NONLINEAR DISPERSIVE WAVES
IN NONLINEAR OPTICS

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ABSTRACT

A study is made of solutions of the macroscopic Maxwell equations in nonlinear media. Both nonlinear and dispersive terms are responsible for effects that are not taken into account in the geometrical optics approximation. The nonlinear terms can, depending on the nature of the nonlinearity, cause plane waves to focus when the amplitude varies across the wavefront. The dispersive terms prevent the singularities that nonlinearity alone would produce. Solutions are found which describe periodic plane waves in fully nonlinear media. Equations describing the evolution of the amplitude, frequency and wave number are generated by means of averaged Lagrangian techniques. The equations are solved for near linear media to produce the form of focusing waves which develop a singularity at the focal point. When higher dispersion is included nonlinear and dispersive effects can balance and one finds amplitude profiles that propagate with straight rays.

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CHAPTER I

INTRODUCTION AND GOVERNING EQUATIONS

I.1. Introduction

The interest in nonlinear effects in electromagnetic theory at optical frequencies has evolved from several roots. The most obvious one is the practical application of these effects that is now possible due to the invention of the laser, a high intensity source of coherent light, without which the effects are too small to be detected. Some aspects of the theory were developed before the laser, however, as the study of electromagnetic waves at radio frequencies involves the use of nonlinear effects. The similarity between the optical effects and those found in fluid mechanics is striking; many researchers are producing material in both fields. The pioneering work in self-focusing beams, with which this thesis is concerned, was done in the context of the propagation of electromagnetic waves in plasma, a field which gained impetus from the study of the relationship between the ionosphere and long range communication. In that respect it might be said that the study of electric discharges in gases and the invention of the vacuum tube have played a part in the evolution of interest in nonlinear optics.

In this first chapter the equations for the classical theory of electromagnetic waves are introduced. These are accompanied by differential equations describing the model chosen for the medium. The medium is represented by a continuum of oscillating dipoles, the negatively charged particle responding to the electric field and to the potential well created by the positive core of the atom. The potentials for the electric and magnetic quantities are introduced so that a Lagrangian may be found. The Lagrangian and associated variational principle will be the main tool in the analysis of the nonlinear phenomena.

In Chapter II, periodic plane wave solutions to the governing equations are derived. The special cases of linear and circular polarization are discussed and various special forms of the nonlinear terms are studied in detail. For cubic restoring forces, exact solutions in the form of Jacobian elliptic functions are found. The case in which the cubic term is small is then dealt with using a much simpler approximate method where the periodic waves are taken as sinusoidal. This case is referred to as near linear. For purposes of illustration the general solution is expanded for small values of the cubic term of the restoring force to reproduce the near linear solution.

The averaged Lagrangian technique is introduced in chapter III. This technique is used to obtain equations governing the slow variation of amplitudes, wave number and frequency, quantities that are constant in the case of periodic plane waves. The consequences are studied for the time-dependent modulations of nonlinear plane waves. These are concerned with the way in which modulations and wave packets propagate and how nonlinearity affects the more familiar linear results.

In chapter IV methods are applied to beams where self-focusing is the phenomenon of most interest. They may be considered as spatial modulations similar to the time-like modulations of chapter III. The features of the general two-dimensional and three-dimensional radially symmetric cases are discussed, and the near linear problem is used as an illustration. An analogy is drawn between these equations and the equations of fluid mechanics; methods taken from the theory of fluid mechanics are then adapted to optics. The equations are solved to produce representations of self-focusing beams. A solution is produced for the case of a near linear beam of large width whose rays bend significantly before focusing and a review is made of treatments of thin beams whose rays never deviate greatly from being parallel to the axis of propagation.

The first order theory of modulations leads to singularities which are resolved by including higher order effects of dispersion. These questions are taken up in chapter V. Averaged Lagrangians are found both in the case of linear and circular polarization. Special solutions are found which have straight parallel rays but whose amplitudes vary in a variety of ways. For example, time-independent solutions are found where the beam is localized about the axis of propagation and propagates without distortion and also solitary wave envelopes are found.

I.2. The Equations for the Classical Model

The problem that will be dealt with throughout the thesis is concerned with the propagation of classical electromagnetic waves through a medium which will be modeled by a continuous distribution of electric dipoles. The electromagnetic fields will be described by the macroscopic Maxwell equations in vacuum. The medium will be felt through the source terms.

The equations take the form

$$\nabla \times \underline{H} = \underline{J} + \epsilon_o \frac{\partial \underline{E}}{\partial t} , \qquad \nabla \cdot \underline{H} = 0 ,$$
$$\nabla \times \underline{E} = - \mu_o \frac{\partial \underline{H}}{\partial t} , \qquad \nabla \cdot \underline{E} = \frac{\rho}{\epsilon_o} . \tag{I.1}$$

All the dependent variables are functions of x, y, z and t. The dipole moment at each point is that of a finite sized dipole with the positive particle fixed at the point and the negatively charged particle located in a potential well centered at the positive particle. The only force on the negative particle considered is that of the electric field. The magnetic effects, the distortion of the lattice and dissipation are neglected. The potential at a displacement $\underline{R}$ is given by the general function U(R). The form of the power series expansion of U(R) is crucial in determining the effects that occur. When U is quadratic, leading to a linear restoring force, this is the Lorentz theory of dispersion. When the series contains higher than quadratic terms the problem is nonlinear. Chiao et al. [1] indicate some of the physical effects responsible for the nonlinear terms and list the magnitudes for some of the materials that show nonlinear properties.

The expression of Newton's law for the oscillators is

$$m \frac{\partial^2 R_i}{\partial t^2} + \frac{\partial U}{\partial R_i} = - eE_i \,, \qquad (\text{for } i = 1,2,3)$$

where m is the mass of the electron and e is its charge. It is convenient to work with the polarization, $\underline{P} = - Ne\underline{R}$, where N is the density of oscillators. In terms of the polarization Newton's law becomes

$$\frac{\partial^2 P_i}{\partial t^2} + \frac{\partial V}{\partial P_i} = \epsilon_o \omega_p{}^2 E_i \,, \tag{I. 2}$$

where we have defined

$$V(P) = \frac{N^2 e^2}{m} U(R)$$

and

$$\omega_p{}^2 = \frac{Ne^2}{\epsilon_o m} \,.$$

It remains to relate $\underline{P}$ to the sources $\underline{J}$ and ρ and then (I. 1) and (I. 2) will form a complete set of equations.

Current density is due to the motion of the negative particles, hence

$$\underline{J} = - Ne \frac{\partial \underline{R}}{\partial t}$$

$$= \frac{\partial \underline{P}}{\partial t} \,.$$

Then the first equation of (I. 1) gives

$$\epsilon_o \nabla \cdot \underline{E} = - \nabla \cdot \underline{P}$$

which replaces the last equation of (I.1). The substitution of $\underline{J}$ into Maxwell's equations produces the terms that normally arise from the constitutive properties of the medium. The equations governing the electromagnetic fields now stand as

$$\begin{aligned} \nabla \times \underline{H} &= \epsilon_o \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t}, & \nabla \cdot \underline{H} &= 0, \\ \nabla \times \underline{E} &= - \mu_o \frac{\partial H}{\partial t}, & \epsilon_o \nabla \cdot \underline{E} &= - \nabla \cdot \underline{P}. \end{aligned} \tag{I.3}$$

I.3. The Potential Representation and the Lagrangian

The equations will be placed into potential form so that a Lagrangian may be found. Using the $\nabla \cdot \underline{H}$ equation of (I.3), $\underline{H}$ may be written in terms of a vector $\underline{A}$ such that

$$\mu_o \underline{H} = \nabla \times \underline{A} .$$

Using the $\nabla \times \underline{E}$ equation and the potential representation for $\underline{H}$, $\underline{E}$ may be determined by $\underline{A}$ and a scalar ϕ such that

$$\underline{E} = - \frac{\partial A}{\partial t} - \nabla \phi . \tag{I.4}$$

By means of the potentials $\underline{A}$ and ϕ, two of the equations (I.3) are satisfied identically and the remaining two give the equations

$$\nabla \times \nabla \times \underline{A} = - \epsilon_o \mu_o \frac{\partial^2 A}{\partial t^2} - \epsilon_o \mu_o \frac{\partial}{\partial t} \nabla \phi + \mu_o \frac{\partial P}{\partial t},$$

$$\nabla^2 \phi + \frac{\partial}{\partial t} \nabla \cdot \underline{A} = \frac{1}{\epsilon_o} \nabla \cdot \underline{P} .$$

A gauge condition for $\underline{A}$ is permitted since only $\nabla \times \underline{A}$ is determined. Here the Lorentz gauge is chosen:

$$\nabla \cdot \underline{A} = - \epsilon_o \mu_o \frac{\partial \phi}{\partial t} .$$

Using this condition and the identity

$$\nabla \times \nabla \times \underline{A} = \nabla \nabla \cdot \underline{A} - \nabla^2 \underline{A} ,$$

a final form of the working equations is produced:

$$\begin{aligned}
&\nabla^2 A_i = - \epsilon_o \mu_o \frac{\partial^2 A_i}{\partial t^2} + \mu_o \frac{\partial P_i}{\partial t} = 0 , \\
&\nabla^2 \phi - \epsilon_o \mu_o \frac{\partial^2 \phi}{\partial t^2} - \frac{1}{\epsilon_o} \frac{\partial P_i}{\partial x_i} = 0 , \qquad \text{(I.5)} \\
&\frac{\partial^2 P_i}{\partial t^2} + \frac{\partial V}{\partial P_i} = - \epsilon_o \omega_p^2 \left(\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x_i} \right) .
\end{aligned}$$

The Lagrangian for this system is

$$L = \frac{\epsilon_o}{2} (A_{i,t}^{\;2} - c^2 A_{i,x_k}^{\;2}) - A_{i,t} P_i - \frac{\epsilon_o}{2c^2} (\phi_t^{\;2} - c^2 \phi_{x_k}^{\;2}) - \phi_{x_k} P_k$$

$$+ \frac{1}{\epsilon_o \omega_p^2} \left(\frac{P_{i,t}^{\;2}}{2} - V(P) \right) , \qquad \text{(I.6)}$$

where $\epsilon_o \mu_o = \frac{1}{c^2}$.

For a two-dimensional problem $\underline{E}$, $\underline{P}$, $\underline{A}$ and $\underline{R}$ have only z components and are functions of x, y and t; the scalar potential φ is zero. The equations reduce to

$$
\begin{aligned}
&E_{tt} - c^2(E_{xx} + E_{yy}) = -\frac{1}{\epsilon_o} P_{tt} ,\\
&P_{tt} + V'(P) = \epsilon_o \omega_p^2 E .
\end{aligned}
\tag{I.7}
$$

The scalars E and P are the magnitudes of the z components of $\underline{E}$ and $\underline{P}$. The Lagrangian reduces to

$$
L = \frac{\epsilon_o}{2}\left(A_t^2 - c^2(A_x^2 + A_y^2)\right) - A_t P + \frac{1}{\epsilon_o \omega_p^2}\left(\frac{P_t^2}{2} - V(P)\right) , \tag{I.8}
$$

where $E = -A_t$, and the scalar A is the magnitude of the z component of $\underline{A}$.

CHAPTER II

PERIODIC PLANE WAVES

II.1. Linearly Polarized Waves

In chapter I the equations for electromagnetic waves in non-linear media were derived. We now wish to study exact periodic solutions to these equations in detail. These will be referred to as fundamental solutions since they play an important role in methods to be used later in the study of more general solutions to these equations. We shall find periodic waves propagating in the x-direction. When we come to use these fundamental solutions in the techniques of chapter III we shall note that waves propagating in any direction produce the same averaged Lagrangian, hence the approach taken here is sufficiently general.

The equations for the one-dimensional problem are:

$$E_{tt} - c^2 E_{xx} = - \frac{1}{\epsilon_o} P_{tt} ,$$

$$P_{tt} + V'(P) = \epsilon_o \omega_p{}^2 E .$$

This is the case of linear polarization since the path traced out by the head of the vector $\underline{E}$ is a straight line. We constrain E and P to be functions of θ where

$$\theta = \kappa x - \omega t ,$$

κ and ω being given constants. Enforcing this constraint results in the new form of the working equations

$$(\omega^2 - c^2\kappa^2)E_{\theta\theta} = -\frac{\omega^2}{\epsilon_o} P_{\theta\theta} , \tag{II.1}$$

$$\omega^2 P_{\theta\theta} + V'(P) = \epsilon_o \omega_p^{\ 2} E . \tag{II.2}$$

Equation (II.1) is integrated twice to produce

$$(\omega^2 - c^2\kappa^2)E = -\frac{\omega^2}{\epsilon_o} P + A\theta + B . \tag{II.3}$$

For E and P to be periodic the secular term must be suppressed by setting $A = 0$. For $E \equiv 0$ and $P \equiv 0$ to satisfy the system, B must be set to zero, hence B represents the displacement due to a constant electric field. We shall disregard for the moment, the case where this constant field is present. Having set A and B equal to zero in equation (II.3), we use this result to eliminate E in equation (II.2). One integration then leads to

$$\omega^2 P_\theta^{\ 2} + \frac{\omega^2 \omega_p^{\ 2}}{\omega^2 - c^2\kappa^2} P^2 + 2V(P) = M . \tag{II.4}$$

The integration constant M determines the amplitude of $P(\theta)$. The equation is solved for P_θ and integrated to give the following implicit form for $P(\theta)$:

$$\int_{P_o}^{P} \frac{\omega \, dP}{\sqrt{M - \frac{\omega^2 \omega_p^{\ 2}}{\omega^2 - c^2\kappa^2} P^2 - 2V(P)}} = \theta . \tag{II.5}$$

Examination of (II.4) shows that $P(\theta)$ oscillates between simple zeros of the denominator of the integrand of (II.5). Limiting cases where zeros coincide correspond to profiles where P approaches a constant as θ becomes infinite and are not oscillatory. Such "solitary waves" will appear in a different connection later. The lower limit of integration, P_o, fixes the phase. The period in θ is not yet normalized and on choosing it arbitrarily to be 2π one has

$$\oint \frac{\omega\, dP}{\sqrt{M - \frac{\omega^2 \omega_p^{\ 2}}{\omega^2 - c^2\kappa^2} P^2 - 2V(P)}} = 2\pi \ , \tag{II. 6}$$

where the notation $\oint$ denotes integration through the values of P over one complete cycle. Different normalizations of θ are compensated by changes in the meaning of κ and ω. (II-6) is a statement of the dispersion relation and relates κ, ω and M. When κ, ω and M are chosen to satisfy (II. 6), then (II. 5) gives a periodic solution of (II. 1) and (II. 2) where κ indicates the number of waves contained in a distance 2π in the x-direction and the frequency ω indicates the number of waves passing a fixed point in a time 2π.

A typical case is the form

$$V(P) = \frac{\omega_o^{\ 2} P^2}{2} - \frac{\gamma P^4}{4} \ . \tag{II. 7}$$

For this case exact solutions are known in terms of Jacobian elliptic functions. This case also arises as the first two terms in the near linear approximation for V when V is symmetric. Placing (II. 7) into (II. 5) we examine the form of the functions to illustrate the periodic features of the exact solutions.

The exact solution to (II. 1) and (II. 2) is given implicitly by

$$\int_{P_o}^{P} \frac{\omega\, dP}{\sqrt{M - \left(\frac{\omega^2 \omega_p^{\ 2}}{\omega^2 - c^2 \kappa^2} + \omega_o^{\ 2}\right) P^2 + \frac{\gamma P^4}{2}}} = \theta . \tag{II.8}$$

The particular Jacobian elliptic function that the solution becomes depends on the signs of the zeros of the quadratic form in P^2 in the denominator of the integrand of (II.8). We define these roots by

$$R_{1,2}^{\ 2} = \frac{D \pm \sqrt{D^2 - 2\gamma M}}{\gamma} , \qquad \text{where} \qquad D = \omega_o^{\ 2} + \frac{\omega_p^{\ 2} \omega^2}{\omega^2 - c^2 \kappa^2} .$$

Since the relevant cases in later developments are for small γ, we take $D^2 > 2|\gamma| M$ in this example. The sign of γ is determined by the medium, hence a medium supports only one of the following waves. For $\gamma > 0$ a periodic solution is

$$P = R_2 \,\mathrm{sn}\left(\frac{\sqrt{M}\,\theta}{\omega R_2} , \frac{R_2^{\ 2}}{R_1^{\ 2}}\right) .$$

Of course translations of the origin also yield solutions. The second argument of the elliptic function is the modulus squared. For $\gamma < 0$, one of the roots for P^2 is negative. We set $S_2^{\ 2} = - R_2^{\ 2}$, S_2 being a positive real quantity. Then a solution is

$$P = R_1 \,\mathrm{cn}\left(\frac{\sqrt{M(R_1^{\ 2} + S_2^{\ 2})}\,\theta}{\omega R_1 S_2} , \frac{R_1^{\ 2}}{R_1^{\ 2} + S_2^{\ 2}}\right) .$$

Again translations of the origin merely change the phase.

II. 2. The Direct Approach for Near Linear Equations

Periodic solutions to the near linear equations may be found in a more direct manner than by the general procedure of section II. 1. In anticipation that the solutions are nearly sinusoidal, we substitute sinusoids directly into the equations of motion. The near linear equations are found by placing potential (II. 7) into the equations (II. 1) and (II.2), and γ is taken as a small parameter. They become

$$(\omega^2 - c^2\kappa^2)E_{\theta\theta} = -\frac{\omega^2}{\epsilon_o} P_{\theta\theta}$$

$$\omega^2 P_{\theta\theta} + \omega_o^2 P - \gamma P^3 = \epsilon_o \omega_p^2 E \, . \qquad \text{(II. 9)}$$

The periodic solutions are represented by Fourier series

$$E = \sum_{n=1}^{\infty} (a_n \sin n\theta + c_n \cos n\theta)$$

$$P = \sum_{n=1}^{\infty} (b_n \sin n\theta + d_n \cos n\theta) \; ,$$

and we anticipate that successive coefficients involve increasing powers of γ. The constant terms have been omitted as in the fully nonlinear case. The effect is to suppress terms where n is even. Further, fixing the phase by setting $c_1 = 0$, $d_1 = 0$ suppresses all the cosine terms. Placing the first two remaining terms of each series in (II. 9) and equating coefficients of $\sin\theta$ and $\sin 3\theta$ results in the equations

$$(\omega^2 - c^2\kappa^2)a_1 = -\frac{1}{\epsilon_o}\omega^2 b_1 ,$$

$$(\omega^2 - c^2\kappa^2)a_3 = -\frac{1}{\epsilon_o}\omega^2 b_3 ,$$

$$(\omega_o{}^2 - \omega^2)b_1 - \gamma(\tfrac{3}{4}b_1{}^3 - \tfrac{3}{4}b_1{}^2 b_3 + \tfrac{3}{2}b_1 b_3{}^2) = \epsilon_o\omega_p{}^2 a_1 ,$$

$$(\omega_o{}^2 - \omega^2)b_3 - \gamma(\tfrac{1}{4}b_1{}^3 - \tfrac{3}{2}b_1{}^2 b_3 - \tfrac{3}{4}b_3{}^2) = \epsilon_o\omega_p{}^2 a_3 .$$

Elimination of a_1 and a_3 produces, to $O(\gamma)$,

$$b_3 = \frac{\gamma}{32\omega^2} b_1{}^3 ,$$

$$c^2\kappa^2 = \omega^2\left(1 + \frac{\omega_p{}^2}{\omega_o{}^2 - \omega^2}\right) + \frac{3\gamma\omega_p{}^2\omega^2 b_1{}^2}{4(\omega_o{}^2 - \omega^2)^2} . \qquad \text{(II. 10)}$$

(II. 10) is the near linear dispersion relation which must be the approximate form of (II. 6) for small γ, and the Fourier series is similarly expected to be the expanded form of (II. 5) for small γ. This expansion for the near linear approximation of the fully nonlinear forms will now be carried out for illustrative purposes at this stage of the problem. It will be mentioned in later sections when fully nonlinear forms arise, that the expansion for the near linear case produces the same results as the direct use of leading terms of Fourier series in the relevant procedures. Each situation involves the expansion of elliptic functions and elliptic integrals and follows the same course that is presented in section II. 3.

II.3. Expansion of the Fully Nonlinear Solutions for the Near Linear Case

As stated in section II.1, an exact periodic solution to equations (II.1) and (II.2) is given by (II.5). To reduce to the near linear situation the potential V is expanded in a Maclaurin series and the first two terms are retained. The case of interest is when V(P) is an even function of P; hence the quadratic and quartic terms are kept, the quartic term being small. As in section II.2, we use the parameter γ to represent the small quantity in the quartic term of V and we expand with respect to γ. If the potential V were not symmetric the first term leading to a nonlinear restoring force would be cubic and the periodic solutions would still be expressible in terms of Jacobian elliptic functions. We now proceed to expand the exact solutions for small γ to reproduce the solutions obtained by direct substitution of sinusoids into the equations. Repeating the implicit form of the exact solution with the near linear potential (II.7), we have

$$\int_{P_o}^{P} \frac{\omega\, dP}{\sqrt{M - \left(\frac{\omega^2 \omega_p^2}{\omega^2 - c^2 \kappa^2} + \omega_o^2\right) P^2 + \frac{\gamma P^4}{2}}} = \theta \qquad \text{(II.8)}$$

and the zeros of the denominator of the integrand were given by

$$R_{1,2}^2 = \frac{D \pm \sqrt{D^2 - 2\gamma M}}{\gamma}, \quad \text{where} \quad D = \frac{\omega^2 \omega_p^2}{\omega^2 - c^2 \kappa^2} + \omega_o^2 .$$

We set $P_o = 0$ to coincide with the choice of phase in the direct approach. The substitution

$$P = R_2 u \tag{II.11}$$

simplifies (II.8) to the convenient form

$$\frac{R_2\omega}{\sqrt{M}}\int_0^u \frac{du}{\sqrt{1-u^2}\sqrt{1-\frac{R_2^{\,2}}{R_1^{\,2}}u^2}} = \theta \ .$$

Now it is apparent that both R_2 and $\left(1 - \frac{R_2^{\,2}}{R_1^{\,2}} u^2\right)^{-1/2}$ are to be expanded for small γ. The latter expansion of a function of u within the integral leads to integration of a series term by term. Retaining terms of $O(\gamma)$, one obtains a formula for θ, which when inverted and P resubstituted for u, becomes

$$P = \left(\sqrt{\frac{M}{D}} + \frac{9}{32}\frac{\gamma M^{3/2}}{D^{5/2}}\right)\sin\left\{\frac{\theta}{\frac{\omega}{\sqrt{D}}\left(1+\frac{3\gamma M}{8D^2}\right)}\right\}$$

$$+ \frac{\gamma M^{3/2}}{32 D^{5/2}}\sin\left\{\frac{3\theta}{\frac{\omega}{\sqrt{D}}\left(1+\frac{3\gamma M}{8D^2}\right)}\right\} + O(\gamma^2) \ . \tag{II.12}$$

The normalization of the period to 2π is carried out by setting

$$\frac{\omega}{\sqrt{D}}\left(1 + \frac{3\gamma M}{8D^2}\right) = 1 \ . \tag{II.13}$$

For comparison with the result of section II.2 we denote the amplitude of the sinusoid by b and use it as a parameter in place of M; hence we set

$$\sqrt{\frac{M}{D}} + \frac{9}{32}\,\frac{\gamma M^{3/2}}{D^{5/2}} = b \qquad \text{(II. 14)}$$

When (II. 14) and the definition of D are used in (II. 12) we have

$$c^2\kappa^2 = \omega^2\left(1 + \frac{\omega_p^{\,2}}{\omega_o^{\,2} - \omega^2}\right) + \frac{3\gamma\omega^2\omega_p^{\,2} b^2}{4(\omega_o^{\,2} - \omega^2)^2} + O(\gamma^2)\ ;$$

this is the near linear dispersion relation and is exactly the form of (II. 10). Now (II. 12) reads

$$P = b \sin\theta + \frac{\gamma b^3}{32\omega^2}\sin 3\theta + O(\gamma^2)$$

and the coefficient of sin 3θ coincides with b_3 of the direct approach.

Of course (II. 10) must also result from expansion of the exact dispersion relation (II. 6). Using the change of variable (II. 11), the loop integral becomes

$$\frac{4R_2\omega}{\sqrt{M}}\int_0^1 \frac{du}{\sqrt{1-u^2}\,\sqrt{1-\dfrac{R_2^{\,2}}{R_1^{\,2}}u^2}} = 2\pi\ .$$

Expansion of this complete elliptic integral for small γ in the same manner as before, and the use of (II. 14) to eliminate M, produce (II. 10) once again as expected.

II.4. Orbitally Polarized Uniform Plane Waves in Fully Nonlinear Media

So far in chapter II we have dealt with the case of linear polarization, the z-component of the electric field being the only non-zero component. The case of circular polarization occurs when the head of the electric field vector traces out a circle, and similarly, when an ellipse is traced out, the polarization is called elliptical. In a nonlinear medium the figure traced out by the electric field vector may not be closed; in fact, only exceptional cases are closed. While the magnitude of the electric field is periodic in x and t, the individual y and z components are not in general. We name this orbital polarization due to the similarity with the trajectory of a particle in a central force field.

