I. NONEXISTENCE OF LOOPING TRAJECTORIES IN HYDROMAGNETIC WAVES OF FINITE AMPLITUDE

II. BREAKING OF WAVES IN A COLD COLLISION-FREE PLASMA IN A MAGNETIC FIELD

III. ON STABILITIES OF PERIODIC WAVES IN A COLD COLLISION-FREE PLASMA IN A MAGNETIC FIELD

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ABSTRACT

This dissertation consists of three parts. In Part I, it is shown that looping trajectories cannot exist in finite amplitude stationary hydromagnetic waves propagating across a magnetic field in a quasi-neutral cold collision-free plasma. In Part II, time-dependent solutions in series expansion are presented for the magnetic piston problem, which describes waves propagating into a quasi-neutral cold collision-free plasma, ensuing from magnetic disturbances on the boundary of the plasma. The expansion is equivalent to Picard's successive approximations. It is then shown that orbit crossings of plasma particles occur on the boundary for strong disturbances and inside the plasma for weak disturbances. In Part III, the existence of periodic waves propagating at an arbitrary angle to the magnetic field in a plasma is demonstrated by Stokes expansions in amplitude. Then stability analysis is made for such periodic waves with respect to side-band frequency disturbances. It is shown that waves of slow mode are unstable whereas waves of fast mode are stable if the frequency is below the cutoff frequency. The cutoff frequency depends on the propagation angle. For longitudinal propagation the cutoff frequency is equal to one-fourth of the electron's gyrofrequency. For transverse propagation the cutoff frequency is so high that waves of all frequencies are stable.

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PART I

NONEXISTENCE OF LOOPING TRAJECTORIES IN HYDROMAGNETIC WAVES OF FINITE AMPLITUDE

1. INTRODUCTION

The existence of hydromagnetic finite amplitude solitary waves propagating across a magnetic field in a quasi-neutral collision-free cold plasma was shown by Adlam and Allen [1]. The wave velocity lies between the Alfvén speed and twice the Alfvén speed. The lower bound ensures that the wave exists as a solitary pulse, while the upper bound ensures that the particle trajectories do not loop. When the wave velocity exceeds twice the Alfvén speed, Adlam and Allen's solution indicates the formation of a single, symmetric loop in a particle's trajectory, contrary to the original formulation for the problem which assumes a single stream at each point of space for the ions and electrons. Besides the solitary waves, there exist also periodic waves, discussed by Davis, Lüst, and Schlüter [2]. The validity of their solution is also restricted to certain ranges of the parameters characterizing the momentum and energy of the waves. Outside that range, their solution indicates the formation of an infinite train of loops. It is interesting to investigate whether a solution in which the particles execute loops exists or not. Our analysis shows that such a solution with nonoverlapping loops does not exist.

2. EQUATIONS

We assume the trajectories for electrons and ions in the shock frame have a loop not overlapped with neighboring loops. Then we proceed to demonstrate a contradiction. Quasi-neutrality of the plasma requires that loops in the electron trajectories must be accompanied by corresponding loops in the ion trajectories. We consider the region of x, where there is an isolated loop. Suppose the loop occurs in the region $x_1 < x < x_2$ (see Fig. 1). So in the region $x<x_1$, there is only one stream for each fluid, subscripted by 1 henceforth. In the region $x_1 < x < x_2$, there are three streams subscripted by 1, 2, and 3. In the region $x > x_2$, there is again only one stream, subscripted by 3. Superscript + refers to ions and superscript - refers to electrons. Summation $\sum_k$ will mean $\sum_{k=1}^{1}$ in $x < x_1$, $\sum_{k=1}^{3}$ in $x_1 < x < x_2$, and $\sum_{k=3}^{3}$ in $x > x_2$. We confine attention to the case in which the trajectories lie in the x-y plane, the magnetic field B is in the z-direction, and the electric field has x-component E and y-component F.

Referred to the shock frame which moves with the wave velocity U, the wave is stationary, and all variables are independent of time and functions of x alone. The equations in rationalized mks units to be satisfied are:

Continuity equations

$$d(n_k^+ u_k^+)/dx = 0 , \tag{1}$$

$$d(n_k^- u_k^-)/dx = 0 ; \tag{2}$$

Momentum equations

$$m^+ u_k^+ du_k^+/dx = q^+(E + v_k^+ B) , \tag{3}$$

$$m^+ u_k^+ dv_k^+/dx = q^+(F - u_k^+ B) , \tag{4}$$

$$m^- u_k^- du_k^-/dx = -q^-(E + v_k^- B) , \tag{5}$$

$$m^- u_k^- dv_k^-/dx = -q^-(F - u_k^- B) ; \tag{6}$$

Maxwell's equations (with the quasi-neutral approximation)

$$0 = \sum_k (q^+ n_k^+ u_k^+ - q^- n_k^- u_k^-) , \tag{7}$$

$$dB/dx = -\mu \sum_k (q^+ n_k^+ v_k^+ - q^- n_k^- v_k^-) , \tag{8}$$

$$dF/dx = 0 , \tag{9}$$

$$0 = \sum_k (q^+ n_k^+ - q^- n_k^-) ; \tag{10}$$

where n is the particle density, u is the x-component velocity, v is the y-component velocity, m is the particle mass, q is the particle charge, and μ is the permeability of vacuum. Equation (9) gives $F = F_0$. Equations (1), (2), and (7) and the fact that stream 2 is the continuation of stream 1 and stream 3 is the continuation of stream 2 give

$$(-)^{k+1} q^+ n_k^+ u_k^+ = (-)^{k+1} q^- n_k^- u_k^- = G_0 .$$

Both F_0 and G_0 are constants characterizing the wave.

3. CONDITIONS AT THE LOOP EDGES

At the loop edges, two streams of ions and electrons have zero velocities in the x-direction; hence, their particle densities are infinite. However, the infinities are integrable, being inversely proportional to the square root of the distance from the loop edge, and yield no accumulation of charges and currents at the loop edges. Therefore, both the electric field and the magnetic field are continuous there. To show this, we observe that by Eqs. (3) and (5), as $x \to x_1+0$, we have

$$u_2^+ \to -[2(q^+/m^+)(E+v_2^+B)(x-x_1)]^{\frac{1}{2}} ,$$

$$u_3^+ \to [2(q^+/m^+)(E+v_3^+B)(x-x_1)]^{\frac{1}{2}} ,$$

$$u_2^- \to -[-2(q^-/m^-)(E+v_2^-B)(x-x_1)]^{\frac{1}{2}} ,$$

$$u_3^- \to [-2(q^-/m^-)(E+v_3^-B)(x-x_1)]^{\frac{1}{2}} ,$$

$$v_2^+ \to v_3^+ , \qquad v_2^- \to v_3^- .$$

The corresponding charge densities are of the order of $(x-x_1)^{-\frac{1}{2}}$ which is integrable at $x = x_1$.

Similarly, as $x \to x_2 - 0$, we have

$$u_1^+ \to [-2(q^+/m^+)(E+v_1^+B)(x_2-x)]^{\frac{1}{2}} ,$$

$$u_2^+ \to -[-2(q^+/m^+)(E+v_2^+B)(x_2-x)]^{\frac{1}{2}} ,$$

$$u_1^- \to [2(q^-/m^-)(E+v_1^-B)(x_2-x)]^{\frac{1}{2}} ,$$

$$u_2^- \to -[2(q^-/m^-)(E+v_2^-B)(x_2-x)]^{\frac{1}{2}} ,$$

$$v_1^+ \to v_2^+ , \qquad v_1^- \to v_2^- .$$

Differentiation of Eq. (10) and substitution of Eqs. (3) and (5) give

$$\frac{E}{B} = -\frac{\sum_k (-)^{k+1}[(q^+/m^+)v_k^+(u_k^+)^{-3} + (q^-/m^-)v_k^-(u_k^-)^{-3}]}{\sum_k (-)^{k+1}[(q^+/m^+)(u_k^+)^{-3} + (q^-/m^-)(u_k^-)^{-3}]} \quad . \tag{11}$$

In the regions where there is only one stream of ions and electrons, $u_1^+ = u_1^-$ in $x < x_1$ and $u_3^+ = u_3^-$ in $x > x_2$; hence,

$$\frac{E}{B} = -\frac{(q^+/m^+)v_1^+ + (q^-/m^-)v_1^-}{q^+/m^+ + q^-/m^-} \qquad \text{at } x = x_1 \ ,$$

$$\frac{E}{B} = -\frac{(q^+/m^+)v_3^+ + (q^-/m^-)v_3^-}{q^+/m^+ + q^-/m^-} \qquad \text{at } x = x_2 \ .$$

With the conditions at the loop edges, Eq. (11) gives

$$\frac{E}{B} = -\frac{[-(E/B+v_2^-)/(E/B+v_2^+)]^{\frac{3}{2}}v_2^+ + (m^-q^+/m^+q^-)^{\frac{1}{2}}v_2^-}{[-(E/B+v_2^-)/(E/B+v_2^+)]^{\frac{3}{2}} + (m^-q^+/m^+q^-)^{\frac{1}{2}}} \qquad \text{at } x = x_1, x_2 \ .$$

Solving for E/B , we get

$$\frac{E}{B} = -\frac{(q^+/m^+)v_2^+ + (q^-/m^-)v_2^-}{q^+/m^+ + q^-/m^-} \qquad \text{at } x = x_1, x_2 \ .$$

Therefore, by continuity of E/B at the loop edges, we have

$$\frac{q^+}{m^+}(v_2^+ - v_1^+) = \frac{q^-}{m^-}(v_1^- - v_2^-) \qquad \text{at } x = x_1 \ , \tag{12}$$

$$\frac{q^+}{m^+}(v_3^+ - v_2^+) = \frac{q^-}{m^-}(v_2^- - v_3^-) \qquad \text{at } x = x_2 \ . \tag{13}$$

4. PROOF OF A CONTRADICTION

In the looping region, the time for a particle in the stream k to traverse its trajectory is

$$T_k^+ = \int_{x_1}^{x_2} \frac{(-)^{k+1}}{u_k^+} dx \qquad \text{or} \qquad T_k^- = \int_{x_1}^{x_2} \frac{(-)^{k+1}}{u_k^-} dx .$$

Equation (10) gives

$$\sum_k \frac{(-)^{k+1}}{u_k^+} = \sum_k \frac{(-)^{k+1}}{u_k^-} .$$

Integration from $x = x_1$ to $x = x_2$ gives

$$T_1^+ + T_2^+ + T_3^+ = T_1^- + T_2^- + T_3^- = T \quad \text{say.} \tag{14}$$

That is to say, it takes the same amount of time for either an ion or an electron to traverse its whole loop.

Equation (4) gives

$$\frac{d(v_1^+ - v_2^+)}{dx} = \frac{q^+}{m^+} F_0 \left(\frac{1}{u_1^+} - \frac{1}{u_2^+} \right) ,$$

$$\frac{d(v_3^+ - v_2^+)}{dx} = \frac{q^+}{m^+} F_0 \left(\frac{1}{u_3^+} - \frac{1}{u_2^+} \right) .$$

Integration from $x = x_1$ to $x = x_2$ gives

$$v_2^+ - v_1^+ \Big|_{x=x_1} = (q^+/m^+) F_0 (T_1^+ + T_2^+) , \tag{15}$$

$$v_3^+ - v_2^+ \Big|_{x=x_2} = (q^+/m^+) F_0 (T_3^+ + T_2^+) . \tag{16}$$

Similarly, Eq. (6) gives

$$\frac{d(v_1^- - v_2^-)}{dx} = - \frac{q^-}{m^-} F_0 \left(\frac{1}{u_1^-} - \frac{1}{u_2^-} \right) ,$$

$$\frac{d(v_3^- - v_2^-)}{dx} = -\frac{q^-}{m^-} F_0 \left(\frac{1}{u_3^-} - \frac{1}{u_2^-} \right) .$$

Integration from $x = x_1$ to $x = x_2$ gives

$$v_1^- - v_2^- \Big|_{x=x_1} = (q^-/m^-) F_0 (T_1^- + T_2^-) , \tag{17}$$

$$v_2^- - v_3^- \Big|_{x=x_2} = (q^-/m^-) F_0 (T_3^- + T_2^-) . \tag{18}$$

Combining Eqs. (12) - (18), we get

$$(q^+/m^+)^2 (T_1^+ + T_2^+) = (q^-/m^-)^2 (T_1^- + T_2^-) , \tag{19}$$

$$(q^+/m^+)^2 (T_3^+ + T_2^+) = (q^-/m^-)^2 (T_3^- + T_2^-) . \tag{20}$$

From Eqs. (14), (19), and (20), we get

$$(q^+/m^+)^2 (T + T_2^+) = (q^-/m^-)^2 (T + T_2^-) ;$$

hence,

$$T_2^+ = \left[\left(\frac{m^+ q^-}{m^- q^+} \right)^2 - 1 \right] T + \left(\frac{m^+ q^-}{m^- q^+} \right)^2 T_2^- . \tag{21}$$

For $m^+ q^- / m^- q^+ > \sqrt{2}$ as is the case for plasmas in physical reality, the above equation cannot be true because its left-hand side is less than T while its right-hand side is greater than T. Thus we obtain a contradiction.

5. CONCLUSIONS

For solitary waves, a symmetric looping trajectory would imply that the final state behind the wave is identical to the initial state ahead of the wave, and an asymmetric looping trajectory would imply a shock transition. We have shown that trajectories with nonoverlapping loops, either symmetric or asymmetric, cannot exist in hydromagnetic waves propagating across a magnetic field in a quasi-neutral cold collision-free plasma. Thus, when the wave speed has such a value that the Adlam-Allen solution or the Davis-Lüst-Schlüter solution no longer holds, the trajectories must be either multi-looped with neighboring loops overlapped or, as is more likely, the waves must be unsteady.

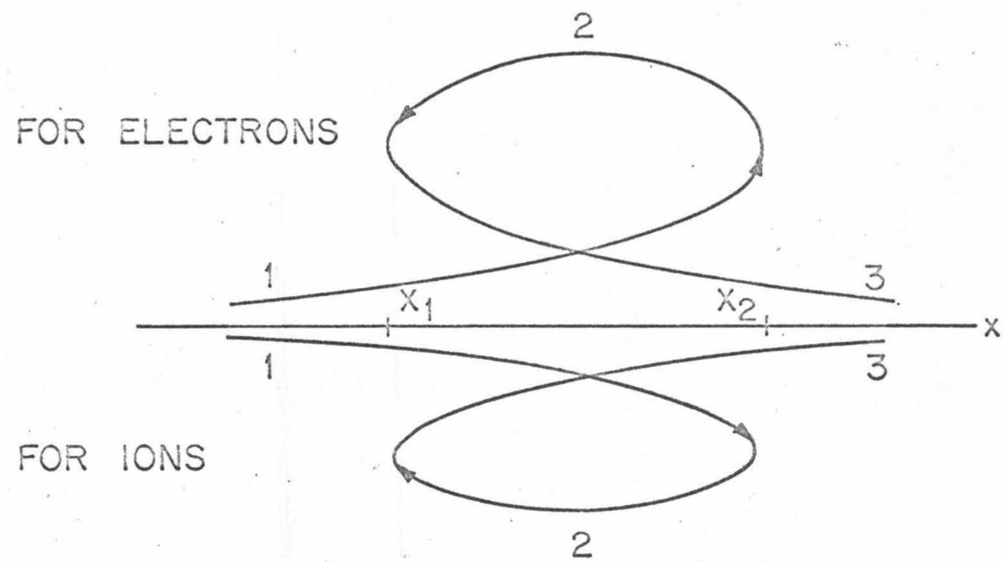

Figure 1. Assumed Trajectories for Ions and Electrons.

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PART II

BREAKING OF WAVES IN A COLD COLLISION-FREE PLASMA IN A MAGNETIC FIELD

1. INTRODUCTION

Wave propagation in a cold collision-free plasma in a magnetic field has been studied by many authors in recent years. Because of mathematical difficulties, so far the analyses have been restricted to steady solutions. Adlam and Allen [1] have found transverse solitary waves, Davis, Lüst and Schlüter [2] have found transverse periodic waves, propagating across a magnetic field. Saffman [3] has found longitudinal solitary waves, and Ferraro [4] has found longitudinal periodic waves, propagating along a magnetic field. As to unsteady solutions, the previous work has been done by numerical integrations of partial differential equations. Adlam and Allen [5] have calculated a numerical solution for a magnetic disturbance on the boundary which increases linearly in time, and found that orbit crossings of plasma particles occur on the plasma boundary. Auer, Hurwitz and Kilb [6], Jones and Rossow [7] have calculated numerical solutions for various disturbances on the boundary, and found that orbit crossings occur inside the plasma in their examples. All these calculations are for the case of transverse propagation.

In this paper we consider the magnetic piston problem for a plasma. The time-dependent solutions represent plasma motions ensuing from magnetic disturbances on the plasma boundary. In Chapter 2, we formulate the governing equations of the plasma motion in the Lagrangian coordinates, as well as the proper initial conditions and the boundary conditions. In particular, we show that the magnetic disturbance on the plasma boundary will set up an electric field immediately inside a quasi-neutral plasma. This initial electric field is a part of the initial conditions, which provides the initial acceleration to the charged particles. In Chapter 3, using Picard's iteration method† we present the solutions in series expansion. The same results are obtainable by formal expansion of the solution in a dummy parameter which serves the only purpose of grouping various terms in the expansion. The boundary data enter the solution in the form of various convolutions. For large disturbances, the partial sum of the first few terms of the series is a good approximation to the sum of the series. We are not concerned with the uniform validity in time of this approximation, because at finite time the wave breaks, thereafter the equations no longer describe the motion. In Chapter 4, we extend the method to the general case in which the initial magnetic field is at an arbitrary angle to the direction of propagation of the disturbance.

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† I am indebted to Professor Whitham who pointed out that the formal expansion in a dummy parameter is equivalent to Picard's successive approximations.

The magnetic piston problem for a plasma is an analogy to the piston problem for a gas. In the piston problem for a gas, neutral particles are accelerated by the pressure gradient, whereas in the magnetic piston problem for a plasma, charged particles are accelerated by the Lorentz forces in an electromagnetic field. The initial electric field set up immediately by the magnetic disturbance on the plasma boundary initiates the motion of the charged particles from rest. Once the particles are in motion, the Lorentz force due to the magnetic field will enhance the motion. Since the initial electric field decays exponentially inside the plasma as a sheath phenomenon of a plasma, particles near the boundary are accelerated more in the initial stage, so that particles on the boundary will overtake inner particles if the magnetic disturbance is sufficiently strong. For weak disturbances, orbit crossings of particles might occur inside the plasma. In an ordinary gas a shock wave begins to develop at the point where orbit crossing of particles occurs. In a cold plasma multiple streams appear as a result of orbit crossings. It is plausible that an avalanche of such wave breakings will lead to a plasma turbulence.

2. THE GOVERNING EQUATIONS OF PLASMA MOTION

We consider one-dimensional unsteady motion of a plasma in a magnetic field. The plasma is treated as a cold collision-free, two-fluid mixture of ion gas and electron gas. The governing equations of the plasma motion are the continuity equations, the momentum equations and Maxwell's equations. In mks units, these are

$$\frac{\partial}{\partial T}\rho_{+} + \frac{\partial}{\partial X}\rho_{+}u_{+} = 0,$$

$$\frac{\partial}{\partial T}\rho_{-} + \frac{\partial}{\partial X}\rho_{-}u_{-} = 0,$$

$$\frac{\partial}{\partial T}\rho_{+}\vec{u}_{+} + \frac{\partial}{\partial X}\rho_{+}\vec{u}_{+}u_{+} = \frac{q_{+}}{m_{+}}\rho_{+}\ (\vec{E} + \vec{u}_{+} \times \vec{B}),$$

$$\frac{\partial}{\partial T}\rho_{-}\vec{u}_{-} + \frac{\partial}{\partial X}\rho_{-}\vec{u}_{-}u_{-} = -\frac{q_{-}}{m_{-}}\rho_{-}\ (\vec{E} + \vec{u}_{-} \times \vec{B}),$$

$$\frac{\partial}{\partial T}\vec{B} + \frac{\partial}{\partial X} \times \vec{E} = 0,$$

$$-\epsilon\frac{\partial}{\partial T}\vec{E} + \frac{1}{\mu}\frac{\partial}{\partial X}\times\vec{B} = \frac{q_{+}}{m_{+}}\rho_{+}\vec{u}_{+} - \frac{q_{-}}{m_{-}}\rho_{-}\vec{u}_{-},$$

$$\frac{\partial}{\partial X}\cdot\vec{B} = 0,$$

$$\epsilon\frac{\partial}{\partial X}\cdot\vec{E} = \frac{q_{+}}{m_{+}}\rho_{+} - \frac{q_{-}}{m_{-}}\rho_{-},$$

where T and X are the time and distance variables in the Eulerian coordinates, $m_{\pm}$ and $q_{\pm}$ are the mass and the charge of an ion or an electron, μ and ϵ are the magnetic permeability and the electric permitivity of vacuum, $\rho_{\pm}$ and

$\vec{u}_\pm = (u_\pm, v_\pm, w_\pm)$ are the mass density and the velocity of the ion gas or the electron gas, $\vec{B} = (G, B, H)$ and $\vec{E} = (E_1, E_2, E_3)$ are the magnetic field and the electric field.

