SOME PROBLEMS IN NONLINEAR ELASTICITY

Thesis by

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ABSTRACT

Two separate problems are discussed: axisymmetric equilibrium configurations of a circular membrane under pressure and subject to thrust along its edge, and the buckling of a circular cylindrical shell.

An ordinary differential equation governing the circular membrane is imbedded in a family of n-dimensional nonlinear equations. Phase plane methods are used to examine the number of solutions corresponding to a parameter which generalizes the thrust, as well as other parameters determining the shape of the nonlinearity and the undeformed shape of the membrane. It is found that in any number of dimensions there exists a value of the generalized thrust for which a countable infinity of solutions exist if some of the remaining parameters are made sufficiently large. Criteria describing the number of solutions in other cases are also given.

Donnell-type equations are used to model a circular cylindrical shell. The static problem of bifurcation of buckled modes from Poisson expansion is analyzed using an iteration scheme and pertubation methods. Analysis shows that although buckling loads are usually simple eigenvalues, they may have arbitrarily large but finite multiplicity when the ratio of the shell's length and circumference is rational. A numerical study of the critical buckling load for simple eigenvalues indicates that the number of waves along the axis of the deformed shell is roughly proportional to the length of the shell, suggesting the possibility of a "characteristic length." Further numerical work indicates that initial post-buckling curves are typically steep, although the load may increase or decrease. It is shown that either a sheet of solutions or two distinct branches bifurcate from a double eigenvalue. Furthermore, a shell may be subject to a uniform torque, even though one is not prescribed at the ends of the shell, through the interaction of two modes with the same number of circumferential waves. Finally, multiple time scale techniques are used to study the dynamic buckling of a rectangular plate as well as a circular cylindrical shell; transition to a new steady state amplitude determined by the nonlinearity is shown. The importance of damping in determining equilibrium configurations independent of initial conditions is illustrated.

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INTRODUCTION

This study is concerned with two separate problems. The first is motivated by equations which model the behavior of a circular membrane. The resulting equation is imbedded in a class of equations, and the existence of multiple solutions is analyzed for this class. The second problem is to study the static and dynamic buckling of a circular cylindrical shell under axial loading.

For the first problem, we are concerned with studying the possibility of multiple equilibrium configurations of a circular shallow elastic membrane whose surface is subject to an axisymmetric pressure. A radial thrust is specified along the membrane's edge, and the edge is restrained from deforming normal to its midplane. Only axisymmetric deformations of the membrane are considered.

In chapter 1 we study the case of an initially flat membrane under a variable pressure. The situation of a flat membrane under constant pressure has been studied by A. Callegari, E. Reiss, and H. Keller [2]. In chapter 2 we consider a membrane which is not initially flat and is subjected to a variable pressure.

The reader is referred to the references [1, 2] for a derivation of the membrane theory. Notes on the final formulation of the problem are given in Appendix C. The resulting equation is

$$\frac{d}{dr}\left(r^3 \frac{du}{dr}\right) + \lambda^3 \frac{G}{(1-u)^2} = \lambda B r \phi^2 \qquad (C.6)$$

The boundary conditions are

$$\frac{du}{dr} = 0 \quad \text{at } r = 0 \qquad (C.7a)$$

$$u(1) = 0 \tag{C.7b}$$

When G(r) and $\phi(r)$ are of the form to be prescribed in chapters 1 and 2, it is found that equation (C.6) can be transformed into a second order autonomous system which is amenable to phase plane analysis. This method was first used by Gel′fand [4] to study solution multiplicity in certain problems arising in the theory of chemical reactors, viz.

$$\left.\begin{aligned} &\frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1} \frac{du}{dr}\right) + \lambda e^u = 0 \quad , \quad n = 1, 2, 3 \\ &\frac{du}{dr} = 0 \quad \text{at } r = 0 \\ &u(1) = 0 \end{aligned}\right\} \tag{0.1}$$

He found that there exists a value $\lambda_* > 0$ such that there are

(a) no solutions when $\lambda > \lambda_*$ $(n = 1, 2, 3)$

(b) one solution when $\lambda = \lambda_*$ $(n = 1, 2, 3)$

(c) two solutions when $0 < \lambda < \lambda_*$ $(n = 1, 2)$

(d) a countable infinity of solutions when $\lambda = \lambda_\infty = 2$ and $n = 3$

(e) a finite but large number of solutions when $n = 3$ and $|\lambda - \lambda_\infty|$ is small.

A. Callegari, E. Reiss, and H. Keller [2] applied this method to study an initially flat circular membrane under constant pressure, modeled by

$$\frac{1}{r^3} \frac{d}{dr}\left(r^3 \frac{du}{dr}\right) + \lambda (1-u)^{-2} = 0$$

Here the differential operator is a Laplacian in four dimensions $(n = 4)$. They found the behavior can be described by (a), (b), (d), and (e) with $\lambda_\infty = \sqrt[3]{16/9}$.

Joseph and Lundgren [8] studied (0.1) and

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{du}{dr}\right) + \lambda(1-\alpha u)^{1-\beta} = 0 \qquad (0.2)$$

for arbitrary positive integers n and for $\beta > 1$, $\alpha > 0$. For (0.1) they found that (a) and (b) hold for $n \geq 1$, that (d) and (e) hold for $2 < n < 10$ with $\lambda_\infty = n(n-2)$, and that for $n \geq 10$ there exists one solution for $\lambda < 2(n-2)$. For (0.2) they found (a) and (b) hold for $n \geq 1$, (d) and (e) hold when $n-2 < f(\beta)$, and for $n-2 \geq f(\beta)$ there is only one solution in $0 < \lambda < \lambda_*$. Here

$$f(\beta) = \frac{4(\beta-1)}{\beta} + 4\sqrt{\frac{\beta-1}{\beta}}$$

(Note: [8] also includes a similar study for $\alpha < 0$ and $\beta < 1$).

In chapters 1 and 2 of this study we find that equation (C.6), for appropriate functions G and ϕ, is of the type

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{du}{dr}\right) + \lambda^\beta r^\mu (1-\alpha u)^{1-\beta} = \lambda A\, r^{2+(\mu+2)/\beta} \qquad (0.3)$$

with $\beta > 1$ and $\alpha \neq 0$. We investigate solutions of (0.3) for all real λ, thereby generalizing the results of [8]. Of particular interest is the result that for $\alpha > 0$ there exist values μ^* (for $n \geq 1$) and A_0 (for $n \geq 3$) such that if $\mu > \mu^*$ or $A > A_0$, then the situation may be described by (a), (b), (d), and (e) with appropriate λ_∞. (The cases of A large and n = 1, 2 are not investigated here.) From this we see that the possibility of an infinity of solutions persists in all dimensions. We summarize our results below.

<u>For A = 0</u> there exist λ_* and μ^* such that, for $n \geq 1$, there are

(a) no solutions for $\alpha\lambda^\beta > \alpha\lambda_*^\beta > 0$

(b) one solution for $\alpha\lambda^\beta < 0$

(c) finitely many solutions for $0 < \alpha\lambda^{\beta} < \alpha\lambda_*^{\beta}$ if $\mu \leq \mu^*$

(d) a countable infinity of solutions when $\lambda = \lambda_\infty$ if $\mu > \mu^*$

(e) a finite but large number of solutions if $|\lambda - \lambda_\infty| \neq 0$ is small and $\mu > \mu^*$.

Here μ^* is such that $\Phi(\mu^* + 2) = 0$, where

$$\Phi(\nu) = -4(\beta-1)[\nu + \tfrac{1}{2}\beta(n-2)]^2 + \beta^3(n-2)^2 \tag{1.23}$$

For $A \neq 0$ we restrict β to integral values and take $n > 2$. When $|A| \ll 1$ the situation is the same as for $A = 0$, except that μ^* depends on A. For $|A|$ sufficiently large we find

(a) β odd, $A > 0$, $\alpha > 0$: For $\lambda > 0$ there exist λ_∞ and λ_* as above. For $\lambda < 0$ either there exists one solution for all λ or λ_*^- exists such that no solutions exist for $\lambda < \lambda_*^-$ and finitely many exist for $\lambda_*^- < \lambda < 0$.

(b) β odd, $A < 0$, $\alpha > 0$: For $\lambda > 0$, λ_* exists but there is no λ_∞ and hence there are only finitely many solutions. For $\lambda < 0$, there exists one solution for all λ.

(c) β even, $A > 0$, $\alpha > 0$: λ_*, λ_∞, and λ_*^- as above all exist.

(d) β even, $A > 0$, $\alpha < 0$: For $\lambda > 0$ either there exists one solution for all λ or λ_* exists but λ_∞ does not. For $\lambda < 0$ there exists one solution for all λ.

The cases omitted may be found by transforming $\alpha \to -\alpha$, $\lambda \to -\lambda$ for β odd and $A \to -A$, $\lambda \to -\lambda$ for β even.

For the second problem, we are concerned with studying the buckling of a circular cylindrical shell under axial loading. The fundamental equations for a Donnell-type model are developed in Appendix A.

In chapter 3 we consider the static problem. The classical solution known as Poisson expansion is introduced, and the problem of the bifurcation of equilibrium states from this solution is formulated. We analyze the multiplicity (i.e., the number of independent eigenfunctions) of eigenvalues or buckling loads and find that although they are typically simple, an arbitrarily large albeit finite multiplicity is possible when the ratio of the shell's circumference and length is rational. A numerical study is made of the mode corresponding to the critical buckling load, and it is found that the number of waves along the axis of the shell is roughly proportional to the length of the shell, suggesting the possibility of a "characteristic length" over which buckling occurs. An iteration scheme developed by H. Keller and W. Langford [13] is utilized to calculate the initial post-buckling curve for simple eigenvalues. We find that the load may increase or decrease, but regardless, the load-deflection curve is usually very steep. A pertubation scheme is used to study the number of bifurcating branches when the buckling load is a double eigenvalue. Several possibilities occur: there may exist a one or two-parameter "sheet" of solutions, or else there exist precisely two branches of solutions. A final calculation shows that, through the interaction of two modes with the same number of circumferential waves, the shell may be subjected to non-vanishing uniform torque even though no torque is prescribed at the ends of the shell.

Chapter 4 treats the dynamic problem when the load is such that Poisson expansion is unstable. The load is taken to be a "small distance" into the unstable regime, and a pertubation scheme employing multiple time scales is utilized. This method was first used by B. Matkowsky[14]. We first apply it to the dynamic buckling of a rectangular plate, and the results are compared to those of a study by Reiss and Matkowsky[15] of the buckling of rods. The equation governing the amplitude of the unstable mode is found to be a second order autonomous equation in the absence of damping. However, the equilibrium points depend on the initial conditions, which contradicts the fact that equilibrium configurations satisfy the time-independent steady state equations. When damping is present, the terms depending on the initial conditions vanish exponentially, and bounded solutions are shown to be asymptotic to the critical points of a reduced autonomous system. A similar discussion applies to the problem of a circular cylindrical shell. Qualitatively the two problems differ in that the reduced system for a plate is two dimensional, but that of a cylindrical shell is four dimensional. Also the plate has two physically distinct stable equilibrium configurations, but the cylindrical shell has only one. (For both problems we assume that the critical eigenvalue is simple.)

CHAPTER 1: INITIALLY FLAT MEMBRANES

In this chapter we will analyze the number of equilibrium configurations of an initially flat membrane subject to a pressure distribution of the form

$$p(r^*) = p_{max}(r^*/R)^c$$

for $c \geq 0$. Substituting this into the formulae given in equations (C. 1) results in

$$G(r) = r^{\mu+3}$$
$$P = \frac{2}{(\mu+4)^2}\left(\frac{p_{max}}{E}\right)^2\left(\frac{R}{h}\right)^3$$

where we have set $\mu = 2c$. A flat membrane is described by $\phi(r) \equiv 0$; hence equation (C. 6) becomes

$$\frac{d^2u}{dr^2} + \frac{3}{r}\frac{du}{dr} + \lambda^3 \frac{r^\mu}{(1-u)^2} = 0$$

subject to boundary conditions (C. 7).

Recognizing $\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr}$ as a spherically symmetric Laplacian in four dimensions motivates the following simple generalization of the membrane problem:

$$\frac{d^2u}{dr^2} + \frac{n-1}{r}\frac{du}{dr} + \lambda^\beta r^\mu(1-\alpha u)^{1-\beta} = 0 \quad , \quad 0 < r < 1 \tag{1.1}$$

$$\lim_{r\to 0}\left|\frac{1}{r}\frac{du}{dr}\right| < \infty \tag{1.2}$$

$$u(1) = 0 \tag{1.3}$$

The regularity condition (1. 2) only mildly strengthens the previous boundary condition (C. 7a). We assume $\alpha \neq 0$ so that the problem is truly nonlinear; furthermore, we retain the assumption that $\mu \geq 0$. Finally, we restrict β such that $\beta > 1$; this restriction on the form of the nonlinearity will play a strategic role in certain of the arguments to follow.

We seek a solution $u \in C^2(0, 1)$, so equation (1.1) implies that $1-\alpha u(r) \neq 0$ for $0 < r < 1$. $u(1) = 0$ and continuity then imply that $1-\alpha u > 0$ in $(0, 1]$. We extend this and require

$$1-\alpha u(r) > 0 \quad , \quad 0 \leq r \leq 1 \tag{1.4}$$

Elementary considerations using the theory of Lie lead to the following change of variables:

$$x = \log r \tag{1.5}$$

$$v(x) = (1-\alpha u)r^{\gamma} \tag{1.6}$$

where

$$\gamma \equiv -(\mu+2)/\beta \tag{1.7}$$

Note that $\gamma < 0$. These new variables transform equation (1.1) into the equivalent autonomous equation

$$\frac{d^2 v}{dx^2} - (2\gamma+2-n)\frac{dv}{dx} + \gamma(\gamma+2-n)v - \alpha\lambda^{\beta} v^{1-\beta} = 0$$

or

$$\frac{d^2 v}{dx^2} - (\gamma+\theta)\frac{dv}{dx} + \gamma\theta v - \alpha\lambda^{\beta} v^{1-\beta} = 0 \tag{1.8}$$

where we have defined $\theta \equiv \gamma - (n-2)$ (1.9)

Boundary conditions (1.2) and (1.3) become respectively

$$\lim_{x\to -\infty} e^{-(\gamma+2)x} \left| \frac{dv}{dx} - \gamma v \right| < \infty \tag{1.10}$$

and

$$v = 1 \quad \text{at } x = 0 \tag{1.11}$$

Remark also that condition (1.4) implies

$$v \to +\infty \quad \text{as } x \to -\infty \tag{1.12}$$

since $\gamma < 0$.

Although it is possible to study equation (1.8) in the phase plane directly, one last change of variables proves to greatly simplify the analysis. Set

$$y(x) = \alpha\lambda^{\beta} v^{-\beta} \tag{1.13}$$

$$z(x) = \frac{1}{v}\frac{dv}{dx} \tag{1.14}$$

We find that equation (1.8) is equivalent to the system

$$\dot{y} = -\beta yz \equiv f(y, z) \tag{1.15a}$$

$$\dot{z} = y - (z-\gamma)(z-\theta) \equiv g(y, z) \tag{1.15b}$$

where differentiation with respect to x is indicated by a dot. The boundary conditions become

$$\lim_{x \to -\infty} e^{-(\nu+2)x}(z-\gamma)|y|^{-1/\beta} < \infty \tag{1.16}$$

$$y(0) = \alpha\lambda^{\beta} \tag{1.17}$$

Furthermore, (1.12), (1.13) and the hypothesis $\beta > 1$ imply

$$y \to 0 \quad \text{as} \quad x \to -\infty \tag{1.18}$$

In making the transformation (1.13) we have tacitly assumed that $\lambda \neq 0$. When $\lambda = 0$ it is easily verified that equations (1.1), (1.2) and (1.3) have the unique solution

$$u(r) = 0 \quad , \quad 0 \leq r \leq 1$$

for $n > 0$. In the remainder of the chapter $\alpha\lambda^{\beta} \neq 0$ will be presumed.

Depending on the dimension, three cases arise in the phase plane analysis, namely: $n > 2$, $n = 2$ and $n = 1$.

<u>The case $n > 2$</u>

From (1.9) we see that $\theta < \gamma < 0$. System (1.15) has three critical points in the finite plane:

$$P_1: \; y = 0, \; z = \gamma$$

$$P_2: \; y = 0, \; z = \theta$$

$$P_3: \; y = \gamma\theta, \; z = 0$$

In figure 2 we indicate the field of tangent vectors corresponding to system (1.15), including the locus Γ where $\dot{z} = g(y, z) = 0$. Note that $y \equiv 0$, $\dot{z} = -(z-\gamma)(z-\theta)$ provides three exact solutions whose trajectories completely cover the z axis; consequently no trajectory can cross the z axis.

Next consider the local behavior about each critical point. We readily compute $f_y = -\beta z$, $f_z = -\beta y$, $g_y = 1$, $g_z = -2z+\gamma+\theta$ and so the equation for the characteristic exponent ℓ at the critical point (y_0, z_0) is

$$\begin{vmatrix} -\beta z_0 - \ell & -\beta y_0 \\ 1 & -2z_0 + \gamma + \theta - \ell \end{vmatrix} = 0 \qquad (1.19)$$

For P_1, $y_0 = 0$, $z_0 = \gamma$ and (1.19) becomes

$$\begin{vmatrix} -\beta\gamma - \ell & 0 \\ 1 & \theta - \gamma - \ell \end{vmatrix} = 0$$

which has roots $\ell = \ell_1 = -\beta\gamma$ and $\ell = \ell_2 = \theta - \gamma$. Now $-\beta\gamma = \mu+2 > 0$ and $\theta - \gamma = -(n-2) < 0$ for $n > 2$. Consequently P_1 is always a saddle point.

For P_2, $y_0 = 0$, $z_0 = \theta$ and the roots are $\ell_1 = -\beta\theta$ and $\ell_2 = \gamma - \theta$. $\beta > 1$ and $\theta < 0$ mean $\ell_1 > 0$. $\gamma - \theta = n-2 > 0$ for $n > 2$ mean $\ell_2 > 0$. Consequently, P_2 is an unstable node. Furthermore, $\beta > 1$ and $n > 2$ imply that P_2 is an improper node, for recalling the definitions of γ and θ we find

$$\ell_1 = -\beta\theta = -\beta\gamma + \beta(n-2) = (\mu+2) + \beta(n-2)$$
$$> n-2 = \gamma - \theta = \ell_2$$

To describe the behavior near P_2 in more detail, set

$$y = \xi \quad , \quad z = \theta + \zeta$$

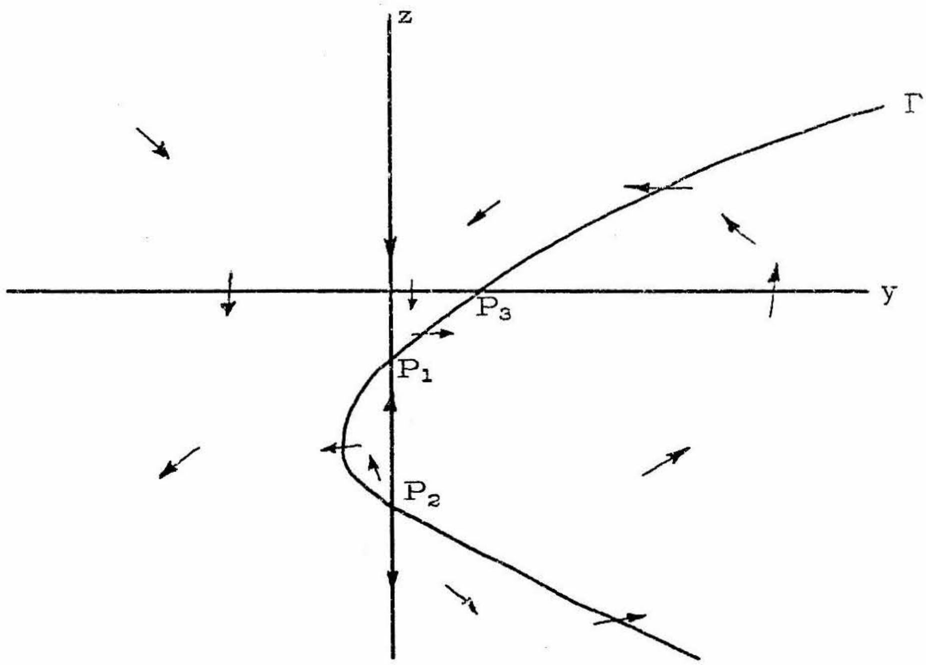


Figure 2 The field of tangent vectors for n > 2

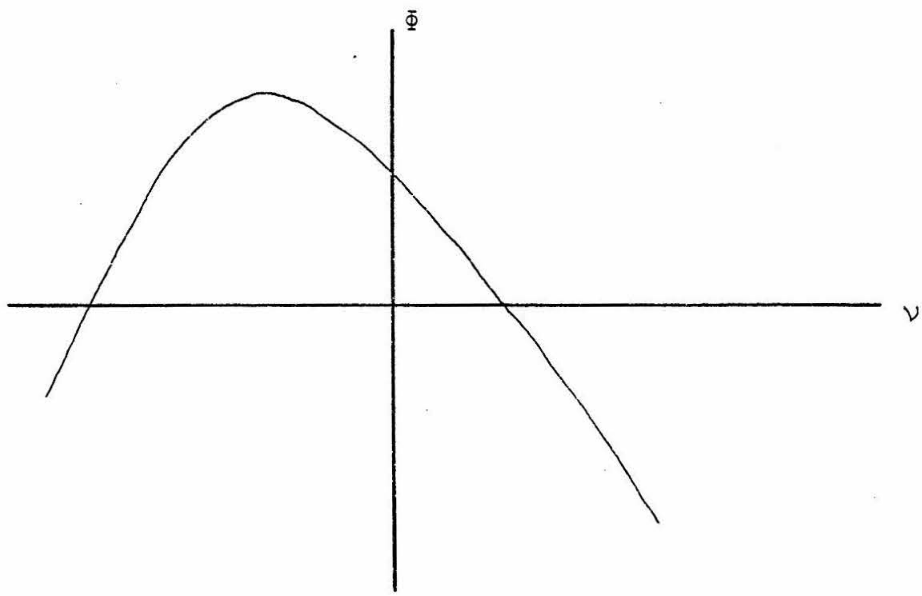


Figure 3 $\Phi(\nu)$ for $\beta > 1$, $n > 2$

We obtain the linearized form of equations (1.15)

$$\dot{\xi} = -\beta\theta\xi$$
$$\dot{\zeta} = \xi + (n-2)\zeta$$

which has solutions

$$\left.\begin{aligned} \xi &= a\,e^{-\beta\theta x} \\ \zeta &= b\,e^{(n-2)x} - \frac{a}{\beta\theta + n-2}\,e^{-\beta\theta x} \end{aligned}\right\} \quad (1.20)$$

Since P_2 is an unstable node, trajectories approach P_2 for $x \to -\infty$. This, and the relation $-\beta\theta > n-2 > 0$ imply that trajectories are tangent to the ζ (or z) axis unless $b = 0$, in which case there exist two trajectories tangent to the line $\xi + (\beta\theta + n-2)\zeta = 0$.

For P_3, $y_0 = \gamma\theta$, $z_0 = 0$ and the characteristic equation is

$$\ell^2 - (\gamma+\theta)\ell + \beta\gamma\theta = 0$$

which has roots

$$\left.\begin{aligned} \ell_1 &= \tfrac{1}{2}(\gamma+\theta) + \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4\beta\gamma\theta} \\ \ell_2 &= \tfrac{1}{2}(\gamma+\theta) - \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4\beta\gamma\theta} \end{aligned}\right\} \quad (1.21)$$

Now $4\beta\gamma\theta = -4(\mu+2)\theta > 0$ so that the real parts of ℓ_1 and ℓ_2 always have the same sign; consequently P_3 is always an attractor (i.e., a spiral point or a node). Furthermore, $\gamma+\theta < 0$ so that P_3 is a stable attractor.

Consider first when P_3 is a spiral point, i.e.,

$$(\gamma+\theta)^2 - 4\beta\gamma\theta < 0 \quad .$$

This relation is equivalent to the following inequality in terms of the original parameters β, μ and n:

$$\Phi(\mu+2) < 0 \qquad (1.22)$$

where

$$\Phi(\nu) \equiv -4(\beta-1)[\nu + \tfrac{1}{2}\beta(n-2)]^2 + \beta^3(n-2)^2 \qquad (1.23)$$

Figure 3 shows the qualitative behavior of $\Phi(\nu)$ for $\beta > 1$, $n > 2$. In the region $\nu > 0$, Φ decreases monotonically, so for given β and n, we have a spiral for all $\mu \geq \mu^*$ if and only if $\Phi(\mu^*+2) < 0$. In particular, P_3 is a spiral point for all $\mu \geq 0$ if $\Phi(2) < 0$, and this can be shown to be equivalent to

$$\beta^2(n-2)(n-10) + 8\beta(n-4) + 16 < 0 \tag{1.24}$$

For the membrane problem originally proposed, $n = 4$ and $\beta = 3$; it is a simple matter to verify that (1.24) is indeed satisfied for these values of the parameters. More generally, for given values of $n > 2$ and $\beta > 1$ there exists a value μ^* such that for all $\mu > \mu^*$, P_3 is a spiral point.

Next we consider the structure of the phase plane near P_3 when it is a node. (It will turn out that the local structure about P_3 critically determines the number of equilibrium solutions of the membrane problem.) Set $y = \gamma\theta + \xi$, $z = \zeta$; the linearized form of equations (1.15) is now

$$\begin{pmatrix} \dot{\xi} \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma\theta \\ 1 & \gamma+\theta \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} .$$

Eigenfunctions

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = e^{\ell x} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

satisfy

$$\begin{pmatrix} -\ell & -\beta\gamma\theta \\ 1 & \gamma+\theta-\ell \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $c_1 = [\ell-(\gamma+\theta)]c_2$. Note from equations (1.21) that

$$\ell_1 - (\gamma+\theta) = -\ell_2$$

$$\ell_2 - (\gamma+\theta) = -\ell_1$$

The linearized theory in the neighborhood of P_3 thus provides the approximate relations

$$\begin{pmatrix}\xi\\ \zeta\end{pmatrix} \doteq a e^{\ell_1 x}\begin{pmatrix}-\ell_2\\ 1\end{pmatrix} + b e^{\ell_2 x}\begin{pmatrix}-\ell_1\\ 1\end{pmatrix}$$

for some constants a, b for a given trajectory. Since $\ell_2 < \ell_1 < 0$, we can conclude from this that in a neighborhood of P_3 all trajectories are tangent to the line $y - \gamma\theta + \ell_2 z = 0$, except for one pair of trajectories which is tangent to the line $y - \gamma\theta + \ell_1 z = 0$.

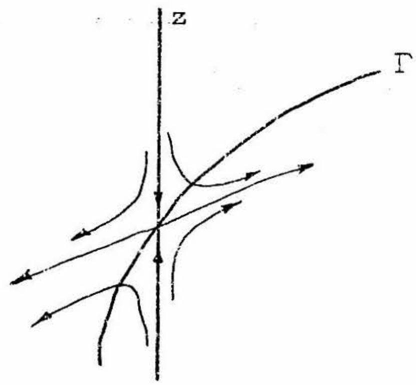
In the special case that $\ell_2 = \ell_1 < 0$, there is only one eigen-vector, and the general theory for critical points shows that P_3 is again an improper node with all trajectories tangent to the same line.

In figure 4 we summarize the above results about the behavior of trajectories in a neighborhood of each of the finite critical points. Some reference to figure 2 may also be helpful. We note in passing that the tangent line to Γ at P_3 is $y - \gamma\theta + (\gamma+\theta)z = 0$. A justification that the linearized theory does in fact accurately describe the behavior of the solutions of the full nonlinear equations in neighborhoods of each of the critical points can be found in a standard reference, such as Coddington and Levinson [3].

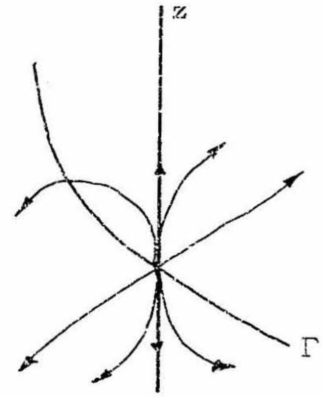
By combining the results of figures 2, 4a, and 4b we can ascertain that the phase plane for $y \leq 0$ has the qualitative form of figure 5, regardless of the behavior at P_3. We now concentrate on completing the phase plane for $y > 0$. First we show that no limit cycles exist. Introduce $\eta = \log y$ and compute

$$\left.\begin{aligned} \dot{\eta} &= \dot{y}/y = -\beta z \equiv F(\eta, z) \\ \dot{z} &= y - (z-\gamma)(z-\theta) = e^{\eta} - (z-\gamma)(z-\theta) \equiv G(\eta, z) \end{aligned}\right\} \quad (1.25)$$

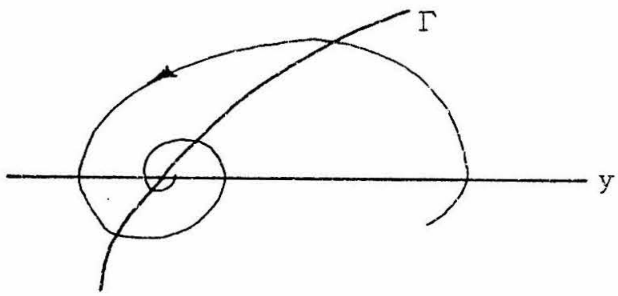
Suppose a limit cycle exists (necessarily it must lie entirely within the region $y > 0$). Then its image in the η, z plane must also be a simple



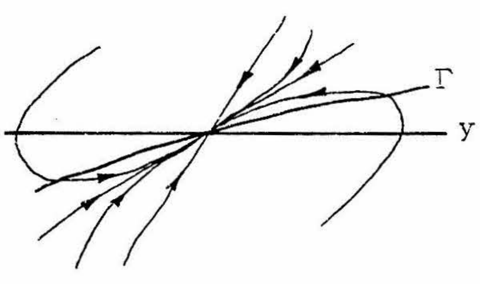
(a) near P_1



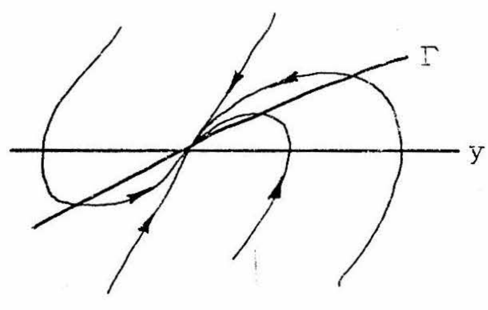
(b) near P_2



(c) near P_3, spiral case



(d) near P_3, node with distinct eigenvalues



(e) near P_3, node with a double eigenvalue

Figure 4 Phase plane structure near the critical points

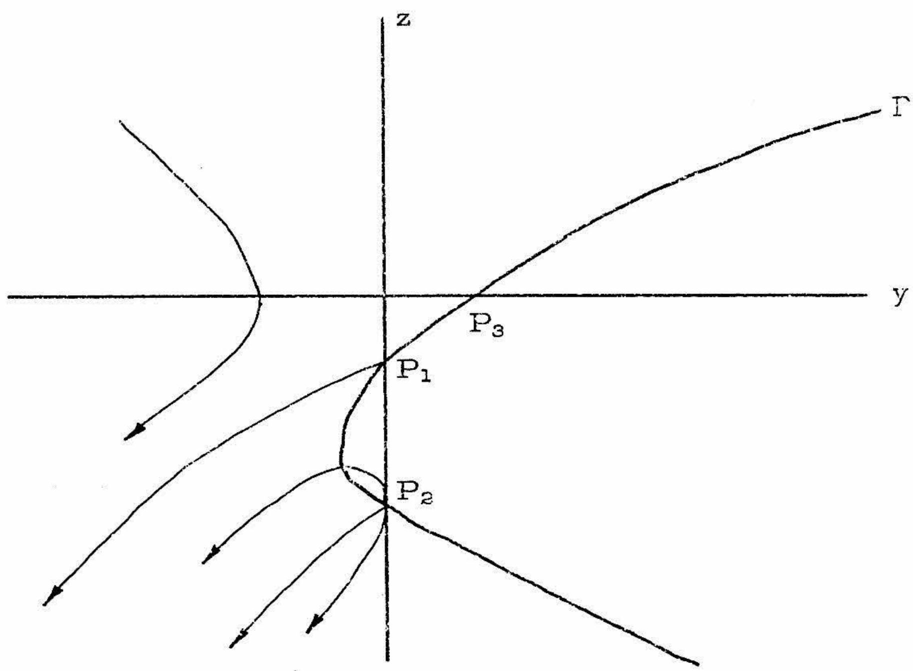


Figure 5 Phase plane for $y \leq 0$

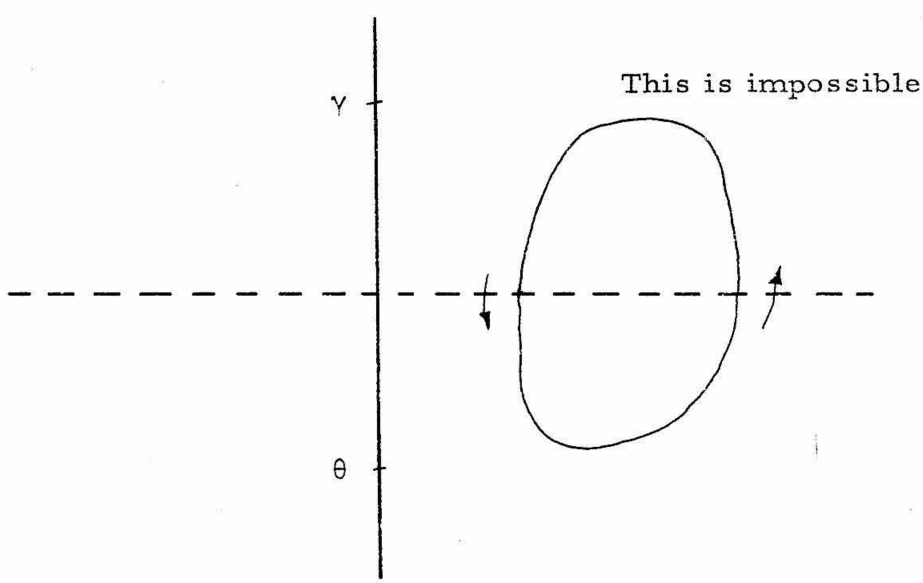


Figure 6 Non-existence of limit cycles

closed curve, say C, inclosing an area S. We calculate, integrating over one period in x:

$$0 = \int_C (\dot{\eta}\ddot{z} - \dot{z}\ddot{\eta})dx = \int_C (\dot{\eta}dz - \dot{z}d\eta) = \int_C (Fdz - Gd\eta)$$

$$= \int_S \nabla(\eta, z) \cdot (F, G)d\eta dz \qquad \text{(using Green's theorem)}$$

$$= \int_S (-2z + \gamma + \theta)d\eta dz$$

Consequently the limit cycle cannot lie wholly above nor wholly below the line $z = \frac{1}{2}(\gamma+\theta)$. (This is true in either phase plane, since η is merely a rescaling of the y axis.) However from either equation (1.15) or figure 2 we conclude that $\dot{z} > 0$ along the half-line $y > 0$, $z = \frac{1}{2}(\gamma+\theta)$. This means that a trajectory can only cross this line in one sense (Cf. figure 6). It follows that no limit cycles exist.

Now consider the (unique) trajectory eminating from the saddle point P_1 into the region $y > 0$. From the vector field we see that $\dot{y} > 0$, $\dot{z} > 0$ initially. Now either this trajectory intersects the y axis in some finite x, or else $z < z_0 \leq 0$ for all x, in which case $z \to z_0$ as $x \to +\infty$. But then we must also have $\dot{z} \to 0$ and $y \to +\infty$. This is inconsistent with $\dot{z} = y - (z-\gamma)(z-\theta)$, so in fact this trajectory must intersect the y axis at a finite point (necessarily to the right of the spiral point P_3). From the vector field (cf. figure 2) we can see that the trajectory then moves upward to the left, intersects Γ in the region $0 < z$, continues downward to the left, intersects the y axis in $0 < y < \gamma\theta$, and then continues downward to the right, intersecting Γ in the region $\gamma < z < 0$. Finally, the trajectory again moves upward to the right. We can now argue that the trajectory is bounded and, in the absence of limit cycles, necessarily spirals into P_3.

If we take any point on Γ in the region $y > 0$, $z < \theta$ and follow the trajectory through such a point as $x \to +\infty$, exactly analogous arguments apply. If we follow such a trajectory for $x \to -\infty$, the vector field forces it directly into the unstable node P_2. This discussion and the results illustrated in figures 2 and 4 permit us to construct the phase plane illustrated in figure 7 when P_3 is the spiral point. When P_3 is a node, the plane remains qualitatively the same except in the neighborhood of P_3, where trajectories either tend directly into the critical point or spiral about it at most a finite number of times before tending into it.

