COMPRESSIBLE FLOWS AT SMALL REYNOLDS NUMBERS

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ABSTRACT

The problem of the slow viscous flow of a gas past a sphere is considered. The fluid cannot be treated incompressible in the limit when the Reynolds number Re, and the Mach number M, tend to zero in such a way that Re $\sim o(M^2)$. In this case, the lowest order approximation to the steady Navier-Stokes equations of motion leads to a paradox discovered by Lagerstrom and Chester. This paradox is resolved within the framework of continuum mechanics using the classical slip condition and an iteration scheme that takes into account certain terms in the full Navier-Stokes equations that drop out in the approximation used by the above authors. It is found however that the drag predicted by the theory does not agree with R. A. Millikan's classic experiments on sphere drag.

The whole question of the applicability of the Navier-Stokes theory when the Knudsen number $\frac{M}{Re}$ is not small is examined. A new slip condition is proposed. The idea that the Navier-Stokes equations coupled with this condition may adequately describe small Reynolds number flows when the Knudsen number is not too large is looked at in some detail. First, a general discussion of asymptotic solutions of the equations for all such flows is given. The theory is then applied to several concrete problems of fluid motion. The deductions from this theory appear to interpret and summarize the results of Millikan over a much wider range of Knudsen numbers (almost up to the free molecular or kinetic limit) than hitherto believed possible by a purely continuum theory. Further experimental tests are suggested and certain interesting applications to the theory of dilute suspensions in gases are noted. Some of the questions raised in the main body of the work are explored further in the appendices.

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Chapter I

Introduction

§1. The problem of the flow of a viscous fluid past finite bodies is one of the central problems of fluid mechanics. Stokes was the first to solve a particular case of it. If the Reynolds number of the flow is large, the problem is complicated by many difficulties and in the present state of knowledge, very poorly understood. On the other hand, when the Reynolds number is small, some progress can be made. When the Reynolds number is small, a compressible fluid is characterized by the Mach number of the flow in addition to several other, less important, parameters. The Mach number measures the compressibility effects on the flow as well as the continuum nature of the fluid. From a kinetic point of view, the ratio M/Re is equal to $\overline{\lambda}/L$, where $\overline{\lambda}$ is the mean free path of the fluid and L is a typical linear dimension of the finite body around which the fluid flows. This ratio, which is also called the Knudsen number, must be small if the flow is regarded as the flow of a continuous medium around the body. Stokes himself gave a theory which has since been extended and developed by others, of flows for which $Re << 1$ and $\frac{M}{Re} << 1$. An account of this theory may be found in [1]. The question now arises whether we can give a theory of flow past finite bodies when $\frac{M}{Re}$ is O(1) or even larger. Traditionally, the statement is made that in this regime, the continuum hypothesis does not hold any more and that therefore the Navier-Stokes equations are inapplicable. Kinetic theory has been used under these conditions to calculate flow properties like drag coefficients when $\frac{M}{Re} >> 1$.

§2. The usual calculations of kinetic theory (not involving the actual solution of the Boltzmann equation subject to given boundary conditions) neglect the effect of the body on the mean velocity of the flow. Again, they do not lead to any details of the flow field. It is therefore of interest to examine whether the Navier-Stokes equations provide a useful picture of the flow. There are reasons to believe that this may be so. If one writes down the Boltzmann equation and tries to solve it by the method of moments, the simplest set of moment equations are the Navier-Stokes equations. In fact, it has been found that including a few higher moments usually yields results that are not as good as the Navier-Stokes results. A discussion of these and experimental results obtained by R. A. Millikan and other workers are given in [2]. The limits of continuum theory are discussed in [3], where it is stated that the problem of the transition of continuum to free molecular flow is still unclarified.

§3. The first attempt to solve the Navier-Stokes equations for a flow past a sphere when $Re << 1$, $M << 1$, and $\frac{Re}{M^2} << 1$ was made by Lagerstrom and Chester [4]. This attempt will be briefly discussed in the following pages. It ran into certain paradoxes. These paradoxes could be removed by a modification of their technique if one assumes a "slip condition" given in [2]. It will, however, be shown that the solution so obtained contradicts available experimental evidence.

§4. At this stage, a new slip condition is proposed and the Navier-Stokes equations are solved for various Re, M. Re is always restricted to small values. The results will be compared with experiment, and with kinetic theory, where appropriate. This new condition is very simple and goes over smoothly into the classical no-slip condition at the appropriate limit. It also appears to be a rather natural one.

The contents of the following chapters are briefly summarized here. Chapter II contains a statement of the equations to be used and a brief description of the Lagerstrom-Chester theory. In Chapter III the modification of this theory to resolve the paradoxes in it is described. The results so obtained are found to be in disagreement with experiment. Chapter IV contains a full discussion of a new slip condition and an asymptotic theory of low Reynolds number flows of a compressible fluid past finite bodies. In Chapter V we discuss the "outer" or Oseen expansions of such flows. Applications of these methods are made to concrete cases in Chapter VI. Comparison with experiment is also made.

Chapter II

The Formulation of Lagerstrom-Chester Theory

§5. In the following chapters we shall use the non-dimensional form of the Navier-Stokes equations for a perfect gas. We shall assume the Prandtl number to be a constant. The coefficients of viscosity and the ratio of specific heats vary somewhat with temperature but we shall replace them with constants, an approximation which is probably justified for most steady flow problems. We shall also restrict the Reynolds number of the flow (to be defined below) to be small and shall in fact use it as a perturbation parameter. Only steady flows past finite sized bodies will be considered. The boundary condition at infinity is simply a uniform flow. The conditions at the body will be discussed as they arise naturally in the problem. The Navier-Stokes equations are discussed in detail in [1]. External force fields in addition to the body as well as heat sources could be easily included in the theory but will be left out in the interests of simplicity.

§6. The equations of steady flow of a gas with the properties stated above are

$$\nabla \cdot \rho \vec{u} = 0 , \tag{6.1}$$

$$\rho \vec{u} \cdot \nabla \vec{u} + \nabla p = \nabla(\lambda \nabla \cdot \vec{u}) + \nabla \cdot (\mu(\mathrm{def}\, \vec{u})) , \tag{6.2a}$$

$$= \mu \nabla^2 \vec{u} + (\lambda+\mu) \nabla(\nabla \cdot \vec{u}) , \tag{6.2b}$$

$$\rho \vec{u} \cdot \nabla (e + \frac{p}{\rho}) - \vec{u} \cdot \nabla p = k \nabla^2 T + \Phi , \tag{6.3}$$

$$\Phi = \lambda (\nabla \cdot \vec{u})^2 + \frac{\mu}{2} \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 , \tag{6.4}$$

$$p = R \rho T , \tag{6.5}$$

$$h = C_p T \quad ; \quad e = C_v T . \tag{6.6}$$

We non-dimensionalize these equations in the following way. Let L be the typical linear dimension of the body and let ρ_∞, T_∞, U, p_∞ denote the values of the flow variables at infinity. Define,

$$x^* = \frac{x}{L} , \quad \rho^* = \frac{\rho}{\rho_\infty} , \quad p^* = \frac{p}{p_\infty} , \quad T^* = \frac{T}{T_\infty} ,$$

$$Pr = \frac{\mu C_p}{k} , \quad \frac{C_p}{C_v} = \gamma , \quad \frac{\mu}{\lambda + \mu} = \alpha , \quad Re = \frac{\rho U L}{\mu} ,$$

$$\frac{U^2}{\gamma T_\infty R} = M^2 , \quad \Phi^* = \frac{L^2}{\mu U^2} \Phi . \tag{6.7}$$

We find then,

$$\nabla^* \cdot \rho^* \vec{u}^* = 0 , \tag{6.8}$$

$$Re \, \rho^* \vec{u}^* \cdot \nabla^* \vec{u}^* + \frac{Re}{\gamma M^2} \nabla^* p^* = \nabla^{*2} \vec{u}^* + \frac{1}{\alpha} \nabla^* (\nabla^* \cdot \vec{u}^*) , \tag{6.9}$$

$$\mathrm{Re}\,\rho^{*}\,\vec{u}^{*}\cdot\nabla^{*}T^{*} - \left(\frac{\gamma-1}{\gamma}\right)\mathrm{Re}\,\vec{u}^{*}\cdot\nabla^{*}p^{*} = \frac{1}{\mathrm{Pr}}\nabla^{*2}T^{*} + (\gamma-1)M^{2}\Phi^{*}, \quad (6.10)$$

where Re, γ, M, Pr, α are parameters. We also have

$$p^{*} = \rho^{*}T^{*}. \quad (6.11)$$

At Infinity,

$$\vec{u}^{*} \to \vec{i} \quad \text{and} \quad \rho^{*} \to 1\ ,\ T^{*} \to 1\ . \quad (6.12)$$

The solutions of these equations will be sought for fixed γ, Pr, α and Re $<<1$.

§7. The stage is now set for a discussion of the results of Lagerstrom and Chester [4]. We shall use equations (6.8), (6.9), (6.10), (6.11) and (6.12) omitting the stars, however. The variables used henceforth will be non-dimensional unless stated otherwise. We therefore consider a steady flow past a sphere of unit radius together with the following assumptions.

$$\mathrm{Re} << 1\ ;\quad M^2 << 1\ ;\quad \frac{\mathrm{Re}}{\gamma M^2} = \epsilon << 1\ . \quad (7.1)$$

$$U_R = 0 \text{ on } R = 1\ ;\qquad U_\theta = 0 \text{ on } R = 1\ ; \quad (7.2)$$

$$T = 1 \text{ on } R = 1\ . \qquad \text{(For example)}\ . \quad (7.3)$$

To the lowest order we obtain the following equations (if for the moment we retain the ϵ term):

$$\nabla \cdot \rho \vec{u} = 0 \, , \qquad (7.4)$$

$$\nabla^2 \vec{u} + \frac{\nabla(\nabla \cdot \vec{u})}{\alpha} = \epsilon \nabla p \, , \qquad (7.5)$$

$$\nabla^2 T = 0 \, , \qquad (7.6)$$

$$p = \rho T \, . \qquad (7.7)$$

These are the so-called inner equations of perturbation. The neglected inertia terms may not be neglected at infinity and hence the solution of the full equations to leading order, in order to be uniformly valid, must include an outer part which would be matched with this one. A brief discussion of the outer part will be given later.

Since the temperature equation may be solved for separately, we will not discuss it any further. Lagerstrom and Chester solved (7.4), (7.5) together with the boundary conditions assuming that $\epsilon \nabla p$ is a small term. However, they were led to a density ρ which became infinite on the surface of the sphere in such a way as to make the flux into the sphere finite: clearly a physical contradiction.

Taking a spherical co-ordinate system as shown in Fig. 1. and assuming the boundary conditions stated above to hold,

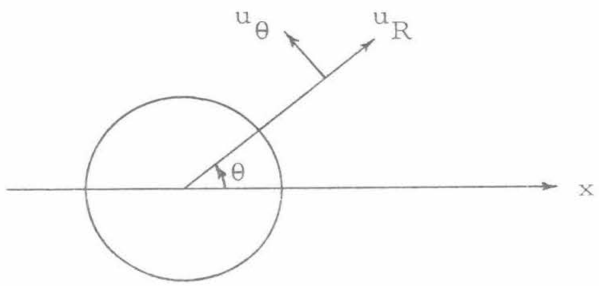

Figure 1

Lagerstrom and Chester arrived at these flow fields.

$$U_R^{(L)} = \left(1 - \frac{3(1+\alpha)}{2+3\alpha}\left(\frac{1}{R}\right) + \frac{1}{2+3\alpha}\left(\frac{1}{R^3}\right)\right)\cos\theta \quad , \tag{7.8}$$

$$U_\theta^{(L)} = \left(-1 + \frac{3(1+2\alpha)}{2(2+3\alpha)}\left(\frac{1}{R}\right) + \frac{1}{2(2+3\alpha)}\left(\frac{1}{R^3}\right)\right)\sin\theta \ , \tag{7.9}$$

$$\rho^{(L)} = (R-R_2)^{A_2}\ (R-R_3)^{A_3}\ (R-1)^{-1} , \tag{7.10}$$

$$R_2 + R_3 = \frac{1}{2+3\alpha} \quad , \qquad R_2 R_3 = -\frac{1}{2+3\alpha} \quad ; \tag{7.11}$$

$$A_2 = \frac{(2+3\alpha)R_2+1}{(2+3\alpha)(R_2-R_3)} \ , \qquad A_3 = \frac{(2+3\alpha)R_3+1}{(2+3\alpha)(R_3-R_2)} \quad ; \tag{7.12}$$

From these expressions, the following facts emerge. The density has a singularity at $R = 1$ and the flux into the sphere is finite at any point.

The sphere is not a stream-line but is a locus of stagnation points. Any attempt to improve the solution by iteration fails as $\nabla\rho$ has a worse singularity and the resulting equations for $\vec{u}$ have no solution. This situation arose from neglecting the pressure term in (7.5). If one could solve (7.4), (7.5) with this term, the solution might not give a physically meaningless result. However, if this term is included, a non-linear, elliptic-hyperbolic system results. It is doubtful if an exact solution could be found. In the next chapter an iterative method of solving these equations will be presented and the results of a leading order calculation will be given. In Appendix IV another approximate method equivalent to the iteration scheme is given.

Chapter III

The Solution of the Lagerstrom-Chester Paradox

§8. In the regime of flow we are considering, physical experience indicates that the fluid slips at the surface of the sphere. The mathematical expression of this fact derives from certain considerations in kinetic theory. In the literature (for example in [2]) the boundary condition is always quoted in the form (see also [5]):

$$\beta\, U_{tangential} = \text{Tangential Stress} . \tag{8.1}$$

where β is a surface constant. $\beta \rightarrow \infty$ gives the no-slip condition. For the present chapter we shall assume this condition and attempt to solve the Lagerstrom-Chester problem. The approximate solution obtained will then be compared with experiment and will be shown inadequate. We will then examine (8.1) more carefully and find an improved slip condition which leads to agreement with experiment. The present chapter is interesting from a mathematical standpoint, however.

§9. Assuming the thermal equation to have been solved already, the equations to be solved are:

$$\nabla \cdot \rho \vec{u} = 0 , \tag{9.1}$$

$$\nabla^2 \vec{u} + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{u}) = \epsilon \nabla \rho T , \tag{9.2}$$

$$\vec{u} = \vec{i} \text{ at } \infty \quad ; \quad \rho = 1 \text{ upstream, } \infty \text{ ,} \tag{9.3}$$

$$u_R = 0 \text{ on } R = 1 \text{; } \beta u_\theta = \left\{ R \frac{\partial}{\partial R}\left(\frac{u_\theta}{R}\right) + \frac{1}{R}\frac{\partial u_R}{\partial \theta} \right\}_{R=1} \text{;}$$

$$\therefore \; u_\theta = \tau \frac{\partial u_\theta}{\partial R} \text{, where } \tau = \frac{1}{1+\beta} \text{ ,} \qquad \text{with } \epsilon << 1 \text{ .}$$

Before we proceed to the method of solution of (9.1), (9.2), (9.3) we collect a few general facts. Write ,

$$\vec{u} = \vec{v} + \nabla \phi \text{ ,} \tag{9.4}$$

with

$$\nabla \cdot \vec{v} = 0 \text{ .} \tag{9.5}$$

This decomposition is unique up to a harmonic function. Substituting in (9.2), we have

$$\nabla^2 \vec{v} + \left(1 + \frac{1}{\alpha}\right) \nabla \nabla^2 \phi = \epsilon \nabla \rho T \text{ ,} \tag{9.6}$$

$$\therefore \qquad \nabla^2 \vec{v} = \nabla \chi \text{ .} \tag{9.7}$$

Where,

$$\nabla^2 \chi = 0 \text{ ,} \tag{9.8}$$

since

$$\nabla \cdot \vec{v} = 0 \text{ .} \tag{9.9}$$

We therefore have the following "correlation" with Stokes equations of incompressible flow:

$$\vec{u} = \vec{v} + \nabla\phi , \tag{9.10}$$

with

$$\nabla \cdot \vec{v} = 0 ; \tag{9.11}$$

$$\nabla^2 \vec{v} = \nabla\chi ; \tag{9.12}$$

and

$$\chi + (1 + \frac{1}{\alpha}) \nabla^2 \phi = \epsilon(\rho T - 1). \tag{9.13}$$

χ is a harmonic function $\rightarrow 0$ at ∞. Although $\vec{v}$ satisfies the Stokes equations, it is coupled to ϕ through the boundary conditions. By a suitable choice of the decomposition, it is possible to have $v_R = 0$ on $R = 1$. But v_θ and ϕ are still coupled via the boundary condition. These correlation equations generalize earlier results of Lagerstrom and Chester. In fact, the fields given in (7.8), (7.9), (7.10) may be obtained rather simply from (9.10), (9.11), (9.12), (9.13). Dropping the pressure term, we solve,

$$\nabla^2 \vec{v} = \nabla\chi , \tag{9.12}$$

$$\nabla \cdot \vec{v} = 0 , \tag{9.11}$$

subject to $\vec{v} \rightarrow C\vec{i}$ at ∞ where C is an undetermined constant, and $\vec{v} = 0$ on $R = 1$. This gives χ. We then solve

$$\nabla^2 \phi + \frac{\alpha}{\alpha+1} \chi = 0 , \qquad (9.13)$$

subject to

$$\phi = 0 \ ; \ \phi_R = 0 \quad \text{on} \quad R = 1. \qquad (9.14)$$

$$\nabla\phi \text{ thus obtained} \rightarrow \frac{C\alpha}{2(1+\alpha)} \vec{i} \text{ at infinity}. \qquad (9.15)$$

Hence we have

$$C = \frac{2(1+\alpha)}{2+3\alpha} . \qquad (9.16)$$

The solution with the slip condition proceeds in a similar manner. The velocity fields so obtained (by dropping the ϵ term) have a remarkable fore-and aft symmetry. u_R is odd in $\frac{\pi}{2} - \theta$ and u_θ is even in $\frac{\pi}{2} - \theta$. ρ is spherically symmetric. The equations (9.1), (9.2), (9.3) do not possess this symmetry.

§10. The equations (9.10) etc. are complicated not only on account of their non-linearity but also because the boundary conditions mix ϕ , and $\vec{v}$. Moreover, the "inner equations" obtained by dropping the ϵ term in (9.13) have a singular density even in the case of finite slip. However, for the case of finite slip, the sphere is a stream-line and not a locus of stagnation points. The search for a straightforward perturbation procedure runs into great difficulties (see, however, Appendix IV for such a scheme). An alternative iterative procedure is suggested here. For the lowest order calculation, the procedure leads to a physically meaningful density. Owing to the great complexity, higher order calculations were not carried out though the procedure itself is well-defined.

We write the equations in the following form.

$$\nabla^2 \vec{v} = \nabla \chi . \tag{10.1}$$

$$\nabla \cdot \vec{v} = 0 . \tag{10.2}$$

$$(\vec{v} + \nabla\phi) \cdot \nabla \rho + \rho \left\{ \frac{\epsilon}{1+\frac{1}{\alpha}} (\rho T - 1) - \frac{\chi}{1+\frac{1}{\alpha}} \right\} = 0 . \tag{10.3}$$

$$(1+\frac{1}{\alpha}) \nabla^2 \phi = \epsilon (\rho T - 1) - \chi . \tag{10.4}$$

The basic idea is the following. The term $\epsilon(\rho T-1)$ is important in governing the behaviour of density gradients near the sphere; it does not affect the velocities near the sphere very much but it does affect the divergence of the velocity to which it is closely related. We now proceed to solve (10.1), (10.2), (10.3), (10.4) in an iterative way. Note that (10.3) involves only the velocities explicitly (and, of course, χ). We determine ρ to the lowest order from (10.3) by using the Lagerstrom-Chester approximation to the velocity (with slip) and to χ. This differs from the Lagerstrom-Chester treatment in two essential respects: i) the approximate velocity fields used correspond to slip flow ii) the term $\epsilon(\rho T-1)$ which governs the behaviour of ρ near the sphere, is taken into account in (10.4).

It will now be indicated how one might improve the solution. Let $\vec{v}_n$, ϕ_n, ρ_n be the fields at any stage of the iteration. To get the fields for the next stage one solves the following set of equations.

$$\nabla^2 \vec{v}_{n+1} = \nabla \chi_{n+1} \; ; \tag{10.5}$$

$$\nabla \cdot \vec{v}_{n+1} = 0 \quad ; \tag{10.6}$$

$$(1 + \frac{1}{\alpha}) \nabla^2 \phi_{n+1} = \epsilon (\rho_n T - 1) - \chi_{n+1} \; ; \tag{10.7}$$

$$(\vec{v}_{n+1} + \nabla\phi_{n+1}) \cdot \nabla\rho_{n+1} + \rho_{n+1} \left\{ \frac{\epsilon}{1+\frac{1}{\alpha}} (\rho_{n+1} T - 1) - \frac{\chi_{n+1}}{1+\frac{1}{\alpha}} \right\} = 0 \; . \tag{10.8}$$

The equations for the velocities are always linear, inhomogeneous. The equation for ρ_{n+1} is a quasi-linear partial differential equation, the characteristics of which are the stream-lines corresponding to the velocity field $\vec{u}_{n+1}$. This quasi-linear equation is in fact a linear equation for a variable $\sigma_{n+1} = \frac{1}{\rho_{n+1}}$. Then, it reads,

$$(\vec{v}_{n+1} + \nabla\phi_{n+1}) \cdot \nabla\sigma_{n+1} = \left\{ \frac{\epsilon T}{1+\frac{1}{\alpha}} - \left(\frac{\epsilon + \chi_{n+1}}{1+\frac{1}{\alpha}}\right) \sigma_{n+1} \right\} \cdot \tag{10.9}$$

This is a linear inhomogeneous p.d.e. One integrates it as an o.d.e. along the characteristics from the upstream end. We will have obtained a solution if this iteration scheme converges. This appears to be quite difficult to establish. The method of solving (10.5), (10.6), (10.7) is by calculating the Green's function which may be obtained by a slight modification of a technique given by Proudman and Pearson. This is done in Appendix II.