The approximation of quasi-neutrality of the plasma makes possible an equivalent one-fluid description of the plasma in terms of the mass density ρ defined by $\rho = \rho_+ + \rho_-$, the velocity $\vec{u} = (u, v, w)$ defined by $\rho\vec{u} = \rho_+\vec{u}_+ + \rho_-\vec{u}_-$, the current density $\vec{J}$ defined by $\vec{J} = \frac{q_+}{m_+}\rho_+\vec{u}_+ - \frac{q_-}{m_-}\rho_-\vec{u}_-$, the magnetic field $\vec{B}$, and the electric field $\vec{E}$. We shall use σ to denote the inverse of ρ. With this approximation, the ion gas and the electron gas have equal longitudinal velocities $u = u_+ = u_-$, because zero displacement current gives $\frac{q_+}{m_+}\rho_+u_+ - \frac{q_-}{m_-}\rho_-u_- = 0$ while zero charge density gives $\frac{q_+}{m_+}\rho_+ - \frac{q_-}{m_-}\rho_- = 0$. Thus, the governing equations become

$$\frac{\partial}{\partial T}\rho + \frac{\partial}{\partial X}\rho u = 0,$$

$$\frac{\partial}{\partial T}\rho\vec{u} + \frac{\partial}{\partial X}\rho\vec{u}u = \vec{J}\times\vec{B},$$

$$\frac{\partial}{\partial T}\vec{J} + \frac{\partial}{\partial X}\vec{J}u = \frac{q_+q_-}{m_+m_-}\rho(\vec{E} + \vec{u}\times\vec{B}) - \left(\frac{q_-}{m_-} - \frac{q_+}{m_+}\right)\vec{J}\times\vec{B},$$

$$\frac{\partial}{\partial T}\vec{B} + \frac{\partial}{\partial X}\times\vec{E} = 0,$$

$$\frac{1}{\mu}\frac{\partial}{\partial X}\times\vec{B} = \vec{J}.$$

There is no longitudinal current as a consequence of the quasi-neutrality approximation. In the one-dimensional problem, the longitudinal magnetic field is constant because both its time derivative and its spatial derivative vanish.

We shall normalize the above equations with respect to some reference values. Suppose ρ_{ref} and B_{ref} are typical values of the mass density and the magnitude of the magnetic field in the problem, we may use them as the reference values for the mass density and the magnetic field. Accordingly the reference values for other variables are defined in terms of them as follows.

$$X_{ref} = \sqrt{\frac{m_+ m_-}{q_+ q_-} \frac{1}{\mu \rho_{ref}}} ,$$

$$T_{ref} = \sqrt{\frac{m_+ m_-}{q_+ q_-}} \frac{1}{B_{ref}} ,$$

$$\sigma_{ref} = \frac{1}{\rho_{ref}} ,$$

$$u_{ref} = \frac{B_{ref}}{\sqrt{\mu \rho_{ref}}} ,$$

$$J_{ref} = \sqrt{\frac{q_+ q_-}{m_+ m_-} \frac{\rho_{ref}}{\mu}} B_{ref} ,$$

$$E_{ref} = \frac{B^2_{ref}}{\sqrt{\mu \rho_{ref}}} .$$

Using these reference values as normalization factors, we obtain the following set of normalized equations. Henceforth all notations are normalized quantities.

$$\frac{\partial}{\partial T} \rho + \frac{\partial}{\partial X} \rho u = 0,$$

$$\frac{\partial}{\partial T} \rho \vec{u} + \frac{\partial}{\partial X} \rho \vec{u} u = \vec{J} \times \vec{B},$$

(1) $$\frac{\partial}{\partial T} \vec{J} + \frac{\partial}{\partial X} \vec{J} u = \rho(\vec{E} + \vec{u} \times \vec{B}) - \Gamma \vec{J} \times \vec{B},$$

$$\frac{\partial}{\partial T} \vec{B} + \frac{\partial}{\partial X} \times \vec{E} = 0,$$

$$\frac{\partial}{\partial X} \times \vec{B} = \vec{J},$$

where $\Gamma = \sqrt{\frac{m_+ q_-}{m_- q_+}} - \sqrt{\frac{m_- q_+}{m_+ q_-}}$.

We shall transform these equations from the Eulerian coordinates (T, X) to the Lagrangian coordinates (t, x) by the contact transformation defined by

$$dt = dT \quad \text{and} \quad dx = \rho(dX - u\,dT).$$

The inverse transformation is

$$dT = dt \quad \text{and} \quad dX = \sigma\,dx + u\,dt.$$

Accordingly, $\frac{\partial}{\partial t} = \frac{\partial}{\partial T} + u \frac{\partial}{\partial X}$ and $\frac{\partial}{\partial x} = \frac{1}{\rho} \frac{\partial}{\partial X}$.

In the Lagrangian coordinates, equations (1) become

$$\frac{\partial}{\partial t} \sigma - \frac{\partial}{\partial x} u = 0,$$

$$\frac{\partial}{\partial t} \vec{u} = \sigma \vec{J} \times \vec{B},$$

$$(2) \qquad \frac{\partial}{\partial t}\,\sigma\,\vec{J} = \vec{E} + \vec{u} \times \vec{B} - \quad \sigma\,\vec{J} \times \vec{B},$$

$$\sigma \frac{\partial}{\partial t}\,\vec{B} - u \frac{\partial}{\partial x}\,\vec{B} + \frac{\partial}{\partial x} \times \vec{E} = 0,$$

$$\frac{\partial}{\partial x} \times \vec{B} = \sigma\,\vec{J}.$$

Upon elimination of $\vec{J}$ and $\vec{E}$, we obtain the following equations in scalar components.

$$\frac{\partial}{\partial t}\,\sigma - \frac{\partial}{\partial x}\,u = 0,$$

$$\frac{\partial}{\partial t}\,u + \frac{1}{2}\frac{\partial}{\partial x}\,(B^2 + H^2) = 0,$$

$$\frac{\partial}{\partial t}\,v - G\frac{\partial}{\partial x}\,B = 0,$$

$$(3) \qquad \frac{\partial}{\partial t}\,w - G\frac{\partial}{\partial x}\,H = 0,$$

$$\frac{\partial}{\partial t}\,\left(\frac{\partial^2}{\partial x^2}\,B - \sigma B\right) + G\,\left(\Gamma\frac{\partial^2}{\partial x^2}\,H + \frac{\partial}{\partial x}\,v\right) = 0,$$

$$\frac{\partial}{\partial t}\,\left(\frac{\partial^2}{\partial x^2}\,H - \sigma H\right) + G\,\left(-\,\Gamma\frac{\partial^2}{\partial x^2}\,B + \frac{\partial}{\partial x}\,w\right) = 0.$$

Upon further elimination of u, v and w, we obtain the following equations for σ, B and H.

$$\frac{\partial^2}{\partial t^2}\,\sigma + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,(B^2 + H^2) = 0,$$

$$(4) \qquad \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2}\,B - \sigma\,B\right) + \frac{\partial^2}{\partial x^2}\left(\Gamma G\,\frac{\partial}{\partial t}\,H + G^2 B\right) = 0,$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2}\,H - \sigma\,H\right) + \frac{\partial^2}{\partial x^2}\left(-\,\Gamma G\,\frac{\partial}{\partial t}\,B + G^2 H\right) = 0.$$

We are interested in the plasma motion caused by a magnetic disturbance on the boundary of a plasma filling the right half space at rest initially. In the quasi-neutrality approximation, the disturbance on the boundary sets up a transverse electric field instantaneously inside the plasma, as shown later. This is expected, because this approximation is tantamount to putting permitivity zero, hence the propagation velocity of the electromagnetic disturbance is infinite. The charged particles in the plasma are accelerated by this electric field from rest, and then also accelerated by the magnetic field when they are in motion. In the Lagrangian coordinates, each moving transverse plane inside the plasma is described by a constant value of x. We shall let x = 0 be the moving boundary of the plasma. Outside the plasma, the magnetic field has no spatial variation in virtue of Maxwell's equations in vacuum.

We use the initial values of the mass density and the magnitude of the magnetic field as the reference values. Thus the initial conditions of the plasma can be described as

$$\sigma\Big|_{t=0} = 1, \qquad \vec{u}\Big|_{t=0} = 0,$$

$$\vec{B}\Big|_{t=0} = (\cos\theta,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi\,).$$

where θ is the propagation angle between the direction of the initial magnetic field and the direction of the propagation of the disturbance, ϕ is the inclination angle of the initial transverse magnetic field in the transverse coordinates. With the quasi-neutrality assumption, the electric field and the current density are determined by the mass density, the velocity and the magnetic field according to the third and fifth equations of (2). Suppose the magnetic disturbance on the plasma boundary has a magnitude f(t) with an inclination angle $\psi(t)$, then the boundary conditions are

$$\vec{B}\Big|_{x=0} = (\cos\theta,\ \sin\theta\cos\phi + f\cos\psi,\ \sin\theta\sin\phi + f\sin\psi),$$

$$\vec{B}\Big|_{x=\infty} = (\cos\theta\ ,\ \sin\theta\cos\phi\ ,\ \sin\theta\sin\phi\ ).$$

To show the immediate setting up of an electric field inside the plasma by the magnetic disturbance on the boundary, we substitute the initial conditions that $\sigma = 1$, and $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x}, \frac{\partial B}{\partial x}, \frac{\partial H}{\partial x} = 0$ into the following equations

$$\frac{\partial^2}{\partial x^2}\frac{\partial B}{\partial t} - \sigma\frac{\partial B}{\partial t} - B\frac{\partial u}{\partial x} + \cos\theta\frac{\partial v}{\partial x} + \Gamma\cos\theta\frac{\partial^2 H}{\partial x^2} = 0,$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial H}{\partial t} - \sigma\frac{\partial H}{\partial t} - H\frac{\partial u}{\partial x} + \cos\theta\frac{\partial w}{\partial x} - \Gamma\cos\theta\frac{\partial^2 B}{\partial x^2} = 0,$$

which are obtained from (3), and we get

$$\left(\frac{\partial^2}{\partial x^2} - 1\right)\left(\frac{\partial B}{\partial t}\right)_{t=0} = 0,$$

$$\left(\frac{\partial^2}{\partial x^2} - 1\right)\left(\frac{\partial H}{\partial t}\right)_{t=0} = 0.$$

Using the boundary conditions that

$$\left(\frac{\partial \vec{B}}{\partial t}\right)_{t=0}\Bigg|_{x=0} = \left(0,\ \left(\frac{d}{dt} f \cos\psi\right)_{t=0},\ \left(\frac{d}{dt} f \sin\psi\right)_{t=0}\right),$$

$$\left(\frac{\partial \vec{B}}{\partial t}\right)_{t=0}\Bigg|_{x=\infty} = 0,$$

we obtain

$$\left(\frac{\partial \vec{B}}{\partial t}\right)_{t=0} = \left(0,\ \left(\frac{d}{dt} f \cos\psi\right)_{t=0},\ \left(\frac{d}{dt} f \sin\psi\right)_{t=0}\right)e^{-x}.$$

Then substituting it in the following equation

$$\left(\frac{\partial \vec{B}}{\partial t}\right)_{t=0} + \frac{\partial}{\partial x} \times (\vec{E})_{t=0} = 0$$

which is obtained from the fourth equation of (2) with the initial conditions, we obtain

$$(\vec{E})_{t=0} = \left(0,\ \left(\frac{d}{dt} f \sin\psi\right)_{t=0},\ -\left(\frac{d}{dt} f \cos\psi\right)_{t=0}\right) e^{-x}$$

because $\vec{E}$ vanishes at $x = \infty$. That the longitudinal component is zero comes from the third equation of (2).

Therefore, equations (4) are supplemented by the initial conditions

$$\sigma\Big|_{t=0} = 1, \quad B\Big|_{t=0} = \sin\theta\cos\phi, \quad H\Big|_{t=0} = \sin\theta\sin\phi,$$

(5)

$$\frac{\partial\sigma}{\partial t}\Big|_{t=0} = 0, \qquad \frac{\partial B}{\partial t}\Big|_{t=0} = \left(\frac{d}{dt} f\cos\psi\right)_{t=0} e^{-x},$$

$$\frac{\partial H}{\partial t}\Big|_{t=0} = \left(\frac{d}{dt} f\sin\psi\right)_{t=0} e^{-x}.$$

and the boundary conditions

$$B\Big|_{x=0} = \sin\theta\cos\phi + f\cos\psi, \quad H\Big|_{x=0} = \sin\theta\sin\phi + f\sin\psi,$$

(6)

$$B\Big|_{x=\infty} = \sin\theta\cos\phi, \qquad H\Big|_{x=\infty} = \sin\theta\sin\phi.$$

3. THE TRANSVERSE CASE

Equations (3) permit solutions with $\vec{u} = (u, o, o)$, $\vec{J} = (o, o, J)$, $\vec{B} = (o, B, o)$ and $\vec{E} = (E_1, o, E_3)$ which describe waves propagating across a magnetic field. The magnetic field has no longitudinal component and the transverse component is plane polarized. Accordingly equations (3) reduce to

$$\frac{\partial}{\partial t}\sigma - \frac{\partial}{\partial x}u = 0,$$

(7) $$\frac{\partial}{\partial t}u + \frac{1}{2}\frac{\partial}{\partial x}B^2 = 0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}B - \sigma B\right) = 0.$$

Upon elimination of u, we obtain

$$\frac{\partial^2}{\partial t^2}\sigma + \frac{1}{2}\frac{\partial^2}{\partial x^2}B^2 = 0,$$

(8)

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}B - \sigma B\right) = 0.$$

With $\theta = \frac{\pi}{2}$, $\phi = 0$ and $\psi = 0$, the initial conditions (5) become

(9) $$\sigma\Big|_{t=0} = 1, \qquad \frac{\partial\sigma}{\partial t}\Big|_{t=0} = 0, \qquad B\Big|_{t=0} = 1;$$

and the boundary conditions (6) become

(10) $$B\Big|_{x=0} = 1 + f(t), \qquad B\Big|_{x=\infty} = 1.$$

The second equation of (8) can be integrated once, thus we have

$$
(11) \qquad \begin{aligned} &\frac{\partial^2}{\partial t^2}\,\sigma + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,B^2 = 0, \\ &\frac{\partial^2}{\partial x^2}\,B - \sigma\,B = -1. \end{aligned}
$$

Equations (11) may be written as

$$
(12) \qquad \frac{\partial^2}{\partial t^2}\,\sigma = -\frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,B^2,
$$

with the initial conditions

$$
\sigma\Big|_{t=0} = 1, \qquad \frac{\partial\sigma}{\partial t}\Big|_{t=0} = 0,
$$

and

$$
(13) \qquad \left(\frac{\partial^2}{\partial x^2} - 1\right) B = -1 + B\,(\sigma - 1),
$$

with the boundary conditions

$$
B\Big|_{x=0} = 1 + f(t), \qquad B\Big|_{x=\infty} = 1.
$$

Substituting

$$
\sigma = 1 - \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\int_o^t dt'\int_o^{t'} B^2(x,\ t'')dt''
$$

obtained from (12) into (13), we have

$$
(14) \qquad \left(\frac{\partial^2}{\partial x^2} - 1\right) B = -1 - \frac{1}{2}\,B\,\frac{\partial^2}{\partial x^2}\int_o^t dt'\int_o^{t'} B^2(x,\ t'')dt''.
$$

Picard's iteration method may be used to solve equation (14). Many iteration schemes are at our disposal. We take

$$
(15) \qquad \begin{aligned} &\sigma^{(o)} = 1, \\ &B^{(o)} = 1 + f(t)\,e^{-x} \end{aligned}
$$

as the initial approximation, and define the successive approximations by the solutions of the following equations.

(16) $$\frac{\partial^2}{\partial t^2}\sigma^{(n)} = P^{(n)},$$

with the initial conditions

$$\sigma^{(n)}\Big|_{t=0} = 1, \qquad \frac{\partial\sigma^{(n)}}{\partial t}\Big|_{t=0} = 0,$$

and

(17) $$\left(\frac{\partial^2}{\partial x^2} - 1\right) B^{(n)} = Q^{(n)},$$

with the boundary conditions

$$B^{(n)}\Big|_{x=0} = 1 + f(t), \qquad B^{(n)}\Big|_{x=\infty} = 1,$$

in which $P^{(n)}$ and $Q^{(n)}$ are specified functions determined by the chosen iteration scheme. In terms of the increments of iteration defined as

$$\sigma_o = \sigma^{(o)},$$
$$B_o = B^{(o)},$$
$$\sigma_n = \sigma^{(n)} - \sigma^{(n-1)} \quad \text{for } n\geq 1,$$
$$B_n = B^{(n)} - B^{(n-1)} \quad \text{for } n\geq 1,$$

we have

$$\sigma^{(n)} = \sum_{m=0}^{n} \sigma_m ,$$

$$B^{(n)} = \sum_{m=0}^{n} B_m ;$$

and as $n \to \infty$

$$\sigma = \sum_{m=0}^{\infty} \sigma_m ,$$

$$B = \sum_{m=0}^{\infty} B_m .$$

Subtracting (16) and (17) with n replaced by n-1 from (16) and (17) respectively, we obtain the following equations for the increments.

(18) $$\frac{\partial^2}{\partial t^2} \sigma_n = P_n$$

with the initial conditions

$$\sigma_n \Big|_{t=0} = 0, \qquad \frac{\partial \sigma_n}{\partial t}\Big|_{t=0} = 0 ,$$

and

(19) $$\left(\frac{\partial^2}{\partial x^2} - 1\right) B_n = Q_n$$

with the boundary conditions

$$B_n \Big|_{x=0} = 0 , \qquad B_n \Big|_{x=\infty} = 0 ,$$

in which

$$P_n = P^{(n)} - P^{(n-1)} ,$$

$$Q_n = Q^{(n)} - Q^{(n-1)} .$$

The right-hand sides of (12) and (13) suggest the following iteration scheme

$$P^{(n)} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} B^{(n-1)} B^{(n-1)} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \Big(\sum_{m=1}^{n} B_{m-1} \Big)^2 ,$$

$$Q^{(n)} = -1 + B^{(n-1)}(\sigma^{(n)} - 1) = -1 + (\sum_{m=1}^{n} B_{m-1})(\sum_{m=1}^{n} \sigma_m);$$

and correspondingly

$$P_n = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sum_{m=1}^{n} B_{m-1}B_{n-m} + \sum_{k=n+1}^{2n-1}\sum_{m=k-n+1}^{n} B_{m-1}B_{k-m} - \sum_{k=n}^{2n-3}\sum_{m=k-n+2}^{n-1} B_{m-1}B_{k-m}\right],$$

$$Q_n = \sum_{m=1}^{n}\sigma_m B_{n-m} + \sum_{k=n+1}^{2n-1}\sum_{m=k-n+1}^{n}\sigma_m B_{k-m} - \sum_{k=n}^{2n-3}\sum_{m=k-n+2}^{n-1}\sigma_m B_{k-m},$$

for $(\sum_{m=1}^{n} B_{m-1})^2$ and $(\sum_{m=1}^{n} B_{m-1})(\sum_{m=1}^{n}\sigma_m)$ can be written as

$$\sum_{k=1}^{n}\sum_{m=1}^{k} B_{m-1}B_{k-m} + \sum_{k=n+1}^{2n-1}\sum_{m=k-n+1}^{n} B_{m-1}B_{k-m} \quad \text{and}$$

$$\sum_{k=1}^{n}\sum_{m=1}^{k}\sigma_m B_{k-m} + \sum_{k=n+1}^{2n-1}\sum_{m=k-n+1}^{n}\sigma_m B_{k-m}.$$

Instead, we shall use an iteration scheme such that

$$P_n = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{m=1}^{n} B_{m-1}B_{n-m}$$

(20)

$$Q_n = \sum_{m=1}^{n}\sigma_m B_{n-m}$$

corresponding to

$$P^{(n)} = -\frac{1}{2}\frac{\partial}{\partial x^2}\left[B^{(0)}B^{(n-1)} + \sum_{m=1}^{n-1}(B^{(m)} - B^{(m-1)})B^{(n-m-1)}\right]$$

$$= -\frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{k=1}^{n}\sum_{m=1}^{k} B_{m-1}B_{k-m},$$

(21)

$$Q^{(n)} = -1 + \sum_{m=1}^{n}(\sigma^{(m)} - \sigma^{(m-1)})B^{(n-m)}$$

$$= -1 + \sum_{k=1}^{n}\sum_{m=1}^{k}\sigma_m B_{k-m}$$

which approach $-\frac{1}{2}\frac{\partial^2}{\partial x^2}B^2$ and $-1+B(\sigma-1)$ respectively as $n\to\infty$, because $\sum_{k=1}^{n}\sum_{m=1}^{k}B_{m-1}B_{k-m}$ has the same limit as $(\sum_{m=1}^{n}B_{m-1})^2$, and $\sum_{k=1}^{n}\sum_{m=1}^{k}\sigma_m B_{k-m}$ has the same limit as $(\sum_{m=1}^{n}B_{m-1})(\sum_{m=1}^{n}\sigma_m)$.

The same results using this second iteration scheme can be obtained by substituting the following formal expansions

$$(22)\qquad \begin{aligned} \sigma &= \sigma_o + \sum_{m=1}^{\infty} a^m\sigma_m , \\ B &= B_o + \sum_{m=1}^{\infty} a^m B_m , \end{aligned}$$

into the following equations

$$(23)\qquad \begin{aligned} &\frac{\partial^2}{\partial t^2}\sigma + \frac{1}{2}a\frac{\partial^2}{\partial x^2}B^2 = 0 , \\ &\frac{\partial^2}{\partial x^2}B - \sigma B = -1 , \end{aligned}$$

which are equations (11) with the insertion of a dummy parameter a. This dummy parameter serves in grouping various terms in the expansions only and eventually we shall put a=1. Equating the resultant coefficients of various powers of a in equations (23) after substitution of (22) to zero, we obtain

$$(24)\qquad \frac{\partial^2}{\partial t^2}\sigma_o = 0$$

with the initial conditions

$$\sigma_o \Big|_{t=0} = 1, \qquad \frac{\partial \sigma_o}{\partial t}\Big|_{t=0} = 0 ,$$

and

(25) $$\left(\frac{\partial^2}{\partial x^2} - \sigma_o\right) B_o = -1$$

with the boundary conditions

$$B_o \Big|_{x=0} = 1 + f(t) , \qquad B_o \Big|_{x=\infty} = 1,$$

as well as equations (18) and (19) with P_n and Q_n given by (20).