We recall the three-dimensional equations (I.5):

$$\begin{aligned}
&\frac{\partial^2 A_i}{\partial t^2} - c^2\nabla^2 A_i = \frac{1}{\epsilon_o}\frac{\partial P_i}{\partial t} \\
&\frac{\partial^2 \phi}{\partial t^2} - c^2\nabla^2\phi = -c^2\frac{\partial P_i}{\partial x_i} \qquad \text{(II.15)} \\
&\frac{\partial^2 P_i}{\partial t^2} + \frac{\partial V(P)}{\partial P_i} = -\epsilon_o\omega_p^2\left(\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x_i}\right).
\end{aligned}$$

We wish to find the form of a uniform plane wave propagating in the x-direction where we are permitting the electric field to have non-zero y and z components. To do this we set

$$\underline{A} = (0, A_2(\theta), A_3(\theta))$$

where $\theta = \kappa x - \omega t$. It will turn out that $A_2(\theta)$ and $A_3(\theta)$ are not in general periodic in θ, but we expect that they oscillate. From the first equation of (II.15) we see that the form of $\underline{P}$ must be

$$\underline{P} = (0, P_2(\theta), P_3(\theta)) \, .$$

From the Lorentz gauge condition we find that ϕ_t must be zero, and choosing ϕ to be zero is consistent with (II.15). We obtain from the first and third equations of (II.15)

$$\begin{aligned}
(\omega^2 - c^2\kappa^2)A_{2,\theta\theta} &= -\frac{\omega}{\epsilon_o} P_{2,\theta} \, , \\
(\omega^2 - c^2\kappa^2)A_{3,\theta\theta} &= -\frac{\omega}{\epsilon_o} P_{3,\theta} \, , \\
\omega^2 P_{2,\theta\theta} + \frac{\partial V}{\partial P}\frac{P_2}{P} &= \omega\epsilon_o\omega_p^{\,2} A_{2,\theta} \, , \\
\omega^2 P_{3,\theta\theta} + \frac{\partial V}{\partial P}\frac{P_3}{P} &= \omega\epsilon_o\omega_p^{\,2} A_{3,\theta} \, .
\end{aligned} \tag{II.16}$$

As in section (II.1) we integrate the first two equations of (II.16) and set the constants of integration equal to zero as they represent secular terms relating $\underline{A}$ and $\underline{P}$. These integrated equations are placed into the third and fourth equations of (II.15), eliminating $A_{2,\theta}$ and $A_{3,\theta}$, to produce

$$\begin{aligned}
\omega^2 P_{2,\theta\theta} + \frac{\partial V}{\partial P}\frac{P_2}{P} + \frac{\omega_p^{\,2}\omega^2}{\omega^2 - c^2\kappa^2} P_2 &= 0 \, , \\
\omega^2 P_{3,\theta\theta} + \frac{\partial V}{\partial P}\frac{P_3}{P} + \frac{\omega_p^{\,2}\omega^2}{\omega^2 - c^2\kappa^2} P_3 &= 0 \, .
\end{aligned} \tag{II.17}$$

At this point the set of equations is in the form of a single particle with displacement P, orbiting in a central force field of strength $\frac{\partial V}{\partial P} + \frac{\omega_p^2 \omega^2}{\omega^2 - c^2 \kappa^2} P$. To integrate these equations we set

$$P_2 = P \cos \psi$$

$$P_3 = P \sin \psi$$

and now we shall have that $P(\theta)$ is a periodic function of θ, while $\psi(\theta)$ has a term linear in θ and a term periodic in θ. Placing these definitions in (II.17), two independent equations become:

$$\omega^2 (P_{\theta\theta} - P\psi_\theta^{\,2}) + \frac{\partial V}{\partial P} + \frac{\omega_p^2 \omega^2 P}{\omega^2 - c^2 \kappa^2} = 0$$

$$2P_\theta \psi_\theta + P\psi_{\theta\theta} = 0 \; . \tag{II.18}$$

The second equation of (II.18) has the integral

$$P^2 \psi_\theta = h \; , \tag{II.19}$$

where h is a constant which is equivalent to angular momentum in mechanics. Using (II.19) to eliminate ψ_θ, the first equation of (II.18) takes the form

$$\omega^2 P_{\theta\theta} - \frac{\omega^2 h^2}{P^3} + \frac{\partial V}{\partial P} + \frac{\omega_p^2 \omega^2 P}{\omega^2 - c^2 \kappa^2} = 0 .$$

One integration gives

$$\omega^2 P_\theta^{\,2} + \frac{\omega^2 h^2}{P^2} + 2V + \frac{\omega_p^2 \omega^2 P^2}{\omega^2 - c^2 \kappa^2} = M . \tag{II.20}$$

As in the case of linear polarization, the constant of integration, M, determines the amplitude of $P(\theta)$. Now (II.20) is solved for P_θ and integrated a final time to give

$$\int_{P_o}^{P} \frac{\omega\, dP}{\sqrt{M - \frac{\omega^2 h^2}{P^2} - 2V(P) - \frac{\omega_p^{\,2}\omega^2 P^2}{\omega^2 - c^2\kappa^2}}} = \theta \, . \qquad \text{(II.21)}$$

This is an implicit form for $P(\theta)$. (II.19) may be written

$$d\psi = \frac{h}{P^2}\, d\theta \, ,$$

and (II.20) gives $d\theta$ in terms of dP. Placing (II.20) into the equation above, and integrating, results in

$$\psi = \int_{P_o}^{P} \frac{\omega h\, dP}{P^2\sqrt{M - \frac{\omega^2 h^2}{P^2} - 2V(P) - \frac{\omega_p^{\,2}\omega^2 P^2}{\omega^2 - c^2\kappa^2}}} \qquad \text{(II.22)}$$

Finally, the period in θ is normalized to 2π; thus

$$\oint \frac{\omega\, dP}{\sqrt{M - \frac{\omega^2 h^2}{P^2} - 2V(P) - \frac{\omega_p^{\,2}\omega^2 P^2}{\omega^2 - c^2\kappa^2}}} = 2\pi \qquad \text{(II.23)}$$

is the dispersion relation.

(II.21) and (II.22) give the magnitude and angle of the polarization vector. It is clear from (II.19) that ψ_θ is of one sign, hence the rotation always continues in one direction. This indicates that ψ has a secular and a periodic term. The magnitude P oscillates between the zeros of the denominator of (II. 21). Due to the singularity at $P = 0$ in that expression, P oscillates between positive values and does not go to zero when h is not zero. The speed of precession of the orbit may be found from (II. 22). When P oscillates from a maximum to a minimum and back to a maximum the angle through which ψ moves is

$$\psi_o = \oint \frac{\omega h \, dP}{P^2 \sqrt{M - \frac{\omega^2 h^2}{P^2} - 2V(P) - \frac{\omega_p^2 \omega^2 P^2}{\omega^2 - c^2 \kappa^2}}}$$

It may be shown that ψ_o is 2π if $V(P)$ is quadratic in P or proportional to $1/P$ and not otherwise. [See Landau and Lifshitz [2]]. The angle of precession per period in P is $\psi_o - 2\pi$. If $\psi = \frac{m}{n} 2\pi$, where m and n are integers, then the orbit closes after n rotations of $\underline{P}$.

In the special case of circular polarization P is constant and ψ is linear in θ. The integrated forms (II. 21) and (II. 21) become degenerate since P does not vary. Rather than using (II. 23) to define κ and ω we shall set $\psi \equiv \theta$ and then κ and ω will measure the number of oscillations of each component of $\underline{P}$ in an interval of 2π in x or t. The solutions are now

$$P_2 = P \cos \theta \ , \qquad P_3 = P \sin \theta \ .$$

The first equation of (II. 18) gives the dispersion relation

$$- \omega^2 P + \frac{\partial V}{\partial P} + \frac{\omega_p^2 \omega^2 P}{\omega^2 - c^2 \kappa^2} = 0 \ . \qquad \text{(II. 24)}$$

This form exhibits the nonlinear behavior that we have seen before. When V is quadratic the phase velocity becomes independent of the amplitude P.

We now examine the special case of potential (II. 7) which leads to a cubic central force:

$$V(P) = \frac{\omega_o{}^2 P^2}{2} - \frac{\gamma P^4}{4} .$$

The implicit form, (II. 21), for $P(\theta)$ becomes

$$\int_{P_o}^{P} \frac{\omega \, dP}{\sqrt{M - \frac{\omega^2 h^2}{P^2} - \omega_o{}^2 P^2 + \frac{\gamma P^4}{2} - \frac{\omega_p{}^2 \omega^2 P^2}{\omega^2 - c^2 \kappa^2}}} = \theta .$$

Multiplying the numerator and denominator by P we obtain

$$\int_{P_o{}^2}^{P^2} \frac{\frac{1}{2} \omega \, d(P^2)}{\sqrt{MP^2 - \omega^2 h^2 - \omega_o{}^2 P^4 + \frac{\gamma P^6}{2} - \frac{\omega_p{}^2 \omega^2 P^4}{\omega^2 - c^2 \kappa^2}}} = \theta .$$

The denominator of the integrand is the square root of a cubic in P^2, hence the inverse form $P^2(\theta)$ may be written in terms of Jacobian elliptic functions. For example if we choose $P_o{}^2 = b$ and write the contents of the square root in the factored form

$$\frac{\gamma}{2} (a - P^2)(P^2 - b)(P^2 + c) ,$$

then we make the substitution

$$P^2 = (b + c) \mathrm{nd}^2 u - c ,$$

where the modulus of the elliptic function is $k^2 = \frac{a-b}{a+c}$. The integral collapses to give

$$\theta = \frac{\frac{1}{2}\omega}{\sqrt{\frac{\gamma}{2}}} \frac{2}{\sqrt{a-b}}(u-b) .$$

Then the elimination of u gives

$$P^2(\theta) = -c + (b+c)\mathrm{nd}^2\left(b + \frac{\sqrt{\frac{\gamma}{2}}\sqrt{a-b}\,\theta}{\omega}\right) .$$

The function nd oscillates between a minimum of one and a maximum greater than one. From the factored form, P oscillates between a maximum of $\sqrt{a}$ and a minimum of $\sqrt{b}$. ψ is given by the integral of (II.19) and becomes a very complicated function.

In the case of circular polarization we use the special form of V(P) in (II.24) to obtain the dispersion relation

$$\omega_o^2 - \omega^2 + \frac{\omega_p^2\omega^2}{\omega^2 - c^2\kappa^2} - \gamma P^2 = 0 . \qquad \text{(II.25)}$$

II.5. Uniform Plane Waves with Circular Polarization in Near Linear Media

Uniform plane waves in near linear media are found by the direct approach by inserting sinusoids, as the first term of a Fourier series, into the three-dimensional equations. We found in section II.4 that the exact solutions for the components of $\underline{P}$ are sinusoids, and hence we have the correct forms already. The near linear solutions are

$$A_2 = a \sin\theta , \qquad P_2 = b \cos\theta ,$$

$$A_3 = - a \cos\theta , \qquad P_3 = b \sin\theta .$$

The dispersion relation is just (II.25) expanded for small γ. This result is

$$c^2\kappa^2 = \omega^2\left(1 + \frac{\omega_p^2\omega^2}{\omega_o^2 - \omega^2}\right) + \frac{\gamma\omega_p^2\omega^2 b^2}{\omega_o^2 - \omega^2} . \qquad \text{(II.26)}$$

By comparison with (II.10), the near linear dispersion relation for linear polarization, we see that the dispersion relation for circular polarization differs only by a numerical factor in the amplitude dependent term.

CHAPTER III

THE LAGRANGIAN APPROACH AND MODULATIONS

The goal of this chapter is to develop the equations governing the amplitude, frequency and wave number for nearly periodic wave trains. The idea to be used is due to Whitham [3], [4], [5] and consists of insertion of the periodic functions found in the last section, into the Lagrangian and the averaging of this quantity over one period, the parameters being regarded as constants. The averaged Lagrangian is then used to generate differential equations in the parameters which are then regarded as slowly varying. The last paper [5] puts this idea into the framework of two-timing and the averaging becomes the integration of the Lagrangian over one of the domains, the fast time.

The first three sections deal with the near linear problem where it is known beforehand that the fundamental solutions are sinusoidal. These are substituted directly into the Lagrangian and the averaging over one period is performed with known functions. Variation with respect to the amplitudes then gives equations relating them to the other parameters. Variation of the frequency and wave number produce a further equation, the variation being constrained by the requirement that these quantities be derived from a phase.

The second part of the chapter deals with a fully nonlinear approach in which the integrals of the Euler equations are used in the averaged Lagrangian to produce an integrable form. The equations resulting from the variation of the averaged Lagrangian relate the slowly varying frequency, wave number and the integration constants that measure amplitude. The fast oscillations are now the Jacobian elliptic functions found in chapter II. For the case of linearly polarized waves, we follow the work of Knight and Peterson [6]. We then deal with orbital polarization.

In chapter II we disregarded the possibility of a constant component of electric field in addition to the oscillations. Now that the amplitude, frequency and wave number are allowed to vary slowly, it is not clear that no constant field is generated even if it is absent at some initial time. This is the case if the potential V(P) is an even function of P and we leave to the appendix the demonstration that this is so. Knight and Peterson correctly disregard the possibility of a constant field. The taking into account of constant fields requires the introduction of pseudo-frequencies which are also explained by Whitham [3], [4], [5].

III.1. The Near Linear Formulation for Linearly Polarized Waves

We use potential (II.7) with γ a small parameter to form the near linear problem. Inserting this into Lagrangian (I.8), we obtain the near linear Lagrangian for linearly polarized waves:

$$L = \frac{\epsilon_o}{2}\left\{A_t^{\,2} - c^2(A_x^{\,2} + A_y^{\,2})\right\} - A_t P + \frac{1}{\epsilon_o \omega_p^{\,2}}\left\{\frac{P_t^{\,2}}{2} - \frac{\omega_o^{\,2} P^2}{2} + \frac{\gamma P^4}{4}\right\}. \quad \text{(III.1)}$$

Following the theory, we substitute the periodic solutions into the Lagrangian. We know from chapter II that the periodic solutions are nearly sinusoid, hence we set

$$A = a\cos\theta + \dots ,$$
$$P = b\sin\theta + \dots . \qquad \text{(III. 2)}$$

Temporarily we set $\theta = \kappa x - \omega t$ where κ and ω are constants. Defining the averaged Lagrangian by

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L\, d\theta ,$$

we insert (III. 2) into (III. 1) and perform the integration. The result is

$$\mathcal{L} = \frac{\epsilon_o}{4}(\omega^2 - c^2\kappa^2)a^2 - \frac{\omega ab}{2} + \frac{1}{4\epsilon_o\omega_p^2}\left[(\omega^2 - \omega_o^2)b^2 + \frac{3\gamma b^4}{8}\right]. \qquad \text{(III. 3)}$$

The averaged Lagrangian was derived by inserting into the two-dimensional Lagrangian the form of a periodic plane wave propagating in the x-direction and averaging over one period. Had we set $\theta = \kappa_1 x + \kappa_2 y - \omega t$ and defined $\kappa^2 = \kappa_1^2 + \kappa_2^2$, then the same result (III. 3) would have been reached. The averaged Lagrangian is independent of direction of propagation and direction of polarization of the fundamental solution and hence the correct general form has been found.

$\mathcal{L}$ is now a function of the constants ω, κ, a and b. At this point we relax our view of these quantities and allow them to be slowly varying. The slowly varying definitions of $\underline{\kappa}$ and ω are generalizations of their definitions as constants. We shall take

$$\kappa_1 = \frac{\partial\theta}{\partial x}, \quad \kappa_2 = \frac{\partial\theta}{\partial y}, \quad \kappa_3 = \frac{\partial\theta}{\partial z}, \quad \omega = -\frac{\partial\theta}{\partial t}.$$

The new form agrees with the original one when κ_1, κ_2, κ_3, ω are constants. Consistency relations resulting from these definitions are:

$$\frac{\partial \underline{\kappa}}{\partial t} + \frac{\partial \omega}{\partial \underline{x}} = 0, \quad \nabla \times \underline{\kappa} = 0 . \tag{III.4}$$

The expression (III.3) for $\mathcal{L}$ is regarded as a Lagrangian for the functions a, b, ω, κ_1, κ_2, κ_3 and the consistency relations become side conditions that must be enforced when variations of $\mathcal{L}$ are performed. The Euler equations for $\mathcal{L}$, subject to (III.4) are

$$\mathcal{L}_a = 0, \qquad \mathcal{L}_b = 0,$$

$$-\frac{\partial}{\partial t}\mathcal{L}_\omega + \frac{\partial}{\partial x}\mathcal{L}_{\kappa_1} + \frac{\partial}{\partial y}\mathcal{L}_{\kappa_2} + \frac{\partial}{\partial z}\mathcal{L}_{\kappa_3} = 0 . \tag{III.5}$$

One of the amplitudes a and b may be elimina ted from the Lagrangian by means of the Euler equations. If we choose to eliminate a then we would obtain a new Lagrangian, $\mathcal{L}(\omega, \kappa, b)$. The Euler equations

$$\mathcal{L}_b = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial \underline{x}}\mathcal{L}_\kappa = 0$$

are, respectively, the dispersion relation (II.10) and the wave action. It is more convenient, however, to retain a rather than b. The second equation of (III.5) is used to reduce $\mathcal{L}$ to

$$\mathcal{L} = \frac{\epsilon_o}{4}\left(\omega^2 - c^2\kappa^2 + \frac{\omega_p^2\omega^2}{\omega_o^2 - \omega^2}\right)a^2 + \frac{3\gamma\epsilon_o^3\omega_p^6\omega^4a^4}{32(\omega_o^2 - \omega^2)^4}$$

and this form is varied to produce the working equations

$$\mathcal{L}_a = 0\,, \qquad \frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial \underline{x}}\mathcal{L}_{\underline{\kappa}} = 0.$$

These are the dispersion relation and wave action and take the form

$$c^2\kappa^2 = \omega^2\left(1 + \frac{\omega_p^{\,2}}{\omega_o^{\,2} - \omega^2}\right) + \frac{3\,\epsilon_o\omega_p^{\,6}\omega^4 a^2}{4(\omega_o^{\,2} - \omega^2)^4}$$

$$\frac{\partial}{\partial t}\left\{\left(1 + \frac{\omega_o^{\,2}\omega_p^{\,2}}{(\omega_o^{\,2} - \omega^2)^2}\right)\omega a^2\right\} + \frac{\partial}{\partial x}(c^2\kappa_1 a^2) + \frac{\partial}{\partial y}(c^2\kappa_2 a^2) + \frac{\partial}{\partial z}(c^2\kappa_3 a^2) = 0. \tag{III.6}$$

It should be mentioned that this procedure may be carried through to any degree of accuracy. The form (III.2) could have been chosen to be an entire Fourier series with undetermined coefficients. The independent variation of each of these coefficients then produces a sufficient number of equations to solve for each in terms of the coefficient of the lowest mode. For example, the form of the next term as found in (II.10) is easily produced. The Lagrangian which retains one more term in the Fourier series is

$$\mathcal{L}_1 = \frac{\epsilon_o}{4}(\omega^2 - c^2\kappa^2)(a_1^{\,2} + 9a_3^{\,2}) - \frac{\omega}{2}(a_1b_1 + 3a_3b_3)$$
$$+ \frac{1}{\epsilon_o\omega_p^{\,2}}\left\{\frac{\omega^2}{4}(b_1^{\,2} + 9b_3^{\,2}) - \frac{\omega_o^{\,2}}{4}(b_1^{\,2} + b_3^{\,2}) + \frac{\gamma}{4}\left(\frac{3}{8}b_1^{\,4} - \frac{b_1^{\,3}b_3}{2}\right.\right.$$
$$\left.\left. + \frac{3b_1^{\,2}b_3^{\,2}}{2} + \frac{3}{8}b_3^{\,4}\right)\right\}.$$

The equations obtained by varying a_3 and b_3 are

$$\frac{9}{2}\epsilon_o(\omega^2 - c^2\kappa^2)a_3 - \frac{3\omega}{2} b_3 = 0 ,$$

$$-\frac{3\omega}{2} a_3 + \frac{1}{\epsilon_o\omega_p^2}\left\{\frac{9}{2}\omega^2 b_3 - \frac{\omega_o^2 b_3}{2} + \frac{\gamma}{4}\left(-\frac{b_1^3}{2} + 3b_1^2 b_3 + \frac{3b_3^2}{2}\right)\right\} = 0.$$

Elimination of a_3 and retention of only significant terms produces $b_3 = \gamma b_1^3/32\omega^2$ as required to agree with (II.10).

III.2. The Near Linear Formulation for Circular Polarization

In section III.1 the averaged Lagrangian for linearly polarized waves was found by substituting a periodic plane wave propagating in the x-direction into the Lagrangian and averaging over one period. It was noted that the resulting form was independent of the direction. We now produce the averaged Lagrangian for circular polarization by substituting a circularly polarized plane wave propagating in the x-direction, into the three-dimensional Lagrangian (I.6), with $V(P) = (\omega_o^2 P^2/2) - (\gamma P^4/4)$. The Lagrangian is

$$L = \frac{\epsilon_o}{2}\left(A_{i,t}^2 - c^2 A_{i,x_\kappa}^2\right) - A_{i,t}P_i - \frac{\epsilon_o}{2c^2}(\phi_t^2 - c^2\phi_{x_i}^2) - \phi_{x_i}P_i$$
$$+ \frac{1}{\epsilon_o\omega_p^2}\left(\frac{P_{i,t}^2}{2} - \frac{\omega_o^2 P_i^2}{2} + \frac{\gamma(P_i^2)^2}{4}\right).$$

The plane wave solutions from section (II.5) are:

$$A_1 = 0 , \qquad P_1 = 0 ,$$

$$A_2 = a\sin\theta, \qquad P_2 = b\cos\theta , \qquad \phi \equiv 0 ,$$

$$A_3 = -a\cos\theta \qquad P_3 = b\sin\theta ,$$

where $\theta = \kappa x - \omega t$. Temporarily regarding a, b, κ and ω as constants, we perform the integration

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L\, d\theta$$

to produce

$$\mathcal{L} = \frac{\epsilon_o}{2}(\omega^2 - c^2\kappa^2)a^2 - \omega ab + \frac{1}{2\epsilon_o\omega_p^2}\left[(\omega^2 - \omega_o^2)b^2 + \frac{\gamma b^4}{2}\right].$$

Elimination of a from the Lagrangian by means of an Euler equation, and then variation of the new Lagrangian with respect to b, would reproduce dispersion relation (II.26). We wish, however, to retain a rather than b, as in the case of linear polarization. We use the Euler equation $\mathcal{L}_b = 0$ to eliminate b, giving the reduced form of $\mathcal{L}$:

$$\mathcal{L} = \left(\frac{\epsilon_o}{2}\left(\omega^2 - c^2\kappa^2\right) + \frac{\omega_p^2\omega^2}{\omega_o^2 - \omega^2}\right)a^2 + \frac{\gamma\epsilon_o^3\omega_p^6\omega^4 a^4}{8(\omega_o^2 - \omega^2)^4} .$$

The wave action is the same as in (III.6). The dispersion relation differs by a numerical factor in the nonlinear term and we absorb this into a new small parameter γ_1. We rewrite (III.6) as

$$c^2\kappa^2 = \omega^2\left(1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}\right) + \frac{\gamma_1\epsilon_o^2\omega_p^6\omega^3 a^2}{(\omega_o^2 - \omega^2)^4},$$

$$\frac{\partial}{\partial t}\left\{\left(1 + \frac{\omega_o^2\omega_p^2}{(\omega_o^2 - \omega^2)^2}\right)\omega a^2\right\} + \frac{\partial}{\partial x}(c^2\kappa_1 a^2) + \frac{\partial}{\partial y}(c^2\kappa_2 a^2) + \frac{\partial}{\partial z}(c^2\kappa_3 a^2) = 0, \tag{III.7}$$

where

$$\gamma_1 = \frac{3}{4}\gamma \quad \text{for linearly polarized waves,}$$

$$= \gamma \quad \text{for circularly polarized waves.}$$

III. 3. The One-Dimensional Time-Dependent Problem

We now neglect the y and z dependence in (III. 7) to produce the problem of a uniform plane wave that varies only in its direction of propagation. The more interesting steady two- and three-dimensional problems are left to chapter IV as the symmetries in those equations produce tractable problems that lead to some closed form solutions.