Two possible families of trajectories may exist which have not yet been discussed, their locations are indicated in figure 7 by region I and region II. First we consider a point in region I. Necessarily as $x \to +\infty$ the trajectory through such a point tends to P_3. We have not argued, however, that such a trajectory intersects the curve Γ (in the region $y > 0$, $z > 0$) as $x \to -\infty$. If this does not occur, then $y \to +\infty$ and $z \to +\infty$ as $x \to -\infty$. Similarly, the trajectory through a point in region II must tend to P_2 as $x \to -\infty$. We have not argued that such a trajectory intersects Γ (in the region $y > 0$, $z < \theta$) as $x \to +\infty$. Should this not occur, then $y \to +\infty$ and $z \to -\infty$ as $x \to +\infty$. Fortunately, it will turn out that these two possibilities are irrelevant to the boundary value problem posed, and so further investigation is unnecessary.

We now search for trajectories in the full plane satisfying boundary conditions (1.16) and (1.17), as well as the derived condition (1.18). Consider first (1.18), viz. $y(-\infty) = 0$. This eliminates all trajectories except the two eminating from the saddle point P_1 and those emanating from the (unstable) node P_2. In particular, this

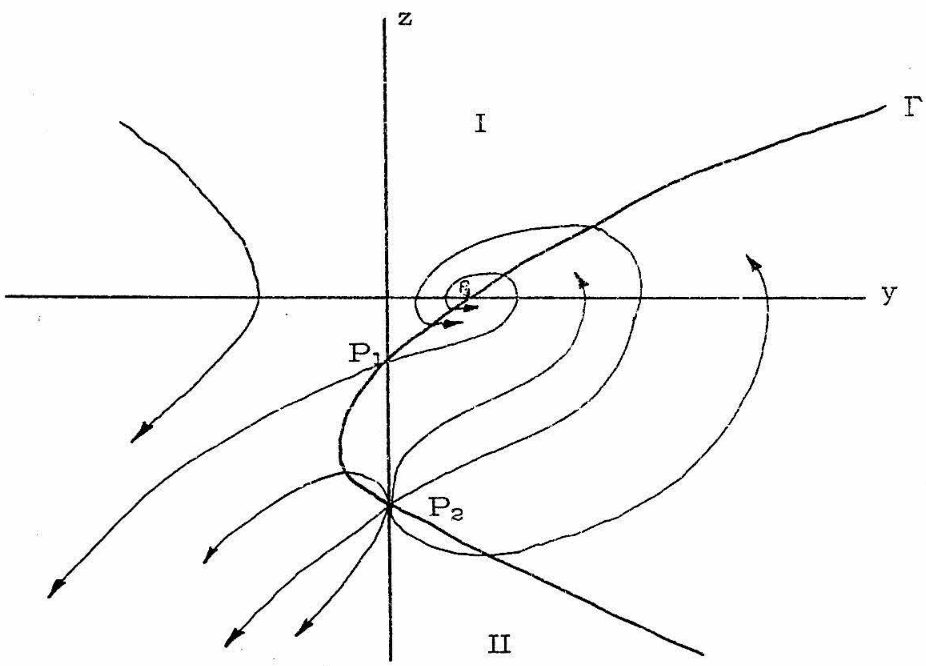


Figure 7 The phase plane

eliminates the possible trajectories in region I mentioned above.

Consider trajectories tending to P_2 as $x \to -\infty$. Along them $z - \gamma \to \theta - \gamma \neq 0$, so meeting boundary condition (1.16) is equivalent to satisfying $e^{-x(\gamma+2)}|y|^{-1/\beta} < \infty$ as $x \to -\infty$. We derived in equation (1.20) the asymptotic relation $y = \xi \sim a e^{-\beta\theta x}$ near P_2. The trajectories satisfying $a = 0$ locally are, in fact, segments of the z axis with $y \equiv 0$; hence they cannot satisfy (1.17). Near P_2 boundary condition (1.16) thus reduces to

$$e^{-(\gamma+2)x}\left(e^{-\beta\theta x}\right)^{-1/\beta} = e^{-(\gamma+2)x+\theta x}$$

$$= e^{-(\gamma+2)x + (\gamma-n+2)x} = e^{-nx} < \infty \text{ as } x \to -\infty$$

But this is not possible for $n > 2$, so that no trajectory tending to P_2 as $x \to -\infty$ can satisfy (1.16). In particular, this also eliminates the possible trajectories in region II described above.

Our only hope for a solution, then, lies with the two trajectories emanating from the saddle point P_1. With $y = \xi$, $z = \gamma + \zeta$ the linearized equations for (ξ, ζ) are

$$\dot{\xi} = -\beta\gamma\xi = (\mu+2)\xi$$

$$\dot{\zeta} = \xi + (\theta-\gamma)\zeta = \xi - (n-2)\zeta$$

with solutions

$$\xi = a e^{(\mu+2)x} = a e^{-\beta\gamma x}$$

$$\zeta = b e^{-(n-2)x} + \frac{a}{\mu+n} e^{(\mu+2)x}$$

In the parameter domain $n > 2$ trajectories tending to $\xi = \zeta = 0$ as $x \to -\infty$ must have $b = 0$. Thus as $x \to -\infty$

$$y \sim a e^{(\mu+2)x} = a e^{-\beta\gamma x}$$

$$z \sim \gamma + \frac{a}{n+\mu} e^{(\mu+2)x}$$

and

$$e^{-(\gamma+2)x}|y|^{-1/\beta}|z-\gamma| \sim e^{-(\gamma+2)x}e^{\gamma x}e^{(\mu+2)x} = e^{\mu x} < \infty,$$

as $x \to -\infty$ for $\mu \geq 0$. Thus these two trajectories do indeed satisfy the boundary condition at $x = -\infty$ (corresponding to $r = 0$).

In figure 8 we graph these two trajectories when P_3 is a spiral point. In figures 9 a, b, and c, we graph some typical examples when P_3 is a node. In these latter cases the trajectory in the right half-plane may tend directly into P_3 or may spiral about (at most) a finite number of times before tending into P_3. The analysis presented here is insufficient to determine precisely how many times P_3 is encircled in the case of a node.

The remaining boundary condition, (1.17), is trivial to satisfy; it merely requires that the solution trajectory intersect the line $y = \alpha\lambda^{\beta}$ when $x = 0$. If a trajectory intersects the line $y = \alpha\lambda^{\beta}$, then the translation invariance of the autonomous system (1.15) implies that a solution exists for which $x = 0$ at the point of intersection.

The question of the number of solutions to equation (1.1) with boundary conditions (1.2) and (1.3) is thus reduced to counting the number of intersections of the trajectories emanating from P_1 with the line $y = \alpha\lambda^{\beta}$ (each distinct point of intersection corresponds to a different transformation back to the independent variable $0 \leq r \leq 1$ and hence a distinct solution).

Regardless of whether P_3 is a spiral point or a node, precisely one solution exists for every value of $\alpha\lambda^{\beta} < 0$. We noted earlier in the chapter that when $\lambda = 0$ the unique solution $u(r) \equiv 0$ exists. It is equally simple to treat the linear problem resulting when $\alpha = 0$ to get

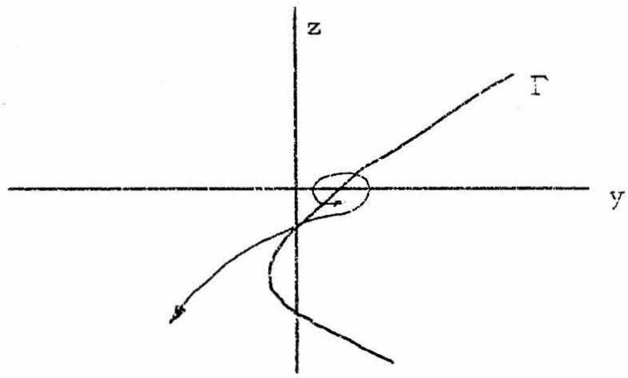
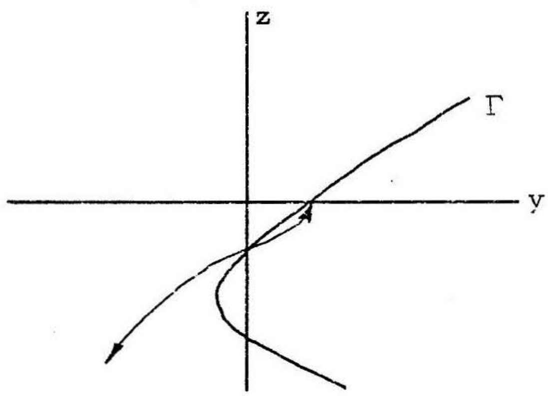
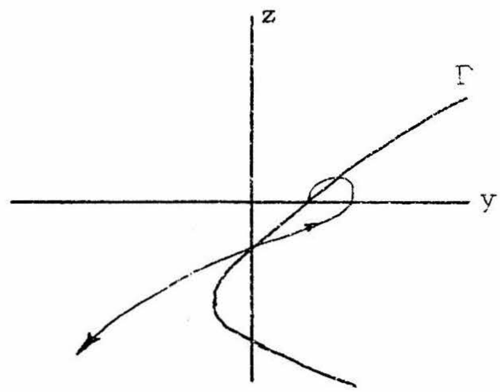


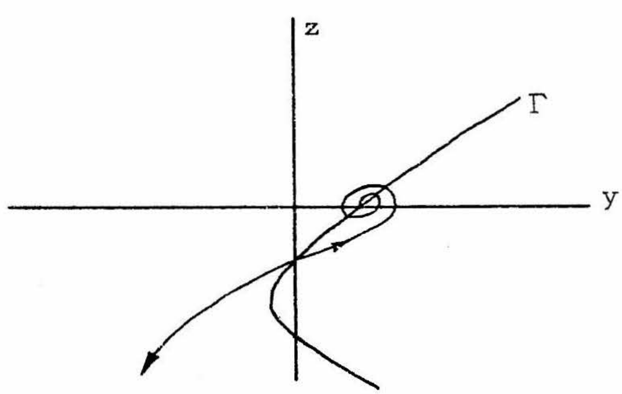
Figure 8 Trajectories satisfying (1.16) and (1.18), P_3 is a spiral point



(a)



(b)



(c)

Figure 9 Trajectories satisfying (1.16) and (1.18), P_3 is a node the unique solution

$$u(r) = \frac{\lambda}{(\mu+2)(\mu+n)} (r^{\mu+2} - 1) \quad .$$

Hence we see that even when $\alpha\lambda^{\beta} = 0$ a unique solution always exists.

The number of solutions for $\alpha\lambda^{\beta} > 0$ is of considerably more interest--particularly when P_3 is a spiral point (i.e. equation (1.22) is satisfied). Inspection of figures 8 and 11 shows that there exists a sequence of numbers $\{m_j\}$

$$m_0 = -\infty < 0 < m_2 < m_4 < \ldots < m_\infty < \ldots < m_3 < m_1 < m_{-1} = +\infty$$

such that

for $\alpha\lambda^{\beta} = m_j$	j solutions exist $j = 1, 2, \ldots \infty$
for $m_{2k} < \alpha\lambda^{\beta} \leq m_{2k+2}$	2k+1 solutions exist $k = 0, 1, 2, \ldots$
for $m_{2k+1} < \alpha\lambda^{\beta} < m_{2k-1}$	2k solutions exist $k = 0, 1, 2, \ldots$

In figure 10 these results are summarized graphically. It is evident that m_∞ is the abscissa of P_3, i.e. for $\alpha\lambda^{\beta} = m_\infty = \gamma\theta$ a countable infinity of solutions exists.

We remark in passing that it was found earlier that for the flat membrane P_3 is a spiral point for all values of μ. Consequently, figure 10 depicts the distribution of equilibrium configurations for various edge thrusts and pressure distributions.

When P_3 is a node the situation for $\alpha\lambda^{\beta} > 0$ is somewhat less dramatic and, unfortunately, more vague. As can be seen from figures 9a, b, and c, the separatrix connecting P_1 and P_3 can tend directly into P_3 without any spiral behavior, or it can spiral a finite number of times before tending into P_3. Consequently, a similar sequence of m_j can be constructed, with the difference that only a finite number of m_j exist and the number of solutions for arbitrary

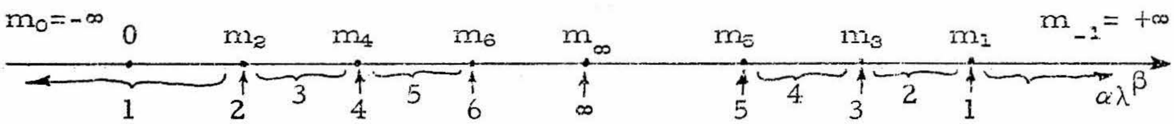


Figure 10 Number of solutions for each value of $\alpha\lambda^\beta$, P_3 is a spiral

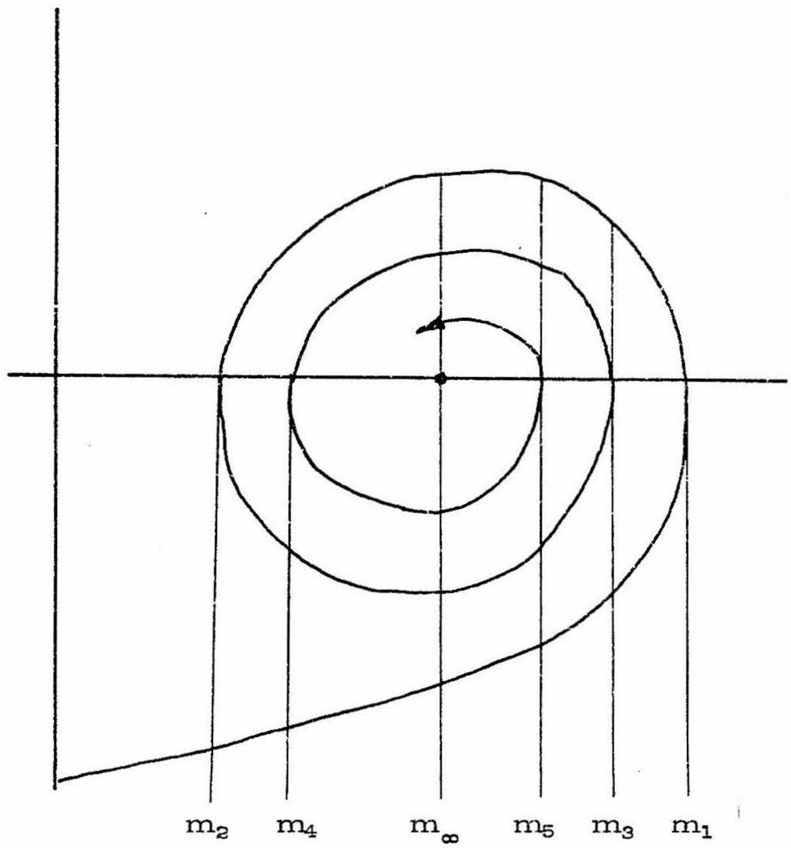


Figure 11 Location of the m_j

values of $\alpha\lambda^{\beta}$ is bounded for given values of β, n, and μ.

<u>The case n = 2</u>

In this case $\theta = \gamma < 0$ (cf. equation (1.9)) and the critical points P_1 and P_2 coalesce into a single point, say P^*, with coordinates $y = 0$, $z = \gamma$. The characteristic exponents at P^* are $\ell_1 = \mu + 2 > 0$ and $\ell_2 = 0$; since one of the exponents vanishes P^* is not an elementary critical point and a special analysis will be necessary.

The characteristic exponents at P_3 are given by equations (1.21) with $\theta = \gamma$; this yields

$$\ell_{1,2} = \gamma \mp \gamma\sqrt{-\beta+1}$$

Since $-\beta+1 < 0$, P_3 is always a spiral point, unlike the case $n > 2$. The previous argument that no limit cycles exist is still valid.

The tangent field is illustrated in figure 12. To discuss the behavior near P^* set $y = \xi$, $z = \gamma + \zeta$. Then

$$\left.\begin{aligned} \dot{\xi} &= -\beta\gamma\xi - \beta\xi\zeta \\ \dot{\zeta} &= \xi - \zeta^2 \end{aligned}\right\} \quad (1.26)$$

It is convenient to introduce local polar coordinates

$$\xi = r\cos\phi \quad , \quad \zeta = r\sin\phi$$

in terms of which (1.26) becomes

$$\begin{aligned} \dot{r} &= r(-\beta\gamma\cos^2\phi + \cos\phi\sin\phi) + r^2(-\beta\cos^2\phi\sin\phi - \sin^3\phi) \\ &\equiv r\,R(\phi) + r^2\rho(\phi) \\ \dot{\phi} &= (\cos^2\phi + \beta\gamma\cos\phi\sin\phi) + r(\beta-1)(\sin^2\phi\cos\phi) \\ &\equiv S(\phi) + r\,\sigma(\phi) \end{aligned} \quad (1.27)$$

The ζ axis is covered by two trajectories satisfying $\xi \equiv 0$, $\dot{\zeta} = -\zeta^2$, so that all other trajectories lie entirely within either the half-plane $\xi > 0$ or the half-plane $\xi < 0$. Consider trajectories tending to P^* as

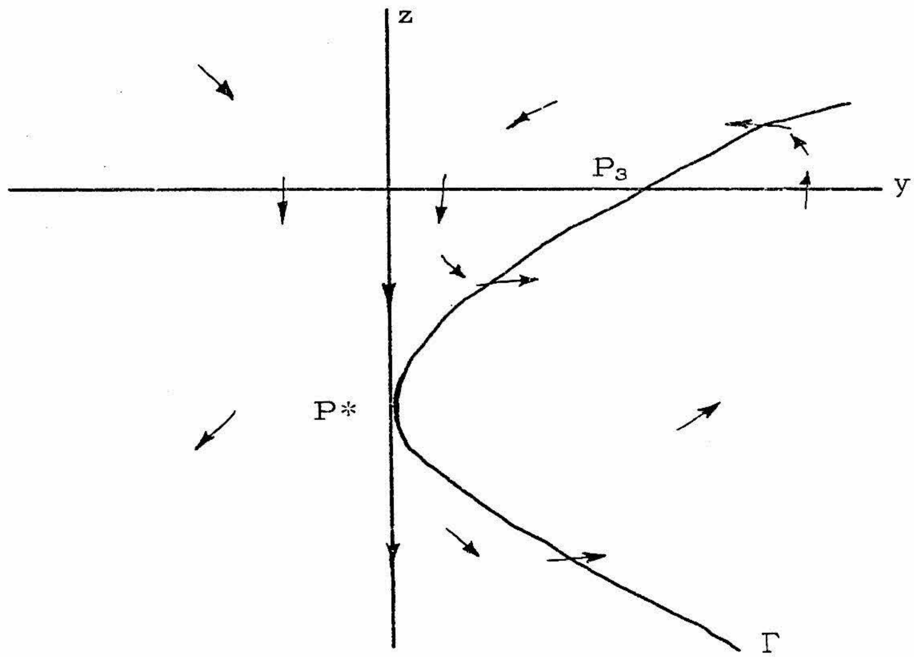


Figure 12 Tangent field, n = 2

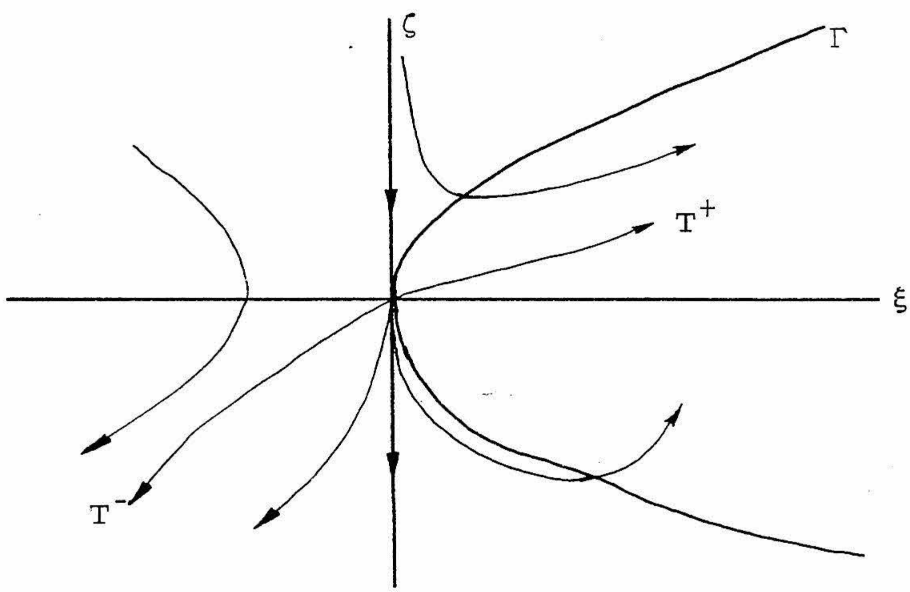


Figure 13 Phase plane in a neighborhood of P*

$x \to \pm\infty$. It is a standard result (e.g. see Hartman [5]) that as $r(x) \to 0$ either $|\phi(x)| \to \infty$ or else $\phi(x) \to \phi_0$, where ϕ_0 is a solution of $S(\phi_0) = 0$. The first alternative, that a trajectory spirals into P*, is impossible since no trajectory can cross the ζ axis. To find two possible angles of approach we solve

$$0 = S(\phi_0) = \cos\phi_0 (\cos\phi_0 + \beta\gamma \sin\phi_0)$$

to conclude either $\cos\phi_0 = 0$ or $\tan\phi_0 = -1/\beta\gamma = 1/(\mu+2)$. The first possibility yields $\phi_0 = \pm\pi/2$. We have already observed that the positive ζ axis is covered by a trajectory tending to P* as $x \to +\infty$ along $\phi_0 = \pi/2$, and the negative ζ axis is covered by a trajectory tending to P* as $x \to -\infty$ along $\phi_0 = -\pi/2$.

It is also a standard result that at least one trajectory exists which approaches P* along $\phi = \phi_0$ if, in addition to $S(\phi_0) = 0$, it is true that $S'(\phi_0) \neq 0$. Suppose that $\tan\phi_0 = -1/\beta\gamma$ and in addition

$$0 = S'(\phi_0) = -2\cos\phi_0 \sin\phi_0 + \beta\gamma\cos 2\phi_0$$

or $\tan 2\phi_0 = \beta\gamma$. The trigonometric identity

$$\tan 2\alpha = 2\tan\alpha \,/(1-\tan^2\alpha)$$

yields

$$\beta\gamma = (-2/\beta\gamma)/(1-1/\beta^2\gamma^2)$$

which is impossible for $0 < -1/\beta\gamma = 1/(\mu+2) < 1$. Consequently, taking into account the directions of the tangent field, we can conclude that there exists at least one trajectory T^+ in the half-plane $\xi > 0$ and at least one trajectory T^- in $\xi < 0$ such that $T^\pm$ tends to P* along the line $\tan\phi_0 = 1/(\mu+2)$ as $x \to -\infty$.

We can show that the trajectories $T^\pm$ are, in fact, unique by evoking a theorem due to K. A. Keil (see Sansone and Conti [7], page

257). Consider the system for x(t), y(t)

$$\dot{x} = kx + f(x, y)$$

$$\dot{y} = g(x, y)$$

where $k \neq 0$. If the origin is an isolated singular point of this system, if f, $g \in C^1$ in a neighborhood of the origin, and if in addition $f = g = f_x = f_y = g_x = g_y = 0$ at the origin, then there exist two and only two trajectories with equations $y = y(x)$ defined to the right and to the left of $x = 0$, respectively, tangent to the x axis at the origin. It is simple to show that equations (1.26) satisfy the hypotheses of this theorem under the transformation

$$x = \xi \quad , \; y = \xi + \beta\gamma\zeta$$

where the x axis corresponds to the line $\xi + \beta\gamma\zeta = 0$.

We are now in a position to construct the local phase portrait about P* shown in figure 13. A solution trajectory must satisfy (1.16), (1.17), and (1.18). Equation (1.18) implies that the trajectory emanates from P*. Consider first trajectories tangent to $\phi = -\pi/2$ as $x \to -\infty$. Set $\phi = -\pi/2 + \phi^*$. As $x \to -\infty$, $\phi^* \to 0$; hence $\cos\phi \sim \phi^*$ and $\sin\phi \sim -1$. From equation (1.27) we get asymptotically

$$\dot{r} \sim r\phi^* + r^2 \qquad (1.28a)$$

$$\dot{\phi}^* \sim -\beta\gamma\phi^* \qquad (1.28b)$$

Equation (1.28b) implies

$$\phi^* \sim e^{-\beta\gamma x} \quad \text{as } x \to -\infty \quad . \qquad (1.29)$$

Substituting this into (1.28a) we get the further asymptotic relation

$$\dot{r} \sim r^2 \qquad (1.30)$$

which implies

$$r \sim (x_0 - x)^{-1} \sim |x|^{-1} \quad \text{as} \quad x \to -\infty \qquad (1.31)$$

The result that r decays algebraically is consistent with the assumption that

$$r\phi^* = o(r^2)$$

which was made in deriving (1.30) from (1.28a). Also

$$\xi = r\cos\phi \sim r\phi^*$$

$$\zeta = r\sin\phi \sim -r \quad .$$

We can now use equations (1.29) and (1.31) to check whether or not trajectories tangent to $\phi = -\pi/2$ as $x \to -\infty$ satisfy (1.16).

$$e^{-(\gamma+2)x}(z-\gamma)\,|y|^{-1/\beta} = e^{-(\gamma+2)x}\zeta|\xi|^{-1/\beta}$$

$$\sim e^{-(\gamma+2)x}\, r\,(r\phi^*)^{-1/\beta}$$

$$\sim e^{-(\gamma+2)x}\, r^{(\beta-1)/\beta}\, e^{\gamma x}$$

$$\sim e^{-2x}\,|x|^{-(\beta-1)/\beta} \to \infty \quad \text{as } x \to -\infty \quad .$$

Thus such trajectories do not satisfy boundary condition (1.16).

Next consider trajectories $T^{\pm}$ which are tangent to the line $\xi = -\beta\gamma\zeta$ as $r \to 0$. Equations (1.26) yield

$$\dot{\xi} \sim -\beta\gamma\xi + \xi^2/\gamma$$

$$\sim -\beta\gamma\xi$$

$$\dot{\zeta} \sim -\beta\gamma\zeta - \zeta^2$$

$$\sim -\beta\gamma\zeta$$

which imply

$$\xi \sim e^{-\beta\gamma x}\,, \quad \zeta \sim e^{-\beta\gamma x} = e^{(\mu+2)x}$$

Checking equation (1.16), we calculate

$$e^{-(\gamma+2)x}\zeta\,|\xi|^{-1/\beta} \sim$$

$$e^{-(\gamma+2)x}e^{(\mu+2)x}e^{\gamma x} =$$

$$e^{\mu x} < \infty \quad \text{as } x \to -\infty$$

for $\mu \geq 0$. Thus trajectories $T^{\pm}$ meet this boundary condition.

We can now see that the case $n = 2$, for all values of $\beta > 1$ and $\mu \geq 0$, is identical with the case $n > 2$ when P_3 is a spiral point and can be summarized by figure 10.

<u>The case n = 1</u>

In this case $\theta = \gamma + 1$ and there are three subcases to consider, depending on whether $\theta < 0$ $(\gamma < -1)$, $\theta > 0$ $(-1 < \gamma < 0)$, or $\theta = 0$ $(\gamma = -1)$. In the first two cases there exist three distinct critical points (the same notation as before will be used); in the third case points P_2 and P_3 coalesce into P with coordinates $y = z = 0$. Only the salient features of the discussion will be mentioned since most of the details are similar to arguments used for $n \geq 2$.

For all three subcases, P_1 $(y = 0,\ z = \gamma)$ is an unstable improper node with characteristic exponents $\ell_1 = -\beta\gamma = \mu + 2 > 0$ and $\ell_2 = \theta - \gamma = 1$. All trajectories except one pair are tangent to the z axis as they approach P_1. Only the pair of trajectories tangent to the line $y = (\mu + 1)(z - \gamma)$ satisfy boundary condition (1.16). Now consider the subcase $\theta < 0$. Note that this is equivalent to $0 < -\beta\theta = -\beta(\gamma + 1)$ or

$$\beta < \mu + 2$$

Critical point P_2 $(y = 0,\ z = \theta)$ has characteristic exponents $\ell_1 = -\beta\theta > 0$ and $\ell_2 = \gamma - \theta = -1$ and is a saddle point. The separatrices are tangent to the lines $y = (1 - \beta\theta)(z - \theta)$ and $y = 0$ respectively. P_3 $(y = \gamma\theta,\ z = 0)$ is either a stable node or a stable spiral point, depending on the sign of the function $\Phi(\mu + 2)$ (evaluated with $n = 1$). Just as for the case $n > 2$, there exists a value of $\mu = \mu^*$ such that for $\mu > \mu^*$ P_3 is always a spiral point. The reader is referred back to figure 7 for the phase plane portrait, only with the labels for the points P_1 and P_2 exchanged. Existence and multiplicity are completely analogous to the case $n > 2$.

Next consider the subcase $\theta > 0$ ($\beta > \mu + 2$). Then the characteristic exponents of P_2 are $\ell_1 = -\beta\theta < 0$ and $\ell_2 = -1$, so it is a stable node. The characteristic exponents of P_3 are

$$\ell_+ = \tfrac{1}{2}(\gamma + \theta) + \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 + 4\theta(\mu+2)} > 0$$

$$\ell_- = \tfrac{1}{2}(\gamma + \theta) - \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 + 4\theta(\mu+2)} < 0$$

since $\theta(\mu + 2) > 0$. Hence P_3 is a saddle point. From an earlier discussion we know that the separatrix corresponding to ℓ_+ is tangent to the line $(y - \gamma\theta) + \ell_- z = 0$ and the separatrix corresponding to ℓ_- is tangent to the line $(y - \gamma\theta) + \ell_+ z = 0$. Taking into account the tangent field, we are in a position to construct the phase plane portrait in figure 14a. In figure 14b we have isolated the two trajectories that satisfy conditions (1.16) and (1.18). Note that for $\alpha\lambda^\beta \leq 0$ precisely one solution exists, and that there exists a number $m_1 > 0$ such that for $0 < \alpha\lambda^\beta < m_1$ two solutions exist, for $\alpha\lambda^\beta = m_1$, one solution exists, and for $\alpha\lambda^\beta > m_1$ no solutions exist.

Finally, consider the limiting case $\theta = 0$ ($\mu + 2 = \beta$). Critical points P_2 and P_3 coalesce into $\hat{P}$ with $y = z = 0$. Since this is not an elementary critical point, we introduce polar coordinates

$$y = r\cos\phi \quad , \quad z = r\sin\phi$$

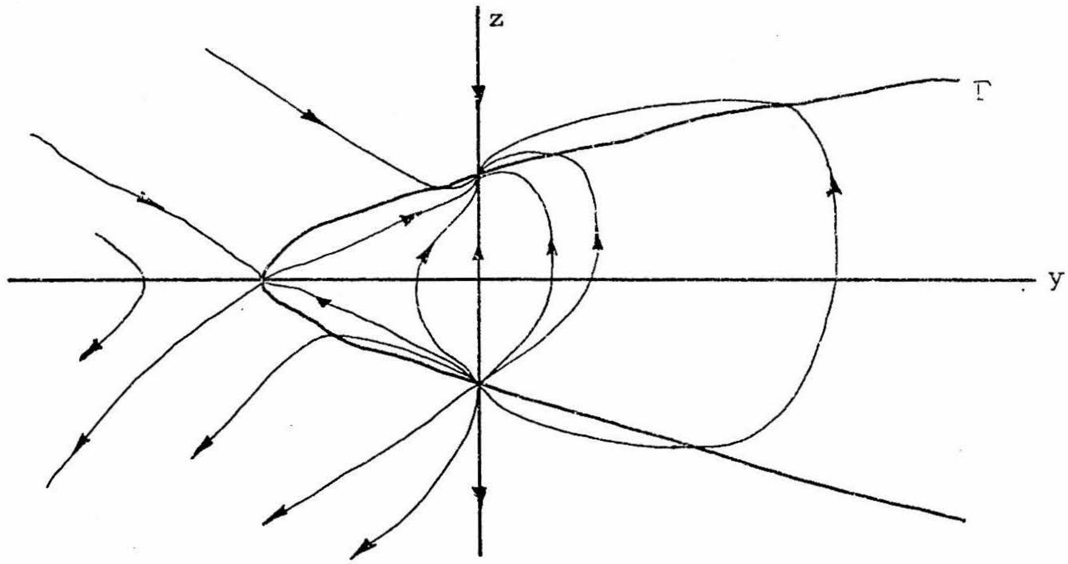
and find that equations (1.15) become

$$\dot{r} = r(\cos\phi\sin\phi + \gamma\sin^2\phi) - r^2(\beta\cos^2\phi\sin\phi + \sin^3\phi)$$

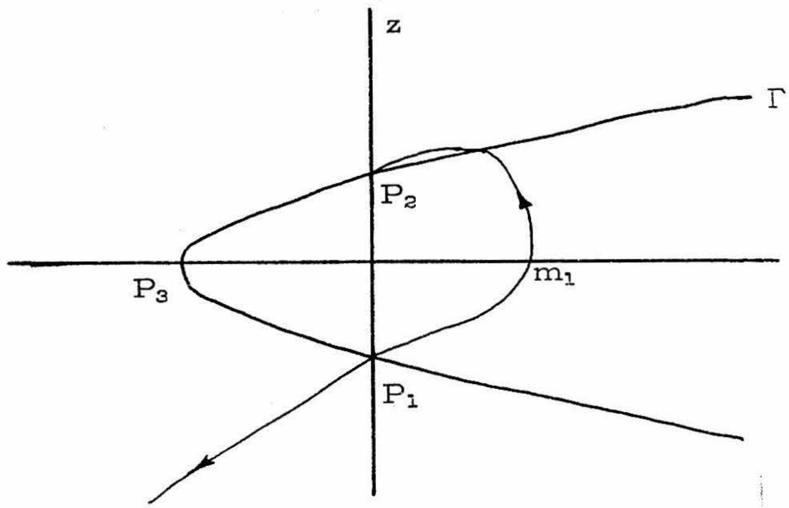
$$\equiv r\,R(\phi) + r^2\rho(\phi)$$

$$\dot{\phi} = (\cos^2\phi + \gamma\cos\phi\sin\phi) + r(\beta-1)\sin^2\phi\cos\phi$$

$$\equiv S(\phi) + r\,\sigma(\phi)$$



(a) The phase plane



(b) Solution separatrices only

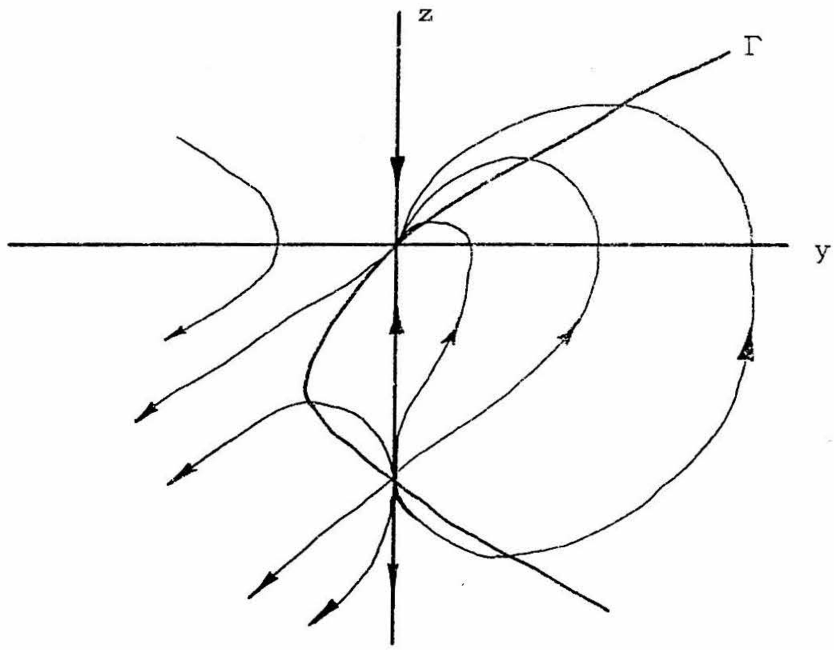
Figure 14 Phase plane for $n = 1$, $\theta > 0$

The z axis is covered by three trajectories, so $\hat{P}$ cannot be a spiral point. The possible angles of approach satisfying $S(\phi_0) = 0$ are

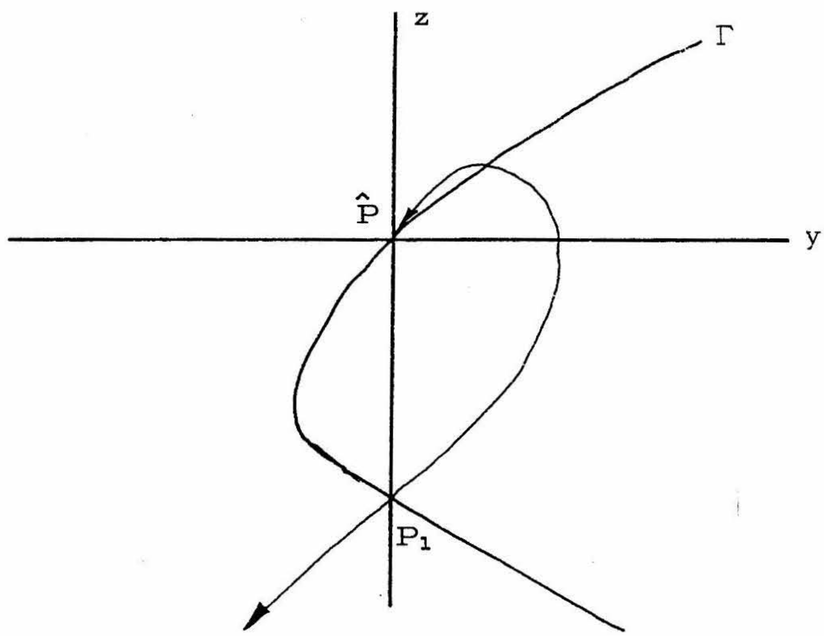
$$\phi_0 = \pm\pi/2 \qquad \text{and} \qquad \tan\phi_0 = -1/\gamma = 1$$

Using arguments similar to those for n = 2, it is possible to show that trajectories exist which tend to $\hat{P}$ for all four angles of approach, and that any trajectory which tends to $\hat{P}$ as $x \to -\infty$ does not satisfy boundary condition (1.16). In figure 15a we construct the phase plane portrait, taking the tangent field into consideration; in figure 15b we isolate the two trajectories satisfying (1.16) and (1.18). Note that the multiplicity of solutions is qualitatively the same as for the subcase $\theta > 0$.

