I. SINGULAR PERTURBATION PROBLEMS INVOLVING SINGULAR POINTS AND TURNING POINTS

II. ON THE AVERAGED LAGRANGIAN TECHNIQUE FOR NONLINEAR DISPERSIVE WAVES

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Abstract

In Part I a class of linear boundary value problems is considered which is a simple model of boundary layer theory. The effect of zeros and singularities of the coefficients of the equations at the point where the boundary layer occurs is considered. The usual boundary layer techniques are still applicable in some cases and are used to derive uniform asymptotic expansions. In other cases it is shown that the inner and outer expansions do not overlap due to the presence of a turning point outside the boundary layer. The region near the turning point is described by a two-variable expansion. In these cases a related initial value problem is solved and then used to show formally that for the boundary value problem either a solution exists, except for a discrete set of eigenvalues, whose asymptotic behaviour is found, or the solution is non-unique. A proof is given of the validity of the two-variable expansion; in a special case this proof also demonstrates the validity of the inner and outer expansions.

Nonlinear dispersive wave equations which are governed by variational principles are considered in Part II. It is shown that the averaged Lagrangian variational principle is in fact exact. This result is used to construct perturbation schemes to enable higher order terms in the equations for the slowly varying quantities to be calculated. A simple scheme applicable to linear or near-linear equations is first derived. The specific form of the first order correction terms is derived for several examples. The stability of constant solutions to these equations is considered and it is shown that the correction terms lead to the instability cut-off found by Benjamin. A general stability criterion is given which explicitly demonstrates the conditions under which this cut-off occurs. The corrected set of equations are nonlinear dispersive equations and their stationary solutions are investigated. A more sophisticated scheme is developed for fully nonlinear equations by using an extension of the Hamiltonian formalism recently introduced by Whitham. Finally the averaged Lagrangian technique is extended to treat slowly varying multiply-periodic solutions. The adiabatic invariants for a separable mechanical system are derived by this method.

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I. SINGULAR PERTURBATION PROBLEMS INVOLVING SINGULAR POINTS AND TURNING POINTS

1. Introduction

In recent years singular perturbation problems have appeared in many branches of Applied Mathematics. These problems occur when a limit process expansion (see [1]) fails to approximate the solution uniformly throughout the region of interest.

There are two main types of such problems and a different technique has evolved to treat each. A typical problem of the first kind appears when the limit process expansion fails to satisfy one of the boundary conditions of the problem. One then constructs another limit process expansion valid in the boundary layer near to this boundary, and satisfying the conditions there. Both of these expansions will be under-determined in general and they are matched (see [2] for the fundamental theory) when both are fully determined. A uniform expansion can then be constructed from these two expansions. This is not of the limit process type.

The second type of problem involves the long time behaviour of slowly varying oscillatory solutions. The limit process expansion then breaks down over large times due to cumulative effect of the slow variation. Here a uniform expansion is constructed directly using the two-variable expansion procedure (see [1]), which is not of the limit process type.

In Section 2 we consider problems which are apparently of the first type but where the method of matching limit process expansions breaks down. It is shown in Section 3 how asymptotic solutions can still be constructed using two-variable and limit process expansions and the concept of matching these expansions.

The difficulty here is an unusual turning point problem. The idea of using two variable and limit process expansions, and matching these expansions in turning point problems appears in [1]. In the problem considered there the procedure is equivalent to the W.K.B. procedure used in Quantum Mechanics [3]. Fowkes [4] has recently shown how this procedure can be refined so that only a two-variable expansion is needed in that problem. Although we have also independently used a similar refinement, it is shown that one still needs to use the idea of matching in the more complicated problem considered here. An equivalent remark is that the W.K.B. procedure fails in this case.

2. Failure of limit process expansions

We consider the problem of constructing a uniform asymptotic expansion, as $\varepsilon \downarrow 0$, of the solution $y(x,\varepsilon)$ to the two-point boundary value problem

$$\varepsilon y'' + p(x)y' + q(x)y = 0, \tag{1}$$

$$y(0,\varepsilon) = 0, \; y(1,\varepsilon) = 1. \tag{2}$$

Here ε is a small positive parameter and the expansion is to be uniform for $0 \leq x \leq 1$ (unless otherwise stated we shall restrict x to this interval in Part I). We require that $p > 0$ for $x > 0$, p and q being sufficiently smooth in this range. It is assumed that

$$\left.\begin{aligned} p(x) &\sim x^{\alpha-1} (1 + p_1 x + \dots), \\ q(x) &\sim x^{\beta-2} (q_o + q_1 x + \dots), \end{aligned}\right\} \tag{3}$$

as $x \downarrow 0$, where the powers of x in the leading terms have been labeled in a convenient manner. The further assumption is made that the corresponding expansion for p' is obtained by differentiating (3). The equation has been scaled so that $p_o = 1$.

We attempt to solve this problem using limit process expansions in the manner described in [1], Chapter 2, whose terminology we also adopt. The first step is to construct an outer expansion

$$y \sim f_o(x) + \varepsilon f_1(x) + \dots ,$$

as $\varepsilon \downarrow 0$, which we expect to be uniformly valid for $0 < A \leq x \leq 1$. Here, and throughout Part I, we denote arbitrary constants by upper case letters and known constants by lower case Greek letters, unless otherwise indicated. In the above we have anticipated that the boundary layer will be at $x = 0$ since $p(x) > 0$ for $x > 0$ and this implies that it is impossible to match any other boundary layer to the outer expansion.

It is found that

$$f_o(x) = \exp\left(\int_x^1 \frac{q(t)}{p(t)}\,dt\right),$$

$$f_n(x) = f_o(x) \int_x^1 \frac{1}{p(t)f_o(t)} \{f_{n-1}(t)\}''dt$$

where $f_o(1) = 1$, $f_n(1) = 0$.

We note that

$$\begin{aligned} f_o &\sim \mu &&, \quad \alpha < \beta, \\ &\sim \gamma x^{-q_o} &&, \quad \alpha = \beta, \\ &\sim \exp\left(\frac{q_o\{1+o(1)\}}{(\alpha-\beta)\, x^{\alpha-\beta}}\right), && \quad \alpha > \beta. \end{aligned}$$

as $x \downarrow 0$. When $\alpha \geq \beta$, $q_o < 0$ we find $f_o(o) = 0$. In this special case the outer expansion is the required uniform expansion to $0(\varepsilon^n)$ for all n when $\alpha > \beta$ or $\alpha = \beta$, $\alpha \leq 0$; and for $n < (-q_o/\alpha) \leq n + 1$ when $\alpha = \beta$, $\alpha > 0$.

The remaining cases are

$$f_o \sim \mu \quad , \quad \alpha < \beta \quad , \tag{4a}$$

$$\sim \gamma x^{-q_o} \quad , \quad \alpha = \beta,\ q_o > 0, \tag{4b}$$

$$\sim \exp\left(\frac{q_o\{1+o(1)\}}{(\alpha-\beta)\, x^{\alpha-\beta}}\right), \quad \alpha > \beta,\ q_o > 0. \tag{4c}$$

The outer expansion is not uniformly valid near $x = 0$ in these cases and we attempt to describe this region by an inner expansion. Thus we put

$$\bar{x} = \frac{x}{\delta(\varepsilon)},$$

and (1) becomes

$$\frac{d^2y}{d\bar{x}^2} + \frac{\delta^\alpha}{\varepsilon}\bar{x}^{\alpha-1}\frac{dy}{d\bar{x}} + \frac{\delta^\beta}{\varepsilon}\bar{x}^{\beta-2}\, y \sim 0 \ . \tag{5}$$

We attempt to choose $\delta(\varepsilon)$ so that $\delta^\alpha/\varepsilon$ or δ^β/ε is 1, whichever is the dominant order. The inner expansion is then

$$y \sim \mu_o(\varepsilon)g_o(\bar{x}) + \mu_1(\varepsilon)g_1(\bar{x}) + \ldots. \ , \tag{6}$$

where the $\{\mu_n(\varepsilon)\}$ form an asymptotic sequence. The $\{\mu_n(\varepsilon)\}$ and any unknown constants in the expansion are to be determined by matching with the outer expansion. When this procedure is successful a uniform expansion can be constructed from the two expansions.

The problem, for the cases where $f_o(0) \neq 0$, now falls naturally into three main classes defined by certain ranges of α and β. The first class is that range for which the inner expansion can be found and matched to the outer expansion. The second is where no such inner expansion can be found and the third is when the inner expansion can be found but cannot be matched to the outer expansion.

The first class corresponds to (4a) and (4b) when $\alpha > 0$. Here $\delta^\alpha/\varepsilon$ is of equal or larger order than δ^β/ε; therefore in (5) we choose

$$\delta = \varepsilon^{1/\alpha} .$$

The inner-outer procedure outlined above then goes through in the usual manner, the details are given in Appendix A.

The second class arises in all the cases (4a), (4b), (4c), when $\alpha \leq 0$ or $\beta \leq 0$ according to which is the smaller. The difficulty here is due to the strong singularities of p and q at $x = 0$ and, therefore, we examine whether such equations even admit non-trivial solutions which are zero at $x = 0$.

We discuss this question for the equation

$$\varepsilon y'' + x^{\alpha-1}y' + q_o x^{\beta-2}y = 0 \tag{7}$$

using the theory of regular and irregular singular points (see [5]). Since (1) behaves like (7) near $x = 0$ we speculate that the results we derive for (7) can be extended to (1). When α and β are rationals the equation can be transformed by a new independent variable $z = x^{\sigma}$ to one in which the coeficients are integer powers of z (we assume it is sufficient to consider this case). The above theory then confirms that, with one exception, no non-trivial solution exists which is zero at $x = 0$. The exception is (4c) with $\alpha > 0$, $\beta \leq 0$. In the exceptional case the solutions are zero at the origin but are not identically zero; this case will be studied in Section 3. Apart from this, we conclude that no solution of the boundary value problem exists for this class.

We now concentrate on the third class, which is the case (4c) with $\beta > 0$. We choose

$$\delta(\varepsilon) = \varepsilon^{1/\beta} ,$$

when it is found that

$$g_o(\bar{x}) = C\bar{x}^{1/2} J_{1/\beta} \{(2q_o^{1/2}/\beta)\bar{x}^{-\beta/2}\} \tag{8}$$

However, here the outer expansion increases exponentially as $x \downarrow 0$ while the inner expansion performs damped oscillations as $\bar{x} \uparrow \infty$ and so the two expansions cannot be matched, in contrast to the first class of problems. In view of the fact that this technique copes with the latter class, one might query the existence of $y(x,\varepsilon)$ for the former class. We note, however, that an initial value problem for the same equation is well posed and can be handled by the same technique for the problems in the first class while it again breaks down in the present case. Thus we conclude that the difficulty is due to an incorrect perturbation scheme.

A few remarks can be made about the existence of y. We first transform to $v(x,\varepsilon)$ so that the problem for v involves an inhomogeneous equation with homogeneous boundary conditions, the homogeneous problem being self-adjoint. Thus a solution v exists, and is unique, if no solution to the homogeneous problem exists, that is if there are no eigenvalues $\{\varepsilon_n\}$ for sufficiently small ε. Standard arguments show that there is at most an enumerable set of eigenvalues with a limit point at $\varepsilon = 0$, at least when p' and q are continuous for $0 \leq x \leq 1$ (see [5]). The correct perturbation scheme applied to an appropriate initial value problem will then decide the existence of these eigenvalues.

We examine the failure of the matching in the hope this will lead us to the correct scheme. One can interpret the failure in two ways:

either the regions of validity of the first terms of the two expansions cannot be extended sufficiently so that they overlap, or the expansions are not in fact asymptotic to the exact solution as assumed. The fact that the expansions can be extended is guaranteed by Kaplun's Extension Theorem [2], but this does not ensure overlap.

We expect the outer expansion to be asymptotic to the exact solution since it is valid away from $x = 0$ and is not influenced by the behaviour of p and q at $x = 0$. Since the inner expansion is oscillatory we examine its validity by discussing the oscillatory nature of the solutions of (1).

We first reduce (1) to standard form by writing

$$y(x,\varepsilon) = \exp\left[-\int_0^x \frac{p(s)}{2\varepsilon} ds\right] u(x,\varepsilon) \,, \tag{9}$$

when it follows that

$$u'' + \left\{ \frac{-p^2}{4\varepsilon^2} + \frac{[q-(p'/2)]}{\varepsilon} \right\} u = 0 \,. \tag{10}$$

Sturm's oscillation theorems (see [5]) then show that, at least when p' and q are continuous for $0 \leq x \leq 1$, the solutions oscillate in the region $0 \leq x \leq 0(\varepsilon^{1/(2\alpha-\beta)})$ when $\alpha \geq \beta$, $q_o > 0$.

If we denote the number of oscillations by $n(\varepsilon)$ then $n(\varepsilon) = 0(1)$ when $\alpha = \beta$ while $n(\varepsilon) = 0(\varepsilon^{-(\alpha-\beta)/(2\alpha-\beta)})$ for $\alpha > \beta$. Unless otherwise stated it is understood that $q_o > 0$ for the rest of Part I. In the latter case $n(\varepsilon) = 0(1)$ for the range $0 \leq x \leq 0(\varepsilon^{1/\beta})$ which is exactly the property of the inner expansion. The analysis therefore indicates that the inner expansion is asymptotic to the exact solution and the difficulty is the non-overlapping of the expansions due to a turning point at $x = O(\varepsilon^{1/(2\alpha-\beta)})$.

3. Expansions near turning points

Here we define a turning point of order m as a zero of order m of the coefficient of u in (10). We note that for the Sturm-Liouville type equation

$$u'' + \{ \frac{r(x)}{\varepsilon^2} + s(x) \} u = 0 \qquad (11)$$

the turning points are defined in [6] to be the zeros of $r(x)$. Our perturbation scheme will be based on the knowledge of the turning points and it is crucial to use our definition here. For (11) it is shown in Appendix B that the two definitions are equivalent.

We can write the coefficient of u in (10) as $x^{\beta-2}F(x,\varepsilon)$ where $F(0,\varepsilon)$ is finite and non-zero and F has a simple zero at $x = x_o(\varepsilon)$, $x_o = O(\varepsilon^{1/(2\alpha-\beta)})$.

Thus we have a turning point of order one at $x = x_o$ and a turning point or singularity of order $|\beta-2|$ at $x = 0$, corresponding to $\beta > 2$ or $\beta < 2$ respectively. The alternative definition would not give a turning point at $x = x_o$.

The distinction between the cases $\alpha = \beta$ and $\alpha > \beta$ now becomes clear. In the former the turning point at $x_o = O(\varepsilon^{1/(2\alpha-\beta)}) = O(\varepsilon^{1/\alpha})$ occurs inside the boundary layer where $x = O(\varepsilon^{1/\alpha})$. Thus in the region between the two expansions both are monotonic functions and matching is possible. In the latter case, however, this turning point is outside the boundary layer. It is clear that neither of the expansions can describe the solution near $x = x_o$ and so the expansions do not overlap. We need another expansion to describe the region near $x = x_o$.

In Appendix B the asymptotic form of the solutions to (11) in a region including a first order turning point is derived. If the turning point is at $x = x_1$ then a two-variable expansion (see [1], Chapter 3) is used with fast variable $\theta = \Theta(x)/\varepsilon$ and slow variable x, where $\Theta(x_1) = 0$.

We attempt to use a similar procedure here and we first change to a new independent variable s such that if $s = s_o(\varepsilon)$ corresponds to $x = x_o(\varepsilon)$ then $s_o = O(1)$. Thus we write

$$s = \frac{x}{\varepsilon^{1/(2\alpha-\beta)}} , \tag{12}$$

and (10) becomes

$$\frac{d^2u}{ds^2} - \frac{h(s,\varepsilon)}{\varepsilon^{2(\alpha-\beta)/(2\alpha-\beta)}} u = 0 . \tag{13}$$

Here

$$h(s,\varepsilon) = \frac{s^{\beta-2}}{4} (s^{2\alpha-\beta} - 4q_o) + O(\varepsilon^{(\alpha-\beta)/(2\alpha-\beta)}, \varepsilon^{1/(2\alpha-\beta)}) , \tag{14}$$

where the order symbol holds uniformly for $s = O(1)$.

We then write

$$u = U(\theta,s,\varepsilon)$$

where

$$\theta = \frac{\Theta(s,\varepsilon)}{\varepsilon^{2(\alpha-\beta)/3(2\alpha-\beta)}} , \quad \Theta' > 0 . \tag{15}$$

Equation (13) then becomes

$$\frac{\partial^2 U}{\partial\theta^2} - \frac{h}{\varepsilon^{2\mu/3}(\Theta')^2} U = - \frac{2\varepsilon^{2\mu/3}}{\Theta'} \frac{\partial^2 U}{\partial\theta\partial s} - \frac{\varepsilon^{2\mu/3}\Theta''}{(\Theta')^2} \frac{\partial U}{\partial\theta} - \frac{\varepsilon^{4\mu/3}}{(\Theta')^2} \frac{\partial^2 U}{\partial s^2} , \tag{16}$$

where $\mu = (\alpha-\beta)/(2\alpha-\beta)$.

We choose

$$(\Theta')^2 \, \Theta = h, \tag{17}$$

so that (16) becomes

$$\frac{\partial^2 U}{\partial \theta^2} - \theta U = O(\varepsilon^{2\mu/3}) \, , \tag{18}$$

where the right hand side of (18) coincides with that of (16).

The solution of (17) is

$$\Theta = \operatorname{sgn}\,(s-s_o) \left\{ \frac{3}{2} \left| \int_{s_o}^{s} |h|^{1/2} \, dt \right| \right\}^{2/3} \tag{19}$$

where $\Theta(s_o, \varepsilon) = 0$. Here we could replace h by $h_o(s,\varepsilon)$ where $h = h_o + o(\varepsilon^{\mu})$ uniformly for $s = O(1)$. One might have expected that $h-h_o = o(\varepsilon^{2\mu/3})$ would be sufficient but in fact a secular term $\theta^{1/2}\varepsilon^{\mu/3}$ occurs and the more stringent error bound is required. The order symbol in (14) also indicates that h_o will change as $\alpha - \beta - 1$ increases through integer values. These points, together with the observation that using h simplifies the higher order terms, lead us to use h rather than h_o.

We solve (18) asymptotically by considering θ and s as independent variables and requiring the expansion to be uniform for $\theta = O(\varepsilon^{-2\mu/3})$, $s = O(1)$. This expansion is then a uniform expansion of (13) for $s = O(1)$.

Thus we assume

$$U \sim \nu(\varepsilon) \, \{U_o(\theta,s) + \varepsilon^{2\mu/3} \, U_1(\theta,s) + \dots \} \, ,$$

and we find

$$U_o = \frac{C\,Ai(\theta) + D\,Bi(\theta)}{(\theta')^{1/2}} , \tag{20}$$

where Ai, Bi are the Airy functions of the first and second kind respectively.

It is seen, however, that

$$U_o \sim \sigma s^{-(\beta-2)/4} ,$$

as $s \downarrow 0$, and therefore either U_o or U_{os} are singular at $s = 0$ for $\beta > -2$, unless $\beta = 2$. This breakdown is due to the turning point or singularity at $s = 0$. The region near $s = 0$ is described by the inner expansion for $\beta > 0$. In the case $\beta = 2$, (20) contains the inner expansion and is uniformly valid at least for $o \leq s \leq E < \infty$. Thus for $\beta > 0$ we have completely described the region $o \leq x \leq 1$, since these expansions can now be matched and a uniformly valid expansion constructed.

We now discuss the case $\alpha > 0$, $\beta \leq 0$, for which no inner expansion can be constructed. It is seen that the behaviour of u_o near the origin is precisely that of the exact solution, as predicted by the theory of singular points, and so we expect u_o to be valid near the origin for this case.

We note that in fact the asymptotic expansion of the general solution to (1) can be constructed by these methods. The outer expansion is replaced by a two-variable expansion with fast variable $\int_1^x (p/2\varepsilon)dt$ and slow variable x. One linearly independent solution is f_o and the other is exponentially small. Each expansion now contains two arbitrary constants and so we can examine any boundary or initial value problem using these techniques.

