NONLINEAR DISPERSIVE WAVE PROBLEMS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1966

(Submitted April 4, 1966)

ACKNOWLEDGMENTS

The author expresses his thanks to Professor G. B. Whitham, who suggested the problems treated in this dissertation, and has made available various preliminary calculations, preprints of papers, and lecture notes. In addition, Professor Whitham has given freely of his time for useful and enlightening discussions and has offered timely advice, encouragement, and patient supervision.

Others who have assisted the author by suggestions or constructive criticism are Mr. G. W. Bluman and Mr. R. Seliger of the California Institute of Technology and Professor J. Moser of New York University.

The author gratefully acknowledges the National Science Foundation Graduate Fellowship that made possible his graduate education.

J. C. L.

ABSTRACT

The nonlinear partial differential equations for dispersive waves have special solutions representing uniform wavetrains. An expansion procedure is developed for slowly varying wavetrains, in which full nonlinearity is retained but in which the scale of the nonuniformity introduces a small parameter. The first order results agree with the results that Whitham obtained by averaging methods. The perturbation method provides a detailed description and deeper understanding, as well as a consistent development to higher approximations. This method for treating partial differential equations is analogous to the "multiple time scale" methods for ordinary differential equations in nonlinear vibration theory. It may also be regarded as a generalization of geometrical optics to nonlinear problems.

To apply the expansion method to the classical water wave problem, it is crucial to find an appropriate variational principle. It was found in the present investigation that a Lagrangian function equal to the pressure yields the full set of equations of motion for the problem. After this result is derived, the Lagrangian is compared with the more usual expression formed from kinetic minus potential energy. The water wave problem is then examined by means of the expansion procedure.

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PART I

A PERTURBATION METHOD FOR NONLINEAR DISPERSIVE WAVES

1. Introduction

Special solutions representing infinitely long, periodic wavetrains are readily obtainable for many partial differential equations of interest in water wave theory, plasma dynamics, and other fields. For the usual linear examples these are sinusoidal, and a general solution may then be constructed by superposition of these wavetrains in a Fourier integral. Since the different uniform wavetrains generally have different velocities of propagation, a local disturbance expressed in this way tends to break up, or disperse, into its various component waves. The saddle-point or stationary-phase approximation shows for typical examples that a nearly uniform wavetrain eventually develops in any locality.

Knowledge of the family of uniform wavetrain solutions also aids examination of such dispersive wave behavior for nonlinear equations, but an approximation becomes necessary since superposition methods do not apply. The approximation of interest here is the case of a nearly uniform wavetrain. Whitham [1]

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1. G. B. Whitham, Proc. Roy. Soc. A, vol. 283 (1965), pp. 238-261.

showed one way to obtain approximate equations for the slow, large-scale variation of amplitude, wavenumber, etc., for such a wavetrain. A subsequent method [2] allows these results to be obtained in a simpler and more satisfying manner by application of an averaging procedure to the Lagrangian of the original system. The desired equations then arise directly as the Euler equations of the averaged Lagrangian. The averaged Lagrangian method is shown in §2 for a simple example.

For more precise and deeper understanding of the averaged Lagrangian method it is necessary to know how the same results may be obtained directly from the differential equation as the first approximation in a formal perturbation expansion procedure. The present dissertation provides this for certain classes of equations. It should be remarked, however, that the averaged Lagrangian method gives the result in an elegant form that is not immediately evident from the expansion procedure.

The background material in §3 explains why a certain form of expansion was chosen, but §4 contains the essential ideas of the expansion method. A simple example is worked in detail and the result compared with that given by the averaged Lagrangian method. In Part II, the material of §7, which gives a

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2. G. B. Whitham, J. Fluid Mech., vol. 22 (1965), pp. 273-283.

variational principle for the water wave problem, is really a separate item and may be read independently of the rest [3].

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3. The material of §§4, 5, and 6 is to appear in Proc. Roy. Soc. A, (1966). The material of §7 has been submitted to J. Fluid Mech.

2. The Averaged Lagrangian Method for Nonlinear Dispersive Waves

For comparison with results to be obtained in §4 by an expansion method, Whitham's [4] averaged Lagrangian method will now be applied to the simple variational problem

$$\delta\iint L\,dx\,dt = \delta\iint\left(\tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 - V(u)\right)dx\,dt = 0, \tag{2.1}$$

for which the Euler equation is

$$u_{tt} - u_{xx} + V'(u) = 0. \tag{2.2}$$

(One may visualize $u(x,t)$ as the displacement of a stretched string subject to a nonlinear restoring force $V'(u)$ per unit length; however, other nonlinear terms are sometimes of importance in such a problem.)

First consider special solutions of (2.2) of the form

$$u(x,t) = U(\theta), \tag{2.3}$$

the phase variable θ being set equal to $\kappa x - \omega t$ for some constants κ and ω. Such uniform solutions, in which u is constant on lines of constant θ, are typical in problems that are invariant under x and t translation. Substitution of (2.3) in (2.2) gives

$$(\omega^2 - \kappa^2)\,U_{\theta\theta} + V'(U) = 0. \tag{2.4}$$

Equation (2.4) is nonlinear, but may be solved implicitly with

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4. Loc. cit.

the aid of an energy integral. Multiplication of (2.4) by U_θ and integration gives

$$\tfrac{1}{2}(\omega^2 - \kappa^2) U_\theta^2 + V(U) = E, \tag{2.5}$$

where E is an energy-like integration constant. Then solution of (2.5) shows that

$$\theta = (\omega^2 - \kappa^2)^{\frac{1}{2}} \int^U \left\{2(E - V(U'))\right\}^{-\frac{1}{2}} dU' - \eta, \tag{2.6}$$

where η is a phase-like integration constant. Inversion of (2.6) gives a family of solutions of (2.4), which may be denoted by

$$U(\theta) = f(\theta + \eta, E, \omega^2 - \kappa^2). \tag{2.7}$$

In the case of interest, U oscillates between two zeros of $E - V(U)$ and is periodic in θ. In the linear case, where $V(u) = \frac{1}{2} u^2$, (2.7) is simply a sinusoid.

Through use of the above family of uniform, periodic solutions, one may now examine the behavior of nearly uniform wavetrains, in which the amplitude, frequency, and wavenumber change slowly over large distances and times. Whitham [5] obtained equations for the slow variation of these quantities by applying an averaging procedure to the Lagrangian in (2.1). In this procedure, E, κ, and ω are temporarily treated as fixed parameters, and L is averaged over one oscillation of the uniform

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5. Ibid.

solution to give

$$\mathcal{L}(E,\kappa,\omega) = (\omega^2 - \kappa^2)^{\frac{1}{2}} \oint \left\{2(E - V(U))\right\}^{\frac{1}{2}} dU - E. \qquad (2.8)$$

Then E, κ, and ω are allowed to be slowly varying functions of x and t. For convenience this will be expressed here by writing $E(X,T)$, $\kappa(X,T)$, and $\omega(X,T)$ as functions of "stretched coordinates" X and T. Finally, the averaged Lagrangian (2.8) is interpreted as giving a new variational principle for the quantities $E(X,T)$ and $\Theta(X,T)$, where $\kappa = \Theta_X$ and $\omega = -\Theta_T$. The Euler equation for E-variations is

$$(\omega^2 - \kappa^2)^{\frac{1}{2}} \oint \left\{2(E - V(U))\right\}^{-\frac{1}{2}} dU = 1. \qquad (2.9)$$

In the linear case, $V(u) = \frac{1}{2} u^2$, the functional relation (2.9) between E, κ, and ω reduces to the usual dispersion relation $4\pi^2(\omega^2 - \kappa^2) = 1$, with no dependence on E. The Euler equation

$$\frac{\partial}{\partial T} \mathcal{L}_\omega - \frac{\partial}{\partial X} \mathcal{L}_\kappa = 0$$

for Θ-variations is

$$\frac{\partial}{\partial T} \left\{\omega(\omega^2 - \kappa^2)^{-\frac{1}{2}} \oint \left(2(E - V(U))\right)^{\frac{1}{2}} dU\right\}$$

$$+ \frac{\partial}{\partial X} \left\{\kappa(\omega^2 - \kappa^2)^{-\frac{1}{2}} \oint \left(2(E - V(U))\right)^{\frac{1}{2}} dU\right\} = 0. \qquad (2.10)$$

Equations (2.9) and (2.10), together with the condition

$$\kappa_T + \omega_X = 0, \qquad (2.11)$$

form a set of coupled equations for E, κ, and ω.

3. Differential Equations with Slowly Varying Parameters

Numerous expansion methods have been used for ordinary differential equations with slowly varying parameters. The objective is to generalize one of these methods to partial differential equations and then to obtain, as the lowest approximation of the expansion, the same results as in §2. Several expansion methods are discussed below from the standpoint of such a generalization.

Because of its applications in quantum mechanics and plasma physics, the theory of adiabatic invariants has been widely examined. The traditional treatments of this problem depend heavily on the formalism of Hamiltonian mechanics. A well-known result [6] is that, for certain systems, an "adiabatic invariant" of the form $\oint p\, dq$ remains nearly constant as parameters of the system are slowly varied. Thus, for example, when a pendulum weight swings from a string of length S, the amplitude of small oscillations varies approximately as $S^{\frac{1}{4}}$ if S is gradually changed. The theory of adiabatic invariants is also applicable to the full nonlinear problem.