Remark that for n = 1 and fixed β, there always exists a value μ^* such that for all $\mu > \mu^*$ a countable infinity of solutions exists for some value of $\alpha\lambda^\beta$ depending on μ.



(a) The phase plane



(b) Solution separatrices only

Figure 15 Phase plane for $n = 1$, $\theta = 0$

CHAPTER 2: INITIALLY CURVED MEMBRANES

Recall from the introduction that the symmetric deformation of a circular membrane can be described by

$$\frac{d}{dr} \quad r^3 \frac{du}{dr} \quad + \lambda^3 \frac{G}{(1-\mu)^2} = \lambda \, Br \phi^2 \qquad (C.6)$$

In chapter 1 we considered the situation when the membrane is initially flat, $\phi \equiv 0$, and is subjected to a pressure of the form $p = p_{max} r^{\mu/2}$. Recognizing $r^{-3}\frac{d}{dr}(r^3 \frac{du}{dr})$ as a spherically symmetric Laplacian, we generalized the problem to

$$\frac{1}{r^{n-1}} \frac{d}{dr} \quad r^{n-1} \frac{du}{dr} \quad + \lambda^{\beta} r^{\mu} (1-\alpha u)^{1-\beta} = 0 \qquad (1.1)$$

for $\mu \geq 0$, $\beta > 1$, $\alpha \neq 0$.

We next consider the situation in which the pressure distribution remains the same, but the initial configuration of the membrane is given by $\phi = a\, r^b$, $b \geq 0$. The natural generalization of equation (C.6) is

$$\frac{1}{r^{n-1}} \frac{d}{dr} \quad r^{n-1} \frac{du}{dr} + \lambda^{\beta} r^{\mu} (1-\alpha u)^{1-\beta} = \frac{\lambda A r^{2b+1}}{r^{n-1}} = \lambda A r^{2(b+1)-n} \qquad (2.1)$$

with $A = Ba^2$.

The analysis of chapter 1 was possible because the transformations

$$x = \log r \qquad (1.5)$$

$$v(x) = (1-\alpha u) r^{\gamma} \qquad (1.6)$$

with

$$\gamma \equiv -(\mu + 2)/\beta \qquad (1.7)$$

yield a second order automonous equation. This is still true when the membrane is initially curved if the exponent b satisfies

$$2b = n - 4 - \gamma \quad .$$

Note that for the membrane problem $n = 4$, $\beta = 3$, and so

$$b = (\mu + 2)/6 > 0 \quad .$$

Consequently $\phi(0) = 0$ and the membrane is indeed flat at its apex. The response of the curved membrane is governed by

$$\frac{d^2 v}{dx^2} - (\gamma + \theta)\frac{dv}{dx} + \gamma\theta v - \alpha\lambda^{\beta} v^{1-\beta} + \alpha\lambda A = 0 \tag{2.2}$$

To facilitate analysis in the phase plane we introduce

$$w = \lambda/v \tag{2.3}$$

$$z = \frac{1}{v}\frac{dv}{dz} \tag{2.4}$$

Then equation (2.2) is equivalent to the system

$$\dot{w} = -wz \qquad \equiv f(w, z) \tag{2.5a}$$

$$\dot{z} = \alpha w^{\beta} - \alpha A w - (z-\gamma)(z-\theta) \equiv g(w, z) \tag{2.5b}$$

where differentiation with respect to x is indicated by a dot. The boundary and regularity conditions (1.10), (1.11), and (1.12) become

$$e^{-(\gamma+2)x} w^{-1} |z-\gamma| < \infty \text{ as } x \to -\infty \tag{2.6}$$

$$w = \lambda \quad \text{at} \quad x = 0 \tag{2.7}$$

$$w \to 0 \text{ as } x \to -\infty \tag{2.8}$$

In this chapter $A \neq 0$ so that the membrane is indeed curved, and $\alpha \neq 0$ so that the problem is truly nonlinear. Equation (2.1) can be solved exactly for a unique solution satisfying the appropriate boundary conditions when $\lambda = 0$, so we will also assume $\lambda \neq 0$.

In chapter 1 we found that the cases $n = 1$ and $n = 2$ required special analysis, although the results were not startlingly different than for $n > 2$. In this chapter we will limit ourselves to the case $n > 2$ insofar as the introduction of the new parameter A provides a multitude of possibilities to consider. In particular

$$\theta = \gamma - (n-2) < \gamma < 0$$

will always hold. Finally, to simplify the discussion, we will restrict β to integer values.

Consider the critical points of system (2.5). As before there exist points P_1 and P_2 given by $w = 0$, $z = \gamma$ and $w = 0$, $z = \theta$ respectively. Any remaining critical point is of the form $w = W$, $z = 0$ where W is a root of

$$p(w) \equiv \alpha w^{\beta} - \alpha A w - \gamma\theta = 0 \tag{2.9}$$

Although it is not possible, in general, to give explicit formulae for such roots, we can derive much qualitative information graphically. First note that $p(0) = -\gamma\theta < 0$ always. Suppose $0 = p'(\hat{w}) = \alpha\beta\hat{w}^{\beta-1} - \alpha A$ or

$$\hat{w}^{\beta-1} = A/\beta \tag{2.10}$$

If β is even there always exists precisely one point where $p' = 0$. If β is odd, there exist two values of $\hat{w}$ (equal in magnitude but opposite in sign) where $p' = 0$ when $A > 0$ and no such $\hat{w}$ when $A < 0$. Finally, $p'' = 0$ only at $w = 0$, but $p'(0) \neq 0$ so there exist no inflection points.

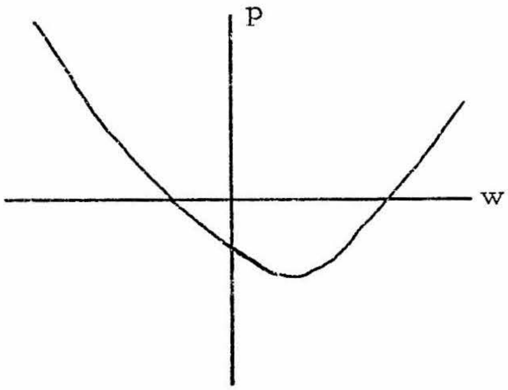
Using these simple facts we readily construct the various possible graphs of p(w) in figures 16. Note that when β is even we consider only $A > 0$, and when β is odd we consider only $\alpha > 0$. This is sufficient to give the qualitative behavior of the phase plane in all possible cases, because system (2.5) is invarient under

$$(w, z, \alpha, A) \rightarrow (-w, z, \alpha, -A)$$

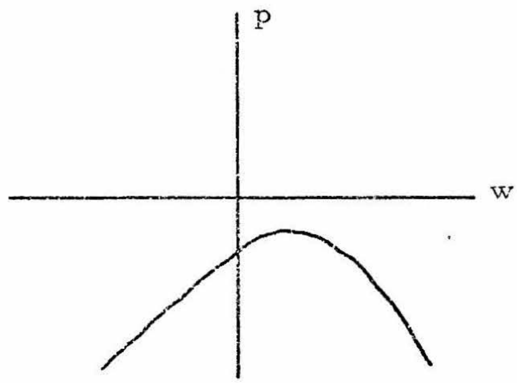
when β is even and

$$(w, z, \alpha, A) \rightarrow (-w, z, -\alpha, A)$$

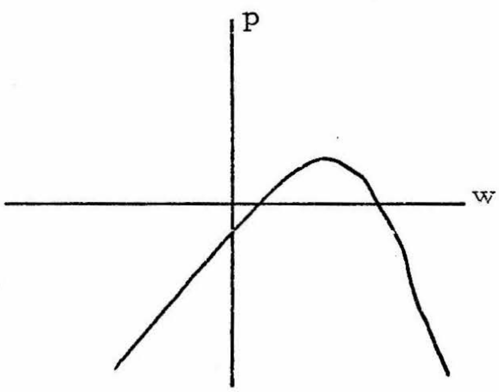
when β is odd. Consequently, the phase portraits not discussed can be obtained by simple reflection about the z axis.



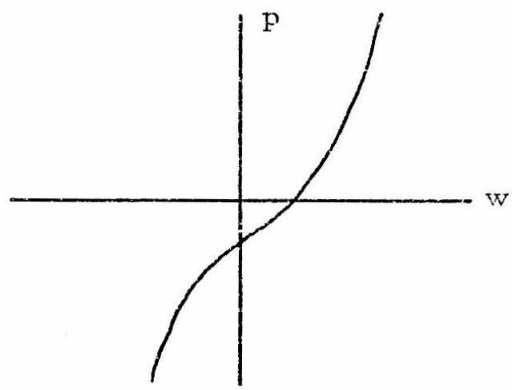
(a) β even, $A > 0$, $\alpha > 0$



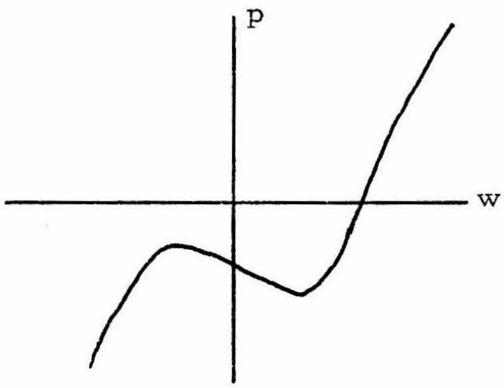
(b) β even, $1 \gg A > 0$, $\alpha < 0$



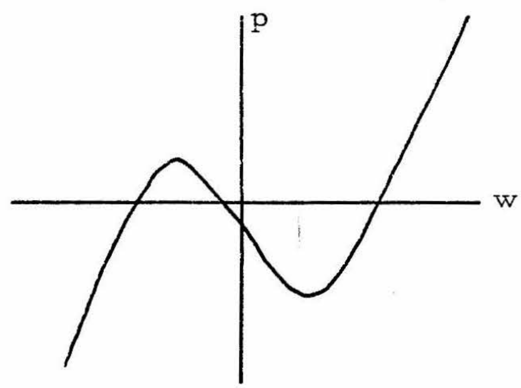
(c) β even, $A \gg 1$, $\alpha < 0$



(d) β odd, $A < 0$, $\alpha > 0$



(e) β odd, $1 \gg A > 0$, $\alpha > 0$



(f) β odd, $A \gg 1$, $\alpha > 0$

Figure 16 p(w) versus w

Note that for β even, $A > 0$ and $\alpha < 0$ there may exist no root, one, or two roots, depending on whether the maximum of p is negative, zero, or positive, respectively. Combining (2.9) and (2.10), the condition for two roots is

$$\alpha (A/\beta)^{\beta/(\beta-1)} - \alpha A(A/\beta)^{1/(\beta-1)} - \gamma\theta > 0$$

or simplifying

$$A > A^* \equiv \beta[\gamma\theta/\alpha(1-\beta)]^{(\beta-1)/\beta} \qquad (2.11)$$

Likewise, for $A = A^*$ one root exists, and for $0 < A < A^*$ no roots exist.

Similarly, for β odd, $A > 0$ and $\alpha > 0$ there may exist one, two, or three roots, depending on the sign of the local maximum of p. The local maximum occurs at $\hat{w} = -(A/\beta)^{1/(\beta-1)}$ and an analogous calculation shows that for

$A > A^*$	three roots exist
$A = A^*$	two roots exist
$0 < A < A^*$	one root exists

where A^* is again defined by (2.11).

Next consider the behavior locally about P_1 and P_2. Near P_1 we may set $w = \xi$, $z = \gamma+\zeta$ to obtain the linearized equations

$$\dot{\xi} = -\gamma\xi$$
$$\dot{\zeta} = -\alpha A\xi - (\gamma-\theta)\zeta = -\alpha A\xi - (n-2)\zeta$$

with solutions

$$\xi = a e^{-\gamma x}, \quad \zeta = b e^{-(n-2)x} + a_1 e^{-\gamma x}$$

where $a_1 = -a\alpha A/(n-2-\gamma) = a\alpha A/\theta$. Since $-\gamma > 0$ and $-(n-2) < 0$, this is a saddle point.

$$\zeta/\xi = (b/a)e^{\theta x} + \alpha A/\theta \quad .$$

We designate the trajectory emanating from P_1 into $w > 0$ by T^+ and by T^- for $w < 0$. If, near P_2, we set $w = \xi$, $z = \theta + \zeta$ and linearize, we obtain

$$\dot{\xi} = -\theta\xi$$

$$\dot{\zeta} = -\alpha A\xi - \zeta(\theta-\gamma) = -\alpha A\xi + (n-2)\zeta$$

with solutions

$$\xi = a e^{-\theta x}, \quad \zeta = b e^{(n-2)x} + a_1 e^{-\theta x}$$

where

$$a_1 = a\alpha A/(n-2+\theta) = a\alpha A/\gamma$$

Since $-\theta > 0$ and $n-2 > 0$, P_2 is an unstable node.

$$\zeta/\xi = (b/a)e^{\theta x} + \alpha A/\gamma$$

Graphs of the phase plane in a neighborhood of P_1 and P_2 are summarized in figure 17.

To study the behavior in the neighborhood of a critical point $w = W$, $z = 0$ (should one exist) we examine the characteristic exponents ℓ which satisfy

$$0 = \begin{vmatrix} f_w - \ell & f_z \\ g_w & y_z - \ell \end{vmatrix} = \begin{vmatrix} -\ell & -W \\ \alpha\beta W^{\beta-1} - \alpha A & \gamma + \theta - \ell \end{vmatrix}$$

or

$$\ell^2 - (\gamma+\theta)\ell + \alpha\beta W^{\beta} - \alpha A W = 0 \quad .$$

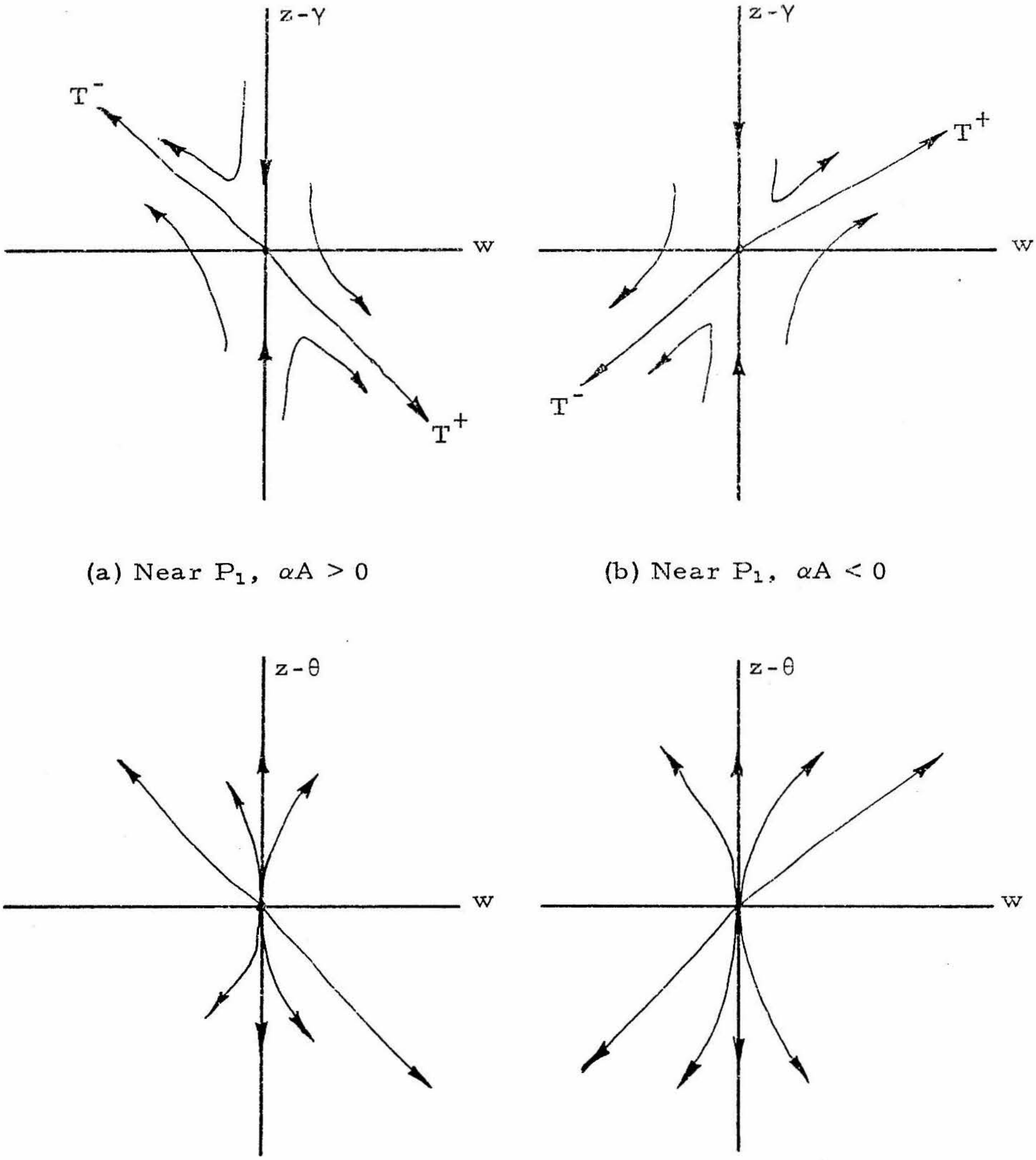
Using $p(W) = 0$ we simplify this to

$$\ell^2 - (\gamma+\theta)\ell + (\beta-1)\alpha A W + \beta\gamma\theta = 0$$

which yields the characteristic exponents

$$\ell_{\pm} = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4[(\beta-1)\alpha A W + \beta\gamma\theta]} \qquad (2.12)$$

To analyze this in more detail, we shall consider the cases of small amplitude and large amplitude initial configurations. First, however, there are several more relevant observations.



(a) Near P_1, $\alpha A > 0$ (b) Near P_1, $\alpha A < 0$

(c) Near P_2, $\alpha A > 0$ (d) Near P_2, $\alpha A < 0$

Figure 17 Local phase plane behavior near P_1 and P_2

Just as in the case of the initially flat membrane, the z axis is completely covered by trajectories satisfying

$$\dot{w} \equiv 0 \qquad \dot{z} = -(z-\gamma)(z-\theta) \quad .$$

Consequently, no trajectory can cross the z axis. In particular, if any limit cycles exist, they must lie entirely within the right half-plane or the left. By making the transformation $\eta = \log w$, an argument completely analogous to that of chapter 1 shows that any limit cycle must necessarily intersect the line $z = \frac{1}{2}(\gamma + \theta)$.

Arguments regarding the boundary conditions are also similar to those of chapter 1 and will not be repeated. Condition (2.8) requires that a solution trajectory emanate from P_1 or P_2. Condition (2.6) further requires that the solution trajectory be a separatrix emanating from the saddle point P_1, viz. T^+ or T^-. Condition (2.7) requires that we choose the separatrix in the right half-plane for $\lambda > 0$ and in the left half-plane for $\lambda < 0$.

Small amplitude pertubations, $0 < |A| \ll 1$

Case 1: β even, $A > 0$, $\alpha > 0$

From figure 16a it is clear that there exist two roots to p(w) and hence two critical points on the w axis. If W is one such root, expand

$$W = W_0 + AW_1 + A^2 W_2 + \ldots \tag{2.13}$$

and substitute into (2.9) to get

$$\alpha W_0{}^{\beta} - \gamma\theta = 0 \ , \quad \alpha\beta W_0{}^{\beta-1} W_1 - \alpha W_0 = 0$$

or

$$W_0 = \pm(\gamma\theta/\alpha)^{1/\beta} \tag{2.14a}$$

$$W_1 = \frac{1}{\beta}(\alpha/\gamma\theta)^{(\beta-2)/\beta} > 0 \tag{2.14b}$$

Thus to leading order in A

$$\ell_{\pm} = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4\beta\gamma\theta} \qquad (2.15a)$$

i.e., the result is the same as for the unperturbed case: either a stable spiral or a stable node. In the special case that $(\gamma+\theta)^2 - 4\beta\gamma\theta = 0$, we have

$$\ell_{\pm} = \tfrac{1}{2}(\gamma+\theta) \pm \sqrt{-(\beta-1)\alpha A W_0} \qquad (2.15b)$$

so that the unperturbed critical point, a node, becomes a spiral in the right half-plane and remains a node in the left half-plane.

Each intersection of a separatrix from P_1 with the line $w = \lambda$ corresponds to a distinct solution of the boundary value problem. In the unperturbed problem there was no difference between λ and $-\lambda$. In the perturbed problem, when the critical points are spirals, for example, there still exist values λ_{∞}^{+} and λ_{∞}^{-} such that there exist a countable infinity of solutions when λ attains either of these values, but it is no longer true that $\lambda_{\infty}^{+} = -\lambda_{\infty}^{-}$. However, for $|\lambda|$ small there still exists a unique solution to the boundary value problem, and for $|\lambda|$ large no solution exists. In the special case that $(\gamma+\theta)^2 - 4\beta\gamma\theta = 0$, an arbitrarily high multiplicity of solutions is possible only for $\lambda > 0$; for $\lambda < 0$ the multiplicity is bounded for given values of β, n, α, μ, and A. We note in passing that there cannot be a limit cycle about either of the critical points because, as is clear from the tangent field, the separatrices from P_1 bound the critical point s away from the line $z = \frac{1}{2}(\gamma + \theta)$. This will be true in all cases to be considered, for $A \gg 1$ and for $0 < |A| \ll 1$, so no further comment regarding the nonexistence of limit cycles will be made. In figure 18 the phase plane portrait is given. The locus of points where $\dot{w} = f(w, z) = 0$ is merely the w and z axes. The locus of points where $\dot{z} = g(w, z) = 0$ is denoted by Γ and

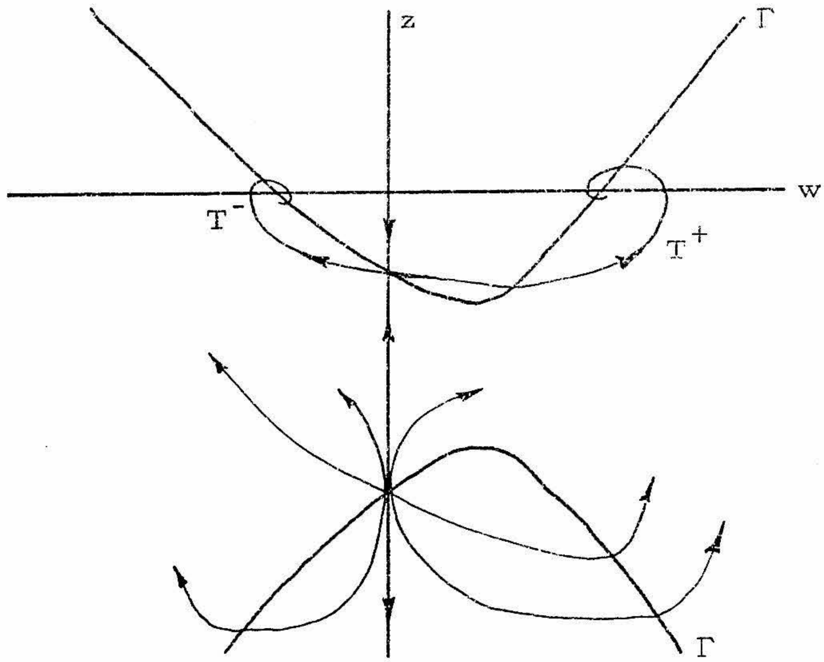


Figure 18 Phase plane for β even, 1 >> A > 0, $\alpha > 0$

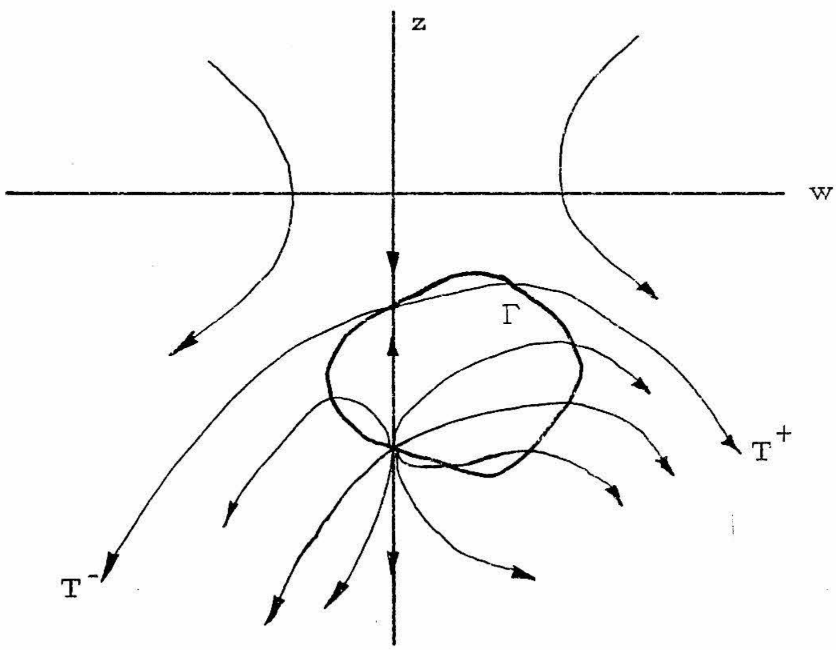


Figure 19 Phase plane for β even, 1 >> A > 0, $\alpha < 0$

has been included for clarity. We can also invert $g(w, z) = 0$ to get

$$z = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 + 4p(w)} \tag{2.16}$$

Reference to figure 16 and the fact that $z = \gamma$ or $z = \theta$ when $w = 0$ make it easy to sketch Γ. Recall that $z^+ = 0$ when $p = 0$.

Case 2: β even, $A > 0$, $\alpha < 0$

Refer to figure 16b. $p(w)$ has no roots, Γ is a simple closed curve enclosing the region of the phase plane where $g > 0$ and we can construct the phase plane portrait of figure 19. It is clear that precisely one solution exists for all values of λ.

Case 3: β odd, $A < 0$, $\alpha > 0$

From figure 16d we see that p has one root. This root may be given to leading order by (2.14a), except that it must be positive.

$$W_0 = (\gamma\theta/\alpha)^{1/\beta} > 0 \tag{2.17}$$

As in case 1, the nature of the corresponding critical point is unchanged from that for the unperturbed system, except when $(\gamma+\theta)^2 - 4\beta\gamma\theta = 0$. However, even in this instance we find that the point remains a stable node (Cf. equation (2.15b)). The phase plane is given in figure 20. For all $\lambda \leq 0$, a unique solution exists. In fact, there exist numbers $m_1 > m_2 > 0$ such that for $\lambda < m_2$ a unique solution exists, and for $\lambda > m_1$ no solutions exist. The situation for $m_2 \leq \lambda \leq m_1$ depends on whether $(w, 0)$ is a spiral or a node. Depending on this, the appropriate discussion of chapter 1 applies (e.g. Cf. figure 10).

Case 4: β odd, $A > 0$, $\alpha > 0$

This is essentially the same as case 3. From figure 16e we see that p has one root which, to leading order, is given by equation

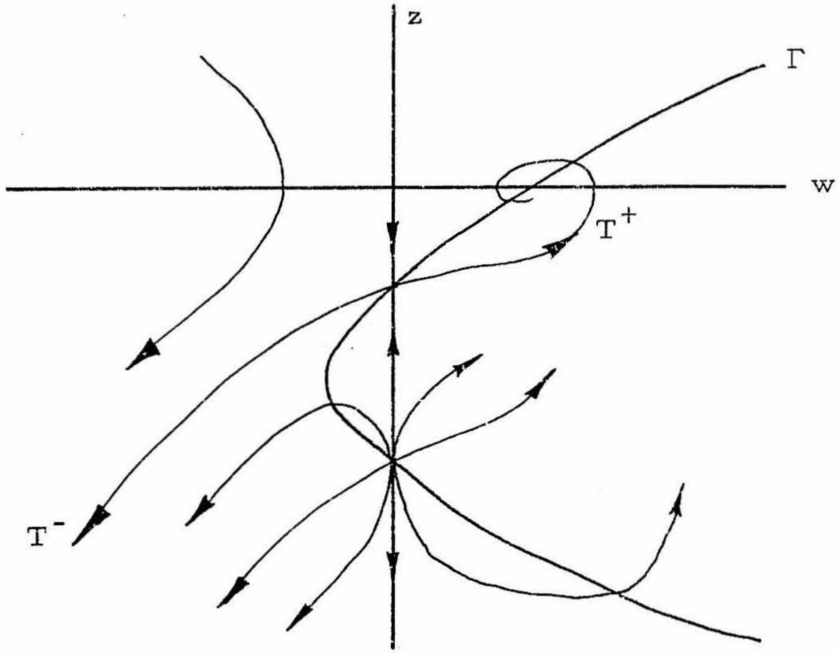


Figure 20 Phase plane for β odd, A < 0, α > 0

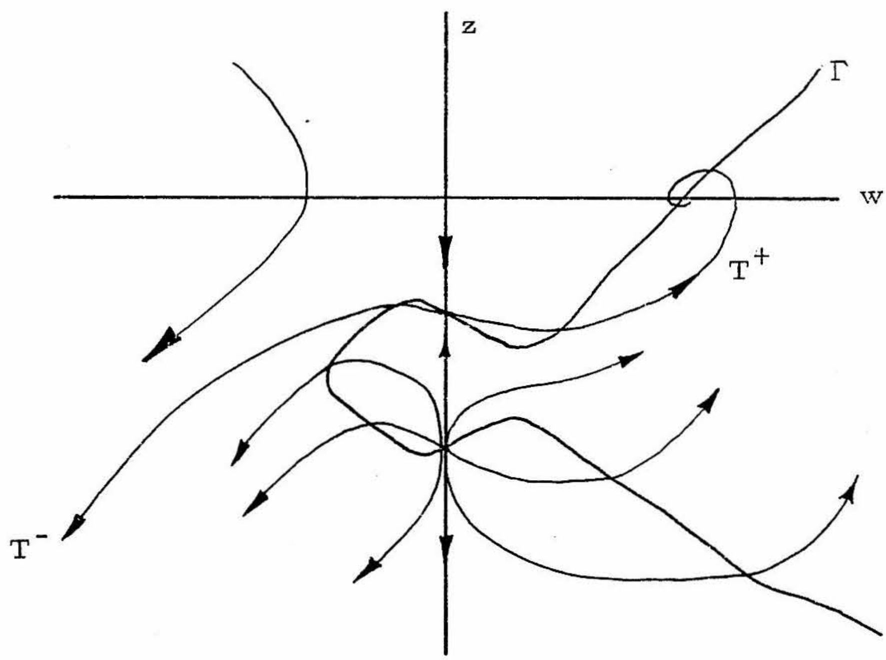


Figure 21 Phase plane for β odd, 1 >> A > 0, α > 0

(2.17). Although the phase plane, shown in figure 21, is slightly altered, the trajectories T^+ and T^-, the nature of the critical point $(w, 0)$, and the possible number of solutions for a given λ are qualitatively the same as for case 3. The only exception is that when $(\gamma+\theta)^2 - 4\beta\gamma\theta = 0$, the unperturbed critical point $(W, 0)$ is a node, but for $A > 0$ it becomes a spiral point (and hence there exist λ with arbitrarily many solutions).

Large amplitude pertubations, $|A| >> 1$

Case 1: β odd, $A > 0$, $\alpha > 0$

From figure 16f we see that $p(w)$ has three roots, say W^*, W^-, and W^+ where

$$W^- < W^* < 0 < W^+ \quad .$$

Because $p'(0) = -\alpha A \to -\infty$ as $A \to +\infty$ we expect $W^* \to 0$. The local extrema of p occur at $w = \pm(A/\beta)^{1/(\beta-1)}$ (Cf. equation (2.10)) which tend to $\pm\infty$ as $A \to +\infty$; consequently $W^+ \to +\infty$ and $W^- \to -\infty$.

To find the roots W we substitute

$$W = U A^{1/(\beta-1)} \tag{2.18}$$

into equation (2.9) to get

$$U^\beta - U = (\gamma\theta/\alpha)\,\varepsilon \tag{2.19}$$

where

$$\varepsilon = A^{-\beta/(\beta-1)} \tag{2.20}$$

and $0 < \varepsilon << 1$ for $A >> 1$. We now expand

$$U = U_0 + \varepsilon U_1 + \ldots \tag{2.21}$$

and substitute to get

$$U_0^\beta - U_0 = 0 \tag{2.22a}$$

$$\beta U_0^{\beta-1} U_1 - U_1 = \gamma\theta/\alpha \tag{2.22b}$$

We are only interested in the roots to leading order. Equation

(2.22a) yields solutions $U_0 = +1, -1$, and 0. The first two provide us with

$$W^+ \sim A^{1/(\beta-1)} \quad (2.23)$$

$$W^- \sim -A^{1/(\beta-1)} \quad (2.24)$$

If we take $U_0 = 0$ and substitute into (2.22b) we get

$$U_1 = -\gamma\theta/\alpha$$

and hence

$$W^* \sim -(\gamma\theta/\alpha)\epsilon A^{1/(\beta-1)} = -\gamma\theta/\alpha A \quad (2.25)$$

The characteristic exponents for each critical point are given by equation (2.12). If we substitute the values from (2.23), (2.24), and (2.25), we get, to leading order:

$$\text{at } (W^+, 0) \qquad \ell_\pm = \tfrac{1}{2}(\gamma+\theta) \pm i\sqrt{(\beta-1)\alpha A^{\beta/(\beta-1)}} \quad (2.26)$$

$$\text{at } (W^-, 0) \qquad \ell_\pm = \tfrac{1}{2}(\gamma+\theta) \pm \sqrt{(\beta-1)\alpha A^{\beta/(\beta-1)}} \quad (2.27)$$

$$\text{at } (W^*, 0) \qquad \ell_\pm = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}(\gamma-\theta) = \gamma \text{ or } \theta \quad (2.28)$$

Consequently, $(W^+, 0)$ is a stable spiral, $(W^-, 0)$ is a saddle point, and $(W^*, 0)$ is a stable improper node. In figures 22 abc the three compatible phase planes are illustrated. They differ in that the separatrix T^- may tend into $(W^-, 0)$, it may tend into $(W^*, 0)$, or it may be unbounded. For clarity, only T^+ is shown in the right half-plane. Note that the locus Γ where $g(w, z) = 0$ consists of two branches: a closed curve to the left and a parabolic-like curve to the right. If we only consider the vector field, it is conceivable that T^+ tends to infinity in the fourth quadrant without ever intersecting Γ. Were this not to occur, T^+ could not spiral into $(W^+, 0)$ as shown.