We consider first the initial value problem

$$y(x,0) = 0, \quad y'(x,0) = R(\varepsilon) \ , \tag{21}$$

when it is sufficient to use the outer expansion

$$y \sim \gamma(\varepsilon) \{\bar{f}_o(x) + \varepsilon \bar{f}_1(x) + \dots \} \ ,$$

where

$$\bar{f}_n(x) = F_n f_n(x) \ .$$

In performing the matching a further classification has to be made since h_o needs to be defined in matching the two-variable and outer expansions. The matching has only been carried out for the cases $o < \alpha - \beta < 1$, $1 < \alpha - \beta < 2$, as the computation involved increases rapidly with $\alpha - \beta$ but no theoretical difficulty was foreseen for the cases not treated.

The matching shows that y is exponentially small for $x > O(\varepsilon^{1/\alpha})$ unless $R(\varepsilon) \geq O[\exp(\tau/\varepsilon^{\mu})]$, where $\tau > 0$. Thus in order to solve the boundary value problem we need $R(\varepsilon)$ to be exponentially large. It is seen that, by choosing an appropriate value of R, a formal solution to the boundary value problem can be constructed to first order except when

$$\varepsilon = \varepsilon_n \sim \varepsilon_{n_o} = \left\{ \frac{\int_0^{s_o} (-h)^{1/2} \, dt}{\pi[n+(\beta+1)/2\beta]} \right\}^{1/\mu} , \tag{22}$$

for $\beta > 0$. Here $\varepsilon_n \sim \varepsilon_{n_o}$ as $n \uparrow \infty$.

We note that this sequence has zero as a limit point and so we have formally demonstrated the existence of the eigenvalues discussed earlier.

In general s_o will not be known exactly and we can replace it by a known approximation. Thus for $o < \alpha - \beta < 1$ we write

$$s_o = (4q_o)^{1/(2\alpha-\beta)} + 0\ (\varepsilon^{\mu})$$

and to the order considered we find

$$\varepsilon_{n_o} = \frac{\gamma_o}{n^{1/\mu}} + \frac{\gamma_1}{n^{(1/\mu+1)}} ,$$

where

$$\gamma_o = \left\{ \frac{1}{\pi}\int_o^{(4q_o)^{1/(2\alpha-\beta)}} \frac{1}{t}[q_o t^{\beta} - t^{2\alpha}/4]^{1/2} dt \right\}^{(2\alpha-\beta)/(\alpha-\beta)} ,$$

$$\gamma_1 = \frac{\gamma_o}{\mu}\left\{ \frac{\pi(\beta+1)}{2(\beta+2)} + \frac{1}{4}\int_o^{(4q_o)^{1/(2\alpha-\beta)}} \frac{(\alpha-1)t^{\alpha-1}}{[q_o t^{\beta} - t^{2\alpha}/4]} dt \right\} .$$

A similar expansion for the case $1 < \alpha - \beta < 2$ results in an intermediate term of $O(n^{-\{(1/\mu)-(1/[\alpha-\beta])\}})$ in the expression for ε_{n_o}. We also note that a similar expansion can be used to eliminate s_o from Θ for the cases $o < \alpha - \beta < 1$, $1 < \alpha - \beta < 3/2$. However, for the case $3/2 \leq \alpha - \beta < 2$ although s_o can be replaced by a known approximation, the expansion in terms of integrals of $(s^{2\alpha}/4 - q_o s^{\beta})^{1/2}$ becomes non-uniform when $s - s_o = O(\varepsilon^{1/(2\alpha-\beta)})$. A uniform expansion involves integrals of $\{s^{2\alpha}/4 - q_o s^{\beta} + \varepsilon^{1/(2\alpha-\beta)}\ (p_1 s^{2\alpha+1}/2 - q_1 s^{\beta})\}^{1/2}$. For any given value of s these integrals may be expanded further but there is no uniform expansion of these integrals for the whole range. To explain this further classification we note that near $s = s_o$, y can be approximated by a limit process expansion in terms of

$$z = \frac{s - (4q_o)^{1/(2\alpha-\beta)}}{\varepsilon^{2\mu/3}} ,$$

when $o < \alpha - \beta < 3/2$, and in terms of

$$z = \frac{s - (4q_o)^{1/(2\alpha-\beta)} - \zeta\varepsilon^{1/(2\alpha-\beta)}}{\varepsilon^{2\mu/3}} ,$$

for $3/2 \leq \alpha - \beta < 2$.

Here integrals of the type

$$I = \int_o^{so} (-h)^{1/2} dt$$

are expanded by $O(\varepsilon^{\mu})$ by noting that

$$I = \int_o^{s_o-\delta(\varepsilon)} (-h)^{1/2} dt + O(\delta^{3/2}), \; \delta^{3/2} = o(\varepsilon^{\mu}), \; 0<\alpha-\beta<1 ,$$

and choosing δ so that $(-h)^{1/2}$ can be expanded uniformly in terms of $(q_o t^{\beta} - t^{2\alpha}/4)^{1/2}$ in the remaining integral. Thus we require $\varepsilon^{\mu} << \delta << \varepsilon^{2\mu/3}$. Upon expanding $(-h)^{1/2}$, integrals of the type

$$J = \int_o^{s_o-\delta} \{q_o t^{\beta} - t^{2\alpha}/4\} \, dt$$

appear, which can be rewritten as

$$J = \int_o^{(4q_o)^{1/(2\alpha-\beta)}} \{q_o t^{\beta} - t^{2\alpha}/4\} \, dt ,$$

to the order considered. For the range $3/2 \leq \alpha - \beta < 2$ we cannot find such a δ for integrals of the type $\int_s^{s_o} (-h)^{1/2} dt$. Here we put $s_o - \zeta\varepsilon^{1/(2\alpha-\beta)} - \delta$ as the limit of integration when $s = 0$. We shall discuss a more sophisticated procedure for expanding such integrals in Part II. The difficulties in the case $3/2 \leq \alpha - \beta < 2$ do not appear there to the order considered due to the fact that here the expansion involves integral and non-integral powers of ε^{μ} while the integrals in Part II contain only integral powers of ε.

It finally remains to treat the case $\alpha > \beta$, $\alpha > 0$, $\beta \leq 0$. Here the two-variable expansion is valid near the origin and both the leading terms in this expansion are zero at $x = 0$. In fact, in terms of the boundary value problem, it is seen that D (see (20)) is determined by matching with the outer expansion while C is still arbitrary. Here the term $Ai(\theta)/(\Theta')^{1/2}$ is an eigenfunction in the sense that it is zero at $x = 0$ and is exponentially small at $x = 1$. Thus we see that formally there is a continuous spectrum of eigenvalues and that a solution $y(x,\varepsilon)$ exists but is non-unique.

In Appendix C we give a proof of the validity of the two-variable expansion. For a special case this also confirms the validity of the inner and outer expansions.

We note that if we choose initial conditions so that $D = 0$ and $R(\varepsilon) = O(1)$ then u is exponentially small for $x > O(\varepsilon^{1/(2\alpha-\beta)})$. This rapid damping can be expressed in a physical manner by noting that the equation

$$\varepsilon y'' + y' + q_o x^{\beta-2} y = 0, \quad q_o > 0, \quad \beta < 1, \quad \beta \neq 0 ,$$

when expressed in terms of u, can be rewritten as

$$\frac{d^2u}{dz^2} + \frac{\{E - V(z)\}u}{\bar{\varepsilon}^2} = 0 \tag{23}$$

where $\beta - 2 = -\gamma$, $\bar{\varepsilon} = \varepsilon^{(\gamma-1)/\gamma}$, $E = -q_o{}^{2/(2-\gamma)}/4$, $z = x/\varepsilon^{1/\gamma}q_o{}^{1/(\alpha-2)}$ and $V(z) = -1/z^{\gamma}$.

Equation (23) is the non-dimensional Schrodinger equation (see [3]) for the wave function u describing the stationary states of the one-dimensional motion of a particle in a potential V(z) with an energy E. It also describes the s-states of a particle in a central force field with potential V(z), where z is the radial co-ordinate. Since $E < 0$ we expect that bound states are possible and this corresponds to $D = 0$. The region $z > O(1)$, or $x > O(\varepsilon^{1/(2\alpha-\beta)})$ is the classically excluded region and we expect that u will be small there since $\bar{\varepsilon}$ is small and so the motion is quasi-classical. The rapid damping is therefore due to an extremely steep potential well. For any energy of O(1) the particle has very little chance of being in the region $x = O(1)$.

Appendix A

Here we display the results for the cases $\beta > \alpha > 0$ and $\alpha = \beta$, $\alpha > 0$, $q_o > 0$ which can be treated by using the inner and outer expansions.

We consider first the case $\alpha = \beta$, $\alpha > 0$, $q_o > 0$ where we find that

$$\mu_o(\varepsilon) = \varepsilon^{-q_o/\alpha} ,$$

$$g_o(\bar{x}) = \eta\bar{x}\ {}_1F_1 \left\{ \frac{1 + q_o}{\alpha}, \frac{1 + \alpha}{\alpha}, -\frac{\bar{x}^\alpha}{\alpha} \right\} ,$$

where ${}_1F_1$ is the confluent hypergeometric function. (The notation is that of (6)).

Thus we note that for the initial value problem where $R(\varepsilon) = O(\varepsilon^{-1/\alpha})$, and hence $\mu_o = 1$, the outer expansion will be $O(\varepsilon^{q_o/\alpha})$. Thus the outer expansion diverges algebraically as $x \downarrow 0$ and is algebraically small for an appropriate initial value problem, in direct analogy to the case $\alpha > \beta$, $q_o > 0$, when "algebraic" is replaced by "exponential".

In the remaining case we find

$$\mu_o(\varepsilon) = 1 ,$$

$$g_o(\bar{x}) = \tau \int_0^{\bar{x}} \exp[-\frac{t^\alpha}{\alpha}]\ dt .$$

The interest here is in the second term of the expansions. We can require that all the g_i, $i > 1$, satisfy inhomogeneous equations by writing

$$\mu_o(\varepsilon) = 1 + \bar{\mu}_1(\varepsilon) + \ldots. ,$$

and choosing the μ_i to ensure matching.

It is then found that

$$\mu_1(\varepsilon) = \varepsilon^{1/\alpha} \quad , \quad \alpha - \beta \leq -1,$$

$$= \varepsilon^{(\beta-\alpha)/\alpha}, \quad -1 \leq \alpha - \beta < 0.$$

We consider the matching to $O(\mu_1)$ and it is seen that $\bar{\mu}_1$ is not needed to match f_o. However,

$$f_1(x) = f_o(x) \int_x^1 \left\{ \frac{q^2}{p^3} - \frac{1}{p} \left(\frac{q}{p}\right)' \right\} dt,$$

and therefore

$$f_1 \sim \delta_o \, x^{\beta-2\alpha} \quad , \qquad \beta < 2\alpha,$$

$$\sim \delta_1 \ln \frac{1}{x} + \delta_2, \qquad \beta = 2\alpha,$$

$$\sim \delta_3 \quad , \qquad \beta > 2\alpha.$$

as $x \downarrow 0$.

In the first of these three cases it is seen that $\bar{\mu}_1 = o(\mu_1)$. The second case corresponds to $\beta - \alpha = \alpha$ and we find $\bar{\mu}_1 = o(\mu_1)$ for $\alpha > 1$ and

$$\bar{\mu}_1 = F\,\varepsilon \ln \frac{1}{\varepsilon},$$

for $0 < \alpha \leq 1$, where $\mu_1 = \varepsilon$.

Finally in the case $\beta > 2\alpha$, corresponding to $\beta - \alpha > \alpha$, we find

$$\bar{\mu}_1 = G\,\varepsilon$$

for $-1 < \alpha - \beta < 0$ and $\alpha - \beta \leq -1$, $0 < \alpha \leq 1$; $\bar{\mu}_1 = o(\mu_1)$ for the remaining cases.

We note that in general we have

$$g_1 = H\,g_o - \int_o^{\bar{x}} \exp[-\frac{t^\alpha}{\alpha}] \left\{ \int_o^t \exp[\frac{\bar{t}^\alpha}{\alpha}] \, (\nu_1 q_o \bar{t}^{\beta-2} g_o(\bar{t}) + \nu_2 p_1 \bar{t}^\alpha g_o'(\bar{t}))d\bar{t} \right\} dt$$

where $\nu_1 = 1$ for $-1 \leq \alpha - \beta < 0$, $\nu_2 = 1$ for $\alpha - \beta \leq -1$ and otherwise are zero.

All the unknown constants have been determined by matching.

Appendix B

We consider here the derivation of the asymptotic form of the general solution to (11). A derivation is given in [1] for the case where $r(x)$ has a simple zero at $x = 0$. We shall first briefly review this and then show how a more simple form can be derived.

We first note (11) can be rewritten as

$$\frac{d^2u}{d\bar{x}^2} + [r(x) + \varepsilon^2 s(x)]u = 0, \quad \bar{x} = \frac{x}{\varepsilon}, \tag{B-1}$$

and therefore, if $r(o) \neq 0$, a limit process expansion with $\bar{x}$ fixed exists but is non-uniform when $x = O(1)$. Thus we assume a two-variable expansion

$$u \sim U_o(\theta, x) + \ldots ,$$

where

$$\theta = \frac{\Theta(x)}{\varepsilon}, \quad \Theta(o) = 0, \quad \Theta' > 0,$$

and we note $\theta = O(\bar{x})$. For $r > 0$ we require u to be periodic in θ of period 2π. The period must be independent of x to preserve uniformity [7]. If $r < 0$ we demand that to $O(1)$ the equation must be independent of x, again to preserve uniformity.

It then follows that

$$\Theta(x) = \int_o^x |r(t)|^{1/2}\, dt,$$

and it is found that, for $r > 0$,

$$U_o = \frac{A \cos \theta + B \sin \theta}{r^{1/4}} .$$

Thus the expansion breaks down at the zeros of $r(x)$, as might have been expected since the original limit process expansion in $\bar{x}$ is no longer valid and $\Theta' = 0$ there.

We consider the case where r has a simple zero at $x = 0$. The region near $x = 0$ is then described by a limit process expansion based on $\tilde{x} = x/\varepsilon^{2/3}$. The two expansions are then matched and a uniformly valid expansion is constructed for $x = 0(1)$.

We note that in the above two-variable expansion the dependence of U_o on θ is exactly that of the first term of the corresponding limit process expansion on $\bar{x}$. Thus since the above expansion in $\tilde{x}$ has as first terms Airy functions of negative arguments, we expect to replace this by a two-variable expansion

$$u = U(\theta, x, \varepsilon), \qquad \theta = \frac{\Theta(x)}{\varepsilon^{2/3}}, \quad \Theta' > 0 ,$$

which is uniformly valid for $x = 0(1)$ and where

$$\frac{\partial^2 U}{\partial \theta^2} + \theta U = 0(\varepsilon^{2/3}).$$

Equation (B-1) then becomes

$$\frac{\partial^2 U}{\partial \theta^2} + \frac{[r(x) + \varepsilon^2 s(x)]U}{(\Theta')^2 \varepsilon^{2/3}} = - \frac{2\varepsilon^{2/3}}{\Theta'} \frac{\partial^2 U}{\partial \theta \partial s} - \frac{\varepsilon^{2/3}\Theta''}{(\Theta')^2} \frac{\partial U}{\partial \theta} - \frac{\varepsilon^{4/3}}{(\Theta')^2} \frac{\partial^2 U}{\partial s^2} , \quad \text{(B-2)}$$

and upon choosing

$$(\Theta')^2\Theta = r,$$

we find

$$\frac{\partial^2 U}{\partial\theta^2} + \theta U = \frac{-2\varepsilon^{2/3}}{\Theta'}\frac{\partial^2 U}{\partial\theta\partial s} - \frac{\varepsilon^{2/3}\Theta''}{(\Theta')^2}\frac{\partial U}{\partial\theta} + O(\varepsilon^{4/3}) \quad , \qquad \text{(B-3)}$$

as required. Here

$$\Theta = \operatorname{sgn} x \left\{\frac{3}{2}\left|\int_0^x |r(t)|^{1/2}dt\right|\right\}^{2/3}.$$

We now solve (B-3) asymptotically by assuming

$$U \sim U_o(\theta,x) + \varepsilon^{2/3}U_1(\theta,x) + \ldots. ,$$

and it is found that

$$U_o = \frac{C\,Ai(\theta) + D\,Bi(\theta)}{(\Theta')^{1/2}} .$$

We note that $\Theta' > 0$ and thus U_o has no singularities. Thus it is seen that U_o is a uniformly valid first approximation to u. This result has also recently been obtained by Fowkes [4] for essentially the same equation. We also note that the higher order terms can be simplified by replacing r(x) by $r(x) + \varepsilon^2 s(x)$ in the equation for Θ.

It is easily seen that in general if r(x) has an nth order zero at the origin then we define

$$\theta = \frac{\Theta(x)}{\varepsilon^{2/(n+2)}} , \qquad \Theta^n(\Theta')^2 = r(x),$$

when it is seen that

$$\frac{\partial^2 U}{\partial\theta^2} + \theta^n U = O(\varepsilon^{2/(n+2)}) .$$

In Section 3 we noted that the two definitions of turning points are equivalent for this equation and we now discuss this point. It is obvious that if $\hat{x}$ is in the region $\tilde{x} = O(1)$ then the limit process expansion with $\hat{x}$ fixed is equivalent to that based on $\tilde{x}$, and it is obtained by writing $\tilde{x} = \tilde{x}\ (\hat{x})$. Thus if we can show that the turning points defined by our method are inside the region $\tilde{x} = O(1)$ then either definition shows that a limit process expansion based at the turning point together with a two-variable expansion leads to a uniformly valid expansion. This is equivalent in the problem just considered to noting that either $r(x)$ or $r(x) + \varepsilon^2 s(x)$ can be used in the equation for Θ.

If $r(x) + \varepsilon^2 s(x) \sim x^n + \varepsilon^2 x^m$ for $x \downarrow 0$, where $n > 0$, $m \geq 0$, then our definition shows there are turning points at $x = 0$, $m > 0$, and at $x = O(\varepsilon^{2/(n-m)})$ for $n > m$, considering $x = o(1)$ only. The other definition shows that $\tilde{x} = x/\varepsilon^{2/(n+2)}$ and the result follows.

We note that the method of using the two-variable expansion together with the limit process expansion is a formal way of describing the W.K.B. technique (see [3]) while the second method involved is essentially a formal description of Langer's ideas [8]. It is seen therefore that the problem considered in Section 3 cannot be treated by the

W.K.B. technique while the idea of matching, which is essential, is not used in the Langer theory; the idea there being to construct a uniform expansion directly, but it is not at all obvious how this could be done.

We now make a few remarks in general about turning point problems for the equation $u'' - \varepsilon^{-2} q(x,\varepsilon) u = 0$. One first constructs the two-variable expansion valid away from the turning points. This will break down at the turning points according to the other definition. If our definition does not introduce any new turning points outside the region where the limit process expansions around these turning points are valid, then these expansions can be matched to the two-variable expansion. If only one new turning point is introduced outside these regions on either side of the turning point, then we proceed as in Section 3. In more complicated cases an application of the ideas of Section 3 should enable the problem to be solved.

Appendix C

We give here a proof of the validity of the two-variable expansion. For the special equation

$$\varepsilon y'' + x^{\alpha-1}y' + y = 0, \qquad \alpha \geq 4, \quad \alpha = 3 , \tag{C-1}$$

this expansion is uniformly valid over the whole range and in proving this we demonstrate the validity of the inner and outer expansions.

We note that two formal linearly independent solutions to the equation for u were shown to be of the form .

$$u_1 = \frac{Ai(\theta)}{(\theta')^{1/2}} + O(\varepsilon^{4\mu/3})\, Ai'(\theta) ,$$

$$u_2 = \frac{Bi(\theta)}{(\theta')^{1/2}} + O(\varepsilon^{4\mu/3})\, Bi'(\theta)$$

Thus we wish to prove

$$u_1 = \frac{A_i(\theta)}{(\theta')^{1/2}} \{1 + O(\varepsilon^{\mu})\} , \quad s_o \leq s \leq K < \infty ,$$

$$= \frac{A_i(\theta)}{(\theta')^{1/2}} \{1 + O(\varepsilon^{\mu})\} + \frac{B_i(\theta)}{(\theta')^{1/2}} O(\varepsilon^{\mu}) , \quad 0 < L \leq s_o \leq s ,$$

with a similar expression for u_2.