§11. Now we give the results for the lowest order. Take,

$$\rho_0 = \frac{1}{T} . \tag{11.1}$$

$$\nabla^2 \vec{v}_1 = \nabla \chi_1 ; \tag{11.2}$$

$$\nabla \cdot \vec{v}_1 = 0 ; \tag{11.3}$$

$$(1+\frac{1}{\alpha}) \nabla^2 \phi_1 = - \chi_1 \quad ; \qquad \vec{v}_1 + \nabla\phi_1 \to \vec{i} \text{ at } \infty ; \tag{11.4}$$

$$v_{1R} + \frac{\partial \phi_1}{\partial R} = 0 \text{ on } R = 1 . \tag{11.5a}$$

$$v_{1\theta} + \frac{1}{R}\frac{\partial \phi_1}{\partial \theta} = \tau \frac{\partial}{\partial R}(v_{1\theta} + \frac{1}{R}\frac{\partial \phi_1}{\partial \theta}) \text{ on } R=1 . \tag{11.5b}$$

$$\vec{u}_1 \cdot \nabla \rho_1 + \{ \epsilon \frac{(\rho_1 T - 1)}{1+\frac{1}{\alpha}} - \frac{\chi_1}{1+\frac{1}{\alpha}} \} = 0 . \tag{11.6}$$

Solving (11.2), (11.3), (11.4) subject to (11.5) we get for $\vec{u}_1 = \vec{v}_1 + \nabla\phi_1$,

$$u_{1R} = (1 + \frac{B}{R} + \frac{C}{R^3}) \cos\theta , \tag{11.7}$$

$$u_{1\theta} = (-1 - \frac{2\alpha+1}{2(\alpha+1)} \frac{B}{R} + \frac{C}{2R^3}) \sin\theta , \tag{11.8}$$

$$\chi_1 = \frac{B \cos\theta}{R^2} , \tag{11.9}$$

where
$$B = - \frac{3(1+\tau)(\alpha+1)}{(2+3\alpha) + \tau(4+5\alpha)} , \tag{11.10}$$

and
$$C = \frac{1 - \tau(1+2\alpha)}{(2+3\alpha) + \tau(4+5\alpha)} . \tag{11.11}$$

In this form of the scheme, it is not necessary to assume $\epsilon << 1$ so long as we start with some initial density distribution. However, we shall proceed to calculate ρ_1 assuming ϵ to be small. The stream lines corresponding to $\vec{u_1}$ are given by

$$R^{A_0} (R-1)^{A_1} (R-\alpha_2)^{A_2} (R-\alpha_3)^{A_3} \sin\theta = \text{const.} ; \tag{11.12}$$

where $\Sigma A_i = 1$, $A_0 = -\frac{1}{2}$, $A_1 > 0$ and $\rightarrow 0$ as $\tau \rightarrow 0$.

$$R-\alpha_2 \;,\; R-\alpha_3 \quad \text{are factors of} \quad R^3 + BR^2 + C . \tag{11.13}$$

the constant in (11.12) may be interpreted as $r^{A_0}(r-1)^{A_1}(r-\alpha_2)^{A_2}(r-\alpha_3)^{A_3}$ where r is the distance of the point of intersection of the stream-line with the plane $\theta : \pi/2$. We may therefore introduce r as the labelling parameter and use s as the distance along the stream-line from $(\theta = \pi/2 \;,\; r)$ to (θ, R).

$$\frac{dR}{u_{1R}} = \frac{R d\theta}{u_{1\theta}} = + \frac{ds}{\sqrt{u^2_{1R} + u^2_{1\theta}}} = \frac{ds}{v(r,s)} . \tag{11.14}$$

The stream-line $r = 1$ is a limiting case and will be treated separately. s is negative for $\theta \epsilon [\pi, \pi/2]$ and positive in the aft part. Incidentally, all known functions of R, θ may now be expressed in terms of the stream-line coordinates (r, s). In the following we will assume this to have been done.

The equation for σ_1 now reads,

$$\frac{d\sigma_1}{ds} = \frac{1}{1+\frac{1}{\alpha}} \left[\epsilon\, T(r,s) - (\epsilon + \chi_1(r,s))\sigma_1 \right] \frac{1}{v(r,s)} \cdot \qquad (11.15)$$

In (11.15), r is merely a parameter. The boundary condition is $\sigma_1 \to 1$ as $s \to -\infty$. For $r > 1$ $v(r,s) > 0$ and $\epsilon + \chi_1(r,s) > 0$ for $s < 0$. $\epsilon + \chi_1$ may be positive or negative for $s > 0$.

Introducing the notations :

$$\frac{\epsilon + \chi_1}{v(r,s)(1+\frac{1}{\alpha})} = H(r,s) \quad , \qquad \frac{\epsilon\, T(r,s)}{v(r,s)(1+\frac{1}{\alpha})} = G(r,s) \, , \qquad (11.16)$$

the solution for $r > 1$ is given explicitly by the following expressions:

$$\sigma_1 e^{\int_0^s H(r,t)dt} = \int_0^s G(r,t) e^{\int_0^t H(r,u)du} dt + F(r) \, , \qquad (11.17)$$

where $F(r)$, a function of r alone is found from the boundary condition to be given by the following expression:

$$F(r) = \int_0^\infty G(r,\chi) e^{-\int_0^\chi H(r,-u)du} d\chi \; . \qquad (11.18)$$

The integral exists as a positive number for any $r > 1$. The solution is known for $s > 0$ and $\rightarrow 1$ as $s \rightarrow +\infty$. $\frac{1}{F(r)}$ is the value of ρ_1 at $(r, \pi/2)$.

There remains the case of the limiting stream-line. This is done in three steps. We integrate first from upstream to the front stagnation point. Using the stagnation pressure so determined, we integrate around the sphere. Finally, we continue the solution downstream. The solution for the first step is done most simply by considering the first and the third equations of (11.14); with $\theta = \pi$. We therefore have,

$$\frac{d\sigma_1}{dR} + H(R)\sigma_1 = G(R) \quad ; \tag{11.19}$$

$$H(R) = \frac{-(\epsilon - \frac{B}{R^2})}{(1 + \frac{B}{R} + \frac{C}{R^2})} \frac{1}{1 + \frac{1}{\alpha}} \quad ; \tag{11.20a}$$

$$G(R) = - \frac{\epsilon T(R, \pi)}{(1 + \frac{1}{\alpha})(1 + \frac{B}{R} + \frac{C}{R^2})} \quad . \tag{11.20b}$$

The solution may be shown to be,

$$\sigma_1(R) W(R) = \int_R^{\infty} G(t) W(t) \, dt \quad ; \tag{11.21}$$

where

$$W'(R) = H(R) W(R) \, . \tag{11.22}$$

We find $\sigma_1(\infty) = 1$ and

$$\sigma_1(1) = \frac{T(1,\pi)}{1-\frac{B}{\epsilon}} > 0 \; . \tag{11.23}$$

The solution from the stagnation point at front to the aft stagnation point is obtained by considering,

$$\frac{R\,d\theta}{u_{1\theta}} = \frac{d\sigma_1}{(\epsilon\,T - \epsilon\,\sigma_1 - \chi_1\sigma_1)\frac{1}{1+\frac{1}{\alpha}}} \; ; \tag{11.24}$$

where $R = 1$ and

$$\sigma_1(\pi) = \frac{T(1,\pi)}{1-\frac{B}{\epsilon}} \; , \tag{11.25}$$

$$\frac{d\sigma_1}{d\theta} - \sigma_1 F(\theta) = -\epsilon\, Q(\theta) \; ; \tag{11.26}$$

where

$$F(\theta) = \frac{\epsilon + B\cos\theta}{(1+\frac{1}{\alpha})U_1\sin\theta} \; ; \qquad Q(\theta) = \frac{T(1,\theta)}{(1+\frac{1}{\alpha})U_1\sin\theta} \; ; \tag{11.27}$$

where

$$-u_{1\theta} = U_1\sin\theta \qquad ; \qquad U_1 = \frac{3\tau(1+\alpha)}{(2+3\alpha)+\tau(4+5\alpha)} \, . \tag{11.28}$$

Setting $x = \cos\theta$, the solution may be written explicitly.

$$\sigma_1 \exp\left[\frac{1}{2U_1(1+\frac{1}{\alpha})}\left\{\epsilon\log\left(\frac{1+x}{1-x}\right) - B\log(1-x^2)\right\}\right] = \frac{\epsilon}{U_1(1+\frac{1}{\alpha})}\int_{-1}^{x}\frac{T(t)}{1-t^2}\,e^{\zeta(t)}\,dt \, . \tag{11.29}$$

$$\zeta'(t) = \frac{1}{U_1(1+\frac{1}{\alpha})}\left(\frac{\epsilon + Bt}{1-t^2}\right) . \tag{11.30}$$

The integral is convergent at both ends.

$\sigma_1 \to \infty$ at $x = 1$ but is otherwise > 0. From the equation for ρ, we see that we must put $\rho \equiv 0$ for the stream-line from the rear-stagnation point to downstream to fit this solution. This completes the calculation of the lowest order density.

The limit of this solution for the no-slip case is discussed in Appendix III. Let us now discuss the qualitative features of this first iterate. For one thing, the velocity field is identical to the Lagerstrom-Chester solution with slip. The density field however, is quite different. It is always positive (except on the stream-line through the rear stagnation point where it is zero). It is finite over the entire flow field. For small ϵ the stagnation pressure is $O(\frac{1}{\epsilon})$. The dependence of the density on ϵ is quite complicated. The density also possesses a complicated wake structure. At infinity, this density tends to 1 as $\frac{1}{R^2}$.

This approximate solution must be matched with an appropriate solution of the Oseen equations. Such a matching will be carried out in Chapter VI. To compare this solution with experiment, we estimate the drag. From the form of the density on the sphere, it is clear that there will be some form drag due to $\epsilon\rho$ term in addition to the viscous skin drag due to the stresses.

§12. The skin drag provides a lower estimate on the total drag. The form drag due to the pressure in the slip case appears to be of lower order than the skin drag. The skin friction is estimated as follows.

$$\nabla \cdot \overline{\tau} = 0 \,, \tag{12.1}$$

where

$$\overline{\tau} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \left(\frac{1}{\alpha} - 1\right) \nabla \cdot \vec{u}\, \delta_{ij} \,. \tag{12.2}$$

Defining

$$\tau'_{RR} = \frac{2\partial u_R}{\partial R} \quad ; \tag{12.3}$$

$$\tau'_{R\theta} = \left(\frac{1}{R}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right) ; \tag{12.4}$$

$$\nabla \cdot \vec{u} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, u_\theta) \,. \tag{12.5}$$

From (12.1) it follows that,

$$\text{Drag} = \int_S \left(\frac{1}{\alpha} - 1\right) \nabla \cdot \vec{u} \cos\theta \, dS + \left(\tau'_{RR}\cos\theta - \tau'_{R\theta}\sin\theta\right) dS \,, \tag{12.6}$$

where S is a sphere of very large radius. From (12.6), using (11.7), (11.8), (11.10), (11.11) we obtain,

$$D = -4\pi B \,,$$

$$= \frac{12\pi(1 + \tau)(\alpha + 1)}{(2 + 3\alpha) + \tau(4 + 5\alpha)} \,. \tag{12.6}$$

This corresponds to the Stokes drag on a sphere of unit radius when the velocity at infinity is $-\frac{2B}{3}$. From the correlation theorems of §9 we see that the drag arises entirely from the solenoidal field $\vec{v}$ and that $\nabla\phi$, the irrotational component does not contribute to the drag, even though it does contribute to the stresses.

Putting back dimensions, the drag force on a sphere of radius a, is

$$D = \frac{12\pi(1+\tau)(\alpha+1)}{(2+3\alpha)+\tau(4+5\alpha)} \; (Ua\mu) \; . \qquad (12.7)$$

The drag coefficient is,

$$C_D = \frac{D}{\frac{1}{2}\rho u^2 a^2} = \left(\frac{12\pi(1+\tau)(\alpha+1)}{(2+3\alpha)+\tau(4+5\alpha)}\right)\left(\frac{2}{Re}\right) , \qquad (12.8)$$

where $Re, M^2 << 1$ and $\frac{Re}{\gamma M^2} << 1$.

Now, kinetic theory and experiments [2], require that the drag coefficient of a body in this regime be independent of the Reynolds number and vary with M like $\frac{Const.}{M}$. Equation (12.8) does not agree with this result for any positive value of τ since $\frac{24\pi(1+\tau)(\alpha+1)}{2+3\alpha+\tau(4+5\alpha)}$ varies between two constants for any positive τ.

When $\frac{Re}{\gamma M^2}$ is large, the continuum theory may be expected to apply. When $\frac{Re}{\gamma M^2}$ is large <u>and</u> Kn is small, Basset [5] has given the following formula for the drag coefficient:

$$C_D = \frac{12\pi}{Re} \left(\frac{1+\tau}{1+2\tau}\right) . \qquad (12.9)$$

This formula agrees with experiment when Kn is small and τ is taken proportional to Kn. However, if one extrapolates it for moderate or large Kn, keeping $\frac{Re}{\gamma M^2}$ large, the agreement no longer exists. The formula (12.9) is also derived in a systematic way under more general conditions in [10].

Since density does not appear in (12.7), it would seem that we have a finite drag on the sphere as the gas became increasingly rarified. This of course contradicts facts as well as physical intuition. Goldberg [9] refers to this as an inherent paradox of the Navier-Stokes theory. In the next chapter we will show how to use the Navier-Stokes equations as they stand and yet obtain drag coefficients that are in qualitative agreement with the experimental results by the use of a new slip condition. We note here that another consequence of the theory is a high stagnation pressure at the forward stagnation point. This certainly appears to contradict physical intuition about flow of a rarified gas past a sphere.

Chapter IV

A New Slip Condition

§13. In this rather long chapter, we propose to discuss a general theory of low Reynolds number flows presumably valid for 'arbitrary' Knudsen numbers. We noted that for a particular class of flows, the Navier-Stokes equations and the classical slip condition imply drag coefficients in disagreement with experiment. We have two alternative hypotheses before us: i) the Navier-Stokes equations are inapplicable to all flows where $Kn << 1$ is not satisfied, whatever be the boundary condition, ii) the Navier-Stokes equations may be used so long as the mean free path is of a larger order than the molecular dimensions (we are explicitly discussing gases) of the fluid molecules which are also assumed to be much smaller than the finite body immersed in the flow. Up to the present time it has been assumed that i) is correct. It is one of the aims of this thesis to question this position and to suggest that hypothesis ii) may in fact be valid. It is clear that such a suggestion must, in order to be in harmony with established experimental facts, supply some boundary condition different from the slip condition previously used. We shall discuss such a condition and endeavour to show that hypothesis ii) does lead to results which are in agreement with facts.

§14. The objections to a Navier-Stokes theory for moderate or large Knudsen numbers are two-fold. First, we may discuss the à priori objection that for moderate Knudsen numbers, the medium is no longer a continuum and therefore a microscopic theory should be used. It is clear however that a gas, say air, does not lose its overall continuum properties if we immerse in it an object (say, a Millikan oil drop) whose dimensions are much larger than that of a gas molecule but is of the same order or even smaller than the mean free path of the gas. To be sure, the flow near the object will differ a great deal from the case when the Knudsen number is $<< 1$. The difference consists in the different interaction of the gas molecules with the surface of the object in the two cases. In other words, the presence of the boundary will affect the gas differently in the two cases. The Navier-Stokes equations describe the mean motion of the gas adequately if we do not look for microscopic fluctuations. These microscopic fluctuations will be small in a gas so long as the number of molecules in the microscopic volume we look at is large. The 'microscopic' volumes therefore have to be much larger than molecular dimensions but should be much smaller than the dimensions of the finite body. This is possible in many cases and for such cases it is difficult to see why i) should be true à priori. The calculations using Navier-Stokes equations in a tremendous variety of situations have been remarkably accurate. In fact, (see [2] p. 32) the Navier-Stokes <u>approximation</u> with suitable boundary conditions appears to be much better than higher moment theories of the

Boltzmann equation. This may be due to the fact that Navier-Stokes equations may be derived independently of any molecular theory from laws of conservation of mass, momentum, energy, the laws of thermodynamics, isotropy and homogeniety of space and the additional hypothesis of linear stress-rate of strain relations. In such a derivation, there is always an implicit neglect of fluctuations. We have seen above that moderate or not too high Knudsen number flows do not contradict such an assumption.

The second objection to the Navier-Stokes theory is that the results it gives with the standard slip condition do not agree with experiment. In fact, this objection must be qualified. For flat plates (see [2], [3]) the agreement is good. For spheres, as we have seen, the agreement is very poor. The case of the flat plates has been dismissed as a 'coincidence.' This may be true. The aim of this investigation is to show that it may not be a coincidence or at least, if it is one, then it is a very striking one.

§15. In this section, we discuss the modifications in the slip condition. The proposals here are in no sense a derivation from some fundamental theory. They are more in the nature of phenomenological statements. They must stand or fall according as they agree with or contradict existing experimental evidence. In addition we expect to make new predictions which may be experimentally tested.

Consider first a general formulation of the slip condition of the preceding chapters. The Navier-Stokes stress tensor (in dimensional variables)

$$S_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(-p + \lambda \frac{\partial u_k}{\partial x_k}\right) \delta_{ij} . \tag{15.1}$$

Consider an elementary area dA on the boundary with normal n_i drawn into the fluid.

$S_{ij} n_j = F_i$ is the total stress vector at dA . (15.2)

From F_i one constructs the tangential stress vector G_i through

$$G_i = F_i - (F_j n_j) n_i . \tag{15.3}$$

The boundary condition (8.1) is therefore stated in the general form

$$\beta u_i(A) = G_i . \tag{15.4}$$

G_i is not linear in n_i and β has the dimensions $\frac{\mu}{L}$. For flat surfaces, with $u_i n_i = 0$,

$$u_i \propto \frac{\partial u_i}{\partial n} .$$

From this case, it follows that β should be positive to be physically meaningful. As we have seen, this condition agrees with experiment for flat plates only, and many 'kinetic' derivations of it apply to the flat plate case.

We now notice that (15.4) is not the simplest boundary condition that can be stated in terms of u_i, its first partial derivatives and n_i, which would give correct results for flat plates.

Consider, for example,

$$\frac{\vec{u}(A)}{L(A)} = (\text{curl } \vec{u}) \times \vec{n} . \qquad (15.5)$$

Here L(A) is some length which may vary from point to point on the boundary. For $L(A) > 0$, we have from (15.5) that $\vec{u}(A) \cdot \vec{n} = 0$. For flat plates we have

$$u_t \propto \frac{\partial u_t}{\partial n}$$

as before.

The fact that $(\text{curl } \vec{u}) \times \vec{n}$ vanishes for irrotational flow implies that for $L > 0$, the vorticity and the velocity must vanish simultaneously on the surface. For $L \to \infty$, $\text{curl } \vec{u} \to 0$ on the boundary. For $L \to 0$, the no-slip condition is recovered. A consideration of the following theorem shows that the vortex traction $\vec{T} = (\text{curl } \vec{u}) \times \vec{n}$ arises quite naturally.

Theorem: Consider an incompressible viscous flow past a finite object S such that the velocities $\to 0$ sufficiently rapidly at infinity and (15.5) holds on S. The kinetic energy is a monotone decreasing function of time if $\vec{\omega} \not\equiv 0$.

Proof: The governing equations are

$$\nabla \cdot \vec{u} = 0 , \tag{15.6}$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} = -\nabla \sigma - \mu \operatorname{curl} \vec{\omega} . \tag{15.7}$$

Taking a dot product of (15.7) with $\vec{u}$ and integrating over the infinite volume of fluid,

$$\frac{\partial}{\partial t} \int_V \frac{\vec{u}^2}{2} dv = -\mu \int_V \vec{u} \cdot \operatorname{curl} \vec{\omega} \; dv . \tag{15.8}$$

Now,

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \operatorname{curl} \vec{A} - \vec{A} \cdot \operatorname{curl} \vec{B} . \tag{15.9}$$

Hence,

$$\frac{\partial}{\partial t} \int_V \frac{\vec{u}^2}{2} dv = + \mu \int_V \nabla \cdot (\vec{u} \times \vec{\omega}) dv - \mu \int_V \vec{\omega}^2 dv , \tag{15.10}$$

$$= - \mu \int_S (\vec{u} \times \vec{\omega}) \cdot \vec{n} \, ds - \mu \int_V \vec{\omega}^2 dv , \tag{15.11}$$

where the surface integral over the infinite sphere vanishes by hypothesis and $\vec{n}$ is the normal drawn into the fluid. Hence from (15.5),

$$\frac{\partial}{\partial t} \int \frac{\vec{u}^2}{2} dv = - \mu \int_S L(s) \, \vec{u}^2 ds - \mu \int_V \vec{\omega}^2 dv . \tag{15.12}$$

The theorem is proved.

Cor: From (15.11) it follows that if the Navier-Stokes equations (15.6), (15.7) have an irrotational solution with $\frac{\partial \phi}{\partial n} = 0$ on S and satisfying the conditions of the theorem at infinity, the kinetic energy is conserved. In fact, for irrotational solutions of the equations (15.6), (15.7) the following conservation law holds.

$$\frac{\partial}{\partial t}\left(\frac{\vec{u}^{\,2}}{2}\right) + \nabla \cdot \left(\vec{u}\left(\frac{\vec{u}^{\,2}}{2} + p\right)\right) = 0 \; . \qquad (15.13)$$

The slip boundary condition may be formulated in general as follows.

$$\vec{u}(A) = k_1 \vec{T} + k_2' \vec{G} + k_3\left(\nabla T - \vec{n}\,\frac{\partial T}{\partial n}\right) \cdots \qquad (15.14)$$

where k_1, k_2', k_3 are surface parameters. $\vec{T} = \vec{G}$ if $\vec{u} = 0$. (This is proved, for example, in [1] p. 46). In general, they will be linearly independent. The last term describes a phenomenon known as 'thermal creep.' Similar conditions apply to temperature. For flat plates, the two different vectors $\vec{T}$ and $\vec{G}$ give the same results.