However, by examining the nature of the phase plane at infinity, we can verify that T^+ indeed intersects Γ. The method to be used was

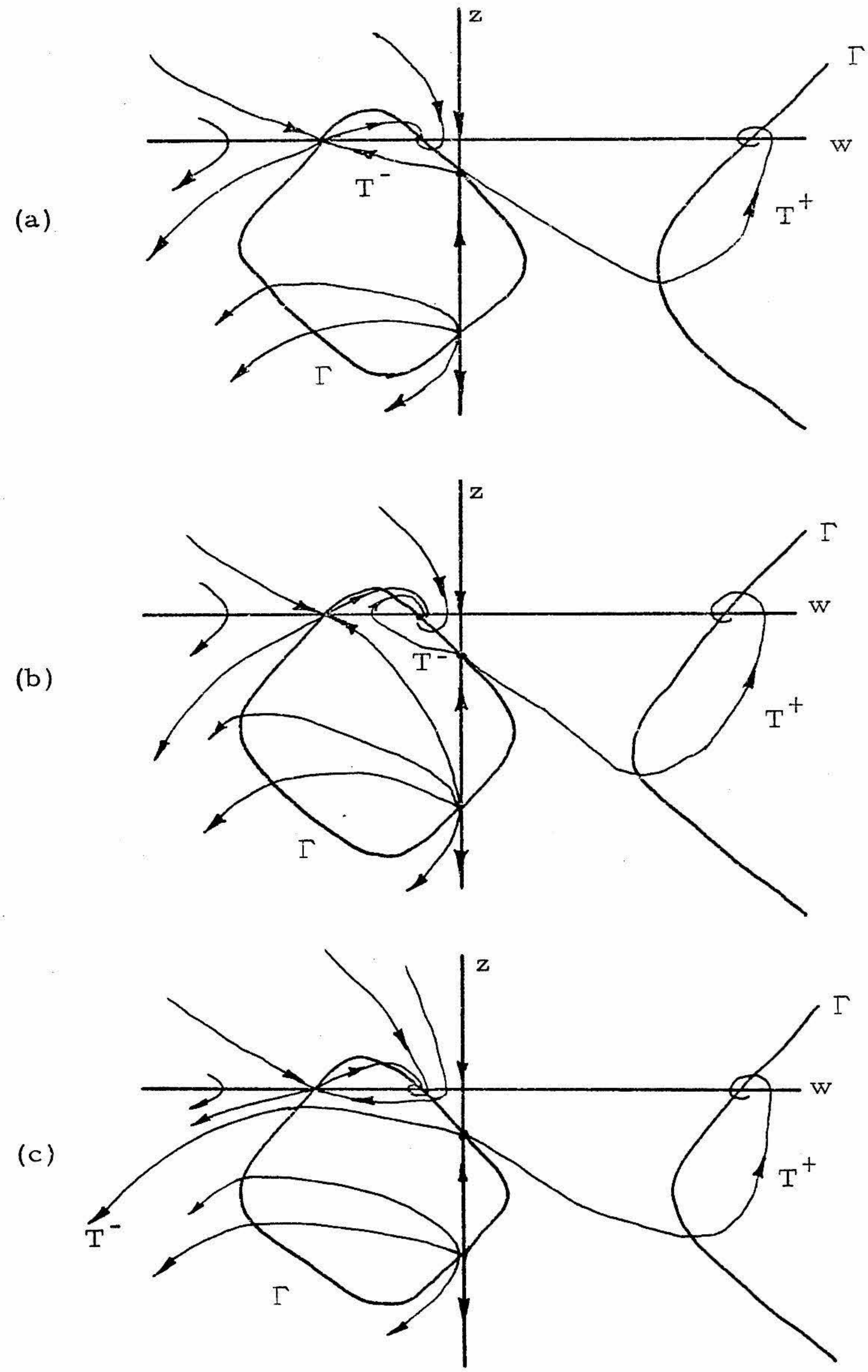


Figure 22 Phase plane for β odd, A >> 1, $\alpha > 0$

first introduced by Poincaré and may be found in the references [6], [7]. A point (w, z) in the plane can be represented in projective coordinates by (w, z, 1), or more generally, by $(\tilde{w}, \tilde{z}, \tilde{v})$ where

$$\tilde{w}/\tilde{v} = w \qquad \tilde{z}/\tilde{v} = z \qquad \tilde{v} \neq 0.$$

Points with $\tilde{v} = 0$ lie on the circle at infinity. If we take $\tilde{v} \equiv 1$, equations (2.5) can be rewritten as

$$\dot{\left(\frac{\tilde{w}}{\tilde{v}}\right)} = f\left(\frac{\tilde{w}}{\tilde{v}}, \frac{\tilde{z}}{\tilde{v}}\right) \tag{2.29a}$$

$$\dot{\left(\frac{\tilde{z}}{\tilde{v}}\right)} = g\left(\frac{\tilde{w}}{\tilde{v}}, \frac{\tilde{z}}{\tilde{v}}\right) \tag{2.29b}$$

Poincaré introduces

$$\begin{aligned} f^* &= \tilde{v}^2 f\left(\frac{\tilde{w}}{\tilde{v}}, \frac{\tilde{z}}{\tilde{v}}\right) \\ &= -\tilde{w}\tilde{z} \end{aligned} \tag{2.30a}$$

$$\begin{aligned} g^* &= \tilde{v}^\beta g\left(\frac{\tilde{w}}{\tilde{v}}, \frac{\tilde{z}}{\tilde{v}}\right) \\ &= \alpha\tilde{w}^\beta - \alpha A\tilde{w}\tilde{v}^{\beta-1} - \tilde{z}^2\tilde{v}^{\beta-2} - \gamma\theta\tilde{v}^\beta + (\gamma+\theta)\tilde{z}\tilde{v}^{\beta-1} \end{aligned} \tag{2.30b}$$

where f is a polynomial of degree 2 and g is a polynomial of degree β. Equations (2.5) can be imbedded in the equation

$$0 = \begin{vmatrix} d\tilde{w} & d\tilde{z} & d\tilde{v} \\ \tilde{w} & \tilde{z} & \tilde{v} \\ \tilde{v}^{\beta-2} f^* & g^* & 0 \end{vmatrix} \tag{2.31}$$

which, when expanded, is

$$\begin{aligned} &-\tilde{v}[\alpha\tilde{w}^\beta - \alpha A\tilde{w}\tilde{v}^{\beta-1} - \tilde{z}^2\tilde{v}^{\beta-2} - \gamma\theta\tilde{v}^\beta + (\gamma+\theta)\tilde{z}\tilde{v}^{\beta-1}]\,d\tilde{w} - \tilde{v}^{\beta-1}\tilde{w}\tilde{z}\,d\tilde{z} \\ &\quad + \{\tilde{w}[\alpha\tilde{w}^\beta - \alpha A\tilde{w}\tilde{v}^{\beta-1} - \tilde{z}^2\tilde{v}^{\beta-2} - \gamma\theta\tilde{v}^\beta + (\gamma+\theta)\tilde{z}\tilde{v}^{\beta-1}] + \tilde{v}^{\beta-2}\tilde{w}\tilde{z}^2\}\,d\tilde{v} \\ &= 0 \end{aligned} \tag{2.31'}$$

Critical points are characterized by the simultaneous vanishing of the coefficients of $d\tilde{w}$, $d\tilde{z}$, and $d\tilde{v}$. In addition to the finite points determined above with $\tilde{v} = 1$, we find points on the circle at infinity by setting $\tilde{v} = 0$. By examining the coefficient of $d\tilde{v}$, we conclude there is only one such point, viz.

$$(\tilde{w}, \tilde{z}, \tilde{v}) = (0, 1, 0)$$

We introduce coordinates

$$W = \tilde{w}/\tilde{z}, \quad V = \tilde{v}/\tilde{z}, \quad \tilde{z} \equiv 1$$

Note that points (w, z, 1) with $w > 0$, $z < 0$ corresponds to points (W, V) with $W < 0$, $V < 0$, and that $(W, V) = (0, 0)$ corresponds to infinity in the (w, z) plane.

With $\tilde{z} \equiv 1$, equation (2.31') is equivalent to the system

$$\dot{W} = \alpha W^{\beta-1} - \alpha A W^2 V^{\beta-1} - \gamma\theta W V^{\beta} + (\gamma+\theta) W V^{\beta-1} \tag{2.32a}$$

$$\dot{V} = \alpha W^{\beta+1} - \alpha A V^{\beta} W - V^{\beta-1} - \gamma\theta V^{\beta+1} + (\gamma+\theta) V^{\beta} \tag{2.32b}$$

We wish to study behavior in the neighborhood of the origin. Note that $W \equiv 0$, $\dot{V} = -V^{\beta-1} - \gamma\theta V^{\beta+1} + (\gamma+\theta)V^{\beta}$ and $V \equiv 0$, $\dot{W} = \alpha W^{\beta+1}$ provide exact solutions which completely cover the V and W axes, respectively. In particular, this implies that trajectories in the quadrant $W < 0$, $V < 0$ can only leave this quadrant by tending to the origin (infinity in the w-z plane), by tending to infinity (which means crossing the line $z = 0$ in the w-z plane), or by tending to one of the other finite critical points (which all correspond to finite critical points in the w-z plane).

Since $\beta > 1$, the origin is not an elementary critical point of equations (2.32). To study the behavior of trajectories, we introduce polar coordinates

$$W = r\cos\phi, \qquad V = r\sin\phi$$

which result in the system

$$\dot{r} = -r^{\beta-1}\sin^{\beta}\phi + (\gamma+\theta)r^{\beta}\sin^{\beta-1}\phi$$
$$+ r^{\beta+1}(\alpha\cos^{\beta}\phi - \alpha A\cos\phi\sin^{\beta-1}\phi - \gamma\theta\sin^{\beta}\phi) \quad (2.33a)$$
$$\dot{\phi} = -r^{\beta-2}\cos\phi\sin^{\beta-1}\phi \quad (2.33b)$$

Since trajectories cannot cross the W axis and the V axis, and hence cannot spiral into the origin, they must approach the origin along angles ϕ_0 satisfying

$$\cos\phi_0\sin^{\beta-1}\phi_0 = 0$$

This leaves only $\phi_0 = 0, \pi/2, \pi, 3\pi/2$, so that a trajectory can only approach the origin tangent to one of the axes.

Now consider a trajectory through a point in the quadrant $W < 0$, $V < 0$. There $\pi < \phi < 3\pi/2$ so $\cos\phi < 0$. Since β is odd, $\sin^{\beta-1}\phi > 0$. Consequently, $\dot{\phi} > 0$ and the trajectory must approach the -V axis. Near the origin

$$r \sim -r^{\beta-1}\sin^{\beta}\phi > 0$$

since $\sin\phi < 0$ and β is odd. This implies that as the trajectory approaches the -V axis, it must move away from the origin. In short, as regards the quadrant $W < 0$, $V < 0$, the origin is a saddle point! We conclude that a trajectory in this quadrant must tend to a finite critical point other than the origin or must tend to infinity. Correspondingly, a trajectory in the quadrant $w > 0$, $z < 0$ must tend to a finite critical point or must cross the line $z = 0$. Since the only possible limit point as $x \to +\infty$ is a spiral point on the +w axis, the trajectory necessarily crosses the line $z = 0$. This in turn implies that the trajectory crosses the curve Γ and the argument is completed.

Having confirmed the behavior shown in figure 22, we summarize: For $\lambda > 0$ the behavior can be described by figure 10, including the existence of numbers m_1, m_2, and m_∞ such that

for $0 < \lambda < m_2$ one solution exists

for $\lambda > m_1$ no solutions exist

for $\lambda = m_\infty$ a countable infinity of solutions exist.

For $\lambda < 0$ one of three alternatives occurs, viz.

either there exists a number $m < 0$ such that

for $m < \lambda < 0$ one solution exists

for $\lambda \leq m$ no solutions exist (Cf. figure 22a)

or there exist numbers m_1, m_2 such that

for $m_1 < \lambda < 0$ one solution exists

for $m_2 < \lambda < m_1$ finitely many solutions exist, but more than one

for $\lambda = m_2$ one solution exists

for $\lambda < m_2$ no solutions exist (Cf. figure 22b)

or for all $\lambda < 0$ precisely one solution exists (Cf. figure 22c).

Case 2: β odd, $A < 0$, $\alpha > 0$

From figure 16d we see that p has one root W which tends to zero as $A \to -\infty$. Referring to the discussion of case 1, we conclude

$$W = W^* \sim -\gamma\theta/\alpha A > 0 \tag{2.25}$$

and the characteristic exponents are

$$\ell_+ \doteq \gamma \text{ and } \ell_- \doteq \theta \quad . \tag{2.28}$$

The corresponding point is a stable node. Figure 20 accurately describes the phase plane, and figure 10 describes the multiplicities, with the restriction that there exist only finitely many numbers m_k.

Case 3: β even, $A > 0$, $\alpha > 0$

From figure 16a we see that p has two roots, viz.

$$W^+ \sim A^{1/(\beta-1)} > 0 \text{ and } W^* \sim -\gamma\theta/\alpha A < 0.$$

The characteristic exponents of the corresponding critical points are, respectively,

$$\ell_\pm \doteq \tfrac{1}{2}(\gamma+\theta) \pm i\sqrt{(\beta-1)\alpha A^{\beta/(\beta-1)}} \tag{2.26}$$

and

$$\ell_+ \doteq \gamma, \quad \ell_- \doteq \theta \quad .$$

Hence $(W^+, 0)$ is a stable spiral point and $(W^*, 0)$ is a stable improper node. The phase plane is described in figure 23. An argument similar to that used in case 1 confirms that T^+ indeed intersects Γ. For $|\lambda|$ large no solutions exist, and for $|\lambda|$ sufficiently small a unique solution exists. For $\lambda < 0$ the multiplicity of any λ is bounded, whereas for $\lambda > 0$ the multiplicity is unbounded; in particular, there exists a unique value $\lambda = m_\infty = W^+$ for which a countable infinity of solutions exists.

Case 4: β even, $A > 0$, $\alpha < 0$

From figure 16e we see that p has two positive roots, which must be

$$W^+ \sim A^{1/(\beta-1)} > 0 \text{ and } W^* \sim -\gamma\theta/\alpha A > 0$$

The characteristic exponents of the corresponding critical points are, respectively,

$$\ell_\pm \doteq \tfrac{1}{2}(\gamma+\theta) \pm \sqrt{-(\beta-1)\alpha A^{\beta/(\beta-1)}}$$

and

$$\ell_+ \doteq \gamma \ , \quad \ell_- \doteq \theta$$

so that $(W^+, 0)$ is a saddle point and $(W^*, 0)$ is a stable node. There exist three alternate phase planes, differing in the region $w > 0$ just as the phase planes for case 1 differ for $w < 0$. In fact, the possible multiplicities of $\lambda > 0$ for case 4 are analogous to those for $\lambda < 0$ in

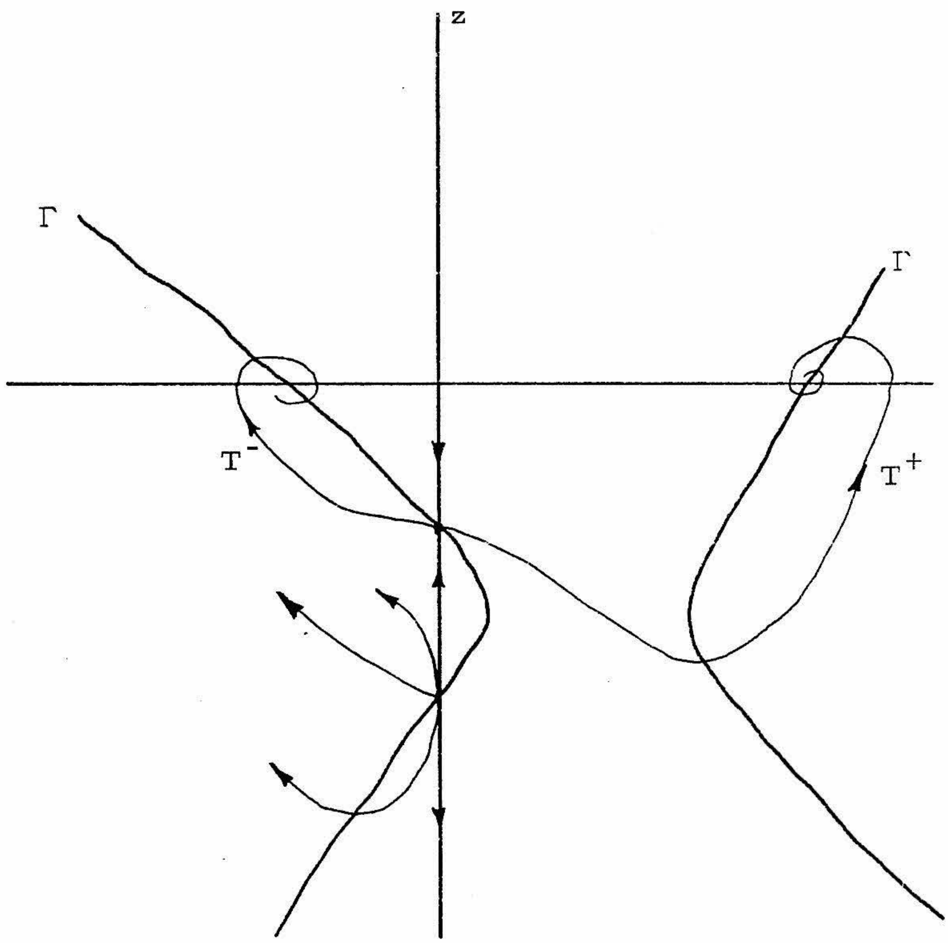


Figure 23 Phase plane for β even, A >> 1, $\alpha > 0$

case 1. For $\lambda < 0$, case 4 always has a unique solution. The phase planes are illustrated in figure 24.

Infinite multiplicities

When $A > 0$ and $\alpha > 0$ there always exists an attractor in the region $w > 0$. When this attractor is a spiral point, there exists a value λ_∞ with a countable infinity of solutions. When $A \ll 1$ this attractor may be a node, but by making μ sufficiently large it can be made a spiral point. When $A \gg 1$ it is always a spiral point. The question arises: if the attractor is a spiral point for $A = A_0$, is it a spiral point for $A > A_0$? We answer this in the affirmative.

Denote the coordinates of the attractor by $w = W$, $z = 0$. From figures 16a, e, f it is clear that

$$W > \hat{w} > 0 \tag{2.27}$$

where $\hat{w}$ is the (positive) root of $\frac{dp}{dw}$. Recall that $\hat{w}$ satisfies

$$\hat{w}^{\beta-1} = A/\beta \tag{2.10}$$

By differentiating the relation $p(W) = 0$ we obtain

$$\frac{dW}{dA} = W/(\beta W^{\beta-1} - A) \tag{2.28}$$

From (2.27) it is clear that W is an increasing function of A. Now the characteristic exponents at (W, 0) are

$$\ell_\pm = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4[(\beta-1)\alpha A W + \beta\gamma\theta]} \tag{2.12}$$

For $\beta > 1$ and $\alpha > 0$

$$(\gamma+\theta)^2 - 4[(\beta-1)\alpha A W + \beta\gamma\theta]$$

is a decreasing function of A; if it is negative for $A = A_0$, it remains negative for $A > A_0$. Hence the desired result is shown.

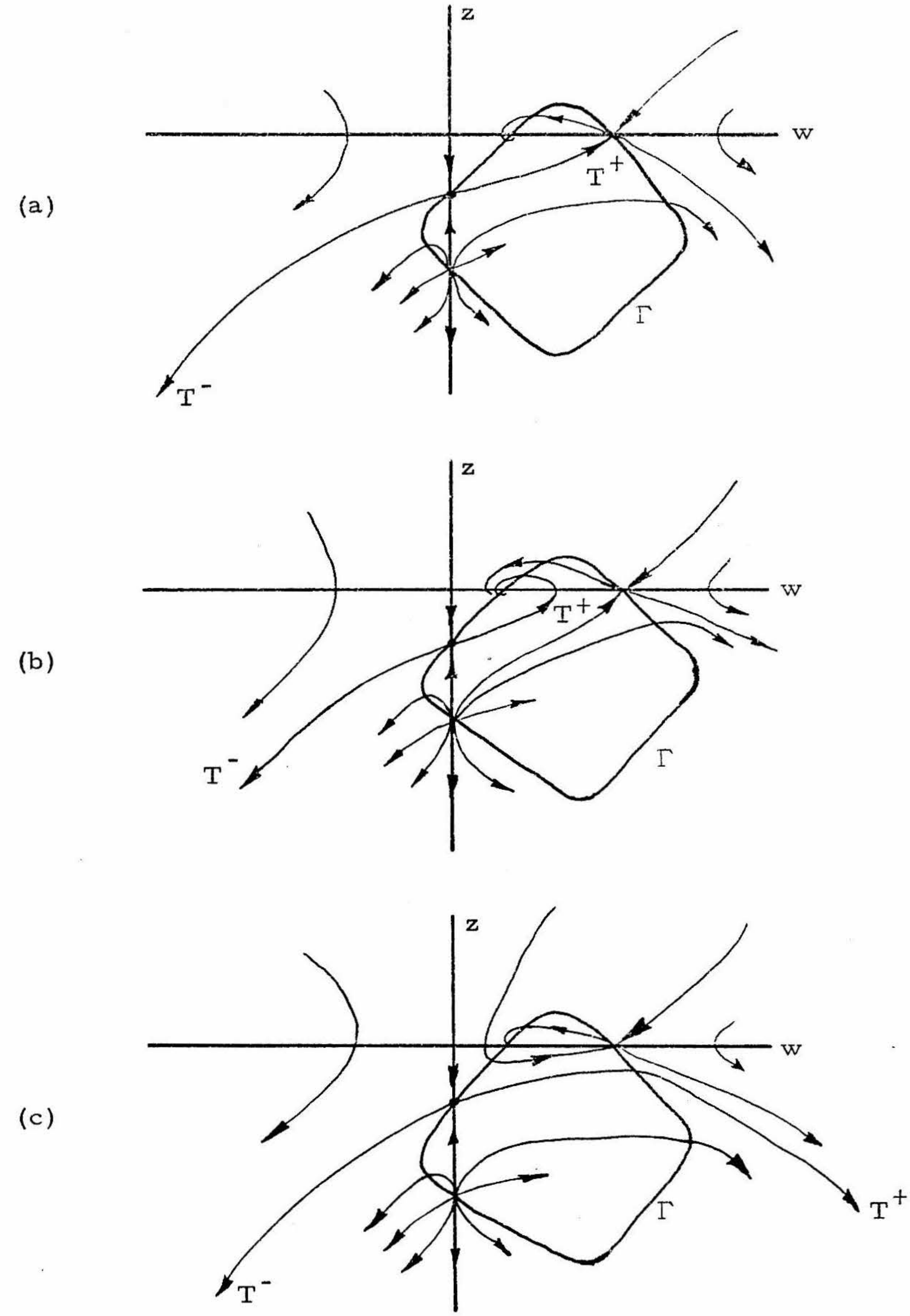


Figure 24 Phase plane for β even, A >> 1, $\alpha < 0$

CHAPTER 3

CIRCULAR CYLINDRICAL SHELLS: THE STATIC PROBLEM

The following two chapters are concerned with the problem of the buckling of a circular cylindrical shell under axial loading. In this chapter we treat the static problem as an example of bifurcation phenomena, using primarily methods from pertubation theory. In the next chapter we use multi-time scaling methods to analyze the dynamic buckling behavior of such a shell.

Donnell-type equations are used to model the cylinder. A derivation of these equations is in Appendix A. We assume that the shell is made from a homogeneous isotropic medium, and that locally the body may be assumed to be in a state of plane stress. The resulting equations are given below.

The shell is described by axial, circumferential, and radial coordinates x, y, and r, respectively; the corresponding displacements are denoted by u, v, and w. The components σ_r, σ_{xr}, σ_{yr} of the stress tenor vanish by the assumption of plane stress. In terms of the remaining components we define the axial, circumferential, and shear forces per unit width

$$N_x = h\sigma_x \ , \quad N_y = h\sigma_y \ , \quad N_{xy} = h\sigma_{xy}$$

where h is the thickness of the shell. For a body in a state of plane stress, Hooke's law becomes

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\,\epsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\,\epsilon_x) \end{aligned}\right\} \qquad (3.1)$$

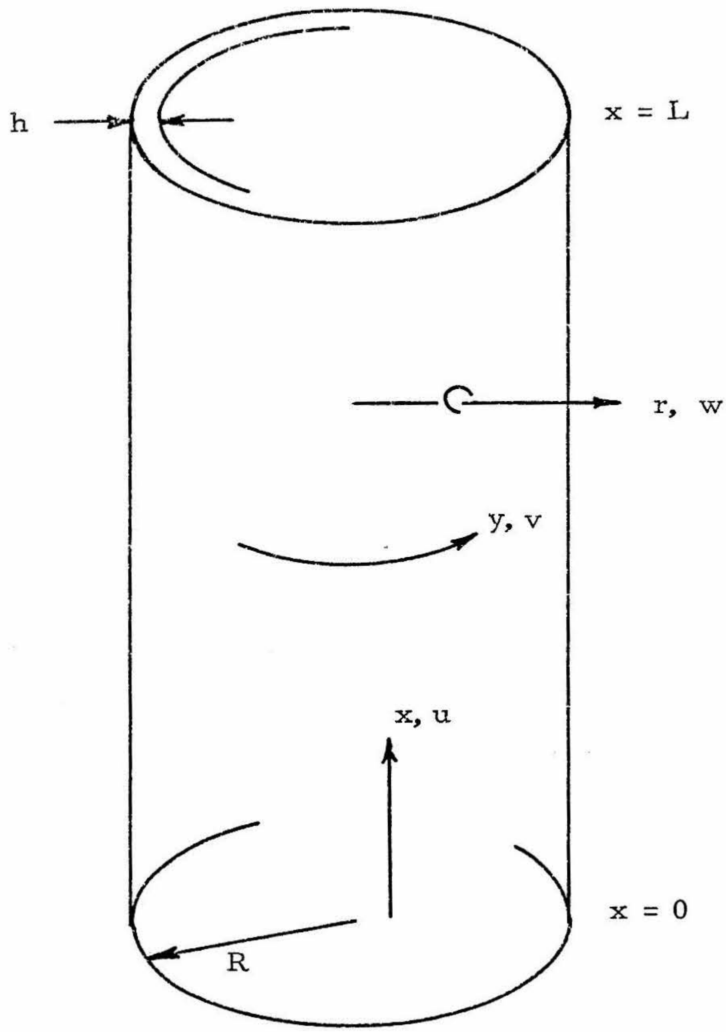


Figure 25 Geometry of the shell

$$\sigma_{xy} = \frac{E}{1-\nu} \epsilon_{xy}$$

where E is Young's modulus, ν is Poisson's ratio, and the strains are approximated (Cf. Appendix A) by

$$\left.\begin{aligned} \epsilon_x &= u_x + \tfrac{1}{2} w_x^2 \\ \epsilon_y &= v_y + \frac{1}{R} w + \tfrac{1}{2} w_y^2 \\ 2\epsilon_{xy} &= v_x + u_y + w_x w_y \end{aligned}\right\} \quad (3.2)$$

With the deflectional rigidity defined by

$$D = h^3 E/12(1-\nu^2)$$

the equations of equilibrium are

$$\frac{\partial}{\partial x} N_x + \frac{\partial}{\partial y} N_{xy} = 0 \quad (3.3a)$$

$$\frac{\partial}{\partial y} N_y + \frac{\partial}{\partial x} N_{xy} = 0 \quad (3.3b)$$

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w - \left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{R} N_y = 0 \quad (3.3c)$$

Equations (3.3a) and (3.3b) are satisfied if we introduce the Airy stress function F such that

$$N_x = \partial^2 F/\partial y^2 \qquad N_y = \partial^2 F/\partial x^2 \qquad N_{xy} = -\partial^2 F/\partial x \partial y$$

Then equation (3.3c) becomes

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w - \left(\frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{R} \frac{\partial^2 F}{\partial x^2} = 0 \quad (3.4)$$

Equations (3.1) and (3.2) may be viewed as a system of three equations in the four unknowns F, u, v, and w. If we eliminate u and v from these, we get a second equation in w and F, commonly known as the compatability relation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 F = (hE)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{R}\frac{\partial^2 w}{\partial x^2}\right] \tag{3.5}$$

The problem may be made dimensionless by introducing

$$x = L\tilde{x}, \quad y = L\tilde{y} \text{ for } 0 \le x \le L, \quad 0 \le y \le 2\pi R$$

$$u = L\tilde{u}, \quad v = L\tilde{v}, \quad w = R\tilde{w}$$

$$\sigma_x = E\tilde{\sigma}_x, \quad \sigma_y = E\tilde{\sigma}_y, \quad \sigma_{xy} = E\tilde{\sigma}_{xy}$$

$$F = L^2 hE\tilde{F}$$

$$\tilde{h}^2 \equiv h^2/12(1-\nu^2)L^2$$

$$\omega \equiv L/R, \quad \Omega \equiv 2\pi/\omega$$

If we now suppress the ~ notation, we get the following non-dimensional statement of the static problem on $0 \le x \le 1$, $0 \le y \le \Omega$:

$$\left.\begin{aligned} \sigma_x &= \frac{1}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\ \sigma_y &= \frac{1}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \\ \sigma_{xy} &= \frac{1}{1+\nu}\epsilon_{xy} \end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} \epsilon_x &= u_x + \frac{1}{2\omega^2} w_x^2 \\ \epsilon_y &= v_y + w + \frac{1}{2\omega^2} w_y^2 \\ 2\epsilon_{xy} &= v_x + u_y + \frac{1}{\omega^2} w_x w_y \end{aligned}\right\} \tag{3.7}$$

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{3.8}$$

Equilibrium:

$$h^2 \Delta^2 w + \omega^2 \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{3.9}$$

Compatability:

$$\omega^2 (\Delta^2 F - \frac{\partial^2 w}{\partial x^2}) = (\frac{\partial^2 w}{\partial x \partial y})^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \tag{3.10}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian.

Various boundary conditions are possible. We assume that the cylinder is constrained against radial expansion or contraction at the ends and is simply supported there so

$$w = 0 \quad \text{at } x = 0, 1 \tag{3.11a}$$

$$w_{xx} = 0 \quad \text{at } x = 0, 1 \tag{3.11b}$$

Also the edges are restrained from twisting, which implies

$$v = 0 \quad \text{at } x = 0, 1 \tag{3.11c}$$

A prescribed uniform axial load is applied at the ends, say

$$\sigma_x = -\sigma \equiv \text{constant} \quad \text{at } x = 0, 1 \tag{3.11d}$$

Finally, all physical quantities (and their derivatives) must be periodic in y:

$$u, v, w, \sigma_x, \sigma_y, \sigma_{xy} \text{ have period } \Omega \text{ in } y \tag{3.11e}$$

Decomposition of the Airy stress function

The requirement that σ_x, σ_y, and σ_{xy} be periodic in y places restrictions on the structure of F. Define

$$k(x, y) \equiv F(x, y+\Omega) - F(x, y)$$

Then

$$k_{yy}(x, y) = F_{yy}(x, y+\Omega) - F_{yy}(x, y) = \sigma_x(x, y+\Omega) - \sigma_x(x, y) = 0$$

and we can write

$$k(x, y) = k_0(x) + yk_1(x) \quad .$$

Now define f(x, y) by

$$F(x, y) = f(x, y) + y\left[\frac{1}{\Omega}k_0(x) - \tfrac{1}{2}k_1(x)\right] + y^2\frac{1}{2\Omega}k_1(x) \quad .$$

We compute

$$\begin{aligned} k_0(x) + yk_1(x) &= k(x, y) \\ &= F(x, y+\Omega) - F(x, y) \\ &= \left\{f(x, y+\Omega) + (y+\Omega)\left[\frac{1}{\Omega}k_0(x) - \tfrac{1}{2}k_1(x)\right] + (y+\Omega)^2\frac{1}{2\Omega}k_1(x)\right\} \\ &\qquad - \left\{f(x, y) + y\left[\frac{1}{\Omega}k_0(x) - \tfrac{1}{2}k_1(x)\right] + y^2\frac{1}{2\Omega}k_1(x)\right\} \\ &= f(x, y+\Omega) - f(x, y) + \left[k_0(x) - \frac{\Omega}{2}k_1(x)\right] + \left(y + \frac{\Omega}{2}\right)k_1(x) \\ &= f(x, y+\Omega) - f(x, y) + k_0(x) + yk_1(x) \quad . \end{aligned}$$

This implies

$$f(x, y+\Omega) = f(x, y)$$

so that f is necessarily periodic in y. If we set $m_0(x) = \frac{1}{\Omega}k_0(x) - \frac{1}{2}k_1(x)$ and $m_1(x) = \frac{1}{\Omega}k_1(x)$, then

$$F(x, y) = f(x, y) + ym_0(x) + \tfrac{1}{2}y^2 m_1(x) \quad .$$

Thus far we have only used the periodicity of $F_{yy} = \sigma_x$. Now $-\sigma_{xy} = F_{xy} = f_{xy} + m_0'(x) + ym_1'(x)$ must be periodic, so $m_1'(x) = 0$. Furthermore, $\sigma_y = F_{xx} = f_{xx} + ym_0''(x)$, so $m_0''(x) = 0$. Hence there exist constants M_0, M_1, M_2 such that

$$m_1(x) = M_1 \qquad m_0(x) = M_0 + xM_2 \quad .$$

In summary,

$$F(x, y) = f(x, y) + M_0 y + \tfrac{1}{2}M_1 y^2 + M_2 xy \quad .$$

Note that F is arbitrary within any function G such that $G_{xx} = G_{yy} = G_{xy} = 0$; i.e., if F_1 is an Airy stress function which solves a particular problem, then so is

$$F_2 = F_1 + G$$

where

$$G = ax + by + c$$

for constants a, b, and c. The freedom to choose a, b, and c permits us to specify that

$$M_0 = 0$$

and, as we will find convenient below,

$$\int_0^\Omega f(x, y)dy = 0 \qquad \text{for } x = 0 \text{ and } x = 1 \quad .$$

Now at $x = 0$ and $x = 1$, boundary condition (3.11d) becomes

$$-\sigma = \sigma_x = F_{yy} = f_{yy} + M_1$$

Periodicity of f and hence f_y then imply

$$-\sigma\Omega = \int_0^\Omega (f_{yy} + M_1)dy = f_y\Big|_0^\Omega + M_1\Omega = M_1\Omega$$

or $M_1 = -\sigma$ and $f_{yy} = 0$ for $x = 0$ and $x = 1$. Setting $M_2 = \ell$, we have

$$F(x, y) = f(x, y) - \tfrac{1}{2}\sigma y^2 + \ell_{xy} \tag{3.12}$$

For $x = 0$ and $x = 1$ expand

$$f(x, y) = \tfrac{1}{2}a_0(x) + \sum_{n>0} a_n(x) \cos n\omega y + b_n(x) \sin n\omega y \quad .$$

Then $f_{yy} = 0$ implies

$$0 = \int_0^\Omega f_{yy} \cos n\omega y\, dy = -(n\omega)^2 \int_0^\Omega f \cos n\omega y\, dy$$

where we have integrated by parts and used the periodicity of f. For $n > 0$ we conclude

$$a_n(x) = 0 \qquad \text{for } x = 0, 1$$

and similarly $b_n(x) = 0$. As noted above, without any loss of generality we can require that

$$a_o(x) = \frac{2}{\Omega}\int_o^\Omega f(x, y)\,dy = 0 \qquad \text{for } x = 0 \text{ and } x = 1 \;.$$

Thus we conclude

$$f(x, y) = 0 \quad \text{for } x = 0 \text{ and } x = 1 \;. \tag{3.13}$$

It remains to see what (3. 11c) implies about f at $x = 0$ and $x = 1$. For the moment relax condition (3. 11a) to the more general condition

$$w = \text{constant} \qquad \text{at } x = 0, 1 \tag{3.11a'}$$

Now $v = 0$ and $w =$ constant imply $v_y = w_y = 0$. The second of the strain relations (3. 7) simplifies to

$$\epsilon_y = w \quad \text{at } x = 0, 1 \;.$$

Invert the first two equations of Hooke's law (3. 6) to obtain

$$\epsilon_y = \sigma_y - \nu\sigma_x = F_{xx} - \nu F_{yy} = f_{xx} - \nu f_{yy} + \nu\sigma = f_{xx} + \nu\sigma$$

or

$$f_{xx} = w - \nu\sigma \quad \text{at } x = 0 \text{ and } x = 1 \tag{3.14}$$

Symmetric solutions

We now seek solutions angularly symmetric about the shell axis, i. e., assume $u, v, w, \sigma_x, \sigma_y$, and σ_{xy} are independent of y. First note that the strain relations simplify to

$$\left.\begin{aligned} \epsilon_x &= u_x + \frac{1}{2\omega^2} w_x^2 \\ \epsilon_y &= w \\ 2\epsilon_{xy} &= v_x \end{aligned}\right\} \tag{3.7S}$$

We calculate

$$\frac{\partial}{\partial x}\sigma_x = \frac{\partial}{\partial x}F_{yy} = \frac{\partial}{\partial y}F_{xy} = -\frac{\partial}{\partial y}\sigma_{xy} = 0$$

so that $\sigma_x =$ constant. From boundary condition (3. 11d) we conclude

$$\sigma_x = -\sigma$$

Similarly

$$\frac{\partial}{\partial x}\sigma_{xy} = -\frac{\partial}{\partial x}F_{xy} = -\frac{\partial}{\partial y}F_{xx} = -\frac{\partial}{\partial y}\sigma_{y} = 0$$

so that σ_{xy} = constant. From Hooke's law (3.6) we see that ϵ_{xy} is constant, and hence (3.7S) implies that v is linear in x. Boundary condition (3.11c) in turn requires that v vanish identically

$$v = 0$$

from which we infer that $\epsilon_{xy} = 0$ and hence

$$\sigma_{xy} = 0$$

From (3.6)and (3.7S) calculate

$$w = \epsilon_y = \sigma_y - \nu\sigma_x = \sigma_y + \nu\sigma$$

or

$$\sigma_y = w - \nu\sigma$$

Also

$$\sigma_x - \nu\sigma_y = \epsilon_x = u_x + \frac{1}{2\omega^2}w_x^2 = -\sigma - \nu w + \nu^2\sigma$$

Integrating this yields

$$u(x) = u(0) - \int_0^x (1-\nu^2)\sigma + \nu w + \frac{1}{2\omega^2}w_x^2 \, dx$$

All that remains is to find w(x). The equilibrium equation (3.9) reduces to

$$Lw \equiv h^2 w_{xxxx} + \sigma w_{xx} + \omega^2 w = \omega^2 \nu\sigma \qquad (3.15)$$

subject to boundary conditions (3.11a, b). The linear operator L is self-adjoint; consequently (3.15) has a unique solution unless there exists a non-trivial solution to

$$Lz = 0$$

satisfying (3.11a, b). In that case (3.15) has no solution or a continuum of solutions, depending on whether $\langle z, 1\rangle \neq 0$ or $\langle z, 1\rangle = 0$, respectively.