The proof follows the same lines as that given by Langer [8] for (11) when $s = O(1)$. We note that $A_i(\theta)/(\theta')^{1/2}$ and $B_i(\theta)/(\theta')^{1/2}$ are the two linearly independent solutions to the equation

$$\frac{d^2u}{ds^2} - \left\{ \frac{h(s,\varepsilon)}{\varepsilon^{2\mu}} - g(s,\varepsilon) \right\} u = 0 ,$$

where

$$g = (\Theta')^{1/2} \left\{\frac{1}{(\Theta')^{1/2}}\right\}'' .$$

We then rewrite (13) in the form

$$\frac{d^2u}{ds^2} - \left\{ \frac{h(s,\varepsilon)}{\varepsilon^{2\mu}} - g(s,\varepsilon) \right\} u = g(s,\varepsilon)u ,$$

and solve by variation of parameters to find

$$u_1 = \frac{Ai(\theta)}{(\Theta')^{1/2}} - \frac{\pi\varepsilon^{2\mu/3}}{(\Theta')^{1/2}} \int_K^s \frac{\{Bi(\theta)Ai(\theta_r) - Ai(\theta)Bi(\theta_r)\}}{(\Theta'_r)^{1/2}} g(r,\varepsilon)u_1(r,\varepsilon)dr. \tag{C-2}$$

Here we have written $\Theta_r = \Theta(r,\varepsilon)$ and $\theta_r = \Theta_r/\varepsilon^{2\mu/3}$. A similar equation can be derived for u_2. The lower limit of the integral is then s_o in order for the ensuing method to work. We assume p''' and q'' are continuous for $x > 0$ when it is seen that the kernel of the Volterra integral equation (C-2) is continuous and the general theory of such equations ensures that there exists a unique continuous solution (see [9]). Such a solution satisfies (13).

We first consider the region $s \geq s_o$ and we put

$$u_1(s,\varepsilon) = v_1(s,\varepsilon) \frac{Ai(\theta)}{(\Theta')^{1/2}} ,$$

when (C-2) becomes

$$v_1 = 1 - \pi\, \varepsilon^{2\mu/3} \int_K^s \frac{\{Bi(\theta)Ai(\theta_r) - Bi(\theta_r)Ai(\theta)\}}{(\theta'_r)^{1/2} Ai(\theta)}$$

$$\times\, Ai(\theta_r) g(r,\varepsilon) v_1(r,\varepsilon)\, dr, \qquad \text{(C-3)}$$

Since $|v_1|$ is continuous on a closed interval $s_o \leq s \leq K$ it attains its maximum $N(\varepsilon)$ at some point s'. Thus we find

$$N(\varepsilon) \leq 1 + \pi \varepsilon^{2\mu/3} N(\varepsilon) \int_{s'}^K |Q(s',r,\varepsilon)|\, dr$$

where we have written the kernel of (C-3) as $Q(s,r,\varepsilon)$. Upon using the bounds

$$|B_i(\theta_r) A_i(\theta_r)| \leq \frac{M}{\theta_r^{1/2}},$$

$$\left| \frac{B_i(\theta_{s'})\, A_i^2(\theta_r)}{A_i(\theta_{s'})} \right| \leq \frac{M}{\theta_r^{1/2}}, \quad \theta_{s'} = \theta(s',\varepsilon),$$

where M denotes a generic constant, we find

$$|Q| \leq \frac{|g|}{h^{1/2}}\, \varepsilon^{\mu/3}.$$

It is then seen that $|g|$ is uniformly bounded over the whole range while h has a simple zero at $s = s_o$ and is uniformly bounded away from zero otherwise. Thus we conclude that $N(\varepsilon)$ is bounded and hence (C-3) can be written as

$$|v_1 - 1| \le M \varepsilon^{\mu} ,$$

as required.

We now examine the region $o < L \le s \le s_o$. The above procedure fails here since $Ai(\theta)$ has zeros in this region. We revert to (C-2) and write

$$\int_K^s Q(s,r,\varepsilon)\, u_1(r,\varepsilon) dr = \int_{s_o}^s Q\, u_1\, dr - \int_{s_o}^K Q\, u_1\, dr ,$$

and in the second integral we use the bound

$$|u_1| < M \frac{A_i(\theta)}{(\Theta')^{1/2}} ,$$

just derived. Here we use the facts that $Ai(\theta)$, $Bi(\theta)$ and g are uniformly bounded for $L \le s \le s_o$, while Θ' is uniformly bounded away from zero, to deduce that u_1 is uniformly bounded in the range considered.

Thus we find

$$|u_1 - \frac{Ai(\theta)}{(\Theta')^{1/2}}| \le M\, \varepsilon^{2\mu/3} \int_s^{s_o} |Q| dr + \frac{\varepsilon^{\mu}}{(\Theta')^{1/2}}$$

$$\times \{M_1 |Ai(\theta)| + M_2 |Bi(\theta)|\} ,$$

and we use the bounds

$$|Ai(\theta_r)|,\ |Bi(\theta_r)| < \frac{M}{(-\theta_r)^{1/4}} ,$$

to deduce

$$|Q| < \frac{\varepsilon^{\mu/3}}{(-h)^{1/2}} .$$

The required result then follows and it is equally well written

$$u_1 = \frac{Ai(\theta)}{(\theta')^{1/2}} + O(\varepsilon^{\mu}), \quad o < L \le s \le s_o .$$

In exactly the same manner the results for u_2 can be obtained. The change in the lower limit is to ensure that $Ai(\theta s')Bi^2(\theta r)/Bi(\theta s')$ is bounded, for $s \ge s_o$.

We now restrict ourselves to (C-1). Here we replace K in (C-2) by $\varepsilon^{-1/2(\alpha-1)}$, corresponding to x = 1. Over this extended range it is again seen that g and $\int_{s_o}^{\varepsilon^{-1/2(\alpha-1)}} h^{-1/2} dr$ are uniformly bounded and hence u_1 is uniformly bounded. For the range $s_o \le s \le \varepsilon^{-1/2(\alpha-1)}$ we again arrive at the result

$$|v_1 - 1| \le M\, \varepsilon^{\mu} \int_{s}^{\varepsilon^{1/2(1-\alpha)}} \frac{|g|}{h^{1/2}}\, dr .$$

Thus upon using the sharper estimates

$$|g| \le \frac{M}{r^2} ,$$

$$|h| \ge M\, r^{2(\alpha-1)} , \quad r > 2^{1/(\alpha-1)} + \delta,\ \delta > 0,$$

the result

$$|v_1 - 1| \le \frac{M\, \varepsilon^{\mu}}{s^{\alpha}} , \tag{C-4}$$

follows for $s > 2^{1/(\alpha-1)} + \delta$. For $s_o \le s \le 2^{1/(\alpha-1)} + \delta$ the above integral is bounded and so we find (C-4) holds uniformly for $s_o \le s \le \varepsilon^{-1/2(\alpha-1)}$. The range $o \le s \le s_o$ is treated in the same manner as before.

Thus we have shown that

$$u_1 = \frac{Ai(\theta)}{(\Theta')^{1/2}} \{1 + \frac{O(\varepsilon^{\mu})}{s^{\alpha}}\} , \qquad s_o \le s \le \varepsilon^{-1/2(\alpha-1)} ,$$

and hence in particular when $x = O(1)$, that is $s = O(\varepsilon^{1/2(\alpha-1)})$, we find

$$u_1 = \frac{Ai(\theta)}{(\Theta')^{1/2}} \{1 + O(\varepsilon)\} .$$

It is finally seen that

$$y_1 = \gamma(\varepsilon)x^{1-\alpha} \exp[-\frac{x^{\alpha}}{\alpha\varepsilon} - \frac{1}{(\alpha-2)x^{\alpha-2}}] \{1 + O(\varepsilon)\} ,$$

which is exactly the exponentially small part of the two-variable expansion which replaces the outer expansion, with $O(\varepsilon)$ error as required.

Similarly we have

$$u_1 = \frac{Ai(\theta)}{(\Theta')^{1/2}} + O(\varepsilon^{\mu}), \qquad o \le s \le s_o ,$$

and we deduce

$$y_1 = \bar{\gamma}(\varepsilon) \{\nu_1(\varepsilon) \sin \bar{x} + \nu_2(\varepsilon) \cos \bar{x} + O(\varepsilon^{\mu})\} ,$$

where $\bar{x}$ is the inner variable and ν_1, ν_2 are $O(1)$. Thus we have also displayed the validity of the inner expansion.

In attempting to derive the same results for u_2 the change in the lower limit forces us to accept the weaker result,

$$u_2 = \frac{Bi(\theta)}{(\theta')^{1/2}} \{1 + O(\varepsilon^{\mu}) + \frac{O(\varepsilon^{\mu})}{s^{\alpha}}\} .$$

It seems likely that in fact the integral equation (C-2) defines $\sigma(\varepsilon)u_2$, where $\sigma(\varepsilon) = 1 + O(\varepsilon^{\mu})$, and not u_2, but this is only speculation. Perhaps a more careful examination of the error would eliminate the $O(\varepsilon^{\mu})$ term; however, we can still deduce

$$y_2 = \tau(\varepsilon) \{f_o(x) + O(\varepsilon^{\mu}) + O(\varepsilon)\} ,$$

and therefore we have displayed the validity of the outer expansion with a weakened error bound. We deduce the same form for the inner expansion from y_2 as for y_1.

In general the validity of the inner and outer expansions will be proved in a similar way but with other integral equations. The formal procedure used is then justified if it can be shown that the various expansions overlap.

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II. ON THE AVERAGED LAGRANGIAN TECHNIQUE FOR NONLINEAR DISPERSIVE WAVES

1. Introduction

Linear dispersive systems are distinguished by the fact that all dependent variables have solutions of the form

$$u = a \cos \theta, \quad \theta = \kappa x - \omega t + \eta, \quad \omega = \omega(\kappa) \,, \tag{1}$$

for arbitrary wave number κ, amplitude a and phase shift η. The general solution is then given by Fourier integrals. Here we assume ω is real and $\omega''(\kappa) \not\equiv 0$. Each Fourier component (1) travels with a different phase velocity $c(\kappa) = \omega/\kappa$ and hence a local disturbance tends to 'disperse' into its various components.

To obtain information from the Fourier integrals the method of stationary phase may be used to derive an asymptotic expansion for large x and t, x/t fixed. The resulting form is

$$u \sim a \cos \theta, \quad k = \theta_x, \quad \omega = -\theta_t, \quad \omega = \omega(k), \tag{2}$$

where a(x,t) and k(x,t) are slowly varying functions of x and t. The sense in which a, k are slowly varying will be made precise later. Thus for large x and t the general solution is given locally by (1).

It is therefore natural to look <u>directly</u> for solutions which are locally of the form (1) for all x and t, but which vary slowly over large x and t. Such solutions can be found approximately by the W.K.B. technique. That is we assume

$$u \sim (a_o + \varepsilon a_1 + \ldots\ldots) \cos \frac{\Theta}{\varepsilon} \tag{3}$$

and upon substituting (3) into the differential equations we derive partial differential equations for the slowly varying functions, κ, ω and the a_i, where $\kappa = \Theta_x/\varepsilon$, $\omega = -\Theta_t/\varepsilon$. Such concepts as group velocity can then be developed from these slow modulation equations.

We now discuss how these ideas can be extended to nonlinear dispersive equations. The equations considered admit uniform periodic wavetrain solutions of the form

$$u = U_o(\theta, A_i, \omega, \kappa) \qquad \theta = \kappa x - \omega t + \eta\ , \tag{4}$$

where U_o is periodic in θ, and the A_i are constants of integration. We cannot superpose solutions here and so we look for solutions which are locally of the form (4) but which are slowly varying, as in the linear case.

Thus we assume

$$u \sim U_o(\theta, A_i, \omega, \kappa) + \ldots\ldots\ , \tag{5}$$

where θ is no longer a linear function of x and t and $\kappa = \theta_x$, $\omega = -\theta_t$ and the A_i are slowly varying. Whitham [1] has shown how to derive the slow modulation equations by averaging the differential equations expressed in conservation form. Subsequently he developed a simpler and more general approach by formulating the original problem as a variational principle [2]. The slow modulation equations then follow from an averaged variational principle whose Lagrangian is the average of the original Lagrangian over θ. In particular for a linear system the averaged

Lagrangian can be written down explicitly and this results in an improved view of the linear theory. All this discussion is limited to the first approximation.

In [1] Whitham treated several examples and restricted his discussion to the cases where the slow modulation equations were hyperbolic. These equations have multi-valued solutions and he argued that in some cases these would correspond to shock waves. In [3] Whitham showed that constant solutions to the modulation equation were stable or unstable according to whether these equations were hyperbolic or elliptic. He used this result to examine the full water waves problem and to derive the change in stability of the Stokes waves at a finite depth from instability in deep water to stability in shallow water. However, he did not find the cut-off in the instability for sufficiently large side-band wave numbers as predicted by Benjamin [4] using the near-linear interaction theory.

In Part II we are concerned with constructing higher approximations to the averaged Lagrangian used by Whitham in such a way that these lead to corrections to his slow modulation equations rather than a sequence of such equations. These corrections involving higher order derivations of ω, κ and the A_i will be important in the study of the shock waves and stability results mentioned above. The perturbation schemes are applied directly to the variational principle. Two schemes are developed, one a simple scheme for linear or near-linear problems the other a more sophisticated theory for fully nonlinear problems.

Luke [5] had previously devised a perturbation scheme for fully nonlinear problems to apply directly to the original differential equations. In principle this leads to a sequence of slow modulation equations but he only displayed the first order equations. He did, however, show that these equations are equivalent to the Euler equations of the first order averaged Lagrangian. The procedure was continued to higher orders for the nonlinear Klein-Gordon equation. Certain integrals were set equal to zero which upon using the θ dependence of the solution would, in principle, yield the sequence of slow modulation equations, but this last step is laborious to perform. Thus although Luke placed the first order averaged Lagrangian in a consistent perturbation scheme he left undecided the question of how to improve upon this approximate Lagrangian.

In Section 2 we derive the result that the averaged Lagrangian is exact if the exact solution is used in the averaging. This forms the basis for the perturbation schemes we use.

Sections 3 and 4 contain a derivation of the corrected set of equations for typical near-linear examples where the Lagrangian depends either on one function or on two functions, one being a potential. The perturbation scheme here involves expanding the original Lagrangian in a Fourier series. The Benjamin cut-off is found where predicted by that theory, the present derivation is simpler than the interaction approach, however.

In particular the theory is applied to the problem in nonlinear optics discussed by Ostrovskii [6] who seems to be the first to discover the Benjamin cut-off by the corrected equations approach. However, his equations were derived by an ad-hoc averaging technique and they are criticized by comparing with the results of the present method.

We remark that the approach taken here enables a general stability criterion to be given which explicitly shows the circumstances under which the cut-off occurs. It is also seen that the stationary solutions found for a simple example also apply to a wide class of problems.

In Section 5 we consider fully nonlinear equations using the Hamiltonian formalism developed by Whitham [7]. This formalism is extended to the exact solution, rather than the first approximation considered by Whitham. The exact form of the averaged Lagrangian is then deduced. The perturbation scheme used is unusual in that this exact Lagrangian is expanded rather than the actual solution. Also the solution is not explicitly solved for in this calculation, which is in the spirit of Whitham's original theory.

The Hamiltonian formalism was also introduced independently by Bisshopp [8] who also derived the result that the averaged Lagrangian is exact. He then performed an iteration scheme directly on the equations and derived a set of integral conditions similar to Luke's. Each iterated set are then shown to be the Euler equations of an iterated averaged Lagrangian. However, he does not derive any slow modulation equations and his scheme still involves solving explicitly for the solution. It is shown in Section 5 that a direct attack on the variational principle leads to a much simplified and more fundamental theory. We finally note that Bisshopp's analysis is restricted to Lagrangians depending on a single function, while our approach extends to more general Lagrangians.

Finally in Section 6 a few remarks are made about the extension of the method to deal with slowly varying multiply-periodic solutions. In particular the adiabatic invariants for a separable mechanical system are derived.

2. The exact averaged Lagrangian

We consider dispersive wave equations which can be derived from a variational principle. All known dispersive equations are of this type. To begin with we consider the class of equations governed by a variational principle of the form

$$\delta \iint L(u, u_x, u_t)\, dx\, dt = 0 . \tag{6}$$

(Throughout Part II we restrict ourselves to one space dimension, but the extension is straightforward).

The corresponding Euler equation is

$$\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) - \frac{\partial L}{\partial u} = 0 , \tag{7}$$

and we first give a review of Luke's procedure for solving (7). A two-variable expansion is used and we put

$$u(x, t, \varepsilon) = U(\theta, X, T, \varepsilon) , \tag{8}$$

where

$$\theta = \frac{\Theta(X,T)}{\varepsilon},\ X = \varepsilon x,\ T = \varepsilon t,\ \kappa = \Theta_X,\ \omega = -\Theta_T,\ \Theta(0,0) = 0, \tag{9}$$

and U is periodic in θ. Here ε is small positive parameter which is supposed introduced through the initial or boundary conditions. We note that $\theta = O(x,t)$ and so the slow variations occur over times and distances of $O(\varepsilon^{-1})$. Thus ε is the ratio of the time scale of the oscillations to that of the modulations.

Upon using (8) we find (7) becomes

$$\kappa \frac{\partial}{\partial\theta}\left(\frac{\partial L}{\partial u_x}\right) + \varepsilon \frac{\partial}{\partial X}\left(\frac{\partial L}{\partial u_x}\right) - \omega \frac{\partial}{\partial\theta}\left(\frac{\partial L}{\partial u_t}\right) + \varepsilon \frac{\partial}{\partial T}\left(\frac{\partial L}{\partial u_t}\right) - \frac{\partial L}{\partial u} = 0 \quad ,$$

or

$$\frac{\partial}{\partial\theta}\left(\frac{\partial L}{\partial U_\theta}\right) + \frac{\partial}{\partial X}\left(\frac{\partial L}{\partial U_X}\right) + \frac{\partial}{\partial T}\left(\frac{\partial L}{\partial U_T}\right) - \frac{\partial L}{\partial U} = 0 \quad , \tag{10}$$

where

$$L\ (u,\ u_x,\ u_t) = L\ (U,\ \kappa U_\theta + \varepsilon U_X,\ -\ \omega U_\theta + \varepsilon U_T)\ .$$

The idea is to consider (10) as an equation in the three independent variables θ, X and T. A uniform expansion of U is then constructed for $\theta = O(\varepsilon^{-1})$; $X,T = O(1)$ which along the surface $\theta = \Theta(\varepsilon x, \varepsilon t)/\varepsilon$. $X = \varepsilon x$, $T = \varepsilon t$, is seen to be a uniform expansion of (7) for $x,t = O(\varepsilon^{-1})$, as required. The expansion is kept uniform by requiring that U be periodic in θ of period 2π.

Luke then assumes

$$U \sim U_o\ (\theta,\ X,\ T) + \varepsilon\, U_1\ (\theta,\ X,\ T) + \ldots . \quad ,$$

and requires that each U_n be periodic in θ of period 2π. This requirement leads to the nonlinear dispersion relation when applied to U_o. Each U_n can be solved for explicitly but is not automatically periodic. The explicit form contains two integrals of the form $\int_o^\theta F_n^{(j)} d\theta$, $j = 1,2$, and in order for U_n to be periodic of period 2π we must ensure that

$$\int_0^{2\pi} F_n^{(j)} d\theta = 0 \,, \qquad (11)$$

where $F_n^{(j)}$ depends on the Ui, i < n, and the derivatives of U_{n-1} and U_{n-2}. These conditions are called secular conditions and when the θ dependence of the $F_n^{(j)}$ is used to evaluate the integrals they give rise to the sequence of slow modulation equations mentioned previously. The dispersion relation together with the above condition for $F_1^{(1)}$ forms the set of slow modulation equations for ω, κ and $A = A(X,T)$, where A is the first integral of the equation for U_o.

