PART I

ON THE BREAKING OF NONLINEAR DISPERSIVE WAVES

PART II

VARIATIONAL PRINCIPLES IN CONTINUUM MECHANICS

Thesis by

Robert L. Seliger

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1968

(Submitted November 10, 1967)

ACKNOWLEDGEMENTS

The author wishes to thank Professor G. B. Whitham for suggesting these topics and for providing guidance, many useful suggestions and invaluable criticism during the course of this research.

The financial support of the author's graduate education was attained through the Fellowship Programs of the Hughes Aircraft Company. The author is sincerely grateful for having participated in these programs and would like to thank all the people who made this possible. The author especially thanks the Fellowship representative with whom he had most contact, Mrs. D. McClure of the Hughes Research Laboratories, for consistently providing accurate, efficient and outstandingly pleasant administration of the Programs. Special thanks also go to Mr. J. H. Molitor of the Hughes Research Laboratories for complementing the author's academic education with insight into the real world and for being so generously receptive to the toils and troubles of students.

Finally, the author acknowledges an extremely competent and cheerful typist, Miss Ann Law of the Hughes Research Laboratories for typing this dissertation.

ABSTRACT

A model equation for water waves has been suggested by Whitham to study, qualitatively at least, the different kinds of breaking. This is an integro-differential equation which combines a typical nonlinear convection term with an integral for the dispersive effects and is of independent mathematical interest. For an approximate kernel of the form $e^{-b|x|}$, it is shown first that solitary waves have a maximum height with sharp crests and secondly that waves which are sufficiently asymmetric break into "bores." The second part applies to a wide class of bounded kernels, but the kernel giving the correct dispersion effects of water waves has a square root singularity and the present argument does not go through. Nevertheless the possibility of the two kinds of breaking in such integro-differential equations is demonstrated.

Difficulties arise in finding variational principles for continuum mechanics problems in the Eulerian (field) description. The reason is found to be that continuum equations in the original field variables lack a mathematical "self-adjointness" property which is necessary for Euler equations. This is a feature of the Eulerian description and occurs in non-dissipative problems which have variational principles for their Lagrangian description. To overcome this difficulty a "potential representation" approach is used which consists of transforming to new (Eulerian) variables whose equations are self-adjoint. The transformations to the

velocity potential or stream function in fluids or the scaler and vector potentials in electromagnetism often lead to variational principles in this way. As yet no general procedure is available for finding suitable transformations. Existing variational principles for the inviscid fluid equations in the Eulerian description are reviewed and some ideas on the form of the appropriate transformations and Lagrangians for fluid problems are obtained. These ideas are developed in a series of examples which include finding variational principles for Rossby waves and for the internal waves of a stratified fluid.

TABLE OF CONTENTS

PART		PAGE
I.	ON THE BREAKING OF NONLINEAR DISPERSIVE WAVES	
	1. Introduction	1
	2. Solutions with Sharp Crests	3
	3. Solutions which Break into Bores	7
II.	VARIATIONAL PRINCIPLES IN CONTINUUM MECHANICS	
	4. Introduction	12
	5. The Self-Adjointness Condition of Vainberg . . .	18
	6. A Variational Formulation of Inviscid Fluid Mechanics	25
	7. Variational Principles for Rossby Waves in a Shallow Basin and in the "β-Plane" Model . . .	37
	8. The Variational Formulation of a Plasma	49
	9. Variational Principles for the Internal Waves of a Stratified Fluid	53
	APPENDIX A — The Singularity in $K_g(x)$	59
	APPENDIX B — An Estimate of $\phi(x, t)$ for a Breaking Wave	61

PART I

ON THE BREAKING OF NONLINEAR DISPERSIVE WAVES

1. Introduction

An integro-differential equation has been proposed by Whitham [1] which offers an improvement on the well known Korteweg-de Vries equation for water waves. This integral equation can be considered an extension of the (relatively long wave) Korteweg-de Vries model to include all the dispersion present in linear water wave theory. The Korteweg-de Vries equation [2] is

$$\eta_t + (\alpha\eta + c_o)\,\eta_x + \gamma\,\eta_{xxx} = 0,$$

where η is the elevation of the water surface above the undisturbed depth h_o, $\alpha = 3c_o/2h_o$, $c_o = \sqrt{gh_o}$, and $\gamma = \frac{1}{6}c_o h_o^2$. This equation is valid for water waves whose typical amplitude a and wavelength λ are such that a/h_o and h_o^2/λ^2 are comparable small quantities. The form of the proposed extension is

$$\eta_t + \alpha\eta\eta_x + \int_{-\infty}^{\infty} K(x-\xi)\,\eta_\xi(\xi,t)\,d\xi = 0\,. \qquad (1)$$

---

1. G.B. Whitham, Proc. Roy. Soc. A, Vol. 299 (1967) pp. 6-25.
2. D.J. Korteweg and G. de Vries, Phil. Mag., Vol. 39 (1895), pp. 422-443.

For water waves the appropriate kernel is the Fourier transform of the linear water wave phase speed $c(k)$;

$$K(x) = K_g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{ikx} dk, \quad c(k) = \left(\frac{g}{k} \tanh k h_o\right)^{1/2} .$$

If only the first two terms $c_o - \gamma k^2$, in the long wave $(kh_o \ll 1)$ expansion of $c(k)$ are used, equation (1) reduces to the Korteweg-de Vries equation. It is known that two kinds of breaking are observed for water waves: formation of sharp crests and breaking into bores. Although the Korteweg-de Vries equation has solitary wave and cnoidal wave train solutions, breaking of solutions into sharp crests or bores has not been found. Apparently the η_{xxx} term prevents these high frequency effects from occurring. No theory has yet been given which demonstrates both types of breaking. The more general equation (1) was proposed with the hope for finding these high frequency effects.

The water wave kernel $K_g(x)$ has a square root singularity [3] which makes the integral in (1) difficult to handle. To gain insight into the behavior of the solutions to such integral equations, the kernel will be approximated by functions of the form $e^{-b|x|}$.

---

3. See Appendix A.

2. Solutions with Sharp Crests

Uniformly propagating solutions of (1) are found with $\eta = \eta(\chi)$, where $\chi = x - Ut$. For these solutions (1) can be integrated once to the form

$$A - U\eta + \frac{1}{2} a\eta^2 + \int_{-\infty}^{\infty} K(\chi - \xi)\, \eta(\xi)\, d\xi = 0 , \tag{2}$$

where A is the constant of integration. Oscillatory and solitary wave solutions can be found for (2). But unlike the Korteweg-de Vries equation, (2) has maximum amplitude wave solutions which have sharp crests. Solitary wave solutions can be distinguished from oscillatory wave trains by requiring that $\eta > 0$ and $\eta \to 0$ as $|\chi| \to \infty$. Thus, to study solitary waves we take $A = 0$. Then, by writing (2) in the form

$$\left(\frac{1}{2} a\eta - U\right)\eta + \int_{-\infty}^{\infty} K(\chi - \xi)\, \eta(\xi)\, d\xi = 0 , \tag{3}$$

it is seen that for a positive kernel, the height of solitary wave solutions must be less than $2\frac{U}{a}$.

More detailed investigations of (3) can be made for special kernels. Whitham [4] discusses the occurrence of a sharp crest in the maximum amplitude solitary wave solution to (2) for a kernel of the form $e^{-b|x|}$. This limiting solution can be found explicitly.

---

4. *Ibid.*

For the kernel $e^{-b|x|}$ (3) becomes a second order ordinary differential equation when operated on by $\left(\frac{d^2}{dx^2} - b^2\right)$. The resulting equation can be integrated once and written in the form

$$\left(\eta - \frac{U}{a}\right)^2 \eta'^2 = \frac{1}{4} b^2 \eta^2 (\eta - \eta_1)(\eta - \eta_2), \tag{4}$$

where the constant of integration is again zero (for solitary waves). If the precise kernel used to obtain (4) is

$$K_{ab} = \frac{3}{4} ab\, e^{-b|x|},$$

then η_1 and η_2 are the two roots of the quadratic equation

$$(a\eta)^2 + 4(a - U)\, a\, \eta + 2U(2U - 3a) = 0$$

and are given by the formulas

$$\eta_{\substack{1\\2}} = \frac{2a}{a} \left\{ \left(\frac{U}{a} - 1\right) \pm \sqrt{1 - \frac{U}{2a}} \right\}.$$

Solitary wave solutions of (4) occur for real roots of the same sign. It can be shown that a family of smooth solitary wave solutions exists when the speed U lies in the range

$$\frac{3}{2} a < U < 2a .$$

This corresponds to roots η_1, η_2 which satisfy

$$0 < \eta_2 < \frac{U}{\alpha} < \eta_1 .$$

These smooth solutions can be given implicitly in terms of Jacobian elliptic functions and have maximum height equal to η_2. The speed $U = \frac{3}{2}a$ gives $\eta_2 = 0$ and the trivial solution. As $U \to 2a$ both roots approach the value U/α. When $U = 2a$ the maximum amplitude solitary wave occurs. Equation (4) reduces to

$$\eta'^2 = \frac{1}{4} b^2 \eta^2$$

which gives a solution of the form $e^{-\frac{b}{2}|x|}$. It can be verified directly that

$$\eta = \frac{U}{\alpha} e^{-\frac{b}{2}|x|}, \quad U = 2a \tag{5}$$

is a solution of the integral equation (3) for the kernel $K_{ab}(x)$. This solitary wave solution has a finite angle at the crest and agrees qualitatively with observed water wave profiles [5].

