ON THE SETTLING SPEED OF DILUTE ARRAYS OF SPHERES

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ABSTRACT

A method is developed to calculate the settling speed of dilute arrays of spheres for the three cases of: I, a random array of freely moving particles; II, a random array of rigidly held particles; and III, a cubic array of particles. The basic idea of the technique is to give a formal representation for the solution and then manipulate this representation in a straightforward manner to obtain the result. For infinite arrays of spheres, our results agree with the results previously found by other authors, and the analysis here appears to be simpler. This method is able to obtain more terms in the answer than was possible by Saffman's unified treatment for point particles. Some results for arbitrary two sphere distributions are presented, and an analysis of the wall effect for particles settling in a tube is given. It is expected that the method presented here can be generalized to solve other types of problems.

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I. INTRODUCTION

When a collection of particles is allowed to settle under the action of gravity, it is observed experimentally that the mean settling speed of the cloud depends on the concentration of the particles, as well as the size, shape, and excess weight of each particle. If the dimensions of the cloud are less than the size of the bounding container (Figure 1), the settling speed increases as more particles are added to the cloud; while if the cloud is uniform throughout the container (Figure 2), the settling speed decreases with increasing particle concentration.

The mechanism for this can be seen by considering the two types of interactions between particles. A settling particle causes a downward velocity in the neighboring fluid, and, as a consequence, any particle in that region will experience an increase in its settling speed. For Stokes flow, it is known that this increase is asymptotically proportional to $\frac{1}{r}$ where r is the distance from the particle. This is a direct particle interaction.

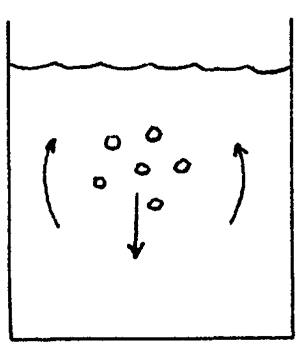

Figure 1. Non-uniform dispersion.

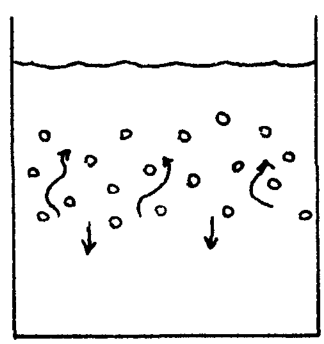

Figure 2. Uniform dispersion.

On the other hand, when particles are sedimenting in a closed container, the net flux of fluid plus solid must be zero through any fixed horizontal surface by conservation of mass, so that the downward flux of each particle and its neighboring fluid is compensated by a diffuse return flow that is spread throughout the fluid. This return flow is due to the additional pressure gradient in the fluid caused by the presence of the particle, and it is an indirect particle interaction.

It is evident that for points near the particle, the direct interaction is dominant. Hence, in order to preserve zero net flow through the instantaneous horizontal plane of the particle, this indirect interaction must be dominant for distant points in that plane.

Thus, for the cloud of small dimension, we expect the direct interactions are dominant, and the speed of the cloud to increase as particles are added. However, when the dispersion is uniform throughout the container, there is a decrease in settling speed because there are many more particles far away than near any given particle. (This statement will be made rigorous later.) This phenomenon is known as "hindered settling." It is necessary that the concentration of the dispersion be uniform for this effect; otherwise, small regions of high concentration will tend to behave in the same way relative to regions of low concentration as a small cloud behaves relative to the surrounding fluid.

Hindered settling has been observed for particles of all shapes and sizes, and for a range of Reynolds numbers from 10^{-4} to 10^3 [Richardson and Zaki (1954)]. (The high Reynolds number experiments are done by fluidization rather than sedimentation.) The associated theoretical problem is to predict the decrease in settling speed as a function of the concentration of particles. This decrease will be measured relative to the terminal velocity, U_0 , of a single particle falling in an unbounded fluid. We assume that the fluid is incompressible, and that the particles are equi-sized spheres large enough not to be affected by Brownian motion. We also assume that the Reynolds number is small enough that inertia forces can be neglected, and that the Stokes equation may be used. The dependence on the volume concentration, c , may be complex, so we only look at the limit with c small. As a final assumption, the effect of the wall is ignored.

It is this last assumption that causes mathematical difficulty, for now we are considering a uniform dispersion in an infinite fluid, with the condition that the average flux through any horizontal surface is zero, or equivalently, that the mean velocity of the dispersion (fluid plus solid) is zero at any point. Attempts to sum the direct particle interactions lead to divergent sums, because the fluid velocity due to a particle of radius a moving at speed U_0 asymptotically decays like $U_0 \frac{a}{r}$. Similarly, the diffuse return flow from a single particle,which was well defined in a bounded container, becomes infinitesimal in the unbounded fluid.

Three different types of methods were developed to cope with this problem. The first method is the cell method (Figure 3) where it is assumed that each sphere is surrounded by a region of fluid and that the effect of the other particles can be represented by some boundary condition on the surface of the cell. The size of the cell is usually taken as proportional to the mean interparticle distance, $n^{-1/3}$, where n is the number of particles per unit volume. A special case of this method is obtained when it is assumed the particles are in some regular pattern, for instance, a cubic array.

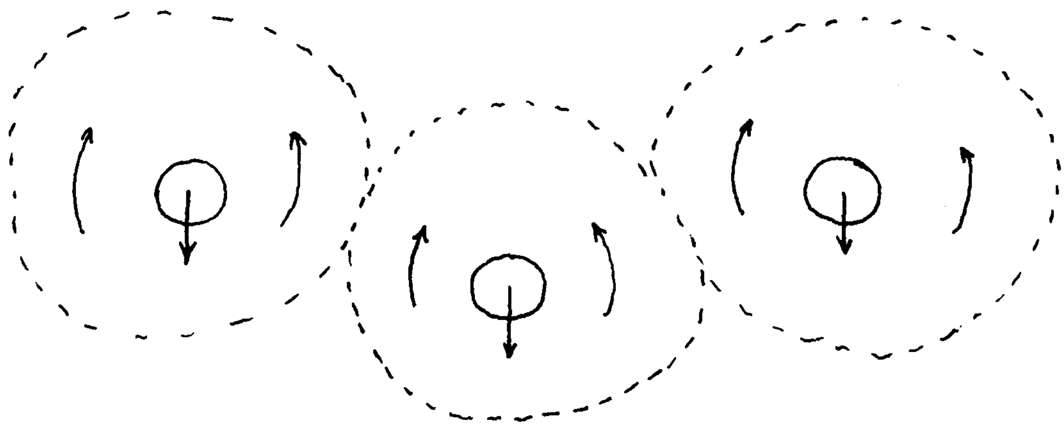

Figure 3. The cell model of interactions.

The second method is to assume the surrounding particles act as a porous medium, and so consider the motion of a single particle in a porous medium. This was Brinkman's idea (1947), and it has since been updated by T. S. Lundgren (1972).

The third method is to directly use statistical analytical methods on the dispersion, and to manipulate the variables in some systematic manner in order to obtain convergent results. This method was used by Burgers (1942), Pyun and Fixman (1964), J. B. Keller (unpublished), and Batchelor (1972).

However, these three methods of solving the same problem gave qualitatively different results. To leading order, they predicted that the settling speed was hindered by quantities proportional to $c^{1/3}$, $c^{1/2}$, and c respectively. There was disagreement in the literature as to which theoretical result was correct, and it was not until the last few years that this problem was resolved. The correct answer for this sedimentation problem is the order c result obtained by the third method, and it is now understood that the other two methods inherently make incorrect assumptions about the settling of freely moving spheres. The assumption made by the cell model is that the spheres are widely spaced, and hence there can be no close interactions between spheres. This is not true for the slow sedimentation problem. The more subtle assumption made by the porous medium model is that the spheres may not move relative to each other; this model actually describes the flow past a fixed random array of spheres.

Thus, the physical sedimentation problem created three problems of mathematical interest, namely, the settling of a random array of freely moving spheres, and flow past both a cubic array and a random fixed array. The known solutions for these three problems are

Random Free Array (statistical model), Batchelor (1972)

$$\underline{U} = \underline{U}_0 (1 - 6.55c + \dots)$$

Regular (Cubic) Array, Hasimoto (1959)

$$\underline{U} = \underline{U}_0 (1 - 1.76c^{1/3} + c + \dots)$$

Random Fixed Array (porous medium model), Childress (1972)

$$\underline{U} = \underline{U}_0 \left(1 - \frac{3}{\sqrt{2}} c^{1/2} - \frac{135}{64} c \log c + O(c)\right)$$

where each result is expressed in the reference frame where the net velocity of fluid plus solid is zero. In each case, the mean settling velocity, $\underline{u}$, is given in terms of the volume concentration, c , of particles, and the terminal velocity, $\underline{u}_0$, of a single sphere in an unbounded fluid.

These three results were obtained by different methods, and in order to understand how these problems were related to each other, it became desirable to have a unified method that would solve all three problems. Saffman (1973) used a Fourier transform technique to do this and derived the $1.76\, c^{1/3}$ in the cubic array and the $\frac{3}{\sqrt{2}}\, c^{1/2}$ term in the random fixed array, and showed that the lead term in the free array was $O(c)$. He pointed out that the random fixed array is basically a different problem than the other two because, in the former case, the drag force on each particle is different, whereas the drag force is the same for all the particles in the other two cases. Saffman also showed the difference between the random free array and the cubic array was kinematical in nature. Previously, Batchelor (1972) had noted that the difference between a cell model and his statistical model was that the cell model assumes a characteristic length for the problem, namely, $n^{-1/3}$, that should not be there. Lundgren (1972), using a model for porous interfaces developed by Saffman (1971), showed why the random fixed array was more hindered than the random free array. His method is interesting in that it treats a single sphere as surrounded by a medium with some average properties, while other workers consider the sum of individual sphere interactions. A brief summary of his method is given in Appendix A.

However, the Fourier transform method used by Saffman cannot be easily extended to find the order c terms in the settling speeds, and the major thrust of this thesis is to develop a method that gives these order c (and higher order) terms. This method is described in Chapter II. Although we will only use it to solve these settling speed problems, it is expected that this method can be extended to solve a wide variety of problems. For example, Batchelor (1974) has described how the statistical model he used for sedimenting spheres can be used to find the transport properties of two phase materials with random structure. These properties include electrical conductivity, magnetic permeability, and shear viscosity, just to name a few. We expect the method given in this paper can be used in all these problems. In addition, this method can solve problems with strong particle interactions, as in the random fixed array, that cannot be done by Batchelor's method.

In Chapter III we find the settling velocity of a random free array of spheres, and our result agrees with Batchelor's result. The method makes it easy to set up the problem for solution, and it is found that Batchelor's concept of the "mean deviatoric stress" in the fluid is not needed. Furthermore, we treat all the particles simultaneously, and avoid all the arguments of treating N particles and letting $N \to \infty$.

Chapter IV describes the flow past a random fixed array of spheres, and our result agrees with that obtained by Childress. The method involves truncating a hierarchy of equations with the approximation proposed by Saffman, and solving the resulting integral equation to obtain an implicit solution amenable to iteration.

In Chapter V we consider the settling of a cubic lattice of spheres. The main difficulty in this problem is summing the resulting triple infinite series. A simple method to do this using the mean value theorem is given here. We also obtain, for an arbitrary lattice, a sufficient condition for the coefficient of the order c term to be equal to one. While our solution agrees with Hasimoto's solution, the relation between the two is not trivial, and this difference is discussed at the end of the chapter.

Chapter VI gives some results for arbitrary two sphere distribution functions, and it is shown that simple relations hold between the results for different distributions. We also demonstrate that a sufficient condition for $c^{1/3}$ hindrance in either the free or fixed arrays is that no two particles be closer than a distance of order $n^{-1/3}$. In particular, this means that any model where the particles are arranged in a quasi-regular manner, e.g. a cell model, will exhibit a $c^{1/3}$ reduction in settling speed. It also implies that strong interparticle repulsion will have a large effect on the settling speed.

In Chapter VII we consider the sedimentation of spheres in a cylinder, and estimate the wall effect on the settling speed in terms of the motion of a single particle in a tube. The theory predicts that the wall should slightly decrease the c dependence of the mean settling velocity, and an experiment is proposed to test this prediction.

Finally, Chapter VIII reviews some of the experiments that have been done on the concentration dependence of the settling speed, and it is concluded that none of the experiments has adequately tested the theory.