Suppose $Lz = 0$ and expand $z = \sum_{k=1}^{\infty} c_k \sin k\pi x$.

Integrating by parts, we find

$$0 = \int_0^1 (h^2 z_{xxxx} + \sigma z_{xx} + \omega^2 z) \sin m\pi x \, dx$$

$$= [h^2 (m\pi)^4 - \sigma (m\pi)^2 + \omega^2] \, cm/2$$

Now $z \neq 0$ implies $c_m \neq 0$ for some c_m; hence

$$\sigma = \sigma_{m,1} \equiv h^2 (m\pi)^2 + \omega^2/(m\pi)^2 \tag{3.16}$$

If we treat $m > 0$ as a continuous parameter and graph $\sigma_{m,1}$ as a function of m, it is clear that for a given value of σ there exist at most two values of m such that (3.16) holds. If a (non-unique) solution to (3.15) exists when $\sigma = \sigma_{m,1}$, then necessarily

$$0 = \langle z, 1\rangle = \int_0^1 \sin m\pi x \, dx$$

i.e., m must be even.

When $\sigma \neq \sigma_{m,1}$ for any m, it is a simple exercise to find the (unique) solution to (3.15), viz.

$$\begin{aligned} w(x) = \nu\sigma \Bigg[1 &+ \frac{\mu_2^2}{(\mu_1^2 - \mu_2^2)\sin\mu_1} (\sin\mu_1 x + \sin\mu_1 (1-x)) \\ &+ \frac{\mu_1^2}{(\mu_2^2 - \mu_1^2)\sin\mu_2} (\sin\mu_2 x + \sin\mu_2 (1-x)) \Bigg] \end{aligned} \tag{3.17}$$

where μ_1 and μ_2 are defined by

$$\left.\begin{aligned} \mu_1^2 &= (\sigma + \sqrt{\sigma^2 - 4h^2\omega^2})/2h^2 \\ \mu_2^2 &= (\sigma - \sqrt{\sigma^2 - 4h^2\omega^2})/2h^2 \end{aligned}\right\} \tag{3.18}$$

Numerical evaluation of w suggests that away from the boundaries of the interval $0 < x < 1$, $w(x) \sim \nu\sigma$. We can obtain this result for small values of σ by setting $\sigma = sh$ in (3.18) and letting $h \to 0+$. For s such that $0 < s < 2\omega$, μ_1 and μ_2 are complex and $|\mu_1| = |\mu_2| = O(h^{-\frac{1}{2}})$. Then for $0 < x < 1$

$$\sin \mu_1 x / \sin \mu_1 \to 0 \qquad \sin \mu_1 (1-x)/\sin \mu_1 \to 0 \qquad \text{as } h \to 0$$

and similarly for μ_2, but

$$\mu_2^2/(\mu_1^2 - \mu_2^2) = O(1), \; \mu_1^2/(\mu_2^2 - \mu_1^2) = O(1) \qquad .$$

From equation (3.17) it is clear that $w(x) \sim \nu\sigma$.

Poisson expansion and bifurcation

This motivates seeking a solution of the problem compatible with w = constant. From the equilibrium equation (3.9) we conclude $\sigma_y = F_{xx} = 0$; then (3.3b) implies $\sigma_{xy,x} = 0$ and $\sigma_{xy} = \alpha(y)$ for some function α. From equation (3.3a) we obtain $\sigma_{x,x} = -\alpha'(y)$ or $\sigma_x = \beta(y) - \alpha'(y)x$ for some function $\beta(y)$. The boundary condition $\sigma_x = -\sigma$ at $x = 0, 1$ implies $\beta(y) = -\sigma$ and $\alpha'(y) = 0$, whence $\sigma_x = -\sigma$ and $\sigma_{xy} = \alpha$ = constant. Now

$$\epsilon_y = \sigma_y - \nu\sigma_x = \nu\sigma$$

$$= v_y + w + \frac{1}{2\omega^2} w_y^2 = v_y + w$$

Hence $v = (\nu\sigma - w)y + \gamma(x)$ for some function $\gamma(x)$. Periodicity of v in y forces $w = \nu\sigma$, and $v = 0$ at $x = 0, 1$ then gives $\gamma(0) = \gamma(1) = 0$.

$$\epsilon_x = \sigma_x - \nu\sigma_y = -\sigma$$

$$= u_x + \frac{1}{2\omega^2} w_x^2 = u_x$$

which implies $u = -\sigma x + \delta(y)$ for some function $\delta(y)$.

$$2\epsilon_{xy} = 2(1+\nu)\sigma_{xy} = 2(1+\nu)\alpha$$

$$= v_x + u_y + \frac{1}{\omega^2} w_x w_y = \gamma'(x) + \delta'(y)$$

Therefore $\gamma'(x)$ = constant and $\delta'(y)$ = constant. Since $\gamma(0) = \gamma(1) = 0$, we conclude $\gamma = 0$. Periodicity of u in y forces δ = constant which can be chosen arbitrarily by a solid body translation along the x axis. Finally δ = constant and $\gamma = 0$ yield $\alpha = 0$. In summary, the only solution compatible with w = constant is

$$w = \nu\sigma \quad u = -\sigma(x-\tfrac{1}{2}) \quad v = 0$$

$$\sigma_x = -\sigma \quad \sigma_y = 0 \quad \sigma_{xy} = 0$$

$$f = 0 \quad \ell = 0$$

The solution is known as Poisson expansion; it satisfies the problem with boundary condition (3.11a) replaced by

$$w = \nu\sigma \quad \text{at } x = 0, 1 \quad . \tag{3.11A}$$

Note that with (3.11A), condition (3.14) becomes

$$f_{xx} = 0 \quad \text{at } x = 0, 1 \tag{3.14'}$$

Equations (3.9) and (3.10) are two equations in w and F; they can be converted into equations in w and f if we find an appropriate expression for ℓ. From (3.6) and (3.7) we calculate

$$2\epsilon_{xy} = 2(1+\nu)\sigma_{xy} = -2(1+\nu)F_{xy} = -2(1+\nu)(f_{xy} + \ell)$$

and

$$2\epsilon_{xy} = v_x + u_y + \frac{1}{\omega^2} w_x w_y$$

Periodicity of u and f imply

$$\int_o^\Omega f_{xy} dy = \int_o^\Omega u_y dy = 0$$

while boundary condition (3.11c) yields

$$\int_o^1 v_x dx = 0$$

By integrating by parts and using the periodicity of w, we have

$$\int_0^{\Omega} w_x w_y dy = -\int_0^{\Omega} w w_{xy} dy$$

Putting all this together, we can conclude

$$\ell = \frac{1}{4\pi\omega(1+\nu)} \int_0^{\Omega}\int_0^1 w w_{xy} dx dy \tag{3.19}$$

For the remainder of this chapter we will be concerned with the bifurcation of solutions from Poisson expansion. We will have to solve a hierarchy of constant coefficient linear equations with inhomogeneous terms which are known explicitly at each step. If, instead of Poisson expansion, we took the axisymmetric solution (3.17) as the state from which bifurcation occurs, the relevant system would be

$$h^2\Delta^2 w + \sigma w_{xx} + \omega^2 f_{xx} - f_{yy} w_{s,xx} - f_{s,xx} w_{yy}$$

$$= f_{yy} w_{xx} - 2f_{xy} w_{xy} + f_{xx} w_{yy} - 2\ell w_{xy}$$

$$\omega^2(\Delta^2 f - w_{xx}) + w_{s,xx} w_{yy} = w_{xy}^2 - w_{xx} w_{yy}$$

Here (w_s, f_s) denotes the solution of (3.17) and (w_s+w, f_s+f) is the solution of the full problem. The linearized equations for the perturbations w and f have variable coefficients and require numerical solution insofar as an explicit analytic solution is not possible. Numerical studies by Almroth [9] have shown that the value of the critical load σ_0 at which buckling (or bifurcation) occurs is not changed sufficiently by the boundary conditions to explain the well-known discrepancy between theory and test data. Consequently we will use Poisson expansion as the pre-buckling state.

Let $w = \nu\sigma + \hat{w}$; then drop the ^ notation so that w represents the displacement from Poisson expansion. Equations (3.9), (3.10), and (3.12), as well as boundary conditions (3.11A), (3.11b), (3.13), (3.14'), and (3.11e), are transformed into the following final formulation of the problem (equation (3.19) is unchanged under this transformation):

$$h^2\Delta^2 w + \sigma w_{xx} + \omega^2 f_{xx} = f_{yy}w_{xx} - 2f_{xy}w_{xy} + f_{xx}w_{xy} - 2\ell w_{xy} \qquad (3.20)$$

$$\omega^2(\Delta^2 f - w_{xx}) = w_{xy}^2 - w_{xx}w_{yy} \qquad (3.21)$$

$$w = w_{xx} = 0 \quad \text{at } x = 0, 1 \qquad (3.22a)$$

$$f = f_{xx} = 0 \quad \text{at } x = 0, 1 \qquad (3.22b)$$

$$w \text{ and } f \text{ have period } \Omega \text{ in } y \qquad (3.22c)$$

We propose to attack this problem by seeking solutions which bifurcate from Poisson expansion (which corresponds to $w = f = 0$ in this notation). We seek solutions of the form:

$$\left.\begin{aligned} w &= \epsilon w_1 + \epsilon^2 w_2 + \dots \\ f &= \epsilon f_1 + \epsilon^2 f_2 + \dots \\ \sigma &= \sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \dots \\ \ell &= \epsilon^2 \ell_2 + \dots \end{aligned}\right\} \qquad (3.23)$$

where the small parameter ϵ is defined by

$$\epsilon^2 = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 w^2 + f^2\,dxdy \qquad (3.24)$$

Substituting (3.23) into equations (3.19), (3.20), and (3.21) yields the following hierarchy:

$$O(\epsilon): \; h^2\Delta^2 w_1 + \sigma_0 w_{1_{xx}} + \omega^2 f_{1_{xx}} = 0 \qquad (3.25a)$$

$$\omega^2(\Delta^2 f_1 - w_{1_{xx}}) = 0 \qquad (3.25b)$$

$$O(\varepsilon^2) \qquad h^2\Delta^2 w_2 + \sigma_0 w_{2xx} + \omega^2 f_{2xx} = f_{1yy}w_{1xx} - 2f_{1xy}w_{1xy} + f_{1xx}w_{1yy} - \sigma_1 w_{1xx} \tag{3.26a}$$

$$\omega^2(\Delta^2 f_2 - w_{2xx}) = w_{1xy}^2 - w_{1xx}w_{1yy} \tag{3.26b}$$

$$\ell_2 = \frac{1}{4\pi\omega(1+\nu)} \int_0^\Omega\int_0^1 w_1 w_{1xy}\, dxdy \tag{3.26c}$$

$$O(\varepsilon^3) \qquad h^2\Delta^2 w_3 + \sigma_0 w_{3xx} + \omega^2 f_{3xx} = f_{1yy}w_{2xx} + f_{2yy}w_{1xx} - 2f_{1xy}w_{2xy} - 2f_{2xy}w_{1xy} + f_{1xx}w_{2yy} + f_{2xx}w_{1yy} - \sigma_1 w_{2xx} - \sigma_2 w_{1xx} - 2\ell_2 w_{1xy} \tag{3.27a}$$

$$\omega^2(\Delta^2 f_3 - w_{3xx}) = 2w_{1xy}w_{2xy} - w_{1xx}w_{2yy} - w_{2xx}w_{1yy} \tag{3.27b}$$

$$\ell_3 = \frac{1}{4\pi\omega(1+\nu)} \int_0^\Omega\int_0^1 w_1 w_{2xy} + w_2 w_{1xy}\, dxdy \tag{3.27c}$$

The normalization condition (3.24) yields

$$\begin{aligned} \Omega/4 &= \int_0^\Omega\int_0^1 w_1{}^2 + f_1{}^2\, dxdy \\ 0 &= \int_0^\Omega\int_0^1 w_1 w_2 + f_1 f_2\, dxdy \\ 0 &= \int_0^\Omega\int_0^1 w_2{}^2 + 2w_1 w_3 + f_2{}^2 + 2f_1 f_3\, dxdy \end{aligned} \tag{3.28}$$

All the w_j and f_j inherit the linear homogeneous boundary conditions (3.22).

Recall $\omega = 2\pi/\Omega$ and $0 \le y \le \Omega$. The set of functions

$$\begin{aligned} \psi_{mn}(x, y) &= \sin m\pi x \cos n\omega y \\ \hat{\psi}_{mn}(x, y) &= \sin m\pi x \sin n\omega y \end{aligned} \qquad \begin{cases} m = 1, 2, 3, \ldots \\ n = 0, 1, 2, \ldots \end{cases}$$

is complete on $[0, 1] \times [0, \Omega]$. If we multiply equations (3.25) by one such function and integrate by parts, we obtain

$$\left.\begin{aligned} h^2 Q^2_{mn} a_{mn} - \sigma_0 (m\pi)^2 a_{mn} - \omega^2 (m\pi)^2 A_{mn} &= 0 \\ Q^2_{mn} A_{mn} + (m\pi)^2 a_{mn} &= 0 \end{aligned}\right\} \quad (3.29)$$

and similarly for b_{mn}, B_{mn}, where

$$a_{mn} = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 w_1 \sin m\pi x \cos n\omega y \, dxdy$$

$$A_{mn} = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 f_1 \sin m\pi x \cos n\omega y \, dxdy$$

$$b_{mn} = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 w_1 \sin m\pi x \sin n\omega y \, dxdy$$

$$B_{mn} = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 f_1 \sin m\pi x \sin n\omega y \, dxdy$$

and we define

$$Q_{mn} \equiv (m\pi)^2 + (n\omega)^2 \quad (3.30)$$

Equations (3.29) have a non-trivial solution if and only if

$$\sigma_0 = \omega^2 (m\pi)^2 / Q^2_{mn} + h^2 Q^2_{mn} / (m\pi)^2 \quad (3.31)$$

If there exists a unique integer pair M, N such that $M > 0$, $N \geq 0$, and $\sigma_0 = \sigma_0(M, N)$ we say σ_0 is a simple eigenvalue. Note, however, that for $N > 0$ there exist two eigenfunctions corresponding to (M, N). However, if $\{w(x, y), f(x, y)\}$ is a solution of the problem, then the translation invariance in y of the equations and boundary conditions implies that $\{w(x, y+y_0), f(x, y+y_0)\}$ is also a solution for any constant y_0. These merely correspond to a solid body rotation of the cylinder. Consequently, even though the solution of (3.25)is, for some constants c_1 and c_2,

$$w_1(x, y) = c_1 Q^2_{MN} \sin M\pi x \cos N\omega y + c_2 Q^2_{MN} \sin M\pi x \sin N\omega y$$

$$f_1(x, y) = -c_1 (M\pi)^2 \sin M\pi x \cos N\omega y - c_2 (M\pi)^2 \sin M\pi x \sin N\omega y$$

we may take $c_2 = 0$ without any loss of generality. The normalization condition (3.28) then yields

$$c_1 = \pm 1/\sqrt{Q^4_{MN} + (M\pi)^4}$$

Multiplicity of the eigenvalues

If we write $\sigma_0(t) = \omega^2 t + h^2/t$, then the eigenvalues are $\sigma_0(t_{mn})$ where

$$t_{mn} = (m\pi/Q_{mn})^2 \quad .$$

From the convexity of the graph of $\sigma_0(t)$ for $t > 0$ it follows that there exist at most two values t_1 and t_2 such that $\sigma_0(t_1) = \sigma_0(t_2)$. For distinct t_1 and t_2, a short algebraic manipulation shows that $\sigma_0(t_1) = \sigma_0(t_2)$ is equivalent to

$$t_1 t_2 = h^2/\omega^2 \quad .$$

Since the values t_{mn} depend on ω but not on h, given t_{mn} and $t_{m'n'}$ distinct it is possible to choose h (in a unique fashion) so that $\sigma_0(t_{mn}) = \sigma_0(t_{m'n'})$. Thus we see that σ_0 may have "multiplicity" at most two in regard to the number of corresponding values of t_{mn}, and indeed "multiplicity" two does occur. We can guarantee that $\sigma_0(t_{mn})$ has "multiplicity" one by choosing $h = t_{mn}\omega$, for in that case t_{mn} occurs at the global minimum of $\sigma(t)$ in $t > 0$ ($\sigma = 2\omega h$ there). Consequently, the question of the actual multiplicity of σ_0 (i.e., the number of integer pairs $m > 0$, $n \geq 0$ such that $\sigma_0 = \sigma_0(m, n)$ can be reduced to studying the multiplicity of t_{mn}.

Suppose $t_{m_1 n_1} = t_{m_2 n_2}$; then $m_1 \pi/Q_{m_1 n_1} = m_2 \pi/Q_{m_2 n_2}$, or

$$\omega^2/\pi^2 = m_1 m_2 (m_1 - m_2)/(m_1 n_2^2 - m_2 n_1^2)$$

so that ω^2/π^2 must be rational. Thus the irrationality of ω^2/π^2 is sufficient to assure that all t_{mn} are simple, and consequently the

$\sigma_0(m, n)$ are simple (except when $\sigma_0(t_{mn}) = \sigma_0(t_{m'n'})$ for $t_{mn} \neq t_{m'n'}$).

To further investigate the possible multiplicity of σ_0, consider when $q = \omega/\pi$ is rational. Suppose $t^* = t_{MN}$; we wish to find all other (m, n) such that $t^* = t_{mn}$. Manipulating,

$$t^* = (m\pi/Q_{mn})^2 = (m\pi)^2/[(m\pi)^2 + (n\omega)^2]^2$$

leads to

$$(m^2 + n^2q^2)/m = 1/\pi\sqrt{t^*} \equiv 2c$$

Note that c is prescribed by q (or ω) and an integer pair (M, N); also c is rational. Under these circumstances, determing the multiplicity of σ_0 is reduced to finding the number of integer pairs (m, n) with $m > 0$, $n \geq 0$, such that

$$(m-c)^2 + (nq)^2 = c^2 \quad .$$

We will show that there exist q and c (and hence $\sigma_0(M, N)$) with arbitrarily large (but nonetheless finite) multiplicity. This will be done by construction, using integral values of c and special values of q. When c is a positive integer it is possible, for any value of q, to find a pair (M, N) such that $c = c(q, M, N)$; simply let $M = 2c$ and $N = 0$. Designate $\mu = m-c$ so that $-c < \mu \leq c$ and $n \geq 0$. When $q = 1$ there exists a one-to-one correspondence between such pairs (μ, n) satisfying

$$\mu^2 + n^2 = c^2$$

and the eigenfunctions ψ_{mn} (disregarding the translational invariance in y) corresponding to $\sigma_0(t^*)$.

The key to solving this problem is a standard result from the theory of numbers (Cf. reference [12], sections 16.9 and 16.10).

We define R(C) as the number of representations of C in the form $C = A^2 + B^2$, where A and B are integers (not necessarily positive).

We count representations as distinct even when they differ only trivially, i.e., in the sign or order of A and B. For example,

$$0 = 0^2 + 0^2 \qquad R(0) = 1$$

$$1 = (\pm 1)^2 + 0^2 = 0^2 + (\pm 1)^2 \qquad R(1) = 4$$

$$5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2 \qquad R(5) = 8$$

The main result is as follows:

write $C = 2^{\alpha} \prod p_k^{r_k} \prod q_j^{s_j}$ where the p_k are distinct prime numbers of the form $4m + 1$ and the q_j are distinct prime numbers of the form $4m + 3$. Then

$$R(C) = \begin{cases} 4 \prod (r_k + 1) & \text{if all } s_j \text{ are even} \\ 0 & \text{otherwise} \end{cases}$$

To apply this to the problem $\mu^2 + n^2 = c^2$, write $c = 2^{\alpha'} \prod p_k^{r'_k} \prod q_j^{s'_j}$. Then $C = c^2 = 2^{2\alpha'} \prod p_k^{2r'_k} \prod q_j^{2s'_j}$ so that

$$\alpha = 2\alpha' \quad r_k = 2r'_k \quad s_j = 2s'_j \quad .$$

Clearly all the s_j are even, so

$$R(c^2) = 4 \prod (2r'_k + 1)$$

However, $R(c^2)$ gives the number of pairs A, B satisfying

$$A^2 + B^2 = c^2 \qquad -c \leq A, \ B \leq c$$

Identify μ with A and n with B. Obviously $A = \pm c$, $B = 0$ are solutions. Therefore there are R-2 solutions with $B \neq 0$. Of these $\frac{1}{2}(R-2)$ are solutions with $B > 0$ by symmetry. Now $B > 0$ implies $-c < A < c$, but we must also include the case $\mu = A = c$, $n = B = 0$ (and exclude the case $A = -c$, $B = 0$). This results in $\frac{1}{2}(R-2) + 1$ cases of pairs μ, n within the given bounds. Simplifying, the multiplicity of t^* for $q = 1$ and c an integer is

$$2\prod(2r_k' + 1)$$

which can be made arbitrarily large by merely taking enough prime factors of the form $4m + 1$.

The above construction is limited to multiplicities which are twice an odd number. Next consider $q = 2$ and c an odd integer. Set $\nu = 2n$ so that

$$\mu^2 + \nu^2 = c^2 \quad .$$

When c is odd, precisely one of the numbers μ, ν must be even; this must always be ν. Since the function R treats A and B symmetrically, there are $\frac{1}{2}R$ cases with B (or ν) even. Except for the case $\nu = 0$, they occur in symmetric pairs with $\nu > 0$ and $\nu < 0$; this leaves $\frac{1}{2}(\frac{1}{2}R-2)$ cases with $\nu > 0$ and ν even. Finally, we add the case $\mu = A = c$, $\nu = B = 0$, resulting in $\frac{1}{2}(\frac{1}{2}R-2)+1$ pairs μ, ν. Simplifying, the multiplicity of t^* for $q = 2$ and c an odd integer is

$$\prod(2r_k' + 1)$$

which can be set equal to an arbitrary odd number.

We can construct σ_0 with an arbitrary finite multiplicity. To achieve an even multiplicity, decompose the even number into a sum of two odd numbers and find corresponding values t_1 and t_2 (with, say, $q = 2$); then choose h so as to satisfy $t_1 t_2 = h^2/\omega^2$.

There is a physical significance to q being rational. $q = \omega/\pi = L/R\pi$, where L is the length of the cylinder and R is its radius. When q is rational, L and $2\pi R$ are commensurable, i.e., there exists a unit of length which divides both the length and the circumference of the cylinder an integral number of times. In this situation alone is it possible to cover the cylinder's surface with squares.

The buckling mode

The smallest buckling load

$$\sigma_{min} = \min_{m>0,\ n\geq 0} \sigma_0(t_{mn})$$

is of the most physical interest. In table 1 the values of M and N corresponding to σ_{min} for various combinations of (dimensional) h, R, and L are given, assuming that $\nu = 0.3$ (in all cases σ_{min} is simple). The values of N show no particular pattern; however for a fixed thickness and radius, the values of M tend to increase somewhat linearly with the length. This suggests the existence of a characteristic length over which buckling occurs. Indeed, buckling (or bifurcating) from Poisson expansion is characterized by a rather large number of waves along the entire axis of the shell for common values of L/R. In experiments, however, one typically observes only a few tiers (~ 2) located roughly midway along the axis (see reference [18]). We noted earlier that symmetric solutions with the ends restrained (but simply supported) tend to undergo Poisson expansion away from the ends of the shells; however, as the load increases, so does the width of the boundary layers. Near buckling only a fraction of the length of the cylinder is undergoing Poisson expansion, and we might conjecture that it is this effective length which undergoes the deformations observed in buckling.

Bifurcation for simple eigenvalues

We return to the problem of computing $\sigma = \sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \ldots$ along the bifurcating solution branch when σ_0 is a simple eigenvalue. Although it is possible to continue with the scheme indicated by equations (3.23) through (3.28), an iterative scheme exists which has

L/R \ R/h	0.1	0.25	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
1,000	1, 29	4, 19	8, 19	16, 19	24, 19	32, 19	44, 11	41, 25	43, 27	67, 16	72, 19	88, 11
900	1, 27	1, 23	5, 27	17, 8	26, 2	15, 27	32, 24	52, 2	11, 21	67, 10	78, 2	64, 24
800	1, 25	4, 8	7, 18	14, 18	20, 20	28, 18	31, 22	40, 20	22, 25	21, 24	8, 16	81, 5
700	1, 23	2, 24	4, 24	8, 24	12, 24	16, 24	38, 4	37, 19	53, 5	61, 3	32, 24	76, 4
600	1, 20	3, 16	3, 22	6, 22	9, 22	12, 22	15, 22	18, 22	21, 22	24, 22	27, 22	51, 20
500	1, 17	3, 11	5, 17	10, 17	8, 20	20, 17	19, 20	16, 20	35, 17	35, 19	24, 20	38, 20
400	1, 12	2, 17	4, 17	5, 18	17, 5	23, 3	26, 11	15, 18	40, 4	45, 6	52, 1	33, 18
300	1, 1	2, 13	5, 1	10, 1	14, 8	19, 7	25, 1	28, 8	5, 11	38, 7	42, 8	43, 11

TABLE 1: VALUES OF M, N ($\nu = 0.3$) h, R, L dimensional corresponding to σ_{min}

rigorously been shown to be convergent for simple eigenvalues (see reference [13]). When N = 0, equations (3.19)-(3.22) reduce to

$$h^2 w_{xxxx} + \sigma w_{xx} + \omega^2 w = 0$$

which has a continuum of solutions of arbitrary amplitude for $\sigma = \sigma_0$; since we can solve for this bifurcating branch exactly, we will assume $N > 0$ without any loss of generality.

Introduce the vector $\underline{u} = (u_1, u_2, u_3)$, where $u_1 = w$, $u_2 = f$, and $u_3 = \ell$, and define an inner product by

$$<\underline{u}, \underline{v}> = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 u_1 v_1 + u_2 v_2 \, dxdy + u_3 v_3 \quad .$$

The problem may be formulated as $G(\underline{u}, \sigma) = 0$ where

$$G(\underline{u}, \sigma) \equiv \begin{pmatrix} h^2\Delta^2 w + \sigma w_{xx} + \omega^2 f_{xx} - f_{yy} w_{xx} + 2 f_{xy} w_{xy} - f_{xx} w_{yy} + 2\ell w_{xy} \\ \omega^2 \Delta^2 f - \omega^2 w_{xx} + w_{xx} w_{yy} - w^2_{xy} \\ 4\pi\omega(1+\nu)\ell - \int_0^{\Omega}\int_0^1 w\, w_{xy} \, dxdy \end{pmatrix} \tag{3.32}$$

The linearized problem about $\underline{u} = \underline{0}$ is given, for $\underline{\tilde{\phi}} = (\tilde{w}, \tilde{f}, \tilde{\ell})$, by

$$G^o_{\underline{u}}\underline{\tilde{\phi}} = \begin{pmatrix} h^2\Delta^2\tilde{w} + \sigma_0 \tilde{w}_{xx} + \omega^2 \tilde{f}_{xx} \\ \omega^2\Delta^2\tilde{f} - \omega^2\tilde{w}_{xx} \\ 4\pi\omega(1+\nu)\tilde{\ell} \end{pmatrix} = 0 \tag{3.33}$$

This coincides with our pertubation analysis. A nontrivial solution exists if and only if $\sigma_0 = \sigma_0(t_{MN})$ for some (M, N). We assume this is the case, and we further assume σ_0 is simple. If we remove the translational arbitrariness of the solution, we have

$$\underline{\tilde{\phi}} = (\rho Q^2 \sin M\pi x \cos N\omega y, \; -\rho(M\pi)^2 \sin M\pi x \cos N\omega y, \; 0) \tag{3.34}$$

where

$$\rho = 1/\sqrt{Q^4 + (M\pi)^4} \quad , \quad Q = Q_{MN} \quad .$$

The factor ρ was chosen so that

$$\| \underline{\phi} \|^2 = \langle \underline{\phi}, \underline{\phi} \rangle = 1$$

The adjoint operator is

$$G_{\underline{u}}^{o\,\dagger} \underline{\phi}^* = \begin{pmatrix} h^2 \Delta^2 w^* + \sigma_0 w^*_{xx} - \omega^2 f^*_{xx} \\ \omega^2 \Delta^2 f^* + \omega^2 w^*_{xx} \\ 4\pi\omega(1+\nu)\, \ell^* \end{pmatrix} \tag{3.35}$$

where $\underline{\phi}^* = (w^*, f^*, \ell^*)$; the appropriate member of the null space is

$$\underline{\phi}^* = (Q^2 \sin M\pi x \cos N\omega y,\ (M\pi)^2 \sin M\pi x \cos N\omega y,\ 0) \tag{3.36}$$

We seek a functional $\Lambda(\underline{u})$ satisfying

$$\langle \underline{\phi}^*, G(\underline{u}, \Lambda(\underline{u})) - G_{\underline{u}}^o \underline{u} \rangle = 0 \quad .$$

Substituting and carrying out the required manipulations yields

$$\Lambda(\underline{u}) = \sigma_0 + N(\underline{u}) / D(\underline{u})$$

where

$$N(\underline{u}) = \int_0^{\Omega} \int_0^1 [Q^2 (2\ell w_{xy} - f_{yy} w_{xx} + 2 f_{xy} w_{xy} - f_{xx} w_{yy}) + (M\pi)^2 (w_{xx} w_{yy} - w_{xy}^2)] \sin M\pi x \cos N\omega y \, dx dy$$

and

$$D(\underline{u}) = Q^2 (M\pi)^2 \int_0^{\Omega} \int_0^1 w \sin M\pi x \cos N\omega y \, dx dy$$

The iteration scheme yields the solution $(\underline{u}, \sigma)$ as the limit of $\{(\underline{u}^k, \sigma^k)\}$, where

$$\underline{u}^k = \epsilon(\underline{\phi} + \epsilon \underline{v}^k) \tag{3.38a}$$

$$\sigma^{k+1} = \Lambda(\underline{u}^k) \tag{3.38b}$$

and $\underline{v}^{k+1}$ solves

$$G_{\underline{u}}^o \underline{v}^{k+1} = \epsilon^{-2} \{ G_{\underline{u}}^o \underline{u}^k - G(\underline{u}^k, \sigma^{k+1}) \}, \quad \langle \underline{\phi}^*, \underline{v}^{k+1} \rangle = 0 \tag{3.38c}$$

with $\underline{v}^o = 0$. Furthermore, the error decreases as ϵ^{k+1}.

With this notation $\underline{u}^o = \tilde{\underline{\phi}}$, as defined in (3.34); substituting this into (3.38b) yields $\sigma^1 = \sigma_0$. For the next iterate set $\underline{v}^1 = (w_1, f_1, \ell_1)$. After a little simplification, equation (3.38c) can be written as

$$h^2\Delta^2 w_1 + \sigma_0 w_{1xx} + \omega^2 f_{1xx} = \tilde{w}_{xx}\tilde{f}_{yy} - 2\tilde{w}_{xy}\tilde{f}_{xy} + \tilde{w}_{yy}\tilde{f}_{xx} \qquad (3.39a)$$

$$\omega^2(\Delta^2 f_1 - w_{1xx}) = \tilde{w}^2_{xy} - \tilde{w}_{xx}\tilde{w}_{yy} \qquad (3.39b)$$

$$4\pi\omega(1+\nu)\,\ell_1 = \int_0^\Omega\int_0^1 \tilde{w}\,\tilde{w}_{xy}\,dxdy \qquad (3.39c)$$

along with the normalization condition

$$\int_0^\Omega\int_0^1 w_1 w^* + f_1 f^*\,dxdy = 0 \quad . \qquad (3.39d)$$

We can evaluate the right sides of equations (3.39a, b, c) using (3.34); the results are

$$h^2\Delta^2 w_1 + \sigma_0 w_{1_{xx}} + \omega^2 f_{1_{xx}} = \rho^2 (M\pi)^4 (N\omega)^2 Q^2 (\cos 2M\pi x - \cos 2N\omega y)$$

$$\omega^2(\Delta^2 f_1 - w_{1_{xx}}) = \tfrac{1}{2}\rho^2 (M\pi)^2 (N\omega)^2 Q^4 (\cos 2M\pi x - \cos 2N\omega y)$$

$$\ell_1 = 0$$

A particular solution for w_1 and f_1 is

$$\left.\begin{aligned} w_1 &= a_0(x) + a_2(x)\cos 2N\omega y \\ f_1 &= A_0(x) + A_2(x)\cos 2N\omega y \end{aligned}\right\} \qquad (3.40)$$

with

$$\left.\begin{aligned} h^2 a_0^{iv} + \sigma_0 a_0'' + \omega^2 A_0'' &= \rho^2 (M\pi)^4 (N\omega)^2 Q^2 \cos 2M\pi x \\ \omega^2(A_0^{iv} - a_0'') &= \tfrac{1}{2}\rho^2 (M\pi)^2 (N\omega)^2 Q^4 \cos 2M\pi x \end{aligned}\right\} \qquad (3.41)$$

and

$$h^2[a_2^{iv} - 2(2N\omega)^2 a_2'' + (2N\omega)^4 a_2] + \sigma_0 a_2'' + \omega^2 A_2'' = -\rho^2 (M\pi)^4 (N\omega)^2 Q^2$$
$$\qquad (3.42)$$
$$\omega^2[A_2^{iv} - 2(2N\omega)^2 A_2'' + (2N\omega)^4 A_2 - a_2''] = -\tfrac{1}{2}\rho^2 (M\pi)^2 (N\omega)^2 Q^4$$

Note that a_0, a_2, A_0, A_2 satisfy the boundary conditions

$$a_0 = a_{0_{xx}} = 0 \quad \text{at} \quad x = 0, 1, \quad \text{etc.}$$

This particular solution is in fact the desired solution, for it satisfies (3.39d) with $\underline{\phi}^*$ given by (3.36), and also with $\underline{\phi}^*$ given by

$$\underline{\psi}^* = (Q^2 \sin M\pi x \sin N\omega y, \quad (M\pi)^2 \sin M\pi x \sin N\omega y \;, \; 0) \tag{3.36'}$$

($\underline{\phi}^*$ and $\underline{\psi}^*$ together span the nullspace of $G_{\underline{u}}^{o\dagger}$). The assumption that σ_0 is a simple eigenvalue assures us that there exist no non-trivial solutions to the homogeneous forms of equations (3.41) and (3.42) satisfying the boundary conditions; consequently the equations are invertible and we are guaranteed that a_0, a_2, A_0, and A_2 exist.