To check the properties of this solution quantitatively Whitham [6] chooses $a = 2/3$ and $b = \pi/2$ so that the kernel $K_{ab}(x)$ models the water wave kernel $K_g(x)$. (Of course, the approximation is poor near $x = 0$ where K_g is singular.) The value $\alpha = 3/2$ is taken to agree with the Korteweg-de Vries equation

---

5. See, for example, J. J. Stoker, Water Waves, sec. 10.10. New York: Interscience (1957).

6. Ibid.

and $g = h_o = 1$. Now the limiting solitary wave solution (5) becomes

$$\eta = \frac{8}{9} e^{-\frac{\pi}{4}|x|}$$

which has a crest angle of 110° and a maximum height of 8/9. Whitham points out that although these values agree reasonably well with Stokes' 120° angle and McCowan's maximum height 0.78, the angle result should not be taken seriously since the angle size depends on the local behavior of the kernel near $x = 0$. The maximum height result depends on the whole kernel and may be taken more seriously.

3. Solutions which Break into Bores

For approximate kernels progress can also be made on proving that solutions of (1) can break asymmetrically into bores. The analysis is carried through for kernels which are bounded, integrable, even functions of x which approach zero monotonically as x approaches infinity. This includes a kernel such as $e^{-b|x|}$ but does not include $K_g(x)$ because of the singularity at $x = 0$. Sufficient conditions on the initial profile are found which guarantee that the corresponding solution to (1) breaks. The approach is motivated from the hyperbolic equation

$$\eta_t + \alpha\eta\eta_x + \beta\eta = 0, \quad \beta > 0 . \tag{6}$$

In this case, a method for seeing how solutions can break is to study the differential equation for the maximum value of $-\eta_x$. Let $m(t)$ denote this most negative slope and $m(t) = \eta_x(X(t), t)$, where $X(t)$ satisfies $\eta_{xx}(X(t), t) = 0$. The ordinary differential equation for $m(t)$

$$\frac{dm}{dt} = -m(\alpha m + \beta), \tag{7}$$

is then obtained by differentiating (6) with respect to x and setting $x = X(t)$. The solution of (7) is

$$m(t) = \frac{\beta}{\alpha} \frac{m_o}{\left(m_o + \frac{\beta}{\alpha}\right)e^{\beta t} - m_o} ,$$

where $m_o = m(0)$ is the most negative slope on the initial profile $\eta(x, 0)$. It is seen that if $m_o < \frac{-\beta}{\alpha}$ breaking occurs $(m(t) \to \infty)$ as $t \to \frac{1}{\beta} \log \left\{(\alpha m_o)/(\alpha m_o + \beta)\right\}$. The significance of the breaking condition $m_o < -\beta/\alpha$ is that it ensures that the right hand side of (7), and thus dm/dt, is negative initially. With $m(0)$ and $dm(0)/dt$ negative, $m(t)$ continues to decrease and the dominant term $-\alpha m^2$ gives $m \to -\infty$ in a finite time.

This same kind of dynamical breaking is found for the nonlinear dispersive equation (1), but the initial conditions required to start the steepening process are different and the argument has to be extended. It is clear that important differences come in when one takes into account that (1), unlike (6), has nonbreaking solutions with steady profiles. The oscillatory nature of these solutions indicates that consideration of the maximum positive slope will also be required. Let $m_-(t)$ be the most negative slope of the solution $\eta(x, t)$ to (1). As before $m_-(t) = \eta_x(X_-(t), t)$, where $X_-(t)$ satisfies $\eta_{xx}(X_-(t), t) = 0$. However, now m_- satisfies the equation

$$\frac{dm_-}{dt} = -\alpha m_-^2 - \phi_-(t)\,, \tag{8}$$

where

$$\phi(x, t) = \int_{-\infty}^{\infty} K(\xi)\eta_{xx}(x-\xi, t)d\xi, \quad \phi_-(t) = \phi(X_-(t), t)\,.$$

Equation (8) governs breaking for the dispersive equation (1) as (7) does for the hyperbolic equation (6). If steepening is to occur in a similar way the $-\alpha m_-^2$ term must dominate $\phi_-(t)$. Bounds on ϕ_- are therefore needed for this study and, since $\phi(x, t)$ is linear in η the situation looks hopeful. Using the second mean value theorem[7], $\phi(x, t)$ can be written as

$$\phi(x, t) = K(0) \left| \eta_x(\xi_1, t) - \eta_x(\xi_2, t) \right| ,$$

where ξ_1 and ξ_2 are two numbers which depend on x and t and satisfy $\xi_1 \le x$, $\xi_2 \ge x$. A linear bound for $\phi(x, t)$ in terms of $m_-(t)$ and the maximum value of the positive slope $m_+(t)$ is then

$$|\phi(x, t)| \le K(0) \; (m_+(t) - m_-(t)) \; ; \tag{9}$$

this inequality holds for all x. The essential difference between (7) and (8) is the presence of $m_+(t)$ in the bound for ϕ_-. The function $m_+(t)$ enters in such a way that it makes the right hand side of (8) more positive and therefore deters breaking. An estimate for $m_+(t)$ is found by considering its equation,

$$\frac{dm_+}{dt} = -\alpha m_+^2 - \phi_+(t) , \tag{10}$$

---

7. See, for example, E. W. Hobson, The Theory of Functions of a Real Variable, Vol. 1, p. 618. New York: Dover (1927).

where $\phi_+(t) = \phi(X_+(t), t)$ and $X_+(t)$ satisfies $\eta_{xx}(X_+(t), t) = 0$. From the estimate in (9), the derivatives of $m_-(t)$ and $m_+(t)$ can be bounded by

$$\frac{dm_-}{dt} \le -am_-^2 + K(0)(m_+ - m_-), \tag{11}$$

$$\frac{dm_+}{dt} \le -am_+^2 + K(0)(m_+ - m_-). \tag{12}$$

Then

$$\frac{d}{dt}(m_+ + m_-) \le (m_+ - m_-)\Big(2K(0) + a(m_+ + m_-)\Big) - 2am_+^2 .$$

If $m_+ + m_- < -2K(0)/a$ initially, then $m_+ + m_-$ decreases further and

$$m_+ + m_- < -\frac{2K(0)}{a} \quad \text{for all } t > 0. \tag{13}$$

Using this estimate for m_+ in (11), we have

$$\frac{dm_-}{dt} \le -am_-^2 - 2K(0)m_- - \frac{2K(0)^2}{a} = -a\left(m_- + \frac{K(0)}{a}\right)^2 - \frac{K(0)^2}{a},$$

from which

$$\frac{d}{dt}\left(m_- + \frac{K(0)}{a}\right) \le -a\left(m_- + \frac{K(0)}{a}\right)^2$$

follows. Let $q(t) = -\left(m_-(t) + \frac{K(0)}{a}\right)$ and note that $q > 0$ when (13) is satisfied. Therefore $dq/dt \ge aq^2$ and

$$\frac{d}{dt}\left(\frac{1}{q}\right) \leq -\alpha ,$$

$$\frac{1}{q} \leq -\alpha t + \frac{1}{q_o} ,$$

where $q = q_o > 0$ at $t = 0$. Finally, for q we have

$$q \geq \frac{q_o}{1 - \alpha q_o t} \rightarrow \infty \text{ as } t \rightarrow \frac{1}{\alpha q_o}$$

We conclude that if the initial wave profile is sufficiently asymmetrical so that $m_+(0) + m_-(0) < -2K(0)/\alpha$, the slope $m_-(t)$ will become infinite in a time less than $|\alpha m_-(0) + K(0)|^{-1}$. This is a sufficient condition only, not a sharp criterion.

For the kernel $K_g(x)$ a proof that solutions of (1) can break into bores is not yet available. One would like to carry through similar arguments, but the extension of the proof is difficult; the singularity in $K_g(x)$ prevents the a priori estimate (9) of the integral ϕ_- in terms of slopes. However, even with the infinity in $K_g(x)$ it seems likely that ϕ_- is dominated by the term $-\alpha m_-^2$ in (8) and can not prevent breaking of sufficiently asymmetric waves. A partial indication of this is that if a typical breaking profile is substituted into the integral it can be shown [8] that $\phi_- = o(|m_-|^{7/4})$ as $|m_-| \rightarrow \infty$.

---

8. See Appendix B.

PART II

VARIATIONAL PRINCIPLES IN CONTINUUM MECHANICS

4. Introduction

The variational formulation of classical mechanics provides an elegant setting in which the deep ideas of dynamics, such as canonical transformations and adiabatic invariants, can be concisely expressed. Hopefully, a variational formulation of continuum mechanics would be equally as fruitful. But the slow progress in this direction indicates that new problems arise. When Hamilton's principle is applied to the particles of a continuum, a variational formulation does result which correctly governs motion of the continuum as described by Lagrangian (or particle) coordinates. But in many continuum problems, for example in fluid mechanics, it is preferable to study the equations which correspond to Eulerian (or field) coordinates. To obtain these equations as they stand as the Euler equations of some variational principle is generally impossible. The reason is that in their original form the continuum equations for field quantities generally do not satisfy a self-adjointness condition. This condition is an extension of the usual self-adjointness property to nonlinear operators and was established as a necessary and sufficient condition for Euler equations by Vainberg [9]. This is why existing variational

---

9. M.M. Vainberg, Variational Methods for the Study of Nonlinear Operators. Transl. by A. Feinstein, Holden-Day (1964).

principles usually appear in terms of auxiliary quantities (e.g., the velocity potential and stream function in fluids or the vector and scaler potentials in electromagnetism) whose equations are self-adjoint.