In the following, we shall not consider the most general condition (15.14) but propose to discuss the consequences of the one parameter theory given by,

$$\vec{u}(A) = k_1 \vec{T} \; . \qquad (15.15)$$

One reason for this specialization is that the theory with as few arbitrary parameters as possible is not only simpler to handle mathematically but is also adapted very well to experimental testing. If necessary, other refinements could be added on as corrections. Once and for all, we start with (15.15).

We noted that k_1 had the dimension of a length. If u be a typical velocity and L a typical radius of curvature of the body, the vorticity is of the general order $\frac{u}{L}$ near the body. The length k_1 is a sort of interaction length between the surface and the fluid molecules. Its general order depends on the mean free path of the gas. Rarer the gas, the more difficult it is for the body to transfer vorticity into the fluid. We may therefore write,

$$\frac{\vec{u}(A)}{\bar{\lambda}} = k_1(\text{curl}\,\vec{u}) \times \vec{n} \tag{15.16}$$

where k_1 is a surface roughness parameter and $\bar{\lambda}$ is the mean free path. k_1 is now non-dimensional. In non-dimensional variables,

$$\vec{u}^*(A) = k_1 \frac{M}{Re} (\text{curl}^* \vec{u}^*) \times \vec{n}. \tag{15.17}$$

k_1 here is essentially arbitrary but it may in fact be some function (depending on the shape of the body) of the Knudsen number $\frac{M}{Re} = Kn$. It is clear that for small Knudsen numbers we have Stokes flow. For large Knudsen numbers, the flow is one of complete slipping and unlike the classical slip condition we obtain irrotational flow as the limit. The explicit dependence of the slip coefficient on Mach number is not necessary but the fact that the Knudsen number enters explicitly in (15.17) will affect the drag coefficient in a very fundamental manner.

The boundary condition on the temperature is the usual one

$$T(A) = T_{WALL}(A) + k_2 \frac{M}{Re} \frac{\partial T}{\partial n} , \tag{15.18}$$

in non-dimensional variables.

We shall proceed to study the solutions (6.8) etc. subject to these conditions, with the general restriction Re << 1.

§16. We first consider the case when Re << 1 and Kn is O(1). The equations are then the following:

$$\nabla \cdot \rho\vec{u} = 0 , \tag{16.1}$$

$$Re\, \rho\vec{u} \cdot \nabla\vec{u} + \frac{1}{\gamma Kn M} \nabla p = \nabla^2 \vec{u} + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{u}) , \tag{16.2}$$

$$Re\, \rho\vec{u} \cdot \nabla T - \frac{(\gamma-1)}{\gamma} Re\, \vec{u} \cdot \nabla p = \frac{1}{Pr} \nabla^2 T + (\gamma-1) M^2 \Phi , \tag{16.3}$$

$$p = \rho T , \tag{16.4}$$

$$\left.\begin{aligned} &\vec{u} = k_1 Kn\, \mathrm{curl}\, \vec{u} \times \vec{n} \quad \text{on S.} \\ &\vec{u} = \vec{i} \quad \text{at } \infty . \\ &\rho, T = 1 \text{ at } \infty . \\ &T = T_W + k_2 Kn \frac{\partial T}{\partial n} \quad \text{on S.} \end{aligned}\right\} \tag{16.5}$$

To solve these equations we assume the formal expansions,

$$\begin{aligned} T &= T_0 + f_1(M)T_1 + f_2(M)T_2 + \cdots , \\ \rho &= \rho_0 + f_1(M)\rho_1 + \cdots , \\ \vec{u} &= \vec{u_0} + f_1(M)\vec{u_1} + \cdots . \end{aligned} \quad (16.6)$$

We shall show only the methods of obtaining the leading terms. These asymptotic expansions are done in just the usual way and no new ideas are needed to calculate in principle the higher approximations. Note that $Re << 1\ Kn \cdot O(1) \Rightarrow M << 1$. $Re = \frac{M}{Kn}$.

Substitution of the expansions gives the following leading order equations.

$$\nabla^2 T_0 = 0 , \quad (16.7)$$

with $T_0 = 1$ at infinity and

$$T_0 = T_W + k_2 Kn \frac{\partial T_0}{\partial n} \quad \text{on} \quad S , \quad (16.8)$$

$$p = 1 + f_1(M)\, p_1 . \quad (16.9)$$

Hence,

$$\rho_0 = \frac{1}{T_0} . \quad (16.10)$$

We then have the following:

$$f_1(M) = KnM \, , \tag{16.11}$$

$$\nabla \cdot \rho_0 \vec{u_0} = 0 \, , \tag{16.12}$$

$$\nabla^2 \vec{u_0} + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{u_0}) = \frac{1}{\gamma} \nabla p_1 \, . \tag{16.13}$$

(16.12), (16.13) are solved subject to the boundary conditions on the velocity since T_0 may be obtained by solving (16.7), (16.8). The equations are linear though their solution may be quite complicated in the general case. These solutions are to be matched with appropriate outer expansions. The next order equations are somewhat long to write out, though they involve no difficulty of principle.

For the following particular case of a slightly heated body, the solution is more manageable. Let

$$T_W = 1 + \theta T_{W_1} + \theta^2 T_{W_2} + \cdots \, , \tag{16.14}$$

where θ is a small parameter. We then expand T_0 in powers of θ.

$$T_0 = 1 + \theta T_0^{(1)} + \theta^2 T_0^{(2)} + \cdots \, . \tag{16.15}$$

$$\therefore \qquad \rho_0 = 1 + \theta \rho_0^{(1)} + \theta^2 \rho_0^{(2)} + \cdots \, . \tag{16.16}$$

(16.12), (16.13) reduce after expanding $\vec{u_0}$

$$\vec{u}_0 = \vec{u}_0^{(0)} + \theta\,\vec{u}_0^{(1)} + \cdots \quad , \tag{16.17}$$

to the Stokes equations:

$$\nabla \cdot \vec{u}_0^{(0)} = 0 \quad , \tag{16.18}$$

$$\nabla^2 \vec{u}_0^{(0)} = \frac{1}{\gamma} \nabla p_1^{(0)} \; , \tag{16.19}$$

with

$$\begin{aligned} \vec{u}_0^{(0)} &= \vec{i} \text{ at } \infty \; , \\ \vec{u}_0^{(0)} &= k_1 \mathrm{Kn}(\mathrm{curl}\, \vec{u}_0^{(0)}) \times \vec{n} \; . \end{aligned} \tag{16.20}$$

These equations are tractable for simple geometries. If a solution exists it must be unique.

<u>Theorem</u>: Let ,

$$\nabla \cdot \vec{u} = 0 \; , \tag{16.21}$$

$$- \mathrm{curl}\,\vec{\omega} = \frac{1}{\gamma} \nabla p \; , \tag{16.22}$$

have a solution $\vec{u} \to 0$ at ∞ sufficiently rapidly and let $\vec{u} = \beta(\mathrm{curl}\vec{u}) \times \vec{n}$ on S. Then $\vec{u} \equiv 0$. $(\beta > 0)$.

<u>Proof</u>:
$$\int_V \vec{u} \cdot \mathrm{curl}\,\vec{\omega}\, dv = 0 \; .$$

$$\therefore \qquad \int_S (\vec{u} \times \vec{\omega}) \cdot \vec{n}\, dS + \int \vec{\omega}^2\, dv = 0 \; . \tag{16.23}$$

This is possible iff

$$\vec{\omega} \equiv 0 \quad \text{and} \quad \vec{u} = 0 \text{ on } S \,. \tag{16.24}$$

This implies for sufficiently smooth S, $\vec{u} \equiv 0$.

We shall give explicit solutions to some of these problems for certain simple bodies in the next chapter. The case of the heated body where $T_W > 1$ is of interest but the equations are quite complicated.

The next case we consider is when $Re << 1$ $\frac{Re}{\gamma M^2} = \epsilon$ fixed. Here $Kn = \frac{M}{Re}$ is $O(\frac{1}{\epsilon M})$ and $M << 1$. We write,

$$\vec{u} = \vec{u}_0 + M\vec{u}_1 + \cdots \,, \tag{16.25}$$

$$T = T_0 + MT_1 + \cdots \,, \tag{16.26}$$

$$\rho = \rho_0 + M\rho_1 + \cdots \,. \tag{16.27}$$

We substitute these expansions into (note $Re = O(\epsilon M^2)$).

$$\nabla \cdot \rho\vec{u} = 0 \,, \tag{16.28}$$

$$Re\, \rho\vec{u} \cdot \nabla\vec{u} + \epsilon\nabla p = \nabla^2\vec{u} + \frac{1}{\alpha}\nabla(\nabla \cdot \vec{u}) \,, \tag{16.29}$$

$$Re\, \rho\vec{u} \cdot \nabla T - \left(\frac{\gamma-1}{\gamma}\right) Re\, \vec{u} \cdot \nabla p = \frac{1}{Pr}\nabla^2 T + (\gamma-1)M^2\Phi \,, \tag{16.30}$$

$$p = \rho T \,. \tag{16.31}$$

To leading order we have

$$\nabla \cdot \rho_0 \vec{u}_0 = 0 , \qquad (16.32)$$

$$\epsilon \nabla p_0 = \nabla^2 \vec{u}_0 + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{u}_0) , \qquad (16.33)$$

$$\frac{1}{Pr} \nabla^2 T_0 = 0 . \qquad (16.34)$$

The boundary conditions on S reduce to

$$\gamma M \vec{u} = \frac{k_1}{\epsilon} \operatorname{curl} \vec{u} \times \vec{n} , \qquad (16.35)$$

$$(T - T_W) \gamma M = \frac{k_2}{\epsilon} \frac{\partial T}{\partial n} . \qquad (16.36)$$

Hence we have

$$\operatorname{curl} \vec{u}_0 \times \vec{n} = 0 \quad \text{on} \quad S \quad \text{with} \quad \vec{u}_0 \cdot \vec{n} = 0 . \qquad (16.37)$$

We also have

$$\frac{\partial T_0}{\partial n} = 0 \quad \text{on} \quad S . \qquad (16.38)$$

The leading term of the solution is very simple and is written $\rho_0 = 1$, $T_0 = 1$, $p_0 = 1$,

$$\vec{u}_0 = \nabla \phi_0 ; \qquad (16.39)$$

where ϕ_0 is a harmonic function. In fact, $\vec{u_0}$ is the potential flow solution past a finite body with $\vec{u_0} \to \vec{i}$ at infinity.

As the leading order is so trivial (no drag for instance) we calculate the next order. The boundary conditions dictate that we should have an O(M) term. The equations we get are the following:

$$\nabla \cdot (\rho_1 \nabla \phi_0) + \nabla \cdot \vec{u_1} = 0 , \tag{16.40}$$

$$\nabla^2 \vec{u_1} + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{u_1}) = \epsilon \nabla p_1 , \tag{16.41}$$

$$\frac{1}{Pr} \nabla^2 T_1 = 0 , \tag{16.42}$$

$$p_1 = \rho_1 + T_1 , \tag{16.43}$$

$$\gamma \vec{u_0} = \frac{k_1}{\epsilon} \operatorname{curl} \vec{u_1} \times \vec{n} , \tag{16.44}$$

$$(1 - T_W)\gamma = \frac{k_2}{\epsilon} \frac{\partial T_1}{\partial n} , \tag{16.45}$$

$$\vec{u_1},\ T_1,\ \text{etc} \to 0 \text{ at } \infty . \tag{16.46}$$

In fact, (16.46) may have to be replaced by a matching condition; the T_1 equation is a standard problem in harmonic functions. We may eliminate p_1 from (16.40), (16.41) and attempt to solve for $\vec{u_1}$ and ρ_1.

We split

$$\vec{u}_1 = \vec{v}_1 + \nabla \phi_1 , \tag{16.47}$$

where we put as usual

$$\nabla \cdot \vec{v}_1 = 0 . \tag{16.48}$$

Substitution gives the following:

$$\nabla^2 \vec{v}_1 = \nabla \chi_1 , \tag{16.49}$$

$$\chi_1 + (1 + \frac{1}{\alpha}) \nabla^2 \phi_1 = \epsilon (\rho_1 + T_1) , \tag{16.50}$$

$$\nabla \phi_0 \cdot \nabla \rho_1 + \nabla^2 \phi_1 = 0 . \tag{16.51}$$

The problem for $\vec{v}_1$ is a Stokes problem - with

$$\begin{aligned} \vec{v}_1 \cdot \vec{n} &= 0 \text{ on } S , \\ \epsilon \gamma \vec{u}_0 &= k_1 \operatorname{curl} \vec{v}_1 \times \vec{n} , \end{aligned} \tag{16.52}$$

with $\vec{v}_1 \to 0$ at infinity. This gives χ_1. We eliminate $\nabla^2 \phi_1$ from (16.51) using (16.50). We obtain the linear first order p.d.e.

$$\nabla \phi_0 \cdot \nabla \rho_1 + \left\{ \epsilon (\rho_1 + T_1) - \chi_1 \right\} \frac{1}{1 + \frac{1}{\alpha}} = 0 . \tag{16.53}$$

The particular integral of this equation $\to 0$ at ∞ is found. We then solve (16.50) with $\frac{\partial\phi}{\partial n} = 0$ on S and $\nabla\phi \to 0$ at ∞. From an order of magnitude estimate, we see that the drag coefficient must be of the form, (to leading order)

$$C_D = \text{const} \cdot \frac{\epsilon M}{k_1 Re} \quad \text{i.e.} \ \frac{\text{const.}}{k_1 M} , \qquad (16.54)$$

where the constant depends on the shape of the body. These results replace the results of the previous chapter where we found $C_D \sim \frac{\text{const}}{Re}$.

§17. As a final case we consider the following situation. $Re << 1$ M is not necessarily small. This is a somewhat academic case as the flow is at the extreme free molecular limit. It is interesting however, to see just what results the equations of fluid mechanics and our slip condition predict in this case. Here, we have the small parameter Re and accordingly our asymptotic expansions are in terms of Re.

$$\rho = \rho_0 + f_1(Re)\rho_1 + \cdots , \qquad (17.1)$$

$$\vec{u} = \vec{u_0} + f_1(Re)\,\vec{u_1} + \cdots , \qquad (17.2)$$

$$T = T_0 + f_1(Re)T_1 + \cdots . \qquad (17.3)$$

Substitution into the equations of fluid mechanics gives, (to leading order),

$$\nabla \cdot \rho_0 \vec{u}_0 = 0 , \tag{17.4}$$

$$\nabla^2 \vec{u}_0 + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{u}_0) = 0 , \tag{17.5}$$

$$\nabla \cdot (\frac{1}{Pr} \nabla T_0) + (\gamma - 1) M^2 \Phi_0 = 0 ; \tag{17.6}$$

$$\left.\begin{aligned} \text{curl } \vec{u}_0 \times \vec{n} &= 0 \quad \text{on} \quad S ; \\ \vec{u}_0 \cdot \vec{n} &= 0 \quad \text{on} \quad S ; \end{aligned}\right. \tag{17.8}$$

$$\left.\begin{aligned} &T_0 = T_W + \frac{M}{Re} k_2 \frac{\partial T}{\partial n} , \\ &\text{implies } \frac{\partial T}{\partial n} = 0 \text{ on } S . \end{aligned}\right. \tag{17.9}$$

The condition at infinity must be determined (as also the asymptotic sequence) from matching as usual with suitable outer solutions.

We note that the nonlinear (only formally nonlinear) equations (17.4), (17.5) may be solved for $\vec{u}_0 = \vec{i}$ at ∞ and (17.7), (17.8) in the following manner. Set

$$\vec{u}_0 = \nabla \phi_0 , \tag{17.10}$$

$$\rho_0 = 1 . \tag{17.11}$$

The equations and the boundary conditions are satisfied if

$$\nabla^2 \phi_0 = 0 , \tag{17.12}$$

with,

$$\nabla\phi_0 \rightarrow \vec{i} \text{ at } \infty \text{ and } \frac{\partial\phi_0}{\partial n} = 0 . \tag{17.13}$$

But this is the classic irrotational, incompressible flow of a uniform stream past the finite body S. One then has the Poisson equation (17.6) with an adiabatic condition at the wall. We may impose the condition $T_0 \rightarrow 1$ at infinity. Since the Mach number is no longer negligible, we must take into account the heat produced by the stresses in the fluid. This is precisely why we have a source term in (17.6).

If we set $f_1(\mathrm{Re}) = \mathrm{Re}$, we obtain the next set of equations.

$$\nabla \cdot \vec{u_1} + \nabla\rho_1 \cdot \nabla\phi_0 = 0 , \tag{17.14}$$

$$\nabla\left(\frac{\nabla\phi_0{}^2}{2} + \frac{T_0}{\gamma M^2}\right) = \nabla^2\vec{u_1} + \frac{1}{\alpha}\nabla(\nabla\cdot\vec{u_1}) , \tag{17.15}$$

with,

$$\vec{u_1}\cdot\vec{n} = 0 , \tag{17.16}$$

$$\vec{u_0} = k_1 M \text{ curl } \vec{u_1} \times \vec{n} , \tag{17.17}$$

$$\nabla\phi_0 \cdot \nabla T_0 - \frac{(\gamma-1)}{\gamma}\nabla\phi_0 \cdot \nabla T_0 = \frac{1}{\mathrm{Pr}}\nabla^2 T_1 + (\gamma-1)M^2\Phi_1 , \tag{17.18}$$

$$T_0 - T_W = Mk_2 \frac{\partial T_1}{\partial n} \cdot \qquad (17.19)$$

The velocity-density equations can be solved simply in the following way. If

$$\vec{u_1} = \vec{v_1} + \nabla \phi_1 , \qquad (17.20)$$

such that

$$\nabla \cdot \vec{v_1} = 0 , \qquad (17.21)$$

then,

$$\nabla^2 \vec{v_1} = \nabla \chi_1 ; \qquad (17.22)$$

with,

$$\frac{\nabla \phi_0^2}{2} + \frac{T_0}{\gamma M^2} = \chi_1 + (1 + \frac{1}{\alpha}) \nabla^2 \phi_1 , \qquad (17.23)$$

$$\nabla \rho_1 \cdot \nabla \phi_0 + \nabla^2 \phi_1 = 0 \cdot \qquad (17.24)$$

Here, χ_1 is a harmonic function with a constant value at infinity given by $\frac{1}{2} + \frac{1}{\gamma M^2}$. It tends to this value like $\frac{1}{R^2}$.

Now we solve the Stokes problem for $\vec{v_1}$ with the conditions

$$\vec{v_1} \to 0 \text{ at } \infty . \qquad (17.25)$$

$$\vec{v_1} \cdot \vec{n} = 0 \text{ on S and } \vec{u_0} = k_1 M \text{ curl } \vec{v_1} \times \vec{n} . \qquad (17.26)$$

This gives χ_1. However, in equation (17.23), $\nabla\phi_1$ may not in general $\rightarrow 0$ at ∞ but to a constant. The outer solution must be found to match this constant. This constant arises from the fact that our perturbation expansion is not uniformly valid at infinity and if the left hand side of (17.23) $\rightarrow 0$ at infinity like $\frac{1}{R}$. There is no solution which will make $\nabla\phi_1 \rightarrow 0$ at ∞. This is the analogue of Whitehead's paradox and it is solved in the usual way. We now solve the thermal equation (17.18) using the fields already found. The density may be found independently of the velocity from (17.14) and (17.23), and it too will in general have non-uniform behaviour at infinity. Certain difficulties also arise for R = 1. (See § 25). This concludes the discussion of the inner equations of these small Reynolds number flows. In the next chapter we will consider the relatively simpler problem of the finding the outer solutions corresponding to these problems and the matching to leading order. It must be mentioned however that the simplicity is only in the framework which reduces to solving classical types of equations. These equations are linear but sometimes their solutions can be very complicated. Only the simplest concrete problems are dealt with in details but the general framework is given.

Chapter V

Formal Theory of the Oseen Equations

§18. In this chapter we discuss methods for finding appropriate outer solutions to the Navier-Stokes equations. These solutions must then be matched with the inner solutions in the usual manner to obtain uniformly valid solutions. For information concerning the Oseen equations the following references are useful: [1], [6], [10], [11], and [12]. We shall only indicate the methods for constructing the leading order terms or terms at most one order higher than the leading order. Many of the formal techniques used here are described in [13]. The notation of generalized functions will be systematically exploited in connection with certain fundamental singular solutions. The use of these functions can be made quite rigorous. (See, for instance, Chapter II in [14]).