II. FORMULATION OF THE PROBLEM

For slow viscous motion in the dispersion, the appropriate equations of motion are:

$$\begin{cases} -\mu \nabla^2 \underline{u} + \nabla p = 0 & (2.1a) \\ \nabla \cdot \underline{u} = 0 & (2.1b) \end{cases}$$

with

$$\underline{u} = \underline{V}^{\alpha} + \underline{\Omega}^{\alpha} \times (\underline{r} - \underline{r}^{\alpha}) \quad \text{on} \quad |\underline{r} - \underline{r}^{\alpha}| = a . \qquad (2.2)$$

Thus, we want to satisfy the no-slip condition on the surface of each sphere α with center at $\underline{r}^{\alpha}$ and radius a. We are assuming the "zero Reynolds number" Stokes equations are valid, which is equivalent to stating that all significant interactions occur at a distance less than some characteristic length, ℓ, where

$$Re \ll a/\ell .$$

Here, we will use the reference frame where the mean speed of the dispersion (fluid plus solid) is zero, although other frames of reference are also useful, e.g. the frame with the spheres held fixed (flow past an array). It is simple to translate results between reference frames, so we consider only the first reference frame here. For the random free array we specify the excess weight of the particles, and find the mean speed of the particles. For the random fixed array the velocity is given, and we must find the average force per particle. For the regular array, either approach is meaningful.

Following Saffman (1973) we replace the particles by multipole distributions of forces at the center of each particle, and suppose that equation (2.1) holds throughout all space. Then (2.1a) becomes, for the i^{th} component,

$$-\mu \nabla^2 u_i + \partial_i p = \sum_{\alpha} \left\{ F_i^{\alpha} \delta(\underline{r}-\underline{r}^{\alpha}) + F_{ij}^{\alpha} \partial_j \delta(\underline{r}-\underline{r}^{\alpha}) + \ldots \right\} \tag{2.3}$$

Here, Greek superscripts refer to particles and the subscripts refer to coordinate axes. The summation convention is used for subscripts, and the simplified notation

$$\partial_i \equiv \frac{\partial}{\partial r_i}$$

is employed.

The F's are determined by satisfying the no-slip boundary conditions (2.2), and are related to the forces and moments on the particle; for instance, the drag on D_i^{α} and torque T_i^{α} acting on particle α are $-F_i^{\alpha}$ and $-\epsilon_{ijk} F_{jk}^{\alpha}$, respectively. The excess weight of particle α is given by

$$F_i^{\alpha} = \frac{4}{3}\pi a^3 (\rho_s - \rho) g_i \tag{2.4}$$

where ρ_s and ρ are the densities of the sphere and fluid, a is the particle radius, and $\underline{g}$ is the acceleration due to gravity.

§1. Results for a Single Sphere

For a single spherical particle settling in an unbounded fluid, the terminal velocity $\underline{U}_0$ is found by equating the excess weight and the drag force,

$$F_i^{\alpha} = -D_i^{\alpha} = 6\pi a \mu U_{0i} \tag{2.5}$$

which implies

$$\underline{U}_0 = \frac{2}{9}\frac{a^2}{\mu}(\rho_s - \rho)\underline{g} \tag{2.6}$$

It is known that for a single particle settling without rotation the no-slip boundary condition is satisfied by choosing

$$F_{ijk} = \frac{a^2}{6}\delta_{jk}F_i \tag{2.7}$$

with the corresponding solution valid everywhere outside the particle:

$$u_i(\underline{r}) = \frac{F_j}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right) + \frac{F_j a^2}{24\pi\mu}\left(\frac{\delta_{ij}}{r^3} - 3\frac{r_i r_j}{r^5} - \frac{8}{3}\delta_{ij}\delta(\underline{r})\right) \tag{2.8}$$

$$p(\underline{r}) = \frac{1}{4\pi}\frac{F_j r_j}{r^3} + \frac{a^2}{6\pi}F_j\partial_j\delta(\underline{r}) \tag{2.9}$$

where $\underline{r}$ is measured relative to the center of the particle. This solution may also be written in terms of the Stokeslet $S_{ij}(\underline{r})$ defined by

$$S_{ij}(\underline{r}) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right) . \tag{2.10}$$

Then (2. 8) becomes

$$u_i(\underline{r}) = F_j S_{ij}(\underline{r}) + \frac{2}{3}a^2\nabla^2 F_j S_{ij}(\underline{r}) . \tag{2.11}$$

The Stokeslet is the Green's function or fundamental solution of the equation in the sense that the solution $u_i(\underline{r})$ of

$$\begin{cases} -\mu \nabla^2 u_i + \partial_i p = F_i \, \delta(\underline{r} - \underline{r}^{\alpha}) \\ \nabla \cdot \underline{u} = 0 \\ \underline{u} \to 0 \text{ as } |\underline{r} - \underline{r}^{\alpha}| \to \infty \end{cases}$$

is just

$$u_i(\underline{r}) = F_j \, S_{ij}(\underline{r} - \underline{r}^{\alpha}) \, .$$

When a finite number of particles are settling in an unbounded fluid, the velocity of each particle can be found by superposing the single particle solutions, using Faxen's laws, and adjusting the F 's in order to satisfy the boundary conditions. This procedure will be illustrated by considering the settling of two particles in an unbounded fluid, and it will be done in some detail because it is the basis of the method for infinite dispersions.

§2. The Settling of Two Distant Spheres; Faxen's Law

It is desired to find the terminal settling speeds of two identical spheres with centers instantaneously located at $\underline{\ell}_1$ and $\underline{\ell}_2$ (where $|\underline{\ell}_2 - \underline{\ell}_1| \geq 2a$ so the spheres do not intersect). We know that each particle has the same excess weight F_i , and that the equations of motion are

$$\begin{cases} -\mu \nabla^2 u_i + \partial_i p = \sum_{n=1,2} \left\{ F_i \, \delta(\underline{r} - \underline{\ell}_n) + F_{ij}^{n} \partial_j \delta(\underline{r} - \underline{\ell}_n) + \ldots \right\} \\ \nabla \cdot \underline{u} = 0 . \\ \underline{u} \to 0 \text{ at } \infty . \end{cases} \qquad (2.12)$$

If the particles are far apart, a good first approximation to the particle velocities $\underline{V}_1, \underline{V}_2$ is that each moves as if it were settling alone in an unbounded fluid. Thus, we take

$$\underline{V}_1 = \underline{U}_0 + \underline{V}_1'$$
$$\underline{V}_2 = \underline{U}_0 + \underline{V}_2' \qquad (2.13)$$

We assume that $\underline{V}_1'$ and $\underline{V}_2'$ are small relative to $\underline{U}_0$, so the first approximation to the F's is given by the single particle solution

$$\begin{cases} F_{ijk}^{n} = F_{ijk}^{S.P.} + F_{ijk}^{n'} \, ; \; n = 1,2 \quad \text{and} \quad F_{ijk}^{S.P.} = \frac{a^2}{6}\delta_{jk} F_i \\ \text{other } F\text{'s small (except the lead term } F_i \text{)} \end{cases} \qquad (2.14)$$

since then we have

$$U_{o_i} = F_j S_{ij}(\underline{r} - \underline{l}_n) + F_{jlm}^{S.P.} \partial_l \partial_m S_{ij}(\underline{r} - \underline{l}_n) \quad \text{for} \quad |\underline{r} - \underline{l}_n| = a .$$

To proceed further, we need to know how well the boundary conditions are satisfied -- in particular, we need to approximate the movement of each sphere.

Suppose that a fluid has velocity $\underline{u}(\underline{x})$ in the presence of some boundaries, and then a rigid sphere replaces a spherical mass of fluid with center $\underline{x}_0$. The presence of this sphere will cause adjustments in the entire flow field. Faxen's laws state that if the other boundary conditions are disregarded for the moment, the translation $\underline{V}$ and rotation $\underline{\Omega}$ of the sphere are given by:

$$\underline{F} = 6\pi\mu a\left(\underline{V} - [\underline{u}(\underline{x})]_{\underline{x}=\underline{x}_0}\right) - \mu\pi a^3 [\nabla^2 \underline{u}(\underline{x})]_{\underline{x}=\underline{x}_0} \qquad (2.15a)$$

$$\underline{T} = 8\pi\mu a^3 \left(\underline{\Omega} - \frac{1}{2}[\nabla \times \underline{u}(\underline{x})]_{\underline{x}=\underline{x}_0}\right) \qquad (2.15b)$$

where $\underline{F}$ is the force and $\underline{T}$ is the torque exerted by the sphere on the fluid [Happel and Brenner (1965)]. This assumes the new flow field with the sphere present will be the sum of $\underline{u}(\underline{x})$ and the velocity field created by the sphere in the absence of other boundaries. In reality, this sum will not satisfy the other boundary conditions, but, if the resulting error on the other boundaries is small, Faxen's laws give a good approximation to the action motion of the sphere.

In the case we are considering, where gravity is the sole external force, there is no torque exerted by the sphere, and the body force is simply its excess weight. Hence, $\underline{T} = \underline{0}$, $\underline{F} = 6\pi a \mu \underline{U}_0$, and Faxen's laws become

$$\underline{V} = \underline{U}_0 + [\underline{u}(\underline{x})]_{\underline{x}=\underline{x}_0} + \frac{a^2}{6}[\nabla^2 \underline{u}(\underline{x})]_{\underline{x}=\underline{x}_0}, \qquad (2.16a)$$

$$\underline{\Omega} = \frac{1}{2}[\nabla \times \underline{u}(\underline{x})]_{\underline{x}=\underline{x}_0}. \qquad (2.16b)$$

However, there is a more natural way to interpret Faxen's laws when one considers only the multipole forces exerted on a fluid. Suppose the exact solution for the fluid velocity $\underline{w}(\underline{x})$ in the presence of a number of spheres is known, where all the boundary conditions are solved exactly. Then in the neighborhood of one particular sphere (which is represented by a multipole series), the velocity field has a singular part from the sphere's multipole representation, and a regular part, $\underline{v}(\underline{x})$, from the contributions to $\underline{w}(\underline{x})$ from the multipole representations of the other spheres. We now interpret Faxen's law with the regular part of the local velocity field $\underline{v}(\underline{x})$ replacing $\underline{u}(\underline{x})$ in the above equations (2.16a), (2.16b). Then, by definition, in the presence of the sphere, all the boundary conditions are satisfied exactly. Thus, Faxen's law gives the exact translational and rotational velocities of a sphere in terms of the regular part of the velocity field near the sphere.

Now apply Faxen's law to the two sphere problems. Consider sphere 1 first. The approximate velocity field $\underline{u}(\underline{r})$ that has been computed so far is the sum of the two single particle solutions:

$$u_i(\underline{r}) = \sum_{n=1,2} \left\{ F_j S_{ij}(\underline{r}-\underline{\ell}_n) + F_{j\ell m}^{S.P.} \partial_\ell \partial_m S_{ij}(\underline{r}-\underline{\ell}_n) \right\}$$

The regular part of the velocity field near $\underline{\ell}_1$, the center of sphere 1, is defined as:

$$v_i(\underline{r};\underline{\ell}_1) \equiv u_i(\underline{r}) - \left\{ F_j S_{ij}(\underline{r}-\underline{\ell}_1) + F_{j\ell}^{1} \partial_\ell S_{ij}(\underline{r}-\underline{\ell}_1) + \ldots \right\}, \tag{2.17}$$

which, in this case, is

$$v_i(\underline{r};\underline{\ell}_1) = F_j S_{ij}(\underline{r}-\underline{\ell}_2) + F_{j\ell m}^{S.P.} \partial_\ell \partial_m S_{ij}(\underline{r}-\underline{\ell}_2) .$$

The $\underline{\ell}_1$ as argument of v_i means we have eliminated the singular part of the flow from the sphere at $\underline{\ell}_1$. Using Faxen's laws (2.16), the motion of sphere 1 is given by:

$$\left\{ \begin{aligned} V_{1i} &= U_{oi} + v_i(\underline{\ell}_1;\underline{\ell}_1) + \frac{a^2}{6}\left[\nabla^2 v_i(\underline{r};\underline{\ell}_1)\right]_{\underline{r}=\underline{\ell}_1}, \\ \Omega_{1i} &= \frac{1}{2}\left[\epsilon_{ijk}\partial_j v_k(\underline{r};\underline{\ell}_1)\right]_{\underline{r}=\underline{\ell}_1}. \end{aligned} \right. \tag{2.18}$$

In order to simplify the algebra, we note that for $\frac{r}{a}$ large,

$$F_j S_{ij}(\underline{r}) \sim U_{oi}\frac{a}{r}$$

while

$$F^{S.P.}_{j\ell m}\partial_\ell\partial_m S_{ij}(\underline{r}) \sim U_{o_i}\left(\frac{a}{r}\right)^3$$

so that if the particle centers are far apart we need only consider, to leading order in $\frac{a}{r}$,

$$v_i(\underline{r};\underline{\ell}_1) \approx F_j S_{ij}(\underline{r}-\underline{\ell}_2)$$

This is commonly called the point particle approximation for the sphere at $\underline{\ell}_2$. The point particle approximation is the leading term in the asymptotic development for $\frac{a}{|\underline{r}-\underline{\ell}_2|}$ small. Using this approximation in (2. 18) then gives

$$\begin{cases} V_{1_i} \approx U_{o_i} + F_j S_{ij}(\underline{\ell}_1-\underline{\ell}_2) + O\left(\frac{a^3}{|\underline{\ell}_1-\underline{\ell}_2|^3}\right) \\ \Omega_{1_i} \approx \frac{1}{2}\left[\epsilon_{ijk}\partial_j F_m S_{km}(\underline{r}-\underline{\ell}_2)\right]_{\underline{r}=\underline{\ell}_1} + O\left(\frac{a^3}{|\underline{\ell}_1-\underline{\ell}_2|^3}\right) \end{cases}$$

Thus, on the surface of sphere 1, $|\underline{r}-\underline{\ell}_1| = a$, we want the fluid velocity to satisfy the no-slip condition,

$$u_i(\underline{r}) = \underline{V}_{1_i} + \epsilon_{ijk}\Omega_{1_j}(r_k-\ell_{1_k}),$$

and we choose the higher multipoles of sphere 1 to satisfy this condition. The general method for doing this is well known [Lamb (1932)], namely, expand the sphere's velocity and the fluid velocity in spherical harmonics about $\underline{\ell}_1$, and choose the F 's to match the harmonic components of the sphere's velocity. Then the known F 's on sphere 1 are used to compute the regular part of the velocity near $\underline{\ell}_2$, and the method proceeds iteratively.