Equations which are not self-adjoint can only come indirectly from variational principles. Several different approaches to indirect formulations are possible. A formulation in which the Lagrangian for any equation of the form

$$Mu(x) = 0 \tag{14}$$

is immediately known, is to vary u in the functional

$$J[u] = \int (Mu)^2 \, dx \, . \tag{15}$$

Since $J[u]$ takes on its minimum value zero for solutions of (14), its Euler equation must be satisfied by solutions of (14). However, the actual Euler equation for $J[u]$ is more complicated than equation (14). For a linear operator L the functional $J[u]$ becomes the inner product (Lu, Lu) whose Euler equation is $L^*Lu = 0$, where L^* is the operator adjoint to L. In the case of the heat equation,

$$u_{xx} - u_t = 0 \, ,$$

the functional $J[u]$ is

$$\iint (u_{xx} - u_t)^2 \, dxdt$$

and the corresponding Euler equation is

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) u = 0 .$$

Thus, in this approach the functional to be varied is easily formulated and the difficulty is that information about the original equation is hidden in a more complicated Euler equation.

Another indirect variational technique is to vary only part of the Lagrangian. To illustrate this approach we consider the equation

$$\nabla \cdot (\lambda(T) \nabla T) = 0 \tag{16}$$

which arises in problems of steady-state heat conduction in a solid. In such problems (16) is to be solved for the temperature distribution $T_o(\underline{x})$ in a body. The kind of variational principle that we have in mind now is to find a functional $\Phi[T_o, T]$, depending on the two functions T_o and T, for which the condition

$$\left\{\frac{\delta \Phi(T_o, T)}{\delta T}\right\}_{T_o} = 0 \tag{17}$$

leads to equation (16). The meaning of (17) is that T is varied in $\Phi(T_o, T)$ while T_o is treated as a known function. Then, in the resulting Euler equation, T is set equal to T_o. One functional which leads to (16) in this way is

$$\Phi(T_o, T) = \frac{1}{2}\int \lambda_o T_o^2 \left\{\nabla\left(\frac{1}{T}\right)\right\}^2 d\underline{x} , \tag{18}$$

where $\lambda_o = \lambda(T_o)$. Varying T in (18) gives

$$\nabla \cdot \left(\lambda_o T_o^2 \; \nabla\left(\frac{1}{T}\right)\right) = 0 \, ,$$

which becomes (16) with $T = T_o$. Another functional which also leads to (16) by this method is given by the integral

$$\int \lambda_o \, \nabla T_o \cdot \nabla T \, d\underline{x} \; . \qquad (19)$$

The above example is taken from a paper by Glansdorff and Prigogine [10] where a theory is presented which arrives at the functional $\Phi(T_o, T)$ given in (18). Indeed some basis for choosing the most advantageous functional for a given equation would be desirable. In this kind of variational technique the equations obtained are precisely the ones of interest but the functionals used depend on the solutions of the problems.

The approach to be considered here for finding indirect variational principles is to introduce new dependent variables by means of "potential" representations which result in self-adjoint equations. Variational principles in terms of the potentials are then found which have Euler equations equivalent to the original non-self-adjoint set. The ideas of adiabatic invariants and canonical transformations do occur in continuum problems for which potential type variational formulations are used. In his

---

10. P. Glansdorff and I. Prigogine, Physica, Vol. 30 (1964), pp. 351-374.

averaging method for nonlinear dispersive equations, Whitham [11] finds that a prohibitive amount of algebra is avoided if the whole theory is obtained from a variational formalism. The variational principles he uses are of this potential representation type and the ideas of his theory then follow the lines of adiabatic invariants in classical Lagrangian-Hamiltonian mechanics. Clebsch [12] gives the first variational formulation of rotational fluid flow by introducing a velocity representation of the form

$$\underline{u} = \nabla \phi + \alpha \nabla \beta .$$

Under this transformation the new equations of motion can be put in a canonical form which resembles the classical Hamilton equations [13]. The difficult steps in the potential representation method are to find a suitable representation and to find a Lagrangian which leads to the appropriate equations for the potentials. A very general (one to one) representation is not always required. For special problems such as irrotational flow, the limited velocity potential representation $\underline{u} = \nabla\phi$ is adequate and the work in finding the variational formulation is reduced. There

---

11. G.B. Whitham, Proc. Roy. Soc. A, Vol. 283 (1965), pp. 238-261. J. Fluid Mech., Vol. 22 (1965), pp. 273-283.

12. A. Clebsch, J. reine u. angew. math., Vol. 56 (1859), pp. 1-10.

13. See, for example, H. Lamb, Hydrodynamics, 6 ed., p. 249, New York: Dover (1932).

are, however, interesting examples for which more general potential representations are required. This is true for Rossby waves and internal waves which are examples of nonlinear dispersive waves that can only exist in rotational fluid flows. The model equations used to study these waves are approximate forms of general equations for an inviscid fluid. Variational formulations for these examples are found (Section 4, 6) by first studying the variational formulation for a general inviscid fluid (Section 3) and then suitably modifying the potential representation and Lagrangian.

The discussion begins (Section 2) with a brief account of Vainberg's necessary and sufficient self-adjointness condition for Euler equations and its application to differential equations.

5. The Self-Adjointness Condition of Vainberg

Necessary and sufficient conditions for a class of nonlinear equations to be the Euler equations of a variational principle have been given by Vainberg [14]. Although Vainberg's work deals mostly with nonlinear integral operators, his results are easily applied to the partial differential equations of physics with which we are concerned. This theory will now be discussed in terms of a single ordinary differential equation of the form

$$F(u, u_x, u_{xx}, x) = 0 , \tag{20}$$

where F is a polynomial function of the dependent variable u, its first and second derivatives, and the independent variable x. Such a limited model example can be used because extension of the ideas to systems of nonlinear partial differential equations is straightforward. Vainberg found that the conditions for (20) to result from the variation of some functional are analogous to the conditions for a vector function to be the gradient of some scalar function. It is well known that in three dimensions a vector can be expressed as the gradient of a scalar if its curl is zero or, by Stoke's theorem, if the line integral of the vector around any closed curve is zero. To generalize the "curl" condition the notion of the derivative of an operator is needed. Here it is useful to define

$$\lim_{\sigma \to 0} \frac{N(u + \sigma h) - N(u)}{\sigma} \equiv DN(u;h)$$

---

14. Loc. cit.

as the derivative of the operator N at the point u, in the direction h. For operators such as F in (20), one recognizes that $DF(u;h)$ is just the operator F linearized about u operating on h. The analogy to the "curl" condition is that for an operator F to come from varying some functional, F must satisfy the symmetry condition

$$(DF(u;h_1), h_2) = (DF(u;h_2), h_1) \tag{21}$$

at each point u and for all h_1 and h_2. The inner products in (21) are usual integrals over x. Condition (21) means that when F is linearized about any point the resulting operator is self-adjoint. For linear operators $F = Lu$, $DF(u;h) = Lh$, and (21) reduces to the usual definition of self-adjointness. Vainberg also defines the concept of a curvelinear integral in function space for an operator and shows that the independence of this integral on path determines when the operator is an Euler equation. The definitions of these function space concepts will not be needed here. We just note for motivation that if the operator F has the path independence property, its integral along any curve in function space, between the points 0 and u, say, must equal the straight line integral in function space between 0 and u. But this straight line integral can be written explicitly as

$$\int_0^1 (F(\lambda u), u)\, d\lambda = J[u] , \tag{22}$$

where again the inner product is an integral over x. Thus the functional $J[u]$ in (22) should be an excellent candidate for a Lagrangian for F.

It will now be shown that (21) is indeed a necessary and sufficient condition for (20) to come from a variational principle, and that the Lagrangian is given in (22). Suppose that F satisfies the self-adjointness condition (21). Computing $\delta J[u]$ from (22) gives

$$\delta J[u] = \int_0^1 \int_{x_1}^{x_2} \left\{ F(\lambda u, \lambda u_x, \lambda u_{xx}, x)\delta u + \lambda u DF(\lambda u; \delta u) \right\} dx d\lambda \ .$$

By the self-adjointness of F it follows that

$$\int_{x_1}^{x_2} \lambda u \, DF(\lambda u; \delta u) dx = \int_{x_1}^{x_2} \lambda \delta u \, DF(\lambda u; u) dx$$

$$= \int_{x_1}^{x_2} \lambda \frac{d}{d\lambda} F(\lambda u, \lambda u_x, \lambda u_{xx}, x) \, \delta u \, dx;$$

hence

$$\delta J[u] = \int_0^1 \int_{x_1}^{x_2} \left\{ F(\lambda u, \lambda u_x, \lambda u_{xx}, x) + \lambda \frac{d}{d\lambda} F(\lambda u, \lambda u_x, \lambda u_{xx}, x) \right\} \delta u dx d\lambda \ .$$

Assuming that the variations δu vanish at x_1 and x_2, interchange the order of integration and integrate the second term by parts with respect to λ to obtain

$$\delta J[u] = \int_{x_1}^{x_2} F(u, u_x, u_{xx}, x)\, \delta u \, dx.$$

The variational principle $\delta J[u] = 0$ therefore leads to (20) as its Euler equation. Conversely, suppose that $F = 0$ is the Euler equation from the variational principle

$$\delta \int_{x_1}^{x_2} \mathcal{L}(u, u_x, x)\, dx = 0 .$$

Then F must have the form

$$F(u, u_x, u_{xx}, x) = \mathcal{L}_u - \left(\mathcal{L}_{u_x}\right)_x . \tag{23}$$

The derivative of F is now given by

$$DF(u;h) = \mathcal{L}_{uu} h + \mathcal{L}_{uu_x} h_x - \left(\mathcal{L}_{uu_x} h + \mathcal{L}_{u_x u_x} h_x\right)_x .$$

It is a straightforward calculation to show that

$$h_1 \, DF(u;h) = h \, DF(u;h_1) ,$$

from which the self-adjointness of F follows.

