SOME MODIFIED BIFURCATION PROBLEMS

WITH APPLICATION TO IMPERFECTION SENSITIVITY IN BUCKLING

Thesis by

James Paul Keener

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California 91109

1972

(Submitted April 27, 1972)

Acknowledgements

The author wishes to thank Professor Herbert B. Keller for making many helpful suggestions regarding the problems treated in this thesis. He has given freely of his time in offering advice, encouragement, and patient supervision throughout the course of this work.

The author is grateful to Mrs. Vivian Davies for her conscientious and painstaking typing of the manuscript.

The work represented by this thesis would have been considerably more difficult were it not for the constant love and encouragement of his wife Kristine, to whom much credit is due.

The author gratefully acknowledges the Fannie and John Hertz Foundation for their generous support during the course of this study.

Abstract

The branching theory of solutions of certain nonlinear elliptic partial differential equations is developed, when the nonlinear term is perturbed from unforced to forced. We find families of branching points and the associated nonisolated solutions which emanate from a bifurcation point of the unforced problem. Nontrivial solution branches are constructed which contain the nonisolated solutions, and the branching is exhibited. An iteration procedure is used to establish the existence of these solutions, and a formal perturbation theory is shown to give asymptotically valid results. The stability of the solutions is examined and certain solution branches are shown to consist of minimal positive solutions, Other solution branches which do not contain branching points are also found in a neighborhood of the bifurcation point.

The qualitative features of branching points and their associated nonisolated solutions are used to obtain useful information about buckling of columns and arches. Global stability characteristics for the buckled equilibrium states of imperfect columns and arches are discussed. Asymptotic expansions for the imperfection sensitive buckling load of a column on a nonlinearly elastic foundation are found and rigorously justified.

Table of Contents

PART	TITLE	PAGE
I	Introduction	1
II	General Imperfection Theory	7
	II. 1 Notation and Definitions	7
	II. 2 Perturbation Theory for Nonisolated Solutions	13
	II. 3 Existence of Nonisolated Solutions	24
	II. 4 Comparison of Iteration Scheme and Perturbation Procedure	36
	II. 5 Extension of Solution Branch from Nonisolated Solution	46
	II. 6 Stability of Extended Solution Branch	59
	II. 7 Minimal Positive Solutions	65
	II. 8 Other Solution Branches	91
III	Dynamic Buckling of Columns and Arches	101
	III. 1 Introduction	101
	III. 2 Equilibrium States and their Relationship to Imperfection Theory	103
	III. 3 Dynamic Treatment of Global Stability	115
IV	Buckling of an Imperfect Column on a Nonlinearly Elastic Foundation	131
Illustrations		147
References		157

Chapter I

Introduction

Branching is a change in the number of solutions u of an equation

$$g(\lambda, u) = 0 \tag{1.1}$$

produced by a small change in the real parameter λ. Those values λ at which branching occurs are called branching points, and the corresponding solutions are called nonisolated solutions of (1.1). If solutions u of (1.1) are also arbitrarily small in a neighborhood of the branching point and $u \equiv 0$ is a solution for all λ, then the phenomenon is called bifurcation, and the branching point is called a bifurcation point. The problem (1.1) is called "unforced" if $g(\lambda, 0) = 0$ for all real values of λ, and it is called "forced" if $g(\lambda, 0) \neq 0$ for some values of λ. In this thesis, we are concerned with the behavior of branching points and solutions in their neighborhood, as the problem (1.1) is perturbed from an unforced to a forced problem. Letting τ represent a "forcing" parameter, we are interested in finding solutions of

$$G(\lambda, \tau, u) = 0 \tag{1.2}$$

for nonzero values of τ, where $G(\lambda, 0, 0) = 0$ for all real λ and

$G(\lambda, \tau, 0) \neq 0$ when $\tau \neq 0$.

As a simple illustration consider the single algebraic equation given by

$$x + f(\lambda, \tau, x) = 0 \tag{1.3}$$

where $f(\lambda, 0, 0) = 0$ and $f(\lambda, \tau, 0) \neq 0$ if $\tau \neq 0$. When $\tau = 0$, $x = 0$ is a solution of (1.3) for any value of λ. From the implicit function theorem, we know that the identically zero solution is the only arbitrarily small solution of (1.3) in a neighborhood of $\lambda = \lambda_0$, provided the Jacobian of (1.3) evaluated at $(\lambda, \tau, x) = (\lambda_0, 0, 0)$ does not vanish, or symbolically, if

$$J(\lambda_0, 0, 0) = 1 + f_x(\lambda_0, 0, 0) \neq 0. \tag{1.4}$$

If (1.4) does not hold then the point $(\lambda, x) = (\lambda_0, 0)$ is a possible bifurcation point with $\tau = 0$. Similarly if $(\lambda, \tau, x) = (\lambda_1, \tau_1, x_1)$ is a nontrivial solution of (1.3) then by the implicit function theorem we know that there is a unique function $x = x(\lambda)$ with $x(\lambda_1) = x_1$ when $\tau = \tau_1$ is fixed, provided

$$J(\lambda_1, \tau_1, x_1) = 1 + f_x(\lambda_1, \tau_1, x_1) \neq 0 . \tag{1.5}$$

If the Jacobian condition (1.5) is not satisfied at a point $(\lambda, x) = (\lambda_1, x_1)$, then $x = x_1$ is a multiple or nonisolated solution of (1.3), and $\lambda = \lambda_1$ is a possible branch point.

Suppose that for $\lambda = \lambda_0$, equation (1.4) fails to hold, and that $\lambda = \lambda_0$ is a bifurcation point of (1.3) with $\tau = 0$. Then we can find the possible branch points of (1.3) which lie in a neighborhood of $(\lambda, \tau) = (\lambda_0, 0)$ by applying the implicit function theorem to the system

$$(1.6) \qquad \begin{cases} x + f(\lambda, \tau, x) = 0 \\ 1 + f_x(\lambda, \tau, x) = 0 . \end{cases}$$

Since $f_\lambda(\lambda, 0, 0) = 0$ by assumption, we know that there are functions $\lambda = \lambda(x)$ and $\tau = \tau(x)$ which satisfy the system (1.6) for x sufficiently small, whenever

$$(1.7) \qquad f_\tau(\lambda_0, 0, 0) \cdot f_{\lambda x}(\lambda_0, 0, 0) \neq 0 .$$

A condition very similar to (1.7) will be assumed in the more general discussion in Chapter II. The functions $\lambda = \lambda(x)$ and $\tau = \tau(x)$ represent a family of possible branching points of (1.3) emanating from $\lambda = \lambda_0$ and $\tau = 0$. One could now study neighboring solutions to determine if branching occurs.

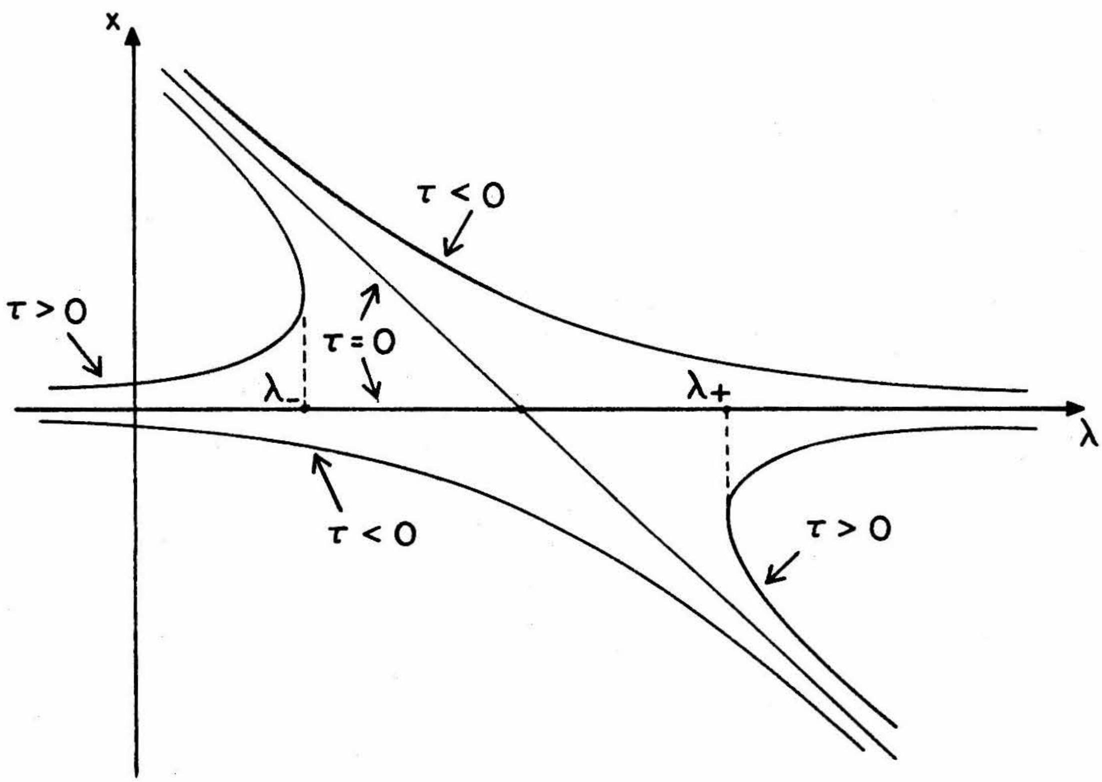
A simple algebraic example possessing characteristics which we will find in other more general problems is given by

$$(1.8) \qquad x - x(x + \lambda) - \tau = 0$$

The solutions of (1.8) are

(1.9) $$x_{\pm} = \frac{1-\lambda}{2} \pm \tfrac{1}{2}\sqrt{(\lambda-1)^2 - 4\tau} .$$

When $\tau = 0$, the solutions reduce to $x = 0$ and $x = 1-\lambda$, so that the point $\lambda = 1$ is a bifurcation point. When $\tau < 0$ the two solutions given by (1.9) are well defined with $x_+ > 0$ and $x_- < 0$ for all values of λ. However, when $\tau > 0$, real valued solutions do not exist for $1 - 2\tau^{1/2} < \lambda < 1 + 2\tau^{1/2}$, and the points $\lambda_{\pm} = 1 \pm 2\tau^{1/2}$ are branching points of equation (1.8). The accompanying plot shows the solution curves (1.9) for various values of τ.



Equation (1.2) can represent very general operator equations. In this thesis we are concerned primarily with nonlinear boundary value problems involving either second or fourth order partial differential operators. It is a simple matter to consider more general operators, such as compact nonlinear operators on a Hilbert space, since most of the changes necessary are notational only. Our primary application is to the buckling of imperfect engineering structures [3]*, [4], [15], where τ represents the amplitude of some imperfection, and the branching point represents the load at which buckling may occur.

Our general results for second order equations make use of a perturbation procedure coupled with an iteration technique used by H. B. Keller [17] for bifurcation problems. The perturbation procedure is used to suggest the proper form of the solution. Then the iteration technique is used to prove the existence of such solutions. In Sections II.2 through II.4 we show the existence of a unique family of nonisolated solutions for certain values of τ sufficiently small. The perturbation procedure is also shown to be asymptotic. In Section II.5, a solution branch is constructed through a nonzero nonisolated solution of (1.2). In Section II.6, the "stability" of the constructed branch is examined, and is simply summarized in Figure 1. Under certain circumstances, given in Section II.7, part of the solution branch constructed in Section II.5 is shown to be a branch of minimal positive solutions, in the sense

*Numbers in square brackets refer to the list of references at the end of the thesis.

of Keller and Cohen [19]. Furthermore, conditions are given under which a branching point is the least upper bound of values λ for which positive solutions of (1.2) exist with τ fixed. The bifurcation diagram is completed in Section II.8, where it is shown that for all values of τ sufficiently small, (1.2) has two distinct solution branches, although some of these branches may not contain branching points. A graphical summary of the main results in Chapter II is given in Figures 2, 3 and 4.

In Chapters III and IV, these ideas are applied to the dynamic buckling of arches and imperfect columns and to the buckling of an imperfect column on a nonlinearly elastic foundation, respectively. In Chapter III, global stability characteristics for the buckled equilibrium states of an imperfect column are studied using the qualitative features of nonisolated solutions discussed in Chapter II. In Chapter IV, an advantage in using the present iteration technique in problems of imperfection sensitivity in buckling is demonstrated. It is a simple consequence of our approach that an approximate solution of the buckling load is asymptotic to the exact solution. Approximation techniques used elsewhere do not have this feature [3].

Equation number (1.1) refers to the first equation of Section 1 of the given chapter. Similarly, Theorem 3-1 refers to the first theorem of Section 3 of the given chapter. When reference is made to an equation or theorem in a different chapter, the other chapter is named explicitly. The meaning of symbols remains unchanged within each chapter, but may differ in different chapters.

Chapter II

General Imperfection Theory

II. 1. Notation and Definitions.

We want to study branching phenomena for elliptic boundary value problems of the form

$$(1.1) \qquad Lu + f(\lambda, \tau, u) = 0 \qquad x \in D$$

$$Bu = 0 \qquad x \in \partial D .$$

Here $x = (x_1, \ldots, x_n)$ and L is the uniformly elliptic second order operator defined on D by

$$(1.2) \qquad Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x) \frac{\partial u}{\partial x_j} - a_0(x) u .$$

The boundary operator B is defined on ∂D by

$$(1.3) \qquad Bu = b_0(x)u + b_1(x) \sum_{j=1}^{n} \beta_j(x) \frac{\partial u}{\partial x_j}$$

where for notational purposes we will denote

$$\frac{\partial u}{\partial \beta} = \sum_{j=1}^{n} \beta_j(x) \frac{\partial u}{\partial x_j} .$$

We denote by $C^{k+\alpha}(\Omega)$ the space of real valued functions which are k times continuously differentiable on a point set Ω, and have Hölder continuous k^{th} derivatives on Ω with Hölder exponent α. We assume that D is a bounded domain in $\mathbb{R}^n$ with boundary ∂D of class $C^{2+\alpha}$. The coefficients $a_{ij}(x)$, $a_j(x)$, $a_0(x) > 0$ are assumed to be in $C^{2+\alpha}(\overline{D})$, $C^{1+\alpha}(\overline{D})$ and $C^{\alpha}(\overline{D})$ respectively, while $b_0(x)$, $b_1(x)$, $\beta_i(x)$ are in $C^{1+\alpha}(\partial D)$ for some $\alpha \in (0,1)$. The uniform ellipticity of L implies that for all unit vectors $y = (y_1, \dots, y_n)$

$$(1.4)\text{-i)} \qquad \sum_{i,j=1}^{n} a_{ij}(x)\, y_i y_j \geqslant a > 0 \qquad x \in D .$$

Taking $n_i(x)$ to be the components of the unit outward normal at $x \in \partial D$, we assume that the coefficients of the boundary operator B satisfy

$$(1.4)\text{-ii)} \qquad \sum_{j=1}^{n} \beta_j(x)\, n_j(x) > 0 \quad , \quad \sum_{j=1}^{n} \beta_j^2(x) = 1$$

and that ∂D can be decomposed into $\partial D = \partial D_1 \cup \partial D_2$ where

$$(1.4)\text{-iii)} \qquad b_0(x) > 0 \qquad b_1(x) \equiv 0 \qquad x \in \partial D_1$$

$$(1.4)\text{-iv)} \qquad b_0(x) \geqslant 0 \qquad b_1(x) > 0 \qquad x \in \partial D_2 .$$

The assumed smoothness assumptions on L and B are sufficient to assure us that, for $F(x) \in C^{\alpha}(\overline{D})$, the linear problem

$$\text{(1.5)} \qquad Lu(x) = F(x) \qquad x \in D$$

$$Bu(x) = 0 \qquad x \in \partial D$$

has a unique solution, $u(x) \in C^{2+\alpha}(\overline{D})$ ([26], pp 134-136). These assumptions further imply that L and B satisfy the strong maximum principle [31] which leads to

Proposition (1): If $\phi(x) \in C'(\overline{D}) \cap C^2(D)$, then

i - $L\phi \leqslant$ on D , $B\phi \geqslant 0$ on $\partial D \Rightarrow \phi(x) \geqslant 0$ on $\overline{D}$

ii - $L\phi <$ on D , $B\phi \geqslant 0$ on $\partial D \Rightarrow \phi(x) > 0$ on D

Furthermore, if $\phi(x) = 0$ for some $x \in \partial D$, then

$$\frac{\partial \phi}{\partial \alpha} < 0 \qquad x \in \partial D$$

where $\frac{\partial}{\partial \alpha}$ is the directional derivative taken in any outward direction.

We will assume that the nonlinearity $f(\lambda, \tau, u)$ satisfies $f(\lambda, 0, 0) = 0$ for all real λ, and $f(\lambda, \tau, 0) \neq 0$ if $\tau \neq 0$. We will assume that $f(\lambda, \tau, u) \in C^{\alpha}(D)$ whenever $u \in C^{2+\alpha}(D)$, and the partial derivative at λ, τ, u satisfies $f_u(\lambda, \tau, u) \in C^{\alpha}(D)$ when $u \in C^{2+\alpha}(D)$. All other derivatives up to and including third order are assumed to be continuous on D if $u \in C^{2+\alpha}(D)$. Although $f(\lambda, \tau, u)$ is allowed to depend on x, this dependence will not be explicitly shown.

The standard bifurcation problem with $\tau = 0$ has been treated in numerous places ([7], [17], [21], [25], [34], [38]). One result of these studies [25] is that branching can occur at a point $(\lambda, u) = (\lambda_0, 0)$ only if there are nontrivial solutions of the problem

$$(1.6)\qquad \begin{aligned} L\phi + f_u(\lambda_0, 0, 0)\phi &= 0 \qquad x \in D \\ B\phi &= 0 \qquad x \in \partial D. \end{aligned}$$

For the forced case with $\tau_0 \neq 0$, a point (λ_0, u_0) can be a branching point of (1.1) only if there are nontrivial solutions to the problem resulting from linearization of (1.1) about the known solution at $\lambda = \lambda_0$. That is, there must be nontrivial solutions to

$$(1.7)\qquad \begin{aligned} L\psi + f_u(\lambda_0, \tau_0, u_0)\psi &= 0 \qquad x \in D \\ B\psi &= 0 \qquad x \in \partial D \end{aligned}$$

where (λ_0, τ_0, u_0) satisfy (1.1). Solutions (u, ψ, λ, τ) satisfying both (1.1) and (1.7) will be referred to as non-isolated solutions of (1.1) corresponding to the point λ.

To provide a starting point for our investigation, we will assume that there is a number λ_0 and a nontrivial function $\phi_0(x) \in C^{2+\alpha}(\overline{D})$ which satisfy (1.6). The quadruple $(u, \psi, \lambda, \tau) = (0, \phi_0, \lambda_0, 0)$ will be referred to as a trivial nonisolated solution of (1.1). We will also assume that all solutions of (1.6) are multiples of $\phi_0(x)$.

By defining the inner product

$$(1.10) \qquad \langle u, v \rangle = \int_D u(x)\ v(x)\,dx$$

we can define adjoint operators L^* and B^* to be those operators satisfying

$$(1.11) \qquad \langle v, Lu \rangle - \langle u, L^* v \rangle = 0$$

whenever $u, v, \in C^{2+\alpha}(\overline{D})$ and $Bu = 0$, $B^* v = 0$. The operators which result from this definition are given by ([6], [13])

$$(1.12) \qquad L^* v = \sum_{i,j=1}^{n} \frac{\partial^2 (a_{ij}(x)v)}{\partial x_i\, \partial x_j} - \sum_{j=1}^{n} \frac{\partial (a_j(x)v)}{\partial x_j} - a_0(x)v\ , \quad x \in D\ .$$

$B^* v = 0$ is defined by requiring

$$(1.13) \qquad P[u, v] = \sum_{i,j=1}^{n} \left[\frac{\partial u}{\partial x_i} a_{ij} v - \frac{\partial}{\partial x_j}(a_{ij} v) u \right] + \sum_{i=0}^{n} a_i uv = 0 \qquad x \in \partial D.$$

when $Bu = 0$. For $\rho(x) \in C^{\alpha}(\overline{D})$, whenever

$$(1.14) \qquad L\phi + \rho(x)\phi = 0 \qquad x \in D$$

$$B\phi = 0 \qquad x \in \partial D$$

has a nontrivial solution, we know from the study of spectral theory for compact operations [11], that the associated adjoint problem

$$L^* \phi^* + \rho(x)\phi^* = 0 \tag{1.15}$$
$$B^* \phi^* = 0$$

also has a nontrivial solution, and the null space of equation (1.14) is of the same dimension as the null space of (1.15). The Fredholm Alternative Theorem [6] holds for solutions of

$$Lv + \rho(x)v = g(x) \ . \tag{1.16}$$

Specifically, this asserts that (1.16) has a solution $v(x) \in C^{2+\alpha}(\overline{D})$ provided $g(x) \in C^{\alpha}(\overline{D})$ and

$$\langle g(x) \ , \ \phi_0^* \rangle = 0 \tag{1.17}$$

where ϕ_0^* is a solution of (1.15). Let $\Lambda(x) \in C(\overline{D})$ be a "weight function" such that $\langle \phi_0(x) \ , \ \phi_0^*(x)\,\Lambda(x) \rangle \neq 0$. We make the stronger assumption that if the solution $v(x)$ in (1.16) is made unique by requiring the orthogonality condition

$$\langle \ v(x) \ , \ \phi_0^*(x) \ \Lambda(x) \ \rangle = 0 \tag{1.18}$$

then there exists a constant $G > 0$ such that

$$\|v\|_\infty \leq G \, \|g\|_\infty \tag{1.19}$$

The notation has been chosen with an eye toward generalizations. If we wanted L to be an operator in a real Hilbert space H, then the inner product (1.10) could be chosen appropriately. The inequality (1.19) could be assumed to hold in the induced norm of H, and many of the results that follow would be true with only a slight change of wording.

II. 2. Perturbation Theory for Non-isolated Solutions.

Formal perturbation theory is often used to obtain useful approximations to solutions of nonlinear boundary value problems. The ideas used in the method originated in the work of Lindstedt and Poincaré [30] on periodic motion in celestial mechanics. Recently it has been applied by J. B. Keller and others [22], [23], [29] to a number of nonlinear boundary value problems arising in such diverse areas as nonlinear optics, heat conduction, and superconductivity.

In this section, we will develop a formal perturbation scheme which indicates the form of nontrivial non-isolated solutions of (1.1). We will show that this scheme is well defined and can be carried out to arbitrary order provided the nonlinearity $f(\lambda, \tau, u)$ is sufficiently differentiable in each of the arguments λ, τ and u. It will be the task of later sections to show the validity of this perturbation scheme.

Suppose that the quadruple $(u, \phi, \lambda, \tau) = (0, \phi_0, \lambda_0, 0)$ is a known nonisolated solution of (1.1). Our hope is that this solution is an element of a branch of nonisolated solutions, and that this branch can be represented parametrically with some parameter ϵ. If this parametric representation is also sufficiently differentiable at the known solution $(u, \psi, \lambda, \tau) = (0, \phi_0, \lambda_0, 0)$, then we can expand the parametric representation in a Taylor series about known solution. We choose the parameter ϵ so that $\big(u(x,\epsilon), \psi(x,\epsilon), \lambda(\epsilon), \tau(\epsilon)\big) = (0, \phi_0, \lambda_0, 0)$ when $\epsilon = 0$.

The first n+1 terms of this Taylor expansion will be referred to as the n^{th} perturbation expansion for nonisolated solutions of (1.1), and will be in the form

$$(2.1) \quad \begin{cases} \tilde{u}^n(x,\epsilon) = \epsilon(u_0 + \epsilon u_1 + \cdots + \epsilon^n u_n) \\ \tilde{\psi}^n(x,\epsilon) = \phi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots + \epsilon^n\psi_n \\ \tilde{\lambda}^n(\epsilon) = \lambda_0 + \epsilon\lambda_1 + \cdots + \epsilon^n\lambda_n \\ \tilde{\tau}^n(\epsilon) = \epsilon(\tau_0 + \epsilon\tau_1 + \cdots + \epsilon^n\tau_n) . \end{cases}$$

There are two equivalent ways to determine the coefficients in (2.1). Since (2.1) is intended to be the Taylor series of solutions of (1.1), (1.7) about $\epsilon = 0$, one could differentiate (1.1) and (1.7) k times, and then set $\epsilon = 0$, thus finding the equations which determine the coefficients of the k^{th} terms as functions of the previously determined coefficients. Alternately, one could substitute expression (2.1) directly into (1.1) and (1.7), expand the nonlinear terms in powers of ϵ, and then equate coefficients of like powers of ϵ. The equations which result will again determine the k^{th} set of coefficients as functions of previously determined coefficients. Since these two methods are equivalent, both require that the nonlinearity $f(\lambda, \tau, u)$ have smooth derivatives of at least order n.

Carrying out the above expansion procedure, we get

$$(2.2) \qquad \begin{aligned} Lu_0 + f_u(\lambda_0, 0, 0)u_0 &= -f_\tau(\lambda_0, 0, 0)\tau_0 \\ Bu_0 &= 0 \end{aligned}$$

$$(2.3) \qquad \left\{ \begin{aligned} Lu_1 + f_u(\lambda_0, 0, 0)u_1 = -\Big\{ & f_{\lambda u}(\lambda_0, 0, 0)\lambda_1 u_0 + f_{\lambda\tau}(\lambda_0, 0, 0)\lambda_1 \tau_0 \\ & + f_{\tau u}(\lambda_0, 0, 0)u_0 \tau_0 + \tfrac{1}{2} f_{uu}(\lambda_0, 0, 0)u_0^2 \\ & + \tfrac{1}{2} f_{\tau\tau}(\lambda_0, 0, 0)\tau_0^2 + f_\tau(\lambda_0, 0, 0)\tau_1 \Big\} \quad x \in D \\ Bu_1 = 0 \qquad x \in \partial D & \end{aligned} \right.$$

$$(2.4) \qquad \begin{aligned} L\phi_0 + f_u(\lambda_0, 0, 0)\phi_0 &= 0 \qquad x \in D \\ B\phi_0 &= 0 \qquad x \in \partial D \end{aligned}$$

$$(2.5) \qquad \left\{ \begin{aligned} L\psi_1 + f_u(\lambda_0, 0, 0)\psi_1 = -\Big\{ & f_{u\lambda}(\lambda_0, 0, 0)\lambda_1 \phi_0 + f_{uu}(\lambda_0, 0, 0)\phi_0 u_0 \\ & + f_{u\tau}(\lambda_0, 0, 0)\tau_0 \phi_0 \Big\} \qquad x \in D \\ B\psi_1 = 0 \qquad x \in \partial D. & \end{aligned} \right.$$

Since the operator $L^* + f_u(\lambda_0, 0, 0)$ has a null space spanned by ϕ_0^*, we know by the Fredholm alternative theorem that equations (2.2) - (2.5) can be solved if and only if the right hand side of each equation is orthogonal to ϕ_0^* as in (1.17). This condition determines the constants λ_1, τ_0 and τ_1 in (2.2) - (2.5). Furthermore these solutions will not be unique, since we may add any multiple of ϕ_0 to the solution. To make the solutions unique, we require

$$(2.6) \qquad \begin{cases} \langle \psi(x), \phi_0^*(x)\, f_{\lambda u}(\lambda_0, 0, 0) \rangle = 1 \\ \\ \langle u(x), \phi_0^*(x)\, f_{\lambda u}(\lambda_0, 0, 0) \rangle = \epsilon \end{cases}$$

This places a restriction on the terms of the perturbation expansion (2.1), requiring that

$$(2.7) \qquad \begin{cases} \langle \phi_0, \phi_0^* f_{\lambda u}(\lambda_0, 0, 0) \rangle = 1 \\ \\ \langle u_0, \phi_0^* f_{\lambda u}(\lambda_0, 0, 0) \rangle = 1 \end{cases}$$

and

$$(2.8) \qquad \begin{cases} \langle \psi_i, \phi_0^* f_{\lambda u}(\lambda_0, 0, 0) \rangle = 0 \\ \\ \langle u_i, \phi_0^* f_{\lambda u}(\lambda_0, 0, 0) \rangle = 0 \end{cases} \qquad i = 1, 2, \ldots$$

hold.

In order to solve (2.2), the Fredholm alternative theorem requires that

$$(2.9) \qquad \tau_0 \langle f_\tau(\lambda_0, 0, 0) , \phi_0^* \rangle = 0 .$$

Assuming that $\langle f_\tau(\lambda_0, 0, 0) , \phi_0^* \rangle \neq 0$, we must have $\tau_0 = 0$. With $\tau_0 = 0$, equations (2.2) and (2.4) are identical so that, applying (2.7),

$$(2.10) \qquad u_0(x) = \phi_0(x) .$$

Using this information, equation (2.3) becomes

$$(2.11) \qquad \begin{cases} Lu_1 + f_u(\lambda_0,0,0)u_1 = -\left\{ f_{\lambda u}(\lambda_0,0,0)\lambda_1\phi_0 + \frac{1}{2} f_{uu}(\lambda_0,0,0)\phi_0^2 \right. \\ \qquad\qquad \left. + f_\tau(\lambda_0,0,0)\tau_1 \right\} . \quad x \in D \\ Bu_1 = 0 \qquad x \in \partial D . \end{cases}$$

Applying the Fredholm alternative theorem to (2.11), we have

$$(2.12) \qquad \lambda_1 \langle f_{\lambda u}(\lambda_0, 0, 0)\phi_0 , \phi_0^* \rangle + \tau_1 \langle f_\tau(\lambda_0, 0, 0), \phi_0^* \rangle = -\tfrac{1}{2} \langle f_{uu}(\lambda_0, 0, 0)\phi_0^2 , \phi_0^* \rangle$$

Similarly, from equation (2.5) we get

$$(2.13) \qquad \lambda_1 \langle f_{\lambda u}(\lambda_0, 0, 0)\phi_0 , \phi_0^* \rangle = -\langle f_{uu}(\lambda_0, 0, 0)\phi_0^2 , \phi_0^* \rangle$$

Equations (2.12) and (2.13) are two linear simultaneous equations for λ_1 and τ_1. The determinant of this system is

(2.14) $$D = \langle f_\tau(\lambda_0, 0, 0)\, \phi_0^* \rangle \cdot \langle f_{\lambda u}(\lambda_0, 0, 0)\, \phi_0, \phi_0^* \rangle$$

so that these equations can be solved provided $D \neq 0$. If $D \neq 0$, the solution of (2.12) - (2.13) is

(2.15) $$\begin{cases} \lambda_1 = -\dfrac{\langle f_{uu}(\lambda_0, 0, 0), \phi_0^2, \phi_0^* \rangle}{\langle f_{\lambda u}(\lambda_0, 0, 0)\ \ \phi_0, \phi_0^* \rangle} \\[2em] \tau_1 = \dfrac{1}{2}\, \dfrac{\langle f_{uu}(\lambda_0, 0, 0)\, \phi_0^2, \phi_0^* \rangle}{\langle f_\tau(\lambda_0, 0, 0), \phi_0^* \rangle} \end{cases}$$

Of interest in many applications is the relationship between λ, the "buckling load," and τ, the "imperfection amplitude." According to (2.1)

$$\tilde{\tau} = \epsilon^2 \tau_1 + O(\epsilon^3)\ ,$$

so if $\tau_1 \neq 0$, we can find $\lambda = \lambda(\tau)$ approximately. In particular,

(2.16) $$\tilde{\tau} = \frac{\epsilon^2}{2}\, \frac{\langle f_{uu}(\lambda_0, 0, 0)\, \phi_0^2, \phi_0^* \rangle}{\langle f_\tau(\lambda_0, 0, 0), \phi_0^* \rangle} + O(\epsilon^3)\ ,$$

and

$$(2.17)\qquad \tilde{\lambda} = \lambda_0 - \epsilon \frac{\langle f_{uu}(\lambda_0, 0, 0)\, \phi_0^2, \phi_0^* \rangle}{\langle f_{\lambda u}(\lambda_0, 0, 0)\, \phi_0, \phi_0^* \rangle} + O(\epsilon^2),$$

can be combined to give

$$(2.18)\qquad \tilde{\lambda} = \lambda_0 \pm \tau^{1/2} \frac{[2\langle f_\tau(\lambda_0,0,0), \phi_0^*\rangle \cdot \langle f_{uu}(\lambda_0,0,0)\, \phi_0^2, \phi_0^*\rangle]^{1/2}}{\langle f_{\lambda u}(\lambda_0,0,0)\, \phi_0, \phi_0^*\rangle} + O(\tau),$$

where τ must be restricted so that λ is real.

In many applications, $f_{uu}(\lambda_0, 0, 0) \equiv 0$, so that (2.18) is not valid. Suppose there is an integer p such that

$$(2.19)\qquad \begin{cases} \dfrac{\partial^k f(\lambda_0, 0, 0)}{\partial u^k} = 0 \qquad 2 \leq k \leq p \\[2ex] \left\langle \dfrac{\partial^{p+1} f(\lambda_0, 0, 0)}{\partial u^{p+1}}\, \phi_0^{p+1}, \phi_0^* \right\rangle \neq 0 . \end{cases}$$

Then the perturbation equations can be shown to reduce to

$$(2.20)\qquad \begin{aligned} Lu_k + f_u(\lambda_0, 0, 0)u_k &= 0 \qquad x \in D \quad k = 0, 1, 2, \ldots, p-1 \\ Bu_k &= 0 \qquad x \in \partial D \end{aligned}$$

$$(2.21)\qquad \begin{cases} Lu_p + f_u(\lambda_0,0,0)u_p = -\Big[\lambda_p f_{\lambda u}(\lambda_0,0,0)u_0 + \dfrac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \dfrac{u_0^{p+1}}{(p+1)!} \\ \qquad\qquad + f_\tau(\lambda_0,0,0)\tau_p\Big], \quad x \in D \\[1ex] Bu_p = 0 \qquad x \in \partial D \end{cases}$$

and

$$(2.22) \qquad L\psi_k + f_u(\lambda_0, 0, 0)\psi_k = 0 \qquad x \in D \qquad k = 0, 1, 2, \dots, p-1$$

$$B\psi_k = 0 \qquad x \in \partial D$$

$$(2.23) \begin{cases} L\psi_k + f_u(\lambda_0, 0, 0)\psi_p = -\left[\lambda_p f_{\lambda u}(\lambda_0, 0, 0)\phi_0 + \dfrac{\partial^{p+1} f(\lambda_0, 0, 0)}{\partial u^{p+1}} \dfrac{\phi_0^{p+1}}{p!}\right], x \in D \\ B\psi_p = 0 \ , \ x \in \partial D \end{cases}$$

and the conditions (2.7) and (2.8) are required to hold.

According to equations (2.20) and (2.21),

$$(2.24) \qquad \begin{cases} u_0(x) = \phi_0(x) \\ u_k(x) = \psi_k(x) = 0 \qquad k = 1, 2, \dots p-1 \ , \end{cases}$$

and the calculations used in deriving (2.20) - (2.23) show that

$$(2.25) \qquad \lambda_k = \tau_k = 0 \qquad k = 1, 2, \dots, p-1 \ .$$

Using (2.24) and (2.25) in (2.1), the form of the solution reduces to

$$(2.26) \qquad \begin{cases} \tilde{u}^p = \epsilon(\phi_0 + \epsilon^p u_p) + O(\epsilon^{p+2}) \\ \tilde{\psi}^p = \phi_0 + \epsilon^p \psi_p + O(\epsilon^{p+1}) \\ \tilde{\lambda}^p = \lambda_0 + \epsilon^p \lambda_p + O(\epsilon^{p+1}) \\ \tilde{\tau}^p = \epsilon^{p+1} \tau_p + O(\epsilon^{p+2}). \end{cases}$$

Invoking the Fredholm alternative theorem in (2.21) and (2.23), we can find λ_p and τ_p. Specifically,

$$(2.27)\begin{cases} \lambda_p \langle f_{\lambda u}(\lambda_0,0,0)\phi_0, \phi_0^* \rangle + \tau_p \langle f_\tau(\lambda_0,0,0), \phi_0^* \rangle \\ \qquad\qquad = -\frac{1}{(p+1)!} \langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^* \rangle \\ \lambda_p \langle f_{\lambda u}(\lambda_0,0,0)\phi_0, \phi_0^* \rangle = -\frac{1}{p!} \langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^* \rangle \end{cases}$$

so that

$$(2.28) \qquad \tau_p = \frac{p}{(p+1)!} \frac{\langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^* \rangle}{\langle f_\tau(\lambda_0,0,0), \phi_0^* \rangle} .$$

At the outset, we assumed conditions (2.19) that assured us that $\tau_p \neq 0$. Now we can solve for $\lambda = \lambda(\tau)$ approximately. Doing so, we get

$$(2.29) \qquad \lambda = \lambda_0 - \frac{\tau^{\frac{p}{p+1}}}{p!} \left(\frac{(p+1)!}{p} \langle f_\tau(\lambda_0,0,0),\phi_0^*\rangle\right)^{\frac{p}{p+1}} \frac{\left(\langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^*\rangle\right)^{\frac{1}{p+1}}}{\langle f_{\lambda u}(\lambda_0,0,0)\phi_0, \phi_0^* \rangle} + O(\tau).$$

Thus, the buckling load λ is altered by imperfections in the order of $\tau^{p/p+1}$ for τ sufficiently small.