Whitham [4] worked with an averaged Lagrangian similar to $\mathcal{L}(\omega, \kappa, a)$ of sections III. 1 and III. 2. He dealt with the dispersion relation in the form

$$\omega = \omega_o(\kappa) + \omega_1(\kappa)a^2$$

but due to the simplicity of κ as a function of ω, we shall reverse the roles of κ and ω and of x and t. Hence we write the first equation of (III. 7), the dispersion relation, as

$$\kappa = \kappa^{(0)}(\omega) + \gamma_1\kappa^{(1)}(\omega)a^2 + O(\gamma_1{}^2)$$

and we transform the second, the wave equation, into the form

$$\frac{\partial a^2}{\partial x} + \frac{\partial}{\partial t}\left(c_o{}^{-1}(\omega)a^2\right) = 0, \tag{III. 8}$$

where

$$C_o^{-1}(\omega) = \frac{\partial \kappa^{(0)}(\omega)}{\partial \omega} .$$

The consistency relation in one dimension is written

$$\frac{\partial \omega}{\partial x} + \frac{\partial}{\partial t}(\kappa^{(0)} + \gamma_1 \kappa^{(1)} a^2) = 0 . \tag{III.9}$$

Now (III.8) and (III.9) are the one-dimensional equations and we can see how the nonlinear features enter since the linear problem is recovered by setting $\gamma_1 = 0$. Then we read that C_o is the characteristic velocity for both equations and that the determination of the characteristic direction is independent of amplitude. Returning to the nonlinear form of (III.8) and (III.9), the system can be placed into the following characteristic form:

$$\frac{\partial a^2}{\partial x} + \left(C_o^{-1} \pm a\sqrt{(C_o^{-1})'\gamma_1 \kappa^{(1)}}\right)\frac{\partial a^2}{\partial t}$$
$$\pm a\sqrt{\frac{(C_o^{-1})'}{\gamma_1 \kappa^{(1)}}}\left\{\frac{\partial \omega}{\partial x} + \left(C_o^{-1} \pm a\sqrt{(C_o^{-1})'\gamma_1 \kappa^{(1)}}\right)\frac{\partial \omega}{\partial t}\right\} = 0.$$

The inverse of the characteristic velocities are given by

$$C^{-1} = C_o^{-1} \pm a\sqrt{(C_o^{-1})'\gamma_1 \kappa^{(1)}} .$$

This is a typical feature of nonlinear wave systems, the splitting of the group velocity. Since the system is hyperbolic when characteristics are real, then it is hyperbolic when $(C_o^{-1})'\gamma_1 \kappa^{(1)} > 0$ and elliptic when $(C_o^{-1})'\gamma_1 \kappa^{(1)} < 0$. When examining this condition one must take care that real values of κ and ω are taken. The dispersion relation is sketched in the diagram and it can be seen that the interval

$\omega_o^2 < \omega^2 < \omega_o^2 + \omega_p^2$ is a forbidden region for the frequency ω. The turning point for hyperbolicity is a complicated expression that falls within this region, hence the system is hyperbolic for

$\gamma_1 > 0, \ \omega < \omega_o$

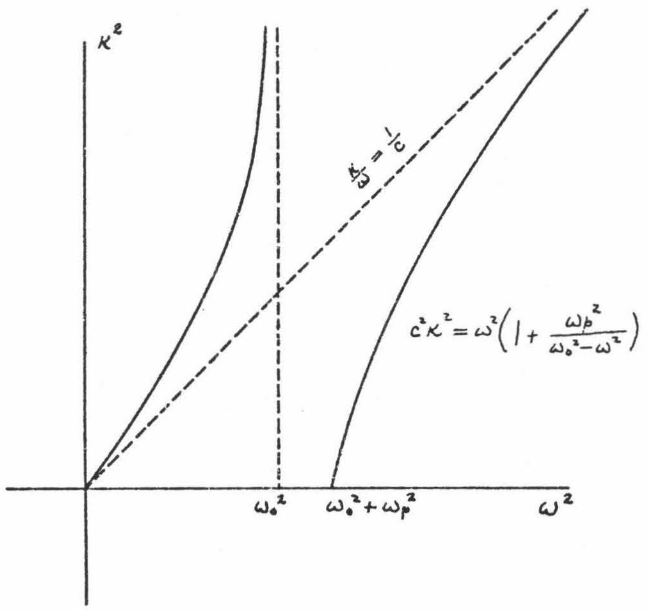
$\gamma_1 < 0, \ \omega > \ \omega_o^2 + \omega_p^2$

and elliptic for

$\gamma_1 > 0, \ \omega > \ \omega_o^2 + \omega_p^2$

$\gamma_1 < 0, \ \omega < \omega_o$.

A feature of the elliptic case is that small sinusoidal modulations grow exponentially and hence are unstable.



III.4. The Fully Nonlinear Formulation for Linear Polarization

We proceed to find the averaged Lagrangian for the fully nonlinear problem for linearly polarized waves. The integrals of the Euler equations are used to manipulate the Lagrangian into an integrable form and this is equivalent to placing the periodic solutions of chapter II directly into the Lagrangian. The fully nonlinear two-dimensional Lagrangian is given in chapter I as

$$L = \frac{\epsilon_o}{2}\left\{A_t^{\ 2} - c^2(A_x^{\ 2} + A_y^{\ 2})\right\} - A_tP + \frac{1}{\epsilon_o\omega_p^{\ 2}}\left\{\frac{P_t^{\ 2}}{2} - V(P)\right\} \qquad \text{(III. 10)}$$

and we shall take $V(P)$ as an even function of P. Considering A and P as periodic functions of $\theta = \kappa x - \omega t$ where κ, ω are temporarily regarded as constants, we examine the Euler equations for the form of A and P to be placed into L. In the same manner as in Chapter II we produce the integrated forms

$$(\omega^2 - c^2\kappa^2)A_\theta = -\frac{\omega}{\epsilon_o}P \qquad \text{(III. 11)}$$

$$\omega^2 P_\theta^{\ 2} + 2V(P) + \frac{\omega_p^{\ 2}\omega^2}{\omega^2 - c^2\kappa^2}P^2 = M \qquad \text{(III. 12)}$$

The constant of integration in (III. 11) has been suppressed to avoid the constant electric field. The definition of the averaged Lagrangian is

$$\mathfrak{L} = \frac{1}{2\pi}\int_0^{2\pi} L\, d\theta$$

and hence its form will be

$$\mathfrak{L} = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\epsilon_o}{2}(\omega^2 - c^2\kappa^2)A_\theta^{\ 2} + \omega A_\theta P + \frac{1}{\epsilon_o\omega_p^{\ 2}}\left[\frac{\omega_o P_\theta^{\ 2}}{2} - V(P)\right]\right\} d\theta \, .$$

A_θ is eliminated by means of (III. 11) which leaves the form

$$\mathfrak{L} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\epsilon_o\omega_p^{\ 2}}\left\{\omega^2 P_\theta^{\ 2} - \frac{\omega_p^{\ 2}\omega^2 P^2}{\omega^2 - c^2\kappa^2} - 2V(P)\right\} d\theta$$

and (III. 12) is used in the following successive forms, first to eliminate $V(P)$ and then to remove P_θ:

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\epsilon_o \omega_p^2}\left\{2\omega^2 P_\theta^2 - M\right\} d\theta$$

$$= \frac{1}{4\pi\epsilon_o \omega_p^2}\left\{\oint 2\omega\sqrt{M - \frac{\omega_p^2 \omega^2}{\omega^2 - c^2\kappa^2}P^2 - 2V(P)}\, dP - 2\pi M\right\}. \quad \text{(III. 13)}$$

The parameters ω, κ and M are now regarded as slowly varying quantities and the averaged Lagrangian is varied with respect to them subject to the constraint that ω and $\underline{\kappa}$ are derived from a phase θ. The Euler equations for this averaged Lagrangian are:

$$\mathcal{L}_M = 0,$$

$$-\frac{\partial}{\partial t}\mathcal{L}_\omega + \frac{\partial}{\partial \underline{x}}\mathcal{L}_{\underline{\kappa}} = 0 . \quad \text{(III. 14)}$$

III. 5. The Two-Timing Approach

We mention at this point how the two-timing method fits into this framework since higher order terms than those included here will be retained in chapter V. We consider the problem as having two time and space scales related by a small parameter ϵ such that

$$X = \epsilon x, \quad Y = \epsilon y, \quad Z = \epsilon z, \quad T = \epsilon t.$$

The slowly varying functions $\underline{\kappa}$, ω and the amplitude of the waves depend on the "slow variables" $\underline{X}$ and T and the oscillations will be periodic functions of the "fast" variable

$$\theta = \frac{1}{\epsilon}\Theta(\underline{X}, T)$$

where

$$\underline{\kappa} = \frac{\partial \Theta}{\partial \underline{X}} \quad \text{and} \quad \omega = -\frac{\partial \Theta}{\partial T} \, .$$

The original variational principle

$$\delta \iint_{-\infty}^{\infty} L \, d\underline{x} \, dt = 0 \qquad \text{(III.15)}$$

is now modified by the standard two-timing procedure of neglecting the relation between the fast and slow scales and regarding them as independent. The intuitive extension of (III.15) is

$$\delta \iint_{-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} L \, d\theta \, d\underline{X} \, dT = 0$$

and Whitham [5] has shown this to be equivalent to the straightforward application of two-timing. The periodic functions being known, the θ integral

$$\mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} L \, d\theta$$

can be performed leaving $\mathcal{L}$ a function of the slowly varying quantities. The new variational principle is then

$$\delta \iint_{-\infty}^{\infty} \mathcal{L} \, d\underline{X} \, dT = 0 \, .$$

III.6. The Equations of the System and the One-Dimensional Time-Dependent Problem

In order to display the form of the Euler equations we use the notation of Knight and Peterson and define

$$J(M,\omega,\kappa) = \frac{1}{2\pi}\oint\sqrt{M - \frac{\omega_p^2\omega^2}{\omega^2 - c^2\kappa^2}P^2 - 2V(P)}\,dP\ .$$

The averaged Lagrangian (III.13) becomes

$$\mathcal{L} = \frac{\omega J(M,\omega,\kappa)}{2\epsilon_o\omega_p^2} - \frac{M}{2\epsilon_o\omega_p^2}\ .$$

The Euler equations (III.14) become

$$\omega J_M = 1\ , \tag{III.16}$$

and

$$-\frac{\partial}{\partial t}(J + \omega J_\omega) + \frac{\partial}{\partial x}(\omega J_{\kappa_1}) + \frac{\partial}{\partial y}(\omega J_{\kappa_2}) + \frac{\partial}{\partial z}(\omega J_{\kappa_3}) = 0. \tag{III.17}$$

Accompanied by the consistency relations

$$\omega_{\underline{x}} + \underline{\kappa}_t = 0, \qquad \nabla \times \underline{\kappa} = 0 \tag{III.18}$$

they form a complete set. The dispersion relation (III.16) is used to eliminate one variable from the differential equations (III.17) and (III.18). Since J is a function of κ/ω it is convenient to introduce the index of refraction

$$n = \frac{c\kappa}{\omega}\ .$$

Now ω is eliminated by (III.16) and the ω/κ dependence becomes dependence on n. In these variables the set of differential equations is

$$\frac{\partial}{\partial t}(nJ_n - J) + \frac{\partial}{\partial x}\left(\frac{cn_1}{n} J_n\right) + \frac{\partial}{\partial y}\left(\frac{cn_2}{n} J_n\right) + \frac{\partial}{\partial z}\left(\frac{cn_3}{n} J_n\right) = 0$$

$$\frac{\partial}{\partial \underline{x}}\left(\frac{1}{J_M}\right) + \frac{\partial}{\partial t}\left(\frac{\underline{n}}{cJ_M}\right) = 0 \tag{III.19}$$

We now examine the one-dimensional time-dependent problem which is produced by neglecting the y and z dependence in equations (III.19). The analysis of Knight and Peterson proceeds by carrying through the differentiation of equations (III.19) to produce

$$(nJ_{Mn} - J_M)\frac{\partial M}{\partial t} + nJ_{nn}\frac{\partial n}{\partial t} + cJ_{Mn}\frac{\partial M}{\partial x} + cJ_{nn}\frac{\partial n}{\partial x} = 0,$$

$$nJ_{MM}\frac{\partial M}{\partial t} + (nJ_{Mn} - J_M)\frac{\partial n}{\partial t} + cJ_{MM}\frac{\partial M}{\partial x} + cJ_{Mn}\frac{\partial n}{\partial x} = 0 .$$

The characteristic form is

$$\frac{\partial M}{\partial t} + \lambda_\pm \frac{\partial M}{\partial x} \pm \sqrt{\frac{J_{nn}}{J_{MM}}}\left(\frac{\partial n}{\partial t} + \lambda_\pm \frac{\partial n}{\partial x}\right) = 0$$

where

$$\lambda_\pm = \frac{c}{n - \dfrac{J_M}{J_{Mn} \pm \sqrt{J_{MM}J_{nn}}}} .$$

This differential equation states that there are two types of characteristic curves given by

$$\frac{dx}{dt} = \lambda_\pm \tag{III.20}$$

and along the corresponding curves, the Riemann invariants $R_\pm(M,n)$, given by

$$dR_{\pm} = dM \pm \sqrt{\frac{J_{nn}}{J_{MM}}}\, dn \qquad \text{(III. 21)}$$

are constant. When the curves (III. 20) are real the system is hyperbolic and when they are complex it is elliptic.

When $J_{MM}J_{nn} < 0$ the system is elliptic and constant solutions are unstable in the sense that small sinusoidal oscillations grow exponentially. When $J_{MM}J_{nn} > 0$ the system is hyperbolic and there are two characteristic velocities given by (III. 20). In the limit of small amplitude both J_{nn} and J_{MM} go to zero leaving only one value of λ, the group velocity of the linear system; hence the nonlinear feature of the splitting of the group velocity is displayed.

Simple waves occur when a wave packet has an initial profile that satisfies one of the conditions

$$\frac{\partial M}{\partial n} \pm \sqrt{\frac{J_{nn}}{J_{MM}}} = 0 \ .$$

Then one of the Riemann invariants, say R_I, will be constant initially, and characteristics of type I, will carry the constant value forward in time, so that R_I is constant in all space and time. Along characteristics of type II, R_{II} is constant. Both R_I and R_{II} being constant requires M and n to be constant, and hence the characteristics of slope λ_{II} are straight. Since the value of R_{II} varies with the choice of characteristic, different characteristics of type II have different constant slopes λ_{II}.

We consider the case where λ_I is the slow velocity and λ_{II} the fast one. For a well behaved system, simple waves of this type steepen toward the front of the pulse since large amplitudes are far from linear and hence travel much faster than the linear wave speed; smaller amplitudes have speeds closer to the linear wave speed. One can imagine more complicated systems in which λ_{II} is not monotonic with amplitude. There would be regions of amplitude values where speed is monotonic with amplitude and the tendency to steepen switches from one end of the pulse to the other when the amplitude enters a new region. When λ_I is the fast velocity and λ_{II} the slow one, steepening occurs at the trailing edge in simple systems. When the characteristics of type II cross, the slopes of the pulse become vertical and multiple valued solutions arise. The solutions are continued by fitting discontinuities, but when this takes place the slowly varying assumptions break down and must be replaced by other physical considerations. Across the shock energy and momentum must be conserved.

III. 7. The Fully Nonlinear Formulation for Orbitally Polarized Waves

The procedure in this section combines the Lagrangian techniques of section III. 4 for the linearly polarized but fully nonlinear waves, and the integration techniques of section II. 4 where orbitally polarized plane waves were found. The three-dimensional Lagrangian (I. 6) is

$$L = \frac{\epsilon_o}{2}(A_{i,t}^{\;2} - c^2 A_{i,x_k}^{\;2}) - A_{i,t}P_i - \frac{\epsilon_o}{2c^2}(\phi_t^{\;2} - c^2\phi_{x_i}^{\;2}) - \phi_{x_i}P_i$$

$$+ \frac{1}{\epsilon_o \omega_p^{\;2}}\left(\frac{P_{i,t}^{\;2}}{2} - V(P)\right).$$

The uniform plane wave propagating in the x-direction takes the form

$$\underline{A} = (0, A_2(\theta), A_3(\theta)), \quad \underline{P} = (0, P_2(\theta), P_3(\theta)), \quad \phi = 0,$$

where $\theta = \kappa x - \omega t$. Placing the plane wave into the Lagrangian, we obtain

$$L = \frac{\epsilon_o}{2}(\omega^2 - c^2\kappa^2)(A_{2,\theta}^{\;2} + A_{3,\theta}^{\;2}) + \omega(A_{2,\theta}P_2 + A_{3,\theta}P_3)$$

$$+ \frac{1}{\epsilon_o \omega_p^{\;2}}\left(\frac{\omega^2}{2}(P_{2,\theta}^{\;2} + P_{3,\theta}^{\;2}) - V(P)\right).$$

The Euler equations are the same as (II.15). The first two equations, those produced by variation of A_2 and A_3, are

$$(\omega^2 - c^2\kappa^2)A_{2,\theta\theta} = -\frac{\omega}{\epsilon_o}P_{2,\theta},$$

$$(\omega^2 - c^2\kappa^2)A_{3,\theta\theta} = -\frac{\omega}{\epsilon_o}P_{3,\theta}.$$

As in chapter II we integrate each equation once with respect to θ and suppress the constant of integration since we desire A_2, A_3, P_2 and P_3 to have bounded oscillations. (The bounds will be slowly varying when we are finished, of course.) A_2, A_3, P_2 and P_3 will not be periodic in general. The integrated forms

$$(\omega^2 - c^2\kappa^2)A_{2,\theta} = -\frac{\omega}{\epsilon_o} P_2 ,$$

$$(\omega^2 - c^2\kappa^2)A_{3,\theta} = -\frac{\omega}{\epsilon_o} P_3 ,$$

are placed into L to eliminate $A_{2,\theta}$ and $A_{3,\theta}$. The new form is

$$L = \frac{1}{\epsilon_o \omega_p^2}\left(\frac{\omega^2}{2}(P_{2,\theta}^{\ 2} + P_{3,\theta}^{\ 2}) - V(P)\right) - \frac{\omega^2(P_2^{\ 2} + P_3^{\ 2})}{2\epsilon_o(\omega^2 - c^2\kappa^2)} .$$

The variables P_2 and P_3 are expressed in polar coordinates:

$$P_2 = P\cos\psi ,$$

$$P_3 = P\sin\psi ,$$

and this form is placed into L to give

$$L = \frac{1}{\epsilon_o \omega_p^2}\left(\frac{\omega^2}{2}(P_\theta^{\ 2} + \psi_\theta^{\ 2}P^2) - V(P)\right) - \frac{\omega_p^{\ 2}\omega^2 P^2}{2(\omega^2 - c^2\kappa^2)} .$$

Variation with respect to P and ψ give the Euler equations (II.17). We use the two-timing formalism to set up the form of the slowly varying quantities. We found in chapter II that the uniform plane wave solutions were of the form

$$P = P(\theta), \qquad \psi = \nu\theta + \Psi(\theta), \qquad \theta = \kappa x - \omega t .$$

P and Ψ are periodic functions of θ with period 2π and ν, κ and ω are constants. We wish to consider solutions that are slowly varying in the sense that, if

$$T = \epsilon t, \qquad \underline{X} = \epsilon \underline{x}$$

are slow scales and ϵ is a small parameter determined by the rate of change of boundary data in time and space, then we have solutions of the form

$$P = P(\theta, \underline{X}, T)$$

$$\psi = \epsilon^{-1} \zeta(\underline{X}, T) + \Psi(\theta, \underline{X}, T) .$$

P and Ψ are periodic in the fast time θ, and

$$\nu(\underline{X}, T) = \zeta_{\Theta}, \quad \omega(\underline{X}, T) = -\Theta_T, \quad \kappa(\underline{X}, T) = \Theta_X, \qquad \text{(III. 22)}$$

where the slowly varying function Θ is given by

$$\Theta(\underline{X}, T) = \epsilon \theta ,$$

but this last relationship is neglected in the two-timing procedure, θ and Θ are considered as independent quantities. The "two-timed" Lagrangian becomes

$$L = \frac{1}{\epsilon_o \omega_p^2} \left\{ \frac{\omega^2}{2} P_\theta^2 + \frac{\omega^2}{2} (\nu + \Psi_\theta)^2 P^2 - V(P) - \frac{\omega_p^2 \omega^2 P^2}{2(\omega^2 - c^2 \kappa^2)} \right\} .$$

The Euler equations,

$$\omega^2 P_{\theta\theta} - \omega^2 (\nu + \Psi_\theta)^2 P + V'(P) + \frac{\omega_p^2 \omega^2 P}{\omega^2 - c^2 \kappa^2} = 0$$

$$\frac{\partial}{\partial \theta} (P^2 (\nu + \Psi_\theta)) = 0 ,$$

are integrated to give

$$\frac{\omega^2 P_\theta{}^2}{2} + \frac{\omega^2 h^2}{2P^2} + V(P) + \frac{\omega_p{}^2 \omega^2 P^2}{2(\omega^2 - c^2\kappa^2)} = \frac{M}{2} \tag{III. 23}$$

$$P^2(\nu + \Psi_\theta) = h\ , \tag{III. 24}$$

where the "constants" of integration are functions of $\underline{X}$, T. The averaged Lagrangian is given by

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\epsilon_o \omega_p{}^2}\left\{\frac{\omega^2}{2} P_\theta{}^2 + \frac{\omega^2}{2}(\nu + \Psi_\theta)^2 P^2 - V(P) - \frac{\omega_p{}^2 \omega^2 P^2}{2(\omega^2 - c^2\kappa^2)}\right\} d\theta\ .$$

First (III. 24) is used to eliminate Ψ_θ from $\mathcal{L}$. That form is

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\epsilon_o \omega_p{}^2}\left\{\frac{\omega^2 P_\theta{}^2}{2} + \frac{\omega^2 h^2}{2P^2} - V(P) - \frac{\omega_p{}^2 \omega^2 P^2}{2(\omega^2 - c^2\kappa^2)}\right\} d\theta\ .$$

Then (III. 23) is used to produce the following form:

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\epsilon_o \omega_p{}^2}\left\{\omega^2 P_\theta{}^2 + \frac{\omega^2 h^2}{P^2} - \frac{M}{2}\right\} d\theta\ .$$

We use (III. 23) to convert one factor of P_θ in the first term into an expression in P; the other converts $P_\theta\, d\theta$ into dP. In the second term, (III. 24) is used to eliminate P^2 and the constant third term is integrated directly. $\mathcal{L}$ becomes

$$\mathcal{L} = \frac{1}{2\pi}\oint \frac{1}{\epsilon_o \omega_p{}^2}\left\{M - \frac{\omega^2 h^2}{P^2} - 2V(P) - \frac{\omega_p{}^2 \omega^2 P^2}{\omega^2 - c^2\kappa^2}\right\}^{1/2} dP$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{\omega^2 h(\nu + \Psi_\theta)}{\epsilon_o \omega_p{}^2}\, d\theta - \frac{M}{2\epsilon_o \omega_p{}^2}\ .$$

Since Ψ is periodic in θ with period 2π, the integral over Ψ_θ vanishes. The final form of $\mathfrak{L}$ is

$$\mathfrak{L} = \frac{1}{2\pi} \oint \frac{\omega}{\epsilon_o \omega_p^2} \left\{ M - \frac{\omega^2 h^2}{P^2} - 2V(P) - \frac{\omega_p^2 \omega^2 P^2}{\omega^2 - c^2 \underline{\kappa}^2} \right\}^{1/2} dP$$

$$+ \frac{1}{\epsilon_o \omega_p^2} \left(\omega^2 h - \frac{M}{2} \right) . \qquad \text{(III. 25)}$$

To produce the Euler equations we must vary $\mathfrak{L}$ with respect to the slowly varying functions M, h, $\underline{\kappa}$, ω, ν, subject to the consistency relations which result from conditions (III. 22). These relations are

$$\frac{\partial \omega}{\partial \underline{x}} + \frac{\partial \underline{\kappa}}{\partial t} = 0$$

$$\frac{\partial}{\partial \underline{x}} (\nu \omega) + \frac{\partial}{\partial t} (\nu \underline{\kappa}) = 0 .$$

It becomes convenient to subtract the first from the second and use that in place of the second. The ones we shall use are, therefore,

$$\frac{\partial \omega}{\partial \underline{x}} + \frac{\partial \underline{\kappa}}{\partial t} = 0$$

$$\omega \frac{\partial \nu}{\partial \underline{x}} + \underline{\kappa} \frac{\partial \nu}{\partial t} = 0. \qquad \text{(III. 26)}$$

We enforce (III. 26) as side conditions on the variation of $\mathfrak{L}$ by means of Lagrange multipliers. We write

$$\mathfrak{L}^* = \mathfrak{L} + \underline{\lambda} \cdot (\omega \nu_{\underline{x}} + \underline{\kappa} \nu_t) + \underline{\mu} \cdot (\omega_{\underline{x}} + \underline{\kappa}_t) .$$

Now $\mathfrak{L}^*$ must be varied freely with respect to M, h, $\underline{\kappa}$, ω, ν, $\underline{\lambda}$ and

$\underline{\mu}$. The resulting equations are

$$\mathcal{L}_M = 0 \quad \text{(i)} \qquad \mathcal{L}_\nu - \omega \nabla \cdot \underline{\lambda} - \underline{\kappa} \cdot \underline{\lambda}_t = 0 \quad \text{(v)}$$

$$\mathcal{L}_h = 0 \quad \text{(ii)} \qquad \omega \nu_{\underline{x}} + \underline{\kappa} \nu_t = 0 \quad \text{(vi)}$$

$$\mathcal{L}_{\underline{\kappa}} + \underline{\lambda} \nu_t - \underline{\mu}_t = 0 \quad \text{(iii)} \qquad \omega_{\underline{x}} + \underline{\kappa}_t = 0 \,. \quad \text{(vii)}$$

$$\mathcal{L}_\omega + \underline{\lambda} \cdot \nu_{\underline{x}} - \mu_{\underline{x}} = 0 \quad \text{(iv)}$$

$\underline{\lambda}$ and $\underline{\mu}$ must be eliminated from (iii), (iv) and (v) to produce the equations of motion. First, elimination of $\underline{\mu}$ from (iii) and (iv) produces

$$-\frac{\partial}{\partial t} \mathcal{L}_\omega + \nabla \cdot \mathcal{L}_{\underline{x}} - \underline{\lambda}_t \cdot \nu_{\underline{x}} + \nabla \cdot \underline{\lambda} \nu_t = 0. \quad \text{(viii)}$$

Now condition (vi) indicates that the terms in $\underline{\lambda}_t$ and $\nabla \cdot \underline{\lambda}$ in equation (v) are proportional to those in equation (viii). Elimination of the common quantities from (v) and (viii) using (vi) gives

$$-\frac{\partial}{\partial t} \mathcal{L}_\omega + \nabla \cdot \mathcal{L}_{\underline{\kappa}} + \frac{\nu_t}{\omega} \mathcal{L}_\nu = 0 \,.$$

The equations of the system are the Euler equations

$$\mathcal{L}_M = 0 \qquad \mathcal{L}_h = 0$$

$$-\frac{\partial}{\partial t} \mathcal{L}_\omega + \frac{\partial}{\partial \underline{x}} \mathcal{L}_{\underline{\kappa}} + \frac{\nu_t}{\omega} \mathcal{L}_\nu = 0 \,, \qquad \text{(III. 27)}$$

together with the consistency conditions

$$\frac{\partial \omega}{\partial \underline{x}} + \frac{\partial \underline{\kappa}}{\partial t} = 0 \qquad \omega \frac{\partial \nu}{\partial \underline{x}} + \underline{\kappa} \frac{\partial \nu}{\partial t} = 0 \,.$$

Chapter IV

Beams

IV.1 The Nonlinear Equations in Two and Three Dimensions

The time-independent equations can be treated in some detail and one setting is the important problem of nonlinear effects on beams. When the phase velocity decreases as amplitude increases, a localized beam travels more slowly at its center. Accordingly, a surface of constant phase progresses slowly at the center of the beam and the faster outskirts are caused more and more to converge toward the axis of the beam. When a focusing point occurs, the second derivatives of amplitude which were neglected as small in the last chapter become important terms and must be included to continue the solution. This will be done in Chapter V.