When F is the expression in (23) $J[u]$ becomes

$$J[u] = \int_0^1 \int_{x_1}^{x_2} u \left\{ \mathcal{L}_u(\lambda u, \lambda u_x, x) - \left(\mathcal{L}_{u_x}(\lambda u, \lambda u_x, x) \right)_x \right\} dx\, d\lambda .$$

The second term can be integrated by parts to give

$$J[u] = \int_0^1 \int_{x_1}^{x_2} \left\{ u\mathcal{L}_u + u_x \mathcal{L}_{u_x} - \left(u\mathcal{L}_{u_x} \right)_x \right\} dx\, d\lambda$$

$$= \int_0^1 \int_{x_1}^{x_2} \left\{ \frac{d}{d\lambda} \mathcal{L}(\lambda u, \lambda u_x, x) - \left(u\mathcal{L}_{u_x} \right)_x \right\} dx\, d\lambda$$

$$= \int_{x_1}^{x_2} \mathcal{L}(u, u_x, x) dx - \int_{x_1}^{x_2} \mathcal{L}(0, 0, x) dx - \int_0^1 \Big[u\mathcal{L}_{u_x}(\lambda u, \lambda u_x, x) \Big]_{x_1}^{x_2} d\lambda .$$

Since the second and third integrals do not contribute to the variations, $J[u]$ is equivalent to the original Lagrangian.

To extend the above theory to systems of nonlinear partial differential equations it is convenient to introduce vector notation. Denote a system of partial differential equations by

$$\underline{F}(\underline{u}, \underline{x}) = 0 \tag{24}$$

where

$$\underline{F}(\underline{u}, \underline{x}) = (F_1(\underline{u}, \underline{x}),\ F_2(\underline{u}, \underline{x}),\ \ldots,\ F_n(\underline{u}, \underline{x}))\ ,$$

$$\underline{u} = (u_1, u_2, \ldots, u_n),$$

$$\underline{x} = (x_1, x_2, \ldots, x_n)\ .$$

The F_i are again polynomial functions of the dependent variables $u_1, u_2, \ldots, u_n$, their derivatives and the independent variables $x_1, x_2, \ldots, x_n$. The derivative of the vector operator $\underline{F}$ is now defined by

$$D\underline{F}(\underline{u};\underline{h}) = (DF_1(\underline{u};\underline{h}),\ DF_2(\underline{u};\underline{h}),\ \ldots,\ DF_n(\underline{u};h))\ ,$$

$$DF_i(\underline{u};\underline{h}) = \lim_{\sigma \to 0} \frac{F_i(\underline{u} + \sigma\,\underline{h}) - F_i(\underline{u})}{\sigma}\ .$$

The self-adjointness condition for $\underline{F}$ is then

$$(D\underline{F}(\underline{u};\underline{h}_1), \underline{h}_2) = (D\underline{F}(\underline{u};\underline{h}_2), \underline{h}_1)\ ,$$

where the inner product is defined by

$$(\underline{g}, \underline{h}) = \int \cdots \int \left(\sum_{i=1}^{n} g_i h_i \right) dx_1\, dx_2 \ldots dx_n\ .$$

Using this definition of the inner product, the generalization of the functional J[u] in (22) is

$$J[u] = \int_0^1 (\underline{F}(\lambda \underline{u}), \underline{u}) \, d\lambda \, .$$

With these definitions the results obtained for (20) are also true for (24).

6. A Variational Formulation of Inviscid Fluid Mechanics

Variational principles for non-dissipative fluid flows described in terms of Lagrangian (or particle) coordinates are known to be the generalization of the classical Hamilton's principle for a system of particles. The appropriate Lagrangian density has the form of kinetic energy — internal energy for a fluid particle and is a function of the position and velocity of the particle. In these variational principles the particle paths themselves are varied to give the necessarily self-adjoint Lagrangian equations for the fluid. Variational principles for fluid flows described in the more usual Eulerian (field) coordinates are more difficult. The fluid equations in Eulerian coordinates are not self-adjoint as they stand and therefore cannot be the Euler equations of a variational principle in which the field quantities are varied. For this reason the most common variational principles in fluid mechanics appear for special flows in which new variables, such as the velocity potential or stream function, can be introduced. The advantage is that when the Eulerian equations are written in terms of these auxiliary variables, they often become self-adjoint. Although one usually looks at specialized flows, knowing a change in variables which leads to variational principles for the general fluid equations is useful when treating new problems.

Clebsch [15] introduced the representation

---

15. <u>Loc. cit.</u>

$$\underline{u} = \nabla\phi + \alpha \nabla\beta \tag{25}$$

and was successful in finding a variational formulation for incompressible rotational flow. He was motivated to introduce the potentials ϕ, α, β in the form of (25) from the results of Pfaff's theorem [16] which states that the differential form

$$udx + vdy + wdz \tag{26}$$

is reducible to a form

$$d\phi + \alpha d\beta . \tag{27}$$

Equating the expressions in (26) and (27) leads to Clebsch's representation (25). Clebsch finds that a variational formulation of the equations

$$\left.\begin{aligned} \frac{D\underline{u}}{Dt} &= -\nabla p , \\ \nabla \cdot \underline{u} &= 0 , \end{aligned}\right\} \tag{28}$$

where

$$\underline{u} = (u, v, w) ,$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla ,$$

---

16. See, for example, A.R. Forsyth, Theory of Differential Equations, Vol. 1, New York: Dover (1959).

is obtained by introducing the representation

$$\underline{u} = \nabla\phi + \alpha \nabla \beta , \tag{29}$$

$$p = - \left\{ \phi_t + \alpha \beta_t + \frac{1}{2} (\nabla \phi + \alpha \nabla \beta)^2 \right\} \tag{30}$$

for the dependent variables. From (28) it follows that the equations to be satisfied by ϕ, α, β are

$$\left.\begin{aligned} \frac{D\alpha}{Dt} &= 0 \\ \frac{D\beta}{Dt} &= 0 \\ \nabla\cdot \underline{u} &= 0 \end{aligned}\right\} \tag{31}$$

where $\underline{u}$ stands for the expression in (29). Equations (31) follow from varying ϕ, α, β in the variational principle

$$\delta \iint \left\{ \phi_t + \alpha\beta_t + \frac{1}{2} (\nabla\phi + \alpha\nabla\beta)^2 \right\} d\underline{x}\, dt = 0 .$$

Here, as in all of the subsequent variational principles considered, the Lagrangian is an integral over a fixed region of $(\underline{x}, t)$ space and the variations are assumed to vanish on the boundary of this region. This is the first instance in which the pressure appears as the Lagrangian density. As will be seen in the later examples, the pressure, expressed in terms of the potentials used, is the desired Lagrangian density for problems in fluid mechanics described by Eulerian coordinates, much the way the kinetic energy — potential energy, expressedin terms of coordinates and velocities, is the Lagrangian for problems in classical mechanics.

The potentials α, β account for circulation in the flow. Expressions for the vorticity components in terms of α and β are

$$\xi = w_y - v_z = \frac{\partial(\alpha, \beta)}{\partial(y, z)} ,$$

$$\eta = u_z - w_x = \frac{\partial(\alpha, \beta)}{\partial(z, x)} ,$$

$$\zeta = v_x - u_y = \frac{\partial(\alpha, \beta)}{\partial(x, y)} ,$$

$$\underline{\omega} = (\xi, \eta, \zeta) = \nabla\alpha \times \nabla\beta .$$

The intersection of surfaces α = constant, β = constant represents a vortex line. The first two equations in (31) indicate that α and β are constant following particles. Therefore vortex lines move with the fluid and always carry the same particles. The role of the potentials α, β as canonical variables is discussed by Lamb [17]. This property is also discussed by

---

17. Loc. cit.

Clebsch [18] with his ideas on canonical transformations of these potentials. Here, it will only be remarked that if an arbitrary function $H(\alpha, \beta, t)$ is added to the pressure in (30) the equations to be satisfied by the new potentials take on the canonical form

$$\frac{D\alpha}{Dt} = H_\beta(\alpha, \beta, t) ,$$

$$\frac{D\beta}{Dt} = - H_\alpha(\alpha, \beta, t) ,$$

and follow from varying the integral of the new expression for the pressure.

An extension of the above variational formulation to a compressible fluid is given by Bateman [19]. His results are for a so-called baratropic fluid in which the pressure is a function of the density. The equations for compressible flow are

$$\rho \frac{D\underline{u}}{Dt} = - \nabla p , \qquad (32)$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0 , \qquad (33)$$

---

18. Loc. cit.

19. H. Bateman, Partial Differential Equations, p. 164, Cambridge: Cambridge University Press (1964).

and for convenience $p(\rho)$ is taken to be of the form

$$p = \rho f'(\rho) - f(\rho) \quad . \tag{34}$$

To obtain a variational formulation for (32) and (33), Bateman uses the representation

$$\underline{u} = \nabla\phi + \alpha\nabla\beta \, , \tag{35}$$

$$-p = \rho\left\{\phi_t + \alpha\beta_t + \frac{1}{2}(\nabla\phi + \alpha\nabla\beta)^2\right\} + f(\rho) \quad . \tag{36}$$

Now the potentials ϕ, α, β must satisfy the equations

$$\left.\begin{aligned} \frac{D\alpha}{Dt} &= 0 \, , \\ \frac{D\beta}{Dt} &= 0 \, , \\ \frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} &= 0 \, , \end{aligned}\right\} \tag{37}$$

where $\underline{u}$ stands for the expression in (35). Again, the variation of ϕ, α and β in the integral of the expression for the pressure in (36) leads to the set (37). The variation of ρ in this integral leads to (34).