The treatment of adiabatic invariants given by

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6. See, for example, L. D. Landau and E. M. Lifshitz, Mechanics. London: Pergamon Press (1960).

Gardner [7] is an example of the conciseness obtainable by Hamiltonian methods. In that treatment, a canonical transformation is used to map the q-p plane onto the q'-p' plane in such a way that each level curve of the Hamiltonian function $H(q,p,\lambda(t))$ maps to a circle of corresponding area. If the parameter λ changes slowly with time, the level curves of the transformed Hamiltonian $H'(q',p',\lambda(t))$ deviate only slightly from the circles, and it then follows that $\oint p\, dq$ is an adiabatic invariant. Repetition of the procedure shows that the result also holds to higher approximations if the comparison of amplitudes is made between successive times at which all derivatives of λ are zero.

It appears that methods like that of Gardner cannot be extended readily to partial differential equations, where several independent variables would be involved, for Rüssmann [8] has determined that only trivial canonical transformations (the point transformations) are available when Hamilton's equations are generalized to more than one independent variable. For this reason, some direct approaches that avoid canonical transformations will now be considered.

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7. C. S. Gardner, Phys. Rev., vol. 115 (1959), pp. 791-794.

8. H. Rüssmann, Arch. Rat. Mech. Anal., vol. 8 (1961), pp. 353-357.

The methods of Krylov and Bogoliubov [9] are a well-known direct approach to slightly nonlinear ordinary differential equations. For the equation

$$\frac{d^2u}{dt^2} + k(T)\, u = \epsilon\, f(T,u,\frac{du}{dt}), \tag{3.1}$$

where ϵ is a small parameter and $T = \epsilon t$ is the "slow time," an expansion of the form

$$u = a \cos\theta + \epsilon\, U_1(\theta,a,T) + \epsilon^2\, U_2(\theta,a,T) + \ldots, \tag{3.2}$$

$$\frac{da}{dt} = \epsilon\, A_1(a,T) + \epsilon^2\, A_2(a,T) + \ldots, \tag{3.3}$$

$$\frac{d\theta}{dt} = \omega(T) + \epsilon\, B_1(a,T) + \epsilon^2\, B_2(a,T) + \ldots, \tag{3.4}$$

is typically used. The quantities B_1, B_2,... are determined in such a way as to eliminate secular terms proportional to time, which would otherwise destroy the uniform validity of the expansion. Because the lowest approximation in (3.2) is arbitrarily taken as a sinusoidal function, the method is limited to problems that are only slightly nonlinear.

For linear partial differential equations, the expansion

$$e^{i\epsilon^{-1}S}(A_0 + \epsilon A_1 + \epsilon^2 A_2 + \ldots) \tag{3.5}$$

is commonly used to derive the results of geometrical optics

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9. See, for example, N. N. Bogoliubov and Y. A. Mitropolsky, Asymptotic Methods in the Theory of Non-linear Oscillations. Delhi: Hindustan Publishing Co. (1961).

from the wave equation or to obtain equations of classical mechanics as an approximation to wave mechanics. Expansions similar to (3.2) and (3.5) have also been used for simple, slightly nonlinear partial differential equations [10], but such a procedure quickly leads to complicated algebra.

To combine some of the advantages of the previous methods, it seems reasonable to use a direct expansion resembling (3.2) and (3.5), but to attempt the full nonlinear problem as in the examination of adiabatic invariants. Kuzmak [11] has done this for a class of ordinary differential equations. In that approach, the same ϵ dependence is kept as in (3.2), but the cosine function is replaced. Kuzmak considered asymptotic solutions of the equation

$$\frac{d^2u}{dt^2} + \epsilon\, g(T,u)\, \frac{du}{dt} + G(T,u) = 0, \qquad (3.6)$$

where ϵ is a small parameter, $T = \epsilon t$ is the "slow time," and g and G are prescribed functions. The dependence of G on T represents the slow variation of an external parameter, as in the problem of adiabatic invariants, and the function g

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10. D. Montgomery and D. A. Tidman, Phys. Fluids, vol. 7 (1964), pp. 242-249.

11. G. E. Kuzmak, Prikl. Mat. Mekh., Akad. Nauk SSSR, vol. 23 (1959), pp.515-526. (Translated in Appl. Math. Mech., vol. 23 (1959), pp. 730-744.)

gives a small frictional damping effect. Following Kuzmak, let us use an expansion of the form

$$u(t) = U(\theta,T) + \epsilon\, U_1(\theta,T) + \ldots, \tag{3.7}$$

$$\theta_t = \omega(T). \tag{3.8}$$

The expanded terms

$$u_t = U_\theta\omega + \epsilon(U_T + U_{1\theta}\omega) + O(\epsilon^2),$$

$$u_{tt} = U_{\theta\theta}\omega^2 + \epsilon(U_\theta\omega_T + 2U_{\theta T}\omega + U_{1\theta\theta}\omega^2) + O(\epsilon^2),$$

$$G(T,u) = G(T,U) + \epsilon\, G_u(T,U)\, U_1 + O(\epsilon^2),$$

are substituted into (3.6) and the various powers of ϵ equated to zero to give

$$U_{\theta\theta}\omega^2 + G(T,U) = 0, \tag{3.9}$$

$$U_{1\theta\theta}\omega^2 + G_u(T,U)\, U_1 = -U_\theta\omega_T - 2U_{\theta T}\omega - g(T,U)\, U_\theta\omega. \tag{3.10}$$

Although integration of (3.8) is eventually to give the dependence of θ on t as

$$\theta = \int^t \omega(\epsilon t')\, dt',$$

the crucial idea of the method is to satisfy (3.9) and (3.10) by treating U, U_1,... as functions of two independent variables, θ and T. Then (3.9) and (3.10) may be regarded as ordinary differential equations with θ as the independent variable, T being a parameter. The appropriate approximating functions U and U_1, as well as $\omega(T)$, are eventually to be determined from these equations.

The solution of (3.9) involves a "constant of integration"

which is a function of T; it may be taken to be essentially the energy $E(T)$. This is undetermined from (3.9). However, in the next order equation (3.10) it turns out that a satisfactory U_1 requires a type of orthogonality restriction on the right hand side of (3.10). This gives a first order differential equation for E as a function of T. If $g = 0$ it leads to adiabatic invariance.

Kuzmak's [12] approach extends readily to partial differential equations, and the analogous problems concern dispersive waves. This extension of the expansion method is the main objective of Part I. The motivation and derivation are so similar to those for ordinary differential equations that we shall give the full discussion for a simple dispersive wave problem, instead of continuing the discussion of Kuzmak's work. Only partial differential equations of a variational nature will be considered, however, and so, for completeness, Kuzmak's treatment of slightly dissipative ordinary differential equations will be indicated at the end of §4.

When the expansion method is applied to partial differential equations, the lowest approximating functions turn out to be just the uniform wavetrains considered in §2. For some

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12. Ibid.

simple problems, these periodic wavetrains are known exactly, for example, in terms of elliptic functions. Then the slowly varying but fully nonlinear problem may be examined. Often, of course, the uniform wavetrains for a nonlinear problem are not known in closed form; typically, only the small amplitude approximations are known. Even in that case, it may still be desirable for clarity and for reduction of algebraic complexity to treat the uniform solutions as known and thus to defer the expansion in terms of small amplitude to the last step.

4. The First Order Expansion for $u_{tt} - u_{xx} + V'(u) = 0$

The simple partial differential equation

$$u_{tt} - u_{xx} + V'(u) = 0 \tag{4.1}$$

will now be examined by means of a perturbation expansion. As shown in §2, for typical choices of the function $V(u)$, equation (4.1) has solutions in the form of uniform, periodic wavetrains: that is,

$$u(x,t) = U(\theta), \tag{4.2}$$

the phase variable θ being set equal to $\kappa x - \omega t$ for some constants κ and ω.

The objective is to describe slow variation of a wavetrain over large distances and times by letting U also depend on "stretched coordinates," $X = \epsilon x$ and $T = \epsilon t$, where ϵ is a small parameter. The form of u is then anticipated as

$$u(x,t) = U(\theta,X,T) + \epsilon\, U_1(\theta, X,T) + \epsilon^2\, U_2(\theta,X,T) + \ldots, \tag{4.3}$$

where $U_1, U_2, \ldots$ are higher order corrections. The derivatives of θ are now also allowed to depend on X and T, so that

$$\theta_x = \kappa(X,T), \qquad \theta_t = -\omega(X,T), \tag{4.4}$$

with

$$\kappa_T + \omega_X = 0; \tag{4.5}$$

equivalently this means $\theta = \epsilon^{-1}\Theta(X,T)$. The idea of the expansion (4.3) is to include the relatively fast "local" oscillations through the dependence on the variable θ, while

dependence on X and T takes care of the slow variation in "macroscopic" quantities like amplitude, frequency, and wave-number. A naïve expansion

$$u(x,t;\epsilon) = u_0(x,t) + \epsilon\, u_1(x,t) + \ldots$$

leads to secular terms in u_1, etc.

