BOUNDARY VALUE PROBLEMS
FOR
STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT

A theory of two-point boundary value problems analogous to the theory of initial value problems for stochastic ordinary differential equations whose solutions form Markov processes is developed. The theory of initial value problems consists of three main parts: the proof that the solution process is markovian and diffusive; the construction of the Kolmogorov or Fokker-Planck equation of the process; and the proof that the transistion probability density of the process is a unique solution of the Fokker-Planck equation.

It is assumed here that the stochastic differential equation under consideration has, as an initial value problem, a diffusive markovian solution process. When a given boundary value problem for this stochastic equation almost surely has unique solutions, we show that the solution process of the boundary value problem is also a diffusive Markov process. Since a boundary value problem, unlike an initial value problem, has no preferred direction for the parameter set, we find that there are two Fokker-Planck equations, one for each direction. It is shown that the density of the solution process of the boundary value problem is the unique simultaneous solution of this pair of Fokker-Planck equations.

This theory is then applied to the problem of a vibrating string with stochastic density.

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I. INTRODUCTION

1.1 Stochastic Boundary Value Problems

Almost all equations used to describe and analyze physical situations are of course only approximations; in particular they often contain parameters or functions which must be determined experimentally, or they may be derived from assumptions such as homogeneity or isotropy which cannot hold exactly. For this reason there has recently been increased interest in stochastic or random versions of these equations; the aims are to investigate the errors made by using the deterministic equations, and possibly to develop a more accurate theory through the modeling by some stochastic process of a complex situation whose exact structure we cannot hope to learn.

Some of these investigations have dealt with boundary value problems and eigenvalue problems for stochastic differential equations. The methods used have been classified [15] into "honest" and "dishonest" methods. An "honest" method is one in which the stochastic equation is solved for all allowable values of the random parameters or functions, and then the given statistics of the stochastic quantities are used to find the statistics of the solution. A "dishonest" method, on the other hand, uses the stochastic equation directly to obtain equations for the desired statistics. Since these derived equations are in general an infinite coupled system, some closure assumption, which most often cannot be justified, is necessary.

For example, let $\mathcal{L}$ be a linear self-adjoint differential operator, and consider the equation

$$\mathcal{L}\, u(x) + \lambda\, h(x)\, u(x) = 0, \tag{1.1}$$

for $0 \leq x \leq 1$ with boundary conditions at $x = 0$ and $x = 1$. Here $h(x)$ is a stochastic function. Equation (1.1) has been treated [4] by a "dishonest" method as follows: Let $G(x, \xi)$ be the Green's function for $\mathcal{L}$ with the given boundary conditions. Taking $h(x) = 1 + \eta(x)$, where $\eta(x)$ is a zero-mean stochastic function, the equivalent Fredholm integral equation is

$$u(x) = \lambda \int_0^1 G(x,\xi)(1 + \eta(\xi))\, u(\xi)\, d\xi. \tag{1.2}$$

The expectation of (1.2) is taken in the form

$$\langle \lambda^{-1} u(x) \rangle = \int_0^1 G(x,\xi) \left(\langle u(\xi) \rangle + \langle \eta(\xi) u(\xi) \rangle \right) d\xi.$$

By assuming that both λ^{-1} and $\eta(x)$ are uncorrelated with $u(x)$, this reduces to

$$\langle \lambda^{-1} \rangle \cdot \langle u(x) \rangle = \int_0^1 G(x,\xi)\, \langle u(\xi) \rangle\, d\xi.$$

Hence, under the above assumptions, the eigenvalues λ^{-1} and eigenfunctions $u(x)$ of the stochastic problem have expectations equal to the eigenvalues and eigenfunctions of the deterministic equation with $\eta(x) \equiv 0$.

Higher moments of $u(x)$ and λ^{-1} can be obtained by taking moments of the iterates of (1.2). The result is of course an infinite coupled system; several truncation methods for such systems have been studied [12,22].

"Honest" analyses of equations such as (1.1) have included the following techniques: If $\eta(x)$ is almost surely bounded, then elementary comparison theorems give bounds on the eigenvalues [10]; classical asymptotic estimates for the large eigenvalues have been used [3,5]; as have variational descriptions of the eigenvalues [3]; and by taking $h(x) = 1 + \alpha \cdot \eta(x)$, a perturbation expansion in the parameter α yields approximations for the eigenfunctions and eigenvalues [3,4].

A somewhat different "honest" method is the use of "stochastic Green's functions" [1,2]. It is assumed that the response $y(t)$ of the system under consideration to a stochastic input $x(t)$ can be written in the form

$$y(t) = \int_{-\infty}^{+\infty} h(\alpha, \beta, \ldots, \tau)\, x(\tau)\, d\tau$$

where $\alpha, \beta, \ldots$ are random parameters. When the process $x(t)$ is stationary and independent of $\alpha, \beta, \ldots$, the spectral densities ϕ_x and ϕ_y of $\{x\}$ and $\{y\}$ satisfy

$$\phi_y(f) = \int_{-\infty}^{+\infty} K_H(s, f)\, \phi_x(s)\, ds$$

The kernel K_H, called a "stochastic Green's function," is the spectral density of the function h. Unfortunately, K_H is seldom easy to determine. Also, this approach is of course limited to linear systems.

One way to bypass the closure problem of "dishonest" methods is the use of a generating or characteristic functional [16].

For example, if we are interested in

$$\mathcal{L} u(x) + h(x) \cdot u(x) = g(x),$$

where $g(x)$ is a deterministic function while $h(x)$ is stochastic, then we can consider the generating functional

$$F[\xi, \eta] = \langle \exp\{(\xi, u) + (\eta, h)\} \rangle .$$

Then we can show formally that

$$\mathcal{L} \frac{\delta F}{\delta \xi(x)} = - \frac{\delta^2 F}{\delta \xi(x)\, \delta \eta(x)} + g(x) F,$$

where $\frac{\delta}{\delta \xi(x)}$ denotes a functional derivative. However, functional differential equations do not seem to be easy to solve (see, for example, [18], in which an approximation which involves only first-order functional derivatives is treated).

1.2 The Initial Value Approach

Now the solution of a boundary value problem for an ordinary differential equation is also the solution of an initial value problem for the same equation — but of course the initial values are not known *a priori*. This idea is the basis of the well-known "shooting method" for the numerical solution of boundary value problems and has also been used to prove existence and uniqueness for solutions of some non-linear boundary value problems [3].

The treatment of boundary value problems via the theory of initial value problems has often been successful because the theory of initial value problems is well-developed. Because a great deal is known about initial value problems for stochastic differential equations whose solutions are Markov processes, we might hope that a theory of boundary value problems for these equations could be constructed by utilizing the known theory of initial value problems for Markov processes. This approach is the one we shall take.

Since initial value problems for stochastic differential equations are customarily discussed with the "time" t as the independent variable, we shall consider stochastic differential equations on a time interval, say $t \in [0,1]$, with boundary conditions at $t = 0$ and $t = 1$. Of course most boundary value problems of interest have "spatial" independent variables, but our choice will make the relation between initial value and boundary value problems clearer.

Because we shall make extensive use of the initial value theory, we summarize its main results at this point.

A stochastic process $\underline{x}_t$ is called a Markov process if its conditional distribution functions P satisfy

$$\begin{aligned} &P(\Gamma_{m+1}, t_{m+1}; \ldots ; \Gamma_{m+n}, t_{m+n} | x_1, t_1; \ldots ; x_m, t_m) \\ (1.3) \qquad & \\ &\qquad = P(\Gamma_{m+1}, t_{m+1}; \ldots ; \Gamma_{m+n}, t_{m+n} | x_m, t_m), \end{aligned}$$

for any $t_1 < t_2 < \ldots < t_m < t_{m+1} < \ldots < t_{m+n}$. Here the Γ_i are sets in the space wherein $\{\underline{x}_{t_i}\}$ takes its values.

One consequence of (1.3) is the Chapman-Kolmogorov equation for the transistion distribution function $P(\Gamma, t \mid \underline{\xi}, \tau)$, $t > \tau$:

$$(1.4) \quad P(\Gamma_3, t_3 \mid x_1, t_1) = \int P(\Gamma_3, t_3 \mid x_2, t_2)\, P(dx_2, t_2 \mid x_1, t_1),$$

for any $t_1 < t_2 < t_3$. Further, if we have a transistion distribution function which satisfies (1.4), then we may construct a consistent set of conditional distribution functions satisfying (1.3), so that (1.4) is essentially equivalent to (1.3).

If we have a markovian transistion distribution function $P(\Gamma, t \mid \underline{\xi}, \tau)$ which has a density $p(\underline{x}, t \mid \underline{\xi}, \tau)$ and which also satisfies the diffusion condition

$$(1.5) \qquad P_r\left(\|\underline{x}_t - \underline{\xi}\| \geq \varepsilon > 0 \mid \underline{\xi}, \tau \right) \to 0 \quad \text{as} \quad t \to \tau$$

then it can be shown [21,23] that the transistion probability density $p(\underline{x}, t \mid \underline{\xi}, \tau)$ satisfies a pair of partial differential equations:

$$\frac{\partial p}{\partial \tau} = -\frac{1}{2} \sum_{i,j} b_{ij}(\underline{\xi}, \tau) \frac{\partial^2 p}{\partial \xi_i \partial \xi_j} - \sum_i a_i(\underline{\xi}, \tau) \frac{\partial p}{\partial \xi_i}$$

and

$$\frac{\partial p}{\partial t} = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(b_{ij}(\underline{x}, t) p \right) - \sum_i \frac{\partial}{\partial x_i} \left(a_i(\underline{x}, t)\, p \right)$$

These equations are called, respectively, the backward and forward Kolmogorov equations; the forward equation is also known as the Fokker-Planck equation.

The coefficients in the Kolmogorov equations are the incremental moments of the process $\{\underline{x}_t\}$:

$$\underline{a}(\underline{\xi}, t) = (a_i) = \lim_{\Delta \to 0} \frac{1}{\Delta} E\left[\underline{x}_{t+\Delta} - \underline{\xi} \,|\, \underline{\xi}, t\right],$$

$$B(\underline{\xi}, t) = (b_{ij}) = \lim_{\Delta \to 0} \frac{1}{\Delta} E\left[(\underline{x}_{t+\Delta} - \underline{\xi})(\underline{x}_{t+\Delta} - \underline{\xi})^T \,|\, \underline{\xi}, t\right].$$

Consider now a vector ordinary differential equation with independent variable t,

$$(1.6) \qquad \overset{\circ}{\underline{x}} = \underline{f}(\underline{x}, t) + \underline{g}(\underline{x}, t) N(t)$$

and initial condition

$$\underline{x}(\tau) = \underline{\xi}.$$

Here $N(t)$ is gaussian white noise with the formal properties $E\left[N(t)\right] = 0$, $E\left[N(t) \cdot N(\tau)\right] = 2D\,\delta(t-\tau)$, where D is a constant. The concept of white noise can be made precise in several ways.

Usually we say that (1.6) is a shorthand notation for the integral equation

$$(1.7) \qquad \underline{x}(t) = \underline{\xi} + \int_\tau^t \underline{f}(\underline{x}(s), s)\, ds + \int_\tau^t \underline{g}(\underline{x}(s), s)\, dW(s),$$

where $\{w(t)\}$ is the Weiner process (whose formal derivative has the properties of white noise). The last integral in (1.7) is called a stochastic integral and has been given precise definitions by Ito [7] and by Stratonovich [22]. From (1.7) the method of successive approximations shows, under certain regularity conditions on the function $\underline{f}$ and $\underline{g}$, that there is a Markov process $\{\underline{x}(t)\}$ which satisfies (1.7) and has continuous sample paths with probability one [7]. Further, this solution process is diffusive.

Then we immediately have the Kolmogorov equations; the incremental moments which appear in these equations can be calculated directly from the stochastic equation (1.6). Finally it can be shown without further assumptions that the Kolmogorov equations with the initial condition

$$p(\underline{x},t\,|\,\underline{\xi},\tau) \;=\; \delta(\underline{x}-\underline{\xi}) \quad \text{at} \quad t=\tau$$

have unique solutions [11]. To complete the theory of initial value problems for some specific stochastic differential equation, we need only solve the appropriate Fokker-Planck equation.

Therefore, if we are to develop an analogous theory for boundary value problems, the answers to the following questions are crucial:

(1) When, if ever, is the solution of a stochastic boundary value problem a Markov process? Since this solution clearly must depend on its value at a time in the future, namely $t = 1$, it is not obvious that the solution will be markovian.

(2) If the process is markovian, is it diffusive? That is, does its transistion probability density satisfy the appropriate Kolmogorov equations?

(3) If the solution if markovian and diffusive, how do its Kolmogorov equations differ from those for the initial value

problem for the stochastic equation under consideration?

Specifically, then, our aim in the following is to answer these three questions.

1.3 Notations and Conventions

For simplicity, we shall work with probability densities -- that is, the assumption or proof that some distribution exists will also mean that it is absolutely continuous with respect to Lebesgue measure.

Any integral written without limits is over eudlidean space R^n; the dimension n will be clear from the context. We assume that all integrals are sufficiently well-behaved that the order of integration in multiple integrals and the order of integration and differentiation may be interchanged.

II. THE GAUSSIAN BOUNDARY VALUE PROBLEM

2.1 The Boundary Value Density

We begin our discussion of boundary value problems for stochastic differential equations with the special case of a gaussian process. In this case we can explicitly display all the quantities of interest and find the equations they satisfy. Further, the equations raised in §1.2 can be reduced to a set of matrix equations.

Let

$$\mathcal{L} = \sum_{j=0}^{n} \alpha_j(t) \frac{d^j}{dt^j} , \quad \alpha_n = 1 ,$$

be a linear n-th order differential operator with infinitly differentiable (deterministic) coefficients α_j on the interval $t \in [0,1]$, and let $\mathcal{B}$ be some set of functions in $C^{(n-1)}[0,1]$ determined by linear homogeneous unmixed boundary conditions at $t = 0$ and $t = 1$ such that

$$\mathcal{L}\, y(t) = 0 \quad , \quad y \in \mathcal{B} ,$$

implies

$$y(t) \equiv 0 \quad , \quad t \in [0,1] .$$

Let $h(t, \tau)$ be the Green's function for $\mathcal{L}$ with the boundary conditions $\mathcal{B}$:

$$\mathcal{L}\, h(t,\tau) = \delta(t-\tau) ,$$

where $h(t,\tau) \in \mathcal{B}$ for each $0 < \tau < 1$ except that $h(t,\tau)$ is only in $C^{(n-1)}[0,\tau)$ and $C^{(n-1)}(\tau,1]$.

Then the system

$$\mathcal{L}\,x(t) = N(t), \quad x \in \mathcal{B},$$

has the unique solution

$$x(t) = \int_0^1 h(t,\tau)\, N(\tau)\, d\tau \tag{2.1}$$

Here $N(t)$ is gaussian white noise with the formal properties $E[N(t)] = 0$, $E[N(t)\cdot N(\tau)] = 2D\,\delta(t-\tau)$. Of course (2.1) is only a formal solution, since the integral will not exist as a Lebesgue-Stieltjes integral. To be completely rigorous, we would need either to define a new type of stochastic integral (the Ito [7] and Stratonovich [22] integrals are not appropriate, since the upper limit on the integral here is not t); or to let $N(t)$ be a process with small but non-zero correlation time and to let the correlation time tend to zero. Either approach will of course yield the same results as the formal calculations we shall make; it is only when a white noise process appears as a coefficient in the operator that care must be exercised (cf. [6]).

From (2.1), which we call the <u>boundary value process</u>, we easily obtain the means:

$$E\left[\frac{d^j x(t)}{dt^j}\right] = 0, \quad j = 0(1)(n-1),$$

and the covariances:

$$k_{ij} = E\left[\frac{d^i x(t)}{dt^i} \cdot \frac{d^j x(t)}{dt^j} \right]$$

$$= 2D \int_0^1 \frac{\partial^i h(t,\tau)}{\partial t^i} \cdot \frac{\partial^j h(t,\tau)}{\partial t^j} \, d\tau , \quad i,j = 0(1)(n-1).$$

The probability density $p_b(\underline{x})$ of the vector process $\underline{x} = (x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}})$ is then the normal density with zero mean and covariance matrix $K_t = (k_{ij})$. We shall assume that K_t is non-singular for $0 < t < 1$.

For example, the boundary value problem

$$\left.\begin{aligned} \mathring{x} &= y \\ \mathring{y} &= N(t) \end{aligned}\right\} \quad x(0) = x(1) = 0 \tag{2.2}$$

has covariance matrix

$$K_t = 2D \begin{pmatrix} t^2(1-t)^2/3 & t(t-1)(2t-1)/3 \\ t(t-1)(2t-1)/3 & t^2 - t + 1/3 \end{pmatrix}$$

with

$$\det(K_t) = 4D^2/9 \cdot t^3(1-t)^3.$$

Now this probability density is just the quantity we would like to find in all cases; from it we can calculate any moment desired. However, the above calculations can be carried out only in the gaussian case. Ideally, we desire an equation which the probability density of the boundary value process will satisfy -- and if the boundary value process were markovian, the corresponding Fokker-Planck equation would be the equation to consider. In the gaussian case, we can proceed in reverse -- that is, find the transistion probability density of the boundary value process, determine whether or not it is markovian, and then find its Fokker-Planck equation. The boundary value density $p_b(\underline{x})$ should also satisfy this Fokker-Planck equation.