The solutions of (24) and (25) are (15). By induction, $-\frac{1}{2} \frac{\partial^2}{\partial x^2} \sum_{m=1}^{n} B_{m-1} B_{n-m}$ and $\sum_{m=1}^{n} \sigma_m B_{n-m}$ are finite combinations of the form $\sum_{k=1} \sum_{\ell=0} a_{k,\ell} \, x^{\ell} e^{-kx}$ and $\sum_{k=1} \sum_{\ell=0} b_{k,\ell} \, x^{\ell} e^{-kx}$. Therefore the solutions for (18) and (19) have the following form†.

$$\sigma_n = \sum_{k=1} \sum_{\ell=0} \left[\int_o^t dt' \int_o^{t'} a_{k,\ell}(t'')dt'' \right] x^{\ell} e^{-kx}$$

(26) $$= \sum_{k=1} \sum_{\ell=0} \left[t \circ a_{k,\ell}(t) \right] x^{\ell} e^{-kx} ,$$

$$B_n = \sum_{k=1} \sum_{\ell=0} b_{k,\ell}(t) \left[S_{k,\ell}(x) \, e^{-kx} - S_{k,\ell}(o) \, e^{-x} \right] ,$$

---

† The convolution operator is defined as

$$f(t) \circ g(t) = \int_o^t f(t') \, g(t-t')dt'$$

where

$$S_{k,\ell}(x) = \begin{cases} -\dfrac{\ell!}{2^{\ell+2}} \displaystyle\sum_{n=1}^{\ell+1} 2^n \frac{x^n}{n!} & \text{if } k=1, \\ \dfrac{\ell!}{2} \displaystyle\sum_{n=0}^{\ell} \left[\left(\frac{1}{k-1}\right)^{\ell-n+1} - \left(\frac{1}{k+1}\right)^{\ell-n+1} \right] \frac{x^n}{n!} & \text{if } k \geq 2, \end{cases}$$

are polynomials which satisfy

$$\left(\frac{d^2}{dx^2} - 1\right) S_{k,\ell}(x)\, e^{-kx} = x^{\ell} e^{-kx}.$$

We display the first few terms of the solution as follows

$$B_o = 1 + f\, e^{-x} ,$$

$$B_1 = t \circ f \left(\tfrac{1}{2} x e^{-x}\right) + t \circ f^2 \left(\tfrac{2}{3} e^{-x} - \tfrac{2}{3} e^{-2x}\right)$$

$$+ f(t \circ f)\left(\tfrac{1}{3} e^{-x} - \tfrac{1}{3} e^{-2x}\right) + f(t \circ f^2)\left(\tfrac{1}{4} e^{-x} - \tfrac{1}{4} e^{-3x}\right),$$

and

$$\sigma_o = 1 ,$$

$$\sigma_1 = - t \circ f\, e^{-x} - 2\, t \circ f^2 e^{-2x} ,$$

$$\sigma_2 = t \circ t \circ f \left(1 - \tfrac{1}{2} x\right) e^{-x} + t \circ t \circ f^2 \left[- \tfrac{2}{3} e^{-x} + \tfrac{8}{3} e^{-2x}\right]$$

$$+ t \circ f (t \circ f) \left[- \tfrac{1}{3} e^{-x} + \left(\tfrac{10}{3} - 2x\right) e^{-2x}\right]$$

$$+ t \circ f (t \circ f^2) \left[- \tfrac{1}{4} e^{-x} - \tfrac{8}{3} e^{-2x} + \tfrac{33}{4} e^{-3x}\right]$$

$$+ t \circ f^2 (t \circ f) \left[- \tfrac{4}{3} e^{-2x} + 3 e^{-3x}\right]$$

$$+ t \circ f^2 (t \circ f^2) \left[- e^{-2x} + 4 e^{-4x}\right]$$

On the boundary x=0, we have

$$
\begin{aligned}
\sigma_0(0,\ t) &= 1, \\
\sigma_1(0,\ t) &= -(t\circ f + 2t\circ f^2), \\
\sigma_2(0,\ t) &= t\circ(t\circ f + 2t\circ f^2) + t\circ f(3t\circ f + \tfrac{16}{3}t\circ f^2) \\
&\quad + t\circ f^2(\tfrac{5}{3}t\circ f + 3t\circ f^2), \\
\sigma_3(0,\ t) &= -t\circ t\circ(t\circ f + 2t\circ f^2) - t\circ t\circ f(3t\circ f + \tfrac{16}{3}t\circ f^2) \\
&\quad - t\circ t\circ f^2(\tfrac{5}{3}t\circ f + 3t\circ f^2) \\
&\quad - t\circ[\tfrac{1}{4}(t\circ f)^2 + \tfrac{2}{3}(t\circ f)(t\circ f^2) + \tfrac{4}{9}(t\circ f^2)^2] \\
&\quad - t\circ f[\tfrac{2}{9}(t\circ f)^2 + \tfrac{17}{24}(t\circ f)(t\circ f^2) + \tfrac{8}{15}(t\circ f^2)^2 \\
&\qquad + t\circ(\tfrac{11}{4}t\circ f + \tfrac{46}{9}t\circ f^2) \\
&\qquad + t\circ f(\tfrac{67}{9}t\circ f + \tfrac{919}{72}t\circ f^2) \\
&\qquad + t\circ f^2(\tfrac{71}{18}t\circ f + \tfrac{104}{15}t\circ f^2)] \\
&\quad - t\circ f^2[\tfrac{1}{18}(t\circ f)^2 + \tfrac{1}{5}(t\circ f)(t\circ f^2) + \tfrac{1}{6}(t\circ f^2)^2 \\
&\qquad + t\circ(\tfrac{14}{9}t\circ f + \tfrac{26}{9}t\circ f^2) \\
&\qquad + t\circ f(\tfrac{151}{36}t\circ f + \tfrac{107}{5}t\circ f^2) \\
&\qquad + t\circ f^2(\tfrac{11}{15}t\circ f + \tfrac{23}{6}t\circ f^2)].
\end{aligned}
$$

Our main interest is concerned with the crossing of the particle trajectories. Trajectory crossing occurs at the point where $\sigma = 0$, i.e. the mass density becomes infinite. Because the mass confined between two transverse planes moving rigidly with the particle's longitudinal motion is constant before they penetrate each other, and at the moment of crossing, the distance between them is zero, hence the mass density is infinite there. After the crossing occurs, two streams of particles penetrate each other, the equations formulated to describe the plasma under the assumption of a single stream at each space point are no longer valid. So the wave breaks at the crossing point. To find whether the wave breaks, and when and where breaking occurs we look for the first zero of σ.

We shall confine our attention to the range of the disturbance strength such that the partial sum of the first few terms in (22) is a good approximation to the sum of the infinite series up to the occurence of wave breaking. The criterion for the accuracy of the approximation is that the last retained term should be small compared with the preceding terms. Thus σ is reasonably approximated by $1 + \sigma_1 + \sigma_2 + \sigma_3$ provided that the magnitude of σ_3 is small compared with σ_1 and σ_2.

Our calculations below show that breaking will occur eventually on the plasma boundary if it does not occur inside the plasma previously, provided the disturbance is sufficiently strong. To show this, we consider responses to step function disturbances and linear function disturbances.

For step function disturbances: $f(t) = \beta$ for $t>0$.

$$\sigma_1 = -\beta\frac{t^2}{2!}e^{-x} - 2\beta^2\frac{t^2}{2!}e^{-2x} ,$$

$$\sigma_2 = \beta\frac{t^4}{4!}(1 - \tfrac{1}{2}x)e^{-x} + \beta^2\frac{t^4}{4!}[-e^{-x} + (6-2x)e^{-2x}]$$

$$+\beta^3\frac{t^4}{4!}[-\tfrac{1}{4}e^{-x} - 4e^{-2x} + \tfrac{45}{4}e^{-3x}] + \beta^4\frac{t^4}{4!}[-e^{-2x} + 4e^{-4x}],$$

$$\sigma_3 = \beta\frac{t^6}{6!}(-1 + \tfrac{7}{8}x - \tfrac{1}{8}x^2)e^{-x}$$

$$+ \beta^2\frac{t^6}{6!}[(\tfrac{8}{9} - \tfrac{1}{2}x)e^{-x} + (-\tfrac{365}{36} + \tfrac{95}{6}x - \tfrac{7}{2}x^2)e^{-2x}]$$

$$+ \beta^3\frac{t^6}{6!}[(-\tfrac{121}{96} - \tfrac{1}{8}x)e^{-x} + (\tfrac{254}{9} - 14x)e^{-2x} + (-\tfrac{1709}{32} + \tfrac{105}{2}x)e^{-3x}]$$

$$+ \beta^4\frac{t^6}{6!}[-e^{-x} + (-\tfrac{87}{8} - \tfrac{7}{2}x)e^{-2x} + 105e^{-3x} + (-\tfrac{763}{6} + 28x)e^{-4x}]$$

$$+ \beta^5\frac{t^6}{6!}[-\tfrac{5}{24}e^{-x} - 10e^{-2x} + \tfrac{105}{4}e^{-3x} + 56e^{-4x} - \tfrac{2225}{24}e^{-5x}]$$

$$+ \beta^6\frac{t^6}{6!}[-\tfrac{19}{12}e^{-2x} + 14e^{-4x} - \tfrac{69}{4}e^{-6x}].$$

Accordingly,

$$\sigma_1(0,\ t) = -(\beta + 2\beta^2)\frac{t^2}{2!},$$

$$\sigma_2(0,\ t) = (\beta + 5\beta^2 + 7\beta^3 + 3\beta^4)\frac{t^4}{4!},$$

$$\sigma_3(0,\ t) = -(\beta + \tfrac{37}{4}\beta^2 + \tfrac{238}{9}\beta^3 + \tfrac{817}{24}\beta^4 + \tfrac{62}{3}\beta^5 + \tfrac{29}{6}\beta^6)\frac{t^6}{6!}.$$

Now, $1 + \sigma_1(0,\ t) + \sigma_2(0,\ t)$ vanishes when $t = \frac{\sqrt{4-\sqrt{8}}}{\beta} + 0(\frac{1}{\beta^2})$, while $\sigma_3(0,\ t)$ is equal to $-\frac{29}{6}\frac{(4-\sqrt{8})^3}{6!} + 0(\frac{1}{\beta})$. Therefore, for strong disturbances, namely $\beta \gg 1$, σ will vanish on the boundary approximately when $t = \frac{1.082}{\beta}$ and at that time, the magnitudes of the various terms are $\sigma_1 = -1.1716$, $\sigma_2 = -0.1716$ and $\sigma_3 = -0.0108$.

For linear function disturbances: $f(t) = \beta t$ for $t > 0$.

$$\sigma_1 = -\beta\frac{t^3}{3!}e^{-x} - 4\beta^2\frac{t^4}{4!}e^{-2x}$$

$$\sigma_2 = \beta\frac{t^5}{5!}(1-\tfrac{1}{2}x)e^{-x}$$
$$+\beta^2\frac{t^6}{6!}[-\tfrac{8}{3}e^{-x} + (\tfrac{56}{3} - 8x)e^{-2x}]$$
$$+\beta^3\frac{t^7}{7!}[-\tfrac{5}{2}e^{-x} - \tfrac{160}{3}e^{-2x} + \tfrac{285}{2}e^{-3x}]$$
$$+\beta^4\frac{t^8}{8!}[-60e^{-2x} + 240e^{-4x}] ,$$

$$\sigma_3 = \beta\frac{t^7}{7!}(-1 + \tfrac{7}{8}x - \tfrac{1}{8}x^2)e^{-x}$$
$$+\beta^2\frac{t^8}{8!}[(\tfrac{14}{9} - \tfrac{4}{3}x)e^{-x} + (-\tfrac{703}{18} + 63x - 13x^2)e^{-2x}]$$
$$+\beta^3\frac{t^9}{9!}[(-\tfrac{335}{18} - \tfrac{5}{4}x)e^{-x}+(\tfrac{1240}{3} -224x)e^{-2x}+(-\tfrac{1809}{2}+\tfrac{4011}{4}x)e^{-3x}]$$
$$+\beta^4\frac{t^{10}}{10!}[-\tfrac{620}{9}e^{-x}+(-\tfrac{9520}{9}-320x)e^{-2x}+9768e^{-3x}+(-12320+3136x)e^{-4x}]$$
$$+\beta^5\frac{t^{11}}{11!}[-\tfrac{165}{2}e^{-x}-5840e^{-2x}+15525e^{-3x}+34560e^{-4x}-\tfrac{117125}{2}e^{-5x}]$$
$$+\beta^6\frac{t^{12}}{12!}[-6450e^{-2x}+ 61200e^{-4x} - 79650e^{-6x}].$$

Accordingly,

$$\sigma_1(0,\ t) = -\beta\frac{t^3}{3!} - 4\beta^2\frac{t^4}{4!} ,$$

$$\sigma_2(0,\ t) = \beta\frac{t^5}{5!} + 16\beta^2\frac{t^6}{6!} + \frac{260}{3}\beta^3\frac{t^7}{7!} + 180\beta^4\frac{t^8}{8!} ,$$

$$\sigma_3(0,\ t) = -\beta\frac{t^7}{7!} - \frac{75}{2}\beta^2\frac{t^8}{8!} - \frac{4588}{9}\beta^3\frac{t^9}{9!}$$
$$- \frac{11036}{3}\beta^4\frac{t^{10}}{10!} - 14400\beta^5\frac{t^{11}}{11!} - 24900\beta^6\frac{t^{12}}{12!} .$$

Now, $1 + \sigma_1(0,\ t) + \sigma_2(0,\ t)$ vanishes when $t = \sqrt[4]{\frac{56-1120}{3}}\,\frac{1}{\sqrt{\beta}} + O(\frac{1}{\beta})$ while $\sigma_3(0,\ t)$ is equal to

$- \frac{24910}{12!}(\frac{56 - \sqrt{1120}}{3})^3 + O(\frac{1}{\sqrt{\beta}})$. Therefore, for strong disturbances, namely $\beta \gg 1$, σ will vanish on the boundary approximately when $t = \frac{1.655}{\sqrt{\beta}}$, and at that time the magnitudes of the various terms are $\sigma_1 = -1.2518$, $\sigma_2 = 0.2518$ and $\sigma_3 = -0.0220$.

Numerical calculations, displayed in Figures 2 through 6, show that breakings occur on the boundary for strong disturbances (compared with the initial magnetic field) and inside the plasma for weak disturbances. Figure 1 displays the magnitudes of the first three terms in the series on the boundary for a strong step function disturbance. Figures 2 through 6 are σ versus x plots with time as parameter. When the magnetic disturbance increases linearly in time as f(t) = t, our calculation, displayed in Figure 4, shows that breaking occurs at t=1.67. For the same disturbance, by direct numerical integration of partial differential equations, Adlam and Allen [5] obtained that the wave breaks at the boundary shortly after t=1.6, using a mesh size of $\Delta t = 0.2$. Both calculations are in good agreement.

4. THE GENERAL CASE

When the magnetic field has a non-zero longitudinal component, its transverse component can not be of plane polarization. We shall solve equations (4) with $G = \cos\theta$ supplemented by the initial conditions (5) and the boundary conditions (6). Following the procedure in the transverse case, we write the first equation of (4) as

$$\frac{\partial^2}{\partial t^2}\sigma = -\frac{1}{2}\frac{\partial^2}{\partial x^2}(B^2 + H^2) ,$$

with the initial conditions

$$\sigma\Big|_{t=0} = 1 , \qquad \frac{\partial\sigma}{\partial t}\Big|_{t=0} = 0 ,$$

write the second equation of (4) as

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} - 1\right) B = \frac{\partial^2}{\partial t^2}B(\sigma - 1) - \frac{\partial^2}{\partial x^2}\left(\Gamma\cos\theta \frac{\partial}{\partial t}H + \cos^2\theta\, B\right) ,$$

with the initial and boundary conditions

$$B\Big|_{t=0} = \sin\theta\cos\phi , \quad \frac{\partial B}{\partial t}\Big|_{t=0} = \left(\frac{d}{dt}f\cos\psi\right)_{t=0} e^{-x} ,$$

$$B\Big|_{x=0} = \sin\theta\cos\phi + f\cos\psi , \quad B\Big|_{x=\infty} = \sin\theta\cos\phi ,$$

and write the third equation of (4) as

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} - 1\right) H = \frac{\partial^2}{\partial t^2}H(\sigma - 1) + \frac{\partial^2}{\partial x^2}\left(\Gamma\cos\theta\frac{\partial}{\partial t}B - \cos^2\theta H\right) ,$$

with the initial and boundary conditions

$$H\Big|_{t=0} = \sin\theta\sin\phi , \quad \frac{\partial H}{\partial t}\Big|_{t=0} = \left(\frac{d}{dt}f\sin\psi\right)_{t=0} e^{-x} ,$$

$$H\Big|_{x=0} = \sin\theta\sin\phi + f\sin\psi , \quad H_{x=\infty} = \sin\theta\sin\phi .$$

To use Picard's iteration method, we take

$$\begin{aligned} \sigma^{(o)} &= 1 , \\ B^{(o)} &= \sin\theta\cos\phi + f\cos\psi\, e^{-x} , \qquad (27) \\ H^{(o)} &= \sin\theta\sin\phi + f\sin\psi\, e^{-x} \end{aligned}$$

as the initial approximation, and define the successive approximations by the solutions of the following equations.

$$\frac{\partial^2}{\partial t^2}\sigma^{(n)} = P^{(n)} \qquad (28)$$

with the initial conditions

$$\sigma^{(n)}\Big|_{t=0} = 1 , \qquad \frac{\partial\sigma^{(n)}}{\partial t}\Big|_{t=0} = 0 ,$$

and

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} - 1\right) B^{(n)} = Q^{(n)} \qquad (29)$$

with the initial and boundary conditions

$$B^{(n)}\Big|_{t=0} = \sin\theta\cos\phi , \quad \frac{\partial B^{(n)}}{\partial t}\Big|_{t=0} = \left(\frac{d}{dt}f\cos\psi\right)_{t=0} e^{-x} ,$$

$$B^{(n)}\Big|_{x=0} = \sin\theta\cos\phi + f\cos\psi , \quad B^{(n)}\Big|_{x=\infty} = \sin\theta\cos\phi ,$$

and

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} - 1\right) H^{(n)} = R^{(n)} \qquad (30)$$

with the initial and boundary conditions

$$H^{(n)}\Big|_{t=0} = \sin\theta\sin\phi , \quad \frac{\partial H^{(n)}}{\partial t}\Big|_{t=0} = \left(\frac{d}{dt}f\sin\psi\right)_{t=0} e^{-x} ,$$

$$H^{(n)}\Big|_{x=0} = \sin\theta\sin\phi + f\sin\psi , \quad H^{(n)}\Big|_{x=\infty} = \sin\theta\sin\phi ,$$

in which $P^{(n)}$, $Q^{(n)}$ and $R^{(n)}$ are specified functions determined by the chosen iteration scheme. In terms of the increments of iteration defined as

$$\sigma_o = \sigma^{(o)},$$
$$B_o = B^{(o)},$$
$$H_o = H^{(o)},$$
$$\sigma_n = \sigma^{(n)} - \sigma^{(n-1)} \qquad \text{for } n \geqslant 1,$$
$$B_n = B^{(n)} - B^{(n-1)} \qquad \text{for } n \geqslant 1,$$
$$H_n = H^{(n)} - H^{(n-1)} \qquad \text{for } n \geqslant 1,$$

we have

$$\sigma^{(n)} = \sum_{m=0}^{n} \sigma_m ,$$
$$B^{(n)} = \sum_{m=0}^{n} B_m ,$$
$$H^{(n)} = \sum_{m=0}^{n} H_m ;$$

and as $n \to \infty$

$$\sigma = \sum_{m=0}^{\infty} \sigma_m$$
$$B = \sum_{m=0}^{\infty} B_m$$
$$H = \sum_{m=0}^{\infty} H_m$$

Subtracting (28), (29) and (30) with n replaced by n-1 from (28), (29) and (30) respectively, we obtain the following equations for the increments.

$$(31) \qquad \frac{\partial^2}{\partial t^2}\sigma_n = P_n$$

with the initial conditions

$$\sigma_n\Big|_{t=0} \;, \qquad \frac{\partial \sigma_n}{\partial t}\Big|_{t=0} = 0 \;,$$

and

$$(32) \qquad \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} - 1\right) B_n = Q_n$$

with the initial and boundary conditions

$$B_n\Big|_{t=0} \;, \quad \frac{\partial B_n}{\partial t}\Big|_{t=0} \;, \quad B_n\Big|_{x=0} \;, \quad B_n\Big|_{x=\infty} = 0 \;,$$

and

$$(33) \qquad \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} - 1\right) H_n = R_n$$

with the initial and boundary conditions

$$H_n\Big|_{t=0} \;, \quad \frac{\partial H_n}{\partial t}\Big|_{t=0} \;, \quad H_n\Big|_{x=0} \;, \quad H_n\Big|_{x=\infty} = 0 \;,$$

in which

$$P_n = P^{(n)} - P^{(n-1)} \;,$$

$$Q_n = Q^{(n)} - Q^{(n-1)} \;,$$

$$R_n = R^{(n)} - R^{(n-1)} \;.$$

We use an iteration scheme such that

$$P_n = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{m=1}^{n}\left(B_{m-1}B_{n-m} + H_{m-1}H_{n-m}\right) \;,$$

$$(34)\qquad Q_n = \frac{\partial^2}{\partial t^2}\sum_{m=1}^{n}\sigma_m B_{n-m} - \frac{\partial^2}{\partial x^2}(\Gamma\cos\theta\,\frac{\partial}{\partial t}H_{n-1} + \cos^2\theta B_{n-1}),$$

$$R_n = \frac{\partial^2}{\partial t^2}\sum_{m=1}^{n}\sigma_m H_{n-m} + \frac{\partial^2}{\partial x^2}(\Gamma\cos\theta\,\frac{\partial}{\partial t}B_{n-1} - \cos^2\theta H_{n-1}),$$

corresponding to

$$P^{(n)} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[B^{(o)}B^{(n-1)} + H^{(o)}H^{(n-1)} + \sum_{m=1}^{n-1}(B^{(m)}-B^{(m-1)})B^{(n-m-1)} + \sum_{m=1}^{n-1}(H^{(m)}-H^{(m-1)})H^{(n-m-1)}\Big]$$

$$(35)\qquad = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{k=1}^{n}\sum_{m=1}^{k}(B_{m-1}B_{k-m} + H_{m-1}H_{k-m}),$$

$$Q^{(n)} = \frac{\partial^2}{\partial t^2}\sum_{m=1}^{n}(\sigma^{(m)}-\sigma^{(m-1)})B^{(n-m)} - \frac{\partial^2}{\partial x^2}(\Gamma\cos\theta\frac{\partial}{\partial t}H^{(n-1)}+\cos^2\theta B^{(n-1)})$$

$$= \frac{\partial^2}{\partial t^2}\sum_{k=1}^{n}\sum_{m=1}^{k}\sigma_m B_{k-m} - \frac{\partial^2}{\partial x^2}(\Gamma\frac{\partial}{\partial t}\sum_{m=1}^{n}H_{m-1} + \sum_{m=1}^{n}B_{m-1}),$$

$$R^{(n)} = \frac{\partial^2}{\partial t^2}\sum_{m=1}^{n}(\sigma^{(m)}-\sigma^{(m-1)})H^{(n-m)} + \frac{\partial^2}{\partial x^2}(\Gamma\cos\theta\frac{\partial}{\partial t}B^{(n-1)} - \cos^2\theta H^{(n-1)})$$

$$= \frac{\partial^2}{\partial t^2}\sum_{k=1}^{n}\sum_{m=1}^{k}\sigma_m H_{k-m} + \frac{\partial^2}{\partial x^2}(\Gamma\frac{\partial}{\partial t}\sum_{m=1}^{n}B_{m-1} - \sum_{m=1}^{n}H_{m-1}),$$

which approach $-\frac{1}{2}\frac{\partial^2}{\partial x^2}(B^2+H^2)$, $\frac{\partial^2}{\partial t^2}B(\sigma-1)-\frac{\partial^2}{\partial x^2}(\Gamma\cos\theta\frac{\partial}{\partial t}H+\cos^2\theta B)$ and $\frac{\partial^2}{\partial t^2}H(\sigma-1)+\frac{\partial^2}{\partial x^2}(\Gamma\cos\theta\,\frac{\partial}{\partial t}B-\cos^2\theta H)$ respectively as $n\to\infty$, because $\sum_{k=1}^{n}\sum_{m=1}^{k}B_{m-1}B_{k-m}$ has the same limit as $(\sum_{m=1}^{n}B_{m-1})^2$, and $\sum_{k=1}^{n}\sum_{m=1}^{k}\sigma_m B_{k-m}$ has the same limit as $(\sum_{m=1}^{n}\sigma_m)(\sum_{m=1}^{n}B_{m-1})$.