It is crucial for our scheme that we modify Luke's formalism and write

$$\theta = \Theta\ (X,\ T,\ \varepsilon)/\varepsilon \quad , \quad A = A\ (X,\ T,\ \varepsilon) \quad ,$$

when it is seen that $\Theta(X,T,\varepsilon)$ and $A(X,T,\varepsilon)$ are the two slowly varying functions of the problem. Luke's procedure involved a sequence of slowly varying functions which result from expanding Θ and A for small ε.

We now apply those considerations directly to the variational principle. Thus we look for extremals of (6) which are of the form (8). In analogy with the above we search for a variational principle in the three independent variables θ,X,T, whose extremals evaluated along the appropriate surface are extremals of (6). Intuitively one would consider

$$\delta \iint\int_0^{2\pi} L\ (U,\ \kappa U_\theta + \varepsilon U_X,\ -\ \omega U_\theta + \varepsilon U_T)\ d\theta\ dX\ dT = 0 \qquad (12)$$

and we show that the Euler equation of (12), corresponding to the variation of U, is (10), thus confirming our intuitive ideas. In fact (12) becomes

$$\delta \iint\int_0^{2\pi} \{\frac{\partial L}{\partial U} \delta U + \frac{\partial L}{\partial U_\theta} \delta U_\theta + \frac{\partial L}{\partial U_X} \delta U_X + \frac{\partial L}{\partial U_T} \delta U_T\} \, d\theta dX dT = 0 \quad ,$$

and by the usual arguments we deduce (10), for arbitrary δU which are periodic of period 2π in θ and vanish on the boundary of the (X,T) region, which is mapped from the original (x,t) boundary. A direct derivation of (12) from (6) is given in Appendix D.

We now consider (12) as the basic variational principle of the problem. We note that the normalization of the period to 2π restricts the class of functions Θ, this restriction can be found by varying Θ independently of U in (12). Since $\Theta = \theta(X,T,\varepsilon)$ is independent of θ we find

$$\iint [\frac{\partial}{\partial \kappa} (\int_0^{2\pi} L d\theta) \, \delta\Theta_X - \frac{\partial}{\partial \omega} (\int_0^{2\pi} L d\theta) \, \delta\Theta_T] \, dX dT = 0 \quad ,$$

and therefore we deduce

$$\frac{\partial}{\partial X} (\int_0^{2\pi} \frac{\partial L}{\partial \kappa} d\theta) - \frac{\partial}{\partial T} (\int_0^{2\pi} \frac{\partial L}{\partial \omega} d\theta) = 0 \quad , \tag{13}$$

for arbitrary $\delta\Theta$ vanishing on the boundary of the (X,T) region. We note (13) is an <u>exact</u> secular condition. If we expand Θ and A for small ε then (13) leads to a sequence of secular conditions of the type derived by Luke. If the dependence of U on θ can be found then (13) becomes an exact equation for the slowly varying functions Θ and A.

We note from (12), in a similar manner, that if the θ dependence of U is known then the exact slow modulation equations are given by

$$\delta \iint \mathcal{L} \, dX dT = 0 \quad , \tag{14}$$

where

$$\mathcal{L} = \int_0^{2\pi} L \, d\theta \ , \tag{15}$$

and the right hand side of (15) is evaluated using the known functional dependence of U on θ. Thus $\mathcal{L}$ is not a functional of U but a function of the slow varying quantities. In general it is seen that $\mathcal{L}$ depends on Θ,A and their derivatives. This will become clear in the next section.

We remark that (15) will only be used as it stands for linear or near-linear problems. For fully nonlinear problems the Hamiltonian formalism mentioned earlier will be developed.

3. A near-linear Klein-Gordon equation

The concepts introduced in the last section are further developed and applied to a near-linear Klein-Gordon equation. We note that (14) is deceptively simple in that an exact solution is rarely known and the asymptotic form of U is usually very complicated. The idea is to avoid explicitly solving the Euler equation for U as a function of θ.

For a near-linear equation this is achieved by using a Fourier series expansion of U. This is practical since to first order only one harmonic appears and a limited number of harmonics appear at the next order. Obviously this procedure will be impractical for a fully nonlinear equation.

To illustrate the ideas we consider the linear Klein-Gordon equation [9]

$$u_{tt} - u_{xx} + u = 0 \quad , \tag{16}$$

where

$$L = \frac{1}{2}\,(u_t^2 - u_x^2 - u^2) \quad ,$$

or

$$L = \frac{\mu}{2}\,U_\theta^2 - \frac{1}{2}\,U^2 - \varepsilon U_\theta(\omega U_T + \kappa U_X) + \frac{\varepsilon^2}{2}\,(U_T^2 - U_X^2) \quad , \tag{17}$$

where $\mu = \omega^2 - \kappa^2$.

The exact form for U is

$$U = a(X,\ T,\varepsilon)\cos\theta \quad , \tag{18}$$

when it is seen that

$$\mathcal{L} = \frac{\pi a^2}{2}(\mu - 1) + \frac{\pi\varepsilon^2}{2}(a_T^2 - a_X^2) \quad , \tag{19}$$

is the exact averaged Lagrangian. Thus the exact Euler equations of (14) are

$$(\mu - 1)a + \varepsilon^2(a_{XX} - a_{TT}) = 0 \ , \tag{20}$$

$$\frac{\partial}{\partial X}(\kappa a^2) + \frac{\partial}{\partial T}(\omega a^2) = 0 \ , \tag{21}$$

corresponding to the variations of a and Θ respectively.

We pause here to note that the Euler equation of (17), corresponding to variations of U, is

$$\mu\frac{\partial^2 U}{\partial\theta^2} + U - \varepsilon\left\{2\omega\frac{\partial^2 U}{\partial\theta\partial T} + 2\kappa\frac{\partial^2 U}{\partial\theta\partial X} + (\kappa_X + \omega_T)\frac{\partial U}{\partial\theta}\right\} + \varepsilon^2\left\{\frac{\partial^2 U}{\partial T^2} - \frac{\partial^2 U}{\partial X^2}\right\} = 0 \ , \tag{22}$$

and upon using (18) in this equation we deduce (20) and (21). This is a slight variation on the usual W.K.B. technique in that we have not expanded $a(X, T, \varepsilon)$ for small ε and derived a sequence of equations, and that we have let Θ depend on ε which enables both first harmonics to be treated at the same time. These are trivial differences for the linear theory but are important for the near-linear theory.

The only correction in (20) and (21) to the equations discussed by Whitham is the $O(\varepsilon^2)$ term in (20).

We must add the consistency relation

$$\kappa_T + \omega_X = 0 \ , \tag{23}$$

to complete the set of equations. We do not discuss these equations here as a more general set will be examined in connection with the near linear equation which we consider next.

We first consider the nonlinear equation

$$\bar{u}_{tt} - \bar{u}_{xx} + \bar{u} + 2n\gamma \, \bar{u}^{2n-1} = 0 \text{ , n integer, } n > 1 \text{ ,} \tag{24}$$

for small amplitude motions

$$\bar{u} = \alpha u, \quad u = 0(1) \text{ ,}$$

where γ is a constant and α is a small positive parameter. Equation (24) then becomes

$$u_{tt} - u_{xx} + u + \alpha^{2n-1} \, 2n\gamma \, u^{2n-1} = 0 \text{ ,} \tag{25}$$

which is the required near-linear equation.

Thus

$$L = \frac{\mu}{2} U_\theta^2 - \frac{1}{2} U^2 - \varepsilon U_\theta (\omega U_T + \kappa U_X) + \varepsilon^2 (U_T^2 - U_X^2) - \alpha^{2n-1} \gamma U^{2n}$$

and we write

$$U = a(X, T, \varepsilon) \cos\theta + a_o + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

where the a_i, b_i are functions of X, T, ε and are $0(\alpha^{2n-1})$. Therefore

$$\mathcal{L} = \frac{\pi a^2}{2} (\mu - 1) + \frac{\pi\varepsilon^2}{2} (a_T^2 - a_X^2) - \alpha^{2n-1} \pi\gamma a^{2n} + 0 \, (\varepsilon\alpha^{2n-1}, \alpha^{2(2n-1)}) \tag{26}$$

or

$$\mathcal{L} = \bar{\mathcal{L}} - \pi\alpha^{2n-1}\,\bar{\gamma}a^{2n} + 0(\varepsilon\alpha^{2n-1},\ \alpha^{2(2n-1)}) ,$$

where (19) is written as $\mathcal{L} = \bar{\mathcal{L}}$, and

$$\bar{\gamma} = \frac{\gamma}{\pi} \int_0^{2\pi} \cos^{2n}\theta \, d\theta \quad .$$

Thus to the order considered $\mathcal{L}$ is calculated using the linear solution. We consider the case where $\alpha^{2n-1} = \varepsilon^2$, when the nonlinear terms and the dispersion terms enter at the same order, and we work to $0(\varepsilon^2)$ in what follows. The corresponding Euler equations are then

$$(\mu - 1)a + \varepsilon^2(a_{XX} - a_{TT}) - \varepsilon^2 2n\bar{\gamma}a^{2n-2} = 0 , \tag{27}$$

and (21). Thus the only effect of the nonlinearity is to modify the dispersion relation. We eliminate ω from (21) and (22), using (27), to leave

$$\kappa_T + \frac{\kappa}{\sqrt{(\kappa^2+1)}}\kappa_X + \varepsilon^2 \frac{2\bar{\gamma}(n-1)a^{2n-3}}{\sqrt{(\kappa^2+1)}} a_X - \frac{\varepsilon^2}{2}\frac{\partial}{\partial X}\left(\frac{a_{XX} - a_{TT}}{a\sqrt{(\kappa^2+1)}}\right) = 0, \tag{28}$$

$$\frac{\partial}{\partial X}(\kappa a^2) + \frac{\partial}{\partial T}\left(\sqrt{(\kappa^2+1)}a^2 + \varepsilon^2\frac{n\bar{\gamma}a^{2n}}{\sqrt{(\kappa^2+1)}} - \varepsilon^2\frac{a}{2\sqrt{(\kappa^2+1)}}(a_{XX} - a_{TT})\right) = 0. \tag{29}$$

These equations can be considerably simplified by considering a restricted class of solutions corresponding to a modulated monochromatic wave. We illustrate how this simplification arises by writing

$$\Theta(X, T, \varepsilon) = \Theta_o(X, T) + \varepsilon\,\bar{\Theta}(X, T, \varepsilon), \qquad \bar{\Theta} = 0(1) ,$$

We are free to choose $\Theta_o(X, 0)$ as this still leaves $A(X, 0, \varepsilon)$ and $\bar{\Theta}(X, 0, \varepsilon)$ to satisfy initial conditions. The class of solutions mentioned above corresponds to choosing

$$\Theta_o(X, 0) = \bar{\kappa}_o X ,$$

or

$$\kappa_o(X, 0) = \bar{\kappa}_o, \qquad \kappa_o = \Theta_{oX} . \tag{30}$$

Equation (28) also shows that

$$\kappa_{oT} + \frac{\kappa_o}{\sqrt{(\kappa_o^2+1)}} \kappa_{oX} = 0 \tag{31}$$

and the solution of (31), subject to (30), is

$$\kappa_o(X, T) = \bar{\kappa}_o . \tag{32}$$

Similarly

$$\omega_o(X, T) = \sqrt{(\bar{\kappa}_o + 1)} = \bar{\omega}_o ,$$

and therefore

$$\theta = \bar{\kappa}_o x - \bar{\omega}_o t + \bar{\Theta}(X, T, \varepsilon) . \tag{33}$$

Finally we drop the bars on $\bar{\kappa}_o$ and $\bar{\omega}_o$ and we write

$$\kappa = \kappa_o + \varepsilon\bar{\kappa} , \tag{34}$$

when (28) and (29) become

$$\bar{\kappa}_T + \left(\frac{\kappa_o}{\omega_o} + \varepsilon\frac{\bar{\kappa}}{\omega_o^3}\right)\bar{\kappa}_X + \varepsilon\frac{2\bar{\gamma}n(n-1)a^{2n-3}}{\omega_o}a_X - \frac{\varepsilon}{2\omega_o}\frac{\partial}{\partial X}\left(\frac{a_{XX} - a_{TT}}{a}\right) = 0 , \tag{35}$$

$$a_T + \left(\frac{\kappa_o}{\omega_o} + \varepsilon\frac{\bar{\kappa}}{\omega_o^3}\right)a_X + \varepsilon\frac{a}{2\omega_o}\left(\frac{\kappa_o}{\omega_o}\bar{\kappa}_T + \bar{\kappa}_X\right) = 0 , \tag{36}$$

where we have ignored the $O(\varepsilon^2)$ terms in these equations. It is therefore seen that for solutions of the form (34) it is sufficient to use the first order equation of the linear theory

$$\frac{\partial}{\partial X}(\kappa a^2) + \frac{\partial}{\partial T}[\sqrt{(\kappa^2 + 1)}\; a^2] = 0 , \tag{37}$$

instead of (29), to the order considered.

Stability

We note that $\bar{\kappa} = 0$, $a = a_o =$ constant are solutions of these equations and we examine the stability of such solutions (any constant value of $\bar{\kappa}$ can be absorbed into κ_o). Thus we put

$$\bar{\kappa} = \hat{\kappa}\, e^{i(RX - ST)}, \qquad a = a_o + \hat{a}\, e^{i(RX - ST)} ,$$

and keep terms linear in $\hat{\kappa}$ and $\hat{a}$ in (35) and (36). Such solutions exist if the dispersion relation

$$\left(S - \frac{\kappa_o}{\omega_o}R\right)^2 = \frac{\varepsilon^2}{4\omega_o^4}\{4\bar{\gamma}n(n-1)a_o^{2n-2} + S^2 - R^2\} \tag{38}$$

is satisfied, where we have ignored $O(\varepsilon^4)$. We solve for $S = S(R, \varepsilon)$ to $O(\varepsilon^2)$ when it is seen that S is real, and hence the solutions are stable, if

$$R^2 + 4\bar{\gamma}n(n-1)\omega_o^2\, a_o^{2n-2} > 0 \,. \tag{39}$$

Whitham [10] previously gave a general stability analysis for equations of the type considered where the effect of the higher order derivatives was not included. Thus his theory corresponds to ignoring the R^2 term in (39) and shows that the solutions are stable if $\gamma > 0$ and unstable if $\gamma < 0$.

The extension of Whitham's theory given in (39) still predicts that the solutions are stable for $\gamma > 0$. However, for $\gamma < 0$ the higher order derivatives stabilize the previous instability for sufficiently high wave numbers $R > R_o$ where

$$R_o^2 = 4n(n-1)(-\bar{\gamma})\; \omega_o^2\, a_o^{2n-2} \,. \tag{40}$$

This is the cut-off effect corresponding to the Benjamin side-band instability.

Stationary solutions

We now investigate the stationary solutions of (35) and (36). Thus we put

$$\bar{\kappa} = \bar{\kappa}(\eta), \quad a = a(\eta), \quad \eta = X - V(\varepsilon)T\,, \tag{41}$$

and we choose

$$V(\varepsilon) = \frac{\kappa_o}{\omega_o} + \varepsilon V_1 \quad , \tag{42}$$

when (36) integrates to

$$\bar{\kappa} = \frac{C}{a^2} + \omega_o^3 V_1 \quad . \tag{43}$$

Upon using (42) and (43) it is found that (35) reduces to

$$a'' = \frac{\hat{\gamma} a^{2n+2} - Da^4 + C^2}{a^3} , \quad \hat{\gamma} = 2\bar{\gamma} n(n-1)\omega_o^2 , \qquad (44)$$

where C, $D(V_1)$ are constants of integration. This equation is analyzed in the phase plane, which is symmetrical about $a = 0$, $a' = 0$.

It is immediately seen that if $C \neq 0$ then no solutions exist at $a = 0$. For $\bar{\gamma} < 0$ all solutions are periodic. Such solutions are also found for $\bar{\gamma} > 0$ together with solitary wave solutions which tend exponentially to a_1 as $\eta \to \pm \infty$ where $a = a_1$ is the larger positive singular point (all singular points are on $a' = 0$).

We now consider the case $C = 0$, when $\bar{\kappa} = V_1 \omega_o^3$, where solutions exist at $a = 0$. For $\bar{\gamma} D \leq 0$ all solutions are periodic oscillating in $|a| \leq a_2$ for every $a_2 > 0$. We also find such solutions for $\gamma D > 0$ with restricted ranges of a_2. In the case $\gamma < 0$, $D < 0$ there are also periodic solutions not passing through $a = 0$ and solitary waves tending exponentially to zero as $\eta \to \pm \infty$. Finally in the case when $\gamma > 0$, $D > 0$, there are step function type solutions which tend to a_3 and $-a_3$ as η tends to $+\infty$ and $-\infty$ respectively, where $a = a_3$ is the singular point away from $a = 0$.

Since we have found periodic solutions it follows that (35) and (36) are nonlinear dispersive equations. It is therefore rather surprising to find the step function type solution. Its presence here is due to the fact that the dispersive terms are not linear and contain the nonlinearity a^{-1}. However, we note that in terms of $A = a^2$ these solutions are solitary waves, and since A is related to the physical energy the step function nature of the solutions in a has no physical significence.

We note that since (44) has the form of a nonlinear oscillator it can be solved implicitly to yield η as a function of a, X, T. Then the corresponding dispersion relations for the periodic solutions can be found.

Finally we note a few points about the perturbation scheme. As mentioned earlier, Luke's formulation involved expanding $a(X, T, \varepsilon)$ and $\Theta(X, T, \varepsilon)$ as limit process expansions

$$a \sim a_o(X, T) + \varepsilon a_1(X, T) + \ldots ,$$

$$\Theta \sim \Theta_o(X, T) + \varepsilon \Theta_1(X, T) + \ldots .$$

The above solutions can also be expanded in this manner but the expansions become non-uniform in general when $X, T = O(\varepsilon^{-1})$. Thus our procedure appears capable of extending the uniformity of the expansion over X, T larger than $O(\varepsilon^{-1})$. In this respect it seems to generalize the ideas of Cole and Kevorkian [11] for near-linear ordinary differential equations.

In Appendix E we describe an approximate method of solving (35) and (36) which reduces the set to a single equation, whose stationary solutions are studied. The analysis is based on the derivation of the Korteweg-de Vries equation from the Boussinesq equations [12].

We now briefly discuss the situation when the nonlinearity appears at a lower order than the dispersion terms. Thus we return to (25) and put $\alpha^{2n-1} = \varepsilon$; we work to $O(\varepsilon^2)$ as before. We again write

$$U = a \cos \theta + a_o + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

where the a_i and b_i are $O(\varepsilon)$. However, since

$$\cos^{2n-1}\theta = \sum_{j=0}^{n-1} C_{2j+1} \cos (2j + 1)\, \theta \, , \qquad (45)$$

it follows that the only harmonics which contribute to $\int_0^{2\pi} u^{2n}\, d\theta$ to $O(\varepsilon^2)$ are those displayed in (45). Thus we write

$$U = a \cos\theta + \varepsilon \sum_{j=1}^{n-1} \bar{a}_{2j+1} \cos(2j+1)\theta + O(\varepsilon^2), \quad a_{2j+1} = \varepsilon\bar{a}_{2j+1}, \tag{46}$$

when it is seen that

$$\mathcal{L} = \frac{\pi a^2}{2}(\mu - 1) + \varepsilon^2 \frac{\pi}{2}(a_T^2 - a_X^2) - \varepsilon\pi\bar{\gamma}a^{2n}$$

$$+ \varepsilon^2 \frac{\pi}{2} \sum_j \{[\mu(2j-1)^2 - 1]\, a_{2j+1}^2 - \bar{\gamma}_j \bar{a}_{2j+1} a^{2n-1}\}, \tag{47}$$

where we only need to know u to $O(\varepsilon)$ to calculate $\mathcal{L}$ to $O(\varepsilon^2)$. Here

$$\bar{\gamma}_j = \frac{2n\gamma}{\pi} \int_0^{2\pi} \cos^{2n-1}\theta \cos(2j+1)\theta\, d\theta .$$

The variations of each $\bar{a}_{2j+1}$ are then used to reduce (47) to

$$\mathcal{L} = \bar{\mathcal{L}} - \varepsilon\pi\bar{\gamma}a^{2n} - \varepsilon^2 \frac{\pi\gamma'}{2} a^{4n-2}, \tag{48}$$

where

$$\gamma' = \sum_j \frac{\bar{\gamma}_j^2}{[\mu(2j-1)^2 - 1]} .$$

Thus the dispersion relation is

$$(\mu - 1)a - \varepsilon\, 2n\bar{\gamma}a^{2n-2} - \varepsilon^2\, \pi\gamma'(2n-1)a^{4n-3} + \varepsilon^2(a_{XX} - a_{TT}) = 0 ,$$

and we solve this equation for ω. Thus we can eliminate ω from (23) and the Euler equation corresponding to variations in a, to leave two equations in $\bar{\kappa}$ and a. We note that (23) then takes the form

$$\bar{\kappa}_T + \frac{\kappa_o}{\omega_o}\bar{\kappa}_X + \frac{2n(n-1)\bar{\gamma}a^{2n-3}}{\omega_o} a_X = O(\varepsilon) \ ,$$

and therefore the nonlinearity enters in the equation at $O(1)$. One then proceeds to analyze these equations in the same manner as above.