2.2 The Boundary Value Transistion Probability Density

The calculation of the transistion probability density of the boundary value process (which we shall henceforth refer to as the b.v.t.p.d.) is straightforward. Let $\underline{x}$ be the n-vector $(x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}})$ at time t, and let $\underline{x}_0$ be this vector at time $t_0 < t$. From the formal solution (2.1) of the boundary value problem, we obtain as before the mean and covariance of the 2n-dimensional process $(\underline{x}, \underline{x}_0)$. If $p_b(\underline{x}, \underline{x}_0)$ is the corresponding normal density, and if $p_b(\underline{x}_0)$ is the boundary value density evaluated at time t_0, then Bayes' law gives the b.v.t.p.d. $p_b^+(\underline{x}\ \underline{x}_0)$ as*

$$p_b^+(\underline{x} | \underline{x}_0) = \frac{p_b(\underline{x}, \underline{x}_0)}{p_b(\underline{x}_0)} .$$

The details of this calculation are in Appendix A.1; p_b^+ is of course gaussian, and we find that its mean is given by

$$E[\underline{x} | \underline{x}_0] = M(t, t_0)\, K_{t_0}^{-1}$$

* The "+" notation means that p_b^+ is the density for transitions from time t_0 to time t for $t > t_0$.

and its covariance matrix by

$$L = K_t - M K_{t_0}^{-1} M^T \tag{2.3}$$

Here K_t is again the covariance of $\underline{x}$ with itself, and $M = M(t,t_0)$ is the covariance of $\underline{x}$ and $\underline{x}_0$.

Now setting $\underline{\lambda} = (\lambda_j) = K_{t_0}^{-1}\underline{x}_0$ we have

$$E[x \mid \underline{x}_0] = \sum_{j=0}^{n-1} m_{0j}\lambda_j$$

$$= \int_0^1 \left(2D \sum_{j=0}^{n-1} \frac{\partial^j h(t_0,\tau)}{\partial t_0^j}\right) h(t,\tau)\, d\tau$$

$$\in \mathcal{B} \qquad \text{as a function of } t,$$

and similarly for the expectation of the derivatives of x. Thus, $E[\underline{x} \mid \underline{x}_0]$ satisfies the appropriate boundary conditions. (Although if we have evaluated the integrals giving the elements of $M(t,t_0)$ for $t > t_0$, then only the boundary conditions at $t = 1$ will be satisfied by the resulting expressions.) The same is clearly true for the covariance matrix L.

To show that the boundary value process is markovian, we need only show that p_b^+ satisfies the Chapman-Kolmogorov equation

$$p_b^+(\underline{x} \mid \underline{x}_0) = \int p_b^+(\underline{x} \mid \underline{x}_\tau)\, p_b^+(\underline{x}_\tau \mid \underline{x}_0)\, d\underline{x}_\tau \tag{2.4}$$

for any $t_0 < \tau < t$. It is easily shown that (2.4) will hold if and only if

$$M(t,\tau)K_\tau^{-1} = M(t,t_0)M(\tau,t_0)^{-1}\ , \quad t_0 < \tau < t \tag{2.5}$$

Letting $Q(t,\tau) = M(t,\tau)K_\tau^{-1}$, (2.5) becomes

$$Q(t,\tau)\,Q(\tau,t_0) = Q(t,t_0) \tag{2.6}$$

Now $Q(t,t) = I$, so if the matrix

$$\Phi = \Phi(t,\tau) = \frac{\partial Q}{\partial t} Q^{-1} = \frac{\partial M}{\partial t} M^{-1} \tag{2.7}$$

is independent of τ, then $Q(t,\tau)$ will be the fundamental matrix solution of

$$\frac{\partial Q}{\partial t} = \Phi Q \quad , \quad Q\big|_{t=\tau} = I \ ,$$

and (2.6) and (2.4) will hold.

In fact, (2.6) does hold; the proof is in Appendix A.2. There we also find the the elements φ_{ij} of Φ are given by

$$\varphi_{ij} = \begin{cases} \delta_{i+1,j} & , \quad i = 0(1)(n-2) \\ -\alpha_j(t) + \beta_j(t), & i = n-1 \end{cases} ,$$

where the β_j are determined by

$$\sum_{k=0}^{n-1} \beta_k m_{kj} = 2D\,\frac{\partial^j h(t_0,t)}{\partial t_0^j}\ , \quad j = 0(1)(n-1)$$

and also satisfy the relation

$$\sum_{k=0}^{n-1} \beta_k \frac{\partial^k h(t,t_0)}{\partial t^k} = 0 \quad , \quad t > t_0 .$$

For example, for the system (2.2) we find

$$\beta_0 = \frac{-3}{(1-t)^2} \quad , \quad \beta_1 = \frac{-3}{(1-t)}$$

and since

$$h(t,t_0) = t_0(t-1) \quad , \quad t > t_0 ,$$

we have

$$\sum_0^1 \beta_k \frac{\partial^k h(t,t_0)}{\partial t^k} = \frac{-3}{(1-t)}(t_0 - t_0) = 0 .$$

2.3 The Upward Equation

Now if the boundary value process were diffusive, then, being markovian, it would have a Fokker-Planck equation [21]. Roughly speaking, a diffusive process has no jumps in its sample paths; the precise condition is given by equation (1.5). Rather than verify the diffusive nature of the boundary value process, we shall instead show that p_b^+ satisfies its formal Fokker-Planck equation. Since the diffusive property is only a sufficient condition for the validity of the corresponding Fokker-Planck equation, this approach will show that the boundary value process is what we shall call weakly diffusive: the appropriate Fokker-Planck equation holds.

The formal Fokker-Planck equation for p_b^+ has the form

$$\frac{\partial p_b^+}{\partial t} = \frac{1}{2}\sum_{i,j=0}^{n-1} \frac{\partial^2}{\partial x_i \partial x_j}(b_{ij} p_b^+) - \sum_{i=0}^{n-1} \frac{\partial}{\partial x_i}(a_i p_b^+)$$

where $(x_i) = \underline{x}$ and the a_i and b_{ij} are the incremental moments of the boundary value process $\underline{x}$. We can calculate these moments explicitly:

$$\underline{a}(\underline{x},t) = \lim_{\Delta \to 0} \frac{1}{\Delta}\left[M(t+\Delta,t)K_t^{-1} - I\right]\underline{x}$$

$$= \lim_{\Delta \to 0} \frac{M(t+\Delta,t) - M(t,t)}{\Delta} K_t^{-1}\,\underline{x}$$

$$= \left.\frac{\partial M(t,\tau)}{\partial t}\right|_{\tau=t} \cdot K_t^{-1}\,\underline{x}$$

$$= \Phi(t)\,\underline{x}$$

As we have seen, $\varphi_{ij} = \delta_{i+1,j}$ for $i \neq n-1$. Thus except for a_{n-1} the incremental means a_i will be the same as in the initial value Fokker-Planck equation:

$$a_i = x_{i+1} \quad , \quad i = 0(1)(n-2)$$

The matrix B of second incremental moments is easily found from the formula (2.3) for the covariance of p_b^+ :

$$\begin{aligned} B = \lim_{\Delta \to 0} \frac{1}{\Delta}\Big[& K_{t+\Delta} - M(t+\Delta,t)K_t^{-1} M(t+\Delta,t)^T \\ & + M(t+\Delta,t)K_t^{-1}(\underline{x}\,\underline{x}^T)K_t^{-1} M(t+\Delta,t)^T \\ & - (\underline{x}\,\underline{x}^T)K_t^{-1} M(t+\Delta,t)^T \\ & - M(t+\Delta,t)K_t^{-1}(\underline{x}\,\underline{x}^T) \quad + (\underline{x}\,\underline{x}^T)\Big] \end{aligned}$$

Since $M(t,t) = K_t$, this becomes

$$B = \lim_{\Delta \to 0} \frac{1}{\Delta} \left[M(t+\Delta, t+\Delta) - M(t+\Delta, t) K_t^{-1} M(t+\Delta, t)^T \right]$$

$$= \frac{\partial}{\partial \tau} M(t,\tau) \Big|_{\tau = t} - \frac{\partial}{\partial t} M(t,\tau)^T \Big|_{\tau = t} .$$

The only non-zero element of B is $b_{n-1,n-1}$. Because

$$\frac{\partial}{\partial \tau} m_{n-1,n-1} = 2D \frac{\partial^{n-1} h(t,\tau)}{\partial t^{n-1}} - \sum_{k=0}^{n-1} \alpha_k(\tau) m_{k,n-1} ,$$

$$\frac{\partial}{\partial t} m_{n-1,n-1} = 2D \frac{\partial^{n-1} h(\tau,t)}{\partial \tau^{n-1}} - \sum_{k=0}^{n-1} \alpha_k(t) m_{k,n-1} ,$$

and since the jump in $\dfrac{\partial^{n-1} h(t,\tau)}{\partial t^{n-1}}$ across $t = \tau$ is 1, we have

$$b_{n-1,n-1} = 2D$$

Thus we have the formal Fokker-Planck equation

$$(2.8) \quad \frac{\partial p_b^+}{\partial t} = D \frac{\partial^2 p_b^+}{\partial x_{n-1}^2} - \sum_{i=0}^{n-2} x_{i+1} \frac{\partial p_b^+}{\partial x_i} - \frac{\partial}{\partial x_{n-1}} \left[(\underline{\varphi}, \underline{x}) p_b^+ \right] ,$$

where the vector $\underline{\varphi}$ is the last row of $\Phi(t)$, and has k-th component $-\alpha_k(t) + \beta_k(t)$, $k = 0(1)n-1$.

For example, the system (2.2) yields

$$(\underline{\varphi}, \underline{x}) = -\frac{3x}{(1-t)^2} - \frac{3y}{(1-t)}$$

To show that p_b^+ does indeed satisfy (2.8), that is, that the boundary value process is weakly diffusive, we note that since p_b^+ is gaussian, we need only verify that the first and second moment equations implied by (2.8) are satisfied by the moments of p_b^+. These equations are easily obtained by integration of (2.8):

$$(2.9)\qquad \frac{\partial}{\partial t} E[\underline{x} \mid \underline{x}_0] = \frac{\partial}{\partial t} \int \underline{x}\, p_b^+(\underline{x} \mid \underline{x}_0)\, d\underline{x} = \Phi(t) \cdot E[\underline{x} \mid \underline{x}_0]$$

and

$$(2.10)\qquad \frac{\partial}{\partial t} E[\underline{x}\,\underline{x}^T \mid \underline{x}_0] = B + \Phi \cdot E[\underline{x}\,\underline{x}^T \mid \underline{x}_0] + E[\underline{x}\,\underline{x}^T \mid \underline{x}_0] \cdot \Phi .$$

Of course, (2.9), being equivalent to the markovian nature of the process, holds. Since by (2.3)

$$E[\underline{x}\,\underline{x}^T \mid \underline{x}_0] = K_t - M K_{t_0}^{-1} M^T + M K_{t_0}^{-1} (\underline{x}_0 \underline{x}_0^T) K_{t_0}^{-1} M^T$$

and since

$$\begin{aligned}
\frac{\partial}{\partial t} K_t &= \frac{\partial}{\partial t} M(t,t) \\
&= \left[\frac{\partial}{\partial t} M(t,\tau) + \frac{\partial}{\partial \tau} M(t,\tau) \right]_{\tau = t} \\
&= \frac{\partial}{\partial \tau} M(t,\tau)\Big|_{\tau=t} - \frac{\partial}{\partial t} M(t,\tau)^T\Big|_{\tau=t} \\
&\quad + \frac{\partial}{\partial t} M(t,\tau)\Big|_{\tau=t} + \frac{\partial}{\partial t} M(t,\tau)^T\Big|_{\tau=t} \\
&= B + \Phi K_t + K_t \Phi^T
\end{aligned}$$

we also have (2.10) and conclude that the boundary value process is weakly diffusive.

Now there are two essential questions we must ask about the boundary value Fokker-Planck equation (2.8):

(1) Is the boundary value density p_b of §2.1 a unique solution of (2.8) and the appropriate boundary conditions? If so, then we could use this equation to determine all the properties of boundary value processes, using the same techniques that are applied in initial value problems.

(2) How may we find the incremental moments that appear in (2.8) without knowing the transistion probability density _a priori_? This question will be answered in chapters III and IV. For the present, it is interesting to note that the only difference between (2.8) and the initial value Fokker-Planck equation for the same stochastic differential equation is the extra term $\left(\sum_{k=0}^{n-1} \beta_k x_k \right)$; this term can be interpreted as the conditioned mean of the (zero-mean) noise process $N(t)$. For a boundary value process, then, the conditioned mean is _not an average across all samples_, since the conditioning variables contain information about the samples under consideration. However, the conditioned variance of $N(t)$ is apparently unchanged, for it yields the same second-derivative terms in the Fokker-Planck equation.

The answer to question (1) is in fact _no_: (2.8) with the appropriate boundary condtions does not uniquely determine p_b. As before, we need only consider the moment equations, which are just (2.9) and (2.10) without the conditioning. Since K_t, the actual covariance of p_b, satisfies (2.10), we set

$$N = E[\underline{X}\,\underline{X}^T] - K_t$$

Then we have

$$\frac{d}{dt} E[\underline{X}] = \Phi(t)\, E[\underline{X}] \tag{2.11a}$$

and

$$\frac{dN}{dt} = \Phi N + N \Phi^T \tag{2.11b}$$

Thus p_b, which has mean zero, will uniquely satisfy (2.8) if and only if the two equations (2.11) with the appropriate boundary conditions have only the trivial solutions.

Unfortunately, the general solution of (2.11a) is

$$E[\underline{X}] = Q(t,0)\,\underline{e}_0$$

for an arbitrary vector $\underline{e}_0$. As we have seen (§2.2), $Q(1,0)\underline{e}_0$ will satisfy the boundary conditions at $t = 1$ for any $\underline{e}_0$; and the boundary conditions at $t = 0$ will of course not determine $\underline{e}_0$ uniquely (unless we have the degenerate case where the boundary value problem is actually an initial value problem).

In fact, we could satisfy many boundary conditions at $t = 0$ by choosing $\underline{e}_0$ appropriately. It will turn out that p_b^+ and its Fokker-Planck equation (2.8) are <u>independent</u> of the conditions imposed at $t = 0$ (see chapter IV).

Also the general solution of (2.11b) is

$$N = \alpha_0 \, M(t,0)\, M(t,0)^T$$

for any constant α. This expression matches the boundary conditions at both $t = 0$ and $t = 1$, since $M(t,0)^T = M(0,t)$.

Therefore we shall need more information, preferably another equation for p_b, in order to determine the boundary value density. Up to this point, we have not taken into account one of the essential differences between boundary value and initial value problems: For a boundary value problem, there is <u>no preferred time direction</u>. Hence, it should be possible to carry out all the above analysis for transistions from time t_0 to time t for $t_0 > t$.

For this reason p_b^+ will be called the <u>upward</u> b.v.t.p.d. and (2.8) the <u>upward equation</u>; there should also be a <u>downward equation</u>, that is, the Fokker-Planck equation for the <u>downward</u> b.v.t.p.d., which will be denoted by p_b^-.