The first variational formulation of fluid mechanics in the Eulerian description which includes all the effects of compressibility, circulation and entropy variation was given by C. C. Lin [20] who has a somewhat constructive technique for finding the potential representation. The equations of interest are

$$\left.\begin{aligned} \rho \frac{D\underline{u}}{Dt} &= -\nabla p , \\ \frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} &= 0, \\ \frac{DS}{Dt} &= 0 , \\ p &= p_o(\rho, S) . \end{aligned}\right\} \quad (38)$$

Lin proposes that the Lagrangian have the form kinetic energy — internal energy as in the case where a Lagrangian (particle) description of the fluid is used. Expecting that the variation of this Lagrangian should lead to the momentum equations, Lin uses the other two equations, conservation of mass and entropy, as side conditions. A third side condition, which Lin calls "conservation of the identity of particles," is also included to account for the fact that the "unnatural" Eulerian description is being used. This condition has the form $D\underline{X}/Dt = 0$, where $\underline{X} = \underline{X}(\underline{x}, t)$ denotes

---

20. C. C. Lin, Scoula internazionale di fisica, Varenna Italy, no. 21, p. 91, Academic Press (1963). See also J. Serrin, Encyclopedia of Physics (ed. S. Flugge), Vol. VIII/1, p. 125. Berlin: Springer-Verlag (1959).

the initial position of the particle which is at position $\underline{x}$ at time t. The side conditions can be built into the variational principle by using Lagrange multipliers. The resulting variational principle has the form

$$\delta \iint \left[L + \phi \left(\rho_t + \nabla \cdot (\rho \underline{u}) \right) - \rho \beta \frac{DS}{Dt} - \rho \underline{\gamma} \cdot \frac{D\underline{X}}{Dt} \right] d\underline{x}\, dt = 0 , \tag{39}$$

where ϕ, β, $\underline{\gamma}$ are Lagrange multipliers,

$$L = \frac{1}{2} \rho \underline{u}^2 - \rho E(\rho, S)$$

and $E(\rho, S)$ is the specific internal energy. The independent variations of $\underline{u}$, ρ, S, $\underline{X}$ then lead to Euler equations which can be put in the form

$$\left.\begin{aligned}
\underline{u} &= \nabla \phi + \beta \nabla S + \nabla \underline{X} \cdot \underline{\gamma} , \\
h &= - \left(\phi_t + \beta S_t + \underline{\gamma} \cdot \underline{X}_t + \frac{1}{2} \underline{u}^2 \right) \\
\frac{D\beta}{Dt} &= E_S = T , \\
\frac{D\underline{\gamma}}{Dt} &= 0 ,
\end{aligned}\right\} \tag{40}$$

where $h = E + p/\rho$ is the enthalpy and use has been made of the thermodynamic equation

$$\left[\frac{\partial E(\rho, S)}{\partial \rho}\right]_S = \frac{p(\rho, S)}{\rho^2} \quad .$$

A direct calculation shows that the momentum equations follow from the set (40). Herivel [21] gives the formulation in (39) and (40) without the condition $D\underline{X}/Dt = 0$. He arrives at the velocity representation $\underline{u} = \nabla\phi + \beta \nabla S$ and notes that it is a particular case of the form $\underline{u} = \nabla\phi + \alpha \nabla \beta$ given by Clebsch [22]. Although Herivel's representation does allow for some rotational flows, in cases of uniform entropy only potential flows are represented. It was this restriction which Lin [23] wanted to overcome when he added the third side condition $D\underline{X}/Dt = 0$ to the variational principle (39). In the more general representation

$$\underline{u} = \nabla\phi + \beta \nabla S + \nabla \underline{X} \cdot \underline{\gamma}$$

which results, the term $\beta \nabla S$ leads to vorticity due to entropy variation and the three terms $\nabla\underline{X} \cdot \underline{\gamma}$ account for other sources of rotation.

It can be shown that the Lin formulation goes through with only a single equation as the third side condition. That is, when the scalar side condition $DX/Dt = 0$ is used in place of $D\underline{X}/Dt = 0$

---

21. J. W. Herivel, Proc. Cambridge Phil. Soc., Vol. 51 (1955) pp. 344-349.

22. Loc. cit.

23. Loc. cit.

in (39), with a scalar Lagrange multiplier γ, the expression for the velocity in (40) becomes

$$\underline{u} = \nabla\phi + \beta\nabla S + \gamma\nabla X . \tag{41}$$

The rest of the equations in (40) remain unchanged and the momentum equations still follow.

An alternate view of Lin's formulation can be taken in which the side conditions are eliminated from the variational principle. For this approach one begins by assuming the first two equations in (40) as representations for $\underline{u}$ and p. Instead of using these equations as they stand, the more concise expression for $\underline{u}$ given in (41) is used and the notation for the potentials is changed so that the representation is assumed in the form

$$\left.\begin{aligned} \underline{u} &= \nabla\phi + \alpha\nabla\beta + \xi\nabla\eta , \\ S &= \xi , \\ p &= -\rho\left\{E(\rho,\xi) + \phi_t + \alpha\beta_t + \xi\eta_t + \tfrac{1}{2}\underline{u}^2\right\} \end{aligned}\right\} \tag{42}$$

where E is a known function which satisfies $\rho^2 E_\rho(\rho, S) = p_o(\rho, S)$. From equations (38), it follows that the potentials $\phi, \alpha, \beta, \xi, \eta$ must satisfy the equations

$$\left.\begin{aligned}
\frac{D\alpha}{Dt} &= 0 \ , \\
\frac{D\beta}{Dt} &= 0 \ , \\
\frac{D\xi}{Dt} &= 0 \ , \\
\frac{D\eta}{Dt} &= -E_{\xi} \ , \\
\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} &= 0 \ ,
\end{aligned}\right\} \qquad (43)$$

where $\underline{u}$ stands for the expression in (42). The set (43) follows from varying ϕ, α, β, ξ, η in the variational principle

$$\delta\iint\rho\left\{E(\rho,\xi) + \phi_t + \alpha\beta_t + \xi\eta_t + \frac{1}{2}(\nabla\phi + \alpha\nabla\beta + \xi\nabla\eta)^2\right\} d\underline{x}\, dt = 0, \qquad (44)$$

where again the Lagrangian density is the pressure. The variation of ρ in (44) gives the equation of state $p = p_o(\rho, S)$. This variational formulation includes the formulations of Clebsch and Bateman described earlier in this section.

Two important ideas which appear in the general fluid formulation and will be used for finding variational formulations for the examples which follow are: (i) use a velocity representation of the form

$$\underline{u} = \nabla\phi + \sum_i \alpha_i \nabla\beta_i \ ,$$

where pairs of potentials α_i, β_i are used for the different sources of vorticity in the problem ; (ii) the Lagrangian density is the pressure suitably expressed in terms of the potentials.

7. Variational Principles for Rossby Waves in a Shallow Basin and in the "β-Plane" Model

Rossby waves, also called planetary waves, provide an example of wave motion whose existence depends explicitly on the rotation of a fluid. To give some idea of the physical significance of these waves, we will briefly note some results of past investigations. Planetary waves were first found by Laplace and Hough in their work on the theory of tides. Hough [24] shows that the small oscillations of a thin layer of fluid on a rotating sphere under the action of gravitational and Coriolis forces, are divided into two classes. The first are the usual gravity waves modified to some extent by the rotation. The others, called "oscillations of the second class" by Hough, are distinguished from gravity waves by the properties that their frequency is less than twice the angular frequency Ω of the sphere and is proportional to Ω in the limit $\Omega \rightarrow 0$. Rossby [25] shows that these second-class oscillations are important in the theory of atmosphere flow. He was able to isolate the second-class modes by considering horizontal nondivergent flow in a coordinate system which roughly approximates a spherical globe. By means of this so-called "β-plane" model, Rossby gave the first clear physical interpretation of these waves: the northward displacement of a

---

24. S.S. Hough, Phil. Trans.A, Vol. 191 (1898), pp. 139-185.

25. C.G. Rossby, J. Mar. Res., Vol. 2 (1939), pp. 38-55.

fluid element gives rise to a restoring force due to the increase in the vertical component of angular velocity. Extensions of Rossby's model were made by Haurwitz [26], and more recently by Longuet-Higgins [27], who also studies the validity of the β-plane approximation.

Rossby waves can also occur in rotating basins of variable depth. Lamb [28] shows that slow second-class oscillations are included in the solutions of the linearized shallow water equations when applied to a rotating basin. For the special basin whose depth is proportional to its radius squared, Phillips [29] finds that the solution of Lamb's equations is expressible in terms of elementary functions and that for low frequencies the dispersion relation is similar to the one for Rossby waves in the β-plane.

As regards finding a variational formulation for Rossby waves, it is convenient to begin with the shallow basin model. A variational principle is first given for the shallow water equations. Then the rotation is put in by changing to a coordinate system rotating with the fluid. The transformed equations and variational principle are applicable to Rossby waves. Finally, by noting the

---

26. B. Haurwitz, J. Mar. Res., Vol. 3 (1940), pp. 35-50.

27. M.S. Longuet-Higgins, Proc. Roy. Soc. A, Vol. 279 (1964), pp. 446-473. Proc. Roy. Soc. A, Vol. (1965), pp. 40-68.

28. Loc. cit. p. 326.

29. N.A. Phillips, Tellus, Vol. XVII (1965), pp. 295-301.

similarity between the β-plane equations for Rossby waves and the rotating shallow water equations, a variational formulation of the β-plane model is given.

