = \left(\frac{\bar{p}-p_0}{\mu_0}\right)^2 = \frac{2}{\pi}\ \frac{k_B T}{\mu_0}\ . \tag{23}$$

Of course, the $\bar{p} > p_0$ solution is used for $\overset{+}{\sigma}$ and the $\bar{p} < p_0$ solution for $\bar{\sigma}$. This is the same procedure used in [7] and probably in [8], and produces values for $\bar{p}$ that fall close to the center of the contributing p range. The calculation of B and B' is not very sensitive to the particular method used to determine the average value of p used to evaluate (21). Changes in B and B' with T come entirely from the quantities N_r and I shown in Eqs.(27) and not from the cross sections.

After removing the average σ's from (20) we have

$$f_x(p>p_0) = \overline{\sigma}^{+}_{\parallel}(\bar{p}) \iiint\limits_{p>p_0} \frac{dp}{h^3}\, n(\underline{p}, \underline{p}\cdot\underline{u})\left(\frac{p-p_0}{\mu_0}\right) p \sin\psi \cos\varphi$$

$$+ \overline{\sigma}^{+}_{\perp}(\bar{p}) \iiint\limits_{p>p_0} \frac{dp}{h^3}\, n(\underline{p}, \underline{p}\cdot\underline{u})\left(\frac{p-p_0}{\mu_0}\right) p \sin\psi \sin\varphi \qquad (24)$$

Noting that

$$\underline{p}\cdot\underline{i}_x = p \sin\psi \cos\varphi ,$$

and

$$\underline{p}\cdot\underline{i}_y = -p \sin\psi \sin\varphi ,$$

we recognize the two integrals in (24) as the x and -y components of

$$\iiint\limits_{p>p_0} \frac{dp}{h^3}\, n(\underline{p}, \underline{p}\cdot\underline{u})\, v_G\, \underline{p} \quad . \qquad (25)$$

Such integrals can be shown to only have a component along $\underline{u}$, and since $\underline{u} = u\,\underline{i}_x$ the second integral in (24) must equal zero. This can also be verified by direct integration.

The integrals in Eqs. (24) or (25) can be evaluated in a straightforward manner using the techniques of Khalatnikov [14]. This involves an expansion of $n(\underline{p}, \underline{p}\cdot\underline{u})$ in powers of u. Assuming u to be small and retaining only the first nonvanishing term in (24) we find

$$
\left.
\begin{aligned}
f_x(p > p_0)\underline{i}_x &= \overset{+}{\sigma}_{\parallel} \iiint_{p>p_0} \frac{d\underline{p}}{h^3}\, n(\underline{p}, \underline{p}\cdot\underline{u})\left(\frac{p-p_0}{\mu_0}\right)\underline{p} \\
&= \overset{+}{\sigma}_{\parallel} \frac{p_0^4}{3\sqrt{2\pi}} \frac{N_R}{(\mu_0 k_B T)^{3/2}} \overset{+}{I}(\underline{\bar{v}}_n - \underline{\bar{v}}_s) \\
&\triangleq \overset{+}{\sigma}_{\parallel} \overset{+}{A}(\underline{\bar{v}}_n - \underline{\bar{v}}_s) .
\end{aligned}
\right\} \tag{26}
$$

N_R and $\overset{+}{I}$ in (26) are given by

$$
\left.
\begin{aligned}
N_R &= \frac{2(2\pi)^{3/2}(\mu_0 k_B T)^{1/2} p_0^2}{h^3} e^{-\frac{\Delta}{k_B T}}, \\
\text{and} \qquad \overset{+}{I} &= \int_1^{\infty} \eta^4 (\eta - 1) e^{-\frac{\alpha}{T}(\eta-1)^2} d\eta ,
\end{aligned}
\right\} \tag{27}
$$

where

$$
\alpha = \frac{p_0^2}{2\mu_0 k_B} .
$$

The quantities in (27) are easily evaluated.