2.4 The Downward Equation

We proceed as before, with the downward b.v.t.p.d. given by

$$p_b^-(\underline{x} \mid \underline{x}_0) = \frac{p_b(\underline{x}, \underline{x}_0)}{p_b(\underline{x}_0)} \qquad \text{for } t_0 > t.$$

Then p_b^- has mean

$$\tilde{e} = M(t_0,t)^T K_{t_0}^{-1} \underline{x}_0$$

and covariance

$$\tilde{L} = K_t - M(t_0,t)^T K_{t_0}^{-1} M(t_0,t),$$

and satisfies the Chapman-Kolmogorov equation

$$p_b^-(\underline{x}|\underline{x}_0) = \int p_b^-(\underline{x}|\underline{x}_\tau)\, p_b^-(\underline{x}_\tau|\underline{x}_0)\, d\underline{x}_\tau \quad , \; t<\tau<t_0.$$

From this we obtain in the usual way [23] the Fokker-Planck equation for p_b^- :

$$-\frac{\partial p_b^-}{\partial t} = \frac{1}{2}\sum_{i,j=0}^{n-1} \frac{\partial^2}{\partial x_i \partial x_j}(\tilde{b}_{ij}\, p_b^-) - \sum_{i=0}^{n-1} \frac{\partial}{\partial x_i}(\tilde{a}_i\, p_b^-),$$

where

$$\tilde{\underline{a}} = (a_i) = \lim_{\Delta\to 0} \frac{1}{\Delta} E[\underline{x}_{t-\Delta} - \underline{x}_t | \underline{x}_t],$$

$$\tilde{B} = (b_{ij}) = \lim_{\Delta\to 0} \frac{1}{\Delta} E[(\underline{x}_{t-\Delta}-\underline{x}_t)(\underline{x}_{t-\Delta}-\underline{x}_t)|\underline{x}_t].$$

Defining

$$\tilde{\Phi}(t) = \frac{\partial M(t_0,t)^T}{\partial t} \cdot \left(M(t_0,t)^T\right)^{-1}$$

we find that the elements $\tilde{\varphi}_{ij}$ of $\tilde{\Phi}$ are given by

$$\tilde{\varphi}_{ij} = -\,\delta_{i+1,j} \quad , \quad i = 0(1)(n-1).$$

Also as before, the matrix $\tilde{B}$ has only one non-zero element, which is

$$\tilde{b}_{n-1,n-1} = 2D.$$

Then the downward equation (which is an equation of backward parabolic type) is

$$\text{(2.12)} \qquad \frac{\partial p_b^-}{\partial t} = -D\frac{\partial^2 p_b^-}{\partial x_{n-1}^2} - \sum_{i=0}^{n-1} x_{i+1}\frac{\partial p_b^-}{\partial x_i} + \frac{\partial}{\partial x_{n-1}}\left[(\underline{\tilde{\varphi}},\underline{x})p_b^-\right]$$

where $\underline{\tilde{\varphi}}$ is the last row of $\underline{\Phi}$.

For example, the system (2.2) yields

$$(\underline{\tilde{\varphi}}, \underline{x}) = \frac{3x}{t^2} - \frac{3y}{t}$$

It is clear that we also could have obtained (2.12) by finding the upward equation for the system produced from the original stochastic differential equation by the transformation $t \longrightarrow 1-t$ and then inverting this transformation.

Of course (2.12) does not have a unique solution any more than the upward equation (2.8) did. However, they do have a unique simultaneous solution, as is easily seen:

The moment equations from (2.12) are, corresponding to (2.11),

$$\text{(2.13a)} \qquad \frac{d}{dt} E[\underline{x}] = -\tilde{\Phi}\cdot E[\underline{x}]$$

$$\text{(2.13b)} \qquad \frac{dN}{dt} = -\tilde{\Phi}\cdot N - N\cdot\tilde{\Phi}^T$$

Since clearly $\underline{\varphi} \neq -\underline{\tilde{\varphi}}$ ($\underline{\varphi}$ is singular at $t = 1$ and $\underline{\tilde{\varphi}}$ at $t = 0$), the only simultaneous solution of (2.11) and (2.13) with the appropriate boundary conditions is $E[\underline{x}] = 0$ and $N = 0$. This means that the unique simultaneous solution to (2.8) and (2.12) with the appropriate boundary conditions is just p_b, the boundary value density.

2.5 Summary of the Gaussian Case

The results obtained from consideration of gaussian boundary value processes may be summarized as follows:

(1) The boundary value process is markovian and weakly diffusive.

(2) The boundary value density p_b is the simultaneous solution of two Fokker-Planck equations, an upward equation and a downward equation.

(3) The upward and downward equations are parabolic equations of forward and backward type, respectively. They are identical with the corresponding initial-value Fokker-Planck equations except that the conditioned mean of the driving noise process is non-zero.

Our aim is to show that these three statements also hold in the non-gaussian case.

III. FIRST-ORDER FINAL VALUE PROBLEMS

3.1 The Final Value Densities

In searching for a way to extend the properties enumerated in §2.5 to non-gaussian boundary value problems, it is natural to consider the simplest possible case. As we have noted in §2.3, the gaussian b.v.t.p.d. p_b^+ and the upward equation which it satisfies are independent of the boundary conditions at $t = 0$. This suggests the study of what we shall call <u>final value problems</u>: a Markov process $\{\underline{x}_t\}$ for t increasing, $t < 1$, but with boundary conditions at $t = 1$. For simplicity we treat first-order final value problems.

Consider the system

$$\dot{x} + f(x) = N(t), \tag{3.1}$$

for $t < 1$, with the final value

$$x(1) = x_1 .$$

The <u>final value density</u> $p_f(x_t)$ will of course be just the transistion probability density for the process in reverse time,

$$\overset{\circ}{y} - f(y) = N(t). \tag{3.2}$$

That is, if $q(y,t \mid y_0)$ is the transistion probability density for (3.2) with $t > 0$, then

$$p_f(x_t) = q(x_t, 1-t \mid x_1).$$

However, our main interest is in the transistion probability density p_f^+ of the final value process. Letting $p(\cdot \mid \cdot)$ be the indicated densities for the process (3.1), we have

$$p_f^+(x_t \mid x_{t_0}) = p(x_t \mid x_{t_0}, x_1), \quad t > t_0. \tag{3.3}$$

We assume that (3.1), as an initial value process, defines a Markov process.* Then (3.3) becomes

$$p_f^+(x_t \mid x_{t_0}) = \frac{p(x_1, 1-t \mid x_t)\; p(x_t, t-t_0 \mid x_{t_0})}{p(x_1, 1-t_0 \mid x_{t_0})}. \tag{3.4}$$

Here $p(x,t \mid x_0)$ is the (initial value) transistion probability density for (3.1); we note for later use that is satisfies the backward Kolmogorov equation

$$\frac{\partial}{\partial t} p(x,t \mid x_0) = D \frac{\partial^2 p}{\partial x_0^2} - f(x_0) \frac{\partial p}{\partial x_0} \tag{3.5}$$

as well as the Fokker-Planck equation

$$\frac{\partial}{\partial t} p(x,t \mid x_0) = D \frac{\partial^2 p}{\partial x_0^2} + \frac{\partial}{\partial x}\left[f(x) p \right]. \tag{3.6}$$

The Chapman-Kolmogorov equation for p_f^+ follows immediately from the representation (3.4), since for $t_0 < \tau < t$,

---

* This has been proven only when $f(x)$ is essentially linear — i.e., $|f(x)| \leq$ (const.) $(1+x^2)^{1/2}$ [7].

$$\int p_f^+(x_t|x_\tau)\, p_f^+(x_\tau|x_0)\, dx_\tau$$

$$= \frac{p(x_1|x_t)}{p(x_1|x_{t_0})} \cdot \int p(x_t|x_\tau)\, p(x_\tau|x_{t_0})\, dx_\tau$$

$$= \frac{p(x_1|x_t)}{p(x_1|x_{t_0})} \cdot p(x_t|x_{t_0})$$

$$= p_f^+(x_t|x_{t_0}).$$

Hence the final value process is markovian; we must now find its Fokker-Planck equation and show that the process is markovian.

3.2 The Upward Equation

To find the formal Fokker-Planck equation for p_f^+ (which will be an equation of upward type), we need only find the incremental moments of the final value process. Because we shall later make use of uniqueness theorems for the solution of Fokker-Planck equations, it is not sufficient to find any parabolic equation which p_f^+ satisfies; we must construct its Fokker-Planck equation.

The incremental mean is

$$a(x,t) = \lim_{\Delta \to 0} \frac{1}{\Delta} E[x_{t+\Delta} - x_t | x_t]$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int \xi\, p(\xi,\Delta|x)\, \frac{p(x_1, 1-t-\Delta|\xi)}{p(x_1, 1-t|x)}\, d\xi - x \right\}$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int \xi \, p(\xi, \Delta | x) \frac{p(x_1, 1-t | \xi)}{p(x_1, 1-t | x)} d\xi \quad - x \right\}$$

$$+ \lim_{\Delta \to 0} \int \xi \, p(\xi, \Delta | x) \frac{\frac{\partial}{\partial t} p(x_1, 1-t | \xi)}{p(x_1, 1-t | x)} d\xi \, .$$

Thus we have

$$a(x,t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int \xi \, p(\xi, \Delta | x) \frac{p(x_1, 1-t | \xi)}{p(x_1, 1-t | x)} d\xi - x \right\}$$

$$(3.7) \qquad + x \cdot \frac{\frac{\partial}{\partial t} p(x_1, 1-t | x)}{p(x_1, 1-t | x)} \, .$$

To evaluate this limit, we need a small-t expansion of $p(x,t \mid x_0)$. For this purpose we note that p satisfies the integral equation

$$(3.8) \qquad p(x,t | x_0) = \Gamma(x - x_0, t) + \int_0^t \int \Gamma(x-\xi, t-\sigma) \frac{\partial}{\partial \xi} \left[f(\xi) p(\xi, \sigma | x_0) \right] d\xi \, d\sigma,$$

where Γ is the density of the Weiner process,

$$\Gamma(x,t) = \exp\left\{ -x^2/4Dt \right\} / (2\sqrt{\pi D t}) \, .$$

The integral equation (3.8) is related to the parametrix method for solving parabolic equations (see [8,9]) and has also appeared in the theory of Weiner measure [14]. It is clear that the Neumann series for (3.8) will yield a series in t.

We have

$$p(x,t | x_0) = \Gamma(x - x_0, t) + \int_0^t \int \Gamma(x-\xi, t-\sigma) \frac{\partial}{\partial \xi} \left[f(\xi) \Gamma(\xi - x_0, \sigma) \right] d\xi \, d\sigma$$

$$+ O(t^2)$$

Integrating by parts,

$$p(x,t|x_0) = \Gamma(x-x_0,t) + \frac{\partial}{\partial x}\int_0^t \int \Gamma(x-\xi, t-\sigma)\, f(\xi)\, \Gamma(\xi - x_0, \sigma)\, d\xi\, d\sigma$$

$$+ \; O(t^2)$$

Upon using the relation

$$\Gamma(x-\eta, t)\,\Gamma(\eta - x_0, \sigma) = \Gamma(x-x_0, t+\sigma)\,\Gamma\left(\eta - \frac{\sigma x + t x_0}{t+\sigma}, \frac{\sigma t}{t+\sigma}\right),$$

we obtain

$$(3.9) \qquad p(x,t|x_0) = \Gamma(x-x_0,t) + t\frac{\partial}{\partial x}\left[\Gamma(x-x_0,t)\, Q(x,x_0,t)\right]$$

$$+ \; O(t^2)$$

where

$$Q(x,x_0,t) = \int_0^1 \int f(\xi)\, \Gamma\big(\xi - [\tau x + (1-\tau)x_0],\; t\tau[1-\tau]\big)\, d\xi\, d\tau$$

$$= f(x_0) + O(t).$$

Substituting the approximation (3.9) into the formula (3.7) for the incremental mean, we have

$$a(x,t) = \lim_{\Delta\to 0} \frac{1}{\Delta}\left\{ \int \xi\, \Gamma(\xi - x, \Delta)\, \frac{p(x_1, 1-t|\xi)}{p(x_1,1-t|x)}\, d\xi - x \right\}$$

$$+ \lim_{\Delta\to 0} \int \xi \frac{\partial}{\partial \xi}\left[f(x)\,\Gamma(\xi - x, \Delta)\right] \frac{p(x_1,1-t|\xi)}{p(x_1,1-t|x)}\, d\xi$$

$$+ \; x\, \frac{\frac{\partial}{\partial t} p(x_1,1-t|x)}{p(x_1,1-t|x)}$$

$$a(x,t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \int \frac{\xi\, p(x_1, 1-t|\xi) - x\, p(x_1,1-t|x)}{p(x_1, 1-t|x)} \Gamma(\xi - x, \Delta) d\xi$$

$$- f(x) \lim_{\Delta \to 0} \int \frac{\frac{\partial}{\partial \xi}\left[\xi\, p(x_1, 1-t|\xi)\right]}{p(x_1, 1-t|x)} \Gamma(\xi - x, \Delta) d\xi$$

$$+ \ x \cdot \frac{\partial}{\partial t} p(x_1, 1-t|x) \Big/ p(x_1, 1-t|x)$$

$$= \frac{\frac{\partial}{\partial x}\left[x\, p(x_1, 1-t|x)\right]}{p(x_1, 1-t|x)} \cdot \lim_{\Delta \to 0} \int \frac{\xi - x}{\Delta} \Gamma(\xi - x, \Delta) d\xi$$

$$+ \frac{\frac{1}{2} \frac{\partial^2}{\partial x^2}\left[x\, p(x_1, 1-t|x)\right]}{p(x_1, 1-t|x)} \cdot \lim_{\Delta \to 0} \int \frac{(\xi - x)^2}{\Delta} \Gamma(\xi - x, \Delta) d\xi$$

$$- f(x) \frac{\frac{\partial}{\partial x}\left[x\, p(x_1, 1-t|x)\right]}{p(x_1, 1-t|x)}$$

$$+ \frac{\frac{\partial}{\partial t}\left[x\, p(x_1, 1-t|x)\right]}{p(x_1, 1-t|x)}$$

$$= \frac{x}{p(x_1, 1-t|x)} \cdot \Big[D \frac{\partial^2}{\partial x^2} p(x_1, 1-t|x)$$

$$- f(x) \frac{\partial}{\partial x} p(x_1, 1-t|x) + \frac{\partial}{\partial t} p(x_1, 1-t|x) \Big]$$

$$+ \frac{2D \frac{\partial}{\partial x} p(x_1, 1-t|x)}{p(x_1, 1-t|x)}$$

$$- \ f(x)$$

$$a(x,t) = 2D \frac{\frac{\partial}{\partial x} p(x_1, 1-t \mid x)}{p(x_1, 1-t \mid x)} - f(x),$$

since the expression in brackets at the bottom of page 31 is just the Kolmogorov equation (3.5) for $p(x_1, 1-t \mid x)$.

For example, the linear final value problem

$$\mathring{x} + \beta x = N(t), \quad t < 1 \; ; \quad x(1) = x_1$$

has incremental mean

$$a(x,t) = \frac{2\beta\left(x_1 - x e^{-\beta(1-t)}\right) e^{-\beta(1-t)}}{1 - e^{-2\beta t}} - \beta x$$

$$= -\beta x \cdot \left[\frac{e^{2\beta(1-t)} + 1}{e^{2\beta(1-t)} - 1} \right] + \frac{2\beta}{e^{\beta(1-t)} - e^{-\beta(1-t)}} \cdot x_1$$

The evaluation of the second incremental moment proceeds in a similar manner:

$$b(x,t) = \lim_{\Delta \to 0} \frac{1}{\Delta} E\left[(x_{t+\Delta} - x_t)^2 \mid x_t \right]$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int \xi^2 p(\xi, \Delta \mid x) \frac{p(x_1, 1-t-\Delta \mid \xi)}{p(x_1, 1-t \mid x)} d\xi - x^2 \right\}$$

$$- 2x \lim_{\Delta \to 0} \frac{1}{\Delta} E\left[x_{t+\Delta} - x_t \mid x_t \right]$$

$$b(x,t) = \lim_{\Delta\to 0}\frac{1}{\Delta}\left\{\int \xi^2\, p(\xi,\Delta|x)\,\frac{p(x_1,1-t|\xi)}{p(x_1,1-t|x)}\,d\xi - x^2\right\}$$

$$+\; x^2\;\frac{\frac{\partial}{\partial t}p(x_1,1-t|x)}{p(x_1,1-t|x)}$$

$$-\,2x\left[2D\;\frac{\frac{\partial}{\partial x}p(x_1,1-t|x)}{p(x_1,1-t|x)} - f(x)\right]$$

$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\int \frac{\xi^2\,p(x_1,1-t|\xi) - x^2\,p(x_1,1-t|x)}{p(x_1,1-t|x)}\,\Gamma(\xi-x,\Delta)\,d\xi$$

$$+\, f(x)\lim_{\Delta\to 0}\int \xi^2\;\frac{\partial\Gamma(\xi-x,\Delta)}{\partial\xi}\cdot\frac{p(x_1,1-t|\xi)}{p(x_1,1-t|x)}\,d\xi$$

$$+\; x^2\,\frac{\partial p}{\partial t}\Big/p \;-\;4Dx\,\frac{\partial p}{\partial x}\Big/p \;+\; 2x\,f(x)$$

$$= D\cdot\frac{\partial^2}{\partial x^2}(x^2p)\Big/p \;-\; f(x)\frac{\partial}{\partial x}(x^2p)\Big/p$$

$$+\; x^2\,\frac{\partial p}{\partial t}\Big/p \;-\;4Dx\,\frac{\partial p}{\partial x}\Big/p \;+\; 2x\,f(x)$$

$$= x^2\left[D\frac{\partial^2 p}{\partial x^2} - f(x)\frac{\partial p}{\partial x} + \frac{\partial p}{\partial t}\right]\Big/p$$

$$+\;2D$$

$$=\;2D$$

since once again the expression in brackets is the Kolmogorov equation (3.5) for $p = p(x_1,1-t\,|\,x)$.