§19. When we consider the outer problem, we are confronted with the following physical situation. We have a uniform stream in which a very small object is immersed. The perturbation of the flow due to the object will depend on the size of the object. Moreover, beyond a certain distance, the inertia terms are important in the equation. With these observations in mind, we look at the equations of §16 in terms of a new space variable. Define,

$$\vec{\tilde{x}} = \mathrm{Re}\,\vec{x}. \qquad (19.1)$$

The tilde will henceforth denote Oseen variables. If $F(\vec{x})$ is any function of $\vec{x}$, we define,

$$F(\vec{x}) = \tilde{F}(\tilde{x}). \qquad (19.2)$$

In terms of this new variable, the typical linear dimension of the body is O(Re). This of course implies that the body is a small object in the uniform stream. In lieu of equations (16.1) etc. we have the following:

$$\tilde{\nabla} \cdot \tilde{\rho} \vec{\tilde{u}} = 0 , \qquad (19.3)$$

$$\tilde{\rho} \vec{\tilde{u}} \cdot \tilde{\nabla} \vec{\tilde{u}} + \frac{1}{\gamma M^2} \tilde{\nabla} \tilde{p} = \tilde{\nabla}^2 \vec{\tilde{u}} + \frac{1}{\alpha} \tilde{\nabla} (\tilde{\nabla} \cdot \vec{\tilde{u}}), \qquad (19.4)$$

$$\tilde{\rho} \vec{\tilde{u}} \cdot \tilde{\nabla} \tilde{T} - \frac{(\gamma-1)}{\gamma} \vec{\tilde{u}} \cdot \tilde{\nabla} \tilde{p} = \frac{1}{Pr} \tilde{\nabla}^2 \tilde{T} + (\gamma-1) M^2 \tilde{\Phi} , \qquad (19.5)$$

$$\tilde{p} = \tilde{\rho} \tilde{T}. \qquad (19.6)$$

The boundary conditions at infinity are the free stream conditions:

$$\vec{\tilde{u}} = \vec{i} , \qquad (19.7)$$

$$\tilde{\rho} = 1 , \qquad (19.8)$$

$$\tilde{T} = 1. \qquad (19.9)$$

In fact, these values actually satisfy the equations. The typical dimension of the object is Re and hence one expects the solution of these equations to have an asymptotic expansion with the small parameter Re. We assume therefore,

$$\vec{\tilde{u}} = \vec{i} + \mathrm{Re}\,\vec{g}_1 + f_2(\mathrm{Re})\,\vec{g}_2 + \cdots, \tag{19.10}$$

$$\tilde{p} = 1 + \mathrm{Re}\,\omega_1 + \cdots, \tag{19.11}$$

$$\tilde{\rho} = 1 + \mathrm{Re}\,S_1 + \cdots, \tag{19.12}$$

$$\tilde{T} = 1 + \mathrm{Re}\,\theta_1 + \cdot \tag{19.13}$$

The Oseen equations for the first order perturbations are:

$$\omega_1 = S_1 + \theta_1, \tag{19.14}$$

$$\frac{\partial S_1}{\partial \tilde{x}} + \tilde{\nabla} \cdot \vec{g}_1 = 0, \tag{19.15}$$

$$\frac{\partial \vec{g}_1}{\partial \tilde{x}} + \frac{1}{\gamma M^2} \tilde{\nabla}\,\omega_1 = \tilde{\nabla}^2 \vec{g}_1 + \frac{1}{\alpha} \tilde{\nabla}(\tilde{\nabla} \cdot \vec{g}_1), \tag{19.16}$$

$$\frac{\partial \theta_1}{\partial \tilde{x}} - \frac{\gamma-1}{\gamma} \frac{\partial \omega_1}{\partial \tilde{x}} = \frac{1}{\mathrm{Pr}} \tilde{\nabla}^2 \theta_1 . \tag{19.17}$$

Our task will be to find solutions for these equations which match up with appropriate inner solutions. Henceforth we will discuss only genuine three-dimensional problems. The special case of cylinders with radii O(Re) will be considered separately in the next chapter. If the inner equations of the previous chapters have solutions going to free stream values, we already have matching O(1). These solutions tend to the free stream values in general as $\frac{1}{R}$. This means that to match them O(Re), we must solve (19.14), (19.15), (19.16), and (19.17) in a suitable manner. We shall indicate how this is done for various cases.

For the first case we have $\frac{M}{Re} = Kn$ is O(1). From (19.16) it follows that $\omega_1 = 0$. If however we write,

$$\tilde{p} = 1 + Re^3 \omega_3 + \cdots , \tag{19.18}$$

the outer equations reduce to

$$S_1 + \theta_1 = 0 , \tag{19.19}$$

$$\frac{\partial S_1}{\partial \tilde{x}} + \tilde{\nabla} \cdot \vec{g}_1 = 0 , \tag{19.20}$$

$$\frac{\partial \vec{g}_1}{\partial \tilde{x}} + \frac{1}{\gamma} \tilde{\nabla} \omega_3 = \tilde{\nabla}^2 \vec{g}_1 + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{g}_1) , \tag{19.21}$$

$$\frac{\partial \theta_1}{\partial \tilde{x}} - \frac{\gamma - 1}{\gamma} Re^2 \frac{\partial \omega_3}{\partial \tilde{x}} = \frac{1}{Pr} \tilde{\nabla}^2 \theta_1 , \text{ i.e. } \frac{\partial \theta_1}{\partial \tilde{x}} = \frac{1}{Pr} \tilde{\nabla}^2 \theta_1 . \tag{19.22}$$

These equations correspond to (16.10), (16.12), (16.13) and (16.7) respectively. They ultimately reduce to

$$\frac{\partial \theta_1}{\partial \tilde{x}} = \frac{1}{Pr} \tilde{\nabla}^2 \theta_1 , \tag{19.23}$$

$$\tilde{\nabla} \cdot \vec{g}_1 - \frac{\partial \theta_1}{\partial \tilde{x}} = 0 , \qquad (19.24)$$

$$\frac{\partial \vec{g}_1}{\partial \tilde{x}} + \frac{1}{\gamma} \tilde{\nabla} \omega_3 = \tilde{\nabla}^2 \vec{g}_1 + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{g}_1) . \qquad (19.25)$$

The method of matching should now be intuitively clear. Certain solutions of these equations written in terms of R will look like the expansion of the inner solutions for large R. Arbitrary constants in the solutions will be found by the matching principle. [13]. We now introduce the fundamental solutions of the linear equations (19.23), (19.24), (19.25). These are immediate generalisations of potentials due to point charges in Potential Theory [15]. These fundamental solutions are solutions of the following equations:

$$\tilde{\nabla} \cdot \vec{g}_1 - \frac{\partial \theta_1}{\partial \tilde{x}} = 0 , \qquad (19.26)$$

$$\frac{\partial \vec{g}_1}{\partial \tilde{x}} + \frac{1}{\gamma} \tilde{\nabla} \omega_3 = \vec{a}\ \delta(\vec{\tilde{x}}) + \tilde{\nabla}^2 \vec{g}_1 + \frac{1}{\alpha} \tilde{\nabla}(\tilde{\nabla} \cdot \tilde{g}_1) , \qquad (19.27)$$

$$\frac{\partial \theta_1}{\partial \tilde{x}} = C\, \delta(\vec{\tilde{x}}) + \frac{1}{Pr} \tilde{\nabla}^2 \theta_1 , \qquad (19.28)$$

where $\delta(\vec{\tilde{x}})$ is the 3-dimensional delta function and $\vec{a}$, is a constant vector and C a constant number. If we utilize the interpretation of the equations as conservation laws, we see that these equations describe the response (to the highest order) of a free stream subjected to a concentrated volume force Re $\vec{a}\,\delta(\vec{x})$ and a heat source ReC $\delta(\vec{x})$.

The component of $-\vec{a}$ parallel to $\vec{i}$ is known as the drag and the perpendicular component is the lift. Physically we are replacing the tiny body by a point force on the fluid and are describing the fact that it is heated by a point thermal source. Mathematically, these are the simplest singularities one may put in. We shall get other singular solutions by the principle of superposition in the same way one obtains multipole potentials from monopole potentials by differentiation.

First, let us specialize to the case of axisymmetric flow for which the lift is zero by symmetry. The singular force is then $F\vec{i}\ \delta(\vec{\tilde{x}})$. We solve the equations by splitting the velocity and the force fields into their scalar and vector potentials.

We have then,

$$\vec{i}\ \delta(\vec{\tilde{x}}) = \tilde{\nabla} \times \vec{f} + \tilde{\nabla}\phi \ ; \tag{19.29}$$

where,

$$\vec{f} = \tilde{\nabla} \times \frac{\vec{i}}{4\pi\tilde{R}} \qquad ; \qquad \phi = -\frac{\partial}{\partial\tilde{x}}\left(\frac{1}{4\pi\tilde{R}}\right) . \tag{19.30}$$

(This decomposition is unique up to an arbitrary gradient in $\vec{f}$). The last equations may immediately be verified on noting that

$$\tilde{\nabla}^2 \left(\frac{1}{4\pi\tilde{R}}\right) = -\delta(\vec{\tilde{x}}) . \tag{19.31}$$

Set,

$$\overrightarrow{g_1} = \widetilde{\nabla} \times \overrightarrow{A}_1 + \widetilde{\nabla} \psi_1 . \tag{19.32}$$

We obtain the following set of equations.

$$\widetilde{\nabla}^2 \psi_1 - \frac{\partial \theta_1}{\partial \widetilde{x}} = 0 , \tag{19.33}$$

$$\frac{\partial \psi_1}{\partial \widetilde{x}} + \frac{1}{\gamma} \omega_3 = - \frac{F}{4\pi} \frac{\partial}{\partial \widetilde{x}} (\frac{1}{\widetilde{R}}) + (1 + \frac{1}{\alpha}) \widetilde{\nabla}^2 \psi_1 , \tag{19.34}$$

$$\frac{\partial \overrightarrow{A}_1}{\partial \widetilde{x}} = F \overrightarrow{f} + \widetilde{\nabla}^2 \overrightarrow{A}_1 , \tag{19.35}$$

$$\frac{\partial \theta_1}{\partial \widetilde{x}} = C \delta(\overrightarrow{\widetilde{x}}) + \frac{1}{Pr} \widetilde{\nabla}^2 \theta_1 . \tag{19.36}$$

We may reduce these equations to a more familiar form with the substitutions,

$$\psi_1 = - \frac{1}{4\pi} \frac{F}{\widetilde{R}} + \psi_1' , \tag{19.37}$$

$$\overrightarrow{A}_1 = \widetilde{\nabla} \times \overrightarrow{i} \sigma . \tag{19.38}$$

We then obtain the following:

$$\widetilde{\nabla}^2 \psi_1' - \frac{\partial \theta_1}{\partial \widetilde{x}} = - \delta(\overrightarrow{\widetilde{x}}) F , \tag{19.39}$$

$$\frac{\partial \psi_1'}{\partial \widetilde{x}} + \frac{1}{\gamma} \omega_3 = (1 + \frac{1}{\alpha}) \widetilde{\nabla}^2 \psi_1' + F(1 + \frac{1}{\alpha}) \delta(\overrightarrow{x}) , \tag{19.40}$$

$$\frac{\partial \sigma}{\partial \tilde{x}} = \frac{F}{4\pi \tilde{R}} + \tilde{\nabla}^2 \sigma . \tag{19.41}$$

Setting

$$\tilde{\nabla}^2 \sigma = \frac{\partial \sigma}{\partial \tilde{x}} - \frac{F}{4\pi \tilde{R}} = \chi , \tag{19.42}$$

we get,

$$\tilde{\nabla}^2 \chi - \frac{\partial \chi}{\partial \tilde{x}} = F\, \delta(\vec{\tilde{x}}), \tag{19.43}$$

$$\begin{aligned} \tilde{\nabla} \times \vec{A}_1 &= \tilde{\nabla} \frac{\partial \sigma}{\partial \tilde{x}} - \vec{i}\, \tilde{\nabla}^2 \sigma , \\ &= F\, \tilde{\nabla} \frac{1}{4\pi \tilde{R}} + \tilde{\nabla}\chi - \vec{i}\, \chi . \end{aligned} \tag{19.44}$$

$$\therefore \qquad \vec{g}_1 = \tilde{\nabla}\chi - \vec{i}\,\chi + \tilde{\nabla}\psi_1' . \tag{19.45}$$

We shall call equations (19.39), (19.40), (19.43) and (19.45) the defining equations of Lamb correlation. They are extensions to compressible flow of Lamb's theory of the Oseen equations of incompressible flow. ([5], p. 610). Equation (19.40) may be simplified further.

$$\frac{\partial \psi_1'}{\partial \tilde{x}} + \frac{1}{\gamma}\, \omega_3 = \left(1 + \frac{1}{\alpha}\right) \frac{\partial \theta_1}{\partial \tilde{x}} . \tag{19.46}$$

If we solved for ψ_1', θ_1, χ using

$$\tilde{\nabla}^2 \chi - \frac{\partial \chi}{\partial \tilde{x}} = F\ \delta(\vec{\tilde{x}}) ,$$

$$\tilde{\nabla}^2 \psi_1' - \frac{\partial \theta_1}{\partial \tilde{x}} = -F\ \delta(\vec{\tilde{x}}) ,$$

$$\frac{\partial \theta_1}{\partial \tilde{x}} - \frac{1}{Pr} \tilde{\nabla}^2 \theta_1 = C\ \delta(\vec{\tilde{x}}),$$

ω_3 would be determined from (19.46). In fact these equations are quite simply solved utilizing the fundamental solution of (19.43). This equation has a solution,

$$\chi = - \frac{F}{4\pi\tilde{R}} e^{-\frac{1}{2}(\tilde{R}-\tilde{x})} . \qquad (19.47)$$

Utilizing essentially the same result,

$$\theta_1 = \frac{C\,Pr}{4\pi\tilde{R}} e^{-\frac{1}{2}Pr(\tilde{R}-\tilde{x})} . \qquad (19.48)$$

We also have,

$$\begin{aligned} \psi_1' &= \frac{1}{Pr} \theta_1 + \frac{(F-C)}{4\pi\tilde{R}} , \\ &= \frac{C}{4\pi\tilde{R}} e^{-\frac{1}{2}Pr(\tilde{R}-\tilde{x})} + \frac{(F-C)}{4\pi\tilde{R}} . \end{aligned} \qquad (19.49)$$

The flow field is given by the following formulae.

$$\vec{g}_1 = \tilde{\nabla}\chi - \vec{i}\,\chi + \tilde{\nabla}\psi_1' , \qquad (19.50)$$

$$\chi = -\frac{F}{4\pi\widetilde{R}} e^{-\frac{1}{2}(\widetilde{R}-\widetilde{x})} , \tag{19.51}$$

$$\psi_1{}' = \frac{C}{4\pi\widetilde{R}} e^{-\frac{1}{2}\Pr(\widetilde{R}-\widetilde{x})} + \frac{(F-C)}{4\pi\widetilde{R}} , \tag{19.52}$$

$$\theta_1 = \frac{C\Pr}{4\pi\widetilde{R}} e^{-\frac{1}{2}\Pr(\widetilde{R}-\widetilde{x})} , \tag{19.53}$$

$$\omega_3 = \gamma \frac{\partial}{\partial\widetilde{x}} \left[(1+\frac{1}{\alpha})\theta_1 - \psi_1{}'\right] . \tag{19.54}$$

If the inner equations can be solved and the solution is expanded for large values of R, we can determine F, C by writing these variables in terms of the inner variables. For example, if $T_W = 1$. $T_0 = 1$. This implies that $C = 0$. F is then given directly by the drag formulae of the inner solution. In general, to match, the outer solution does not consist simply of the above fundamental solution. There may be higher multipoles involved in the match. We note that in the overlap region, to match the leading term of the Stokes expansion to O(1), one needs in general more than one term in the Oseen expansion. Once we match, we may be sure that the error is uniformly O(Re). We shall discuss the higher multipoles briefly here.

§20. For the sake of simplicity we restrict ourselves to the axisymmetric case. The higher multipoles are certain singular solutions (with singularities of certain types at $\widetilde{R} = 0$) of the equations (19.14) through (19.17). Consider the following formal equations:

$$\omega_1 = S_1 + \theta_1 , \tag{20.1}$$

$$\frac{\partial S_1}{\partial \tilde{x}} + \tilde{\nabla} \cdot \vec{g_1} = \sum_{k=0}^{\infty} \rho_k \frac{\partial^k}{\partial \tilde{x}^k} (\delta (\vec{\tilde{x}})) , \tag{20.2}$$

$$\frac{\partial \vec{S_1}}{\partial \tilde{x}} + \frac{1}{\gamma M^2} \tilde{\nabla} \omega_1 = \vec{i} \sum_{k=0}^{\infty} f_k \frac{\partial^k}{\partial \tilde{x}^k} (\delta (\vec{\tilde{x}}))$$

$$+ \nabla \left(\sum_{k=0}^{\infty} v_k \frac{\partial^k}{\partial \tilde{x}^k} (\delta (\vec{\tilde{x}})) \right.$$

$$+ \tilde{\nabla}^2 \vec{g_1} + \frac{1}{\alpha} \tilde{\nabla} (\tilde{\nabla} \cdot \vec{g_1}) , \tag{20.3}$$

$$\frac{\partial \theta_1}{\partial \tilde{x}} - \frac{\gamma-1}{\gamma} \frac{\partial \omega_1}{\partial \tilde{x}} = \frac{1}{Pr} \tilde{\nabla}^2 \theta_1 + \sum_{k=0}^{\infty} t_k \frac{\partial^k}{\partial \tilde{x}_k} (\delta (\vec{\tilde{x}})) , \tag{20.4}$$

where ρ_k, f_k , v_k , t_k are constants and the formal series are usually finite. These equations define an axisymmetric Oseen flow with mass, force, potential and thermal multipoles.

As usual, we write

$$\vec{g_1} = \nabla \chi_1 - \vec{i} \chi_1 + \tilde{\nabla} \psi_1' ; \tag{20.5}$$

$$\tilde{\nabla} \cdot \vec{g_1} = \tilde{\nabla}^2 \chi_1 - \frac{\partial \chi_1}{\partial \tilde{x}} + \tilde{\nabla}^2 \psi_1' . \tag{20.6}$$

Set,

$$\tilde{\nabla}^2 \chi_1 - \frac{\partial \chi_1}{\partial \tilde{x}} = \sum_{k=0}^{\infty} f_k \frac{\partial^k}{\partial \tilde{x}^k} (\delta (\vec{\tilde{x}})) . \tag{20.7}$$

The equations then lead to

$$\frac{\partial S_1}{\partial \tilde{x}} + \tilde{\nabla}^2 \psi_1' = \sum_{k=0}^{\infty} (\rho_k - f_k) \frac{\partial^k}{\partial \tilde{x}^k} (\delta (\vec{\tilde{x}})) , \tag{20.8}$$

$$\frac{\partial \psi_1'}{\partial \tilde{x}} + \frac{1}{\gamma M^2} \omega_1 = (1 + \frac{1}{\alpha}) \tilde{\nabla}^2 \psi_1 + \sum_{k=0}^{\infty} \left[(1 + \frac{1}{\alpha}) f_k + v_k \right] \frac{\partial^k}{\partial \tilde{x}^k} (\delta (\vec{\tilde{x}})) , \tag{20.9}$$

$$\frac{\partial \theta_1}{\partial \tilde{x}} - (\frac{\gamma - 1}{\gamma}) \frac{\partial \omega_1}{\partial \tilde{x}} = \frac{1}{Pr} \tilde{\nabla}^2 \theta_1 + \sum_{k=0}^{\infty} t_k \frac{\partial^k}{\partial \tilde{x}^k} (\delta (\vec{\tilde{x}})) . \tag{20.10}$$

The solution of these equations can be obtained by Fourier transforms. An example is given in Appendix V.

The general method of matching is now clear. If one has the general solutions of the above, one writes the flow fields in terms of the inner variables and gets the constants ρ_k, f_k, v_k, t_k from matching the far field expansion of the Stokes solutions. This gives a uniformly valid solution to leading order. Examples of this process will be given in the next chapter.

Chapter VI

Applications

§21. In this final chapter we shall apply the methods of the preceding chapters and find explicit solutions to some simple problems. The results will be compared with experiment where possible. We shall find that the slip condition introduced in Chapter IV is simpler to handle than the classical one. First we shall find an outer expansion for the Lagerstrom-Chester solution.

§22. A general investigation of the problem shows the following features. The inner solution (approximate, of course) matches with the free stream to zero order. We may try to go one step further and put in multipoles in the outer solution and match up to O(Re). We are only matching an approximate, zeroth iterate. This fully matched 'solution' will in general differ from the exact solution of the equations by terms depending on $\epsilon = \frac{Re}{\gamma M^2}$. In view of this fact and in view of the complexity of the general case, we consider the following special problem of isothermal flow. We assume that $Pr \to 0$ so that the heat conduction dominates in the entire flow field.

The outer equations for this simplified problem are:

$$\tilde{\nabla} \cdot \vec{g_1} + \frac{\partial S_1}{\partial \tilde{x}} = 0 \ , \qquad (22.1)$$

$$\frac{\partial \vec{g}_1}{\partial \tilde{x}} + \frac{\tilde{\nabla} S_1}{\gamma M^2} = \tilde{\nabla}^2 \vec{g}_1 + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{g}_1) . \tag{22.2}$$

For $\tilde{R} \neq 0$. We set

$$\vec{g}_1 = \nabla \chi_1 - \vec{i} \chi_1 + \nabla \psi_1' , \tag{22.3}$$

with ,

$$\nabla^2 \chi_1 - \frac{\partial \chi_1}{\partial \tilde{x}} = 0 \; ; \quad \text{for } \tilde{R} \neq 0 . \tag{22.4}$$

$$\therefore \qquad \tilde{\nabla}^2 \psi_1' + \frac{\partial S_1}{\partial \tilde{x}} = 0 . \tag{22.5}$$

$$\frac{\partial \psi_1'}{\partial \tilde{x}} + \frac{S_1}{\gamma M^2} = (1 + \frac{1}{\alpha}) \tilde{\nabla}^2 \psi_1' . \tag{22.6}$$

If we set

$$S_1 = \frac{\partial G_1}{\partial \tilde{x}} \; ; \; \psi_1' = - (1 + \frac{1}{\alpha}) \frac{\partial G_1}{\partial \tilde{x}} - \frac{1}{\gamma M^2} G_1 \; ; \tag{22.7}$$

$$(1 + \frac{1}{\alpha}) \tilde{\nabla}^2 \frac{\partial G_1}{\partial \tilde{x}} - \frac{\partial^2 G_1}{\partial \tilde{x}^2} + \frac{1}{\gamma M^2} \tilde{\nabla}^2 G_1 = 0 , \tag{22.8}$$

for $\tilde{R} \neq 0$.