The shallow water equations written in cylindrical polar coordinates (r, θ) are

$$\left.\begin{aligned} \eta_t + \frac{1}{r}\left\{ru(h+\eta)\right\}_r + \frac{1}{r}\left\{v(h+\eta)\right\}_\theta &= 0, \\ u_t + uu_r + \frac{vu_\theta}{r} - \frac{v^2}{r} &= -g\eta_r , \\ v_t + \frac{vv_\theta}{r} + \frac{uv}{r} + uv_r &= -\frac{1}{r} g \eta_\theta , \end{aligned}\right\} \quad (45)$$

where (u, v) are the velocity components of the fluid in the directions (r, θ), η is the free surface elevation and h is depth of the bottom. A variational formulation for these equations is found by applying the ideas discussed in the previous section. The last two equations in (45) look like momentum equations for an incompressible flow with pressure $g\eta$. Accordingly, the Clebsch type representation (see Section 6)

$$\left.\begin{aligned} u &= \phi_r + \alpha\beta_r \\ v &= \frac{1}{r}(\phi_\theta + \alpha\beta_\theta) \\ g\eta &= -\left\{\phi_t + \alpha\beta_t + \frac{1}{2}(u^2+v^2)\right\} \end{aligned}\right\} \quad (46)$$

is introduced. It follows from (45) that the equations to be satisfied by the potentials α, β, ϕ are

$$\left.\begin{aligned} &\eta_t + \frac{1}{r}\left\{ru(h+\eta)\right\}_r + \frac{1}{r}\left\{v(h+\eta)\right\}_\theta = 0, \\ &\alpha_t + u\alpha_r + \frac{v}{r}\alpha_\theta = 0, \\ &\beta_t + u\beta_r + \frac{v}{r}\beta_\theta = 0, \end{aligned}\right\} \quad (47)$$

where u, v, η are replaced by their representations given in (46). The appropriate Lagrangian density for system (47) is the pressure. In the shallow water approximation the pressure p is the integral of the hydrostatic pressure $g(\eta - z)$, i.e.,

$$p = \int_{-h}^{\eta} g(\eta - z)dz = \frac{1}{2}g(h+\eta)^2 .$$

Equations (47) are obtained when ϕ, α, β are varied in the variational principle

$$\delta\iiint \frac{1}{2}g\left\{h - \frac{1}{g}\left|\phi_t + \alpha\beta_t + \frac{1}{2}(\phi_r + \alpha\beta_r)^2 + \frac{1}{2r^2}(\phi_\theta + \alpha\beta_\theta)^2\right|\right\}^2 r\,dr\,d\theta\,dt = 0.$$

To include the rotation, let the bottom be a concave surface of revolution which rotates about its axis with angular velocity Ω. Under the transformation to (primed) coordinates rotating with the fluid

$$\left.\begin{aligned} u' &= u , & v' &= v - \Omega r \\ \eta' &= \eta - \frac{\Omega^2 r^2}{2g} , & h' &= h + \frac{\Omega^2 r^2}{2g} \\ \theta' &= \theta - \Omega t , & t' &= t, \quad r' = r , \end{aligned}\right\} \quad (48)$$

the shallow water equations (45) become

$$\left.\begin{aligned} &\eta_t + \frac{1}{r}\left\{ru(h+\eta)\right\}_r + \frac{1}{r}\left\{v(h+\eta)\right\}_\theta = 0 , \\ &u_t + uv_r + \frac{vu_\theta}{r} - \frac{v^2}{r} - 2\Omega v = -g\eta_r , \\ &v_t + \frac{vv_\theta}{r} + \frac{uv}{r} + uv_r + 2\Omega u = -\frac{1}{r} g\eta_\theta , \end{aligned}\right\} \quad (49)$$

where the primes have been dropped and now u, v are the radial and axial components of the velocity in a frame (r, θ) rotating with the fluid; η is the free surface elevation above the undisturbed surface $\Omega^2 r^2/2g$ and h is the depth of the basin below the undisturbed free surface. When linearized, equations (49) contain Rossby wave solutions. A variational formulation for this system is found by using the modified representation [30]

---

30. For clarity, primes are kept on the potentials which represent the variables in the rotating frame.

$$\left.\begin{aligned} u &= \phi'_r + \alpha'\beta'_r \,, \\ v &= \frac{1}{r}\,(\phi'_\theta + \alpha'\beta'_\theta) - \Omega r \\ g\eta &= -\left\{\phi'_t + \alpha'\beta'_t + \frac{1}{2}\,(u^2+v^2)\right\} . \end{aligned}\right\} \quad (50)$$

From (49) it follows that the equations to be satisfied by ϕ', α', β' are the same as equations (47) provided that u, v, η stand for their new expressions in (50). These equations for the primed potentials still come from varying the integral of the pressure which now has the form

$$\frac{1}{2}g\left\{h - \frac{1}{g}\left[\phi'_t + \alpha'\beta'_t + \frac{1}{2}(\phi'_r + \alpha'\beta'_r)^2 + \frac{1}{2r^2}(\phi'_\theta + \alpha'\beta'_\theta - \Omega r^2)^2\right]\right\}^2 r\,dr\,d\theta\,dt \,.$$

Disregarding for the moment that the variables in equations (45) and (49) stand for different quantities, we note that equations (45) are identical to equations (49) when the Coriolis terms are added. Moreover, the complete variational formulation of (49) is obtained from the formulation of (45) by only a minor change (the term $-\Omega r$ in the expression for v) in the representation.

The equations for Rossby waves in an elementary "β-plane" model are [31]

---

31. Loc. cit. Longuet-Higgins.

$$\left.\begin{aligned} u_t + uu_x + vu_y - 2\,\Omega\, f(y)\, v &= -p_x , \\ v_t + uv_x + vv_y + 2\,\Omega\, f(y)\, u &= -p_y , \\ u_x + v_y &= 0 . \end{aligned}\right\} \qquad (51)$$

They are an attempt to locally approximate a thin shell of fluid located between two concentric rotating spheres, by a flow between two parallel planes. Thus, u, v are the velocity components of the fluid in the directions of the rectangular coordinates x, y which increase to the east and north respectively. The Coriolis parameter $2\,\Omega\, f(y)$ is an approximation to the true one $2\,\Omega \cos \phi$ in spherical coordinates which varies with the latitude angle ϕ. Rossby [32] notes that this variation is small near the equator and takes $2\,\Omega\, f'(y) = \beta$ a constant, in his model. Here, and in the next paragraph, β is used as this constant of the "β-plane" model and should not be confused with the potential β.

The last equation in (51) is satisfied identically when the velocity (u, v) is derived from a stream function ψ by the equations

$$u = \psi_y, \qquad v = -\psi_x .$$

---

32. Loc. cit.

Eliminating the pressure then leads to the equation

$$\nabla^2 \psi_t + \psi_y \nabla^2 \psi_x - \psi_x \nabla^2 \psi_y + \beta \psi_x = 0. \tag{52}$$

It is interesting to observe that equation (52) has exact plane wave solutions of the form $e^{i(kx + my - \omega t)}$; the nonlinear terms cancel out. A uniform eastward flow can also be included so that (52) has solutions of the form

$$\psi = Uy - A \cos my \sin k(x - ct),$$

where

$$c = U - \frac{\beta}{m^2 + k^2}$$

Rossby considered the case $m = 0$ and noted that standing waves occur at the critical wave number $k_s = \sqrt{\beta/U}$. Waves with wavelengths greater than $2\pi/k_s$ move to the west, and shorter waves move to the east.

A variational principle for (52) would be most convenient for studying Rossby waves. But, as it stands, not even the linear terms in (52) are self-adjoint. By the substitution $x = \zeta_t$ the linear terms become

$$\nabla^2 \zeta_{tt} + \beta \zeta_{xt}$$

which are self-adjoint and come from a variational principle of the form

$$\delta \iiint \frac{1}{2} \left\{ (\nabla \zeta_t)^2 - \beta \zeta_x \zeta_t \right\} dxdydt = 0 .$$

The nonlinear terms become

$$\zeta_{yt} \nabla^2 \zeta_{xt} - \zeta_{xt} \nabla^2 \zeta_{yt};$$

but in spite of the symmetry of the derivatives, they are not self-adjoint and therefore cannot come from a variational principle in terms of ζ.

A variational formulation has been found for Rossby waves in the β-plane model by introducing a Clebsch-type representation. Aside from the Coriolis terms, equations (51) are just incompressible flow equations. As in the rotating basin problem, the modification needed to include the Coriolis terms occurs when a velocity corresponding to a vorticity equal to $2\Omega f(y)$ is subtracted from the standard Clebsch representation. Accordingly, either velocity representation

$$\text{(i)} \quad \begin{bmatrix} u = \phi_x + \alpha\beta_x \\ v = \phi_y + \alpha\beta_y - 2\Omega x f(y) \end{bmatrix} \quad \text{or} \quad \text{(ii)} \quad \begin{bmatrix} u = \phi_x + \alpha\beta_x + 2\Omega F(y) \\ v = \phi_y + \alpha\beta_y, \; f(y) = F'(y) \end{bmatrix}$$

can be used along with

$$p = - \left\{ \phi_t + \alpha\beta_t + \frac{1}{2}(u^2 + v^2) \right\}$$

to obtain a variational formulation for (51). In either case the Lagrangian density is the pressure and both sets of potentials must satisfy equations of the form

$$\alpha_t + u\alpha_x + v\alpha_y = 0 ,$$

$$\beta_t + u\beta_x + v\beta_y = 0 ,$$

$$u_x + v_y = 0 ,$$

with u and v appropriately given by (i) or (ii).