We would like to show that the perturbation scheme given by (2.1) is well defined, and that the k^{th} terms of the expansion are determined as solutions of linear equations involving only the previously determined k terms. If we assume that $f(\lambda, \tau, u)$ is at least n+2 times continuously differentiable in all variables, then expanding $f(\lambda, \tau, u)$ in a multivariable Taylor series with remainder near $\lambda = \lambda_0$, $\tau = 0$, $u = 0$, and substituting (2.1) into the Taylor series, it is easy to see that

$$(2.30)\quad \begin{cases} f(\tilde{\lambda}^n, \tilde{\tau}^n, \tilde{u}^n) = \epsilon\Big(f_u(\lambda_0,0,0)u_0 + f_\tau(\lambda_0,0,0)\tau_0\Big) \\ \qquad +\epsilon\sum_{k=1}^{n}\epsilon^k\Big[f_u(\lambda_0,0,0)u_k + f_{\lambda u}(\lambda_0,0,0)u_0\lambda_k + f_\tau(\lambda_0,0,0)\tau_k \\ \qquad + P_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\}\Big] + O(\epsilon^{n+2}) \end{cases}$$

$$(2.31)\quad \begin{cases} f_u(\tilde{\lambda}^n, \tilde{\tau}^n, \tilde{u}^n) = f_u(\lambda_0,0,0) + \sum_{k=1}^{n}\epsilon^k\Big[f_{u\lambda}(\lambda_0,0,0)\lambda_k \\ \qquad + Q_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\}\Big] \\ \qquad + O(\epsilon^{n+1}) \end{cases}$$

On substituting (2.30) and (2.31) into (1.1), (1.7) we get

$$(2.32)\quad \begin{cases} L\tilde{u}^n + f_u(\lambda_0,0,0)\tilde{u}^n = -\Big[\epsilon f_{\lambda u}(\lambda_0,0,0)u_0(\tilde{\lambda}^n - \lambda_0) + f_\tau(\lambda_0,0,0)\tilde{\tau}^n \\ \qquad + \epsilon\sum_{k=1}^{n}\epsilon^k P_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\}\Big] \\ \qquad + O(\epsilon^{n+2}) \quad x\in D \\ B\tilde{u}^n = 0, \quad x\in\partial D \end{cases}$$

and

$$(2.33)\quad \begin{cases} L\tilde{\psi}^n + f_u(\lambda_0,0,0)\tilde{\psi}^n = -\sum_{k=1}^{n}\epsilon^k\Big[\sum_{j=1}^{k}\Big(f_{u\lambda}(\lambda_0,0,0)\lambda_j + Q_j\{\cdots\}\Big)\psi_{k-j}\Big] + O(\epsilon^{n+1}) \\ \qquad x\in D \\ B\tilde{\psi}^n = 0, \quad x\in\partial D \end{cases}$$

Equating like powers of ϵ, we have

$$(2.34)\begin{cases} Lu_k + f_u(\lambda_0 0,0)u_k = -\left[f_{\lambda u}(\lambda_0,0,0)u_0\lambda_k + f_\tau(\lambda_0,0,0)\tau_k + P_k\{\cdots\}\right] & x\in D \\ Bu_k = 0 \qquad x\in\partial D \qquad k=1,2,\cdots,n\,. \end{cases}$$

$$(2.35)\begin{cases} L\psi_k + f_u(\lambda_0,0,0)\psi_k = -\sum_{j=1}^{k}\left[f_{u\lambda}(\lambda_0,0,0)\lambda_j + Q_j\{\cdots\}\right]\psi_{k-j} & x\in D \\ B\psi_k = 0\,, \quad x\in\partial D \qquad k=1,2,\ldots,n\,. \end{cases}$$

As before, equations (2.34) and (2.35) can be solved only if the Fredholm alternative is satisfied. Using that $u_0 = \psi_0 = \phi_0$, the resulting equations are

$$(2.36) \qquad \lambda_k\langle f_{\lambda u}(\lambda_0,0,0)\phi_0, \phi_0^*\rangle + \tau_k\langle f_\tau(\lambda_0,0,0), \phi_0^*\rangle = -\langle P_k\{\cdots\}, \phi_0^*\rangle$$

and

$$(2.37)\ \lambda_k\langle f_{\lambda u}(\lambda_0,0,0)\phi_0, \phi_0^*\rangle = -\sum_{j=1}^{k-1}\langle\left(f_{u\lambda}(\lambda_0,0,0)\lambda_j + Q_j\{\cdots\}\right)\psi_{k-j}, \phi_0^*\rangle - \langle Q_k\{\cdots\}\phi_0, \phi_0^*\rangle\,.$$

Notice now that using equations (2.34) - (2.37) determines the k^{th} terms of the expansion (2.1), u_k, ψ_k, λ_k and τ_k as functions of the previously determined k terms. The equations (2.34) - (2.37) are linear, and involve the same differential operator and matrix operator for each term of the expansion. This assures us that the procedure can be carried out indefinitely, provided the determinant

D of (2.14) does not vanish, and provided $f(\lambda, \tau, u)$ is sufficiently differentiable. The condition (2.8) makes the procedure unique. The solutions of (2.36) and (2.37) are

$$\lambda_k = \frac{-1}{\langle f_{\lambda u}(\lambda_0,0,0)\phi_0, \phi_0^* \rangle} \Big[\sum_{j=1}^{k-1} \langle \big(f_{u\lambda}(\lambda_0,0,0)\lambda_j + Q_j\{u_0,\cdots,u_{j-1};\lambda_1\cdots,\lambda_{j-1};\tau_0,\cdots,\tau_{j-1}\}\big) \psi_{k-j}, \phi_0^* \rangle + \langle Q_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\}\phi_0, \phi_0^* \rangle \Big] \tag{2.38}$$

and

$$\tau_k = \frac{1}{\langle f_\tau(\lambda_0,0,0), \phi_0^* \rangle} \Big[\sum_{j=1}^{k-1} \langle \big(f_{u\lambda}(\lambda_0,0,0)\lambda_j + Q_j\{u_0,\cdots,u_{j-1};\lambda_1,\cdots,\lambda_{j-1};\tau_0,\cdots,\tau_{j-1}\}\big) \psi_{k-j}, \phi_0^* \rangle + \langle Q_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\}\phi_0 - P_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\}, \phi_0^* \rangle \Big].$$

When the coefficients u_k, ψ_k, λ_k and τ_k are substituted into (2.1), the resulting expansion is an asymptotic solution of (1.1) for ϵ sufficiently small. This fact will be shown in Section 4.

II.3. Existence of Non-isolated Solutions.

In Section II.2, we were able to develop a perturbation scheme which gave rise to expressions which we hope are approximate solutions of (1.1), (1.7). At this stage, however, we do not even know that (1.1), (1.7) have "nontrivial" solutions. In order to show that such solutions exist, we look for solutions of (1.1), (1.7) in a form suggested by the perturbation method, namely

$$(3.1)\qquad \begin{cases} u(x,\epsilon) = \epsilon\phi_0 + \epsilon^2 v(x,\epsilon) , \\ \psi(x,\epsilon) = \phi_0 + \epsilon\,\chi(x,\epsilon) , \\ \lambda(\epsilon) = \lambda_0 + \epsilon\mu(\epsilon) , \\ \tau(\epsilon) = \epsilon^2\eta(\epsilon) , \end{cases}$$

where $\phi_0(x)$ satisfies (1.6). In addition, we require that

$$(3.2)\qquad \begin{cases} \langle v(x,\epsilon) , \phi_0^*(x)\, f_{\lambda u}(\lambda_0,0,0)\rangle = 0 \\ \\ \langle \chi(x,\epsilon) , \phi_0^*(x)\, f_{\lambda u}(\lambda_0,0,0)\rangle = 0 . \end{cases}$$

We must show that for some nontrivial range of the parameter ϵ, $0 \leq |\epsilon| \leq \epsilon_0$, the functions $v(x,\epsilon)$, $\chi(x,\epsilon)$, $\mu(\epsilon)$, $\eta(\epsilon)$ exist and are bounded uniformly in ϵ. If this can be shown, then as ϵ approaches zero, the solutions (3.1) approach the trivial solution $(u,\psi,\lambda,\tau) = (0,\phi_0,\lambda_0,0)$ continuously. Furthermore, the solutions (3.1) constitute a family of nonisolated solutions of (1.1) depending continuously on the parameter ϵ.

To carry out the analysis for this problem, we will make use of the identity

$$(3.3)\qquad g(a) - g(b) = (a-b)\int_0^1 \frac{dg}{dx}(sa + (1-s)b)\, ds$$

provided the derivative $\frac{dg}{dx}$ exists and is continuous for $x \in [a,b]$.

To use this identity for (1.1), (1.7) we will assume that $f(\lambda, \tau, u)$ has at least three continuous derivatives in λ, τ and u. Substituting (3.1) into (1.1) and (1.7) gives

$$
(3.4)\quad \begin{cases} Lv + f_u(\lambda_0,0,0)v = -\dfrac{1}{\epsilon^2}\left[f(\lambda,\tau,u) - f_u(\lambda_0,0,0)u\right] \\ \quad = -\left[\eta(\epsilon)\displaystyle\int_0^1 f_\tau(\lambda, s\tau, u)ds + \mu(\phi_0+\epsilon v)\int_0^1\!\!\int_0^1 f_{\lambda u}(\lambda_0+s\epsilon\mu, 0, tu)ds\,dt \right. \\ \quad \left. + (\phi_0+\epsilon v)^2 \displaystyle\int_0^1\!\!\int_0^1 f_{uu}(\lambda_0, 0, stu)\, s\,dt\,ds\right] \qquad x \in D \\ \quad = P(v, \mu, \eta, \epsilon; x) \\ Bv = 0, \quad x \in \partial D \end{cases}
$$

and

$$
(3.5)\quad \begin{cases} L\chi + f_u(\lambda_0,0,0)\chi = -\dfrac{1}{\epsilon}\left[f_u(\lambda,\tau,u) - f_u(\lambda_0,0,0)\right]\psi \\ \quad = -\left[\mu\displaystyle\int_0^1 f_{\lambda u}(\lambda_0+s\epsilon\mu, 0, u)ds + \epsilon\eta\int_0^1 f_{\tau u}(\lambda, s\tau, u)ds \right. \\ \quad \left. + (\phi_0+\epsilon v)\displaystyle\int_0^1 f_{uu}(\lambda_0, 0, su)ds\right](\phi_0+\epsilon\chi) \\ \quad = Q(v, \chi, \mu, \eta, \epsilon; x) \qquad x \in D \\ B\chi = 0 \qquad x \in \partial D \end{cases}
$$

Equations (3.4) and (3.5) are of the form (1.16) and can be solved for v and χ only if the orthogonality conditions

$$
(3.6)\quad \begin{aligned} \langle P(v, \mu, \eta, \epsilon; x), \phi_0^* \rangle &= 0 \\ \langle Q(v, \chi, \mu, \eta, \epsilon; x), \phi_0^* \rangle &= 0 \end{aligned}
$$

hold. These solutions, if they exist, are only determined to within an additive multiple of ϕ_0, unless the conditions (3.2) are satisfied.

The orthogonality condition (3.6) provides the method by which we intend to solve (3.4) and (3.5). We will solve them iteratively, first by choosing values of η and μ so that (3.6) holds, and then solving (3.4) and (3.5) for the functions v and χ. With the new functions v and χ, we must choose new values of η and μ so that (3.6) again holds, and the process continues indefinitely. If we can show that this process converges, then roughly speaking, we will have found a solution of (3.4) and (3.5). This iteration scheme is a modification of the standard technique of Lyapunov and Schmidt [38] suggested by the treatment in [17] of the bifurcation problem (1.1) with $\tau = 0$.

To formulate the contraction mapping we introduce the sets of functions

(3.7) $$\mathcal{B}_K = \{y(x) \,|\, y(x) \in C^{2+\alpha}(D), \|y\| \leq K \ , \ \langle y(x), \phi_0^*(x) f_{\lambda u}(\lambda_0, 0, 0)\rangle = 0\} \ ,$$

and the real interval

(3.8) $$\mathcal{I}_K = \{\eta \,|\, |\eta| \leq K\} \ .$$

In addition, we introduce the set

$$(3.9)\quad S_1(\rho,\Gamma) = \Big\{(\lambda,\tau,u\ ;\ x)\ \Big|\ \lambda = \lambda_0 + \epsilon\mu,\ \tau = \epsilon^2\eta\ ,\ u = \epsilon\phi_0 + \epsilon^2 v,\ x\in \overline{D}\ ,$$
$$v\in \mathcal{B}_\Gamma\ ;\ \mu,\eta\in \mathcal{J}_\Gamma\ ;\ 0\leq|\epsilon|\leq\rho\Big\}$$

Notice that $S_1(\rho,\Gamma)$ depends on ρ and $\rho\Gamma$ but not on Γ alone. For each $v(x), \chi(x)\in \mathcal{B}_K$ and $\eta,\mu\in\mathcal{J}_K$, a transformation T_ϵ is defined for each ϵ in $0\leq|\epsilon|\leq\epsilon_1$ by

$$T_\epsilon[v,\chi,\mu,\eta] = [\tilde{v},\tilde{\chi},\tilde{\mu},\tilde{\eta}]$$

where

$$(3.10)\ \begin{cases} \tilde{\mu}\ \langle(\phi_0+\epsilon\chi)\int_0^1 f_{\lambda u}(\lambda_0+s\epsilon\mu,0,u)ds,\phi_0^*\rangle \\ \qquad = -\ \langle(\phi_0+\epsilon v)(\phi_0+\epsilon\chi)\int_0^1 f_{uu}(\lambda_0,0,su)ds,\phi_0^*\rangle \\ \qquad\quad -\epsilon\eta\ \langle(\phi_0+\epsilon\chi)\int_0^1 f_{\tau u}(\lambda,s\tau,u)ds,\phi_0^*\rangle\ , \end{cases}$$

$$(3.11)\ \begin{cases} \tilde{\eta}\langle\int_0^1 f_\tau(\lambda,s\tau,u)ds,\phi_0^*\rangle = -\tilde{\mu}\langle(\phi_0+\epsilon v)\int_0^1\int^1 f_{\lambda u}(\lambda_0+s\epsilon\mu,0,tu)ds\,dt,\phi_0^*\rangle \\ \qquad -\langle(\phi_0+\epsilon v)^2\int_0^1\int_0^1 f_{uu}(\lambda_0,0,stu)s\,dt\,ds,\phi_0^*\rangle\ , \end{cases}$$

$$(3.12)\ \begin{cases} L\tilde{v} + f_u(\lambda_0,0,0)\tilde{v} = -\Big[\tilde{\eta}\int_0^1 f_\tau(\lambda,s\tau,u)ds \\ \qquad + \tilde{\mu}(\phi_0+\epsilon v)\int_0^1\int_0^1 f_{\lambda u}(\lambda_0+s\epsilon\mu,0,tu)ds\,dt \\ \qquad + (\phi_0+\epsilon v)^2\int_0^1\int_0^1 f_{uu}(\lambda_0,0,stu)s\,dt\,ds\Big]\quad x\in D \\ B\tilde{v} = 0\ ,\ x\in\partial D \qquad \langle\tilde{v},\phi_0^* f_{\lambda u}(\lambda_0,0,0)\rangle = 0\ , \end{cases}$$

$$
(3.13)\begin{cases}
L\widetilde{\chi}+f_u(\lambda_0,0,0)\widetilde{\chi} = -\Big[\widetilde{\mu}(\phi_0+\epsilon\chi)\int_0^1 f_{\lambda u}(\lambda_0+s\epsilon\mu,\,0,\,u)ds \\
\qquad + \epsilon\eta\,(\phi_0+\epsilon\chi)\int_0^1 f_{\tau u}(\lambda,s\tau,u)ds \\
\qquad + (\phi_0+\epsilon v)(\phi_0+\epsilon\chi)\int_0^1 f_{uu}(\lambda_0,0,su)ds\Big] \quad x\in D \\
B\widetilde{\chi} = 0\ ,\ x\in\partial D \\
\langle\widetilde{\chi},\overset{*}{\phi_0}f_{\lambda u}(\lambda_0,0,0)\rangle = 0\ .
\end{cases}
$$

This definition of T_ϵ induces an iteration procedure in a natural way. Suppose we let an initial iterate be $\Big(v^0(\epsilon,x),\ \chi^0(\epsilon,x),\ \mu^0(\epsilon),\ \eta^0(\epsilon)\Big)$. Then we define the sequence of iterates $\Big\{\Big(v^\nu(\epsilon,x),\ \chi^\nu(\epsilon,x),\ \mu^\nu(\epsilon),\ \eta^\nu(\epsilon)\Big)\Big\}$ by

$$
(3.14)\qquad \Big[v^{\nu+1},\ \chi^{\nu+1},\ \mu^{\nu+1},\ \eta^{\nu+1}\Big] = T_\epsilon\Big[v^\nu,\chi^\nu,\mu^\nu,\eta^\nu\Big]\ .
$$

We are now able to state and prove the following

Theorem 3-1: Let $S_1 = S_1(\rho,\Gamma)$ for some fixed $\rho\leq 1$, $\rho\Gamma\leq 1$. Suppose that

$$
(3.15)\qquad \begin{cases}
f(\lambda,\tau,u)\in C^\alpha(S_1)\ ,\ f_u(\lambda,\tau,u)\in C^\alpha(S_1), \\
\text{and} \\
f_\tau(\lambda,\tau,u), f_\lambda, f_{\lambda u}, f_{uu}, f_{\tau u}, f_{uuu}, f_{\lambda uu}, f_{\tau uu}, f_{\tau\tau u}, f_{\lambda\lambda u} \\
\quad f_{\lambda\tau u}, f_{\tau\tau}\in C(S_1)\ .
\end{cases}
$$

and that $\langle f_T(\lambda_0, 0, 0), \phi_0^* \rangle \cdot \langle \phi_0 f_{\lambda u}(\lambda_0, 0, 0), \phi_0^* \rangle \neq 0$. Then there are real positive constants ϵ_0 and K , $\epsilon_0 \leq \rho$, $\epsilon_0 K \leq \rho\Gamma$ such that the mapping T_ϵ given by (3.10) - (3.13) maps $U_K = (\mathcal{B}_K \times \mathcal{B}_K \times \mathcal{I}_K \times \mathcal{I}_K)$ into U_K , and T_ϵ is a contraction on U_K for all ϵ $0 \leq |\epsilon| \leq \epsilon_0$. Furthermore, the problem (1.1), (1.7) has a nontrivial solution of the form (3.1) where $v(x,\epsilon)$, $\chi(x,\epsilon)$, $\mu(\epsilon)$, $\eta(\epsilon)$ satisfy (3.4) - (3.6) and are the limits of the iteration scheme generated by T_ϵ for any initial iterates in U_K .

Proof: For notational purposes, define

(3.16) $$\|g\|_s = \sup_{\omega \in S_1} |g(\omega)| .$$

Since S_1 depends on the numbers ρ and $\rho\Gamma$ but not on Γ alone, we can use the norm (3.16) without knowing Γ. We need only require that $\epsilon_0 \leq \rho$, $\epsilon_0 K \leq \rho\Gamma$. By requiring $\epsilon_0 \leq \max\{1, 1/K\}$ we can use the norm $\|g\|_s$ with $\rho = 1$, $\rho K = 1$.

By virtue of the smoothness assumptions we made about inverting the operator $L + f_u(\lambda_0, 0, 0)$ (cf. (1.16)), to show that T_ϵ maps U_K into U_K , we need only find appropriate constants K and ϵ_1 that define $\mathcal{B}_K$ and $\mathcal{I}_K$.

We assumed that $|\langle \phi_0 f_{u\lambda}(\lambda_0,0,0), \phi_0^* \rangle| = \alpha \neq 0$ and $|\langle f_T(\lambda_0,0,0), \phi_0^* \rangle| = \beta \neq 0$. Notice that in (2.7) we assumed $\alpha = 1$ without loss of generality. We restrict ϵ_1 to be small enough so that

$$(3.17)\quad \begin{cases} \left| \left\langle (\phi_0 + \epsilon\chi) \int_0^1 f_{u\lambda}(\lambda_0 + \epsilon s\mu, 0, u)ds, \phi_0^* \right\rangle \right| \geq \frac{\alpha}{2}, \\ \left| \left\langle \int_0^1 f_\tau(\lambda, s\tau, u)ds, \phi_0^* \right\rangle \right| \geq \frac{\beta}{2} \end{cases}$$

for $(\lambda, \tau, u) \in S_1$. Then if $(v, \chi, \mu, \eta) \in U_K$ for some ϵ_1, K with $\epsilon_1 \leq 1$, $\epsilon_1 K \leq 1$, we have

$$(3.18)\quad \begin{cases} |\tilde{\mu}| \leq \frac{2}{\alpha} \Phi(\|\phi\|_\infty + \epsilon_1 K)\left[(\|\phi\|_\infty + \epsilon_1 K)\|f_{uu}\|_s + \epsilon_1 K \|f_{u\tau}\|_s\right] \\ \qquad \leq A_1 + \epsilon_1 B_1(\epsilon_1, K), \end{cases}$$

$$(3.19)\quad \begin{cases} |\tilde{\eta}| \leq \frac{2}{\beta} \Phi\left[|\tilde{\mu}|(\|\phi\|_\infty + \epsilon_1 K)\|f_{u\lambda}\|_s + (\|\phi\|_\infty + \epsilon_1 K)^2 \frac{\|f_{uu}\|_s}{2}\right] \\ \qquad \leq A_2 + \epsilon_1 B_2(\epsilon_1, K), \end{cases}$$

$$(3.20)\quad \begin{cases} \|\tilde{v}\|_\infty \leq G\left[|\tilde{\eta}| \; \|f_\tau\|_s + |\tilde{\mu}|(\|\phi\|_\infty + \epsilon_1 K)\|f_{u\lambda}\|_s \right. \\ \qquad \left. + (\|\phi\|_\infty + \epsilon_1 K)^2 \; \|f_{uu}\|_s\right] \leq A_3 + \epsilon_1 B_3(\epsilon_1, K), \end{cases}$$

and

$$(3.21)\quad \begin{cases} \|\tilde{\chi}\|_\infty \leq G(\|\phi\|_\infty + \epsilon_1 K)\left[|\tilde{\mu}| \; \|f_{u\lambda}\|_s + (\|\phi\|_\infty + \epsilon_1 K)\|f_{uu}\|_s \right. \\ \qquad \left. + \epsilon_1 K \|f_{u\tau}\|_s\right] \leq A_4 + \epsilon_1 B_4(\epsilon_1, K), \end{cases}$$

where $\Phi = \langle 1, |\phi_0^*| \rangle$. The positive numbers A_i do not depend on ϵ_1 or K, and the positive numbers $B_i(\epsilon_1, K)$ are bounded on compact sets of (ϵ_1, K). Our goal is to find $K > 0$ such that

$$A_i + \epsilon_1 B_i(\epsilon_1, K) \leq K \qquad i = 1,2,3,4 \; . \tag{3.22}$$

This is easily accomplished, since $B_i(\epsilon_1, K)$ depend continuously on ϵ_1 and K, we can pick $K > \max_{i=1,2,3,4} \{A_i\}$, and then find an $\epsilon_1 > 0$ so that (3.22) holds. Letting $\epsilon_2 = \min\{1, \frac{1}{K}, \epsilon_1\}$ we have that $T_\epsilon : U_K \to U_K$ for $0 \leq |\epsilon| \leq \epsilon_2$. The second part of the proof involves finding $\epsilon_0 \leq \epsilon_2$ so that T_ϵ is a contraction on U_K for $0 \leq |\epsilon| \leq \epsilon_0$. Suppose we let $w_1 = (v, \chi, \mu, \eta) \in U_K$ and $w_2 = (y, \zeta, \nu, \kappa) \in U_K$. Then we can show that there exists a positive constant M such that

$$\|\tilde{w}_1 - \tilde{w}_2\| \leq (|\epsilon| M) \|w_1 - w_2\| \quad \text{whenever } |\epsilon| \leq \epsilon_2 \tag{3.23}$$

where

$$\|w\| = \max\{\|v\|_\infty, \|\chi\|_\infty, |\mu|, |\eta|\}, \; w \in U_K \; .$$

In particular, with some straight forward calculations, it is easily shown that

$$|\tilde{\mu} - \tilde{\nu}| \leq \frac{2\Phi}{\alpha} |\epsilon| \left[A_{11} \|v-y\|_\infty + A_{12} \|\chi - \zeta\|_\infty + A_{13} |\mu - \nu| + A_{14} |\eta - \kappa| \right], \tag{3.24}$$

$$|\tilde{\eta} - \tilde{\kappa}| \leq \frac{2\Phi}{\beta} |\epsilon| \left[A_{21} \|v-y\|_\infty + A_{22} |\mu - \nu| + A_{23} |\eta - \kappa| \right] + \frac{2\Phi}{\beta} A_{24} |\tilde{\mu} - \tilde{\nu}| \, , \tag{3.25}$$

$$(3.26)\qquad \|\tilde{v}-\tilde{y}\|_{\infty} \leq G|\epsilon|\left[A_{31}\|v-y\|_{\infty} + A_{32}|\mu-\nu| + A_{33}|\eta-\kappa|\right] + G A_{34}|\tilde{\eta}-\tilde{\kappa}| + G A_{35}|\tilde{\mu}-\tilde{\nu}| ,$$

and

$$(3.27)\qquad \|\tilde{\chi}-\tilde{\zeta}\|_{\infty} \leq G|\epsilon|\left[A_{41}\|v-y\|_{\infty} + A_{42}\|\chi-\zeta\|_{\infty} + A_{43}|\mu-\nu| + A_{44}|\eta-\kappa|\right] + G A_{45}|\tilde{\mu}-\tilde{\nu}| ,$$

where

$$(3.28)\qquad \begin{cases}
A_{11} = A_{41} = (\|\phi\|_{\infty}+1)\left[\|f_{uu}\|_s + \|f_{uu\lambda}\|_s + \frac{\epsilon_2}{2}(\|\phi\|_{\infty}+1)\|f_{uuu}\|_s + \epsilon_2\|f_{uu\tau}\|_s\right. , \\
A_{12} = A_{42} = K\|f_{u\lambda}\|_s + \|f_{u\tau}\|_s + (\|\phi\|_{\infty}+1)\|f_{uu}\|_s , \\
A_{13} = A_{43} = (\|\phi\|_{\infty}+1)\left[\epsilon_2\|f_{u\lambda\tau}\|_s + \frac{K}{2}\|f_{u\lambda\lambda}\|_s\right], \\
A_{14} = A_{44} = (\|\phi\|_{\infty}+1)\left[\|f_{u\tau}\|_s + \frac{\epsilon_2}{2}\|f_{u\tau\tau}\|_s\right], \\
A_{21} = (\|\phi\|_{\infty}+1)\left[\|f_{uu}\|_s + \frac{\epsilon_2}{6}(\|\phi\|_{\infty}+1)\|f_{uuu}\|_s + \frac{1}{2}\|f_{uu\lambda}\|_s\right] + K\|f_{u\lambda}\|_s + \|f_{\tau u}\|_s , \\
A_{22} = A_{32} = K\left[\frac{1}{2}(\|\phi\|_{\infty}+1)\|f_{u\lambda\lambda}\|_s + \|f_{\tau\lambda}\|_s\right] , \\
A_{23} = A_{33} = \frac{1}{2}\|f_{\tau\tau}\|_s ,\ A_{24} = A_{35} = A_{45} = (\|\phi\|_{\infty}+1)\|f_{u\lambda}\|_s ,\ A_{34} = \|f_{\tau}\|_s , \\
A_{31} = \|f_{\tau u}\|_s + K\|f_{u\lambda}\|_s + (\|\phi\|_{\infty}+1)\left[\|f_{uu}\|_s + \frac{1}{2}\|f_{uu\lambda}\|_s + (\|\phi\|_{\infty}+1)\frac{\|f_{uuu}\|_s}{6}\right] .
\end{cases}$$

Clearly, (3.24) - (3.28) imply the existence of a constant M such that (3.23) holds. By choosing $0 < \epsilon_3 M < 1$, the mapping T_ϵ is a contraction on U_K for $0 \leq |\epsilon| \leq \epsilon_0$ where $\epsilon_0 = \min(\epsilon_2, \epsilon_3)$.

We have now shown that for $|\epsilon| \leq \epsilon_0$, T_ϵ maps U_K into itself and is a contraction. But this is not sufficient to show that the iteration scheme generated by T_ϵ converges to a solution of (3.4) - (3.6). We know by virtue of the contraction that the sequences $\{v^\nu(\epsilon, x)\}$ and $\{\chi^\nu(\epsilon, x)\}$ converge uniformly on $\overline{D}$ and that $\{\eta^\nu(\epsilon)\}$ and $\{\mu^\nu(\epsilon)\}$ converge. By a simple induction we also know that $v^\nu(\epsilon, x)$, $\chi^\nu(\epsilon, x) \in C^{2+\alpha}(\overline{D})$. This allows us to apply the Compactness Theorem 12.2 of Agmon, Douglis and Nirenberg [1], which justifies taking the limit $\nu \to \infty$ in (3.14). Q.E.D.

It is easy to see that a solution of the form (3.1) is unique. If it were not unique, then there would be two solutions, say $w_1 \neq w_2$ which both satisfy (3.4) - (3.6). Thus, both w_1 and w_2 are fixed points of the mapping T_ϵ given by (3.10) - (3.13), so that $\tilde{w}_1 = w_1$ and $\tilde{w}_2 = w_2$. Applying (3.23) we see that

$$\|w_1 - w_2\| < \|w_1 - w_2\| \tag{3.29}$$

whenever $|\epsilon| \leq \epsilon_0$ which is a contradiction. Thus, $w_1 = w_2$, and the solution is unique.

The proof of Theorem 3.1 assured us that nonisolated solutions of (1.1) are of the form (3.1), where $v(\epsilon, x)$, $\chi(\epsilon, x)$, $\mu(\epsilon)$, and $\eta(\epsilon)$ are uniformly bounded by K for $|\epsilon| \leq \epsilon_0$. To know more about the quantitative behavior of the solution, we would like to know more about $\mu(\epsilon)$ and $\eta(\epsilon)$. We know that $\mu(\epsilon)$ and $\eta(\epsilon)$ are fixed points of (3.10) and (3.11) respectively. Suppose that there is an integer p such that

$$(3.30)\quad \begin{cases} \dfrac{\partial^k f(\lambda_0, 0, 0)}{\partial u^k} = 0 \qquad 2 \le k \le p \\[2ex] \left\langle \dfrac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1} , \phi_0^* \right\rangle \neq 0 \end{cases}$$

holds, and assume that all third derivatives of $f(\lambda, \tau, u)$ exist and are continuous. Then applying the identity (3.3) to (3.10) and (3.11) we find

$$(3.31)\quad \mu\Big(\langle \phi_0 f_{\lambda u}(\lambda_0,0,0), \phi_0^* \rangle + O(\epsilon)\Big) + \eta\, O(\epsilon) = -\frac{\epsilon^{p-1}}{p!} \left\langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^* \right\rangle + O(\epsilon^p),$$

and

$$(3.32)\quad \begin{aligned} &\eta\Big(\langle f_\tau(\lambda_0,0,0), \phi_0^* \rangle + O(\epsilon)\Big) + \mu\Big(\langle \phi_0 f_{\lambda u}(\lambda_0,0,0), \phi_0^* \rangle + O(\epsilon)\Big) \\ &= -\frac{\epsilon^{p-1}}{(p+1)!} \left\langle \phi_0^{p+1} \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}}, \phi_0^* \right\rangle + O(\epsilon^p), \end{aligned}$$

Although (3.31) and (3.32) include implicit dependence on $\mu(\epsilon)$ and $\eta(\epsilon)$ in the $O(\epsilon)$ and $O(\epsilon^p)$ terms, we know that $|\mu(\epsilon)| \le K$, $|\eta(\epsilon)| \le K$ for $|\epsilon| \le \epsilon_0$, and this permits the determination of the asymptotic form of $\mu(\epsilon)$ and $\eta(\epsilon)$ as $|\epsilon| \to 0$.

The system (3.31) - (3.32) can be solved for ϵ sufficiently small, to give

$$\mu(\epsilon) = -\frac{\epsilon^{p-1}}{p!} \frac{\left\langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^* \right\rangle}{\langle \phi_0 f_{\lambda u}(\lambda_0,0,0), \phi_0^* \rangle} + O(\epsilon^p) \tag{3.33}$$

$$\eta(\epsilon) = \epsilon^{p-1} \frac{p}{(p+1)!} \frac{\left\langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^* \right\rangle}{\langle f_\tau(\lambda_0,0,0), \phi_0^* \rangle} + O(\epsilon^p) \tag{3.34}$$

Coupling (3.33) and (3.34) with the form of the solution (3.1), we see that the perturbation solution (2.26) - (2.28) is asymptotic to the solution (3.1) as $\epsilon \to 0$. In section 4, we will show that this is true for the perturbation scheme with any number of terms.

II.4. Comparison of Iteration Scheme and Perturbation Procedure.

In Section 3 we found a mapping T_ϵ whose fixed point gave rise to solutions of (1.1), (1.7) for each ϵ, $0 \leq |\epsilon| \leq \epsilon_0$. The iterations generated by T_ϵ were found to converge to the fixed point for all initial iterates in U_K.

In this section, we will examine the iterations generated by the initial iterate

$$w^0 = \left(v^0(\epsilon,x), \chi^0(\epsilon,x), \mu^0(\epsilon), \eta^0(\epsilon)\right) = (0,0,0,0) . \tag{4.1}$$

To estimate the errors of the k^{th} iterate w^k, we apply (3.23) to get

$$\|w^{k+1} - w^k\| \leq (|\epsilon| M)\|w^k - w^{k-1}\| \ , \tag{4.2}$$

where ϵ_0 is chosen so that $\epsilon_0 M < 1$. Applying (4.2) recursively we find that

$$\left\{ \begin{aligned} \|w^{k+1} - w^k\| &\leq (|\epsilon|M)^k \ \|w' - w^0\| \\ &\leq (|\epsilon|M)^k K \ , \end{aligned} \right. \tag{4.3}$$

A simple application of the triangle inequality implies

$$\begin{aligned} \|w^{k+m} - w^k\| &\leq K(|\epsilon|M)^k \left[(|\epsilon|M)^{m-1} + (|\epsilon|M)^{m-2} + \cdots + (|\epsilon|M) + 1\right] \\ &= K(|\epsilon|M)^k \ \frac{1 - (|\epsilon|M)^m}{1 - |\epsilon|M} \ , \end{aligned} \tag{4.4}$$

and passing to the limit as $m \to \infty$, we get

$$\|w - w^k\| \leq K \ \frac{(|\epsilon|M)^k}{1 - |\epsilon|M} \ , \tag{4.5}$$

where $w = \big(v(\epsilon, x),\ \chi(\epsilon, x),\ \mu(\epsilon),\ \eta(\epsilon)\big)$ is a solution of (3.4) - (3.6). Writing this another way, as $\epsilon \to 0$, we have

$$\|w - w^k\| = O(|\epsilon|^k) \ . \tag{4.6}$$

We can interpret this information in terms of the solutions of (1.1), (1.7) in the form (3.1). The sequence $\{w^k\}$ corresponds for ϵ fixed to finding a sequence $\left(u^k(\epsilon, x),\ \psi^k(\epsilon, x),\ \mu^k(\epsilon),\ \eta^k(\epsilon)\right)$ where

$$
(4.7) \qquad \begin{cases}
u^k(\epsilon, x) = \epsilon\,\phi_0 + \epsilon^2 v^k(\epsilon, x)\ , \\
\psi^k(\epsilon, x) = \phi_0 + \epsilon\,\chi^k(\epsilon, x)\ , \\
\lambda^k(\epsilon) = \lambda_0 + \epsilon\,\mu^k(\epsilon)\ , \\
\tau^k(\epsilon) = \epsilon^2 \eta^k(\epsilon)\ ,
\end{cases}
$$

with initial iterate

$$
(4.8) \qquad (u^0, \psi^0, \lambda^0, \tau^0) = (\epsilon\,\phi_0, \phi_0, \lambda_0, 0)\ .
$$

Furthermore, (4.6) tells us that

$$
(4.9) \qquad \begin{cases}
\|u - u^k\| = O(|\epsilon|^{k+2})\ , \\
\|\psi - \psi^k\| = O(|\epsilon|^{k+1})\ , \\
\|\lambda - \lambda^k\| = O(|\epsilon|^{k+1})\ , \\
\|\tau - \tau^k\| = O(|\epsilon|^{k+2})\ .
\end{cases}
$$

We would now like to show that the perturbation method described in Section 2 gives an expansion which is asymptotic as

$\epsilon \to 0$. Specifically, we will show that (4.9) holds for the iterates (4.7) and also for the perturbation terms (2.1). To do so we prove the following

Theorem 4-1: Let the hypotheses of Theorem 3-1 hold, and let (2.30) and (2.31) be satisfied for all ϵ, $|\epsilon| \leq \epsilon_0$. Let $(\tilde{u}^n, \tilde{\psi}^n, \tilde{\lambda}^n, \tilde{\tau}^n)$ be of the form (2.1) with $u_i(x)$, $\psi_i(x)$ bounded on $\overline{D}$ for $i = 1, 2, \cdots n$. Then the iterates $(u^n, \psi^n, \lambda^n, \tau^n)$ of (4.7) and the perturbation expansions $(\tilde{u}^n, \tilde{\lambda}^n, \tilde{\lambda}^n, \tilde{\tau}^n)$ of (2.1) and (2.34) - (2.37) satisfy

$$
(4.10) \quad \begin{cases}
\| u^n(\epsilon, x) - \tilde{u}^n(\epsilon, x) \| = O(|\epsilon|^{n+2}) \\
\| \psi^n(\epsilon, x) - \tilde{\psi}^n(\epsilon, x) \| = O(|\epsilon|^{n+1}) \\
| \lambda^n(\epsilon) - \tilde{\lambda}^n(\epsilon) | = O(|\epsilon|^{n+1}) \\
| \tau^n(\epsilon) - \tilde{\tau}^n(\epsilon) | = O(|\epsilon|^{n+2}) .
\end{cases}
$$

Note that applying the triangle inequality with (4.9), (4.10) assure us that the perturbation method is asymptotic to the known solution as $\epsilon \to 0$.