The expansion and collection of powers of ϵ is straightforward, all functions, of course, being assumed sufficiently smooth. Substituting the expanded terms

$$\left.\begin{aligned} u_{xx} &= U_{\theta\theta}\kappa^2 + \epsilon(U_\theta \kappa_X + 2U_{\theta X}\kappa + U_{1\theta\theta}\kappa^2) + O(\epsilon^2), \\ u_{tt} &= U_{\theta\theta}\omega^2 + \epsilon(-U_\theta \omega_T - 2U_{\theta T}\omega + U_{1\theta\theta}\omega^2) + O(\epsilon^2), \\ V'(u) &= V'(U) + \epsilon\, U_1 V''(U) + O(\epsilon^2) \end{aligned}\right\} \quad (4.6)$$

into (4.1) and equating the various powers of ϵ to zero gives the equations

$$(\omega^2 - \kappa^2)\, U_{\theta\theta} + V'(U) = 0 \qquad (4.7)$$

and

$$(\omega^2 - \kappa^2)\, U_{1\theta\theta} + V''(U)\, U_1 = F_1(\theta,X,T), \qquad (4.8)$$

where

$$F_1 = 2\omega U_{\theta T} + 2\kappa U_{\theta X} + \omega_T U_\theta + \kappa_X U_\theta. \qquad (4.9)$$

Equations (4.7) and (4.8) are essentially similar to equations (3.9) and (3.10) used by Kuzmak for the examination of ordinary differential equations.

Although integration of (4.4) is eventually to determine the dependence of θ on x and t, the crucial idea in the method is to satisfy (4.7) and (4.8) by treating U, U_1,... as

functions of three independent variables, θ, X, and T. Then (4.7) and (4.8) may be regarded as ordinary differential equations with θ as the independent variable, X and T being parameters.

Equation (4.7) has already been examined in §2 and the solution indicated in (2.7); however, the quantities U, κ, and ω, as well as the integration "constants" E and η, now depend on the parameters X and T. Thus, the form of the solution for U may be written as

$$U(\theta,X,T) = f(\theta + \eta,E,\omega^2 - \kappa^2), \tag{4.10}$$

in which the function f is implicitly known but the functions $\eta(X,T)$, $E(X,T)$, $\omega(X,T)$ and $\kappa(X,T)$ are as yet undetermined.

In §2, the equations governing the large-scale behavior of E, κ, and ω arose by an averaging technique; the same equations will now arise in the expansion method as conditions necessary for the uniform validity of the expansion. The periodicity of U with respect to θ means that

$$U(\theta + nP,X,T) = U(\theta,X,T), \tag{4.11}$$

where P is the period and n is any integer. Partial differentiation of (4.11) with respect to X and θ, and combination of these two results shows that

$$U_X(\theta + nP,X,T) = U_X(\theta,X,T) - nP_X U_\theta(\theta,X,T). \tag{4.12}$$

Unless P_X is zero, $U_X(\theta + nP,X,T)$ will be unbounded for large

n, and hence for large θ, and the uniform validity of the assumed expansion (4.3) will be destroyed. Thus the period P must not be allowed to depend on X or T, and may be normalized to unity. From (2.6) this condition implies the relation

$$(\omega^2 - \kappa^2)^{\frac{1}{2}} \oint \left\{2(E - V(U))\right\}^{-\frac{1}{2}} dU = 1 \qquad (4.13)$$

between ω, κ, and E. Requirement (4.13) is simply the dispersion relation given before in (2.9).

The next requirement arises from (4.8); however, the full solution of (4.8) is not needed until later, when higher order results will be considered. The information needed for the first order result may be obtained quickly by deriving one condition that is necessary to avoid unbounded secular terms. The key to this is to note from (4.7) that U_θ is a solution of the homogeneous equation for U_1, that is,

$$(\omega^2 - \kappa^2)\, U_{\theta\theta\theta} + V''(U)\, U_\theta = 0. \qquad (4.14)$$

Then a substitution $U_1 = wU_\theta$ in (4.8) leads to a first order equation for w_θ:

$$(\omega^2 - \kappa^2)(U_\theta w_{\theta\theta} + 2U_{\theta\theta} w_\theta) = F_1,$$

or, better,

$$(\omega^2 - \kappa^2)\, \frac{\partial}{\partial\theta}\, (U_\theta^2 w_\theta) = U_\theta F_1.$$

On replacing w by U_1/U_θ, this shows that (4.7) and (4.8) imply

$$(\omega^2 - \kappa^2)\, \frac{\partial}{\partial\theta}\, (U_{1\theta} U_\theta - U_1 U_{\theta\theta}) = U_\theta F_1. \qquad (4.15)$$

Now, it is clear from (4.15) that U_1 or $U_{1\theta}$ will be unbounded

unless the integral

$$\int_0^{\theta} U_\theta F_1 \, d\theta \tag{4.16}$$

is bounded for large θ. Since U is periodic in θ, it follows from (4.9) that $U_\theta F_1$ is periodic in θ, so that (4.16) is bounded only if

$$\int_0^{1} U_\theta F_1 \, d\theta = 0. \tag{4.17}$$

This condition, involving the integral over one period, is thus necessary so that secular terms proportional to θ may be avoided.

Unless condition (4.17) holds, the unboundedness of U_1 or $U_{1\theta}$ will destroy the uniform validity of the assumed expansion (4.3); however, the functions ω, κ, and E satisfy only two relations (4.5) and (4.13) so far, and (4.17) can be imposed. This is the crucial step in the analysis. With the expression for F_1 substituted from (4.9), condition (4.17) becomes

$$\frac{\partial}{\partial T}\left\{\omega \int_0^{1} U_\theta{}^2 d\theta\right\} + \frac{\partial}{\partial X}\left\{\kappa \int_0^{1} U_\theta{}^2 \, d\theta\right\} = 0. \tag{4.18}$$

Requirement (4.18) is equivalent to the equation (2.10) given previously, as follows by a change of the variable of integration from θ to U and by the use of (2.5). Thus, when applied to the simple partial differential equation (4.1), the

expansion method agrees with Whitham's averaged Lagrangian method and yields the set of coupled equations (2.9) - (2.11).

When using the direct expansion method presented here, one should beware of a possible source of confusion that is inherent in the approach: namely, to assure that the solution is correct to a certain order of precision, it is not enough to satisfy the original equation to the same precision. It turns out that it would be sufficient to satisfy the original equation to a precision two orders higher, although this is by no means necessary. The full pattern is explained at the end of §5, but, as an example, let us consider now the present status of the procedure if $\kappa(X,T)$, $\omega(X,T)$, and $E(X,T)$ are regarded as determined by (2.9) - (2.11). Equation (4.7) has already been satisfied. So that the original equation (4.1) may be satisfied except for terms of order ϵ^2, we should also determine a function U_1 so as to satisfy (4.8). Here caution is required, for at this stage in the procedure, the determination of U_1 is not unique; two undetermined functions of X and T arise in the integration of (4.8). If the objective is merely to satisfy the original equation to the stated precision, these undetermined functions may be set to zero (or arbitrarily specified), but it must be remembered that this does not necessarily give the correct expression for U_1. To

assure this would require the use (although not the full solution) of the next two higher order equations following (4.7) and (4.8). Actually, at the present stage in the procedure, even the lowest order approximation $U(\theta,X,T)$ is not fully known, for the function $\eta(X,T)$ that appears in (4.10) is not yet determined.

Knowledge of the family of uniform wavetrains provides enough information so that it is easy to integrate (4.8) in closed form; in fact, the solution, U_1, may be obtained in several alternate forms. One of these is obtained by performing an indefinite integral on (4.15) and then solving the resulting first order, linear differential equation in the usual way. In this form only one solution, U_θ, of the homogeneous equation is used. This solution, U_θ, is related to the dependence of the family of wavetrains on the parameter η. Two solutions of the homogeneous equation are available, however, and if both of them are used, U_1 assumes a second, more convenient form. The higher approximations will be derived in this way in §5. The second solution of the homogeneous equation is related to the parameter E; because of this, the alternate form for U_1 seems to use more systematically the information provided by the family of uniform wavetrains.

The first form for U_1, which was used by Kuzmak [13] in the treatment of ordinary differential equations, would be awkward for higher approximations because the factor U_θ^{-2} causes difficulties of an artificial nature at the zeros of U_θ.

For the slightly dissipative ordinary differential equation (3.6), Kuzmak obtained the result

$$\frac{d}{dT}\left\{\omega \int_0^1 U_\theta^2 \, d\theta\right\} + \omega \int_0^1 g U_\theta^2 \, d\theta = 0, \tag{4.19}$$

which is analogous to (4.18) and follows by similar steps. If $g(T,U) = g(T)$, this may be integrated to

$$\omega \int_0^1 U_\theta^2 \, d\theta = (\text{constant}) \exp\left\{ - \int g \, dT\right\}, \tag{4.20}$$

which shows how the "adiabatic invariant" on the left side changes due to the effect of the damping term.

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13. Ibid.

5. Higher Order Approximations

In more detail, substitution of (4.3) into (4.1) leads to the set of equations

$$\mu U_{\theta\theta} + V'(U) = 0, \tag{5.1}$$

$$\mu U_{n\theta\theta} + V''(U)\, U_n = F_n(\theta,X,T), \text{ for } n=1,2,\ldots, \tag{5.2}$$

where

$$\mu = \omega^2 - \kappa^2, \tag{5.3}$$

$$F_1 = 2\omega U_{\theta T} + 2\kappa U_{\theta X} + \omega_T U_\theta + \kappa_X U_\theta, \tag{5.4}$$

and

$$F_2 = 2\omega U_{1\theta T} + 2\kappa U_{1\theta X} + \omega_T U_{1\theta} + \kappa_X U_{1\theta} + U_{XX} - U_{TT} - \tfrac{1}{2} V'''(U)\, U_1^2. \tag{5.5}$$

For $n = 2,3,\ldots,$ $F_{(n+1)}$ is of the form

$$F_{(n+1)} = 2\omega U_{n\theta T} + 2\kappa U_{n\theta X} + \omega_T U_{n\theta} + \kappa_X U_{n\theta} + U_{(n-1)XX} - U_{(n-1)TT} - U_1 U_n V'''(U) + \tilde{F}_{(n+1)}(\theta,X,T), \tag{5.6}$$

where $\tilde{F}_{(n+1)}$ is a polynomial in $U_1, U_2, \ldots, U_{(n-1)}$, and the derivatives of the function $V(U)$.