Now suitable solutions of (22.5) (22.6) (or equivalently (22.8) must be found for matching. These equations correspond to (20.7) (20.8) and (20.9) with $\omega_1 = S_1$. Let us note the following facts.

Consider the function $W(\tilde{x})$ defined by the integral,*

$$W(\tilde{x}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} e^{it\tilde{x}} K_0(m\tilde{\rho})dt \; ; \; \tilde{\rho}^2 = \tilde{y}^2 + \tilde{z}^2 \, , \tag{22.9}$$

with ,

$$m^2 = \frac{t^3 - \dfrac{it^2(1-\gamma M^2)}{(1+\frac{1}{\alpha})\gamma M^2}}{t - \dfrac{i}{\gamma M^2(1+\frac{1}{\alpha})}} \; . \tag{22.10}$$

$K_0(m\rho)$ is the solution of

$$\frac{d^2 K_0}{d\rho^2} + \frac{1}{\rho}\frac{dK_0}{d\rho} - m^2 K_0 = 0 \; . \tag{22.11}$$

This function $W(\tilde{x})$ is discussed in Appendix V. For values of $\tilde{R} \sim O(Re)$, we have

$$W(\vec{\tilde{x}}) \sim \frac{1}{4\pi}\left[\frac{1}{\tilde{R}} + \frac{1}{1+\frac{1}{\alpha}}\frac{\partial}{\partial\tilde{x}}\left(\frac{\tilde{R}}{2}\right) + \cdots\right] . \tag{22.12}$$

We also define $Y(\vec{\tilde{x}})$ by the solution of,

$$\left(1+\frac{1}{\alpha}\right)\frac{\partial Y}{\partial\tilde{x}} + \frac{Y}{\gamma M^2} = -\frac{\partial W}{\partial\tilde{x}} \; , \tag{22.13}$$

such that $Y \rightarrow 0$ at infinity.

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* This work follows the methods given in a different context in [10].

For small $\widetilde{R}$,

$$Y(\vec{\widetilde{x}}) \sim -\frac{1}{4\pi}\frac{1}{(1+\frac{1}{\alpha})}\frac{1}{\widetilde{R}} . \tag{22.14}$$

We note that

$$\widetilde{\nabla}^2 W + \frac{\partial Y}{\partial \widetilde{x}} = -\delta(\vec{\widetilde{x}}) , \tag{22.15}$$

$$\frac{\partial W}{\partial \widetilde{x}} + \frac{Y}{\gamma M^2} = (1+\frac{1}{\alpha})\,\widetilde{\nabla}^2 W + (1+\frac{1}{\alpha})\,\delta(\vec{\widetilde{x}}) . \tag{22.16}$$

In the notation of the previous chapter, χ , W, Y are solutions with $f_0 = 1$ where χ is the solution of

$$\widetilde{\nabla}^2 \chi - \frac{\partial \chi}{\partial \widetilde{x}} = \delta(\vec{\widetilde{x}}) . \tag{22.17}$$

$\frac{\partial W}{\partial \widetilde{x}}$, $\frac{\partial Y}{\partial \widetilde{x}}$ are solutions of the same equations for $\rho_1 = -1$ $v_1 = (1+\frac{1}{\alpha})$ with other parameters zero. We now assume the solution in the following form:

$$\vec{g}_1 = \widetilde{\nabla}\chi_1 - \vec{i}\,\chi_1 + \widetilde{\nabla}\psi' , \tag{22.18a}$$

$$\chi_1 = -\frac{C''}{4\pi}\,(\frac{1}{\widetilde{R}})\,e^{-\frac{1}{2}(\widetilde{R}-\widetilde{x})} , \tag{22.18b}$$

$$\psi_1' = A''W(\vec{\widetilde{x}}) + B''\frac{\partial W}{\partial \widetilde{x}} . \tag{22.19}$$

For $\widetilde{R} \sim O(Re)$,

$$\vec{\tilde{u}} = \vec{i} + \mathrm{Re}\left[\frac{C''}{4\pi}\left(\frac{1}{\tilde{R}}\, e^{-\frac{1}{2}(\tilde{R}-\tilde{x})}\right)\vec{i} - \frac{C''}{4\pi}\tilde{\nabla}\left(\frac{1}{\tilde{R}}\, e^{-\frac{1}{2}(\tilde{R}-\tilde{x})}\right)\right.$$

$$+ \frac{A''}{4\pi}\tilde{\nabla}\left\{\frac{1}{\tilde{R}} + \frac{1}{1+\frac{1}{\alpha}}\frac{\partial}{\partial\tilde{x}}\left(\frac{\tilde{R}}{2}\right) + \cdots\right\}$$

$$\left.+ \frac{B''}{4\pi}\tilde{\nabla}\left\{\frac{\partial}{\partial\tilde{x}}\left(\frac{1}{\tilde{R}}\right) + \frac{1}{1+\frac{1}{\alpha}}\frac{\partial^2}{\partial\tilde{x}^2}\left(\frac{\tilde{R}}{2}\right)\cdots\right\}\right]. \qquad (22.20)$$

This may be written in the form,

$$\vec{\tilde{u}} = \vec{i}\left[1 + (\mathrm{Re})\left(\frac{C''}{4\pi}\right)\left(\frac{1}{\tilde{R}}\right) - (\mathrm{Re})\left(\frac{C''}{8\pi}\right)\left(\frac{1}{\tilde{R}}\right) + (\mathrm{Re})\left(\frac{A''}{4\pi}\right)\left(\frac{1}{2(1+\frac{1}{\alpha})}\right)\left(\frac{1}{\tilde{R}}\right) - (\mathrm{Re})\left(\frac{B''}{4\pi}\right)\left(\frac{1}{\tilde{R}^3}\right)\right]$$

$$- (\mathrm{Re})\left(\frac{C''}{4\pi}\right)\tilde{\nabla}\left(\frac{1}{\tilde{R}}\right) - (\mathrm{Re})\left(\frac{C''}{8\pi}\right)\tilde{x}\,\tilde{\nabla}\left(\frac{1}{\tilde{R}}\right)$$

$$+ (\mathrm{Re})\left(\frac{A''}{4\pi}\right)\tilde{\nabla}\left(\frac{1}{\tilde{R}}\right) + (\mathrm{Re})\left(\frac{A''}{4\pi}\right)\left(\frac{1}{2(1+\frac{1}{\alpha})}\right)\tilde{x}\,\tilde{\nabla}\left(\frac{1}{\tilde{R}}\right)$$

$$- (\mathrm{Re})\left(\frac{B''}{4\pi}\right)\tilde{x}\,\tilde{\nabla}\left(\frac{1}{\tilde{R}^3}\right) + (\mathrm{Re})\left(\frac{B''}{4\pi}\right)\left(\frac{1}{2(1+\frac{1}{\alpha})}\right)\tilde{\nabla}\frac{\partial^2}{\partial\tilde{x}^2}(\tilde{R}) . \qquad (22.21)$$

One of the simplest ways to match this with (11.7), (11.8) is to resolve (22.21) in the radial direction and write $\tilde{R} = (\mathrm{Re})R$.

We then obtain the following results.

$$A'' = C'' \cdots \qquad (22.22a)$$

$$\tilde{u}_R((\mathrm{Re})R) = \left[1 + \left(\frac{A''}{4\pi}\right)\left(\frac{1}{R}\right) + \left(\frac{2B''}{4\pi}\right)\left(\frac{1}{\mathrm{Re}^2}\right)\left(\frac{1}{R}\right)\right]\cos\theta , \qquad (22.22b)$$

Thus,

$$- A'' = \frac{12\pi(1+\tau)(1+\alpha)}{(2+3\alpha)+\tau(4+5\alpha)} , \tag{22.23a}$$

and

$$B'' = \frac{2\pi[1-\tau(1+2\alpha)]}{(2+3\alpha)+\tau(4+5\alpha)} \; (Re^2) . \tag{22.23b}$$

This match checks with u_θ , and ρ expansions for large R. The errors are now uniformly O(Re). This concludes the discussion of the Lagerstrom-Chester flow at small Prandtl numbers to leading order.

§23. We consider now the approximate solution of flow past a slightly heated sphere using the new slip condition. Consider first of all Kn $\sim$ O(1). We assume the following expansions:

$$\vec{u} = \vec{u_0} + Re\vec{u_1} + \cdots , \tag{23.1}$$

$$T = T_0 + Re\, T_1 + \cdots , \tag{23.2}$$

$$\rho = \rho_0 + Re\rho_1 + \cdots , \tag{23.3}$$

$$p = p_0 + Re\, p_1 + \cdots , \tag{23.4}$$

Substitution gives the leading order equations:

$$\nabla \cdot \rho_0 \vec{u_0} = 0 , \tag{23.5}$$

$$\frac{1}{\gamma Kn^2} \nabla p_1 = \nabla^2 \vec{u_0} + \frac{1}{\alpha} \nabla (\nabla \cdot \vec{u_0}), \tag{23.6}$$

$$\nabla^2 T_0 = 0 , \tag{23.7}$$

$$p_0 = 1 = \rho_0 T_0 . \tag{23.8}$$

The next order equations are

$$\nabla \cdot \rho_0 \vec{u_1} + \nabla \cdot \rho_1 \vec{u_0} = 0 , \tag{23.9}$$

$$\rho_0 \vec{u_0} \cdot \nabla \vec{u_0} + \frac{1}{\gamma Kn^2} \nabla p_2 = \nabla^2 \vec{u_1} + \frac{1}{\alpha} \nabla (\nabla \cdot \vec{u_1}) , \tag{23.10}$$

$$\rho_0 \vec{u_0} \cdot \nabla T_0 - \left(\frac{\gamma - 1}{\gamma}\right) \vec{u_0} \cdot \nabla p_0 = \frac{1}{Pr} \nabla^2 T_1 . \tag{28.11}$$

Consider first the case of the unheated sphere $T_W = 1$. $\nabla^2 T_0 = 0$.

$$T_0 (R = 1) = 1 + k_2 Kn \frac{\partial T_0}{\partial R} (R = 1) . \tag{23.12}$$

Hence,

$$T_0 = 1 \; ; \; \rho_0 = 1 . \tag{23.13}$$

We are thus left with Stokes equations for $\vec{u_0}$, p_1 and the boundary condition on the sphere is (R = 1):

$$u_{0R} = 0 \; ; \; u_{0\theta} = k_1 \mathrm{Kn}\left(\frac{1}{R}\frac{\partial}{\partial R}(R u_{0\theta}) - \frac{1}{R}\frac{\partial u_{0R}}{\partial \theta}\right). \tag{23.14}$$

The solution with the free stream velocity at infinity is :

$$u_{0R} = \left[1 - \left(\frac{3}{2R}\right)\left(\frac{1}{1+k_1\mathrm{Kn}}\right) + \left(\frac{\frac{1}{2} - k_1\mathrm{Kn}}{1+k_1\mathrm{Kn}}\right)\left(\frac{1}{R^3}\right)\right]\cos\theta\,, \tag{23.15}$$

$$u_{0\theta} = \left[-1 + \left(\frac{3}{4R}\right)\left(\frac{1}{1+k_1\mathrm{Kn}}\right) + \left(\frac{\frac{1}{2} - k_1\mathrm{Kn}}{1+k_1\mathrm{Kn}}\right)\left(\frac{1}{2R^3}\right)\right]\sin\theta\,, \tag{23.16}$$

$$p_1 = -\frac{3}{2}\frac{\gamma \mathrm{Kn}^2}{1+k_1\mathrm{Kn}}\left(\frac{\cos\theta}{R^2}\right). \tag{23.17}$$

Drag $= \frac{6\pi}{1+k_1\mathrm{Kn}}$ (in the Stokes variables). The outer equations are derived from the full Navier-Stokes equations in the variable $\widetilde{R} = \mathrm{Re}R$. We now have the expansions

$$\widetilde{\rho} = \widetilde{\rho}_0 + \mathrm{Re}\,\widetilde{\rho}_1 + \cdots\,, \tag{23.18}$$

$$\widetilde{T} = \widetilde{T}_0 + \mathrm{Re}\,\widetilde{T}_1 + \cdots\,, \tag{23.19}$$

$$\widetilde{u} = \widetilde{u}_0 + \mathrm{Re}\,\widetilde{u}_1 + \cdots\,, \tag{23.20}$$

$$\widetilde{p} = \widetilde{p}_0 + (\mathrm{Re})\,\widetilde{p}_1 + f_2(\mathrm{Re})\,\widetilde{p}_2\,. \tag{23.21}$$

We find that $\widetilde{\rho}_0 = 1$, $\widetilde{T}_0 = 1$, $\widetilde{u}_0 = \vec{i}$ and $\widetilde{p}_0 = 1$. We also find that $\widetilde{p}_1 = 0$ and $\widetilde{p} = 1 + (\mathrm{Re})^3\,\widetilde{p}_2$. This implies

$$\tilde{\rho}_1 + \tilde{T}_1 = 0 . \tag{23.22}$$

The outer equations for the perturbations are:

$$\frac{\partial \tilde{\rho}_1}{\partial \tilde{x}} + \tilde{\nabla} \cdot \tilde{u}_1 = 0 , \tag{23.23}$$

$$\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{1}{\gamma Kn^2} \tilde{\nabla} \tilde{p}_2 = \tilde{\nabla}^2 \tilde{u}_1 + \frac{1}{\alpha} \tilde{\nabla}(\tilde{\nabla} \cdot \tilde{u}_1) , \tag{23.24}$$

$$\frac{\partial \tilde{T}_1}{\partial \tilde{x}} = \frac{1}{Pr} \tilde{\nabla}^2 \tilde{T}_1 . \tag{23.25}$$

To match with the inner solution, we set

$$\tilde{\rho}_1 = 0 \quad \tilde{T}_1 = 0 . \tag{23.26}$$

Then $\tilde{\nabla} \cdot \tilde{u}_1 = 0$.

$$\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{1}{\gamma Kn^2} \tilde{\nabla} \tilde{p}_2 = \tilde{\nabla}^2 \tilde{u}_1 .$$

If ,

$$\tilde{u}_1 = \tilde{\nabla} \chi - \vec{i} \chi + \tilde{\nabla} \psi' , \tag{23.27}$$

where,

$$\tilde{\nabla}^2 \chi - \frac{\partial \chi}{\partial \tilde{x}} = 0 \quad , \qquad \tilde{R} \neq 0 \quad , \tag{23.28a}$$

and

$$\tilde{\nabla}^2 \psi' = 0 . \tag{23.28b}$$

We have

$$\widetilde{p}_2 = -\gamma Kn^2 \frac{\partial\psi'}{\partial\widetilde{x}} \cdot \qquad (23.29)$$

We choose,

$$\psi' = -\left(\frac{3}{2(1+k_1 Kn)}\right)\left(\frac{1}{\widetilde{R}}\right) . \qquad (23.30)$$

Then,

$$\widetilde{p}_2 = -\left(\frac{3\gamma Kn^2}{2(1+k_1 Kn)}\right)\left(\frac{\cos\theta}{\widetilde{R}^2}\right) .$$

Hence $1+(Re)^3\,\widetilde{p}_2$ written in inner variables becomes $1+(Re)p_1$.

We also choose χ according to the equation:

$$\chi = \left(\frac{6\pi}{1+k_1 Kn}\right)\left(\frac{1}{4\pi\widetilde{R}}\right) e^{-\frac{1}{2}(\widetilde{R}-\widetilde{x})} . \qquad (23.31)$$

It turns out that we need a doublet to make the error uniformly O(Re). Take,

$$\psi' = \frac{A'}{\widetilde{R}} + B'\frac{\partial}{\partial\widetilde{x}}\left(\frac{1}{\widetilde{R}}\right) , \qquad (23.32)$$

$$\chi = \frac{C'}{\widetilde{R}} e^{-\frac{1}{2}(\widetilde{R}-\widetilde{x})} . \qquad (23.33)$$

$$\vec{i} + Re\,\vec{u}_1 = \vec{i} + (Re)\Big[C' e^{-\frac{1}{2}(\widetilde{R}-\widetilde{x})}\widetilde{\nabla}\left(\frac{1}{\widetilde{R}}\right) + \frac{C'}{\widetilde{R}}\widetilde{\nabla} e^{-\frac{1}{2}(\widetilde{R}-\widetilde{x})} - \vec{i}\,\frac{C'}{\widetilde{R}} e^{-\frac{1}{2}(\widetilde{R}-\widetilde{x})} + A'\widetilde{\nabla}\left(\frac{1}{\widetilde{R}}\right)$$

$$+ B'\widetilde{\nabla}\frac{\partial}{\partial\widetilde{x}}\left(\frac{1}{\widetilde{R}}\right)\Big] . \qquad (23.34)$$

For $\widetilde{R} \sim O(Re)$, we have,

$$\widetilde{\vec{u}}(\widetilde{R}) \sim \vec{i} + (Re)(C') \widetilde{\nabla}(\frac{1}{\widetilde{R}}) + \vec{i}(Re)(\frac{C'}{2})(\frac{1}{\widetilde{R}})$$

$$+ (\tfrac{1}{2}C')(Re)\widetilde{x}\widetilde{\nabla}(\frac{1}{\widetilde{R}}) - \vec{i}(Re)(C')(\frac{1}{\widetilde{R}})$$

$$+ (Re)A'\widetilde{\nabla}(\frac{1}{\widetilde{R}}) - \vec{i}(B')(Re)(\frac{1}{\widetilde{R}^3})$$

$$- \widetilde{x}\widetilde{\nabla}(\frac{1}{\widetilde{R}^3})(Re)(B') + O(Re) .$$

Resolving in the radial direction and writing $\widetilde{R} = (Re)R$,

$$\widetilde{u}_R((Re)R) \sim \left[1 - C'(\frac{1}{R}) + (\frac{2B'}{Re^2})(\frac{1}{R^3})\right]\cos\theta ,$$

with ,

$$C' = -A' .$$

Comparing with (23.15) we get ,

$$C' = \frac{3}{2}\left(\frac{1}{1+k_1 Kn}\right) , \quad A' = -\frac{3}{2}\left(\frac{1}{1+k_1 Kn}\right) , \tag{23.35}$$

and

$$B' = (Re^2)\left(\frac{1-2k_1 Kn}{4(1+k_1 Kn)}\right) . \tag{23.36}$$

$\widetilde{u}_\theta((Re)R)$ checks automatically.

$$\therefore \left(\frac{-1}{\gamma Kn^2}\right)\tilde{p}_2 = -\left(\frac{3}{2}\right)\left(\frac{1}{1+k_1 Kn}\right)\frac{\partial}{\partial\tilde{x}}\left(\frac{1}{\tilde{R}}\right) + \frac{(Re^2)}{4}\left[\frac{(1-2k_1 Kn)}{4(1+k_1 Kn)}\right]\frac{\partial^2}{\partial\tilde{x}^2}\left(\frac{1}{\tilde{R}}\right). \quad (23.37)$$

Hence

$$1 + (Re)^3\,\tilde{p}_2((Re)R) \sim 1 - (Re)\left[-\frac{3}{2}\left(\frac{1}{1+k_1 Kn}\right)\frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \frac{(Re)}{4}\left\{\frac{(1-2k_1 Kn)}{(1+k_1 Kn)}\right\}\frac{\partial^2}{\partial x^2}\left(\frac{1}{R}\right)\right](\gamma Kn^2).$$

We can therefore be sure that the pressure matches the Stokes value with the error uniformly O(Re).

For a slightly heated sphere we have $(h \ll 1)$,

$$T_W = 1 + h\,\Delta_1(\theta) + h^2\Delta_2(\theta). \quad (23.38)$$

We write,

$$\vec{u_0} = \sum_{k=0}^{\infty} \zeta_k(\vec{x})\,h^k, \quad (23.39)$$

$$\rho_0 = \sum_{k=0}^{\infty} \xi_k(\vec{x})\,h^k, \quad (23.40)$$

$\xi_0 = 1$; $\vec{\zeta_0}$ is the solution given by (23.15), (23.16).

$$T_0 = 1 + \sum_{k=1}^{\infty} \eta_k(\vec{x})\,h^k ; \quad (23.41)$$

where

$$\nabla^2\eta_k = 0 \quad ; \quad \eta_k \to 0 \text{ at } \infty ,$$

and

$$\eta_k(\theta) = \Delta_k(\theta) + k_2 \mathrm{Kn} \frac{\partial \eta_k}{\partial R} \; (R=1, \theta) \; , \tag{23.42}$$

$$\eta_k = \sum_{m=0}^{\infty} P_m(\cos\theta)\left(\frac{E_{mk}}{R^{m+1}}\right) . \tag{23.43}$$

E_{mk} are related simply to the Legendre expansion of Δ_k . Now that T_0 is known in terms of the boundary data, the equation $\rho_0 = \frac{1}{T_0}$ determines ξ_k in (23.40). We will indicate the solution of (23.5), (23.6) when ρ_0 is given in the form (23.40). We have already determined ξ_0.

$$p_1 = \sum_{k=0}^{\infty} B_k(\vec{x})\, h^k \; ; \tag{23.44}$$

$$\frac{1}{\gamma \mathrm{Kn}^2} \nabla B_k = \nabla^2 \vec{\zeta}_k + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{\zeta}_k) \; ; \tag{23.45}$$

$$\nabla \cdot \sum_{\mu+\nu=k} \xi_\mu \vec{\zeta}_\nu = 0 \quad ; \tag{23.46}$$

$\vec{\zeta}_1$, B_1 satisfy the equations:

$$\nabla \cdot \vec{\zeta}_1 + \vec{\zeta}_0 \cdot \nabla \xi_1 = 0 \; , \tag{23.47}$$

$$\frac{1}{\gamma \mathrm{Kn}^2} \nabla B_1 = \nabla^2 \vec{\zeta}_1 + \frac{1}{\alpha} \nabla(\nabla \cdot \vec{\zeta}_1) \, . \tag{23.48}$$

For a specific ξ_1, these equations may be solved by a correlation method. For example, let $\xi_1 = \frac{1}{R}$. Let $\vec{\zeta}_1 = \vec{v} + \nabla\phi$ with $\nabla \cdot \vec{v} = 0$

$$\nabla^2 \phi + \nabla \cdot \frac{\vec{\zeta}_0}{R} = 0 \,. \qquad (23.49)$$

In spherical coordinates,

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \phi}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{U_R}{R}\right)$$

$$+ \frac{1}{R \sin\theta}\frac{\partial}{\partial \theta}\left(\frac{u_\theta \sin\theta}{R}\right) = 0 \,. \qquad (23.50)$$

Here u_R , u_θ are given by (23.15), (23.16). If we put $\phi = f(R)\cos\theta$, we get,

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{df}{dR}\right) - \frac{2f(R)}{R^2} + H(R) = 0 \,. \qquad (23.61)$$

Where $H(R) = \frac{1}{R^2}\frac{d}{dR}(Ru) + \frac{2v}{R^2}$ where $u(R)$, $v(R)$ are functions of R in the equations (23.15), (23.16). Actually H(R) is of the form,

$$H(R) = \frac{c_1}{R^2} + \frac{c_2}{R^3} + \frac{c_3}{R^5} \,. \qquad (23.56)$$

A particular integral of the form $D_1 + \frac{D_2}{R} + \frac{D_3}{R^3}$ exists where D_2, D_2, D_3 are simple functions of c_1, c_2, c_3. Now all that is necessary is to solve

$$\nabla^2 \vec{v} = \nabla \chi \,,$$

$$\nabla \cdot \vec{v} = 0 \,,$$

$$\vec{v} \to 0 \text{ at } \infty \text{ and}$$

$$\vec{v} + \nabla\phi = k_1 Kn \,\text{curl}\, \vec{v} \times \vec{n} \text{ on the sphere };$$

$$v_R + \frac{df}{dR}\cos\theta = 0 \text{ on } R = 1 \,, \qquad (23.57)$$

$$v_\theta + \left(\frac{-1}{R} f \sin\theta\right) = k_1 \mathrm{Kn} \left(\frac{1}{R}\frac{\partial}{\partial R}(R v_\theta) - \frac{1}{R}\frac{\partial v_R}{\partial \theta}\right) . \qquad (23.58)$$

These equations fully determine the field now. We have,

$$\frac{1}{\gamma \mathrm{Kn}^2} B_1 = \chi - (1 + \frac{1}{\alpha}) \nabla^2 \phi . \qquad (23.59)$$

The details will not be given.