Therefore, the upward equation for the first-order final value problem (3.1) is

$$(3.10) \qquad \frac{\partial p_f^+}{\partial t} = D\,\frac{\partial^2 p_f^+}{\partial x^2} + \frac{\partial}{\partial x}\left\{\left[\cdot f(x) - 2D\,\frac{\frac{\partial}{\partial x}p(x_1,1-t\,|\,x)}{p(x_1,1-t\,|\,x)}\right]p_f^+\right\}.$$

It is easily seen that the representation (3.4) for p_f^+ satisfies (3.10), upon taking account of the equations (3.5) and (3.6). Thus the final value process is weakly diffusive.

Also, we note that the term

$$2D \;\; \frac{\partial}{\partial x}p(x_1,1-t\,|\,x) \,/\, p(x_1,1-t\,|\,x)$$

can be intepreted as the conditioned mean of the white noise process N(t), and that the conditioned variance of N(t) is the same as for the initial value problem, at least insofar as it affects the Fokker-Planck equation.

We note that the term we have identified as the conditioned mean of N(t) vanishes at any point which is a relative minimum or maximum in x of $p(x_1,1-t \mid x)$. This observation allows us to make a further interpretation of this term:

Consider a process $\{x_t\}$ in decreasing time $t<1$ starting from x_1 at $t = 1$. Suppose that <u>all</u> samples pass through some point x_t at time t. Then if we consider $\{x_t\}$ as a final value process in increasing time, the expectation, conditioned on x_t, of the zero mean process N(t) will be zero, because the expectation will be over all samples. Also, if <u>no</u> samples pass through x_t at time t, then the conditional expectation of the noise will again vanish, since in this case the conditioning set is empty. Therefore the term $2D \cdot \frac{\frac{\partial}{\partial x} p(x_1, 1-t|x)}{p(x_1, 1-t|x)}$ is seen to be the generalization of these situations to the case wherein more samples pass through some points than through others.

We now have a procedure for finding the upward equation from a representation such as (3.4) for the upward transistion probability density. Thus we are prepared to consider two-point boundary value problems.

IV. SECOND-ORDER BOUNDARY VALUE PROBLEMS

4.1 The Boundary Value Process

In chapters II and III we have obtained several results which we expect may be true for stochastic boundary value problems in general, and we have developed some methods for verifying these results. We now consider a second-order stochastic two-point boundary value problem,

$$\left.\begin{aligned} \mathring{x} &= y \\ \mathring{y} &= f(x, \underline{z}_t) \end{aligned}\right\} \quad 0 < t < 1 \tag{4.1}$$

with the boundary conditions

$$x(0) = x_0, \quad x(1) = x_1 \tag{4.2}$$

Here $\{\underline{z}_t\}$ is an n-dimensional diffusive Markov process; we take a fixed initial condition $\underline{z}(0) = \underline{z}_0$ for $\{\underline{z}_t\}$ so that $f(x,\underline{z}_t)$ will be, for each sample of $\{\underline{z}_t\}$, a given function of x and $t \in [0,1]$. As usual, we assume that (4.1) as an initial value problem defines an (n+2)-dimensional diffusive Markov process $\{(x_t, y_t, \underline{z}_t)\}$. Let $\underline{x}_t = (x_t, y_t, \underline{z}_t)$.

Our first requirement is that (4.1) with (4.2) shall have unique solutions for each sample of $\{\underline{z}_t\}$.* The simplest condition to insure this is [13] that $\frac{\partial}{\partial x} f(x, \underline{z}_t)$ be jointly continuous for all x and all $t \in [0,1]$, and be non-negative there. If $\frac{\partial^2 f}{\partial x^2}$ is jointly continuous for all x and $t \in [0,1]$,

* Actually we only need unique solutions for almost all samples.

then [17] it is sufficient to require $\frac{\partial f}{\partial x} > -\pi^2$.* If $f(x,\underline{z}) = g(\underline{z})\cdot x + h(\underline{z})$, then a sufficient condition for uniqueness is $g(\underline{z}_t)$ and $h(\underline{z}_t)$ continuous for $t \in [0,1]$ and either $g(\underline{z}_t) > -\pi^2$ or $-n^2\pi^2 > g(\underline{z}_t) > -(n+1)^2\pi^2$ for some integer n, for all $t \in [0,1]$.

When we do have unique solutions to (4.1) with (4.2), then the boundary value process is just the unique solution of the integral equations

$$x_t = x_0 + t\left[x_1 - x_0 + \int_0^1 (\tau-1)\, f(x_\tau, \underline{z}_\tau)\, d\tau\right] + \int_0^t (t-\tau)\, f(x_\tau, \underline{z}_\tau)\, d\tau$$

$$y_t = x_1 - x_0 + \int_0^1 (\tau-1)\, f(x_\tau, \underline{z}_\tau)\, d\tau + \int_0^t f(x_\tau, \underline{z}_\tau)\, d\tau$$

(assuming that these integrals exist and are well-behaved). In particular, for fixed x_0, x_1, and $\underline{z}_0$ y_0 will be the random variable

$$y_0 = x_1 - x_0 + \int_0^1 (\tau-1)\, f(x_\tau, \underline{z}_\tau)\, d\tau$$

4.2 The Boundary Value Densities

Let (4.1) with the boundary conditions (4.2) have unique solutions. Then the density $p(y_0 | x_0, x_1, \underline{z}_0)$ will exist, and for $0 < t < 1$ we will have the following relations between

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* In [17] it is assumed that $f(x,\underline{z}_t)$ is infinitely differentiable, but only two derivatives are need for the proof of existence and uniqueness given there.

the indicated densities

$$p(\underline{x}_t, y_0 | x_0, x_1, \underline{z}_0) = p(\underline{x}_t | \underline{x}_0; x_1)\, p(y_0 | x_0, x_1, \underline{z}_0)$$

$$= p(\underline{x}_t ; x_1 | \underline{x}_0) \cdot \frac{p(y_0 | x_0, x_1, \underline{z}_0)}{p(x_1 | \underline{x}_0)}$$

$$= p(x_1 | \underline{x}_t)\, p(\underline{x}_t | \underline{x}_0)\, \frac{p(y_0 | x_0, x_1, \underline{z}_0)}{p(x_1 | \underline{x}_0)}$$

where we have used the markovian nature of $\{\underline{x}_t\}$. The density p_b of the boundary value process is of course just the marginal density

$$p_b(\underline{x}_t) = \int p(\underline{x}_t ; y_0 | x_0, x_1, \underline{z}_0)\, dy_0$$

Therefore

$$(4.3) \qquad p_b(\underline{x}_t) = p(x_1 | \underline{x}_t) \cdot \int p(\underline{x}_t | x_0, y_0, \underline{z}_0)\, \frac{p(y_0 | x_0, x_1, \underline{z}_0)}{p(x_1 | x_0, y_0, \underline{z}_0)}\, dy_0$$

Now (4.3) will also hold if we replace $\underline{x}_t$ by the $2(n+2)$-vector $(\underline{x}_t, \underline{x}_{t_0})$ for $t > t_0$. This allows us to find the upward b.v.t.p.d. p_b^+, since

$$p_b^+(\underline{x}_t | \underline{x}_{t_0}) = \frac{p_b(\underline{x}_t, \underline{x}_{t_0})}{p_b(\underline{x}_{t_0})}$$

Now

$$p_b(\underline{x}_t, \underline{x}_{t_0}) = p_b^+(\underline{x}_t | \underline{x}_{t_0}) \cdot p_b(\underline{x}_{t_0})$$

$$= p(x_1 | \underline{x}_t) \cdot p(\underline{x}_t | \underline{x}_{t_0})$$

$$\cdot \int p(\underline{x}_{t_0} | x_0, y_0, \underline{z}_0) \frac{p(y_0 | x_0, x_1, \underline{z}_0)}{p(x_1 | x_0, y_0, \underline{z}_0)} dy_0 .$$

Using the representation (4.3) for $p_b(\underline{x}_{t_0})$, we obtain

$$p_b^+(\underline{x}_t | \underline{x}_{t_0}) = \frac{p(x_1 | \underline{x}_t) \cdot p(\underline{x}_t | \underline{x}_{t_0})}{p(x_1 | \underline{x}_{t_0})} \tag{4.4}$$

which has exactly the form of the corresponding expression (3.4) for the first-order final value problems. In particular, p_b^+ is independent of x_0 and $\underline{z}_0$, the boundary values at $t = 0$. In fact, (4.4) would have followed immediately (by the same argument that led in §3.1 to (3.4)) if we had known this a priori. Further, the Chapman-Kolmogorov equation is again immediate.

We shall also need the downward b.v.t.p.d p_b^- ; here there are two final conditions, $x(0) = x_0$ and $\underline{z}(0) = \underline{z}_0$, to be met. Letting $q(\cdot)$ be the densities for (4.1) in decreasing time,* we obtain

* Although reversal of the sense of time in a Markov process yields a Markov process, the process in reverse time does not necessarily have a transistion function. To obtain the existence of the densities q in our case, we may either apply our usual assumptions to the stochastic differential equation (4.1) in reversed time; or we may note that the existence of the densities p implies the existence of the transistion distributions for the process in reversed time [19].

(4.5) $$p_b(\underline{x}_t) = q(x_0, \underline{z}_0 | \underline{x}_t) \iint q(\underline{x}_t | x_1, y_1, \underline{z}_1) \cdot \frac{q(y_1, \underline{z}_1 | x_0, x_1, \underline{z}_0)}{q(x_0, \underline{z}_0 | x_1, y_1, \underline{z}_1)} dy_1 d\underline{z}_1$$

and for $t < t_0$,

(4.6) $$p_b^-(\underline{x}_t | \underline{x}_{t_0}) = \frac{q(x_0, \underline{z}_0 | \underline{x}_t)\, q(\underline{x}_t | \underline{x}_{t_0})}{q(x_0, \underline{z}_0 | \underline{x}_{t_0})} .$$

Clearly p_b^- is also markovian.

To obtain the backward equation satisfied by the (initial value) transistion probability density q, we derive it in the usual manner [21] from the Chapman-Kolmogorov equation

$$q(\underline{x}_0, t_0 | \underline{x}, t) = \int q(\underline{x}_0, t_0 | \underline{\xi}, \tau)\, q(\underline{\xi}, \tau | \underline{x}, t)\, d\underline{\xi}$$

which holds for any $t_0 < \tau < t$. Then

$$q(\underline{x}_0, t_0 | \underline{x}, t+\Delta) - q(\underline{x}_0, t_0 | \underline{x}, t)$$

$$= \int \left\{ q(\underline{x}_0, t_0 | \underline{\xi}, t) - q(\underline{x}_0, t_0 | \underline{x}, t) \right\} q(\underline{\xi}, t | \underline{x}, t+\Delta)\, d\underline{\xi}$$

and we obtain

(4.7) $$\frac{\partial}{\partial t} q(\underline{x}_0, t_0 | \underline{x}, t) = \frac{1}{2} \sum_{i,j} b_{ij} \frac{\partial^2 q}{\partial x_i \partial x_j} + \sum_i a_i \frac{\partial q}{\partial x_i}$$

where

$$a_i = \lim_{\Delta \to 0} \frac{1}{\Delta} E\left[x_i(t) - x_i(t+\Delta) \,|\, \underline{x}_{t+\Delta} \right]$$

$$b_{ij} = \lim_{\Delta \to 0} \frac{1}{\Delta} E\left[(x_i(t) - x_i(t+\Delta)) \cdot (x_j(t) - x_j(t+\Delta)) \,|\, \underline{x}_{t+\Delta} \right]$$

Thus b_{ij} and $-a_i$ are just the incremental moments of $p(\underline{x},t \mid \underline{x}_0)$, whose Fokker-Planck equation is

$$(4.8) \qquad \frac{\partial p}{\partial t} = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (b_{ij}\, p) + \sum_i \frac{\partial}{\partial x_i} (a_i\, p) .$$

4.3 The Upward and Downward Equations

For simplicity, we assume that $\{\underline{z}_t\}$ is a one-dimensional diffusive Markov process $\{z_t\}$ with infinitesimal generator (backward Kolmogorov operator)

$$b(z) \frac{\partial^2}{\partial z^2} + a_0(z) \frac{\partial}{\partial z}$$

We shall use the vector notations $\underline{x} = (x,y,z)$, $\underline{x}_0 = (x_0,y_0,z_0)$ and $\underline{\xi} = (\xi, \eta, \zeta)$.

The incremental mean of z is

$$\begin{aligned} a(\underline{x},t) &= \lim_{\Delta \to 0} \frac{1}{\Delta} E\left[z_{t+\Delta} - z_t \mid \underline{x}_t \right] \\ &= \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int \zeta\, p(\underline{\xi}, \Delta \mid \underline{x}) \frac{p(x_1, 1-t \mid \underline{\xi})}{p(x_1, 1-t \mid \underline{x})} d\underline{\xi} - z \right\} \\ &\quad + z \frac{\frac{\partial}{\partial t} p(x_1, 1-t \mid \underline{x})}{p(x_1, 1-t \mid \underline{x})} . \end{aligned}$$

In order to proceed as we did for first-order final value problems, we need a small-t expansion of $p(\underline{x},t \mid \underline{x}_0)$, which satisfies the Fokker-Planck equation (4.8); this equation now becomes

$$(4.9) \qquad \frac{\partial p}{\partial t} = \frac{\partial^2}{\partial z^2}\left(b(z)p\right) - \frac{\partial}{\partial z}\left(a_0(z)p\right) - y\frac{\partial p}{\partial x} - f(x,z)\frac{\partial p}{\partial y}$$

Unfortunately, the only third-order systems whose Fokker-Planck equations have been solved are linear systems. Accordingly, we assume that $\{z_t\}$ is an Ornstein-Uhlenbeck process with

$$b(z) = D, \quad a_0(z) = -\beta z$$

Let $\Gamma(\underline{x},t \mid \underline{x}_0)$ be the (gaussian) density of the initial value problem

$$(4.10) \qquad \left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -\beta z + N(t) \end{aligned}\right\} \quad \underline{x}(0) = \underline{x}_0$$

Then from (4.9) we see that $p(\underline{x},t \mid \underline{x}_0)$ satisfies the integral equation

$$p(\underline{x},t \mid \underline{x}_0) = \Gamma(\underline{x},t \mid \underline{x}_0) + \int_0^t \int \Gamma(\underline{x},t-\sigma \mid \underline{\xi})\, F(\underline{\xi})\, \frac{\partial p(\underline{\xi},\sigma \mid \underline{x}_0)}{\partial \eta}\, d\underline{\xi}\, d\sigma$$

$$= \Gamma(\underline{x},t \mid \underline{x}_0) - \int_0^t \int F(\underline{\xi})\, \Gamma(\underline{\xi},\sigma \mid \underline{x}_0)\, \frac{\partial}{\partial \eta}\Gamma(\underline{x},t-\sigma \mid \underline{\xi})\, d\underline{\xi}\, d\sigma$$

$$+ \; O(t^2),$$

where $F(\underline{x}) = z - f(x,z)$. Now the gaussian density $\Gamma(\underline{x},t \mid \underline{x}_0)$ has mean

$$E[\underline{x}\,|\,\underline{x}_0] = \begin{pmatrix} 0 & t & (e^{-\beta t} - 1 + \beta t)/\beta^2 \\ 0 & 1 & (1 - e^{-\beta t})/\beta \\ 0 & 0 & e^{-\beta t} \end{pmatrix} \underline{x}_0$$

and therefore

$$\frac{\partial}{\partial y_0} \Gamma(\underline{x}, t\,|\,\underline{x}_0) = \frac{\partial \Gamma}{\partial y} + t\,\frac{\partial \Gamma}{\partial x}$$