To solve (3.41) we first need the characteristic exponents of the corresponding homogeneous constant coefficient system. Setting $a_0 = c\,e^{i\mu x}$, $A_0 = C\,e^{i\mu x}$, we find for nontrivial c and C that

$$\mu^4 (h^2 \mu^4 - \sigma_0 \mu^2 + \omega^2) = 0$$

which has roots $\mu = 0$ and $\mu = \pm\mu_+$, $\mu = \pm\mu_-$ where we define $\mu_+ > 0$ and $\mu_- > 0$ by

$$\mu_+^2 = (\sigma_0 + \sqrt{\sigma_0^2 - 4\omega^2 h^2}\,)/\,2h^2$$

$$\mu_-^2 = (\sigma_0 - \sqrt{\sigma_0^2 - 4\omega^2 h^2}\,)/\,2h^2$$

(Remark that μ_+ and μ_- are real because $\sigma_0 = \sigma_0(t_{MN}) \geq 2\omega h$ insofar as $2\omega h$ is the global minimum of $\sigma(t)$ for $t > 0$.) This generates eight linearly independent solutions of the homogeneous system:

$$\begin{pmatrix} a_0 \\ A_0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} -\mu_+^2 \cos \mu_+ x \\ \cos \mu_+ x \end{pmatrix}, \begin{pmatrix} -\mu_+^2 \sin \mu_+ x \\ \sin \mu_+ x \end{pmatrix}, \begin{pmatrix} -\mu_-^2 \cos \mu_- x \\ \cos \mu_- x \end{pmatrix}, \begin{pmatrix} -\mu_-^2 \sin \mu_- x \\ \sin \mu_- x \end{pmatrix}$$

A particular solution can be found as a constant vector times $\cos 2M\pi x$. Now equations (3.41) and their associated boundary conditions admit a solution which is symmetric about $x = \frac{1}{2}$; consequently we can find constants $\beta_1, \ldots, \beta_6$ such that

$$\begin{pmatrix} a_0(x) \\ A_0(x) \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta_3 \cos\mu_+(x-\tfrac{1}{2}) \begin{pmatrix} -\mu_+^2 \\ 1 \end{pmatrix} + \beta_4 \cos\mu_-(x-\tfrac{1}{2}) \begin{pmatrix} -\mu_-^2 \\ 1 \end{pmatrix}$$

$$+ \begin{pmatrix} \beta_5 \\ \beta_6 \end{pmatrix} \cos 2M\pi x$$

β_5 and β_6 are determined by the inhomogeneous terms; then β_1, β_2, β_3, and β_4 are determined from the boundary conditions. After carrying out the indicated algebra, we find

$$\beta_1 = -\lambda_2/\omega^2(2M\pi)^2$$

$$\beta_2 = (\lambda_1\omega^2 + \lambda_2\sigma_0)/\omega^4(2M\pi)^2$$

$$\beta_3 = (2M\pi)^2(\beta_5 + \beta_6\mu_-^2)/(\mu_+^2 - \mu_-^2)\mu_+^2 \cos\tfrac{1}{2}\mu_+$$

$$\beta_4 = (2M\pi)^2(\beta_5 + \beta_6\mu_+^2)/(\mu_-^2 - \mu_+^2)\mu_-^2 \cos\tfrac{1}{2}\mu_-$$

$$\beta_5 = (\lambda_1(2M\pi)^2 + \lambda_2)/D$$

$$\beta_6 = (\lambda_2(2M\pi)^2 h^2 - \lambda_2\sigma_0 - \omega^2\lambda_1)/\omega^2 D$$

where

$$\lambda_1 = \rho^2(M\pi)^4(N\omega)^2 Q^2 \quad , \qquad \lambda_2 = \tfrac{1}{2}\rho^2(M\pi)^2(N\omega)^2 Q^4$$

$$D = (2M\pi)^2[h^2(2M\pi)^4 - \sigma_0(2M\pi)^2 + \omega^2]$$

We solve equations (3.42) in a similar fashion. It is easy to see that a particular solution is $a_{2p} = -\lambda_1/h^2(2N\omega)^4$, $A_{2p} = -\lambda_2/\omega^2(2N\omega)^4$. Solutions of the homogeneous system exist of the form $a_2 = c\,e^{i\nu x}$, $A_2 = C\,e^{i\nu x}$, where ν must satisfy

$$h^2[\nu^2 + (2N\omega)^2]^4 - \sigma_0[\nu^2 + (2N\omega)^2]^2\nu^2 + \omega^2\nu^4 = 0 \ .$$

Again the solution is symmetric about $x = \frac{1}{2}$, and we write

$$\begin{pmatrix} a_2(x) \\ A_2(x) \end{pmatrix} = \begin{pmatrix} -\lambda_1/h^2(2N\omega)^4 \\ -\lambda_2/\omega^2(2N\omega)^4 \end{pmatrix} + \sum_{j=1}^{4} \gamma_j \cos \nu_j(x-\tfrac{1}{2}) \begin{pmatrix} Q_j^2 \\ -\nu_j^2 \end{pmatrix} .$$

where $Q_j = \nu_j^2 + (2N\omega)^2$ and γ_1, γ_2, γ_3, γ_4 are constants to be determined by the boundary conditions. In general, there exist four distinct complex values for ν^2. Although an exact solution exists for the roots of a quartic equation, the answer is unwieldly and impracticable. For the purposes of numerical computation, it was found to be easier to find roots ν^2 by Newtonian iteration. When $\sigma_0 = \sigma_{min} = 2\omega h + \delta^2$ with $\delta^2 << 2\omega h$, good first estimates for two of the roots are roots of

$$h[\nu^2 + (2N\omega)^2]^2 - \omega\nu^2 = 0$$

viz.,

$$\nu^2 = \frac{\omega}{2h} - (2N\omega)^2 \pm \sqrt{\frac{\omega^2}{4h^2} - \frac{\omega}{h}(2N\omega)^2} .$$

Once two values of ν^2 have been found by iteration, the quartic may be reduced to a managable quadratic in the remaining roots. Alternately, an analytic expression may be found for ν^2 by expanding as a power series in δ.

Finally, $\underline{v}^1$ is known, and thence $\underline{u}^1$; substituting into (3.38b) we have an expression for the second iterate, σ^2, which after considerable simplification can be written as

$$\sigma^2 = \sigma_0 + \epsilon^2(N\omega)^2 H/Q^2 = \sigma_0 + \epsilon^2\sigma_2$$

where

$$H = \int_0^1 -8(M\pi)^2 a_0(x)\cos 2M\pi x + 4Q^2 A_0(x)\cos 2M\pi x + 4(M\pi)^2 a_2(x) - 2Q^2 A_2(x)\, dx .$$

The integrations indicated can be carried out explicitly:

$$\int_0^1 a_0(x)\cos 2M\pi x\, dx = \frac{2\beta_3\mu_+^3}{(2M\pi)^2-\mu_+^2} + \frac{2\beta_4\mu_-^3}{(2M\pi)^2-\mu_-^2} + \frac{\beta_5}{2}$$

$$\int_0^1 A_0(x)\cos 2M\pi x\, dx = \frac{2\beta_3\mu_+}{\mu_+^2-(2M\pi)^2} + \frac{2\beta_4\mu_-}{\mu_-^2-(2M\pi)^2} + \frac{\beta_6}{2}$$

$$\int_0^1 a_2(x)\, dx = -\lambda_1/h^2(2N\omega)^4 + \sum_{j=1}^{4} 2\gamma_j Q_j^2 \sin(\nu_j/2)/\nu_j$$

$$\int_0^1 A_2(x)\, dx = -\lambda_2/\omega^2(2N\omega)^4 - \sum_{j=1}^{4} 2\gamma_j \nu_j \sin(\nu_j/2)$$

Table 2 lists values of σ_2 computed for the modes (M, N) of table 1. The data reflect few common features. They may be either negative or positive, although these occur in proportion four to one. Also, they vary in magnitude from 11 to 21, 116, 509. However, most values are rather large--on the order of 10^3 or more--indicating that the bifurcating branch is steep and that σ changes value rapidly.

Bifurcation for double eigenvalues

We next investigate the ramifications of σ_0 being a double eigenvalue. Assume that there exist distinct pairs (M_1, N_1) and (M_2, N_2) such that $\sigma_0 = \sigma_0(M_1, N_1) = \sigma_0(M_2, N_2)$ and that there exist no other pairs. We further assume that $N_1 > 0$, for if both $N_1 = 0$ and $N_2 = 0$, then the leading order solution which bifurcates from Poisson expansion is symmetric; however, for symmetric solutions the equations become linear and we get a two parameter family of solutions of arbitrary amplitude, all corresponding to $\sigma = \sigma_0$.

We refer back to the pertubation expansion used to derive

R/h \ L/R	0.1	0.25	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
1,000	93,085	-1,018	-1,822	-561	-457	-707	-2,048	13,659	-1,107	-929	-485	-3,042
900	16,883	-267	-32,005	-2,271	1,124	14,114	348	1,114	-324	-35,232	-119	3,234
800	4,023	-452	-542	-518	-669	-836	-18,182	-6,059	-7,865	-1,303	-50	-7,955
700	782	-335,820	-183,605	-29,864	2,116,509	135,476	-39,388	-376	-1,499	-292,157	-579,578	-1,072
600	-212	-412	146,988	-3,981	-19,430	-3,853	2,555	-108	-5,671	-12,126	-1,802	7,229
500	-505	-1,990	-70	-16,132	2,305	-296	-3,775	14,223	-951	-294	-1,601	-10,968
400	-555	13,302	-303	1,871	-1,707	-626	-1,740	6,273	-630	-829	-11	-2,447
300	-1,106	-37	-60	-16	-132	-1,141	-14	-1,148	268	-20,007	-1,045	-152

TABLE 2: VALUES OF σ_2 $(\nu = 0.3)$ h, R, L dimensional

equations (3.25) and (3.26). Since $N_1 > 0$, we can remove the translational indeterminancy in y by suppressing the mode $\sin M_1 \pi x \sin N_1 \omega y$ from the leading order solution. Hence, for some constants c_1, c_2, d_2

$$\left.\begin{aligned} w_1 &= c_1 Q_1^2 \sin M_1 \pi x \cos N_1 \omega y + c_2 Q_2^2 \sin M_2 \pi x \cos N_2 \omega y \\ &\qquad + d_2 Q_2^2 \sin M_2 \pi x \sin N_2 \omega y \\ \phi_1 &= -c_1 (M_1\pi)^2 \sin M_1 \pi x \cos N_1 \omega y - c_2 (M_2\pi)^2 \sin M_2 \pi x \cos N_2 \omega y \\ &\qquad - d_2 (M_2\pi)^2 \sin M_2 \pi x \sin N_2 \omega y \end{aligned}\right\} \quad (3.43)$$

where $Q_i = (M_i\pi)^2 + (N_i\omega)^2$, $i = 1, 2$. The normalization condition (3.28) integrates to yield

$$1 = c_1^2 [Q_1^4 + (M_1\pi)^4] + c_2^2 [Q_2^4 + (M_2\pi)^4] + d_2^2 [Q_2^4 + (M_2\pi)^4] \quad (3.44)$$

Using (3.43), equations (3.26a) and (3.26b) become, after some simplification:

$$h^2 \Delta^2 w_2 + \sigma_0 w_{2_{xx}} + \omega^2 f_{2_{xx}} = R_1(x, y) \quad (3.45a)$$

where

$$\begin{aligned} R_1(x, y) \equiv\ & \sin^2 N_1\omega y \cos M_1 \pi x [2 c_1^2 Q_1^2 (N_1\omega)^2 (M_1\pi)^4] \\ & - \cos^2 N_1 \omega y \sin^2 M_1 \pi x [2 c_1^2 Q_1^2 (N_1\omega)^2 (M_1\pi)^4] \\ & + \sin^2 N_2 \omega y \cos^2 M_2 \pi x [2 c_2^2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4] \\ & - \cos^2 N_2\omega y \sin^2 M_2 \pi x [2 c_2^2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4] \\ & + \cos^2 N_2 \omega y \cos^2 M_2 \pi x [2 d_2^2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4] \\ & - \sin^2 N_2\omega y \sin^2 M_2 \pi x [2 d_2^2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4] \\ & - \cos N_1 \omega y \cos N_2 \omega y \sin M_1 \pi x \sin M_2 \pi x \ c_1 c_2 \\ & \quad [(N_1\omega)^2 (M_1\pi)^2 Q_2^2 (M_2\pi)^2 + (M_1\pi)^4 Q_2^4 (N_2\omega)^2 \\ & \quad + Q_1^2 (M_1\pi)^2 (N_2\omega)^2 (M_2\pi)^2 + Q_1^2 (N_1\omega)^2 (M_2\pi)^4] \\ & - \cos N_1 \omega y \sin N_2 \omega y \sin M_1 \pi x \sin M_2 \pi x \ c_1 d_2 \\ & \quad [Q_1^2 (M_1\pi)^2 (N_2\omega)^2 (M_2\pi)^2 + Q_1^2 (N_1\omega)^2 (M_2\pi)^4 \\ & \quad + (N_1\omega)^2 (M_1\pi)^2 Q_2^2 (M_2\pi)^2 + (M_1\pi)^4 Q_2^{\,2} (N_2\omega)^2] \end{aligned}$$

$$-\cos N_2\omega y \sin N_2\omega y \sin^2 M_2 \pi x [4 c_2 d_2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4]$$

$$+\sin N_1\omega y \sin N_2\omega y \cos M_1 \pi x \cos M_2 \pi x \; 2 c_1 c_2 (N_1\omega)(N_2\omega)(M_1\pi)(M_2\pi)[Q_1^2 (M_2\pi)^2 + Q_2^2 (M_1\pi)^2]$$

$$-\sin N_1\omega y \cos N_2\omega y \cos M_1 \pi x \cos M_2 \pi x \; 2 c_1 d_2 (N_1\omega)(N_2\omega)(M_1\pi)(M_2\pi)[Q_1^2 (M_2\pi)^2 + Q_2^2 (M_1\pi)^2]$$

$$-\sin N_2\omega y \cos N_2\omega y \cos^2 M_2 \pi x [4 c_2 d_2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4]$$

$$+\cos N_1\omega y \sin M_1 \pi x [\sigma_1 c_1 Q_1^2 (M_1\pi)^2]$$

$$+\cos N_2\omega y \sin M_2 \pi x [\sigma_1 c_2 Q_2^2 (M_2\pi)^2]$$

$$+\sin N_2\omega y \sin M_2 \pi x [\sigma_1 d_2 Q_2^2 (M_2\pi)^2]$$

and

$$\omega^2 (\Delta^2 f_2 - w_{2_{xx}}) = R_2(x, y) \tag{3.45b}$$

where

$$\begin{aligned}
R_2(x, y) \equiv\; & \sin^2 N_1\omega y \cos^2 M_1 \pi x [c_1^2 Q_1^4 (N_1\omega)^2 (M_1\pi)^2] \\
& -\cos^2 N_1\omega y \sin^2 M_1 \pi x [c_1^2 Q_1^4 (N_1\omega)^2 (M_1\pi)^2] \\
& +\sin^2 N_2\omega y \cos^2 M_2 \pi x [c_2^2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2] \\
& -\cos^2 N_2\omega y \sin^2 M_2 \pi x [c_2^2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2] \\
& +\cos^2 N_2\omega y \cos^2 M_2 \pi x [d_2^2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2] \\
& -\sin^2 N_2\omega y \sin^2 M_2 \pi x [d_2^2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2] \\
& -\cos N_1\omega y \cos N_2\omega y \sin M_1 \pi x \sin M_2 \pi x \; c_1 c_2 Q_1^2 Q_2^2 \\
& \qquad [(N_1\omega)^2 (M_2\pi)^2 + (M_1\pi)^2 (N_2\omega)^2] \\
& -\cos N_1\omega y \sin N_2\omega y \sin M_1 \pi x \sin M_2 \pi x \; c_1 d_2 Q_1^2 Q_2^2 \\
& \qquad [(N_1\omega)^2 (M_2\pi)^2 + (M_1\pi)^2 (N_2\omega)^2] \\
& -\cos N_2\omega y \sin N_2\omega y \sin^2 M_2 \pi x [2 c_2 d_2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2] \\
& +\sin N_1\omega y \sin N_2\omega y \cos M_1 \pi x \cos M_2 \pi x \\
& \qquad [2 c_1 c_2 Q_1^2 Q_2^2 (N_1\omega)(N_2\omega)(M_1\pi)(M_2\pi)] \\
& -\sin N_1\omega y \cos N_2\omega y \cos M_1 \pi x \cos M_2 \pi x \\
& \qquad [2 c_1 d_2 Q_1^2 Q_2^2 (N_1\omega)(N_2\omega)(M_1\pi)(M_2\pi)]
\end{aligned}$$

$$-\sin N_2\omega y \cos N_2\omega y \cos^2 M_2 \pi x [2 c_2 d_2 Q_2^4 (N_2\omega)^2 (M_2 \pi)^2]$$

The linear operator

$$L\begin{pmatrix} w \\ f \end{pmatrix} \equiv \begin{pmatrix} h^2 \Delta^2 w + \sigma_0 w_{xx} + \omega^2 f_{xx} \\ \omega^2 (\Delta^2 f - w_{xx}) \end{pmatrix}$$

is not invertible. The nullspace of its adjoint is spanned by the four vectors

$$\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} = (Q_1^2 \cos N_1\omega y \sin M_1 \pi x, \ \ (M_1 \pi)^2 \cos N_1\omega y \sin M_1 \pi x)^t$$

$$\begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} = (Q_1^2 \sin N_1\omega y \sin M_1 \pi x, \ \ (M_1 \pi)^2 \sin N_1\omega y \sin M_1 \pi x)^t$$

$$\begin{pmatrix} \phi_3 \\ \psi_3 \end{pmatrix} = (Q_2^2 \cos N_2\omega y \sin M_2 \pi x, \ \ (M_2 \pi)^2 \cos N_2\omega y \sin M_2 \pi x)^t$$

$$\begin{pmatrix} \phi_4 \\ \psi_4 \end{pmatrix} = (Q_2^2 \sin N_2\omega y \sin M_2 \pi x, \ \ (M_2 \pi)^2 \sin N_2\omega y \sin M_2 \pi x)^t$$

We write equations (3.45) as

$$L\begin{pmatrix} w_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

the Fredholm Alternative [16] implies that (3.45) has a solution if and only if

$$\left\langle \begin{pmatrix} \phi_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \right\rangle = 0 \ , \quad k = 1, 2, 3, 4 \qquad (3.46)$$

where the inner product is defined by

$$\left\langle \begin{pmatrix} v \\ g \end{pmatrix}, \begin{pmatrix} u \\ f \end{pmatrix} \right\rangle \equiv \frac{4}{\Omega} \int_0^\Omega \int_0^1 uv + fg \ \ dxdy \quad .$$

Equations (3.46) and (3.44) provide the additional information needed to calculate c_1, c_2, and d_2. A table of integrals (see Table 3) facilitates

F	$\int_0^1 F \sin M\pi x \, dx$
$\sin^2 M\pi x$	0 M even , $4/3 M\pi$ M odd
$\cos^2 M\pi x$	0 M even , $2/3 M\pi$ M odd
$\sin^2 m\pi x$	0 M even , $4m^2/\pi M(4m^2-M^2)$ M odd
$\cos^2 m\pi x$	0 M even , $(4m^2-2M^2)/M\pi(4m^2-M^2)$ M odd
$\sin M\pi x \sin m\pi x$	0 m even , $4M^2/\pi m(4M^2-m^2)$ m odd
$\cos M\pi x \cos m\pi x$	0 m even , $2M/\pi(4M^2-m^2)$ m odd

for $n > 0$, $N > 0$:

F	$\int_0^\Omega F \cos N\omega y \, dy$	$\int_0^\Omega F \sin N\omega y \, dy$	$\int_0^\Omega F \cos n\omega y \, dy$	$\int_0^\Omega F \sin n\omega y \, dy$
$\sin^2 N\omega y$	0	0	$-\Omega/4$, $n = 2N$	0
$\cos^2 N\omega y$	0	0	$\Omega/4$, $n = 2N$	0
$\cos N\omega y \cos n\omega y$	$\Omega/4$, $n = 2N$	0	$\Omega/4$, $N = 2n$	0
$\cos N\omega y \sin n\omega y$	0	$\Omega/4$, $n = 2N$	0	$-\Omega/4$, $N = 2n$
$\cos n\omega y \sin n\omega y$	0	$\Omega/4$, $N = 2n$	0	0
$\sin N\omega y \sin n\omega y$	$\Omega/4$, $n = 2N$	0	$\Omega/4$, $N = 2n$	0
$\sin N\omega y \cos n\omega y$	0	$-\Omega/4$, $n = 2N$	0	$\Omega/4$, $N = 2n$

TABLE 3: "HANDY INTEGRALS"
(Note: if $n \neq 2N$, etc., integrals vanish)

the computation in (3.46). Equations (3.46) fall into four cases.

Case 1: M_1 and M_2 even, or $N_1 \neq 2N_2$ and $N_2 \neq 2N_1$ (but $N_2 > 0$), or both

This is the simplest case; one of the equations in (3.46) is vacuous, and the remaining three give

$$\sigma_1 c_1 = \sigma_1 c_2 = \sigma_1 d_2 = 0$$

From the normalization condition (3.44) it follows that $c_1 = c_2 = d_2 = 0$ is impossible. Consequently $\sigma_1 = 0$ and these three equations are satisfied for arbitrary values of c_1, c_2, and d_2. The normalization relation puts one constraint on these parameters and, in addition, guarantees that they are bounded. The result is a two-parameter family of solutions, even after the translational indeterminancy in y has been removed.

Case 2: $N_2 = 0$ (Recall $N_1 > 0$ always)

Take $d_2 = 0$ insofar as the corresponding terms in w_1 and f_1 are absent. Two of equations (3.46) are vacuous.

Subcase 2a: M_2 even

The remaining equations reduce to

$$\sigma_1 c_1 = \sigma_1 c_2 = 0$$

$$B_1 c_1^2 + B_2 c_2^2 = 1$$

where

$$B_1 = Q_1^4 + (M_1\pi)^4 \qquad B_2 = Q_2^4 + (M_2\pi)^4$$

Again $\sigma_1 = 0$; for this case we find a one-parameter family of solutions.

Subcase 2b: M_2 odd

Now the remaining equations reduce to

$$\sigma_1 c_1 = c_1 c_2 A_1 \qquad \sigma_1 c_2 = c_1^2 A_2$$

$$B_1 c_1^2 + B_2 c_2^2 = 1$$

where

$$A_1 = \pi[Q_1^2 + 2(M_1\pi)^2(M_2\pi)^2](N_1\omega)^2(2M_2)^3/Q_1^2(4M_1^2 - M_2^2)$$

$$A_2 = 2[Q_1^2 + 2(M_1\pi)^2(M_2\pi)^2]Q_1^2(N_1\omega)^2 M_1^2/(M_2\pi)^7(4M_1^2 - M_2^2)$$

The solutions of this system are:

(i) $c_1 = 0 \quad , \quad c_2 = 1/\sqrt{B_2} \quad , \quad \sigma_1 = 0$

(Note: $c_2 = -1/\sqrt{B_2}$ corresponds to the same branch with $\varepsilon < 0.$)

(ii) $c_1 = \sqrt{A_1/(A_1B_1 + A_2B_2)} \qquad c_2 = \sqrt{A_2/(A_1B_1 + A_2B_2)}$

$$\sigma_1 = A_1\sqrt{A_2/(A_1B_1 + A_2B_2)}$$

(iii) $c_1 = -\sqrt{A_1/(A_1B_1 + A_2B_2)} \qquad c_2 = \sqrt{A_2/(A_1B_1 + A_2B_2)}$

$$\sigma_1 = A_1\sqrt{A_2/(A_1B_1 + A_2B_2)}$$

The solution represented by (ii) has the form

$$w_1 = c_1 Q_1^2 \cos N_1\omega y \sin M_1\pi x + c_2 Q_2^2 \sin M_2\pi x \quad .$$

If we translate $y \to y + \pi/N_1\omega$ this solution transforms into

$$w_1 = -c_1 Q_1^2 \cos N_1\omega y \sin M_1\pi x + c_2 Q_2^2 \sin M_2\pi x$$

which is the solution given in (iii). Consequently, we see that for this case there exist two physically distinct solution branches that bifurcate from Poisson expansion: one corresponding to $\sigma_1 = 0$ and another corresponding to $\sigma_1 = A_1\sqrt{A_2/(A_1B_1 + A_2B_2)}$.

Case 3: $N_2 = 2N_1$

Subcase 3a: M_2 even

One of equations (3.46) is vacuous, the remaining equations and (3.44) yield:

$$\sigma_1 c_1 = \sigma_1 c_2 = \sigma_1 d_2 = 0$$

$$B_1 c_1^2 + B_2 c_2^2 + B_2 d_2^2 = 1$$

which is the same as case 1.

Subcase 3b: M_2 odd

The relevant equations become

$$\sigma_1 c_1 = c_1 c_2 A_3 \qquad \sigma_1 c_2 = c_1{}^2 A_4$$

$$\sigma_1 d_2 = 0 \qquad c_1 d_2 = 0$$

$$B_1 c_1{}^2 + B_2 c_2{}^2 + B_2 d_2{}^2 = 1$$

where

$$A_3 = \frac{4\pi\omega^2 (Q_1^2 M_2^2 + 2Q_2^2 M_1^2)(N_2^2 M_1^2 - N_1 N_2 M_2^2 + N_1^2 M_2^2)}{(4M_1^2 - M_2^2) Q_1^2 M_2}$$

$$= 4\pi (N_1\omega)^2 (Q_1{}^2 M_2{}^2 + 2Q_2{}^2 M_1{}^2)/Q_1{}^2 M_2$$

$$A_4 = 2\pi Q_1{}^2 (N_1\omega)^2 M_1{}^2 (Q_1{}^2 M_2{}^2 + 2Q_2{}^2 M_1{}^2)/Q_2{}^4 M_2{}^3$$

The solutions fall into three sets:

(i) a one-parameter family of solutions described by

$$\sigma_1 = 0 \ , \quad c_1 = 0 \qquad c_2{}^2 + d_2{}^2 = 1/B_2$$

(ii) a branch described by

$$c_1 = \sqrt{A_3/(A_3 B_1 + A_4 B_2)} \qquad c_2 = \sqrt{A_4/(A_3 B_1 + A_4 B_2)}$$

$$d_2 = 0 \qquad \sigma_1 = A_3 \sqrt{A_4/(A_3 B_1 + A_4 B_2)}$$

(iii) a branch described by

$$c_1 = -\sqrt{A_3/(A_3 B_1 + A_4 B_2)} \qquad c_2 = \sqrt{A_4/(A_3 B_1 + A_4 B_2)}$$

$$d_2 = 0 \qquad \sigma_1 = A_3 \sqrt{A_4/(A_3 B_1 + A_4 B_2)}$$

It should be noted that solution (i) has reintroduced the translational indeterminancy, since setting $c_1 = 0$ removes one mode. Thus we are free to set $d_2 = 0$, for example, in which case (i) represents the non-degenerate bifurcation of a branch given by

$$c_1 = 0 \ , \quad c_2 = 1/\sqrt{B_2} \ , \quad d_2 = 0 \ , \quad \sigma_1 = 0 \quad .$$

As in case 2b, solutions (ii) and (iii) represent the same physical solution. Consequently, there exist precisely two physically distinct bifurcation branches.

Case 4: $N_1 = 2N_2$

Subcase 4a: M_1 even

This case is identical with cases 1 and 3a.

Subcase 4b: M_1 odd

Equation (3.44) and simplified forms of equations (3.46) are

$$\sigma_1 c_2 = A_5 c_1 c_2 \qquad \sigma_1 d_2 = -A_5 c_1 d_2$$

$$\sigma_1 c_1 = A_6 (c_2^2 - d_2^2) \qquad c_2 d_2 = 0$$

$$B_1 c_1^2 + B_2 c_2^2 + B_2 d_2^2 = 1$$

where

$$A_5 = 4\pi(N_2\omega)^2 (Q_2^2 M_1^2 + 2Q_1^2 M_2^2)/Q_2^2 M_1$$

$$A_6 = 2\pi Q_2^2 (N_2\omega)^2 M_2^2 (Q_2^2 M_1^2 + 2Q_1^2 M_2^2)/Q_1^4 M_1^3$$

There are five solution sets

(i) $c_1 = 1/\sqrt{B_1}\ , \quad c_2 = 0\ , \quad d_2 = 0\ , \quad \sigma_1 = 0$

(ii) $c_1 = \sqrt{A_6/(A_6 B_1 + A_5 B_2)}\ , \quad c_2 = \sqrt{A_5/(A_6 B_1 + A_5 B_2)}$

$d_2 = 0\ , \quad \sigma_1 = A_5 \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$

(iii) $c_1 = \sqrt{A_6/(A_6 B_1 + A_5 B_2)}\ , \quad c_2 = -\sqrt{A_5/(A_6 B_1 + A_5 B_2)}$

$d_2 = 0\ , \quad \sigma_1 = A_5 \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$

(iv) $c_1 = -\sqrt{A_6/(A_6 B_1 + A_5 B_2)}\ , \quad c_2 = 0$

$d_2 = \sqrt{A_5/(A_6 B_1 + A_5 B_2)}\ , \quad \sigma_1 = A_5 \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$

(v) $c_1 = -\sqrt{A_6/(A_6 B_1 + A_5 B_2)}\ , \quad c_2 = 0$

$d_2 = -\sqrt{A_6/(A_6 B_1 + A_5 B_2)}\ , \quad \sigma_1 = A_5 \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$

The translation $y \to y + \pi/N_2\omega$ shows that solutions (ii) and (iii) coincide, and that solutions (iv) and (v) coincide. Using the fact that $N_1 = 2N_2$, we find that the translation $y \to y + \pi/N_1\omega$ takes solution (ii)

$$w_1 = c_1 Q_1^2 \cos N_1\omega y \sin M_1 \pi x + c_2 Q_2^2 \cos N_2\omega y \sin M_2 \pi x$$

into

$$w_1 = -c_1 Q_1^2 \cos N_1\omega y \sin M_1 \pi x - c_2 Q_2^2 \sin N_2\omega y \sin M_2 \pi x$$

which is solution (v). Consequently, solutions (ii)-(v) all represent the same physical solution, and so two branches bifurcate from Poisson expansion. Actually this just serves as a verification of case 3, since the two are physically symmetric to one another.

Reference [13] also provides a rigorous justification for the branches found by the pertubation expansion when c_1, c_2, d_2, and σ_1 are isolated roots of the algebraic bifurcation equations. This is true for cases 2b, 3b, and 4b.

Multiple eigenvalues can exhibit a curious effect not possible for simple eigenvalues. Recall that for a simple eigenvalue, ℓ vanishes to $O(\varepsilon^2)$. Using equations (3.26c) and (3.43) we find that for a double eigenvalue, $\ell_2 = 0$ unless $N_1 = N_2 = N > 0$ and $M_1 + M_2$ is odd. Under these special circumstances we calculate

$$\omega(1+\nu)\ell_2 = c_1 d_2 Q_1^2 Q_2^2 N M_1 M_2/(M_1^2 - M_2^2) \quad .$$

From our pertubation analysis (case 1) there exists a two-parameter family of values c_1, c_2, d_2; consequently, we can find values with $c_1 d_2 \neq 0$. It follows that the cylinder can be subjected to a uniform torque, even though there is no tangential displacement ($v = 0$) at the edges.

CHAPTER 4

CIRCULAR CYLINDRICAL SHELLS: THE DYNAMIC PROBLEM

In this chapter we propose to study the dynamic buckling of a circular cylindrical shell using multi-time scale perturbation methods [10]. When studying the stability of a solution to a nonlinear problem, one commonly considers the linearized equations for small perturbations to the solution. If all such perturbations vanish (exponentially) for large time, the solution is said to be stable; but if even one perturbation grows (exponentially), it is said to be unstable. In the latter case the linearized equations become an invalid approximation as the solution grows in magnitude. Matkowsky [14] has found that it is sometimes possible to examine the effect that nonlinearities have on curbing such growth for parameters which are only "a small distance" into the unstable regime. Reiss and Matkowsky [15] have applied this method to study the buckling of rods.

We will first illustrate the method by applying it to study the buckling of a rectangular plate. Although the governing equations are closely related to those for a circular cylindrical shell, the computations are considerably simpler, thus rendering the exposition clearer. It will turn out that the equation describing the nonlinear growth is almost identical with that for rods; however, we will make several observations not found in [15]. Following that, we proceed to apply the method to the problem of the cylinder.

The rectangular plates

The static equations governing the buckling of plate of length L_x and width L_y may be obtained from the local equations (3.1)-(3.5) for a circular cylindrical shell by letting the radius of curvature R become infinite. We assume that the plate is subjected to an end thrust directed along its length, and that the edges are simply supported. We make the problem dimensionless by measuring in units of length L_x; this necessitates introducing a parameter

$$\rho \equiv L_x/L_y$$

The dynamic equations are obtained by adding to equilibrium equation (3.3c) a term representing the acceleration and one representing damping effects. One obtains the non-dimensional equations for $0 \le x \le 1,\ 0 \le y \le \rho$

$$w_{tt} + 2\Gamma w_t + h^2 \Delta^2 + \sigma w_{xx} =$$

$$f_{yy} w_{xx} - 2f_{xy} w_{xy} + f_{xx} w_{yy} - 2\ell w_{xy} \qquad (4.1a)$$

$$\Delta^2 f = w_{xy}^2 - w_{xx} w_{yy} \qquad (4.1b)$$

$$2(1+\nu)\rho\ell = \int_0^\rho \int_0^1 w\, w_{xy}\, dxdy \qquad (4.1c)$$

subject to

$$\left.\begin{array}{llll} w = w_{xx} & = 0 & \text{at} & x = 0, 1 \\ w = w_{yy} & = 0 & \text{at} & y = 0, \rho \\ f = f_{xx} & = 0 & \text{at} & x = 0, 1 \\ f = f_{yy} & = 0 & \text{at} & y = 0, \rho \end{array}\right\} \qquad (4.2)$$

We will be primarily concerned with behavior when damping is small, i.e. $\Gamma = \epsilon\gamma$ for some small parameter $\epsilon > 0$.

The equilibrium configuration whose stability we analyze is $w = 0,\ f = 0,\ \ell = 0$. (For small loads, a plate remains unaltered, whereas a cylinder undergoes Poisson expansion.) The linearized form of equations (4.1) about this state is

$$w_{tt} + h^2 \Delta^2 w + \sigma w_{xx} = 0$$
$$\Delta^2 f = 0 \tag{4.3}$$

where we have assumed $w = O(\epsilon)$ and $f = O(\epsilon)$. A complete set of functions satisfying (4.2) is $\{Y_{mn}\}$, where

$$Y_{mn}(x, y) \equiv \sin m\pi x \sin n\omega y$$

with $\omega = \pi/\rho$ now. Let

$$w = \Sigma\, w_{mn}(t)\, Y_{mn}$$
$$f = \Sigma\, f_{mn}(t)\, Y_{mn}$$

From (4.3) we conclude

$$w_{mn,tt} + (h^2 Q^2_{mn} - \sigma(m\pi)^2) w_{mn} = 0$$
$$Q^2_{mn} f_{mn} = 0$$

where we have retained the notation $Q_{mn} = (m\pi)^2 + (n\omega)^2$. It follows that $f = 0$ and $w_{mn} = a \cos\lambda_{mn} t + b \sin\lambda_{mn} t$, with

$$\lambda^2_{mn} = h^2 Q^2_{mn} - \sigma(m\pi)^2$$

λ^2_{mn} is a monotonically decreasing function of σ. If, for some (m, n) $\lambda^2_{mn} < 0$, then w_{mn} grows exponentially. Hence the solution $w = 0$, $f = 0$ is unstable for $\sigma > \sigma_o$, with

$$\sigma_o = \min_{m,n} \sigma_{mn} \tag{4.4a}$$

$$\sigma_{mn} = h^2 Q^2_{mn}/(m\pi)^2 = h^2/t_{mn} \tag{4.4b}$$

using the same definition for t_{mn} as in Chapter 3. We note in passing that the same arguments apply regarding the multiplicity of t_{mn} and consequently the multiplicity of σ_{mn} as an eigenvalue.

To find the behavior in the region $\sigma > \sigma_o$, we expand

$$\left.\begin{aligned} \sigma &= \sigma_0 + \sigma_1 \varepsilon + \sigma_2 \varepsilon^2 + \cdots \\ w &= w_1 \varepsilon + w_2 \varepsilon^2 + \cdots \\ f &= f_1 \varepsilon + f_0 \varepsilon^2 + \cdots \\ \ell &= \varepsilon^2 \ell_2 + \cdots \end{aligned}\right\} \tag{4.5}$$

Thus we are perturbing away from the transition boundary between stability and instability. We introduce multiple time scales t_k defined by

$$t_k = t\,\varepsilon^k \quad ; \quad k = 0, 1, 2, \ldots \tag{4.6}$$

and treat them formally as independent variables. The differential operator $\partial/\partial t$ transforms according to

$$\partial_t = \partial_{t_0} + \varepsilon\, \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \cdots \tag{4.7}$$

To simplify notation we shall write $t_0 = \tau$, $t_1 = s$, $t_2 = \eta$. Substituting (4.5) and (4.7) into equations (4.1) generates the following hierarchy:

$$O(\varepsilon) \qquad \partial^2_\tau w_1 + h^2 \Delta^2 w_1 + \sigma_0\, \partial^2_x w_1 = 0 \tag{4.8a}$$

$$\Delta^2 f_1 = 0 \tag{4.8b}$$

$O(\varepsilon^2)$ $\quad \partial_T^2 w_2 + 2\partial_{TS} w_1 + 2\gamma\partial_T w_1 + h^2\Delta^2 w_2$

$$+ \sigma_0 \partial_x^2 w_2 + \sigma_1 \partial_x^2 w_1 =$$

$$\sigma_y^2 f_1 \; \partial_x^2 w_1 - 2\partial_{xy} f_1 \; \partial_{xy} w_1 + \partial_x^2 f_1 \, \partial_y^2 w_1 \qquad (4.9a)$$

$$\Delta^2 f_2 = (\partial_{xy} w_1)^2 - \partial_x^2 w_1 \; \sigma_y^2 w_1 \qquad (4.9b)$$

$$2(1+\nu)\rho \, \ell_2 = \int_0^t \int_0^1 w_1 \; \partial_{xy} w_1 \, dxdy \qquad (4.9c)$$

$O(\varepsilon^3)$ $\quad \partial_T^2 w_3 + 2\,\partial_{TS} w_2 + \partial_S^2 w_1 + 2\partial_{T\eta} w_1 + 2\gamma\,\partial_T w_2$

$$+ 2\gamma\partial_S w_1 + h^2\Delta^2 w_3 + \sigma_0 \partial_x^2 w_3 + \sigma_1 \, \sigma_x^2 \, w_2 + \sigma_2 \partial_x^2 w_1$$

$$+ \ell_2 \; \partial_{xy} w_1 =$$

$$\partial_y^2 f_2 \, \partial_x^2 w_1 + \partial_y^2 f_1 \; \partial_x^2 w_2 - 2\partial_{xy} f_2 \; \partial_{xy} w_1 - 2\partial_{xy} f_1 \, \partial_{xy} w_2$$

$$+ \partial_x^2 f_2 \; \partial_y^2 w_1 + \partial_x^2 f_1 \; \partial_y^2 w_2 \qquad (4.10a)$$

$$\Delta^2 f_3 = 2\partial_{xy} w_1 \, \partial_{xy} w_2 - \partial_x^2 w_1 \; \partial_y^2 w_2 - \partial_x^2 w_2 \; \partial_y^2 w_1 \qquad (4.10b)$$

$$2(1+\nu)\rho \, \ell_3 = \int_0^\rho \int_0^1 w_1 \; \partial_{xy} w_2 + w_2 \; \partial_{xy} w_1 \; dxdy \qquad (4.10c)$$

We assume that σ_o is a simple eigenvalue; i.e. there exists a unique integer pair (M, N) with $M \geq 1$, $N \geq 1$ such that $\sigma_o = \sigma_o(M, N)$. Now

$$\sigma_o = \min_{m,n} h^2 Q_{mn}^2/(m\pi)^2$$

$$= \min_{m,n} h^2 \left[(m\pi)^2 + (n\omega)^2 \right]^2 / (m\pi)^2$$

But Q_{mn} is an increasing function of n, so that necessarily $N = 1$.