Repeating the derivation for the $p < p_0$ case and for the y component of $\underline{f}$ results in the final force per unit length on a Feynman vortex. We find

$$
\underline{f} = D(\underline{\bar{v}}_n - \underline{\bar{v}}_s) + D' \underline{i}_z \times (\underline{\bar{v}}_n - \underline{\bar{v}}_s) , \tag{28}
$$

where

$$D = \overset{+}{\sigma}_{\|} \overset{+}{A} - \overset{-}{\sigma}_{\|} \overset{-}{A} ,$$

$$D' = \overset{+}{\sigma}_{\perp} \overset{+}{A} - \overset{-}{\sigma}_{\perp} \overset{-}{A} . \tag{29}$$

$\overset{-}{A}$ has the same form as $\overset{+}{A}$ except that the integral $\overset{+}{I}$ is replaced by $\bar{I}$, where

$$\bar{I} = \int_0^1 \eta^4 (\eta-1)\, e^{-\frac{\alpha}{T}(\eta-1)^2} \, d\eta .$$

From Eq. (28), we calculate the force per unit volume $\underline{F}$. For the rotating bucket experiment we can immediately conclude that the length of line per unit volume is given by Eq. (3). So then

$$\underline{F} = \frac{2\omega_0}{\Gamma_0} \underline{f} = \frac{2\omega_0}{\Gamma_0} D(\underline{\bar{v}}_n - \underline{\bar{v}}_s) + \frac{2\omega_0}{\Gamma_0} D' \underline{i}_z \times (\underline{\bar{v}}_n - \underline{\bar{v}}_s). \tag{30}$$

In order to conform with the established notation, we rewrite (30) as

$$F = -B \frac{\rho_s \rho_n}{\rho} \frac{\underline{\omega}_0 \times [\underline{\omega}_0 \times (\underline{\bar{v}}_n - \underline{\bar{v}}_s)]}{\omega_0} - B' \frac{\rho_s \rho_n}{\rho} [\underline{\omega}_0 \times (\underline{\bar{v}}_n - \underline{\bar{v}}_s)] , \tag{31}$$

where

$$\left.\begin{aligned} B &= \frac{2\rho}{\rho_s \rho_n \Gamma_0} D , \\ B' &= -\frac{2\rho}{\rho_s \rho_n \Gamma_0} D' . \end{aligned}\right\} \tag{32}$$

B and B' were computed from the preceding equations for

four temperatures between 1.1°K and 2.0°K. About 400-500 rotons were run on the computer and used to calculate the σ's. It was found that $\overset{+}{\sigma}_{\|} \simeq 10$ Å, while all the other cross sections had magnitudes in the 2Å-3.5 Å range. In Table 1, we collect our computed values of B and B' for $a_0 = 3$ Å and L = 150 Å. Various data from Donnelly [12] were used in some of the numerical calculations.

T(°K)	B	B'
1.1°	1.30	.62
1.4°	1.35	.55
1.7°	1.75	.70
2.0°	4.00	1.35

Table 1.

For comparison, we list, in Table 2, the experimental values for B found by Hall and Vinen [3], and for B' found by Snyder and Linekin [4]. Note the large error bounds on B'.

T(°K)	B	B'
1.2°	1.5	.73±.15
1.4°	1.4	.47±.25
1.7°	.9	.23±.15
2.0°	1.0	.36±.25

Table 2.

4. Discussion.

We have plotted various experimental and theoretical values for B and B' in Fig. 4.

Our results, curves d and e, compare well with the experiments for $1.1^\circ \leq T < 1.6^\circ$, but differ considerably up near 2.0°K, where the roton gas is more than ten times as dense as it is at 1.4°K. Of course, it is not surprising that our simple kinetic picture fails at such high densities. From curve b, we can see that Hall and Vinen had the same problem. They tried to correct this by taking into account a Magnus effect, which is completely negligible for $T \leq 1.8^\circ$K, and the "dragging of the normal fluid near the core," which improved their agreement considerably.

Use of this "dragging" effect would also improve our high temperature results, but we find this correction disagreeable. We could also lower our values for high T by taking into account the fact that L decreases as the roton density increases. At 1.9°K L is about 1/10 of its value at 1.4°K. Such a drastic reduction in L would decrease our cross sections somewhat and improve our experimental agreement. However, it is not really possible to use our simple kinetic picture with a mean free path of about 10 Å. Also, Eqs. (5) and (6) may not hold for high roton densities.

The existence of a nonzero B' from the Hall and Vinen theory (curve c) is a consequence of the Magnus effect. They found the roton scattering to be symmetric, and this would give $B' \equiv 0$ without the Magnus correction.