When the spheres are close together, convergence of this method is slow, and other methods are better. Goldman, Cox, and

Brenner (1966), and Stimson and Jefferey (1926) used bipolar coordinates to find the motion of two neighboring spheres. In particular, the first paper gives numerical results for the translation and rotation of two spheres in Stokes flow as a function of the relative separation and orientation of the sphere centers. It is found that the relative position of the spheres remains the same for two spheres in Stokes flow. When more than two spheres are present, there is unsteady relative motion among the spheres [Happel and Brenner (1965)], and this makes the theoretical problem much more difficult. The usual approach is to assume the spheres are far apart (so that the effect of multi-particle interactions is small) and to approximate the motion of each particle by its single particle motion plus the sum of its two particle interactions. This is the approximation Batchelor (1972) used, and we will use, to find the mean settling velocity of the random free array.

Thus, it is straightforward in principle to solve for the settling velocity of finite number of spheres. The single particle solutions are superposed, and the higher multipole coefficients for each sphere are found by satisfying the boundary conditions. If the velocity field created by a single particle decayed as fast as $\frac{1}{r^{3+\epsilon}}$, the problem for an infinite number of spheres could be solved in the same manner. However, the Stokes velocity field only decays as fast as $\frac{1}{r}$; thus, directly superposing the Stokes solution in the infinite array case leads to a divergent sum. The method that will be used for the infinite array is a direct extension of the solution for a finite number of spheres. The single particle solutions will be superposed, and then a homogeneous solution of the equation (2.1) will be added to make the result convergent.

§ 3. The Settling of an Infinite Number of Particles

Consider first the point particle problem:

$$\begin{cases} -\mu \nabla^2 u_i + \partial_i p = \sum_\alpha F_i \, \delta(\underline{r}-\underline{r}^\alpha) & (2.19a) \\ \nabla \cdot \underline{u} = 0, & (2.19b) \end{cases}$$

$$\begin{cases} \overline{u_i} = 0, \text{ where } \overline{u_i} \text{ is average of } u_i(\underline{r}) \text{ over all space,} \\ \overline{p} = 0 \, . \end{cases} \qquad (2.20)$$

We represent the solution in the following form:

$$u_i(\underline{r}) = \sum_\alpha F_j S_{ij}(\underline{r}-\underline{r}^\alpha) - n \int_V F_j S_{ij}(\underline{r}-\underline{s}) \, d\underline{s} \qquad (2.21a)$$

$$p(\underline{r}) = \sum_\alpha \frac{F_j}{4\pi} \frac{(r_j - r_j^\alpha)}{|\underline{r}-\underline{r}^\alpha|^3} - n \int_V \frac{F_j}{4\pi} \frac{(r_j - s_j)}{|\underline{r}-\underline{s}|^3} \, d\underline{s} \qquad (2.21b)$$

where

n = number of particles per unit volume,

V = entire region accessible to the sphere centers.

Neither the sums nor integrals converge, but the difference between a sum and its corresponding integral does converge.

Note this formal solution is the direct sum of the individual point particle solutions, and a solution of the homogeneous form of (2.19). It still must be demonstrated that this solution satisfies the boundary conditions (2.20). Consider the points $\{\underline{r}^\alpha\}$ fixed. To compute $\overline{u_i}$, we want to average $u_i(\underline{r})$ over all points $\underline{r}$.

This means we are averaging over every possible position relative to each fixed point $\underline{r}^{\alpha}$. Since there are n particles per unit volume, this implies

$$\overline{\sum_{\alpha} F_j S_{ij}(\underline{r}-\underline{r}^{\alpha})} = n \int_V F_j S_{ij}(\underline{r}-\underline{s})\, d\underline{s}$$

so that

$$\overline{u_i(\underline{r})} = 0 \, .$$

The same result is true for the mean pressure $\overline{p(\underline{r})}$. The above form (2.21) of the solution for the point particle problem was first used by J. B. Keller (unpublished).

In our method of solution, we need only an expression for $u_i(\underline{r})$ to solve the problem, and so the equation for the pressure (2.21b) will be ignored for the remainder of the paper.

The solution for the velocity (2.21a) may be interpreted in another way using Saffman's (1973) paper. He Fourier transforms the equations (2.19) and applies the boundary condition $\overline{u_i} = 0$ by saying that $\hat{u}_i(\underline{k})$, the Fourier transform of $u_i(\underline{r})$, has no part proportional to $\delta(\underline{k})$. This assures the mean velocity of the dispersion is zero. The equation he obtains for $\hat{u}_i(\underline{k})$ (equation (2.20) of his paper) is essentially the Fourier transform of the solution (2.21a) presented here. Unfortunately, while this transform method works well for the point particle problem, it cannot easily be extended to the complete multipole problem.

Consider now the complete problem:

$$\left\{\begin{array}{l} -\mu\nabla^2 u_i + \partial_i p = \sum_\alpha \left\{ F_j^\alpha \delta(\underline{r}-\underline{r}^\alpha) + F_{ij}^\alpha \partial_j \delta(\underline{r}-\underline{r}^\alpha) + \ldots \right\} \\ \nabla\cdot\underline{u} = 0 \\ \overline{u_i(\underline{r})} = 0, \end{array}\right. \tag{2.22}$$

where the superscripts α are a reminder that different spheres will in general have different multipole representations. As before, the solution for $u_i(\underline{r})$ is written as:

$$\begin{aligned} u_i(\underline{r}) = {} & \sum_\alpha F_j^\alpha S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int_V \overline{F}_j S_{ij}(\underline{r}-\underline{s})d\underline{s} \\ & + \sum_\alpha F_{j\ell}^\alpha \partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int_V \overline{F}_{j\ell}\partial_\ell S_{ij}(\underline{r}-\underline{s})d\underline{s} \\ & + \sum_\alpha F_{j\ell m}^\alpha \partial_\ell\partial_m S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int_V \overline{F}_{j\ell m}\partial_\ell\partial_m S_{ij}(\underline{r}-\underline{s})d\underline{s} \\ & + \ldots \end{aligned} \tag{2.23}$$

where the overbar indicates an average over all particles. Once again, averaging over $\underline{r}$ shows $\overline{u_i} = 0$. The regular part of the velocity field in the neighborhood of a sphere center is now easily found by subtracting the singular terms from $u_i(\underline{r})$ at that point; e. g. near $\underline{r} = \underline{r}^\beta$ we have

$$V_i(\underline{r};\beta) \equiv u_i(\underline{r}) - \left\{ F_j^\beta S_{ij}(\underline{r}-\underline{r}^\beta) + F_{j\ell}^\beta \partial_\ell S_{ij}(\underline{r}-\underline{r}^\beta) + \ldots \right\}$$

or, in the form of (2.23),

$$v_i(\underline{r};\beta) = \sum_\alpha{}' F_j^\alpha S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int_V \overline{F}_j S_{ij}(\underline{r}-\underline{s})d\underline{s}$$
$$+\sum_\alpha{}' F_{j\ell}^\alpha \partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int_V \overline{F}_{j\ell}\partial_\ell S_{ij}(\underline{r}-\underline{s})d\underline{s}$$
$$+\sum_\alpha{}' F_{j\ell m}^\alpha \partial_\ell\partial_m S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int_V \overline{F}_{j\ell m}\partial_\ell\partial_m S_{ij}(\underline{r}-\underline{s})d\underline{s} \qquad (2.24)$$
$$+ \dots$$

where $\sum_\alpha{}'$ means $\alpha=\beta$ is excluded from the sum. In order to proceed further, the detailed specifications of the problem must be known. In the following chapters, the forms (2.23) and (2.24) will be used to find the settling speeds of an infinite random array of freely moving particles, an infinite random array of rigidly fixed particles, an infinite cubic array, and a random array of freely moving particles in a tube.

III. SEDIMENTATION OF FREELY MOVING SPHERES

In this chapter we consider the sedimentation of a dilute infinite dispersion of identical spheres where the translation and rotation of each sphere is determined by the local fluid velocity. The mean settling velocity of the spheres is found to be

$$\langle \underline{V} \rangle = \underline{U}_0 (1 - 6.55c + o(c)), \quad c = \text{volume concentration of spheres}$$

which agrees with Batchelor's (1972) result. His method, and methods used by other authors, will be briefly summarized at the end of this section.

We are given that the drag force $-F_i^{\alpha}$ on each particle is the same, and that the positions of the particle centers $\underline{r}^{\alpha}$ are random variables homogeneously distributed throughout an infinite domain. Thus, the equations of motion (2.22) become

$$\begin{cases} -\mu \nabla^2 u_i + \partial_i p = \sum_{\alpha} \left\{ F_i \delta(\underline{r} - \underline{r}^{\alpha}) + F_{ij}^{\alpha} \partial_j \delta(\underline{r} - \underline{r}^{\alpha}) + \dots \right\} \\ \nabla \cdot \underline{u} = 0 \\ \overline{u_i} = 0 \end{cases} \tag{3.1}$$

We wish to find the mean settling speed of the spheres. This means that the speed of each sphere would need to be obtained and then the average speed of the spheres computed. Instead, we will look at ensemble averages of the positions of the surrounding spheres relative to a given sphere. It is assumed these two averages are equal, using the idea that any single realization of the dispersion contains each possible configuration relative to some sphere in the dispersion.

Ensemble averages are denoted by the angle brackets $\langle \dots \rangle$. Two types of averages will be used; the unconditional average at a point, and the conditional average given these is a particle at a point $\underline{r}^{\beta}$. The latter ensemble average will have a subscript β. For example, let $f(\underline{r})$ be a function defined on every particle. Then

$$\langle \sum_{\alpha} f(\underline{r}^{\alpha}) \rangle \equiv \int_V f(\underline{s}) p(\underline{s}) d\underline{s}$$

where $p(\underline{s})$ is the probability density of the location of particle centers. For the unconditional average, we take $p(\underline{s}) = n$, where n is the (uniform) number density of particles. Hence,

$$\langle \sum_{\alpha} f(\underline{r}^{\alpha}) \rangle = n \int_V f(\underline{s}) d\underline{s} \,. \tag{3.2}$$

Similarly, for the conditional average we define

$$\langle \sum_{\alpha} f(r^{\alpha}) \rangle_{\beta} \equiv \int_V f(\underline{s}) q(\underline{s}) d\underline{s} \tag{3.3}$$

and choose the probability density $q(\underline{s})$ to have the form:

$$q(\underline{s}) = n\left[1 - G(\underline{s}-\underline{r}^{\beta})\right] \tag{3.4}$$

where $G(\underline{s}-\underline{r}^{\beta}) \to 0$ as $|\underline{s}-\underline{r}^{\beta}| \to \infty$. For example, if we do not want the other spheres to intersect the sphere at $\underline{r}^{\beta}$, a logical choice for $G(\underline{s}-\underline{r}^{\beta})$ is:

$$G(\underline{s}-\underline{r}^{\beta}) = \begin{cases} 1, & |\underline{s}-\underline{r}^{\beta}| < 2a \\ 0, & |\underline{s}-r^{\beta}| \geq 2a \,. \end{cases}$$

Then we have

$$\langle {\sum_{\alpha}}' f(\underline{r}^{\alpha}) \rangle_{\beta} = n \int_{|\underline{s}-\underline{r}^{\beta}| \geq 2a} f(\underline{s}) d\underline{s} \,. \tag{3.5}$$

The above choice of G is the one we will use for the conditional ensemble averages unless otherwise stated. It is a reasonable choice provided there are no repulsive forces between particles. If $f(\underline{r})$ does not decay fast enough as $\underline{r} \to \infty$, the above integrals are divergent. Nevertheless, they will be used in a meaningful way in the formal representation of the solution.