4. Extension to systems of equations

We now extend the ideas discussed in the previous sections to the more general case where

$$L = L(u, u_x, u_t, \phi_x, \phi_t) . \tag{49}$$

In (49) we have followed Whitham [2] in noting that for all known Lagrangians of nonlinear dispersive systems, which depend on n functions, (n-1) of these are potentials.

The corresponding Euler equations are (7) together with

$$\frac{\partial}{\partial x} \left(\frac{\partial L}{\partial \phi_x}\right) + \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \phi_t}\right) = 0 . \tag{50}$$

The slowly varying solution for ϕ is

$$\phi = \frac{\Psi(X, T, \varepsilon)}{\varepsilon} + \Phi(\theta, X, T, \varepsilon) \quad , \tag{51}$$

where $\int_0^{2\pi} \Phi d\theta = 0$, with the same form for u as before. This choice is dictated by the requirement that L be periodic in θ, and O(1). This implies ϕ_x and ϕ_t must satisfy these conditions and (51) then follows. We define

$$\Psi_X = \beta(X, T, \varepsilon), \qquad \Psi_T = -\gamma(X, T, \varepsilon) \ , \tag{52}$$

so that

$$L(u, u_x, u_t, \phi_x, \phi_t) = L(U, \kappa U_\theta + \varepsilon U_X, - \omega U_\theta + \varepsilon U_T, \beta + \kappa\Phi_\theta + \varepsilon\Phi_X ,$$

$$- \gamma - \omega\Phi_\theta + \varepsilon\Phi_T) . \tag{53}$$

Thus (50) becomes

$$\frac{\partial}{\partial\theta}\{\kappa\frac{\partial L}{\partial\phi_x} - \omega\frac{\partial L}{\partial\phi_t}\} + \varepsilon\frac{\partial}{\partial X}(\frac{\partial L}{\partial\phi_x}) + \varepsilon\frac{\partial}{\partial T}(\frac{\partial L}{\partial\phi_t}) = 0 ,$$

or

$$\frac{\partial}{\partial\theta}(\frac{\partial L}{\partial\Phi_\theta}) + \frac{\partial L}{\partial X}(\frac{\partial L}{\partial\Phi_X}) + \frac{\partial}{\partial T}(\frac{\partial L}{\partial\Phi_T}) = 0 . \qquad (54)$$

It is seen that (12) is still the required variational principle with L defined by (53). Then (14) determines the slow modulation equations with $\mathcal{L}$ as the average of (53).

We illustrate the ideas with a linear example, as before. The equation considered is

$$v_t + v_{xxx} = 0 , \qquad (55)$$

which is the Korteweg-de Vries equation linearized about $v = 0$. Whitham [10] has shown that

$$L = \frac{1}{2}\phi_x\phi_t + \phi_x u_x + \frac{1}{2}u^2 , \qquad (56)$$

where $u = v_x$, $v = \phi_x$. It then follows that

$$L = -\frac{1}{2}(\beta + \kappa\Phi_\theta)(\gamma + \omega\Phi_\theta) + \kappa U_\theta(\beta + \kappa\Phi_\theta) + \frac{U^2}{2} + \varepsilon\{\frac{1}{2}\Phi_T(\beta + \kappa\Phi_\theta)$$
$$- \frac{1}{2}\Phi_X(\gamma + \omega\Phi_\theta) + U_X(\beta + \kappa\Phi_\theta) + \kappa U_\theta\Phi_X\} + \varepsilon^2\{\frac{1}{2}\Phi_X\Phi_T + \Phi_X U_X\} . \qquad (57)$$

An exact solution is

$$U = a(X, T, \varepsilon)\cos\theta , \qquad (58)$$

$$\Phi = b(X, T, \varepsilon)\cos\theta + \bar{b}(X, T, \varepsilon)\sin\theta , \qquad (59)$$

and it then follows that

$$\mathcal{L} = -\beta\gamma - \frac{\omega\kappa}{2}(b^2 + \bar{b}^2) + \kappa^2 ab + \frac{a^2}{2} + \varepsilon\left\{\frac{\kappa}{2}(\bar{b}b_T - \bar{b}_T b) + \frac{\omega}{2}(b\bar{b}_X - b_X\bar{b})\right.$$
$$\left. + \kappa(a_X\bar{b} - a\bar{b}_X)\right\} + \varepsilon^2\left\{\frac{1}{2}b_X b_T + \frac{1}{2}\bar{b}_X\bar{b}_T + a_X b_X\right\} . \qquad (60)$$

The variation of Ψ implies that

$$\beta_T - \gamma_X = 0 , \qquad (61)$$

and using this together with the consistency relation,

$$\beta_T + \gamma_X = 0 ,$$

enables Ψ to be determined. That the equivalent of (61) uncouples Ψ is seen to be true for all linear problems. In fact here

$$\Psi = \Psi_o(X, \varepsilon) + \Psi_1(T, \varepsilon) .$$

The variation of $\bar{b}$ shows that $\bar{b} = O(\varepsilon)$ and to eliminate $\bar{b}$ from $\mathcal{L}$ we only need $\bar{b}$ to $O(\varepsilon)$. In a similar manner b is also eliminated from $\mathcal{L}$ to leave

$$\mathcal{L} = \left(\frac{\kappa^3}{\omega} + 1\right)\frac{a^2}{2} + \varepsilon^2\left\{\frac{3\kappa}{2\omega}a_X^2 + \frac{3\kappa^2}{2\omega^2}a_X a_T + \frac{\kappa^3}{2\omega^3}a_T^2\right\} + \varepsilon^2 g + O(\varepsilon^3) \qquad (62)$$

In (62) we have anticipated that we shall only consider solutions of the type (34) when g, which is a homogeneous function of the derivatives of ω and κ, can be neglected to $O(\varepsilon^2)$. This is due to the fact that the $O(\varepsilon^2)$ term in $\mathcal{L}$ is neglected in the Θ variation (see(37)) and g will be $O(\varepsilon)$ in the a variation. This is not necessary to the method but simplifies the algebra. In the remainder of this section we work to $O(\varepsilon^2)$.

We now consider the nonlinear equation

$$\bar{v}_t + \bar{v}_{xxx} + \tau(n + 1)(n + 2)\bar{v}^n\bar{v}_x = 0$$

for small amplitude motion, when it becomes

$$v_t + v_{xxx} + \alpha^n\tau(n + 1)(n + 2)v^n v_x = 0 \ , \qquad (63)$$

with the same notation as in Section 3. Thus n = 1 corresponds to the Korteweg-de Vries equation while n = 2 is the model equation used to demonstrate the side-band instability.

Thus we have

$$L = \frac{1}{2}\,\phi_x\phi_t + \phi_x u_x + \frac{1}{2}\,u^2 + \alpha^n\tau\phi_x^{n+2} \qquad (64)$$

and we begin by considering the case $\alpha^n = \varepsilon^2$, when the nonlinearity appears at the same order as the dispersion terms. It is then seen that, as in the previous section, we evaluate $\mathcal{L}$ using the linear solutions (58) and (59). Thus we find

$$\mathcal{L} = \bar{\mathcal{L}} + \varepsilon^2\,\frac{\bar{\tau}\kappa^{2n+4}}{\omega^{n+2}}\,a^{n+2} \quad , \qquad (65)$$

where (62) is written as $\mathcal{L} = \bar{\mathcal{L}}$ and

$$\bar{\tau} = (-1)^{n+2}\,\frac{\tau}{\pi}\int_o^{2\pi} \sin^{n+2}\theta d\theta \ .$$

Here we have written $\beta = \varepsilon\bar{\beta}$, $\gamma = \varepsilon\bar{\gamma}$, so that, to $O(\varepsilon^2)$, $\bar{\beta}$ and $\bar{\gamma}$ satisfy (61) and can then be ignored in $\mathcal{L}$. Thus we are considering solutions $v = -\kappa b \sin\theta + O(\varepsilon)$. We note that $\bar{\tau}$ is zero if n is odd and in order to treat this case we must take $\alpha^n = \varepsilon$ to bring in nonlinear effects.

The corresponding Euler equations are

$$\left(\frac{\kappa^3}{\omega} + 1\right)a + \varepsilon^2\, \bar{\tau}(n+2)\, \frac{\kappa^{2n+4}}{\omega^{n+2}}\, a^{n+1} - \varepsilon^2 \left\{ \frac{3\kappa}{\omega}\, a_{XX} + \frac{3\kappa^3}{\omega^2}\, a_{XT} + \frac{\kappa^3}{\omega^3}\, a_{TT} \right\} = 0, \quad (66)$$

$$\frac{\partial}{\partial X}\left(\frac{3\kappa^2 a^2}{2\omega}\right) + \frac{\partial}{\partial T}\left(\frac{\kappa^3 a^2}{2\omega^2}\right) = 0(\varepsilon^2) , \quad (67)$$

and we note that these have the same form as (27) and (37). We show later that (65) is the general type of Lagrangian for near linear equations with nonlinearities of $0(\varepsilon^2)$, when solutions of the form (34) are considered. Thus we defer a discussion of (66) and (67) until then.

We now briefly treat the case $\alpha^n = \varepsilon$. We use the same reasoning as for the single equation case to write

$$U = a \cos\theta + \varepsilon \sum_{j=1}^{(n+1)/2} a_{2j} \cos 2j\theta + 0(\varepsilon^2) , \quad (68)$$

$$\Phi = b \cos\theta + \bar{b} \sin\theta + \varepsilon \sum_{j=1}^{(n+1)/2} b_{2j} \cos 2j\theta + 0(\varepsilon^2) , \quad (69)$$

where $b_{2j} = b_{2j}(X, T, \varepsilon)$.

It follows that

$$\mathcal{L} = \bar{\mathcal{L}} - \varepsilon^2\bar{\beta}\bar{\gamma} + \varepsilon^2\bar{\tau}_o\kappa^{n+1}\,\bar{\beta}\, b^{n+1} - \varepsilon^2 2\kappa \sum_j \{2\omega^{n+1}\, b^{n+1}\, j^2\, \bar{\tau}_{2j}\, b_{2j} + \omega j^2\, b_{2j}^2\}$$
$$+ \frac{\varepsilon^2}{2}\{a_{2j}^2 + 8j^2\kappa^2 \sum_{j=1}^{(n+1)/2} a_{2j}\, b_{2j}\} , \quad (70)$$

where

$$\bar{\tau}_o = \frac{\tau(n+2)}{\pi} \int_o^{2\pi} \sin^{n+1}\theta\, d\theta, \quad \bar{\tau}_{2j} = \frac{\tau(n+2)}{\pi} \int_o^{2\pi} \sin^{n+1}\theta \sin 2j\theta\, d\theta .$$

The variation of Ψ leads to the equation

$$\frac{\partial}{\partial X}\left\{\bar{\gamma} - \frac{\bar{\tau}_o \kappa^{2(n+1)}}{\omega^{n+1}} a^{n+1}\right\} - \frac{\partial \bar{\beta}}{\partial T} = 0 , \tag{71}$$

and this equation no longer completely uncouples $\bar{\beta}$ and $\bar{\gamma}$. We use

$$\bar{\beta}_T + \bar{\gamma}_X = 0$$

to find

$$\bar{\gamma} = \frac{\bar{\tau}_o \kappa^{2(n+1)}}{2\omega^{n+1}} a^{n+1} .$$

In a similar manner we eliminate $\bar{\gamma}_X$ from (71) and use (67) to express a_X in terms of a_T, to find

$$\bar{\beta} = \frac{\bar{\tau}_o \kappa^{2n+3}}{6\omega^{n+2}} a^{n+1} .$$

The variations of each b_{2j} allow them to be expressed in terms of a and we finally find

$$\mathcal{L} = \bar{\mathcal{L}} + \varepsilon^2 \hat{\tau} a^{n+1} , \tag{72}$$

where $\hat{\tau} > 0$ and $\hat{\tau} = \hat{\tau}(\tau^2)$. We defer a discussion of the Euler equations of (72) until the general theory is discussed. It is noted, however, that (72) has the same form as (65).

Some general results

We now discuss the general form of $\mathcal{L}$ for a linear system of dispersive equations governed by (49). Thus L is a quadratic form, the terms $u\, u_x$, $u\, u_t$ being omitted as they are divergence expressions. In order for the system to be dispersive L cannot contain terms both of the type $u_x\phi_t$ and $u\phi_t$. Thus we have two classes of Lagrangians: one in which

$$L = L_o(u_x, u_t, \phi_x, \phi_t) - \nu_o \frac{u^2}{2} , \tag{73}$$

where L_o is a quadratic form and the other in which

$$L = L_1(u_x, u_t) + L_2(\phi_x, \phi_t) + \lambda_o u\phi_x + \tau_o u\phi_t - \frac{\rho_o u^2}{2} \tag{74}$$

where L_1 and L_2 are quadratic forms, ν_o, λ_o, τ_o, ρ_o being constants.

In both cases an exact form is

$$U = a \cos \theta \quad , \tag{75}$$

$$\Phi = b \cos \theta + \bar{b} \sin \theta , \tag{76}$$

but in the first class $\bar{b} = O(\varepsilon)$ while in the second class $b = O(\varepsilon)$. We shall display the analysis for the first class of Lagrangians, the extension to the second class is immediate. Here we put $\Psi = 0$ since the equation for Ψ uncouples in general for a linear system to admit this solution. Thus we consider the first class and we write

$$\Phi = b \cos \theta + \varepsilon\bar{b} \sin \theta, \quad \bar{b} = O(1) . \tag{77}$$

The results we derive also hold for the second class when we write

$$\Phi = b \sin \theta + \varepsilon\bar{b} \cos \theta , \tag{78}$$

The final form of the Lagrangian will be seen to be the same in both cases; here we work to $O(\varepsilon^2)$ as before.

It is then seen that

$$\mathcal{L} = f_1 \frac{a^2}{2} + f_2 ab + f_3 \frac{b^2}{2} + \varepsilon^2 \{f_4 \frac{\bar{b}^2}{2} + g_1 \bar{b} + g_2 \bar{b}_X + g_3 \bar{b}_T + h_1\} , \qquad (79)$$

where $f_i = f_i(\omega, \kappa)$; g_1 is a linear form in a_X, a_T, b_X, b_X; g_2 and g_3 are similar functions of a and b, and h_1 is a quadratic form in a_X, a_T, b_X, b_T. Here, and in what follows, all functions g_i have coefficients depending on ω and κ.

We can then rewrite $\mathcal{L}$ in the form

$$\mathcal{L} = f_1 \frac{a^2}{2} + f_2\, ab + f_3 \frac{b^2}{2} + \varepsilon^2 \{f_4 \frac{\bar{b}^2}{2} + g_4 \bar{b} + h_1\} ,$$

where g_4 is a linear form in the derivatives of a, b, ω, κ with coefficients depending on a and b. The variation of $\bar{b}$ is used to eliminate $\bar{b}$ to leave

$$\mathcal{L} = f_1 \frac{a^2}{2} + f_2\, ab + f_3 \frac{b^2}{2} + \varepsilon^2 g_5 ,$$

where g_5 is a quadratic form in the derivatives of a, b, ω, κ with coefficients depending on a and b. The variation of b then shows

$$b = -\frac{f_2}{f_3}\, a + \varepsilon^2 \hat{b} ,$$

and it then follows that

$$f_2\, ab + f_3 \frac{b^2}{2} = \frac{-f_2^2 a^2}{2 f_3} ,$$

to the order considered. Thus upon eliminating b from $\mathcal{L}$ we find

$$\mathcal{L} = f_4 \frac{a^2}{2} + \varepsilon^2 g_6 , \qquad (80)$$

where g_6 is a quadratic form in the derivatives of a, ω, κ with coefficients depending on a. Thus $f_4 = 0$ is the dispersion relation. In particular if we restrict ourselves to solutions of the form (34) then, in deriving the equations for $\bar{\kappa}$ and a to $O(\varepsilon)$, we use

$$\mathcal{L} = f_4 \frac{a^2}{2} + \frac{\varepsilon^2}{2} \{f_5 a_X^2 + f_6 a_X a_T + f_7 a_T^2\} . \tag{81}$$

If we now consider such a system of equations with nonlinear terms of $O(\varepsilon^2)$ it is seen that

$$\mathcal{L} = \bar{\mathcal{L}} + \varepsilon^2 h(a, \omega, \kappa) , \tag{82}$$

where we assume that $\beta = \varepsilon\bar{\beta}$, $\gamma = \varepsilon\bar{\gamma}$ as before, and (81) is written as $\mathcal{L} = \bar{\mathcal{L}}$.

The corresponding Euler equations are

$$f_4 a + \varepsilon^2 h_a - \varepsilon^2 \{f_5 a_{XX} + f_6 a_{XT} + f_7 a_{TT}\} = 0 , \tag{83}$$

$$\frac{\partial}{\partial X} (f_{4\kappa} a^2) - \frac{\partial}{\partial T} (f_{4\omega} a^2) = O(\varepsilon^2) . \tag{84}$$

Equation (83) is solved for ω to leave

$$\omega = \omega_o(\kappa) + \varepsilon^2 \omega_2 , \tag{85}$$

where

$$\omega_2 = \frac{\{j_1 a_{XX} + j_2 a_{XT} + j_3 a_{TT}\}}{a} + \hat{h} (a, \kappa), \; j_i = j_i(\kappa) , \tag{86}$$

Whitham [10] has shown that (84) can be rewritten as

$$\frac{\partial}{\partial T} (a^2) + \frac{\partial}{\partial X} (\omega_o' (\kappa)\, a^2) = O(\varepsilon^2) . \tag{87}$$

In terms of $\bar{\kappa}$, (85) and (87) become

$$\omega = \omega_o(\kappa_o) + \varepsilon\omega_o'(\kappa_o)\bar{\kappa} + \frac{\varepsilon^2}{2}\,\omega_o''(\kappa_o)\bar{\kappa}^2 + \varepsilon^2\omega_2 \ , \tag{88}$$

$$\frac{\partial}{\partial T}(a^2) + \frac{\partial}{\partial X}\,(\{\omega_o'(\kappa_o) + \varepsilon\,\omega_o''(\kappa_o)\bar{\kappa}\}\,a^2) = 0 \ . \tag{89}$$

Thus the equations in $\bar{\kappa}$ and a are

$$\bar{\kappa}_T + \{\omega_o'(\kappa_o) + \varepsilon\omega_o''(\kappa_o)\bar{\kappa}\}\ \bar{\kappa}_X + \varepsilon\omega_{2X} = 0 \ , \tag{90}$$

$$a_T + \{\omega_o'(\kappa_o) + \varepsilon\omega_o''(\kappa_o)\bar{\kappa}\}\ a_X + \frac{\varepsilon\omega_o''(\kappa_o)}{2}\ a\ \bar{\kappa}_X = 0 \ , \tag{91}$$

where

$$\omega_2 = \frac{(c_o a_{XX} + c_1 a_{XT} + c_2 a_{TT})}{a} + \bar{h}(a) \ ,$$

the c_i being known constants.