The same results using this iteration scheme can be obtained by substituting the following formal expansions

$$\sigma = \sigma_o + \sum_{m=1}^{\infty} a^m \sigma_m$$

(36) $$B = B_o + \sum_{m=1}^{\infty} a^m B_m$$

$$H = H_o + \sum_{m=1}^{\infty} a^m H_m$$

into the following equations

$$\frac{\partial^2}{\partial t^2}\sigma + \frac{1}{2}a\frac{\partial^2}{\partial x^2}(B^2 + H^2) = 0$$

(37) $$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2}B - \sigma B\right) + a\frac{\partial^2}{\partial x^2}\left(\Gamma\cos\theta\frac{\partial}{\partial t}H + \cos^2\theta B\right) = 0$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{a^2}{ax^2}H - \sigma H\right) + a\frac{\partial^2}{\partial x^2}\left(-\Gamma\cos\theta\frac{\partial}{\partial t}B + \cos^2\theta H\right) = 0$$

which are equations (4) with the insertion of a dummy parameter a. This dummy parameter serves in grouping various terms in the expansions only and eventually we shall put a=1. Equating the resultant coefficients of various powers of a in equations (37) after substitution of (36) to zero, we obtain

(38) $$\frac{\partial^2}{\partial t^2}\sigma_o = 0$$

with the initial conditions

$$\sigma_o\Big|_{t=0} = 1 , \qquad \frac{\partial\sigma_o}{\partial t}\Big|_{t=0} = 0 ,$$

and

(39) $$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} - \sigma_o\right) B_o = 0$$

with the initial and boundary conditions

$$B_o\Big|_{t=0} = \sin\theta\cos\phi\ , \quad \frac{\partial B_o}{\partial t}\Big|_{t=0} = \left(\frac{d}{dt}f\ \cos\psi\right)_{t=0} e^{-x}\ ,$$

$$B_o\Big|_{x=0} = \sin\theta\cos\phi + f\ \cos\psi\ , \quad B_o\Big|_{x=\infty} = \sin\theta\cos\phi,$$

and

(40) $$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} - \sigma_o\right) H_o = 0$$

with the initial and boundary conditions

$$H_o\Big|_{t=0} = \sin\theta\sin\phi\ , \quad \frac{\partial H_o}{\partial t}\Big|_{t=0} = \left(\frac{d}{dt}f\ \sin\psi\right)_{t=0} e^{-x}\ ,$$

$$H_o\Big|_{x=0} = \sin\theta\sin\phi + f\sin\psi\ , \quad H_o\Big|_{x=\infty} = \sin\theta\sin\phi\ ,$$

as well as equations (31), (32) and (33) with P_n, Q_n and R_n given by (34).

The solutions of (38), (39) and (40) are (27). By induction, P_n, Q_n and R_n are finite combinations of the form $\sum_{k=1}\sum_{\ell=0} a_{k,\ell}(t)x^\ell e^{-kx}$, $\sum_{k=1}\sum_{\ell=0} b_{k,\ell}(t)x^\ell e^{-kx}$, and $\sum_{k=1}\sum_{\ell=0} h_{k,\ell}(t)x^\ell e^{-kx}$. Therefore the solutions of (31), (32) and (33) have the following form

$$\sigma_n = \sum_{k=1}\sum_{\ell=0}[\,t\circ a_{k,\ell}(t)]x^\ell e^{-kx},$$

$$B_n = \sum_{k=1}\sum_{\ell=0}[\,t\circ b_{k,\ell}(t)][S_{k,\ell}(x)e^{-kx} - S_{k,\ell}(o)e^{-x}]\ ,$$

$$H_n = \sum_{k=1} \sum_{\ell=0} [t\circ h_{k,\ell}(t)][S_{k,\ell}(x)e^{-kx} - S_{k,\ell}(o)e^{-x}] ,$$

where $S_{k,\ell}(x)$ are polynomials defined in Chapter 3.

We display the first few terms of the solution as follows

$$B_o = \sin\theta \cos\phi + f \cos\psi e^{-x}$$

$$\begin{aligned} B_1 = &[\sin^2\theta\cos\phi\ t\circ f\cos(\psi-\phi)+\cos^2\theta t\circ f\cos\psi+\Gamma\cos\theta\circ f\sin\psi](\tfrac{1}{2}xe^{-x}) \\ &+ \sin\theta\cos\phi(t\circ f^2)(\tfrac{2}{3}e^{-x} - \tfrac{2}{3}e^{-2x}) \\ &+ f\cos\psi\ [t\circ f\sin\theta\cos(\psi-\phi)](\tfrac{1}{3}e^{-x} - \tfrac{1}{3}e^{-2x}) \\ &+ f\cos\psi\ (t\circ f^2)(\tfrac{1}{4}e^{-x} - \tfrac{1}{4}e^{-3x}) , \end{aligned}$$

$$H_o = \sin\theta\sin\phi + f\sin\psi e^{-x}$$

$$\begin{aligned} H_1 = &[\sin^2\theta\sin\phi\ t\circ f\cos(\psi-\phi)+\cos^2\theta t\circ f\sin\psi-\Gamma\cos\theta\circ f\cos\psi](\tfrac{1}{2}xe^{-x}) \\ &+ \sin\theta\sin\phi\ (t\circ f^2)(\tfrac{2}{3}e^{-x} - \tfrac{2}{3}e^{-2x}) \\ &+ f\sin\psi\ [t\circ f\sin\theta\cos(\psi-\phi)](\tfrac{1}{3}e^{-x} - \tfrac{1}{3}e^{-2x}) \\ &+ f\sin\psi(t\circ f^2)(\tfrac{1}{4}e^{-x} - \tfrac{1}{4}e^{-3x}) ; \end{aligned}$$

and

$$\sigma_o = 1,$$

$$\sigma_1 = -[t\circ f\sin\theta\cos(\psi-\phi)]e^{-x} - 2(t\circ f^2)e^{-2x} ,$$

$$\begin{aligned} \sigma_2 = &[t\circ t\circ f\sin\theta\cos(\psi-\phi)+\Gamma\circ t\circ f\sin\theta\cos\theta\sin(\psi-\phi)](1-\tfrac{1}{2}x)e^{-x} \\ &+ t\circ f\cos^2\theta\ [\cos\psi(t\circ f\cos\psi)+ \sin\psi(t\circ f\sin\psi)](2-2x)e^{-2x} \\ &+ t\circ f\cos\theta[\cos\psi(\Gamma\circ f\sin\psi) - \sin\psi(\Gamma\circ f\cos\psi)](2-2x)e^{-2x} \\ &+ t\circ t\circ f^2\sin^2\theta[-\tfrac{2}{3}e^{-x} + \tfrac{8}{3}e^{-2x}] \\ &+ t\circ f\sin\theta\cos(\psi-\phi)\ [t\circ f\sin\theta\cos(\psi-\phi)][-\tfrac{1}{3}e^{-x}+(\tfrac{10}{3}-2x)e^{-2x}] \\ &+ t\circ f\sin\theta\cos(\psi-\phi)(t\circ f^2)\ [-\tfrac{1}{4}e^{-x} - \tfrac{8}{3}e^{-2x} + \tfrac{33}{4}e^{-3x}] \\ &+ t\circ f^2[t\circ f\sin\theta\cos(\psi-\phi)][-\tfrac{4}{3}e^{-2x} + 3e^{-3x}] \end{aligned}$$

$$+ t\circ f^2(t\circ f^2)[-e^{-2x} + 4e^{-4x}] .$$

For strong disturbances, the dominant parts of σ_1, σ_2 and σ_3 are

$$\sigma_1 \approx t\circ f^2(-2e^{-2x}) ,$$

$$\sigma_2 \approx t\circ f^2(t\circ f^2)(-e^{-2x} + 4e^{-4x}) ,$$

$$\sigma_3 \approx t\circ f^2[t\circ f^2(t\circ f^2)] (\tfrac{1}{6}e^{-2x} + 2e^{-4x} - 6e^{-6x}) ,$$

because they are the parts which contain the strongest dependence on the disturbance f(t) and g(t). By induction, using the following recursive equations

$$\frac{\partial^2}{\partial t^2} \sigma_n \approx - \frac{\partial^2}{\partial x^2}(f\cos\psi e^{-x}B_{n-1} + f\sin\psi e^{-x}H_{n-1}) ,$$

$$\frac{\partial^2}{\partial t^2}(\frac{\partial^2}{\partial x^2} - 1) B_n \approx \frac{\partial^2}{\partial t^2} f\cos\psi e^{-x}\sigma_n ,$$

$$\frac{\partial^2}{\partial t^2}(\frac{\partial^2}{\partial x^2} - 1) H_n \approx \frac{\partial^2}{\partial t^2} f\sin\psi e^{-x}\sigma_n ,$$

with σ_n, $\frac{\partial\sigma_n}{\partial t}$, B_n, $\frac{\partial B_n}{\partial t}$, H_n, $\frac{\partial H_n}{\partial t} = 0$ at $t=0$ and B_n, $H_n = 0$ at $x=0$, it can be shown that

$$\sigma_n \approx (-)^n t\circ f^2[t\circ f^2(\ldots t\circ f^2 \ldots)]$$
$$\cdot[2ne^{-2nx} - (n-1)e^{-(2n-2)x} - \tfrac{1}{6}(n-2)e^{-(2n-4)x} - \tfrac{1}{12}(n-3)e^{-(2n-6)x} - \ldots]$$

with n times of convolutions

We conclude that breaking will occur approximately at the first zero of the following expression.

$$1 - 2t\circ f^2e^{-2x} + t\circ f^2(t\circ f^2)(-e^{-2x} + 4e^{-4x})$$

provided the disturbance is sufficiently strong.

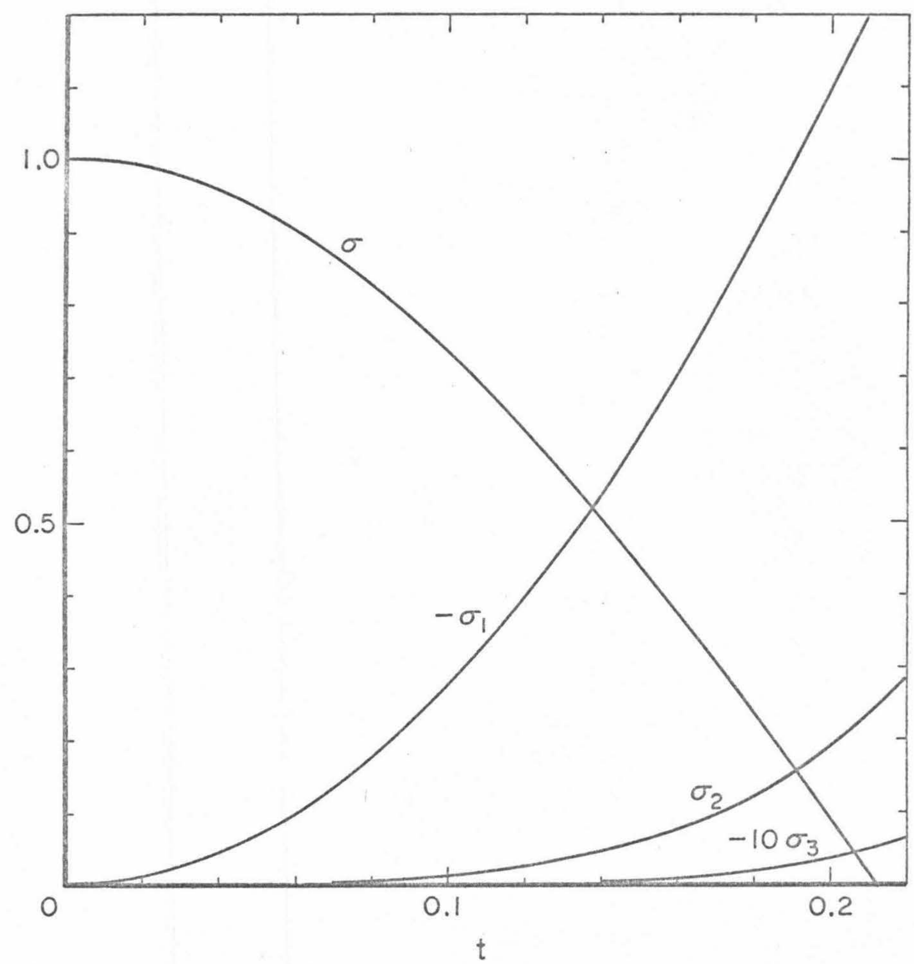

Figure 1. Plots of σ, $-\sigma_1$, σ_2 and $-10\sigma_3$ versus t at $x = 0$ for a step function disturbance $f = 5$ for $t > 0$.

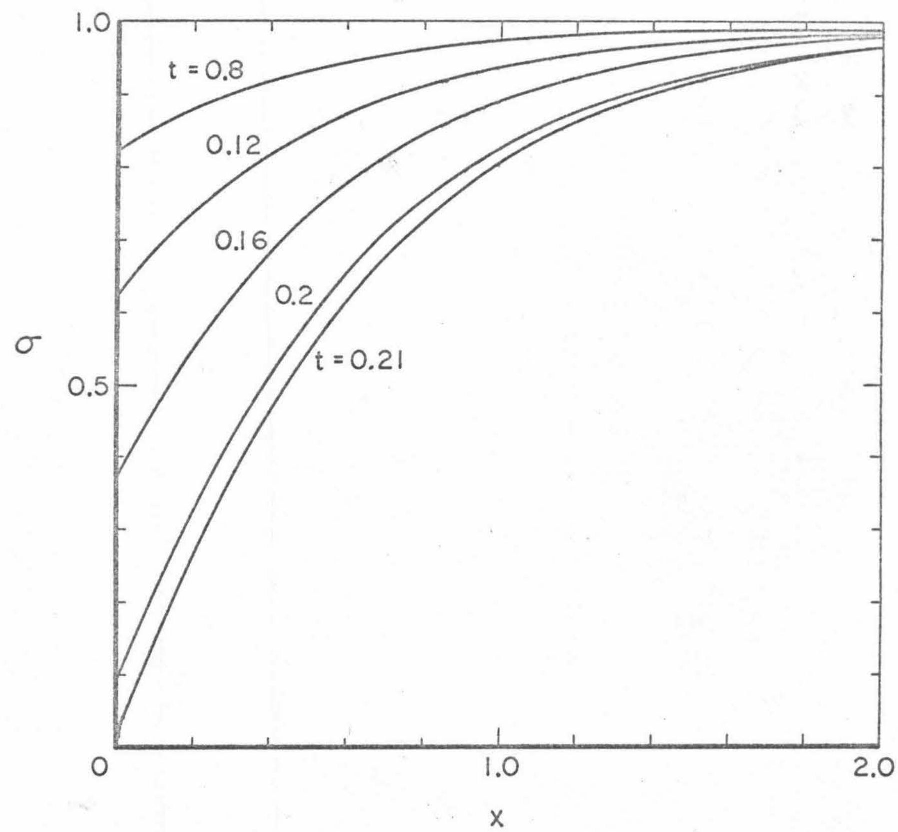

Figure 2. Plots of σ versus x for a strong step function disturbance f = 5 for t > 0.

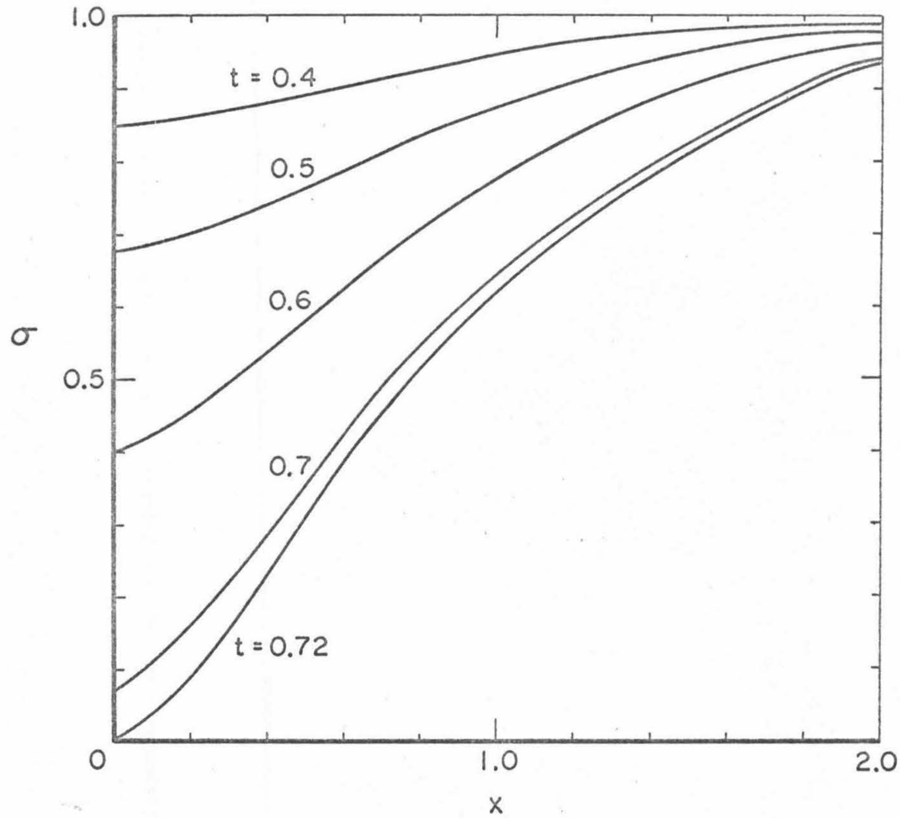

Figure 3. Plots of σ versus x for a strong linear function disturbance $f = 5t$.

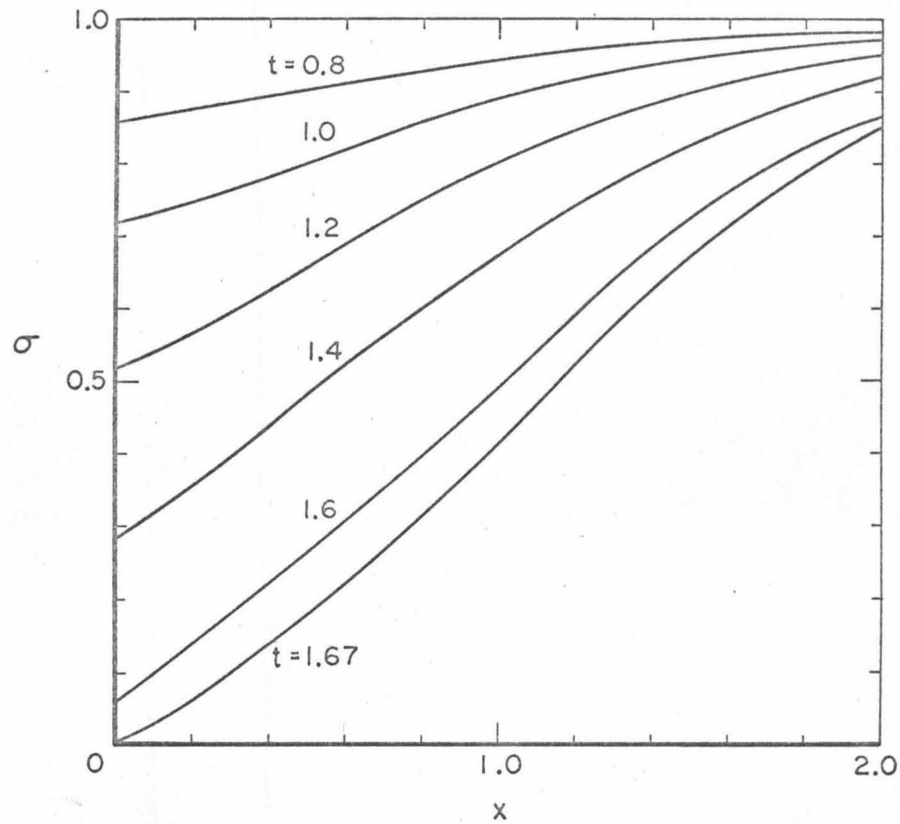

Figure 4. Plots of σ versus x for a strong linear function disturbance f = t.