IV.1.i The Two-Dimensional Equations

The fully nonlinear equations of Chapter III are examined in two dimensions with their time derivatives set equal to zero. The form in which they now stand is:

$$\mathcal{L}_a = 0 \ , \tag{IV.1}$$

$$\frac{\partial}{\partial x}\left(\frac{\mathcal{L}_\kappa}{\kappa}\kappa_1\right) + \frac{\partial}{\partial y}\left(\frac{\mathcal{L}_\kappa}{\kappa}\kappa_2\right) = 0 \ , \tag{IV.2}$$

$$\frac{\partial \kappa_2}{\partial x} - \frac{\partial \kappa_1}{\partial y} = 0 \, , \tag{IV.3}$$

$$\omega = \text{constant.}$$

Since $\mathcal{L}$ depends only upon a and κ, a convenient procedure is to set $\rho = |\mathcal{L}_\kappa|/\kappa$ and to use ρ as a variable instead of a. The dispersion relation (IV.1) is solved for ρ as a function of κ. In the special case of near linear the averaged Lagrangian is

$$\mathcal{L} = \frac{\epsilon_0}{4}\left[\omega^2\left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right) - c^2\kappa^2\right] a^2 + \frac{\gamma_1 \epsilon_0^3 \omega_p^6 \omega^4 a^4}{8(\omega_0^2 - \omega^2)^4} \, .$$

Variation (IV.1) gives

$$c^2\kappa^2 = \omega^2\left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right) + \frac{2\gamma_1 \epsilon_0 \omega_p^6 \omega^4 \rho}{(\omega_0^2 - \omega^2)^2 c^2} \, .$$

The near linear set of equations takes the form

$$\frac{\partial}{\partial x}(\rho\kappa_1) + \frac{\partial}{\partial y}(\rho\kappa_2) = 0 \, ,$$

$$\frac{\partial \kappa_2}{\partial x} - \frac{\partial \kappa_1}{\partial y} = 0 \, , \tag{IV.4}$$

$$\kappa^2 = K^2(1 + 2\tau\rho) \, , \tag{IV.5}$$

where K is the constant $\frac{\omega^2}{c^2}\left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right)$ and τ is the small parameter

$\dfrac{\gamma_1 \epsilon_0 \omega_p^{\,6} \omega^4}{\omega^2 c^4 \left(1+\dfrac{\omega_p^{\,2}}{\omega_0^{\,2}-\omega^2}\right)(\omega_0^{\,2}-\omega^2)^2}$. τ has the same sign as γ, the parameter fixed by the medium. Equations (IV.4) hold for the fully nonlinear system. The near linear form of the dispersion relation (IV.5) fixes attention on the near linear problem.

We place the system (IV.4) into characteristic form to find its type. That form is

$$\frac{\partial \kappa_1}{\partial x} + C \frac{\partial \kappa_1}{\partial y} + \frac{\rho_\kappa \dfrac{\kappa_1 \kappa_2}{\kappa} \pm \sqrt{-\rho^2 - \rho \rho_\kappa \kappa}}{\rho_\kappa \dfrac{\kappa_1^{\,2}}{\kappa} + \rho} \left(\frac{\partial \kappa_2}{\partial x} + C \frac{\partial \kappa_2}{\partial y}\right) = 0 \quad \text{(IV.6)}$$

where

$$C = \frac{\rho_\kappa \dfrac{\kappa_1 \kappa_2}{\kappa} \mp \sqrt{-\rho^2 - \rho \rho_\kappa \kappa}}{\rho_\kappa \dfrac{\kappa_1^{\,2}}{\kappa} + \rho} .$$

The system is hyperbolic when there exist real curves

$$\frac{dy}{dx} = C$$

along which equations (IV.6) become ordinary differential equations. That condition is that C be real, hence the system is hyperbolic when

$$\frac{\kappa}{\rho} \frac{\partial \rho}{\partial \kappa} + 1 < 0$$

and elliptic when

$$\frac{\underline{\kappa}}{\rho}\frac{\partial\rho}{\partial\kappa} + 1 > 0 \ .$$

The condition for the near linear system is

$$\frac{1 + 2\tau\rho}{\tau\rho} + 1 < 0$$

for the set to be hyperbolic. Sence τ is small and ρ is positive, the sign of τ always determines the type; a given medium supports a system of equations of only one type. When $\tau > 0$ the system is elliptic and when $\tau < 0$ the system is hyperbolic.

We note that the phase velocity is found by moving with the waves so that θ is held constant. The magnitude of this velocity is given by $\frac{\omega}{\kappa}$ and it is the speed that a wave front moves perpendicular to itself. The direction of the velocity is the direction if $\underline{\kappa}$. We see from (IV.5) that for ω constant, when $\tau > 0$,

$$\frac{\omega}{\kappa} \text{ decreases with increasing } \rho$$

and when $\tau < 0$,

$$\frac{\omega}{\kappa} \text{ increases with increasing } \rho \ .$$

It is apparent from the heuristic description at the beginning of this section that the case in which beams focus is $\tau > 0$ and the system is elliptic; the defocusing case is $\tau < 0$ and the system is hyperbolic.

IV.1.ii The Three-Dimensional Equations

We now wish to consider the radially symmetric problem. The variational principle is

$$\delta \int\int \mathcal{L} \, r \, dr \, d\tilde{\omega} = 0 \, .$$

The Euler equations for the time-independent problem are

$$\mathcal{L}_a = 0 \quad ,$$

$$\frac{\partial}{\partial r}\left(r \frac{\kappa_2}{\kappa} \mathcal{L}_\kappa\right) + \frac{\partial}{\partial x}\left(r \frac{\kappa_1}{\kappa} \mathcal{L}_\kappa\right) = 0 \; ,$$

$$\frac{\partial \kappa_2}{\partial x} - \frac{\partial \kappa_1}{\partial r} = 0 \; ,$$

$$\omega = \text{constant} \quad .$$

κ_2 is now the component of $\underline{\kappa}$ in the radial direction. The only difference between this case and the two-dimensional case is the appearance of the factor r in the second equation. Its effect is to add a forcing function to the right side of the characteristic equations, which, using ρ as before, now read:

$$\frac{\partial \kappa_1}{\partial x} + C\frac{\partial \kappa_1}{\partial r} + \frac{\frac{\rho_\kappa \kappa_1 \kappa_2}{\kappa} \pm \sqrt{-\rho^2 - \rho\rho_\kappa \kappa}}{\rho_\kappa \frac{\kappa_1^2}{\kappa} + \rho}\left(\frac{\partial \kappa_2}{\partial x} + C\frac{\partial \kappa_2}{\partial r}\right) = -\frac{\kappa_2}{r}\frac{\rho\kappa}{\rho\kappa + \rho_\kappa \kappa_1},$$

where

$$C = \frac{\rho_\kappa \frac{\kappa_1 \kappa_2}{\kappa} \mp \sqrt{-\rho^2 - \rho\rho_\kappa \kappa}}{\rho_\kappa \frac{\kappa_1^2}{\kappa} + \rho}.$$

The conditions on the type are the same. We shall see that in the focusing case, focusing is faster than the two-dimensional problem, but in the defocusing case, divergence is slower.

IV.1.iii Analogy with Fluid Flow

The equations of two-dimensional, irrotational, isentropic, steady fluid flow are

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0,$$

$$\tfrac{1}{2}q^2 + \frac{\gamma \rho^{\gamma-1}}{\gamma-1} = \tfrac{1}{2}q^2_{max},$$

where u and v are the x and y components of particle velocity, ρ is the density and $q^2 = u^2 + v^2$. The similarity between the first two equations and (IV.4) is immediate. The third equation gives the form of $\rho(q)$. Thus a number of methods developed in fluid mechanics can be adapted here. Each of the following two sections outlines a method taken from fluid mechanics and pursues the solutions that result.

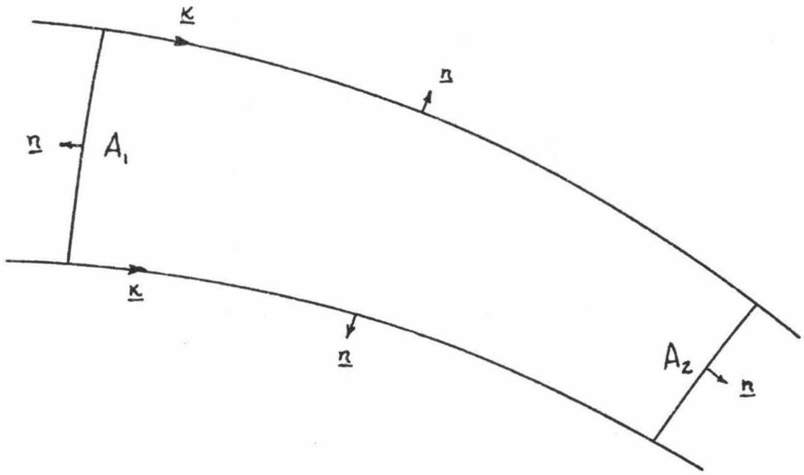
IV. 2 Solution of the Beam Equations

We now outline a method which produces solutions in which rays are allowed to bend significantly from being parallel to the axis of the beam. In contrast, in Section IV. 3 we shall study a method which treats "thin beams," where the angle of the rays never deviates far from parallel to the axis.

IV. 2. i Method of Shock Dynamics

An approach to problems dealing with the propagation of shock waves in channels has been developed by Whitham [8] for use in the equations of gas dynamics. We shall use it to obtain solutions to the two-dimensional problem.

The essential idea of the method is to transform the coordinates x and y to a coordinate system in which the first equation of (IV. 4) is algebraic. Lines whose direction is always in the direction of $\underline{\kappa}$ are called rays. The diagram shows a two-dimensional ray tube bounded by two neighboring rays and having cross-sections at the ends of A_1 and A_2. Applying the divergence theorem over the area of the ray tube and using the first of (IV. 4),



we find

$$0 = \int\int \nabla \cdot (\underline{\kappa}\rho)dxdy = \oint \underline{\kappa}\rho \cdot \underline{n}\, d\ell \ .$$

Since $\underline{\kappa} \cdot \underline{n} = 0$ on the sides of the ray tube, only the end contributions to the line integral remain and we have

$$(\kappa\rho A)_2 - (\kappa\rho A)_1 = 0 \ .$$

Thus $\kappa\rho A$ is constant along a ray tube. We choose the rays as a coordinate system since the first equation of (IV.4) becomes algebraic. The quantity $\kappa\rho A$ depends only upon the coordinate orthogonal to the rays. Since the ray pattern is stationary, the successive positions of a peak, say, of the oscillations will be the surfaces of constant phase perpendicular to the rays. Using α as a length scale along a ray we must make $d\alpha$ proportional to dt so that the new value of α still corresponds to a constant phase surface. Choosing

$$d\alpha = \omega \, dt$$

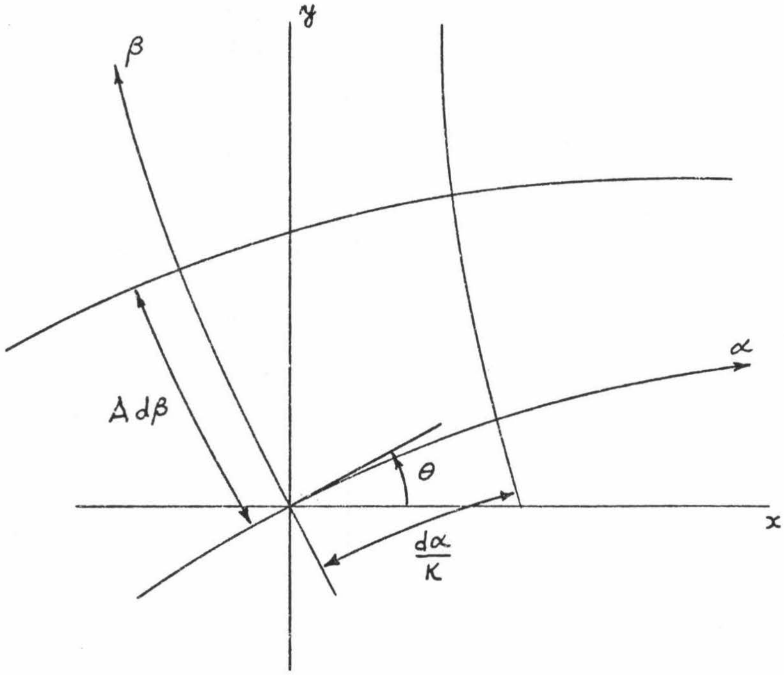
we find that the physical distance measured by $d\alpha$ is the phase velocity multiplied by dt.

$$ds = \frac{\omega}{\kappa} dt$$

$$= \frac{d\alpha}{\kappa} \quad .$$

The other coordinate is β, measured along the surface of constant phase. The width of the ray tube is $A d\beta$, A being the area density of the two-dimensional ray tube. The angle θ is the angle that the rays make with the x-axis. The local change of variables is a rotation given by

$$\begin{aligned} dx &= \frac{\cos\theta}{\kappa} d\alpha - A \sin\theta \, d\beta \quad , \\ dy &= \frac{\sin\theta}{\kappa} d\alpha + A \cos\theta \, d\beta \quad . \end{aligned} \qquad \text{(IV. 7)}$$



On introducing a variable A, the first equation of (IV. 4) gave

$$\rho\kappa A = f(\beta) \qquad \text{(IV. 8)}$$

and this is the condition that A conform to the width of a ray tube. We obtain another equation by using (IV. 8) to eliminate $\rho\kappa\cos\theta$ and $\rho\kappa\sin\theta$ from the first equation of (IV. 4) and converting to α, β coordinates by means of (IV. 7). This results in

$$\frac{\partial\theta}{\partial\beta} - \kappa \frac{\partial A}{\partial\alpha} = 0 \ . \tag{IV.9}$$

Transformation of the second equation of (IV.4) to α, β coordinates by (IV.7) gives

$$\frac{\partial\theta}{\partial\alpha} - \frac{1}{\kappa^2 A}\frac{\partial\kappa}{\partial\beta} = 0 \ . \tag{IV.10}$$

Equations (IV.9) and (IV.10) are geometrical statements about the coordinates. Accompanied by (IV.8) and the equation giving the form of $\rho(\kappa)$, (IV.5), they form a complete set of equations. The two algebraic equations will be used to eliminate two dependent variables leaving two first order equations in two unknowns. Having solved this set, the original coordinates are recovered by integration along rays and surfaces of constant phase:

$$\begin{aligned} x(\alpha,\beta) &= \int_0^\alpha \frac{\cos\theta(\alpha',\beta)}{\kappa(\alpha',\beta)}\, d\alpha' - \int_0^\beta A(0,\beta')\sin\theta(0,\beta')d\beta' \\ y(,,\beta) &= \int_0^\alpha \frac{\sin\theta(\alpha',\beta)}{\kappa(\alpha',\beta)}\, d\alpha' + \int_0^\beta A(0,\beta')\cos\theta(0,\beta')d\beta' \ . \end{aligned} \tag{IV.11}$$

IV.2.ii Solution by Separation of Variables

The method of Section IV.2.i has reduced the awkward set of equations (IV.4) to equations that are more easily handled and we note here that these methods apply equally well to the fully nonlinear equations derived in Chapter III as to the near linear ones. The resulting equations for the near linear problem, however, have some closed form solutions which we proceed to derive.

The complete set of equations in α, β coordinates that we have produced is:

$$\kappa^2 = K^2 (1 + 2\tau\rho) \tag{IV.5}$$

$$\rho\kappa A = f(\beta) \tag{IV.8}$$

$$\frac{\partial\theta}{\partial\beta} - \kappa \frac{\partial A}{\partial\alpha} = 0 \tag{IV.9}$$

$$\frac{\partial\theta}{\partial\alpha} - \frac{1}{\kappa^2 A} \frac{\partial\kappa}{\partial\beta} = 0 \tag{IV.10}$$

We have introduced the coordinates α and β which parameterize distance along the rays and along the surfaces of constant phase, the angle θ that rays make with the x-axis, the area density of ray tubes, A, and the arbitrary function $f(\beta)$ which fixes A for some initial value of α. The choice $f(\beta) \equiv 1$ is used here since it has been found to produce all the separated solutions that other choices can produce. We now use the two algebraic equations (IV.5) and (IV.8) to eliminate κ and A from the two differential equations leaving only θ and ρ as dependent variables. The exact form is not required, however, as (IV.5) contains a small parameter. The new form is

$$\frac{\partial\theta}{\partial\alpha} - (\tau\rho + O(\tau^2)\ )\ \frac{\partial\rho}{\partial\beta} = 0\ ,$$

$$\frac{\partial\theta}{\partial\beta} + (\frac{1}{\rho^2} + O(\tau)\ )\ \frac{\partial\rho}{\partial\alpha} = 0\ . \tag{IV.12}$$

The coefficient of $\frac{\partial\rho}{\partial\beta}$ in the first equation has a leading term proportional to τ due to the differentiation of κ in (IV.10), its O(1) term being constant, and in this way τ becomes a parameter of the problem. The terms written as "$O(\tau)$" and "$O(\tau^2)$" are neglected as they display more accuracy than the near linear equations contain.

The effect of transforming form x, y to α, β coordinates has been to consolidate the equations to two terms each, a form in which the variables separate. The set (IV.12) is nonlinear so that a separated solution is a special one. Solutions for different values of the separation constant cannot be superposed and no general solution is found by this method. We note, however, that reversing the roles of dependent and independent variables changes the form of the derivatives but leaves the coefficients unchanged, the coefficients now containing only independent variables, and the new set is linear. This approach is discussed in Section IV.2.iii.

We reduce (IV.12) to one second order equation in one dependent variable by satisfying one equation by a potential and substituting in the other. It makes no difference which equation is identically satisfied by the potential and no further solutions are generated by doing it both ways.

We set

$$\chi_\alpha = \theta \, , \qquad \chi_\beta = 1/\rho \tag{IV.13}$$

to satisfy the second equation of (IV.12). Substitution in the first produces

$$\chi_{\alpha\alpha} + \frac{\tau}{\chi_\beta^3}\, \chi_{\beta\beta} = 0 \, .$$

We set

$$\chi = U(\alpha)\; V(\beta)$$

and separate variables to find the equations

$$\begin{aligned} U^2 U'' + C &= 0 \, , \\ \frac{\tau V''}{V(V')^3} &= C \, , \end{aligned} \tag{IV.14}$$

where C is the separation constant. We require $C > 0$ to form a beam and for $C > 0$ we find the first integrals of (IV.14):

$$\begin{aligned} U' &= -\sqrt{2C}\sqrt{\frac{1}{U} + D} \, , \\ V' &= \frac{1}{\rho_0 - \frac{CV^2}{2\tau}} \, . \end{aligned} \tag{IV.15}$$

A further integration gives the implicit forms of the solution. The form of the integrals depends on the sign of D.

When $D > 0$

$$\alpha - \alpha_0 = \frac{1}{\sqrt{2C}\, D^{3/2}} \left(\log(\sqrt{U} + \sqrt{1/D + U}) - \sqrt{UD}\sqrt{1+UD} \right) \tag{IV.16}$$

where α_0 is found by setting $\alpha = 0$, $U = 1$.

When $D < 0$

$$\alpha - \alpha_0 = \frac{1}{\sqrt{2C}\; |D|^{3/2}} \left(\cos^{-1}\sqrt{|D|U} + \sqrt{|D|U}\sqrt{1-|D|U} \right) \tag{IV.17}$$

In both cases the other integral is

$$\beta = \rho_0 V - \frac{CV^3}{6\tau} . \tag{IV.18}$$

We have chosen U to be 1 and $\alpha = 0$ so that the constant of integration ρ_0 in (IV.18) is the maximum value of ρ at the initial surface $\alpha = 0$. While the hodograph method was not used to obtain these solutions, the implicit form is typical of that method and multivalued solutions have been studied extensively in this connection. The inversion of (IV.18) gives a multivalues function for V and hence ρ and θ are multivalued. The edges of the fold are called limiting lines and they form the edges of the finite beam. We accept the branch of the solution that is symmetric about $\beta = 0$ and fit a zero solution to extend from the edge of the beam to infinity.

From the definition of χ, (IV.13), we find

$$\theta = U'(\alpha)\ V(\beta) \quad , \quad \rho = \frac{1}{U(\alpha)\ V'(\beta)} \ . \tag{IV.19}$$

The analytical formulas are best found by using U and V in place of α and β in (IV.11) to return to x, y coordinates, but before doing that we shall examine the qualitative features of the solution displayed in (IV.16), (IV.17) and (IV.18).

The following diagrams illustrate the two cases which we have labelled A for $D > 0$ and B for $D < 0$. The functions U, U', V and V' are sketched in figures 1 to 4. In figure 2A, U' is always negative while in 2B U' crosses zero. Figures 3 and 4 are the same for both cases. In figure 3 the cross strokes indicate branches that are extraneous; the branch that is symmetric about the origin is the desirable one. Then using (IV.19) we obtain sketches of θ and ρ. θ is the product of figures 2 and 3, and ρ is the inverse of the product of figures 1 and 4. Finally, figure 7 indicates the ray pattern which results. The successive profiles of ρ are mapped onto the successive surfaces of constant phase which are perpendicular to the rays. All the rays focus at a single point and a singularity in amplitude occurs there. Figure 7A illustrates that rays are always convergent while in 7B the rays are parallel somewhere and have a focal point before and after that plane.