Proof: The proof of a similar fact for the bifurcation problem (when $\tau = 0$) has been given by Keller and Langford [20]. The proof uses a standard inductive argument. By (2.9), (2.10) and (4.8), we see that (4.10) holds trivially for $n = 0$. For $n > 0$, the iterates are generated according to (3.10) - (3.13). Without using the identity (3.3), these can be written as

$$(4.11)\quad \left(\frac{\lambda^{\nu+1}-\lambda_0}{\lambda^{\nu}-\lambda_0}\right)\langle f_u(\lambda^{\nu},0,u^{\nu})\psi^{\nu}-f_u(\lambda_0,0,u^{\nu})\psi^{\nu},\phi_0^*\rangle = -\langle f_u(\lambda^{\nu},\tau^{\nu},u^{\nu})\psi^{\nu}-f_u(\lambda^{\nu},0,u^{\nu})\psi^{\nu},\phi_0^*\rangle$$
$$-\langle f_u(\lambda_0,0,u^{\nu})\psi^{\nu}-f_u(\lambda_0,0,0)\psi^{\nu},\phi_0^*\rangle$$

$$(4.12)\quad \frac{\tau^{\nu+1}}{\tau^{\nu}}\langle f(\lambda^{\nu},\tau^{\nu},u^{\nu})-f(\lambda^{\nu},0,u^{\nu}),\phi_0^*\rangle + \left(\frac{\lambda^{\nu+1}-\lambda_0}{\lambda^{\nu}-\lambda_0}\right)\langle f(\lambda^{\nu},0,u^{\nu})-f(\lambda_0,0,u^{\nu}),\phi_0^*\rangle$$
$$= -\langle f(\lambda_0,0,u^{\nu}) - f_u(\lambda_0,0,0)u^{\nu},\phi_0^*\rangle ,$$

$$(4.13)\quad \begin{cases} Lu^{\nu+1}+f_u(\lambda_0,0,0)u^{\nu+1} = -\Big[\dfrac{\tau^{\nu+1}}{\tau^{\nu}}\Big(f(\lambda^{\nu},\tau^{\nu},u^{\nu})-f(\lambda^{\nu},0,u^{\nu})\Big) \\ \qquad + \dfrac{\lambda^{\nu+1}-\lambda_0}{\lambda^{\nu}-\lambda_0}\Big(f(\lambda^{\nu},0,u^{\nu})-f(\lambda_0,0,u^{\nu})\Big) \\ \qquad + f(\lambda_0,0,u^{\nu})-f_u(\lambda_0,0,0)u^{\nu}\Big] , \\ Bu^{\nu+1} = 0 , \\ \langle u^{\nu+1},\phi_0^* f_{\lambda u}(\lambda_0,0,0)\rangle = \epsilon , \end{cases}$$

$$(4.14)\quad \begin{cases} L\psi^{\nu+1}+f_u(\lambda\ ,0,0)\psi^{\nu+1} = -\Big[\dfrac{\lambda^{\nu+1}-\lambda_0}{\lambda^{\nu}-\lambda_0}\Big(f_u(\lambda^{\nu},0,u^{\nu})-f_u(\lambda_0,0,u^{\nu})\Big) \\ \qquad + f_u(\lambda^{\nu},\tau^{\nu},u^{\nu})-f_u(\lambda^{\nu},0,u^{\nu}) \\ \qquad + f_u(\lambda_0,0,u^{\nu})-f_u(\lambda_0,0,0)\Big]\psi^{\nu} , \\ B\psi^{\nu+1} = 0 , \\ \langle \psi^{\nu+1},\phi_0^* f_{\lambda u}(\lambda_0,0,0)\rangle = 1 , \end{cases}$$

provided $\tau^\nu \neq 0$ and $\lambda^\nu \neq \lambda_0$. If $\tau^\nu = 0$, the expression $\dfrac{f(\lambda^\nu,\tau^\nu,u^\nu)-f(\lambda^\nu,0,u^\nu)}{\tau^\nu}$ is replaced by $f_\tau(\lambda^\nu,0,u^\nu)$ in (4.12) and (4.13). Similarly, if $\lambda^\nu = \lambda_0$, the expression $f_u(\lambda^\nu, 0, u^\nu) - f_u(\lambda_0, 0, u^\nu) / \lambda^\nu - \lambda_0$ is replaced by $f_{\lambda u}(\lambda_0, 0, u^\nu)$ in (4.11) and (4.14), and $f(\lambda^\nu, 0, u^\nu) - f(\lambda_0, 0, u^\nu)/\lambda^\nu - \lambda_0$ is replaced by $f_\lambda(\lambda_0, 0, u^\nu)$ in (4.12) and (4.13).

Suppose that (4.10) holds for some $n > 0$. This implies

$$
(4.15)\quad \begin{cases}
u^n = \tilde{u}^n + \epsilon^{n+2} e_n(\epsilon), & \|e_n\| = O(1), \\
\psi^n = \tilde{\psi}^n + \epsilon^{n+1} \theta_n(\epsilon), & \|\theta_n\| = O(1), \\
\lambda^n = \tilde{\lambda}^n + \epsilon^{n+1} \mu_n(\epsilon), & \|\mu_n\| = O(1), \\
\tau^n = \tilde{\tau}^n + \epsilon^{n+2} \eta_n(\epsilon), & \|\eta_n\| = O(1).
\end{cases}
$$

Applying (4.15) with (2.30) and (2.31) we see that

$$
(4.16)\quad \begin{cases}
f(\lambda^n, \tau^n, u^n) = f_u(\lambda_0,0,0)u^n + f_\tau(\lambda_0,0,0)\tau^n + \epsilon\, f_{\lambda u}(\lambda_0,0,0)u_0(\lambda^n - \lambda_0) \\
\qquad + \epsilon \displaystyle\sum_{k=1}^{n+1} \epsilon^k P_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\} + O(\epsilon^{n+3}),
\end{cases}
$$

and

$$
(4.17)\quad \begin{cases}
f_u(\lambda^n, \tau^n, u^n) = f_u(\lambda_0,0,0) + f_{u\lambda}(\lambda_0,0,0)(\lambda^n - \lambda_0) \\
\qquad + \displaystyle\sum_{k=1}^{n+1} \epsilon^k\, Q_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\} + O(\epsilon^{n+2}).
\end{cases}
$$

If we assume that

$$(4.18) \qquad \lambda^{\nu+1} = \lambda_0 + \sum_{k=0}^{n+1} \epsilon^k \beta_k + O(\epsilon^{n+2}) ,$$

then (4.17), (4.18) combined with (4.11) give

$$(4.19) \begin{cases} \displaystyle\sum_{k=1}^{n+1} \epsilon^k \langle f_{u\lambda}(\lambda_0,0,0) \sum_{j=1}^{k} \beta_j \psi_{k-j}, \overset{*}{\phi}_0 \rangle + \frac{\sum_{k=1}^{n+1} \epsilon^k \beta_k}{\sum_{k=1}^{n} \epsilon^k \lambda_k + \epsilon^{n+1}\mu_n} \sum_{k=2}^{n+1} \epsilon^k q_k \\ \displaystyle = - \sum_{k=1}^{n+1} \epsilon^k \sum_{j=1}^{k} \langle Q_j\{u_0,\cdots,u_{j-1};\lambda_1,\cdots,\lambda_{j-1};\tau_0,\cdots,\tau_{j-1}\} \psi_{k-j}, \overset{*}{\phi}_0 \rangle + \sum_{k=2}^{n+1} \epsilon^k q_k + O(\epsilon^{n+2}) \\ \text{where} \\ \displaystyle q_k = \sum_{j=2}^{k} \langle (Q_j\{u_0,\cdots,u_{j-1};\lambda_1,\cdots,\lambda_{j-1};0,\cdots,0\} - Q_j\{u_0,\cdots,u_{j-1};0,0;0,\cdots,0\})\psi_{k-j}, \overset{*}{\phi}_0 \rangle , \end{cases}$$

where λ_k, τ_k, u_k and ψ_k are the coefficients of the perturbation scheme given by (2.34), (2.35), (2.38) and (2.39). Suppose that $\lambda_k = 0$ for $k = 1, \cdots p-1$ and $\lambda_p \neq 0$. Then $q_k = 0$ for $k \leq p$. If $p \geq n+1$, then the polynomial $\sum_{k=2}^{n+1} \epsilon^k q_k$ vanishes identically, and the corresponding terms involving this polynomial are not present in (4.19). If $p \leq n$, then $\sum_{k=2}^{n+1} \epsilon^k q_k \Big/ \sum_{k=1}^{n} \epsilon^k \lambda_k + \epsilon^{n+1} \mu_n$ is a polynomial of order ϵ. In either of these cases, equating the coefficients of ϵ in (4.19) gives

$$(4.20) \qquad \beta_1 \langle f_{u\lambda}(\lambda_0,0,0)\, \phi_0, \overset{*}{\phi}_0 \rangle = - \langle Q_1\{u_0;\tau_0\}\, \phi_0 , \overset{*}{\phi}_0 \rangle .$$

Comparing this with (2.38) we see that $\beta_1 = \lambda_1$. In fact it is easily seen that $\beta_k = \lambda_k$ for $k = 1, 2, \cdots, \min(p, n+1)$.

If $p \leq n$ we must still determine β_k for $k \geq p+1$. Suppose that for some $k \geq p+1$, $\beta_\nu = \lambda_\nu$ for $\nu < k$. Then we observe

$$(4.20) \qquad \frac{\sum_{k=1}^{n+1} \epsilon^k \beta_k}{\sum_{k=1}^{n} \epsilon^k \lambda_k + \epsilon^{n+1} \mu_n} = 1 + O(\epsilon^{k-p}) .$$

Since $\sum_{k=2}^{n+1} \epsilon^k q_k$ is a polynomial of order ϵ^{p+1}, equating the coefficients of ϵ^k in (4.19)

$$(4.21) \quad \beta_k \langle f_{u\lambda}(\lambda_0,0,0)\psi_0, \phi_0^* \rangle = -\sum_{j=1}^{k} \langle \Omega_j\{u_0,\cdots,u_{j-1};\lambda_1,\cdots,\lambda_{j-1};\tau_0,\cdots,\tau_{j-1}\}\ \psi_{k-j}, \phi_0^* \rangle$$
$$- \sum_{j=1}^{k-1} \beta_j \langle f_{u\lambda}(\lambda_0,0,0)\psi_{k-j}, \phi_0^* \rangle$$

which upon comparison with (2.38), shows that $\beta_k = \lambda_k$. This process can be carried out for all $p+1 \leq k \leq n+1$, which completes the induction necessary to show that

$$(4.22) \qquad \lambda^{n+1} = \lambda_0 + \sum_{k=1}^{n+1} \epsilon^k \lambda_k + O(\epsilon^{n+2}) = \tilde{\lambda}^{n+1} + O(\epsilon^{n+2}) .$$

In a similar manner, combining (4.12), (4.16) with (4.20) and (4.22) gives

$$(4.23)\quad \begin{cases} \epsilon \sum_{k=0}^{n+1} \epsilon^k \gamma_k \langle f_\tau(\lambda_0,0,0), \phi_0^* \rangle = -\epsilon \sum_{k=1}^{n+1} \epsilon^k \lambda_k \langle f_{u\lambda}(\lambda_0,0,0)\phi_0, \phi_0^* \rangle \\ \qquad - \epsilon \sum_{k=1}^{n+1} \epsilon^k \langle P_k\{u_0,\cdots,u_{k-1},\lambda_1,\cdots,\lambda_{k-1},0,\cdots,0\}, \phi_0^* \rangle \\ \qquad + \epsilon \dfrac{\sum_{k=0}^{n+1} \epsilon^k \gamma_k}{\sum_{k=0}^{n} \epsilon^k \tau_k + \epsilon^{n+1}\eta_n} \sum_{k=1}^{n+1} \epsilon^k p_k + O(\epsilon^{n+3}) \\ \text{where} \\ p_k = \langle P_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\} - P_k\{u_0,\cdots,u_{k-1};\lambda_1,\cdots,\lambda_{k-1};0,0\}, \phi_0^* \rangle , \end{cases}$$

where we have assumed $\tau^{n+1}(\epsilon)$ to be of the form

$$(4.24)\qquad \tau^{n+1} = \epsilon \sum_{k=0}^{n+1} \epsilon^k \gamma_k + O(\epsilon^{n+3}) .$$

The argument is now exactly the same as the argument given above and will not be repeated. The result of the argument is that

$$(4.25)\quad \gamma_k \langle f_\tau(\lambda_0,0,0), \phi_0^* \rangle = -\lambda_k \langle f_{u\lambda}(\lambda_0,0,0)\phi_0, \phi_0^* \rangle - \langle P_k\{u_0,\cdots,u_{k-1};\lambda_1,\lambda_{k-1};\tau_0,\cdots,\tau_{k-1}\}, \phi_0^* \rangle$$

for $k = 0, 1, \cdots n+1$, or that

$$(4.26)\qquad \tau^{n+1} = \epsilon \sum_{k=1}^{n+1} \epsilon^k \tau_k + O(\epsilon^{n+3}) = \tilde{\tau}^{n+1} + O(\epsilon^{n+3}).$$

The final step of the inductive argument involves substituting (4.16), (4.17), (4.22) and (4.26) into (4.13) and (4.14). In light of (4.20) and the similar relationship for the quotient $\tau^{\nu+1}/\tau^{\nu}$, it is easy to see that

(4.27) $$L(u^{n+1}-\tilde{u}^{n+1}) + f_u(\lambda_0,0,0)(u^{n+1}-\tilde{u}^{n+1}) = O(\epsilon^{n+3}) \qquad x\in D ,$$

$$B(u^{n+1}-\tilde{u}^{n+1}) = 0 \qquad x\in \partial D ,$$

and

(4.28) $$L(\psi^{n+1}-\tilde{\psi}^{n+1}) + f_u(\lambda_0,0,0)(\psi^{n+1}-\psi^{n+1}) = O(\epsilon^{n+2}) \qquad x\in D ,$$

$$B(\psi^{n+1}-\tilde{\psi}^{n+1}) = 0 \qquad x\in \partial D.$$

The right hand sides of (4.27) and (4.28) consist of the differences of right hand sides of (4.13) and (2.32) and of (4.14) and (2.33) respectively. Since each right hand side expression is orthogonal to ϕ_0^*, so also must their differences be orthogonal. By (1.19) the inverse of the differential operator $L + f_u(\lambda_0,0,0)$ is bounded, so that

(4.29) $$\| u^{n+1}-\tilde{u}^{n+1} \|_\infty = O(\epsilon^{n+3}) ,$$

and

(4.30) $$\| \psi^{n+1}-\tilde{\psi}^{n+1} \|_\infty = O(\epsilon^{n+2}) .$$

The expressions (4.22), (4.26), (4.29) and (4.30) are of the form (4.10), so the induction argument, and hence the proof of the theorem, is complete. Q.E.D.

II.5. Extension of Solution Branch from Nonisolated Solution.

In the previous sections, we showed that there are non-trivial nonisolated solutions of (1.1) depending continuously on a parameter ϵ for $|\epsilon| \leq \epsilon_0$. In this section we want to show circumstances under which a nonisolated solution of (1.1) is an element of a nontrivial solution branch with τ fixed. To do so we will construct the solution branch of (1.1) which contains a given non-isolated solution. A similar problem has been treated by Dean and Chambré [8], [9].

Suppose $\tau_0 \neq 0$ is fixed arbitrarily. If we make the identification

$$g(\lambda, u) = f(\lambda, \tau_0, u) \,, \tag{5.1}$$

equation (1.1) becomes

$$\begin{aligned} Lu + g(\lambda, u) &= 0 \qquad x \in D \,, \\ Bu &= 0 \qquad x \in \partial D \,, \end{aligned} \tag{5.2}$$

where we assume that $g(\lambda, 0) \neq 0$. Suppose that $u = w_0(x)$ is a nonisolated solution of (5.2) for $\lambda = \mu_0$. Accordingly, there exist functions $\psi_0(x)$ and $\overset{*}{\psi}_0(x)$ which satisfy

$$(5.3) \qquad L\psi + g_u(\mu_0, w_0(x))\psi = 0 \qquad x \in D ,$$
$$B\psi = 0 \qquad x \in \partial D ,$$

and

$$(5.4) \qquad \overset{*}{L}\overset{*}{\psi} + g_u(\mu_0, w_0(x))\overset{*}{\psi} = 0 \qquad x \in D ,$$
$$\overset{*}{B}\overset{*}{\psi} = 0 \qquad x \in \partial D ,$$

respectively, where $g_u(\mu_0, w_0(x))$ is the partial derivative of $g(\lambda, u)$ at $(\mu_0, w_0(x))$, and $\overset{*}{L}$, $\overset{*}{B}$ are adjoint operators defined previously. We will assume that all solutions of (5.3) and (5.4) can be represented as multiples of $\psi_0(x)$ and $\overset{*}{\psi}_0(x)$ respectively. With these assumptions, the Fredholm alternative theorem (1.16) - (1.19) is applicable when solving equations such as (5.3) with a nonzero right hand side.

We want to find solution sets $(\mu, w(x))$ of (5.2), if they exist, such that $\mu - \mu_0$ and $w(x) - w_0(x)$ are small. A natural way to proceed is to use the perturbation method to suggest the form of such solutions, and then to construct a contraction mapping which shows that the suggested form leads to solutions. Suppose we assume an expansion of (μ, w) in powers of δ which has the form

$$(5.5) \quad \begin{cases} \widetilde{w}(x, \delta) = w_0(x) + \delta w_1(x) + \delta^2 w_2(x) + \cdots, \\ \widetilde{\mu}(\delta) = \mu_0 + \delta\mu_1 + \delta^2\mu_2 + \cdots. \end{cases}$$

Substituting (5.5) into (5.2), expanding $g(\widetilde{\mu}, \widetilde{w})$ in powers of δ, and equating the coefficients of like powers of δ, leads to the equations

$$(5.6) \quad \begin{aligned} Lw_0 + g(\mu_0, w_0) &= 0 \qquad x \in D, \\ Bw_0 &= 0 \qquad x \in \partial D, \end{aligned}$$

$$(5.7) \quad \begin{aligned} Lw_1 + g_u(\mu_0, w_0)w_1 &= -g_\lambda(\mu_0, w_0)\mu_1 \qquad x \in D, \\ Bw_1 &= 0 \qquad x \in \partial D, \end{aligned}$$

and

$$(5.8) \quad \begin{cases} Lw_2 + g_u(\mu_0, w_0)w_2 = -\Big[g_\lambda(\mu_0, w_0)\mu_2 + \tfrac{1}{2}g_{uu}(\mu_0, w_0)w_1^2 \\ \qquad + g_{\lambda u}(\mu_0, w_0)\mu_1 w_1 + \tfrac{1}{2}g_{\lambda\lambda}(\mu_0, w_0)\mu_1^2\Big] \quad x \in D, \\ Bw_2 = 0 \quad x \in \partial D, \end{cases}$$

provided the derivatives g_λ, $g_{\lambda\lambda}$, $g_{\lambda u}$ and g_{uu} exist and are continuous. In order that w_2 be uniquely determined we require that

$$(5.9) \quad \langle w_2, \psi_0^* g_{\lambda u}(\mu_0, w_0) \rangle = 0$$

hold.

Equation (5.6) is automatically satisfied by our definition of μ_0 and $w_0(x)$. Because ψ_0 satisfies (5.3), the Fredholm alternative theorem implies that (5.7) can be solved only if

$$(5.9) \qquad \mu_1 \langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle = 0 .$$

If we assume that $\langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle \neq 0$, then (5.9) implies that

$$(5.10) \qquad \mu_1 = 0 \quad , \quad w_1(x) = \psi_0(x) .$$

Finally, the Fredholm alternative theorem applied to (5.8) gives us that

$$(5.11) \qquad \mu_2 \langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle = -\tfrac{1}{2} \langle g_{uu}(\mu_0, w_0)\psi_0^2, \psi_0^* \rangle .$$

Thus, the perturbation method indicates that solutions of (5.2) are of the form

$$(5.12) \qquad \begin{cases} \tilde{w}(x, \delta) = w_0(x) + \delta\psi_0(x) + O(\delta^2) , \\ \tilde{\mu}(\delta) = \mu_0 + \delta^2 \mu_2 + O(\delta^3) , \\ \text{where } \mu_2 = -\tfrac{1}{2} \dfrac{\langle g_{uu}(\mu_0, w_0)\psi_0^2, \psi_0^* \rangle}{\langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle} \end{cases} .$$

Motivated by the results of the perturbation method (5.12), we propose to look for solutions of (5.2) of the form

$$(5.13)\quad \begin{cases} w(x,\delta) = w_0(x) + \delta\psi_0(x) + \delta^2 y(x,\delta) , \\ \mu(\delta) = \mu_0 + \delta^2 \nu(\delta) , \\ \quad \text{where} \quad \langle y(x,\delta), \psi_0^*(x)\, g_{\lambda u}(\mu_0, w_0)\rangle = 0 . \end{cases}$$

On substituting (5.13) into (5.2) we get

$$(5.14)\quad \begin{cases} Ly + g_u(\mu_0,w_0)y = -\dfrac{1}{\delta^2}\Big[g(\mu,w) - g(\mu_0,w_0) - g_u(\mu_0,w_0)(w-w_0)\Big] \\ \qquad = -\Big[\nu \displaystyle\int_0^1 g_\lambda(\mu_0 + \delta^2 s\nu_1 w)ds \\ \qquad\quad + (\psi_0 + \delta y)^2 \displaystyle\int_0^1\int_0^1 g_{uu}\Big(\mu_0, w_0 + \delta st(\psi_0 + \delta y)\Big)\, s\,dt\,ds\Big] \\ \qquad = P(y,\nu,\delta;x), \qquad x \in D , \\ By = 0 \qquad x \in \partial D , \\ \langle y, \psi_0^*\, g_{\lambda u}(\mu_0, w_0)\rangle = 0 . \end{cases}$$

Equation (5.14) is of the form (1.16) and can be solved for y only if the orthogonality condition

$$(5.15)\qquad \langle P(y,\nu,\delta;x), \psi_0^* \rangle = 0$$

holds.

As before, we expect that we shall be able to find a solution to (5.14) for δ sufficiently small by employing an iteration procedure.

To set up such a procedure we introduce the set of functions

$$(5.16)\quad \mathcal{B}_k=\{y(x)\,|\,y(x)\in C^{2+\alpha}(D),\ \|y\|_\infty\leq K,\ \langle y(x),\psi_0^*(x)\,g_{\lambda u}(\mu_0,w_0)\rangle=0\}\ ,$$

and the real interval

$$(5.17)\qquad \mathcal{I}_K=\{\eta\,\big|\,|\eta|\leq K\}\ .$$

In addition we introduce the set

$$S_2(\rho,\Gamma)=\Big\{(\mu,w;x)\,\Big|\,\mu=\mu_0+\delta^2\nu,\ w=w_0+\delta\psi_0+\delta^2 y,\quad x\in\overline{D}\ ,$$

$$0\leq|\delta|\leq\rho,\ \nu\in\mathcal{I}_\Gamma,\ w\in\mathcal{B}_\Gamma\Big\}\ .$$

For each y(x) in $\mathcal{B}_K$ and $\nu\in\mathcal{I}_K$ we define the mapping T_δ for each δ in $0\leq|\delta|\leq\delta_1$ by

$$T_\delta(y,\nu)=\left(\tilde{y},\tilde{\nu}\right)\ ,$$

where

$$(5.18)\quad \tilde{\nu}\Big\langle\int_0^1 g_\lambda(\mu_0+\delta^2 s\nu,w)\,ds,\psi_0^*\Big\rangle=-\Big\langle(\psi_0+\delta y)^2\int_0^1\!\!\int_0^1 g_{uu}\Big(\mu_0,w_0+\delta st(\psi_0+\delta y)\Big)s\,dt\,ds,\psi_0^*\Big\rangle\ ,$$

and

$$(5.19)\quad \begin{cases} L\tilde{y} + g_u(\mu_0, w_0)\tilde{y} = -\Big[\tilde{\nu}\int_0^1 g_\lambda(\mu_0+\delta^2 s\nu, w)ds \\ \qquad + (\psi_0+\delta y)^2 \int_0^1\int_0^1 g_{uu}\big(\mu_0, w_0+\delta st(\psi_0+\delta y)\big)s\,dt\,ds\Big],\ x \in D\ , \\ B\tilde{y} = 0\ ,\quad x \in \partial D\ , \\ \langle \tilde{y}\ ,\ \psi_0^* g_{\lambda u}(\mu_0, w_0)\rangle = 0\ . \end{cases}$$

Then, after picking some initial iterate (y^0, ν^0), a sequence of iterates $\{y^k, \nu^k\}$ will be generated by

$$(5.20)\qquad \left(y^{k+1}, \nu^{k+1}\right) = T_\delta\left(y^k, \nu^k\right)\ .$$

We now state and prove the following

Theorem 5-1: <u>Let $S_2 = S_2(\rho, \Gamma)$ for some fixed $\rho \leq 1$, $\rho\,\Gamma \leq 1$. Suppose that</u>

<u>(5.21) $g(\lambda, u), g_u(\lambda, u) \in C^\alpha(S_2)$, g_λ, g_{uu}, $g_{\lambda u}$, $g_{\lambda\lambda}$, $g_{uuu} \in C(S_2)$</u>

<u>and that $\qquad \langle g_\lambda(\mu_0, w_0), \psi_0^*\rangle \neq 0$.</u>

<u>Then $\exists$ real positive constants δ_0 and M, $\delta_0 \leq \rho$, $\delta_0 M \leq \rho\,\Gamma$ such that the mapping T_δ given by (5.18), (5.19) maps $W_M = (\mathcal{B}_M \times \mathcal{I}_M)$ into W_M, and T_δ is a contraction on W_M for all δ, $0 \leq |\delta| \leq \delta_0$.</u>

Furthermore, the problem (5.2) has a solution of the form (5.13) for all δ $0 \leq |\delta| \leq \delta_0$, where $y(x, \delta)$ and $\nu(\delta)$ are the limits of the iterate $\{y^k, \nu^k\}$ generated by (5.20) for any initial iterate in W_M.

Proof: The proof is similar to the proof of Theorem 3-1. We need to show that the mapping T_δ given by (5.18), (5.19), is a contraction mapping of W_M into W_M for appropriate constants δ_0 and M. Again we will use the norm

$$(5.22) \qquad \|g\|_s = \sup_{w \in S} |g(w)| .$$

Because of the smoothness assumptions we have placed on $g(\lambda, u)$, to show that T_δ maps W_M into W_M we need only find the constants M and δ_0 which define $\mathcal{B}_M$, $\mathcal{G}_M$.

Since we assumed that $\left| \langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle \right| = \gamma \neq 0$, we can restrict δ_0 to be sufficiently small so that

$$(5.23) \qquad \left| \langle \int_0^1 g_\lambda(\mu_0 + \delta \epsilon \nu, w) ds, \psi_0^* \rangle \right| \geq \gamma/2 .$$

Suppose that $(y, \nu) \in W_M$ for some M, δ_0. Then

$$(5.24) \qquad |\tilde{\nu}| \leq \frac{\Phi}{\gamma} (\|\psi\|_\infty + \delta M)^2 \|g_{uu}\|_s \leq A_1 + \delta B_1(M, \delta) ,$$

and

$$(5.25) \qquad \|\tilde{y}\|_\infty \leq G\{|\tilde{\nu}| \, \|g_\lambda\|_s + (\|\psi\|_\infty + \delta M)^2 \frac{\|g_{uu}\|_s}{2}\} \leq A_2 + \delta B_2(M, \delta) ,$$

where

$$\Phi = \langle 1, |\psi^*| \rangle \qquad .$$

The positive numbers A_1 and A_2 do not depend on M or δ, and the numbers $B_1(M, \delta)$ and $B_2(M, \delta)$ are bounded on compact sets of (M, δ). We can easily find (M, δ_1) so that

$$(5.26) \qquad A_i + \delta_1 B_i(M, \delta_1) \leq M \qquad i = 1, 2,$$

by picking $M > \max\{A_1, A_2\}$, and then finding the largest δ_1 for which (5.26) holds. By picking $\delta_2 = \min\{1, \frac{1}{M}, \delta_1\}$, we have that $T_\delta : W_M \to W_M$ for $0 \leq |\delta| \leq \delta_2$.

To show that T_δ is a contraction for $|\delta| \leq \delta_0$, assume that $w_1 = (y, \nu)$ and $w_2 = (z, \mu)$ are in W_M. Then for $|\delta| \leq \delta_2$ we have

$$(5.27) \qquad |\tilde{\nu} - \tilde{\mu}| \leq \frac{2\Phi}{\gamma} |\delta| \left[A_{11} \|y - z\|_\infty + A_{12} |\nu - \mu| \right] ,$$

and

$$(5.28) \qquad \|\tilde{y} - \tilde{z}\|_\infty \leq G|\delta| \left[A_{21} \|y - z\|_\infty + A_{22} |\nu - \mu| \right] + G \|g_\lambda\|_s |\tilde{\nu} - \tilde{\mu}| \quad ,$$

where

$$(5.29) \qquad \begin{cases} A_{11} = A_{21} = (\|\psi\|_\infty + 1) \|g_{uu}\|_s + \delta_2 (\|\psi\|_\infty + 1)^2 \dfrac{\|g_{uuu}\|_s}{6} + \|g_{\lambda u}\|_s , \\[2ex] A_{12} = A_{22} = \dfrac{M}{2} \|g_{\lambda\lambda}\|_s . \end{cases}$$

Clearly, (5. 27) - (5. 29) implies the existence of a constant $C > 0$ such that

$$(5.30) \qquad \|\tilde{w}_1 - \tilde{w}_2\| \leq (|\delta| C) \|w_1 - w_2\| ,$$

where $\qquad \|w\| = \max\{\|y\|_\infty, |\nu|\}, \quad w \in W_M.$

Now, by choosing $0 < \delta_3 C < 1$, the mapping T_δ is a contraction on W_M for $0 \leq |\delta| \leq \delta_0$, where $\delta_0 = \min(\delta_2, \delta_3)$.

To complete the proof we need only observe that the compactness Theorem of Agmon, Douglis and Nirenberg [1] applies, as it did in the proof of Theorem 3-1, and justifies taking the limit as $k \to \infty$ in (5. 20). Q.E.D.

The solution given by (5.13) is unique in the sense that there is only one solution of that form in $S_2(\epsilon_0, M)$. If there were two solutions $w_1 \neq w_2$, each would be fixed points of T_δ, and (5.30) implies that

$$(5.31) \qquad \|w_1 - w_2\| \leq (|\delta| C) \|w_1 - w_2\| .$$

For $|\delta| \leq \delta_0$, this cannot hold, so that $w_1 = w_2$ is unique.

We could compare the iteration procedure (5. 20) with the perturbation scheme (5.12). Once again we would find that the perturbation scheme is asymptotic to the iteration scheme, and that the iteration scheme is asymptotic to the solution as $\delta \to 0$. Rather than carrying out the details of such a proof, we will examine the asymptotic expansion of the solution $(y(\delta, x), \nu(\delta))$, which are fixed points of (5.18), (5.19).

Examining (5.18), it is easy to see that

$$\nu(\delta) = \frac{-\langle g_{uu}(\mu_0, w_0)\psi_0^2, \overset{*}{\psi_0}\rangle}{2\langle g_\lambda(\mu_0, w_0), \overset{*}{\psi_0}\rangle} + O(\delta)\ . \tag{5.32}$$

Substituting (5.32) into (5.13) we see that to the order which we have taken the solution, the exact solution (5.13) and the perturbation solution (5.12) agree asymptotically as $\delta \to 0$.

Knowing the form of the solution (5.13) gives us information about those parameter values μ for which solutions of (5.2) exist. Since $\mu = \mu_0 + \delta^2 \nu(\delta)$, if $\nu(0) > 0$, then solutions of (5.2) exist in the neighborhood of (w_0, μ_0) for which $\mu > \mu_0$. If $\nu(0) < 0$, then solutions of (5.2) exist in the neighborhood of (w_0, μ_0) for which $\mu < \mu_0$. In either case, the point $\mu = \mu_0$ is a branching point where the number of solutions of (1.1) changes from zero to two or from two to zero as μ changes from $\mu < \mu_0$ to $\mu > \mu_0$, in the respective cases $\nu(0) > 0$ and $\nu(0) < 0$. Figure 1 gives plots of $\mu(\delta)$ versus δ when $\langle g_\lambda(\mu_0, w_0), \overset{*}{\psi_0}\rangle > 0$.

We have now shown circumstances under which a nonisolated solution of (1.1) is an element of a solution branch of (1.1) for τ fixed. Since in Section 3 we were able to show that nontrivial nonisolated solutions of (1.1) do exist, it is natural to ask how Theorem 5-1 applies to the results of Section 3.

If ϵ_0 found in Section 3 is sufficiently small, then the resulting eigenvalue $\lambda = \lambda_0 + \epsilon\,\mu(\epsilon)$ in (3.1) remains isolated, so that the null space of (1.7) remains one-dimension for $|\epsilon| \leq \epsilon_0$. The hypotheses of Theorem 3-1 are sufficient to insure that the hypotheses of Theorem 5-1 hold in $S_1 = S_1(\rho, \Gamma)$ for certain nonzero ρ, Γ, for some fixed ϵ, $|\epsilon| \leq \epsilon_0$. Applying Theorem 5-1, we substitute into (5.13) for $w_0(x)$, $\psi_0(x)$ and μ_0, the nonisolated solutions of (1.1) found in Theorem 3-1 and given in the form (3.1). The resulting solutions of (1.1) are

$$(5.33)\quad \begin{cases} u = (\epsilon+\delta)\phi_0(x) + \epsilon^2 v(x,\epsilon) + \epsilon\,\delta\,\chi(x,\epsilon) + \delta^2 y(x,\epsilon,\delta), \\ \lambda = \lambda_0 + \epsilon\,\mu(\epsilon) + \delta^2 \nu(\epsilon,\delta), \\ \tau = \epsilon^2 \eta(\epsilon), \end{cases}$$

where ϵ is fixed, $|\epsilon| \leq \epsilon_0$.

The solution of (1.1) given by (5.33) is valid only if $|\delta| < \delta_0$. However, the number δ_0 is not independent of the number ϵ. Notice in the proof of Theorem 5-1 that δ_0 was chosen (cf. (5.30)) so that

$$(5.34)\qquad \delta_0 < \frac{1}{C} = \frac{\gamma}{\tilde{C}}$$

where

$$\gamma = \left| \langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle \right| .$$

Since (μ_0, w_0) are related to ϵ by

$$(5.35)\qquad \begin{cases} w_0(x) = u(x,\epsilon)\ , \\ \mu_0 = \lambda(\epsilon)\ , \end{cases}$$

where $\big(u(x,\epsilon)\ ,\ \lambda(\epsilon)\big)$ are of the form (3.1), we can find the dependence of γ on ϵ. Specifically

$$(5.36)\qquad \begin{aligned} \langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle &= \langle f_\lambda(\lambda, \tau, u), \psi_0^* \rangle \\ &= \epsilon\ \langle f_{\lambda u}(\lambda_0, 0, 0)\phi_0, \phi_0^* \rangle + O(\epsilon^2). \end{aligned}$$

Note that in (5.36) we have used that $\psi_0^* = \phi_0^* + O(\epsilon)$. This can be shown to be true in the same way that it was shown in Section 3 that $\psi_0 = \phi_0 + O(\epsilon)$, using that $(\phi_0^*)^* = \phi_0$.

Since the constant $\tilde{C}$ in (5.34) is bounded away from zero when $|\epsilon| \leq \epsilon_0$, (5.34) coupled with (5.36) imply that

$$(5.37)\qquad \delta_0 = O(\epsilon).$$

Clearly, as ϵ approaches zero, the range of validity of (5.33) decreases. This decrease in the range of validity is not unexpected. As seen in Figures 2, 3 and 4, for $\tau = 0$, the bifurcation solution has a sharp "corner" at $\lambda = \lambda_0$. As $\epsilon \to 0$, the solution branch (5.33) with $\tau \neq 0$ approaches this "corner." But since (5.33) is a smooth function of δ, it cannot have a "corner" when $\epsilon = 0$, so that $\delta_0(\epsilon)$

must approach zero as $\epsilon \to 0$.