For initial conditions we take

$$w = \varepsilon \, \phi, \; w_t = \varepsilon^2 \psi \;\; \text{at} \;\; t = 0 \qquad (4.11)$$

The motivation for taking $w_t = O(\varepsilon^2)$ will be explained in the course of the calculations. Expressed in terms of the perturbation expansion, (4.11) becomes

$$\left.\begin{array}{lll} w_1 = \phi & \partial_\tau w_1 = 0 & \\ w_2 = 0 & \partial_\tau w_2 + \partial_s w_1 = \psi & \\ w_3 = 0 & \partial_\tau w_3 + \partial_s w_2 + \partial_\eta w_1 = 0 & \end{array}\right\} \text{at } t = 0 \qquad (4.12)$$

Introduce notation for the Fourier coefficients of a function g_k by

$$g^k_{mn} \equiv \frac{4}{\rho} \int_o^\rho \int_o^1 g_k Y_{mn} \, dxdy$$

Then equations (4.8) imply

$$\partial^2_\tau w^1_{mn} + \lambda^2_{mn} w^1_{mn} = 0 \qquad (4.13a)$$

$$Q^2_{mn} f^1_{mn} = 0 \qquad (4.13b)$$

where

$$\lambda^2_{mn} = h^2 Q^2_{mn} - \sigma_o (m\pi)^2 \qquad (4.14)$$

The solutions to (4.8) are

$$\left.\begin{array}{l} w_1 = \sum w^1_{mn} Y_{mn} \\ f_1 = 0 \end{array}\right\} \qquad (4.15)$$

where

$$w^1_{mn} = a^1_{mn} \cos \lambda_{mn} \tau + b^1_{mn} \sin \lambda_{mn} \tau \qquad (4.16a)$$

for $(m, n) \neq (M, 1)$ (in which case $\lambda^2_{mn} > 0$)

and

$$w^1_{M1} = w^1_{M1} + b^1_{M1}\tau \tag{4.16b}$$

since $\lambda^2_{M1} = 0$ by construction. We seek a solution which is bounded for all time; consequently, we conclude that $b^1_{M1} = 0$. Note that this in turn implies that $\partial_\tau w^1_{M1} = 0$. This is the reason for choosing $\partial_t w = O(\varepsilon^2)$; otherwise we would have to require that one particular mode is absent from the initial velocity. Rather than place such an awkward constraint on the initial data, we find that we can circumvent the difficulty using this simple device.

The solution satisfying initial conditions (4.12) is that of (4.16) where

$$a^1_{mn} = a^1_{mn}(s, \eta, \ldots)$$

$$b^1_{mn} = b^1_{mn}(s, \eta, \ldots)$$

and

$$a^1_{mn} = \phi_{mn} \qquad b^1_{mn} = 0 \qquad \text{at } t = 0 \ . \tag{4.17}$$

It is worth noting that choosing $b^1_{M1} \equiv 0$ is not the only possible resolution to the problem of keeping w^1_{M1} bounded--it is merely the simplest, and consequently, the most natural to try first. If we retain b^1_{M1} and later discover, for example, that it decays exponentially in s, this would also be sufficient.

Using (4.15), equation (4.9a) reduces to

$$\partial^2_\tau w_2 + 2\partial_{\tau s} w_1 + 2\gamma\partial_\tau w_1 + h_2 \Delta^2 w_2 + \sigma_0 \partial^2_x w_1 + \sigma_1 \partial^2_x w_1 = 0$$

which can be Fourier analyzed to give

$$\partial_\tau^2 w_{mn}^2 + h^2 \Omega_{mn}^2 w_{mn}^2 - \sigma_0 (m\pi)^2 w_{mn}^2 = \partial_\tau^2 w_{mn}^2 + \lambda_{mn}^2 w_{mn}^2 =$$

$$-2\, \partial_{\tau s} w_{mn}^1 - 2\gamma\, \partial_\tau w_{mn}^1 + \sigma_1 (m\pi)^2 w_{mn}^1 \qquad (4.18)$$

In order that w_{mn}^2 be bounded in τ, one applies a familiar argument to "suppress secular terms" [8]. Integrating by parts, we compute

$$\frac{1}{T}\int_0^T (\partial_\tau^2 w_{mn}^2 + \lambda_{mn}^2 w_{mn}^2)(\alpha \cos \lambda_{mn}\tau + \beta \sin \lambda_{mn}\tau)d\tau =$$

$$\frac{1}{T}\, [\, \partial_\tau\, w_{mn}^2(\alpha \cos\lambda_{mn}\tau + \beta\sin \lambda_{mn}\tau)$$

$$+ \lambda_{mn} w_{mn}^2(\alpha \sin \lambda_{mn}\tau - \beta\cos\lambda_{mn}\tau)]\Big|_0^T$$

$\to 0$ as $T \to \infty$ if w_{mn}^2 and $\partial_\tau\, w_{mn}^2$ are bounded. Hence it is necessary that

$$0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T (2\partial_{\tau s} w_{mn}^1 + 2\gamma\partial_\tau w_{mn}^1 - \sigma_1 (m\pi)^2 w_{mn}^1)(\alpha\cos\lambda_{mn}\tau$$

$$+ \beta\sin\lambda_{mn}\tau)d\tau \qquad (4.19)$$

Consider first the (M, 1) mode. $w_{M1}^1 = a_{M1}^1(s, \dots)$, $\lambda_{M1} = 0$, and so with $\alpha = 1$, $\beta = 0$, equation (4.19) becomes

$$\lim_{T\to\infty} \sigma_1 (m\pi)^2\, a_{M1}^1(s, \dots) = 0$$

which implies $\sigma_1 = 0$. Then for $(m, n) \neq (M, 1)$ equation (4.16a) yields

$$0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T (\partial_{\tau s} w_{mn}^1 + \gamma\partial_\tau w_{mn}^1)(\alpha\cos\lambda_{mn}\tau + \beta\sin\lambda_{mn}\tau)d\tau$$

$$= -\tfrac{1}{2}\beta\lambda_{mn}(\partial_s a_{mn}^1 + \gamma a_{mn}^1) + \tfrac{1}{2}\alpha\, \lambda_{mn}(\partial_s b_{mn}^1 + \gamma b_{mn}^1)$$

or since α and β are independent,

$$\partial_s a^1_{mn} + \gamma a^1_{mn} = 0 \qquad \partial_s b^1_{mn} + \gamma b^1_{mn} = 0$$

Applying initial conditions (4.17), we find for $(m, n) \neq M1$

$$a^1_{mn} = \hat{a}_{mn} e^{-\gamma s} \qquad b^1_{mn} = \hat{b}_{mn} e^{-\gamma s} \tag{4.20}$$

with

$$\hat{a}_{mn} = \hat{a}_{mn}(\eta, \dots) \qquad \hat{b}_{mn} = \hat{b}_{mn}(\eta, \dots)$$

$$\hat{a}_{mn} = \phi_{mn}, \quad \hat{b}_{mn} = 0 \quad \text{at} \quad t = 0 \tag{4.21}$$

Using (4.16) and (4.20), equation (4.18) simplifies to

$$\partial^2_\tau w^2_{mn} + \lambda^2_{mn} w^2_{mn} = 0$$

with solutions

$$w^2_{mn} = a^2_{mn} \cos\lambda_{mn}\tau + b^2_{mn} \sin\lambda_{mn}\tau, (m, n) \neq (M, 1) \tag{4.22a}$$

$$w^2_{M1} = a^2_{M1}(s, \eta, \dots) \tag{4.22b}$$

From equation (4.9b) we derive

$$Q^2_{mn} f^2_{mn} = \frac{4}{\rho} \int_0^\rho\int_0^1 (\partial_{xy} w_1 \, \partial_{xy} w_1 - \partial^2_x w_1 \, \partial^2_y w_1) Y_{mn} dxdy \tag{4.23}$$

We can characterize (to leading order) the large time behavior of w_1 by (4.15) and (4.16) once we find a^1_{M1}. To do this we must proceed to the $O(\epsilon^3)$ equations. With $f_1 = 0$ and $\sigma_1 = 0$, equation (4.10a) is

$$\partial^2_\tau w_3 + h^2 \Delta^2 w_3 + \sigma_0 \partial^2_x w_3 + 2\partial_{\tau s} w_2 + \partial^2_s w_1$$

$$+ 2 \partial_{\tau\eta} w_1 + 2\gamma\partial_\tau w_2 + 2\gamma\partial_s w_1 + \sigma_2 \partial^2_x w_1 =$$

$$2 \partial^2_y f_2 \partial^2_x w_1 - 2\partial_{xy} f_2 \sigma_{xy} w_1 + \partial^2_x f_2 \partial^2_y w_1 - \ell_2 \partial_{xy} w_1$$

Using the facts that $\lambda_{M1} = 0$ and $\partial_\tau w^1_{M1} = \partial_\tau w^2_{M1} = 0$, one readily calculates

$$\partial^2_\tau w^3_{M1} + \partial^2_s w^1_{M1} + 2\gamma \partial_s w^1_{M1} - \sigma_2 (M\pi)^2 w^1_{M1} = K \qquad (4.24a)$$

where

$$K = \frac{4}{\rho} \int_0^\rho \int_0^1 (\partial^2_y f_2 \partial^2_x w_1 - 2\partial_{xy} f_2 \partial_{xy} w_1 + \partial^2_x f_2 \partial^2_y w_1 - \ell_2 \partial_{xy} w_1) Y_{M1} dxdy \qquad (4.24b)$$

Up to this point ε has remained undefined; we have only used $\varepsilon > 0$. We effectively define ε by requiring that $\sigma_2 = 1$. Note then that

$$\sigma = \sigma_o + \varepsilon^2 + O(\varepsilon^3) > \sigma_o \quad \text{for} \quad O < \varepsilon << 1$$

so that $\varepsilon > 0$ puts σ into the unstable region.

Next we again "suppress secular terms," assuming that $\partial_\tau w^3_{M1}$ is bounded. Writing $w^1_{M1} = a(s, \eta, \ldots)$, we have

$$\partial^2_s a + 2\gamma \partial_s a - (M\pi)^2 a = H \qquad (4.25a)$$

with

$$H = \lim_{T \to \infty} \frac{1}{T} \int_o^T K \, d\tau \qquad (4.25b)$$

The actual calculation of H is lengthy and is left to Appendix B. We cite the result here, derived under the further assumption that all the λ_{mn} are distinct. Introduce the notation

$$s_k = \sin k\pi x \qquad c_k = \cos k\pi x$$
$$S_k = \sin k\omega y \qquad C_k = \cos k\omega y$$

and define the operators M and J by

$$M[g] = \lim_{T \to \infty} \frac{1}{T} \int_0^T g(\tau)d\tau \qquad (4.26)$$

$$J[G] = \frac{4}{\rho} \int_0^{\rho}\int_0^1 G(x, y)dxdy \qquad (4.27)$$

Then it can be shown that

$$M\{J[\partial_y^2 f_2\, \partial_x^2 w_1 - 2\partial_{xy} f_2\, \partial_{xy} w_1 + \partial_x^2 f_2\, \partial_y^2 w_1)Y_{M1}]\}$$

$$= k_1\, a + k_2\, a^3 \qquad (4.28a)$$

with

$$k_1 = \tfrac{1}{2}\pi^4\omega^4 \sum_{\substack{mn \neq M1 \\ \underline{mn}}} N^1_{mn} Q^{-2}_{\underline{mn}} m^2 n^2 J_1 ((M^2\underline{n}^2 + \underline{m}^2)J_2 - 2M\underline{m}\,\underline{n}\,J_3)$$

$$- N^1_{mn} Q^{-2}_{\underline{mn}} ((m^2\underline{n}^2 + \underline{m}^2 n^2)J_4 - 2mn\,\underline{mn}\,J_5)\ x$$

$$((M^2 n^2 + m^2)J_6 - 2M_{mn} J_7)$$

$$k_2 = \pi^4\omega^4 \sum_{mn} ((M^2 n^2 + m^2)J_8 - 2M_{mn} J_9)Q^{-2}_{mn} M^2 J_{10} \qquad (4.28c)$$

The notation in (4.28b, c) is

$$N^1_{mn} = (a^1_{mn})^2 + (b^1_{mn})^2\ , \quad (m, n) \neq (M, 1) \qquad (4.28d)$$

and

$$J_1 = J[(c_m^2\, C_n^2 - s_m^2\, S_n^2)s_{\underline{m}}\, S_{\underline{n}}] \qquad (4.28e)$$

$$J_2 = J[s_{\underline{m}}\, s_M^2\, S_{\underline{n}}\, S_1^2] \quad J_3 = J[c_{\underline{m}} c_M s_M C_{\underline{n}} C_1 S_1]$$

$$J_4 = J[s_{\underline{m}}\, s_m s_M S_{\underline{n}} S_n S_1] \quad J_5 = J[c_{\underline{m}} c_m s_M C_{\underline{n}}\, C_n S_1]$$

$$J_6 = J[s_{\underline{m}}\, s_m s_M S_{\underline{n}} S_n S_1] \quad J_7 = J[s_{\underline{m}} c_m c_M C_n C_1 S_{\underline{n}}]$$

$$J_8 = J[s_m\, s_M^2\, S_n\, S_1^2] \qquad J_9 = J[c_m c_M s_M\, C_n C_1 S_1]$$

$$J_{10} = J[(c_M^2\, C_1^2 - s_M^2\, S_1^2)s_m\, S_n]$$

It can also be shown that

$$M\{J[\ell_2\ \partial_{xy}\ w_1\ Y_{M1}]\} = k_3\ a \tag{4.29a}$$

with

$$k_3 = \frac{\pi^2\ \omega^2}{16(1+\nu)}\ \Sigma_{mn\neq M1}\ m^2 n^2\ N^1_{mn}\ J^2_{11} + M_{mn} N^1_{mn} J_{12}\ J_{13} \tag{4.29b}$$

and

$$J_{11} = J[s_M c_m\ S_1\ C_n] \quad J_{12} = J[s_m c_M S_n C_1] \quad J_{13} = J[s_M c_m S_1\ C_n] \tag{4.29c}$$

Now from (4.20) and (4.28d)

$$N^1_{mn} = [(\hat{a}_{mn})^2 + (\hat{b}_{mn})^2]e^{-2\gamma s}\ , \quad (m, n) \neq (M, 1)$$

and so we can set

$$k_1 + k_3 = -\ \alpha\ e^{-2\gamma s} \qquad k_2 = -\beta$$

with α and β independent of τ and s. Equation (4.25) becomes

$$\partial^2_s\ a + 2\gamma\ \partial^2_s a - (M\pi)^2 a + \alpha\ e^{-2\gamma s}\ a + \beta\ a^3 = 0 \tag{4.30}$$

Recall $a^1_{M1} = a = \phi_{M1}$ at $t = 0$ (equation (4.17)). To get the second initial condition, note that (4.12) implies $\partial_\tau\ w^2_{mn} + \partial_s\ w^1_{mn} = \psi_{mn}$ at $t = 0$. Using (4.22b) we conclude $\partial_s a = \psi_{M1}$ initially.

Several comments are in order. First remark that equation (4.30) is essentially the same as the equation governing a rod derived in [15]. However, in [15] the constants α and β are clearly positive, whereas for the plate this no longer seems to be true. Consequently, the solutions of (4.30) need not be bounded, and the perturbation scheme may fail to show how the nonlinearity stops the exponential growth of small perturbations.

In the case of no damping ($\gamma = 0$) (4.30) is autonomous and may be analyzed by phase plane methods. However, the N^1_{mn}, and hence α, depend on the initial conditions. It follows that the location of the equilibrium points (critical points) varies with the initial data. This runs counter to general experience, since equilibrium configurations are typically properties of the differential equations. The corresponding term in the equation for a rod also depends on the initial data, but the authors of [15] do not comment on the significance of this. It appears that damping is necessary for this model to yield physically meaningful results.

We are interested in bounded solutions of (4.30) when $\gamma > 0$ (when solutions are unbounded this model is no longer an accurate model of large time behavior). An energy relation shows that solutions are bounded when $\alpha > 0$ and $\beta > 0$:

$$\frac{\partial}{\partial s}\left(\frac{1}{2}\left(\frac{\partial a}{\partial s}\right)^2 - \frac{(M\pi)^2}{2} a^2 + \frac{1}{2}\alpha e^{-2\gamma s} a^2 + \frac{1}{4}\beta a^4\right)$$
$$= -2\gamma\left(\frac{\partial a}{\partial s}\right)^2 - \alpha\gamma e^{-2\gamma s} a^2 < 0 \qquad (4.31)$$

For large s it is tempting to ignore the term $\alpha e^{-2\gamma s}$ in (4.30) asymptotically. A simple geometric argument shows that this approximation is indeed valid. Introduce $b = a'$ and $g = \alpha e^{-2\gamma s}$, where $'$ denotes ∂_s. Solutions of (4.30) are contained among trajectories of the autonomous system

$$\left.\begin{aligned} a' &= b \\ b' &= (M\pi)^2 a - 2\gamma b - ga - \beta a^3 \\ g' &= -2\gamma g \end{aligned}\right\} \qquad (4.32)$$

In the region $g \neq 0$ the g-component of every tangent vector is directed towards the a-b plane. Consequently, the limit points of any trajectory lie within the a-b plane. The limit points of a bounded trajectory are a closed connected set consisting of either a limit point, a limit cycle, or a separatrix connecting limit points [5]. Now the reduced system

$$\left.\begin{aligned} a' &= b \\ b' &= (M\pi)^2 a - 2\gamma b - \beta a^3 \end{aligned}\right\} \qquad (4.33)$$

always has a saddle point at the origin (the characteristic exponents there are $\lambda = -\gamma \pm \sqrt{\gamma^2 + (M\pi)^2}$). If $\beta \leq 0$ there are no other critical points and trajectories are unbounded. Hence $\beta > 0$ for bounded solutions. In that case there exist two attractors (stable nodes or spirals) at $a = \pm M\pi/\beta^{\frac{1}{2}}$, $b = 0$ with characteristic exponents $\lambda = -\gamma \pm \sqrt{\gamma^2 - 2(M\pi)^2}$) at either one. There are no limit cycles and all trajectories tend to one or another of the attractors as $s \to \infty$. Substituting this result into our earlier calculations shows that for $\sigma \sim \sigma_o + \varepsilon^2$, $\gamma > 0$, and $\beta > 0$, solutions of (4. 1) with small initial displacement and velocity tend to one of the states

$$w \sim \pm (M\pi/\beta^{\frac{1}{2}}) Y_{M1}$$

to leading order as $t \to \infty$. Note that when damping is present the equilibrium points do not depend on the initial data.

The circular cylindrical shell

We proceed to study the dynamic buckling of a circular cylindrical shell for small initial displacements and velocities when the load is a "small distance" into the unstable regime. The appropriate equations, with small damping $\Gamma = \varepsilon\gamma$, are

$$w_{tt} + 2\varepsilon\gamma\, w_t + h^2 \Delta^2 w + \sigma w_{xx} + \omega^2 f_{xx} =$$

$$f_{yy}\, w_{xx} - 2f_{xy}\, w_{xy} + f_{xx}\, w_{yy} - 2\ell\, w_{xy} \qquad (4.34a)$$

$$\omega^2\, \Delta^2 f - \omega^2\, w_{xx} = w_{xy}^2 - w_{xx}\, w_{yy} \qquad (4.34b)$$

$$4\pi\omega(1+\nu)\,\ell = \int_0^{\Omega}\int_0^{1} w\, w_{xy}\, dxdy \qquad (4.34c)$$

The initial conditions and boundary conditions are

$$w = \varepsilon\,\phi, \qquad w_t = \varepsilon^2\psi \qquad \text{at} \qquad t = 0 \qquad (4.35)$$

$$w = w_{xx} = f = f_{xx} = 0 \qquad \text{at} \qquad x = 0, 1 \qquad (4.36)$$

$$w, f \text{ have period } \Omega = 2\pi/\omega \text{ in } y \qquad (4.37)$$

The small parameter ε will be determined in the course of the calculations. We seek solutions of the form

$$w = \varepsilon\, w_1 + \varepsilon^2\, w_2 + \cdots$$

$$f = \varepsilon f_1 + \varepsilon^2\, f_2 + \cdots$$

$$\ell = \varepsilon^2\, \ell_2 + \cdots$$

$$\sigma = \sigma_0 + \varepsilon\,\sigma_1 + \varepsilon^2\,\sigma_2 + \cdots$$

where σ_0 is the smallest load for which the state $w = f = 0$ is unstable to infinitesimal perturbations. Introduce time scales $t_k = \varepsilon^k t$ $(k = 0, 1, 2, \ldots)$, and set $t_0 = \tau$, $t_1 = s$, $t_2 = \eta$ for convenience. As for the rectangular plate, these expansions lead to a hierarchy of equations:

$$O(\epsilon) \qquad \partial_\tau^2 w_1 + h^2 \Delta^2 w_1 + \sigma_0 \partial_x^2 w_1 + \omega^2 \partial_x^2 f_1 = 0 \tag{4.38a}$$

$$\omega^2 \Delta^2 f_1 - \omega^2 \partial_x^2 w_1 = 0 \tag{4.38b}$$

$$O(\epsilon^2) \quad \partial_\tau^2 w_2 + h^2 \Delta^2 w_2 + \sigma_0 \partial_x^2 w_2 + \omega^2 \partial_x^2 f_2 + 2\partial_{\tau s} w_1 + 2\gamma \partial_\tau w_1 + \sigma_1 \partial_x^2 w_1$$

$$= \partial_y^2 f_1 \, \partial_x^2 w_1 - 2\partial_{xy} f_1 \, \partial_{xy} w_1 + \partial_x^2 f_1 \, \partial_y^2 w_1 \tag{4.39a}$$

$$\omega^2 \Delta^2 f_2 - \omega^2 \partial_x^2 w_2 = (\partial_{xy} w_1)^2 - \partial_x^2 w_1 \, \partial_y^2 w_1 \tag{4.39b}$$

$$4\pi\omega(1+\nu)\ell_2 = \int_0^\Omega \int_0^1 w_1 \, \partial_{xy} w_1 \, dxdy \tag{4.39c}$$

$$O(\epsilon^3) \quad \partial_\tau^2 w_3 + h^2 \Delta^2 w_3 + \sigma_0 \partial_x^2 w_3 + \omega^2 \partial_x^2 f_3 + 2\partial_{\tau s} w_2 + \partial_s^2 w_1 + 2\partial_{\tau\eta} w_1$$

$$+ 2\gamma \partial_\tau w_2 + 2\gamma \partial_s w_1 + \sigma_1 \partial_x^2 w_2 + \sigma_2 \partial_x^2 w_1 =$$

$$\partial_y^2 f_2 \, \partial_x^2 w_1 + \partial_y^2 f_1 \, \partial_x^2 w_2 - 2\partial_{xy} f_2 \, \partial_{xy} w_1 - 2\partial_{xy} f_1 \, \partial_{xy} w_2$$

$$+ \partial_x^2 f_2 \, \partial_y^2 w_1 + \partial_x^2 f_1 \, \partial_y^2 w_2 - 2\ell_2 \, \partial_{xy} w_1 \tag{4.40a}$$

$$\omega^2 \Delta^2 f_3 - \omega^2 \partial_x^2 w_3 = 2\partial_{xy} w_1 \, \partial_{xy} w_2 - \partial_x^2 w_1 \, \partial_y^2 w_2 - \partial_x^2 w_2 \, \partial_y^2 w_1 \tag{4.40b}$$

Equations (4.38) are identical with the equations for a linearized stability analysis of the state $w = f = 0$; consequently we determine σ_0 so that the solution of (4.38) is conditionally stable.

If we expand

$$\left.\begin{aligned} w_1 &= \sum \xi_n w^1_{mn}(\tau, s, \eta, \ldots) Y_{mn} + \bar{w}^1_{mn} \bar{Y}_{mn} \\ f_1 &= \sum \xi_n f^1_{mn}(\tau, s, \eta, \ldots) Y_{mn} + \bar{f}^1_{mn} \bar{Y}_{mn} \end{aligned}\right\} \tag{4.41}$$

where $\xi_n = \frac{1}{2}$ for $n = 0$, $\xi_n = 1$ for $n > 0$

and $Y_{mn}(x, y) = \sin m\pi x \cos n\omega y$

$\bar{Y}_{mn}(x, y) = \sin m\pi x \cos n\omega y$

for $m = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots,$ then (4.38) implies

$$\partial_\tau^2 w^1_{mn} + (h^2 Q^2_{mn} - \sigma_o (m\pi)^2) w^1_{mn} - (m\pi)^2 \omega^2 f^1_{mn} = 0 \qquad (4.42a)$$

$$\omega^2 Q^2_{mn} f^1_{mn} + \omega^2 (m\pi)^2 (m\pi)^2 w^1_{mn} = 0 \qquad (4.42b)$$

Here

$$Q_{mn} = (m\pi)^2 + (n\omega)^2 \qquad (3.30)$$

A pair of equations analogous to (4.42) holds for $\bar{w}^1_{mn}$ and $\bar{f}^1_{mn}$.
From (4.42) we conclude

$$\partial_\tau^2 w^1_{mn} + \lambda^2_{mn} w^1_{mn} = 0 \qquad (4.43a)$$

$$f^1_{mn} = -\left((m\pi)^2 / Q^2_{mn} \right) w^1_{mn} \qquad (4.43b)$$

with

$$\lambda^2_{mn} = (m\pi)^2 (\sigma_{mn} - \sigma_o) \qquad (4.44)$$

and

$$\sigma_{mn} = h^2 Q^2_{mn} / (m\pi)^2 + \omega^2 (m\pi)^2 / Q^2_{mn} \qquad (4.45)$$

Solutions of (4.43) grow exponentially if $\lambda^2_{mn} < 0$, so conditional stability occurs for $\lambda^2_{mn} = 0$. The smallest value σ_o such that $\lambda^2_{mn} = 0$ for some pair (m, n) is

$$\sigma_o = \min_{m,n} \sigma_{mn} \qquad (4.46)$$

We will assume that σ_o is simple, i.e. that there exists a unique pair (M, N) such that $\sigma_o = \sigma_{MN}$.

We continue to denote Fourier components by the notation

$$g^k_{mn} = \frac{4}{\Omega} \int_0^\Omega \int_0^1 g_k \, Y_{mn} \, dxdy$$

$$\bar{g}^k_{mn} = \frac{4}{\Omega} \int_0^\Omega \int_0^1 g_k \, \bar{Y}_{mn} \, dxdy \, .$$

In general, for each equation in a variable g^k_{mn} there exists a symmetric equation in $\bar{g}^k_{mn}$. We will suppress the second equation in most instances for brevity.

Initial conditions (4.35), when expanded as in (4.12), lead to

$$\left.\begin{aligned} w^1_{mn} &= \phi_{mn} & \partial_\tau w^1_{mn} &= 0 \\ w^2_{mn} &= 0 & \partial_\tau w^2_{mn} + \partial_s w^1_{mn} &= \psi_{mn} \\ w^3_{mn} &= 0 & \partial_\tau w^3_{mn} + \partial_s w^2_{mn} + \partial_\eta w^1_{mn} &= 0 \end{aligned}\right\} \text{ at } t = 0 \qquad (4.47)$$

Thus the solution of (4.43) is

$$\left.\begin{aligned} w^1_{mn} &= a^1_{mn} \cos \lambda_{mn}\tau + b^1_{mn} \sin \lambda_{mn}\tau \quad (m, n) \neq (M, N) \\ w^1_{MN} &= a^1_{M1} + b^1_{M1}\,\tau \end{aligned}\right\} \qquad (4.48)$$

where

$$a^1_{mn} = a^1_{mn}(s, \eta, \ldots) \text{ etc.}$$

and

$$a^1_{mn} = \phi_{mn} \quad b^1_{mn} = 0 \quad \text{at } t = 0 \qquad (4.49)$$

Boundedness of w^1_{MN} implies that $b^1_{M1} \equiv 0$.

To determine the behavior of the a^1_{mn} and b^1_{mn} we proceed to the $O(\varepsilon^3)$ equations. We introduce the operators

$$J[g] = \frac{4}{\Omega} \int_0^\Omega \int_0^1 g \, dxdy$$

$$J_{mn}[g] = J[g\, Y_{mn}] \quad \bar{J}_{mn}[g] = J[g\, \bar{Y}_{mn}].$$

Equations (4.39 a, b) yield

$$\partial_\tau^2 w^2_{mn} + (h^2 Q^2_{mn} - \sigma_o (m\pi)^2) w^2_{mn} - (m\pi)^2 \omega^2 f^2_{mn}$$

$$+ 2\,\partial_{\tau s} w^1_{mn} + 2\gamma\,\partial_\tau w^1_{mn} - \sigma_1 (m\pi)^2 w^1_{mn} = P^1_{mn} \tag{4.50a}$$

$$\omega^2 Q^2_{mn} f^2_{mn} + \omega^2 (m\pi)^2 w^2_{mn} = P^2_{mn} \tag{4.50b}$$

with

$$P^1_{mn} = J_{mn}[\partial_y^2 f_1\, \partial_x^2 w_1 - 2\,\partial_{xy} f_1\, \partial_{xy} w_1 + \partial_x^2 f_1\, \partial_y^2 w_1]$$

$$P^2_{mn} = J_{mn}[(\partial_{xy} w_1)^2 - \partial^2 w_1\, \partial_y^2 w_1]$$

Setting $P_{mn} = P^1_{mn} + (m\pi/Q_{mn})^2 P^2_{mn}$, we have

$$\partial_\tau^2 w^2_{mn} + \lambda^2_{mn} w^2_{mn} = P_{mn} - 2\,\partial_{\tau s} w^1_{mn} - 2\gamma\,\partial_\tau w^1_{mn} + \sigma_1 (m\pi)^2 w^1_{mn} \tag{4.51}$$

Using the operator M defined in (4.26), the "suppression of secular terms" follows from

$$M[(\partial_\tau^2 w^2_{mn} + \lambda^2_{mn} w^2_{mn}) z] = 0$$

where z is any (bounded) solution of the (adjoint) equation

$$\partial_\tau^2 z + \lambda^2_{mn} z = 0\,. \tag{4.52}$$

Thus

$$M[(2\partial_{\tau s} w^1_{mn} + 2\gamma\,\partial_\tau w^1_{mn} - \sigma_1 (m\pi)^2 w^1_{mn}) z] = M[P_{mn} z] \tag{4.53}$$

Consider $(m, n) = (M, N)$; $\lambda_{MN} = 0$ and so $z = 1$. From (4.48) we know that $\partial_\tau w^1_{MN} = 0$, so (4.53) simplifies to

$$-\sigma_1 (M\pi)^2 w^1_{MN} = M[P_{MN}]$$

Now if $M[P_{MN}] \neq 0$, σ_1 will depend on the initial conditions.

Since we seek a solution which is valid for arbitrary initial conditions, this possibility must be excluded. Consequently, for this perturbation scheme to be valid, it is necessary that

$$M[P_{MN}] = 0 \tag{4.54}$$

from which we conclude

$$\sigma_1 = 0 \tag{4.55}$$

Note that nothing equivalent to (4.54) was required for the problem of a rectangular plate. It can be shown (cf. Appendix B) that condition (4.54) is

$$O = \sum_{mn \neq MN} \left(\frac{Q^2_{MN} m^4 n^2}{Q^2_{mn}} + \frac{M^2 m^2 n^2}{2} \right) [A_{mn}(\alpha_m + \beta_n) + \overline{A}_{mn}(\alpha_m + \gamma_n)] \tag{4.56}$$

where

$$A_{mn} = (a^1_{mn})^2 + (b^1_{mn})^2 \quad \overline{A}_{mn} = (\overline{a}^1_{mn})^2 + (\overline{b}^1_{mn})^2$$

and

$$\alpha_m = J[s^2_m s_M C_N] \quad \beta_n = J[s_M S^2_n C_N] \quad \gamma_n = J[s_M C^2_n C_N] .$$

Here we are again using the notation

$$s_k = \sin k\pi x \qquad c_k = \cos k\pi x$$

$$S_k = \sin k\omega y \qquad C_k = \cos k\omega y$$

In order that (4.54) or (4.56) hold, it is sufficient that α_m, β_n, and γ_n vanish for all m and n. This will be the case if either N is odd or M is even. In the general case, the contradiction inherent in (4.54) can be circumvented by taking $w_1 \equiv 0$, $f_1 \equiv 0$ and assuming $w_2 \neq 0$,

$f_2 \neq 0$. For the remainder of this investigation we will assume that N is odd.

With $\sigma_1 = 0$, it remains to calculate (4.53) in detail for $(m, n) \neq (M, N)$. Details of the computation are indicated in Appendix B. We note that one further assumption is needed for the calculation. A sum of the form

$$\lambda_{mn} \pm \lambda_{m_1 n_1} \pm \lambda_{m_2 n_2}$$

can vanish if $(m, n) = (M, N)$ and $(m_1, n_1) = (m_2, n_2)$ since $\lambda_{MN} = 0$. We assume that this is the only way in which such a sum can vanish. In particular, this implies that

$$\lambda_{m_1 n_1} = \lambda_{m_2 n_2}$$

if and only if $(m_1, n_1) = (m_2, n_2)$.

Using $z = \cos\lambda_{mn}\tau$ and $z = \sin\lambda_{mn}\tau$, we find that for $(m, n) \neq (M, N)$, equation (4.53) yields

$$\partial_s a^1_{mn} + \gamma a^1_{mn} = 0 \qquad \partial_s b^1_{mn} + \gamma b^1_{mn} = 0$$

when N is odd (or M is even). Hence

$$a^1_{mn} = \hat{a}_{mn} e^{-\gamma s} \qquad b^1_{mn} = \hat{b}_{mn} e^{-\gamma s} \quad (m, n) \neq (M, N) \tag{4.57}$$

where

$$\hat{a}_{mn} = \hat{a}_{mn}(\eta, \ldots) \qquad \hat{b}_{mn} = \hat{b}_{mn}(\eta, \ldots)$$

and

$$\hat{a}_{mn} = \phi_{mn}, \quad \hat{b}_{mn} = 0 \qquad \text{at } t = 0 .$$

To determine the behavior of a^1_{MN} we proceed to the $O(\varepsilon^3)$ equations. For this we need the solutions w_2 and f_2, or equivalently, we need w^2_{mn} and f^2_{mn}. With the results (4.55) and (4.57), equation (4.51) simplifies to

$$\partial^2_\tau w^2_{mn} + \lambda^2_{mn} w^2_{mn} = P_{mn} \tag{4.58}$$

From (4.47) and (4.57) we can determine the initial conditions

$$w^2_{mn} = 0, \quad \partial_\tau w^2_{mn} = \psi_{mn} + \gamma \phi_{mn} \quad \text{at } t = 0$$

when $(m, n) \neq (M, N)$. We also know $w^2_{MN} = 0$ initially. The initial value of $\partial_\tau w^2_{MN}$ requires more subtle consideration. Solutions of

$$\partial^2_\tau w^2_{MN} = P_{MN}$$

can become unbounded due to two sources: P_{MN} may contain a "resonant term," even though it is bounded itself, or w^2_{MN} may contain a term linear in τ which satisfies the homogeneous equation. The first possibility was eliminated by (4.54). Removing the homogeneous solution proportional to τ fixes the initial value of $\partial_\tau w^2_{MN}$.

Fortunately, it turns out that it is not necessary to explicitly find w^2_{mn} in order to determine the equation for a^1_{MN}. It will suffice to be able to evaluate

$$M[w^1_{\underline{mn}} \, w^2_{mn}]$$

when $(\underline{m}, \underline{n}) \neq (m, n)$ (and hence, by assumption, $\lambda_{\underline{mn}} \neq \lambda_{mn}$). From (4.43a)

$$\int_0^T \partial^2_\tau w^1_{\underline{mn}} \, w^2_{mn} \, d\tau = -\lambda^2_{\underline{mn}} \int_0^T w^1_{\underline{mn}} \, w^2_{mn} \, d\tau .$$

Using (4.58) and integrating by parts, we also calculate

$$\int_0^T \partial_\tau^2 w^1_{\underline{mn}} w^2_{mn} d\tau$$

$$= [\partial_\tau w^1_{\underline{mn}} w^2_{mn} - w^1_{\underline{mn}} \partial_\tau w^2_{mn}]\Big|_0^T + \int_0^T w^1_{\underline{mn}} \partial_\tau^2 w^2_{mn} d\tau$$

$$= [\ldots]\Big|_0^T + \int_0^T w^1_{\underline{mn}} (P_{mn} - \lambda^2_{mn} w^2_{mn}) d\tau .$$

Hence

$$(\lambda^2_{mn} - \lambda^2_{\underline{mn}}) \int_0^T w^1_{\underline{mn}} w^2_{mn} d\tau = [\ldots]\Big|_0^T + \int_0^T w^1_{\underline{mn}} P_{mn} d\tau .$$

Multiply by T^{-1} and take the limit as $T \to \infty$. The result is

$$M[w^1_{\underline{mn}} w^2_{mn}] = (\lambda^2_{mn} - \lambda^2_{\underline{mn}})^{-1} M[w^1_{\underline{mn}} P_{mn}] \tag{4.59}$$

when $\lambda_{mn} \neq \lambda_{\underline{mn}}$.