It was found that our results were more sensitive to the value of

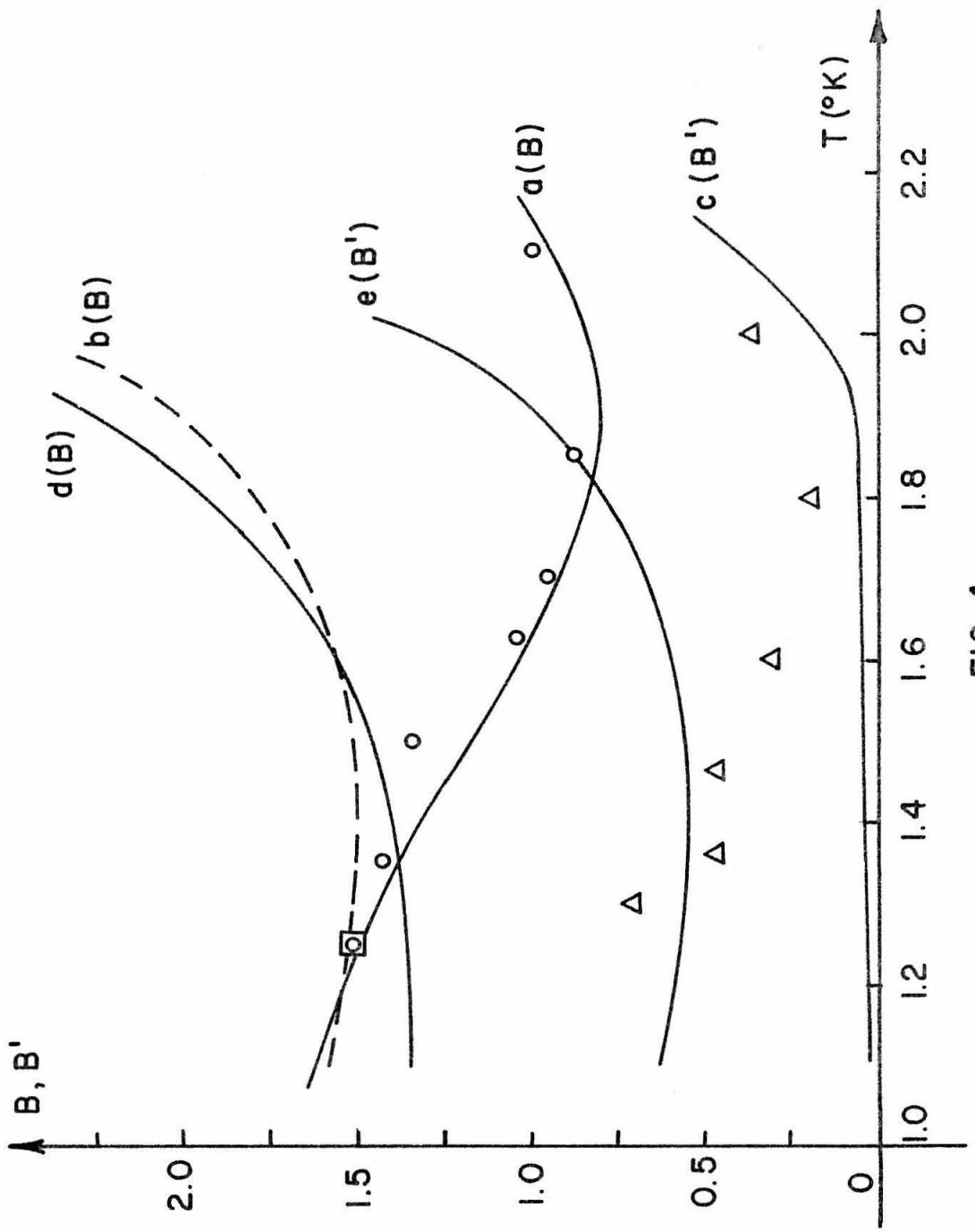

FIG. 4

Fig. 4

o are experimental values for B from Hall and Vinen [3].

Δ are experimental values for B' from Snyder and Linekin [4].

Curve a(B) are the theoretical values of B obtained by Hall and Vinen [7] from a derivation which includes corrections for Magnus effect and normal fluid dragging near the core.

Curve b(B) are the theoretical values of B obtained in [7] without their Magnus and dragging corrections.

Both curves a and b have been fitted to the experimental point [o] by an appropriate choice of an arbitrary parameter in the Hall and Vinen theory.

Curve c(B') are the theoretical values of B' from [7].

Curves d(B) and e(B') are the theoretical values of B and B' calculated in Section 3.

Γ_0 than to either a_0 or L. There are two sources of Γ_0 in our calculations, Eqs. (3) and (22). Our work then seems to indicate that Γ_0 cannot differ appreciably from $\frac{h}{m_{He}}$ and that Eq. (3) for the line density in a rotating vessel must be essentially correct.

The closest things to arbitrary parameters in our calculations are a_0 and L, both of which were chosen a priori in a rational manner. Physical considerations limit these quantities to values that must be near the ones that were picked. Our calculation is very insensitive to the choice of L (probably a logarithmic dependence), and we estimate that a different reasonable choice of a_0 (say 4 Å) would not change our values of B or B' by more than 10-15%. Most known estimates put $2\,\text{Å} \leq a_0 \leq 4\,\text{Å}$ for the temperatures we are considering.