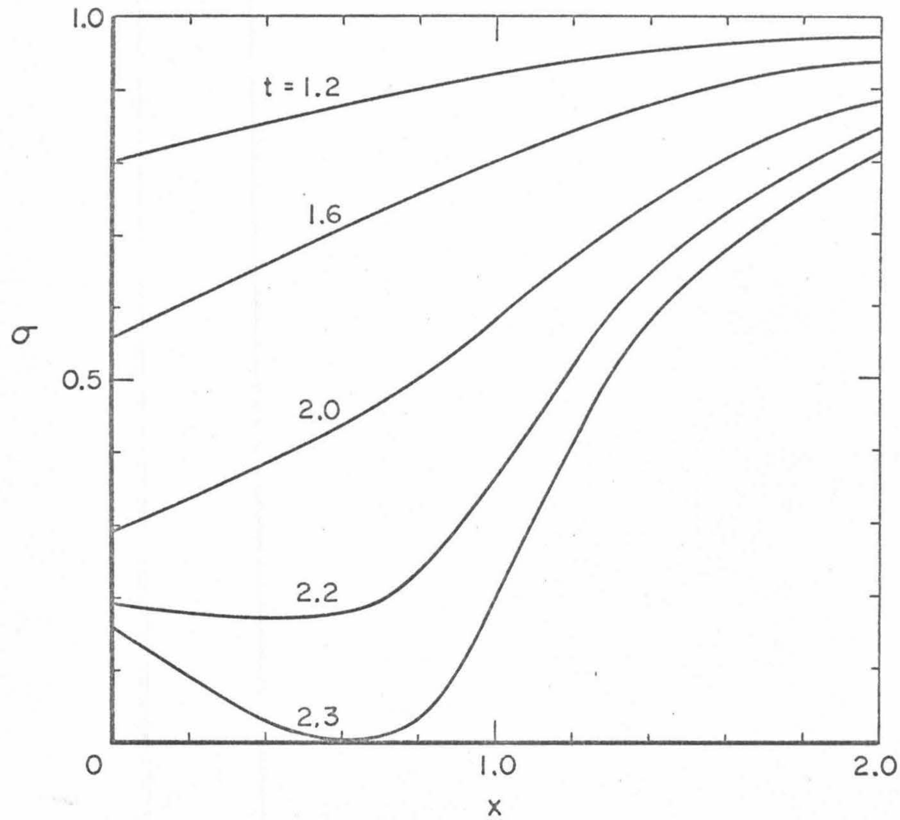

Figure 5. Plots of σ versus x for a weak linear function disturbance $f = \frac{1}{2}t$.

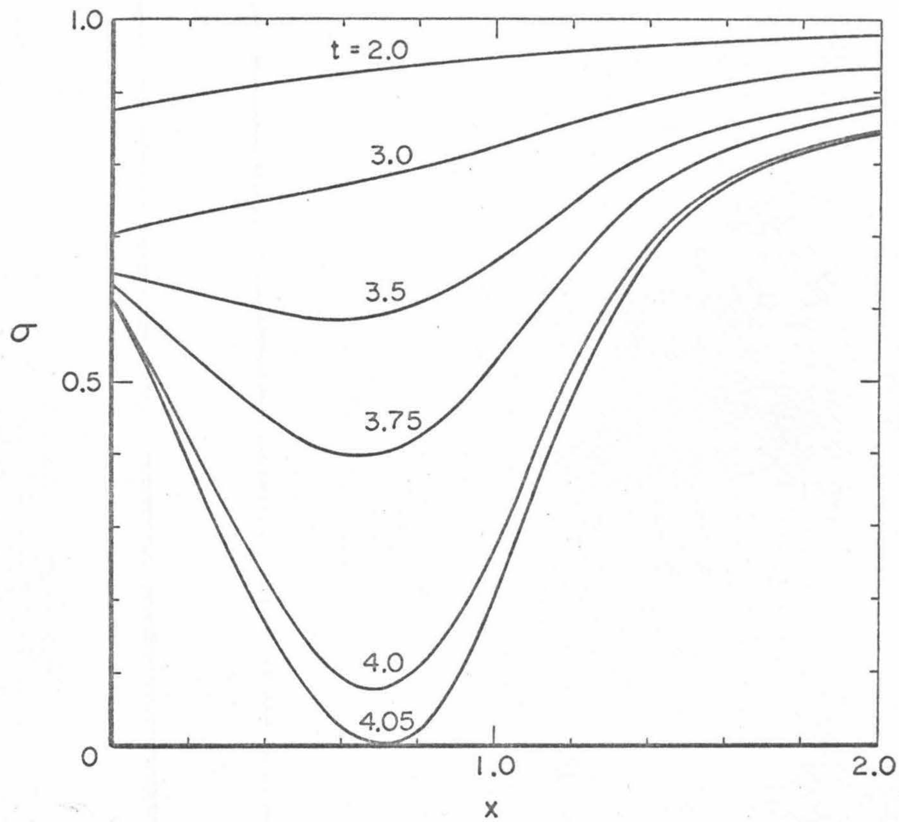

Figure 6. Plots of σ versus x for a weak linear function disturbance $f = \frac{1}{10}t$.

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PART III

ON STABILITIES OF PERIODIC WAVES IN A COLD COLLISION-FREE PLASMA IN A MAGNETIC FIELD

1. INTRODUCTION

Recently there have been some discussions on weak interactions of nonlinear dispersive waves. Benjamin and Feir [1] have found the instability of Stokes waves on deep water with respect to a disturbance of sideband frequencies. Zakharov [2] has found the instability of Langmuir oscillations in an isothermal plasma without a magnetic field and the instability of ion-sound waves in a cold ion gas imbedded in an isothermal electron gas. For Stokes waves on water of finite depth, Whitham [3] and Benjamin [4] have found a cutoff frequency such that the wave is stable or unstable according to whether the wave frequency is less or greater than the cutoff frequency.

In this paper we analyze the stabilities of periodic waves in a cold plasma in a magnetic field. In the limit of linear theory, there exist two modes of waves in a cold quasi-neutral plasma: slow waves and fast waves, characterized by different dispersion relations. In general when the main wave and the disturbance waves are in different modes or when their frequencies differ largely, the coupling between them is weak so that the main wave is stable from the disturbances. However when they are in the same mode and their frequencies differ only slightly, the coupling between them may be strong enough to cause large transfer of energy from the main wave to the disturbance waves with the consequence of instability of the main wave. Our results show that slow waves are always unstable, whereas fast waves have a cutoff frequency for their stabilities.

In Chapter 2, we show that waves of permanent form in the Eulerian coordinates also appear as waves of permanent form in the Lagrangian coordinates and vice versa.This allows us to do the stability analysis of periodic waves in either coordinates. It is simpler to do the calculations in the Lagrangian coordinates, since in those coordinates the equations of plasma motion can be reduced to those for the inverse mass density and the magnetic field only. In Chapter 3, we demonstrate the existence of periodic waves, which are propagated at an oblique angle to the magnetic field, by constructing their Stokes expansions in amplitude. The stability analysis is presented in Chapter 4.

Both water waves and plasma waves have a cutoff frequency, beyond which they are unstable to sideband disturbances. The cutoff frequency depends on the water depth or on the electron's gyrofrequency. In this parallelism plasma waves propagating across a magnetic field correspond to water waves on shallow water, both are stable. Indeed this parallelism is clear from Whitham's paper [5] in which he has shown that the averaged equations for Boussinesq equations for shallow water waves and those for plasma waves of transverse propagation are identical in structure.

2. WAVES OF PERMANENT FORM

The governing equations for one-dimensional unsteady motion of a plasma in a magnetic field have been derived as (1) and (2) in PART II. The contact transformation employed between the Eulerian coordinates and the Lagrangian coordinates, together with the continuity equation gives such a kinematic property that waves of permanent form in the Eulerian coordinates also appear as waves of permanent form in the Lagrangian coordinates and vice versa.

Suppose a wave of permanent form in the Eulerian coordinates is described in terms of a phase variable

$$\Phi = WT + KX$$

in which W and K are the frequency and wavenumber in the Eulerian coordinates. Then

(1) $$d\Phi = W\,dT + K\,dX\,.$$

Using the contact transformation that $dT = dt$ and $dX = \sigma\,dx + u\,dt$, we get

$$\rho\,d\Phi = \rho(W + Ku)\,dt + K\,dx\,.$$

The continuity equation in the Eulerian coordinates

$$\frac{d}{d\Phi}(W\rho + K\rho u) = 0$$

can be integrated to

$$\rho(W + Ku) = W + KU$$

in which U is the value of u at the place where ρ is equa to 1. Accordingly

(2) $$\rho\,d\Phi = (W + KU)\,dt + K\,dx\,.$$

Therefore, from (1) and (2), $d\Phi = 0$ gives the phase velocity in the Eulerian coordinates

(3) $$\frac{dX}{dT} = -\frac{W}{K}$$

and the corresponding phase velocity in the Lagrangian coordinates

(4) $$\frac{dx}{dt} = -\frac{W}{K} - U$$

which is a constant. Hence the wave also has permanent form in the Lagrangian coordinates.

Conversely suppose a wave of permanent form in the Lagrangian coordinates is described by a phase variable

$$\phi = \omega t + kx$$

in which ω and k are the frequency and wavenumber in the Lagrangian coordinates. Then

(5) $$d\phi = \omega\, dt + k\, dx .$$

Using the inverse transformation that $dt = dT$ and $dx=\rho(dX - u\, dT)$, we get

$$\sigma\, d\phi = (\omega\sigma - ku)\, dT + k\, dX.$$

The continuity equation in the Lagrangian coordinates

$$\frac{d}{d\phi}(\omega\sigma - ku) = 0$$

can be integrated to

$$\omega\sigma - ku = \omega - kU.$$

using the fact that $u=U$ at the place where $\sigma = \frac{1}{\rho} = 1$. Accordingly

(6) $\sigma\, d\phi = (\omega - kU)\, dT + k\, dX.$

Therefore, from (5) and (6), $d\phi=0$ gives the phase velocity in the Lagrangian coordinates

(7) $$\frac{dx}{dt} = -\frac{\omega}{k}$$

and the corresponding phase velocity in the Eulerian coordinates

(8) $$\frac{dX}{dT} = -\frac{\omega}{k} + U$$

which is a constant. Hence the wave also has permanent form in the Eulerian coordinates.

From (3) and (4), or (7) and (8), we have

$$\frac{dX}{dT} = \frac{dx}{dt} + U .$$

Thus the phase velocities in the two coordinates differ by U which may be identified as the streaming velocity of the plasma. If we identify the frequency in the Lagrangian coordinates with the frequency in the Eulerian coordinates

$$\omega = W ,$$

then from (5) and

$$\frac{W}{W + KU}\rho\, d\Phi = W\, dt + \frac{WK}{W + KU}\, dx ,$$

it is seen that the wavenumber in the Lagrangian coordinates is related to the wavenumber in the Eulerian coordinates by

$$k = \frac{W}{W + KU} K$$

and the two phase variables are related by

$$d\phi = \frac{W}{W + KU} \rho \, d\Phi$$

which can be integrated to give ϕ as a function of Φ , because ρ is a function of Φ .

3. PERIODIC WAVES OF OBLIQUE PROPAGATION

It is well known [6] that waves of permanent form can be propagated across a magnetic field. All such waves of transverse propagation are periodic waves with symmetry at wave crests and wave troughs, including the solitary waves as the limit case when the wavelength is infinite. It is also known [7] that periodic waves can be propagated along a magnetic field. We shall show that the equations of plasma motion also admit a solution in the form of Stokes expansion in amplitude which represents a periodic wave propagating at an oblique angle to the magnetic field.

For waves of permanent form with phase variable $\phi=\omega t + kx$, equations (3) in PART II become

$$
(9)\quad
\begin{aligned}
&\frac{d}{d\phi}(\omega\sigma - ku) = 0 \quad ,\\
&\frac{d}{d\phi}[\omega u + \tfrac{1}{2}k(B^2 + H^2)] = 0 \quad ,\\
&\frac{d}{d\phi}(\omega v - kGB) = 0 \quad ,\\
&\frac{d}{d\phi}(\omega w - kGH) = 0 \quad ,\\
&\frac{d}{d\phi}(\omega k^2\frac{d^2}{d\phi^2}B - \omega\sigma B + \Gamma k^2 G\frac{d}{d\phi}H + kGv) = 0 \ ,\\
&\frac{d}{d\phi}(\omega k^2\frac{d^2}{d\phi^2}H - \omega\sigma H - \Gamma k^2 G\frac{d}{d\phi}B + kGw) = 0 \ .
\end{aligned}
$$

Suppose u, v, w, B, H take the values U, V, W, b, h at the place where σ is equal to 1, then equations (9) have the following integrals

$$\omega\sigma - ku = \omega - kU ,$$

$$\omega u + \frac{1}{2}k(B^2 + H^2) = \omega U + \frac{1}{2}k(b^2 + h^2) ,$$

$$\omega v - kGB = \omega V - kGb ,$$

$$\omega w - kGH = \omega W - kGh .$$

Accordingly,

$$\omega^2\sigma + \frac{1}{2}k^2(B^2 + H^2) = \omega^2 + \frac{1}{2}k^2(b^2 + h^2) ,$$

$$(10) \quad \frac{d}{d\phi}\left\{\omega^2k^2\frac{d^2}{d\phi^2}B + \Gamma\omega k^2G\frac{d}{d\phi}H + [-\omega^2+k^2(G^2 - \frac{b^2+h^2}{2} + \frac{B^2+H^2}{2})]B\right\} = 0 ,$$

$$\frac{d}{d\phi}\left\{\omega^2k^2\frac{d^2}{d\phi^2}H - \Gamma\omega k^2G\frac{d}{d\phi}B + [-\omega^2+k^2(G^2 - \frac{b^2+h^2}{2} + \frac{B^2+H^2}{2})]H\right\} = 0 .$$

We shall use the quiescent values of the mass dendity and the magnitude of the magnetic field as the reference values in normalization. Hence $G = \cos\theta$, $b = \sin\theta\cos\Psi$, $h = \sin\theta\sin\Psi$ in which θ is the propagation angle.

Equations (10) are invariant under the transformation $B \to B$, $H \to -H$ and $\phi \to -\phi$, hence with the condition $\left.\frac{dB}{d\phi}\right|_{\phi=0}=0$ and $H\big|_{\phi=0}=0$ the solutions are reflectively symmetric with respect to the plane $\phi=0$. Among these symmetric solutions, there are periodic solutions, as shown later, which represent periodic waves propagating at an oblique angle to the magnetic field.

To construct periodic solutions, we substitute the following expansions

$$B = b + a\,B^{(1)} + a^2B^{(2)} + \dots ,$$

(11) $$H = h + a\,H^{(1)} + a^2 H^{(2)} + \dots ,$$
$$\omega = \Omega + a\,\Omega^{(1)} + a^2\Omega^{(2)} + \dots ,$$

into equations (10). The equations satisfied by $B^{(1)}$ and $H^{(1)}$ are

$$\frac{d}{d\phi}\left[\left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2+k^2G^2+k^2b^2\right)B^{(1)}+\left(\Gamma\Omega k^2G\frac{d}{d\phi} +k^2bh\right)H^{(1)}\right] = 0 ,$$

$$\frac{d}{d\phi}\left[\left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2+k^2G^2+k^2h^2\right)H^{(1)}+\left(-\Gamma\Omega k^2G\frac{d}{d\phi} + k^2bh\right)B^{(1)}\right]= 0 ,$$

which can be combined to give

(12) $$\frac{d}{d\phi}\Big[\left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2+k^2G^2+k^2b^2\right)\left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2+k^2G^2+k^2h^2\right)$$
$$+ \Gamma^2\Omega^2k^4G^2\frac{d^2}{d\phi^2} - k^4b^2h^2\Big]B^{(1)} = 0 .$$

Equation (12) admits sinusoidal functions as solutions. We may normalize the period to 2π by making Ω satisfy the following dispersion relation

$$[\Omega^2(k^2+1)-k^2(G^2+b^2)][\Omega^2(k^2+1)-k^2(G^2+h^2)]-\Gamma^2\Omega^2k^4G^2-k^4b^2h^2 = 0 .$$

With Ω so satisfied, the general solution of equation (12) is a linear combination of 1, $\cos\phi$, $\sin\phi$, $\cos\nu\phi$ and $\sin\nu\phi$ in which $\nu = \sqrt{(1+G^2+\Gamma^2G^2)/\Omega^2 - 2/k^2 - 1}$. Periodicity of period 2π rules out the appearance of either $\cos\nu\phi$ or $\sin\nu\phi$ in the solution. Hence the solution can be written as

$$B^{(1)} = b_{10} + \mathrm{Re}\, b_{11}e^{i\phi}$$

and accordingly,

$$H^{(1)} = h_{10} + \mathrm{Re}\, ih_{11}e^{i\phi}$$

where b_{10} and h_{10} are undetermined real-valued constants, b_{11} is an undetermined complex-valued constant, and

$$h_{11} = \frac{-\Gamma\Omega k^2G - ik^2bh}{\Omega^2(k^2+1) - k^2(G^2+h^2)}b_{11} = \frac{-\Omega^2(k^2+1)+k^2(G^2+b^2)}{\Gamma\Omega k^2G - ik^2bh}b_{11} .$$

The equations satisfied by $B^{(2)}$ and $H^{(2)}$ are

$$\frac{d}{d\phi}\left[\left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2+k^2G^2+k^2b^2\right)B^{(2)}+\left(\Gamma\Omega k^2G\frac{d}{d\phi} + k^2bh\right)H^{(2)}+P^{(2)}\right]=0,$$

(13)

$$\frac{d}{d\phi}\left[\left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2+k^2G^2+k^2h^2\right)H^{(2)}+\left(-\Gamma\Omega k^2G\frac{d}{d\phi} +k^2bh\right)B^{(2)}+Q^{(2)}\right]=0,$$

where

$$P^{(2)}=\Omega^{(1)}\left[2\Omega\left(k^2\frac{d^2}{d\phi^2}-1\right)B^{(1)}+\Gamma k^2G\frac{d}{d\phi}H^{(1)}\right]+k^2\left[b\left(\tfrac{3}{2}B^{(1)}B^{(1)}+\tfrac{1}{2}H^{(1)}H^{(1)}\right)+hB^{(1)}H^{(1)}\right]$$

$$= P_{20} + \mathrm{Re}(P_{21}e^{i\phi} + P_{22}e^{i2\phi}) ,$$

$$Q^{(2)}=\Omega^{(1)}\left[2\Omega\left(k^2\frac{d^2}{d\phi^2}-1\right)H^{(1)}-\Gamma k^2G\frac{d}{d\phi}B^{(1)}\right]+k^2\left[h\left(\tfrac{3}{2}H^{(1)}H^{(1)}+\tfrac{1}{2}B^{(1)}B^{(1)}\right)+bH^{(1)}B^{(1)}\right]$$

$$= Q_{20} + \mathrm{Re}(iQ_{21}e^{i\phi} + iQ_{22}e^{i2\phi}) ,$$

in which

$$P_{21}=-\Omega^{(1)}[2\Omega(k^2+1)b_{11}+\Gamma k^2Gh_{11}]+k^2[b(3b_{10}b_{11}+ih_{10}h_{11})+h(h_{10}b_{11}+ib_{10}h_{11})],$$

$$P_{22}= k^2[b(\tfrac{3}{4}b_{11}^2- \tfrac{1}{4}h_{11}^2) + i\tfrac{1}{2}hb_{11}h_{11}] ,$$

$$Q_{21}=-\Omega^{(1)}[2\Omega(k^2+1)h_{11}+\Gamma k^2Gb_{11}]+k^2[h(3h_{10}h_{11}-ib_{10}b_{11})+b(b_{10}h_{11}-ih_{10}b_{11})],$$

$$Q_{22}= k^2[ih(\tfrac{3}{4}h_{11}^2 - \tfrac{1}{4}b_{11}^2) + \tfrac{1}{2}bh_{11}b_{11}] .$$

Equations (13) can be combined to give

$$\frac{d}{d\phi}\left\{\left[\left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2 + k^2G^2 + k^2b^2\right)\left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2 + k^2G^2 + k^2h^2\right) + \Gamma^2\Omega^2k^4G^2\frac{d^2}{d\phi^2} - k^4b^2h^2\right]B^{(2)} + R^{(2)}\right\} = 0$$

where

$$R^{(2)} = \left(\Omega^2k^2\frac{d^2}{d\phi^2} - \Omega^2 + k^2G^2 + k^2h^2\right)P^{(2)} - \left(\Gamma\Omega k^2G\frac{d}{d\phi} + k^2bh\right)Q^{(2)}$$

$$= \mathrm{Re}\ (R_{20} + R_{21}e^{i\phi} + R_{22}e^{i2\phi}) \ ,$$

in which

$$R_{21} = -[\Omega^2(k^2+1) - k^2(G^2+h^2)]P_{21} + (\Gamma\Omega k^2G - ik^2bh)Q_{21} \ ,$$

$$R_{22} = -[\Omega^2(4k^2+1) - k^2(G^2+h^2)]P_{22} + (2\Gamma\Omega k^2G - ik^2bh)Q_{22} \ .$$