We again examine the stability of the solutions $\bar{\kappa} = 0$, $a = a_o$, and the resulting dispersion relation is seen to be

$$(S - \omega_o'R)^2 = \frac{\varepsilon^2}{2}\ \omega_o''R^2\ \{a_o\bar{h}'(a_o) - c_oR^2 + c_1RS - c_2S^2\} \ . \tag{92}$$

The solutions are then stable if

$$\omega_o''\ \{a_o\bar{h}'(a_o) - R^2[c_o - c_1\omega_o' + c_2(\omega_o')^2]\} > 0 \ . \tag{93}$$

In Whitham's previous theory [10] he considered

$$\bar{h}(a) = a^2\bar{\omega}_2(\kappa_o) \ ,$$

and, ignoring the term in R^2, (93) reduces to his result

$$\omega_o''\bar{\omega}_2 > 0. \tag{94}$$

It is obvious from (93) that a cut-off of some type exists when

$$\frac{a_o \bar{h}'(a_o)}{c_o - c_1\omega_o' + c_2(\omega_o')^2} > 0 . \tag{95}$$

It is seen that there is the possibility that the higher derivatives can be de-stabilizing but no examples of this have been found. It is possible that the implicit restriction that the system is dispersive rules this out.

In particular if $\omega_o'' a_o \bar{h}'(a_o) < 0$ and (95) holds, it is seen that the solutions are stable for $R > R_o$, where

$$R_o^2 = \frac{a_o \bar{h}'(a_o)}{c_o - c_1\omega_o' + c_2(\omega_o')^2} , \tag{96}$$

thus the previous instability in (94) is stabilized for sufficiencly high modulation wavenumbers.

For the Lagrangian in (65) it follows that (93) becomes

$$R^2 > \frac{\bar{\tau} n(n+2) a_o^n}{3\kappa_o^n} , \tag{97}$$

and thus the equation is stable for $\tau < 0$ and for sufficiently high modulation wavenumbers when $\tau > 0$.

In particular for the case $n = 2$, $\tau = 1/4$, which corresponds to

$$v_t + v_{xxx} + 3\varepsilon^2 v^2 v_x = 0 , \tag{98}$$

(97) becomes

$$R^2 > \frac{a_o^2}{2\kappa_o^2} \quad . \tag{99}$$

We note that, since $u = v_x$, we find to $O(1)$

$$v = \frac{a}{\kappa_o} \sin\theta = \bar{a} \sin\theta ,$$

when (99) becomes

$$R^2 > \frac{\bar{a}^2}{2} ,$$

which is exactly the result of the Benjamin theory, (see [10]).

Similarly for the Lagrangian in (72) it is seen that the solutions are stable for $\hat{\tau} > 0$ and hence are stable independent of the sign of τ. In particular this result holds for the Korteweg-de Vries equation.

We finally examine the stationary solutions of (90) and (91). Thus we put

$$\bar{\kappa} = \bar{\kappa}(\eta), \quad a = a(\eta), \quad \eta = X - V(\varepsilon)T , \tag{100}$$

and we choose

$$V = \omega_o' + \varepsilon V_1 \quad , \tag{101}$$

when equation (91) integrates to

$$\bar{\kappa} = \frac{C}{a^2} + \frac{V_1}{\omega_o''} \quad . \tag{102}$$

Equation (90) then becomes

$$\frac{\omega_o'' C^2}{2a^4} + \omega_2 = D(V_1)$$

or

$$a'' = \frac{(2a^4\bar{h}/\omega_o'') - \hat{D}a^4 + C^2}{\bar{\mu}a^3} \quad , \tag{103}$$

where

$$\bar{\mu} = \frac{2}{\omega_o''}\left(c_1\omega_o' - c_o - c_2(\omega_o')^2\right) .$$

In particular if $\bar{h} = \sigma a^m$ then (103) is precisely (44) if $\bar{\mu} > 0$. We note that $\bar{\mu}$ depends only on $\mathcal{L}$ and for (62) we have $\bar{\mu} > 0$. Hence the corresponding equations for (63), derived from (65) and (72), have exactly the same solutions as (44). Other cases arise if $\bar{\mu} < 0$ since C^2 is replaced by $-C^2$ in (44).

A physical example

We now apply these general results to the problem in nonlinear optics considered by Ostrovskii [6].

The governing equations are

$$E_{tt} - c^2 E_{xx} + P_{tt} = 0 \ , \tag{104}$$

$$P_{tt} + \hat{\omega}^2 P - \bar{\alpha} P^3 = \omega_p^2 E \ , \tag{105}$$

where E is the electric field, $P/4\Pi$ is the polarization, $\hat{\omega}, \omega_p$ are constants and $\bar{\alpha}$ is a small parameter.

The corresponding Lagrangian is formulated in terms of the vector potential A, $E = A_t$, and is found to be

$$L = \frac{A_t^2}{2} - \frac{c^2A_x^2}{2} + A_tP + \frac{P_t^2}{2\omega_p^2} - \frac{\hat{\omega}^2P^2}{2\omega_p^2} + \frac{\bar{\alpha}P^4}{4\omega_p^2} . \quad (106)$$

We note that this belongs to the second class of Lagrangians. We consider $\bar{\alpha} = 0(\varepsilon^2)$ and as before we evaluate $\mathcal{L}$ to $0(\varepsilon^2)$ using the linear solutions

$$A = a \cos \theta , \quad (107)$$

$$P = b \sin \theta + \varepsilon\bar{b} \cos \theta . \quad (108)$$

In exactly the same manner as described in the general theory, we deduce

$$\mathcal{L} = \frac{a^2}{2} \left\{ c^2\kappa^2 - \omega^2 - \frac{\omega^2\omega_p^2}{\hat{\omega}^2 \ \omega^2} \right\} + \frac{\varepsilon^2}{2} \left\{ c^2a_X^2 - sa_T^2 \right\} - \frac{g\omega^4a^4}{8} , \quad (109)$$

where

$$s = 1 + \frac{\hat{\omega}^2\omega_p^2 \ (3\omega^2 + \hat{\omega}^2)}{(\hat{\omega}^2 - \omega^2)^3} ,$$

$$g = \frac{3\bar{\alpha}\omega_p^6}{2(\hat{\omega}^2 - \omega^2)^4} = \frac{3\alpha\omega_p^6}{2(4\Pi)^2 \ (\hat{\omega}^2 - \omega^2)^4} , \quad g = 0(\varepsilon^2) ,$$

are defined in order to compare with Ostrovskii's work.

The variation of a then leads to the equation

$$\omega^2 + \frac{\omega^2\omega_p^2}{\hat{\omega}^2 - \omega^2} = c^2\kappa^2 - \varepsilon^2 \frac{(c^2a_{XX} - sa_{TT})}{a} - \frac{g\omega^4a^2}{2} . \quad (110)$$

We solve this equation for ω to find

$$\omega = \omega_o(\kappa) - \frac{\omega_o'(\kappa)}{2\kappa c} \left\{ \varepsilon^2 \frac{(c^2a_{XX} - a_{TT})}{a} + \frac{g\omega^4a^2}{2} \right\} , \quad (111)$$

where it is seen that the condition for stability (93) becomes

$$\left\{ \frac{S^2}{\omega_o^2} - \frac{4g\bar{A}^2}{\chi} \right\} > 0, \quad \bar{A} = \frac{\omega_o a}{2} , \tag{112}$$

which is the result obtained by Ostrovskii, corrected for a slight algebraic error. Thus the O(1) equations which are unstable for $(g/\chi) > 0$ are stabilized at sufficiently high modulation frequencies by the higher order derivative terms.

We also note that $\bar{\mu} > 0$ and thus the stationary solutions are exactly the solutions of (44).

In Ostrovskii's paper he considered the case $\alpha = O(\varepsilon)$ but he still used the solutions (107) and (108). However, as seen earlier, we must write

$$A = a \cos\theta + \varepsilon a_3 \cos 3\theta + O(\varepsilon^2) ,$$

$$P = b \sin\theta + \varepsilon\bar{b} \cos\theta + \varepsilon b_3 \sin 3\theta + O(\varepsilon^2) ,$$

and therefore we find

$$\mathcal{L} = \hat{\mathcal{L}} + \varepsilon\alpha\gamma a^6 , \tag{113}$$

where we have written (109) as $\mathcal{L} = \hat{\mathcal{L}}$ and γ is a known constant. The new term in (113) does not appear in Ostrovskii's equations, which therefore seem to be inconsistent. The stability result is not affected by this new term.

5. The Hamiltonian approach

We now consider how to derive higher approximations to (15) for fully nonlinear equations. We begin by considering Lagrangians of the form (6). The procedure used is based on the Hamiltonian formalism developed by Whitham [10]. As the analysis there was only given to the first order we shall review the theory and remove this restriction.

The basic variational principle is (12) and we define

$$\Pi = \frac{\partial L}{\partial U_\theta}, \quad L = L(U, \kappa U_\theta + \varepsilon U_X, -\omega U_\theta + \varepsilon U_T) . \tag{114}$$

This equation is then solved for

$$U_\theta = U_\theta(\Pi, U, \varepsilon U_X, \varepsilon U_T, \omega, \kappa) ,$$

which is used in defining

$$H(\Pi, U, \varepsilon U_X, \varepsilon U_T, \omega, \kappa) = \Pi U_\theta - L. \tag{115}$$

An equivalent variational principle to (12) is then

$$\delta \iint_0^{2\pi} (\Pi U_\theta - H)\, d\theta dX dT = 0 , \tag{116}$$

where Π, U, Θ are to be varied independently.

The equivalence of (116) and (12) is easily seen from the Euler equations. The independent variations of Π and U in (116) lead to the equations

$$U_\theta = \frac{\partial H}{\partial \Pi} , \tag{117}$$

$$\Pi_\theta = - \frac{\partial H}{\partial U} + \frac{\partial}{\partial X}\left(\frac{\partial H}{\partial U_X}\right) + \frac{\partial}{\partial T}\left(\frac{\partial H}{\partial U_T}\right) . \tag{118}$$

Since H is explicitly independent of θ we deduce the conservation law

$$\frac{\partial H}{\partial \theta} = \frac{\partial}{\partial X}\left(U_\theta \frac{\partial H}{\partial U_X}\right) + \frac{\partial}{\partial T}\left(U_\theta \frac{\partial H}{\partial U_T}\right), \tag{119}$$

by using Noether's theorem (see[13]). These equations have also been derived independently by Bisshopp [8].

It follows from (119) that

$$H = A(X, T, \varepsilon) + \frac{\partial}{\partial X}\left(\int_0^\theta U_\theta \frac{\partial H}{\partial U_X} d\theta\right) + \frac{\partial}{\partial T}\left(\int_0^\theta U_\theta \frac{\partial H}{\partial U_T} d\theta\right) . \tag{120}$$

We then consider the averaged variational principle

$$\delta \iint \mathcal{L}\, dXdT = 0 , \tag{121}$$

where

$$\mathcal{L} = \int_0^{2\pi} (\Pi U_\theta - H)\, d\theta . \tag{122}$$

From (120) it is seen that

$$\int_0^{2\pi} H d\theta = 2\pi A + \frac{\partial}{\partial X}\left\{\int_0^{2\pi} \int_0^\theta U_\theta \frac{\partial H}{\partial U_X} d\theta' d\theta\right\} + \frac{\partial}{\partial T}\left\{\int_0^{2\pi} \int_0^\theta U_\theta \frac{\partial H}{\partial U_T} d\theta' d\theta\right\},$$

and therefore

$$\mathcal{L} = \oint \Pi dU - 2\pi A , \tag{123}$$

since a divergence expression does not contribute to the Euler equations. In (123) it is assumed that we can solve for Π as a function of U, the slow variables ω, κ and A and their derivatives.

We also note that the variation of Θ in (116) leads to the exact secular condition

$$\frac{\partial}{\partial X}\left(\int_0^{2\pi}\frac{\partial H}{\partial \kappa}\,d\theta\right)-\frac{\partial}{\partial T}\left(\int_0^{2\pi}\frac{\partial H}{\partial \omega}\,d\theta\right)=0\ . \tag{124}$$

It is seen, however, that $\partial L/\partial\kappa = -\partial H/\partial\kappa$, $\partial L/\partial\omega = -\partial H/\partial\omega$ and therefore (124) is just (13).

At this stage another exact secular condition can be deduced. If we can also solve for U_X, U_T, as functions of Π and the $\alpha_i(\omega, \kappa, A$ and their derivatives), then (117) expresses U_θ as a function of the same variables. Thus the periodicity condition

$$\oint \frac{1}{U_\theta}\,dU = 2\pi\ , \tag{125}$$

is the other exact secular condition. Here $\oint$ denotes the integral over a complete period of U.

We remark here that since $\partial H/\partial\kappa = \varepsilon^{-1}U_\theta(\partial H/\partial U_X)$ and $\partial H/\partial\omega = -\varepsilon^{-1}U_\theta(\partial H/\partial U_T)$ we can rewrite (124) in the form

$$\frac{\partial}{\partial X}\left(\oint\frac{\partial H}{\partial U_X}\,dU\right)+\frac{\partial}{\partial T}\left(\oint\frac{\partial H}{\partial U_T}\,dU\right)=0\ . \tag{126}$$

Thus (125) and (126) form the complete set of slow modulation equations for A and Θ, or for A, ω, κ when the consistency relation (23) is added. We shall discuss later how these conditions can be expanded to yield higher approximations to the slow modulation equations.

We now discuss the first approximation to (123), we ignore $O(\varepsilon)$ throughout this discussion. The idea is to solve (120) for $\Pi = \Pi(U, \alpha_i)$, where to higher orders we also have to solve (117) for U_X, U_T as functions of the same variables. To first order it suffices to solve

$$H_o(\Pi, U, \omega, \kappa) = A\ , \tag{127}$$

where

$$H_o(\Pi, U, \omega, \kappa) = [H]_{\varepsilon=0} .$$

We assume (127) is solved for Π resulting in

$$\Pi = \Pi_o(U, A, \omega, \kappa) .$$

Thus (123) becomes

$$\mathcal{L} = \oint \Pi_o dU - 2\pi A , \tag{128}$$

and we assume that Π_o is zero at the zeros of $U_{o\theta}$, $U = U_o + O(\varepsilon)$. In the integral U varies "there and back" between the values, $\bar{U}_{o1}(A, \omega, \kappa)$, $\bar{U}_{o2}(A, \omega, \kappa)$, of U at these zeros.

The corresponding Euler equations are

$$\oint \frac{\partial \Pi_o}{\partial A} dU = 2\pi , \tag{129}$$

$$\frac{\partial}{\partial X} \left(\oint \frac{\partial \Pi_o}{\partial \kappa} dU\right) - \frac{\partial}{\partial T} \left(\oint \frac{\partial \Pi_o}{\partial \omega} dU\right) = 0 , \tag{130}$$

and we show that these are the first approximations to (125) and (126) respectively.

We differentiate (127) with respect to A to find

$$\frac{\partial H_o}{\partial \Pi_o} = \frac{1}{\partial \Pi_o / \partial A} ,$$

and upon using (117) the required result for (129) follows.

In a similar manner we deduce

$$\frac{\partial H_o}{\partial \Pi_o}\frac{\partial \Pi_o}{\partial \kappa} = -\frac{\partial H_o}{\partial \kappa}, \quad \frac{\partial H_o}{\partial \Pi_o}\frac{\partial \Pi_o}{\partial \omega} = -\frac{\partial H_o}{\partial \omega},$$

and the corresponding result for (130) then follows from (123), or from (126) upon relating $\partial H_o/\partial\kappa$, $\partial H_o/\partial\omega$ to $\partial H/\partial U_X$, $\partial H/\partial U_T$ as before.

We now show how to construct higher approximations to (128) and to illustrate the procedure we consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + V'(u) = 0 \tag{131}$$

where V(u) is a potential allowing stationary periodic solutions.

Thus we have

$$L = \frac{1}{2}(u_t^2 - u_x^2 - V(u)) ,$$

or

$$L = \frac{\mu}{2} U_\theta^2 - V(U) - \varepsilon U_\theta(\omega U_T + \kappa U_X) + \frac{\varepsilon^2}{2}(U_T^2 - U_X^2) , \tag{132}$$

where $\mu = \omega^2 - \kappa^2$, as before.

We then find

$$H = \frac{\Pi^2}{2\mu} + V(U) + \frac{\varepsilon\Pi}{\mu}(\omega U_T + \kappa U_X) + \frac{\varepsilon^2}{2\mu}(\omega U_T + \kappa U_X)^2 - \frac{\varepsilon^2}{2}(U_T^2 - U_X^2) , \tag{133}$$

and we rewrite (120) in the form

$$H = A + U_X \frac{\partial H}{\partial U_X} + U_T \frac{\partial H}{\partial U_T} + \frac{\partial}{\partial X}\left(\int_{\bar{U}_1}^{U} \frac{\partial H}{\partial U_X} dU\right) + \frac{\partial}{\partial T}\left(\int_{\bar{U}_1}^{U} \frac{\partial H}{\partial U_T} dU\right) \tag{134}$$

where $\bar{U}_1$ and $\bar{U}_2$ are zeros of U_θ.

In (134) it is understood that Π, U_X, U_T are expressed in terms of U and the α_i. Thus $\partial/\partial X$ is now evaluated keeping U and T fixed, similarly for $\partial/\partial T$. Thus upon using (133) it is seen (134) becomes

$$\frac{\Pi^2}{2} + V(U) = A + \varepsilon \frac{\partial}{\partial X} \left(\frac{\kappa}{\mu} \int_{\bar{U}_1}^{U} \Pi dU\right) + \varepsilon \frac{\partial}{\partial T} \left(\frac{\omega}{\mu} \int_{\bar{U}_1}^{U} \Pi dU\right)$$

$$+ \varepsilon^2 \frac{\partial}{\partial X} \left(\frac{\omega}{\mu} \int_{\bar{U}_1}^{U} [\omega U_X + \kappa U_T]\, dU\right) + \varepsilon^2 \frac{\partial}{\partial T} \left(\frac{\kappa}{\mu} \int_{\bar{U}_1}^{U} [\omega U_X + \kappa U_T]\, dU\right). \quad (135)$$

We note that if we attempt to solve for Π, for small ε, from (135) then the expansion becomes non-uniform near the zeros of $\Pi_o = \{2\mu[A - V(U)]\}^{1/2}$. Terms eventually appear which have non-integrable singularities at these zeros and therefore this expansion cannot be used to evaluate $\oint \Pi dU$. Also there is the added difficulty that the zeros of Π_o may not lie between $\bar{U}_1$ and $\bar{U}_2$, the zeros of U_θ. It is therefore seen that we must expand $\oint \Pi dU$ for small ε, this is achieved using contour integration.

We first note that U_o oscillates between two simple zeros of $U_{o\theta}$. It is then seen that similarly U oscillates between two simple zeros of U_θ. Thus U_θ is a double valued function of U, U_θ^2 being single valued. We now show that U_X and U_T are single valued functions of U. We assume that U_θ is known so that

$$U_\theta = F(U, \alpha_i) \ , \quad (136)$$

and we solve for

$$U_X = G(U, \alpha_i) \ . \quad (137)$$

We note that (136) can be integrated to yield

$$\theta = \int_{U_1}^{U} \frac{1}{F}\, dU \ .$$

We then write the identity

$$U = \bar{U}(\theta[U, \alpha_i], X, T) ,$$

and differentiate both sides with respect to X, keeping U and T fixed. Thus we find

$$\bar{U}_\theta \, \theta_X + \bar{U}_X = 0 ,$$

or

$$U_X = -F \frac{\partial}{\partial X} \left(\int_{\bar{U}_1}^{U} \frac{1}{F} \, dU \right) . \tag{138}$$

Similarly

$$U_T = -F \frac{\partial}{\partial T} \left(\int_{\bar{U}_1}^{U} \frac{1}{F} \, dU \right) , \tag{139}$$

and it follows that U_X and U_T are single valued when the periodicity condition (125) is used.