On the other hand, the analyses of Hall and Vinen and of Lifshitz and Pitaevskii contain very important adjustable parameters whose choice is equivalent to choosing values for their cross sections to fit the experimental data. It should be noted that the cross sections used in their calculations for B and B' are not equal to those that they derive from the Born and quasi-classical approximations.

In its original form, the quasi-classical analysis of Lifshitz and Pitaevskii produces poor values for B and B', yet our calculation gives good results. Lifshitz and Pitaevskii give no details of their analysis, but the difference between their results and ours seems to stem from the fact that they did not treat the very strong interactions near the core. They improve matters by hypothesizing

that rotons passing near the core must give almost all their momentum to the vortex, and on the basis of this they simply add an arbitrary constant to one of their cross sections. The constant is then chosen to fit the experimental data. If, in our analysis, we exclude rotons with impact parameters less than about an atomic spacing ($\simeq$ 4 Å) then we get the same poor values that Lifshitz and Pitaevskii found at first. Our good results therefore seem to be a consequence of the strong snap-backs and roton absorptions that occur in the core region. Of course, we do not claim that our model is strictly correct from a quantum mechanical standpoint; but, like so many other semi-classical models used in the study of liquid Helium II, it seems to be perfectly adequate from a phenomenological standpoint.

Appendix

Small Angle Roton-Vortex Line Scattering

For large impact parameter b, the vortex field is weak and the rotons are deflected through very small angles. In this section we shall analyze small angle roton-QVL scattering and show that the cross section $\overset{+}{\sigma}_{\perp}$ (Eq. (21)) diverges logarithmically as $b \to \infty$. We follow the method of Landau and Lifshitz [15].

The QVL shall be considered stationary and the x-axis taken as the direction of incidence of a roton in the initial state $\underline{p}$ (see Fig. A-1). The angle of deflection θ_f is given by

$$\theta_f \simeq \sin \theta_f = \frac{p_{fy}}{p_f} \simeq \frac{p_{fy}}{p} \tag{A-1}$$

in the small angle approximation.

Using the roton Hamiltonian

$$H = \epsilon(p) + \underline{p} \cdot \underline{v}_s \triangleq \epsilon + V ,$$

we can integrate the Hamilton equation

$$\frac{dp_y}{dt} = - \frac{\partial H}{\partial y} = - \frac{\partial V}{\partial y} ,$$

to obtain

$$p_{fy} = \int_{-\infty}^{\infty} - \frac{\partial V}{\partial y} \, dt . \tag{A-2}$$