To determine an equation for a^1_{M1}, we apply "suppression of secular terms" to the $O(\varepsilon^3)$ equations. From (4.40) we deduce

$$\partial_\tau^2 w^3_{MN} + (h^2 Q^2_{MN} - \sigma_o (M\pi)^2) w^3_{MN} - (M\pi)^2 \omega^2 f^3_{MN}$$

$$+ 2\,\partial_{\tau s} w^2_{MN} + 2\gamma \partial_\tau w^2_{MN} + \partial_s^2 w^1_{MN} + 2\,\partial_{\tau\eta} w^1_{MN}$$

$$+ 2\gamma \partial_s w^1_{MN} - \sigma_2 (M\pi)^2 w^1_{MN} = P^3_{MN} \tag{4.60a}$$

$$\omega^2 Q^2_{MN} f^3_{MN} + \omega^2 (M\pi)^2 f^3_{MN} = P^4_{MN} \tag{4.60b}$$

with

$$P^3_{MN} = J_{MN}[\partial_y^2 f_2\, \partial_x^2 w_1 + \partial_y^2 f_1\, \partial_x^2 w_2 - 2\partial_{xy} f_2\, \partial_{xy} w_1 - 2\partial_{xy} f_1\, \partial_{xy} w_2$$

$$+ \partial_x^2 f_2\, \partial_y^2 w_1 + \partial_x^2 f_1\, \partial_y^2 w_2 - 2\, \ell_2\, \partial_{xy} w_1]$$

$$P^4_{MN} = J_{MN}[2\partial_{xy} w_1\, \partial_{xy} w_2 - \partial_x^2 w_1\, \partial_y^2 w_2 - \partial_x^2 w_2\, \partial_y^2 w_1] .$$

(ℓ_2 is given by (4.39c).) Setting $R = P^3_{MN} + (M\pi/Q_{MN})^2 P^4_{MN}$ we have, since $\lambda_{MN} = 0$,

$$\partial^2_\tau w^3_{MN} + 2\partial_{\tau s} w^2_{MN} + 2\gamma\partial_\tau w^2_{MN} + 2\partial_{\tau\eta} w^1_{MN}$$
$$+ \partial^2_s w^1_{MN} + 2\gamma\partial_s w^1_{MN} - \sigma_2 (M\pi)^2 w^1_{MN} = R \qquad (4.61)$$

We effectively define ϵ at this point by requiring that $\sigma_2 = 1$, and so $\sigma \sim \sigma_o + \epsilon^2$. Write $w^1_{MN} = a^1_{MN} = A$. Operating on equation (4.61) with M results in

$$\partial^2_s A + 2\gamma\partial_s A - (M\pi)^2 A = M[R] \qquad (4.62)$$

The explicit calculation of M[R] is lengthy. Relevant details are indicated in Appendix B; we only cite the results here. Recall that, corresponding to the mode $\overline{Y}_{MN}$, there also exists a Fourier coefficient $\overline{w}^1_{MN} = \overline{a}^1_{MN} = B$. We find

$$\left.\begin{aligned}\partial^2_s A + 2\gamma\partial_s A - (M\pi)^2 A + (c_1 A + c_2 B)e^{-2\gamma s} + k_1 A^3 + k_2 AB^2 = 0\\ \partial^2_s B + 2\gamma\partial_s B - (M\pi)^2 B + (c_2 A + c_1 B)e^{-2\gamma s} + k_2 A^2 B + k_1 B^3 = 0\end{aligned}\right\} \qquad (4.63)$$

where the c_j and k_j are constants with respect to τ and $s (j = 1, 2)$. Furthermore, the c_j depend on the initial conditions, but k_j do not. The expressions for the k_j are unwieldy, and it is not clear whether or not they are positive or negative or both.

We propose to analyze solutions of (4.63) in phase space. When $\gamma = 0$ the system is autonomous, but the equilibrium points (or critical points) depend on the initial conditions through the c_j. Consequently we will only be concerned with the more physical case $\gamma > 0$. Furthermore, we will only discuss bounded solutions insofar as unbounded solutions of this model do not depict large time behavior accurately. Denote ∂_s by a prime ' and introduce $a = \partial_s A$, $b = \partial_s B$, and $g = e^{-2\gamma s}$. Then solutions of (4.63) are contained among trajectories satisfying

$$\left.\begin{aligned} A' &= a \\ a' &= (M\pi)^2 A - 2\gamma a - g(c_1 A + c_2 B) - k_1 A^3 - k_2 A B^2 \\ B' &= b \\ b' &= (M\pi)^2 B - 2\gamma b - g(c_2 A + c_1 B) - k_2 A^2 B - k_1 B^3 \\ g' &= -2\gamma g \end{aligned}\right\} \qquad (4.64)$$

This system is analogous to (4.32) which describes buckling of a rectangular plate. We argue that all tangent vectors are directed towards the AaBb hyperplane, and consequently the limit points of any trajectory lie within this hyperplane. As before, the limit points of a bounded trajectory constitute a closed connected set consisting of either a limit point, a limit cycle, or a separatrix connecting limit points [5]. Thus to describe large time behavior we are led to consider the reduced system

$$\left.\begin{aligned} A' &= a \\ a' &= (M\pi)^2 A - 2\gamma a - k_1 A^3 - k_2 AB^2 \\ B' &= b \\ b' &= (M\pi)^3 B - 2\gamma b - k_2 A^2 B - k_1 B^3 \end{aligned}\right\} \qquad (4.65)$$

Remark that if $B = b = 0$ initially, a solution to (4.65) satisfies

$$A' = a$$
$$a' = (M\pi)^2 A - 2\gamma a - k_1 A^3$$
$$B' = 0$$
$$b' = 0$$

From this we conclude that $k_1 > 0$ if solutions are to be bounded. The qualitative behavior of (4.65) is unaltered under the transformation $A \to A/\sqrt{k_1}$, $a \to a/\sqrt{k_1}$, $B \to B/\sqrt{k_1}$, $b \to b/\sqrt{k_1}$. Setting $k_2/k_1 = \alpha$, one obtains the canonical form of (4.65)

$$\left.\begin{aligned} A' &= a \\ a' &= (M\pi)^2 A - 2\gamma a - A^3 - \alpha A B^2 \\ B' &= b \\ b' &= (M\pi)^2 B - 2\gamma b - \alpha A^2 B - B^3 \end{aligned}\right\} \qquad (4.66)$$

To find equilibrium configurations of the shell, we investigate the critical points of (4.66). Necessarily $a = b = 0$, so the problem reduces to studying the characteristic exponents associated with roots of

$$A^3 + \alpha A B^2 = (M\pi)^2 A, \quad B^3 + \alpha A^2 B = (M\pi)^2 B$$

(i) <u>A=B=0</u>: There are two double exponents $\lambda_{1,2} = -\gamma + \sqrt{\gamma^2 + (M\pi)^2}$ and $\lambda_{3,4} = -\gamma - \sqrt{\gamma^2 + (M\pi)^2}$. This generalizes a saddle point; the origin is an unstable equilibrium point.

(ii) <u>A=0, B=±Mπ; A=±Mπ, B=0</u>: At each of these four points, the characteristic exponents are $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - 2(M\pi)^2}$ and $\lambda_{3,4} = -\gamma \pm \sqrt{\gamma^2 + (1-a)(M\pi)^2}$. $\mathrm{Re}\{\lambda_{1,2}\} < 0$. If $\alpha > 1$, $\mathrm{Re}\{\lambda_{3,4}\} < 0$ and these points are stable; if $a < 1$, $\lambda_3 > 0$ and $\lambda_4 < 0$ and these points are unstable.

(iii) <u>$A \neq 0,\ B \neq 0$</u>: Then we have

$$A^2 + \alpha B^2 = (M\pi)^2 \qquad B^2 + \alpha A^2 = (M\pi)^2 \quad .$$

<u>If $\alpha = 1$</u> there exists a one-dimensional locus of (non-isolated) critical points. This merely reflects the translational degeneracy of the solutions. These solutions contain case (ii).

<u>If $\alpha \neq 1$</u>, we solve to find $A^2 = B^2 = (M\pi)^2/(1+\alpha)$. For $\alpha \leq -1$ no such solutions exist, but for $\alpha > -1$ four solutions exist. In the latter case, characteristic exponents satisfy

$$\lambda^2 + 2\gamma\lambda + 2(M\pi)^2 = 0$$

and

$$\lambda^2 + 2\gamma\lambda + 2(M\pi)^2(1-\alpha)/(1+\alpha) = 0$$

The first equation gives rise to roots $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - 2(M\pi)^2}$, so $\mathrm{Re}\{\lambda_{1,2}\} < 0$. The second equation has roots

$$\lambda_{3,4} = -\gamma \pm \sqrt{\gamma^2 - 2(M\pi)^2(1-\alpha)/(1+\alpha)}$$

For stability we must have $2(M\pi)^2(1-\alpha)/(1+\alpha) > 0$ or

$$1 > \alpha > -1$$

This is the opposite of case (ii). We classify the four solutions of (iii) as physically identical under translation.

Summarizing the behavior of (i)-(iii), we conclude that if solutions are bounded, there exists only one physical equilibrium configuration.

APPENDIX A: DERIVATION OF SHELL EQUATIONS

In this appendix we derive Donnell-type equations describing a circular cylindrical shell in equilibrium. In equilibrium the potential energy V has a stationary value

$$\delta V = 0 \tag{A.1}$$

The potential energy can be computed from the strain energy (or internal energy) S and the applied work W

$$V = S - W \tag{A.2}$$

We assume that the strain energy is the same as in linear theory, viz.

$$S = \frac{1}{2}\int \sigma_{ij}\,\varepsilon_{ij}\,d\underline{x} \tag{A.3}$$

Here σ_{ij} and ε_{ij} are the physical components of the stress and strain tensors, respectively; summation convention is used, and integration is carried out over the volume of the shell walls. Using tensor calculus one can derive [11] the exact relations

$$\left.\begin{aligned}
\varepsilon_r &= \frac{\partial w}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial r}\right)^2\right] \\
\varepsilon_\theta &= \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{1}{2r^2}\left[\left(\frac{\partial w}{\partial \theta} - v\right)^2 + \left(\frac{\partial v}{\partial \theta} + w\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right] \\
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] \\
\varepsilon_{r\theta} &= \frac{1}{2r}\left[r\frac{\partial v}{\partial r} + \frac{\partial w}{\partial \theta} - v + \frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial \theta} - v\right) + \frac{\partial v}{\partial r}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta}\right] \\
\varepsilon_{rx} &= \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial u}{\partial x}\right] \\
\varepsilon_{\theta x} &= \frac{1}{2r}\left[r\frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial \theta} - v\right) + \frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial \theta}\right]
\end{aligned}\right\} \tag{A.4}$$

Here $\theta = y/r$ is the angle about the axis of the cylinder.

The stresses can be determined using Hooke's law, which becomes, for a homogeneous isotropic medium [17],

$$\sigma_{ij} = \lambda \Theta\, \delta_{ij} + 2\mu\varepsilon_{ij} \tag{A.5}$$

Here $\Theta = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$ and the 1, 2, 3 directions refer to some cartesian coordinates. Also

$$\lambda = E\nu/(1+\nu)(1-2\nu) \qquad \mu = E/2(1+\nu)$$

where E is Young's modulus and ν is Poisson's ratio. A body is said to be in a state of plane stress parallel to the x_1, x_2 plane when $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. From (A. 5) and $\sigma_{33} = 0$ one can deduce

$$\varepsilon_{33} = \frac{-\lambda}{\lambda+2\mu}(\varepsilon_{11} + \varepsilon_{22}) \tag{A. 6}$$

From (A. 6), it follows that

$$\left.\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}) \\ \sigma_{22} &= \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}) \\ \sigma_{12} &= \frac{E}{1+\nu}\varepsilon_{12} \end{aligned}\right\} \tag{A. 7}$$

We assume that locally the cylinder is in a state of plane stress parallel to the tangent plane; hence (A. 7) holds with

$$\sigma_{11} = \sigma_x \qquad \sigma_{22} = \sigma_\theta \qquad \sigma_{12} = \sigma_{x\theta} \qquad \sigma_{33} = \sigma_r$$

$$\varepsilon_{11} = \varepsilon_x \qquad \varepsilon_{22} = \varepsilon_\theta \qquad \varepsilon_{12} = \varepsilon_{x\theta}$$

Symmetry of the stress tensor yields

$$\sigma_{21} = \sigma_{12} \qquad \sigma_{31} = \sigma_{13} = 0 \qquad \sigma_{32} = \sigma_{23} = 0$$

We can now approximate (A. 3) by

$$S = \frac{1}{2}\int (\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + 2\sigma_{x\theta}\varepsilon_{x\theta})\, d\underline{x} \tag{A. 8}$$

If we substitute (A. 4) and (A. 7) into (A. 8), the result is still rather unwieldly. Assume further that displacements and their gradients are small compared to 1. If it is also assumed (on intuitive grounds) that most of the change occurs in the radial direction, one can argue that

$$\left|\frac{\partial w}{\partial y}\right| >> \left|\frac{\partial u}{\partial y}\right| , \qquad \left|\frac{\partial w}{\partial x}\right| >> \left|\frac{\partial v}{\partial x}\right|$$

Then (A. 4) simplifies to

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \varepsilon_\theta = \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$2\varepsilon_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial w}{\partial x} \tag{A. 9}$$

Recall R designates the distance from the axis to the midsurface of the undeformed shell. Set $r-R = \rho$ and expand, for small ρ,

$$\left.\begin{aligned} u(r,\theta,x) &\doteq u_0(\theta,x) + \rho\, u_1(\theta,x) \\ v(r,\theta,x) &\doteq v_0(\theta,x) + \rho\, v_1(\theta,x) \\ w(r,\theta,x) &\doteq w_0(\theta,x) \end{aligned}\right\} \qquad \text{(A.10)}$$

To find u_1 we use

$$0 = 2\varepsilon_{rx} \doteq \frac{\partial w}{\partial x} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}\frac{\partial v}{\partial x} \doteq \frac{\partial w_0}{\partial x} + u_1 + v_1\frac{\partial v_0}{\partial x}$$

Since $v_1 \frac{\partial v_0}{\partial x}$ is small to second order, we have approximately

$$u_1 \doteq -\frac{\partial w_0}{\partial x} \qquad \text{(A.11a)}$$

Similarly, from $\varepsilon_{r\theta} = 0$ we conclude

$$v_1 \doteq -\frac{1}{r}\frac{\partial w_0}{\partial \theta} \qquad \text{(A.11b)}$$

(Note: $\varepsilon_{rx} = \varepsilon_{r\theta} = 0$ follows from $\sigma_{13} = \sigma_{23} = 0$ and Hooke's law.)

Approximate $1/r = 1/R$. Then (A.9, 10, 11) imply

$$\varepsilon_x = \varepsilon_{x0} + \rho\,\varepsilon_{x1} \qquad \varepsilon_\theta = \varepsilon_{\theta 0} + \rho\,\varepsilon_{\theta 1} \qquad \varepsilon_{\theta x} = \varepsilon_{\theta x0} + \rho\,\varepsilon_{\theta x1} \qquad \text{(A.12a)}$$

with

$$\begin{aligned} \varepsilon_{x0} &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 & \varepsilon_{x1} &= -\frac{\partial^2 w_0}{\partial x^2} \\ \varepsilon_{\theta 0} &= \frac{w_0}{R} + \frac{1}{R}\frac{\partial v_0}{\partial \theta} + \frac{1}{2R^2}\left(\frac{\partial w_0}{\partial \theta}\right)^2 & \varepsilon_{\theta 1} &= -\frac{1}{R^2}\frac{\partial^2 w_0}{\partial \theta^2} \qquad \text{(A.12b)} \\ 2\varepsilon_{\theta x0} &= \frac{\partial v_0}{\partial x} + \frac{1}{R}\frac{\partial u_0}{\partial \theta} + \frac{1}{R}\frac{\partial w_0}{\partial \theta}\frac{\partial w_0}{\partial x} & \varepsilon_{\theta x1} &= -\frac{1}{R}\frac{\partial^2 w_0}{\partial \theta\,\partial x} \end{aligned}$$

For a shell of thickness h we have

$$\begin{aligned} S &= \frac{1}{2}\int_0^L\int_0^{2\pi}\int_{R-h/2}^{R+h/2}(\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + 2\sigma_{x\theta}\varepsilon_{x\theta})\,r\,dr\,d\theta\,dx \\ &\doteq \frac{1}{2}\int_0^L\int_0^{2\pi}\int_{-h/2}^{h/2}(\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + 2\sigma_{x\theta}\varepsilon_{x\theta})\,R\,d\rho\,d\theta\,dx \end{aligned} \qquad \text{(A.8')}$$

We substitute (A.7) and (A.12) into (A.8') and carry out the indicated integration in ρ; then we calculate δS and simplify by integrating by parts, using the boundary conditions

$$\left.\begin{aligned} w_0 &= w_{0,xx} = 0 \\ \sigma_{x0} &= -\sigma_0 = \text{constant} \\ v_0 &= 0 \end{aligned}\right\} \text{ at } x = 0, L \qquad (A.13)$$

The result is

$$\begin{aligned} \delta S = \frac{E}{1-\nu^2} Rh \int_0^L \int_0^{2\pi} d\theta\, dx \Big\{ \Big[&\frac{1}{R}\varepsilon_{\theta 0} - \frac{1}{R^2}\frac{\partial}{\partial\theta}\Big(\varepsilon_{\theta 0}\frac{\partial w}{\partial\theta}\Big) - \frac{1}{12}\frac{h^2}{R^2}\frac{\partial^2\varepsilon_{\theta 1}}{\partial\theta^2} \\ &+ \frac{\nu}{R}\varepsilon_{x0} - \frac{\nu}{R^2}\frac{\partial}{\partial\theta}\Big(\varepsilon_{x0}\frac{\partial w_0}{\partial\theta}\Big) - \frac{\nu}{12}\frac{h^2}{R^2}\frac{\partial^2\varepsilon_{x1}}{\partial\theta^2} - \frac{\partial}{\partial x}\Big(\varepsilon_{x0}\frac{\partial w_0}{\partial x}\Big) \\ &- \nu\frac{\partial}{\partial x}\Big(\varepsilon_{\theta 0}\frac{\partial w_0}{\partial x}\Big) - \frac{1}{12}h^2\frac{\partial^2\varepsilon_{x1}}{\partial x^2} - \frac{\nu}{12}h^2\frac{\partial^2\varepsilon_{\theta 1}}{\partial x^2} \\ &- \frac{(1-\nu)}{R}\frac{\partial}{\partial\theta}\Big(\varepsilon_{\theta x0}\frac{\partial w_0}{\partial x}\Big) - \frac{(1-\nu)}{R}\frac{\partial}{\partial x}\Big(\varepsilon_{\theta x0}\frac{\partial w_0}{\partial\theta}\Big) \\ &- \frac{(1-\nu)}{12}\frac{h^2}{R}\frac{\partial^2\varepsilon_{\theta x1}}{\partial\theta\,\partial x}\Big]\delta w_0 - \Big[\frac{\partial}{\partial x}\varepsilon_{x0} + \nu\frac{\partial}{\partial x}\varepsilon_{\theta 0} + \frac{(1-\nu)}{R}\frac{\partial}{\partial\theta}\varepsilon_{\theta x0}\Big]\delta u_0 \\ &- \Big[\frac{1}{R}\frac{\partial}{\partial\theta}\varepsilon_{\theta 0} + \frac{\nu}{R}\frac{\partial}{\partial\theta}\varepsilon_{x0} + (1-\nu)\frac{\partial}{\partial x}\varepsilon_{\theta x0}\Big]\delta v_0 \Big\} \\ + \Big[\frac{E}{1-\nu^2} Rh \int_0^{2\pi} (\varepsilon_{x0} + \nu\varepsilon_{\theta 0})\, \delta u_0\, d\theta\Big]\Big|_{x=0}^{x=L} & \end{aligned} \qquad (A.14)$$

But

$$\Big[\frac{E}{1-\nu^2} Rh \int_0^{2\pi} (\varepsilon_{x0} + \nu\varepsilon_{\theta 0})\,\delta u_0\, d\theta\Big]\Big|_{x=0}^{x=L} = \Big[\int_0^{2\pi} \sigma_0\,\delta u_0\, h R\, d\theta\Big]\Big|_{x=0}^{x=L} = \delta W$$

so that only the surface integral in (A.14) contributes to δV. The variations δu_0, δv_0, δw_0 are independent and give rise to three equations. Using Hooke's law (A.7) we can write these equations as

$$\frac{\partial}{\partial x}\sigma_{x0} + \frac{1}{R}\frac{\partial}{\partial\theta}\sigma_{\theta x0} = 0 \qquad (A.15a)$$

$$\frac{1}{R}\frac{\partial}{\partial\theta}\sigma_{\theta 0} + \frac{\partial}{\partial x}\sigma_{\theta x0} = 0 \qquad (A.15b)$$

$$\begin{aligned} &\frac{1}{R}\sigma_{\theta 0} - \frac{1}{R^2}\frac{\partial}{\partial\theta}\Big(\sigma_{\theta 0}\frac{\partial w_0}{\partial\theta}\Big) - \frac{\partial}{\partial x}\Big(\sigma_{x0}\frac{\partial w_0}{\partial x}\Big) - \frac{1}{R}\frac{\partial}{\partial\theta}\Big(\sigma_{\theta x0}\frac{\partial w_0}{\partial x}\Big) \\ &- \frac{1}{R}\frac{\partial}{\partial x}\Big(\sigma_{\theta x0}\frac{\partial w_0}{\partial\theta}\Big) - \frac{1}{12}\frac{E}{1-\nu^2}\Big[\frac{h^2}{R^2}\frac{\partial^2}{\partial\theta^2}(\varepsilon_{\theta 1} + \nu\varepsilon_{x1}) \\ &+ h^2\frac{\partial^2}{\partial x^2}(\varepsilon_{x1} + \nu\,\varepsilon_{\theta 1})\Big] - \frac{1}{6}\frac{E}{1+\nu}\frac{h^2}{R}\frac{\partial^2}{\partial\theta\,\partial x}\varepsilon_{\theta x1} = 0 \end{aligned} \qquad (A.15c)$$

We can use (A.12b) and (A.15a, b) to simplify (A.15c). Also let $y = R\theta$

and write $w = w_o$, $\sigma_x = \sigma_{xo}$, $\sigma_y = \sigma_{\theta o}$, $\sigma_{xy} = \sigma_{\theta xo}$. Then equations (A.15) become

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\sigma_x + \frac{\partial}{\partial y}\sigma_{xy} = 0 \\ &\frac{\partial}{\partial y}\sigma_y + \frac{\partial}{\partial x}\sigma_{xy} = 0 \\ &\frac{h^2 E}{12(1-\nu^2)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w - \left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\sigma_{xy}\frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) \\ &\qquad + \frac{1}{R}\sigma_y = 0 \end{aligned}\right\} \quad (A.16)$$

APPENDIX B: SOME CALCULATIONS FOR CHAPTER 4

We indicate how to calculate

$$H = M[K] \tag{4.25b}$$

where

$$K = J[(\partial_y^2 f_2 \, \partial_x^2 w_1 - 2\,\partial_{xy} f_2 \, \partial_{xy} w_1 + \partial_x^2 f_2 \, \partial_y^2 w_1 - \ell_2 \, \partial_{xy} w_1) \, Y_{M1}] \quad . \tag{4.24b}$$

The operators M and K are defined in (4.26) and (4.27), respectively.

Recall the notation

$$s_k = \sin k\pi x \qquad c_k = \cos k\pi x$$
$$S_k = \sin k\omega y \qquad C_k = \cos k\omega y \quad .$$

Now

$$M \circ J[\partial_y^2 f_2 \, \partial_x^2 w_1 \, s_M \, S_1] =$$
$$M \circ J[(-\Sigma (\underline{n}\omega)^2 f^2_{\underline{m}\underline{n}} s_{\underline{m}} S_{\underline{n}})(-\Sigma (\underset{\sim}{m}\pi)^2 w'_{\underset{\sim}{m}\underset{\sim}{n}} s_{\underset{\sim}{m}} S_{\underset{\sim}{n}}) \, s_M S_1] =$$
$$\Sigma \, (\underline{n}\omega)^2 (\underset{\sim}{m}\pi)^2 \, M[f^2_{\underline{m}\underline{n}} \, w'_{\underset{\sim}{m}\underset{\sim}{n}}] \; J[s_{\underline{m}} s_{\underset{\sim}{m}} s_M S_{\underline{n}} S_{\underset{\sim}{n}} S_1] \tag{B.1}$$

Recall

$$Q^2_{\underline{m}\underline{n}} f^2_{\underline{m}\underline{n}} = J[(\partial_{xy} w_1 \, \partial_{xy} w_1 - \partial_x^2 w_1 \, \partial_y^2 w_1) \, s_{\underline{m}} S_{\underline{n}}] \tag{4.23}$$

Thus we are led to consider

$$M \circ J[(\partial_{xy} w_1 \; \partial_{xy} w_1 \; s_{\underline{m}} S_{\underline{n}}) \, w'_{\underset{\sim}{m}\underset{\sim}{n}}] \tag{B.2}$$

When $(\underset{\sim}{m}, \underset{\sim}{n}) = (M, 1)$ we have

$$w'_{\underset{\sim}{m}\underset{\sim}{n}} = a'_{M1}(s, \eta, \ldots)$$

and (B.2) becomes

$$a'_{M1} \, M \circ J[\partial_{xy} w_1 \; \partial_{xy} w_1 \; s_{\underline{m}} S_{\underline{n}}] =$$
$$a'_{M1} \, M \circ J[(\Sigma \, w'_{m_1 n_1} (m_1\pi)(n_1\omega) \, c_{m_1} C_{n_1})(\Sigma \, w'_{m_2 n_2} (m_2\pi)(n_2\omega) \, c_{m_2} C_{n_2}) \, s_{\underline{m}} S_{\underline{n}}] =$$
$$a'_{M1} \, \Sigma \, (m_1 m_2 \pi^2)(n_1 n_2 \omega^2) \, M[w'_{m_1 n_1} w'_{m_2 n_2}] \, J[c_{m_1} c_{m_2} s_{\underline{m}} C_{n_1} C_{n_2} S_{\underline{n}}] \tag{B.3}$$

Since

$$w'_{mn} = a'_{mn} \cos \lambda_{mn} \tau + b'_{mn} \sin \lambda_{mn} \tau \tag{4.16a}$$

it follows from the assumption

$$\lambda_{m_1 n_1} = \lambda_{m_2 n_2} \text{ if and only if } (m_1, n_1) = (m_2, n_2)$$

that

$$\begin{aligned} M[w'_{m_1 n_1} w'_{m_2 n_2}] &= 0 & (m_1, n_1) \neq (m_2, n_2) \\ M[w'_{mn} w'_{mn}] &= \tfrac{1}{2}[(a'_{mn})^2 + (b'_{mn})^2] & (m,n) \neq (M,1) \\ M[w'_{M1} w'_{M1}] &= (a'_{M1})^2 \end{aligned} \qquad \text{(B. 4)}$$

Using (B. 4) we simplify (B. 3) to

$$a'_{M1} \sum (m\pi)^2 (n\omega)^2 \, M[(w'_{mn})^2] \;\; J[\, c_m^2 \, s_{\underset{\sim}{m}} \, C_n^2 \, S_{\underset{\sim}{n}}] \qquad \text{(B. 3')}$$

From elementary trigonometric formulae one can show

$$\left.\begin{aligned} 4\,C_p C_q C_r &= C_{p+q+r} + C_{-p+q+r} + C_{p-q+r} + C_{p+q-r} \\ 4\,S_p S_q S_r &= S_{-p+q+r} + S_{p-q+r} + S_{p+q-r} - S_{p+q+r} \\ 4\,S_p S_q C_r &= C_{-p+q+r} + C_{p-q+r} - C_{p+q-r} - C_{p+q+r} \\ 4\,C_p C_q S_r &= S_{p+q+r} + S_{-p+q+r} + S_{p-q+r} - S_{p+q-r} \end{aligned}\right\} \qquad \text{(B. 5)}$$

This implies that

$$M[w'_{m_1 n_1} \, w'_{m_2 n_2} \, w'_{m_3 n_3}] \neq 0$$

only if a sum vanishes of the form

$$\pm \lambda_{m_1 n_1} \pm \lambda_{m_2 n_2} \pm \lambda_{m_3 n_3} = 0 \quad . \qquad \text{(B. 6)}$$

Clearly (B. 6) holds if

$$\lambda_{m_1 n_1} = 0 \qquad \lambda_{m_2 n_2} = \lambda_{m_3 n_3} \quad .$$

We further assume that this is the only way (B. 6) can be true. With this assumption we can evaluate (B. 2) when $(\underset{\sim}{m}, \underset{\sim}{n}) \neq (M, 1)$. We are led to

$$\sum (m_1 m_2 \pi^2)(n_1 n_2 \omega^2) \; M[w'_{\underset{\sim}{m}\underset{\sim}{n}} \, w'_{m_1 n_1} \, w'_{m_2 n_2}] \; J[\, c_{m_1} c_{m_2} s_{\underset{\sim}{m}} \, C_{n_1} C_{n_2} S_{\underset{\sim}{n}}] \, . \quad \text{(B. 7)}$$

Since $(\underset{\sim}{m}, \underset{\sim}{n}) \neq (M, 1)$, there are only two possible terms which can be non-vanishing, viz.

$$(m_1, n_1) = (\underset{\sim}{m}, \underset{\sim}{n}) \; , \quad (m_2, n_2) = (M, 1)$$

and $$(m_2, n_2) = (\underset{\sim}{m}, \underset{\sim}{n}) \quad , \quad (m_1, n_1) = (M, 1) \quad .$$

Thus when (m, n) ≠ (M, 1) we find that (B.2) reduces to

$$a'_{M1}\,[(a'_{\underset{\sim}{m}\underset{\sim}{n}})^2 + (b'_{\underset{\sim}{m}\underset{\sim}{n}})^2]\,(M\underset{\sim}{m}\pi^2)(\underset{\sim}{n}\omega^2)\;J[c_{\underset{\sim}{m}} c_M s_{\underline{m}}\, C_{\underset{\sim}{n}} C_1 S_{\underline{n}}] \quad . \qquad \text{(B. 7')}$$

In a similar fashion we calculate

$$M \cdot J\,[\,(\partial_x^2 w_1\, \partial_y^2 w_1\, s_{\underline{m}}\, S_{\underline{n}})\, w'_{\underset{\sim}{m}\underset{\sim}{n}}\,] \quad .$$

This gives us

$$M\,[\,f^2_{mn}\, w'_{\underset{\sim}{m}\underset{\sim}{n}}]$$

which, as we see from (B. 1), is the quantity necessary to calculate H.

The arguments needed to calculate equations (4.53), (4.54), (4.56), and (4.62) (describing the buckling of circular cylindrical shells) are completely analogous to the above arguments (for a rectangular plate) and can be carried out by the persevering reader.

APPENDIX C: NOTES ON THE MEMBRANE EQUATIONS

The reader is referred to the references [1, 2] for a derivation of the membrane theory; only the final formulation of the problem is given here. The midsurface of the undeformed membrane is generated by rotating a curve C about the axis of symmetry (see figure 1). This surface extends a distance R from the axis. The curve C can be described by prescribing the angle $\theta(r^*)$ between the normal to the surface (at distance r^* from the axis, $0 \le r^* \le R$) and the axis of rotation. We will assume that $\theta(0) = 0$ so that the membrane is not pointed at the apex. The surface is deformed by a pressure $p(r^*)$ which is normal to the midsurface; p is positive if it is directed toward the center of curvature. $\sigma_r(r^*)$ is the radial stress, h is the thickness, and E is Young's modulus. Then with the definitions

$$\left.\begin{aligned}
& r \equiv r^*/R \,, \qquad 0 \le r \le 1 \\
& Q \equiv \tfrac{1}{2} h^2 R^2 \\
& P \equiv \max_{0 \le r \le 1} \frac{Q}{r}\left[\int_0^{Rr} \frac{p(\xi)}{E}\,\xi\, d\xi\right]^2 \\
& \sigma(r) \equiv \sigma_r(r^*)/EP^{1/3} \\
& \phi(r) \equiv \theta(r^*) \\
& G(r) \equiv \frac{Q}{Pr}\left[\int_0^{Rr} \frac{p(\xi)}{E}\,\xi\, d\xi\right]^2 \\
& B \equiv \tfrac{1}{2} P^{1/3}
\end{aligned}\right\} \qquad (C.1)$$

the problem of interest can be formulated in terms of dimensionless variables as follows:

$$\frac{d}{dr}\left(r^3 \frac{d\sigma}{dr}\right) + \frac{G}{\sigma^2} = Br\phi^2 \qquad (C.2)$$

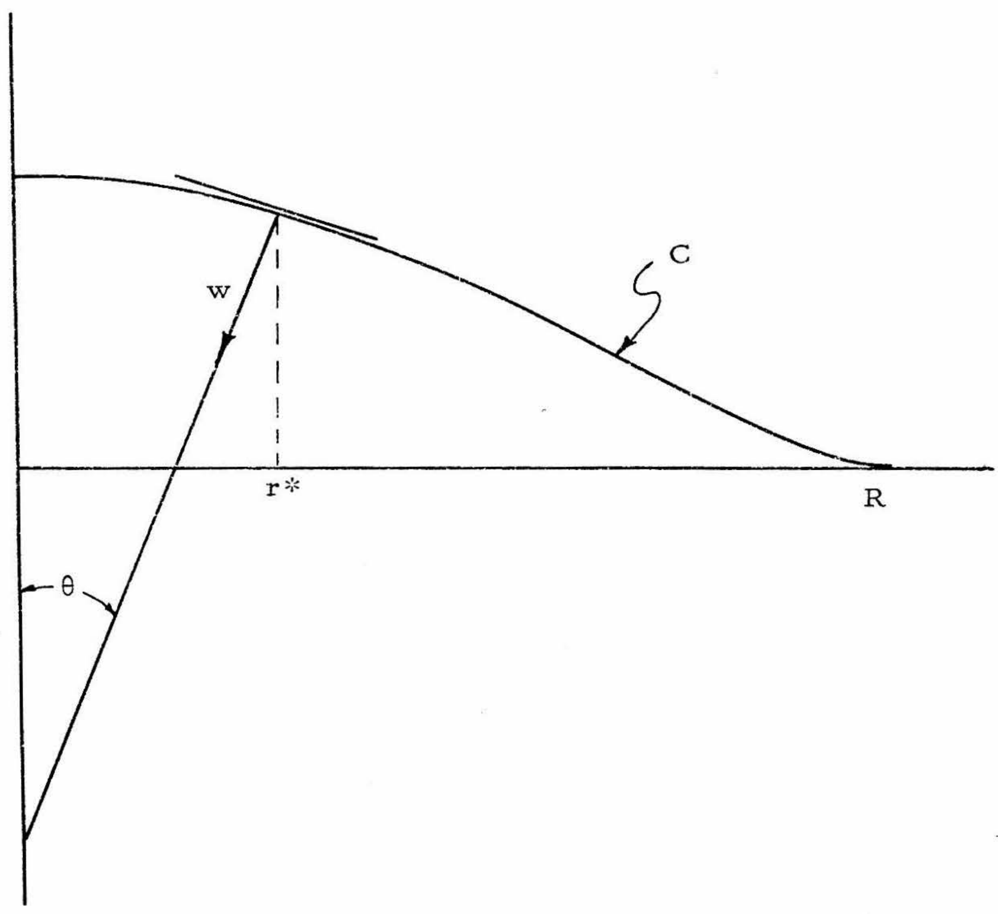


Figure 1 Geometry of the undeformed midsurface

Symmetry and boundedness of the stresses and displacements at $r = 0$ imply

$$|\sigma(0)| < \infty \tag{C.3a}$$

and

$$\frac{d\sigma}{dr} = 0 \quad \text{at} \quad r = 0 \tag{C.3b}$$

The prescribed radial stress at the edge yields the further boundary condition

$$\sigma(1) = -S \equiv T/EP^{1/3} \tag{C.4}$$

($T < 0$ when the stress is compressive.) The normal displacement of the midsurface $W(r^*) = Rw(r)$ can be regained from

$$\frac{dw}{dr} = \frac{-1}{\sigma}\left(2P^{1/3}\ \frac{G}{r}\right)^{\frac{1}{2}} - \phi \tag{C.5a}$$

$$w(1) = 0 \tag{C.5b}$$

Instead of studying σ directly we prefer to introduce

$$u \equiv S^{-1}\sigma + 1$$

and

$$\lambda \equiv S^{-1}$$

which results in the formulation

$$\frac{d}{dr}\left(r^3 \frac{du}{dr}\right) + \lambda^3 \frac{G}{(1-u)^2} = \lambda B r \phi^2 \tag{C.6}$$

The boundary conditions are then

$$\frac{du}{dr} = 0 \qquad \text{at } r = 0 \tag{C.7a}$$

$$u(1) = 0 \tag{C.7b}$$

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