The velocity of a sphere (chosen to be at the origin) in a particular realization of the dispersion is given by Faxen's law (2.16a) as:

$$V_i^o = \mathcal{U}_{o_i} + \left[\left(1+\frac{a^2}{6}\nabla^2\right) v_i(\underline{r};0)\right]_{\underline{r}=\underline{0}} .$$

Upon taking the conditional ensemble average, we have

$$\langle V_i^o \rangle_o = \mathcal{U}_{o_i} + \left(1+\frac{a^2}{6}\nabla^2\right)\langle v_i(\underline{r};\underline{0})\rangle_o \tag{3.6}$$

where the notation $\left(1+\frac{a^2}{6}\nabla^2\right)\langle v_i(\underline{r};\underline{0})\rangle_o \equiv \left[\left(1+\frac{a^2}{6}\nabla^2\right)\langle v_i(\underline{r};0)\rangle\right]_{\underline{r}=\underline{0}}$

is employed, and we have used the identity

$$\left\langle \left[\nabla^2 v_i(\underline{r};\underline{0})\right]_{\underline{r}=\underline{0}} \right\rangle_o = \left[\nabla^2 \langle v_i(\underline{r};\underline{0})\rangle_o\right]_{\underline{r}=\underline{0}}$$

which is true because $v_i(\underline{r};\underline{0})$ is a smooth function of $\underline{r}$ near $\underline{r} = \underline{0}$.

The value of $\langle v_i(\underline{r};\underline{0})\rangle_o$ is now obtained by taking an ensemble average of (2.24), but first we note that

$$\left\langle \sum_\alpha{}' F_{j\ell}^\alpha \partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha)\right\rangle_o = \left\langle \sum_\alpha{}' \langle F_{j\ell}^\alpha\rangle_o \partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha)\right\rangle_o \tag{3.7}$$

$$\left(\text{where } \langle F_{j\ell}^\alpha\rangle_o \equiv \langle F_{j\ell}(\underline{r}^\alpha)\rangle_o\right)$$

since for any fixed $\underline{r}^\alpha$ the surrounding particles take all possible configurations relative to $\underline{r}^\alpha$ while they take all possible configurations relative to $\underline{O}$. From (2.24) and (3.7) we now obtain:

$$\langle v_i(\underline{r};\underline{O})\rangle_o = \langle {\sum_\alpha}' F_j S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_o - n\int F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$$

$$+\langle {\sum_\alpha}' \langle F_{j\ell}^\alpha\rangle_o \partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_o - n\int \overline{F}_{j\ell}\,\partial_\ell S_{ij}(\underline{r}-\underline{s})\,d\underline{s} \quad (3.8)$$

$$+\langle {\sum_\alpha}' \langle F_{j\ell m}^\alpha\rangle \partial_\ell\partial_m S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_o - n\int \overline{F}_{j\ell m}\,\partial_\ell\partial_m S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$$

$$+ \ldots .$$

Using the definition (3.5) of ensemble averages we have

$$\langle {\sum_\alpha}' F_j S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_o = n\int_{|\underline{s}|\geq 2a} F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$$

so that

$$\langle {\sum_\alpha}' F_j S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_o - n\int F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s} = -n\int_{|\underline{s}|<2a} F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$$

The last integral is convergent and has the value (valid for $|\underline{r}|\leq a$)

$$-n\int_{|\underline{s}|<2a} F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s} = \frac{F_j\,\delta_{ij}}{6\pi\mu a}\left[-6c + \tfrac{4}{5}\pi a n r^2 - \tfrac{2}{5}\pi a n r_i^2\right] \quad (3.9)$$

$$(\text{no sum on } i)$$

Similarly, for the next two terms of (3.8) we have

$$\langle {\sum_\alpha}' \langle F_{j\ell}^\alpha\rangle_o \partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_o - n\int \overline{F}_{j\ell}\,\partial_\ell S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$$

$$= \langle {\sum_\alpha}' \left[\langle F_{j\ell}^\alpha\rangle_o - \langle F_{j\ell}\rangle\right]\partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_o - n\int_{|\underline{s}|<2a}\langle F_{j\ell}\rangle\,\partial_\ell S_{ij}(\underline{r}-\underline{s})\,d\underline{s} \quad (3.10)$$

where we have used the fact that $\langle F_{j\ell}\rangle = \overline{F}_{j\ell}$. The last integral of (3.10) represents the net effect of the average Stokeslet derivatives surrounding the sphere at $\underline{O}$, and hence its value is related to the derivative of (3. 9). In particular, we have

$$\int_{|\underline{s}|<2a} \partial_\ell S_{ij}(\underline{r}-\underline{s})\,d\underline{s} \;=\; \partial_\ell \int_{|\underline{s}|<2a} S_{ij}(\underline{r}-\underline{s})\,d\underline{s}\,,$$

with similar relations true for the integrals of higher derivatives of the Stokeslets. The above steps simplify (3. 8) to read

$$\begin{aligned}\langle v_i(\underline{r};\underline{0})\rangle_0 = -n\left(F_j + \langle F_{j\ell}\rangle \partial_\ell + \dots\right)\int_{|\underline{s}|<2a} S_{ij}(\underline{r}-\underline{s})\,d\underline{s} \\ + \left\langle \sum_\alpha{}' \left[\langle F^\alpha_{j\ell}\rangle_0 - \langle F_{j\ell}\rangle\right]\partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha)\right\rangle_0 \qquad (3.11) \\ + \quad \dots\,,\end{aligned}$$

and to proceed further an estimate is needed for the ensemble averages of the F's.

Since the dispersion is assumed to be dilute, the first approximation to each of the unconditional averages is evidently the single particle value. In the first part of (3. 11), this assumption is equivalent to $\langle F_{j\ell m}\rangle = F^{S.P.}_{j\ell m}$, since (3. 9) shows the coefficients of the other unknown ensemble averages are zero when $\underline{r} = \underline{0}$. This approximation has a maximum error of $o(c)$ since $F^{S.P.}_{j\ell m}$ gives an order c term itself, and the average correction to it from the non-zero second derivative of the regular part of the velocity field is of smaller order.

The other terms of (3. 11) are approximated by relating the differences in ensemble averages to two particle and single particle forces. In particular, for each $\underline{r}^\alpha$, the approximations

$$\langle F_{j\ell}(\underline{r}^{\alpha})\rangle_0 - \langle F_{j\ell}\rangle = F_{j\ell}^{T.P.}(\underline{r}^{\alpha}) - F_{j\ell}^{S.P.}$$

$$\langle F_{j\ell m}(\underline{r}^{\alpha})\rangle_0 - \langle F_{j\ell m}\rangle = F_{j\ell m}^{T.P.}(\underline{r}^{\alpha}) - F_{j\ell m}^{S.P.} \qquad (3.12)$$

$$\vdots$$

are used, where the superscript T. P. denotes the two particle value for two spheres in relative position $\underline{r}^{\alpha}$ in an unbounded fluid, and S. P. denotes the value for a single particle in an unbounded fluid. This approximation ignores the effects of many-particle interactions. This neglected contribution is estimated below.

Consider a finite sphere, α, surrounded by point particles. In any realization, the multipole coefficients of α can be written in the form, e. g. for $F_{j\ell m}^{\alpha}$,

$$F_{j\ell m}^{\alpha} = F_{j\ell m}^{S.P.} + \sum_{\gamma}{}' F_{j\ell m}^{\alpha\gamma},$$

where $F_{j\ell m}^{\alpha\gamma}$ is the change in $F_{j\ell m}^{\alpha}$ from the point particle at $\underline{r}^{\gamma}$. (The point particle approximation for the particles γ gives the leading contribution of the stress changes on α from the other finite particles at $\underline{r}^{\gamma}$.) Taking the ensemble average gives:

$$\langle F_{j\ell m}^{\alpha}\rangle_{\alpha} = F_{j\ell m}^{S.P.} + \langle \sum_{\gamma}{}' F_{j\ell m}^{\alpha\gamma}\rangle_{\alpha} . \qquad (3.13)$$

Now suppose there is a (point) particle at β. Then

$$F_{j\ell m}^{\alpha} = F_{j\ell m}^{S.P.} + F_{j\ell m}^{\alpha\beta} + \sum_{\gamma\neq\beta}{}' F_{j\ell m}^{\alpha,\gamma} .$$

Taking the ensemble average with both α and β fixed gives

$$\langle F^{\alpha}_{j\ell m} \rangle_{\alpha,\beta} = F^{S.P.}_{j\ell m} + \langle F^{\alpha\beta}_{j\ell m} \rangle_{\alpha,\beta} + \langle {\sum_{\gamma \neq \beta}}' F^{\alpha\gamma}_{j\ell m} \rangle_{\alpha,\beta} . \tag{3.14}$$

Combining (3. 13) and (3. 14) gives:

$$\langle F^{\alpha}_{j\ell m} \rangle_{\alpha,\beta} - \langle F^{\alpha}_{j\ell m} \rangle_{\alpha} = \langle F^{\alpha\beta}_{j\ell m} \rangle_{\alpha,\beta} + \langle {\sum_{\gamma \neq \beta}}' F^{\alpha\gamma}_{j\ell m} \rangle_{\alpha,\beta} - \langle {\sum_{\gamma}}' F^{\alpha\gamma}_{j\ell m} \rangle_{\alpha} . \tag{3.15}$$

But the only difference between the last two ensemble averages in (3. 15) is that $\langle {\sum_{\gamma \neq \beta}}' F^{\alpha\gamma}_{j\ell m} \rangle_{\alpha,\beta}$ excludes the possibility of a particle in a sphere about $\underline{r}^{\beta}$. This happens with probability $8c$, and the effect of a particle in that sphere on $F^{\alpha}_{j\ell m}$ is nearly $F^{\alpha\beta}_{j\ell m}$. Hence,

$$\langle F^{\alpha}_{j\ell m} \rangle_{\alpha,\beta} - \langle F^{\alpha}_{j\ell m} \rangle_{\alpha} \approx \langle F^{\alpha,\beta}_{j\ell m} \rangle_{\alpha,\beta} (1 - 8c) \tag{3.16}$$

By definition, $\langle F^{\alpha\beta}_{j\ell m} \rangle_{\alpha,\beta} = F^{T.P.}_{j\ell m}(\underline{r}^{\alpha} - \underline{r}^{\beta}) - F^{S.P.}_{j\ell m}$, and so the approximation (3. 12) is correct to order c.

With the above approximations for the ensemble averages, equation (3. 11) becomes:

$$\begin{aligned} \langle v_i(\underline{r};\underline{0}) \rangle_0 \approx & -n\left(F_j + \frac{a^2}{6} F_j \delta_{\ell m} \partial_\ell \partial_m \right) \int_{|\underline{s}|<2a} S_{ij}(\underline{r}-\underline{s})\, d\underline{s} \\ & + n \int_{|\underline{s}| \geq 2a} \left(F^{T.P.}_{j\ell}(\underline{s}) - F^{S.P.}_{j\ell} \right) \partial_\ell S_{ij}(\underline{r}-\underline{s})\, d\underline{s} \\ & + n \int_{|\underline{s}| \geq 2a} \left(F^{T.P.}_{j\ell m}(\underline{s}) - F^{S.P.}_{j\ell m} \right) \partial_\ell \partial_m S_{ij}(\underline{r}-\underline{s})\, d\underline{s} + \ldots . \end{aligned} \tag{3.17}$$

From the two particle problem, $F^{T.P.}_{j\ell}(\underline{s}) - F^{S.P.}_{j\ell} \sim \frac{1}{s^2}$ as $|\underline{s}| \to \infty$, so that the second integral above is convergent. Similarly, it is found that the other integrals in (3.17) are convergent, and $\langle v_i(\underline{r};\underline{0})\rangle_0$ can now be evaluated. In particular, the value of $\left(1+\frac{a^2}{6}\nabla^2\right)\langle v_i(\underline{0};\underline{0})\rangle_0$ is desired in order to compute $\langle V_i^0\rangle_0$ from Faxen's law (3.6). The first term on the right hand side of (3.17) is simple to compute using (3.9), with the result

$$-n\left(F_j+\frac{a^2}{6}F_j\nabla^2\right)\int_{|\underline{s}|<2a} S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$$
$$=\frac{F_i}{6\pi\mu a}\left[-\frac{11}{2}c+\frac{4}{5}\pi a n r^2-\frac{2}{5}\pi a n r_i^2\right]. \quad (\text{no sum on } i) \tag{3.18}$$

Then operating on (3.18) with $\left(1+\frac{a^2}{6}\nabla^2\right)$ and evaluating at $\underline{r}=\underline{0}$ gives

$$\left[\left(1+\frac{a^2}{6}\nabla^2\right)\frac{F_i}{6\pi\mu a}\left(-\frac{11}{2}c+\frac{4}{5}\pi a n r^2-\frac{2}{5}\pi a n r_i^2\right)\right]_{\underline{r}=\underline{0}}$$
$$=\frac{F_i}{6\pi\mu a}(-5c)=-5c\,\mathcal{U}_{0i}. \tag{3.19}$$

To complete the solution of the problem, only the value at $\underline{r}=\underline{0}$ of

$$\left(1+\frac{a^2}{6}\nabla^2\right)\left\{n\int_{|\underline{s}|\geq 2a}\left[\left(F_{jl}^{T.P.}(\underline{s})-F_{jl}^{S.P.}\right)\partial_l S_{ij}(\underline{r}-\underline{s})+\ldots\right]d\underline{s}\right\} \tag{3.20}$$

is needed. This can be found in terms of the numerical results for the motion of two finite spheres using a method devised by Batchelor (1972). His argument is presented below in our notation.