The inverted form of U and V indicates that they are the convenient variables to use in recovering x and y rather than using

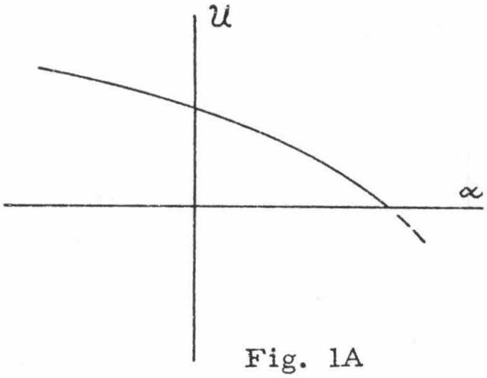


Fig. 1A

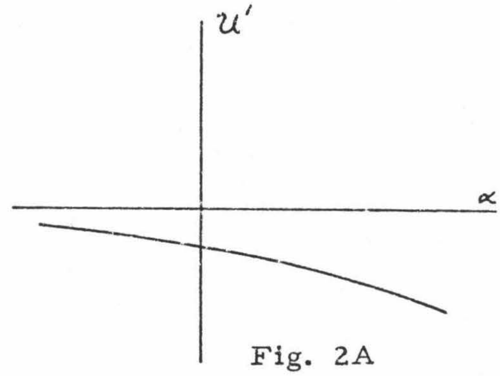


Fig. 2A

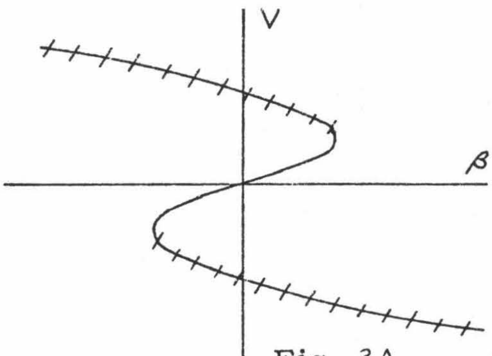


Fig. 3A

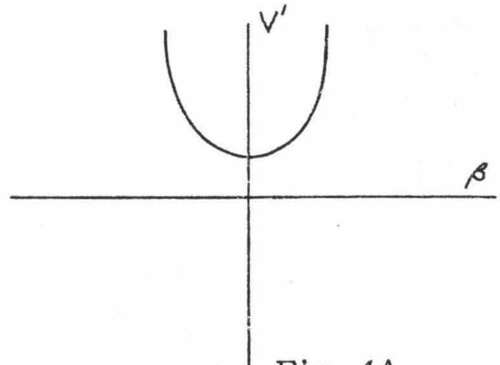


Fig. 4A

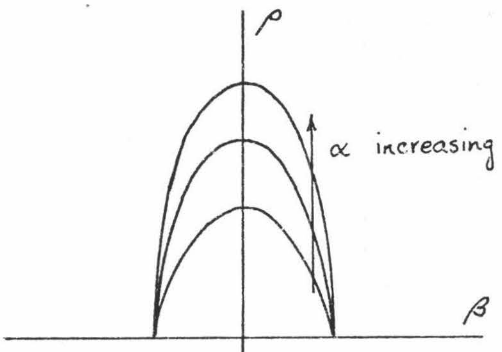


Fig. 5A

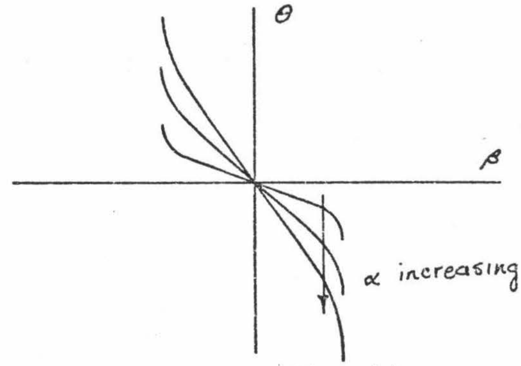


Fig. 6A

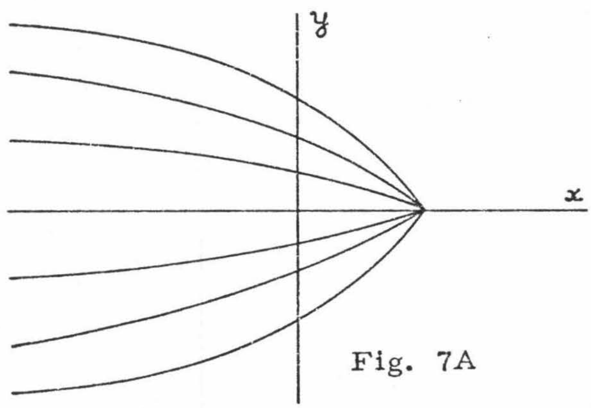


Fig. 7A

Sketches of Solution for $D > 0$

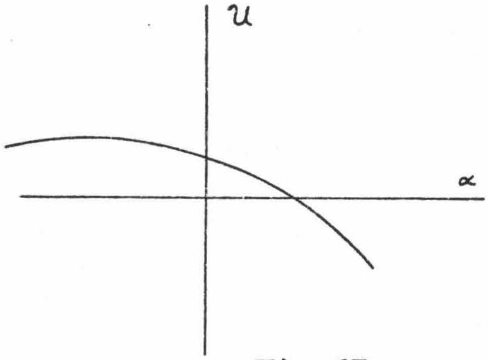


Fig. 1B

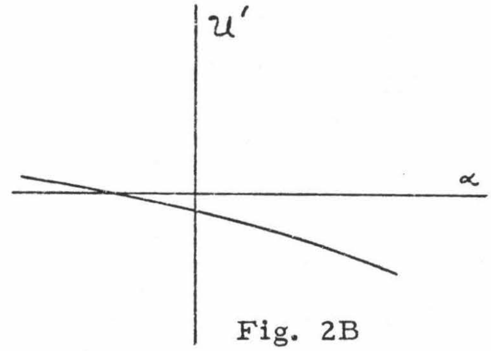


Fig. 2B

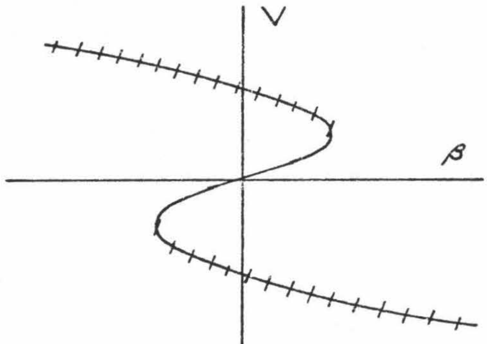


Fig. 3B

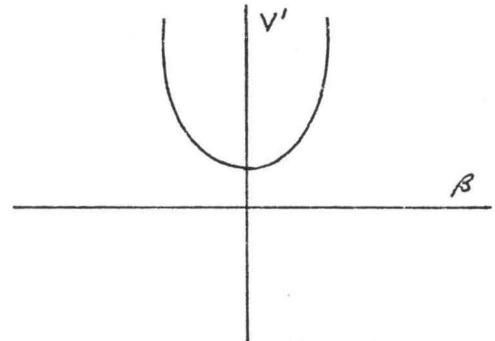


Fig. 4B

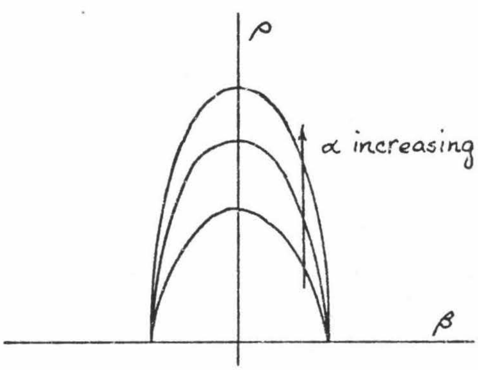


Fig. 5B

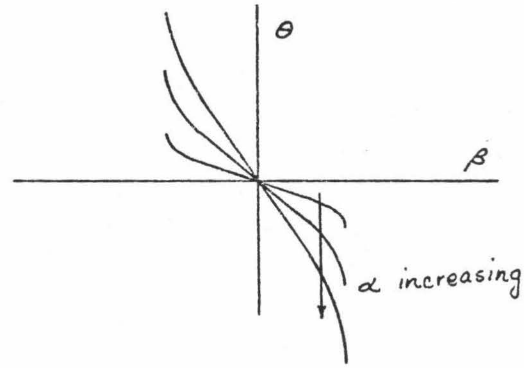


Fig. 6B

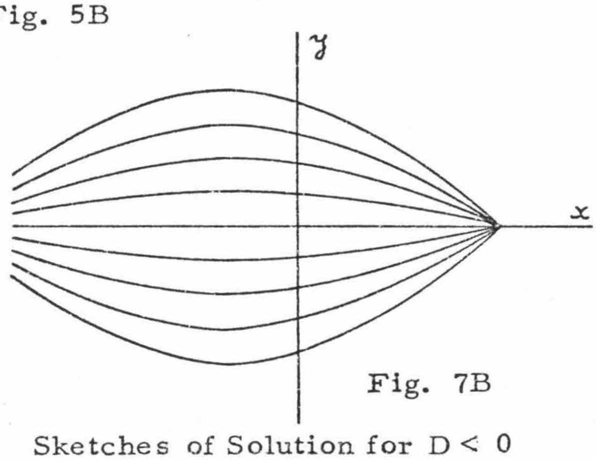


Fig. 7B

Sketches of Solution for $D < 0$

α and β. Then (IV.11) becomes

$$x = \int_1^U \frac{\cos\theta(U,V)}{\kappa(U,V)} \frac{d\alpha}{dU}\, dU - \int_0^V A(1,V)\sin\theta(1,V)\frac{d\beta}{dV}\, dV ,$$

$$y = \int_1^U \frac{\sin\theta(U,V)}{\kappa(U,V)} \frac{d\alpha}{dU}\, dU + \int_0^V A(1,V)\cos\theta(1,V)\frac{d\beta}{dV}\, dV .$$

Now we use integrals (IV.15) to produce the parametric form of the solution:

$$\theta = -\sqrt{2C}\,\sqrt{1/U+D}\; V ,$$

$$\rho = \frac{\rho_0 - \dfrac{CV^2}{2\tau}}{U} ,$$

$$x = \frac{1}{K\sqrt{2C}} \int_U^1 \frac{\cos\left(\sqrt{2C(1/U+D)}\; V\right) dU}{\sqrt{1/U+D}} - \frac{1}{K}\int_0^V \sin\left(\sqrt{2C(\tfrac{1}{u}+D)}V\right) dV , \qquad \text{(IV.20)}$$

$$y = -\frac{1}{K\sqrt{2C}} \int_U^1 \frac{\sin\left(\sqrt{2C(1/U+D)}\; V\right)}{\sqrt{1/U+D}}\, dU + \frac{1}{K}\int_0^V \cos\left(\sqrt{2C(\tfrac{1}{u}+D)}V\right) dV .$$

The integrals do not have a closed form for $V \neq 0$ but may be performed by expanding the sine and cosine in a Maclaurin series and integrating term by term.

The interesting feature that may be found exactly is the focusing distance. On the axis, V = 0 and under this condition the second and third equations of (IV. 20) give $x(\rho)$ in closed form. That form is: for $D > 0$

$$x = \frac{1}{K\sqrt{2C}\,D^{3/2}}\left\{\log\left(\frac{1+\sqrt{1/D+1}}{\sqrt{U}+\sqrt{1/D+U}}\right)-\sqrt{D(1+D)}+\sqrt{UD(1+UD)}\right\},$$

and for $D < 0$ (IV. 21)

$$x = \frac{1}{K\sqrt{2C}\,D^{3/2}}\left\{\cos^{-1}\sqrt{|D|U}-\cos^{-1}\sqrt{|D|}+\sqrt{|D|U(1-|D|U)}-\sqrt{|D|(1-|D|)}\right\}$$

and $\rho = \rho_0/U$. At the focal point $\rho \rightarrow \infty$ and the focusing distance is given by setting U = 0 in (IV. 21). Thus for $D > 0$

$$x_f = \frac{1}{K\sqrt{2C}\,D^{3/2}}\left\{\log(\sqrt{D}+\sqrt{1+D})-\sqrt{D(1+D)}\right\} \qquad \text{(IV.22)}$$

and for $D < 0$

$$x_f = \frac{1}{K\sqrt{2C}\,|D|^{3/2}}\left\{\frac{\pi}{2}-\cos^{-1}\sqrt{|D|}-\sqrt{|D|(1-|D|)}\right\} .$$

All beams for which $D < 0$ may be classified as "thin" beams which are beams in which gradients in y are steeper than gradients in x. This occurs for $D < 0$ because there is a value of x for which the rays are parallel and the changes in angle after that point are due to the interaction between gradients in y and nonlinearity. In near linear theory these changes must be small. The only way to have other than a thin beam is to have the rays sufficiently convergent at minus infinity and then amplitudes at the origin can be moderate while the angles of the rays may be large. This is the case when $D = O(1/\tau)$ in our equations.

In Section IV. 3 we shall deal with thin beams and, as it happens, we shall produce the same thin beam solution that we have just found. As a point of comparison we set $D = -1$ which makes the rays parallel at the origin. Then

$$\rho = \rho_0 - \frac{CK^2}{2\tau} y^2 \tag{IV. 23}$$

follows from setting $U = 1$ in (IV. 20) as the initial amplitude profile. In Section IV. 3 we shall, in fact, look for quadratic profiles directly. If we take w as the half width of the beam then (IV. 23) gives

$$w = \sqrt{\frac{2\tau\rho_0}{CK^2}}$$

and this may be used to eliminate C from (IV. 22) to give a focusing distance of

$$x_f = \frac{\pi w}{2\sqrt{\tau\rho_0}}$$

which we shall compare to other thin beams.

We conclude that the theory just presented for use with moderate amplitude beams entering nonlinear media is necessary when the beam has passed through a converging lens. In most other circumstances the thin beam theory should be sufficiently accurate. For fully nonlinear beams, however, (higher than "moderate" amplitude) the theory just presented is essential.

IV.2.iii The Hodograph Method

Equations such as (IV.12) may be transformed into linear equations by the hodograph transformation in which α and β are treated as functions of ρ and θ. This has been much developed in fluid mechanics. While the resulting equations are linear and solutions may be superposed, it is not clear how to form a solution representing a beam. We note the first steps just to indicate what would be involved.

We recall that equations (IV.12) are

$$\frac{\partial\theta}{\partial\alpha} - \tau\frac{\partial\rho}{\partial\beta} = 0 \; ,$$

$$\frac{\partial\theta}{\partial\beta} + \frac{1}{\rho^2}\frac{\partial\rho}{\partial\alpha} = 0 \; . \tag{IV.12}$$

The roles of the variables are reversed by the transformation

$$\rho_\alpha = \frac{\beta_\theta}{J} \; , \qquad \theta_\alpha = -\frac{\beta_\rho}{J} \; ,$$

$$\rho_\beta = -\frac{\alpha_\theta}{J}\ , \qquad \theta_\beta = \frac{\alpha_\rho}{J}\ ,$$

with $J = \alpha_\rho\beta_\theta - \alpha_\theta\beta_\rho$. Since the set (IV.12) is homogeneous in derivatives the Jacobian cancels out leaving the equations

$$\frac{\partial\alpha}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial\beta}{\partial\theta} = 0\ ,$$

$$\frac{\partial\beta}{\partial\rho} - \tau\rho\frac{\partial\alpha}{\partial\theta} = 0\ . \tag{IV.24}$$

They are now linear and solutions may be superposed. Separation of variables is used here and a suitable approach is again to introduce a potential to satisfy one of the equations. Again the choice of which equation to satisfy identically is irrelevant; no further separated solutions are obtained by reversing the roles of the equations. The introduction of χ such that

$$\chi_\theta = \beta\ , \qquad \chi_\rho = \tau\rho\alpha$$

satisfies the second equation of (IV.24) and substitution in the first produces

$$\rho\chi_{\rho\rho} - \chi_\rho + \tau\chi_{\theta\theta} = 0\ .$$

Separation of variables is accomplished by setting

$$\chi = R(\rho)\ T(\theta)$$

with the resulting separated equations being

$$\rho R'' - R' - c_1 R = 0 \ ,$$

$$T'' + \frac{c_1}{\tau} T = 0 \ , \qquad \text{(IV. 25)}$$

where c_1 is the separation constant. Regarding c_1 as positive we solve these equations and then augment the solutions by those found with c_1 negative. For convenience we set $c_2 = -c_1$ in the latter case. The first equation of (IV. 25) is placed into standard form by the substitution

$$\rho = \frac{\zeta^2}{4c_1} \quad , \quad R = \zeta^2 u \ ,$$

with the result

$$u_{\zeta\zeta} + \frac{u_\zeta}{\zeta} + (1 - \frac{4}{\zeta^2})\, u = 0$$

which is a Bessel Equation of the second order. The second of (IV. 25) has sinusoid solutions so that elementary solutions for the separated factors of χ take the form

$$R = \rho J_2(2\sqrt{c_1\rho}\ ) \qquad \text{and} \qquad R = \rho Y_2(2\sqrt{c_1 \rho})$$

$$T = \sin\left(\sqrt{\frac{c_1}{\tau}}\,\theta\right) \quad \text{and} \quad T = \cos\left(\sqrt{\frac{c_1}{\tau}}\,\theta\right) .$$

The form of α and β is arrived at using Bessel function identities. The elementary solutions are

$$\alpha = \frac{1}{\sqrt{\tau\rho}}\; J_1(2\sqrt{c_1\rho})\cos\left(\sqrt{\frac{c_1}{\tau}}\,\theta\right) ,$$

$$\beta = -\rho J_2(2\sqrt{c_1\rho})\,\sin\left(\sqrt{\frac{c_1}{\tau}}\,\theta\right)$$

and

$$\alpha = \frac{1}{\sqrt{\tau\rho}}\; Y_1(2\sqrt{c_1\rho})\,\cos\left(\sqrt{\frac{c_1}{\tau}}\,\theta\right) ,$$

$$\beta = -\rho\; Y_2(2\sqrt{c_1\rho})\,\sin\left(\sqrt{\frac{c_1}{\tau}}\,\theta\right) .$$

The cases of β being proportional to $\cos\left(\sqrt{\frac{c_1}{\tau}}\,\theta\right)$ have been discarded since we wish to deal with ray patterns which are symmetric about the x-axis.

In the same way we operate with $-c_1 = c_2 > 0$ to produce

$$\alpha = \frac{1}{\sqrt{\tau\rho}}\; J_1(2\sqrt{c_2\rho})\cosh\left(\sqrt{\frac{c_2}{\tau}}\,\theta\right) ,$$

$$\beta = \rho\; J_2(2\sqrt{c_2\rho})\;\sinh\left(\sqrt{\frac{c_2}{\tau}}\,\theta\right)$$

and

$$\alpha = \frac{1}{\sqrt{\tau\rho}} \; Y_1(2\sqrt{c_2\rho}\;) \cosh\left(\sqrt{\frac{c_2}{\tau}}\,\theta\right) ,$$

$$\beta = \rho Y_2(2\sqrt{c_2\rho}\,) \sinh\left(\sqrt{\frac{c_2}{\tau}}\,\theta\right) .$$

These elementary solutions turn out not to resemble a beam. To form a beam from them would presumably take a complicated superposition of them.

IV.2.iv The Radially Symmetric Problem

We shall not attempt to solve the radially symmetric three-dimensional problem by the preceding method. We note that the equations that one obtains in ray coordinates are

$$\rho\kappa A = \frac{1}{r} \quad ,$$

$$\kappa^2 = K^2(1 + 2\tau\rho) \; .$$

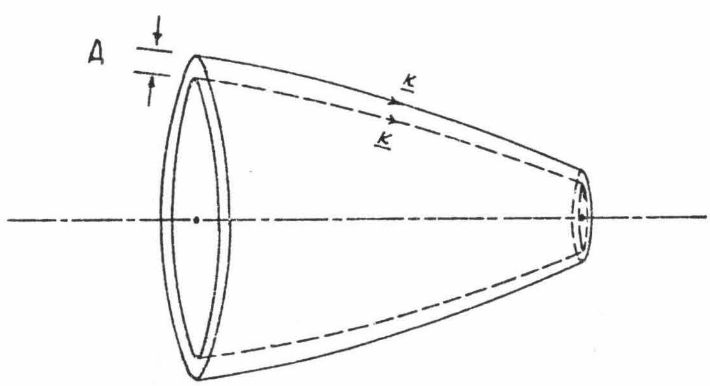
$$\frac{\partial\theta}{\partial\alpha} - \frac{1}{\kappa^2 A}\frac{\partial\kappa}{\partial\beta} = 0 \; ,$$

$$\frac{\partial\theta}{\partial\beta} - \kappa\frac{\partial A}{\partial\alpha} = 0 \; .$$

r is eliminated from the first equation by the integral transformation from α, β coordinates,

$$r = \int_0^{\alpha} \frac{\sin\theta}{\kappa}\, d\alpha + \int_0^{\beta} A(0,\beta)\cos\theta(0,\beta)d\beta \; .$$

The resulting system is very complicated. We observe from the first equation, however, that energy density within ray tubes decreases as the inverse of distance from the axis of the beam. If the beam is heading toward the axis, the change in energy density due to nonlinear effects is augmented by an increase in energy density due to a geometrical decrease in the area of the ray tube. (A ray tube is now the region between the surfaces of revolution of close rays.) The focusing effect is increased over that of the two-dimensional case. Similarly, when a ray heads away from the axis of the beam, amplitude decreases disproportionately from the two-dimensional case.



A Three-Dimensional Ray Tube

IV.3 Solution of the Thin Beam Equations

A method that we now outline was devised in the fluid context for treating a steady flow that differs little from a mean flow and will be adapted to beams whose rays are always close to parallel. In contrast to the first approach, we shall obtain a solution for the radially symmetric beam as well as the two-dimensional beam.

IV.3.i The Thin Beam Equations

We recall the system of equations

$$\frac{\partial}{\partial x}(r^m \rho \kappa_1) + \frac{\partial}{\partial r}(r^m \rho \kappa_2) = 0 ,$$

$$\frac{\partial \kappa_1}{\partial r} - \frac{\partial \kappa_2}{\partial x} = 0 , \qquad \text{(IV.26)}$$

$$\kappa^2 = K^2(1 + 2\tau\rho) ,$$

where $m = 0$ for two dimensions,

$m = 1$ for three dimensions and radial symmetry.

We introduce the potential ϕ to satisfy the second equation of (IV.26) in the form

$$\kappa_1 = K\left(1 + \frac{\partial \phi}{\partial x}\right) ,$$

$$\kappa_2 = K \frac{\partial \phi}{\partial r} . \qquad \text{(IV.27)}$$

We seek solutions of the problem in such form that the derivatives of ϕ are small corrections to a mean wave number K for waves propagating in the x-direction. The dispersion relation, the third equation of (IV.26), gives

$$\kappa_2 = K^2(1+2\tau\rho) = K^2\left[1+2\frac{\partial\phi}{\partial x}+\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2\right]. \tag{IV.28}$$

A straight linearization would take $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial r}$ as comparable small quantities and (IV.28) would require that we set $\tau\rho = \frac{\partial\phi}{\partial x}$. This leads to ρ being constant and does not exhibit the crucial nonlinear effects. These effects are included by looking for solutions where $O\left(\frac{\partial\phi}{\partial x}\right) = O\left(\frac{\partial^2\phi}{\partial r^2}\right)$, and then the leading term of each component of $\underline{\kappa}$ in (IV.28) is related to ρ. The significance is that we become interested in beams where gradients in r are steeper than gradients in x. This is the case of thin beams and is analagous to the case of thin jets in fluid mechanics. The significant terms of (IV.28) give

$$\tfrac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{\partial\phi}{\partial x} = \tau\rho\ . \tag{IV.29}$$

Now retaining first order x derivatives and second order r derivatives in the first equation of (IV.27) gives

$$\frac{\partial\rho}{\partial x} + \frac{m\rho}{r}\frac{\partial\phi}{\partial r} + \frac{\partial\rho}{\partial r}\frac{\partial\phi}{\partial r} + \rho\frac{\partial^2\phi}{\partial r^2} = 0\ . \tag{IV.30}$$

The set (IV.29) and (IV.30) are the thin beam equations that have been analysed by Russian researchers [10]. There are several interesting exact solutions to these equations which we outline in the next two sections.

The essential difference between this method and the one outlined in section IV.2 using ray coordinates is seen from (IV.28).

In section IV.2 (while we did not display this form) the three derivatives of ϕ in (IV.28) were allowed to be O(1) quantities but their sum had to be the small quantity $2\tau\rho$; the speed was nearly constant but the direction was allowed to change significantly. Here, in the thin beam case, the derivatives of ϕ are all small; the rays are nearly parallel and the phase velocity does not change significantly over all space.

We proceed with the solutions of the set (IV.29) and (IV.30). Most of the work has been contributed by Akhmanov et al. [10], who derived the equations directly as the thin beam approximation of the oscillatory equations rather than an approximation to the averaged ones. We have filled in several computational details that they have omitted. The first method treats the two-dimensional equations by means of hodograph transformations and separation of variables. The second one, which applies also to the three-dimensional cylindrical beam, involves a two term power series in the radial coordinate and turns out to produce an exact solution to the thin beam equations.