We would like to be able to further understand the nature of the solution given by (5.33). Suppose we examine the expression for $\lambda = \lambda(\epsilon, \delta)$. Recall that $\lambda = \lambda_0 + \epsilon\,\mu(\epsilon) + \delta^2 \nu(\epsilon,\delta)$ where $\nu(\epsilon,\delta)$ is given by

$$(5.32) \qquad \nu(\epsilon,\delta) = -\tfrac{1}{2}\,\frac{\langle g_{uu}(\mu_0, w_0)\psi_0^2, \psi_0^* \rangle}{\langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle} + O(\delta).$$

Since we know the form of μ_0, w_0 and ψ_0 as functions of ϵ, we can rewrite (5.32) as

$$(5.38) \qquad \nu(\epsilon,\delta) = -\frac{\epsilon^{p-2}}{2(p-1)!}\,\frac{\left\langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^* \right\rangle}{\langle f_{\lambda u}(\lambda_0,0,0)\phi_0, \phi_0^* \rangle} + O(\epsilon^{p-1}) + O(\delta)$$

where the integer p is defined in (3.30). We now see that the non-isolated solution (3.1) corresponds to a branching point for ϵ sufficiently small, since according to (3.30), $\left\langle \frac{\partial^{p+1} f(\lambda_0,0,0)}{\partial u^{p+1}} \phi_0^{p+1}, \phi_0^* \right\rangle \neq 0$ and hence $\nu(\epsilon, 0) \neq 0$.

II.6. Stability of Extended Solution Branch.

In the previous four sections, we have studied various aspects of steady state solutions of the more general time dependent problem for $y(x,t)$

$$\frac{\partial y}{\partial t} = Ly + f(\lambda, \tau, y) \qquad (x,t) \in D \times [0,\infty) \ ,$$

(6.1) $$By = 0 \qquad (x,t) \in \partial D \times [0,\infty) \ ,$$

$$y(x,0) = h(x) \qquad x \in D \ .$$

We can examine the stability of steady state solutions of (6.1) by looking at the behavior of (6.1) in a neighborhood of the steady state solution with λ and τ fixed. The resulting theory is the so-called linear stability theory.

Suppose that $u(x)$ is a steady state solution of (6.1) with λ and τ fixed. If we assume that solutions of (6.1) have the form

(6.2) $$y(x,t) = u(x) + \alpha \zeta(x) e^{-\gamma t},$$

where α is assumed to be small, then we can substitute (6.2) into (6.1) and linearize the resulting equation by keeping only the terms which are lowest order in α. The equation which results is

(6.3) $$L\zeta + \left[\gamma + f_u(\lambda, \tau, u)\right]\zeta = 0 \qquad x \in D \ ,$$

$$B\zeta = 0 \qquad x \in \partial D \ .$$

At this point it is helpful if we state our definition of stability.

Definition 6-1: A steady state solution $u(x)$ of (6.1) is said to be linearly stable if $\|y(x,t) - u(x)\| \to 0$ as $t \to \infty$ for $y(x,t)$ given by (6.2).

u(x) is said to be linearly unstable if $\|y(x,t)-u(x)\| \to \infty$ as $t \to \infty$, and u(x) is said to have neutral stability if u(x) is not linearly stable, but $\|y(x,t)-u(x)\|$ is bounded for all time.

Because of (6.3) we can make the equivalent

Definition 6-2: Let γ_1 be the principal (smallest) eigenvalue of (6.3). A steady state solution u(x) of (6.1) is said to be linearly stable, neutrally stable or linearly unstable if $\gamma_1 > 0$, $\gamma_1 = 0$ or $\gamma_1 < 0$ respectively.

Throughout this section stability or instability will actually mean linear stability or linear instability. We will not examine the more difficult question of global stability. We will also assume that the operators L and B are self adjoint. Then it is possible to classify instabilities in the following manner:

Definition 6-3: Let γ_k be the k^{th} eigenvalue of (6.3) counting multiplicities, $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k$. If $\gamma_k < 0$ and $\gamma_{k+1} \geq 0$ then the steady state solution is said to be k-mode unstable or is said to have a k-mode instability. Furthermore, if $\gamma_{k+1} = 0$, then u(x) is also said to be neutrally stable in the $k{+}1^{st}$ mode. A solution which is 0-mode unstable is linearly stable.

Immediately we realize that if u(x) is a nonisolated solution of (1.1), then $\gamma_p = 0$ for some $p > 0$, and the nonisolated solution is neutrally stable in the p^{th} mode. In either case, when we have some type of neutral stability, we would like to know how this stability characteristic changes as we move along the solution branch which contains the nonisolated solution.

In the light of previous sections, the most natural approach is to use, if possible, a perturbation technique, which can then be justified using a contraction mapping. By now it is clear how to find the correct contraction mapping, and how to give the corresponding existence proof once the perturbation technique has been applied. Thus, in this section we will only examine the results of the perturbation technique, and will not give the details of its justification.

As we did in Section 5, suppose for $\tau \neq 0$ fixed arbitrarily we make the identification

$$g(\lambda, u) = f(\lambda, \tau, u) . \tag{6.4}$$

In Section 5 we found steady state solutions of

$$\begin{aligned} Lw + g(\mu, w) &= 0 \qquad x \in D , \\ Bw &= 0 \qquad x \in \partial D , \end{aligned} \tag{5.2}$$

to be of the form

$$\begin{cases} w(x, \delta) = w_0(x) + \delta\psi_0(x) + \delta^2 y(x, \delta) , \\ \mu(\delta) = \mu_0 + \delta^2 \nu(\delta) , \end{cases} \tag{5.5}$$

where $(w_0(x), \mu_0)$ is a nonisolated solution of (5.2). Furthermore, it was noted that

$$(5.32) \qquad \nu(\delta) = -\frac{1}{2} \frac{\langle g_{uu}(\mu_0, w_0)\psi_0^2, \psi_0^* \rangle}{\langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle} + O(\delta) ,$$

where $\psi_0(x)$, $\psi_0^*(x)$ satisfy (5.3) and (5.4) respectively. Since we know that $\delta = 0$ implies $\gamma = 0$, we try a solution of (6.3) of the form

$$(6.5) \qquad \begin{cases} \zeta(x,\delta) = \psi_0(x) + \delta\zeta_1(x) + \delta^2\zeta_2(x) + O(\delta^2) , \\ \gamma(\delta) = \delta\gamma_0 + \delta^2\gamma_1 + O(\delta^3) . \end{cases}$$

To show that this assumed form is valid, one must employ the contraction mapping technique outlined before. Upon substituting (6.5) into (6.3) we find that the perturbation equations are

$$(6.6) \qquad \begin{aligned} L\psi_0 + g_u(\mu_0, w_0)\psi_0 &= 0 , \qquad x \in D , \\ B\psi_0 &= 0 , \qquad x \in \partial D , \end{aligned}$$

and

$$(6.7) \qquad \begin{aligned} L\zeta_1 + g_u(\mu_0, w_0)\zeta_1 &= -\left[\gamma_0 + g_{uu}(\mu_0, w_0)\psi_0\right]\psi_0(x), \quad x \in D, \\ B\zeta_1 &= 0 , \qquad x \in \partial D . \end{aligned}$$

In order that $\zeta_i(x)$ be uniquely determined we add the condition

$$(6.8) \qquad \langle \zeta_i(x) , g_{\lambda u}(\mu_0, w_0)\psi_0^*(x) \rangle = 0 .$$

The function $\psi_0(x)$ was chosen so that (6.6) is automatically satisfied. To solve (6.7), the Fredholm alternative theorem must hold, namely

(6.9) $$\gamma_0 \langle \psi_0(x), \psi_0^*(x) \rangle + \langle g_{uu}(\mu_0, w_0)\, \psi_0^2, \psi_0^* \rangle = 0 .$$

If $\langle \psi_0, \psi_0^* \rangle \neq 0$, (6.9) implies that

(6.10) $$\gamma_0 = - \frac{\langle g_{uu}(\mu_0, w_0) \psi_0^2, \psi_0^* \rangle}{\langle \psi_0, \psi_0^* \rangle} .$$

Substituting (6.10) into (6.5) we can see the stability characteristics of the steady state solution $w(x, \delta)$ immediately. Suppose, for simplicity, that $\langle \psi_0, \psi_0^* \rangle > 0$, that $\langle g_\lambda(\mu_0, w_0), \psi_0^* \rangle > 0$, and that $\gamma = 0$ is the k^{th} eigenvalue of (6.3). If $\langle g_{uu}(\mu_0, w_0) \psi_0^2, \psi_0^* \rangle > 0$, solutions of (5.2) occur for $\mu < \mu_0$ since $\mu(\delta) = \mu_0 + \delta^2 \nu(\delta)$, and $\nu(\delta) < 0$ for δ sufficiently small. By (6.5), (6.10), $\gamma(\delta) = \delta \gamma_0 + O(\delta^2)$ and $\gamma_0 < 0$, so that $\gamma(\delta) < 0$ for $\delta > 0$ sufficiently small, while $\gamma(\delta) > 0$ for $\delta < 0$ sufficiently small. Since $\gamma(\delta)$, the k^{th} eigenvalue of (6.3), is negative when $\delta > 0$, (6.3) has k negative eigenvalues and the corresponding solutions are, by Definition 6-3, k-mode unstable. With $\delta < 0$, the k^{th} eigenvalue $\gamma(\delta)$ is positive, so that the corresponding solutions of (5.2) are k-1-mode unstable. On the other hand, if $\langle g_{uu}(\mu_0, w_0) \psi_0^2, \psi_0^* \rangle < 0$, solutions of (5.2) occur for $\mu > \mu_0$ since $\nu(\delta) > 0$ when δ is sufficiently small. The k^{th} eigenvalue is $\gamma(\delta) = \delta\gamma_0 + O(\delta^2)$ where $\gamma_0 > 0$. Therefore, for $\delta < 0$ sufficiently small, $\gamma(\delta)$ is negative and the corresponding solution is k-mode unstable. The solution is k-1 mode unstable when $\delta > 0$, since $\gamma(\delta)$ becomes positive. Of course, when $\gamma = 0$ is the principal eigenvalue of (6.2), a k-1 mode unstable solution is really a stable solution, by Definition 6-3. These results are summarized for k=1 in Figure 1.

Finally, we should remark that this stability characterization is not limited to $|\delta| < \delta_0$. In fact, since the solution branch (5.5) can be extended in either direction to the next nonisolated solution [28], the stability characterization of the solution branch is the same for all steady state solutions lying on any interval of a branch with no nonisolated solutions. This is clear, since the stability characterization can change only at a nonisolated solution where $\gamma_k = 0$ for some k.

II.7. Minimal Positive Solutions

Many problems of physical interest involve finding solutions of (1.1) which are positive. See, for example [18], [19], [24], [27] and [36]. In this section we would like to show that, under certain circumstances, the solution branch found in Section 5 is a branch of positive solutions, and that certain of these solutions are minimal positive solutions.

Discussion of minimal positive solutions have been given by Keller and Cohen [19], Amann [2] and Sattinger [35]. For our discussion we consider the problem

$$(7.1) \qquad Lu + g(\lambda, u) = 0 \qquad x \in D \,, \qquad Bu = 0 \qquad x \in \partial D \,,$$

where $g(\lambda, u)$ is always assumed to have some continuity properties as in Section 5. As necessary, we will also assume that

$$(7.2) \qquad g(\lambda, 0) > 0 \quad \text{provided} \quad \lambda > 0 \,,$$

$$(7.3) \qquad g(\lambda, u) < g(\lambda', u) \quad \text{for} \quad \lambda < \lambda' \,, \; u \geqslant 0 \,,$$

and

$$(7.4) \qquad g(0, u) = 0 \quad \text{for} \quad u \geqslant 0 \,.$$

No assumptions regarding monotonicity or concavity-convexity in u of $g(\lambda, u)$ will be made at this time. The operators L and B of (7.1) are those given in Section 1, and have the associated strong maximum principle [31], which can be used to give

Proposition (1'): If $\phi(x) \in C^1(\overline{D}) \cap C^2(D)$, then for any $\Omega \geqslant 0$, $\Omega \in C^{\alpha}(\overline{D})$,

$$(7.5) \qquad \begin{cases} L\phi - \Omega\phi \geqslant 0 \text{ on } D \,, \quad B\phi \leqslant 0 \text{ on } \partial D \Longrightarrow \phi(x) \leqslant 0 \text{ on } \overline{D} \,, \\ L\phi - \Omega\phi > 0 \text{ on } D \,, \quad B\phi \leqslant 0 \text{ on } \partial D \Longrightarrow \phi(x) < 0 \text{ on } D \,. \end{cases}$$

Furthermore, if $\phi(x) = 0$ for some $x \in \partial D$, then

$$(7.6) \qquad \frac{\partial \phi(x)}{\partial \alpha} < 0 \qquad x \in \partial D,$$

where α is any outward direction at $x \in \partial D$. A minimal positive solution $\underline{u}(x)$ of (7.1) is a solution of (7.1) satisfying $\underline{u}(x) \leq u(x)$ for all positive $u(x)$ satisfying (7.1).

We now develop the facts which we will use later. This part of the discussion gives a generalization of the results of Keller and Cohen [19] using assumptions (7.2) - (7.4).

Theorem 7-1: Under assumptions (7.3) and (7.4) equation (7.1) can have positive solutions only for positive λ.

Proof. Suppose $u(x) > 0$, $x \in D$ is a solution of (7.1) and $\lambda < 0$. By (7.3) and (7.4), $g(\lambda, u) < 0$ for $x \in D$. Hence

$$Lu = -g(\lambda, u) \geq 0 \ , \quad x \in D \ ,$$
$$Bu = 0 \qquad x \in \partial D \ ,$$

and Proposition 1 implies that $u \leq 0$ for $x \in D$ which contradicts the assumption that $u(x) > 0$ for $x \in D$. If $\lambda = 0$ then $u \equiv 0$, and the proof is complete. Q.E.D.

The existence of minimal positive solutions of (7.1) was established in [19] by making use of a monotone sequence generated by an iteration scheme. The function $g(\lambda, u)$ was required to be monotone increasing in u in order to insure that the sequence was monotone. The iteration scheme used here, which does not require a monotonic nonlinearity $g(\lambda, u)$, was used in [6] for nonlinear equations involving the Laplacian. It has since been used by Keller

[16], Amann [2] and Sattinger [35] for more general operators L. The iteration scheme which we use defines the sequence $\{u_n(x)\}$ by

$$
(7.7)\qquad \begin{cases} u_0(x) \equiv 0 \\ Lu_{n+1} - \Omega u_{n+1} = -\left[g(\lambda, u_n) + \Omega u_n \right] & x \in D \\ Bu_{n+1} = 0 \quad x \in \partial D, \quad n = 0, 1, 2, \cdots & , \end{cases}
$$

for any $\lambda > 0$ and any $\Omega(x) \geq 0$, $\Omega(x) \in C^{\alpha}(\overline{D})$. Using this iteration scheme we have

Theorem 7-2: Let $g(\lambda, u)$ satisfy (7.2).

a) If, for $\lambda > 0$ and $\Omega(x) \geq 0$ fixed, the sequence $\{u_n(x)\}$ is a monotone sequence, and if it is uniformly bounded by some constant $M > 0$, then the sequence $\{u_n(x)\}$ converges to a solution of (7.1).

b) If a positive solution $u(\lambda, x) \geq 0$ of (7.1) exists for a given $\lambda > 0$, then $\exists\ \Omega(x) \geq 0$ such that the sequence (7.7) converges monotonely and uniformly to its limit, say,

$$
\underline{u}(\lambda, x) = \lim_{n \to \infty} u_n(\lambda, x)
$$

where $\underline{u}(\lambda, x)$ is the minimal positive solution of (7.1).

Proof: The proof of part a) was given by Keller [16] using the compactness result of Agmon, Douglis and Nirenberg [1] to justify passing to the limit $n \to \infty$ in (7.7).

To see part b), since $g_u(\lambda, u) \in C^{\alpha}(D)$, we can choose a constant $\Omega > 0$ so that

$$(7.8) \qquad \frac{g(\lambda, z)-g(\lambda, y)}{z-y} \geq -\Omega, \quad x \in D, \text{ for all } y, z, 0 \leq y, \; z \leq \|u(\lambda,x)\|_{\infty}$$

since $\|u(\lambda, x)\|_{\infty} < \infty$. With this choice of Ω, the sequence $u_n(x)$ is monotone. In fact,

$$\begin{aligned} Lu_1 - \Omega u_1 &= -g(\lambda, 0) < 0, \quad x \in D, \\ Bu_1 &= 0, \qquad\qquad x \in \partial D, \end{aligned}$$

so that $u_1(x) \geq 0 = u_0(x)$. Since $u_0(x) = 0 \leq u(\lambda, x)$, suppose that $u_k(x) \leq u(\lambda, x)$ for $k = 0, 1, \cdots, n$, and that $u_n(x) \geq u_{n-1}(x)$. Then

$$\begin{aligned} L(u_{n+1}-u_n)-\Omega(u_{n+1}-u_n) &= -\Big[\big(g(\lambda, u_n)+\Omega u_n\big) - \big(g(\lambda,u_{n-1})+\Omega u_{n-1}\big)\Big] \leq 0 \\ & \qquad\qquad x \in D \\ B(u_{n+1}-u_n) &= 0 \quad x \in \partial D, \end{aligned}$$

by virtue of (7.8). Proposition(1') implies that $u_{n+1}(x) \geq u_n(x)$. Furthermore

$$\begin{aligned} L\big(u_{n+1}-u(\lambda, x)\big) -\Omega\big(u_{n+1}-u(\lambda,x)\big) &= -\Big[\big(g(\lambda,u_n)+\Omega u_n\big) - \big(g(\lambda,u)+\Omega u\big)\Big] \geq 0 \quad x \in D, \\ B(u_{n+1}-u) &= 0 \quad x \in \partial D. \end{aligned}$$

again because of (7.8). This implies that $u_{n+1}(x) \leq u(\lambda,x)$ which completes the inductive proof that the sequence $\{ u_n(x) \}$ is monotone and uniformly bounded, and part a) applies. $\underline{u}(\lambda, x) = \lim_{n\to\infty}\{u_n(x)\}$ must be the minimal positive solution, since by the last part of the induction $u_{n+1}(x) \leqslant u(\lambda, x)$, where $u(\lambda, x)$ is any solution of (7.1). Passing to the limit gives

$$\underline{u}(\lambda, x) \leqslant u(\lambda, x) \quad , \quad x \in \overline{D} \quad ,$$

which completes the proof. Q.E.D.

Theorem 7-2 is not the same as Theorem 3.2 of Keller and Cohen [19], since we have not given necessary conditions for the existence of $\underline{u}(\lambda, x)$. Such necessary conditions are explored in [16] and [2], and are not included in this discussion.

The basic comparison result which we use is

<u>Theorem 7-3</u>: <u>Let $g(\lambda, u)$ satisfy (7.2). Suppose $G(\lambda, \phi)$ is a given function which satisfies</u>

$$(7.9) \qquad G(\lambda, \phi) \geqslant g(\lambda, \phi) \quad \text{for} \quad \lambda \geqslant 0 \ , \ \phi \geqslant 0 \ .$$

<u>Suppose there is a function $y_0(x) \geqslant 0$ and $\lambda_0 > 0$ such that</u>

$$(7.10) \qquad \begin{aligned} Ly_0 + G(\lambda_0, y_0) &= 0 \qquad x \in D \ , \\ By_0 &= 0 \qquad x \in \partial D \ . \end{aligned}$$

<u>Then (7.1) has a minimal positive solution for $\lambda = \lambda_0$, and</u>

$$\underline{u}(\lambda_0, x) \leqslant y_0(x) \qquad x \in \overline{D}.$$

Proof: In order to apply the results of Theorem 7-2, choose $\Omega \geqslant 0$ so that (7.8) holds for all y, z, $0 \leqslant y, z \leqslant \|y_0(x)\|_\infty$ and for $\lambda = \lambda_0$. We need to show that $u_n(x) \leqslant y_0(x)$ for all n. Clearly,

$$L\Big(u_1(x)-y_0(x)\Big) - \Omega\Big(u_1(x)-y_0(x)\Big) = -g(\lambda,0) + G(\lambda, y_0) + \Omega y_0$$

$$= -\Big(g(\lambda, y_0)-G(\lambda, y_0)\Big)$$

$$+ \Big[\Big(g(\lambda, y_0)+\Omega y_0\Big) - g(\lambda,0)\Big] \geqslant 0 \quad x \in D,$$

$$B(u_1 - y_0) = 0 \qquad x \in \partial D,$$

so that $u_1(x) \leqslant y_0(x)$. If $u_n(x) \leqslant y_0(x)$, then

$$L(u_{n+1}-y_0) - \Omega(u_{n+1}-y_0) = -\Big(g(\lambda, u_n) + \Omega u_n\Big) + G(\lambda, y_0) + \Omega y_0$$

$$= G(\lambda, y_0) - g(\lambda, y_0)$$

$$+ \Big(g(\lambda, y_0) + \Omega y_0\Big) - \Big(g(\lambda, u_n) + \Omega u_n\Big) \geqslant 0 \quad x \in D,$$

$$B(u_{n+1}-y_0) = 0, \qquad x \in \partial D,$$

which implies that $u_{n+1} \leqslant y_0$, $x \in \overline{D}$. Since $u_n(x) \leqslant y_0(x)$ for all n, then (7.8) holds for each element of the sequence $\{u_n(x)\}$ which in turn, implies that the sequence $\{u_n(x)\}$ is monotone increasing. Theorem 7-2 is applicable, so that passing to the limit as $n \to \infty$ implies

$$u(\lambda, x) \leqslant y_0(x). \qquad \text{Q.E.D.}$$

Using this theorem, we establish the very useful

Corollary 7-4. Let $g(\lambda, u)$ satisfy (7.2) and (7.3) and suppose a solution $u(\lambda_0, x)$ exists for $\lambda = \lambda_0$. Then for each λ in $0 < \lambda \leqslant \lambda_0$ the minimal positive solution $\underline{u}(\lambda, x)$ of (7.1) exists and is a pointwise increasing function of λ.

Proof: For any fixed value of λ in $0 < \lambda \leqslant \lambda_0$ define

$$G(\lambda, \phi) = g(\lambda_0, \phi) = g\left(\lambda\left(\frac{\lambda_0}{\lambda}\right), \phi\right)$$

By (7.3) $G(\lambda, \phi) = g(\lambda_0, \phi) \geqslant g(\lambda, \phi)$. The hypotheses of Theorem 7-3 are satisfied with the choice $y_0(x) = \underline{u}(\lambda_0, x)$ which exists by Theorem 7-2. We conclude that

$$\underline{u}(\lambda, x) \leqslant y_0(x) = \underline{u}(\lambda_0, x)$$

To see that the inequality is strict in the interior of D, notice that, for $\Omega \geqslant 0$ chosen as in Theorem 7-3,

$$\begin{aligned} &L\left(\underline{u}(\lambda_0, x)-\underline{u}(\lambda, x)\right) - \Omega\left(\underline{u}(\lambda_0, x)-\underline{u}(\lambda, x)\right) \\ &\quad = -\left(g\left(\lambda_0, \underline{u}(\lambda_0, x)\right) - g\left(\lambda, \underline{u}(\lambda, x)\right)\right) - \Omega\left(\underline{u}(\lambda_0, x)-\underline{u}(\lambda, x)\right) \\ &\quad = g\left(\lambda_0, \underline{u}(\lambda, x)\right) + \Omega\,\underline{u}(\lambda, x) - \left(g\left(\lambda_0, \underline{u}(\lambda_0, x)\right) + \Omega\,\underline{u}(\lambda_0, x)\right) \\ &\quad\quad + g\left(\lambda, \underline{u}(\lambda, x)\right) - g\left(\lambda_0, \underline{u}(\lambda, x)\right) \leqslant g\left(\lambda, \underline{u}(\lambda, x)\right) - g\left(\lambda_0, \underline{u}(\lambda, x)\right) < 0 \qquad x \in D \end{aligned}$$

whenever $\lambda < \lambda_0$, according to (7.3). By (7.5),

$$\underline{u}(\lambda, x) < \underline{u}(\lambda_0, x) \quad \text{for} \quad x \in D. \qquad \text{Q.E.D.}$$

With these basic facts established we now examine problem (7.1) when more is known about the nonlinearity $g(\lambda, u)$. We will use, when necessary, the additional restrictions

(7.11) $$g_u(0, u) = 0 \quad \text{for } u(x) \geqslant 0$$

(7.12) $$g_u(\lambda, u) > g_u(\lambda', u) \quad \text{for } \lambda > \lambda', \ u(x) \geqslant 0$$

(7.13) $$g_{uu}(\lambda, u) > 0 \quad \text{for } u(x) > 0 \quad \text{(convex)}$$

(7.14) $$g_{uu}(\lambda, u) < 0 \quad \text{for } u(x) > 0 \quad \text{(concave)}$$

Notice that together, (7.11) and (7.12) imply that $g_u(\lambda, u) > 0$ whenever $u(x) \geqslant 0$ and $\lambda > 0$. Although until now we have purposely avoided assuming this condition, it makes certain matters which follow more tolerable if we allow (7.11) and (7.12) to hold. This is not a serious assumption. In fact, the foregoing results show that a smooth function $g(\lambda, u)$ can always be made to look like a monotone function on any compact set of (λ, u) by adding Ωu to the function and subtracting Ωu from the operator L, for some appropriately chosen constant $\Omega > 0$. In other words, we can assume that $g(\lambda, u)$ is monotone in u without loss of generality.

For the results that follow we need to assume that the operator L is such that, for $\rho(x) > 0$, when an eigenfunction $\phi_0(x)$ of

$$(7.15) \qquad L\phi + \mu\rho(x)\phi = 0 \qquad x \in D, \qquad B\phi = 0 \qquad x \in \partial D,$$

is positive on D, then the corresponding adjoint eigenfunction ϕ_0^* is also positive. This is obviously true when L is self adjoint, since $\phi_0 = \phi_0^*$. When L is not self adjoint, some sufficient conditions implying that $\phi_0^* > 0$ are given in

<u>Lemma 7-5</u>: <u>Let the differential operator L and boundary operator B be given by (1.2) - (1.4), and let the associated adjoint operators L^* and B^* be given by</u>

$$(7.16) \qquad L^*v = \sum_{i,j=1}^{n} a_{ij} \frac{\partial v}{\partial x_i \partial x_j} + \sum_{j=1}^{n} \hat{a}_j \frac{\partial v}{\partial x_j} - \hat{a}_0(x)v$$

<u>and</u>

$$(7.17) \qquad B^*v = \hat{b}_0(x)v + \hat{b}_1(x) \sum_{i=1}^{n} \hat{\beta}_i(x) \frac{\partial v}{\partial x_i} .$$

<u>Then, if</u>

$$(7.18) \qquad \hat{a}_0(x) \geqslant 0 \text{ on } D, \quad \hat{b}_0(x) \geqslant 0, \quad \hat{b}_1(x) \geqslant 0,$$

<u>and</u>

$$(7.19) \qquad \sum_{i=1}^{n} \hat{\beta}_i(x)\, n_i(x) > 0, \quad \max \{\hat{b}_0, \hat{b}_1\} > 0 \qquad x \in \partial D,$$

where $n_i(x)$ are the components of the unit outward normal at $x \in \partial D$, and if $\phi_0(x) > 0$ is a solution of (7.15) for $\mu = \mu_0$, then the solution $\phi_0^*(x)$ of

$$L^*\phi^* + \mu\rho(x)\phi^* = 0 \qquad (7.20)$$

$$B^*\phi^* = 0$$

for $\mu = \mu_0$ is positive on D.

Proof: When (7.18), (7.19) hold, the operator L^* with boundary operator B^* satisfies the strong maximum principle of Proposition (1). Thus the operator L^{*-1} of $C^{\alpha}(\overline{D})$ into $C^{2+\alpha}(\overline{D}) \cap \{u(x) \mid B^*u = 0\}$ is a strongly positive compact operator. By the theorem of Krein-Rutman [25], a strongly positive compact operator has a positive eigenfunction $\phi_0^*(x)$ corresponding to a simple, minimal positive eigenvalue $\mu = \mu_0^*$. By the spectral theory of compact operators [11], the eigenvalues of L and L^* are identical, and a nontrivial solution of (7.20) exists for $\mu = \mu_0$.

If $\phi_0(x) > 0$ is a solution of (7.15) for $\mu = \mu_0$, then μ_0 is a simple, minimal eigenvalue of L. But since the eigenvalues of L and L^* are identical, and both μ_0 and μ_0^* are the minimal eigenvalues, we must have $\mu_0 = \mu_0^*$. Therefore, since μ_0^* is simple, the eigenfunction $\phi_0^*(x)$ corresponding to $\phi_0(x) > 0$ is positive. Q.E.D.

Define λ^* to be the least upper bound of the values λ for which positive solutions of (7.1) exist. For each λ for which $\underline{u}(\lambda,x)$

exists, define $\mu = \mu_1(\lambda)$ to be that value of μ, if there is one, which admits positive solutions to

$$(7.21)\quad \begin{cases} L\phi + g_u\Big(\mu, \underline{u}(\lambda, x)\Big)\phi = 0, & x \in D, \\ B\phi = 0\ , & x \in \partial D. \end{cases}$$

The corresponding value of μ will be referred to as the principal eigenvalue of (7.21). Unless (7.12) holds, $\mu_1(\lambda)$ is not necessarily well defined. In the following theorems, we will show that if (7.13) holds (convex) and if there is a value $\lambda_0 = \mu_1(\lambda_0)$, then $\lambda_0 = \lambda^*$, and by definition, λ^* corresponds to a nonisolated solution of (7.1). However, if (7.14) (concave) holds, then it will be shown that the equation $\lambda = \mu_1(\lambda)$ has no solutions, so that the branch of minimal positive solutions has no nonisolated solutions.

<u>Theorem 7-6</u>: <u>Let $g(\lambda, u)$ satisfy (7.2), (7.3), (7.11)-(7.13) (convex). If the pair $(u(\lambda_0, x), \lambda_0)$ is any positive solution of (7.1), and in addition there exists a $\phi(x) > 0$ on D satisfying</u>

$$(7.22)\quad \begin{cases} L\phi + g_u\Big(\lambda_0, u(\lambda_0, x)\Big)\phi = 0, & x \in D, \\ B\phi = 0\ , & x \in \partial D, \end{cases}$$

<u>then $\lambda_0 = \lambda^*$. Furthermore the solution $u(\lambda_0, x)$ is the unique positive solution of (7.1) for $\lambda = \lambda_0$.</u>

<u>Proof:</u> Suppose $\lambda_0 \neq \lambda^*$. Then there exists a positive function

v(x) and a number $\lambda > \lambda_0$ which together satisfy (7.1). Thus

$$0 = Lu + g(\lambda_0, u) = Lv + g(\lambda, v) , \quad x \in D ,$$

so that

$$L(u-v) + g_u(\lambda_0, u)(u-v) = g(\lambda, v) - g(\lambda_0, v)$$

$$(7.23) \qquad + (v-u)^2 \int_0^1 \int_0^1 g_{uu}\Big(\lambda_0, stv + (1-st)u\Big) s\,dt\,ds, \quad x \in D,$$

$$B(u-v) = 0, \quad x \in \partial D$$

The function $\phi(x) > 0$ satisfies (7.22) so that the Fredholm alternative theorem requires

$$(7.24) \quad 0 = \langle g(\lambda, v) - g(\lambda_0, v), \phi_0^* \rangle + \langle (v-u)^2 \int_0^1 \int_0^1 g_{uu}\Big(\lambda_0, stv + (1-st)u\Big) s\,dt\,ds, \phi_0^* \rangle .$$

In addition, $\phi_0^* > 0$ so that (7.24) can be satisfied only if

$$(7.25) \qquad \langle g(\lambda, v) - g(\lambda_0, v), \phi_0^* \rangle < 0 .$$

But, according to (7.3), (7.25) can hold only if $\lambda < \lambda_0$. If $u = v$, then we must have $\lambda = \lambda_0$. In either case, $\lambda \leq \lambda_0$, which contradicts our original assumption. The uniqueness of the solution u(x) when $\lambda = \lambda_0$ is obvious from (7.24). Q.E.D.

Corollary 7-7: Under the hypotheses of Theorem 7-6, λ^* is the unique solution of $\lambda = \mu_1(\lambda)$.

Proof: Clearly $\lambda^* = \mu_1(\lambda^*)$. Suppose there exists $\lambda^{**} \neq \lambda^*$ satisfying $\lambda^{**} = \mu_1(\lambda^{**})$. If $\lambda^{**} > \lambda^*$, then λ^* is not an upper bound of the numbers λ for which positive solutions of (7.1) exist, which contradicts the definition of λ^*. If $\lambda^{**} < \lambda^*$, then by the proof of Theorem 7-6, all λ for which positive solutions of (7.1) exist satisfy $\lambda \leqslant \lambda^{**}$. But $\lambda^* > \lambda^{**}$ gives a contradiction. Q.E.D.

Corollary 7-8: Let $g(\lambda, u)$ satisfy (7.2), (7.3), (7.11), (7.12), and (7.14) (concave). Then there is no value λ for which $\lambda = \mu_1(\lambda)$.

According to Keller and Cohen [19], $\lambda \leqslant \mu_1(\lambda)$. This corollary is simply one way of saying that when $g(\lambda, u)$ is concave in u, the branch of minimal positive solutions has no nonisolated solutions, and hence no branching points.

Proof: Suppose $\left(\underline{u}(\lambda_0, x), \lambda_0\right)$ exist so that $\lambda_0 = \mu_1(\lambda_0)$. By Corollary 7-4, for every $\lambda < \lambda_0$, there is a $\underline{u}(\lambda, x) < \underline{u}(\lambda_0, x)$ satisfying (7.1). Then, (7.24) holds with $v = \underline{u}(\lambda, x)$. Since (7.12) holds, (7.24) implies

$$\langle g(\lambda, v) - g(\lambda_0, v), \phi_0^* \rangle > 0 \tag{7.26}$$

which by (7.3) cannot hold for $\lambda < \lambda_0$, and gives a contradiction. Q.E.D.

We now prove a result which enables us to identify the minimal positive solution of (7.1). We first give the following "non-ordering" theorem.

Theorem 7-9. Let $g(\lambda, u)$ satisfy (7.11), (7.12) and either (7.13) or

<u>(7.14). For fixed $\lambda > 0$, (1.1) does not possess three distinct solutions $u_1(x)$, $u_2(x)$ and $u_3(x)$ which are ordered $u_1 \leq u_2 \leq u_3$, $\forall x \in \overline{D}$.</u>

<u>Proof</u>: The proof of this theorem has been given previously by Laetsch [27] and Fujita [12]. The proof given here is valid for operators L which are self adjoint.

Suppose $0 \leq u_1(x) \leq u_2(x) \leq u_3(x)$ satisfy (7.1). Letting $w_1(x) = u_2 - u_1$, $w_2(x) = u_3 - u_2$ we have

$$Lw_1 + G(\lambda, u_1, u_2)w_1 = 0, \quad x \in D,$$

$$Bw_1 = 0, \quad x \in \partial D,$$

and

$$Lw_2 + G(\lambda, u_2, u_3)w_2 = 0, \quad x \in D,$$

$$Bw_2 = 0, \quad x \in \partial D,$$

where
$$G(\lambda, u, v) = \int_0^1 g_u\big(\lambda, su + (1-s)v\big)\, ds .$$

When $g(\lambda, u)$ satisfies (7.13) (convex),

$$G(\lambda, u_1, u_2) \leq G(\lambda, u_2, u_3),$$

where the inequality is strict somewhere on D. Since the functions w_1 and w_2 are nontrivial positive functions, they must be principal eigenfunctions for the problems

$$Lw + \mu G(\lambda, u_1, u_2)w = 0 , \quad x \in D,$$
$$Bw = 0 , \quad x \in \partial D ,$$

and

$$Lw + \mu G(\lambda, u_2, u_3)w = 0 , \quad x \in D ,$$
$$Bw = 0 , \quad x \in \partial D,$$

respectively, with principal eigenvalue $\mu_1 = 1$. Since L is a self adjoint operator, the variational characterization of the principal eigenvalue is valid [6]. That is

$$\mu_1 = \min_{\psi \in C} \frac{\langle \psi, L\psi \rangle}{\langle \psi, G(\lambda, u, v)\psi \rangle}$$

where C is the class of admissible functions

$$C = \{\psi(x) \,|\, \psi(x) > 0 \quad x \in D, \ \psi \in C(\overline{D}) \cap C'(D), \ \psi(x) = 0 \text{ on } \partial D_1\}.$$

With this formulation of μ_1 we have

$$1 = \min_{\psi \in C} \frac{\langle \psi, L\psi \rangle}{\langle \psi, G(\lambda, u_1, u_2)\psi \rangle} = \frac{\langle w_1, Lw_1 \rangle}{\langle w_1, G(\lambda, u_1, u_2)w_1 \rangle}$$

$$> \frac{\langle w_1, Lw_1 \rangle}{\langle w_1, G(\lambda, u_2, u_3)w_1 \rangle} \geq \min_{\psi \in C} \frac{\langle \psi, L\psi \rangle}{\langle \psi, G(\lambda, u_2, u_3)\psi \rangle} = 1$$

Clearly this is a contradiction. When $g(\lambda, u)$ satisfies (7.14) (concave), the inequalities are simply reversed, and the proof is complete. Q.E.D.