Consider two settling spheres with centers instantaneously at $\underline{0}$ and $\underline{s}$. The velocity of the sphere at $\underline{0}$ is given by Faxen's law

$$V_i^{T.P.}(\underline{s}) = U_{o_i} + \left[\left(1 + \frac{a^2}{6}\nabla^2\right) v_i(\underline{r};\underline{0})\right]_{\underline{r}=\underline{0}} .$$

Upon representing $v_i(\underline{r};\underline{0})$ in terms of the multipole expansion of the sphere at $\underline{s}$, we have

$$V_i^{T.P.}(\underline{s}) = U_{o_i} + \left[\left(1 + \frac{a^2}{6}\nabla^2\right)\left(F_j S_{ij}(\underline{r}-\underline{s}) + F_{j\ell}^{T.P.}(\underline{s})\partial_\ell S_{ij}(\underline{r}-\underline{s}) + \ldots\right]_{\underline{r}=\underline{0}} \quad (3.21)$$

Now suppose there were no sphere at $\underline{0}$, but the sphere at $\underline{s}$ remained. Then the fluid velocity at $\underline{0}$ is calculated from the single particle solution for a sphere with center at $\underline{s}$, namely

$$u_i(\underline{0}) = \left[F_j S_{ij}(\underline{r}-\underline{s}) + F_{j\ell m}^{S.P.}\partial_\ell\partial_m S_{ij}(\underline{r}-\underline{s})\right]_{\underline{r}=\underline{0}} .$$

Using the relation $F_{j\ell m}^{S.P.} = \frac{a^2}{6} F_j \delta_{\ell m}$ this can be rewritten

$$u_i(\underline{0}) = \left[\left(1 + \frac{a^2}{6}\nabla^2\right) F_j S_{ij}(\underline{r}-\underline{s})\right]_{\underline{r}=\underline{0}} . \quad (3.22)$$

Now take the difference of (3.21) and (3.22). This yields

$$V_i^{T.P.}(\underline{s}) - U_{o_i} - \left(1+\frac{a^2}{6}\nabla^2\right)u_i(\underline{0}) = \left[\left(1 + \frac{a^2}{6}\nabla^2\right)\left(\left[F_{j\ell}^{T.P.}(\underline{s}) - F_{j\ell}^{S.P.}\right]\partial_\ell S_{ij}(\underline{r}-\underline{s}) + \ldots\right)\right]_{\underline{r}=\underline{0}} \quad (3.23)$$

The left hand side of (3.23) is known; $V_i^{T.P.}(\underline{s})$ is known numerically from the two particle analysis of Goldman, Cox, and Brenner (1966), and U_{o_i} and $u_i(\underline{0})$ are known from theory. Upon integrating (3.23) over all space, the right hand side of (3.23) becomes (3.20), and the left hand side was computed by Batchelor to be

$$n \int_{|\underline{s}| \geq 2a} \left[V_i^{T.P.}(\underline{s}) - U_{0_i} - \left(1 + \frac{a^2}{6}\nabla^2\right) u_i(\underline{0}) \right] d\underline{s} = -1.55\, c\, U_{0_i} . \tag{3.24}$$

Finally, substituting the results (3.19) and (3.24) into Faxen's law (3.6) we have, as the final answer,

$$\langle \underline{V} \rangle = \langle \underline{V}^{\circ} \rangle_0 = \underline{U}_0 \left(1 - 6.55c + o(c)\right), \tag{3.25}$$

in the reference frame where the mean velocity of the dispersion is zero. In summary, the essential features of the method of solution were to specify the particle distribution in some way, use the representation (2.24) for the regular part of the velocity field, and apply Faxen's law (3.6). In this particular problem, the bulk of the analysis was relating the dispersion problem to the two sphere problem. The methods used by other authors are briefly summarized below.

Batchelor (1972) overcomes the difficulty of the divergent integrals by choosing quantities that have the same long range dependence as the variables he wants to calculate, and considers only the difference between them. He finds $\langle v_i(\underline{0};\underline{0})\rangle_0$ essentially by considering $\langle v_i(\underline{0};\underline{0})\rangle_0 - \langle u_i(\underline{0})\rangle$ where he knows $\langle u_i(\underline{0})\rangle = 0$. This amounts to considering the difference of the ensemble averages of (2.23) and (2.24). Then he calculates $\langle \nabla^2 v_i(\underline{0};\underline{0})\rangle_0$ by considering the mean deviatoric stress in the dispersion instead of merely computing $\nabla^2 \langle v_i(\underline{0};\underline{0})\rangle_0$ as was done here. His concluding two particle analysis is the one used above.

Pyun and Fixman (1964) expand the sphere velocity $\mathbf{V_s}$ and fluid velocity v_f in terms of powers of the small number density n. They keep only the leading term in n giving:

$$v_s = v_s(1) + n \int_{|\underline{r}| \geq 2a} [v_s(1,2) - v_s(1)]\, d\underline{r} + \ldots$$

and

$$v_f = v_f(1) + n \int_{|\underline{r}| \geq a} [v_f(1,2) - v_f(1)]\, d\underline{r} + \ldots .$$

where $v_f(1) \equiv 0$ is the average fluid velocity, $v_s(1)$ is the velocity of a single sphere, $v_s(2,1)$ is the velocity of sphere 1 in the presence of a second sphere 2 , etc. Neither of the above integrals is convergent, but only the difference $v_s - v_f$ is considered. Then,

$$v_s - v_f = v_s(1) + n \left\{ - \int_{a \leq |\underline{r}| < 2a} v_f(1,2)\, d\underline{r} + \int_{|\underline{r}| \geq 2a} [v_s(1,2) - v_s(1) - v_f(1,2)]\, d\underline{r} \right\}$$

The second integral above is identically (3. 24) of our solution. The first integral is (3. 18) evaluated at $\underline{r} = \underline{0}$. They did not use Faxen's law for the velocity of the sphere, and consequently did not find the $+\frac{1}{2} c \underline{U}_0$ contribution that is obtained from operating on (3. 18) with $\left(1 + \frac{a^2}{6} \nabla^2\right)$ to give (3. 19). An unusual feature of their solution is the analysis of a case where the spheres may intersect and together move as a rigid body, which they used as a model of a polymer solution.

Finally, J. B. Keller (unpublished) considered the corresponding point particle problem, and using a similar analysis to the one presented here, arrived at the leading term of (3. 9), i. e., a hindrance of $-6c\underline{U}_0$. Of course, since the finite size of the spheres is an order c effect, the point particle problem cannot give the correct answer to order c .

IV. VISCOUS FLOW PAST A RANDOM FIXED ARRAY

In this example, it is assumed that the locations of particle centers $\underline{r}^{\alpha}$ are random variables, and the particles are held rigidly, i.e., they may not translate or rotate. We wish to find the expected force acting on a sphere in terms of the mean fluid velocity in the array. The problem will be solved in the reference frame where the mean velocity of the fluid plus solid is zero, and the result will be related to the solution when the particles are held fixed. It is assumed that the particles are identical spheres and that the volume concentration of particles, c , is small. Only the point particle approximation will be considered, but it is used here in the sense of a leading order approximation to the complete problem. This means that the ensemble average (3.5) will be used here since, for the finite sphere problem, we want to exclude intersections of spheres. In the "strict" point particle problem, however, the particles are points and no such exclusion is necessary.

The physical difference between this example and the settling of free spheres (Chapter III) is that here the spheres are subjected to random forces dependent on the statistics of the array, while the free spheres were all subjected to the same drag force $\underline{D}$. The former effect is often called "shielding" since the drag force on a given sphere is reduced if another sphere is nearby.

Childress (1972) solved this problem to order c with an analysis based on partial summation of the formal power series in for finite distributions. His result was

$$\underline{F} = \left(1 + \frac{3}{\sqrt{2}} c^{1/2} + \frac{135}{64} c \log c\right) \underline{F}_0 + c\left(\underline{\underline{I}} \cdot \underline{F}_0\right) + \ldots$$

where $\underline{F}_0 = 6\pi a \mu \mathcal{U}_0$ and $\underline{\underline{I}}$ is expressed in terms of the forces in a two sphere Stokes flow. This result to order c is for the case when the conditional ensemble average (3.5) is used. Saffman (1973) obtained the same $c^{1/2}$ term as Childress, but did not proceed any further. The solution presented here obtains the $c \log c$ term using the solution (2.24), and directly derives the truncations proposed by Saffman. The effects of the two sphere distribution function in the array will be mentioned here and more fully discussed in Chapter VI.

Using the point particle approximation, the velocity (2.23) can be written

$$u_i(\underline{r}) = \sum_\alpha F_j^\alpha S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int \langle F_j \rangle S_{ij}(\underline{r}-\underline{s})\, d\underline{s} \qquad (4.1)$$

where

$$\langle u_i(\underline{r}) \rangle = 0 . \qquad (4.2)$$

Similarly, the regular part of the velocity field in the neighborhood of $\underline{r}^\beta$ is given by (2.24) as:

$$v_i(\underline{r};\beta) = \sum_\alpha{}' F_j^\alpha S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int \langle F_j \rangle S_{ij}(\underline{r}-\underline{s})\, d\underline{s} . \qquad (4.3)$$

As was done in the free array, it is assumed that the ensemble average about $\underline{r}^\beta$ equals the array average for any realization of the array. Then, taking the conditional ensemble average of (4.3) yields

$$\langle v_i(\underline{r};\beta)\rangle_\beta = \langle \sum_\alpha{}' \langle F_j^\alpha \rangle_\beta S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_\beta - n\int \langle F_j \rangle S_{ij}(\underline{r}-\underline{s})\, d\underline{s} \qquad (4.4)$$

To obtain an expression for $\langle F_j^\alpha \rangle_\beta$, we use the point particle version of Faxen's law (2.15a) found by setting $a = 0$, *i.e.*,

$$v_j(\underline{r}^\alpha;\alpha) = V_j^\alpha - \frac{F_j^\alpha}{6\pi\mu a} \quad . \tag{4.5}$$

The velocity V_j^α is the same for each particle α in this problem. Consequently, the superscript on V_j^α will be dropped. Taking the conditional ensemble average of (4.5) yields

$$\langle v_j(\underline{r}^\alpha;\alpha)\rangle_\beta = V_j - \frac{\langle F_j^\alpha\rangle_\beta}{6\pi\mu a} \quad . \tag{4.6}$$

Equation (4.6) relates $\langle F_j^\alpha\rangle_\beta$ to $\langle v_j(\underline{r}^\alpha;\alpha)\rangle_\beta$. An expression for the latter is now obtained by averaging (4.3) (and renaming the subscripts and superscripts) with the result:

$$\langle v_j(\underline{r}^\alpha;\alpha)\rangle_{\alpha,\beta} = \langle {\sum_\gamma}' \langle F_k^\gamma\rangle_{\alpha,\beta} S_{jk}(\underline{r}^\alpha - \underline{r}^\gamma)\rangle_{\alpha,\beta} - n\int \langle F_k\rangle S_{jk}(\underline{r}^\alpha - \underline{\xi})d\underline{\xi} \tag{4.7}$$

The double subscript α,β means both α and β are fixed in this conditional ensemble average. The presence of the term $\langle F_k^\gamma\rangle_{\alpha,\beta}$ in (4.7) indicates a developing hierarchy of equations -- in order to find the one particle conditional average $\langle F_j^\alpha\rangle_\beta$, the two particle conditional average $\langle F_j^\alpha\rangle_{\beta,\gamma}$ is needed. It is convenient to truncate the hierarchy with an approximation for the ensemble average velocities rather than the forces.