The function $f(\psi,E,\mu)$ was defined in (2.7) so that

$$U(\theta,X,T) = f(\theta + \eta,E,\mu)$$

satisfies (5.1) for all θ, and for general E and μ. For concreteness, let the lower limit of the integral in (2.6) be chosen as a zero of $E - V(U)$, so that from (2.5) $f_\psi(0,E,\mu) = 0$, and $f(-\psi,E,\mu) = f(\psi,E,\mu)$.

The full solution of (5.2) follows from the observation that $f_\psi(\theta + \eta, E, \mu)$ and $f_E(\theta + \eta, E, \mu)$ are both solutions of the corresponding homogeneous equation, since differentiation of (5.1) shows that

$$\mu \frac{\partial^2 f_\psi}{\partial \theta^2} + V''(U) f_\psi = 0, \tag{5.7}$$

$$\mu \frac{\partial^2 f_E}{\partial \theta^2} + V''(U) f_E = 0. \tag{5.8}$$

(Here the differentiation with respect to E is with E and μ independent, not related by (4.13).) To obtain the solution by variation of parameters, it would be sufficient to employ f_ψ to reduce the order of (5.2) as indicated in §4; however, the full solution assumes a more convenient form if both f_ψ and f_E are used in a trial solution of the form

$$U_n = \alpha f_\psi + \beta f_E. \tag{5.9}$$

Here, both parameters α and β are allowed to depend on ψ, but the relation

$$\alpha_\psi f_\psi + \beta_\psi f_E = 0 \tag{5.10}$$

is imposed between them to make the expression for $U_{n\theta}$ simplify to

$$U_{n\theta} = \alpha f_{\psi\psi} + \beta f_{E\psi}. \tag{5.11}$$

Then calculation of $U_{n\theta\theta}$ and substitution in (5.2) shows that, in addition to (5.10), α and β need only satisfy

$$\mu\alpha_\psi f_{\psi\psi} + \mu\beta_\psi f_{E\psi} = F_n. \tag{5.12}$$

After solution of (5.10) and (5.12) for α and β, substitution into (5.9) gives the general solution of (5.2) in the form

$$U_n(\theta,X,T) = U_n(\psi - \eta,X,T) =$$

$$\frac{f_\psi}{\mu W}\int_0^\psi F_n f_E \, d\psi' + f_\psi \eta_n - \frac{f_E}{\mu W}\int_0^\psi F_n f_\psi \, d\psi' + f_E E_n, \tag{5.13}$$

where the integration constants $\eta_n(X,T)$ and $E_n(X,T)$ are not yet determined, and where the Wronskian

$$W = f_E f_{\psi\psi} - f_{E\psi} f_\psi \tag{5.14}$$

is independent of ψ, as follows by calculation of W_ψ with the aid of (5.7) and (5.8).

The integrals in (5.13) obviously lead to secular terms unless conditions are imposed; the first of these conditions,

$$\int_0^1 F_1 f_\psi \, d\psi = 0,$$

was obtained in §4. In the linear case, where $V(u) = \frac{1}{2}u^2$ and

$$f = (2E)^{\frac{1}{2}} \cos(\psi\mu^{-\frac{1}{2}}),$$

the boundedness conditions are simply the orthogonality conditions

$$\int_0^1 F_n \cos 2\pi\psi \, d\psi = 0, \qquad \int_0^1 F_n \sin 2\pi\psi \, d\psi = 0.$$

However, in the nonlinear case, it is not sufficient to require the two integrals in (5.13) to be zero over one period, since it turns out that f_E itself is no longer bounded. For, partial differentiation of

$$f(\psi + P(E,\mu),E,\mu) = f(\psi,E,\mu) \tag{5.15}$$

with respect to E and ψ implies that

$$f_E(\psi + P,E,\mu) = f_E(\psi,E,\mu) - f_\psi(\psi,E,\mu)\, P_E. \tag{5.16}$$

(As noted after equation (5.8), f_E must be calculated before the normalization condition $P = 1$, relating E and μ by (4.13) is applied.) Then (5.16) shows that the function g defined by

$$g(\psi,E,\mu) = f_E(\psi,E,\mu) + \psi\, P_E\, P^{-1}\, f_\psi(\psi,E,\mu) \tag{5.17}$$

is bounded and periodic, and the expression of f_E in the form

$$f_E = g - \psi\, P_E\, P^{-1} f_\psi \tag{5.18}$$

shows explicitly the secular term proportional to ψ. In the linear case, of course, P is independent of E and the secular term in f_E does not arise. After substitution of (5.18) into (5.13) to eliminate f_E, and after integration by parts to remove ψ from within the integral, (5.13) assumes the form

$$U_n = f_\psi \left\{ \eta_n + \int_0^\psi \left(\frac{F_n g}{\mu W} - \frac{P_E E_n}{P} + \frac{P_E}{P\mu W} \int_0^{\psi'} F_n f_\psi \, d\psi'' \right) d\psi' \right\}$$

$$+ g \left\{ E_n - \frac{1}{\mu W} \int_0^\psi F_n f_\psi \, d\psi' \right\}. \qquad (5.19)$$

The conditions for avoidance of secular terms follow from (5.19). The coefficients of f_ψ and g may be treated separately since the zeros of these functions do not coincide. If F_n has the same periodicity as f, the integrand $F_n f_\psi$ is also periodic, and so the integral over one period is required to be zero:

$$\int_0^1 F_n f_\psi \, d\psi = 0. \qquad (5.20)$$

If this condition on the coefficient of g is satisfied, the requirement

$$\int_0^1 \left(\frac{F_n g}{\mu W} - \frac{P_E E_n}{P} + \frac{P_E}{P\mu W} \int_0^{\psi'} F_n f_\psi \, d\psi'' \right) d\psi' = 0 \qquad (5.21)$$

then follows from the periodicity of the integrand of this integral. If the above conditions hold, U_n is bounded and periodic in ψ.

The recursive nature of the expansion procedure now becomes clear; two undetermined functions, E_n and η_n, arise at each

stage, but the two conditions (5.20) and (5.21) for the avoidance of secular terms provide equations to relate functions previously left undetermined. In the first order result obtained in §4, the dispersion relation (4.13) and condition (5.20) for $n=1$ provide equations to relate E, κ, and ω. (To make the pattern more evident, the phase function Θ might instead have been labeled η_{-1}.) In the second approximation, condition (5.21) for $n=1$ and condition (5.20) for $n=2$ relate E_1 and η. Similarly, condition (5.21) for $n=N$ and condition (5.20) for $n=N+1$ provide a set of equations for E_N and η_{N-1}, although it is not immediately clear that (5.20) does not also involve η_N.

For approximations higher than the first, the attempt to obtain explicit coupled pairs of equations by substitution in (5.20) and (5.21) leads to tedious algebra, which will be indicated only briefly. To show that, for $n=N+1$, (5.20) indeed does not involve η_N, one may substitute in (5.20)

$$U_N(\theta,X,T) = U_\theta(\theta,X,T)\,\eta_N + \tilde{U}_N(\theta,X,T),$$

where the function $\tilde{U}_N$ is regarded as known at the corresponding stage of the procedure. Then after appropriate derivatives of (4.7) and (4.8) are used to eliminate $V'''(U)$ and $V''(U)$ from the integrand, all terms involving η_N integrate to periodic

functions, so that the integral of these terms around one cycle vanishes. The further algebra is then straightforward; however, it should be remarked that the results simplify somewhat because $f(\psi,E,\mu)$ is an even function in ψ. This causes several of the integrands to be odd, periodic functions, whose integral over one period vanishes. As a result, the second approximation has the trivial solution $E_1 = 0$, $\eta = 0$.

6. General Variational Equation of Second Order

The expansion method might be applied to several more general classes of problems; §6 deals with a single second order equation of general variational form. The case of two or more variational equations is mentioned at the end of §6 but not otherwise treated. Several other questions arise but have not been examined. For example, use of the Hamiltonian rather than the Lagrangian formalism in a direct expansion method would allow comparison with other results given by Whitham [14]. Also, a proof that the perturbation procedure does asymptotically approximate an actual solution of the original equation would be of interest both for ordinary and partial differential equations. Finally, Kuzmak [15] allowed a small frictional damping term in the case of an ordinary differential equation, and this could presumably be done also for partial differential equations.

For a function $u(\underset{\sim}{x}) = u(x_1, x_2, \ldots, x_n)$ consider the variational problem defined by

$$\delta \int\int \cdots \int L(q, \underset{\sim}{r}, \lambda)\, dx_1 dx_2 \ldots dx_n = 0, \tag{6.1}$$

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14. Loc. cit.