The following corollary gives a criterion to find minimal positive solutions.

Corollary 7-10: Let $g(\lambda,u)$ be as in Theorem 7-9. If, for fixed λ, (7.1) has distinct positive solutions $u_1(x) \leqslant u_2(x)$, then $u_1(x) = \underline{u}(\lambda, x)$ is the minimal positive solution of (7.1) for the given λ.

Proof: Suppose $u_1(x)$ is not the minimal positive solution of (7.1). Then $\underline{u}(\lambda, x) \leqslant u_1(x) \leqslant u_2(x)$ where $\underline{u}$, u_1 and u_2 are distinct solutions of (7.1), which contradicts Theorem 7-9, Q.E.D.

In Section 5 we were able to show the existence of branches of solutions of (1.1) which contain the nonisolated solutions of Section 3. These solutions were shown in (5.33) to be of the form

$$(7.27)\qquad \begin{cases} u(x,\epsilon,\delta) = \epsilon\phi_0 + \epsilon^2 v(x,\epsilon) + \delta\psi_0(x,\epsilon) + \delta^2 y(x,\epsilon,\delta), \\ \psi_0(x,\epsilon) = \phi_0 + \epsilon\chi(x,\epsilon), \\ \lambda(\epsilon,\delta) = \lambda_0 + \epsilon\,\mu(\epsilon) + \delta^2 \nu(\epsilon,\delta), \\ \tau(\epsilon) = \epsilon^2\eta(\epsilon), \end{cases}$$

where $\psi_0(x,\epsilon)$ satisfies

$$(7.28)\qquad \begin{aligned} L\psi + f_u\big(\lambda(\epsilon,0),\tau(\epsilon),\ u(x,\epsilon,0)\big)\,\psi &= 0, \qquad x \in D, \\ B\psi &= 0, \qquad x \in \partial D. \end{aligned}$$

We want to show that if $f(\lambda,\tau,u)$ is required to satisfy conditions (7.2) - (7.4) and (7.11) - (7.13) (convex) for $\tau > 0$, and if $\phi_0 > 0$ is the principal eigenfunction of (1.6), then the solution branch (7.27) with $\epsilon > 0$ and $\delta < 0$ is part of the branch of minimal positive solutions in the sense of Keller and Cohen [19]. Our main tool is the following

Lemma 7-11: Let $p(x)$ and $q(x)$ be $C^{2+\alpha}(\overline{D})$ functions satisfying $Bp(x) = 0$, $Bq(x) = 0$ $\forall x \in \partial D$. Suppose that $p(x) > 0$, $x \in D$ and that if $p(x) = 0$ for $x \in \partial D$, then $\frac{\partial p(x)}{\partial \alpha} < 0$ where $\frac{\partial}{\partial \alpha}$ is any outward directional derivative of $x \in \partial D$. Then the function $R(x) = \frac{q(x)}{p(x)}$ is continuous on $\overline{D}$.

Proof: The proof given here is a correction of a proof given by H. B. Keller in [18]. The function $R(x)$ is continuous on D, since $p(x) > 0$ on D. Recall from (1.3), (1.4) that the form of the boundary operator is

$$Bu = b_0(x) + b_1(x) \frac{\partial u}{\partial \beta}$$

where

$$\frac{\partial u}{\partial \beta} = \sum_{j=1}^{n} \beta_j(x) \frac{\partial u}{\partial x_j}$$

and

$$\sum_{j=1}^{n} \beta_j^2(x) = 1 \quad , \quad \sum_{j=1}^{n} \beta_j(x)\, n_j(x) > 0 \ ,$$

where $n_j(x)$ are components of the outward unit normal at $x \in \partial D$. Furthermore, we decompose the boundary ∂D into ∂D_1 and ∂D_2 where

$$b_0(x) > 0 \qquad b_1(x) \equiv 0 \quad \text{for } x \in \partial D_1$$

$$b_0(x) \geqslant 0 \qquad b_1(x) > 0 \quad \text{for } x \in \partial D_2 \qquad \partial D = \partial D_1 \cup \partial D_2 \ .$$

Notice that ∂D_1 is a closed subset of ∂D, since $b_0(x)$ and $b_1(x)$ do not vanish simultaneously.

Suppose $p(x) = 0$ for some $x \in \partial D_2$. Then $\frac{\partial p(x)}{\partial \beta} < 0$ and $b_1(x) > 0$ contradicts that $Bp(x) = 0$ for $x \in \partial D_2$. Thus, $p(x) \neq 0$ for $x \in \partial D_2$, so that $R(x)$ is continuous on $\partial D_2 \cup D$. Define

$$\tilde{R}(x) = \begin{cases} R(x) & x \in D \cup \partial D_2 \\ \dfrac{\partial q(x)}{\partial n} \Big/ \dfrac{\partial p(x)}{\partial n} & x \in \partial D_1 \end{cases} \tag{7.29}$$

where $\frac{\partial}{\partial n}$ represents the outward normal directional derivative. We intend to show that $\tilde{R}(x)$ is continuous on $\overline{D}$.

Suppose $x \in \partial D_1$. For $y \in D$ with $|x-y|$ sufficiently small

$$\begin{cases} p(y) = |x-y| \left[\cos\theta \dfrac{\partial p(x)}{\partial n} + \sin\theta \dfrac{\partial p(x)}{\partial \tau}\right] + O(|x-y|^2), \\ q(y) = |x-y| \left[\cos\theta \dfrac{\partial q(x)}{\partial n} + \sin\theta \dfrac{\partial q(x)}{\partial \tau}\right] + O(|x-y|^2), \end{cases} \tag{7.30}$$

where the vector $x-y$ is assumed to make an angle θ with the normal vector $\vec{n}$ at x, and where $\frac{\partial}{\partial \tau}$ denotes the tangential derivative at x. However, since $p(x) = q(x) = 0$ for all $x \in \partial D_1$, both tangetial derivatives vanish, so that

$$p(y) = |x-y| \cos\theta \frac{\partial p(x)}{\partial n} + O(|x-y|^2) ,$$

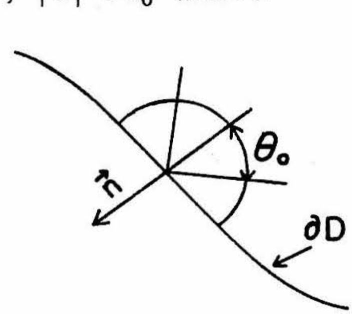
$$q(y) = |x-y| \cos\theta \frac{\partial q(x)}{\partial n} + O(|x-y|^2) ,$$

and

$$\tilde{R}(y) = \frac{q(y)}{p(y)} = \frac{|x-y|\cos\theta \frac{\partial q(x)}{\partial n} + O(|x-y|^2)}{|x-y|\cos\theta \frac{\partial p(x)}{\partial n} + O(|x-y|^2)}$$

$$= \frac{\frac{\partial q(x)}{\partial n}}{\frac{\partial p(x)}{\partial n}} + \frac{O(|x-y|)}{\cos\theta \frac{\partial p(x)}{\partial n}} , \quad \cos\theta \neq 0 .$$

Since $\frac{\partial p(x)}{\partial n} < 0$, if $|\theta| \leq \theta_0 < \pi/2$, we have that for $\epsilon > 0$, $|\tilde{R}(y) - \tilde{R}(x)| < \epsilon/2$ whenever $|x-y| < \delta(x,\epsilon)$, $|\theta| \leq \theta_0$ where $\delta(x,\epsilon) = \cos\theta_0 \left|\frac{\partial p(x)}{\partial n}\right| \Big/ K$ for some $K > 0$. Since ∂D_1 is closed, $\exists$ $\delta_1(\epsilon)$ so that $|\tilde{R}(y) - \tilde{R}(x)| < \frac{\epsilon}{2}$ whenever $|x-y| < \delta_1(\epsilon)$ and $|\theta(x,y)| < \theta_0$ for $x \epsilon\, \partial D_1$.



Since $p(x)$, $q(x) \epsilon\, ^{2+\alpha}(\overline{D})$ and since $\frac{\partial p(x)}{\partial n} < 0$ on ∂D_1, the quotient $\frac{\partial q(x)}{\partial n} \Big/ \frac{\partial p(x)}{\partial n}$ is continuous for $x \epsilon\, \partial D_1$. Therefore, $\exists$ $\delta_2(\epsilon) > 0$ so that for $|x-y| < \delta_2(\epsilon)$, $x, y \epsilon\, \partial D_1$, $|\tilde{R}(x) - \tilde{R}(y)| < \epsilon/2$.

For a given $x \epsilon\, \partial D_1$, there may exist many values of $y \epsilon\, \partial D_2$ satisfying $|x-y| < \delta_2(\epsilon)$. We know that $\tilde{R}(x)$ is continuous on $\partial D_2 \cup D$.

Furthermore, since $b_0(x) > 0$ on ∂D_1, $b_0(x)$ must remain positive for $x \in \partial D_2$ with x "near" ∂D_1. In such neighborhoods

$$R(x) = \frac{q(x)}{p(x)} = \frac{b_0 q(x)}{b_0 p(x)} = \frac{\partial q(x)}{\partial \beta} \Bigg/ \frac{\partial p(x)}{\partial \beta} .$$

For $x \in \partial D_1$, $\frac{\partial p}{\partial \tau} = \frac{\partial q}{\partial \tau} = 0$ so that

$$\frac{\partial p}{\partial \beta} = \alpha_1 \frac{\partial p}{\partial n} + \alpha_2 \frac{\partial p}{\partial \tau} = \alpha \frac{\partial p}{\partial n} ,$$

which implies

$$\frac{\partial q}{\partial n} \Bigg/ \frac{\partial p}{\partial n} = \frac{\partial q}{\partial \beta} \Bigg/ \frac{\partial p}{\partial \beta} \qquad x \in \partial D_1$$

But then $\tilde{R}(x)$ is continuous on $\partial D_1 \cup \partial D_2$. This implies the existence of $\delta_3(\epsilon)$ such that

$$|\tilde{R}(x) - \tilde{R}(y)| < \epsilon/2$$

whenever

$$|x-y| < \delta(\epsilon) \qquad x, y \in \partial D_1 \cup \partial D_2 = \partial D.$$

Continuity of $\tilde{R}(x)$ on $\partial D_2 \cup D$ implies the existence of $\delta_4(\epsilon)$ such that

$$|\tilde{R}(x) - \tilde{R}(y)| < \epsilon/2$$

whenever

$$|x-y| < \delta(\epsilon) \qquad (x,y) \in \partial D_2 \cup D .$$

Let $\delta = \frac{1}{2} \min_{i=1,2,3,4} \{\delta_i(\epsilon)\}$, and suppose $|x-y| < \delta(\epsilon)$ with $x \in \partial D_1$. Then $\exists\, x_1 \in \partial D$ so that $|\theta(x_1, y)| \leq \theta_0$, and regardless of whether $x_1 \in \partial D_1$ or $x_1 \in \partial D_2$, we have

$$|\widetilde{R}(x)-\widetilde{R}(y)| \leq |\widetilde{R}(x)-\widetilde{R}(x_1)| + |\widetilde{R}(x_1)-\widetilde{R}(y)| < \epsilon \quad .$$

which concludes the proof of continuity. Q.E.D.

Using this lemma we can prove

Theorem 7-12: There exists a positive number $d(\epsilon)$ such that for $\epsilon > 0$ sufficiently small, $|\delta| \leq d(\epsilon)$, the solution $u(x,\epsilon,\delta)$ of (1.1) given by (7.27) is positive on D.

Proof: From the perturbation theory for the spectrum of operators [11], we can deduce that the eigenvalues of

$$L\psi + \mu f_u(\lambda, \tau, u)\psi = 0 \qquad x \in D$$

$$B\psi = 0 \qquad x \in \partial D$$

vary continuously as λ, τ, u change continuously. Since $\phi_0(x) > 0$, we know by the Krein Rutman theorem [25] that $\mu = 1$ is the eigenvalue of smallest magnitude, and being simple, remains bounded away from all other eigenvalues as λ, τ, u vary. Hence for $|\epsilon| \leq \epsilon_0$, $\psi(x,\epsilon)$ is the corresponding principal eigenfunction and satisfies $\psi(x,\epsilon) > 0$ for $x \in D$, $\frac{\partial \psi(x,\epsilon)}{\partial \alpha} < 0$ whenever $\psi(x,\epsilon) = 0$ for $x \in \alpha D$, for all outward directions α.

In order to apply Lemma 7-11 to the functions

$$p(x) = \phi_0(x) + \frac{\delta}{\epsilon}\psi(x,\epsilon) \quad , \quad q(x) = v(x,\epsilon) + \left(\frac{\delta}{\epsilon}\right)^2 y(x,\epsilon,\delta) ,$$

notice that by Lemma 7-11, $\frac{\psi(x,\epsilon)}{\phi_0(x)}$ is continuous and is therefore bounded by, say, $M(\epsilon)$. Then $p(x) > 0$ for $|\delta| \leq \frac{1}{2}\frac{|\epsilon|}{M(\epsilon)}$. As noted in the proof of Lemma 7-11, $\psi(x,\epsilon)$ and $\phi_0(x)$ cannot vanish if $x \in \partial D_1$, and must vanish when $x \in \partial D_1$. For $x \in \partial D_1$, $p(x) = 0$ and

$$\frac{\partial p(x)}{\partial \alpha} = \frac{\partial \phi_0(x)}{\partial \alpha} + \frac{\delta}{\epsilon}\frac{\partial \psi(x,\epsilon)}{\partial \alpha}$$

$$= \gamma \frac{\partial \phi_0(x)}{\partial n}\left(1 + \frac{\delta}{\epsilon}\frac{\partial \psi(x,\epsilon)}{\partial n}\Big/\frac{\partial \phi_0(x)}{\partial n}\right) \quad \text{for } \gamma > 0 ,$$

since the tangential derivatives $\frac{\partial \phi_0(x)}{\partial \tau}$ and $\frac{\partial \psi(x,\epsilon)}{\partial \tau}$ vanish for $x \in \partial D_1$. Since $\frac{\partial \phi_0(x)}{\partial n} < 0$ for $x \in \partial D_1$ and ∂D_1 is closed, the function $\frac{\partial \psi(x,\epsilon)}{\partial n}\Big/\frac{\partial \phi_0(x)}{\partial n}$ is bounded on ∂D_1 by $N(\epsilon)$. Thus $\frac{\partial p(x)}{\partial \alpha} < 0$ provided $|\delta| < \frac{1}{2}\frac{|\epsilon|}{N(\epsilon)}$. Notice that since $\psi(x,0) = \phi_0(x)$ and $\psi(x,\epsilon)$ depends continuously on ϵ, it follows that $M(\epsilon)$ and $N(\epsilon)$ are continuous and satisfy $M(0) = N(0) = 1$.

We now conclude from Lemma 7-11 that

$$R(x,\epsilon,\delta) = \frac{q(x)}{p(x)} = \frac{v(x,\epsilon) + (\frac{\delta}{\epsilon})^2\ y(x,\epsilon,\delta)}{\phi_0(x) + \frac{\delta}{\epsilon}\ \psi(x,\epsilon)}$$

is continuous and therefore uniformly bounded on $\overline{D}$ for $|\delta| \leq \frac{1}{2}|\epsilon| \min\left(\frac{1}{M(\epsilon)}, \frac{1}{N(\epsilon)}\right) = d(\epsilon)$. Suppose $|R(x,\epsilon,\delta)| \leq R$ for $x \epsilon \overline{D}$, $|\delta| \leq d(\epsilon)$, $0 \leq |\epsilon| \leq \epsilon_0$. Then

$$u(x,\epsilon,\delta) = \epsilon\left(\phi_0(x) + \frac{\delta}{\epsilon}\psi(x,\epsilon)\right)\left(1 + \epsilon R(x,\epsilon,\delta)\right)$$

is positive on D provided $|\epsilon| \leq \min(\frac{1}{R}, \epsilon_0)$, $|\delta| \leq d(\epsilon)$. Q.E.D.

In a similar manner, Lemma 7-11 is used in the proof of

Theorem 7-13: There exists a positive number $D(\epsilon)$ such that for $\epsilon > 0$ sufficiently small, $u(x,\epsilon,\delta_1) > u(x,\epsilon,\delta_2)$ for $x \epsilon D$ whenever $\phi_0(x) > 0$ and $\delta_1 > 0$, $\delta_2 < 0$, $\max(|\delta_1|, |\delta_2|) \leq D(\epsilon)$.

Proof: As in Theorem 7-12, $\phi_0(x) > 0$ implies $\psi(x,\epsilon) > 0$. Apply Lemma 7-11 to conclude that the function

$$R(x,\epsilon,\delta) = \frac{y(x,\epsilon,\delta)}{\psi(x,\epsilon)}$$

is continuous and bounded on $\overline{D}$. Letting $|R(x,\epsilon,\delta)| \leq R(\epsilon)$ for $x \epsilon \overline{D}$, $|\delta| \leq \delta_0(\epsilon)$, we see that

$$u(x,\epsilon,\delta_1) - u(x,\epsilon,\delta_2) = (\delta_1 - \delta_2)\psi(x,\epsilon)\left[1 + \frac{\delta_1^2}{\delta_1 - \delta_2}R(x,\epsilon,\delta_1) - \frac{\delta_2^2}{\delta_1 - \delta_2}R(x,\epsilon,\delta_2)\right]$$

is positive on D provided $|\delta_i| < \min\left(\frac{1}{2R(\epsilon)}, \delta_0(\epsilon)\right) = D(\epsilon)$, $\delta_1 > 0$, $\delta_2 < 0$. Q.E.D.

The purpose for the last two theorems was to show that under certain circumstances, the solution branch (7.27) is part of the branch of minimal positive solutions of (1.1) when $\epsilon > 0$ and $\delta < 0$. We prove this result in

Theorem 7-14: For each τ, $0 < \tau < \epsilon_0^2 K$, where ϵ_0 and K are found in Theorem 3-1, let $f(\lambda, \tau, u)$ satisfy conditions (7.2)-(7.4), (7.11)-(7.13)(convex). Then the solution branch (7.27) consists of minimal positive solutions of (1.1) for $\epsilon > 0$ and $\delta < 0$ when $\phi_0 > 0$.

Proof: We must first show that the solution (7.27) is applicable to the present situation. Condition (7.2) implies that $f_\tau(\lambda_0, 0, 0) > 0$ on D, condition (7.12) implies that $f_{\lambda u}(\lambda_0, 0, 0) > 0$ on D, and condition (7.13) implies the $\partial^{p+1} f(\lambda_0, 0, 0) / \partial u^{p+1} > 0$ on D. Applying these conditions in Theorem 3-1, we find that there are nonisolated solutions of (1.1) of the form (3.1) for $0 \leq |\epsilon| \leq \epsilon_0$, and that the corresponding $\tau(\epsilon)$ is positive for $\epsilon > 0$ sufficiently small (cf. 2.26, 2.28). Since $\tau(\epsilon)$ is positive for the nonisolated solution we get with $\epsilon > 0$, the extension (7.27) is a valid representation of a solution branch of (1.1).

According to (5.38), $\nu(\delta)$ in (7.27) is negative when $\delta = 0$. But this implies that by choosing $\delta_1 > 0$ and $\delta_2 < 0$, both sufficiently small, we can find $\lambda(\epsilon, \delta_1) = \lambda(\epsilon, \delta_2)$. Applying Theorems 7-12 and 7-13, the corresponding solutions $u(x, \epsilon, \delta_1)$ and $u(x, \epsilon, \delta_2)$ are both positive and satisfy $u(x, \epsilon, \delta_1) > u(x, \epsilon, \delta_2)$ on D.

Applying Corollary 7-10, we see that $u(x, \epsilon, \delta_2)$ is the minimal positive solution of (1.1) for $\lambda = \lambda(\epsilon, \delta_2)$, $\epsilon > 0$, $\delta_2 < 0$. Clearly, this branch with $\epsilon > 0$, $\delta < 0$ is part of the branch of minimal positive solutions. Q.E.D.

The above theorem establishes that the nonisolated solution of (1.1) in question is the minimal positive solution for each $\epsilon > 0$. It is an easy consequence of Theorem 7-6 that the corresponding eigenvalue $\lambda(\epsilon, 0) = \lambda^*(\epsilon)$. Clearly, as $\epsilon \to 0$, $\lambda^*(\epsilon) \to \lambda_0$. One important consequence of this is found in

Theorem 7-15: Assume the hypotheses of Theorem 7-14 hold. Then as $\tau \to 0^+$, the branch of minimal positive solutions which exists for $\lambda \in (0, \lambda^*]$ goes uniformly to the zero solution for $\lambda \in (0, \lambda_0]$.

Proof: Notice that as $\tau \to 0^+$, $\epsilon \to 0^+$ as well, $\epsilon > 0$. But then $\lambda^* \to \lambda_0$ and $\underline{u}(x, \lambda^*) \to 0$, since $\underline{u}(x, \lambda^*)$ is of the form $\underline{u}(x, \lambda^*) = \epsilon\, \phi_0 + \epsilon^2 \chi(x, \epsilon)$. By Corollary 7-14, $\underline{u}(x, \lambda)$ is an increasing function of λ. Thus for $\lambda \in (0, \lambda^*]$

$$0 \leqslant \underline{u}(x, \lambda) \leqslant \underline{u}(x, \lambda^*) \to 0 \quad \text{as } \tau \to 0. \qquad \text{Q.E.D.}$$

The reason that the foregoing discussion centered on minimal positive solutions was for physical reasons only. There are no mathematical reasons why one could not look for maximal negative solution $\overline{u}(x, \lambda) < 0$ on D. To do so simply requires simple inequality reversals in conditions (7.2)-(7.4), (7.11)-(7.14), and then the corresponding changes in the stated results of this section go through using the same proofs.

II-8. Other Solution Branches.

In previous sections we have shown the existence of branches of solutions of (1.1) for τ small. All of the branches contained elements which were nonisolated solutions. Depending on the properties of $f(\lambda, \tau, u)$, for a fixed τ, a given problem may have two, one or possibly no branches with nonisolated solutions. By examining (2.29), we see that if p defined in (2.19) is even, then there is one nonisolated solution for $\tau > 0$ and one for $\tau < 0$. However, if p is odd, τ must be restricted so that λ is real, which means that there will be two nonisolated solutions for τ of one sign, and no nonisolated solutions for τ of the other sign. Work by Simpson and Cohen [36] suggests that this is not the complete story. They find solution branches which have no nonisolated solutions, but points on the branch approach the solution pair $(u, \lambda) = (0, \lambda_0)$ where λ_0 is the principal eigenvalue of (1.6), as $\tau \to 0$. In this section we will show that for all values of τ sufficiently small, (1.1) has at least two distinct solution branches with values of λ in a neighborhood of an eigenvalue λ_0 given by (1.6).

In previous sections we have suggested that the perturbation theory provides a method which will always lead to the desired answer. In this situation, such is not the case. Although we will prove our result by use of a contraction mapping, the mapping is one that is not motivated by a perturbation procedure.

We seek solutions of (1.1) of the form

$$(8.1)\qquad \begin{cases} u(x,\epsilon) = \epsilon\,\phi_0 + \epsilon^2 w(x,\epsilon) \\ \\ \lambda(\epsilon) = \lambda_0 + \epsilon\nu(\epsilon) \end{cases}$$

for τ fixed, $0 \leqslant |\tau| \leqslant \tau_1$. To make $w(x,\epsilon)$ unique we will require

$$(8.2)\qquad \langle w(x,\epsilon),\ \phi_0^* \, f_{\lambda u}(\lambda_0, 0, 0)\rangle = 0$$

Notice that when $\tau \neq 0$, setting $\epsilon = 0$ in (8.1) does not give a solution of (1.1). This leads us to suspect that (8.1) is valid for $0 < \epsilon_1 \leqslant |\epsilon| \leqslant \epsilon_2$ where ϵ_1 and ϵ_2 are related to τ in some way to be determined.

If we substitute (8.1) into (1.1) we find

$$(8.3)\qquad \begin{cases} Lw + f_u(\lambda_0,0,0)w = -\frac{1}{\epsilon^2}\left\{f(\lambda,\tau,u) - f_u(\lambda_0,0,0)u\right\} \\ \qquad = -\Big[\frac{\tau}{\epsilon^2}\int_0^1 f_\tau(\lambda, s\tau, u)ds \\ \qquad\quad + \nu(\phi_0+\epsilon w)\int_0^1 f_{\lambda u}(\lambda_0+\epsilon s\nu, 0, tu)ds\,dt \\ \qquad\quad + (\phi_0+\epsilon w)^2\int_0^1\int_0^1 f_{uu}(\lambda_0, 0, stu)s\,dt\,ds\Big] = R(\nu,\tau,w;\epsilon), \\ Bw = 0\,, \\ \langle w, \phi_0 f_{\lambda u}(\lambda_0, 0, 0)\rangle = 0\,. \end{cases}$$

Equation (8.3) is again of the form (1.12) and can be solved only if the condition

$$(8.4) \qquad \langle R(\nu, \tau, w; \epsilon), \; \overset{*}{\phi}_0 \rangle = 0$$

is satisfied.

To formulate the contraction mapping, we again use the set of functions $\mathcal{B}_K$ of (3.7) and the real interval $\mathcal{I}_K$ of (3.8). We also introduce the set

$$(8.5) \qquad S_3(\rho, \Gamma) = \{ (\lambda, u; x) \,|\, \lambda = \lambda_0 + \epsilon\nu, \; u = \epsilon\phi_0 + \epsilon^2 w, \quad x \in \overline{D}$$
$$0 \leq |\epsilon| \leq \rho, \quad \nu \in \mathcal{I}_\Gamma, \; w \in \mathcal{B}_\Gamma \}.$$

We define the mapping T_ϵ in the natural way suggested by (8.3), (8.4). That is, for each ϵ in $\epsilon_1 \leq |\epsilon| \leq \epsilon_2$, define $T_\epsilon[w, \nu] = [\tilde{w}, \tilde{\nu}]$ by

$$(8.6) \qquad \tilde{\nu} \, \langle (\phi_0 + \epsilon w) \int_0^1 f_{\lambda u}(\lambda_0 + s\epsilon\nu, 0, tu)\, ds\, dt, \; \overset{*}{\phi}_0 \rangle = -\frac{\tau}{\epsilon^2} \langle \int_0^1 f_\tau(\lambda, s\tau, u)\, ds, \; \overset{*}{\phi}_0 \rangle$$
$$+ \langle (\phi_0 + \epsilon w)^2 \int_0^1 \int_0^1 f_{uu}(\lambda_0, 0, stu)\, s\, dt\, ds, \; \overset{*}{\phi}_0 \rangle ,$$

and

$$(8.7) \qquad \begin{cases} L\tilde{w} + f_u(\lambda_0, 0, 0)\tilde{w} = -\Big[\tilde{\nu}(\phi_0 + \epsilon w) \int_0^1 f_{\lambda u}(\lambda_0 + \epsilon s\nu, 0, tu)\, ds\, dt \\ \qquad + \frac{\tau}{\epsilon^2} \int_0^1 f_\tau(\lambda, s\tau, u)\, ds \\ \qquad + (\phi_0 + \epsilon w)^2 \int_0^1 \int_0^1 f_{uu}(\lambda_0, 0, stu)\, s\, dt\, ds \Big], \quad x \in D, \\ B\tilde{w} = 0, \quad x \in \partial D, \\ \langle \tilde{w}, \; \overset{*}{\phi}_0 \, f_{\lambda u}(\lambda_0, 0, 0) \rangle = 0 . \end{cases}$$

The mapping T_ϵ induces a natural iteration procedure, which for some initial iterate (w^0, ν^0) is given by

$$(8.8) \qquad \left[w^{k+1}, \nu^{k+1}\right] = T_\epsilon \left[w^k, \nu^k\right] \qquad k = 0, 1, 2, \cdots .$$

With this machinery available, we state and prove the following

Theorem 8-1: Let $S_3 = S_3(\rho, \Gamma)$ for some fixed $\rho \leq 1$, $\rho\Gamma \leq 1$. Suppose that the smoothness assumption (3.15) on $f(\lambda, \tau, u)$ hold on S_3, and that $\langle \phi_0 f_{\lambda u}(\lambda_0, 0, 0), \phi_0^* \rangle \neq 0$. Then there exist real positive constants ϵ_1, ϵ_2, K where $\epsilon_1 < \epsilon_2 \leq \rho$, $\epsilon_2 K \leq \rho\Gamma$, such that the mapping T_ϵ given by (8.6), (8.7) maps $W_K = (\mathcal{B}_K \times \mathcal{I}_K)$ into W_K, and T_ϵ is a contraction on W_K for all ϵ, $0 < \epsilon_1 \leq |\epsilon| \leq \epsilon_2$. Furthermore, the problem (1.1) has nontrivial solutions of the form (8.1), where $w(x, \epsilon)$, $\nu(\epsilon)$ satisfy (8.3), (8.4), and are the limits of the sequence generated by (8.8) for any initial iterate in W_K.

Proof: The machinery introduced for this theorem are such that we only need to show that T_ϵ is a contraction mapping. The smoothness properties of the iterates, and the proof of convergence of the iterates to solutions of (1.1) are the same as in Theorem 3-1 and Theorem 5-1, and will not be repeated.

The proof that T_ϵ is a contraction mapping of W_K into W_K has some important differences from the proofs previously encountered. Since $\langle \phi_0 f_{\lambda u}(\lambda_0, 0, 0), \phi_0^* \rangle = \alpha \neq 0$ we assume that ϵ_2 is chosen small enough so that

$$(8.9)\qquad \left| \left\langle (\phi_0+\epsilon w)\int_0^1 f_{\lambda u}(\lambda_0+\epsilon s\nu, 0, tu)\,ds\,dt,\ \phi_0^* \right\rangle \right| \geq \frac{\alpha}{2}.$$

Then for $(w,\nu)\in W_K$ we have

$$(8.10)\qquad |\nu| \leq \frac{2\Phi}{\alpha}\left[\frac{|\tau|}{\epsilon^2}\|f_\tau\|_s + \tfrac{1}{2}\left(\|\phi_0\|_\infty + |\epsilon|K\right)^2 \|f_{uu}\|_s\right]$$

and

$$(8.11)\qquad \|\tilde{w}\|_\infty \leq G\left[|\tilde{\nu}|\left(\|\phi_0\|_\infty + |\epsilon|K\right)\|f_{\lambda u}\|_s + \frac{|\tau|}{\epsilon^2}\|f_\tau\|_s + \left(\|\phi_0\|_\infty + |\epsilon|K\right)^2 \frac{\|f_{uu}\|_s}{2}\right].$$

In order to find a K and ϵ_2 so that T_ϵ maps W_K into W_K we must bound $|\tilde{\nu}|$ and $\|\tilde{w}\|_\infty$ from above. In other words, we must require $|\tau| / \epsilon^2 \leq \Lambda$ for $0 \leq |\tau| \leq \tau_1$ and for $\epsilon_1 \leq |\epsilon| \leq \epsilon_2$. Since we have yet to determine τ_1 and ϵ_1, we require

$$(8.12)\qquad \tau_1 = \Lambda\epsilon_1^2 ,$$

where Λ is some fixed positive constant. Then (8.10) and (8.11) are of the form

$$(8.13)\qquad \max\left\{|\tilde{\nu}|, \|\tilde{w}\|_\infty\right\} \leq A(\Lambda) + |\epsilon|\ B(\Lambda,\epsilon,K) .$$

Clearly by choosing $K > A(\Lambda)$, we can find $\epsilon_3 > 0$, $\epsilon_3 \leq \rho$, $\epsilon_3 K \leq \rho\Gamma$ so that $\max\{|\tilde{\nu}|, \|\tilde{w}\|_\infty\} \leq K$ whenever $|\epsilon| \leq \epsilon_3$. The choice of Λ in (8.12) is arbitrary. However, we will see later that the value chosen for Λ has a definite effect on the values of τ_1 and ϵ_1, as it has already had an effect on the values of K and ϵ_3.

The second step is to show that the mapping T_ϵ is a contraction mapping for ϵ_2 chosen appropriately. If we let $y_1 = (w_1, \nu_1) \epsilon W_K$ and $y_2 = (w_2, \nu_2) \epsilon W_K$ and denote

$$(8.14) \qquad \|y\| = \max\{\|w\|_\infty, |\nu|\} \qquad y \epsilon U_K ,$$

then, whenever $|\epsilon| \leqslant \epsilon_3$,

$$(8.15) \qquad |\tilde{\nu}_1 - \tilde{\nu}_2| \leqslant \frac{2}{\alpha}\Phi|\epsilon|\left[A_{11}|\nu_1-\nu_2| + A_{12}\|w_1-w_2\|_\infty\right],$$

and

$$(8.16) \qquad \|\tilde{w}_1-\tilde{w}_2\|_\infty \leqslant G|\epsilon|\left[A_{21}|\nu_1-\nu_2| + A_{22}\|w_1-w_2\|_\infty\right] + G A_{23}|\tilde{\nu}_1-\tilde{\nu}_2| ,$$

where

$$(8.17) \qquad \begin{cases} A_{11} = A_{21} = \Lambda\|f_{\tau\lambda}\|_s + \frac{K}{2}\left(\|\phi_0\|_\infty+1\right)\|f_{\lambda\lambda u}\|_s, \\ A_{21} = A_{22} = \left(\|\phi_0\|_\infty+1\right)\left[\|f_{uu}\|_s + \epsilon_3\left(\frac{\|\phi_0\|_\infty+1}{6}\right)\|f_{uuu}\|_s + \frac{1}{2}\|f_{\lambda uu}\|_s\right] \\ \qquad + \Lambda\epsilon_1\|f_{\tau u}\|_s + K\|f_{\lambda u}\|_s , \\ A_{23} = \left(\|\phi_0\|_\infty+1\right)\|f_{\lambda u}\|_s . \end{cases}$$

Equations (8.15)-(8.17) imply the existence of a constant M_2 such that

$$\|\tilde{y}_1-\tilde{y}_2\| \leq |\epsilon| M_2 \|y_1-y_2\| \quad . \tag{8.18}$$

By choosing $\epsilon_2 < \min\{\epsilon_3, \frac{1}{M_2}\}$, we have that $T_\epsilon : W_K \to W_K$ is a contraction mapping on $\epsilon_1 \leq |\epsilon| \leq \epsilon_2$.

Now that ϵ_2 has been chosen, pick $\epsilon_1 < \epsilon_2$ so that the above statement is not vacuous. Doing so forces $\tau_1 = \Lambda\epsilon_1^2$ so that the mapping T_ϵ is a contraction only when $0 \leq |\tau| \leq \tau_1 = \Lambda\epsilon_1^2$. The remainder of the proof is the same as in Theorem 3-1 and will not be given here. Q.E.D.

Our statement at the beginning of this section was that for each value of τ, (1.1) has two solution branches with λ near λ_0. In terms of the form of these solutions branches (8.1), we see that the two branches result as ϵ takes on positive and negative values, $0 < \epsilon_1 \leq |\epsilon| \leq \epsilon_2$.

When a solution branch has a nonisolated solution, that solution can be found using the technique of Section 3. By a proper choice of Λ in the above proof, we can show that the nonisolated solution found in Section 3, is an element of a solution branch found in the above Theorem 8-1.

Suppose we represent the nonisolated solution (3.1) by

$$u_s(x,\epsilon) = \epsilon\phi_0 + \epsilon^2 v_s(x,\epsilon) \ ,$$

(8.19)
$$\lambda_s = \lambda_0 + \epsilon\mu_s(\epsilon) \ ,$$

$$\tau_s = \epsilon^2\eta_s(\epsilon) \ ,$$

$$\psi_s(x,\epsilon) = \phi_0 + \epsilon\,\chi_s(x,\epsilon) \ .$$

Then $y_s = (u_s, \lambda_s)$ is a nonisolated solution of (1.1) for $\tau = \tau_s$.

Theorem 8-2: Let the hypotheses of Theorem 3-1 and Theorem 8-1 hold. Then by a proper choice of Λ in Theorem 8-1, the non-isolated solution $y_s = (u_s, \lambda_s)$ of (8.19) lies on the solution branch (8.1) of (1.1), for $\tau = \tau_s$ whenever $|\epsilon| \leqslant \min\{\epsilon_s, \epsilon_2\}$.