Alternative forms of the above variational principles can be given which lead to equations more suitable for linearization. When the potentials are set to zero in (i) or (ii), fluid motion remains. This motion is eliminated by transforming to new potentials. In (ii), no motion corresponds to

$$\phi = 0, \qquad \alpha = 2\,\Omega F(y), \qquad \beta = -x .$$

Upon introducing the new potentials A, B by the equations

$$\alpha = A + \sqrt{2\Omega}\ F(y), \quad \beta = B - x\sqrt{2\Omega}\ ,$$

representation (ii) can be written in the form

$$\left. \begin{aligned} u &= \Phi_x + AB_x - A\sqrt{2\Omega}\ , \\ v &= \Phi_y + AB_y - B\sqrt{2\Omega}\ f(y)\ , \end{aligned} \right\} \qquad (53)$$

where

$$\Phi = \phi + B\sqrt{2\Omega}\ f(y)\ .$$

The expression for the pressure then becomes

$$p = -\left\{ \Phi_t + AB_t + \frac{1}{2}(u^2 + v^2) \right\}. \qquad (54)$$

The equations to be satisfied by Φ, A, B are

$$\left. \begin{aligned} A_t + uA_x + vA_y + \sqrt{2\Omega}\ f(y)\ v &= 0\ , \\ B_t + uB_x + vB_y - \sqrt{2\Omega}\ u &= 0\ , \\ u_x + v_y &= 0\ , \end{aligned} \right\} \qquad (55)$$

where u, v stand for the expressions in (53). The set (55) follows from the variation of Φ, A, B in the pressure integral:

$$\iiint \Big[\Phi_t + A B_t + \frac{1}{2} (\Phi_x + A B_x - \sqrt{2\Omega}\, A)^2$$

$$+ \frac{1}{2} (\Phi_y + A B_y - \sqrt{2\Omega}\, f(y)\, B)^2 \Big] \, dxdydt$$

8. The Variational Formulation for a Plasma

In the previous section variational formulations were found for examples of fluid flow which contain rotation. The equations of motion for such flows are complicated by the presence of Coriolis terms of the form $\underline{\omega} \times \underline{u}$. Variational formulations were found for these equations by using a velocity representation obtained by subtracting a velocity $\underline{q}$, such that $\underline{\omega} = \text{curl } \underline{q}$, from Clebsch-type representations. The equation of motion for a charged particle moving in a magnetic field $\underline{B}$ will contain a term $\underline{B} \times \underline{u}$ corresponding to the Lorentz force. In view of the previous examples, it is expected that this term could be included in a variational formulation by using a representation of the form

$$\underline{u} = \nabla\phi + \alpha\nabla\beta - \underline{A} ,$$

where $\underline{A}$ is the magnetic vector potential (i.e., $\underline{B} = \text{curl } \underline{A}$). Using this idea we find a variational principle for the equations of a gas which consists of several species of charged particles, when the different species interact only through the electromagnetic fields.

Let $\underline{u}_i(\underline{x}, t)$ and $n_i(\underline{x}, t)$ be the Eulerian velocity and number density of particles of species i which have mass m_i and charge e_i. The equations to be assumed for the particles in species i are the inviscid fluid equations:

$$\left.\begin{aligned}
&\frac{Dn_i}{Dt} + n_i \, \nabla \cdot \underline{u}_i = 0 \, , \\
&m_i \, \frac{D\underline{u}_i}{Dt} + \frac{e_i}{c} \, \underline{B} \times \underline{u}_i = e_i \underline{E} - \frac{1}{n_i} \nabla p_i \\
&\frac{DS_i}{Dt} = 0 \, , \\
&p_i = p_{o_i} \, (n_i, S_i) \, ;
\end{aligned}\right\} \quad (56)$$

no summation is implied by repeated subscripts. Although the pressure terms in (56) can often be neglected in plasma problems, as in the case of the so-called "collisionless" plasma, the terms p_i will be included here since the variational formulation goes through without extra difficulty. The electric and magnetic fields satisfy the Maxwell equations

$$\left.\begin{aligned}
&\nabla \cdot \underline{B} = 0 \, , \\
&\nabla \cdot \underline{E} = 4\pi \sum_i e_i n_i \, , \\
&\nabla \times \underline{E} + \frac{1}{c} \underline{B}_t = 0 \, , \\
&\nabla \times \underline{B} - \frac{1}{c} \underline{E}_t = \frac{4\pi}{c} \sum_i e_i n_i \underline{u}_i \quad .
\end{aligned}\right\} \quad (57)$$

Hopefully, as with the previous examples, having a variational formulation for these general equations will help in finding variational principles for the model equations which arise in specific problems.

A variational principle for equations (56) and (57) is found by representing the variables $\underline{u}_i$, S_i, p_i, $\underline{E}$, $\underline{B}$ in terms of the potentials ϕ_i, α_i, β_i, ξ_i, η_i, $\underline{A}$, ψ via the equations

$$\left.\begin{aligned} \underline{u}_i &= \nabla\phi_i + \alpha_i \nabla\beta_i + \xi_i \nabla\eta_i - \frac{e_i}{m_i}\underline{A}, \\ S_i &= \xi_i \\ p_i &= -n_i\left\{e_i\psi + m_i\left\{\mathcal{E}_i(n_i,\xi_i) + \frac{\partial\phi_i}{\partial t} + \alpha_i\frac{\partial\beta_i}{\partial t} + \xi_i\frac{\partial\eta_i}{\partial t} + \frac{1}{2}\underline{u}_i^2\right\}\right\} \end{aligned}\right\} \quad (58)$$

where each $\mathcal{E}_i$ is a known function which satisfies

$$n_i^2 \frac{\partial\mathcal{E}_i}{\partial n_i}(n_i, S_i) = p_{o_i}(n_i, S_i) .$$

The representations in (58) for $\underline{u}_i$, S_i and p_i are similar to the inviscid fluid case at the end of Section 6. The term $-\frac{e_i}{m_i}\underline{A}$ now in the expression for $\underline{u}_i$ accounts for the "Coriolis-like" term $\frac{e_i}{c}\underline{B}\times\underline{u}_i$ and the term $-n_i e_i\psi$ contributes to the electric force $e_i\underline{E}$. From (56) and (57) it follows that the equations to be satisfied by the potentials are

$$\left.\begin{aligned}
\nabla \cdot \underline{E} &= 4\pi \sum_i e_i n_i \\
\nabla \times \underline{B} - \frac{1}{c}\underline{E}_t &= \frac{4\pi}{c} \sum_i e_i n_i \underline{u}_i \\
\frac{Dn_i}{Dt} + n_i \, \nabla \cdot \underline{u}_i &= 0 \\
\frac{D\alpha_i}{Dt} &= 0 \\
\frac{D\beta_i}{Dt} &= 0 \\
\frac{D\xi_i}{Dt} &= 0 \\
\frac{D\eta_i}{Dt} &= -\frac{\partial \mathcal{E}_i}{\partial \xi_i}(n_i, \xi_i)
\end{aligned}\right\} \tag{59}$$

The set (59) follows from varying the potentials ϕ_i, α_i, β_i, ξ_i, η_i, $\underline{A}$, ψ in the variational principle

$$\delta \iint \left\{ \frac{\underline{E}^2 - \underline{B}^2}{8\pi} + \sum_i p_i \right\} d\underline{x}\, dt = 0 \tag{60}$$

where $\underline{E}$, $\underline{B}$ and the p_i stand for the expressions in (58). The variation of the n_i leads to the equations of state $p_i = p_{o_i}(n_i, S_i)$.

9. Variational Principles for the Internal Waves of a Stratified Fluid.

Another example of wave motion in a rotational fluid flow is provided by the internal waves of a stratified fluid. If a vertical gravitational force $-\rho g \underline{k}$, ($\underline{k}$ a unit vector in the z-direction) is included in the equations of motion of a general compressible fluid (32), new vertical gradients of the variables, density, pressure etc., will appear. The gravitational force stratifies fluid in the undisturbed state into horizontal layers of constant density. In an analysis of the internal waves which can occur the density $\rho_o(z)$ of the undisturbed layers is specified and model equations for the perturbing motion are prescribed. Perhaps the simplest formulation of internal waves is to consider motion of the interface between two incompressible fluids of different density. In the ocean, variation in salt concentration enables a continuous stratification into vertical layers by the gravitational field. Internal waves in a continuously stratified fluid are more easily studied when the Bousinesq approximation is made. This consists of isolating the internal waves from the compressional waves by treating the fluid as incompressible. The equations for internal waves in the Bousinesq approximation are

$$\left.\begin{aligned} \rho \frac{D\underline{u}}{Dt} &= -\nabla p - g\underline{k}, \\ \frac{D\rho}{Dt} &= 0, \\ \nabla \cdot \underline{u} &= 0, \end{aligned}\right\} \quad (61)$$

where p is no longer a given function of ρ, S and initially $\rho = \rho_o(z)$. Long [33] treats equations (61) for the case of two-dimensional steady motion in which $\underline{u} = (u(x,z), 0, w(x,z))$. The stream function ψ is introduced and its equation is integrated once. A variational principle is given for the integrated equation in which ψ is varied. This analysis rests on using an auxiliary variable $z_o(\psi)$, which denotes the height of undisturbed streamlines $\psi(z)$ = constant, and does not go through in the time dependent case.

Variational formulations can be found for two- and three-dimensional non-steady motions by using a more general (Clebsch-type) representation. For equations (61) in two dimensions, $\underline{u} = (u(x,z,t), 0, w(x,z,t))$, we introduce the representation

$$\left.\begin{aligned} u &= \phi_x + \alpha\beta_x , \\ w &= \phi_z + \alpha\beta_z , \\ \rho &= \rho_o(\alpha) , \\ p &= p_o(\alpha) - \rho_o(\alpha)\left\{\phi_t + \alpha\beta_t + \tfrac{1}{2}(u^2 + w^2) + gz\right\} , \end{aligned}\right\} \quad (62)$$

where $\rho_o(z)$ and $p_o(z)$ are the equilibrium density and pressure profiles. It can be shown that the quantities u, v, ρ, p given

33. R.R. Long, Tellus, vol. V, (1953), pp. 42-58.

by the expressions in (62) satisfy (61) when the potentials ϕ, α, β satisfy the equations

$$\left.\begin{aligned} \frac{D\alpha}{Dt} &= 0 , \\ \frac{D\beta}{Dt} &= \frac{p - p_o}{\rho_o^2} \rho_o'(\alpha) - g , \\ u_x + w_z &= 0 , \end{aligned}\right\} \quad (63)$$

where u, w, p stand for the expressions in (62). The set (63) follows from varying ϕ, α, β in the variational principle

$$\delta \iiint \left\{ p_o(\alpha) - \rho_o(\alpha) \left[\phi_t + \alpha\beta_t + \frac{1}{2}(u^2 + w^2) + gz \right] \right\} dx\, dz\, dt = 0 ,$$

where the Lagrangian density is again the pressure. Having the density a function of α in the representation for stratified flow, was suggested by Professor Whitham. If the term $p_o(\alpha)$ is included in the expression for the pressure, the density turns out to be $\rho_o(\alpha)$ and α stands for the height of constant density layers.