IV.3.ii A Special Solution by Hodograph Transformations and Separation of Variables

We consider the two-dimensional case by setting $m = 0$ and $r = y$ in (IV.29) and (IV.30). This set has three terms in each equation, but to separate variables there must be only two. In order to condense the form, Akhmanov made a transformation that allowed terms to be grouped. To make the set homogeneous in derivatives he differentiated the first equation with respect to y and then set $u = \frac{\partial \phi}{\partial y}$ in both equations producing the set

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} - \tau\frac{\partial \rho}{\partial y} = 0 ,$$

$$\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial y} + u\frac{\partial \rho}{\partial y} = 0 .$$

This set is familiar in the context of shallow water theory but differs in the sign of τ. Due to that sign the equations are elliptic rather than hyperbolic, a feature that has been preserved from the full equations. The significance of each term containing one derivative factor is that the roles of dependent and independent variables may be reversed by a hodograph transformation and the Jacobian will cancel out of the equations. (See section IV.2.iii). The resulting form

$$\frac{\partial y}{\partial \rho} - u\frac{\partial x}{\partial \rho} - \tau\frac{\partial x}{\partial u} = 0 ,$$

$$\frac{\partial y}{\partial u} + \rho\frac{\partial x}{\partial \rho} - u\frac{\partial x}{\partial u} = 0 ,$$

may now be condensed by grouping terms. That form is

$$\frac{\partial}{\partial \rho}(y-ux) - \tau\frac{\partial x}{\partial u} = 0 ,$$

$$\frac{\partial}{\partial u}(y-ux) + \frac{\partial}{\partial \rho}(\rho x) = 0 .$$

The change of variables

$$\xi = y-ux , \quad \eta = \rho x \tag{IV.31}$$

leaves the equations in the separable form

$$\rho\frac{\partial \xi}{\partial \rho} - \tau\frac{\partial \eta}{\partial u} = 0 ,$$

$$\frac{\partial \xi}{\partial u} + \frac{\partial \eta}{\partial \rho} = 0 .$$

While the equations are linear and superposition is permitted to form a general solution, the fundamental solutions of this set are found in terms of Bessel functions and do not resemble a beam. A further hodograph transformation leaves the set in separable form and produces a more satisfactory solution but since the new set is nonlinear it is a special one. This transformation leaves the set in separable form and produces a more satisfactory solution but since the new set is nonlinear it is a special one. This transformation again switches the roles of dependent and independent variables to give the form

$$\rho \frac{\partial u}{\partial \eta} - \tau \frac{\partial \rho}{\partial \xi} = 0 \ ,$$

$$\frac{\partial \rho}{\partial \eta} + \frac{\partial u}{\partial \xi} = 0 \ .$$

Using a potential ψ to satisfy the second of these we have

$$u = -\psi_\eta \ , \quad \rho = \psi_\xi$$

and substitution in the first equation produces

$$\frac{\partial \psi}{\partial \xi} \frac{\partial^2 \psi}{\partial \eta^2} + \tau \frac{\partial^2 \psi}{\partial \xi^2} = 0 \ .$$

This is separated by setting $\psi = N(\eta)E(\xi)$ and obtaining the equations

$$N'' = \frac{2\tau}{w} \ , \quad E'' = -\frac{2}{w} E' E \ .$$

Integration produces

$$N = \frac{\tau}{w} \eta^2 + b\eta + \rho_0 w \ ,$$

$$E = \tanh \frac{\xi}{w} \ ,$$

where w, b and ρ_0 are arbitrary constants, w and ρ_0 being positive to produce a beam. Now the potential ψ is given by

$$\psi(\xi,\eta) = \left(\frac{\tau}{w}\,\eta^2 + b\eta + \rho_0 w\right)\tanh\frac{\xi}{w}\ .$$

Using the definition of ψ we obtain

$$\rho = \left(\frac{\tau}{w^2}\,\rho^2 x^2 + \frac{b}{w}\,\rho x + \rho_0\right)\operatorname{sech}^2\left(\frac{y-ux}{w}\right) \quad ,$$

$$u = -\left(\frac{2\tau}{w}\,\rho x + b\right)\tanh\left(\frac{y-ux}{w}\right) \quad .$$

Akhmanov does not display the b term which gives the orientation of the rays at $x = 0$. For $x = 0$ the amplitude profile is

$$\rho = \rho_0 \operatorname{sech}^2 \frac{y}{w}$$

which is localized about the x-axis. In contrast, the separated solution for thicker beams (IV.18) is of finite width being forced to zero by a fold of the function in ray coordinate space. Within the approximation for thin beams, however, these shapes are similar and comparison may be made. Before focusing takes place $u = 0$ on the x-axis, hence at $y = 0$ the expression for ρ is

$$\rho = \frac{\tau}{w^2}\,\rho^2 x^2 + \frac{b}{w}\,\rho x + \rho_0\ .$$

For real values of ρ

$$\left(\frac{b}{w}\,x - 1\right)^2 - \frac{4\rho_0\tau}{w^2}\,x^2 \geq 0\ .$$

When this condition is at the point of breaking down it gives the focal point:

$$x_f = \frac{w}{2\sqrt{\tau\,\rho_0} + b} \quad . \qquad \text{(IV.32)}$$

For $b = 0$ the rays start parallel to the x-axis and the comparison to the thicker beam focusing distance of $\frac{\pi w}{4\sqrt{\tau\rho_0}}$ is unusually close considering that the starting profiles are only roughly similar. Returning to the full form of (IV.32), we see that for $b < -2\sqrt{\rho_0\tau}$ there is no focal point to the right of the given data and the rays "never come down." This is the case where the rays are so divergent at the start that they escape trapping. For $b > 0$ the rays are convergent to begin with and focusing is hastened.

IV.3.iii Solution by Truncated Power Series

We repeat the thin beam equations for convenience:

$$\frac{\partial\phi}{\partial x} + \frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 = \tau\rho \quad ,$$

$$\frac{\partial\rho}{\partial x} + \frac{\partial\phi}{\partial r}\frac{\partial\rho}{\partial r} + \rho\left(\frac{\partial^2\phi}{\partial r^2} + \frac{m}{r}\frac{\partial\phi}{\partial r}\right) = 0 \quad , \tag{IV.33}$$

where $m = 0$ for two dimensions,

$m = 1$ for three dimensions.

Akhmanov noticed that nonlinear terms occur only with second order r derivatives or with a first order r derivative and a factor $1/r$. The significance is that there is an exact solution of the form

$$\phi = f_0(x) + f_1(x)r^2 \quad ,$$

$$\rho = g_0(x) + g_1(x)r^2 \quad ,$$

since any term can accumulate no more than second powers of r. First powers of r in the above solution are not permitted in the cylindrical beam in order to avoid discontinuous derivatives at the center but would be permitted in two dimensions (slab shaped beam) if one were interested in nonsymmetric beams.

When ϕ and ρ are placed into equations (IV.33), equating like powers of r produces the following relations:

$$f_0' = \tau g_0 , \quad \text{(i)} \qquad g_0' + 2(1+m)g_0 f_1 = 0 , \quad \text{(iii)}$$

$$f_1' + 2f_1^2 = \tau g_1 , \quad \text{(ii)} \qquad g_1' + (6+2m)g_1 f_1 = 0 . \quad \text{(iv)}$$

Equations (ii) and (iv) establish f_1 and g_1, and then g_0 and finally f_0 are determined in terms of f_1. We assign the following initial conditions on the beam:

$$g_0(0) = \rho_0 , \quad g_1(0) = -\frac{\rho_0}{w^2} .$$

Then the amplitude profile is parabolic with maximum ρ_0 and has edge at $r = w$. For ϕ we assign

$$f_0(0) = 0 , \quad f_1(0) = \frac{\sqrt{\tau}}{2R} .$$

The value for f_0 is arbitrary but f_1 is chosen so that κ_2 will be $O\left(\sqrt{\tau}\right)$. Using (IV.27) and the relations between f_0, f_1, g_0, g_1 we have

$$\kappa_1(0) = K\left[1 + \tau\left(\rho_0\left(1 - \frac{1}{r_0^2}\right) - \frac{1}{2R^2}\right)\right] ,$$

$$\kappa_2(0) = \sqrt{\tau}\ \frac{Kr}{R} .$$

For $R < 0$ we have an initially converging beam and for $R > 0$ it is initially diverging.

Eliminating f_1 from (ii) by means of (iv), we obtain

$$g_1'' - \frac{4+m}{3+m}\,\frac{g_1'^{\,2}}{g_1} + (6+2m)\tau\, g_1^2 = 0 \ .$$

The awkward second term is eliminated by the substitution

$$g_1 = -\frac{\rho_0}{w^2 \zeta^{3+m}}$$

to give

$$\zeta'' + \frac{2\tau\rho_0}{\zeta^{2+m}\, w^2} = 0 \ .$$

The initial data are $\zeta(0) = 1$, $\zeta'(0) = \frac{\sqrt{\tau}}{R}$. In terms of ζ the other quantities are given by

$$f_1 = \frac{\zeta'}{2\zeta} \ , \ g_0 = \rho_0\, \zeta^{-(1+m)} , \ f_0' = \tau \rho_0{}^{-(1+m)} \ .$$

Integrating the ζ equation we have

$$\zeta' = \pm\sqrt{\frac{4\tau\,\rho_0}{(1+m)\, w^2\, \zeta^{1+m}} + C} \ , \text{ where } C = \frac{\tau}{R^2} - \frac{4\tau\rho_0}{(1+m) w^2} \ ,$$

and another integration gives

$$\pm\int_1^{\zeta} \frac{d\zeta}{\sqrt{\frac{4\tau\,\rho_0}{(1+m)\, w^2 \zeta^{1+m}} + C}} = x - x_0 \ . \qquad \text{(IV.34)}$$

This integral is now evaluated explicitly for $m = 0$ and $m = 1$ and the sign of C must be taken into account. For $m = 0$,

$$\text{when } C > 0, \; x - x_0 = \pm \frac{4\tau \rho_0}{C^{3/2} \, w^2} \left\{ \sqrt{\frac{C w^2 \zeta}{4\tau \rho_0}} \sqrt{1 + \frac{C w^2 \zeta}{4\tau \rho_0}} \right.$$

$$\left. - \log \left(\sqrt{\frac{C w^2 \zeta}{4\tau \rho_0}} + \sqrt{1 + \frac{C w^2 \zeta}{4\tau \rho_0}} \right) \right\}, \tag{IV.35}$$

$$\text{and when } C < 0, \; x - x_0 = \pm \frac{4\tau \rho_0}{C^{3/2} \, w^2} \left\{ \sqrt{\frac{|C| \, w^2 \zeta}{4\tau \rho_0}} \sqrt{1 - \frac{|C| \, w^2 \zeta}{4\tau \rho_0}} \right.$$

$$\left. + \cos^{-1} \left(\frac{|C| \, w^2 \zeta}{4\tau \rho_0} \right) \right\}.$$

x_0 is given by setting $x = 0$, $\zeta = 1$. We note that this is the same expression as (IV.21). For $m = 1$ we have

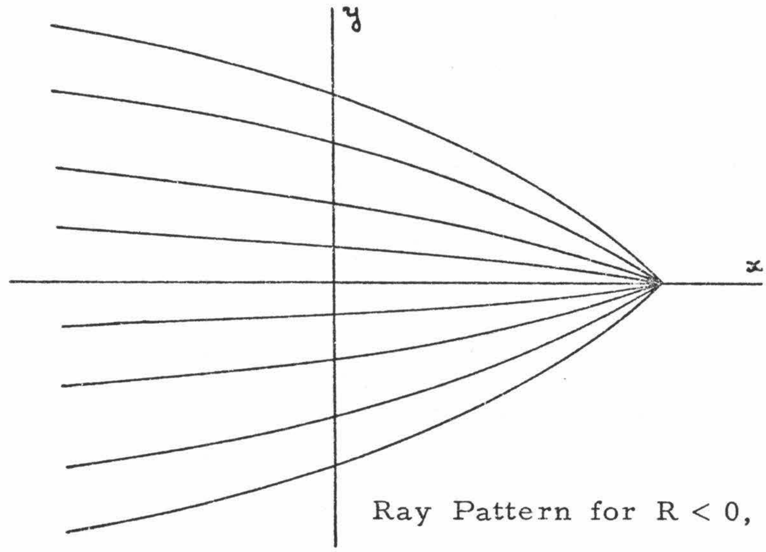
$$x - x_0 = \pm \frac{1}{|C|} \sqrt{\frac{2\tau \rho_0}{w^2} + C\zeta^2} \tag{IV.36}$$

(for either sign of C) and the use of the boundary condition and rearrangement produces

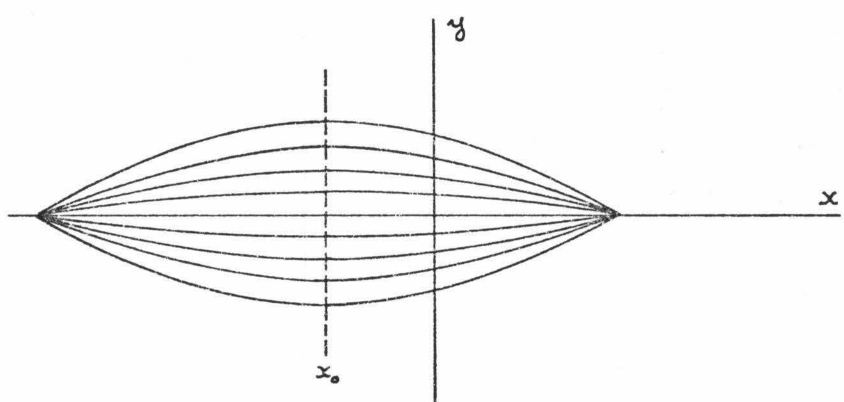
$$\zeta^2 = \left(\frac{\tau}{R^2} - \frac{2\tau \rho_0}{w^2} \right) x^2 + \frac{2\sqrt{\tau}\, x}{R} + 1$$

which can be used to recover the form of ρ and ϕ in closed form.

In this example the focal point occurs where $\zeta = 0$ and the amplitude ρ becomes infinite. (In the first example for thin beams the amplitude at the focal point was finite but became complex beyond the focal point. It need not be singular for thin beams as the conservation of energy within ray tubes is only approximate.) Integral (IV.34) is so arranged that $\zeta' = 0$ for $x = x_0$. This coincides with $\rho' = 0$ and is the value of x for which the rays are parallel. This point can be moved off to infinity for different values of the parameters, hence x_0 is not chosen to be the origin. The situation for $R < 0$ is illustrated.



Ray Pattern for $R < 0$, $C > 0$



Ray Pattern for $R < 0$, $C < 0$

The rays always focus since $\zeta'(0)$ is negative and $\zeta(0)$ is positive. From (IV.34), ζ' becomes more and more negative until $\zeta = 0$ and focusing occurs. Going backwards in x from $x = 0$, if $C > 0$ we find that ζ increases without bound and ζ' becomes asymptotic to $-\sqrt{C}$. If $C < 0$, there is another focal point, hence rays diverging from a focal point may again converge. x_0 is half way between the focal points by the symmetry of (IV.34); the sign of the square root switches at $x = x_0$. The focal distance is the distance from the origin to the focal point.

For $R < 0$, $C > 0$, two dimensions:

$$x_f = \frac{4\tau\rho_0}{C^{3/2} w^2} \left\{ \sqrt{\frac{w^2}{4\rho_0 R^2} - 1} \sqrt{\frac{w^2}{4\rho_0 R^2}} - \log\left(\sqrt{\frac{w^2}{4\rho_0 R^2} - 1} - \sqrt{\frac{w^2}{4\rho_0 R^2}} \right) \right\}$$

and for three dimensions:

$$x_f = \frac{Rw}{\sqrt{\tau}\,(w + \sqrt{2\rho_0}\; R)} \quad .$$

For $R < 0$, $C < 0$, two dimensions:

$$x_0 = -\frac{4\tau\rho_0}{|C|^{3/2} w^2}\left\{\sqrt{1-\frac{w^2}{4\rho_0 R^2}}\ \sqrt{\frac{w^2}{4\rho_0 R^2}}\right.$$

$$\left.+\cos^{-1}\left(\sqrt{1-\frac{w^2}{4\rho_0 R^2}}\right)\right\} ,$$

$$x_1 = \frac{4\tau\rho_0}{|C|^{3/2} w^2} .$$

The positions of the focal points are

$$x_f = x_0 \pm x_1 .$$

In three dimensions

$$x_0 = \frac{1}{|C|}\sqrt{\frac{\tau}{R^2}}$$

$$x_1 = \frac{1}{|C|}\sqrt{\frac{2\tau\rho_0}{w^2}}$$

and the positions of the focal points are

$$x_f = -\sqrt{\frac{\tau}{R^2}} \pm \sqrt{\frac{2\tau\rho_0}{w^2}} .$$

For $R > 0$ the rays are divergent at $x = 0$. The effect is to reverse the sign of x in the solutions already found. The two diagrams reflected in the plane $x = 0$ illustrate this situation.

CHAPTER V

THE LAGRANGIAN APPROACH WITH NONLINEARITY AND HIGHER DISPERSION

It has been discovered in Chapter IV that focusing occurs when the nonlinear theory is applied to waves whose amplitude is not constant. Near the focal point the formulation is invalid since the amplitude is greater than 0(1) and the carrier wave is not sinusoidal. It is noted by Akhmanov et al.[10] that this situation is not remedied by including more terms in the Fourier series, nor does dissipation prevent singularities. We shall study examples in which singularities do not occur when higher derivative terms of the slowly varying parameters are retained.

Solutions will be derived in the two- and three-dimensional cases, for plane waves whose amplitudes vary in time and space. In the case of two dimensions we shall find plane waves whose time-independent envelopes take the form of Jacobian elliptic functions in the direction transverse to the direction of propagation. The limiting form is a beam localized about an axis and which propagates without distortion. We shall also find waves with periodic envelopes that travel with constant speed. These become solitary wave packets in the limit as the period of the envelope oscillations becomes infinite. A combination of these effects produces a solitary wave packet which is localized about the axis of propagation. Similar effects are found in the three-dimensional radially symmetric case but the general form of the envelope of the plane waves is not periodic but decays from a maximum even when the envelope is oscillatory. This case is complicated by not having closed form solutions.

V. 1 Waves with Linear Polarization

When higher dispersion terms are retained the system loses some of the symmetries that it had without these terms. When waves are linearly polarized the direction of polarization becomes a favored direction. There can be two-dimensional waves when the direction of polarization is the direction in which quantities are constant. Similarly, radial symmetry about an axis can occur only with circular polarization. We shall find time-independent solutions for these two distinct cases separately, the two-dimensional beams in this section, the radially symmetric ones in section V. 2. Then having some familiarity with the features of the system, we shall examine some time-dependent solutions for both cases in section V. 3.

V. 1. i The Near Linear Lagrangian in Two Dimensions

We shall proceed in a similar fashion as in Chapter III, substituting sinusoidal forms into the Lagrangian and then averaging over one period. The periodic forms (III. 2) are modified by an additional small term which does not appear in the periodic plane wave solutions.

The Lagrangian in two dimensions is (III. 1):

$$L = \frac{\varepsilon_o}{2}\left\{A_t^2 - c^2(A_x^2 + A_y^2)\right\} - A_t P + \frac{1}{\varepsilon_o \omega_p^2}\left\{\frac{P_t^2}{2} - \frac{\omega_o P^2}{2} + \frac{\gamma P^4}{4}\right\}.$$

The form of the periodic plane wave propagating in the x-direction that we shall use is:

$$\begin{aligned} A &= a \cos \theta, \\ P &= b \sin \theta + \varepsilon \, \overline{b} \cos \theta \end{aligned} \qquad (V. 1)$$

where $\theta = \kappa x - \omega t$, and the slow scales are $\underline{X} = \epsilon \underline{x}$, $T = \epsilon t$. We retain terms of $O(\epsilon^2)$ in the Lagrangian and we take $|\gamma| = \epsilon^2$. The form of the derivatives of A and P is

$$
\begin{aligned}
A_t &= \omega a \sin\theta + \epsilon\, a_T \cos\theta, \\
A_x &= \kappa a \sin\theta + \epsilon\, a_X \cos\theta, \\
A_y &= \epsilon\, a_Y \cos\theta, \\
P_t &= (-\omega b + \epsilon^2 \bar{b}_T)\cos\theta + \epsilon(b_T + \omega\bar{b})\sin\theta \quad .
\end{aligned}
\tag{V.2}
$$

Placing these forms into the Lagrangian and integrating over 2π, we obtain

$$
\begin{aligned}
\mathcal{L} = {} & \frac{\epsilon_o}{4}(\omega^2 - c^2\kappa^2)a^2 - \frac{\omega a b}{2} - \frac{(\omega_o^2 - \omega^2)}{4\epsilon_o\omega_p^2} b^2 \\
& + \epsilon^2 \left\{ \frac{\epsilon_o}{4}\left[a_T^2 - c^2(a_X^2 + a_Y^2)\right] - \frac{a_T \bar{b}}{2} - \frac{1}{4\epsilon_o\omega_p^2}\left(2\omega b\bar{b}_T - (b_T + \omega\bar{b})^2\right.\right. \\
& \left.\left. + \omega_o^2\bar{b}^2\right)\right\} + \frac{3\gamma b^4}{32\epsilon_o\omega_p^2} \quad .
\end{aligned}
$$

To reduce this to a useful form the Euler equations obtained by variation of b and $\bar{b}$ are used to eliminate those variables. Then we are left with a Lagrangian in terms of a, κ and ω which is used in the normal way to generate the dispersion relation and wave action. Since we are looking for plane wave solutions we shall not be concerned with terms involving ω_T and ω_{TT}, hence these are neglected.

Algebraic difficulty is minimized by observing the form of $\mathcal{L}$ and eliminating b first. We write the averaged Lagrangian in the form

$$\mathcal{L} = Q_1 a^2 + 2 Q_2 ab + Q_3 b^2 + \epsilon^2 \mathcal{L}_1(a, b, \bar{b}) \ .$$

Variation with respect to b gives

$$b = -\frac{Q_2}{Q_3} a - \frac{\epsilon^2 \mathcal{L}_{1,b}(a, b, \bar{b})}{2 Q_3} \ .$$

Placing this back into the large terms of $\mathcal{L}$ produces

$$\mathcal{L} = (Q_1 - \frac{Q_2^2}{Q_3}) a^2 + \epsilon^2 \mathcal{L}_1(a, b, \bar{b}) \ ,$$

and only the significant term of b need be substituted into $\mathcal{L}_1$. The full form of $\mathcal{L}$ is now

$$\mathcal{L} = \frac{\epsilon_o}{4}\left[\omega^2\left(1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}\right) - c^2\kappa^2\right]a^2 + \epsilon^2\left\{\frac{\epsilon_o}{4}\left[a_T^2 - c^2(a_X^2 + a_Y^2)\right] - \frac{a_T\bar{b}}{2}\right.$$

$$\left. - \frac{1}{4\epsilon_o\omega_p^2}\left[-\frac{2\epsilon_o\omega_p^2\omega^2 a\,\bar{b}_T}{\omega_o^2 - \omega^2} - \left(\frac{\epsilon_o\omega_p^2\,\omega\, a_T}{\omega_o^2 - \omega^2} + \omega\bar{b}\right)^2 + \omega_o^2\bar{b}^2\right]\right\}$$

$$+ \frac{3\gamma\epsilon_o^3\omega_p^6\omega^4 a^4}{32(\omega_o^2 - \omega^2)^4} \ .$$

Variation with respect to $\bar{b}$ gives

$$\bar{b} = -\frac{\epsilon_o\omega_p^2(\omega_o^2 + \omega^2)}{(\omega_o^2 - \omega^2)^2} a_T \ .$$

Placing this into $\mathcal{L}$ produces the working form

$$\mathcal{L} = \frac{\epsilon_o}{4}\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)-c^2\kappa^2\right]a^2$$
$$+\ \epsilon^2\frac{\epsilon_o}{4}\left[\left(1+\frac{\omega_p^2\,\omega_o^2(\omega_o^2+3\,\omega^2)}{(\omega_o^2-\omega^2)^3}\right)a_T^2-c^2(a_X^2+a_Y^2)\right]$$
$$+\ \frac{3\gamma\epsilon_o^3\,\omega_p^6\,\omega^4 a^4}{32\,(\omega_o^2-\omega^2)^4} \quad . \tag{V.3}$$

The Euler equations resulting from the variation of $\mathcal{L}$ are

$$\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)-c^2\kappa^2\right]a-\left(1+\frac{\omega_p^2\,\omega_o^2(\omega_o^2+3\omega^2)}{(\omega_o^2-\omega^2)^3}\right)a_{tt}$$
$$+\ c^2(a_{xx}+a_{yy})+\frac{3\gamma\,\epsilon_o^2\,\omega_p^6\,\omega^4\,a^3}{4(\omega_o^2-\omega^2)^4} = 0 \quad , \tag{V.4}$$
$$\frac{\partial}{\partial t}\left\{\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)\right]_\omega \frac{a^2}{2}\right\}+\frac{\partial}{\partial x}\left\{c^2\kappa_1 a^2\right\}+\frac{\partial}{\partial y}\left\{c^2\kappa_2 a^2\right\} = 0 \ .$$

The consistency relations remain

$$\frac{\partial\omega}{\partial x}+\frac{\partial\kappa_1}{\partial t} = 0\ , \quad \frac{\partial\omega}{\partial y}+\frac{\partial\kappa_2}{\partial t} = 0, \quad \frac{\partial\kappa_1}{\partial y}-\frac{\partial\kappa_2}{\partial x} = 0 \ . \tag{V.5}$$

V.1.ii Time-Independent Solutions in Two Dimensions

There are convenient solutions which represent plane waves propagating in one direction without distortion. To obtain these solutions we set time derivatives equal to zero to produce a time-independent envelope. To have propagation in only one direction we set $\kappa_2 = 0$ and κ_1 = constant. To satisfy the second equation of (V.4) we must have $a = a(y)$, and now all of (V.5) are satisfied. The remaining equation gives the distribution of amplitude transverse to the direction of propagation. That equation is

$$\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)-c^2\kappa^2\right]a + c^2 a_{yy} + \frac{3\gamma\,\varepsilon_o^2\,\omega_p^6\,\omega^4\,a^3}{4(\omega_o^2-\omega^2)^4} = 0$$

This equation describes the time-independent beam found by Townes [1]. We write this in the form

$$a_{yy} - Da + \tau a^3 = 0 \qquad \text{(V. 6)}$$

where

$$D = \kappa^2 - \frac{\omega^2}{c^2}\left(1 + \frac{\omega_p^2}{\omega_o^2-\omega^2}\right)$$

and

$$\tau = \frac{3\gamma\,\varepsilon_o^2\,\omega_p^6\,\omega^4}{4c^2(\omega_o^2-\omega^2)^4} .$$

D may be positive or negative but τ has the sign of γ and is determined by the medium.