Also, for $0 < \sigma < t$, the following holds for the densities $g(\cdot)$ of any homogeneous markov process $\{\underline{x}_t\}$:

$$g(\underline{x}, t-\sigma\,|\,\underline{\xi}) \cdot g(\underline{\xi}, \sigma\,|\,\underline{x}_0)$$

$$= g(\underline{x}, t;\ \underline{\xi}, \sigma\,|\,\underline{x}_0, 0) \cdot$$

$$= g(\underline{\xi}, \sigma\,|\,\underline{x}, t;\ \underline{x}_0, 0) \cdot g(\underline{x}, t\,|\,\underline{x}_0),$$

where $g(\underline{\xi}, \sigma\,|\,\underline{x}, t;\ \underline{x}_0, 0)$ is just the downward transistion probability density for the process $\{\underline{x}_t\}$ considered as the final value problem $\underline{x}(0) = \underline{x}_0$. Hence, if we let $G(\underline{\xi}, \sigma\,|\,\underline{x}, t; \underline{x}_0, 0)$ be the downward t.p.d. for the system (4.10) with the final value $\underline{x}(0) = \underline{x}_0$, then

$$p(\underline{x}, t\,|\,\underline{x}_0) = \int_0^t \left[\frac{\partial}{\partial y} + (t-\sigma)\frac{\partial}{\partial x}\right] \cdot \left[\Gamma(\underline{x}, t\,|\,\underline{x}_0) \cdot \int F(\underline{\xi}) \cdot G \; d\underline{\xi}\right] d\sigma$$

$$+ \Gamma(\underline{x}, t\,|\,\underline{x}_0) \quad + O(t^2)$$

$$= \Gamma(\underline{x}, t\,|\,\underline{x}_0) \; + \; t\, F(\underline{x}_0) \frac{\partial}{\partial y} \Gamma(\underline{x}, t\,|\,\underline{x}_0)$$

$$+ O(t^2).$$

Therefore the incremental mean of z is

$$a = \lim_{\Delta \to 0} \frac{1}{\Delta} \int \frac{\zeta\, p(x_1, 1-t \mid \underline{\xi}) - z\, p(x_1, 1-t \mid \underline{x})}{p(x_1, 1-t \mid \underline{x})} \Gamma(\underline{\xi}, \Delta \mid \underline{x})\, d\underline{\xi}$$

$$- F(\underline{x}) \lim_{\Delta \to 0} \int \frac{\zeta \frac{\partial}{\partial \eta} p(x_1, 1-t \mid \underline{\xi})}{p(x_1, 1-t \mid \underline{x})} \Gamma(\underline{\xi}, \Delta \mid \underline{x})\, d\underline{\xi}$$

$$+ z \frac{\frac{\partial}{\partial t} p(x_1, 1-t \mid \underline{x})}{p(x_1, 1-t \mid \underline{x})}$$

$$= y \frac{\partial}{\partial x}(zp)/p + z \frac{\partial}{\partial y}(zp)/p$$

$$- \beta z \frac{\partial}{\partial z}(zp)/p + D \frac{\partial^2}{\partial z^2}(zp)/p$$

$$- F(\underline{x})\, z \frac{\partial p}{\partial y}/p + z \frac{\partial p}{\partial t}/p$$

$$= z\Big[D \frac{\partial^2 p}{\partial z^2} + y \frac{\partial p}{\partial x} + f(x, z) \frac{\partial p}{\partial y}$$

$$- \beta z \frac{\partial p}{\partial z} + \frac{\partial p}{\partial t} \Big]/p$$

$$+ 2D \frac{\partial p}{\partial z}/p - \beta z$$

$$= 2D \frac{\frac{\partial}{\partial z} p(x_1, 1-t \mid \underline{x})}{p(x_1, 1-t \mid \underline{x})} - \beta z ,$$

since the expression in brackets is just the backward equation for $p = p(x_1, 1-t \mid \underline{x})$.

Similarly, we find the other incremental means:

$$\lim_{\Delta\to 0} E[x_{t+\Delta}-x_t \mid \underline{x}_t] .$$

$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\int \frac{\xi\, p(x_1, 1-t\mid \underline{\xi}) - x\, p(x_1, 1-t\mid \underline{x})}{p(x_1, 1-t\mid \underline{x})}\,\Gamma(\underline{\xi}, \Delta\mid \underline{x})\,d\underline{\xi}$$

$$+ F(\underline{x})\lim_{\Delta\to 0}\int \xi\,\frac{\partial \Gamma(\underline{\xi}, \Delta\mid \underline{x})}{\partial \eta}\cdot\frac{p(x_1, 1-t\mid \underline{\xi})}{p(x_1, 1-t\mid \underline{x})}\,d\underline{\xi}$$

$$+ \quad x\,\frac{\partial p}{\partial t}\Big/ p$$

$$= \Big\{ y\,\frac{\partial}{\partial x}(xp) + z\,\frac{\partial}{\partial y}(xp) - \beta z\,\frac{\partial}{\partial z}(xp)$$

$$+ D\,\frac{\partial^2}{\partial z^2}(xp) - F(\underline{x})\,x\,\frac{\partial p}{\partial y} + x\,\frac{\partial p}{\partial t}\Big\}\Big/ p$$

$$= \quad y \quad ,$$

and

$$\lim_{\Delta\to 0}\frac{1}{\Delta} E[y_{t+\Delta}-y_t \mid \underline{x}_t]$$

$$= \Big\{ y\,\frac{\partial}{\partial x}(yp) + z\,\frac{\partial}{\partial y}(yp) - \beta z\,\frac{\partial}{\partial z}(yp)$$

$$+ D\,\frac{\partial^2}{\partial z^2}(yp) - F(\underline{x})\,\frac{\partial}{\partial y}(yp) + y\,\frac{\partial p}{\partial t}\Big\}\Big/ p$$

$$= \quad f(x, z) \; .$$

The second incremental moments are also found exactly as they were in the case of first-order final value problems. We then obtain the following upward equation:

$$(4.11)\qquad \begin{aligned}\frac{\partial p_b^+}{\partial t} &= D\frac{\partial^2 p_b^+}{\partial z^2} + \beta\frac{\partial}{\partial z}(z p_b^+) - y\frac{\partial p_b^+}{\partial x}\\ &\quad - f(x,z)\frac{\partial p_b^+}{\partial y} - 2D\frac{\partial}{\partial z}\left[\frac{\frac{\partial}{\partial z}p(x_1,1-t\,|\,\underline{x})}{p(x_1,1-t\,|\,\underline{x})}\,p_b^+\right].\end{aligned}$$

It is easily seen from the representation (4.4) for p_b^+ that the boundary value process is indeed weakly diffusive in the upward direction.

From the representation (4.6) for p_b^- we obtain in the same way the downward equation,

$$(4.12)\qquad \begin{aligned}\frac{\partial p_b^-}{\partial t} &= -D\frac{\partial^2 p_b^-}{\partial z^2} + \beta\frac{\partial}{\partial z}(z p_b^-) - y\frac{\partial p_b^-}{\partial x}\\ &\quad - f(x,z)\frac{\partial p_b^-}{\partial y} + 2D\frac{\partial}{\partial z}\left[\frac{\frac{\partial}{\partial z}q(x_0,z_0\,|\,\underline{x},t)}{q(x_0,z_0\,|\,\underline{x},t)}\,p_b^-\right]\end{aligned}$$

For our special case there is a simple relation between the density $q(x_0,z_0\,|\,\underline{x},t)$ which appears in (4.12) and the upward initial value density $p(\underline{x},t\,|\,\underline{x}_0)$. From the backward equation (4.7) for $q(\underline{x}_0\,|\,\underline{x},t)$, which for the case under consideration is

$$\frac{\partial q(\underline{x}_0|\underline{x},t)}{\partial t} = D\frac{\partial^2 q}{\partial z^2} + \beta z\frac{\partial q}{\partial z} - y\frac{\partial q}{\partial x} - f(x,z)\frac{\partial q}{\partial y}$$

we see that the function $e^{\beta t}\cdot q(\underline{x}_0\,|\,\underline{x},t)$ satisfies the Fokker-Planck equation (4.8) for $p(\underline{x},t\,|\,\underline{x}_0)$. Hence, from the initial condition

$$q(x_0, z_0 | \underline{x}, t)\Big|_{t=0} = \delta(x - x_0)\,\delta(z - z_0),$$

and the uniqueness of solutions to the Fokker-Planck equation [11], we have

$$(4.13) \quad q(x_0, z_0 | \underline{x}, t) = e^{-\beta t} \int p(\underline{x}, t | x_0, y_0, z_0)\, dy_0,$$

and the integral must exist.

We have conjectured that the boundary value density p_b is the unique simultaneous solution of the upward and downward equations together with the appropriate boundary conditions. Hence we must now investigate the simultaneous solutions of (4.11) and (4.12).

4.4 Solutions of the Boundary Value Fokker-Planck Equations

Now both the upward equation (4.11) and the downward equation (4.12) have, as initial value Fokker-Planck equations, no more than one solution [11]. The solution $\tilde{p}_b$ of the upward equation with the initial condition

$$\tilde{p}_b(\underline{x}, t)\Big|_{t=0} = \delta(x - x_0)\,\delta(z - z_0)\,\mu(y)$$

(which is the correct boundary condition for the boundary value density p_b at $t = 0$) is

$$\tilde{p}_b(\underline{x}, t) = \int \delta(\xi - x_0)\,\delta(\zeta - z_0)\,\mu(\eta)\, p_b^{+}(\underline{x}, t | \underline{\xi}, 0)\, d\underline{\xi}$$

or

$$\tilde{p}_b(\underline{x},t) = p(x_1, 1-t \mid \underline{x}) \int \frac{p(\underline{x},t \mid x_0, \eta, z_0)}{p(x_1, 1 \mid x_0, \eta, z_0)} \mu(\eta)\, d\eta$$

for any probability density $\mu(y)$ for which the integral exists and has the derivatives appearing in the upward equation. Of course, $\tilde{p}_b$ also satisfies the appropriate boundary condition at $t = 1$:

$$\tilde{p}_b(\underline{x}, 1) = \delta(x - x_1) \cdot \int \frac{p(\underline{x}, 1 \mid x_0, \eta, z_0)}{p(x_1, 1 \mid x_0, \eta, z_0)} \mu(\eta)\, d\eta .$$

Then we have

$$\frac{\partial \tilde{p}_b}{\partial t} = \frac{\partial p(x_1, 1-t \mid \underline{x})}{\partial t} \cdot \int \frac{p(\underline{x},t \mid x_0, \eta, z_0)}{p(x_1, 1 \mid x_0, \eta, z_0)} \mu(\eta)\, d\eta$$

$$+ \; p(x_1, 1-t \mid \underline{x}) \cdot \int \frac{\partial p(\underline{x},t \mid x_0, \eta, z_0)}{\partial t} \cdot \frac{\mu(\eta)}{p(x_1, 1 \mid x_0, \eta, z_0)}\, d\eta$$

$$= -D \frac{\partial^2 \tilde{p}_b}{\partial z^2} - y \frac{\partial \tilde{p}_b}{\partial x} - f(x,z) \frac{\partial \tilde{p}_b}{\partial y} + \beta \frac{\partial}{\partial z}(z \tilde{p}_b)$$

$$+ \, 2D \frac{\partial}{\partial z} \left[p(x_1, 1-t \mid \underline{x}) \frac{\partial}{\partial z} \int \frac{p(\underline{x},t \mid x_0, \eta, z_0)}{p(x_1, 1 \mid x_0, \eta, z_0)} \mu(\eta)\, d\eta \right],$$

where we have used the backward and forward equations for $p(\underline{x},t \mid \underline{x}_0)$.

Hence $\tilde{p}_b$ will satisfy the downward equation (4.12) if and only if there are functions $\alpha(x,y,t)$ and $\gamma(x,y,t)$ such that

$$(4.14) \qquad \int \frac{p(\underline{x},t\,|\,x_0,\eta,z_0)}{p(x_1,1\,|\,x_0,\eta,z_0)}\,\mu(\eta)\,d\eta \;=\; \alpha\cdot q(x_0,z_0\,|\,\underline{x},t)\,\exp\left\{\gamma\cdot\int_0^z \frac{d\zeta}{\tilde{p}_0(x,y,\zeta,t)}\right\}$$

Now as $z \longrightarrow \pm\infty$, both the left-hand side of (4.14) and $q(x_0,z_0\,|\,\underline{x},t)$ will vanish, and by (4.13) they will go to zero at the same rate. Hence $\gamma = 0$. But from the Fokker-Planck equation for $p(\underline{x},t\,|\,\underline{x}_0)$ and the backward equation for $q(x_0,z_0\,|\,\underline{x},t)$, we see that α must satisfy

$$\frac{\partial\alpha}{\partial t} + y\,\frac{\partial\alpha}{\partial x} + f(x,z)\,\frac{\partial\alpha}{\partial y} = \beta\cdot\alpha(x,y,t).$$

Since α is independent of z, $\frac{\partial\alpha}{\partial y} = 0$, and then $\frac{\partial\alpha}{\partial x} = 0$. Thus $\alpha = \alpha_0 e^{\beta t}$, where α_0 is a constant depending only on x_0, x_1, and z_0.

At $t = 0$, (4.14) becomes

$$\delta(x-x_0)\,\delta(y-y_0)\,\frac{\mu(y)}{p(x_1,1\,|\,x_0,y,z_0)} = \alpha_0\;\delta(x-x_0)\,\delta(y-y_0)$$

Hence $\mu(y) = \alpha_0\, p(x_1,1\,|\,x_0,y,z_0)$; and (4.14) is just (4.13):

$$q(x_0,z_0\,|\,\underline{x},t) = e^{-\beta t}\cdot\int p(\underline{x},t\,|\,x_0,\eta,z_0)\,d\eta.$$

it is necessary and sufficient to require that the initial condition $\mu(y)$ be normalized. That is, we take

$$(4.16) \qquad \alpha_0^{-1} = \int p(x_1, 1 \mid x_0, \eta, z_0)\, d\eta$$

if this integral exists.

Hence, a necessary and sufficient condition for the existence of a unique probability density satisfying both the upward and downward equations as well as the appropriate boundary conditions is the existence of the integral (4.16).* We must now relate this condition to the boundary value problem (4.1).

Suppose that (4.1) has unique solutions. Then we have two alternative representations for the boundary value density p_b :

$$(4.3) \quad p_b(\underline{x}, t) = p(x_1, 1-t \mid \underline{x}) \cdot \int p(\underline{x}, t \mid \underline{x}_0) \frac{p(y_0 \mid x_0, x_1, z_0)}{p(x_1 \mid x_0, y_0, z_0)}\, dy_0$$

$$(4.5) \qquad = q(x_0, z_0 \mid \underline{x}, t) \cdot \iint q(\underline{x}, t \mid \underline{x}_1) \frac{q(y_1, z_1 \mid x_0, x_1, z_0)}{q(x_0, z_0 \mid x_1, y_1, z_1)}\, dy_1\, dz_1$$

both of which of course satisfy the boundary conditions. Now the representation (4.3) satisfies the upward equation, and (4.5) satisfies the downward equation. Since p_b is a probability density, $p_b = \widetilde{p}_b$ is the unique density satisfying both the upward and downward equations.

Furthermore, we see from (4.15b) that

$$(4.17) \qquad \int p(\underline{x}, t \mid x_0, y_0, z_0) \left[\alpha_0 - \frac{p(y_0 \mid x_0, x_1, z_0)}{p(x_1 \mid x_0, y_0, z_0)} \right] dy_0 = 0.$$

---

* From the representation (4.15a) we see that $\widetilde{p}_b$ is sufficiently differentiable to satisfy the equations.

In general, the expression in brackets in (4.17) does not vanish identically, as α_0 is independent of y_0. However, we may interpret (4.17) as an analog of the formula

$$\frac{p(y_0 | x_0, x_1, z_0)}{p(x_1 | x_0, y_0, z_0)} = \frac{p(y_0 | x_0, z_0)}{p(x_1 | x_0, z_0)} = \frac{dx_1}{dy_0}$$

which would hold if there were a relation $x_1 = x_1(y_0)$, with $x_1(y_0)$ a strictly increasing function of y_0, and all these densities existed. This analogy is further strengthened by the observation that for a deterministic boundary value problem of the type (4.1), the existence of a one-to-one function $x_1(y_0)$ is a sufficient condition for the uniqueness of solutions.

Of course we may also obtain the downward versions of (4.16) and (4.17); these are respectively,

$$\alpha_0^{-1} = \iint q(x_0, z_0 | x_1, \eta, \zeta, 1)\, d\eta\, d\zeta \tag{4.18}$$

and

$$\iint q(\underline{x}, t | x_1, \eta, \zeta) \left[\alpha_0 e^{2\beta t} - \frac{q(\eta, \zeta | x_0, x_1, z_0)}{q(x_0, z_0 | x_1, \eta, \zeta)} \right] d\eta\, d\zeta = 0.$$

Most important, however, is the simple expression we now have for the boundary value density,

$$p_b(\underline{x}, t) = \alpha_0 e^{\beta t}\, p(x_1, 1-t | \underline{x})\, q(x_0, z_0 | \underline{x}, t) \tag{4.19}$$

To summarize, we have proven the

<u>Theorem</u>: If the boundary value problem (4.1) has unique solutions, then there is a unique probability density satisfying both the upward and downward equations with the appropriate boundary

conditions; this density is the boundary value density and is given by (4.19).

The converse of this result — that the existence of the integral (4.16) implies that the boundary value problem (4.1) has unique solutions — has not been proven. We might argue that if $p(x_1,1 \mid x_0,y_0,z_0)$ is a density is y_0, then for fixed x_0, x_1, and z_0 there must be a unique y_0 for almost all samples; however, all attempts to express this reasoning in a precise way have failed.