Consider the difference between $\langle v_j(\underline{r}^\alpha;\alpha)\rangle_\beta$ and $\langle u_j(\underline{r}^\alpha)\rangle_\beta$, where, in the second average, the probability that a sphere center is within $2a$ of $\underline{r}^\alpha$ is order c. When $\underline{r}^\alpha$ is far from $\underline{r}^\beta$, the presence or absence of the particle at $\underline{r}^\beta$ makes little difference, so asymptotically as $|\underline{r}^\alpha - \underline{r}^\beta| \to \infty$,

$$\langle v_j(\underline{r}^\alpha;\alpha)\rangle_\beta - \langle u_j(\underline{r}^\alpha)\rangle_\beta \approx \langle v_j(\underline{r}^\alpha;\alpha)\rangle - \langle u_j(\underline{r}^\alpha)\rangle = V_j - \frac{\langle F_j\rangle}{6\pi\mu a}$$

with an order c error. When $\underline{r}^\alpha$ is near $\underline{r}^\beta$, the difference between the averages (which occurs because the presence of the particle at $\underline{r}^\alpha$ shields the nearby particles) remains bounded. Thus, the indicated truncation is:

$$\langle v_j(\underline{r}^\alpha;\alpha)\rangle_\beta - \langle u_j(\underline{r}^\alpha)\rangle_\beta = V_j - \frac{\langle F_j\rangle}{6\pi\mu a}\,. \tag{4.8}$$

Using this truncation, an equation for $\langle u_i(\underline{r})\rangle_\beta$ can be obtained. First, (4.1) is averaged with β fixed, which gives

$$\langle u_i(\underline{r})\rangle_\beta = \langle F_j\rangle S_{ij}(\underline{r}-\underline{r}^\beta) + \langle {\sum_\alpha}' \langle F_j^\alpha\rangle_\beta S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_\beta - n\int \langle F_j\rangle S_{ij}(\underline{r}-\underline{s})\,d\underline{s}\,.$$

Then using (4.8) and (4.6), the unknowns $\langle F_j^\alpha\rangle_\beta$ can be expressed in terms of $\langle u_j(\underline{r}^\alpha)\rangle_\beta$ as

$$\langle F_j^\alpha\rangle_\beta = \langle F_j\rangle - 6\pi\mu a\langle u_j(\underline{r}^\alpha)\rangle_\beta\,. \tag{4.9a}$$

When this approximation for $\langle F_j^\alpha\rangle_\beta$ is substituted into the above equation for $\langle u_i(\underline{r}^\alpha)\rangle_\beta$, the result is

$$\begin{aligned}\langle u_i(\underline{r})\rangle_\beta &= \langle F_j\rangle S_{ij}(\underline{r}-\underline{r}^\beta) + \langle {\sum_\alpha}' \langle F_j\rangle S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_\beta \\ &\quad - 6\pi\mu a\langle {\sum_\alpha}' \langle u_j(\underline{r}^\alpha)\rangle_\beta S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_\beta - n\int \langle F_j\rangle S_{ij}(\underline{r}-\underline{s})\,d\underline{s}\,.\end{aligned} \tag{4.9b}$$

Finally, using (3.5), the ensemble averages are replaced by integrals. This gives the following integral equation for $\langle u_i(\underline{r})\rangle_\beta$,

$$\langle u_i(\underline{r})\rangle_\beta = \langle F_j\rangle S_{ij}(\underline{r}-\underline{r}^\beta) - n\int_{|\underline{s}-\underline{r}^\beta|<2a} \langle F_j\rangle S_{ij}(\underline{r}-\underline{s})\,d\underline{s} - 6\pi\mu a n\int_{|\underline{s}-\underline{r}^\beta|>2a} \langle u_j(\underline{s})\rangle_\beta S_{ij}(\underline{r}-\underline{s})\,d\underline{s}\,, \tag{4.10a}$$

which can be rewritten as

$$\langle u_i(\underline{r})\rangle_\beta = F_j S_{ij}(\underline{r}-\underline{r}^\beta) - 6\pi\mu a n \int \langle u_j(\underline{s})\rangle_\beta S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$$

$$- n \int_{|\underline{s}-\underline{r}^\beta|<2a} F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s} + 6\pi\mu a n \int_{|\underline{s}-\underline{r}^\beta|<2a} \langle u_j(\underline{s})\rangle_\beta S_{ij}(\underline{r}-\underline{s})\,d\underline{s}\,, \tag{4.10b}$$

where $F_j \equiv \langle F_j \rangle$.

The integral equation (4. 10b) has a displacement kernel and is straightforward to solve using Fourier transforms. Define

$$\hat{f}(\underline{k}) = \frac{1}{8\pi^3}\int f(\underline{r})e^{-i\underline{k}\cdot\underline{r}}\,d\underline{r}\,, \qquad f(\underline{r}) = \int \hat{f}(\underline{k})e^{i\underline{k}\cdot\underline{r}}\,d\underline{k}\;. \tag{4. 11}$$

Then the transform of (4. 10) is:

$$\langle \hat{u}_i(\underline{k})\rangle_\beta = F_j \hat{S}_{ij}(\underline{k})\, e^{-i\underline{k}\cdot\underline{r}^\beta} - 6\pi\mu a n \cdot 8\pi^3 \langle \hat{u}_j(\underline{k})\rangle_\beta \hat{S}_{ij}(\underline{k})$$

$$- n \int_{|\underline{s}-\underline{r}^\beta|<2a} F_j e^{-i\underline{k}\cdot\underline{s}} \hat{S}_{ij}(\underline{k})\,d\underline{s} + 6\pi\mu a n \int_{|\underline{s}-\underline{r}^\beta|<2a} \langle u_j(\underline{s})\rangle_\beta e^{-i\underline{k}\cdot\underline{s}} \hat{S}_{ij}(\underline{k})\,d\underline{s} \tag{4. 12}$$

where

$$\hat{S}_{ij}(\underline{k}) = \frac{1}{8\pi^3\mu}\left(\frac{\delta_{ij}}{k^2} - \frac{k_i k_j}{k^4}\right) \tag{4. 13}$$

It is convenient to define

$$P_{ij}(\underline{k}) = \delta_{ij} - \frac{k_i k_j}{k^4}\;. \tag{4. 14}$$

Then using the equation of continuity in the form $P_{ij}(\underline{k})\langle \hat{u}_j(\underline{k})\rangle_\beta = \langle \hat{u}_i(\underline{k})\rangle_\beta$, equation (4. 12) can be written

$$\langle \hat{u}_i(\underline{k})\rangle_\beta \left[1 + \frac{6\pi a n}{k^2}\right] = \frac{F_j}{8\pi^3\mu}\,\frac{P_{ij}}{k^2}\, e^{-i\underline{k}\cdot\underline{r}^\beta}$$

$$+ \frac{1}{8\pi^3\mu}\,\frac{P_{ij}}{k^2} \int_{|\underline{s}-\underline{r}^\beta|<2a} \left[-nF_j + 6\pi\mu a n \langle u_j(\underline{s})\rangle_\beta\right] e^{-i\underline{k}\cdot\underline{s}}\,d\underline{s}\;. \tag{4. 15}$$

When the radius $a = 0$, this reduces to Saffman's equation [his (5.15)]. Remembering that the value of $\langle v_i(\underline{r}^\beta;\beta)\rangle_\beta$ is needed to apply Faxen's law, the substitution

$$\langle \hat{u}_i(\underline{k})\rangle_\beta = \langle \hat{v}_i(\underline{k};\beta)\rangle_\beta + F_j \hat{S}_{ij}(\underline{k})\, e^{-i\underline{k}\cdot\underline{r}^\beta} \tag{4.16}$$

is made on the left hand side of (4.15). (Equation (4.16) is merely the transform of the definition of v_i.) Then solving (4.15) for $\langle \hat{v}_i(\underline{k};\beta)\rangle_\beta$ yields:

$$\begin{aligned}\langle \hat{v}_i(\underline{k};\beta)\rangle_\beta = & -\frac{\lambda^2}{8\pi^3\mu}\frac{P_{ij}F_j}{k^2(k^2+\lambda^2)}\, e^{-i\underline{k}\cdot\underline{r}^\beta} \\ & + \frac{1}{8\pi^3\mu}\frac{P_{ij}}{(k^2+\lambda^2)}\int_{|\underline{s}-\underline{r}^\beta|<2a}\left[-nF_j+\mu\lambda^2\langle u_j(\underline{s})\rangle_\beta\right]e^{-i\underline{k}\cdot\underline{s}}\,d\underline{s}\end{aligned} \tag{4.17}$$

where

$$\lambda^2 = 6\pi a n .$$

The desired answer $\langle v_i(\underline{r};\beta)\rangle_\beta$ is now found by transforming (4.17). In terms of $\underline{r}' \equiv \underline{r} - \underline{r}^\beta$, the result is

$$\begin{aligned}\langle v_i(\underline{r};\beta)\rangle_\beta = & -\frac{F_j}{6\pi\mu a}\cdot\frac{3}{2}a\left(\delta_{ij}A(\lambda r') - \frac{r_i r_j}{r^2}B(\lambda r')\right) \\ & + \int_{|\underline{s}-\underline{r}^\beta|<2a}\left[-nF_j+\mu\lambda^2\langle u_j(\underline{s})\rangle_\beta\right]C_{ij}(\underline{r}-\underline{s})\,d\underline{s}\end{aligned} \tag{4.18a}$$

where

$$\begin{aligned}A(\lambda r) &\equiv \frac{1}{\lambda^2 r^3}\left[1+\tfrac{1}{2}\lambda^2 r^2 - e^{-\lambda r}(1+\lambda r+\lambda^2 r^2)\right] \\ B(\lambda r) &\equiv \frac{3}{\lambda^2 r^3}\left[1-\tfrac{1}{6}\lambda^2 r^2 - e^{-\lambda r}(1+\lambda r+\tfrac{1}{3}\lambda^2 r^2)\right]\end{aligned} \tag{4.18b}$$

and

$$C_{ij}(\underline{r}) = \frac{1}{4\pi\mu}\left[\frac{e^{-\lambda r}}{r}\delta_{ij} + \frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1-e^{-\lambda r}}{r\lambda^2}\right)\right]. \tag{4.18c}$$

This solution has the required generality to be used as the first approximation to the fixed array problem with finite particle size. Note the integral in (4. 18) gives the effect of the finite radius, and the dependence on $\underline{r}'$ is needed to apply the complete Faxen's law (2.15a). Actually, the answer for v_i is given only implicitly because the unknown $\langle u_j(\underline{s})\rangle_\beta$ is on the right hand side. However, $\langle u_j(\underline{s})\rangle_\beta$ is known sufficiently well $\left(\langle u_j(\underline{s})\rangle_\beta \approx F_j S_{ij}(\underline{s}-\underline{r}^\beta)\right)$ near $\underline{s}-\underline{r}^\beta = \underline{0}$ to be a good approximation in computing the left hand side of (4. 18).

We consider only the point particle approximation here; hence, in (4. 18) we set $a = 0$ and evaluate the right hand side at $\underline{r}' = \underline{0}$ with the result

$$\langle v_i(\underline{r}^\beta;\beta)\rangle_\beta = -\frac{F_i}{6\pi\mu a}\lambda a = -\frac{F_i}{6\pi\mu a}\frac{3}{\sqrt{2}}c^{1/2}.$$

An ensemble average of Faxen's law (4. 5) yields

$$\langle v_j(\underline{r}^\beta;\beta)\rangle_\beta = V_j - \frac{F_j}{6\pi\mu a}. \tag{4.19}$$

Therefore, the desired relation between the velocity of the array and the mean drag force per particle is given by:

$$V_j = \frac{F_j}{6\pi\mu a}\left(1 - \frac{3}{\sqrt{2}}c^{1/2}\right). \tag{4.20}$$

There are two interesting features in the above solution that should be mentioned. In finding the answer (4. 20) to order $c^{1/2}$, we have neglected the integral term in (4. 18). This integral is directly related to the conditional ensemble average chosen, and (4.20) is correct only for ensemble averages where the corresponding integral is $o(c^{1/2})$. In the case where (3.5) is used as the ensemble average, the integral is only order c, so the answer stated is correct. For other ensemble averages, the form of that integral changes, and neglecting that integral may not be correct. An example of this is presented in Chapter VI.

The other aspect of the solution is the shielding effect. Saffman (1973) found that with $\alpha = 0$ the solution of (4.15) was

$$\langle u_i(\underline{r})\rangle_\beta = \frac{F_j}{4\pi\mu}\left[\frac{e^{-\lambda r'}}{r'}\delta_{ij} + \frac{\partial^2}{\partial r_i'\partial r_j'}\left(\frac{1-e^{-\lambda r'}}{r'\lambda^2}\right)\right] = F_j C_{ij}(\underline{r}') \quad (4.21)$$

Hence, $\langle u_i(\underline{r})\rangle_\beta$ decays like $\frac{1}{r'^3}$ as $r' \to \infty$, while in the absence of shielding $u_i \sim \frac{1}{r'}$ as $r' \to \infty$. Thus, the presence of the shielding changes the asymptotic dependence of the velocity field due to the presence of a particle.

In order to improve the accuracy of the result, it is necessary to truncate the hierarchy at the next higher level by considering ensemble averages where the positions of two particles are given. This leads to an integral equation for $\langle u_i(\underline{r})\rangle_{\beta,\gamma}$ which is solved as before. Using the result, a better approximation to the one particle ensemble average $\langle F_i^\alpha\rangle_\beta$ is obtained. In turn, this approximation is used in

$$\langle v_i(\underline{r};\beta)\rangle_\beta = \langle \sum_\alpha{}' \langle F_j^\alpha\rangle_\beta S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_\beta - n\int F_j S_{ij}(\underline{r}-\underline{s})d\underline{s} \quad (4.4)$$

to find the relation between the array velocity V_i and the mean force per particle F_i to the next order. We now proceed with the details.