To avoid secular term, it requires that $R_{21}=0$. Then the solutions for (39) are

$$B^{(2)} = b_{20} + \mathrm{Re}\ (b_{21}e^{i\phi} + b_{22}e^{i2\phi}) \ ,$$

$$H^{(2)} = h_{20} + \mathrm{Re}\ (ih_{21}e^{i\phi} + ih_{22}e^{i2\phi}) \ ,$$

where b_{20} and h_{20} are undetermined real-valued constants, b_{21} is an undetermined complex-valued constant, and

$$b_{22} = \frac{-R_{22}}{[\Omega^2(4k^2+1) - k^2(G^2+b^2)][\Omega^2(4k^2+1) - k^2(G^2+h^2)] - 4\Gamma^2\Omega^2k^4G^2 - k^4b^2h^2} \ ,$$

$$h_{21} = \frac{Q_{21} - (\Gamma\Omega k^2G + ik^2bh)\ b_{21}}{\Omega^2(k^2+1) - k^2(G^2+h^2)} \ ,$$

$$h_{22} = \frac{Q_{22} - (2\Gamma\Omega k^2G + ik^2bh)\ b_{22}}{\Omega^2(4k^2+1) - k^2(G^2+h^2)} \ .$$

In general

$$B^{(n)} = b_{no} + \mathrm{Re} \sum_{m=1}^{n} b_{nm} e^{im\phi} ,$$

$$H^{(n)} = h_{no} + \mathrm{Re}\ i \sum_{m=1}^{n} h_{nm} e^{im\phi} ,$$

The four sequences of undetermined constants $\{\Omega^{(1)}, \Omega^{(2)}, \ldots\}$, $\{b_{10}, b_{20}, \ldots\}$, $\{h_{10}, h_{20}, \ldots\}$ and $\{b_{11}, b_{21}, \ldots\}$ are to be determined by three constants which characterize the wave, say the values of B, H and $\frac{dB}{d\phi}$ at $\phi=0$, and the requirement that $R_{n1}=0$ to ensure the boundedness of the solution. Suppose $B\big|_{\phi=0}$, $H\big|_{\phi=0}$ and $\frac{dB}{d\phi}\Big|_{\phi=0}$ have the following expansions

$$B\Big|_{\phi=0} = b + a\,\beta^{(1)} + a^2\beta^{(2)} + \ldots ,$$

$$H\Big|_{\phi=0} = h + a\,\gamma^{(1)} + a^2\gamma^{(2)} + \ldots ,$$

$$\frac{dB}{d\phi}\Big|_{\phi=0} = a\,\beta'^{(1)} + a^2\beta'^{(2)} + \ldots .$$

Then b_{10}, b_{11}, h_{10} and $\Omega^{(1)}$ are determined by the following algebraic equations

$$b_{10} + \mathrm{Re}\, b_{11} = \beta^{(1)} ,$$

$$\mathrm{Re}\ ib_{11} = \beta'^{(1)} ,$$

$$h_{10} + \mathrm{Re}\ ih_{11} = \gamma^{(1)} ,$$

$$R_{21} = 0 ,$$

recalling that h_{11} and R_{21} are known in terms of b_{10}, b_{11}, h_{10} and $\Omega^{(1)}$. And b_{20}, b_{21}, h_{20} and $\Omega^{(2)}$ are determined by

$$b_{20} + \text{Re}\,(b_{21} + b_{22}) = \beta^{(2)} ,$$

$$\text{Re}\; i(b_{21} + 2b_{22}) = \beta'^{(2)} ,$$

$$h_{20} + \text{Re}\; i(h_{21} + h_{22}) = \gamma^{(2)} ,$$

$$R_{31} = 0 .$$

In general, b_{no}, b_{n1}, h_{no} and $\Omega^{(n)}$ are determined by

$$b_{no} + \text{Re} \sum_{m=1}^{n} b_{nm} = \beta^{(n)} ,$$

$$\text{Re}\; i \sum_{m=1}^{n} m b_{nm} = \beta'^{(n)} ,$$

$$h_{no} + \text{Re}\; i \sum_{m=1}^{n} h_{nm} = \gamma^{(n)} ,$$

$$R_{n+1,1} = 0 .$$

Both $b_{n,1}$ and $R_{n+1,1}$ have real parts and imaginary parts. Thus at each stage there are five algebraic equations to determine the five undetermined constants, b_{no}, Re b_{n1}, Im b_{n1}, h_{no} and $\Omega^{(n)}$.

Finally for symmetric waves satisfying the conditions

$$B\Big|_{\phi=0} = \sin\theta + a ,$$

$$H\Big|_{\phi=0} = 0 ,$$

$$\frac{dB}{d\phi}\Big|_{\phi=0} = 0 ,$$

this corresponding to a choice of $\psi=0$, we may make $B^{(n)}$ a linear combination of $\cos n\phi$, $\cos(n-2)\phi, \ldots \; {}^{1}_{\cos}\,\phi$, and $H^{(n)}$

a linear combination of $\sin n\phi$, $\sin(n-2)\phi, \ldots \frac{\sin 2\phi}{\sin \phi}$ if n is $\frac{\text{even}}{\text{odd}}$, and $\Omega^{(n)}$ equal to zero if n is odd by requiring that $\operatorname{Im} R_{n1}=0$ for all n. Then $B^{(n)}\big|_{\phi=\pi} = (-)^n \beta^{(n)}$ and $H^{(n)}\big|_{\phi=\pi} = 0$ which show that $B=\sin\theta - a$, $H=0$ at $\phi=\pi$. We display the results below for the first two terms.

$$[\Omega^2(k^2+1)-k^2][\Omega^2(k^2+1)-k^2\cos^2\theta] - \Gamma^2\Omega^2k^4\cos^2\theta = 0 ,$$

$$b_{10} = 0 \quad , \quad b_{11} = 1 \quad ,$$

$$h_{10} = 0 \quad , \quad h_{11} = \frac{-\Gamma\Omega k^2\cos\theta}{\Omega^2(k^2+1)-k^2\cos^2\theta} \quad ,$$

$$b_{20} = -b_{22} \, , \quad b_{21} = 0 \quad ,$$

$$b_{22} = \frac{k^2[\Omega^2(4k^2+1)-k^2\cos^2\theta](\frac{3}{4} - \frac{1}{4}h_{11}^2)\sin\theta - \Gamma\Omega k^4 h_{11}\sin\theta\cos\theta}{[\Omega^2(4k^2+1)-k^2][\Omega^2(4k^2+1)-k^2\cos^2\theta]-4\Gamma^2\Omega^2k^4\cos^2\theta} ,$$

$$h_{20}=0, \quad h_{21}=0, \quad h_{22}= \frac{\frac{1}{2}k^2h_{11}\sin\theta - 2\Gamma\Omega k^2 b_{22}\cos\theta}{\Omega^2(4k^2+1) - k^2\cos^2\theta} .$$

Using the first equation of (10), we obtain

$$\sigma = 1 + a\,\sigma^{(1)} + a^2\sigma^{(2)} + \ldots$$

where

$$\sigma^{(1)} = -\operatorname{Re} s_{11}e^{i\phi} ,$$

$$\sigma^{(2)} = -\operatorname{Re}(s_{20} + s_{22}e^{i2\phi}) ,$$

in which

$$s_{11} = \frac{k^2}{\Omega^2}\sin\theta \quad ,$$

$$s_{20} = \frac{k^2}{\Omega^2}(\frac{1}{4} + \frac{1}{4}h_{11}^2 + b_{20}\sin\theta) ,$$

$$s_{22} = \frac{k^2}{\Omega^2}(b_{22}\sin\theta + \frac{1}{4} - \frac{1}{4}h_{11}^2) .$$

In general, $\sigma^{(n)}$ is a linear combination of $\cos n\phi$, $\mathrm{cis}(n-2)\phi$,... $\begin{matrix}1\\ \cos\phi\end{matrix}$ if n is $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$.

For the transverse propagation $\theta = \frac{\pi}{2}$, the above results become

$$\Omega^2(k^2+1) - k^2 = 0 \quad ,$$

$$b_{11} = 1 \quad , \quad b_{20} = -\frac{k^2+1}{4k^2} \quad , \quad b_{22} = \frac{k^2+1}{4k^2} \quad ,$$

$$s_{11} = \frac{k^2}{\Omega^2} \quad , \quad s_{20} = -\frac{1}{4\Omega^2} \quad , \quad s_{22} = \frac{2k^2+1}{4\Omega^2} \quad ,$$

which is the expansion of the solution discussed by Davis, Lüst and Schlüter [6] .

For the longitudinal propagation $\theta=0$, the above results become

$$[\Omega^2(k^2+1) - k^2]^2 - \Gamma^2\Omega^2k^4 = 0 \quad ,$$

$$b_{11} = 1 \quad , \quad b_{20} = 0 \quad , \quad b_{22} = 0 \quad ,$$

$$h_{11} = \pm 1 \quad , \quad h_{20} = 0 \quad , \quad h_{22} = 0 \quad ,$$

$$s_{11} = 0 \quad , \quad s_{20} = \frac{1}{2}\frac{k^2}{\Omega^2} \quad , \quad s_{22} = 0 \quad ,$$

which is the expansion of the solution $\sigma = 1 - \frac{1}{2}a^2\,\frac{k^2}{\omega^2}$, $B = a\cos\phi$, $H = \pm a \sin\phi$ with $\omega^2(k^2+1) - k^2(1+\frac{1}{2}a^2) \mp \Gamma\omega k^2 = 0$ discussed by Ferraro [7] .

4. STABILITY ANALYSIS OF PERIODIC WAVES

The governing equations for plasma motion in a magnetic field have been derived in PART II as

$$\frac{\partial^2}{\partial t^2}\sigma + \frac{1}{2}\frac{\partial^2}{\partial x^2}(B^2 + H^2) = 0 ,$$

$$(14) \qquad \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2}B - \sigma B\right) + \frac{\partial^2}{\partial x^2}\left(\Gamma \cos\theta \frac{\partial}{\partial t}H + \cos^2\theta\, B\right) = 0 ,$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2}H - \sigma H\right) + \frac{\partial^2}{\partial x^2}\left(-\Gamma \cos\theta \frac{\partial}{\partial t}B + \cos^2\theta\, H\right) = 0 .$$

We consider wave motions which can be decomposed into components that are periodic in space. Suppose that there are three main components with wavenumbers k_1, k_2 and k_3 related by

$$k_1 = k \;, \quad k_2 = k + \kappa \;, \quad k_3 = k - \kappa$$

Their amplitudes have a magnitude of order $O(a)$. The self-interaction of the component of wavenumber k_n will produce components of wavenumber Nk_n whose amplitudes have a magnitude of order $O(a^N)$. Then the mutual interaction will produce components of wavenumbers $Nk_n \pm Mk_m$ whose amplitudes have a magnitude of order $O(a^{N+M})$. Therefore we may write

$$\sigma = \sigma_o + \mathrm{Re}\left(\sum_{n=1}^{3}\sigma_n e^{ik_n x} + \sum_{nm}\sigma_{nm}e^{ik_{nm}x}\right) + O(a^3) ,$$

$$(15) \qquad B = B_o + \mathrm{Re}\left(\sum_{n=1}^{3}B_n e^{ik_n x} + \sum_{nm}B_{nm}e^{ik_{nm}x}\right) + O(a^3) ,$$

$$H = \mathrm{Re}\left(\sum_{n=1}^{3}H_n e^{ik_n x} + \sum_{nm}H_{nm}e^{ik_{nm}x}\right) + O(a^3) ,$$

in which $\sum_{nm}$ means summation over nm=11, 22, 33, 12, 13, 21, 23, and $k_{nm}=2k_1$, $2k_2$, $2k_3$, k_1+k_2, k_1+k_3, k_2-k_1, k_2-k_3 correspondingly. Later we shall use $k_{31}=k_1-k_3$ and $k_{32}=k_3+k_2$. Also in (15), σ_o and B_o are real-valued constants, which can be written as

$$\sigma_o = 1 + a^2\alpha_o + O(a^4) ,$$

$$B_o = \sin\theta + a^2\beta_o + O(a^4) ,$$

σ_n, B_n and H_n are complex-valued functions of t whose amplitudes have a magnitude of order O(a), σ_{nm}, B_{nm} and H_{nm} are complex-valued functions of t whose amplitudes have a magnitude of order $O(a^2)$.

We substitute (15) in (14), and regroup terms according to e^{ik_1x}, e^{ik_2x}, ... $e^{ik_{23}x}$. The resultant coefficients should vanish, both real parts and imaginary parts separately, because the real part and the imaginary part associated with the same wavenumber represent different components of wave motions which are in quadrature in phase. Thus we obtain the following sets of equations.

$$\frac{d^2}{dt^2}\sigma_n - k_n^2B_oB_n - k_n^2P_n + O(a^5) = 0 ,$$

$$(16)\quad \frac{d^2}{dt^2}[(k_n^2+\sigma_o)B_n+B_o\sigma_n+Q_n]+\Gamma k_n^2\cos\theta\frac{d}{dt}H_n+k_n^2\cos^2\theta B_n+O(a^5)= 0,$$

$$\frac{d^2}{dt^2}[(k_n^2+\sigma_o)H_n+R_n]-\Gamma k_n^2\cos\theta\frac{d}{dt}B_n+k_n^2\cos^2\theta H_n+O(a^5)= 0 ,$$

where

$$P_1 = \frac{1}{2}\left(\sum_{n=1}^{3} B_n^* B_{1n} + B_2 B_{21}^* + B_3 B_{21} + \sum_{n=1}^{3} H_n^* H_{1n} + H_2 H_{21}^* + H_3 H_{21}\right) ,$$

$$P_2 = \frac{1}{2}(B_1^* B_{12} + B_3^* B_{11} + B_2^* B_{22} + B_1 B_{21} + B_3 B_{23})$$

$$+ \frac{1}{2}(H_1^* H_{12} + H_3^* H_{11} + H_2^* H_{22} + H_1 H_{21} + H_3 H_{23}) ,$$

$$P_3 = \frac{1}{2}(B_1^* B_{13} + B_2^* B_{11} + B_3^* B_{33} + B_1 B_{21}^* + B_2 B_{23}^*)$$

$$+ \frac{1}{2}(H_1^* H_{13} + H_2^* H_{11} + H_3^* H_{33} + H_1 H_{21}^* + H_2 H_{23}^*) ,$$

$$Q_1 = \frac{1}{2}\left(\sum_{n=1}^{3} \sigma_n^* B_{1n} + \sigma_2 B_{21}^* + \sigma_3 B_{21}\right) + \frac{1}{2}\left(\sum_{n=1}^{3} B_n^* \sigma_{1n} + B_2 \sigma_{21}^* + B_3 \sigma_{21}\right),$$

$$Q_2 = \frac{1}{2}(\sigma_1^* B_{12} + \sigma_3^* B_{11} + \sigma_2^* B_{22} + \sigma_1 B_{21} + \sigma_3 B_{23})$$

$$+ \frac{1}{2}(B_1^* \sigma_{12} + B_3^* \sigma_{11} + B_2^* \sigma_{22} + B_1 \sigma_{21} + B_3 \sigma_{23}) ,$$

$$Q_3 = \frac{1}{2}(\sigma_1^* B_{13} + \sigma_2^* B_{11} + \sigma_3^* B_{33} + \sigma_1 B_{21}^* + \sigma_2 B_{23}^*)$$

$$+ \frac{1}{2}(B_1^* \sigma_{13} + B_2^* \sigma_{11} + B_3^* \sigma_{33} + B_1 \sigma_{21}^* + B_2 \sigma_{23}^*) ,$$

$$R_1 = \frac{1}{2}\left(\sum_{n=1}^{3} \sigma_n^* H_{1n} + \sigma_2 H_{21}^* + \sigma_3 H_{21}\right) + \frac{1}{2}\left(\sum_{n=1}^{3} H_n^* \sigma_{1n} + H_2 \sigma_{21}^* + H_3 \sigma_{21}\right) ,$$

$$R_2 = \frac{1}{2}(\sigma_1^* H_{12} + \sigma_3^* H_{11} + \sigma_2^* H_{22} + \sigma_1 H_{21} + \sigma_3 H_{23})$$

$$+ \frac{1}{2}(H_1^* \sigma_{12} + H_3^* \sigma_{11} + H_2^* \sigma_{22} + H_1 \sigma_{21} + H_3 \sigma_{23}) ,$$

$$R_3 = \frac{1}{2}(\sigma_1^* H_{13} + \sigma_2^* H_{11} + \sigma_3^* H_{33} + \sigma_1 H_{21}^* + \sigma_2 H_{23}^*)$$

$$+ \frac{1}{2}(H_1^* \sigma_{13} + H_2^* \sigma_{11} + H_3^* \sigma_{33} + H_1 \sigma_{21}^* + H_2 \sigma_{23}^*) ,$$

and

$$\frac{d^2}{dt^2} \sigma_{nm} - k_{nm}^2 B_o B_{nm} - k_{nm}^2 P_{nm} + O(a^4) = 0 ,$$

$$(17)\quad \frac{d^2}{dt^2}\left[(k_{nm}^2+\sigma_o)B_{nm}+B_o\sigma_{nm}+Q_{nm}\right]+\Gamma k_{nm}^2\cos\theta\frac{d}{dt}H_{nm}+k_{nm}^2\cos^2\theta B_{nm}+O(a^4)=0,$$

$$\frac{d^2}{dt^2}\left[(k_{nm}^2+\sigma_o)H_{nm}+P_{nm}\right]-\Gamma k_{nm}^2\cos\theta\frac{d}{dt}B_{nm}+k_{nm}^2\cos^2\theta H_{nm}+O(a^4)=0,$$

where

$$P_{11} = \tfrac{1}{4}(B_1B_1+H_1H_1) + \tfrac{1}{2}(B_3B_2+H_3H_2) ,$$

$$P_{22} = \tfrac{1}{4}(B_2B_2+H_2H_2) ,$$

$$P_{33} = \tfrac{1}{4}(B_3B_3+H_3H_3) ,$$

$$P_{12} = \tfrac{1}{2}(B_1B_2+H_1H_2) ,$$

$$P_{13} = \tfrac{1}{2}(B_1B_3+H_1H_3) ,$$

$$P_{21} = \tfrac{1}{2}(B_2B_1^*+H_2H_1^*+B_3^*B_1+H_3^*H_1) ,$$

$$P_{23} = \tfrac{1}{2}(B_2B_3^*+H_2H_3^*) ,$$

$$Q_{11} = \tfrac{1}{2}(\sigma_1B_1+\sigma_3B_2+B_3\sigma_2) ,$$

$$Q_{22} = \tfrac{1}{2}\sigma_2B_2,$$

$$Q_{33} = \tfrac{1}{2}\sigma_3B_3 ,$$

$$Q_{12} = \tfrac{1}{2}(\sigma_1B_2+B_1\sigma_2) ,$$

$$Q_{13} = \tfrac{1}{2}(\sigma_1B_3+B_1\sigma_3) ,$$

$$Q_{21} = \tfrac{1}{2}(\sigma_2B_1^*+B_2\sigma_1^*+\sigma_3^*B_1+B_3^*\sigma_1) ,$$

$$Q_{23} = \frac{1}{2}(\sigma_2 B_3^* + B_2\sigma_3^*) ,$$

$$R_{11} = \frac{1}{2}(\sigma_1 H_1 + \sigma_3 H_2 + H_3\sigma_2) ,$$

$$R_{22} = \frac{1}{2}\sigma_2 H_2 ,$$

$$R_{33} = \frac{1}{2}\sigma_3 H_3 ,$$

$$R_{12} = \frac{1}{2}(\sigma_1 H_2 + H_1\sigma_2) ,$$

$$R_{13} = \frac{1}{2}(\sigma_1 H_3 + H_1\sigma_3) ,$$

$$R_{21} = \frac{1}{2}(\sigma_2 H_1^* + H_2\sigma_1^* + \sigma_3^* H_1 + H_3^*\sigma_1) ,$$

$$R_{23} = \frac{1}{2}(\sigma_2 H_3^* + H_2\sigma_3^*) .$$

To first order in a , equations (16) are

$$\frac{d^2}{dt^2}\sigma_n - k_n^2 \sin\theta\, B_n + O(a^3) = 0 ,$$

$$(18)\quad [(k_n^2+1)\frac{d^2}{dt^2} + k_n^2\cos^2\theta]B_n + \Gamma k_n^2 \cos\theta \frac{d}{dt}H_n + \sin\theta\frac{d^2}{dt^2}\sigma_n + O(a^3) = 0,$$

$$[(k_n^2+1)\frac{d^2}{dt^2} + k_n^2\cos^2\theta]H_n - \Gamma k_n^2\cos\theta\frac{d}{dt}B_n + O(a^3) = 0 ,$$

which can be combined to give

$$\left\{ [(k_n^2+1)\frac{d^2}{dt^2} + k_n^2][(k_n^2+1)\frac{d^2}{dt^2} + k_n^2\cos^2\theta] + \Gamma^2 k_n^4 \cos^2\theta\frac{d^2}{dt^2}\right\} B_n + O(a^3) = 0.$$

The solutions can be written as

$$B_n = a\,\beta_n A_n e^{i\Omega_n t}$$

where β_n is an undetermined constant, A_n is a slowly varying function of t to be determined later such that $A_n=0(1)$ but $\frac{d}{dt}A_n=0(a^2)$ to account for the term $0(a^3)$ in the equation, and

$$\Omega_n = \Omega(k_n)$$

in which $\Omega(k)$ is a function of k defined implicitly by the following dispersion relation

$$(19)\quad [\Omega^2(k^2+1)-k^2][\Omega^2(k^2+1)- k^2\cos^2\theta]- \Gamma^2\Omega^2k^4\cos^2\theta = 0$$

which defines a four-branch function. Thus $\Omega=\Omega_F$, Ω_S, $-\Omega_F$ and $-\Omega_S$ with

$$\begin{matrix}\Omega_F\\ \Omega_S\end{matrix} = \left\{\frac{k^2}{2(k^2+1)^2}\left[(k^2+1)(1+\cos^2\theta) + \Gamma^2k^2\cos^2\theta \pm\sqrt{(k^2+1)^2\sin^4\theta+ 2\Gamma^2k^2(k^2+1)\cos^2\theta(1+\cos^2\theta)+\Gamma^4k^4\cos^4\theta}\right]\right\}^{\frac{1}{2}} .$$

Accordingly, from (18)

$$H_n = ia\gamma_nA_ne^{i\Omega_nt} + 0(a^3) ,$$

$$\sigma_n = - a\alpha_nA_ne^{i\Omega_nt} + 0(a^3) ,$$

where

$$\gamma_n = \frac{-\Gamma\Omega_nk_n^2\cos\theta}{\Omega_n^2(k_n^2+1) - k_n^2\cos^2\theta}\beta_n ,$$

$$\alpha_n = \frac{k_n^2}{\Omega_n^2}\beta_n\sin\theta .$$

Next, to second order in a, equations (17) are

$$\frac{d^2}{dt^2}\sigma_{nm} - k_{nm}^2 B_{nm}\sin\theta - k_{nm}^2 P_{nm} + O(a^4) = 0 \quad ,$$

(20) $$\left[(k_{nm}^2+1)\frac{d^2}{dt^2}+k_{nm}^2\cos^2\theta\right]B_{nm}+\Gamma k_{nm}^2\cos\theta\frac{d}{dt}H_{nm}+\sin\theta\frac{d^2}{dt^2}\sigma_{nm}+\frac{d^2}{dt^2}Q_{nm}+O(a^4)=0 \; ,$$

$$\left[(k_{nm}^2+1)\frac{d^2}{dt^2}+k_{nm}^2\cos^2\theta\right]H_{nm}-\Gamma k_{nm}^2\cos\theta\frac{d}{dt}B_{nm}+\frac{d^2}{dt^2}R_{nm}+O(a^4) = 0 \; ,$$

which can be combined to give

$$\left\{\left[(k_{nm}^2+1)\frac{d^2}{dt^2}+k_{nm}^2\right]\left[(k_{nm}^2+1)\frac{d^2}{dt^2}+k_{nm}^2\cos^2\theta\right]+\Gamma^2 k_{nm}^4\cos^2\theta\frac{d^2}{dt^2}\right\}B_{nm}$$

$$+\left[(k_{nm}^2+1)\frac{d^2}{dt^2}+k_{nm}^2\cos^2\theta\right]\left(k_{nm}^2P_{nm}\sin\theta+\frac{d^2}{dt^2}Q_{nm}\right)-\Gamma k_{nm}^2\cos\theta\frac{d^3}{dt^3}R_{nm}+O(a^4) = 0 \; .$$

We shall use the following notations

$\Omega_{nm} = \Omega_n+\Omega_m$ for nm= 11, 22, 33, 12, 13, 32

$\Omega_{21} = \Omega_2-\Omega_1$, $\Omega_{23} = \Omega_2-\Omega_3$, $\Omega_{31} = \Omega_1-\Omega_3$.