4.5 A Gaussian Example

The gaussian boundary value problem

$$\left.\begin{aligned} \mathring{x} &= y \\ \mathring{y} &= -\lambda^2 x + z \\ \mathring{z} &= N(t) \end{aligned}\right\} \quad x(0)=x_0,\ x(1)=x_1,\ z(0)=z_0, \tag{4.20}$$

will have unique solutions as long as $\lambda \neq n\pi$, $n = 1,2,\ldots$. Taking $\lambda \neq 0$, we have

$$p(x_1,t \mid \underline{x}_0) = \frac{\exp\left\{-\frac{1}{2\sigma^2}\left[x_1 - x_0\cos\lambda t - y_0\frac{\sin\lambda t}{\lambda} - z_0\frac{1-\cos\lambda t}{\lambda^2}\right]^2\right\}}{\sqrt{2\pi}\cdot\sigma}$$

where

$$\sigma^2 = \frac{D}{\lambda^5}\left[3\lambda t - 4\sin\lambda t + \frac{\sin 2\lambda t}{2}\right].$$

We see that

$$\alpha_0^{-1} = \int p(x_1,1 \mid x_0,y_0,z_0)\,dy_0$$

will exist as long as $\sin\lambda \neq 0$; i.e., as long as (4.20) has unique solutions. Of course, the integral (4.13),

$$\int p(\underline{x}, t \mid \underline{x}_0)\, dy_0$$

must exist for all t and all λ. In the present case, the coefficient of y_0^2 in the exponential of $p(\underline{x}, t \mid \underline{x}_0)$ is

$$-\frac{2D^2/\lambda^6}{\det(K_t)} \cdot \left[(\lambda t)^2 + (\lambda t)\frac{\sin 2\lambda t}{2} - 2\sin^2 \lambda t \right]$$

which vanishes only at $(\lambda t) = 0$; here K_t is the covariance of $\underline{x}_t$ with itself for the initial value process.

V. EXAMPLES

5.1 The Vibrating String with Stochastic Density

We now consider the transverse vibrations of a taut linearized elastic string with constant unit tension and fixed ends on the unit interval $0 \le x \le 1$. This problem has been treated by both "honest" [3] and "dishonest" [4] methods. The displacement $w(x,t)$ of the string satisfies

$$\frac{\partial^2 w}{\partial x^2} = m(x)\frac{\partial^2 w}{\partial t^2}, \quad w(0,t) = w(1,t) = 0.$$

Here t is time, x is the spatial variable, and $m(x)$, the mass per unit length, will be a stochastic process with parameter set $0 \le x \le 1$.

Let $\eta(x,s)$ be the Laplace transform of w,

$$\eta(x,s) = \int_0^\infty w(x,t)\, e^{-st}\, dt .$$

Then we have the ordinary differential equation

$$\text{(5.1a)} \qquad \frac{\partial^2 \eta}{\partial x^2} - s^2 m(x)\eta = m(x)\left\{ s\, w(x,0) + \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} \right\}$$

with the boundary conditions

$$\text{(5.1b)} \qquad \eta(0,s) = \eta(1,s) = 0.$$

We consider the "plucked" string,

$$w(x,0) = \delta(x-\xi), \quad \left.\frac{\partial w}{\partial t}\right|_{t=0} = 0.$$

Our conventional notation for stochastic boundary value problems is obtained by the following transformations:

$$x \longrightarrow t$$
$$\xi \longrightarrow \tau$$
$$m(x) \longrightarrow f(z_t)$$
$$\eta \longrightarrow x$$

Then (5.1) becomes

$$
\begin{aligned}
\mathring{x} &= y \\
\mathring{y} &= s^2 f(z_t)x + s f(z_t)\,\delta(t-\tau) \qquad (5.2)\\
x(0) &= x(1) = 0, \quad z(0) = z_0.
\end{aligned}
$$

Since $f(\underline{z}_t)$ represents a physical density, we must have $f(\underline{z}_t) \geq 0$. Then (5.2) will have unique solutions when the real part of s is positive.

Let $\{\underline{z}_t\}$ be the Wiener process, and take $z_0 = 0$. The boundary value density, p_b, of the process* described by (5.2) is, by (4.19),

$$p_b(\underline{x}, t) = \alpha_0\, p(x_1, 1-t \mid \underline{x})\, q(x_0, z_0 \mid \underline{x}, t)$$

where $\underline{x} = (x,y,z)$, $x_1 = x_0 = z_0 = 0$, and where

$$(5.3a) \quad -\frac{\partial}{\partial t} p(x_1, 1-t \mid \underline{x}) = D\frac{\partial^2 p}{\partial z^2} + y\frac{\partial p}{\partial x} + \left[s^2 f(z)x + s f(z)\,\delta(t-\tau)\right]\frac{\partial p}{\partial y}$$

* Since the solution samples have a jump across $t = \tau$, the process is not diffusive (although it is piecewise diffusive). To be rigorously correct, we should replace the δ-function with a sequence of continuous function tending to the δ-function. Instead we proceed in a formal way, using the δ-function to simplify the calculations.

$$(5.3b) \quad \frac{\partial}{\partial t} q(x_0, z_0 | \underline{x}, t) = D \frac{\partial^2 q}{\partial z^2} - y \frac{\partial q}{\partial x} - \left[s^2 f(z) x + s f(z) \delta(t-\tau) \right] \frac{\partial p}{\partial y}$$

with the initial conditions

$$p|_{t=1} = \delta(x - x_1), \quad q|_{t=0} = \delta(x - x_0)\, \delta(z - z_0)$$

Let $\varphi(\underline{x}, t | \underline{x}_0)$ be the solution of

$$(5.4) \quad \frac{\partial \varphi}{\partial t} = D \frac{\partial^2 \varphi}{\partial z^2} - y \frac{\partial \varphi}{\partial x} - s^2 f(z) x \frac{\partial \varphi}{\partial y}, \quad \varphi|_{t=0} = \delta(\underline{x} - \underline{x}_0).$$

Then for $t < \tau$, $q(x_0, z_0 | \underline{x}, t)$ is given by

$$q := q_1(x_0, z_0; \underline{x}, t) = \int \varphi(\underline{x}, t | x_0, \eta, z_0)\, d\eta .$$

Integrating (5.3b) in t across $t = \tau$, we obtain

$$q|_{t=\tau+} = q|_{t=\tau-} - s f(z) \frac{\partial q}{\partial y}\Big|_{t=\tau} .$$

Hence

$$q(x_0, z_0 | \underline{x}, t) = q_1(x_0, z_0; \underline{x}, t) + H(t-\tau)\, q_2(x_0, z_0; t, \tau, \underline{x})$$

where H is the Heaviside unit step function and where

$$q_2(x_0, z_0; t, \tau, \underline{x}) = -s \int \varphi(\underline{x}, t-\tau | \underline{\xi})\, f(\zeta) \frac{\partial}{\partial \eta} \int \varphi(\underline{\xi}, \tau | x_0, \eta', z_0)\, d\eta'\, d\underline{\xi}$$

Since the function

$$\psi(\underline{x}, t | \underline{x}_0) = \varphi(\underline{x}_0, 1-t | \underline{x})$$

satisfies

$$-\frac{\partial\psi}{\partial t} = D\frac{\partial^2\psi}{\partial z^2} + y\frac{\partial\psi}{\partial x} + s^2 f(z)\frac{\partial\psi}{\partial y}$$

we similarly obtain

$$p(x_1, 1-t\,|\,\underline{x}) = P_1(x_1;\, t, \underline{x}) + H(\tau - t)\, P_2(x_1;\, t, \tau, \underline{x})$$

where

$$P_1(x_1;\, t, \underline{x}) = \iint \varphi(x_1, \eta, \zeta, 1-t\,|\,\underline{x})\, d\eta\, d\zeta$$

and

$$P_2(x_1;\, t, \tau, \underline{x}) = -s\int \varphi(\underline{\xi}, \tau - t\,|\,\underline{x})\, f(\zeta)\frac{\partial}{\partial\eta}\iint \varphi(x_1, \eta', \zeta', 1-\tau\,|\,\underline{\xi})\, d\eta'\, d\zeta'\, d\underline{\xi}\,.$$

It is clear that both p and q are normalized probability densities. Further, we need only solve the equation (5.4) to obtain p_b.

The normalizing factor α_0 for p_b is, by (4.16),

$$\alpha_0^{-1} = \int p(x_1, 1\,|\,x_0, y_0, z_0)\, dy_0$$

$$= \iiint \varphi(x_1, \eta, \zeta, 1\,|\,x_0, y_0, z_0)\, d\eta\, d\zeta\, dy_0$$

$$- s\iint \varphi(\underline{\xi}, \tau\,|\,x_0, y_0, z_0)\cdot$$

$$\cdot f(\zeta)\frac{\partial}{\partial\eta}\iint \varphi(x_1, \eta', \zeta', 1\,|\,\underline{\xi})\, d\eta'\, d\zeta'\, d\underline{\xi}\, dy_0$$

According to (4.18), α_0 is also given by

$$\alpha_0^{-1} = \iint q(x_0, z_0 | x_1, \eta, \zeta, 1)\, d\eta\, d\zeta$$

$$= \iiint \varphi(x_1, \eta, \zeta, 1 | x_0, y_0, z_0)\, d\eta\, d\zeta\, dy_0$$
$$- s \iiint \varphi(x_1, y, z, 1-\tau | \underline{\xi}) f(\zeta) \frac{\partial}{\partial \eta} \int \varphi(\underline{\xi}, \tau | x_0, y_0, z_0)\, dy_0\, d\underline{\xi}\, dy\, dz$$

$$= \iiint \varphi(x_1, \eta, \zeta, 1 | x_0, y_0, z_0)\, d\eta\, d\zeta\, dy_0$$

$$+ s \iint \varphi(\underline{\xi}, \tau | x_0, y_0, z_0) f(\zeta) \frac{\partial}{\partial \eta} \iint \varphi(x_1, y, z, 1-\tau | \underline{\xi})\, dy\, dz\, d\underline{\xi}\, dy_0$$

where the last line is obtained by integrating by parts on η. Therefore the second term in α_0^{-1} vanishes, and we have

$$\alpha_0^{-1} = \iiint \varphi(x_1, \eta, \zeta, 1 | x_0, y_0, z_0)\, d\eta\, d\zeta\, dy_0 \tag{5.5}$$

The boundary value density p_b is now given by

$$p_b(\underline{x}, t) = \alpha_0\, p_1(x_1, t; \underline{x})\, q_1(x_0, z_0; \underline{x}, t)$$

$$+ \alpha_0\, H(t-\tau)\, p_1(x_1; t, \underline{x})\, q_2(x_0, z_0; t, \tau, \underline{x})$$

$$+ \alpha_0\, H(\tau - t)\, p_2(x_1; t, \tau, \underline{x})\, q_1(x_0, z_0; \underline{x}, t).$$

However, the term $\alpha_0 p_1 q_1$ is, aside from normalization, the boundary value density for the problem

$$(5.6)\qquad \left.\begin{aligned} \mathring{x} &= y \\ \mathring{y} &= s^2 f(z)\, x \\ \mathring{z} &= N(t) \end{aligned}\right\} \qquad x(0) = x(1) = z(0) = 0$$

Since the only solution to (5.6) is the trivial one, $x = 0$, $y = 0$, $\mathring{z} = N(t)$, we see that $p_1 q_1$ is proportional to

$$\delta(x)\, \delta(y)\, \Gamma(z,t),$$

where Γ is the density of the Wiener process.

In particular, then, the expected solution, $\langle x \rangle$, of the boundary value problem (5.2) is given by

$$\langle x \rangle = \begin{cases} \alpha_0 \displaystyle\int x\, p(x_1; t, \underline{x})\, q_2(x_0, z_0; t, \tau, \underline{x})\, d\underline{x}, & t > \tau \\[2em] \alpha_0 \displaystyle\int x\, p_2(x_1; t, \tau, \underline{x})\, q_1(x_0, z_0; \underline{x}, t)\, d\underline{x}, & t < \tau \end{cases}$$

Let us write the boundary value density for $t < \tau$ in the form

$$\alpha_0 p_2 q_1 = (\alpha_0) \cdot \left(\int \varphi(\underline{x}, t \mid x_0, \eta, z_0)\, d\eta \right)$$

$$\cdot \left\{ \int \varphi(\underline{\xi}, \tau - t \mid \underline{x}) \left[- s f(\zeta) \frac{\partial}{\partial \eta} \iint \varphi(x_1, \eta', \zeta', 1-\tau \mid \underline{\xi})\, d\eta'\, d\zeta' \right] d\underline{\xi} \right\}$$

We may interpret the various term as follows: The quantity $\varphi(\underline{\xi}, \tau - t \mid \underline{x})$ is of course the density for upward transistions from $(\underline{x},t)$ to $(\underline{\xi},\tau)$, and the integral

$$\iint \varphi(x_1, \eta', \zeta', 1-\tau \mid \underline{\xi})\, d\eta'\, d\zeta'$$

is the density for a further transistion from $(\underline{\xi},\tau)$ to the upper boundary condition $x(1) = x_1$. The expression $-\, s f(\zeta)\frac{\partial}{\partial\eta}$ describes a jump of magnitude $s f(\zeta)$ in the derivative of x. The density p_2, then, is the density for transistion from $(\underline{x},t)$ to $(\underline{\xi},\tau)$, for a jump in y of $s f(\zeta)$ at that point, and for transistion from there to $(x_1,1)$. Since q_1, which is the density for downward transistions from $(\underline{x},t)$ to the lower boundary conditions x_0 and z_0, multiplies p_2, it appears that the transistions upward from $(\underline{x},t)$ and downward from $(\underline{x},t)$ are independent. Finally, the constant α_0 vanishes when these two latter sets of transistions are mutually exclusive. (Actually, we have only shown that $\alpha_0 > 0$ when these two sets are almost never mutually exclusive.)

5.2 The Expected Solution

One of the main drawbacks to the Fokker-Planck approach to stochastic systems (aside from the difficulty of solving parabolic equations) is the problem of modeling physical situations by functions of Markov processes. We shall take

$$f(z) = k_0^2 + \varepsilon g(z) \geq 0 \quad \text{for all } z,$$

where ε is small. Assuming that the expectation of $g(z)$

vanishes, that is

$$\int g(z)\,\Gamma(z,t)\,dz = 0$$

we shall show that the expected solution of the boundary value problem is, up to order ε^2, identical with the solution for $\varepsilon = 0$.

Let $\mu = sk_0$. Then $\varphi(\underline{x},t \mid \underline{x}_0)$ satisfies

$$\frac{\partial \varphi}{\partial t} = D\frac{\partial^2 \varphi}{\partial z^2} - y\frac{\partial \varphi}{\partial x} - \mu^2 x\frac{\partial \varphi}{\partial y} - s^2 \varepsilon\, g(z)\, x\frac{\partial \varphi}{\partial y}.$$

The solution $\varphi_0(\underline{x},t \mid \underline{x}_0)$ of

$$\frac{\partial \varphi_0}{\partial t} = D\frac{\partial^2 \varphi_0}{\partial z^2} - y\frac{\partial \varphi_0}{\partial x} - \mu^2 x\frac{\partial \varphi_0}{\partial y}, \quad \varphi_0\Big|_{t=0} = \delta(\underline{x}-\underline{x}_0),$$

is

$$\varphi_0(\underline{x},t \mid \underline{x}_0) = \Gamma(z-z_0,t)\,\psi_0(x,y,t \mid x_0,y_0)$$

where

$$\psi_0(x,y,t \mid x_0,y_0) = \delta\left(x\frac{e^{-\mu t}+e^{\mu t}}{2} + y\frac{e^{-\mu t}+e^{\mu t}}{2\mu} - x_0\right)$$

$$\cdot\,\delta\left(x\cdot\mu\frac{e^{-\mu t}-e^{\mu t}}{2} + y\frac{e^{-\mu t}+e^{\mu t}}{2} - y_0\right).$$

Then φ is given by

$$\varphi(\underline{x},t \mid \underline{x}_0) = \varphi_0(\underline{x},t \mid \underline{x}_0) + \varepsilon\,\varphi_1(\underline{x},t \mid \underline{x}_0)$$

where

$$\varphi_1(\underline{x}, t \mid \underline{x}_0) = -s^2 \int_0^t \int \varphi_0(\underline{x}, t-\sigma \mid \underline{\xi})\, g(\zeta)\, \xi \frac{\partial}{\partial \eta} \varphi(\underline{\xi}, \sigma \mid \underline{x}_0)\, d\underline{\xi}\, d\sigma.$$