One integration of (V. 6) gives

$$\frac{a_y^2}{2} - \frac{Da^2}{2} + \frac{\tau a^4}{4} = \frac{A}{2} \qquad \text{(V. 7)}$$

and a further integration gives

$$\int_{a_o}^{a} \frac{du}{\sqrt{A + Du^2 - \tau u^4/2}} = y \quad . \qquad \text{(V. 8)}$$

The contents of the square root are factored to give

$$A + Du^2 - \frac{\tau u^4}{2} = A\left(1 - \frac{u^2}{R_1^2}\right)\left(1 - \frac{u^2}{R_2^2}\right) \qquad \text{(V. 9)}$$

where

$$R_{1,2}^2 = \frac{D}{\tau} \pm \sqrt{\frac{D^2}{\tau^2} + \frac{2A}{\tau}} \quad .$$

The form of (V. 7) is the same as the equation governing the oscillations of the polarization (II. 4) and the description in that connection applies here also. The solution oscillates between simple roots of (V. 9). It is important to bear in mind that the solutions found in Chapter II are the fast oscillations while we are dealing here with variations in the envelope of these oscillations.

The zeros of (V. 9) are $u = \pm R_1$ and $u = \pm R_2$. These are the values of a at which $a_y = 0$ and hence simple roots will be extreme values of the wave form. The integral (V. 8) diverges as a goes to the value of a double or higher order root, hence all roots are extreme values but multiple roots may be reached only in the limit as $y \to \pm \infty$ and these solutions are not oscillatory. This is the case of "solitary waves" which we played down when dealing with the fast oscillations in Chapter II. In the present context the solitary wave envelope is the very important case of a localized beam. We itemize the different cases of values of the parameters τ, D and A by the following tables. The arrows indicate the values of the roots of (V. 9) on the sketch of the waveforms.

The first table is for the case $\tau > 0$, the focusing medium, the second for $\tau < 0$, the defocusing medium.

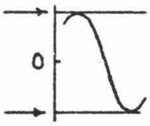
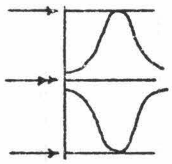
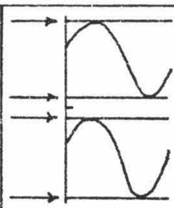
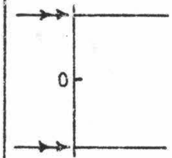
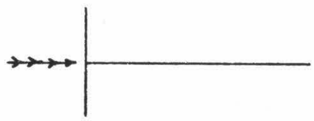
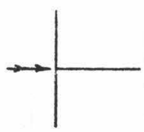
<table>
<tr><th></th><th>$A>0$</th><th>$A=0$</th><th>$0>A>-\frac{D^2}{2\tau}$</th><th>$A=-\frac{D^2}{2\tau}$</th><th>$A<-\frac{D^2}{2\tau}$</th></tr>
<tr><td>$D>0$</td><td rowspan="3">
2 real roots
2 imag. roots
Case (a)</td><td>
2 real roots
2 zero roots
Case (b)</td><td>
4 real roots
Case (c)</td><td>
2 double, real roots</td><td>NO SOLUTION
4 complex roots</td></tr>
<tr><td>$D=0$</td><td colspan="3">
Quadruple zero root</td><td>NO SOLUTION
4 complex roots</td></tr>
<tr><td>$D<0$</td><td>
2 zero roots
2 imag. roots</td><td>NO SOLUTION
4 complex roots</td><td>NO SOLUTION
4 complex roots</td><td>NO SOLUTION
4 complex roots</td></tr>
</table>

Table for $\tau > 0$

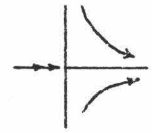
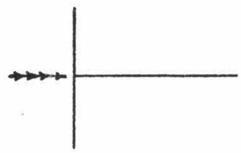
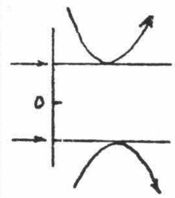
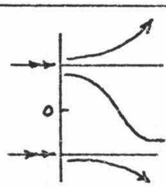
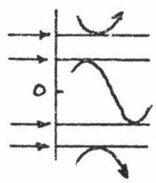
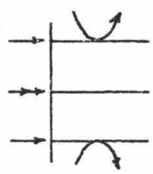
	$A > \frac{D^2}{2\|\tau\|}$	$A = \frac{D^2}{2\|\tau\|}$	$\frac{D^2}{2\|\tau\|} > A > 0$	$A = 0$	$A < 0$
$D > 0$	UNBOUNDED SOLUTION 4 complex roots	UNBOUNDED SOLUTION 4 imag. roots	UNBOUNDED SOLUTION 4 imag. roots	 2 zero roots 2 imag. roots	
$D = 0$	UNBOUNDED SOLUTION 4 complex roots	 4 zero roots			 2 real roots 2 imag. roots
$D < 0$	UNBOUNDED SOLUTION 4 complex roots	 2 double, real roots Case (d)	 4 real roots Case (e)	 2 real roots 2 zero roots	

Table for $\tau < 0$

We now give a more detailed description of the bounded solutions shown in the tables.

Case (a): $\tau > 0$, $D > 0$, $A > 0$; Periodic Wave Form

From (V. 9), $R_1^2 > 0$, $R_2^2 < 0$. We set $S_2^2 = -R_2^2$ and for real a and y we have

$$-R_1 \le a \le R_1 \quad .$$

The implicit form of the solution is

$$\int_a^{R_1} \frac{du}{\sqrt{1 - \frac{u^2}{R_1^2}}\sqrt{1 + \frac{u^2}{S_2^2}}} = \sqrt{A}\; y \; .$$

In terms of Jacobian elliptic functions the solution is

$$a = R_1 \operatorname{cn}\left(\frac{\sqrt{A}\,(R_1^2 + S_2^2)}{R_1 S_2}\; y,\; \frac{R_1^2}{R_1^2 + S_2^2} \right). \tag{V. 10}$$

Case (b): $\tau > 0$, $D > 0$, $A = 0$; Solitary Wave Form

$R_1 > 0$, $R_2 = 0$. For real a, y we have

$$-R_1 \le a \le R_1 \; .$$

The implicit form is

$$\int_a^{R_1} \frac{du}{\sqrt{Du^2 - \frac{\tau u^4}{2}}} = y$$

but in this case the solution has the integrated form

$$a = \sqrt{\frac{2D}{\tau}}\; \operatorname{sech} \sqrt{D}\; y \; . \tag{V. 11}$$

Case (c): $\tau > 0$, $D > 0$, $0 > A > -\frac{D^2}{2\tau}$; Periodic Wave Form

$R_1^2 > 0$, $R_2^2 > 0$. For real a, y there are the regions

$$R_2 \le a \le R_1$$

$$-R_1 \le a \le R_2 \quad .$$

The implicit form is

$$\int_a^1 \frac{du}{\sqrt{1-\frac{u^2}{R_1^2}}\sqrt{\frac{u^2}{R_2^2}-1}} = \sqrt{|A|}\, y \ .$$

Written in the form of Jacobian elliptic functions this is

$$a = R_1 \operatorname{dn}\left(\frac{\sqrt{|A|}}{R_2}\, y, \frac{R_1^2-R_2^2}{R_2^2}\right) . \tag{V.12}$$

This completes the cases for the focusing medium. We emphasize that the preceding waveforms are supported by one type of medium, the following by another.

Case (d): $\tau < 0$, $D < 0$, $A = \frac{D^2}{2|\tau|}$; Solitary Wave Form

$R_1 = R_2 > 0$. For real a, y is real everywhere. There are unbounded solutions for initial conditions outside the region between the roots and a bounded, nonoscillatory solution inside that region. The implicit form is

$$\int_o^a \frac{du}{\sqrt{\frac{|\tau|}{2}\left(\frac{D}{\tau} - u^2\right)}} = y$$

and it integrates to

$$a = \sqrt{\frac{D}{\tau}}\ \tanh\left(\sqrt{\frac{|D|}{2}}\ y\right) \quad .$$

Case (e) $\tau < 0$, $D < 0$, $\frac{D^2}{2|\tau|} > A > 0$; Periodic Wave Form

$R_1 > 0$, $R_2 > 0$. For real a, y is real in the regions

$$a \le -R_1, \quad -R_2 < a < R_2, \quad a \ge R_1 \quad .$$

The central region has periodic solutions and the implicit form is

$$\int_0^a \frac{du}{\sqrt{1 - \frac{u^2}{R_1^2}}\sqrt{1 - \frac{u^2}{R_2^2}}} = \sqrt{A}\ y \quad .$$

The form in terms of Jacobian elliptic functions is

$$a = R_2 \operatorname{sn}\left(\frac{\sqrt{A}}{R_2}\ y, \frac{R_2^2}{R_1^2}\right) \quad . \tag{V. 13}$$

In review, the focusing medium has solutions which oscillate through zero in case (a) and about a positive value in case (c). These oscillatory solutions represent parallel beams propagating without distortion. The solitary wave form of case (b) represents an isolated beam propagating without distortion. Townes [1] has named this a self-trapped beam. In the defocusing medium there is an oscillatory solution in case (e) which likewise represents parallel beams. The solitary wave form is the hyperbolic tangent of case (d) and is nonzero at infinity. It cannot represent an isolated beam; hence, self-trapped beams do not exist for defocusing media.

In heuristic terms, in the focusing medium a beam tends to focus due to nonlinearity but to defocus due to dispersion. When these effects are in balance a beam is self-trapped. In the defocusing medium both nonlinearity and dispersion cause the beam to spread.

V. 2 Waves with Circular Polarization

As we mentioned in the introduction to section V. 1, in this section we shall set up the equations for circularly polarized waves with higher dispersion. We shall examine the time-independent equations to find radially symmetric solutions that represent self-trapped beams.

V. 2. i Averaged Lagrangian for Circular Polarization

The near linear Lagrangian as found in Chapter III is

$$L = \frac{\epsilon_o}{2}(A_{i,t}^2 - c^2 A_{i,x_k}^2) - A_{i,t}P_i - \frac{\epsilon_o}{2c^2}(\phi_t^2 - c^2\phi_{x_i}^2) - \phi_{x_i}P_i$$
$$+ \frac{1}{2\epsilon_o\omega_p^2}\left(P_{i,t}^2 - \omega_o^2 P_i^2 + \frac{\gamma}{2}(P_i^2)^2\right) .$$

We now substitute the form of a circularly polarized plane wave propagating in the x-direction into this Lagrangian. As in the two-dimensional case, the forms found in Chapter III are modified by additional small terms. The solution that we shall take is

$$\begin{aligned} A_1 &= 0 , & P_1 &= 0 , \\ A_2 &= a \sin\theta , & P_2 &= b\cos\theta + \epsilon\,\bar{b}_2 \sin\theta , \\ A_3 &= -a_3 \cos\theta , & P_3 &= b\sin\theta + \epsilon\,\bar{b}_3 \cos\theta , \end{aligned}$$

and $\theta = \kappa x - \omega t$. We must also have

$$\phi = \epsilon\,\phi_1 \sin\theta + \epsilon\,\phi_2 \cos\theta .$$

ϕ_1 and ϕ_2 may be determined from the Lorentz gauge

$$\nabla . \underline{A} = -\frac{1}{c^2}\phi_t \quad .$$

Using the prescribed forms for $\underline{A}$ we find

$$\phi_t = -\varepsilon c^2 (a_Y \sin\theta - a_Z \cos\theta) \ ,$$

and hence

$$\phi = -\frac{\varepsilon c^2}{\omega}(a_Y \cos\theta + a_Z \sin\theta) + O(\varepsilon^2) \ .$$

This is the same result that we would obtain by placing ϕ_1 and ϕ_2 into the Lagrangian and varying with respect to these functions. The derivative terms that are used in the Lagrangian are

$$A_{2,t} = -\omega a \cos\theta + \varepsilon a_T \sin\theta \qquad A_{3,t} = -\omega a \sin\theta - \varepsilon a_T \cos\theta$$

$$A_{2,x} = \kappa a \cos\theta + \varepsilon a_X \sin\theta \qquad A_{3,x} = \kappa a \sin\theta - \varepsilon a_X \cos\theta$$

$$A_{2,y} = \varepsilon a_Y \sin\theta \qquad A_{3,y} = -\varepsilon a_Y \cos\theta$$

$$A_{2,2} = \varepsilon a_Z \sin\theta \qquad A_{3,z} = -\varepsilon a_Z \cos\theta$$

$$\phi_x = \varepsilon\left(\frac{c^2}{\omega}\kappa a_Y \sin\theta - \frac{c^2}{\omega}\kappa a_Z \cos\theta\right) \qquad \phi_y = O(\varepsilon^2) \qquad \phi_2 = O(\varepsilon^2)$$

$$P_{2,t} = \omega b \sin\theta + \varepsilon(b_T - \omega \bar{b}_2)\cos\theta + \varepsilon^2 \bar{b}_{2,T} \sin\theta$$

$$P_{3,t} = -\omega b \cos\theta + \varepsilon(b_T + \omega \bar{b}_3)\sin\theta + \varepsilon^2 \bar{b}_{3,T} \cos\theta$$

$$P_{2,y} = \varepsilon b_Y \cos\theta + \varepsilon^2 \bar{b}_{2,Y} \sin\theta \qquad P_{3,z} = \varepsilon b_Z \sin\theta + \varepsilon^2 \bar{b}_{3,Z} \cos\theta$$

Substituting these forms into the Lagrangian and integrating over 2π, we obtain, to $O(\varepsilon^2)$ and $O(\gamma)$.

$$
\begin{aligned}
\mathcal{L} \;=\; & \frac{\varepsilon_o}{2}\,(\omega^2 - c^2\kappa^2)a^2 + \omega a\, b - \frac{\omega_o^2 - \omega^2}{2\,\varepsilon_o\,\omega_p^2}\, b^2 \\
& + \varepsilon^2 \Bigg\{ \frac{\varepsilon_o}{2}\Big[a_T^2 - c^2(a_X^2 + a_Y^2 + a_Z^2)\Big] - \tfrac{1}{2}(a_T\,\bar{b}_2 - a_T\bar{b}_3) - \frac{c^2\varepsilon_o}{4}\Big(1 - \frac{c^2\kappa^2}{\omega^2}\Big)(a_Y^2 + a_Z^2) \\
& \quad - \frac{c^2}{2\omega}\,(a_Y b_Y + a_Z b_Z) + \frac{1}{2\,\varepsilon_o\omega_p^2}\Big[b_T^2 + 2b\,\omega(\bar{b}_{2,T} - \bar{b}_{3,T}) - \frac{\omega_o^2 - \omega^2}{2}(\bar{b}_2^2 + \bar{b}_3^2)\Big]\Bigg\} \\
& + \frac{\gamma\, b^4}{4\,\varepsilon_o\,\omega_p^2} \quad .
\end{aligned}
$$

In the same way as in the case of linear polarization we observe the form of the Lagrangian in order to simplify the procedure of eliminating the extraneous amplitudes. The form of the Lagrangian is

$$
\mathcal{L} \;=\; Q_1\, a^2 + 2\, Q_2\, ab + Q_3\, b^2 + \varepsilon^2\, \mathcal{L}_1(a, b, \bar{b}_2, \bar{b}_3)\;.
$$

This is identical to the structure of $\mathcal{L}$ in the case of linear polarization. The variation of b and resubstitution produces

$$
\mathcal{L} \;=\; \Big(Q_1 - \frac{Q_2^2}{Q_3}\Big)a^2 + \varepsilon^2\,\mathcal{L}_1 \quad ,
$$

where the first approximation $b = -\dfrac{Q_2}{Q_3}\,a$ is placed into $\mathcal{L}_1$. This form is

$$
\begin{aligned}
\mathcal{L} \;=\; & \frac{\varepsilon_o}{2}\Big[\omega^2\Big(1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}\Big) - c^2\kappa^2\Big]a^2 + \varepsilon^2 \Bigg\{ \frac{\varepsilon_o}{2}\Big[a_T^2 - c^2(a_X^2 + a_Y^2 + a_Z^2)\Big] \\
& - \tfrac{1}{2}(a_T\bar{b}_2 - a_T\bar{b}_3) - \frac{c^2\varepsilon_o}{4}\Big(1 - \frac{c^2\kappa^2}{\omega^2} + \frac{\omega_p^2}{\omega_o^2 - \omega^2}\Big)(a_Y^2 + a_Z^2) \\
& + \frac{1}{2\varepsilon_o\omega_p^2}\Big[\frac{\varepsilon_o^2\,\omega_p^4\,\omega^2 a_T^2}{(\omega_o^2 - \omega^2)^2} + \frac{2\varepsilon_o\omega_p^2\,\omega^2 a}{\omega_o^2 - \omega^2}(\bar{b}_{2,T} - \bar{b}_{3,T}) - \frac{\omega_o^2 - \omega^2}{2}(\bar{b}_2^2 + b_3^2)\Big]\Bigg\} \\
& \qquad + \frac{\gamma\,\varepsilon_o^3\,\omega_p^6\,\omega^4 a^4}{4(\omega_o^2 - \omega^2)^4} \quad .
\end{aligned}
$$

Variation with respect to $\bar{b}_2$ and $\bar{b}_3$ gives

$$-\frac{a_T}{2} - \frac{\omega^2 a_T}{\omega_o^2-\omega^2} - \frac{\omega_o^2-\omega^2}{2\,\epsilon_o\omega_p^2}\,\bar{b}_2 = 0$$

$$\frac{a_T}{2} + \frac{\omega^2 a_T}{\omega_o^2-\omega^2} - \frac{\omega_o^2-\omega^2}{2\,\epsilon_o\,\omega_p^2}\,\bar{b}_3 = 0 \quad .$$

These are used to eliminate $\bar{b}_2$ and $\bar{b}_3$ from $\mathcal{L}$ to give

$$\mathcal{L} = \frac{\epsilon_o}{2}\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)-c^2\kappa^2\right]a^2+\epsilon^2\left\{\frac{\epsilon_o}{2}\left[1+\frac{\omega_p^2\,\omega_o^2(\omega_o^2+3\,\omega^2)}{(\omega_o^2-\omega^2)^3}\right]a_T^2\right.$$

$$\left.-\frac{\epsilon_o\,c^2a_X^2}{2}-\frac{\epsilon_o\,c^2}{4}\left(3-\frac{c^2\kappa^2}{\omega^2}-\frac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)(a_Y^2+a_Z^2)\right\}+\frac{\gamma\,\epsilon_o^3\,\omega_p^6\,\omega^4a^4}{4(\omega_o^2-\omega^2)^4}\,.$$

The Euler equations resulting from the variation of $\mathcal{L}$ are

$$\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)-c^2\kappa^2\right]a-\left(1+\frac{\omega_p^2\omega_o^2(\omega_o^2+3\omega^2)}{(\omega_o^2-\omega^2)^3}\right)a_{tt}$$

$$+c^2a_{xx}+\frac{c^2}{2}\left(3-\frac{c^2\kappa^2}{\omega^2}-\frac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)(a_{yy}+a_{zz})+\frac{\gamma\epsilon_o^2\,\omega_p^6\,\omega^4a^3}{(\omega_o^2-\omega^2)^4} = 0 \qquad \text{(V. 14)}$$

$$\frac{\partial}{\partial t}\left\{\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)\right]_\omega\frac{a^2}{2}\right\}+\frac{\partial}{\partial x}(c^2\kappa_1a^2)+\frac{\partial}{\partial y}(c^2\kappa_2a^2)+\frac{\partial}{\partial z}(c^2\kappa_3a^2) = 0$$

The consistency relations are

$$\frac{\partial\omega}{\partial \underline{x}}+\frac{\partial\kappa}{\partial t} = 0, \quad \nabla\times\underline{\kappa} = 0 \qquad \text{(V. 15)}$$

V. 2. ii Time-Independent Solutions for Circular Polarization

There are two special forms of time-independent solutions for circular polarization. These are the two-dimensional beam that was found for linear polarization and a cylindrically symmetric beam. Disposing of the two-dimensional beam first we set $\kappa_1 = \kappa$, $\kappa_2 = 0$ and take κ and ω as constants in equations (V. 14) and (V. 15). Then if we take a = a(y) the only equation not satisfied identically is the first of (V. 14) which becomes

$$\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)-c^2\kappa^2\right]a+\frac{c^2}{2}\left(3-\frac{c^2\kappa^2}{\omega^2}-\frac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)a_{yy}+\frac{\gamma\varepsilon_o^2\omega_p^6\omega^4a^3}{(\omega_o^2-\omega^2)^4}=0\,.$$

We write this in the form

$$a_{yy} - D\,a + \tau\,a^3 = 0$$

where

$$D = \frac{c^2\kappa^2-\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)}{\frac{c^2}{2}\left(3-\frac{c^2\kappa^2}{\omega^2}-\frac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)}$$

and

$$\tau = \frac{c^2\kappa^2-\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)}{(\omega_o^2-\omega^2)^4c^2\left(3-\frac{c^2\kappa^2}{\omega^2}-\frac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)}\,.$$

This equation is identical in form to (V. 6). While the formulas for D and τ differ slightly, the sign of τ is still the sign of γ. All the solutions that applied in section V. 1 apply here and again self-trapped beams exist only for the focusing medium.

The second and more important time-independent solution has radial symmetry about the x-axis. To obtain this special solution we set $\kappa_1 = \kappa$, $\kappa_2 = \kappa_3 = 0$ and take ω and κ as constants. Setting $r^2 = y^2 + z^2$, we take $a = a(r)$. When the solution is localized about $r = 0$, this represents a cylindrical beam with circular cross-section which propagates without spreading or converging. This beam was also found by Townes [1]. Again all equations of (V.14) and (V.15) are satisfied identically by this simple form except the first of (V.14) which will give the profile $a(r)$. This equation is

$$\left[\omega^2 \left(1 + \frac{\omega_p^2}{\omega_o^2-\omega^2}\right) - c^2\kappa^2\right] a + \frac{c^2}{2}\left(3 - \frac{c^2\kappa^2}{\omega^2} - \frac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)\left(a_{rr} + \frac{1}{r}a_r\right)$$

$$+ \frac{\gamma\,\epsilon_o^2\,\omega_p^6\,\omega^4\,a^3}{(\omega_o^2-\omega^2)^4} = 0 \quad ,$$

which we write as

$$a_{rr} + \frac{1}{r}a_r - Da + \tau a^3 = 0 \qquad \text{(V. 16)}$$

where

$$D = \frac{c^2\kappa^2 - \omega^2\left(1 + \dfrac{\omega_p^2}{\omega_o^2-\omega^2}\right)}{\dfrac{c^2}{2}\left(3 - \dfrac{c^2\kappa^2}{\omega^2} - \dfrac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)} \quad .$$

and

$$\tau = \frac{2\,\gamma\,\epsilon_o^2\,\omega_p^6\,\omega^4}{(\omega_o^2-\omega^2)^4\left(3 - \dfrac{c^2\kappa^2}{\omega^2} - \dfrac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)} \quad .$$

We proceed with the study of the equation

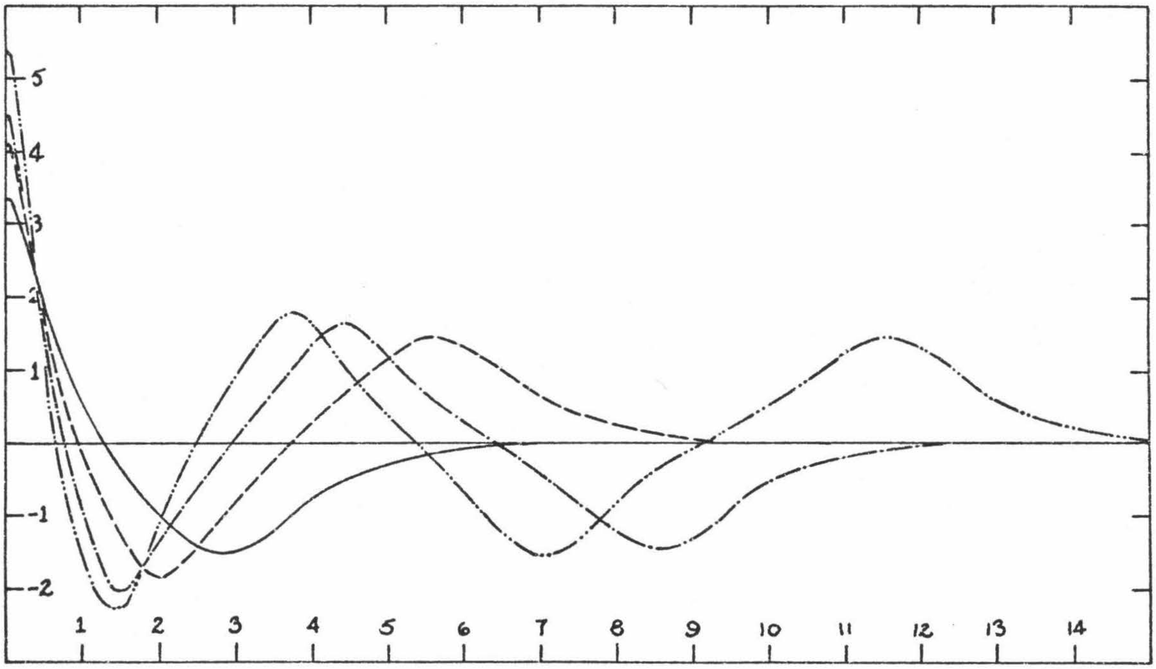
$$a_{rr} + \frac{1}{r} a_r - a + a^3 = 0 \quad . \tag{V.17}$$

We have scaled (V. 16) by the substitution

$$a^* = \sqrt{\frac{\tau}{D}}\; a \; ,$$

$$r^* = \sqrt{D}\; r \; ,$$

and we then drop the asterisks. In Townes' classic paper [1], he discussed (V. 17) and solved it numerically to produce a solution symmetric in r and decaying monotonically to zero. Further work by Haus [12] on an analogue computer produced symmetric profiles that fall from a maximum at r = 0 but cross the zero level a finite number of times, then decay to zero. His work indicates that there is an eigenfunction corresponding to zero, one, two, etc., zero crossings, the higher eigenfunctions having greater amplitude at the center of the beam and greater energy. These cylindrically symmetric beams have a bright central spot and periferal rings that decay in intensity toward the edge. The existence of these eigenfunctions is critically dependent on the quantities D and τ being positive. There are no such self-trapped beams in defocusing media ($\tau < 0$).