§24. The next case is $\frac{Re}{\gamma M^2} = \epsilon$ fixed. $Re << 1$.

$$\mathrm{Kn} = \frac{M}{Re} = \frac{1}{\sqrt{\gamma \epsilon \, Re}} .$$

The boundary conditions on the sphere are: (for the velocity field)

$$u_R = 0 \; ; \; u_\theta = k_1 \frac{1}{\sqrt{\gamma\epsilon\, Re}} \left(\frac{1}{R}\frac{\partial}{\partial R}(R u_\theta) - \frac{1}{R}\frac{\partial u_R}{\partial \theta}\right) . \qquad (24.1)$$

Corresponding to (16. 25), (16. 26) etc. we have,

$$\vec{u} = \vec{u_0} + \sqrt{Re}\, \vec{u_1} + \cdots , \qquad (24.2)$$

$$T = T_0 + \sqrt{Re}\, T_1 + \cdots , \qquad (24.3)$$

$$\rho = \rho_0 + \sqrt{Re}\, \rho_1 + \cdots , \qquad (24.4)$$

$$p = p_0 + \sqrt{Re}\, p_1 + \cdots . \qquad (24.5)$$

Substitution into the equations gives,

$$T_0 = \rho_0 = p_0 = 1 \; ; \; \vec{u}_0 = \nabla \phi_0 \; ; \tag{24.6}$$

where,

$$\phi_0 = R\cos\theta + \frac{1}{2R^2} \cos\theta \; . \tag{24.7}$$

The next order equations (16.40, et seq) are,

$$\nabla \cdot (\rho_1 \nabla \phi_0) + \nabla \cdot \vec{u}_1 = 0 \; , \tag{24.8}$$

$$\nabla^2 \vec{u}_1 + \frac{1}{\alpha} \nabla (\nabla \cdot \vec{u}_1) = \epsilon \nabla p_1 \; , \tag{24.9}$$

$$\frac{1}{Pr} \nabla^2 T_1 = 0 \; , \tag{24.10}$$

$$u_{1R} = 0 \; ; \; \sqrt{\gamma\epsilon} \; u_{0\theta} = k_1 \left(\frac{1}{R} \frac{\partial}{\partial R} (R u_{1\theta}) - \frac{1}{R} \frac{\partial u_{1R}}{\partial \theta} \right) \; ; \tag{24.11}$$

$$(1 - T_w) \sqrt{\gamma\epsilon} = k_2 \frac{\partial T_1}{\partial R} \; ; \tag{24.12}$$

$$p_1 = \rho_1 + T_1 \; . \tag{24.13}$$

Consider the simple case $T_w = 1$. $\therefore$ $T_1 = 0$. $\rho_1 = p_1$. Even for this simple problem, the exact solution of the correlation equations (16.49), (16.50), (16.51) for the sphere is a complicated expression though in <u>principle</u> it is simple to obtain. We will avoid the complicated formulae and give only the vortex part $\vec{v}_1$ of the solution. For this part we have,

$$\nabla^2 \vec{v}_1 = \nabla \chi_1 ,$$

$$\nabla \cdot \vec{v}_1 = 0 ,$$

$$v_{1R} = 0 \quad ; \sqrt{\gamma\epsilon} \ \frac{1}{R} \ \frac{\partial \phi_0}{\partial \theta} = k_1 \left(\frac{1}{R} \ \frac{\partial}{\partial R} (R u_{1\theta}) \right) , \qquad (24.14)$$

with $\vec{v}_1 \to 0$ at ∞ .

$$v_{1R} = \left(\frac{B}{R} + \frac{C}{R^3}\right) \cos\theta , \qquad (24.15)$$

$$v_{1\theta} = \left(- \frac{B}{2R} + \frac{C}{2R^3} \right) \sin\theta , \qquad (24.16)$$

$$B + C = 0 \ ; \ \sqrt{\gamma\epsilon} \left(- \frac{3}{2}\right) = k_1 \left(\frac{B}{2} - \frac{3C}{2} - \frac{B}{2} + \frac{C}{2} \right) ;$$

$$\therefore \quad C = \frac{3}{2k_1} \sqrt{\gamma\epsilon} , \qquad (24.17)$$

$$B = - \frac{3}{2k_1} \sqrt{\gamma\epsilon} , \qquad (24.18)$$

$$\chi_1 = \frac{B \cos\theta}{R^2} ; \qquad (24.19)$$

The drag due to the vortex part is,

$$D = \frac{6\pi \, Re}{k_1 M} . \qquad (24.20)$$

The outer solution is complicated in the general case just as in §22. As before, we will assume now that $Pr \to 0$. The outer expansion now looks like the following:

$$\vec{\tilde{u}} = \vec{i} + Re^{3/2} \vec{\tilde{u}}_1 + \cdots , \tag{24.21}$$

$$\rho = 1 + Re^{3/2} \tilde{\rho}_1 + \cdots , \tag{24.22}$$

$$p = 1 + Re^{3/2} \tilde{p}_1 + \cdots , \tag{24.23}$$

$T = 1$. From which we have $p_1 = \rho_1$.

The outer equations of perturbation are

$$\frac{\partial \tilde{\rho}_1}{\partial \tilde{x}} + \tilde{\nabla} \cdot \vec{\tilde{u}}_1 = 0 , \tag{24.24}$$

$$\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{\epsilon}{Re} \tilde{\nabla} p_1 = \tilde{\nabla}^2 \tilde{u}_1 + \frac{1}{\alpha} \tilde{\nabla}(\tilde{\nabla} \cdot \vec{\tilde{u}}_1) . \tag{24.25}$$

These are the same as (22.1), (22.2). The matching can proceed in a similar manner if the inner solution is known. If ϵ is small as for the Lagerstrom-Chester field, we attempt a perturbation expansion in ϵ.

$$\chi_1 + (1 + \frac{1}{\alpha}) \nabla^2 \phi_1 = \epsilon(\rho_1 + T_1) , \tag{24.26}$$

$$\nabla\phi \cdot \nabla\rho_1 + \nabla^2 \phi_1 = 0 , \tag{24.27}$$

If $T_1 \equiv 0$ ($Pr \to 0$ say), we proceed as follows:

To solve (24.26), (24.27) perturbatively, we expand ϕ_1, ρ_1 in powers of ϵ.

$$\chi_1 + (1 + \frac{1}{\alpha}) \nabla^2 \phi_1^{(0)} = 0 . \tag{24.28}$$

The solution of this equation is similar to (23.49).

$$\chi_1 = \frac{B \cos\theta}{R^2} .$$

$$(1 + \frac{1}{\alpha})\left[\frac{1}{R^2} \frac{d}{dR}(R^2 \frac{dF}{dR}) - \frac{2F(R)}{R^2}\right] + \frac{B}{R^2} = 0 . \tag{24.29}$$

$$F = \frac{+B}{2(1 + \frac{1}{\alpha})} . \tag{24.30}$$

$$\phi_1^{(0)} = \frac{B \cos\theta}{2(1+\frac{1}{\alpha})} + \frac{D \cos\theta}{R^2} . \tag{24.31}$$

Where D is determined from $\frac{\partial \phi_1^{(0)}}{\partial R} = 0$ on $R = 1$. $D = 0$ $\therefore$ $\phi_1^{(0)} = \frac{B\cos\theta}{2(1 + \frac{1}{\alpha})}$. This velocity potential however gives a logarithmic singularity in ρ_1 if one uses (16.51) to compute ρ_1. This is analogous to Lagerstrom's paradox and is resolved in the same way. In this theory, B is $O(\sqrt{\epsilon})$. The stagnation pressure p_1 is $O(\frac{B}{\epsilon})$ i.e. $O(\frac{1}{\sqrt{\epsilon}})$. Hence the real stagnation pressure is $O(\frac{\sqrt{Re}}{\sqrt{\epsilon}}) = \sqrt{\gamma}\, M$ $\therefore$ the stagnation pressure is $O(M)$. This contrasts with the Lagerstrom-Chester theory which gives stagnation pressures of $O(\frac{1}{\epsilon})$.

§25. To illustrate the case of §17, we once again consider $Pr \to 0$ and assume the expansions of §17. We obtain,

$$T_0 = 1 \; ; \tag{25.1}$$

$$\rho_0 = 1 \; ; \; \vec{u}_0 = \nabla\phi_0 \; ; \; \phi_0 = R\cos\theta + \frac{1}{2R^2}\cos\theta \; . \tag{25.2}$$

The equations then give as in §17, $f_1(Re) = Re$.

$$\nabla \cdot \vec{u}_1 + \nabla\rho_1 \cdot \nabla\phi_0 = 0 \; , \tag{25.3}$$

$$\nabla\left(\frac{(\nabla\phi_0)^2}{2} + \frac{T_0}{\gamma M^2}\right) = \nabla^2\vec{u}_1 + \frac{1}{\alpha}\nabla(\nabla\cdot\vec{u}_1) \; , \tag{25.4}$$

$$\vec{u}_1 \cdot \vec{n} = 0 \; ; \; \vec{u}_0 = k_1 M \,\mathrm{curl}\,\vec{u}_1 \times \vec{n} \; . \tag{25.5}$$

The splitting gives,

$$\nabla \cdot \vec{v}_1 = 0 \; , \tag{25.6}$$

$$\nabla^2 \vec{v}_1 = \nabla\chi_1 \; , \tag{25.7}$$

$$\frac{(\nabla\phi_0)^2}{2} + \frac{T_0}{\gamma M^2} = \chi_1 + \left(1 + \frac{1}{\alpha}\right)\nabla^2\phi_1 \; , \tag{25.8}$$

$$\nabla\rho_1 \cdot \nabla\phi_0 + \nabla^2\phi_1 = 0 \; , \tag{25.9}$$

$$v_{1R} = \left(\frac{B}{R} + \frac{C}{R^3}\right)\cos\theta \; , \tag{25.10}$$

$$v_{1\theta} = (-\frac{B}{2R} + \frac{C}{2R^3})\sin\theta , \qquad (25.11)$$

$$C = \frac{3}{2k_1 M} \quad ; \quad B = -\frac{3}{2k_1 M} \quad ; \qquad (25.12)$$

$$\chi_1 = \frac{1}{2} + \frac{1}{\gamma M^2} + \frac{B\cos\theta}{R^2} \quad . \qquad (25.13)$$

Notice now that equation (25.9) has in general a singularity at the stagnation point where $\nabla\phi_0 = 0$.

This possibility was not discussed in the general theory. The reason for this singularity is that in general $\nabla^2\phi_1 \neq 0$ at the stagnation point. Equations (25.8), (25.9) have no solution which is non-singular with the conditions we have imposed. This may imply a genuine breakdown of the continuum theory in this regime. It may also be that the particular perturbation scheme adopted is unsuitable for these boundary conditions. At these extreme free molecular limits ($Kn = O(\frac{1}{Re})$), we feel little inclination to speculate about this matter. It is possible to impose the condition that the pressure be bounded on the sphere instead of $T_0 = 1$.

For example, to make $\nabla^2\phi_1 = 0$ on $R = 1$, we impose the following condition on T_0:

$$\frac{T_0}{\gamma M^2} = \chi_1 = \frac{1}{2} + \frac{1}{\gamma M^2} + B\cos\theta \text{ on } R = 1 \quad ; \qquad (25.14a)$$

we have,

$$T_0 = 1 + \frac{1}{2}(\frac{\gamma M^2}{R}) + \gamma M^2 (\frac{B\cos\theta}{R^2}) \quad . \qquad (25.14b)$$

Then,
$$\frac{(\nabla\phi_0)^2}{2} + \tfrac{1}{2}(\frac{1}{R} - 1) = (1 + \frac{1}{\alpha})\nabla^2\phi_1 \ . \qquad (25.14c)$$

(25.14c) is a Poisson problem. However, $\nabla^2\phi_1 \not\to 0$ at ∞ owing to the $\frac{1}{R}$ term. This has to be matched with a suitable Oseen flow. (25.9) however may be solved now without singularities at $R = 1$. Again, $\rho_1 \not\to 0$ at infinity and must be suitably matched. We will not consider this matter further. It is however an interesting problem from a purely mathematical standpoint and deserves a more careful and thorough investigation.

§26. As an example of a two dimensional flow, we consider the flow past a circular cylinder. We only give the results for $Kn \sim O(1)$. We must solve (16.12),(16.13) for two dimensional flow. For $T_W = 1$, we have $\rho_0 = 1$.

The solution which is appropriate is the following:

$$u_R = (BR^{-2} + C\log R + D)\cos\theta \ , \qquad (26.1)$$

$$u_\theta = (BR^{-2} - C\log R - D - C)\sin\theta \ , \qquad (26.2)$$

$$p_1 = -\frac{2C\gamma}{R}\cos\theta \ , \qquad (26.3)$$

using an obvious cylindrical polar system. B, C, D are arbitrary constants. One has,

$$B + D = 0 \ , \text{ since } \ u_R = 0 \ \text{ on } \ R = 1 \ . \qquad (26.4)$$

$$u_\theta = k_1 Kn\left(\frac{1}{R}\frac{\partial}{\partial R}(Ru_\theta)\right) .$$

$$\therefore \quad (B - D - C) = k_1 Kn(-B-C-D-C) ,$$

$$\therefore \quad B - D = k_1 Kn(-2C) + C ,$$

$$= C(1 - 2k_1 Kn) . \qquad (26.5)$$

Now for matching with Oseen flow $\vec{i}$ at infinity we need,

$$C = \frac{1}{\log \frac{1}{Re}} . \qquad (26.6)$$

$\therefore$ we have ,

$$B + D = 0 ,$$

$$B - D = \frac{1}{\log \frac{1}{Re}} (1 - 2k_1 Kn) .$$

$$\therefore \quad B = \frac{1}{2\log \frac{1}{Re}} (1 - 2k_1 Kn) , \qquad (26.7)$$

$$D = -\frac{1}{2\log \frac{1}{Re}} (1 - 2k_1 Kn) , \qquad (26.8)$$

$$\therefore \quad u_R = \frac{1}{\log \frac{1}{Re}} \left[\log R + \frac{(1-2k_1 Kn)}{2R^2} - \frac{(1-2k_1 Kn)}{2}\right] \cos\theta , \qquad (26.9)$$

$$u_\theta = -\frac{1}{\log \frac{1}{Re}} \left[\log R - \frac{(1-2k_1 Kn)}{2R^2} + \frac{(1+2k_1 Kn)}{2}\right] \sin\theta , \qquad (26.10)$$

$$p_1 = -\left(\frac{2\gamma}{\log \frac{1}{Re}}\right) \frac{\cos\theta}{R} \quad . \tag{26.11}$$

Instead of matching with $\vec{i}$ we leave C for the moment undetermined. Now we go back to the Oseen equations. The method is the same as for a sphere. We write first of all (following Proudman and Pearson),

$$\widetilde{u} = \vec{i} + Re\, \widetilde{u}_1(\widetilde{R}, Re) + \cdots \quad , \tag{26.12}$$

where $\widetilde{u}_1$ is not independent on Re but has coefficients that may depend on Re (even for the sphere this was so). We only require that $O\Big(Re\, \widetilde{u}_1(\widetilde{R}, Re)\Big)$ should be uniformly $o(1)$ when $Re \leqslant \widetilde{R} < \infty$. Following Lamb, we write

$$\widetilde{u}_1(\widetilde{R}, Re) = \widetilde{\nabla}\chi - \vec{i}\chi + \widetilde{\nabla}\psi \quad ; \tag{26.13}$$

where

$$\chi = C(Re)\, e^{\frac{\widetilde{x}}{2}} \int_0^\infty e^{-\frac{\widetilde{R}}{2}\cosh\omega}\, d\omega$$

$$= C(Re)\, K_0\left(\frac{\widetilde{R}}{2}\right) e^{\frac{\widetilde{x}}{2}} \quad ; \tag{26.14a}$$

where C(Re) is a function of Re. Also,

$$\psi = A_0(Re)(\log \widetilde{R}) + \left(\frac{\partial}{\partial \widetilde{x}} \log \widetilde{R}\right) A_1(Re) \quad . \tag{26.14b}$$

Now we look at the inner limit of the velocity (E is Euler's constant).

$$\tilde{u}(\tilde{R} = R(Re), Re) \simeq \vec{i} + (Re)\left[-\frac{C}{2}\left(\tfrac{1}{2} - E - \log\left(\frac{Re}{4}\right)\right)\vec{i} + \frac{C}{2}(\log R)\vec{i} + \vec{i}\,\frac{C}{4}\right]$$

$$- C(\nabla \log R) + C\left(\frac{Re}{4}\right) R^2 \times \nabla\left(\frac{1}{R^2}\right) + A_0(\nabla \log R) + \left(\frac{A_1}{Re}\right)\frac{\vec{i}}{R^2} + \left(\frac{A_1}{Re}\right) \times \nabla\left(\frac{1}{R^2}\right). \qquad (26.15)$$

Resolving in the R direction,

$$\tilde{u}_R \sim \left[\frac{(Re)C}{2}(\log R) + \left(\frac{A_1}{Re}\right)\frac{1}{R^2} - \left(\frac{2A_1}{Re}\right)\frac{1}{R^2} + 1 - (Re)\frac{C}{2}\left(\tfrac{1}{2} - E - \log\left(\frac{Re}{4}\right)\right)\right.$$

$$\left. + (Re)\frac{C}{4} - (Re)\frac{C}{2}\right]\cos\theta , \qquad (26.16)$$

with,

$$A_0 = C . \qquad (26.17)$$

From (26.4), (26.5) we have ,

$$\frac{2B}{C} = 1 - 2k_1 Kn , \qquad (26.18)$$

$$\frac{2D}{C} = -(1 - 2k_1 Kn) . \qquad (26.19)$$

These two equations determine A_1, C.

$$\therefore \qquad (Re)\frac{C}{2} = \Delta = \frac{1}{\frac{1}{2} - E - \log\left(\frac{Re}{4}\right) + k_1 Kn} . \qquad (26.20)$$

Now,

$$A_1 = -(Re)\frac{(1 - 2k_1 Kn)}{2}\Delta \quad , \qquad (26.21)$$

$$A_0 = (\frac{2}{Re})\Delta \quad . \qquad (26.22)$$

The drag is given by $2\pi ReC$.

$$\therefore \qquad Drag = 4\pi\Delta . \qquad (26.23)$$

This is of course drag/unit length. Hence we have the formula

$$D = \frac{4\pi}{\frac{1}{2} - E - \log\frac{Re}{4} + k_1 Kn} . \qquad (26.24)$$

§27. Finally, we shall consider flows past ellipsoids. The solution of Stokes equations with no slip condition is discussed in Lamb [5] (§339). Apparently, the problem has not been solved with slip. We shall give the details of solving equations (23.5), (23.6) for $\rho_0 = 1$ for the following boundary condition. Let

$$F(x,y,z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 , \qquad (27.1)$$

be the equation to the ellipsoid with a, b, c being non-dimensional principal semi-axes.

The problem of solving the equations with

$$\vec{u} = k_1 Kn(\text{curl}\,\vec{u} \times \vec{n}) \qquad (27.2)$$

appears to be quite difficult if k_1 is taken constant. Note that $|\nabla F|$ is O(1) on the ellipsoid. We shall consider the problem of solving the Stokes equations with

$$\vec{u} = k_1{}' Kn(\text{curl}\,\vec{u} \times \nabla F), \qquad (27.3)$$

where $k_1{}'$ is constant. This problem happens to have a very simple solution. The gross features of the flow should not differ from the solution with (27.2). The two Ansatz made here are in a sense indistinguishable macroscopically. The simpler solution may give us a feel for the flow in question.