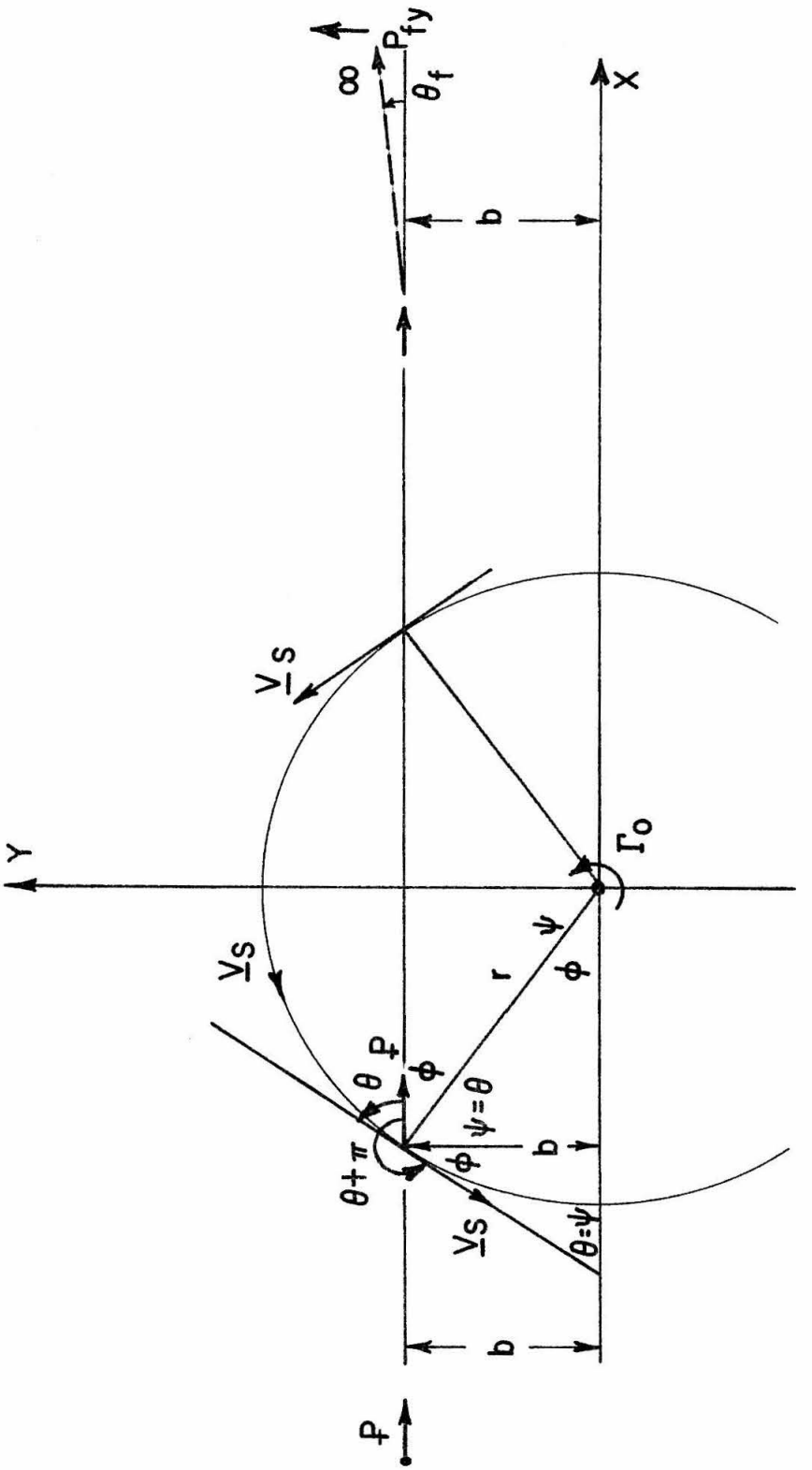
∞
P_{fy}
θ_f
X
b
$\underline{V}_S$
Y
Γ_o
r
ψ
ϕ
θ
P_f
$\psi=\theta$
$\theta+\pi$
$\theta=\psi$
P

FIG. A – 1

Since the vortex field is weak for large b, we can assume the particle is essentially undeflected from its initial path so that

$$y \simeq b$$

and
$$\frac{dx}{dt} = \frac{p-p_0}{\mu_0} = v_G \simeq \text{const} \tag{A-3}$$

for all time.

In our approximation we can rewrite V(x,y) as a function of r only (see Fig. A-1) so that

$$- \frac{\partial V}{\partial y} = - \frac{\partial V}{\partial r} \frac{\partial r}{\partial y} = - \frac{\partial V}{\partial r} \frac{y}{r} \simeq - \frac{\partial V}{\partial r} \frac{b}{r} . \tag{A-4}$$

Using (A-3) and (A-4), (A-2) becomes

$$p_{fy} = - \frac{b}{v_G} \int_{-\infty}^{\infty} \frac{\partial V}{\partial r} \frac{dx}{r} . \tag{A-5}$$

From $V = \underline{p} \cdot \underline{v}_s$, Eq. (8) of Section 2, and Fig. A-1, we find, in this approximation:

$$V = - \frac{\Gamma_0}{2\pi} p \frac{b}{r^2} ,$$

so that

$$\frac{\partial V}{\partial r} = \frac{p\Gamma_0}{\pi} \frac{b}{r^3} . \tag{A-6}$$

Using $r^2 = x^2 + b^2$, we have $dx = \frac{r\,dr}{\sqrt{r^2 - b^2}}$ and $\int_{-\infty}^{\infty} dx \rightarrow 2\int_{b}^{\infty} dr$. Then from (A-5) and (A-6) we have

$$p_{fy} = -\frac{2\Gamma_0 p b^2}{\pi v_G} \int_{b}^{\infty} \frac{dr}{r^3\sqrt{r^2-b^2}} = -p\frac{\Gamma_0}{2v_G}\frac{1}{b}, \qquad \text{(A-7)}$$

i.e. $$p_{fy} \propto \frac{1}{b}.$$

As far as boundedness is concerned, $\sigma_\perp$ is essentially equal to $\int^{L} db\, p_{fy}$ so that

$$\sigma_\perp \rightarrow \ln L \rightarrow \infty, \quad \text{as} \quad L \rightarrow \infty.$$

This proves the statement made at the beginning of this section. The deflection angle can be calculated from Eq. (A-1).

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