Eigenfunctions of Orders 1 to 4

We now derive an approximate analytic solution to (V. 17). The Lagrangian for (V. 17) is

$$\mathcal{L} = r(a_r^2 + a^2 - \frac{a^4}{2})$$

where the leading r is just the Jacobian for polar coordinates. $\mathcal{L}$ is, of course, the averaged Lagrangian subject to the restrictions that we have introduced. We now produce a "best fit" for a trial solution. The nature of the technique involves placing possible solutions into the variational principle with undetermined parameters. Variation with respect to these parameters then produces equations for them. The variational principle for the Lagrangian is

$$\delta J = \delta \int_0^\infty \left\{ r (a_r^2 + a^2 - \frac{a^4}{2}) \right\} dr = 0 \tag{V.18}$$

The trial solution that will be substituted is

$$a = a_1 e^{-b_1 r^2} + a_2 e^{-b_2 r^2} , \tag{V.19}$$

where a_1, a_2, b_1, b_2 are constants to be determined.

Placing (V.19) into (V.20) one uses the integrals

$$\int_0^\infty x e^{-x^2} dx = \tfrac{1}{2} \text{ and } \int_0^\infty x^3 e^{-x^2} dx = \tfrac{1}{2}$$

to obtain

$$\begin{aligned} J = \; & \frac{a_1^2}{2} + \frac{4 a_1 a_2 b_1 b_2}{(b_1+b_2)^2} + \frac{a_2^2}{2} + \frac{a_1^2}{4 b_1} + \frac{a_1 a_2}{b_1+b_2} + \frac{a_2^2}{4b_2} \\ & - \frac{1}{4}\left[\frac{a_1^4}{4b_1} + \frac{4a_1^3 a_2}{3b_1+b_2} + \frac{6 a_1^2 a_2^2}{2(b_1+b_2)} + \frac{4a_1 a_2^3}{b_1+3b_2} + \frac{a_2^4}{4b_2} \right] \end{aligned} \tag{V.20}$$

as the quantity to be minimized. Now the solutions to

$$J_{a_1} = 0, \quad J_{a_2} = 0, \quad J_{b_1} = 0, \quad J_{b_2} = 0$$

provide the numerical values for the parameters in (V.19). This is a complicated system which is reduced considerably by the following substitutions which allow two quantities to be eliminated:

$$a_2 = d a_1 \quad , \quad b_2 = c\, b_1 \quad .$$

The resulting form of (V.20) is

$$J = \frac{a_1^2}{2}\left[1 + \frac{8\,cd}{(1+c)^2} + d^2\right] + \frac{a_1^2}{4\,b_1}\left[1 + \frac{4d}{1+c} + \frac{d^2}{c}\right]$$

$$- \frac{a_1^4}{16\,b_1}\left[1 + \frac{16d}{3+c} + \frac{12d^2}{1+c} + \frac{16d^3}{1+3c} + \frac{d^4}{c}\right]$$

$$= a_1^2\,F_1 + \frac{a_1^2}{b_1}\,F_2 - \frac{a_1^4}{b_1}\,F_3 \qquad \text{(V. 21)}$$

where F_1, F_2 and F_3 depend on c and d. Variation of a_1 and b_1 produce

$$2\,a_1\,F_1 + \frac{2a_1}{b_1}\,F_2 - \frac{4a_1^3}{b_1}\,F_3 = 0$$

and

$$a_1^2\,F_2 - a_1^4\,F_3 = 0 \quad .$$

These equations yield solutions

$$a_1^2 = \frac{F_2}{F_3}\,, \qquad b_1 = \frac{F_2}{F_1} \qquad \text{(V. 22)}$$

which represent the best choices of a_1 and b_1 for given c and d. These solutions are replaced into (V. 21) to eliminate a_1 and b_1. The resulting form

$$J = \frac{F_1F_2}{F_3}$$

is varied with respect to c and d. Thus

$$(F_1F_2)_d\,F_3 - (F_1F_2)F_{3,\,d} = 0$$

$$(F_1F_2)_c\,F_3 - (F_1F_2)F_{3,\,c} = 0$$

must be solved for c and d in some numerical way. The results of calculation on a desk calculator are

$$c = 4.7335, \qquad d = 1.55005 \quad .$$

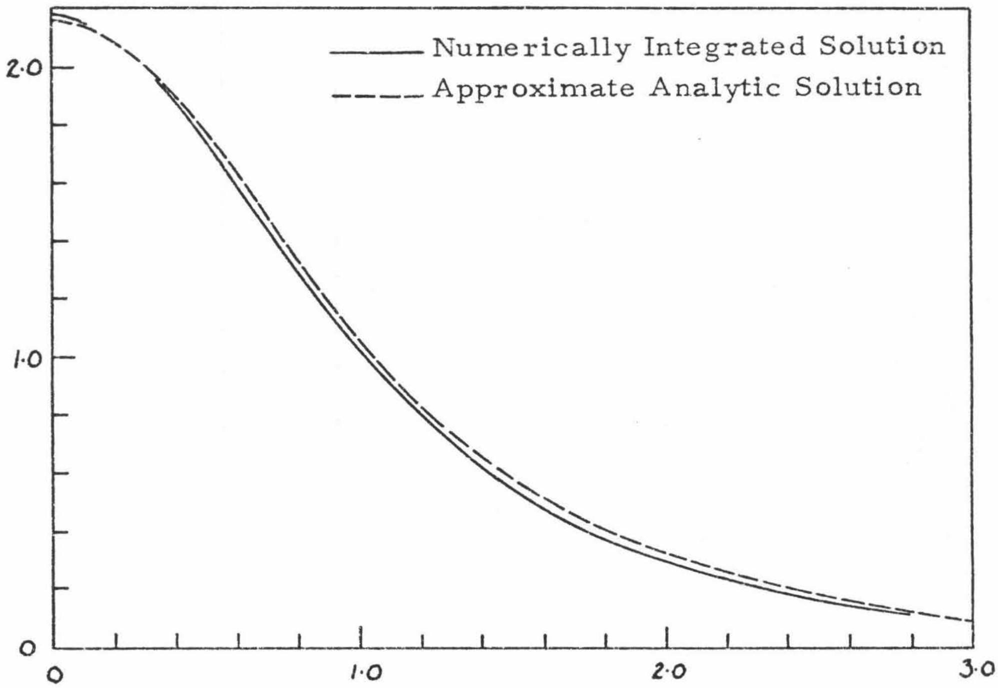
Substitution into (V. 22) produces

$$a_1 = .848763 \text{ and } b_1 = .249505 \quad .$$

Finally the solution has the form

$$a = .848763\, e^{-.249505\, r^2} + 1.315625\, e^{-1.181031\, r^2} \quad .$$

This result is plotted to compare with the numerically integrated result.



Order Zero Eigenfunction

Haus gives the power for the first five modes where power is defined by

$$P = \int_0^\infty a^2 \, r \, dr \quad .$$

These values are

Mode Number	Power
1	1.86
2	12.25
3	31.26
4	58.57
5	94.23

We note that the power for the lowest mode is very much lower than the other modes, hence if a large beam becomes unstable it seems likely that it will break up into beams of the first type. For comparison with Haus' work, the power of our approximate solution is 1.869 compared to his 1.86, while Townes' figure is 1.834.

V.3 Time-Dependent Solutions

Having found some special solutions for the time-independent problems we proceed with some examples of time-dependent solutions. These examples will all be stationary solutions; the envelope pattern will move with a constant speed. The first case will be a one-dimensional wave, all quantities being constant in y and z. We shall then consider solitary waves that are localized near the x-axis and travel with constant speed. These are short pulses that are self-trapped in the nonlinear medium.

V.3.i Stationary One-Dimensional Wave Envelopes

We shall deal with both linearly and circularly polarized waves. Firstly, we consider solutions of the equations of linearly polarized waves, (V.4) and (V.5), which are independent of y and z but vary in a time-dependent manner in the x-direction. If we define a new variable η by

$$\eta = x - Vt$$

where V is constant, then any wave pattern that depends only upon η will translate undistorted in the x-direction with velocity V. We take $\kappa_2 = 0$ and $\kappa_1 = \kappa$ and ω as constants. The second equation of (V.4) becomes

$$-V\left\{\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)\right]_\omega aa_\eta\right\} + 2c^2\kappa\, aa_\eta = 0 \quad .$$

To satisfy this equation we set

$$V = \frac{2c^2\kappa}{\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)\right]_\omega} \quad .$$

The first equation of (V.4) becomes

$$\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)-c^2\kappa^2\right]a + \left[c^2-V^2\left(1+\frac{\omega_p^2\,\omega_o^2(\omega_o^2+3\omega^2)}{(\omega_o^2-\omega^2)^3}\right)\right]a_{\eta\eta}$$

$$+\frac{3\gamma\,\varepsilon_o^2\,\omega_p^6\,\omega^4\,a^3}{4(\omega_o^2-\omega^2)^4} = 0 \quad .$$

This is the equation for stationary waves found by Ostrovskii [11].

We write this as

$$a_{\eta\eta} - D\,a + \tau\,a^3 = 0$$

where

$$D = \frac{c^2\kappa^2 - \omega^2\left(1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}\right)}{c^2 - V^2\left(1 + \frac{\omega_p^2\,\omega_o^2 + 3\,\omega^2)}{(\omega_o^2 - \omega^2)^3}\right)}$$

and

$$\tau = \frac{\dfrac{3\,\gamma\,\epsilon_o^2\,\omega_p^6\,\omega^4}{4(\omega_o^2 - \omega^2)^4}}{c^2 - V^2\left(1 + \frac{\omega_p^2\,\omega_o^2(\omega_o^2 + 3\,\omega^2)}{(\omega_o^2 - \omega^2)^3}\right)} .$$

This is the form of equation (V. 16). Where we had parallel beams in space in section V. 1 we now have a series of pulses and where we had an isolated beam we now have a solitary wave.

We use the same definition of η in the equations of circularly polarized waves, (V. 14) and (V. 15), and we proceed in the same fashion. Exactly the same expressions are found for V and D, and τ differs in that $\frac{3}{4}\gamma$ is replaced by γ. In both cases we find that solitary waves exist only in focusing media.

V. 3. ii Localized Solitary Waves in Two Dimensions

We combine the ideas of the preceding parts and obtain stationary solutions by setting $\eta = x - vt$, but now we seek solutions that are radially symmetric in the variables η and y. The radially symmetric pattern that results propagates with speed V. If the solution is localized in the radial variable then we have a pulse that propagates with constant speed. Again we take $\kappa_2 = 0$ and $\kappa_1 = \kappa$ and ω as constants.

The second equation of (V.4) produces the same value of V as found in section V.3.i:

$$V = \frac{2c^2\kappa}{\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)\right]_\omega} .$$

The first equation of (V.4) produces

$$\left[\omega^2\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)-c^2\kappa^2\right]a - V^2\left[1+\frac{\omega_p^2\omega_o^2(\omega_o^2+3\omega^2)}{(\omega_o^2-\omega^2)^3}\right]a_{\eta\eta}$$

$$+ c^2 a_{\eta\eta} + c^2 a_{yy} + \frac{3\gamma\varepsilon_o^2\omega_p^2\omega^4 a^3}{4(\omega_o^2-\omega^2)^4} = 0 .$$

We set $$r^2 = y^2 + \left(1 - \frac{V^2}{c^2}\left[1+\frac{\omega_p^2\omega_o^2(\omega_o^2+3\omega^2)}{(\omega_o^2-\omega^2)^3}\right]\right)^{-1}\eta^2$$

to obtain the form

$$a_{rr} + \frac{1}{r}a_r - Da + \tau a^3 = 0$$

where $$D = \kappa^2 - \frac{\omega^2}{c^2}\left(1+\frac{\omega_p^2}{\omega_o^2-\omega^2}\right)$$

and $$\tau = \frac{3\gamma\varepsilon_o^2\omega_p^6\omega^4}{4c^2(\omega_o^2-\omega^2)^4} .$$

This is equation (V.16) which we have found to have solutions localized about r = 0. There are such localized pulses only in a focusing medium.

V.3.iii Localized Solitary Waves in Three Dimensions

We make the substitution $\eta = x - Vt$, where V is the same quantity as in V.3.i and V.3.ii, in the equations of circular polarization

(V. 14) and (V. 15). If we assume radial symmetry between y and η with no z dependence, then we obtain a similar two-dimensional solitary wave as in section V. 2. ii. The more interesting case results from assuming spherical symmetry between η, y and z. Here we set

$$r^2 = \left[c^2 - V^2\left(1 + \frac{\omega_p^2\,\omega_o^2(\omega_o^2+3\omega^2)}{(\omega_o^2-\omega^2)^3}\right)\right]^{-1}\eta^2 + \left[\frac{c^2}{2}\left(3 - \frac{c^2\kappa^2}{\omega^2} - \frac{2\,\omega_p^2}{\omega_o^2-\omega^2}\right)\right]^{-1}(y^2+z^2) \; .$$

In this variable we obtain

$$\left[\omega^2\left(1 + \frac{\omega_p^2}{\omega_o^2-\omega^2}\right) - c^2\kappa^2\right]a + a_{rr} + \frac{2}{r}\,a_r + \frac{\gamma\,\varepsilon_o^2\,\omega_p^6\,\omega^4 a^3}{(\omega_o^2-\omega^2)^4} = 0$$

for the first equation of (V. 14), while the rest of equations (V. 14) and (V. 15) are satisfied by the chosen form. We have the case of a pulse, localized about the x-axis, propagating with speed V. We write this as

$$a_{rr} + \frac{2}{r}\,a_r - D\,a + \tau\,a^3 = 0$$

where

$$D = c^2\kappa^2 - \omega^2\left(1 + \frac{\omega_p^2}{\omega_o^2-\omega^2}\right)$$

and

$$\tau = \frac{\gamma\,\varepsilon_o^2\,\omega_p^6\,\omega^4}{(\omega_o^2-\omega^2)^4} \; .$$

This equation has been studied in connection with elementary particle physics. Finkelstein et al. [13] have shown through phase plane arguments that this equation has analytic, symmetric eigensolutions that are asymptotic to zero at infinity and have one, two, three, etc., zero crossings as in the case of solutions to (V. 17). Schiff et al. [14]

display the following approximate solution for the lowest eigensolution obtained by Teshima by Lagrangian techniques in the same manner as we did in section V.2.ii with equation (V.17):

$$a = 2.6060\left[\frac{(e^{-r} - e^{-4.10r})}{r} + (2.33r-1.360)e^{-4.10r}\right].$$

This profile is similar to the lowest eigenfunction solution of equation (V.17) but it has a higher central maximum and is thinner. The higher eigenfunctions represent pulses that are ellipsoidal with sets of ever weakening ellipsoidal shells surrounding them, the whole pattern moving with constant velocity.

APPENDIX

THE QUESTION OF PSEUDO-FREQUENCIES

In Chapter III it was mentioned that a constant electric field cannot build up if its value is zero everywhere at some starting time and the potential $V(P)$ is an even function of P. We now prove this by applying Whitham's procedure [5] including pseudo-frequencies which represent the slowly varying "constant" component of the field. The relevant Lagrangian is (III. 1):

$$L = \frac{\epsilon_o}{2}\left\{A_t^2 - c^2(A_x^2 + A_y^2)\right\} - A_t P + \frac{1}{\epsilon_o \omega_p^2}\left\{\frac{P_t^2}{2} - V(P)\right\}. \qquad (A.1)$$

Following the standard procedure of Whitham, the functions A and P are considered periodic functions of θ, but the potential A is allowed to have a secular function added to it. They are written

$$\begin{aligned} P &= P(\theta, X, Y, T) \quad , \\ A &= \Phi(X, Y, T) + A(\theta, X, Y, T) \end{aligned} \qquad (A.2)$$

where X, Y and T are the slow scales defined by

$$X = \epsilon x, \qquad Y = \epsilon y\,, \qquad T = \epsilon t\;.$$

θ is the fast variable defined so that

$$\theta_x = \kappa_1(X, Y, T), \quad \theta_y = \kappa_2(X, Y, T), \quad \theta_t = -\omega(X, Y, T)$$

and the secular function Φ is defined by pseudo-frequencies β_1, β_2, γ such that

$$\Phi_x = \beta_1(X, Y, T), \; \Phi_y = \beta_2(X, Y, T), \; \Phi_t = -\gamma(X, Y, T).$$

Substitution of (A. 2) into (A. 1) produces the explicit form

$$L = \frac{\varepsilon_o}{2}\left\{(\gamma+\omega A_\theta)^2 - c^2(\beta_1+\kappa_1 A_\theta)^2 - c^2(\beta_2+\kappa_2 A_\theta)^2\right\}$$

$$+ (\gamma+\omega A_\theta)P + \frac{1}{\varepsilon_o\omega_p^2}\left\{\frac{\omega^2 P_\theta^2}{2} - V(P)\right\} .$$

We use Whitham's Hamiltonian formulation to simplify the work. This method produces integrals of the Euler equations quickly so that certain variables in the Lagrangian may be eliminated. One may differentiate the integrals we produce to justify the method. Whitham defines

$$\Pi_1 = \frac{\partial L}{\partial P_\theta}, \quad \Pi_2 = \frac{\partial L}{\partial A_\theta}, \qquad H = P_\theta \Pi_1 + A_\theta \Pi_2 - L$$

and then the integrals of the system are

$$H = \frac{M}{2\varepsilon_o\,\omega_p^2} \quad \text{and} \quad \Pi_2 = B \quad .$$

The explicit form of these functions is

$$\Pi_1 = \frac{\omega^2 P_\theta}{\varepsilon_o\,\omega_p^2} \tag{A. 3}$$

$$\Pi_2 = \varepsilon_o\,\omega(\gamma+\omega A_\theta) - \varepsilon_o c^2\kappa_1(\beta_1+\kappa_1 A_\theta) - \varepsilon_o c^2\kappa_2(\beta_2+\kappa_2 A_\theta) + \omega P = B \tag{A. 4}$$

$$H = \frac{\varepsilon_o}{2}\left\{(\omega^2 - c^2\kappa^2)A_\theta^2 - (\gamma^2 - c^2\beta^2)\right\} - \gamma P + \frac{1}{\varepsilon_o\omega_p^2}\left\{\frac{\omega^2 P_\theta^2}{2} + V(P)\right\}$$

$$= \frac{M}{2\,\varepsilon_o\,\omega_p^2} \tag{A. 5}$$

The integrals (A. 4) and (A. 5) are used to eliminate P_θ and A_θ from (A. 3) for use in the following form of $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2\pi} \oint \Pi_1 \, dP - \frac{M}{2\varepsilon_o \omega_p^2} \quad .$$

Substitution gives the explicit form of $\mathcal{L}$ as

$$\mathcal{L} = \frac{1}{2\pi} \oint \frac{\omega}{\varepsilon_o \omega_p^2} \sqrt{F} \, dP - \frac{M}{2\varepsilon_o \omega_p^2} \tag{A.6}$$

where

$$F = M - 2V(P) + 2\varepsilon_o \omega_p^2 \gamma P + \varepsilon_o^2 \omega_p^2 (\gamma^2 - c^2\beta^2) - \frac{\omega_p^2}{\omega^2 - c^2\kappa^2} \left\{ B - \varepsilon_o [\omega\gamma - c^2(\kappa_1\beta_1 + \kappa_2\beta_2)] - \omega P \right\}^2 .$$

The Euler equations are

$$\begin{aligned} \mathcal{L}_M &= 0 \\ -\frac{\partial}{\partial t}\mathcal{L}_\omega + \frac{\partial}{\partial x}\mathcal{L}_{\kappa_1} + \frac{\partial}{\partial y}\mathcal{L}_{\kappa_2} &= 0 \end{aligned} \tag{A.7}$$

$$\begin{aligned} \mathcal{L}_B &= 0 \\ -\frac{\partial}{\partial t}\mathcal{L}_\gamma + \frac{\partial}{\partial x}\mathcal{L}_{\beta_1} + \frac{\partial}{\partial y}\mathcal{L}_{\beta_2} &= 0 \end{aligned} \tag{A.8}$$

Our purpose is to show that (A. 8) is satisfied identically by setting $B \equiv 0$, $\gamma \equiv 0$, $\beta \equiv 0$. This condition would leave (A. 7) as the set of equations that was dealt with in Chapter III.

The first equation of (A. 8) is

$$-\frac{1}{2\pi} \oint \frac{\omega}{\varepsilon_o(\omega^2 - c^2\kappa^2)} \frac{\{B - \varepsilon_o[\omega\gamma - c^2(\kappa_1\beta_1 + \kappa_2\beta_2)] - \omega P\}}{\sqrt{F}} \, dP + \text{boundary terms} = 0$$

The second is

$$- \frac{\partial}{\partial t} \left\{ \frac{1}{2\pi} \oint \frac{\omega}{\omega_p^2} \frac{1}{\sqrt{F}} \left\{ \omega_p^2 P + \epsilon_o \omega_p^2 \gamma \right. \right.$$

$$\left. \left. + \frac{\omega \omega_p^2}{\omega^2 - c^2 \kappa^2} \left[B - \epsilon_o (\omega\gamma - c^2(\kappa_1\beta_1 + \kappa_2\beta_2)) - \omega P \right] \right\} dP \right\}$$

$$+ \frac{\partial}{\partial x} \left\{ \frac{1}{2\pi} \oint \frac{\omega}{\omega_p^2} \frac{1}{\sqrt{F}} \left\{ -\epsilon_o \omega_p^2 c^2 \beta_1 \right. \right.$$

$$\left. \left. + \frac{c^2 \omega_p^2 \kappa_1}{\omega^2 - c^2 \kappa^2} \left[B - \epsilon_o (\omega\gamma - c^2(\kappa_1\beta_1 + \kappa_2\beta_2)) - \omega P \right] \right\} dP \right\}$$

$$+ \frac{\partial}{\partial y} \left\{ \frac{1}{2\pi} \oint \frac{\omega}{\omega_p^2} \frac{1}{\sqrt{F}} \left\{ -\epsilon_o \omega_p^2 c^2 \beta_2 \right. \right.$$

$$\left. \left. + \frac{c^2 \omega_p^2 \kappa_2}{\omega^2 - c^2 \kappa^2} \left[B - \epsilon_o (\omega\gamma - c^2(\kappa_1\beta_1 + \kappa_2\beta_2)) - \omega P \right] \right\} dP \right\}$$

$$+ \text{boundary terms} = 0$$

Setting $B \equiv 0$, $\gamma \equiv 0$ and $\underline{\beta} \equiv 0$, the integrands in all four loop integrals become odd if $V(P)$ in the factor F is even, so that each loop integral is zero. In the terms that we have called "boundary terms" there is always a factor of $\sqrt{F}$ from the integrand of (A. 6). These terms are evaluated at the roots of F and hence are zero. Thus each term of the equations is zero and the equations are satisfied identically.

REFERENCES

[1] Chiao, R. Y., Garmire, E., and Townes, C. H., Phys. Rev. Letters, 13, 479, (1964).

[2] Landau, L. D., and Lifshitz, E. M., Mechanics, Pergamon Press, 1960.

[3] Whitham, G. B., J. Fluid Mech., 27, 399, (1967).

[4] Whitham, G. B., Proc. Roy. Soc., A 299, 6, (1967).

[5] Whitham, G. B., J. Fluid Mech., 44, 373, (1970).

[6] Knight, B. W., and Peterson, G. A., Proc. Nat. Acad. Sci., 65, 841, (1970).

[7] Lighthill, M. J., The Hodograph Transformation Modern Developments in Fluid Mechanics, L. Howarth, ed., Oxford University Press.

[8] Whitham, G. B., J. Fluid Mech., 2, 145, (1957).

[9] Cumberbatch, E., J. Inst. Maths. Applics., 6, 250, (1970).

[10] Akhmanov, S. A., Sukhorukov, A. P., and Khoklov, R. V., Soviet Physics JETP, 23, 1025, (1966).

[11] Ostrovskii, L. A., Soviet Physics JETP, 24, 797, (1967)

[12] Haus, H. A., App. Phys. Letters, 8, 128, (1966)

[13] Finkelstein, R. J., LeLevier, R., and Ruderman, M., Phys. Rev., 83, 326, (1951).

[14] Betts, D. D., Schiff, H., and Strickfaden, W. B., J. Math. Phys., 4, 334, (1963).