Let the flow at infinity be $\vec{i}$. We wish to solve,

$$\nabla \cdot \vec{u} = 0, \qquad (27.4)$$

$$\nabla^2 \vec{u} = \frac{1}{\gamma Kn^2} \nabla p_1 . \qquad (27.5)$$

Following Lamb, we assume the following form for the solution.

$$\vec{u} = \vec{i} + B x \nabla \chi - \vec{i} B \chi + A \nabla \frac{\partial \Omega}{\partial x}, \qquad (27.6)$$

where, $\nabla^2 \chi = \nabla^2 \Omega = 0$. A, B are undetermined constants.

$$\chi = abc \int_{\lambda}^{\infty} \frac{d\tau}{\Delta(\tau)} \quad , \tag{27.7}$$

where λ is the positive root of

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \; . \tag{27.8}$$

$$\Delta(\tau) = \left\{ (a^2+\tau)\;(b^2+\tau)\;(c^2+\tau)\right\}^{\frac{1}{2}} . \tag{27.9}$$

Ω is the gravitational potential of a homogeneous ellipsoid.

$$\Omega(x,y,z) = \pi\, abc \int_{\lambda}^{\infty} \left(\frac{x^2}{a^2+\tau} + \frac{y^2}{b^2+\tau} + \frac{z^2}{c^2+\tau} - 1\right) \frac{d\tau}{\Delta(\tau)} \; . \tag{27.10}$$

These are classical results due to Dirichlet in the theory of the gravitational potential. We see immediately that by virtue of the fact that χ and Ω are harmonic functions, $\vec{u}$ is solenoidal. Equation (27.5) may also be satisfied. Note that

$$\text{curl}\,\vec{u} = 2B\,\vec{i} \times \nabla\chi = -2B\,\nabla\times(\vec{i}\,\chi) \; . \tag{27.11}$$

Hence ,

$$\begin{aligned} -\,\text{curl curl}\,\vec{u} &= -\nabla(\nabla\cdot\vec{u}) + \nabla^2\vec{u} \\ &= \nabla\left(2B\,\frac{\partial\chi}{\partial x}\right) \; . \end{aligned}$$

$$\therefore \text{ if } \qquad p_1 = 2\,\gamma K n^2 B\,\frac{\partial\chi}{\partial x} \; . \tag{27.12}$$

The Stokes equations will be satisfied by the solution for all A, B. It remains to satisfy the boundary conditions on the ellipsoid. (The free stream condition is readily seen to be satisfied). Now

$$\nabla F = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right) \text{ on } \lambda = 0 . \qquad (27.13)$$

$$\nabla\chi = -\nabla\lambda = -\left(\frac{2a^2}{x}, \frac{2b^2}{y}, \frac{2c^2}{z}\right) \text{ on } \lambda = 0 . \qquad (27.14)$$

$$\therefore \quad \text{curl}\,\vec{u} \times \nabla F = -\nabla F \times (\vec{i} \times \nabla\chi)\, 2B$$

$$= -2B\{\vec{i}\ \nabla F \cdot \nabla\chi - \nabla\chi\ \vec{i} \cdot \nabla F\}$$

$$= -2B\{-12\vec{i} + \frac{2x}{a^2}\nabla\lambda\} \text{ on } \lambda = 0 . \qquad (27.15)$$

$$\frac{\partial\Omega}{\partial x} = 2\pi\alpha\chi , \qquad (27.16)$$

$$\alpha = abc \int_\lambda^\infty \frac{d\tau}{(a^2+\tau)\,\Delta(\tau)} . \qquad (27.17)$$

$$\therefore \quad \vec{u}(\lambda=0) = \vec{i} - B\vec{i}\ \chi_0 - Bx\nabla\lambda + 2\pi\alpha_0 A\vec{i} - \frac{2\pi x}{a^2} A \nabla\lambda$$

$$= k_1{}'Kn\left\{-2B\left[-12\vec{i} + \frac{2x}{a^2}\nabla\lambda\right]\right\} \text{ on } \lambda = 0 . \qquad (27.18)$$

We therefore obtain

$$1 - B\chi_0 + 2\pi\alpha_0 A = 24B k_1'Kn ,$$

$$-B - \frac{2\pi A}{a^2} = -\frac{4B}{a^2} k_1'Kn ,$$

$$\therefore \qquad B\left(\frac{4k_1'Kn}{a^2} - 1\right) = \frac{2\pi A}{a^2} \; ;$$

$$\therefore \qquad A = \frac{a^2}{2\pi}\left(\frac{4k_1'Kn}{a^2} - 1\right) B \; ; \tag{27.19}$$

$$\therefore \qquad 1 = \left\{ \chi_0 - a^2\alpha_0\left(\frac{4k_1'Kn}{a^2} - 1\right) + 24k_1'Kn \right\} B .$$

Hence ,

$$B = \frac{1}{\chi_0 - a^2\alpha_0\left(\frac{4k_1'Kn}{a^2} - 1\right) + 24k_1'Kn} . \tag{27.20}$$

(27.19), (27.20) solve the problem.

$$\chi_0 = abc \int_0^\infty \frac{d\tau}{\Delta(\tau)} , \tag{27.21}$$

$$\alpha_0 = abc \int_0^\infty \frac{d\tau}{(a^2+\tau)\,\Delta(\tau)} , \tag{27.22}$$

These formulae agree with Lamb if Kn = 0.

$$\text{Drag} = 16\pi\, abcB$$

$$= \frac{16\pi\, abc}{\chi_0 - a^2\alpha_0\left(\frac{4k_1'Kn}{a^2} - 1\right) + 24k_1'Kn} . \tag{27.23}$$

It may be noted here that for the case when $\frac{Re}{\gamma M^2} = \epsilon$ is O(1), the vortex part $\vec{v}_1$ may be obtained in an exactly similar manner, once $\vec{u}_0$ has been determined by classical methods. The details of this calculation are not given here. We also note that the case of the free stream flowing at an arbitrary orientation to the ellipsoid is solved simply by superposition. This closes our discussion of applications. In the next and final section we make some comparison with experimental results.

§28. In this concluding section we compare the results deduced in the preceding sections with experiment. The solutions of Navier-Stokes equations with the classical slip conditions are valid only for $Kn << 1$. Some attempts have been made ([9], [20]) to understand phenomena when this condition does not hold. Kinetic theory has proved to be an unqualified success when $Kn >> 1$. The intermediate range has not received an equally consistent treatment. Goldberg's work [9] is based on Grad's moment technique applied to the Boltzmann equation. It is not clear whether this is really applicable when Kn is O(1). In any case, Goldberg's formula does not agree with Millikan's results ([16], [17], [18]) except at the continuum end.

It would appear therefore that there is a need for an interpolating theory that is simple to calculate with and which gives a theoretical interpretation to Millikan's classic researches on spheres. The proposal of this thesis is that the Navier-Stokes equations may be used provided Re and M are restricted to be small with Kn varying from zero to values that are not too large. Kn cannot be $>>1$, for then, the fluctuations would be of the same order as the mean flow quantities. The boundary conditions to be used in conjunction with the equations are assumed to be the ones stated in Chapter IV. These contain two arbitrary parameters k_1, k_2 which may in fact be slowly varying functions of Kn. The consequences of this model have been derived in the preceding chapters and are collected together below.

We concentrate attention on the case $Kn \sim O(1)$. For finite bodies, the drag coefficient C_D is defined as follows:

$$C_D = \frac{\text{Drag}}{\frac{1}{2}\rho u^2 L^2} \quad . \qquad (28.1)$$

From §23 we have,

$$C_D = \frac{12\pi}{Re\left[1 + k_1\left(\frac{M}{Re}\right)\right]} \quad . \qquad (28.2)$$

The non-dimensional pressure coefficient is

$$C_P = \frac{p_s}{p_\infty} - 1 = \frac{3}{2}(Re)\frac{\gamma Kn^2}{1 + k_1 Kn} \quad . \qquad (28.3)$$

Thus,

$$C_D = \frac{8\pi}{\gamma M^2}\, C_P \quad . \qquad (28.4)$$

For cylinders,

$$C_D = \frac{8\pi}{\mathrm{Re}\left[\log(\frac{4}{\mathrm{Re}}) + \frac{1}{2} - E + k_1(\frac{M}{\mathrm{Re}})\right]} . \tag{28.5}$$

$$C_P = \frac{(\mathrm{Re})\, 4\gamma \mathrm{Kn}^2}{\left[\log(\frac{4}{\mathrm{Re}}) + \frac{1}{2} - E + k_1(\frac{M}{\mathrm{Re}})\right]} . \tag{28.6}$$

For ellipsoids (with the conditions of §27) we have,

$$C_D = \frac{32\pi\, abc}{\mathrm{Re}\left[\chi_0 + \alpha_0 a^2 + \mathrm{Kn}(24k_1' - 4\alpha_0 k_1')\right]} ; \tag{28.7}$$

$$C_P = \frac{(\mathrm{Re})\,(4\gamma \mathrm{Kn}^2 a)}{\left[\chi_0 + a^2\alpha_0 + k_1'\mathrm{Kn}(24 - 4\alpha_0)\right]} . \tag{28.8}$$

The main experimental results in this regime are summarized in [2]. R. A. Millikan in a series of classic papers ([16], [17], [18]) determined drag coefficients for spheres for $\mathrm{Re} << 1$, $M << 1$. He was able to summarize his experimental results with a remarkably simple empirical formula that appears to fit the data very well. Apart from Goldberg's work already quoted, no theoretical interpretation for this formula appears to exist. As was already mentioned, Goldberg's formula deviates from the Millikan result in the range we are considering. In the papers cited Millikan gives the empirical formula

$$C_D = \frac{12\pi}{Re\left[1 + \left(A + B\, e^{-\left(\frac{G}{Kn}\right)}\right) Kn\right]} \quad . \qquad (28.9)$$

The constants A, B, G (independent of Re, M or Kn) have been determined to within a few percent. The formula is usually [22] written in terms of $\frac{\bar{\lambda}}{L}$ which differs from Kn by a numerical factor.

$$C_D = \frac{12\pi}{Re\left[1 + \left(A + Be^{-G\left(\frac{L}{\bar{\lambda}}\right)}\right)\left(\frac{\bar{\lambda}}{L}\right)\right]} \quad ; \qquad (28.10)$$

with ,

$$A = 1.23 \; ; \quad B = 0.41 \; ; \quad G = 0.88 \; . \qquad (28.11)$$

According to Millikan, the experiments may in general be fitted by

$$C_D = \frac{12\pi}{Re[1 + f(Kn)]} \quad . \qquad (28.12)$$

In the theory proposed here, we would interpret $\frac{f(Kn)}{Kn}$ as k_1, the surface interaction parameter. While Basset's results agree with (28.10) only for $Kn << 1$, it will be noticed that (28.2) has the same form for all Kn. Since we do have an extra arbitrary parameter k_1 at our disposal this agreement must only be taken to mean that the theory is in qualitative agreement with experiment and summarizes the results in a compact form. The formula (28.4) is a test of the theory since only measurable quantities appear in it. C_P appears not to be available in this regime. It is conceivable that this might be measured by observing motions of balloons in the higher atmosphere. Atassi and Shen [20] use an ansatz on the Boltzmann distribution function and calculate in the continuum approximation the drag coefficient for circular cylinders.

$$C_D = \frac{4\pi}{Re\left[\log(\frac{4}{Re}) - E + \frac{1}{2} + 1.55Kn + c_1 Kn^2\right]} \quad ; \qquad (28.13)$$

for $Kn << 1$, which is almost the same form as (28.5). The specific form of the functional dependence of k_1 on Kn will depend somewhat on the conventional element involved in the definition of Re.

We may mention here certain interesting applications of the foregoing theory to the theory of aerosols. We had already noted that when $Kn \sim O(1)$, the classic Einstein formula for the diffusivity of an aerosol would have to be modified. From (28.2) we see that the new formula would be (if a is the radius of the sphere)

$$D = \left(\frac{RT}{N}\right) \frac{\left(1 + k_1\left(\frac{M}{Re}\right)\right)}{6\pi\mu a} \quad ; \qquad (28.14)$$

$$Kn = \frac{M}{Re} = \left(\frac{\bar{\lambda}}{a}\right)\sqrt{\frac{2}{\pi\gamma}} \quad . \qquad (28.15)$$

(Note that this definition of Kn differs from that of [2] where $\frac{\bar{\lambda}}{a}$ is called K). Thus the diffusivity is enhanced by the factor

$$\left[1 + k_1 \sqrt{\frac{2}{\pi\gamma}} \left(\frac{\bar{\lambda}}{a}\right)\right].$$

If we take $k_1 = \sqrt{\frac{\pi\gamma}{2}} \left(A + B\, e^{-G\, a/\bar{\lambda}}\right)$ following (28.10), we obtain the empirical correction to Cunningham's kinetic formula given by Knudsen and Weber [19].

We also note the results of a calculation (using a method due to Landau and Lifshitz [22] (§22)) of the viscosity of dilute suspensions of very small particles ($Kn \sim O(1)$) in gases. This involves the solution of the Stokes equations for flows past spheres when the flow at infinity is a uniform shear flow. The viscosity of the suspension is denoted by μ_s and has the value:

$$\mu_s = \mu\left[1 + \frac{5}{2}\left(\frac{1}{1 + 3k_1 Kn}\right)\phi\right] \quad ; \qquad (28.16)$$

where ,

$$\phi = \frac{4}{3}\pi a^3 C ,$$

a = radius of the spherical particles ,

C = number of particles/unit volume ,

$$Kn = \left(\frac{\bar{\lambda}}{a}\right)\sqrt{\frac{2}{\pi\gamma}} .$$

Here we see again that Einstein's classic $\frac{5}{2}$ is diminished by the factor $\frac{1}{1+3k_1 Kn}$. We may eliminate $k_1 Kn$ between (28.14) and (28.15) and get a relationship between purely observable quantities. Such a relation ought to provide a sound check on the validity of the model. If we write,

$$D_E = \left(\frac{RT}{N}\right) \frac{1}{6\pi\mu a} \quad ; \tag{28.17}$$

$$D = D_E (1+k_1 Kn) \; ; \tag{28.18}$$

we obtain the following equation:

$$\mu_s = \mu\left[1 + \frac{5}{2}\phi\left(\frac{D_E}{3D-2D_E}\right)\right]. \tag{28.19}$$

Finally, we summarize the results for ϵ taking on moderate or small values. In §12, we mentioned that the Lagerstrom-Chester theory gives a drag coefficient $C_D \propto \frac{1}{Re}$. This result directly contradicts experiment and free-molecular flow theory that applies to this regime. The results of §24 show that the present condition predicts a drag coefficient given by (24.20) that goes over smoothly into the free-molecular value. The leading order potential stresses that exist in this theory contribute no drag. This is an unsatisfactory feature that must be investigated further from a microscopic point of view. Pending such investigation, the results must be looked upon with due suspicion and caution. It will be noticed that the stagnation pressure predicted by the work of §24 are O(M) unlike the $O(\frac{1}{\epsilon})$

stagnation pressure of the Lagerstrom-Chester theory.

In this work, only the leading terms of asymptotic expansions were worked out. These deductions should be unambiguously checked out with experiment and theoretical analysis using kinetic theory where possible, before one proceeds with higher approximations. Certain phenomena like thermal creep have been left out in the interests of clarity and simplicity. There are indications that for certain problems, the more general condition (15.14) is appropriate.

The results on aerosols may conceivably find use in chemical physics or even environmental science. Under certain conditions, it is not impossible that these results could be applied to astrophysical problems involving dust and gas. It would be interesting to extend these ideas about slip flow (particularly, the slip condition) to regimes of flow which may not satisfy $Re << 1$ and also to unsteady problems. A microscopic justification of the slip condition appears to present formidable difficulties and is not attempted.

Appendix I

We collect certain formulae here for reference.

Cylindrical Polars: R, ϕ, z : If u, v, w are the respective velocity components, we have

$$\frac{1}{R}\frac{\partial}{\partial R}(R\rho u) + \frac{1}{R}\frac{\partial}{\partial \phi}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 . \qquad \text{(AI. 1)}$$

The vorticity components are

$$\xi = \frac{1}{R}\frac{\partial w}{\partial \phi} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial R}, \quad \zeta = \frac{1}{R}\frac{\partial}{\partial R}(Rv) - \frac{1}{R}\frac{\partial u}{\partial \phi} . \qquad \text{(AI. 2)}$$

The shear rates of strain are:

$$\tfrac{1}{2}e_{RR} = \frac{\partial u}{\partial R}, \quad \tfrac{1}{2}e_{\phi\phi} = \frac{1}{R}\frac{\partial v}{\partial \phi} + \frac{u}{R}, \quad \tfrac{1}{2}e_{zz} = \frac{\partial w}{\partial z},$$

$$e_{\phi z} = \frac{1}{R}\frac{\partial w}{\partial \phi} + \frac{\partial v}{\partial z}, \; e_{zR} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial R}, \; e_{R\phi} = R\frac{\partial}{\partial R}\left(\frac{v}{R}\right) + \frac{1}{R}\frac{\partial u}{\partial \phi} . \qquad \text{(AI. 3)}$$

The Navier-Stokes momentum equations are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial R} + \frac{v}{R}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} - \frac{v^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{\mu}{\rho}\left(\nabla^2 u - \frac{u}{R^2} - \frac{2}{R^2}\frac{\partial v}{\partial \phi}\right)$$

$$+ \frac{(\lambda+\mu)}{\rho}\frac{\partial}{\partial R}(\nabla \cdot \vec{v}) ,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial R} + \frac{v}{R}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + \frac{uv}{R} = -\frac{1}{\rho}\frac{1}{R}\frac{\partial p}{\partial \phi} + \frac{\mu}{\rho}\left(\nabla^2 v + \frac{2}{R^2}\frac{\partial u}{\partial \phi} - \frac{v}{R^2}\right)$$

$$+ \frac{(\lambda+\mu)}{\rho}\,\frac{1}{R}\,\frac{\partial}{\partial \phi}(\nabla \cdot \vec{v}) \; ,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial R} + \frac{v}{R}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\nabla^2 w + \frac{(\lambda+\mu)}{\rho}\frac{\partial}{\partial z}(\nabla \cdot \vec{v}), \qquad \text{(AI. 4)}$$

where,

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \; . \qquad \text{(AI. 5)}$$

Spherical Polars: R. θ, ϕ, velocities u, v, w respectively

$$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 \rho u) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(\rho v \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}(\rho w) = 0 \; , \qquad \text{(AI. 6)}$$

$$\xi = \frac{1}{R\sin\theta}\left\{\frac{\partial}{\partial \theta}(w\sin\theta) - \frac{\partial v}{\partial \phi}\right\} \; ;$$

$$\eta = \frac{1}{R\sin\theta}\left\{\frac{\partial u}{\partial \phi}\right\} - \frac{1}{R}\frac{\partial}{\partial R}(Rw) \; ; \qquad \text{(AI. 7)}$$

$$\zeta = \frac{1}{R}\frac{\partial}{\partial R}(Rv) - \frac{1}{R}\frac{\partial u}{\partial \theta} \; ;$$

$$\tfrac{1}{2}e_{RR} = \frac{\partial u}{\partial R}\ , \quad \tfrac{1}{2}e_{\theta\theta} = \frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{u}{R}\ , \quad \tfrac{1}{2}e_{\phi\phi} = \frac{1}{R\sin\theta}\ \frac{\partial w}{\partial \phi} + \frac{u}{R} + \frac{v\cot\theta}{R}\ ,$$

$$e_{\theta\phi} = \frac{\sin\theta}{R}\ \frac{\partial}{\partial\theta}\left(\frac{w}{\sin\theta}\right) + \frac{1}{R\sin\theta}\ \frac{\partial v}{\partial\phi}\ ,$$

$$e_{\phi R} = \frac{1}{R\sin\theta}\ \frac{\partial u}{\partial\phi} + R\ \frac{\partial}{\partial R}\left(\frac{w}{R}\right)\ , \tag{AI. 9}$$

$$e_{R\theta} = R\ \frac{\partial}{\partial R}\left(\frac{v}{R}\right) + \frac{1}{R}\ \frac{\partial u}{\partial\theta}\ ,$$

$$\nabla^2 = \frac{1}{R^2}\ \frac{\partial}{\partial R}\left(R^2\ \frac{\partial}{\partial R}\right) + \frac{1}{R^2\sin\theta}\ \frac{\partial}{\partial\theta}\left(\sin\theta\ \frac{\partial}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\ \frac{\partial^2}{\partial\phi^2}\ .$$

The momentum equations are the following:

$$\frac{\partial u}{\partial t} + u\ \frac{\partial u}{\partial R} + \frac{v}{R}\ \frac{\partial u}{\partial\theta} + \frac{w}{R\sin\theta}\ \frac{\partial u}{\partial\phi} - \frac{v^2 + w^2}{R}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{\mu}{\rho}\left(\nabla^2 u - \frac{2u}{R^2} - \frac{2}{R^2}\ \frac{\partial v}{\partial\theta} - \frac{2v}{R^2}\cot\theta - \frac{2}{R^2\sin\theta}\ \frac{\partial w}{\partial\phi}\right)$$

$$+ \frac{(\lambda+\mu)}{\rho}\ \frac{\partial}{\partial R}\ (\nabla\cdot\vec{v})\ , \tag{AI. 10a}$$

$$\frac{\partial v}{\partial t} + u\ \frac{\partial v}{\partial R} + \frac{v}{R}\ \frac{\partial v}{\partial\theta} + \frac{w}{R\sin\theta}\ \frac{\partial v}{\partial\phi} + \frac{uv}{R} - \frac{w^2\cot\theta}{R}$$

$$= -\frac{1}{\rho}\frac{1}{R}\frac{\partial p}{\partial\theta} + \frac{\mu}{\rho}\left(\nabla^2 v + \frac{2}{R^2}\ \frac{\partial u}{\partial\theta} - \frac{v}{R^2\sin^2\theta} - \frac{2\cos\theta}{R^2\sin^2\theta}\ \frac{\partial w}{\partial\phi}\right)$$

$$+ \frac{(\lambda+\mu)}{\rho}\ \frac{1}{R}\ \frac{\partial}{\partial\theta}\ (\nabla\cdot\vec{v})\ , \tag{AI. 10b}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial R} + \frac{v}{R}\frac{\partial w}{\partial \theta} + \frac{w}{R\sin\theta}\frac{\partial w}{\partial \phi} + \frac{wu}{R} + \frac{vw\cot\theta}{R}$$

$$= -\frac{1}{\rho}\left(\frac{1}{R\sin\theta}\right)\frac{\partial p}{\partial \phi} + \frac{\mu}{\rho}\left(\nabla^2 w - \frac{w}{R^2\sin^2\theta} + \frac{2}{R^2\sin\theta}\frac{\partial u}{\partial \phi} + \frac{2\cos\theta}{R^2\sin^2\theta}\frac{\partial v}{\partial \phi}\right)$$

$$+ \frac{(\lambda+\mu)}{\rho}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}(\nabla\cdot\vec{v}) \; . \qquad \text{(AI. 10c)}$$

Methods of writing these equations down for arbitrary coordinate systems will be found in [1]. We will use only the above results.