15. Loc. cit.

where L is a given function, λ is a possible parameter (or set of parameters), and the arguments q and $\underset{\sim}{r}$ are here given by

$$q = u(\underset{\sim}{x}), \qquad r_i = u_{x_i}(\underset{\sim}{x}). \tag{6.2}$$

The corresponding Euler equation is

$$L_q - \frac{\partial}{\partial x_i}\left(L_{r_i}\right) = 0. \tag{6.3}$$

As in the special case considered in §§4 and 5, let $X_i = \epsilon x_i$ be "stretched coordinates" and set

$$u(\underset{\sim}{x}) = u(\theta, \underset{\sim}{X}) = U(\theta, \underset{\sim}{X}) + \epsilon U_1(\theta, \underset{\sim}{X}) + \ldots, \tag{6.4}$$

where

$$\theta_{x_i} = \kappa_i(\underset{\sim}{X}), \tag{6.5}$$

and the functions κ_i satisfy

$$\kappa_{ix_j} - \kappa_{jx_i} = 0. \tag{6.6}$$

No additional complications arise if L exhibits a slow parametric dependence on the independent variables--slow to the same approximation that the quantities κ_i are to be slowly varying. This will be expressed by writing $\lambda = \lambda(\underset{\sim}{X})$. The appearance of such a parameter is of interest, for example, in the problem of waves propagating through a slowly varying medium.

The expansion proceeds much as before. First let $u(\underset{\sim}{x}) = u(\theta, \underset{\sim}{X})$ and, correspondingly, set

$$q = u(\theta,\underset{\sim}{X}), \qquad r_i = u_\theta(\theta,\underset{\sim}{X})\,\kappa_i + \epsilon\, u_{X_i}(\theta,\underset{\sim}{X}) \tag{6.7}$$

in (6.3), which may then be rewritten as

$$L_q - \left(L_{r_i}\right)_\theta \kappa_i - \epsilon\left(L_{r_i}\right)_{X_i} = 0. \tag{6.8}$$

Now set

$$u(\theta,\underset{\sim}{X}) = U(\theta,\underset{\sim}{X}) + \epsilon\, U_1(\theta,\underset{\sim}{X}) + O(\epsilon^2), \tag{6.9}$$

so that

$$\left.\begin{aligned} q &= U(\theta,\underset{\sim}{X}) + \epsilon\, U_1(\theta,\underset{\sim}{X}) + O(\epsilon^2), \\ r_i &= U_\theta(\theta,\underset{\sim}{X})\,\kappa_i + \epsilon(U_{X_i} + U_{1\theta}\kappa_i) + O(\epsilon^2) \end{aligned}\right\} \tag{6.10}$$

in equation (6.8). Taylor series expansion gives the equation

$$\begin{aligned} &L_q + \epsilon L_{qq} U_1 + \epsilon(U_{X_i} + U_{1\theta}\kappa_i)L_{qr_i} \\ &- \left\{L_{r_i} + \epsilon L_{r_i q} U_1 + \epsilon(U_{X_j} + U_{1\theta}\kappa_j)L_{r_i r_j}\right\}_\theta \kappa_i \\ &- \epsilon\left(L_{r_i}\right)_{X_i} = O(\epsilon^2), \end{aligned} \tag{6.11}$$

in which the arguments in $L(q,\underset{\sim}{r},\lambda)$ are now

$$q = U(\theta,\underset{\sim}{X}), \qquad r_i = U_\theta(\theta,\underset{\sim}{X})\,\kappa_i, \qquad \lambda = \lambda(\underset{\sim}{X}). \tag{6.12}$$

On equating the various powers of ϵ to zero in (6.11), the equations

$$L_q - \left(L_{r_i}\right)_\theta \kappa_i = 0 \tag{6.13}$$

and

$$\left(- L_{r_i r_j} U_{1\theta}\right)_\theta \kappa_i \kappa_j + \left\{L_{qq} - \left(L_{r_i q}\right)_\theta \kappa_i \right\} U_1$$

$$+ L_{q r_i} U_{X_i} - \left(L_{r_i r_j} U_{X_j}\right)_\theta \kappa_i - \left(L_{r_i}\right)_{X_i} = 0 \tag{6.14}$$

are obtained.

Solution of these equations proceeds as for (4.7) and (4.8). The energy integral is

$$E = \kappa_i L_{r_i} U_\theta - L. \tag{6.15}$$

Let

$$U_\theta = H(U, E, \underset{\sim}{\kappa}, \lambda) \tag{6.16}$$

indicate the inverse function of (6.15). Then the inverse function of

$$\theta = \int \frac{dU}{H} - \eta(\underset{\sim}{X}) \tag{6.17}$$

gives the desired solution of (6.13), which may be indicated as

$$U(\theta, \underset{\sim}{X}) = f(\theta + \eta(\underset{\sim}{X}), E(\underset{\sim}{X}), \underset{\sim}{\kappa}(\underset{\sim}{X}), \lambda(\underset{\sim}{X})). \tag{6.18}$$

The dispersion relation, corresponding to (4.13), is

$$P(E, \underset{\sim}{\kappa}, \lambda) - 1 \equiv \oint \frac{dU}{H} - 1 = 0. \tag{6.19}$$

Multiplication of (6.14) by U_θ and integration around one cycle then gives the condition

$$\int_0^1 \left\{ L_{qr_i} U_{X_i} - \left(L_{r_i r_j} U_{X_j} \right)_\theta \kappa_i - \left(L_{r_i} \right)_{X_i} \right\} U_\theta \, d\theta = 0, \quad (6.20)$$

which corresponds to (4.16).

Comparison with the averaged Lagrangian approach [16] reveals that (6.20) can be placed in a much more elegant form, which is by no means evident from the above expansion method. The middle term of (6.20), after integration by parts, combines with the first term, so that (6.20) assumes the form

$$\int_0^1 \left\{ \left(L_{r_i} \right)_\theta U_{X_i} - \left(L_{r_i} \right)_{X_i} U_\theta \right\} d\theta = 0. \quad (6.21)$$

Then the first term of (6.21), after integration by parts, combines with the second term, giving

$$- \frac{\partial}{\partial X_i} \left\{ \int_0^1 L_{r_i} U_\theta \, d\theta \right\} = 0, \quad (6.22)$$

in which the arguments of $L(q, \underset{\sim}{r}, \lambda)$ are still given by (6.12). After a change of the variable of integration, (6.22) assumes the form

$$- \frac{\partial}{\partial X_i} \left\{ \oint L_{r_i} \, dU \right\} = 0, \quad (6.23)$$

in which $L(q, \underset{\sim}{r}, \lambda)$ has the arguments

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16. Whitham, loc. cit.

$$q = U, \qquad r_i = \kappa_i\, H(U,E,\underset{\sim}{\kappa},\lambda), \qquad \lambda = \lambda(\underset{\sim}{X}). \tag{6.24}$$

This result agrees with that obtained from the Euler equations of the averaged Lagrangian

$$\mathcal{L}(\underset{\sim}{\kappa},E,\lambda) = \oint L(q,\underset{\sim}{r},\lambda)\, H^{-1} dU + E\left\{\oint H^{-1} dU - 1\right\}, \tag{6.25}$$

where $\kappa_i = \Theta_{X_i}$ and $L(q,\underset{\sim}{r},\lambda)$ has the arguments (6.24). Variation of E gives the Euler equation

$$0 = \mathcal{L}_E \equiv \oint \left(L_{r_j} \kappa_j H_E H^{-1} - L H_E H^{-2}\right) dU$$

$$+ \oint H^{-1} dU - 1 - E \oint H_E H^{-2} dU, \tag{6.26}$$

and variation of Θ gives

$$0 = -\frac{\partial}{\partial X_i}\left(\mathcal{L}_{\kappa_i}\right)$$

$$\equiv -\frac{\partial}{\partial X_i}\left\{\oint\left(L_{r_i} + L_{r_j}\kappa_j H_{\kappa_i} H^{-1} - L H_{\kappa_i} H^{-2}\right)dU - E\oint H_{\kappa_i} H^{-2} dU\right\}. \tag{6.27}$$

Since (6.15) and (6.16) make

$$E = \kappa_j L_{r_j} H - L \tag{6.28}$$

an identity, (6.26) simplifies to (6.19), and (6.27) simplifies to (6.23). The equations (6.6) are implicit in the definition of Θ.

The equations of the higher order approximation have not been derived for the more general case of §6. The symmetry of U about the zeros of U_θ, which made f an even function in the special case of §§4 and 5, does not hold in general, so the corresponding simplifications of the higher order result would not be expected.

The more important examples analyzed by Whitham [17], such as the Boussinesq approximation for waves in shallow water, involve several equations, so the derivation of §6 for a single equation does not apply. In the case of two or more equations, the conditions have not been determined under which the uniform solutions have the desired periodicity, and this appears to be the main difficulty in a thorough examination of the problem. For the Boussinesq case, periodicity results because the velocity potential $\varphi(x,t)$, not being a physical quantity, is absent from the Lagrangian. For such specific examples in which the uniform solutions have the desired periodicity, the application of the perturbation procedure appears to be straightforward.

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17. Ibid.