First, equation (4.1) is averaged keeping two particles fixed with the result

$$\langle u_i(\underline{r})\rangle_{\beta,\gamma} = \langle \sum_{\alpha} \langle F_j^{\alpha}\rangle_{\beta,\gamma} S_{ij}(\underline{r}-\underline{r}^{\alpha})\rangle_{\beta,\gamma} - n\int F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s} \,. \tag{4.22}$$

Then $\langle F_j^{\alpha}\rangle_{\beta,\gamma}$ is found in terms of $\langle v_j(\underline{r}^{\alpha};\alpha)\rangle_{\beta,\gamma}$ by averaging (4.5):

$$\langle v_j(\underline{r}^{\alpha};\alpha)\rangle_{\beta,\gamma} = V_i - \frac{\langle F_j^{\alpha}\rangle_{\beta,\gamma}}{6\pi\mu a} \tag{4.23}$$

By the same reasoning that led to the first level truncation (4.8), the indicated truncation at the second level is

$$\langle v_j(\underline{r}^{\alpha};\alpha)\rangle_{\beta,\gamma} - \langle u_j(\underline{r}^{\alpha})\rangle_{\beta,\gamma} = V_j - \frac{F_j}{6\pi\mu a} \tag{4.24}$$

It is evident by averaging over β and γ that this truncation preserves the zero mean velocity of the dispersion so that the boundary condition (4.2) remains satisfied. We also note that (4.24) is equivalent to the second level truncation proposed by Saffman (1973) [his equation (5.9)]. The truncation (4.24) now replaces the truncation (4.8) used earlier.

Similar to the previous analysis, an integral equation for $\langle u_i(\underline{r})\rangle_{\beta,\gamma}$ is now obtained by substituting (4.24) and (4.23) into (4.22). The result after using the usual conditional ensemble average is:

$$\langle u_i(\underline{r})\rangle_{\beta,\gamma} = \langle F_j^{\beta}\rangle_{\gamma} S_{ij}(\underline{r}-\underline{r}^{\beta}) + \langle F_j^{\gamma}\rangle_{\beta} S_{ij}(\underline{r}-\underline{r}^{\gamma}) - n \int_{|\underline{s}-\underline{r}^{\beta}|<2a} F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$$
$$- 6\pi\mu a n \int \langle u_j(\underline{s})\rangle_{\beta,\gamma} S_{ij}(\underline{r}-\underline{s})\,d\underline{s} + 6\pi\mu a n \int_{|\underline{s}-\underline{r}^{\beta}|<2a} \langle u_j(\underline{s})\rangle_{\beta,\gamma} S_{ij}(\underline{r}-\underline{s})\,d\underline{s}\,. \tag{4.25}$$

This equation is the same as (4.10b) except for the extra Stokeslet here. It is solved in the same manner as the steps leading to (4.15) with the result:

$$\langle \hat{u}_i(\underline{k})\rangle_{\beta,\gamma}\left[1 + \frac{6\pi a n}{k^2}\right] = \frac{\langle F_j^{\beta}\rangle_{\gamma}}{8\pi^3\mu}\frac{P_{ij}}{k^2} e^{-i\underline{k}\cdot\underline{r}^{\beta}} + \frac{\langle F_j^{\gamma}\rangle_{\beta}}{8\pi^3\mu}\frac{P_{ij}}{k^2} e^{-i\underline{k}\cdot\underline{r}^{\gamma}}$$
$$+ \frac{1}{8\pi^3\mu}\frac{P_{ij}}{k^2}\int_{|\underline{s}-\underline{r}^{\beta}|<2a}\left[-nF_j + \mu\lambda^2\langle u_j(\underline{s})\rangle_{\beta,\gamma}\right] e^{-i\underline{k}\cdot\underline{s}}\,d\underline{s} \tag{4.26}$$

Next, the substitution

$$\langle \hat{u}_i(\underline{k})\rangle_{\beta,\gamma} = \langle \hat{v}_i(\underline{k};\beta)\rangle_{\gamma,\beta} + \langle F_j^{\beta}\rangle_{\gamma}\hat{S}_{ij}(\underline{k})\, e^{-i\underline{k}\cdot\underline{r}^{\beta}} \tag{4.27}$$

is made on the left hand side of (4.26), which, in turn, is solved for $\langle \hat{v}_i(\underline{k};\beta)\rangle_{\beta,\gamma}$:

$$\langle \hat{v}_i(\underline{k};\beta)\rangle_{\beta,\gamma} = -\frac{\lambda^2}{8\pi^3\mu}\frac{P_{ij}\langle F_j^{\beta}\rangle_{\gamma}}{k^2(k^2+\lambda^2)} e^{-i\underline{k}\cdot\underline{r}^{\beta}} + \frac{\langle F_j^{\gamma}\rangle_{\beta}}{8\pi^3\mu}\frac{P_{ij}}{k^2+\lambda^2} e^{-i\underline{k}\cdot\underline{r}^{\gamma}}$$
$$+ \frac{1}{8\pi^3\mu}\frac{P_{ij}}{k^2+\lambda^2}\int_{|\underline{s}-\underline{r}^{\beta}|<2a}\left[-nF_j + \mu\lambda^2\langle u_j(\underline{s})\rangle_{\beta,\gamma}\right] e^{-i\underline{k}\cdot\underline{s}}\,d\underline{s} \tag{4.28}$$

As before, this equation is transformed back to $\underline{r}$ space:

$$\langle v_i(\underline{r};\beta)\rangle_{\beta,\gamma} = -\frac{\langle F_j^{\beta}\rangle_{\gamma}}{6\pi\mu a}\cdot\frac{3}{2}a\left[\delta_{ij}A(\lambda r') - \frac{r_i r_j}{r^2}B(\lambda r')\right] + \langle F_j^{\gamma}\rangle_{\beta} C_{ij}(\underline{r}-\underline{r}^{\gamma})$$
$$+ \int_{|\underline{s}-\underline{r}^{\beta}|<2a}\left[-nF_j + \mu\lambda^2\langle u_j(\underline{s})\rangle_{\beta,\gamma}\right] C_{ij}(\underline{r}-\underline{s})\,d\underline{s}\quad. \tag{4.29}$$

where A, B, and C_{ij} are defined in (4.18). Evaluating (4.29) at $\underline{r} = \underline{r}^{\beta}$ gives

$$\langle v_i(\underline{r}^{\beta};\beta)\rangle_{\beta,\gamma} = -\frac{\langle F_i^{\beta}\rangle_{\gamma}}{6\pi\mu a}\lambda a + \langle F_j^{\gamma}\rangle_{\beta} C_{ij}(\underline{r}^{\beta}-\underline{r}^{\gamma}) \qquad (4.30)$$

where the integral has been neglected since it is an order c term.

The above equation provides a relation between the unknowns $\langle F_i^{\beta}\rangle_{\gamma}$, $\langle F_i^{\gamma}\rangle_{\beta}$, and $\langle v_i(\underline{r}^{\beta};\beta)\rangle_{\beta,\gamma}$. Now we will find the conditional averages $\langle F_i^{\gamma}\rangle_{\beta}$. Using the equation

$$\langle v_i(\underline{r}^{\beta};\beta)\rangle_{\beta,\gamma} = V_i - \frac{\langle F_i^{\beta}\rangle_{\gamma}}{6\pi\mu a}$$

the unknown $\langle v_i(\underline{r}^{\beta};\beta)\rangle_{\beta,\gamma}$ can be eliminated from (4.30) with the result

$$V_i = \frac{\langle F_i^{\beta}\rangle_{\gamma}}{6\pi\mu a}(1-\lambda a) + \langle F_j^{\gamma}\rangle_{\beta} C_{ij}(\underline{r}^{\beta}-\underline{r}^{\gamma}). \qquad (4.31a)$$

Another three equations can be obtained by interchanging the particle names β and γ in the above equation:

$$V_i = \frac{\langle F_i^{\gamma}\rangle_{\beta}}{6\pi\mu a}(1-\lambda a) + \langle F_j^{\beta}\rangle_{\gamma} C_{ij}(\underline{r}^{\gamma}-\underline{r}^{\beta}). \qquad (4.31b)$$

Now equations (4.31) are a system of six linear equations in the six unknowns $\langle F_i^{\beta}\rangle_{\gamma}$ and $\langle F_i^{\gamma}\rangle_{\beta}$. However, since $C_{ij}(\underline{r})$ is an even function of $\underline{r}$, the equations are symmetric in the unknowns. This means that $\langle F_i^{\gamma}\rangle_{\beta} = \langle F_i^{\beta}\rangle_{\gamma}$, and the preceding system immediately reduces to

$$V_i = \frac{\langle F_i^{\gamma}\rangle_{\beta}}{6\pi\mu a}(1-\lambda a) + \langle F_j^{\gamma}\rangle_{\beta} C_{ij}(\underline{r}^{\beta}-\underline{r}^{\gamma}) \qquad (4.32)$$

which is a system of three equations for the three unknowns $\langle F_i^\gamma \rangle_\beta$. The solution of these equations is:

$$\begin{cases} \langle F_1^\gamma \rangle_\beta = V_1 \dfrac{6\pi\mu a}{1-\lambda a} \dfrac{1+dD-\frac{r_1^2}{r^2}dE}{(1+dF)(1+dD)} \\ \langle F_2^\gamma \rangle_\beta = V_1 \dfrac{6\pi\mu a}{1-\lambda a} \dfrac{-d\frac{r_1 r_2}{r^2}E}{(1+dF)(1+dD)} \\ \langle F_3^\gamma \rangle_\beta = V_1 \dfrac{6\pi\mu a}{1-\lambda a} \dfrac{-d\frac{r_1 r_3}{r^2}E}{(1+dF)(1+dD)} \end{cases} \qquad (4.33)$$

where the coordinates have been oriented such that $x_1 \parallel \langle \underline{F} \rangle$ and

$$\begin{cases} D \equiv -\dfrac{2}{\lambda}\dfrac{e^{-\lambda r}}{r^2} + \dfrac{2}{\lambda^2}\dfrac{1-e^{-\lambda r}}{r^3} \\ E \equiv D - F \\ F \equiv \dfrac{e^{-\lambda r}}{r} + \dfrac{e^{-\lambda r}}{\lambda r^2} - \dfrac{1-e^{-\lambda r}}{\lambda^2 r^3} \end{cases} \qquad \begin{cases} d \equiv \dfrac{3}{2}\dfrac{a}{1-\lambda a} \\ r \equiv |\underline{r}^\gamma - \underline{r}^\beta| \, . \end{cases} \qquad (4.34)$$

Thus, we have obtained a new approximation for the conditional ensemble average $\langle F_i^\gamma \rangle_\beta$ where one particle is held fixed. This value of $\langle F_i^\gamma \rangle_\beta$ will now be used directly in equation (4.4) to find the expected regular part of the velocity field near a particle.

For simplicity, we suppose the particle β in (4.4) is at $\underline{O}$, and we will only look at the component of $\underline{V}$ in the direction of motion. Then (4.4) becomes

$$\langle v_1(\underline{0};0)\rangle_0 = \langle \sum_\alpha{}' \langle F_j^\alpha \rangle_0 S_{1j}(\underline{r}^\alpha) \rangle_0 - n\int F_j S_{1j}(\underline{r})d\underline{r} \qquad (4.35)$$

The mean value of the force F_j is given from (4.33) as

$$F_1 = V_1 \frac{6\pi\mu a}{1-\lambda a}, \quad F_2 = 0, \quad F_3 = 0. \tag{4.36}$$

Upon substitution (4.33) and (4.36) into (4.35) we have

$$\begin{aligned} \langle v_1(\underline{0};0)\rangle_0 &= n\int_{|\underline{r}|\geq 2a} (\langle F_j^r\rangle_0 - F_j) S_{1j}(\underline{r})\,d\underline{r} - n\int_{|\underline{r}|<2a} F_j S_{1j}(\underline{r})\,d\underline{r} \\ &= V_1 \frac{\mu\lambda^2}{1-\lambda a} \int_{|\underline{r}|\geq 2a} \left[\frac{(1+dD-\frac{r_1^2}{r^2}dE)S_{11}(\underline{r}) - \frac{r_1 r_2}{r^2}dE\,S_{12}(\underline{r}) - \frac{r_1 r_3}{r^2}dE\,S_{13}(\underline{r})}{(1+dF)(1+dD)} - S_{11}(\underline{r}) \right] d\underline{r} \end{aligned} \tag{4.37}$$

where the order c term $n\int_{|\underline{r}|<2a} F_j S_{1j}(\underline{r})\,d\underline{r}$ has been neglected. To simplify this expression first note that by the definition of the Stokeslet

$$\sum_{j=1}^{3} -\frac{r_1 r_j}{r^2} dE\, S_{1j}(\underline{r}) = -\frac{dE}{8\pi\mu}\, 2\frac{r_1^2}{r^3}$$

so that

$$\langle v_1(\underline{0};0)\rangle_0 = \frac{V_1}{8\pi\mu}\frac{\mu\lambda^2}{1-\lambda a}\int_{|\underline{r}|\geq 2a} \frac{-2\frac{r_1^2}{r^2}dE - (dF + d^2 dF)\left(\frac{1}{r} + \frac{r_1^2}{r^3}\right)}{(1+dD)(1+dF)}\, d\underline{r}$$

When the angular integration is carried out, we use the fact that $\int r_1^2 d\Omega = \frac{1}{3}\int r^2 d\Omega$. Then the integral simplifies to

$$\langle v_1(\underline{0};0)\rangle_0 = -\frac{4\pi}{3} n d^2 V_1 \int_{r\geq 2a} \left(\frac{rF}{1+dF} + \frac{rD}{1+dD} \right) dr \tag{4.38}$$

The right hand side of (4.38) can now be expressed as an expansion in the small parameter c. The method involves dividing the region of integration into three parts, and finding an expansion for each region. Details of this calculation can be found in Appendix B, and here we mention only the result:

$$\langle v_1(\underline{0};0)\rangle_0 = V_1\left(-\frac{3}{\sqrt{2}}c^{1/2} - \frac{135}{64}c\log c + O(c)\right).$$

Finally, making use of (4.20), the relation between V_1 and F_1 is given by

$$V_1 = \frac{F_1}{6\pi\mu a}\left(1 - \frac{3}{\sqrt{2}}c^{1/2} - \frac{135}{64}c\log c + O(c)\right). \tag{4.39}$$

This is the best result obtainable by use of the point particle approximation, since the finite size of the particles is an order c effect. As was mentioned earlier, Childress (1972) found the order c terms in the expansion. To obtain order c accuracy by the above method, it would be necessary to keep the other multipole terms in (2.24), and this lengthy analysis was not attempted.