Appendix II

Calculation of an Axisymmetric Green's Function

We wish to solve the following problem:

$$\nabla^2\vec{u} + \frac{1}{\alpha}\nabla(\nabla\cdot\vec{u}) = \vec{g} = \nabla w \ . \qquad \text{(AII. 1)}$$

$\vec{u} \to \vec{i}$ at infinity. $u_R = 0$ on $R = 1$. $u_\theta = \tau \frac{\partial u_\theta}{\partial R}$ $R = 1$. w is a function of R, θ. Here the origin of coordinates is at the centre of the sphere, and the polar angle θ is measured from the positive x-axis. $\vec{i}$ is the unit vector along the x-axis. The correlation equations reduce to

$$\nabla^2 \vec{v} = \nabla\chi \ ,$$

$$\nabla \cdot \vec{v} = 0 \ , \qquad \text{(AII. 2)}$$

$$\chi + (1 + \frac{1}{\alpha}) \nabla^2 \phi = w \ ,$$

where ,

$$\vec{u} = \nabla\phi + \vec{v} \ . \qquad \text{(AII. 3)}$$

The boundary conditions involve both $\vec{v}$ and ϕ. We solve (AII. 2) using a slight adaptation of a method due to Stokes, used extensively by Proudman and Pearson. We split the potential ϕ into two parts ,

$$\phi = \Phi_I + \Phi_{II} . \tag{AII.4}$$

Here ,

$$(1 + \frac{1}{\alpha}) \nabla^2 \Phi_I = w \quad ; \tag{AII.5}$$

with ,

$$\nabla \Phi_I \rightarrow 0 \text{ at } \infty \quad ;$$

and

$$\Phi_I = 0 \quad \text{on} \quad R = 1.$$

We shall assume that the w's we deal with in our work die out sufficiently rapidly at infinity for this problem (Poisson problem in an unbounded domain) to have a unique solution. The solution of this problem utilizes the classic methods of potential theory and will be assumed known.

Now we have to solve:

$$\nabla^2 \vec{v} = \nabla \chi \quad , \tag{AII.6}$$

$$\nabla \cdot \vec{v} = 0 \quad ,$$

$$\chi + (1 + \frac{1}{\alpha}) \nabla^2 \Phi_{II} = 0 \quad ,$$

$$\vec{v} + \nabla\Phi_{II} \to \vec{i} \quad ;$$

$$v_R + \frac{\partial \Phi_{II}}{\partial R} = - \frac{\partial \Phi_I}{\partial R} \quad ;$$

$$v_\theta + \frac{1}{R} \frac{\partial \Phi_{II}}{\partial \theta} = \tau \frac{\partial}{\partial R}\left(v_\theta + \frac{1}{R} \frac{\partial \Phi_{II}}{\partial \theta}\right)R = 1 \quad ;$$

Since w is a function of R, θ, so is Φ_I. The entire flow is axi-symmetric.

Writing,

$$v_R = \frac{1}{R^2 \sin\theta} \frac{\partial \Psi}{\partial \theta} \quad , \tag{AII. 7}$$

$$v_\theta = - \frac{1}{R \sin\theta} \frac{\partial \Psi}{\partial R} \quad , \tag{AII. 8}$$

we have,

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1-x^2}{R^2} \frac{\partial^2}{\partial x^2}\right)^2 \Psi = 0 \quad ; \tag{AII. 9}$$

$x = \cos\theta$. χ is calculated if Ψ is known. The general solution of (AII. 9) subject to appropriate regularity conditions on the sphere is

$$\Psi(R, x) = \sum_{j=1} \psi_j(R) \int_{-1}^{x} P_j(t)dt \quad . \tag{AII. 10}$$

$\{P_j\}$ are the orthonormal Legendre polynomials.

$$\left(\frac{d^2}{dR^2} - \frac{j(j+1)}{R^2}\right) \Psi_j(R) = 0 \; . \qquad \text{(AII. 11)}$$

We also have the expansions,

$$\Phi_{II} = \sum_{j=0} \Phi_j^{(II)}(R) \; P_j(x) \; , \qquad \text{(AII. 12)}$$

$$\chi = \sum_{j=1} \chi_j \; \frac{P_j(x)}{R^{j+1}} \; . \qquad \text{(AII. 13)}$$

Here $\{\chi_j\}$ are constants which are known in terms of the constants of integration in $\Psi_j(R)$. This expression follows from the fact that χ is a harmonic function.

Substitution into (AII. 6) utilizing an expansion of Φ_I similar to (AII. 12), gives ψ_j, Φ_j^{II}, χ_j. We need to solve only simple algebraic equations among the various constants which are in fact related linearly. Since the calculation is the same for any $w(R, \theta)$ we will have effectively calculated the axisymmetric Green's function.

Appendix III

The No-slip Limit of ρ_1 on $R = 1$

According to the point of view adopted in this work, the density distribution ρ_1 on the sphere is to be obtained by taking the limit of $\tau \to 0$ in (II.29) and (II.30). We note that $\zeta(-1) = -\infty$ and $\zeta(+1) = -\infty$ and $\zeta(t)$ has a maximum at $t = -\frac{\epsilon}{B}$. From (II.29) we deduce that for $x < x_0$ where $x_0 = -\frac{\epsilon}{B}$, the leading term in the asymptotic expansion of σ_1 is,

$$\sigma_1 \simeq \frac{\epsilon\, T(x)}{\epsilon + Bx} \quad . \tag{AIII. 1}$$

which is also the leading term outer solution of (II.26) with τ as the small parameter. When $x \geqslant x_0$, it is convenient to use Watson's lemma on (II.29) to get,

$$\sigma_1 e^{\frac{1}{2U_1(1+\frac{1}{\alpha})}\left[\epsilon \log \frac{1+x}{1-x} - B\log(1-x^2)\right]}$$

$$\simeq \frac{\epsilon}{U_1(1+\frac{1}{\alpha})}\left(\frac{T(x_0)}{1-x_0^2}\right)\left(e^{\frac{1}{2U_1(1+\frac{1}{\alpha})}\left[\epsilon \log \frac{1+x_0}{1-x_0} - B\log(1-x_0^2)\right]}\right)$$

$$\times \int_{-\infty}^{\infty} e^{\zeta''(x_0)u^2}\, du \quad . \tag{AIII. 2}$$

$$\zeta''(x_0) = \frac{1}{U_1(1+\frac{1}{\alpha})} \frac{B}{1-x_0^2} < 0 \quad . \qquad \text{(AIII. 3)}$$

$$\therefore \quad \rho_1(x_0) \to 0 \text{ like } \tau^{\frac{1}{2}} \; . \qquad \text{(AIII. 4)}$$

For $x > x_0$, $\rho(x)$ is transcendentally small in τ.

$$\rho_1(x_0) \simeq \frac{1-x_0^2}{T(x_0)} \left(\frac{U_1(1+\frac{1}{\alpha})}{\epsilon}\right) \frac{1}{\sqrt{\pi}} \sqrt{\frac{-B}{U_1(1+\frac{1}{\alpha})(1-x_0^2)}} \; , \qquad \text{(AIII. 5)}$$

$$x_0 = -\frac{\epsilon}{B} \; .$$

This expansion is valid for $\tau \to 0$. We note that the limit process is $\tau \to 0$ for small but <u>fixed</u> ϵ.

Appendix IV

On Certain Asymptotic Expansions

In Chapter III we studied an iterative method for the solution of the system (10.1) to (10.4). An equivalent perturbation scheme is presented here. It appears to be worth studying for its own sake. Consider the following system of equations:

$$\nabla^2 \vec{v} = \nabla\chi ,$$

$$\nabla \cdot \vec{v} = 0 ,$$

$$\chi + (1+\frac{1}{\alpha})\nabla \phi = \epsilon(\frac{1}{\sigma} - 1) , \qquad \text{(AIV.1)}$$

$$(\vec{v} + \nabla\phi)\cdot\nabla\sigma + \frac{\chi\sigma}{1+\frac{1}{\alpha}} = \frac{\Delta(1-\sigma)}{1+\frac{1}{\alpha}} .$$

The boundary conditions are the same as for (10.1) - (10.4). Here $\epsilon << 1$. But the parameter Δ is arbitrary. Suppose we attempt to solve this by the following simple perturbation scheme:

$$\vec{v} = \sum \vec{v}_\nu \epsilon^\nu ,$$

$$\chi = \sum \chi_\nu \epsilon^\nu , \qquad \text{(AIV.2)}$$

$$\sigma = \sum \sigma_\nu \epsilon^\nu ,$$

$$\phi = \sum \phi_\nu \epsilon^\nu ,$$

The functions $\{\chi_\nu\}$, $\{\sigma_\nu\}$ etc. will in general depend on Δ. Suppose that the solution so found is an asymptotic solution to (AIV.1) for all $\Delta > 0$. We would then obtain an asymptotic solution to the set $\{10.2\}$ by setting $\Delta = \epsilon$. The underlying idea in this scheme is the same as in the iteration scheme of Chapter III: we do not wish to neglect the effect of the density fluctuations in the equation of continuity. The equations to leading order are :

$$\nabla^2 \vec{v}_0 = \nabla \chi_0 ,$$

$$\nabla \cdot \vec{v}_0 = 0 , \qquad \text{(AIV.3)}$$

$$\chi_0 + (1 + \frac{1}{\alpha}) \nabla^2 \phi_0 = 0 ,$$

$$(\vec{v}_0 + \nabla\phi_0) \cdot \nabla\sigma_0 + \frac{\chi_0 \sigma_0}{1 + \frac{1}{\alpha}} = \frac{\Delta(1-\sigma_0)}{1 + \frac{1}{\alpha}} .$$

The velocity is the Lagerstrom-Chester field. The next order equations are :

$$\nabla^2 \vec{v}_1 = \nabla \chi_1 ,$$

$$\nabla \cdot \vec{v}_1 = 0 ,$$

$$\chi_1 + (1 + \frac{1}{\alpha}) \nabla^2 \phi_1 = (\frac{1}{\sigma_0} - 1) , \qquad \text{(AIV.4)}$$

$$(\vec{v}_0 + \nabla\phi_0) \cdot \nabla\sigma_1 + \frac{\chi_0 \sigma_1}{1 + \frac{1}{\alpha}} = - \frac{\Delta \sigma_1}{1 + \frac{1}{\alpha}} - (\vec{v}_1 + \nabla\phi_1) \cdot \nabla\sigma_0 - \frac{\chi_1 \sigma_0}{1 + \frac{1}{\alpha}} .$$

Clearly σ_0 is a function of Δ. The higher terms are calculated similarly. The relationship of the iterative procedure to this perturbation procedure is evident. The question of convergence seems just as difficult a problem as before. The equation for σ_0 has an exact solution calculable as in Chapter III. The next order equations are not difficult to solve in theory but are very complicated in practice. The solution to the original problem is obtained by a "confluence" of the solution to (AIV.1) by setting $\Delta = \epsilon$. We note that the dependence on Δ is in general very complicated and hence the resulting expansion is <u>not</u> a simple limit process type expansion in ϵ.

Appendix V

Fourier Transform Solution of the Oseen Equations

We indicate below the simple Fourier transform solution of the Oseen equations. The order of the singularity is deliberately left out. The equations are:

$$\frac{\partial s}{\partial \tilde{x}} + \tilde{\nabla}^2 \psi = 0 \; ,$$

$$\frac{\partial \psi}{\partial \tilde{x}} + \frac{1}{\gamma M^2}\, \omega = (1+\frac{1}{\alpha})\, \nabla^2 \psi \; , \qquad \text{(A V. 1)}$$

$$\frac{\partial \theta}{\partial \tilde{x}} - (\frac{\gamma - 1}{\gamma})\, \frac{\partial \omega}{\partial \tilde{x}} = \frac{1}{Pr}\, \tilde{\nabla}^2 \theta \; ,$$

$$\omega = s + \theta \; .$$

We express s, ψ, ω, θ in terms of their Fourier transforms ,

$$s = \frac{1}{(2\pi)^3} \int e^{i \tilde{k} \cdot \tilde{x}}\, d^3 \tilde{k}\; S(\tilde{k}); \; \tilde{k} = (t, r, s) \; ,$$

$$\psi = \frac{1}{(2\pi)^3} \int e^{i \tilde{k} \cdot \tilde{x}}\, d^3 \tilde{k}\; \Psi(\tilde{k}) \; , \qquad \text{(A V. 2)}$$

$$\theta = \frac{1}{(2\pi)^3} \int e^{i \tilde{k} \cdot \tilde{x}}\, d^3 \tilde{k}\; \Theta(\tilde{k}) \; .$$

If we substitute and get a single equation for $S(\tilde{k})$ we find that we may choose

$$S(\tilde{k}) = \frac{it}{t^2 - (1+\frac{1}{\alpha})it(t^2+k^2) - \frac{1}{\gamma M^2}\left\{1 - \frac{\frac{\gamma-1}{\gamma}}{\frac{it}{\gamma} + \frac{\frac{it}{t^2+k^2}}{Pr}}\right\}(t^2+k^2)} ,$$

$$S(\tilde{k}) = \frac{it}{-\left((1+\frac{1}{\alpha})it + \frac{1}{\gamma M^2}\right)\left[\frac{-(1+\frac{1}{\alpha})it^3 - \frac{t^2(1-\gamma M^2)}{\gamma M^2}}{-(1+\frac{1}{\alpha})it - \frac{1}{\gamma M^2}} + k^2 + F(k,t)\right]} .$$

$$\therefore \quad S(\tilde{k}) = \frac{it}{-\left((1+\frac{1}{\alpha})it + \frac{1}{\gamma M^2}\right)\left[\frac{t - \frac{it^2(1-\gamma M^2)}{(1+\frac{1}{\alpha})\gamma M^2}}{t - \frac{i}{\gamma M^2(1+\frac{1}{\alpha})}} + k^2 + F(k,t)\right]} .$$

Henceforth we will consider,

$$w(\tilde{x}) = \frac{1}{(2\pi)^3}\int \frac{e^{i\tilde{k}\cdot\tilde{x}}\, d^3\tilde{k}}{[m^2 + k^2 + F(k,t)]} . \qquad \text{(A V. 3)}$$

Here the function $F(k,t)$ is defined by the following equation.

$$F(k,t) = \left(\frac{\gamma-1}{\gamma}\right)\frac{1}{\gamma M^2}\left(\frac{it\,(t^2+k^2)}{\frac{it}{\gamma}+\frac{(t^2+k^2)}{Pr}}\right)\left[\frac{i(1+\frac{1}{\alpha})^{-1}}{t-\frac{i}{\gamma M^2\,(1+\frac{1}{\alpha})}}\right],$$

$$= \left(\frac{\gamma-1}{\gamma^2 M^2}\right)\left(\frac{t(t^2+k^2)}{\frac{it}{\gamma}+\frac{(t^2+k^2)}{Pr}}\right)\left[\frac{(1+\frac{1}{\alpha})^{-1}}{\frac{i}{\gamma M^2\,(1+\frac{1}{\alpha})}-t}\right]. \qquad \text{(A V. 4)}$$

If $Pr \to 0$, we get the function defined in Chapter VI. For small Pr, $F(k,t) \simeq \left(\frac{\gamma-1}{\gamma}\right)\frac{Pr}{\gamma M^2}\frac{[-t(1+1/\alpha)]}{t-\frac{i}{\gamma M^2\,(1+1/\alpha)}}$ for large t, k.

Hence for small Pr, $F(k,t)$ can be absorbed in m^2. For large Pr, we should consider a slightly different definition of S and w. We would still be able to express w in terms of an integral involving k_0. For moderate Pr, it is convenient to evaluate the integral over k by residues. The resulting integrals are more complicated. The asymptotic expansion of w near its singularity is obtained by studying the asymptotic expansion of its Fourier transform for large $|\tilde{k}|$. The nature of the singularity can now be explicitly found. From the definition of $S(\tilde{k})$ it is easily seen that it is the Fourier transform of $Y(\tilde{x})$. This concludes this brief discussion of the transform theory of the Oseen equations.

Appendix VI

A Mathematical Model

The Lagerstrom-Chester problem is characterized by the fact that if the pressure term in the momentum equations is dropped entirely, the density is singular. The following mathematical model illustrates the crucial importance of "small" terms in certain equations and the variety of phenomena they may describe. The model can be related to a Lagerstrom-Chester problem where the kinematic viscosities are taken constant.

Consider the simple linear equation ,

$$\frac{dy}{dx} + \frac{\epsilon\, y}{x^2(1-x)} = \frac{1}{1-x} \quad ; \qquad \text{(A VI. 1)}$$

$0 < \epsilon << 1$, $y(0) = 0$, in the range $[0,1]$. If we drop the ϵ term, we obtain ,

$$\frac{dy}{dx} = \frac{1}{1-x} \implies y = -\log(1-x) \quad ; \qquad \text{(A VI. 2)}$$

which apparently fits the initial condition but is infinite at $x = 1$. The exact solution is easily obtained and it shows that $\lim_{x \to 1} y(x) = \frac{1}{\epsilon}$. We also note that this solution cannot be iterated as the singularities compound. It is also true that the solution gives $y(0)$ correctly; however, $y'(0) = 0$ from the exact solution and here we see behaviour similar to the Lagerstrom-Chester velocity field which is a good approximation to the velocity but not to the divergence of the velocity.

We now give a uniform representation. At $x = 0$:

$$\text{Set :} \qquad x = \tilde{x}\epsilon \ ; \ y = \epsilon Y .$$

We get

$$\frac{dY}{d\tilde{x}} + \frac{Y}{\tilde{x}^2} = 1 .$$

$$\therefore \qquad Ye^{-\frac{1}{\tilde{x}}} = \int_0^{\tilde{x}} e^{-\frac{1}{t}} dt . \qquad \text{(A VI. 3)}$$

$$\therefore \qquad y = \epsilon\, e^{+\frac{\epsilon}{x}} \int_0^{\frac{x}{\epsilon}} e^{-\frac{1}{t}} dt . \qquad \text{(A VI. 4)}$$

This is the leading order inner solution. It is easy to see the matching.

At $x = 1$: The layer here is not of a classical type and this illustrates the fact that the usual "stretching and matching" procedure may not always work. Put,

$$x = 1 - e^{\zeta} ;$$

$$\frac{dy}{d\zeta} - \frac{\epsilon\, y}{(1-e^{\zeta})^2} = -1 \ . \qquad \text{(A VI. 5)}$$

Put ,

$$\zeta = -\frac{z}{\epsilon} \ ; \qquad y = \frac{\tilde{y}}{\epsilon} \ ;$$

$$\frac{d\tilde{y}}{dz} + \tilde{y} = +1 \ .$$

$$\therefore \qquad \tilde{y} = 1 + B\, e^{-z} \ .$$

$$\therefore \qquad y = \frac{1}{\epsilon}\left(1 + B(1-x)^{\epsilon}\right) \ .$$

Now for $x \neq 1$ as $\epsilon \to 0$ this tends to $-\log(1-x)$ if $B = -1$. $\therefore$ from matching, in this "transcendental layer",

$$y = \frac{1}{\epsilon}\left(1 - (1-x)^{\epsilon}\right) + O(\epsilon) \ . \qquad \text{(A VI. 6)}$$

$\therefore$ we have the three regions where we have found y to leading order. The last solution shows that $y(1) = \frac{1}{\epsilon}$ as the exact solution predicts. Again, Y is $O(\tilde{x}^2)$ as $\tilde{x} \to 0$.

An equation very similar in structure arises in the following context.

$$\nabla\psi \cdot \vec{u} + \frac{(\epsilon\,\psi - \chi)}{1 + \frac{1}{\alpha}} = 0 \ . \qquad \text{(A VI. 7)}$$

where $\chi = \frac{B\cos\theta}{R^2}$; $\vec{u}$ is the Lagerstrom-Chester field, say. If we consider the limiting streamline from infinity to the stagnation point at the front, the equation along characteristic is very similar to the model if one makes a variable change from R to $R = \frac{1}{x}$. Equation (A VI. 7) is obtained from

$$\nabla \cdot \rho \vec{u} = 0 , \qquad \text{(A VI. 8)}$$

$$\epsilon \nabla \rho = \rho \nu \nabla^2 \vec{u} + \rho(\nu + \kappa) \nabla(\nabla \cdot \vec{u}) . \qquad \text{(A VI. 9)}$$

where ν, κ are constant; the variable $\Psi = \log \rho$ is introduced. This model is only of academic interest.

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