Proof: According to the proof of Theorem 3-1, there are constants ϵ_s, K_s such that (8.19) is a valid representation of the nonisolated solution of (1.1) whenever $0 \leqslant |\epsilon| \leqslant \epsilon_s$, and that $\max\{\|v_s\|, |\mu_s|, |\eta_s|\} \leqslant K_s$. Recall that $S_1(\rho,\Gamma)$ and $S_3(\rho,\Gamma)$ depend solely on our choice of ρ and $\rho\Gamma$. By choosing ρ and $\rho\Gamma$ the same in Theorem 8-1 as for Theorem 3-1, we have that

$$S_3(\rho,\Gamma) = S_1(\rho,\Gamma) \cap \{\tau = \tau_s\} \ . \tag{8.20}$$

Choose $\Lambda = K_s$. The proof of Theorem 8-1 generates a positive number $\epsilon_2(\Lambda)$ so that (8.1) is a solution branch of (1.1) only if $|\epsilon| \leqslant \epsilon_2(\Lambda)$. Now set $\epsilon_1 = |\epsilon|$, where $|\epsilon| \leqslant \min\{\epsilon_s, \epsilon_2\}$. Then

$$(8.21) \qquad |\tau_s| = \epsilon^2 |\eta_s(\epsilon)| \leq \epsilon^2 \Lambda = \epsilon_1^2 \Lambda = \tau_1 \quad .$$

Thus, for any value of ϵ, $|\epsilon| \leq \min\{\epsilon_s, \epsilon_2\}$, Theorem 3-1 and Theorem 8-1 both hold.

The nonisolated solution (8.19) is a fixed point of the mapping (3.10)-(3.13). By picking $\tau = \tau_s$, $y_s = (u_s, \lambda_s)$ must also be a fixed point of (8.6), (8.7). Comparing y_s with $y = (w, \nu)$ we find that

$$(8.22) \qquad \|y_s - y\| \leq |\epsilon| \max\{M_s, M_2\} \|y_s - y\| .$$

But since $|\epsilon| \leq \min\{\epsilon_s, \epsilon_2\} < \min\{\frac{1}{M_s}, \frac{1}{M_2}\}$, Equation (8.22) implies that $y_s = y$, so that the solution (8.1) with $\tau = \tau_s(\epsilon)$ is a nonisolated solution of (1.1) whenever $|\epsilon| \leq \min\{\epsilon_s, \epsilon_2\}$. Q.E.D.

By a simple reorientation, the form of the solution branch (8.1) can be put into the form of (5.33) (the solution branch extended from a nonisolated solution), whenever Theorem 8-2 holds. Since solution branches of the form (5.33) were shown to be unique, the solution branch found here and the solution branch found in Section 5 must be segments of the same branch whenever Theorem 8-2 holds. That is, whenever $\tau = \tau_s$. If there are values τ which do not give rise to nonisolated solution of (1.1) as with the case when p is odd, then it seems reasonable to suspect that the solution branch has no nonisolated solutions.

When the nonlinearity $f(\lambda, \tau, u)$ is a positive monotone increasing, concave function for $\tau > 0$, the results of Simpson and Cohen [36] are applicable. In particular, if $\phi_0(x)$ is the positive eigenfunction of (1.6) and λ_0 the corresponding eigenvalue, then by Lemma 7-11, the solutions (8.1) are positive if τ_1 and ϵ_2 are sufficiently small and $\epsilon > 0$. However, in this situation positive solutions are unique for each λ, so that the solution branch (8.1) must be a segment of the branch of positive solutions found by Simpson and Cohen [36] on which no solutions are nonisolated solutions.

We summarize the results of the foregoing investigation in Figures 2, 3 and 4 by showing some possible solution branches which may occur. The plots are possible for any eigenvalue λ_0, although the minimal positive and maximal negative solutions shown may be included only when $\phi_0 > 0$.

Chapter III

Dynamic Buckling of Columns and Arches

III.1. Introduction.

To show that the ideas and results of Chapter II are relevant to problems which are not second order differential equations, we consider the motion of a slender elastic column subjected to a constant compressive axial displacement and to a transverse load $\tau p(x,t)$. The nondimensional equations of motion are [39]

$$(1.1) \quad \begin{cases} \dfrac{\partial^2 w}{\partial t^2} + \gamma \dfrac{\partial w}{\partial t} + \dfrac{\partial^4 w}{\partial x^4} + \sigma(x,t)\dfrac{\partial^2 w}{\partial x^2} + \tau p(x,t) = 0 \quad x \epsilon (0,1) \\[2ex] \dfrac{\partial \sigma}{\partial x} = 0 \qquad \sigma(x,t) = -\Gamma\left(\dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\right) \\[2ex] u(0,t) = \delta \qquad u(1,t) = -\delta \\ w(0,t) = w(1,t) = 0 \\ \dfrac{\partial^2 w}{\partial x^2}(0,t) = \dfrac{\partial^2 w}{\partial x^2}(1,t) = 0 \,. \end{cases}$$

Here $w(x,t)$ represents transverse displacement, $u(x,t)$ represents axial displacements and δ is the "end shortening." The physical parameter Γ satisfies

$$\Gamma = \frac{L^2 A}{I}$$

where L, A and I are the length, cross-sectional area, and moment of inertia, respectively, of the beam. The system (1.1) was derived assuming that finite deformations occur with small strain, and that Hooke's law is valid. The axial inertia of the column has been neglected, and damping is assumed to be proportional to $\gamma \frac{\partial w}{\partial t}$ for $\gamma > 0$.

Equation (1.1) can be simplified by using $\frac{\partial \sigma}{\partial x} = 0$. In fact, since $\sigma(x,t)$ does not depend on x,

$$\sigma(x,t) = \int_0^1 \sigma(x,t)dx = -\Gamma\int_0^1 \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right) dx$$

$$= 2\Gamma\delta - \frac{\Gamma}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx \quad .$$

Thus equation (1.1) reduces to

$$(1.2)\quad \begin{cases} \frac{\partial^2 w}{\partial t^2} + \gamma \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} + \left(2\Gamma\delta - \frac{\Gamma}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right) \frac{\partial^2 w}{\partial x^2} + \tau\, p(x,t) = 0 \\ w(0,t) = w(1,t) = 0 \\ \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(1,t) = 0 \; . \end{cases}$$

A shallow pinned arch initially stress free with centerline given by $y = \eta_0(x)$ can be treated as a special case of (1.2). The equation of motion for such an arch with transverse loading $q(x,t)$ is

$$
(1.3)\quad \begin{cases} \dfrac{\partial^2 w}{\partial t^2} + \gamma \dfrac{\partial w}{\partial t} + \left(\dfrac{\partial^4 w}{\partial x^4} - \dfrac{\partial^4 \eta_0}{\partial x^4}\right) - \dfrac{\Gamma}{2} \displaystyle\int_0^1 \left[\left(\dfrac{\partial w}{\partial x}\right)^2 - \left(\dfrac{\partial \eta_0}{\partial x}\right)^2 \right] dx \, \dfrac{\partial^2 w}{\partial x^2} + q(x,t) = 0 \\ w(0,t) = w(1,t) = 0 \\ w_{xx}(0,t) = w_{xx}(1,t) = 0 \end{cases}
$$

but this can easily be arranged into the form of (1.2) by making the identification

$$
\tau p(x,t) = q(x,t) - \frac{\partial^4 \eta_0}{\partial x^4}
$$

and

$$
\delta = \frac{1}{4} \int_0^1 \left(\frac{\partial \eta_0}{\partial x}\right)^2 dx \, .
$$

Thus, we will restrict our attention to (1.2). Furthermore, since the initial configuration of an unloaded stress free arch may be viewed as an imperfection in the corresponding end-shortened beam, we will call the parameter τ the imperfection amplitude and $p(x,t)$ the form of the imperfection.

III.2. Equilibrium States and Their Relationship to Imperfection Theory.

The steady state analysis of (1.2) with $\tau = 0$ gives a simple example of bifurcation phenomenon [32]. With no time dependence and $\tau = 0$, (1.2) reduces to

$$
(2.1)\quad \begin{cases} \dfrac{d^4w}{dx^4} + \Gamma\left(2\delta - \frac{1}{2}\int_0^1 \left(\dfrac{dw}{dx}\right)^2 dx\right) \dfrac{d^2w}{dx^2} = 0 \\[2ex] w(0) = w(1) = 0 \\[2ex] \dfrac{d^2w}{dx^2}(0) = \dfrac{d^2w}{dx^2}(1) = 0 \end{cases}
$$

Letting $\lambda^2 = \Gamma\left(2\delta - \frac{1}{2}\int_0^1 \left(\frac{dw}{dx}\right)^2 dx\right)$ we see that

$$
(2.2)\quad \begin{cases} w(x) = w_k \sin k\pi x \qquad \lambda = k\pi \qquad k = 1, 2, \cdots \\[2ex] w(x) = 0 \qquad\qquad \lambda \neq k\pi \end{cases}
$$

When $\lambda = k\pi$, the amplitude w_k is determined by

$$
\begin{aligned} k^2\pi^2 &= 2\Gamma\delta - \frac{\Gamma}{2}\int_0^1 \left(\frac{dw}{dx}\right)^2 dx \\ &= 2\Gamma\delta - \frac{\Gamma}{4} k^2\pi^2 w_k^2 \end{aligned}
$$

or

$$
(2.3)\qquad w_k^2 = \frac{8}{k^2\pi^2}\left(\delta - \frac{k^2\pi^2}{2\Gamma}\right) \qquad \text{provided } \delta \geqslant \delta_k = \frac{k^2\pi^2}{2\Gamma}
$$

We see that for $\delta_k < \delta \leqslant \delta_{k+1}$ there are $2k+1$ solutions of (2.1) given by (2.2), (2.3). The number δ_k is generally referred to as the kth buckling load. A plot of Q vs. δ is given in Figure 5, where $Q_k = kw_k$.

When $\tau \neq 0$, the steady state equation is

$$(2.4)\quad \begin{cases} \dfrac{d^4 w}{dx^4} + \Gamma\left(2\delta - \tfrac{1}{2}\displaystyle\int_0^1 \left(\dfrac{dw}{dx}\right)^2 dx\right) \dfrac{d^2 w}{dx^2} + \tau\, p(x) = 0 \\ w(0) = w(1) = 0 \\ \dfrac{d^2 w}{dx^2}(0) = \dfrac{d^2 w}{dx^2}(1) = 0 \end{cases}$$

As noted above, when $\tau = 0$, bifurcation occurs from the points $\delta_k = \dfrac{k^2 \pi^2}{2\Gamma}$. We expect that these bifurcation points will change as τ changes. We will first use the perturbation method to approximate this relationship.

The operator represented by equation (2.4) is an integro-differential operator. To find the appropriate linear integro-differential eigenvalue problem, suppose that $w(x)$ is a solution of (2.4). Substitute $y = w + \phi$ into equation (2.4) and linearize the equation for small ϕ. The resulting linear equation is

$$(2.5)\quad \begin{cases} \dfrac{d^4 \phi}{dx^4} + \Gamma\left(2\delta - \tfrac{1}{2}\displaystyle\int_0^1 \left(\dfrac{dw}{dx}\right)^2 dx\right) \dfrac{d^2 \phi}{dx^2} - \Gamma \dfrac{d^2 w}{dx^2} \displaystyle\int_0^1 \left(\dfrac{dw}{dx}\right)\left(\dfrac{d\phi}{dx}\right) dx = 0 , \\ \phi(0) = \phi(1) = 0 , \\ \dfrac{d^2 \phi}{dx^2}(0) = \dfrac{d^2 \phi}{dx^2}(1) = 0 . \end{cases}$$

The ideas set forward by the general theory in Chapter II suggest that we look for solutions of the form

$$(2.6)\quad \begin{cases} w(x,\epsilon) = \epsilon\Big(u_0(x) + \epsilon\, u_1(x) + \epsilon^2 u_2(x) + \cdots\Big)\,, \\ \phi(x,\epsilon) = \phi_0(x) + \epsilon\,\phi_1(x) + \epsilon^2\phi_2(x) + \cdots\,, \\ \delta(\epsilon) = \delta_0 + \epsilon\,\delta_1 + \epsilon^2\delta_2 + \cdots\,, \\ \tau(\epsilon) = \epsilon\,(\tau_0 + \epsilon\,\tau_1 + \epsilon^2\tau_2 + \cdots)\,, \end{cases}$$

where $\delta_0 = \dfrac{k^2\pi^2}{2\Gamma}$ for some k, and

$$(2.7)\quad \begin{cases} \int_0^1 u_0(x)\sin k\pi x\,dx = \int_0^1 \phi_0(x)\sin k\pi x\,dx = \frac{\pi}{2}\,, \\ \int_0^1 u_j(x)\sin k\pi x\,dx = 0\,, \quad j = 1, 2, \cdots\,, \\ \int_0^1 \phi_j \sin k\pi x\,dx = 0\,, \quad j = 1, 2, \cdots\,. \end{cases}$$

Substituting (2.6) into (2.4), (2.5) and equating coefficients of like powers of ϵ gives the following equations

$$(2.8)\quad \frac{d^4\phi_0}{dx^4} + 2\Gamma\delta_0\frac{d^2\phi_0}{dx^2} = 0$$

$$(2.9)\quad \frac{d^4\phi_1}{dx^4} + 2\Gamma\delta_0\frac{d^2\phi_1}{dx^2} = -2\Gamma\delta_1\frac{d^2\phi_0}{dx^2}$$

$$(2.10)\quad \frac{d^4\phi_2}{dx^4} + 2\Gamma\delta_0\frac{d^2\phi_2}{dx^2} = -2\Gamma\delta_1\frac{d^2\phi_1}{dx^2} - \left(2\Gamma\delta_2 - \frac{\Gamma}{2}\int_0^1\left(\frac{du_0}{dx}\right)^2 dx\right)\frac{d^2\phi_0}{dx^2} + \Gamma\frac{d^2w_0}{dx^2}\int_0^1\left(\frac{du_0}{dx}\right)\left(\frac{d\phi_0}{dx}\right)dx$$

$$(2.11)\quad \frac{d^4u_0}{dx^4} + 2\Gamma\delta_0\frac{d^2u_0}{dx^2} = -\tau_0\,p(x)$$

$$(2.12)\quad \frac{d^4 u_1}{dx^4} + 2\Gamma\delta_0 \frac{d^2 u_1}{dx^2} = -\tau_1 p(x) - 2\Gamma\delta_1 \frac{d^2 u_0}{dx^2}$$

$$(2.13)\quad \frac{d^4 u_2}{dx^4} + 2\Gamma\delta_0 \frac{d^2 u_2}{dx^2} = -\tau_2 p(x) - 2\Gamma\delta_1 \frac{d^2 u_1}{dx^2} - \Gamma\left(2\delta_2 - \frac{1}{2}\int_0^1 \left(\frac{du_0}{dx}\right)^2 dx\right)\frac{d^2 u_0}{dx^2}$$

The unknown functions u_i , ϕ_i all satisfy the boundary conditions

$$w(0) = w(1) = 0 \ ,$$

$$\frac{d^2 w(0)}{dx^2} = \frac{d^2 w}{dx^2}(1) = 0 \ .$$

In order to have a nontrivial solution for (2.8) we must have $\delta_0 = \frac{k^2\pi^2}{2\Gamma}$ for some k, and then $\phi_0(x) = \sin k\pi x$. We choose k so that

$$\int_0^1 p(x) \sin k\pi x \, dx = \frac{p_k}{2} \neq 0$$

where p_k is the kth Fourier coefficient for the sine expansion of p(x). Since the null space of (2.8) is spanned by $\phi_0(x)$, the Fredholm alternative theorem can be applied to solve equations (2.9) -(2.13). The differential operator in (2.8)-(2.13) is self-adjoint, so we require

$$(2.14)\quad \int_0^1 R(x)\ \phi_0(x) dx = 0$$

where R(x) is the right hand side of the equation to be solved.

Applied to equation (2.9), conditions (2.7) and (2.14) give that

$$\delta_1 = 0 \qquad \phi_1(x) = 0 \ .$$

Equation (2.11) implies that

$$\tau_0 = 0, \qquad u_0(x) = \sin k\pi x \ ,$$

and equation (2.12) gives, using (2.7), that

$$\tau_1 = 0, \qquad u_1(x) = 0.$$

From equation (2.10), the form of (2.14) is

$$-k^2\pi^2\left(2\delta_2 - \tfrac{1}{2}k^2\pi^2\int_0^1 \cos^2 k\pi dx\right)\left(\int_0^1 \sin^2 k\pi x dx\right) + k^4\pi^4\left(\int_0^1 \sin^2 k\pi x dx\right)\left(\int_0^1 \cos^2 k\pi x dx\right) = 0$$

which reduces to imply, again using (2.7), that

$$\delta_2 = \frac{3}{8}k^2\pi^2 \qquad \phi_2(x) = 0 \ .$$

Finally, solving equation (2.13) requires that

$$-\Gamma k^2\pi^2\left(2\delta_2 - \tfrac{1}{2}k^2\pi^2\int_0^1 \cos^2 k\pi x dx\right)\left(\int_0^1 \sin^2 k\pi x dx\right) + \tau_2 \frac{p_k}{2} = 0$$

which implies that $\tau_2 = \dfrac{\Gamma k^4\pi^4}{2p_k}$. Substituting for τ_2 and δ_2 in (2.13), we find that

$$u_2(x) = \frac{k^4\Gamma}{2p_k} \sum_{\substack{j=1 \\ j\neq k}}^{\infty} \frac{p_j}{j^2(j^2-k^2)} \sin j\pi x .$$

where

$$p_j = 2\int_0^1 p(x) \sin j\pi x \, dx .$$

Collecting this information in the form (2.6) gives that

$$(2.15) \quad \begin{cases} w(x,\epsilon) = \epsilon \sin k\pi x + \epsilon^3 \dfrac{k^4\Gamma}{2p_k} \displaystyle\sum_{\substack{j=1 \\ j\neq k}}^{\infty} \dfrac{p_j \sin j\pi x}{j^2(j^2-k^2)} + O(\epsilon^4) , \\ \phi(x,\epsilon) = \sin k\pi x + O(\epsilon^3) , \\ \delta(\epsilon) = \dfrac{k^2\pi^2}{2\Gamma} + \dfrac{3}{8}\epsilon^2 k^2\pi^2 + O(\epsilon^3) , \\ \tau(\epsilon) = \epsilon^3 \dfrac{k^4\pi^4\Gamma}{2p_k} + O(\epsilon^4) . \end{cases}$$

Since we are interested in knowing the relationship between δ and τ, we easily find from (2.15) that

$$(2.16) \qquad \delta = \frac{k^2\pi^2}{2\Gamma} + \frac{3}{4\sqrt[3]{2\Gamma^2}} \left(\frac{\tau p_k}{k\pi}\right)^{2/3} + O(\tau) .$$

At this point, one could verify that (2.15) is asymptotic to the exact solution using a contraction mapping, and then could extend the solution branch from the known solution (2.15) for a fixed value of τ, as in Section II-5. However, it is not necessary to carry out this program, since this problem can be solved exactly using Fourier sine series.

With the given boundary conditions on w(x), the Fourier sine series is complete, so that we let

$$(2.17) \qquad w(x) = \sum_{j=1}^{\infty} w_j \sin j\pi x$$

Operating formally and without justifying term by term differentiation, we substitute (2.17) into (2.4). Using the orthogonality properties of $\sin k\pi x$, we get the infinite set of algebraic equations

$$(2.18) \qquad w_j\, j^4 \pi^4 - \Gamma j^2 \pi^2 \left(2\delta - \frac{\pi^2}{4} \sum_{q=1}^{\infty} q^2 w_q^2\right) w_j + \tau p_j = 0, \quad j = 1,2,3,\cdots.$$

Letting $Q^2 = \sum_{q=1}^{\infty} q^2 w_q^2$, we have

$$(2.19) \qquad w_j = \frac{-4\tau p_j}{j^2 \pi^2 (4j^2 \pi^2 - 8\Gamma\delta + \Gamma\pi^2 Q^2)} \qquad j = 1, 2, \cdots$$

provided $\qquad 4j^2 \pi^2 - 8\Gamma\delta + \Gamma\pi^2 Q^2 \neq 0$

In the special case that $4n^2\pi^2 - 8\Gamma\delta + \Gamma\pi^2 Q^2 = 0$ for some n, we have that

$$(2.20) \qquad \begin{cases} n^2 w_n^2 = 4(2\Gamma\delta - n^2\pi^2) - \dfrac{\tau^2}{\pi^8} \displaystyle\sum_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{p_k^2}{k^2(k^2-n^2)^2} \\ \\ w_k = \dfrac{-4\tau p_k}{\pi^4 k^2 (k^2-n^2)}, \quad k \neq n, \end{cases}$$

and this can occur only when $p_n = 0$, and when δ is sufficiently large so that $w_n^2 \geq 0$ in (2.20).

In general, we use (2.19) to write

$$(2.21) \qquad Q^2 = \sum_{q=1}^{\infty} q^2 w_q^2 = \frac{16\tau^2}{\pi^4} \sum_{j=1}^{\infty} \frac{p_j^2}{j^2(4j^2\pi^2 - 8\Gamma\delta + \Gamma\pi^2 Q^2)^2} = \tau^2 F(\delta, Q^2),$$

which gives a nonlinear equation to be solved for Q^2 when τ and δ are fixed. The roots of (2.21) give the equilibrium states of (2.4) by way of (2.17) and (2.19). Figure 6 gives a graphical interpretation of (2.21) by plotting $y = F(\delta, Q^2)$ and $y = Q^2/\tau^2$. Figure 7 compares the solutions Q as a function of δ with the equilibrium states for the perfect beam shown in Figure 5.

Examining Figure 6, we see that for $\delta \geq \frac{k^2\pi^2}{2\Gamma}$, we have at most 2k+1 solutions Q^2 of (2.21). However, as τ changes, the number of solutions Q^2 of (2.21) may also change. As $\tau \to \infty$, there is only one solution of (2.21), whereas, the maximum number of solutions is obtained as $\tau \to 0$. The branching points are those values of Q^2 and τ which are double roots of (2.21), and are recognized in Figure 6 as points at which $y = F(\delta, Q^2)$ and $y = Q^2/\tau^2$ are tangent at a point of intersection. In Figure 7, branching points are those points with vertical tangents. The kth branching point always occurs for $\delta > \delta_k$ when $\tau \neq 0$. Since a branching point corresponds to a double root of (2.21), it must satisfy

$$(2.22) \qquad 1 = -\frac{32\tau^2\Gamma}{\pi^2} \sum_{j=1}^{\infty} \frac{p_j^2}{j^2(4j^2\pi - 8\Gamma\delta + \Gamma\pi^2 Q^2)^3} .$$

We would like to verify that the equilibrium solution corresponding to a double root of (2.21) is a nonisolated solution of (2.4), in that (2.5) has a nontrivial solution. Assume that w(x) has Fourier coefficients w_j such that $Q^2 = \sum_{q=1}^{\infty} q^2 w_q^2$ satisfies (2.21) and (2.22). Let

$$\phi(x) = \sum_{j=1}^{\infty} b_j \sin j\pi x \tag{2.23}$$

be a solution of (2.5). Substituting (2.23) into (2.5) gives

$$b_j\left[j^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2 Q^2}{4}\right] + \frac{\Gamma j^2\pi^2}{2} w_j \sum_{n=1}^{\infty} w_n b_n = 0 \ , \ j = 1,2,\cdots. \tag{2.24}$$

If we let $B = \sum_{n=1}^{\infty} w_n b_n$, then multiplying (2.24) by w_j and summing over j gives

$$B = -2\Gamma\pi^2 B \sum_{j=1}^{\infty} \frac{j^2 w_j^2}{[4j^2\pi^2 - 8\Gamma\delta + \Gamma\pi^2 Q^2]} \tag{2.25}$$

Using (2.19), this becomes

$$B = -\frac{32\tau^2\Gamma}{\pi^2} B \sum_{j=1}^{\infty} \frac{p_j^2}{j^2[4j^2\pi^2 - 8\Gamma\delta + \Gamma\pi^2 Q^2]^3} \ . \tag{2.26}$$

Since we want $\phi(x)$ to be nontrivial, we choose $B \neq 0$ so that (2.26) can be satisfied only if

$$1 = -\frac{32\tau^2\Gamma}{\pi^2} \sum_{j=1}^{\infty} \frac{p_j^2}{j^2[4j^2\pi^2 - 8\Gamma\delta + \Gamma\pi^2 Q^2]^3} \tag{2.27}$$

which is exactly the condition (2.22), and is satisfied only at a double root of (2.21).

Using (2.21) and (2.22), we can compare the exact solution with the perturbation result (2.15) when τ is small. When τ is small we expect that the nonisolated solutions will also be small so that Q^2 is small. If we let

$$(2.23) \qquad \delta = \frac{k^2\pi^2}{2\Gamma} + \alpha$$

for some k for which $p_k \neq 0$, we expect α to be small. Then (2.21) gives

$$Q^2 = \frac{16\tau^2}{\pi^4}\sum_{j=1}^{\infty} \frac{p_j^2}{j^2[4j^2\pi^2-4k^2\pi^2+\Gamma\pi^2Q^2-8\Gamma\alpha]^2}$$

$$\cong \frac{16\tau^2}{\pi^4}\left[\frac{p_k^2}{k^2\Gamma^2(\pi^2Q^2-8\alpha)^2} + \frac{1}{16\pi^4}\sum_{\substack{j=1\\j\neq k}}^{\infty}\frac{p_j^2}{j^2(j^2-k^2)^2}\right].$$

Keeping only the dominant terms we have

$$(2.29) \qquad \pi^2Q^2(\pi^2Q^2-8\alpha)^2 = \frac{16\tau^2p_k^2}{\pi^2k^2\Gamma^2}.$$

In a similar way (2.22) gives to lowest order

$$(2.30) \qquad 1 = -\frac{32\tau^2}{\pi^2}\frac{p_k^2}{k^2\Gamma^2(\pi^2Q^2-8\alpha)^3}$$

Combining (2.29) and (2.30) we see that

$$(\pi^2 Q^2 - 8\alpha)^3 = -\frac{32\tau^2}{\pi^2}\frac{p_k^{\,2}}{k^2\Gamma^2} = -2\pi^2 Q^2 (\pi^2 Q^2 - 8\alpha)^2$$

which reduces to imply that

$$\pi^2 Q^2 = \frac{8}{3}\alpha \tag{2.31}$$

By (2.30), this implies that

$$\delta = \frac{k^2\pi^2}{2\Gamma} + \frac{3}{4\sqrt[3]{2\Gamma^2}}\left(\frac{\tau p_k}{k\pi}\right)^{\frac{2}{3}} \tag{2.32}$$

to lowest order, which agrees with the perturbation result of (2.16). Knowing Q^2 and α to lowest order, one could also use (2.19) and (2.24) to find w(x) and ϕ(x) to lowest order in τ. Notice that (2.32) also agrees qualitatively with the results of Chapter II. Since the nonlinearity in (2.1) is of the form

$$\Gamma\left(2\delta - \tfrac{1}{2}\int_0^1 \left(\frac{dw}{dx}\right)^2 dx\right)\frac{d^2 w}{dx^2} \quad ,$$

it is composed of terms linear and cubic in w(x), so that in terms of Chapter II, p = 2, and

$$\delta_0 = \delta_0 + k\,\tau^{p/p+1} + \cdots$$

$$= \delta_0 + k\,\tau^{\frac{2}{3}} + \cdots .$$

III.3. Dynamic Treatment of Global Stability

For a fixed end shortening δ, we would like to determine the stability of motion about a given equilibrium state, and to find the initial conditions whose resulting motion approaches a given equilibrium state as $t \to \infty$. A recent study of perfect columns by Reiss and Matkowsky [33] shows, using a two-timing technique, that for δ slightly greater than δ_1, solutions can be expected to go to one of the two buckled states (Figure 5) as $t \to \infty$. In [32] it is shown that for the perfect column, the potential energy of a buckled mode is ordered in the opposite direction from the amplitude of the solution. That is, if $Q_1 > Q_2 > \cdots > Q_n$ correspond to the possible buckled solutions for δ fixed, then $V_1 < V_2 < V_n$, where V_k is the potential energy corresponding to the kth buckled state. The ordering of the potential energies is not sufficient to conclude that the first mode is stable and all others are unstable, and it tells us nothing about how to actually buckle into a higher mode.

From the analysis of Section II-6, a comparison of Figure 1 and Figure 7 suggests that for $\tau \neq 0$ and δ fixed, an equilibrium solution with Q large is "more stable" than one with small Q. More specifically, if these previous stability results are valid in the present situation, we expect the two solutions with largest Q to be stable, and as Q decreases, each pair of solutions encountered will have one more mode in which motion is unstable.

We consider the equation

$$(3.1)\qquad \begin{cases} \dfrac{\partial^2 w}{\partial t^2} + \gamma \dfrac{\partial w}{\partial t} + \dfrac{\partial^4 w}{\partial x^4} + \Gamma\left(2\delta - \frac{1}{2}\int_0^1 \left(\dfrac{\partial w}{\partial x}\right)^2 dx\right)\dfrac{\partial^2 w}{\partial x^2} + \tau p(x,t) = 0 \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = f(x)\ ,\ \dfrac{\partial w}{\partial t}(x,0) = g(x)\ , \qquad \gamma \geqslant 0. \end{cases}$$

We will assume that f(x), g(x) and p(x, t) have finite Fourier sine series expansions of the form

$$(3.2)\qquad \begin{cases} f(x) = \sum_{k=1}^{N} f_k \sin k\pi x \\ g(x) = \sum_{k=1}^{N} g_k \sin k\pi x \\ p(x,t) = \sum_{k=1}^{N} p_k(t) \sin k\pi x \end{cases}$$

We will further assume that for each k, $p_k(t) = p_k + \alpha_k(t)$ where

$$(3.3)\qquad \int_0^\infty \sum_{k=1}^{N} \alpha_k^2(t)\, dt < \infty \text{ and } \alpha_k(t) \to 0 \text{ as } t \to \infty .$$

If we let

$$(3.4)\qquad w(x,t) = \sum_{k=1}^{N} w_k(t) \sin k\pi x$$

and make use of the notation $\frac{\partial u}{\partial t} \equiv \dot{u}$, then equation (3.1) gives N coupled equations of the form

$$(3.5)\quad \begin{cases} \ddot{w}_k + \gamma \dot{w}_k + k^2\pi^2 \left(k^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2}{4} \sum_{q=1}^{N} q^2 w_q^2 \right) w_k + \tau p_k(t) = 0 , \\ \\ w_k(0) = f_k \ , \ \dot{w}_k(0) = g_k \ , \ k = 1, 2, \cdots N. \end{cases}$$

Since f_k , g_k and $p_k(t)$ are nonzero only for $k = 1, 2, \cdots N$, the uniqueness theorem for initial value problems guarantees us that $w_k(t)$ can be nonzero only for $k = 1, 2, \cdots N$, so that (3.4) is a valid representation of $w(x,t)$. Furthermore, if we know a priori that $w_j(t) \equiv 0$ for some j, $1 \leqslant j \leqslant N$, we can renumber the coefficients of $\sin k\pi x$, so that $w_1(t)$ is the first nonzero mode present in (3.4) with x dependence $\sin k_1 \pi x$, and so that all the coefficients $w_j(t)$ in (3.4) are nonzero and have x dependence $\sin k_j \pi x$, where $k_1 < k_2 < k_3 \cdots < k_N$. For simplicity we will assume that $k_j = j$ for $j = 1, 2, \cdots N$.

We can find an energy expression for (3.1) by multiplying the kth equation of (3.5) by $\dot{w}_k$ and adding together all such equations. The resulting equation is

$$0 = \sum_{k=1}^{N} \left[\dot{w}_k \ddot{w}_k + \gamma \dot{w}_k^2 + k^2\pi^2 \left(k^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2}{4} \sum_{q=1}^{N} q^2 w_q^2 \right) w_k \dot{w}_k + \tau p_k(t) \dot{w}_k \right]$$

which can be integrated once to get

$$\sum_{k=1}^{N}\left[\tfrac{1}{2}\dot{w}_k^2+\tfrac{1}{2}k^2\pi^2\left(k^2\pi^2-2\Gamma\delta\right)w_k^2+\tau p_k w_k\right]+\frac{\Gamma\pi^4}{16}\left(\sum_{k=1}^{N}k^2 w_k^2\right)^2 \tag{3.6}$$
$$+\int_0^t\sum_{k=1}^{N}\left[\gamma\dot{w}_k^2+\tau\,\alpha_k(t)\dot{w}_k\right]dt = K(0)\ ,$$

where by K(t) we denote the sum of the kinetic and potential energies

$$K(t) = \tfrac{1}{2}\sum_{k=1}^{N}\dot{w}_k^2+\sum_{k=1}^{N}\left[\tfrac{1}{2}k^2\pi^2\left(k^2\pi^2-2\Gamma\delta\right)w_k^2+\tau\, p_k w_k\right]+\frac{\Gamma\pi^4}{16}\left(\sum_{k=1}^{N}k^2w_k^2\right). \tag{3.7}$$

By completing the square of the integral term and of K(t), we can rewrite (3.6) in the form

$$\gamma\int_0^t\sum_{k=1}^{N}\left(\dot{w}_k+\frac{\tau\alpha_k(t)}{2\gamma}\right)^2 dt = K(0)+\frac{\tau^2}{2\gamma}\int_0^t\sum_{k=1}^{N}\alpha_k^2(t)dt-4\Gamma\delta-\frac{\tau^2}{\pi^4}\sum_{k=1}^{N}\frac{p_k^2}{k^4} \tag{3.8}$$
$$-\left\{\tfrac{1}{2}\sum_{k=1}^{N}\left[\dot{w}_k^2+k^4\pi^4w_k^2+\left(k^2\pi^2w_k+\frac{\tau p_k}{k^2\pi^2}\right)^2\right]\right.$$
$$\left.+\frac{\Gamma}{16}\left(\sum_{k=1}^{N}k^2\pi^2w_k^2-8\delta\right)^2\right\}\ .$$

A proof of the existence of solutions of (3.1) for all $t \geq 0$ has been given by Dickey [10] when $\gamma = \tau = 0$. The proof uses (3.8) with γ and τ set to zero and with $N \to \infty$, in order to show that the solution of (3.1) is bounded for all time. In light of (3.8), an extension of Dickey's existence proof to the present case with $\gamma > 0$ and $\tau > 0$ is straightforward.

Notice that the left hand side of (3.8) is a positive nondecreasing function of time. Since $\int_0^\infty \sum_{k=1}^{N}\alpha_k^2(t)dt < \infty$, we see that

(3.9) $$\int_0^t \sum_{k=1}^{N} \left(\dot{w}_k + \frac{\tau\alpha_k(t)}{2\gamma}\right)^2 dt < \infty \qquad \forall t > 0.$$

Thus the integral in (3.9) is convergent as $t \to \infty$. Furthermore, $\left(\dot{w}_k + \frac{\tau\alpha_k(t)}{2\gamma}\right)^2$ is a smooth function as $t \to \infty$ so that $\dot{w}_k \to 0$ as $t \to \infty$, $k = 1, 2, \cdots N$.

We can write (3.5) as a system of first order differential equations by letting

$$\underset{\sim}{\varphi} = \left(w_1,\ w_2,\ \ldots\ w_N,\ \dot{w}_1,\ \dot{w}_2,\ \ldots\ \dot{w}_N\right).$$

Then (3.5) gives

(3.10) $$\begin{aligned} \dot{\varphi}_j &= \varphi_{j+N} \\ \dot{\varphi}_{j+N} &= -\left\{ \gamma\varphi_{j+N} + j^2\pi^2\left(j^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2}{4}\sum_{q=1}^{N} q^2\varphi_q^2\right)\varphi_j + \tau p_j(t)\right\} \end{aligned}$$

$$j = 1, 2, \cdots N.$$

Since $\dot{w}_k \to 0$ smoothly as $t \to \infty$, the solution must approach a point satisfying $\dot{\underset{\sim}{\varphi}} = 0$. Such a point will be referred to as a critical point of the system (3.10). If a critical point is not approached as $t \to \infty$, then $\ddot{w}_k \not\to 0$ for some k which implies that $\dot{w}_k \not\to 0$ as $t \not\to 0$, a contradiction.

Critical points of the system (3.10) are those points in

2N-dimensional phase space for which the right hand side of (3.10) is identically zero. The first N components of these points are given by $\dot{w}_k = 0$, $k = 1, 2, \cdots N$, and the second N components satisfy

$$\text{(3.11)} \qquad k^2\pi^2\left(k^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2}{4}\sum_{q=1}^{N} q^2 \varphi_q^2\right)\varphi_k + \tau p_k = 0 \quad k=1,2,\cdots,N.$$

As is expected, equations (3.11) are exactly the same as the equations (2.18) which determined the equilibrium states of equation (3.1). In other words, as $t \to \infty$ with $\gamma > 0$, the motion approaches an equilibrium state. It is an analysis of these critical points in 2N-dimensional phase space which allows us to make statements about the global stability of the equilibrium states.