The solutions for B_{nm} corresponding to the response to the excitation are

$$B_{11} = a^2(\beta_{11}A_1A_1e^{i\Omega_{11}t} + \beta_{32}A_3A_2e^{i\Omega_{32}t}) + O(a^4) \; ,$$

$$B_{nm} = a^2\beta_{nm}A_nA_me^{i\Omega_{nm}t} + O(a^4) \quad \text{for nm=22, 33, 12, 13,}$$

$$B_{21} = a^2(\beta_{21}A_2A_1^*e^{i\Omega_{21}t} + \beta_{31}A_3^*A_1e^{i\Omega_{31}t}) + O(a^4) \; ,$$

$$B_{23} = a^2\beta_{23}A_2A_3^*e^{i\Omega_{23}t} + O(a^4) \; ,$$

where

for n=1, 2, 3:

$$\beta_{nn} = \tfrac{1}{4}\Big\{\big[\Omega_{nn}^2(k_{nn}^2+1)-k_{nn}^2\cos^2\theta\big]\big[2\Omega_{nn}^2\alpha_n\beta_n+k_{nn}^2(\beta_n^2-\gamma_n^2)\sin\theta\big] - 2\Gamma\Omega_{nn}^3k_{nn}^2\alpha_n\gamma_n\cos\theta\Big\} \Big/ \Big\{\big[\Omega_{nn}^2(k_{nn}^2+1)-k_{nn}^2\big] \cdot\big[\Omega_{nn}^2(k_{nn}^2+1)-k_{nn}^2\cos^2\theta\big] - \Gamma^2\Omega_{nn}^2k_{nn}^4\cos^2\theta\Big\} \quad ,$$

and for nm=12, 13, 32:

$$\beta_{nn} = \tfrac{1}{2}\Big\{\big[\Omega_{nm}^2(k_{nm}^2+1)-k_{nm}^2\cos^2\theta\big]\big[\Omega_{nm}^2(\alpha_n\beta_m+\beta_n\alpha_m)+k_{nm}^2(\beta_n\beta_m-\gamma_n\gamma_m)\sin\theta\big] - \Gamma\Omega_{nm}^3k_{nm}^2(\alpha_n\gamma_m+\gamma_n\alpha_m)\cos\theta\Big\} \Big/ \Big\{\big[\Omega_{nm}^2(k_{nm}^2+1)-k_{nm}^2\big] \cdot\big[\Omega_{nm}^2(k_{nm}^2+1)-k_{nm}^2\cos^2\theta\big] - \Gamma^2\Omega_{nm}^2k_{nm}^4\cos^2\theta\Big\} \quad ,$$

and for m=1, 3:

$$\beta_{2m} = \tfrac{1}{2}\Big\{\big[\Omega_{2m}^2(k_{2m}^2+1)-k_{2m}^2\cos^2\theta\big]\big[\Omega_{2m}^2(\alpha_2\beta_m+\beta_2\alpha_m)+k_{2m}^2(\beta_2\beta_m+\gamma_2\gamma_m)\sin\theta\big] - \Gamma\Omega_{2m}^3k_{2m}^2(-\alpha_2\gamma_m+\gamma_2\alpha_m)\cos\theta\Big\} \Big/ \Big\{\big[\Omega_{2m}^2(k_{2m}^2+1)-k_{2m}^2\big] \cdot\big[\Omega_{2m}^2(k_{2m}^2+1)-k_{2m}^2\cos^2\theta\big] - \Gamma^2\Omega_{2m}^2k_{2m}^4\cos^2\theta\Big\} \quad ,$$

and

$$\beta_{31} = \tfrac{1}{2}\Big\{\big[\Omega_{31}^2(k_{31}^2+1)-k_{31}^2\cos^2\theta\big]\big[\Omega_{31}^2(\alpha_3\beta_1+\beta_3\alpha_1)+k_{31}^2(\beta_3\beta_1+\gamma_3\gamma_1)\sin\theta\big] - \Gamma\Omega_{31}^3k_{31}^2(\alpha_3\gamma_1-\gamma_3\alpha_1)\cos\theta\Big\} \Big/ \Big\{\big[\Omega_{31}^2(k_{31}^2+1)-k_{31}^2\big] \cdot\big[\Omega_{31}^2(k_{31}^2+1)-k_{31}^2\cos^2\theta\big] - \Gamma^2\Omega_{31}^2k_{31}^4\cos^2\theta\Big\} \quad .$$

Accordingly, from (20)

$$H_{11} = ia^2(\gamma_{11}A_1A_1e^{i\Omega_{11}t} + \gamma_{32}A_3A_2e^{i\Omega_{32}t}) + O(a^4) ,$$

$$H_{nm} = ia^2\gamma_{nm}A_nA_me^{i\Omega_{nm}t} + O(a^4) \quad \text{for } nm=22,\ 33,\ 12,\ 13,$$

$$H_{21} = ia^2(\gamma_{21}A_2A_1^*e^{i\Omega_{21}t} + \gamma_{31}A_3^*A_1e^{i\Omega_{31}t}) + O(a^4) ,$$

$$H_{23} = ia^2\gamma_{23}A_2A_3e^{i\Omega_{23}t} + O(a^4) ,$$

where

$$\gamma_{nn} = \frac{\frac{1}{2}\Omega_{nn}^2\alpha_n\gamma_n - \Gamma\Omega_{nn}k_{nn}^2\beta_{nn}\cos\theta}{\Omega_{nn}^2(k_{nn}^2+1) - k_{nn}^2\cos^2\theta} \quad \text{for } n=1,\ 2,\ 3,$$

$$\gamma_{nm} = \frac{\frac{1}{2}\Omega_{nm}^2(\alpha_n\gamma_m+\gamma_n\alpha_m) - \Gamma\Omega_{nm}k_{nm}^2\beta_{nm}\cos\theta}{\Omega_{nm}^2(k_{nm}^2+1) - k_{nm}^2\cos^2\theta} \quad \text{for } nm=12,\ 13,\ 32,$$

$$\gamma_{2m} = \frac{\frac{1}{2}\Omega_{2m}^2(-\alpha_2\gamma_m+\gamma_2\alpha_m) - \Gamma\Omega_{2m}k_{2m}^2\beta_{2m}\cos\theta}{\Omega_{2m}^2(k_{2m}^2+1) - k_{2m}^2\cos^2\theta} \quad \text{for } m=1,\ 3,$$

$$\gamma_{31} = \frac{\frac{1}{2}\Omega_{31}^2(\alpha_3\gamma_1-\gamma_3\alpha_1) - \Gamma\Omega_{31}k_{31}^2\beta_{31}\cos\theta}{\Omega_{31}^2(k_{31}^2+1) - k_{31}^2\cos^2\theta} ,$$

and

$$\sigma_{11} = -a^2(\alpha_{11}A_1A_1e^{i\Omega_{11}t} + \alpha_{32}A_3A_2e^{i\Omega_{32}t}) + O(a^4) ,$$

$$\sigma_{nm} = -a^2\alpha_{nm}A_nA_me^{i\Omega_{nm}t} + O(a^4) \quad \text{for } nm=22,\ 33,\ 12,\ 13,$$

$$\sigma_{21} = -a^2(\alpha_{21}A_2A_1^*e^{i\Omega_{21}t} + \alpha_{31}A_3^*A_1e^{i\Omega_{31}t}) + O(a^4) ,$$

$$\sigma_{23} = -a^2\alpha_{23}A_2A_3^*e^{i\Omega_{23}t} + O(a^4) ,$$

where

$$\alpha_{nn} = \frac{k_{nn}^2}{\Omega_{nn}^2}\left(\frac{\beta_n\beta_n-\gamma_n\gamma_n}{4} + \beta_{nn}\sin\theta\right) \qquad \text{for } n=1,\ 2,\ 3,$$

$$\alpha_{nm} = \frac{k_{nm}^2}{\Omega_{nm}^2}\left(\frac{\beta_n\beta_m-\gamma_n\gamma_m}{2} + \beta_{nm}\sin\theta\right) \qquad \text{for } nm=12,\ 13,\ 32,$$

$$\alpha_{2m} = \frac{k_{2m}^2}{\Omega_{2m}^2}\left(\frac{\beta_2\beta_m+\gamma_2\gamma_m}{2} + \beta_{2m}\sin\theta\right) \qquad \text{for } m=1,\ 3,$$

$$\alpha_{31} = \frac{k_{31}^2}{\Omega_{31}^2}\left(\frac{\beta_3\beta_1+\gamma_3\gamma_1}{2} + \beta_{31}\sin\theta\right).$$

Now equations (16) can be combined to give

$$\left\{(k_n^2+\sigma_o)^2\frac{d^4}{dt^4} + \left[k_n^2(k_n^2+\sigma_o)(2\cos^2\theta+B_o^2)+\Gamma^2k_n^4\cos^2\theta\right]\frac{d^2}{dt^2}+k_n^4\cos^2\theta(\cos^2\theta+B_o^2)\right\}B_n$$

$$+\left[(k_n^2+\sigma_o)\frac{d^2}{dt^2}+k_n^2\cos^2\theta\right]\left[\frac{d^2}{dt^2}Q_n+k_n^2B_oP_n\right]-\Gamma k_n^2\cos\theta\frac{d^3}{dt^3}R_n+O(a^5) = 0$$

for $n=1,\ 2,\ 3.$

Upon substitution of the preceding expressions of σ_n, B_n, H_n, σ_{nm}, B_{nm} and H_{nm} in terms of A_n and recalling that $\frac{d}{dt}A_n=O(a^2)$ hence

$$\frac{d^2}{dt^2}A_ne^{i\Omega_nt} = \left[-\Omega_n^2A_n + i2\Omega_n\frac{d}{dt}A_n + O(a^4)\right]e^{i\Omega_nt},$$

$$\frac{d^4}{dt^4}A_ne^{i\Omega_nt} = \left[\Omega_n^4A_n - i4\Omega_n^3\frac{d}{dt}A_n + O(a^4)\right]e^{i\Omega_nt},$$

we obtain

$$(21)\quad \begin{aligned}
&i f_1\frac{d}{dt}A_1 + a^2\Big[\Big(f_{10} + \sum_{n=1}^{3} f_{1n}A_n^*A_n\Big)A_1 \\
&\qquad + f_{14}A_1^*A_2A_3 e^{-i(2\Omega_1-\Omega_2-\Omega_3)t}\Big] + O(a^4) = 0 , \\
&i f_2\frac{d}{dt}A_2 + a^2\Big[\Big(f_{20} + \sum_{n=1}^{3} f_{2n}A_n^*A_n\Big)A_2 \\
&\qquad + f_{24}A_1A_1A_3^* e^{i(2\Omega_1-\Omega_2-\Omega_3)t}\Big] + O(a^4) = 0, \\
&i f_3\frac{d}{dt}A_3 + a^2\Big[\Big(f_{30} + \sum_{n=1}^{3} f_{3n}A_n^*A_n\Big)A_3 \\
&\qquad + f_{34}A_1A_1A_2^* e^{i(2\Omega_1-\Omega_2-\Omega_3)t}\Big] + O(a^4) = 0 ,
\end{aligned}$$

in which

for n=1, 2, 3:

$$f_n=\{-4\Omega_n^3(k_n^2+1)^2+2\Omega_n[k_n^2(k_n^2+1)(1+\cos^2\theta)+\Gamma^2k_n^4\cos^2\theta]\}\,\beta_n$$

$$f_{no}=\{2\Omega_n^4(k_n^2+1)\alpha_o-\Omega_n^2k_n^2[\alpha_o(1+\cos^2\theta)+2(k_n^2+1)\beta_o\sin\theta]+2k_n^4\beta_o\sin\theta\cos^2\theta\}\beta_n$$

$$\begin{aligned}
f_{nn}=&-\tfrac{1}{2}[\Omega_n^2(k_n^2+1)-k_n^2\cos^2\theta][\Omega_n^2(\alpha_n\beta_{nn}+\beta_n\alpha_{nn})+k_n^2(\beta_n\beta_{nn}+\gamma_n\gamma_{nn})\sin\theta] \\
&+\tfrac{1}{2}\Gamma\Omega_n^3k_n^2(\alpha_n\gamma_{nn}-\gamma_n\alpha_{nn})\cos\theta
\end{aligned}$$

and for nm=12, 13, 21, 23, 31, 32:

$$\begin{aligned}
f_{nm}=&-\tfrac{1}{2}[\Omega_n^2(k_n^2+1)-k_n^2\cos^2\theta]\{\Omega_n^2[\alpha_m(\beta_{nm}+\beta_{mn})+\beta_m(\alpha_{nm}+\alpha_{mn})] \\
&+k_n^2[\beta_m(\beta_{nm}+\beta_{mn})+\gamma_mX_{nm}]\sin\theta\}+\tfrac{1}{2}\Gamma\Omega_n^3k_n^2(\alpha_mY_{nm}+\gamma_mZ_{nm})\cos\theta
\end{aligned}$$

with

$$X_{12}=\gamma_{12}+\gamma_{21}\,,\qquad Y_{12}=\gamma_{12}-\gamma_{21}\,,\qquad Z_{12}=-\alpha_{12}+\alpha_{21}\,,$$

$$X_{13}=\gamma_{13}-\gamma_{31}\,,\qquad Y_{13}=\gamma_{13}+\gamma_{31}\,,\qquad Z_{13}=-\alpha_{13}+\alpha_{31}\,,$$

$$X_{21} = -\gamma_{21}+\gamma_{12}\ , \quad Y_{21} = \gamma_{21}+\gamma_{12}\ , \quad Z_{21} = \alpha_{21}-\alpha_{12}\ ,$$
$$X_{23} = -\gamma_{23}+\gamma_{32}\ , \quad Y_{23} = \gamma_{23}+\gamma_{32}\ , \quad Z_{23} = \alpha_{23}-\alpha_{32}\ ,$$
$$X_{31} = \gamma_{31}+\gamma_{13}\ , \quad Y_{31} = -\gamma_{31}+\gamma_{13}\ , \quad Z_{31} = \alpha_{31}-\alpha_{13}\ ,$$
$$X_{32} = \gamma_{32}+\gamma_{23}\ , \quad Y_{32} = \gamma_{32}-\gamma_{23}\ , \quad Z_{32} = -\alpha_{32}+\alpha_{23}\ ,$$

and

$$f_{14} = -\frac{1}{2}[(-\Omega_1+\Omega_2+\Omega_3)^2(k_1^2+1)-k_1^2\cos^2\theta]$$
$$\cdot[(-\Omega_1+\Omega_2+\Omega_3)^2(\alpha_1\beta_{32}+\alpha_2\beta_{31}+\alpha_3\beta_{21}+\beta_1\alpha_{32}+\beta_2\alpha_{31}+\beta_3\alpha_{21})$$
$$+k_1^2(\beta_1\beta_{32}+\beta_2\beta_{31}+\beta_3\beta_{21}+\gamma_1\gamma_{32}+\gamma_2\gamma_{31}-\gamma_3\gamma_{21})\sin\theta]$$
$$+\frac{1}{2}\Gamma(-\Omega_1+\Omega_2+\Omega_3)^3k_1^2(\alpha_1\gamma_{32}-\alpha_2\gamma_{31}+\alpha_3\gamma_{21}-\gamma_1\alpha_{32}+\gamma_2\alpha_{31}+\gamma_3\alpha_{21})\cos\theta\ ,$$

$$f_{24} = -\frac{1}{2}[(2\Omega_1-\Omega_3)^2(k_2^2+1)-k_2^2\cos^2\theta]$$
$$\cdot[(2\Omega_1-\Omega_3)^2(\alpha_3\beta_{11}+\alpha_1\beta_{31}+\beta_3\alpha_{11}+\beta_1\alpha_{31})$$
$$+k_2^2(\beta_3\beta_{11}+\beta_1\beta_{31}+\gamma_3\gamma_{11}-\gamma_1\gamma_{31})\sin\theta]$$
$$+\frac{1}{2}\Gamma(2\Omega_1-\Omega_3)^3k_2^2(\alpha_3\gamma_{11}+\alpha_1\gamma_{31}-\gamma_3\alpha_{11}+\gamma_1\alpha_{31})\cos\theta\ ,$$

$$f_{34} = -\frac{1}{2}[(2\Omega_1-\Omega_2)^2(k_3^2+1)-k_3^2\cos^2\theta]$$
$$\cdot[(2\Omega_1-\Omega_2)^2(\alpha_2\beta_{11}+\alpha_1\beta_{21}+\beta_2\alpha_{11}+\beta_1\alpha_{21})$$
$$+k_3^2(\beta_2\beta_{11}+\beta_1\beta_{21}+\gamma_2\gamma_{11}+\gamma_1\gamma_{21})\sin\theta]$$
$$+\frac{1}{2}\Gamma(2\Omega_1-\Omega_2)^3k_3^2(\alpha_2\gamma_{11}-\alpha_1\gamma_{21}-\gamma_2\alpha_{11}+\gamma_1\alpha_{21})\cos\theta\ .$$

To solve (21), we let

$$A_n = a_n e^{i\phi_n} ,$$

$$\phi = (-2\Omega_1 + \Omega_2 + \Omega_3)t + (-2\phi_1 + \phi_2 + \phi_3) ,$$

in which a_n and ϕ_n are real-valued functions of t. Upon separation of real parts and imaginary parts, (21) give

$$f_1\frac{d}{dt}a_1 + a^2 f_{14}a_1a_2a_3\sin\phi + O(a^4) = 0 ,$$

$$f_2\frac{d}{dt}a_2 - a^2 f_{24}a_1a_1a_3\sin\phi + O(a^4) = 0 ,$$

$$f_3\frac{d}{dt}a_3 - a^2 f_{34}a_1a_1a_2\sin\phi + O(a^4) = 0 ,$$

(22)

$$f_1a_1\frac{d}{dt}\phi_1 - a^2(f_{10}+\sum_{n=1}^{3} f_{1n}a_n^2)a_1 - a^2 f_{14}a_1a_2a_3\cos\phi + O(a^4) = 0,$$

$$f_2a_2\frac{d}{dt}\phi_2 - a^2(f_{20}+\sum_{n=1}^{3} f_{2n}a_n^2)a_2 - a^2 f_{24}a_1a_1a_3\cos\phi + O(a^4) = 0,$$

$$f_3a_3\frac{d}{dt}\phi_3 - a^2(f_{30}+\sum_{n=1}^{3} f_{3n}a_n^2)a_3 - a^2 f_{34}a_1a_1a_2\cos\phi + O(a^4) = 0,$$

from which we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\,\frac{f_1}{f_{14}}\,a_1^2\right) = -\,a^2a_1^2a_2a_3\sin\phi + O(a^4) ,$$

$$\frac{d}{dt}\left(\frac{1}{2}\,\frac{f_2}{f_{24}}a_2^2\right) = a^2a_1^2a_2a_3\sin\phi + O(a^4) ,$$

(23)

$$\frac{d}{dt}\left(\frac{1}{2}\,\frac{f_3}{f_{34}}a_3^2\right) = a^2a_1^2a_2a_3\sin\phi + O(a^4) ,$$

$$\frac{d}{dt}(a_1^2a_2a_3\cos\phi) = a^2 f a_1^2a_2a_3\sin\phi + O(a^4) ,$$

where

$$(24)\quad f = \frac{2\Omega_1-\Omega_2-\Omega_3}{a^2} + \left[2\frac{f_{10}}{f_1} - \frac{f_{20}}{f_2} - \frac{f_{30}}{f_3} + \sum_{n=1}^{3}\left(2\frac{f_{1n}}{f_1} - \frac{f_{2n}}{f_2} - \frac{f_{3n}}{f_3}\right)a_n^2\right].$$

The first three equations of (23) can be combined to give

$$\frac{d}{dt}\left(\frac{f_1}{f_{14}} a_1^2 + \frac{1}{2}\frac{f_2}{f_{24}} a_2^2 + \frac{1}{2}\frac{f_3}{f_{34}} a_3^2\right) = O(a^4)$$

which may be interpreted as expressing conservation of energy among the three components of waves.