PART II

THE CLASSICAL WATER WAVE PROBLEM

7. A Variational Principle for a Fluid with a Free Surface

To apply the expansion method to the classical water wave problem, it is crucial to find an appropriate variational principle. It was found in the present investigation that a Lagrangian function equal to the pressure yields the full set of equations of motion for the problem. Several previous authors [18] have applied such a Lagrangian to motion within a fluid, but it seems not to have been explicitly indicated that the free surface boundary conditions follow similarly. The formulation below is also related to that given by Friedrichs [19] and Garabedian and Spencer [20], who used a variational principle to obtain the pressure condition at the free surface in steady flows.

For the case of irrotational motion, let $\varphi(x,y,t)$ be the velocity potential of a fluid lying between $y = 0$ and

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18. A. Clebsch, J. reine u. angew. Math., vol. 56 (1859), pp. 1-10; R. Hargreaves, Phil. Mag. (6), vol. 16 (1908), pp. 436-444; H. Bateman, Partial Differential Equations. Cambridge: Cambridge University Press (1964).

19. K. Friedrichs, Math. Ann., vol. 109 (1933), pp. 60-82.

20. P. R. Garabedian and D. C. Spencer, J. Ratl. Mech. Anal., vol. 1 (1952), pp. 359-409.

$y = h(x,t)$, with gravity acting in the negative y direction. Then the variational principle is

$$\delta J = \delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{0}^{h(x,t)} L \, dy \, dx \, dt = \tag{7.1}$$

$$= \delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{0}^{h(x,t)} \left(\tfrac{1}{2} \varphi_x^2 + \tfrac{1}{2} \varphi_y^2 + \varphi_t + gy\right) dy \, dx \, dt = 0,$$

where $\varphi(x,y,t)$ and $h(x,t)$ are allowed to vary subject to the restrictions $\delta\varphi = 0$, $\delta h = 0$ at x_1, x_2, t_1, and t_2. The only change from earlier formulations using an expression for the pressure is that $h(x,t)$ variations are allowed here.

According to the usual procedure in the calculus of variations, (7.1) becomes

$$\delta J = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \Big\{ \left[\tfrac{1}{2} \varphi_x^2 + \tfrac{1}{2} \varphi_y^2 + \varphi_t + gy\right]_{y=h} \delta h \tag{7.2}$$

$$+ \int_{0}^{h(x,t)} (\varphi_x \delta\varphi_x + \varphi_y \delta\varphi_y + \delta\varphi_t) dy \Big\} \, dx \, dt = 0.$$

Certain natural boundary conditions arise at $y = h$ and $y = 0$ if the integrated terms are carefully retained when (7.2) is integrated by parts. Thus

$$\delta J = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\Big\{\Big[\tfrac{1}{2}\varphi_x^{\,2} + \tfrac{1}{2}\varphi_y^{\,2} + \varphi_t + gy\Big]_{y=h}\delta h$$

$$+ \big[(-h_x\varphi_x + \varphi_y - h_t)\delta\varphi\big]_{y=h} - \big[\varphi_y\delta\varphi\big]_{y=0} \tag{7.3}$$

$$- \int_0^h(\varphi_{xx} + \varphi_{yy})\delta\varphi\ dy\ \Big\}\ dx\ dt = 0.$$

Setting the coefficients of δh and $\delta\varphi$ to zero, we have

$$\tfrac{1}{2}\varphi_x^{\,2} + \tfrac{1}{2}\varphi_y^{\,2} + \varphi_t + gy = 0 \qquad \text{for } y = h, \tag{7.4}$$

$$-h_x\varphi_x + \varphi_y - h_t = 0 \qquad \text{for } y = h, \tag{7.5}$$

$$-\varphi_{xx} - \varphi_{yy} = 0 \qquad \text{for } 0 < y < h, \tag{7.6}$$

$$-\varphi_y = 0 \qquad \text{for } y = 0. \tag{7.7}$$

These are the equations for the classical water wave problem.

No satisfactory solution seems known for the general problem of finding suitable Lagrangian functions. For the water wave problem, in particular, the pressure function used in (7.1) is more productive than the traditional form of the Lagrangian, $\tilde{L}$, equal to kinetic minus potential energy. It is clear that

$$0 = \delta\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)}\tilde{L}\ dy\ dx\ dt =$$

$$= \delta\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)}\left(\tfrac{1}{2}\varphi_x^{\,2} + \tfrac{1}{2}\varphi_y^{\,2} - gy\right)dy\ dx\ dt \tag{7.8}$$

must give the correct equation within the fluid, for the integrals in (7.8) and (7.1) differ by the expression

$$-\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_{0}^{h(x,t)}(2gy + \varphi_t)dy\ dx\ dt, \tag{7.9}$$

which integrates, leaving only boundary terms. However, (7.9) contributes boundary terms at $y = h$, so that (7.8), as it stands, does not give the correct surface conditions.

To see the difference in the boundary conditions, it is necessary instead to relate $\tilde{L}$ to the negative of L. From (7.1) and (7.8), the integral of $L + \tilde{L}$ is

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_{0}^{h(x,t)}(\varphi_x{}^2 + \varphi_y{}^2 + \varphi_t)dy\ dx\ dt, \tag{7.10}$$

which, after integration by parts, becomes

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\Big\{[\varphi(-h_x\varphi_x + \varphi_y - h_t)]_{y=h} - [\varphi\varphi_y]_{y=0}$$

$$-\int_0^h \varphi\,(\varphi_{xx} + \varphi_{yy})dy + \frac{\partial}{\partial x}\int_0^h \varphi\,\varphi_x\,dy + \frac{\partial}{\partial t}\int_0^h \varphi\,dy\Big\}\,dx\,dt. \tag{7.11}$$

The key to the difference then appears to be conservation of mass. If conservation of mass is introduced by varying φ and h only among those functions that satisfy (7.5) - (7.7), the

difference expression (7.11) vanishes except for the last two terms, and the last two terms are of no consequence since they contribute only at the x and t boundaries. In this way $\tilde{L}$ is made equivalent to L and yields (7.4), but only at the expense of assuming the other three equations of motion (7.5) - (7.7) at the outset.

Only the irrotational case has been treated above. For the rotational case, Clebsch [21] expressed the velocity as $\underset{\sim}{u} = \nabla\varphi + \alpha\nabla\beta$. Then the variational principle, in a form similar to that given by Bateman [22] is

$$\delta\int\int\int\int \rho(\varphi_t + \alpha\beta_t + \tfrac{1}{2}\underset{\sim}{u}^2 + gy)dy\,dx\,dz\,dt = 0. \qquad (7.12)$$

Bateman further generalized the variational principle to barotropic flow, that is, to flows in which the pressure is a function of the density alone. To extend his results to free surfaces it is again merely necessary to include the surface elevation among the quantities to be varied.

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21. Loc. cit.; or see J. Serrin, "Mathematical principles of classical fluid mechanics," Encyclopedia of Physics (ed. S. Flügge), vol. VIII/1, p. 125. Berlin: Springer-Verlag (1959).

22. Op. cit., p. 164; H. Bateman, Proc. Roy. Soc. A, vol. 125 (1929), pp. 598-618.

8. Application of the Expansion Method to Water Waves

The theory of linear dispersive waves arose historically from the water wave problem, and the nonlinear development is of corresponding interest. From a mathematical standpoint, the distinctive feature of this problem is that the velocity potential $\varphi(x,t,y)$ depends on the vertical coordinate, y. In the Boussinesq approximation, φ is nearly constant with y, so that only $\varphi(x,t)$ need be considered. By means of the averaged Lagrangian technique, Whitham [23] derived averaged equations for the infinitesimal (Stokes wave) approximation. To explore the generality of the perturbation method, we shall treat the full nonlinear problem below; however, this yields no additional results of practical interest since the functions that characterize the uniform wavetrains are known only approximately. The infinitesimal approximation is used eventually, in §10, to obtain a comparison with Whitham's results.

An initial attempt to carry out the perturbation expansion directly from the Euler equations of the variational principle (7.1) was unproductive; the simplification analogous to (6.20)-(6.23) was by no means evident. It is much easier to consider first the variational problem of the slightly more general form

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23. G. B. Whitham, (To appear).

$$\delta \int \cdots \iint L(\underset{\sim}{s}, u, q, y)\, dy\, d\underset{\sim}{x} = 0, \tag{8.1}$$

with the arguments

$$s_i = \varphi_{x_i}(\underset{\sim}{x}, y), \qquad u = \varphi_y(\underset{\sim}{x}, y), \qquad q = h(\underset{\sim}{x}), \tag{8.2}$$

where $\varphi(\underset{\sim}{x}, y)$ represents the velocity potential, and $h(\underset{\sim}{x})$ represents the surface elevation. Only the vertical coordinate, y, is singled out; the other space coordinates and the time coordinate are given by the vector $\underset{\sim}{x}$. The plan is eventually to set $x_0 = t$, $x_1 = x$, and, if needed, $x_2 = z$. The problem defined by (8.1), (8.2) will be treated in the remainder of §8. Finally, in §9, the results of §8 will be specialized to the water wave problem.

Let us suppose that L is zero for y outside of the region of interest, so that no boundary terms arise from integration by parts. Then the Euler equations of (8.1) are

$$\left.\begin{aligned} \delta\varphi: &\qquad (-L_{s_i})_{x_i} - (L_u)_y = 0, \\ \delta h: &\qquad \frac{\partial}{\partial q} \int L\, dy = 0. \end{aligned}\right\} \tag{8.3}$$

The Euler equation for δh assumes this slightly unusual form because h does not depend on y; with respect to h the integral $\int L\, dy$ rather than the function L might be thought of as the Lagrangian function.