The behavior of a critical point is determined by the eigenvalues of the dynamic problem linearized about the given critical point [5]. When system (3.10) is linearized about some critical point, the resulting equations will be of the form

$$\text{(3.12)} \qquad \begin{pmatrix} \eta \\ \dot{\eta} \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & I \\ -D & -\gamma I \end{pmatrix}\begin{pmatrix} \eta \\ \dot{\eta} \end{pmatrix} = A\begin{pmatrix} \eta \\ \dot{\eta} \end{pmatrix} \quad ,$$

where D is a real symmetric $N \times N$ matrix and I is the $N \times N$ identity matrix. The eigenvalues of the matrix A will determine the local stability, and ultimately, the global stability of the point in question. To determine these eigenvalues, we solve

$$\begin{pmatrix} 0 & I \\ -D & -\gamma I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$Dx + (\lambda\gamma + \lambda^2)Ix = 0 .$$

If eigenvalues of D satisfy $Dx = \mu x$, then the eigenvalues of A satisfy

$$\lambda_{\pm} = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4\mu} . \tag{3.13}$$

If an eigenvalue of D is negative, then

$$\lambda_+ > 0, \quad \lambda_- < 0 ,$$

whereas, if an eigenvalue of D is positive, λ can be either real or complex. If $\gamma^2 - 4\mu \geqslant 0$, then both eigenvalues λ_+ and λ_- are negative. On the other hand, if $\gamma^2 - 4\mu < 0$,

$$\mathrm{Re}(\lambda_{\pm}) = -\frac{\gamma}{2} < 0 .$$

Examining the fundamental solution of (3.12), one sees that for each eigenvalue λ with positive real part, there corresponds an unstable mode. For each eigenvalue λ with negative real part there corresponds an exponentially decaying stable mode. Clearly, if D has N-k eigenvalues μ which are positive and k which are negative, there are 2N-k stable modes and k unstable modes in the solution of (3.12). These modes can equally well be visualized as mutually orthogonal directions on a 2N-dimensional energy surface. Moving "downhill" away from the critical point corresponds to an unstable direction, while moving "uphill" away from the critical point corresponds to a stable direction. With this motivation, a critical point with 2N stable modes will be called a center or well, and a critical point with k unstable modes will be called a saddle point or hyperbolic point of type k. Since the type of a critical point is exactly the number of negative eigenvalues of D, we restrict our attention to determining the eigenvalues of D.

To find the eigenvalues of D, we could plunge blindly ahead to attempt a direct calculation. However, that course of action leads to miserable algebraic computations, and furthermore, ignores the ideas of Chapter II. From the results of Chapter II, we realize that nonisolated solutions are neutrally stable in some mode, and that as τ changes, the locus of nonisolated solutions segments the solution branches into regions with the same stability characteristics. Furthermore, we can find the stability characteristics of a given solution with $\tau \neq 0$, by following a trace of solutions as $\tau \to 0$. If throughout the motion $\tau \to 0$ we have not reached a nonisolated solution, we know that the stability characteristics for a solution with $\tau \neq 0$ are the same as the solution arrived at when τ reaches 0.

With this motivation, we consider the problem for $\tau = 0$. If $\delta_m < \delta < \delta_{m+1}$, where $\delta_k = \frac{k^2\pi^2}{2\Gamma}$, there are 2m+1 equilibrium states and hence, 2m+1 critical points. The coordinates of these 2m+1 critical points are

$$
(3.13)\quad \begin{cases} \underset{\sim}{\phi}_0 = (0,0,\cdots,0)\ , \\ \underset{\sim}{\phi}_{k\pm} = (0,\ \cdots,\ \pm w_k,\ \cdots 0), \quad k=1,2,\cdots m. \\ \qquad w_k^2 = \frac{8}{k^2\pi^2}\left(\delta - \frac{k^2\pi^2}{2\Gamma}\right)\ , \end{cases}
$$

where $\pm w_k$ occurs in the kth position of the 2N-tuple for $\underset{\sim}{\phi}_{k\pm}$. Linearizing (3.10) about the critical point $\underset{\sim}{\phi}_{k\pm}$ we find the matrix

$$
(3.14)\quad \begin{cases} D_0 = (d_{ij}) = \delta_{ij} j^2\pi^2 (j^2\pi^2 - 2\Gamma\delta)\ , \\ D_k = (d_{ij}) = \delta_{ij}\pi^2\left[j^2\pi^2(j^2-k^2) + \delta_{jk}\frac{\Gamma w_k^2}{2}\right] \quad k=1,2,\cdots m. \end{cases}
$$

Since the matrix D is diagonal, its eigenvalues are the diagonal elements.

If $\delta < \delta_1 = \frac{\pi^2}{2\Gamma}$, then $\underset{\sim}{\phi}_0$ is the only critical point of (3.10), and the eigenvalues of D_0 are positive. Thus, the critical point $\underset{\sim}{\phi}_0$ is a center and is stable. If $m>0$, then D_0 has m negative eigenvalues so that $\underset{\sim}{\phi}_0$ is a hyperbolic point of type m. The matrices D_k each have k-1 negative eigenvalues and therefore, $\underset{\sim}{\phi}_{k\pm}$ is a hyperbolic point of type k-1. The first buckled mode $\underset{\sim}{\phi}_1$ has no negative eigenvalues and is therefore a center. In the context of linear stability theory, this implies that at the critical point $\underset{\sim}{\phi}_{k\pm}$, small perturbations in w_j decay to zero if $j \geq k$. However, if $j < k$, then w_j is a "lower" mode than w_k, and small perturbations in w_j will grow away from $\underset{\sim}{\phi}_{k\pm}$. Said another way, the lowest mode included in the motion by the initial data is the only mode which is stable for $m > 0$. All other modes are unstable to perturbation in lower modes.

We can translate this local result into a global result by use of the energy expression (3.8). If $N = 1$, the equation (3.5) reduces to Duffing's equation which has been analyzed thoroughly in [37] and [33]. When $\delta < \delta_1$ there is only one critical point $\underset{\sim}{\phi}_0 = \underset{\sim}{0}$. Since the motion must approach the critical point as $t \rightarrow \infty$, this critical point is globally stable, and the motion resulting from all initial data with $N = 1$ must approach $\underset{\sim}{\phi}_0 = \underset{\sim}{0}$ as $t \rightarrow \infty$. When $\delta > \delta_1$, there are three critical points $\underset{\sim}{\phi}_0$ and $\underset{\sim}{\phi}_{1\pm}$. The points $\underset{\sim}{\phi}_{1\pm}$ are stable and $\underset{\sim}{\phi}_0$ is unstable to local perturbations. Globally, there are interlacing regions of attraction (cf figure 4 in [33]) in initial value space for each of the points $\underset{\sim}{\phi}_{1+}$ and $\underset{\sim}{\phi}_{1-}$. These regions are separated by a set in initial value space of measure zero, the separatrix, for which motion approaches $\underset{\sim}{\phi}_0$ as $t \rightarrow \infty$. Knowledge of the separatrix determines the global properties of the Duffings equation.

For general N, as $t \rightarrow \infty$, the motion will approach one of the two critical points $\underset{\sim}{\phi}_{p\pm}$ corresponding to the lowest mode of motion available, for all initial data, excluding the separatrices, in the 2N-dimensional phase space. Initial data on a separatrix surface will lead to motion which approaches one of the other critical points as $t \rightarrow \infty$. A separatrix corresponds to a set of measure zero.

We can be more specific if we know the initial energy in the system K(0). According to (3.6) the total energy K(t) of the system is a nonincreasing function of time. The potential energy is given by

$$V = \frac{1}{2}\sum_{k=1}^{N} k^2\pi^2(k^2\pi^2 - 2\Gamma\delta)w_k^2 + \frac{\Gamma\pi^4}{16}\left(\sum_{k=1}^{N} k^2 w_k^2\right)^2$$

which evaluated at the kth critical points is

$$V_k = -\frac{\Gamma}{16}(k^2\pi^2\hat{w}_k^2)^2 = -\Gamma\frac{\pi^4 Q_k^4}{16}$$

(3.15)

$$V_0 = 0$$

Clearly, $V_1 < V_2 < \cdots < V_m < V_0 = 0$. Since the total energy is nonincreasing in time, if the initial energy of the system satisfies $K(0) < V_k$, then the critical points $\underset{\sim}{\phi}_{j\pm}$, $k \leq j \leq m$, cannot be approached as $t \to \infty$. If $K(0) < V_2$, then there is only one point which can be approached as $t \to \infty$, since the lowest "pass" between the wells of $\underset{\sim}{\phi}_{1\pm}$ are the points $\underset{\sim}{\phi}_{2\pm}$ with potential energy V_2. Using Figure 8 to illustrate the situation with two modes present, we see that $\underset{\sim}{\phi}_{1+}$ is approached if $f_1 < 0$ (cf. (3.2)) and $\underset{\sim}{\phi}_{1-}$ is approached as $t \to \infty$ if $f_1 > 0$.

When $\tau \neq 0$ we expect a similar situation will hold. Calculations by Hoff and Bruce [14] indeed show this to be true for an arch with two modes $w_1(t)$ and $w_2(t)$ and with $p(x) = p\sin\pi x$. However, motivated by the approach of Chapter II, we are able to get results for more general problems. For notational purposes, we will let Q^2_{k+} and Q^2_{k-} refer respectively to the larger and smaller of the roots of $Q^2 = \tau^2 F(\delta, Q^2)$ which depend continuously on τ (see Figure 7), and which, as $\tau \to 0$

$$Q^2_{k\pm} \to Q^2_k = \frac{4}{\Gamma k^2 \pi^2} (2\Gamma \delta - k^2\pi^2).$$

When $\tau \neq 0$, the kth pair of critical points is

$$\underset{\sim}{\phi}_{k\pm} = (w_1^{k\pm}, \cdots w_m^{k\pm}, 0, 0, \cdots 0)$$

where $w_j^{k\pm}$ are the expressions (2.19), (2.20) evaluated at $Q^2_{k\pm}$.

First calculate the potential energy V at the critical point $\underset{\sim}{\phi}_{k\pm}$, using (2.19)-(2.21) and

$$V = \frac{1}{2} \sum_{j=1}^{N} j^2 \pi^2 (j^2 \pi^2 - 2\Gamma\delta) w_j^2 + \frac{\Gamma \pi^4}{16} \left(\sum_{j=1}^{N} j^2 w_j^2 \right)^2 + \tau \sum_{j=1}^{N} w_j p_j ,$$

to get

$$V_{k\pm} = - \frac{\Gamma \pi^4 Q^4_{k\pm}}{16} , \tag{3.16}$$

which is the same as (3.15). Clearly, $V_{1_+} < V_{1_-} < V_{2_+} \cdots < 0$. That the potential energies are still ordered in the same way with $\tau \neq 0$ as with $\tau = 0$ is no surprise, but it does give credence to our approach. If we look at a plot of the level curves of potential energy with two nonzero modes (Figure 9) we find that the surfaces bear an interesting resemblance to Figure 8 where $\tau = 0$.

We would like to find the type of a given critical point for $\tau \neq 0$ and δ fixed. To do so we will let $\tau \to 0$ keeping δ fixed and show that the critical point has not changed its type. Notice that this is a well-defined proposal, since if a critical point $\underset{\sim}{\phi}_k(\tau)$ exists for a given τ it will also exist for all τ with smaller absolute value. This process is not well defined if we wish to increase $|\tau|$, keeping δ fixed, since the branching point $\delta^* = \delta(\tau)$ is an increasing function of $|\tau|$ (see Figure 7). To accomplish our goal we must examine the eigenvalues of the linearized matrix $D_{k\pm}$ as in (3.12) when $\tau \neq 0$. The linearization is easily accomplished by differentiating the right hand side of (3.10) with respect to ϕ_i, and evaluating the resulting expression at $\underset{\sim}{\phi}_{k\pm}$. Doing so gives the matrix $D = (d_{ij})$ where

$$d_{ij}^{k\pm} = j^2\pi^2\left(j^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2}{4} Q_{k\pm}^2\right)\delta_{ij} + \frac{\Gamma\pi^4}{2} i^2 j^2 w_j^{k\pm} w_i^{k\pm} \tag{3.17}$$

The matrix D is a symmetric matrix of the form

$$D = \begin{pmatrix} A_1 + a_1^2 & a_1 a_2 & \cdots & a_1 a_N \\ a_2 a_1 & A_2 + a_2^2 & \cdots & a_2 a_N \\ \vdots & & & \\ a_N a_1 & a_N a_2 & & A_N + a_N^2 \end{pmatrix}$$

It can be shown inductively that the determinant of a matrix of the form of D is given by

$$(3.18)\qquad |D| = \prod_{j=1}^{N} A_j + \sum_{j=1}^{N} \Big(\prod_{\substack{m=1 \\ m\neq j}}^{N} A_m \Big) \, a_j \, .$$

Since

$$|D| = \prod_{j=1}^{N} \mu_j \, ,$$

where μ_j are the eigenvalues of D, if $|D(\tau)| \neq 0$ for all τ, then, since the eigenvalues of D are real, we know that none of the eigenvalues undergo a change of sign, and that the type of the critical point is preserved for all τ.

Substituting (3.17) into (3.18) gives

$$(3.19)\qquad |D_{k\pm}| = \mathcal{P}_{k\pm} \left[1 + \frac{32\Gamma}{\pi^4} \tau^2 \sum_{j=1}^{N} \frac{p_j^2}{j^2 (4j^2\pi^2 - 8\Gamma\delta + \Gamma\pi^2 Q_{k\pm}^2)^3} \right]$$

where

$$\mathcal{P}_{k\pm} = \prod_{j=1}^{N} j^2 \pi^2 (j^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2}{4} Q_{k\pm}^2)$$

If $p_k \neq 0$, then $\mathcal{P}_{k\pm} \neq 0$. Recalling (2.22), we see that $|D_{k\pm}| = 0$ only if the root $Q_{k\pm}^2$ is a double root of (2.21), or, in other words, if we are at a nonisolated solution. Since by decreasing $|\tau|$ we avoid nonisolated solutions for δ fixed, $|D_{k\pm}(\tau)| \neq 0$ and the type of $\underset{\sim}{\phi}_{k\pm}$ is preserved. If $p_k = 0$, then $Q_{k\pm}^2$ satisfies $k^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2}{4} Q_{k\pm}^2 = 0$

and (3.18) becomes

$$(3.20) \qquad |D_{k\pm}| = \left[\prod_{\substack{j=1 \\ j\neq k}}^{N} j^2\pi^2 (j^2\pi^2 - 2\Gamma\delta + \frac{\Gamma\pi^2}{4} Q^2_{k\pm}) \right] \frac{\Gamma\pi^4 k^4}{2} (w_k^{k\pm})^2$$

which can vanish only when $w_k^{k\pm} = 0$. Once again by (2.20). this can only happen at a branching point which we avoid by requiring $|\tau|$ to decrease.

This calculation also shows us that $|\tau|$ can be allowed to increase without changing the type of the critical point until two critical points merge. At the merger, we know the type of singular point which results. Since Q_{k+} can merge only with Q_{k+1^-}, the resulting point must have k-1 negative eigenvalues, 2N-k positive eigenvalues and one mode with zero eigenvalue, giving "neutral stability" in 2N-dimensional phase space.

We can summarize the global behavior as follows. The two critical points with lowest potential energy are globally stable in that, except for initial data lying on the separatrix, all motion approaches one of these two points as $t \to \infty$. All higher modes have directions in 2N-dimensional phase space that are unstable. Furthermore, if the initial data have initial energy $K(0) < V_i$, then all critical points with $V \geq V_i$ are excluded as possible equilibrium states, provided p(x,t) does not depend on t. If $K(0) < V_{2+}$, then the motion is in the potential well of ϕ_{1+} or ϕ_{1-} for all time, and can approach but one critical point. This is not necessarily true, however, if $p(x,t)$ depends on t, since the load $p(x,t)$ may feed sufficient energy into the system to allow it to reach another critical point.

Chapter IV

Buckling of an Imperfect Column on a Nonlinearly Elastic Foundation

Imperfection Sensitivity and Postbuckling theory have been the subject of extensive literature in recent years [4], [15]. To show that the ideas discussed in Chapter II are applicable to problems of engineering interest, consider a thin uniform column with certain imperfections, resting on a nonlinear elastic foundation, subjected to axial loads. We want to find asymptotic expansions for the buckling load as a function of the imperfection amplitude. A similar problem for an infinitely long column has been treated by Amazigo, Budiansky and Carrier [3], where deterministic and random imperfections were studied and the expansions derived were conjectured to be asymptotic. The more general results given here for deterministic imperfections reduce to results of [3] when specialized to their problem, even though we consider a column of finite length. The method used here includes a proof of the asymptotic behavior of the buckling load as a function of the imperfection amplitude.

We first consider a generalization of the problem for the buckling of a column. This is the boundary value problem

$$
\text{(1.1) a)} \quad \begin{cases} \dfrac{d^4 w}{dx^4} + g(\lambda, \tau, \dfrac{d^2 w}{dx^2}) + f(w) = 0 & x \in (0, \pi) \\ w(0) = w(\pi) = 0 \\ \dfrac{d^2 w}{dx^2}(0) = \dfrac{d^2 w}{dx^2}(\pi) = 0 \end{cases}
$$

where

$$
\text{(1.1) b)} \quad \begin{cases} g(\lambda, 0, 0) = 0 \quad \text{for all } \lambda \\ \dfrac{\partial g}{\partial y}(\lambda, \tau, y) \Big|_{y = \tau = 0} = G(\lambda) \\ f(0) = 0 \\ \dfrac{\partial f}{\partial y}(0) = F = \text{constant} \end{cases}
$$

The functions $g(\lambda, \tau, y)$ and $f(y)$ are also allowed to depend on x for $x \in [0, \pi]$, are at least three times continuously differentiable in λ, τ, and y, and are continuous in x. The numbers F and $G(\lambda)$ do not depend on x. Clearly $(\lambda, \tau, w) = (\lambda, 0, 0)$ is a solution of (1.1a) for all λ as a consequence of (1.1b).

A solution (λ, τ, w) of (1.1) will be nonisolated if there are nontrivial solutions to the linearized problem:

$$(1.2)\quad \begin{cases} \dfrac{d^4\psi}{dx^4} + g_y(\lambda, \tau, \dfrac{d^2 w}{dx^2})\dfrac{d^2\psi}{dx^2} + f_y(w)\psi = 0\ , \\ \psi(0) = \psi(\pi) = 0\ , \\ \dfrac{d^2\psi}{dx^2}(0) = \dfrac{d^2\psi}{dx^2}(\pi) = 0\ . \end{cases}$$

The trivial solution $(\lambda, \tau, w) = (\lambda, 0, 0)$ of (1.1) will be a nonisolated solution whenever λ is such that

$$(1.3)\quad \begin{cases} \dfrac{d^4\phi}{dx^4} + G(\lambda)\dfrac{d^2\phi}{dx^2} + \mathbf{F}\phi = 0 \\ \phi(0) = \phi(\pi) = 0 \\ \dfrac{d^2\phi}{dx^2}(0) = \dfrac{d^2\phi}{dx^2}(\pi) = 0 \end{cases}$$

has nontrivial solutions. These yield the bifurcation points for (1.1) with $\tau = 0$. Nontrivial solution pairs of (1.3) are (ϕ_n, λ_n) where

$$(1.4)\quad \begin{cases} \phi_n(x) = \mathbf{A}\sin nx \qquad \mathbf{A}^2 = \dfrac{2}{\pi} \\ G(\lambda_n) = n^2 + \dfrac{F}{n^2} \qquad n = 1, 2, \cdots, \end{cases}$$

provided such values of λ_n exist. If (1.1) describes a column, the parameter λ corresponds to the axial loading, and the "buckling load" will be the smallest bifurcation value λ_n for which (1.4) is valid. To carry out the required analysis, it is not necessary to know any details about $G(\lambda)$ except that $\lambda_i \neq \lambda_j$ for $i \neq j$. Additional information is used only when we are specifically interested in finding the buckling load for a given imperfection.

The procedure is by now clear. We seek solutions of (1.1) -(1.2) in the form

$$(1.5) \quad \begin{cases} w(x,\epsilon) = \epsilon\phi_0(x) + \epsilon^2 v(x,\epsilon) , \\ \psi(x,\epsilon) = \phi_0(x) + \epsilon\,\chi(x,\epsilon) , \\ \lambda(\epsilon) = \lambda_0 + \epsilon\,\mu(\epsilon) , \\ \tau(\epsilon) = \epsilon^2 \eta(\epsilon) , \end{cases}$$

where

$$(1.6) \quad \begin{cases} \int_0^{\pi} v(x,\epsilon)\,\phi_0(x)dx = 0 , \\ \int_0^{\pi} \chi(x,\epsilon)\,\phi_0(x)\,dx = 0 . \end{cases}$$

Here $\lambda_0 = \lambda_n$, $\phi_0(x) = \phi_n(x)$ are a solution of (1.4) for some n. Substituting (1.5) into (1.1), (1.2) gives equations for $v(x,\epsilon)$ and $\chi(x,\epsilon)$ of the form

$$(1.7)\quad \begin{cases} \dfrac{d^4 v}{dx^4} + (n^2+\dfrac{F}{n^2})\dfrac{d^2 v}{dx^2} + Fv = -\dfrac{1}{\epsilon^2}\Big[g(\lambda,\tau,\dfrac{d^2 w}{dx^2}) - (n^2+\dfrac{F}{n^2})\dfrac{d^2 w}{dx^2} \\ \qquad\qquad + f(w) - Fw\Big], \\ v(0) = v(\pi) = 0, \\ \dfrac{d^2 v}{dx^2}(0) = \dfrac{d^2 v}{dx^2}(\pi) = 0, \end{cases}$$

$$(1.8)\quad \begin{cases} \dfrac{d^4 \chi}{dx^4} + (n^2+\dfrac{F}{n^2})\dfrac{d^2 \chi}{dx^2} + F\chi = -\dfrac{1}{\epsilon}\Big[\Big(g_y(\lambda,\tau,\dfrac{d^2 w}{dx^2}) - (n^2+\dfrac{F}{n^2})\Big)\dfrac{d^2 \psi}{dx^2} \\ \qquad\qquad + (f_y(w) - F)\psi\Big], \\ \chi(0) = \chi(\pi) = 0, \\ \dfrac{d^2 \chi}{dx^2}(0) = \dfrac{d^2 \chi}{dx^2}(\pi) = 0. \end{cases}$$

Of course (1.6) is also required to hold. Once again the Fredholm alternative theorem is used to imply that (1.7) and (1.8) have solutions only if

$$(1.9)\quad \int_0^\pi \Big(g(\lambda,\tau,\frac{d^2 w}{dx^2}) - (n^2+\frac{F}{n^2})\frac{d^2 w}{dx^2} + f(w) - Fw\Big)\phi_n(x)\,dx = 0$$

$$(1.10)\quad \int_0^\pi \Big[\Big(g_y(\lambda,\tau,\frac{d^2 w}{dx^2}) - (n^2+\frac{F}{n^2})\Big)\frac{d^2 \psi}{dx^2} + (f_y(w)-F)\psi\Big]\phi_n(x)\,dx = 0$$

Immediately we notice that the natural iteration scheme will involve solving a boundary value problem of the form

$$(1.11)\quad \begin{cases} \dfrac{d^4 y}{dx^4} + \left(n^2 + \dfrac{F}{n^2}\right)\dfrac{d^2 y}{dx^2} + F\,y = h(x)\ , \\[2ex] y(0) = y(\pi) = 0\ , \\[2ex] \dfrac{d^2 y}{dx^2}(0) = \dfrac{d^2 y}{dx^2}(\pi) = 0\ , \\[2ex] \displaystyle\int_0^\pi y(x)\sin nx\,dx = 0\ . \end{cases}$$

Since (1.11) involves a fourth order differential operator, the mechanics of solving (1.11) are not the same as in Chapter II. However, use of a generalized Green's function $\mathcal{G}(x,\zeta)$ [6] enables us to carry out the analysis necessary in this problem. The generalized Green's function appropriate for solving (1.11) satisfies

$$(1.12)\quad \begin{cases} \dfrac{d^4 \mathcal{G}}{dx^4} + \left(n^2 + \dfrac{F}{n^2}\right)\dfrac{d^2 \mathcal{G}}{dx^2} + F\,\mathcal{G} = -\dfrac{2}{\pi}\sin nx \sin n\zeta\ , \quad x \neq \zeta \\[2ex] \mathcal{G}(0,\zeta) = \mathcal{G}(\pi,\zeta) = 0\ , \\[2ex] \dfrac{d^2\mathcal{G}}{dx^2}(0,\zeta) = \dfrac{d^2\mathcal{G}}{dx^2}(\pi,\zeta) = 0\ , \\[2ex] \dfrac{d^3\mathcal{G}}{dx^3}(\zeta^+,\zeta) - \dfrac{d^3\mathcal{G}}{dx^3}(\zeta^-,\zeta) = 1\ , \\[2ex] \displaystyle\int_0^\pi \mathcal{G}(x,\zeta)\sin nx\,dx = 0\ , \end{cases}$$

where $\dfrac{d^3\mathcal{G}}{dx^3}(\zeta^+,\zeta) = \lim\limits_{\substack{\eta\to\zeta \\ \eta>\zeta}} \dfrac{d^3\mathcal{G}}{dx^3}(\eta,\zeta)$ and $\dfrac{d^3\mathcal{G}}{dx^3}(\zeta^-,\zeta) = \lim\limits_{\substack{\eta\to\zeta \\ \eta<\zeta}} \dfrac{d^3\mathcal{G}}{dx^3}(\eta,\zeta)$.

Knowing the Green's function, the solution of (1.11) can be written as

$$(1.13)\qquad y(x) = \int_0^{\pi} \mathcal{G}(x,\zeta)\, h(\zeta)d\zeta \ ,$$

and $y(x)\in C^4[0,\pi]$ provided $h(x)\in C[0,\pi]$. Furthermore, bounds for $y(x)$ and its first two derivatives follow easily from (1.13) as

$$(1.14)\qquad \begin{cases} \|y\|_\infty \leq \mathcal{G}_0 \|h\|_\infty , \\ \|\frac{dy}{dx}\|_\infty \leq \mathcal{G}_1 \|h\|_\infty , \\ \|\frac{d^2 y}{dx^2}\|_\infty \leq \mathcal{G}_2 \|h\|_\infty , \end{cases}$$

where

$$\mathcal{G}_i = \max_{x\in[0,\pi]} \int_0^{\pi} \left| \frac{d^i \mathcal{G}(x,\zeta)}{dx^i} \right| d\zeta , \quad i = 0,1,2.$$

In the case that n=1, F=1, the Green's function takes the form,

$$(1.15)\qquad \mathcal{G}(x,\zeta)=\begin{cases} \frac{1}{4\pi}\left(x^2+(\zeta-\pi)^2-\frac{1}{3}\pi^2-\frac{3}{2}\right)\sin\zeta\sin x + \frac{x}{2\pi}\sin\zeta\cos x \\ \qquad -(\frac{\zeta-\pi}{2\pi})x\cos\zeta\cos x + (\frac{\zeta-\pi}{2\pi})\cos\zeta\sin x , \quad 0 \leq x < \zeta , \\ \frac{1}{4\pi}\left((x-\pi)^2+\zeta^2-\frac{1}{3}\pi^2-\frac{3}{2}\right)\sin\zeta\sin x + \frac{\zeta}{2\pi}\cos\zeta\sin x \\ \qquad -\frac{\zeta}{2\pi}(x-\pi)\cos\zeta\cos x + (\frac{x-\pi}{2\pi})\sin\zeta\cos x, \quad \zeta < x \leq \pi . \end{cases}$$

The Green's function can be easily calculated for other cases, but the specific form is not important in our discussion.

The proof that (1.5) is a valid form of the solution of (1.1) is only slightly different from the proofs given in Chapter II. The differences arise from the fact that (1.1) is a fourth order equation with nonlinearity involving second derivatives, and are resolved by the existence of a Green's function. To formulate the contraction mapping we introduce the set of functions

$$(1.16) \qquad \mathcal{B}_K = \{ y(x) \mid y(x) \in C^4[0,\pi] \ , \ \|y\| \leq K \ , \ \int_0^\pi y(x) \sin nx \, dx = 0 \}$$

where

$$\|y\| = \max\{ \|y\|_\infty, \ \|\frac{dy}{dx}\|_\infty, \ \|\frac{d^2 y}{dx^2}\|_\infty \}$$

and the real interval

$$(1.17) \qquad \mathcal{I}_K = \{ \eta \mid |\eta| \leq K \} .$$

Using the identity

$$(1.18) \qquad y(b) - y(a) = (b-a) \int_0^1 \frac{dy}{dx}(sb + (1-s)a) \, ds ,$$

we can manipulate (1.7)-(1.10) into a form in which a mapping T_ϵ is naturally suggested. The mapping which results is

$$T_\epsilon(v, \chi, \mu, \eta) = (\tilde{v}, \tilde{\chi}, \tilde{\mu}, \tilde{\eta})$$

where

$$(1.19)\begin{cases} \tilde{\mu}\int_0^{\pi}\left(\int_0^1 g_{\lambda y}(\lambda_0+s\epsilon\mu,0,0)ds\right)\frac{d^2\psi}{dx^2}\sin nx\,dx \\ \quad = -\int_0^{\pi}\left[(-n^2\sin nx+\epsilon\frac{d^2v}{dx^2})\left(\int_0^1 g_{yy}(\lambda,0,s\frac{d^2w}{dx^2})ds\right)\frac{d^2\psi}{dx^2}\right. \\ \qquad + \epsilon\eta\left(\int_0^1 g_{\tau y}(\lambda,s\tau,\frac{d^2w}{dx^2})ds\right)\frac{d^2\psi}{dx^2} \\ \qquad \left. + \left(\int_0^1 f_{yy}(sw)ds\right)\left(\sin nx+\epsilon v\right)\psi(x)\right]\sin nx\,dx \\ \quad = \int_0^{\pi}\mathcal{R}_1(v,\chi,\mu,\eta;x)\sin nx\,dx\ , \end{cases}$$

$$(1.20)\begin{cases} \tilde{\eta}\int_0^{\pi}\left(\int_0^1 g_{\tau}(\lambda,\frac{d^2w}{dx^2},s\tau)ds\right)\sin nx\,dx \\ \quad = \tilde{\mu}\int_0^{\pi}(n^2\sin nx-\epsilon\frac{d^2v}{dx^2})\left(\int_0^1 g_{\lambda y}(\lambda_0+s\epsilon\mu,0,0)ds\right)\sin nx\,dx \\ \qquad - \int_0^{\pi}\left[(n^2\sin nx-\epsilon\frac{d^2v}{dx^2})^2\int_0^1\int_0^1 g_{yy}(\lambda,0,st\frac{d^2w}{dx^2})s\,dt\,ds\right. \\ \qquad\qquad \left. + (\sin nx+\epsilon v)^2\int_0^1\int_0^1 f_{yy}(stw)s\,dt\,ds\right]\sin nx\,dx \\ \quad = \int_0^{\pi}\mathcal{R}_2(v,\chi,\mu,\eta,\tilde{\mu};x)\sin nx\,dx\ , \end{cases}$$

$$(1.21)\qquad \tilde{v}(x,\epsilon) = \int_0^{\pi}\mathcal{G}(x,\zeta)\left[-\tilde{\eta}\int_0^1 g_{\tau}(\lambda,s\tau,\frac{d^2w}{dx^2})ds+\mathcal{R}_2(v,\chi,\mu,\eta,\tilde{\mu};\zeta)\right]d\zeta$$

and

$$(1.22)\qquad \tilde{\chi}(x,\epsilon) = \int_0^{\pi} \mathcal{G}(x,\zeta)\left[-\tilde{\mu}\,\frac{d^2\psi}{dx^2}\int_0^1 g_{\lambda y}(\lambda_0+\epsilon s\mu,0,0)\,ds + \mathcal{R}_1(v,\chi,\mu,\eta;\zeta)\right]d\zeta .$$

Of course this mapping generates a sequence $\{u^n\}$ by

$$u^{n+1} = \left(v^{n+1},\ \chi^{n+1},\ \mu^{n+1},\ \eta^{n+1}\right) = T_{\epsilon}\left(v^n, \chi^n, \mu^n, \eta^n\right)$$

where $u^0 \epsilon\, U_K = \mathcal{B}_K \times \mathcal{B}_K \times \mathcal{I}_K \times \mathcal{I}_K$ for some $K > 0$.

We now see why it is necessary to include $\|\frac{d^2 y}{dx^2}\|_{\infty}$ in the definition of $\mathcal{B}_K$ (1.16). If T_{ϵ} is to map U_K into U_K for some $K > 0$, we must have estimates of $\frac{d^2 v}{dx^2}$ and $\frac{d^2\chi}{dx^2}$ in order to estimate certain parts of (1.19)-(1.22). The estimates (1.14) are necessary to guarantee that $\tilde{v}(x,\epsilon)$ and $\tilde{\chi}(x,\epsilon)$ are in $\mathcal{B}_K$, for some $K > 0$.

The details of finding $K > 0$ and $\epsilon_0 > 0$ so that T_{ϵ} is a contraction mapping of U_K into U_K for $0 \leqslant |\epsilon| \leqslant \epsilon_0$ are the same as in the previous chapters and will not be included here. To complete the proof that (1.5) is a nonisolated solution of (1.1), we need to justify taking the limit of the sequence $\{u^n\}$ as $n \to \infty$.

By induction it is clear that $u^n \epsilon\, \mathcal{B}_K$ for all $n \geqslant 0$. Furthermore the real sequences $\{\mu^n\}$ and $\{\eta^n\}$ converge, and the sequences $\{u^n\}$ and $\{\chi^n\}$ converge in $C^2[0,1]$. In terms of the sequence $\{u^n\}$ we can rewrite (1.21) and (1.22) in the form

$$(1.23)\qquad v^{n+1}(x,\epsilon) = \int_0^{\pi} \mathcal{G}(x,\zeta)\left[-\eta^{n+1}\int_0^1 g_{\tau}(\lambda^n, s\tau^n, \frac{d^2 w^n}{dx^2})\,ds + \mathcal{R}_2(v^n,\chi^n,\mu^n,\eta^n,\mu^{n+1};\zeta)\right]d\zeta$$

and

$$(1.24)\quad \chi^{n+1}(x,\epsilon) = \int_0^{\pi} \mathcal{G}(x,\zeta)\left[-\mu^{n+1}\frac{d^2\psi^n}{dx^2}\int_0^1 g_{\lambda y}(\lambda_0+s\epsilon\mu^n,0,0)ds + \mathcal{R}_1(v^n,\chi^n,\mu^n,\eta^n;\zeta)\right]d\zeta .$$

Both the left and right hand sides of (1. 23) and (1. 24) are uniformly bounded for all n. Therefore, the Lebesque dominated convergence theorem justifies taking the limit as $n \to \infty$.

Now that we know that (1. 7)-(1. 10) have solutions which are uniformly bounded for $|\epsilon| \leqslant \epsilon_0$, we can estimate $\mu(\epsilon)$ and $\eta(\epsilon)$ in (1.9) and (1. 10), and know that the estimates are asymptotically valid for $\epsilon \to 0$. Assume that there are integers p and q such that

$$(1.25)\quad \begin{cases} \dfrac{\partial^k g}{\partial y^k}(\lambda,\tau,y)\Big|_{\substack{\tau=y=0\\ \lambda=\lambda_n}} = 0 & 2 \leqslant k \leqslant p-1 , \\[2ex] \dfrac{\partial^k f}{\partial y^k}(y)\Big|_{y=0} = 0 & 2 \leqslant k \leqslant q-1 , \end{cases}$$

and that

$$(1.26)\quad \begin{cases} \displaystyle\int_0^{\pi} \frac{\partial^p q}{\partial y^p}(\lambda_n,0,0)\,(\sin nx)^{p+1}\,dx \neq 0 \\[2ex] \displaystyle\int_0^{\pi} \frac{\partial^q f}{\partial y^q}(0)(\sin nx)^{q+1}\,dx \neq 0 . \end{cases}$$

Then, keeping only the lowest order terms in (1.19), (1. 20), we have

$$(1.27)\quad \begin{cases} \mu(\epsilon)\Big(n^2 G_\lambda(\lambda_n)+O(\epsilon)\Big) = \dfrac{(-n^2)^p \epsilon^{p-2}}{(p-1)!}\displaystyle\int_0^\pi \dfrac{\partial^p q}{\partial y^p}(\lambda_n,0,0)(A\sin nx)^{p+1}\,dx \\ \qquad + \dfrac{\epsilon^{q-2}}{(q-1)!}\displaystyle\int_0^\pi \dfrac{\partial^q f}{\partial y^q}(0)(A\sin nx)^{q+1}\,dx + O(\epsilon^{p-1})+O(\epsilon^{q-1}) \end{cases}$$

and

$$(1.28)\quad \begin{cases} \eta(\epsilon)\Big(A\displaystyle\int_0^\pi \dfrac{\partial g}{\partial \tau}(\lambda_n,0,0)\sin nx\,dx+O(\epsilon)\Big) - \mu(\epsilon)\Big(n^2 G_\lambda(\lambda_n) + O(\epsilon)\Big) \\ \qquad = -\dfrac{(-n^2)^p \epsilon^{p-2}}{p!}\displaystyle\int_0^\pi \dfrac{\partial^p g}{\partial y^p}(\lambda_n,0,0)(A\sin nx)^{p+1}\,dx \\ \qquad - \dfrac{\epsilon^{q-2}}{q!}\displaystyle\int_0^\pi \dfrac{\partial^q f}{\partial y^q}(0)(A\sin nx)^{q+1}\,dx+O(\epsilon^{p-1})+O(\epsilon^{q-1}) . \end{cases}$$

We can eliminate $\mu(\epsilon)$ from (1. 28) to get

$$(1.29)\quad \begin{cases} \eta(\epsilon)\Big(A\displaystyle\int_0^\pi \dfrac{\partial g}{\partial \tau}(\lambda_n,0,0)\sin nx\,dx + O(\epsilon)\Big) = \\ \dfrac{(-n^2)^p \epsilon^{p-2}(p-1)}{p!}\displaystyle\int_0^\pi \dfrac{\partial^p g}{\partial y^p}(\lambda_n,0,0)(A\sin nx)^{p+1}\,dx + \dfrac{q-1}{q!}\displaystyle\int_0^\pi \dfrac{\partial^q f}{\partial y^q}(0)(A\sin nx)^{q+1}\,dx \\ \qquad + O(\epsilon^{p-1})+O(\epsilon^{q-1}) . \end{cases}$$

If we substitute (1. 27) and (1. 29) into (1. 5) we now know that the resulting approximations of $\lambda(\epsilon)$ and $\tau(\epsilon)$ are asymptotic to the exact solution as $\epsilon \to 0$. We can use these approximations to find $\lambda = \lambda(\tau)$ approximately.