Now

$$\int \varphi(\underline{x}, t \mid \underline{x}_0)\, dz = \psi_0(x, y, t \mid x_0, y_0)$$

$$- s^2 \varepsilon \int_0^t \iint \psi_0(x, y, t-\sigma \mid \xi, \eta)\, \xi \frac{\partial}{\partial \eta} \int g(\zeta) \varphi(\underline{\xi}, \sigma \mid \underline{x}_0)\, d\zeta\, d\xi\, d\eta\, d\sigma$$

and

$$\int g(z) \varphi(\underline{x}, t \mid \underline{x}_0)\, dz = \psi_0(x, y, t \mid x_0, y_0) \cdot \int g(z) \Gamma(z, t)\, dz + O(\varepsilon)$$

$$= O(\varepsilon). \tag{5.7}$$

Thus, the normalizing factor α_0 is, by (5.5),

$$\alpha_0^{-1} = \iiint \varphi(x, y, z, 1 \mid x_0, y_0, z_0)\, dz\, dy\, dy_0$$

$$= \iint \psi_0(0, y, 1 \mid 0, y_0)\, dy_0\, dy + O(\varepsilon^2)$$

$$= \int \delta\left(y \frac{e^{-\mu} - e^{\mu}}{2\mu}\right) dy + O(\varepsilon^2)$$

$$= \left| \frac{2\mu}{e^{-\mu} - e^{\mu}} \right| + O(\varepsilon^2)$$

For real and positive μ we thus have

$$\alpha_0^{-1} = \frac{2\mu}{e^{\mu} - e^{-\mu}} + O(\varepsilon^2).$$

Hence, α_0 is given for $\mathrm{Re}(\mu) \geq 0$ by

$$\alpha_0 = \frac{e^{\mu} - e^{-\mu}}{2\mu} + O(\varepsilon^2).$$

The expected solution for $t < \tau$ is

$$\langle x \rangle = -s\alpha_0\left[E_0 + \varepsilon(E_1 + E_2 + E_3 + E_4)\right],$$

where

(5.8) $$E_0 = \int x \cdot \left\{ \left(\int \varphi_0(\underline{x}, t \mid \underline{x}_0)\, dy_0 \right) \cdot k_0^2 \int \varphi_0(\underline{\xi}, \tau - t \mid \underline{x}) \frac{\partial}{\partial \eta}\left[\iint \varphi_0(x_1, \eta', \zeta', 1-\tau \mid \underline{\xi})\, d\eta' d\zeta' \right] d\underline{\xi} \right\} d\underline{x},$$

$$E_1 = \int x \cdot \left\{ \left(\int \varphi_0(\underline{x}, t \mid \underline{x}_0)\, dy_0 \right) \cdot \int \varphi_0(\underline{\xi}, \tau - t \mid \underline{x})\, g(\zeta) \frac{\partial}{\partial \eta}\left[\iint \varphi_0(x_1, \eta', \zeta', 1-\tau \mid \underline{\xi})\, d\eta' d\zeta' \right] d\underline{\xi} \right\} d\underline{x},$$

$$E_2 = \int x \cdot \left\{ \left(\int \varphi_1(\underline{x}, t \mid \underline{x}_0)\, dy_0 \right) \cdot k_0^2 \int \varphi_0(\underline{\xi}, \tau - t \mid \underline{x}) \frac{\partial}{\partial \eta}\left[\iint \varphi_0(x_1, \eta', \zeta', 1-\tau \mid \underline{\xi})\, d\eta' d\zeta' \right] d\underline{\xi} \right\} d\underline{x},$$

$$E_3 = \int x \cdot \left\{ \left(\int \varphi_0(\underline{x}, t \mid \underline{x}_0)\, dy_0 \right) \cdot \right.$$

$$\left. \cdot k_0^2 \int \varphi_1(\underline{\xi}, \tau - t \mid \underline{x}) \frac{\partial}{\partial \eta} \left[\iint \varphi_0(x_1, \eta', \zeta', 1-\tau \mid \underline{\xi})\, d\eta'\, d\zeta' \right] d\underline{\xi} \right\} d\underline{x},$$

$$E_4 = \int x \cdot \left\{ \left(\int \varphi_0(\underline{x}, t \mid \underline{x}_0)\, dy_0 \right) \cdot \right.$$

$$\left. \cdot k_0^2 \int \varphi_0(\underline{\xi}, \tau - t \mid \underline{x}) \frac{\partial}{\partial \eta} \left[\iint \varphi_1(x_1, \eta', \zeta', 1-\tau \mid \underline{\xi})\, d\eta'\, d\zeta' \right] d\underline{\xi} \right\} d\underline{x} .$$

Then

$$E_1 = \left(\iint g(\zeta)\, \Gamma(\zeta - z, \tau - t)\, \Gamma(z, t)\, dz\, d\zeta \right) \cdot$$

$$\cdot \left[\iint x \cdot \left\{ \left(\int \psi_0(x, y, t \mid x_0, y_0)\, dy_0 \right) \cdot \right. \right.$$

$$\left. \left. \cdot \iint \psi_0(\xi, \eta, \tau - t \mid x, y) \frac{\partial}{\partial \eta} \int \psi_0(x_1, \eta', 1-\tau \mid \xi, \eta)\, d\eta'\, d\xi\, d\eta \right\} dx\, dy \right],$$

and since

$$\int g(\zeta) \int \Gamma(\zeta - z, \tau - t)\, \Gamma(z, t)\, dz\, d\zeta$$

$$= \int g(\zeta)\, \Gamma(\zeta, \tau)\, d\zeta$$

$$= 0,$$

we have

$$E_1 = 0.$$

Since

$$E_2 = k_0^2 \cdot \iint x \cdot \left\{ \int \left[\int \varphi_1(\underline{x}, t | \underline{x}_0)\, dz \right] dy_0 \right.$$

$$\left. \cdot \iint \psi_0(\xi, \eta, \tau - t | x, y) \frac{\partial}{\partial \eta} \left[\int \psi_0(x_1, \eta', 1-\tau | \xi, \eta)\, d\eta' \right] d\xi\, d\eta \right\} dx\, dy.$$

and as we have seen previously (eq. (5.7)) that

$$\int \varphi_1(\underline{x}, t | \underline{x}_0)\, dz = O(\varepsilon),$$

we see that

$$E_2 = O(\varepsilon).$$

Also

$$E_3 = k_0^2 \iint x \left\{ \left(\int \psi_0(x, y, t | x_0, y_0)\, dy_0 \right) \cdot \right.$$

$$\cdot \iiint \left[\iint \Gamma(z, t)\, \varphi_1(\underline{\xi}, \tau - t | \underline{x})\, dz\, d\zeta \right] \cdot$$

$$\left. \cdot \left[\frac{\partial}{\partial \eta} \int \psi_0(x_1, \eta', 1-\tau | \xi, \eta)\, d\eta' \right] d\xi\, d\eta \right\} dx\, dy,$$

and

$$\iint \Gamma(z, t)\, \varphi_1(\underline{\xi}, \tau - t | \underline{x})\, dz\, d\zeta$$

$$= -s^2 \iint \Gamma(z, t) \int_0^{\tau - t} \int \varphi_0(\underline{\xi}, \tau - t - \sigma | \underline{\xi}')\, g(\zeta')\, \xi' \cdot$$

$$\cdot \frac{\partial \psi_0}{\partial \eta'}(\xi', \eta', \sigma | x, y) \cdot \Gamma(\zeta' - z, \sigma)\, d\underline{\xi}'\, d\sigma\, dz\, d\zeta$$

$$+ O(\varepsilon)$$

$$= -s^2 \int_0^{\tau-t} \left\{ \iint \psi_0(\xi,\eta,\tau-t-\sigma \mid \xi',\eta')\, \xi' \frac{\partial \psi_0}{\partial \eta'}(\xi',\eta',\sigma \mid x,y)\, d\xi'\, d\eta' \right\} \cdot$$

$$\cdot \left\{ \iint g(\zeta')\, \Gamma(\zeta-\zeta', \tau-t-\sigma) \cdot \Gamma(\zeta', t+\sigma)\, d\zeta'\, d\zeta \right\} d\sigma$$

$$+ O(\varepsilon)$$

$$= O(\varepsilon).$$

And finally

$$E_4 = k_0^2 \iint x \cdot \left\{ \left(\int \psi_0(x,y,t \mid x_0, y_0)\, dy_0 \right) \cdot \right.$$

$$\cdot \iint \psi_0(\xi,\eta,\tau-t \mid x,y) \cdot$$

$$\cdot \frac{\partial}{\partial \eta} \int \left[\iiint \varphi_1(x_1,\eta',\zeta',1-\tau \mid \underline{\xi})\, \Gamma(\zeta-z, \tau-t) \right.$$

$$\left. \left. \cdot\, \Gamma(z,t)\, dz\, d\zeta\, d\zeta' \right] d\eta'\, d\xi\, d\eta \right\} dx\, dy,$$

$$= O(\varepsilon)$$

since

$$\iiint \varphi_1(x_1,\eta',\zeta',1-\tau \mid \underline{\xi})\, \Gamma(z,t)\, \Gamma(\zeta-z,\tau-t)\, dz\, d\zeta\, d\zeta'$$

$$= \iint \varphi_1(x_1,\eta',\zeta',1-\tau \mid \underline{\xi})\, \Gamma(\zeta,\tau)\, d\zeta\, d\zeta'$$

$$= O(\varepsilon)$$

as in the evaluation above of E_3.

Hence

$$\langle x \rangle = - s\alpha_0 \cdot E_0 + O(\varepsilon^2) .$$

E_0 of course is just the solution for $\varepsilon = 0$, which is known. However, it is reassuring to find that the complicated integrals of (5.8) give the correct result:

$$E_0 = k_0^2 \iint x \cdot \Big\{ \Big(\int \psi_0(x,y,t|x_0,y_0)\, dy_0 \Big) \cdot \iint \psi_0(\xi,\eta,\tau-t|x,y) \cdot \frac{\partial}{\partial\eta}\int \psi_0(x_1,\eta',1-\tau|\xi,\eta)\, d\eta'\, d\xi\, d\eta \Big\}\, dx\, dy$$

$$= -\frac{2\mu k_0^2}{e^{-\mu(1-\tau)} - e^{\mu(1-\tau)}} \cdot \iint x \cdot \Big\{ \delta\Big(x\,\frac{e^{-\mu t}+e^{\mu t}}{2} + y\,\frac{e^{-\mu t}-e^{\mu t}}{2\mu}\Big) \cdot \iint \delta\Big(\xi\,\frac{e^{-\mu(\tau-t)}+e^{\mu(\tau-t)}}{2} + \eta\,\frac{e^{-\mu(\tau-t)}-e^{\mu(\tau-t)}}{2\mu} - x\Big) \cdot \delta\Big(\xi\mu\,\frac{e^{-\mu(\tau-t)}-e^{\mu(\tau-t)}}{2} + \eta\,\frac{e^{-\mu(\tau-t)}+e^{\mu(\tau-t)}}{2} - y\Big) \cdot \delta'\Big(\frac{e^{-\mu(1-\tau)}+e^{\mu(1-\tau)}}{e^{-\mu(1-\tau)}-e^{\mu(1-\tau)}}\cdot\mu\xi - \eta\Big)\, d\xi\, d\eta \Big\}\, dx\, dy$$

$$= -\frac{k_0^2}{2\mu} \cdot \frac{\big(e^{-\mu(1-\tau)} - e^{\mu(1-\tau)}\big)\cdot\big(e^{\mu t} - e^{-\mu t}\big)}{\big(e^{\mu} - e^{-\mu}\big)^2} .$$

Therefore

$$\langle x \rangle = sk_0^2 \frac{\left(e^{-\mu(1-\tau)} - e^{\mu(1-\tau)}\right)\cdot\left(e^{\mu t} - e^{-\mu t}\right)}{2\mu\left(e^{\mu} - e^{-\mu}\right)} + O(\varepsilon^2), \quad t < \tau .$$

The leading term is just the solution of

$$\ddot{\langle x \rangle} = \mu^2 \langle x \rangle + sk_0^2\, \delta(t-\tau), \quad x(0) = x(1) = 0,$$

for $t < \tau$. Similarly we find that for $t > \tau$,

$$\langle x \rangle = sk_0^2 \frac{\left(e^{\mu\tau} - e^{-\mu\tau}\right)\cdot\left(e^{\mu(1-t)} - e^{-\mu(1-t)}\right)}{2\mu\left(e^{\mu} - e^{-\mu}\right)} + O(\varepsilon^2).$$

5.3 The Case of Non-Unique Solutions

We have shown that the normalizing factor α_0 does not vanish when the boundary value problem has unique solutions for almost all samples. As we noted in §4.4, the converse of this result has not been proven. Unfortunately, no Fokker-Planck equations of the form of (5.4) have been solved in any reasonable exact way, so the construction of possible counterexamples to the converse is not feasable.

However, it is possible to solve (5.4) approximately in at least one case of non-uniqueness, and, as we shall show, $\alpha_0 = 0$ in this case. Of course, the important questions are: For an eigenvalue problem, does α_0 vanish identically in a neighborhood of the deterministic eigenvalues, or only at isolated points? And if the latter is the case, how are these points related to the deterministic eigenvalues? Further investigation will be needed to answer these questions.

Consider now the problem

$$
(5.9)\qquad
\begin{aligned}
\mathring{x} &= y \\
\mathring{y} &= s^2(k_0^2 + \varepsilon z)x + s(k_0^2 + \varepsilon z)\delta(t-\tau) \\
\mathring{z} &= N(t) \\
x(0) &= x(1) = z(0) = 0.
\end{aligned}
$$

For some samples of $\{z\}$, $z_t \geq -k_0^2/\varepsilon$ for $0 \leq t \leq 1$, and so (5.9) will have a unique solution for each of these samples, for $\mathrm{Re}(s) \geqslant 0$. However, for some other samples (5.9) will not have unique solutions.

Equation (5.4) becomes in this case

$$
\frac{\partial \varphi}{\partial t} = D\frac{\partial^2 \varphi}{\partial z^2} - y\frac{\partial \varphi}{\partial x} - \mu^2 x\frac{\partial \varphi}{\partial x} - s^2 \varepsilon z x \frac{\partial \varphi}{\partial y},
$$

where $\mu = sk_0$ as before. The transformations

$$
\begin{aligned}
x &= \xi e^{\mu t} + \eta e^{-\mu t} \\
y &= \mu(\xi e^{\mu t} - \eta e^{-\mu t}) \\
\xi_0 &= (\mu x_0 + y_0)/2\mu \\
\eta_0 &= (\mu x_0 - y_0)/2\mu \\
\varepsilon' &= \varepsilon/2k_0 \\
\tau &= st \\
D' &= D/s \\
\xi &= \xi_0 + \varepsilon' a \\
\eta &= \eta_0 + \varepsilon' b
\end{aligned}
$$

and the assumption that φ can be represented by

$$\varphi(\underline{x}, t \mid \underline{x}_0) = \psi_0(a, b, z, \tau) + \varepsilon' \psi_1(a, b, z, \tau) + \cdots$$

results in the equation

$$\frac{\partial \psi_0}{\partial \tau} = D' \frac{\partial^2 \psi_0}{\partial z^2} - z\left(\xi_0 e^{k_0 \tau} + \eta_0 e^{-k_0 \tau}\right)\left(e^{-k_0 \tau} \frac{\partial \psi_0}{\partial a} - e^{k_0 \tau} \frac{\partial \psi_0}{\partial b}\right),$$

with the initial condition

$$\psi_0\Big|_{t=0} = \frac{-1}{2\mu(\varepsilon')^2} \delta(a)\, \delta(b)\, \delta(z - z_0)$$

Now ψ_0 is the density (normalized to $\frac{-1}{2\mu(\varepsilon')^2}$) of the gaussian system

$$\begin{aligned} \overset{\circ}{a} &= (\xi_0 + \eta_0 e^{-2k_0 \tau}) z, & a(0) &= 0, \\ \overset{\circ}{b} &= (\xi_0 e^{2k_0 \tau} + \eta_0) z, & b(0) &= 0, \\ \overset{\circ}{z} &= N(t), & z(0) &= z_0. \end{aligned}$$

Let $\langle a \rangle$, $\langle b \rangle$ be the means of a, b, respectively, and let L be the covariance matrix of a and b. Then the marginal x,y-density of ψ_0 is

$$\varphi_1(x, y, t \mid x_0, y_0, z_0) = \int \psi_0 \, dz$$

$$= \frac{\exp\left\{\frac{-1}{2\varepsilon'^2}\left(L^{-1}\begin{pmatrix} \frac{x + y/\mu}{2} e^{-\mu t} - \xi_0 - \varepsilon'\langle a \rangle \\ \frac{x - y/\mu}{2} e^{\mu t} - \eta_0 - \varepsilon'\langle b \rangle \end{pmatrix}, \begin{pmatrix} \frac{x + y/\mu}{2} e^{-\mu t} - \xi_0 - \varepsilon'\langle a \rangle \\ \frac{x - y/\mu}{2} e^{-\mu t} - \eta_0 - \varepsilon'\langle b \rangle \end{pmatrix}\right)\right\}}{2\pi \,(\det L)^{1/2}\, (-2\mu\, \varepsilon'^2)}.$$

Then the marginal x-density is, for $t = 1$ and $x_0 = x_1 = z_0 = 0$,

$$\varphi_2(y_0) = \int \varphi_1(0, y, 1 \mid 0, y_0, 0)\, dy$$

$$= \frac{\exp\left\{ \dfrac{-1}{2\varepsilon'^2} \cdot \dfrac{(e^{\mu} - e^{-\mu})^2/4\mu^2}{S} \right\}}{\sqrt{2\pi}\ \varepsilon'\ |y_0| \cdot \sqrt{S}}$$

where

$$S = \frac{D}{2\mu^2 k_0^3} \left\{ e^{2\mu} \left(\frac{\mu^3}{3} - \frac{\mu^2}{2} - \frac{\mu}{4} + \frac{5}{8} \right) \right.$$

$$\left. + \left(\frac{2\mu^3}{3} - 2\mu \right) + e^{-2\mu} \left(\frac{\mu^3}{3} + \frac{\mu^2}{2} - \frac{\mu}{4} - \frac{5}{8} \right) \right\}.$$

It is clear from the approximations we have made that our solution will be a reasonable solution only for small μ. Since $S = 0$ at $\mu = 0$ but does not vanish for small $|\mu| \neq 0$, we see that

$$\alpha_0^{-1} = \int \varphi_2(y_0)\, dy_0$$

only exists for $\varepsilon' = 0$, for small $|\mu| \neq 0$. When $\varepsilon' = 0$, we of course have

$$\varphi_2(y_0) = \delta\left(y_0\, \frac{e^{\mu} - e^{-\mu}}{2\mu} \right),$$

and

$$\alpha_0 = \frac{e^{\mu} - e^{-\mu}}{2\mu}.$$

Therefore, in at least this case, α_0 vanishes when the boundary value problem does not have unique solutions for almost all samples.