In our special case

$$\Pi = \mu U_\theta - \varepsilon(\omega U_T + \kappa U_X) , \tag{140}$$

and so Π is a double valued function of U.

Thus we consider $\oint \Pi dU$ as a contour integral in the complex U-plane cut between $\bar{U}_1$ and $\bar{U}_2$, the countour is then around the cut. The contour is then deformed, using Cauchy's theorem, to a contour C on which $A - V(U) = 0(1)$. We then use the expansion of Π to $\mathcal{O}(\varepsilon^2)$ found by solving (135) asymptotically. Here in general it is preferable to use $\hat{U}$ instead of $\bar{U}_1$ as the lower limit in (135), where $V(\hat{U}) = 0$, since the path of integration in the integrals appearing there can then be deformed so that $A - V(U) = O(1)$ on the path. However, it is more convenient here to proceed as in Part I and note that the range of integration where $A - V(U) = O(\varepsilon)$ does not contribute to $O(\varepsilon^2)$. Any integral which has an integrand with a non-integrable singularity at $\bar{U}_{o1}$ and $\bar{U}_{o2}$ is rewritten as the derivative with respect to A of an integral with an integrable integrand at these zeros. Each integral is then evaluated in the U-plane cut between $\bar{U}_{o1}$ and $\bar{U}_{o2}$ by deforming to an integral around the cut. This results in the asymptotic expansion of $\oint \Pi dU$ to $O(\varepsilon^2)$.

Thus we find

$$\oint \Pi dU = \oint \Pi_o dU + \varepsilon\oint\Pi_1 dU + \varepsilon^2\oint\Pi_2 dU + O(\varepsilon^3) \quad , \tag{141}$$

where

$$\Pi_1 = \frac{\mu}{\Pi_o}\left\{\frac{\partial}{\partial X}\left(\frac{\kappa}{\mu}\int_{\bar{U}_{o1}}^{U}\Pi_o dU\right) + \frac{\partial}{\partial T}\left(\frac{\omega}{\mu}\int_{\bar{U}_{o1}}^{U}\Pi_o dU\right)\right\}, \tag{142}$$

$$\Pi_2 = \frac{\mu}{\Pi_o}\left\{\frac{\partial}{\partial X}\left(\frac{1}{\mu}\int_{\bar{U}_{o1}}^{U}[\kappa\Pi_1 + \omega(\omega U_{oX} + \kappa U_{oT})]dU\right)\right.$$

$$\left. + \frac{\partial}{\partial T}\left(\frac{1}{\mu}\int_{\bar{U}_{o1}}^{U}[\omega\Pi_1 + \kappa(\omega U_{oX} + \kappa U_{oT})]dU\right)\right\}. \tag{143}$$

In solving for Π we also have to solve (140) for U_X and U_T using (138) and (139) and we find

$$U_{oX} = \frac{\Pi_o}{\mu}\frac{\partial}{\partial X}\left(\mu\int_{\bar{U}_{o1}}^{U}\frac{1}{\Pi_o}dU\right), \quad U_{oT} = \frac{\Pi_o}{\mu}\frac{\partial}{\partial T}\left(\mu\int_{\bar{U}_{o1}}^{U}\frac{1}{\Pi_o}dU\right) \quad . \tag{144}$$

All integrals now have a range of variation from $\bar{U}_{o1}$ to $\bar{U}_{o2}$ and back again.

We show that $\oint \Pi_1 dU$ is zero by virtue of the fact that Π_1 has the same value at corresponding points on each side of the cut. This would not be true if we used $\hat{U}$ instead of $\bar{U}_1$. In general this is true for functions of the form

$$A(\Pi_o) \int_{\bar{U}_{o1}}^{U} B(\Pi_o)dU \ ,$$

where A, B are odd functions and $\oint BdU = 0$. We note here that all such terms which appear in the Π_i must be such that $\oint BdU = 0$. In our case this is assured by (130) which takes the form

$$\frac{\partial}{\partial X}\left(\frac{\kappa}{\mu} \oint \Pi_o dU\right) + \frac{\partial}{\partial T}\left(\frac{\omega}{\mu} \oint \Pi_o dU\right) = 0 \ ,$$

for our special equation.

It has not been possible, as yet, to work $\oint \Pi_2 dU$ into a manageable form for the fully nonlinear problem. We can make some progress, however, if we agree to evaluate $\oint \Pi_2 dU$ using the linear solution. This will certainly be valid for a near-linear equation and seems reasonable for at least some range of finite amplitudes.

We note that for the linear equation we find

$$\mathcal{L} = 2\pi A(\mu^{1/2} - 1) + \varepsilon^2 \pi \mu^{1/2} \left\{ \frac{A_{TT} - A_{XX}}{2} + \frac{(A_T^2 - A_X^2)}{4A} \right\} \ , \quad (145)$$

to $O(\varepsilon^2)$. In order to compare with the earlier linear theory we note that

$$[H]_{\theta=0} = A \ ,$$

and upon using the linear solution (18) we find

$$A = \frac{a^2}{2} - \frac{\varepsilon^2}{2}(a_T^2 - a_X^2) \ . \quad (146)$$

Then using (146) in (145) it is seen that the corresponding Euler equations are equivalent to those previously derived.

Thus for the nonlinear problem to $0(\varepsilon^2)$ we consider

$$\mathcal{L} = (\mu^{1/2}G(A) - 2\pi A) + \varepsilon^2\pi\mu^{1/2} \left\{\frac{A_{TT} - A_{XX}}{2} + \frac{A_T^2 - A_X^2}{4A}\right\} , \quad (147)$$

when the Euler equations are

$$\mu^{1/2} G'(A) - 2\pi - \frac{\varepsilon^2\pi\mu^{1/2}}{4A^2} (A_T^2 - A_X^2) + \varepsilon^2\pi \frac{\partial}{\partial X}\left[\frac{\mu^{1/2}A_X}{2A}\right] - \varepsilon^2\pi \frac{\partial}{\partial T}\left[\frac{\mu^{1/2}A_T}{2A}\right]$$

$$+ \frac{\varepsilon^2\pi}{2} \left\{\frac{\partial^2}{\partial T^2}(\mu^{1/2}) - \frac{\partial^2}{\partial X^2}(\mu^{1/2})\right\} = 0 , \quad (148)$$

$$\frac{\partial}{\partial X}\left\{\frac{\kappa}{\mu^{1/2}}\left[G(A) + \varepsilon^2\pi\left(\frac{A_{TT} - A_{XX}}{2} + \frac{A_T^2 - A_X^2}{4A}\right)\right]\right\} + \frac{\partial}{\partial T}\left\{\frac{\omega}{\mu^{1/2}}\right.$$

$$\left.\times\left[G(A) + \varepsilon^2\pi\left(\frac{A_{TT} - A_{XX}}{2} + \frac{A_T^2 - A_X^2}{4A}\right)\right]\right\} = 0 , \quad (149)$$

where $G(A) = \oint (\Pi_o/\mu^{1/2})dU$. These are to be solved together with the consistency relation (23). These equations have not been investigated; we note that they are reversible so that it is not clear what relevence they have to the irreversible shocks found in the $0(1)$ equations by Whitham. In investigating shock structure solutions the assumption of slowly varying solutions breaks down and all terms in $\mathcal{L}$ are of equal importance. It may be that in spite of this (148) and (149) do describe the qualitative nature of the solutions but this is only speculation.

We note that (148) and (149) can be derived using the exact secular conditions (125) and (126) where the integrals are expanded in the same manner as $\oint \Pi dU$. Here though one must calculate U_X and U_T to $O(\varepsilon)$ in order to derive (148) and (149). Also the technique of rewriting the integrals as derivatives of other integrals with respect to A appears at this stage. These disadvantages are to be weighed against the fact that after computing $\oint \Pi dU$ one still has to evaluate the Euler equations.

Extension to more general Lagrangians

We now extend the technique just described to cope with Lagrangians of the form (49). The basic variational principle is (12) and we define

$$\Pi_1 = \frac{\partial L}{\partial U_\theta} , \tag{150}$$

$$\Pi_2 = \frac{\partial L}{\partial \Phi_\theta} . \tag{151}$$

These equations are then solved for U_θ and Φ_θ as functions of Π_1, Π_2, U, ω, κ, β, γ, εU_X, εU_T, $\varepsilon \Phi_X$ and $\varepsilon \Phi_T$, and the resulting expressions are used to define

$$H(\Pi_1, \Pi_2; U, \omega, \kappa, \beta, \gamma, \varepsilon U_X, \varepsilon U_T, \varepsilon \Phi_X, \varepsilon \Phi_T) = \Pi_1 U_\theta + \Pi_2 \Phi_\theta - L . \tag{152}$$

An equivalent variational principle to (12) is

$$\delta \iint_0^{2\pi} (\Pi_1 U_\theta + \Pi_2 \Phi_\theta - H) d\theta dX dT = 0 , \tag{153}$$

where Π_1, Π_2, U, Φ, Θ, Ψ are to be varied independently. The Euler equations

$$\Pi_{1\theta} = -\frac{\partial H}{\partial U} + \frac{\partial}{\partial X} \left(\frac{\partial H}{\partial U_X}\right) + \frac{\partial}{\partial T} \left(\frac{\partial H}{\partial U_T}\right) , \tag{154}$$

$$\Pi_{2\theta} = \frac{\partial}{\partial X} \left(\frac{\partial H}{\partial \Phi_X}\right) + \frac{\partial}{\partial T} \left(\frac{\partial H}{\partial \Phi_T}\right) , \tag{155}$$

$$U_\theta = \frac{\partial H}{\partial \Pi_1} \quad , \tag{156}$$

$$\Phi_\theta = \frac{\partial H}{\partial \Pi_2} \quad , \tag{157}$$

correspond to the variations of U, Φ, Π_1 and Π_2. The variation of Θ leads to the exact secular condition (124) while another exact secular condition,

$$\frac{\partial}{\partial X} \left(\int_o^{2\pi} \frac{\partial H}{\partial \beta}\, d\theta\right) - \frac{\partial}{\partial T} \left(\int_o^{2\pi} \frac{\partial H}{\partial \gamma}\, d\theta\right) = 0 \ , \tag{158}$$

follows from the variation of Ψ. In the same manner as before we also deduce the conservation law

$$\frac{\partial H}{\partial \theta} = \frac{\partial}{\partial X} \left(U_\theta \frac{\partial H}{\partial U_X} + \Phi_\theta \frac{\partial H}{\partial \Phi_X}\right) + \frac{\partial}{\partial T} \left(U_\theta \frac{\partial H}{\partial U_T} + \Phi_\theta \frac{\partial H}{\partial \Phi_T}\right) . \tag{159}$$

We integrate (155) and (159) to yield

$$\Pi_2 = B(X, T, \varepsilon) + \frac{\partial}{\partial X} \left(\int_o^{\theta} \frac{\partial H}{\partial \Phi_X}\, d\theta\right) + \frac{\partial}{\partial T} \left(\int_o^{\theta} \frac{\partial H}{\partial \Phi_T}\, d\theta\right) \ , \tag{160}$$

$$H = A(X, T, \varepsilon) + \frac{\partial}{\partial X} \left(\int_o^{\theta} \left\{U_\theta \frac{\partial H}{\partial U_X} + \Phi_\theta \frac{\partial H}{\partial \Phi_X}\right\} d\theta\right)$$
$$+ \frac{\partial}{\partial T} \left(\int_o^{\theta} \left\{U_\theta \frac{\partial H}{\partial U_T} + \Phi_\theta \frac{\partial H}{\partial \Phi_T}\right\} d\theta\right) \quad . \tag{161}$$

Thus we finally conclude

$$\mathcal{L} = \oint \left\{ \Pi_1 + \frac{\Phi_\theta}{U_\theta} \left[\frac{\partial}{\partial X} \left(\int_{\bar{U}_1}^{U} \frac{1}{U_\theta} \frac{\partial H}{\partial \Phi_X}\, dU\right) - \frac{\partial H}{\partial \Phi_X} \frac{\partial}{\partial X} \left(\int_{\bar{U}_1}^{U} \frac{1}{U_\theta}\, dU\right) + \frac{\partial}{\partial T} \left(\int_{\bar{U}_1}^{U} \frac{1}{U_\theta} \frac{\partial H}{\partial \Phi_T}\, dU\right) \right.\right.$$
$$\left.\left. - \frac{\partial H}{\partial \Phi_T} \frac{\partial}{\partial T} \left(\int_{\bar{U}_1}^{U} \frac{1}{U_\theta}\, dU\right) \right] \right\} dU - 2\pi A \ . \tag{162}$$

To first order (160) and (161) become

$$\Pi_2 = B \ , \tag{163}$$

$$H_o(\Pi_1, B, U, \omega, \kappa, \beta, \gamma) = A \ . \tag{164}$$

Thus upon solving (164) for

$$\Pi_1 = \Pi_{1o}(U, B, A, \omega, \kappa, \beta, \gamma) \ , \tag{165}$$

we find

$$\mathcal{L} = \oint \Pi_{1o} dU - 2\pi A \ . \tag{166}$$

The corresponding Euler equations are

$$\oint \frac{\partial \Pi_{1o}}{\partial A} dU = 2\pi \ , \tag{167}$$

$$\oint \frac{\partial \Pi_{1o}}{\partial B} dU = 0 \ , \tag{168}$$

$$\frac{\partial}{\partial X} \left(\oint \frac{\partial \Pi_{1o}}{\partial \kappa} dU\right) - \frac{\partial}{\partial T} \left(\oint \frac{\partial \Pi_{1o}}{\partial \omega} dU\right) = 0 \ , \tag{169}$$

$$\frac{\partial}{\partial X} \left(\oint \frac{\partial \Pi_{1o}}{\partial \beta} dU\right) - \frac{\partial}{\partial T} \left(\oint \frac{\partial \Pi_{1o}}{\partial \gamma} dU\right) = 0 \ . \tag{170}$$

We note here that two more exact secular conditions can be deduced, that is (125) and

$$\oint \frac{\Phi_\theta}{U_\theta} dU = 0 \ , \tag{171}$$

where it is assumed that the integrands are expressed in terms of U and the α_i (A, B, ω, κ, β, γ, and their derivatives).

In the same manner as before we deduce that (167) and (169) are first approximations to (125) and (124). It remains to show that (168) and (170) are the first approximations to (171) and (158). This result follows from the relations

$$\frac{\partial H_o}{\partial \Pi_{1o}} \frac{\partial \Pi_{1o}}{\partial B} + \frac{\partial H_o}{\partial B} = 0 ,$$

$$\frac{\partial H_o}{\partial \Pi_{1o}} \frac{\partial \Pi_{1o}}{\partial \beta} + \frac{\partial H_o}{\partial \beta} = 0, \quad \frac{\partial H_o}{\partial \Pi_{1o}} \frac{\partial \Pi_{1o}}{\partial \gamma} + \frac{\partial H_o}{\partial \gamma} = 0$$

which are derived from (164).

It remains to discuss how to derive higher approximations to (166). The idea is to expand the integrals in $\mathcal{L}$ in the same manner as before, expanding Π_1 and Π_2 from (161) and (160) at the appropriate stage. We also need to solve for U_X and U_T from (156) and Φ_X and Φ_T from (157) as functions of U and the α_i. It remains therefore to solve

$$\Phi_\theta = J(U, \alpha_i), \quad U_\theta = F(U, \alpha_i) ,$$

for Φ_X. It is seen that

$$\Phi = \int_{\bar{U}_1}^{U} \frac{J}{F} \, dU ,$$

and hence

$$\Phi_X = \frac{J}{F} U_X + \frac{\partial}{\partial X} \int_{\bar{U}_1}^{U} \frac{J}{F} \, dU .$$

or

$$\Phi_X = \frac{\partial}{\partial X} \int_{\bar{U}_1}^{U} \frac{J}{F} \, dU - J \frac{\partial}{\partial X} \int_{\bar{U}_1}^{U} \frac{1}{F} \, dU .$$

The analysis now proceeds in the same manner as before. Alternatively we can expand the four exact secular conditons (125), (126), (158), (171).

6. Slowly varying multiply-periodic solutions

We now consider the extension of the previous ideas to deal with equations which admit multiply-periodic solutions. The idea is then to derive more general solutions where the parameters in this solution are allowed to vary slowly. Here we restrict ourselves to a single equation when a multiply-periodic solution takes the form

$$u = U_o(\theta_1, \theta_2 \dots \theta_n) ,$$

where

$$\theta_i = \kappa_i x - \omega_i t, \quad i = 1, 2 \dots n$$

and U_o is periodic in the θ_i. For simplicity of presentation we restrict ourselves to $n = 2$, the extension to arbitrary n is immediate.

The central question here is whether such solutions do in fact exist for a given equation. We shall assume, however, that such a solution is known and display the formalism needed to derive the more general solutions mentioned above.

Thus we look for a solution of the form

$$u = U(\theta_1, \theta_2, X, T, \varepsilon) , \tag{172}$$

where

$$\theta_i = \frac{\Theta_i(X, T, \varepsilon)}{\varepsilon} , \quad \kappa_i = \Theta_{iX} , \quad \omega_i = - \Theta_{iT} ,$$

and U is periodic in the θ_i of period 2π.

The related differential equation is then seen to be

$$\sum_{i=1}^{2} \left\{ \kappa_i \frac{\partial}{\partial \theta_i} \left(\frac{\partial L}{\partial U_x}\right) - \omega_i \frac{\partial}{\partial \theta_i} \left(\frac{\partial L}{\partial U_t}\right) \right\} + \varepsilon \frac{\partial}{\partial X} \left(\frac{\partial L}{\partial U_x}\right) + \varepsilon \frac{\partial}{\partial T} \left(\frac{\partial L}{\partial U_t}\right) - \frac{\partial L}{\partial U} = 0 ,$$

or

$$\sum_{i=1}^{2} \left\{ \frac{\partial}{\partial \theta_i} \left(\frac{\partial L}{\partial U_{\theta i}}\right) \right\} + \frac{\partial}{\partial X} \left(\frac{\partial L}{\partial U_X}\right) + \frac{\partial}{\partial T} \left(\frac{\partial L}{\partial U_T}\right) - \frac{\partial L}{\partial U} = 0 ,$$

where

$$L = L(U, \sum_i \kappa_i U_{\theta i} + \varepsilon U_X, - \sum_i \omega_i U_{\theta i} + \varepsilon U_T) .$$

It is immediately seen that the required related variational principle is

$$\delta \iint \int_0^{2\pi} \int_0^{2\pi} L(U, \sum_i \kappa_i U_{\theta i} + \varepsilon U_X, - \sum_i \omega_i U_{\theta i} + \varepsilon U_T) \, d\theta_1 d\theta_2 dX dT = 0 , \quad (173)$$

where U, Θ_1 and Θ_2 are to be varied independently. Thus if we can determine the dependence of U on θ_1 and θ_2, the slow modulation equations are governed by the averaged Lagrangian

$$\mathcal{L} = \int_0^{2\pi} \int_0^{2\pi} L \, d\theta_1 \, d\theta_2 , \quad (174)$$

although again this form will only be used directly for linear or near-linear problems.

We digress here to consider a class of problems from classical mechanics to illustrate some ideas that may be useful in the above problem. We consider a conservative mechanical system with two degrees of freedom executing a finite motion. Such a system is governed by a Hamiltonian H(p, q). To begin with we restrict ouselves to separable systems, that is systems whose Hamilton-Jacobi equation can be solved by separation of variables.

The existence of multiply-periodic solutions p_i, q_i to such a system is guaranteed by the theory of canonical transformations. It is shown [14] that such a transformation to action-angle variables results in the p_i, q_i being multiply-periodic functions of the angle variables, the action variables being constants. Here we have assumed that the q_i are single valued functions of the position of the system, that is angle type co-ordinates are excluded.

We deduce, from solving the Hamilton-Jacobi equation, that

$$p_i = \bar{P}_i\ (q_i,\ \alpha_j)\ , \qquad i = 1,\ 2\ , \tag{175}$$

where the α_j are constants, and we assume that each $\bar{P}_i$ is zero at the zeros of $q_{i\theta}$, $\bar{P}_i^2$ being single valued.