If we were to continue further without making additional assumptions about p and q, we would be forced to consider the three cases $p < q$, $p = q$ and $p > q$. Rather than maintaining this full generality, we will examine the specific problem treated by Amazigo, Budiansky and Carrier [3]:

$$\frac{d^4 w}{dx^4} + 2\lambda \frac{d^2 w}{dx^2} + 2\lambda\tau \frac{d^2 w_0}{dx^2} + w - w^3 = 0 \tag{1.30}$$

where $w_0(x)$ is the shape of the imperfection. For this example,

$$\begin{cases} g(\lambda, \tau, y) = 2\lambda y + 2\lambda\tau \dfrac{d^2 w_0}{dx^2} , \\ \\ f(w) = w - w^3 . \end{cases} \tag{1.31}$$

Using (1.31), it is easy to see that $q = 3$ and $p = \infty$ and that (1.27) and (1.29) reduce to

$$\begin{cases} \mu(\epsilon) = \dfrac{9}{4n^2 \pi} \epsilon + O(\epsilon^2) \\ \\ \eta(\epsilon) = \dfrac{-3\epsilon}{2\sqrt{2\pi}\, \lambda_n \displaystyle\int_0^\pi \dfrac{d^2 w_0}{dx^2} \sin nx \, dx} + O(\epsilon^2) \end{cases} \tag{1.32}$$

According to (1.4), $\lambda_n = \frac{1}{2}(n^2 + \frac{1}{n^2})$, so that (1.32) combined with (1.5) imply that

$$(1.33)\quad \begin{cases} \lambda(\epsilon) = \frac{1}{2}(n^2+\frac{1}{n^2}) - \frac{9}{4n^2\pi}\epsilon^2 + O(\epsilon^3) \\ \tau(\epsilon) = -\epsilon^3 \dfrac{3n^2}{\sqrt{2\pi}\,(n^4+1)} \dfrac{1}{\int_0^\pi \frac{d^2w_0}{dx^2}\sin nx\,dx} + O(\epsilon^4) \end{cases}$$

The integral $\int_0^\pi \frac{d^2 w_0}{dx^2}\sin nx\,dx$ is proportional to the nth Fourier component of $\frac{d^2w_0}{dx^2}$. If

$$w_0(x) = \sum_{k=1}^{\infty} w_k \sin kx\ ,$$

then $\int_0^\pi \frac{d^2 w_0}{dx^2}\sin nx\,dx = -\frac{n^2\pi}{2} w_n$. Using this information in (1.33) we can solve for λ as a function of τ, finding that

$$(1.34)\quad \lambda(\tau) = \frac{1}{2}(n^2+\frac{1}{n^2}) - \left(\frac{9}{8\sqrt{2}\,n}(n^2+\frac{1}{n^2})\,\tau w_n\right)^{\frac{2}{3}} + O(\tau)$$

For $n=1$, this is exactly the relation found in [3]. However now we also know that this solution is asymptotic to the exact solution as $\tau \to 0$.

Since the generalized Green's function $\mathcal{G}(x,\zeta)$ exists and can be calculated for this problem, one could find additional terms of the expansion (1.34), by calculating one or more of the iterates generated by T_ϵ. After some straightforward calculations, one finds

$$
(1.35)\begin{cases}
w(x,\epsilon) = \epsilon\sqrt{\frac{2}{\pi}}\ \sin nx + \epsilon^3\left[\frac{6}{\pi\sqrt{2\pi}}\ \frac{1}{n^2 w_n}\sum_{k\neq n} w_k \frac{k^2}{(k^2-n^2)(k^2n^2-1)}\sin kx \right. \\
\qquad\qquad \left. -\ \frac{1}{8\pi\sqrt{2\pi}}\ \frac{1}{n^2(9n^4-1)}\ \sin 3nx\right] + O(\epsilon^5)\ , \\
\phi(x,\epsilon) = \sqrt{\frac{2}{\pi}}\ \sin nx - \frac{3\epsilon^2}{\pi\sqrt{2\pi}}\ \frac{1}{8n^2(9n^4-1)}\ \sin 3nx + O(\epsilon^4)\ , \\
\lambda(\epsilon) = (n^2+\frac{1}{n_2}) - \frac{9\epsilon^2}{4n^2\pi} + \frac{3\epsilon^4}{32}\ \frac{1}{\pi^2 n^4(9n^4-1)}\left(54\frac{w_{3n}}{w_n} - \frac{5}{2}\right) + O(\epsilon^6)\ , \\
\tau(\epsilon) = \frac{1}{\pi\sqrt{2\pi}\ (n^4+1)w_n}\left[6\epsilon^3 + \epsilon^5\left(\frac{27}{\pi(n^4+1)} + \frac{3}{8\pi}\ \frac{1}{n^2(9n^4-1)}\ (2-27\frac{w_{3n}}{w_n})\right)\right] \\
\qquad\qquad + O(\epsilon^7)\ ,
\end{cases}
$$

provided $w_n \neq 0$, where w_k is the kth Fourier coefficient of the imperfection $w_0(x)$. This can now be used to find $\lambda = \lambda(\tau)$, and gives

$$
(1.36) \qquad \lambda(\tau) = \tfrac{1}{2}(n^2+\frac{1}{n^2}) - \frac{9}{4n^2\pi}\left(\frac{\tau}{\tau_0}\right)^{\frac{2}{3}}
$$

$$
+ \frac{1}{4n^2\pi^2}\left(\frac{3}{16n^2(9n^4-1)}(54\frac{w_{3n}}{w_n}-1) + \frac{27}{n^4+1}\right)\left(\frac{\tau}{\tau_0}\right)^{\frac{4}{3}} + O\left(\frac{\tau}{\tau_0}\right)^2
$$

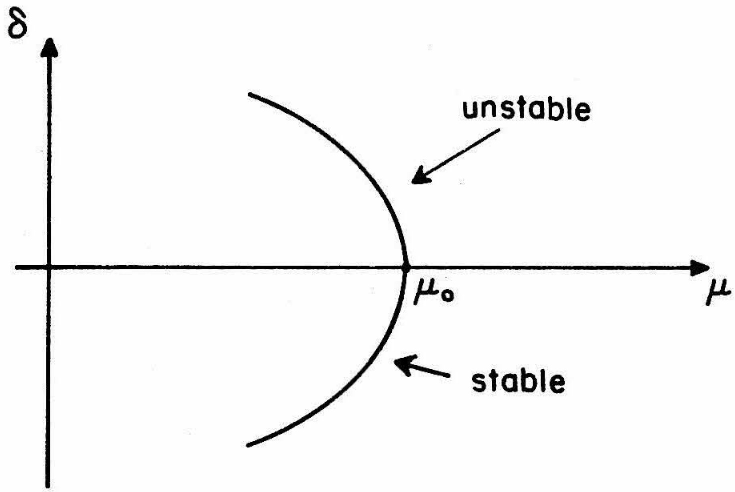
where

$$
\tau_0 = \frac{6}{\pi\sqrt{2\pi}\ (n^4+1)}\ \frac{1}{w_n}\ .
$$

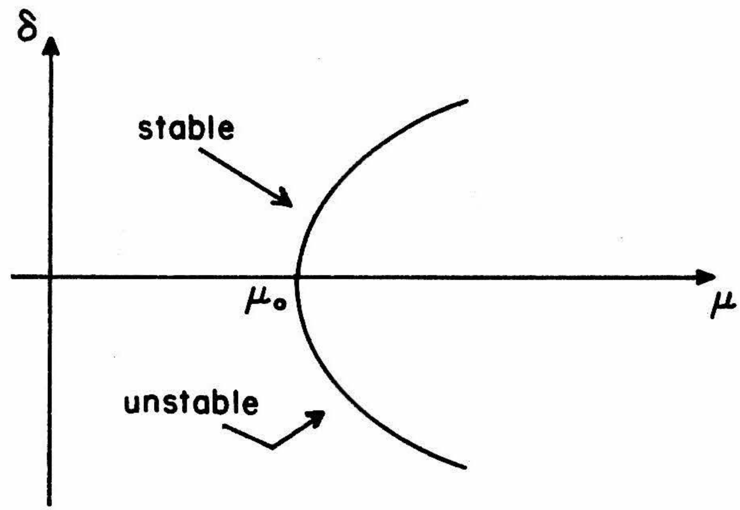
As a final comment, it is clear that one could treat more general equations of the form

$$a(x)\,\frac{d^4 w}{dx^2} + b(x)\,\frac{d^3 w}{dx^3} + g(\lambda, \tau, w, \frac{dw}{dx}, \frac{d^2 w}{dx^2}) = 0 \;. \tag{1.35}$$

where $g(\lambda, 0, 0, 0, 0) = 0$ and $g(\lambda, \tau, 0, 0, 0) \neq 0$ if $\tau \neq 0$, whenever the appropriate generalized Green's function exists. The proof of the existence of a family of nonisolated solutions would be unchanged, since the nonlinearity involves at most the second derivative of w, which is easily estimated in (1.14).

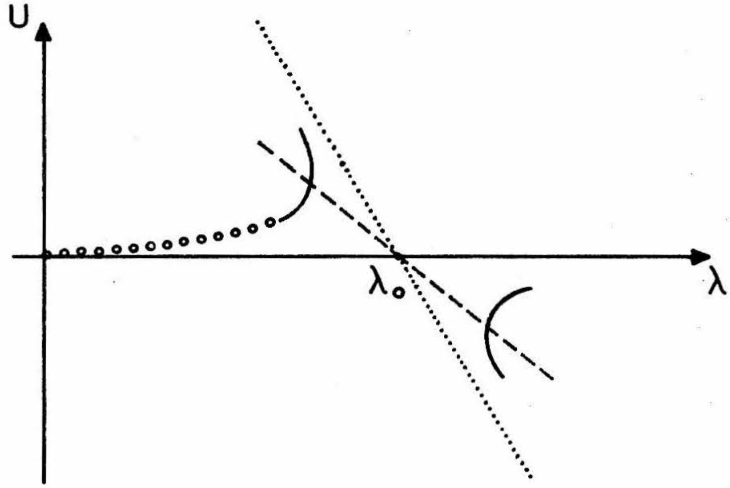


Case (i): $\langle g_{uu}(\mu_o, w_o)\psi_o^2, \psi_o^* \rangle > 0$

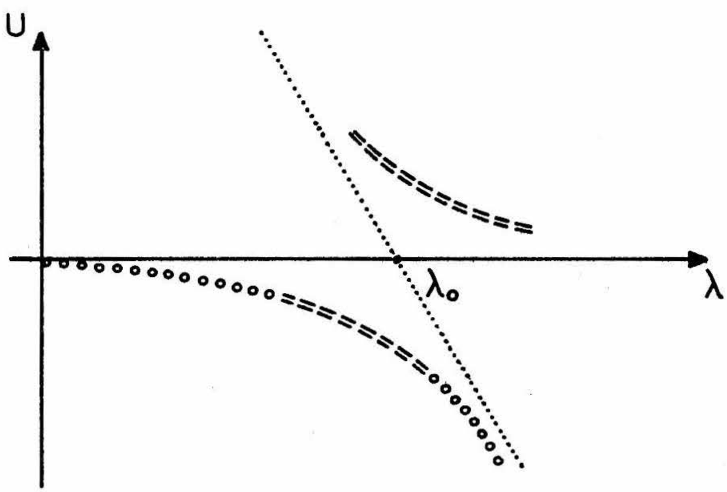


Case (ii): $\langle g_{uu}(\mu_o, w_o)\psi_o^2, \psi_o^* \rangle < 0$

FIG. 1. Plot of $\mu(\delta)$ in (II.5.13) for $\langle g_\lambda(\mu_o, w_o), \psi_o^* \rangle > 0$, with stability indicated when $\langle \psi_o, \psi_o^* \rangle > 0$ and when $\gamma = 0$ is the principal eigenvalue of (II.6.3).

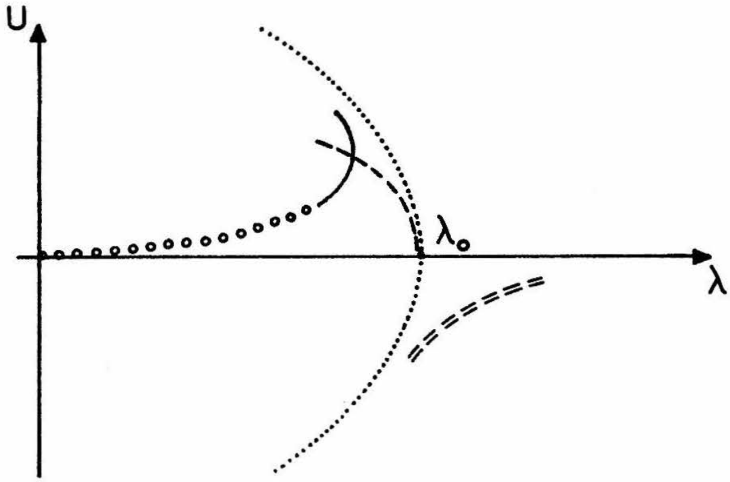


Case (i): sgn(τ) = sgn(T)

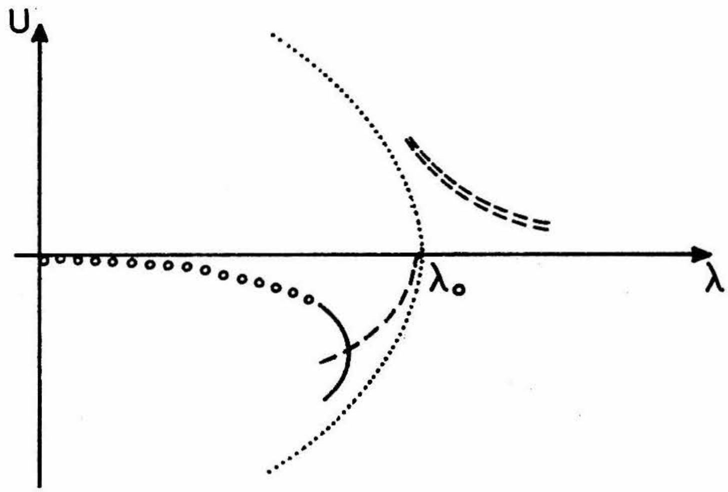


Case (ii): sgn(τ) = − sgn(T)

FIG. 2. Solution branches of (II.1.1) for τ sufficiently small when $\langle f_{uu}(\lambda_o,0,0)\phi_o^2,\phi_o^*\rangle \cdot \langle f_{\lambda u}(\lambda_o,0,0)\phi_o,\phi_o^*\rangle > 0$.
$T = \langle f_{uu}(\lambda_o,0,0)\phi_o^2,\phi_o^*\rangle \cdot \langle f_\tau(\lambda_o,0,0),\phi_o^*\rangle$.

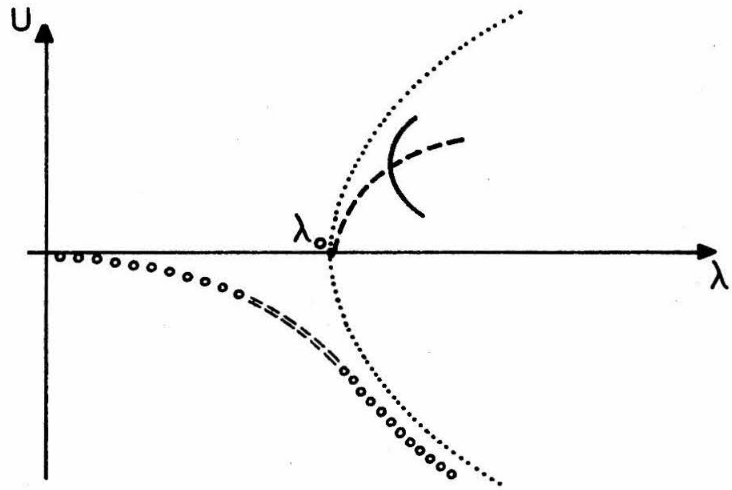


Case(i): $\text{sgn}(\tau) = \text{sgn}(T)$

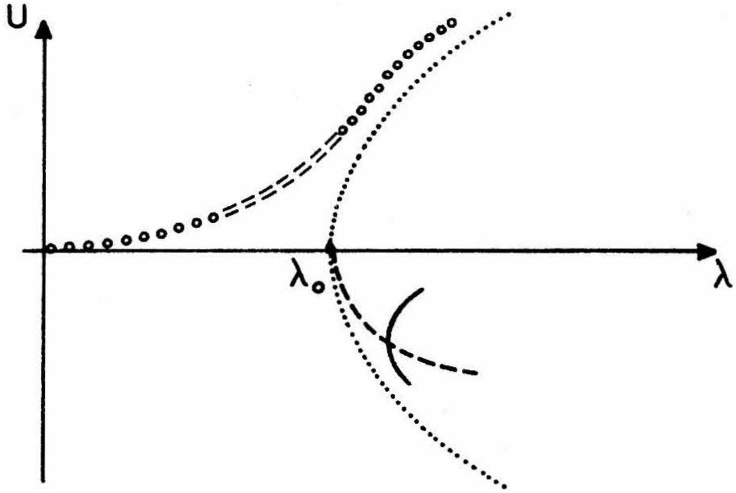


Case (ii): $\text{sgn}(\tau) = - \text{sgn}(T)$

FIG. 3. Solution branches of (II.1.1) when $f_{uu}(\lambda_o, 0, 0) = 0$ and $\langle f_{uuu}(\lambda_o, 0, 0)\phi_o^3, \phi_o^* \rangle \cdot \langle f_{\lambda u}(\lambda_o, 0, 0)\phi_o, \phi_o^* \rangle > 0$ for τ sufficiently small. $T = \langle f_{uuu}(\lambda_o, 0, 0)\phi_o^3, \phi_o^* \rangle \cdot \langle f_\tau(\lambda_o, 0, 0), \phi_o^* \rangle$.



Case (i): $\mathrm{sgn}(\tau) = \mathrm{sgn}(T)$



Case (ii): $\mathrm{sgn}(\tau) = -\,\mathrm{sgn}(T)$

FIG. 4. Solution branches of (II.1.1) when $f_{uu}(\lambda_o,0,0) = 0$ and $\langle f_{uuu}(\lambda_o,0,0)\phi_o^3,\phi_o^*\rangle \cdot \langle f_{\lambda u}(\lambda_o,0,0)\phi_o,\phi_o^*\rangle < 0$ for τ sufficiently small. $T = \langle f_{uuu}(\lambda_o,0,0)\phi_o^3,\phi_o^*\rangle \cdot \langle f_\tau(\lambda_o,0,0),\phi_o^*\rangle$.

KEY for FIGURES 2,3, and 4:

·············· Bifurcation solution when $\tau = 0$.

---- Locus of nonisolated solutions – Theorem II-3-1

—— Extension from nonisolated solution for τ fixed – Theorem II-5-1

∘∘∘∘∘∘∘∘ Minimal positive and maximal negative solutions (included only when $\phi_o(x) > 0$ for $x \in D$) – Section II-7

===== Solution branches with no nonisolated solutions – Theorem II-8-1

$$U = \langle u(x), f_{\lambda u}(\lambda_o, 0, 0)\, \phi_o^*(x) \rangle$$

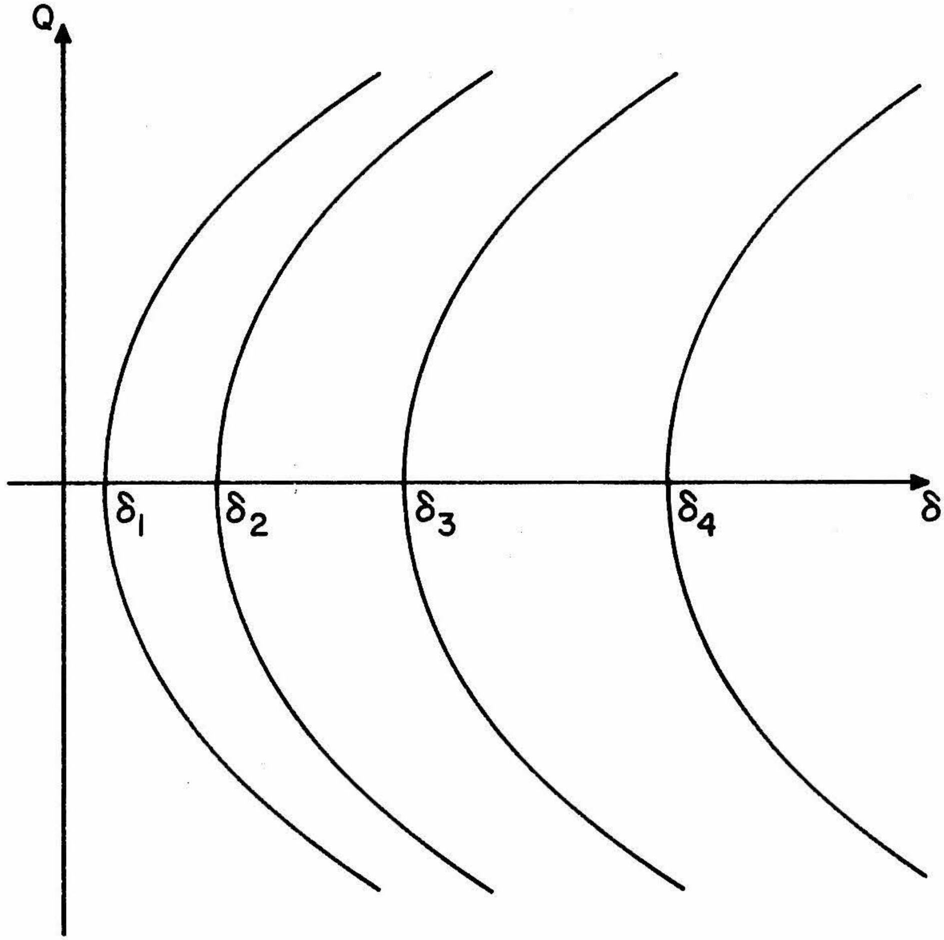


FIG. 5. Amplitude $Q = kw_k(\delta)$ for equilibrium states of a perfect column ($\tau = 0$).

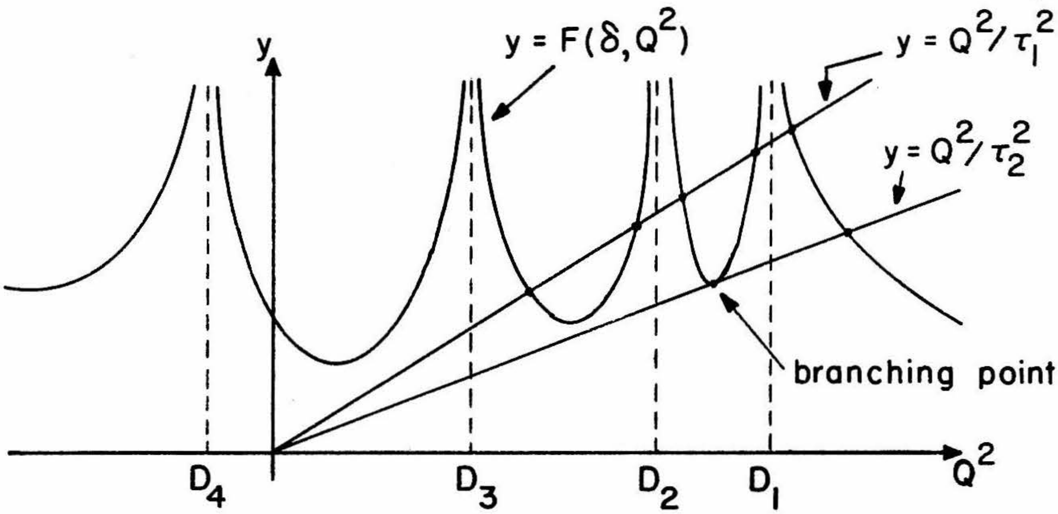


FIG. 6. $y = F(\delta, Q^2)$ and $y = Q^2/\tau^2$ for δ fixed and $\tau_1 < \tau_2$. Points of intersection are solutions of $Q^2 = \tau^2 F(\delta, Q^2)$ given in (III.2.21). $D_j = \frac{4}{\Gamma \pi^2}(2\Gamma\delta - j^2\pi^2)$.

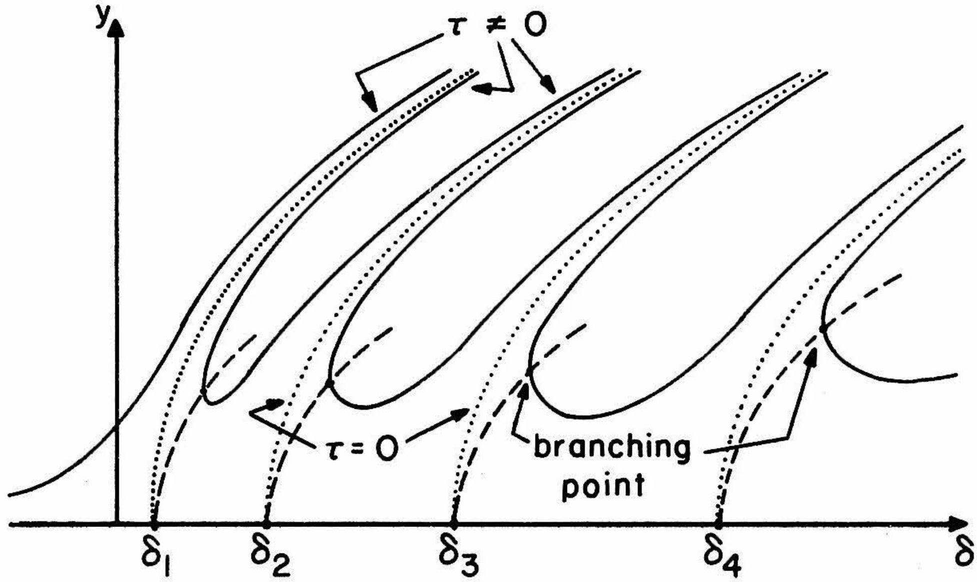


FIG. 7. $y = Q(\delta, \tau)$ for $\tau = 0$ (compare Figure 5) and $\tau \neq 0$ fixed. The loci of branching points are also shown.

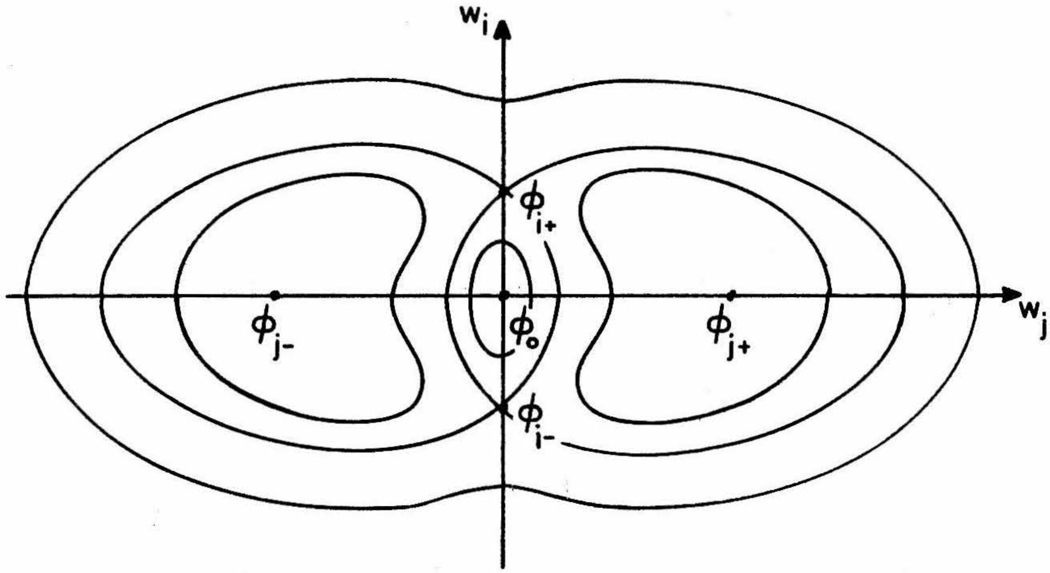


FIG. 8. Level curves of potential energy for the perfect column ($\tau = 0$) with two modes present in the motion; $i > j$.

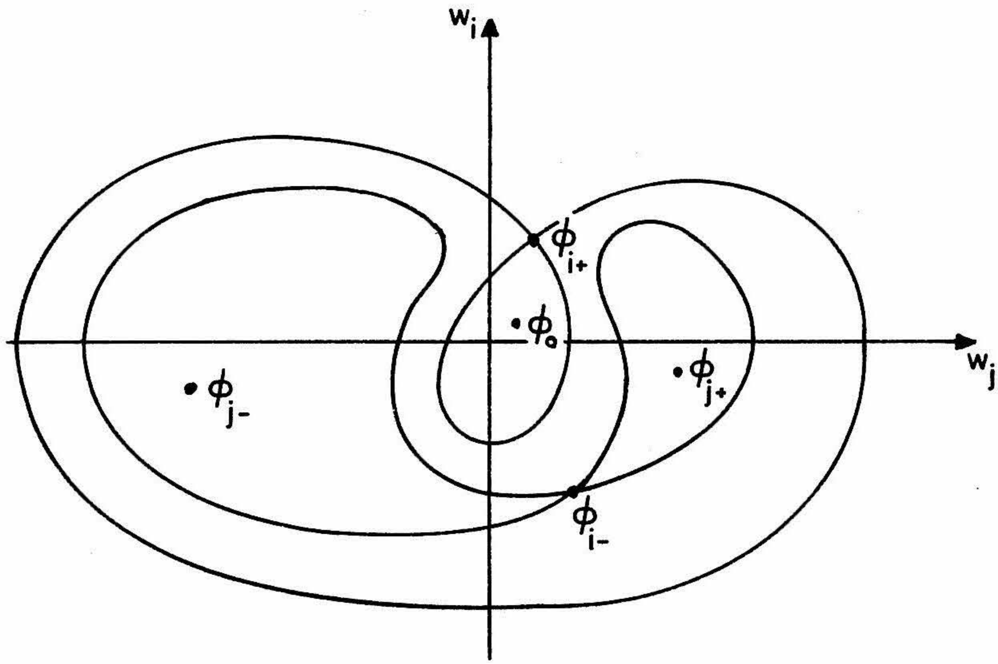


FIG. 9. Level curves of potential energy for an imperfect column ($\tau \neq 0$) with two modes present in the motion; $i > j$.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I. Comm. Pure Appl. Math. 12 (1952), 623-727.

[2] H. Amann, On the Existence of Positive Solutions of Nonlinear Elliptic Boundary Value Problems (to appear Ind. Univ. Math. J.)

[3] J. C. Amazigo, B. Budiansky and G. F. Carrier, Asymptotic Analyses of the Buckling of Imperfect Columns on Nonlinear Elastic Foundations, Int. J. Solids and Struct., 6(1970), 1341-1356.

[4] B. Budiansky and J. W. Hutchinson, A Survey of Some Buckling Problems, AIAA Journal, Vol.4, No. 9 (1966).

[5] E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw Hill, New York, (1955).

[6] R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. I, II, Interscience, New York (1966).

[7] M. G. Crandall and P. H. Rabinowitz, Bifurcation from Simple Eigenvalues, J. Funct. Anal., Vol. 8, 321-340 (1971).

[8] E. T. Dean and P. L. Chambré, Branching of Solutions of Some Nonlinear Eigenvalue Problems, J. Math. Phys., Vol. 11, No. 5, (1970).

[9] E. T. Dean and P. L. Chambré, On the Solutions of the Nonlinear Eigenvalue Problem $Lu + \lambda b(x)u = g(x, u)$, Bull. Amer. Math. Soc., Vol. 76, No. 3 (1970) pp. 595-600.

[10) R. W. Dickey, Free Vibrations and Dynamic Buckling of the Extensible Beam, J. Math. Anal. Appl., Vol. 29, (1970) pp. 443-454.

[11] N. Dunford and J. T. Schwartz, "Linear Operators, Part I," Interscience, New York (1958).

[12] H. Fujita, On the Asymptotic Stability of Solutions of the Equation $v_t = \nabla v + e^v$, Functional Analysis and Related Topics, Univ. of Tokyo (1969).

[13] P. R. Garabedian, "Partial Differential Equations," Wiley (1964).

[14] N. J. Hoff and V. G. Bruce, Dynamic Analysis of the Buckling of Laterally Loaded Flat Arches, J. Math. Phys., Vol. 23, (1953), pp. 276-288.

[15] J. W. Hutchinson and W. T. Koiter, Postbuckling Theory, Appl. Mech. Rev., Vol. 23, No. 12, (1970).

[16] H. B. Keller, Elliptic Boundary Value Problems Suggested by Nonlinear Diffusion Processes, Arch. Rat. Mech. Anal., Vol. 35, No. 5(1969) pp. 363-381.

[17] H. B. Keller, Nonlinear Bifurcation, J. Diff. Eq., Vol. 7, No. 3, (1970).

[18] H. B. Keller, Positive Solutions of Some Nonlinear Eigenvalue Problems, J. Math. Mech., Vol. 19, No. 4, (1969), pp. 279-296.

[19] H. B. Keller, and D. S. Cohen, Some Positone Problems Suggested by Nonlinear Heat Generation, J. Math. Mech., Vol. 16, No. 12 (1967).

[20] H. B. Keller and W. F. Langford, Iterations, Perturbations, and Multiplicities for Nonlinear Bifurcation Problems (to appear in Arch. Rat. Mech. Anal.)

[21] J. B. Keller and S. Antman, "Bifurcation Theory and Nonlinear Eigenvalue Problems," Benjamin, New York (1969).

[22] J. B. Keller and M H. Millman, Perturbation theory of nonlinear electromagnetic wave propagation, Phys. Rev., Vol. 181 (1969) pp. 1730-1747.

[23] J. B. Keller and L. Ting, Periodic vibrations of systems governed by nonlinear partial differential equations, Comm. Pure Appl. Math., Vol. 19 (1966), pp. 371-420.

[24] M. A. Krasnosel'skii, "Positive Solutions of Operator Equations," P. Noordhoff, Ltd., Groningen (1964).

[25] M. A. Krasnosel'skii, "Topological Methods in the Theory of Nonlinear Integral Equations," Machmillan Co. (1964).

[26] O. A. Ladyzenskaja and N. N. Ural'tseva, "Linear and Quasilinear Elliptic Equations," Academic Press (1968).

[27] T. W. Laetsch, "Eigenvalue Problems for Positive Monotonic Nonlinear Operators," Ph.D. thesis, California Institute of Technology (1968).

[28] W. F. Langford, "Bifurcation Theory of Nonlinear Boundary Value Problems," Ph.D. thesis, California Institute of Technology (1968).

[29] M. H. Millman and J. B. Keller, Perturbation Theory of Nonlinear Boundary Value Problems, J. Math. Phys. Vol. 10 (1969), pp. 342-361.

[30] Poincaré, New Methods of Celestial Mechanics, Vol. II, NASA Technical Translation No. F-451 (1967); translated from Les Méthodes Nouvelles de la Mécanique Céleste, Tome II, Gauthier-Villars, Paris (1893).

[31] M. H. Protter and H. F. Weinberger, "Maximum Principles in Differential Equations," Prentice-Hall, Englewood Cliffs, N.J. (1967).

[32] E. L. Reiss, Column Buckling - An Elementary Example of Bifurcation, Lecture I in [20].

[33] E. L. Reiss and B. J. Matkowsky, Nonlinear Dynamic Buckling of a Compressed Elastic Column, Quart. Appl. Math., Vol. 29, No. 2, pp. 245-260.

[34] D. Sather, Branching of Solutions of an Equation in Hilbert Space, Arch. Rat. Mech. Anal., Vol. 36, No. 1 (1970) pp. 47-64.

[35] D. H. Sattinger, Monotone Methods in Nonlinear Elliptic and Parabolic Boundary Value Problems, (to appear in Ind. Univ. Math. J.).

[36] R. B. Simpson and D. S. Cohen, Positive Solutions of Nonlinear Elliptic Eigenvalue Problems, J. Math. Mech. Vol. 19, No. 10, (1970), pp 895-910.

[37] J. J. Stoker, "Nonlinear Vibrations in Mechanical and Electrical Systems," Interscience, New York (1950).

[38] M. M. Vainberg and V. A. Trenogin, The Methods of Lyapunov and Schmidt in the Theory of Non-linear Equations and their Further Development, Russian Math. Surveys, Vol. 17, No. 2 (1962), pp. 1-60.

[39] S. Woinowsky-Kreiger, The Effect of Axial Force on the Vibration of Hinged Bars, J. Appl. Mech., Vol. 17, (1950), pp. 35-36.