APPENDIX: THE DETAILS OF §2.2

A.1 The Mean and Variance of p_b^+

We want to find the mean $\underline{e}$ and covariance matrix L of the conditional gaussian density

$$p_b^+(\underline{x}\mid\underline{x}_0) = \frac{p(\underline{x},\underline{x}_0)}{p(\underline{x}_0)} \tag{A.1}$$

Here $p(\underline{x},\underline{x}_0)$ is a 2n-dimensional gaussian density with zero mean, and $p(\underline{x}_0)$ is its n-dimensional marginal density.

Let the covariance of $p(\underline{x}_0)$ be K_0 and that of $p(\underline{x},\underline{x}_0)$ be K. If we partition K^{-1} into $n \times n$ blocks,

$$K^{-1} = \left(\begin{array}{c|c} K_1 & K_2 \\ \hline K_2^T & K_3 \end{array}\right).$$

Then (A.1) becomes

$$\frac{1}{2\pi\,|L|^{1/2}} \exp\left\{-\frac{1}{2}\left(L^{-1}(\underline{x}-\underline{e}),(\underline{x}-\underline{e})\right)\right\}$$

$$= \frac{|K_0|^{1/2}}{2\pi\,|K|^{1/2}} \cdot \exp\left\{-\frac{1}{2}(K_1\underline{x},\underline{x}) - (K_2\underline{x}_0,\underline{x}) + \frac{1}{2}\left(\left[K_0^{-1}-K_3\right]\underline{x}_0,\underline{x}_0\right)\right\} \tag{A.2}$$

and from this we have

$$\left\{\begin{aligned} L^{-1} &= K_1 \\ \underline{e} &= -(L^{-1}K_2)\underline{x}_0 = -(K_1^{-1}K_2)\underline{x}_0 \end{aligned}\right. \tag{A.3}$$

The other relations we could obtain from (A.2), namely

$$(L^{-1}\underline{e}, \underline{e}) = ([K_3 - K_0^{-1}]\underline{x}_0, \underline{x}_0)$$

and

$$|L| = |K|/|K_0|$$

are of course implied by (A.3) and the fact that $p(\underline{x}_0)$ is the marginal density of $p(\underline{x},\underline{x}_0)$.

In the notation of §2.2, K is partitioned as follows:

$$K = \left(\begin{array}{c|c} K_t & M \\ \hline M^T & K_{t_0} \end{array}\right).$$

The inverse of K in partitioned form is easily found by factoring K :

$$K = \left(\begin{array}{c|c} K_t & O \\ \hline O & M^T \end{array}\right) \cdot \left(\begin{array}{c|c} I & K_t^{-1} M K_{t_0}^{-1} M^T \\ \hline I & I \end{array}\right) \cdot \left(\begin{array}{c|c} I & O \\ \hline O & M^{T-1} K_{t_0} \end{array}\right).$$

Hence

$$K^{-1} = \left(\begin{array}{c|c} I & O \\ \hline O & K_{t_0}^{-1} M^T \end{array}\right) \left(\begin{array}{c|c} (I - K_t^{-1} M K_{t_0}^{-1} M^T)^{-1} & O \\ \hline O & (I - K_t^{-1} M K_{t_0}^{-1} M^T)^{-1} \end{array}\right) \cdot$$

$$\cdot \left(\begin{array}{c|c} I & -K_t M K_{t_0}^{-1} M^T \\ \hline -I & I \end{array}\right) \cdot \left(\begin{array}{c|c} K_t^{-1} & O \\ \hline O & M^{T-1} \end{array}\right),$$

or

$$K^{-1} = \left(\begin{array}{c|c} (K_t - MK_{t_0}^{-1}M^T)^{-1} & -(K_t - MK_{t_0}^{-1}M^T)^{-1}MK_{t_0}^{-1} \\ \hline -K_{t_0}^{-1}M^T(K_t - MK_{t_0}^{-1}M^T)^{-1} & (K_{t_0} - M^T K_t^{-1} M)^{-1} \end{array}\right).$$

Thus we have

$$L = K_t - MK_{t_0}^{-1}M^T$$

and

$$\underline{e} = (MK_{t_0}^{-1})\underline{x}_0 .$$

A.2 Proof of Equation 2.6

We want to show that the matrix

$$\Phi(t,t_0) = \frac{\partial M(t,t_0)}{\partial t} \cdot M(t,t_0)^{-1} \tag{A.4}$$

is independent of t_0. Since $M(t,t) = K_t$ and the elements of M are continuous, there is an $\varepsilon > 0$ such that M is non-singular for $0 \leq t-t_0 < \varepsilon$. When we have shown that (A.4) holds in this range, then we have, writing $Q(t,t_0) = M \cdot K_{t_0}^{-1}$,

$$Q(t,\tau)\,Q(\tau,t_0) = Q(t,t_0), \qquad t_0 < \tau < t .$$

Hence M will then be non-singular for all $0 < t_0 \leq t < 1$, and the proof of (A.4) will be valid for $0 < t_0 \leq t < 1$.

Let $M = (m_{ij})$, $M^{-1} = (m^{ij})$, and $\Phi = (\varphi_{ij})$, where the indices run from 0 to $n-1$. For $i \neq n-1$,

$$\frac{\partial}{\partial t} m_{ij} = 2D \frac{\partial}{\partial t} \int \frac{\partial^i h(t,\tau)}{\partial t^i} \cdot \frac{\partial^j h(t_0,\tau)}{\partial t_0^j} \, d\tau = m_{i+1,j}$$

and so

$$\varphi_{ij} = \sum_{k=0}^{n-1} m_{i+1,k} m^{k,j} = \delta_{i+1,j}, \quad i \neq n-1.$$

Now let

$$\varphi_{n-1,k} = -\alpha_k(t) + \beta_k(t,t_0),$$

where the α_k are the coefficients of the operator

$$\mathcal{L}_t = \sum_{k=0}^{n} \alpha_k(t) \frac{d^k}{dt^k}, \quad \alpha_n = 1,$$

whose Green's function on the space $\mathcal{B}$ is $h(t,t_0)$. We have

$$\frac{\partial}{\partial t} m_{n-1,j} = \sum_{k=0}^{n-1} \varphi_{n-1,k} m_{k,j}$$

$$= -2D \sum_{k=0}^{n-1} \alpha_k(t) \int_0^1 \frac{\partial^k h(t,\tau)}{\partial t^k} \cdot \frac{\partial^j h(t_0,\tau)}{\partial t_0^j} \, d\tau + \sum_{k=0}^{n-1} \beta_k m_{kj}$$

$$= \frac{\partial}{\partial t} m_{n-1,j} - 2D \frac{\partial^j h(t_0,t)}{\partial t_0^j} + \sum_{k=0}^{n-1} \beta_k m_{kj}$$

Thus

$$\sum_{k=0}^{n-1} \beta_k(t,t_0) m_{kj} = 2D \frac{\partial^j h(t_0,t)}{\partial t_0^j}, \quad j = 0(1)n-1. \tag{A.5}$$

Differentiating (A.5) with respect to t_0 for $j \neq n-1$, we obtain

(A.6) $$\sum_{k=0}^{n-1} \frac{\partial \beta_k}{\partial t_0} \cdot m_{kj} = 2D \frac{\partial^{j+1} h(t_0,t)}{\partial t_0^{j+1}} - \sum_{k=0}^{n-1} \beta_k m_{k,j+1} = 0,$$

for $j = 0(1)n-2$.

Now defining

$$F(t,t_0,\tau) = \sum_{k=0}^{n-1} \beta_k(t,t_0) \frac{\partial^k h(t,\tau)}{\partial t^k},$$

(A.5) becomes

$$\int_0^1 F(t,t_0,\tau) \cdot \frac{\partial^j h(t_0,\tau)}{\partial t_0^j} d\tau = \frac{\partial^j h(t_0,t)}{\partial t_0^j}, \quad j = 0(1)n-1.$$

Then we have

$$\mathcal{L}_{t_0}[h(t_0,t)] = \int_0^1 F(t,t_0,\tau) \sum_{k=0}^{n} \alpha_k(t_0) \frac{\partial^k h(t_0,\tau)}{\partial t_0^k} d\tau + \int_0^1 \frac{\partial F(t,t_0,\tau)}{\partial t_0} \cdot \frac{\partial^{n-1} h(t_0,\tau)}{\partial t_0^{n-1}} d\tau$$

(A.7) $$\begin{cases} = F(t,t_0,t_0) + \int_0^1 \frac{\partial F(t,t_0,\tau)}{\partial t_0} \cdot \frac{\partial^{n-1} h(t_0,\tau)}{\partial t_0^{n-1}} d\tau \\ \\ = \delta(t-t_0) \end{cases}$$

We now set

$$\gamma(t_0,t,\sigma) = \int_0^1 F(t,\sigma,\tau)\, h(t_0,\tau)\, d\tau$$

As a function of t_0, $\gamma(t_0,t,\sigma)$ satisfies

(A.8) $$\mathcal{L}_{t_0}(\gamma) = F(t,\sigma,t_0), \quad \gamma \in \mathcal{B}$$

and also

$$\gamma(t_0, t, t_0) = h(t_0, t).$$

Let ψ be any function in the domain $\mathcal{B}^*$ of the adjoint, $\mathcal{L}^*$, of $\mathcal{L}$, and let $\mathcal{L}^*\psi = \mu$. Then we have from (A.8)

$$\int_0^1 F(t,\sigma,t_0)\,\psi(t_0)\,dt_0 = \int_0^1 \mathcal{L}_{t_0}(\gamma)\,\psi(t_0)\,dt_0$$

$$= \int_0^1 \gamma(t_0,t,\sigma)\,\mathcal{L}^*_{t_0}(\psi)\,dt_0$$

$$= \int_0^1 \mu(t_0)\,\gamma(t_0,t,\sigma)\,dt_0 .$$

Now set $\sigma = t_0$:

$$\int_0^1 F(t,t_0,t_0)\,\psi(t_0)\,dt_0 = \int_0^1 \mu(t_0)\,\gamma(t_0,t,t_0)\,dt_0$$

$$= \int_0^1 \mu(t_0)\,h(t_0,t)\,dt_0$$

$$= \psi(t).$$

Since the spaces $\mathcal{B}$ and hence $\mathcal{B}^*$ are determined by linear homogeneous unmixed boundary conditions at $t = 0$ and $t = 1$, it is clear that $\mathcal{B}^*$ contains all the infinitely differentiable functions with compact support in $[0,1]$. Therefore

$$F(t, t_0, t_0) = \delta(t - t_0). \tag{A.9}$$

But then from (A.7),

$$0 = \int_0^1 \frac{\partial F(t, t_0, \tau)}{\partial t_0} \cdot \frac{\partial^{n-1} h(t_0, \tau)}{\partial^{n-1} t_0} d\tau$$

$$= \frac{1}{2D} \sum_{K=0}^{n-1} \frac{\partial \beta_K}{\partial t_0} m_{K,n-1}.$$

This last equation, combined with (A.6) and the non-singularity of M implies

$$\frac{\partial \beta_K}{\partial t_0} = 0, \qquad K = 0(1)n-1.$$

Hence, $\Phi(t, t_0)$ is independent of t_0; and the boundary value process is markovian.

Of course, once we have evaluated the elements of $M(t, t_0)$ for $t > t_0$, then instead of (A.9) we shall only have

$$\sum_{K=0}^{n-1} \beta_K(t) \frac{\partial^K h(t, t_0)}{\partial t^K} = 0,$$

for $t > t_0$.

REFERENCES

[1] Adomian, G., "Linear Stochastic Operators," Rev. Mod. Phys., Vol. 35 (1963), pp. 185-206

[2] Adomian, G., "Stochastic Green's Functions," Proc. Symp. Appl. Math., Vol. 16, Providence: American Mathematical Society, 1964, pp. 1-39

[3] Boyce, W.E., "Random Vibrations of Elastic Strings and Bars," Proc. Fourth U.S. National Congress Appl. Mech., Vol. 1, New York: American Society of Mechanical Engineers, 1962, pp 77-85

[4] Boyce, W.E., "A 'Dishonest' Approach to Certain Stochastic Eigenvalue Problems," SIAM J. Appl. Math., Vol. 15 (1967), pp. 143-152

[5] Boyce, W.E., and B.E. Goodwin, "Random Transverse Vibrations of Elastic Beams," SIAM J. Appl. Math., Vol. 12 (1964), pp. 613-629

[6] Caughey, T.K. and A.H. Gray, "A Controversy in Problems Involving Random Parametric Excitation," J. Math. & Phys., Vol. 44 (1965), pp. 288-296

[7] Doob, J.L., Stochastic Processes, New York: John Wiley & Sons, 1953, pp. 273-291

[8] Feller, W., "Zur Theorie der Stochastichen Prozesse (Existenze- und Eindeutkeitssätze)," Math. Ann., Vol. 113 (1936), pp. 113-159

[9] Friedman, A., Partial Differential Equations of Parabolic Type, Englewood Cliffs: Prentice Hall, 1964, p. 4

[10] Goodwin, B.E. and W.E. Boyce, "The Vibrations of a Random Elastic String: The Method of Integral Equations," Quart. Appl. Math., Vol. 22 (1964), pp. 261-266

[11] Gray, A.H., appendix to T.K. Caughey, "Derivation and Application of the Fokker-Planck Equation to Discrete Nonlinear Dynamic Systems Subjected to White Random Excitation," J. Acoust. Soc. Amer., Vol. 35 (1963), pp. 1683-1692

[12] Haines, C.W., "Heirarchy Methods for Random Vibration of Elastic Strings and Beams," Proc. Fifth U.S. National Congress Appl. Mech., New York: American Society of Mechanical Engineers, 1966, p. 125

[13] Henrici, P., Discrete Variable Methods in Ordinary Differential Equations, New York: John Wiley & Sons, 1962, pp. 347-348

[14] Kac, M., Probabilty and Related Topics in Physical Science, New York: Interscience, 1959, p. 170

[15] Keller, J.B., "Wave Propagation in Random Media," Proc. Symp. Appl. Math., Vol. 13, Providence: American Mathematical Society, 1962, pp. 227-246

[16] Keller, J.B., "Stochastic Equations and Wave Propagation in Random Media," Proc. Symp. Appl. Math., Vol. 16, Providence: American Mathematical Society, 1964, pp. 145-170

[17] Lees, M., "Discrete Methods for Nonlinear Two-Point Boundary Value Problems," in J.H. Bramble, ed., Numerical Solution of Partial Differential Equations, New York: Academic Press, 1966, pp. 59-72

[18] Lewis, R.M., and R.H. Kraichnan, "A Space-Time Functional Formalism for Turbulence," Comm. Pure Appl. Math., Vol. 15 (1962), pp. 397-412

[19] Meyer, P.-A., Processus de Markov, Berlin: Springer-Verlag, 1967, pp. 18-23

[20] Richardson, J.M., "Application of Truncated Heirarchy Techniques," Proc. Symp. Appl. Math., Vol. 16, Providence: American Mathematical Society, 1964, pp. 290-302

[21] Rosenblatt, M., Random Processes, New York: Oxford University Press, 1962, pp. 133-137

[22] Stratonovich, R.L., "A New Representation for Stochastic Integrals and Equations," SIAM J. Control, Vol. 4 (1966), pp. 362-371

[23] Wang, M.C. and G.E. Uhlenbeck, "On the Theory of Brownian Motion II," in N. Wax, ed., Selected Papers on Noise and Stochastic Processes, New York: Dover, 1954, pp. 113-132