The canonical transformation then shows that

$$q_i = \bar{Q}_i\ (\theta_1,\ \theta_2,\ I_1,\ I_2) \tag{176}$$

where

$$\theta_j = \nu_j t + \delta_j\ , \qquad j = 1,\ 2$$

and

$$I_j = \oint p_j dq_j = I_j(\alpha_i)\ . \tag{177}$$

The ν_j and δ_j are constants here.

If ν_1 and ν_2 are commensurable then the p_i, q_i are periodic in t. The system is then said to be degenerate.

We now consider such a system in which the parameters are allowed to vary slowly. The motion is governed by Hamilton's principle

$$\delta \int \{\Sigma p_i q_i - H(p, q, \lambda(\varepsilon t))\}\, dt = 0 . \tag{178}$$

We look for solutions of the form

$$q_i = Q_i(\theta_1, \theta_2, T, \varepsilon) , \tag{179}$$

$$p_i = P_i(\theta_1, \theta_2, T, \varepsilon) , \tag{180}$$

where P_i and Q_i are periodic of period 2π in θ_1 and θ_2 and

$$\theta_i = \frac{\Theta_i\ (T, \varepsilon)}{\varepsilon} , \quad \omega_i = \Theta_i'(T, \varepsilon), \quad T = \varepsilon t .$$

The corresponding variational principle is

$$\delta \int \int_0^{2\pi} \int_0^{2\pi} \left\{ \sum_j P_j(\omega_1 Q_{j\theta_1} + \omega_2 Q_{j\theta_2} + \varepsilon Q_T) - H(P, Q, \lambda) \right\} d\theta_1 d\theta_2 dT = 0 . \tag{181}$$

We note that

$$Q_i = \bar{Q}_i(\theta_1, \theta_2, I_1, I_2; \lambda(T)) + O(\varepsilon) , \tag{182}$$

$$P_i = \bar{P}_i(\bar{Q}_1, \bar{Q}_2, I_1, I_2; \lambda(T)) + O(\varepsilon) , \tag{183}$$

and we work to $O(1)$ from now on.

In the usual manner we deduce two conservation laws from (181) which we rewrite as

$$\frac{\partial H}{\partial \theta_1} = \omega_2 \left\{ \frac{\partial(\bar{P}_1, \bar{Q}_1)}{\partial(\theta_1, \theta_2)} + \frac{\partial(\bar{P}_2, \bar{Q}_2)}{\partial(\theta_1, \theta_2)} \right\} , \tag{184}$$

$$\frac{\partial H}{\partial \theta_2} = - \frac{\omega_1}{\omega_2} \frac{\partial H}{\partial \theta_1} , \tag{185}$$

and upon using (183) we conclude

$$H = E(I_1, I_2, \lambda) . \tag{186}$$

Thus we find

$$\mathcal{L} = \int_0^{2\pi} \int_0^{2\pi} \sum_i \bar{P}_i (\omega_1 \bar{Q}_{i\theta_1} + \omega_2 \bar{Q}_{i\theta_2}) \, d\theta_1 d\theta_2 - 4\pi^2 E \tag{187}$$

We now evaluate the integrals

$$J_{ij} = \int_0^{2\pi} \bar{P}_i \bar{Q}_{i\theta_j} d\theta_j , \qquad j \neq i ,$$

and we shall show that $J_{ij} = 0$. It is shown in [15] that if we increase θ_i by 1 then $\bar{Q}_i$ completes a period but $\bar{Q}_j$ returns to its initial value without completing a period. In J_{ij}, θ_i is fixed, and we convert to an integral over $\bar{Q}_i$ to find

$$J_{ij} = \int_C \bar{P}_i d\bar{Q}_i .$$

Here we assume $\theta_j = 0$ corresponds to a zero of $\bar{P}_i$, say $\bar{Q}_{i1}$, and the region of integration C is then from $\bar{Q}_{i1}$ to $\bar{Q}_{i3}$ and back to $\bar{Q}_{i1}$ where $\bar{Q}_{i3} < \bar{Q}_{i2}$; $\bar{Q}_{i2} > \bar{Q}_{i1}$ being the other zero of $\bar{P}_i$. Thus $\bar{P}_i$ is a single valued function of $\bar{Q}_i$ on C and hence $J_{12} = J_{21} = 0$.

Thus (187) becomes

$$\mathcal{L} = \sum_i \omega_i \oint \bar{P}_i d\bar{Q}_i - 2\pi E , \tag{188}$$

or

$$\mathcal{L} = \sum_i \omega_i I_i - 2\pi E . \tag{189}$$

The Euler equations corresponding to (189) are

$$I_i = \text{constant} \ , \tag{190}$$

$$\omega_i = 2\pi \frac{\partial E}{\partial I_i} = \omega_i(\lambda) \ . \tag{191}$$

The first of these equations expresses the fact that the I_i are the adiabatic invariants of the system, while the second is the same as that predicted by the action-angle variables treatment when λ is constant. Obviously our derivation extends to systems of the same type with n degrees of freedom. We note that from (181) we deduce, to $O(1)$,

$$\int_0^{2\pi} \int_0^{2\pi} \sum_i P_i Q_{i\theta_j} \, d\theta_1 d\theta_2 = \text{constant}, \ j = 1, 2, \tag{192}$$

as the Euler equations corresponding to the variations of Θ_1 and Θ_2. If for a system which is not separable, in the co-ordinates considered, a multiply-periodic solution exists, then (192) is also valid in that case.

We finally note that derivations of (190) are given in [14] and [15]. However, Born restricts his result in general to systems where ω_1 and ω_2 are never commensurable. It seems as though in the cases where ω_1 and ω_2 do not satisfy this condition, that the perturbation scheme breaks down. Whether this is due to the fact that no slowly varying multiply-periodic solution exists, or that the perturbation scheme is incorrect, is not clear.

We now return to the waves problem and note that from (173) we immediately deduce the two exact secular conditions

$$\frac{\partial}{\partial X} \left(\int_0^{2\pi} \int_0^{2\pi} \frac{\partial L}{\partial \kappa_i} \, d\theta_1 d\theta_2 \right) - \frac{\partial}{\partial T} \left(\int_0^{2\pi} \int_0^{2\pi} \frac{\partial L}{\partial \omega_i} \, d\theta_1 d\theta_2 \right) = 0 \ . \tag{193}$$

It is not clear, however, how to evaluate (193) without knowledge of the explicit form of the first order solution for fully nonlinear equations.

Appendix D

Here we give an alternative derivation of the related variational principle (12). We look for extremals of the original variational principle of the form

$$u(x, t, \varepsilon) = U(\theta, X, T, \varepsilon) \quad , \tag{D-1}$$

where U is periodic of period 2π in θ. We can therefore expand U, and hence L, in a Fourier series in θ,

$$L = L(U, \kappa U_\theta + \varepsilon U_X, -\omega U_\theta + \varepsilon U_T) = C_o + \sum_{n=1}^{\infty} \{C_n \cos n\theta + S_n \sin n\theta\} \quad , \tag{D-2}$$

where

$$C_o = \frac{1}{2\pi}\int_o^{2\pi} L d\theta, \; C_n = \frac{1}{\pi}\int_o^{2\pi} L \cos n\theta \, d\theta, \; S_n = \frac{1}{\pi}\int_o^{2\pi} L \sin n\theta \, d\theta \; .$$

Thus C_i and S_i are functionals of U, with X, T as parameters, and functions of ω and κ.

We rewrite (12) as

$$\delta \iint L(U, \kappa U_\theta + \varepsilon U_X, -\omega U_\theta + \varepsilon U_T) \, dXdT = 0 \tag{D-3}$$

and use (D-2), where L depends ultimately on X and T, in the form

$$L = C_o + \sum_{n=1}^{\infty} \{C_n \cos \frac{n\Theta}{\varepsilon} + S_n \sin \frac{n\Theta}{\varepsilon}\}$$

We now show that the extremals of

$$\delta \iint C_o \, dXdT = 0 \quad , \tag{D-4}$$

are also the extremals of

$$\delta \iint \{C_n \cos \frac{n\Theta}{\varepsilon} + S_n \sin \frac{n\Theta}{\varepsilon}\} \, dXdT = 0 \,, \tag{D-5}$$

for every n.

We rewrite (D-5) in the form

$$\delta \iint\int_0^{2\pi} L \cos n \, (\theta - \frac{\Theta}{\varepsilon}) \, d\theta dXdT = 0 \,, \tag{D-6}$$

when the Θ variation implies

$$\frac{\partial}{\partial X} (\int_0^{2\pi} \frac{\partial L}{\partial \kappa} \cos n \, (\theta - \frac{\Theta}{\varepsilon}) d\theta) - \frac{\partial}{\partial T} (\int_0^{2\pi} \frac{\partial L}{\partial \omega} \cos n \, (\theta - \frac{\Theta}{\varepsilon}) \, d\theta)$$

$$- \frac{n}{\varepsilon} \int_0^{2\pi} L \sin n \, (\theta - \frac{\Theta}{\varepsilon}) d\theta = 0 \,,$$

or

$$\int_0^{2\pi} [\frac{\partial}{\partial X} (\frac{\partial L}{\partial \kappa}) - \frac{\partial}{\partial T} (\frac{\partial L}{\partial \omega})] \cos n \, (\theta - \frac{\Theta}{\varepsilon}) d\theta$$

$$+ \frac{n}{\varepsilon} \int_0^{2\pi} [\kappa \frac{\partial L}{\partial \kappa} + \omega \frac{\partial L}{\partial \omega} - L] \sin n \, (\theta - \frac{\Theta}{\varepsilon}) d\theta = 0 \,.$$

The second integral in this equation is then integrated by parts to leave

$$\int_0^{2\pi} U_\theta [\frac{\partial}{\partial \theta} (\frac{\partial L}{\partial U_\theta}) + \frac{\partial}{\partial X} (\frac{\partial L}{\partial U_X}) + \frac{\partial}{\partial T} (\frac{\partial L}{\partial U_T}) - L] \cos n \, (\theta - \frac{\Theta}{\varepsilon}) d\theta = 0$$

which is satisfied by the extremals of (D-4) since the variation of U in (D-4) yields (10).

The variation of U in (D-6) then shows that

$$\cos n \, (\theta - \frac{\Theta}{\varepsilon}) \, [\frac{\partial}{\partial \theta} (\frac{\partial L}{\partial U_\theta}) + \frac{\partial}{\partial X} (\frac{\partial L}{\partial U_X}) + \frac{\partial}{\partial T} (\frac{\partial L}{\partial U_T}) - L]$$

$$- n \sin n \, (\theta - \frac{\Theta}{\varepsilon}) \, [\frac{\partial L}{\partial U_\theta} - \frac{\kappa}{\varepsilon} \frac{\partial L}{\partial U_X} + \frac{\omega}{\varepsilon} \frac{\partial L}{\partial U_T}] = 0 \,.$$

The second bracket vanishes identically and the first vanishes along the extremals of (D-4).

Thus the extremals $U(\theta, X, T)$ of

$$\delta \iiint_{0}^{2\pi} L(U, \kappa U_{\theta} + \varepsilon U_{X}, - \omega U_{\theta} + \varepsilon U_{T}) \, d\theta dX dT = 0 ,$$

when evaluated as functions of X, T, are extremals of (D-3).

Appendix E

We discuss the derivation of a Korteweg-de Vries type equation from (35) and (36) for the case $\gamma > 0$. We combine (35) and (36) so that the first order derivative terms are in characteristic form, that is

$$\bar{\kappa}_T + \left\{ \frac{\kappa_o}{\omega_o} + \varepsilon \left(\frac{\bar{\kappa}}{\omega_o^3} \pm \frac{\beta a^{n-1}}{\omega_o^2} \right) \right\} \bar{\kappa}_X \pm 2\omega_o \beta a^{n-2} \left[a_T + \left\{ \frac{\kappa_o}{\omega_o} + \varepsilon \left(\frac{\bar{\kappa}}{\omega_o^3} \pm \frac{\beta a^{n-1}}{\omega_o^2} \right) \right\} a_X \right]$$

$$= \frac{\varepsilon}{2\omega_o} \frac{\partial}{\partial X} \left(\frac{a_{XX} - a_{TT}}{a} \right) . \qquad \text{(E-1)}$$

We integrate along the backward facing characteristic and assume constant initial conditions to find

$$a = \left\{ \frac{\bar{\kappa} + \alpha}{\beta} \right\}^{1/(n-1)} + 0(\varepsilon) \; , \quad \bar{\beta} = \frac{2\omega_o \beta}{n-1} \; , \quad \beta = \surd[\bar{\gamma} n(n-1)] \; ,$$

where α is a constant.

We then write

$$a = \left\{ \frac{\bar{\kappa} + \alpha}{\beta} \right\}^{1/(n-1)} + \varepsilon b(X, T, \varepsilon) , \qquad \text{(E-2)}$$

and choose b so that (35) and (36) are identical to $0(\varepsilon)$. Thus we find

$$\hat{k}_T + \left[\frac{\kappa_o}{\omega_o} + \frac{\varepsilon\alpha(n-1)}{2\omega_o^3} + \frac{\varepsilon(n+1)\bar{\beta}\hat{k}^{n-1}}{2\omega_o^3} \right] \hat{k}_X = \frac{\varepsilon}{2\omega_o \beta (n-1) \hat{k}^{n-2}} \frac{\partial}{\partial X} \left(\frac{\hat{k}_{XX} - \hat{k}_{TT}}{\hat{k}} \right) , \qquad \text{(E-3)}$$

where

$$\hat{k} = \left(\frac{\bar{\kappa} + \alpha}{\beta} \right)^{1/(n-1)} = a + 0(\varepsilon) \; .$$

An attempt to derive an equation in a leads to an equation without the higher derivative terms. The corresponding value of b is given by

$$b_T + \frac{\kappa_o}{\omega_o} b_X = - \frac{1}{2\omega_o \bar{\beta} \hat{k}^{n-2}} \frac{\partial}{\partial X} \left(\frac{\hat{k}_{XX} - \hat{k}_{TT}}{\hat{k}} \right) . \tag{E-4}$$

We again look for stationary solutions of the form

$$\hat{k} = \hat{k}(\eta) \ , \quad \eta = X - VT \ , \quad V = \frac{\kappa_o}{\omega_o} + \varepsilon V_1 \ ,$$

and it is immediately seen that such solutions cannot lead to stationary solutions for b if b = O(1). Thus although we find stationary solutions for $\bar{\kappa}$ there are no corresponding stationary solutions for a. However, given $\hat{k}$ it is a simple matter to solve (E-4) for b and we have therefore derived more general solutions.

In fact

$$b = \int_o^T F(\sigma + \varepsilon V_1 s) ds \ , \tag{E-5}$$

where $\sigma = X - (\kappa_o/\omega_o)T$ is held fixed in the integration and

$$F(\eta) = - \frac{(1-V^2)}{2\omega_o \bar{\beta} \hat{k}^{n-2}} \left(\frac{\hat{k}''}{\hat{k}} \right)' .$$

The equation governing the stationary solutions is

$$\hat{k}'' = \hat{k} \left\{ \mu \hat{k}^{2(n-1)} + \nu \hat{k}^{(n-1)} + \delta \right\} , \tag{E-6}$$

where $\mu > 0$ is a known constant, $\nu = \nu(V_1)$ and δ is a constant of integration. This equation is analyzed in the phase-plane and similar types of solutions to those found in the previous analysis are discovered. That is, periodic solutions both passing through zero and bounded away from zero;

solitary waves and step function type solutions. Here solutions of the last type exist which tend to zero and ℓ_o as η tends to $\pm\infty$. It is not clear what significence the last solutions have; they cannot be dismissed out of hand here as in the previous case. Their relevence is probably best investigated by considering the case $\alpha^{2n-1} = \varepsilon$, when the nonlinearity appears at $O(1)$ in (E-3). If these solutions persist for that situation then they may have some relevence to the shocks found by Whitham. However, (E-3) will again be reversible while the shocks are irreversible so it is not clear what the connection is. This case has not been investigated, however.

Appendix F

We derive the adiabatic invariant to all orders for a conservative mechanical system with one degree of freedom, executing a finite motion, when a parameter of the system is slowly varied. It is assumed that the co-ordinate describing the system is not an angle type co-ordinate.

The motion is governed by Hamilton's principle

$$\delta \int L \{q, \dot{q}; \lambda(\varepsilon t)\} \, dt = 0 , \tag{F-1}$$

and when λ is constant the motion is periodic. Thus we look for solutions of the form

$$q = Q(\theta, T, \varepsilon) , \tag{F-2}$$

where

$$\theta = \frac{\Theta(T, \varepsilon)}{\varepsilon} , \quad \omega = \Theta' , \quad \Theta' > 0 , \quad T = \varepsilon t ,$$

and Q is periodic of period 2π in θ. The corresponding variational principle is

$$\delta \int \int_0^{2\pi} L(Q, \omega Q_\theta + \varepsilon Q_T, \lambda(T)) \, d\theta dT = 0 . \tag{F-3}$$

We convert to a "pseudo" Hamiltonian formalism by putting

$$R = \omega Q_\theta ,$$

and

$$P = \frac{\partial L}{\partial R} = \frac{\partial L}{\partial \dot{q}} . \tag{F-4}$$

An equivalent variational principle to (F-3) is

$$\delta \int \int_0^{2\pi} (\omega P Q_\theta - H)\, d\theta dT = 0 \ , \tag{F-5}$$

where

$$H(P, Q, \varepsilon Q_T; \lambda) = PR - L(Q, R + \varepsilon Q_T; \lambda) , \tag{F-6}$$

and R is solved as

$$R = R(P, Q, \varepsilon Q_T, \lambda) \ , \tag{F-7}$$

from (F-4), in defining H. We note that P is the usual momentum but that H is not the usual Hamiltonian.

In (F-5) P, Q and Θ are varied independently and the Θ variation gives the exact secular condition

$$\int_0^{2\pi} P Q_\theta d\theta = \text{constant} \ , \tag{F-8}$$

since H is independent of ω. If P can be solved as a function of Q and the α_i (E, ω and their derivatives) then (F-8) becomes

$$\oint P dQ = \text{constant} \ . \tag{F-9}$$

We now show how to solve for P as a function of the above variables and then deduce (F-9) from the averaged Lagrangian.

The fact that H is explicitly independent of θ leads to the conservation law

$$\frac{\partial H}{\partial \theta} = \frac{\partial}{\partial T} (Q_\theta H_{Q_T}) \ , \tag{F-10}$$

or

$$\frac{\partial H}{\partial \theta} = - \varepsilon \frac{\partial}{\partial T} (PQ_\theta) , \qquad \text{(F-11)}$$

since $H_{Q_T} = - L_{Q_T} = - \varepsilon P$. If (F-11) is averaged over a period of θ then (F-8) results.

We integrate (F-11) to find

$$H = E(X, T, \varepsilon) - \varepsilon \frac{\partial}{\partial T} (\int_0^\theta PQ_\theta d\theta) , \qquad \text{(F-12)}$$

and therefore

$$\mathcal{L} = \omega \oint PdQ - 2\pi E . \qquad \text{(F-13)}$$

We then rewrite (F-12) as

$$H = E - \varepsilon PQ_T - \varepsilon \frac{\partial}{\partial T} \int_{\hat{Q}}^{Q} PdQ . \qquad \text{(F-14)}$$

We note, however, that (F-4) implies

$$P = P(Q, R + \varepsilon Q_T, \lambda) ,$$

which is solved for R to yield

$$R = S(P, Q, \lambda) - \varepsilon Q_T .$$

Thus (F-6) becomes

$$H = PS(P, Q, \lambda) - \varepsilon PQ_T - L(Q, S, \lambda) ,$$

$$= \bar{H}(P, Q, \lambda) - \varepsilon PQ_T ,$$

and therefore (F-14) simplifies to

$$\bar{H}(P, Q, \lambda) = E - \varepsilon \frac{\partial}{\partial T} \left(\int_{\hat{Q}}^{Q} P dQ \right) . \tag{F-15}$$

We now solve for P from this equation and we note that P is independent of ω and its derivatives. Thus the variation of Θ in the averaged variational principle leads to (F-9). The expansion of $\oint PdQ$ proceeds as before where we put $\bar{H}(0, \hat{Q}, 0) = 0$.

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