There is no doubt that α contributes to the vorticity but its significance is related to the entropy variations of the fluid. In the spirit of the general velocity representation (see Section 6, eq. (42))

$$\underline{u} = \nabla\phi + \alpha\nabla\beta + \xi\nabla\eta , \tag{64}$$

it would therefore be more appropriate to use the potentials ξ, η in place of α, β for the velocity representation in a stratified flow. Representation (62) differs from those previously given because only three potentials α, β, ϕ are used to represent the four functions u, v, ρ, p. The representation is therefore not completely general and may not be capable of representing all possible flows. This is more easily seen when (62) is generalized to three dimensions by adding a second horizontal velocity. Now equations (61) are being treated with $\underline{u} = (u(x, y, z, t),\ v(x, y, z, t),\ w(x, y, z, t))$. The natural generalization of (62) is obtained by adding the equation

$$v = \phi_y + \alpha\beta_y .$$

Now five variables are represented in terms of three potentials and it is expected that some flows are omitted. This generality question is best settled by the demands of specific problems.

For the last example we consider the variational formulation of the equations for Rossby waves in a stratified fluid. Combining equations (51) for the Rossby waves and equations (61) for the internal waves, gives the equations

$$\left.\begin{aligned}
\frac{Du}{Dt} - f(y)v &= -\frac{1}{\rho}\, p_x , \\
\frac{Dv}{Dt} - f(y)u &= -\frac{1}{\rho}\, p_y , \\
\frac{Dw}{Dt} &= -\frac{1}{\rho}\, p_z - g , \\
\frac{D\rho}{Dt} &= 0 , \qquad \nabla \cdot \underline{u} = 0 ,
\end{aligned}\right\} \qquad (65)$$

where the velocity has components (u, v, w) in the directions (x, y, z) which correspond to east-west, north-south and depth coordinates respectively. The fluid flow described by (65) has vorticity due to both the rotation of the Rossby waves and the entropy variation of the internal waves. A velocity representation of the form (64) is suggested in which α, β account for the Rossby wave effects and ξ, η account for internal wave effects. The desired representation has the form

$$\begin{aligned}
u &= \phi_x + \alpha\beta_x + \xi\eta_x - \sqrt{2\Omega}\ \alpha , \\
v &= \phi_y + \alpha\beta_y + \xi\eta_y - \sqrt{2\Omega}\ f(y)\beta , \\
w &= \phi_z + \alpha\beta_z + \xi\eta_z , \qquad\qquad (66) \\
\rho &= \rho_o(\xi) , \\
p &= p_o(\xi) - \rho_o(\xi)\left\{\phi_t + \alpha\beta_t + \xi\eta_t + \tfrac{1}{2}(u^2 + v^2 + w^2) + gz\right\} .
\end{aligned}$$

The equations to be satisfied by the potentials ϕ, α, β, ξ, η are

$$\left.\begin{aligned}
&\frac{D\alpha}{Dt} + \sqrt{2\Omega}\ f(y)v = 0 \\
&\frac{D\beta}{Dt} - \sqrt{2\Omega}\ u = 0 \\
&\frac{D\xi}{Dt} = 0 \\
&\frac{D\eta}{Dt} = \frac{p - p_o}{\rho_o^2}\ \rho_o'(\xi) - g \\
&\nabla \cdot \underline{u} = 0 \ .
\end{aligned}\right\} \quad (67)$$

The set (67) follows from the variation of the five potentials in the variational principle

$$\delta \iint \left\{ p_o(\xi) - \rho_o(\xi) \left[\phi_t + \alpha\beta_t + \xi\eta_t + \frac{1}{2}(u^2 + v^2 w^2) + gz \right] \right\} d\underline{x}\, dt = 0 \ .$$

APPENDIX A

The Singularity in $K_g(x)$

The kernel $K_g(x)$ appropriate to water waves is given by the integral

$$K_g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{g}{k} \tanh k h_o \right]^{1/2} e^{ikx} dk \ . \qquad \text{(A-1)}$$

An estimate of K_g is obtained by writing

$$K_g(x) = \frac{1}{\pi} \int_{o}^{\infty} \left[\frac{g}{k} \tanh k h_o \right]^{1/2} \cos kx \, dk$$

$$= \frac{1}{\pi} \int_{o}^{\infty} \sqrt{g/k} \cos kx \, dk + f(x) \, , \qquad \text{(A-2)}$$

where

$$f(x) = \frac{1}{\pi} \int_{o}^{\infty} \sqrt{g/k} \left\{ \sqrt{\tanh k h_o} - 1 \right\} \cos kxdk \ .$$

The integral in (A-2) can be evaluated and is equal to $\left| \frac{g}{2\pi x} \right|^{1/2}$.
The function $f(x)$ is bounded by

$$|f(x)| \le \frac{1}{\pi} \int_0^\infty \sqrt{g/k}\,(1 - \sqrt{\tanh k h_o})\, dk$$

$$< \frac{4}{\pi} \int_0^\infty \sqrt{g/k}\; e^{-2kh_o}\, dk = 4\sqrt{g/2\pi h_o}$$

Thus, for $K_g(x)$ we have

$$K_g(x) = \left|\frac{g}{2\pi x}\right|^{1/2} + f(x) \,,$$

where

$$|f(x)| < 4\sqrt{g/2\pi h_o} \;;$$

these formulas hold for all x.

APPENDIX B

An Estimate of $\phi(x, t)$ for a Breaking Wave

To help understand the effect of the singularity in K_g on breaking, we estimate the integral term $\phi(x, t)$ for the singular kernel

$$K(x) = |x|^{-1/2} e^{-b|x|}$$

and a known breaking wave $\eta(x, t)$. Let

$$M_1(t) = \max_x |\eta_x(x, t)|$$

$$M_2(t) = \max_x |\eta_{xx}(x, t)| \quad .$$

The functions $M_1(t)$ and $M_2(t)$ will be determined later for a specific breaking profile. An estimate of $\phi(x, t)$ is obtained by writing

$$\phi(x, t) = \int_{-\infty}^{\infty} |\xi|^{-1/2} e^{-b|\xi|} \eta_{xx}(x-\xi, t)\, d\xi$$

$$= \left(\int_0^{x_1} + \int_{x_1}^{\infty} \right) |\xi|^{-1/2} e^{-b|\xi|} \Big\{ \eta_{xx}(x-\xi, t) + \eta_{xx}(x+\xi, t) \Big\}\, d\xi .$$

Then

$$|\phi(x,t)| < 4M_2 x_1^{1/2} + 2M_1 x_1^{-1/2} , \tag{B-1}$$

where the second mean value theorem was used to estimate the second integral.

A breaking wave which is convenient for finding M_1 and M_2 is provided by the solution of the equation

$$\eta_t + \eta\eta_x = 0 \tag{B-2}$$

with $\eta(x,0) \equiv \eta_o(x) = -\sin x$. The solution to (B-2) is given implicitly by the equations

$$\eta = \eta_o(\zeta) , \qquad \zeta = x - \eta_o t ,$$

from which the formulas

$$\left.\begin{aligned} \eta_x &= \frac{\eta_o'(\zeta)}{1 + \eta_o'(\zeta)t} \\ \eta_{xx} &= \frac{\eta_o''(\zeta)}{\left(1 + \eta_o'(\zeta)t\right)^3} \end{aligned}\right\} \tag{B-3}$$

are derived. With these expressions for η_x and η_{xx} viewed as functions of ζ and t, the lines ζ_{M_1} on which $|\eta_x|$ is a maximum at time t are found to be

$$\zeta_{M_1} = 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \quad . \tag{B-4}$$

The lines ζ_{M_2} on which $|\eta_{xx}|$ is a maximum at time t are found from the equation

$$(1 + t\eta_o') \eta_o''' = 3t\eta_o''^2 \quad . \tag{B-5}$$

For the initial profile $\eta_o(x) = -\sin x$, equation (B-5) becomes

$$2t\left(\cos \zeta_{M_2}\right)^2 + \cos \zeta_{M_2} - 3t = 0$$

which yields

$$\cos \zeta_{M_2} = \frac{1}{4t}\left\{-1 \pm \sqrt{1 + 24t^2}\right\} . \tag{B-6}$$

To study the behavior near $x = 0$, (this behavior is repeated at $x = 2n\pi$) we take $n = 0$ in (B-4) and the positive root in (B-6) to obtain

$$\zeta_{M_1}(t) = 0 \ ,$$

$$\zeta_{M_2}(t) = \left|\frac{2(t-1)}{5}\right|^{1/2} \quad \text{as} \quad t \to 1 \ .$$

Substituting these values in (B-3) then gives

$$M_1(t) = \frac{1}{t - t_b}$$

$$M_2(t) = o(|t - t_b|^{-5/2}) \quad \text{as} \quad t \to t_b ,$$

where $t_b (= 1)$ is the breaking time. Finally, by choosing

$$x_1 = |t - t_b|^{3/2} ,$$

and using the above expressions for M_1 and M_2 in the bound (B-1) we obtain

$$|\phi(x, t)| = o(|t - t_b|^{-7/4}) = o(|M_1|^{7/4}) \quad \text{as} \quad t \to t_b .$$