In the uniform, periodic solutions, the velocity potential φ is generally not periodic; because only the derivatives of φ are physical quantities, an additional term $\beta_i x_i$, linear in x_i, is appropriate in the expression for φ. For slowly varying wavetrains it is then natural to introduce a phase-like variable ψ, just as the phase variable θ was introduced in (4.2) and (6.4) as a generalization of the linear phase term $\kappa_i x_i$. The expansion is then attempted in the form

$$\left.\begin{aligned} \varphi &= \psi + \Phi(\theta,\underset{\sim}{X},y) + \epsilon\, \Phi_1(\theta,\underset{\sim}{X},y) + \cdots, \\ h &= H(\theta,\underset{\sim}{X}) + \epsilon\, H_1(\theta,\underset{\sim}{X}) + \cdots, \end{aligned}\right\} \tag{8.4}$$

where

$$X_i = \epsilon\, x_i, \qquad \theta_{x_i} = \kappa_i(\underset{\sim}{X}), \qquad \psi_{x_i} = \beta_i(\underset{\sim}{X}).$$

The same procedure used previously in §6 gives the lowest order equations

$$\left.\begin{aligned} &\varphi: && (-L_{s_i})_\theta \kappa_i - (L_u)_y = 0, \\ &h: && \int L_q \, dy = 0 \end{aligned}\right\} \tag{8.5}$$

and the first order equations

φ:

$$(-L_{s_is_j}\Phi_{1\theta}\kappa_j)_\theta\kappa_i - (L_{s_iu}\Phi_{1y})_\theta\kappa_i - (L_{us_i}\Phi_{1\theta}\kappa_i)_y - (L_{uu}\Phi_{1y})_y$$
$$-(L_{s_iq}H_1)_\theta\kappa_i - (L_{uq}H_1)_y$$
$$= F, \tag{8.6}$$

h:

$$\int(L_{qs_i}\Phi_{1\theta}\kappa_i + L_{qu}\Phi_{1y} + L_{qq}H_1)\,dy = G \tag{8.7}$$

in which $L(\underset{\sim}{s},u,q,y)$ has the arguments

$$s_i = \beta_i + \Phi_\theta\kappa_i, \qquad u = \Phi_y, \qquad q = H, \tag{8.8}$$

and where

$$\left.\begin{aligned} F &= (L_{s_is_j}\Phi_{X_j})_\theta\kappa_i + (L_{us_i}\Phi_{X_i})_y + (L_{s_i})_{X_i}\,, \\ G &= \int(-L_{qs_i}\Phi_{X_i})dy\,. \end{aligned}\right\} \tag{8.9}$$

The first order equations are of the form

$$\left.\begin{aligned} Op_\varphi[\Phi_1,H_1] &= F(\theta,\underset{\sim}{X},y), \\ Op_h[\Phi_1,H_1] &= G(\theta,\underset{\sim}{X}), \end{aligned}\right\} \tag{8.10}$$

where Op_φ and Op_h represent the two components of a linear operator on $\Phi_1(\theta,\underset{\sim}{X},y)$ and $H_1(\theta,\underset{\sim}{X})$. By integration by parts

it may be verified that this linear operator is self-adjoint, that is, that it satisfies the identity

$$\iint A\ Op_{\varphi}[\tilde{A},\tilde{B}]\ dy\ d\theta + \int B\ Op_{h}[\tilde{A},\tilde{B}]\ d\theta$$

$$= \iint \tilde{A}\ Op_{\varphi}[A,B]\ dy\ d\theta + \int \tilde{B}\ Op_{h}[A,B]\ d\theta \qquad (8.11)$$

for general functions $A(\theta,y)$, $B(\theta)$, $\tilde{A}(\theta,y)$, $\tilde{B}(\theta)$.

It may be remarked, as a digression, that the self-adjointness of the linear operator (8.6),(8.7) is closely related to the variational nature of the problem. Specifically, Op_{φ} and Op_{h} may be obtained by linearization of the lowest order equations (8.5) about Φ and H, and the lowest order equations, in turn, may be derived directly as the Euler equations with respect to Φ and H of the variational principle

$$\left.\begin{aligned} &\delta\iint L(\underline{s},u,q,y)\ dy\ d\theta = 0, \\ s_i = \beta_i + \Phi_{\theta}\kappa_i, \qquad &u = \Phi_y, \qquad q = H. \end{aligned}\right\} \qquad (8.12)$$

Operators obtained by linearization of the Euler equations of a variational principle are, at least for very general classes of problems, necessarily self-adjoint. This is precisely analogous to the fact that if a vector N_{α} is the gradient of a potential function $J(x_1,\ldots,x_n)$, then it satisfies the condition

$$\frac{\partial N_{\alpha}}{\partial x_{\beta}} = \frac{\partial N_{\beta}}{\partial x_{\alpha}}, \qquad \text{for } \alpha,\beta = 1,2,\ldots,n. \qquad (8.13)$$

An alternate statement of (8.13) is that the linearization of N_α about x_β, that is, the linear operator

$$Op_\alpha[A_1,\ldots,A_n] = \frac{\partial N_\alpha}{\partial x_\beta} A_\beta, \tag{8.14}$$

satisfies the identity

$$\tilde{A}_\alpha \, Op_\alpha[A_1,\ldots,A_n] = A_\alpha \, Op_\alpha[\tilde{A}_1,\ldots,\tilde{A}_n]. \tag{8.15}$$

Conversely, if for a given function N_α condition (8.15) holds for linearizations about general $x_1,\ldots,x_n$, then a potential function $J(x_1,\ldots,x_n)$ may be found. Kerner [24] has shown that the analogy extends to Hilbert space. One might ask by analogy whether the self-adjointness of all linearizations of a nonlinear differential operator N is then also a sufficient condition so that a Lagrangian may be found whose Euler equations are $N = 0$. At least for some simple cases this statement holds and the Lagrangian may then be written in terms of integrals. Unfortunately, this procedure seems to be of little use in the practical problem of finding a Lagrangian that yields a given set of equations, because seemingly trivial manipulations of the system of equations can change the operator under consideration and destroy the self-adjointness property.

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24. M. Kerner, Ann. of Math. (2), vol. 34(1933), pp. 546-572.

In the previous applications of the expansion method in §§4 and 6, the family of solutions for uniform wavetrains could be indicated at least implicitly, but the classical water wave problem is essentially more complicated. To obtain secular conditions for the first approximation, however, we do not need to consider the uniform solutions in detail. It is sufficient to note that additive constants in θ and Φ do not affect the lowest order equations (8.5). Thus the family of uniform solutions must depend on two trivial parameters corresponding to these constants. It follows that two solutions of the homogeneous equations

$$\left.\begin{aligned} {}_0p_{\varphi}[\Phi_1, H_1] &= 0, \\ {}_0p_{h}[\Phi_1, H_1] &= 0, \end{aligned}\right\} \qquad (8.16)$$

corresponding to small changes in the two trivial parameters, are

and

$$\left.\begin{aligned} \Phi_1 &= \Phi_\theta, \qquad H_1 = H_\theta, \\ \Phi_1 &= 1, \qquad H_1 = 0. \end{aligned}\right\} \qquad (8.17)$$

Necessary conditions for elimination of secular terms follow through substitution of (8.10) and (8.17) in the self-adjointness relation (8.11). These conditions are

$$\left.\begin{aligned} \iint \Phi_\theta F \, dy \, d\theta + \int H_\theta G \, d\theta &= 0, \\ \iint F \, dy \, d\theta &= 0. \end{aligned}\right\} \qquad (8.18)$$

After simplification by means of the same steps as in equations (6.20) - (6.23), these conditions become

$$\left.\begin{aligned} &\frac{\partial}{\partial X_i}\int_0^P\int L_{s_i}\Phi_\theta \, dy \, d\theta = 0, \\ &\frac{\partial}{\partial X_i}\int_0^P\int L_{s_i} \, dy \, d\theta = 0, \end{aligned}\right\} \qquad (8.19)$$

in which L has the arguments (8.8). This result will be specialized to the water wave problem in §9.

Although the lowest order secular conditions have thus been obtained by clear analogy with previous examples, it is by no means clear whether higher approximations might be solved. In §5, sufficiently many solutions of the homogeneous equation were available so that higher approximations could be constructed directly. This appears not to be true for the classical water wave problem. It is not known whether this is an essential breakdown of the method or merely an inconvenience. If the expansion procedure were to break down at the next stage, the meaningfulness of the first order results might also be in question since at the first stage of the expansion procedure the original equations are not yet satisfied even to the first approximation.