Firstly, we consider the case that the wave motion consists of only the main wave of wavenumber k_1. With $a_2=0$ and $a_3=0$, equations (22) reduce to

$$f_1\frac{d}{dt}a_1 + O(a^4) = 0,$$

$$f_1 a_1 \frac{d}{dt}\phi_1 - a^2(f_{10} + f_{11}a_1^2)a_1 + O(a^4) = 0 .$$

Hence a_1 is constant to third order in a. With the provision of the factor β_1 in the expression for B_1 we may write

$$a_1 = 1 + O(a^4) .$$

Accordingly,

$$\phi_1 = a^2\Omega_1' t + O(a^4)$$

where

$$\Omega_1' = \frac{f_{10} + f_{11}}{f_1} .$$

Therefore

$$\sigma_o = 1 + a^2\alpha_o + O(a^4) ,$$

$$B_o = \sin\theta + a^2\beta_o + O(a^4) ,$$

$$\sigma_1 = -a\alpha_1 e^{i\omega t} + O(a^3) ,$$

$$B_1 = a\beta_1 e^{i\omega t} + O(a^3) ,$$

(25) $$H_1 = ia\gamma_1 e^{i\omega t} + O(a^3) ,$$

$$\sigma_{11} = - a^2\alpha_{11}e^{i2\omega t} + 0(a^4) ,$$

$$B_{11} = a^2\beta_{11}e^{i2\omega t} + 0(a^4) ,$$

$$H_{11} = ia^2\gamma_{11}e^{i2\omega t} + 0(a^4) ,$$

$$\omega = \Omega_1 + a^2\Omega_1' + 0(a^4) .$$

The two sequences of constants $\{\alpha_o, \alpha_1, \alpha_{11}, \ldots.\}$ and $\{\beta_o, \beta_1, \beta_{11}, \ldots\}$ can be determined by two constants that characterize the wave motion, say

$$B \Big|_{\omega t+kx=0} = \sin\theta + a ,$$

$$\sigma \Big|_{\omega t+kx=0} = 1 - \frac{k^2}{\omega^2}(a\sin\theta + \tfrac{1}{2}a^2)$$

$$= 1 - a\frac{k^2}{\Omega^2}\sin\theta - \frac{1}{2}a^2\frac{k^2}{\Omega^2} + 0(a^3) .$$

Hence

$$\beta_1 = 1 , \qquad \beta_{11} + \beta_o = 0 ;$$

$$\alpha_1 = \frac{k^2}{\Omega^2}\sin\theta , \qquad \alpha_{11} - \alpha_o = \frac{1}{2}\frac{k^2}{\Omega^2} .$$

These together with the formulas which define γ_1, α_{11}, β_{11}, γ_{11} determine that α_o, α_1, α_{11}, β_o, β_1, β_{11}, γ_1, γ_{11}, are equal to $-s_{20}$, s_{11}, s_{22}, b_{20}, b_{11}, b_{22}, h_{11}, h_{22} as listed at the end of Chapter 3.

Secondly, we consider the case that the wave motion consists of the main wave of wavenumber k_1 and a pair of disturbance waves of wavenumbers k_2 and k_3. In the initial stage during which the amplitudes of the disturbances are very small compared with that of the main wave, the main wave is almost unaffected and described by (25) but the disturbance waves change slowly. With a_2 and a_3 neglected (24) become

$$f = \frac{2\Omega_1 - \Omega_2 - \Omega_3}{a^2} + 2\Omega_1' - \Omega_2' - \Omega_3'$$

where

$$\Omega_n' = \frac{f_{no} + f_{n1}}{f_n} ,$$

thus the last three equations of (23) can be integrated once to give, denoting the common value by η.

$$(26) \quad \eta = \frac{a_2 a_3 \cos\phi - \epsilon_2\epsilon_3 \cos\phi_o}{f} = \frac{f_2}{f_{24}} \frac{a_2^2 - \epsilon_2^2}{2} + O(a^4)$$

$$= \frac{f_3}{f_{34}} \frac{a_3^2 - \epsilon_3^2}{2} + O(a^4) ,$$

in which ϵ_2, ϵ_3 and ϕ_o are the initial values of a_2, a_3 and ϕ. Substituting these into any of the last three equations of (23), we obtain

$$\left(\frac{d\eta}{dt}\right)^2 = a^4 a_2^2 a_3^2 \sin^2\phi + O(a^6)$$

$$= a^4(a_2^2 a_3^2 - a_2^2 a_3^2 \cos^2\phi) + O(a^6) ,$$

hence

$$(27) \quad \left(\frac{d\eta}{dt}\right)^2 = a^4(D\eta^2 + D'\eta + D''\eta) + O(a^6)$$

where

$$D = 4\frac{f_{24}f_{34}}{f_2 f_3} - f^2 ,$$

$$D' = 2\frac{f_{34}}{f_3}\epsilon_2^2 + 2\frac{f_{24}}{f_2}\epsilon_3^2 - 2f\epsilon_2\epsilon_3\cos\phi_o ,$$

$$D'' = \epsilon_2^2\epsilon_3^2\sin^2\phi_o \quad .$$

We note that from the second and third equations of (23)

$$\frac{d}{dt}\left(\frac{f_2}{f_{24}}a_2^2 - \frac{f_3}{f_{34}}a_3^2\right) + 0(a^4) = 0 \quad .$$

The constancy of $\frac{f_2}{f_{24}}a_2^2 - \frac{f_3}{f_{34}}a_3^2$ to third order in a indicates that the two components in the disturbance will excite each other even if initially one of them is missing.

Equation (27) determines the evolution of the disturbance as time goes from its initial value. Initially η is zero, then it will grow unboundedly with a characteristic time of order $0(\frac{1}{a^2})$ if D is not negative, or it will oscillate between the two zeros of the expression $D\eta^2+D'\eta+D''\eta$ with a period of order $0(\frac{1}{a^2})$ if D is negative. Therefore the stability of the main wave is determined by the algebraic sign of D. In the case that D is negative, the main wave is stable, energy flows to and back between the main wave and the disturbance waves, the amplitudes of the disturbance waves oscillate around their initial values, this is seen from the fact that the two zeros of $D\eta^2+D'\eta+D''\eta$ have opposite signs when D is negative. Otherwise, the main wave is unstable; there is a continual flow of energy from the main wave into the disturbance waves; as the disturbance waves grow, the main wave will disintegrate eventually. Of course after the disturbance becomes large, equations (23) should be used to describe the interaction between the main wave and the disturbance.

Finally we investigate the sign of the quantity D. It is clear that D is negative when the disturbance is in different mode from that of the main wave or when the frequency of the disturbance wave is different largely from that of the main wave so that f is of order $0(\frac{1}{a^2})$ while f_2, f_3, f_{24} and f_{34} are of order 0(1). In these cases, there is weak coupling between the main wave and the disturbance, hence the main wave is stable from the disturbance. However when the frequencies are sufficiently close to each other, say $\kappa = a\delta k$ in which $\delta = 0(1)$, the coupling is strong. Then

$$\Omega_1 = \Omega ,$$

$$\Omega_2 = \Omega + \kappa \frac{d\Omega}{dk} + \frac{1}{2} \kappa^2 \frac{d^2\Omega}{dk^2} + \frac{1}{6} \kappa^3 \frac{d^3\Omega}{dk^3} + 0(\kappa^4) ,$$

$$\Omega_3 = \Omega - \kappa \frac{d\Omega}{dk} + \frac{1}{2} \kappa^2 \frac{d^2\Omega}{dk^2} - \frac{1}{6} \kappa^3 \frac{d^3\Omega}{dk^3} + 0(\kappa^4) .$$

Without loss of generality, we may take β_2, $\beta_3 = 1$, then

$$\alpha_2, \alpha_3 = \alpha_1 + 0(\kappa) ,$$

$$\gamma_2, \gamma_3 = \gamma_1 + 0(\kappa) ,$$

$$\alpha_{12}, \alpha_{13} = 2\alpha_{11} + 0(\kappa) ,$$

$$\beta_{12}, \beta_{13} = 2\beta_{11} + 0(\kappa) ,$$

$$\gamma_{12}, \gamma_{13} = 2\gamma_{11} + 0(\kappa) ,$$

$$\alpha_{21},\ \alpha_{31} = \frac{(\frac{d\Omega}{dk})^2 \alpha_1 \sin\theta + \frac{1+\gamma_1^2}{2}[(\frac{d\Omega}{dk})^2 - \cos^2\theta]}{(\frac{d\Omega}{dk})^2[(\frac{d\Omega}{dk})^2 - 1]}$$

$$\beta_{21},\ \beta_{31} = \frac{(\frac{d\Omega}{dk})^2 \alpha_1 + \frac{1+\gamma_1^2}{2}\sin\theta}{(\frac{d\Omega}{dk})^2 - 1} ,$$

$$\gamma_{21},\ \gamma_{31} = O(\kappa) .$$

Accordingly

$$f_2,\ f_3 = f_1 + O(\kappa) ,$$

$$f_{20},\ f_{30} = f_{10} + O(\kappa) ,$$

$$f_{21},\ f_{31} = f_{11} + g + O(\kappa) ,$$

$$f_{24},\ f_{34} = g + O(\kappa) ,$$

$$\Omega_2',\ \Omega_3' = \Omega_1' + \frac{g}{f_1} + O(\kappa) ,$$

$$f = -2\frac{g}{f_1} - \delta^2 k^2 \frac{d^2\Omega}{dk^2} + O(\kappa) .$$

where

$$f_1 = -4\Omega^3(k^2+1)^2 + 2\Omega[k^2(k^2+1)(1+\cos^2\theta) + \Gamma^2 k^4 \cos^2\theta] ,$$

$$g = -[\Omega^2(k^2+1) - k^2\cos^2\theta][\Omega^2\frac{\alpha_{11}+\alpha_{21}}{2} + k^2(\beta_{11}+\beta_{21})\sin\theta + \frac{1}{2}k^2\gamma_1\gamma_{11}\sin\theta]$$
$$+ \Gamma\Omega^3 k^2(\alpha_1\frac{\gamma_{11}}{2} - \gamma_1\frac{\alpha_{11}-\alpha_{21}}{2})\cos\theta .$$

Hence

$$D = -4\delta^2 k^2 \frac{g}{f_1}\frac{d^2\Omega}{dk^2} - \delta^4 k^4 (\frac{d^2\Omega}{dk^2})^2 + O(\kappa) .$$

Waves of transverse propagation

Using the dispersion relation

$$\Omega^2(k^2 + 1) - k^2 = 0$$

and thereby

$$\frac{d\Omega}{dk} = \frac{\Omega}{k(k^2+1)} ,$$

$$\frac{d^2\Omega}{dk^2} = - \frac{3\Omega}{(k^2+1)^2} ,$$

we obtain

$$f_1 = -2\Omega k^2(k^2+1) ,$$

$$g = \frac{k^2(4k^6+11k^4+15k^2+9)}{8(k^4+3k^2+3)} ,$$

hence $D = - \delta^2 \frac{3k^2(4k^6+11k^4+15k^2+9)}{4(k^2+1)^3(k^4+3k^2+3)} - \delta^4 \frac{9k^6}{(k^2+1)^5} + O(\kappa)$

which is always negative for all values of k. Therefore waves propagating across a magnetic field are stable for all frequencies.

Waves of longitudinal propagation

Using the dispersion relation

$$[\Omega^2(k^2+1) - k^2]^2 - \Gamma^2\Omega^2 k^4 = 0$$

and thereby

$$\frac{d\Omega}{dk} = \frac{2\Omega^3}{k[\Omega^2(k^2+1)+k^2]} ,$$

$$\frac{d^2\Omega}{dk^2} = \frac{2\Omega^5[4\Omega^2(k^2+1)-3(\Gamma^2+4)k^4-4k^2]}{k^2[\Omega^2(k^2+1)+k^2]^3},$$

we obtain

$$f_1 = -4\Omega^3(k^2+1)^2 + 2\Omega\,[2k^2(k^2+1)+\Gamma^2k^4]\ ,$$

$$g = -\frac{\Omega^2[\Omega^2(k^2+1)-k^2]}{(\frac{d\Omega}{dk})^2}\ ,$$

hence $D = \delta^2\dfrac{k^2\Omega^2[-4\Omega^2(k^2+1)+3(\Gamma^2+4)k^4+4k^2]}{[\Omega^2(k^2+1)+k^2]^2} - \delta^4k^4(\dfrac{d^2\Omega}{dk^2})^2 + O(\kappa)$.

Its algebraic sign is the same as that of $-4\Omega^2(k^2+1)+3(\Gamma^2+4)k^4+4k^2$ as $\delta \to 0$.

For fast waves:

$$\Omega_F = \frac{\sqrt{(\Gamma^2+4)k^2+4k^2}+\Gamma k^2}{2(k^2+1)}\ ,$$

$$-4\Omega^2(k^2+1)+3(\Gamma^2+4)k^4+4k^2$$
$$= \frac{k^2}{k^2+1}[3(\Gamma^2+4)k^4+(\Gamma^2+12)k^2-2\Gamma\sqrt{(\Gamma^2+4)k^4+4k^2}\,]$$

which is positive if $k^2 > k_c^2$ and negative if $k^2 < k_c^2$

where $k_c^2 = \dfrac{\Gamma^2-12+\sqrt{\Gamma^4+40\Gamma^2+144}}{6(\Gamma^2+4)} = \dfrac{1}{3} - \dfrac{64}{3\Gamma^4} + O(\dfrac{1}{\Gamma^6})$,

because $[3(\Gamma^2+4)k^4+(\Gamma^2+12)k^2]^2 - 4\Gamma^2[(\Gamma^2+4)k^4+4k^2]$ can be factored to $9(\Gamma^2+4)^2k^2(k^2+1)(k^2-k_c^2)[k^2+\dfrac{16\Gamma^2}{9(\Gamma^2+4)^2k_c^2}]$.

From the positiveness of $\dfrac{d\Omega^2}{dk}$, it is seen that higher wavenumbers correspond to higher frequencies. The cutoff frequency that corresponds to the cutoff wavenumber k_c is $\Gamma[\dfrac{1}{4} + \dfrac{1}{\Gamma^2} - \dfrac{16}{\Gamma^4} + O(\dfrac{1}{\Gamma^6})]$.

For slow waves:

$$\Omega_S = \frac{\sqrt{(\Gamma^2+4)k^2+4k^2}-\Gamma k^2}{2(k^2+1)}\ ,$$

$$-4\Omega^2(k^2+1)+3(\Gamma^2+4)k^4+4k^2$$

$$= \frac{k^2}{k^2+1}\left[3(\Gamma^2+4)k^4+(\Gamma^2+12)k^2+2\Gamma\sqrt{(\Gamma^2+4)k^4+4k^2}\right]$$

which is always positive for all values of k. Therefore slow mode waves are unstable to disturbances.

Waves of oblique propagation

The expression for D is an algebraic function of k, containing two parameters Γ and θ. The value of Γ^2 is large, essentially equal to the ion-to-electron mass ratio for a hydrogen plasma. We shall consider two cases: $\tan\theta$ is much less than or much greater than Γ. The second case covers a narrow range of the propagation angle including the transverse propagation. The first case covers most of the rest of the range of the propagation angle including the longitudinal propagation.

For fast waves: as $\Gamma \to \infty$, we have

$$\Omega^2 = \Gamma^2\frac{k^4}{(k^2+1)^2}\cos^2\theta + \frac{k^2}{k^2+1}(1+\cos^2\theta) + O(\frac{1}{\Gamma^2}) ,$$

$$\frac{1}{\Omega}\frac{d\Omega}{dk} = \frac{2}{k(k^2+1)} - \frac{1}{\Gamma^2}\frac{1}{k^3}\frac{1+\cos^2\theta}{\cos^2\theta} + O(\frac{1}{\Gamma^4}) ,$$

$$\frac{1}{\Omega}\frac{d^2\Omega}{dk^2} = -\frac{2(3k^2-1)}{k^2(k^2+1)^2} + \frac{1}{\Gamma^2}\frac{3k^2-1}{k^4(k^2+1)}\frac{1+\cos^2\theta}{\cos^2\theta} + O(\frac{1}{\Gamma^4})$$

$$f_1 = \Omega\left[-2\Gamma^2k^4\cos^2\theta + O(1)\right] ,$$

$$g = -\frac{1}{4}\Gamma^2k^6(k^2+1)\cos^2\theta + O(1) .$$

Hence $D = \delta^2[\frac{k^2(3k^2-1)}{k^2+1} + O(\frac{1}{\Gamma^2})] - \delta^4 k^4 (\frac{d^2\Omega}{dk^2})^2$

which is negative if $k^2 < k_c^2$ and positive if $k^2 > k_c^2$ as $\delta \to 0$. The cutoff wavenumber

$$k_c = \frac{1}{\sqrt{3}} + O(\frac{1}{\Gamma})$$

corresponds to a cutoff frequency

$$\Omega_c = \frac{1}{4}\Gamma\cos\theta + O(\frac{1}{\Gamma}) .$$

For slow waves: as $\Gamma \to \infty$, we have

$$\Omega^2 = \frac{1}{\Gamma^2} - \frac{1}{\Gamma^4}\frac{k^2+1}{k^2}\frac{1+\cos^2\theta}{\cos^2\theta} + O(\frac{1}{\Gamma^6}) ,$$

$$\frac{1}{\Omega}\frac{d\Omega}{dk} = \frac{1}{\Gamma^2}\frac{1}{k^3}\frac{1+\cos^2\theta}{\cos^2\theta} + O(\frac{1}{\Gamma^4}) ,$$

$$\frac{1}{\Omega}\frac{d^2\Omega}{dk^2} = -\frac{1}{\Gamma^2}\frac{3}{k^4}\frac{1+\cos^2\theta}{\cos^2\theta} + O(\frac{1}{\Gamma^4}) ,$$

$$f_1 = \Omega[2\Gamma^2 k^4 \cos^2\theta + O(1)],$$

$$g = \frac{1}{4}\Gamma^4 k^8 \cos^4\theta + O(\Gamma^2) .$$

Hence $D = \delta^2[\frac{3}{2}k^2(1+\cos^2\theta) + O(\frac{1}{\Gamma^2})] - \delta^4 k^4 (\frac{d^2\Omega}{dk^2})^2$

which is positive as $\delta \to 0$.

For fast waves: as $\theta \to \frac{\pi}{2}$, the limiting fast wave is the transverse wave of plane polarization already discussed, being stable for all frequencies.

For slow waves: as $\theta \to \frac{\pi}{2}$, we have

$$\Omega^2 = \frac{k^2}{k^2+1}\cos^2\theta - \frac{\Gamma^2 k^4}{(k^2+1)^2}\cos^4\theta + \frac{\Gamma^4 k^6 - \Gamma^2 k^4(k^2+1)}{(k^2+1)^3}\cos^6\theta + 0(\cos^8\theta),$$

$$\frac{1}{\Omega}\frac{d\Omega}{dk} = \frac{1}{k(k^2+1)} - \frac{\Gamma^2 k}{(k^2+1)^2}\cos^2\theta + 0(\cos^4\theta) ,$$

$$\frac{1}{\Omega}\frac{d^2\Omega}{dk^2} = -\frac{3}{(k^2+1)^2} + 0(\cos^2\theta) ,$$

$$f_1 = \Omega[2k^2(k^2+1) + 0(\cos^2\theta)] ,$$

$$g = \frac{k^4(2k^4+6k^2+3)}{8\Gamma^2}\frac{1}{\cos^3\theta} + 0(\frac{1}{\cos\theta}) .$$

Hence $D = \delta^2[\frac{3k^4(2k^4+6k^2+3)}{4\Gamma^2(k^2+1)^3}\frac{1}{\cos^3\theta} + 0(\frac{1}{\cos\theta})] - \delta^4 k^4 (\frac{d^2\Omega}{dk^2})^2$

which is positive as $\delta \rightarrow 0$.

In mks units, the cutoff wavenumber is approximately equal to $\frac{1}{\sqrt{3}}$ divided by the geometric mean of the electron's gyro-radius and the ion's gyro-radius, the cutoff frequency is approximately equal to $\frac{1}{4}$ times the electron's gyro-frequency, because the normalization factor for wavenumber is the inverse of the geometric mean of the electron's gyro-radius and the ion's gyro-radius, the normalization factor for frequency is the geometric mean of the electron's gyrofrequency and the ion's gyrofrequency while Γ^2 is almost equal to the ratio of the electron's gyrofrequency to the ion's gyrofrequency.

We conclude that slow waves are always unstable to disturbances while low frequency fast waves are stable but high frequency fast waves are unstable. The cutoff frequency depends on the propagation angle. When the propagation angle is not very close to 90°, the cutoff frequency is about one-fourth of the electron's gyrofrequency in the magnetic field. When the propagation angle is very close to 90°, the cutoff frequency is so high that waves of all frequencies are stable. The transition angle is of the order of arctan Γ, which is 88.5° for a hydrogen plasma for which $\Gamma^2 = 1836$.

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