9. Specialization to the Water Wave Equations

The Lagrangian for the classical water wave problem, as obtained in §7, is integrated only over the region $0 < y < h$ in the variational principle (7.1). Clearly, the derivation of §8 would have been much more tedious if such a variable region of integration had been explicitly included in (8.1). For this reason, the variable boundary was considered there to be incorporated directly into the Lagrangian, L. This may be done with the aid of the Heaviside "step function," S(x), defined by

$$\left.\begin{aligned} S(x) &= 0 \qquad \text{for } x < 0, \\ S(x) &= 1 \qquad \text{for } x \geq 0 \end{aligned}\right\} \tag{9.1}$$

and its "derivative" the Dirac delta function, which will be denoted here by $S'(x)$. Then the variational principle (7.1) may be re-expressed as

$$\delta \iiint (\tfrac{1}{2} \varphi_x^{\,2} + \tfrac{1}{2} \varphi_y^{\,2} + \varphi_t + gy)\, S(h - y)\, S(y)\, dy\, dx\, dt = 0. \tag{9.2}$$

Thus, to use the results of §8, let us set $x_0 = t$, $x_1 = x$, so that, by comparison with (8.2),

$$L(\underline{s}, u, q, y) = (\tfrac{1}{2} s_1^{\,2} + \tfrac{1}{2} u^2 + s_0 + gy)\, S(q - y)\, S(y). \tag{9.3}$$

The Euler equations (8.3) may then be rewritten as

$\delta\varphi$:

$$(-\varphi_x\sigma(y))_x - (\varphi_y\sigma(y))_y - (\sigma(y))_t = 0, \tag{9.4}$$

δh:

$$\int(\tfrac{1}{2}\varphi_x^2 + \tfrac{1}{2}\varphi_y^2 + \varphi_t + gy)\, S'(h - y)\, dy = 0, \tag{9.5}$$

where

$$\sigma(y) = S(h - y)\, S(y).$$

The pressure condition (9.5) is clearly the same as (7.4). By performing the differentiations in (9.4) and equating the terms involving $S'(h - y)$, $S'(y)$, and $\sigma(y)$ to zero, one may obtain the equations for mass conservation in the more usual form given by (7.5) - (7.7). The equivalence has not been rigorously checked, but, presumably, the use of singularity functions in this manner may be regarded as only an alternative notation in which integration by parts is performed under the guise of differentiation.

The main result of §8, the two conditions (8.19) for the avoidance of secular terms, may now be specialized for the water wave problem. Using the arguments (8.8) in (9.3), and noting that $\kappa_0 = -\omega$, $\kappa_1 = \kappa$, $\beta_0 = -\gamma$, $\beta_1 = \beta$, etc., we obtain the two conditions

$$\left.\begin{aligned} &\frac{\partial}{\partial T}\int_0^P\int_0^H \Phi_\theta\, dy\, d\theta + \frac{\partial}{\partial X}\int_0^P\int_0^H (\beta + \Phi_\theta\kappa)\Phi_\theta\, dy\, d\theta = 0, \\ &\frac{\partial}{\partial T}\int_0^P\int_0^H dy\, d\theta + \frac{\partial}{\partial X}\int_0^P\int_0^H (\beta + \Phi_\theta\kappa)\, dy\, d\theta = 0. \end{aligned}\right\} \tag{9.6}$$

The four-parameter family of uniform wavetrains may be described with the aid of the two triads of parameters

$$(\kappa,\omega,a) \qquad \text{and} \qquad (\beta,\gamma,h_o), \tag{9.7}$$

where a is related to the amplitude of the waves and h_o is the mean height. The requirements that Φ be periodic and that θ change by a fixed amount over one period give two functional relations among the six parameters; let us suppose that these are solved to give

$$\left.\begin{aligned} \omega &= \omega(\kappa,a;\beta,h_o), \\ \gamma &= \gamma(\kappa,a;\beta,h_o). \end{aligned}\right\} \tag{9.8}$$

Then (9.6), (9.8), together with the equations

$$\left.\begin{aligned} \kappa_T + \omega_X &= 0, \\ \beta_T + \gamma_X &= 0, \end{aligned}\right\} \tag{9.9}$$

form a set of six equations for the six parameters (9.7).

10. Approximation for Small Amplitude

The results of §9, for the full nonlinear problem, are not immediately useful, because the functions that characterize the uniform wavetrains are known only approximately. To obtain more practical results, we now use the usual infinitesimal (Stokes wave) approximation for these functions [25]. First consider the special class of those uniform solutions for which $\beta = 0$ and $h_o = 0$; the classical water wave problem (7.4) - (7.7) has the approximate solution, which depends on parameters k and a_1:

$$\left.\begin{aligned} &\varphi(x,t,y) = - Ct + \Phi(\theta,y), \\ &h(x,t) = H(\theta), \\ &\theta = k(x - ct), \\ &\Phi(\theta,y) = A_1 \sin\theta \cosh ky + A_2 \sin 2\theta \cosh 2ky \\ &\qquad + O(a_1^3), \\ &H(\theta) = 1 + a_1 \cos\theta + a_2 \cos 2\theta + O(a_1^3), \end{aligned}\right\} \quad (10.1)$$

where

$$\tilde{T} = \tanh k, \qquad c = g^{\frac{1}{2}}k^{-\frac{1}{2}}\tilde{T}^{\frac{1}{2}} + O(a_1^2), \quad C = g + \tfrac{1}{4}k^2A_1^2 + O(a_1^3),$$

$$A_1 = g^{\frac{1}{2}}a_1(\tilde{T}k)^{-\frac{1}{2}}(\cosh k)^{-1}, \quad A_2 = 3g^{\frac{1}{2}}k^{\frac{1}{2}}\tilde{T}^{-\frac{1}{2}} \cdot \frac{a_1^2(1 - \tilde{T}^2)}{8\tilde{T}^3\cosh^2 k},$$

$$a_2 = \tfrac{1}{4}k a_1^2(3 - \tilde{T}^2)\tilde{T}^{-3}.$$

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25. See, for example, J. V. Wehausen and E. V. Laitone, "Surface waves," Encyclopedia of Physics (ed. S. Flügge), vol. IX, p. 446. Berlin: Springer-Verlag (1960).

By virtue of the scaling properties of (7.4) - (7.7), it is possible to generate from (10.1) the full family of uniform wavetrains; we rescale dimensions to make h_o arbitrary and add a constant translational velocity to make β arbitrary. Thus it may be verified that if (10.1) satisfies (7.4) - (7.7), so also does

$$\left.\begin{aligned}
&\varphi = \psi + \Phi(\theta,y),\\
&h = H(\theta),\\
&\theta = kh_o^{-1}x - (k\beta h_o^{-1} + kch_o^{-\frac{1}{2}})\,t,\\
&\psi = \beta x - (\tfrac{1}{2}\beta^2 + h_o C)\,t,\\
&\Phi(\theta,y) = (h_o^3)^{\frac{1}{2}}A_1 \sin\theta \cosh kh_o^{-1}y\\
&\qquad + (h_o^3)^{\frac{1}{2}}A_2 \sin 2\theta \cosh 2kh_o^{-1}y + O(a_1^3),\\
&H(\theta) = h_o + a_1 h_o \cos\theta + a_2 h_o \cos 2\theta + O(a_1^3).
\end{aligned}\right\} \quad (10.2)$$

Writing $k = \kappa h_o$ and $a_1 = a/h_o$, we find that the family of uniform solutions, depending on the four parameters κ, β, a, and h_o, is approximately given by

$$\left.\begin{aligned} \varphi &= \psi + \Phi(\theta,y), \\ h &= H(\theta), \\ \theta &= \kappa x - \omega t, \\ \psi &= \beta x - \gamma t, \end{aligned}\right\} \qquad (10.3)$$

where, for $\tilde{T} = \tanh \kappa h_o$,

$$\left.\begin{aligned} \omega &= \kappa\beta + (\kappa g\tilde{T})^{\frac{1}{2}} + O(a^2), \\ \gamma &= \tfrac{1}{2}\beta^2 + gh_o + \tfrac{1}{2} g\kappa a^2(\sinh 2\kappa h_o)^{-1} + O(a^3), \\ \Phi(\theta,y) &= \frac{g^{\frac{1}{2}}a}{(\tilde{T}\kappa)^{\frac{1}{2}}\cosh \kappa h_o} \sin\theta \cosh \kappa y \\ &\quad + \frac{3\kappa^{\frac{1}{2}}a^2 g^{\frac{1}{2}}}{\tilde{T}^{\frac{1}{2}}\cosh^2\kappa h_o} \cdot \frac{1 - \tilde{T}^2}{8\tilde{T}^3} \sin 2\theta \cosh 2\kappa y \\ &\quad + O(a^3), \\ H(\theta) &= h_o + a\cos\theta + \tfrac{1}{4}\kappa a^2(3 - \tilde{T}^2)\tilde{T}^{-3}\cos 2\theta \\ &\quad + O(a^3). \end{aligned}\right\} \qquad (10.4)$$

Now it is possible to calculate approximately the integrals

$$
\left.
\begin{aligned}
I_0 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{H} dy\, d\theta = h_o, \\
I_1 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{H} \Phi_\theta\, dy\, d\theta = \tfrac{1}{2} g^{\frac{1}{2}} a^2 (\tilde{T}\kappa)^{-\frac{1}{2}} + O(a^4), \\
I_2 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{H} \Phi_\theta^{\,2}\, dy\, d\theta \\
&= \frac{ga^2}{4\kappa^2} \left(1 + \frac{2\kappa h_o}{\sinh 2\kappa h_o} \right) + O(a^4).
\end{aligned}
\right\} \qquad (10.5)
$$

In terms of (10.5), the first order conditions (9.6), for avoidance of secular terms in the expansion of the classical water wave problem, may be rewritten as

$$\frac{\partial I_1}{\partial T} + \frac{\partial}{\partial X} (\beta I_1 + \kappa I_2) = 0, \qquad (10.6)$$

$$\frac{\partial I_0}{\partial T} + \frac{\partial}{\partial X} (\beta I_0 + \kappa I_1) = 0. \qquad (10.7)$$

These agree with averaged equations obtained by Whitham [26].

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26. Loc. cit.