V. CUBIC ARRAY OF SPHERES

This section considers the settling of a cubic array of spheres. The model was first posed by Hasimoto (1959) as an example where flow past an infinite array could be solved in a rigorous way, namely with Fourier series. This avoided all the problems of divergent integrals and sums encountered when solving for flow past a random array of spheres. Hasimoto considered three arrays: the simple cubic lattice, the body-centered cubic lattice, and the face-centered cubic lattice. Only the simple cubic lattice is considered here, and our result for this case,

$$\underline{V} = \underline{U}_0 (1 - 1.76 c^{1/3} + c + \dots),$$

agrees with Hasimoto's result. The method described in Chapter II is straightforward to apply, but the geometry of this problem requires the evaluation of several three-dimensional sums. These sums are the main difficulty of this problem. Hasimoto uses a transformation due to Ewald to sum these series. We will describe a simple method of summation that uses only the mean value theorem. There are a few differences between the definitions used here and the ones used by Hasimoto. These will be mentioned at the end of this section.

Consider now the simple cubic lattice of side b where the particle centers are at the points

$$\underline{r}^{\alpha} = n_i^{\alpha} b \underline{e}_i \qquad (5.1)$$

where $\{n_i^{\alpha}\}$ are integer triads, and $\{e_i\}$ are orthonormal vectors. By symmetry, each sphere has the same multipole representation, so the superscripts on the F's can be omitted. As usual, we choose the reference frame where the mean velocity of the dispersion is zero, and we consider the problem of finding the velocity of the spheres given the excess weight F_i of each sphere.

Thus, we wish to find the value of $\langle v_i(\underline{r};\alpha)\rangle_\alpha$ for a typical particle to obtain the solution. Choose the particle at $\underline{O}$. Then, by (2.24), we have

$$v_i(\underline{r};0) = \sum_\alpha{}' F_j S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int F_j S_{ij}(\underline{r}-\underline{s})d\underline{s}$$
$$+ \sum_\alpha{}' F_{jlm}\partial_l\partial_m S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int F_{jlm}\partial_l\partial_m S_{ij}(\underline{r}-\underline{s})d\underline{s} \qquad (5.2)$$
$$+ \ldots$$

where the odd multipole terms have been set equal to zero by the configuration symmetry about each sphere. The ensemble average of the left hand side is not needed in this problem since the surrounding configuration of spheres is completely specified. Equation (5.2) can be rewritten as

$$v_i(\underline{r};0) = \left[F_j + F_{jlm}\partial_l\partial_m + \ldots\right]\left\{\sum_\alpha{}' S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int S_{ij}(\underline{r}-\underline{s})d\underline{s}\right\} \qquad (5.3)$$

which shows that the complete solution can be found using the fundamental solution of the point particle problem and its derivatives. It is possible to make direct progress with the point particle solution

$$v_i(\underline{r};0) = \sum_\alpha{}' F_j S_{ij}(\underline{r}-\underline{r}^\alpha) - n\int F_j S_{ij}(\underline{r}-\underline{s})d\underline{s} \qquad (5.4)$$

by expanding the right hand side for $\underline{r}$ small, but it is easier to find the periodic function $u_i(\underline{r})$ and use the relation

$$v_i(\underline{r};0) = u_i(\underline{r}) - F_j S_{ij}(\underline{r}) \qquad (5.5)$$

to solve for $v_i(\underline{r};0)$.

Now, $u_i(\underline{r})$ has mean zero by definition, and has the representation

$$u_i(\underline{r}) = \sum_\alpha F_j S_{ij}(\underline{r} - b n_\ell^\alpha \underline{e}_\ell) - n\int F_j S_{ij}(\underline{r}-\underline{s})d\underline{s} \tag{5.6}$$

Formally, the Fourier transform of the sum is

$$\sum_\alpha F_j \hat{S}_{ij}(\underline{k})\, e^{-i\underline{k}\cdot b n_\ell^\alpha \underline{e}_\ell} . \tag{5.7}$$

Using the Poisson summation formula

$$\sum_\alpha e^{-i\underline{k}\cdot b n_\ell^\alpha \underline{e}_\ell} = \frac{8\pi^3}{b^3}\sum_\alpha \delta\left(\underline{k} - \frac{2\pi}{b} n_\ell^\alpha \underline{e}_\ell\right), \tag{5.8}$$

(5.7) can be rewritten as

$$F_j \hat{S}_{ij}(\underline{k})\, \frac{8\pi^3}{b^3}\sum_\alpha \delta\left(\underline{k} - \frac{2\pi}{b} n_\ell^\alpha \underline{e}_\ell\right) . \tag{5.9}$$

The Fourier transform of the (infinite) constant $n\int F_j S_{ij}(\underline{r}-\underline{s})d\underline{s}$ is simply $\delta(\underline{k})$ multiplied by a constant. The latter constant is determined by the fact that $\hat{u}_i(\underline{k})$ can have no part proportional to $\delta(\underline{k})$ because $u_i(\underline{r})$ has zero mean. Hence, the Fourier transform of $n\int F_j S_{ij}(\underline{r}-\underline{s})\,d\underline{s}$ must exactly cancel the $\delta(\underline{k})$ term in the sum (5.9). As a result,

$$\hat{u}_i(\underline{k}) = \frac{8\pi^3}{b^3} F_j \hat{S}_{ij}(\underline{k}) \sum_\alpha{}' \delta\left(\underline{k} - \frac{2\pi}{b} n_\ell^\alpha \underline{e}_\ell\right) \tag{5.10}$$

is the Fourier transform of $u_i(\underline{r})$. We recall that

$$\hat{S}_{ij}(\underline{k}) = \frac{1}{8\pi^3\mu}\left(\frac{\delta_{ij}}{k^2} - \frac{k_i k_j}{k^4}\right) .$$

Then transforming (5.10) yields the Fourier series representation of

$u_i(\underline{r})$:

$$u_i(\underline{r}) = \frac{F_j}{4\pi^2 b\mu} \sum_\alpha{}' \left(\frac{\delta_{ij}}{n^{\alpha 2}} - \frac{n_i^\alpha n_j^\alpha}{n^{\alpha 4}} \right) e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^\alpha} \tag{5.11}$$

where $\underline{n}^\alpha$ denotes a three-dimensional vector of which each component is an integer. The sum is over all lattice points in 3-space excluding the point $\underline{n}^\alpha = \underline{0}$. The required representation of $v_i(\underline{r};0)$ is now obtained by subtracting the Stokeslet at the origin:

$$v_i(\underline{r};0) = \frac{F_j}{4\pi^2 b\mu} \sum_\alpha{}' \left(\frac{\delta_{ij}}{n^{\alpha 2}} - \frac{n_i^\alpha n_j^\alpha}{n^{\alpha 4}} \right) e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^\alpha} - \frac{F_j}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3} \right). \tag{5.12}$$

In order to find the behavior of $v_i(\underline{r};0)$ for $\underline{r}$ small, it is convenient to divide it into two sums defined by

$$v_i(\underline{r};0) = \frac{F_j \delta_{ij}}{4\pi\mu} P(\underline{r}) + \frac{F_j}{4\pi\mu} \partial_i \partial_j R(\underline{r}) \tag{5.13}$$

where

$$P(\underline{r}) \equiv \frac{1}{\pi b} \sum_\alpha{}' \frac{1}{n^{\alpha 2}} e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^\alpha} - \frac{1}{r} \tag{5.14a}$$

and

$$R(\underline{r}) \equiv \frac{b}{4\pi^3} \sum_\alpha{}' \frac{1}{n^{\alpha 4}} e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^\alpha} - \frac{r}{2} . \tag{5.14b}$$

Now

$$\nabla^2 P(\underline{r}) = -\frac{1}{\pi b} \sum_\alpha{}' \left(\frac{2\pi}{b}\right)^2 e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^\alpha} + 4\pi\delta(\underline{r})$$

and using Poisson's summation formula (5.8), this becomes

$$\nabla^2 P(\underline{r}) = \frac{4\pi}{b^3} - 4\pi \sum_{\alpha}{}' \delta(\underline{r} - b\underline{n}^{\alpha}) . \tag{5.15}$$

From (5.15) it is clear that

$$T(\underline{r}) \equiv P(\underline{r}) - \frac{2}{3}\frac{\pi}{b^3} r^2 \tag{5.16}$$

is a harmonic function for $|\underline{r}| < b$, and so the Mean Value Theorem

$$T(\underline{r}) = \frac{1}{4\pi R^2} \iint\limits_{|\underline{s}| = R} T(\underline{r} + \underline{s})\, dS \tag{5.17}$$

may be applied to T [Courant and Hilbert (1962)]. When this theorem is used, it generates a new representation for T with the series part of $T(\underline{r})$ converging more rapidly.

Using this idea, $T(\underline{0})$ can be evaluated easily. Consider the new form $T_1(\underline{r})$ of T defined for $|\underline{r}| < \frac{b}{2}$ by the equation

$$T_1(\underline{r}) = \frac{1}{4\pi R_1^2} \iint\limits_{|\underline{s}| = R_1} T(\underline{r} + \underline{s})\, dS$$

where $R_1 \leq \frac{b}{2}$. For the integral of the sum we have

$$\frac{1}{4\pi R_1^2} \iint\limits_{|\underline{s}| = R_1} \sum_{\alpha}{}' \frac{e^{i(\underline{r}+\underline{s})\cdot \frac{2\pi}{b} \underline{n}^{\alpha}}}{n^{\alpha 2}} R_1^2 \sin\varphi \, d\varphi \, d\theta = \frac{1}{2} \sum_{\alpha}{}' \frac{e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^{\alpha}}}{n^{\alpha 2}} \int_0^{\pi} e^{i\underline{s}\cdot\frac{2\pi}{b}\underline{n}^{\alpha}} \sin\varphi \, d\varphi$$

$$= \frac{b}{2\pi R_1} \sum_{\alpha}{}' \frac{e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^{\alpha}} \sin \frac{2\pi}{b} R_1 n^{\alpha}}{n^{\alpha 3}}$$

where the sum now converges like $\frac{1}{n^{\alpha 3}}$ instead of $\frac{1}{n^{\alpha 2}}$. The result for $T_1(\underline{r})$, after integrating the other terms of T , is then

$$T_1(\underline{r}) = \frac{1}{\pi b}\left(\frac{b}{2\pi R_1}\right)\sum_{\alpha}{}' \frac{e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^{\alpha}} \sin\frac{2\pi}{b}R_1\underline{n}^{\alpha}}{n^{\alpha 3}} - \frac{1}{R_1} - \frac{2\pi}{3b^3}(r^2+R_1^2), \quad (5.18)$$

where we have taken $|\underline{r}| < R_1 \leq \frac{b}{2}$ in evaluating the other integrals. In a similar fashion, using T_1 instead of T, a new representation $T_2(\underline{r})$ can be found which is valid in the smaller region $|\underline{r}| < R_2 \leq \frac{R_1}{2}$. After N iterations the result is

$$T_N(\underline{r}) = -\frac{1}{R_1} - \frac{2\pi}{3b^3}(R_1^2 + R_2^2 + \ldots + R_N^2 + r^2)$$
$$+\frac{1}{\pi b}\left(\frac{b}{2\pi R_1}\right)\cdots\left(\frac{b}{2\pi R_N}\right)\sum_{\alpha}{}' \frac{e^{i\underline{r}\cdot\frac{2\pi}{b}\underline{n}^{\alpha}}\left(\sin\frac{2\pi}{b}R_1 n^{\alpha}\right)\ldots\left(\sin\frac{2\pi}{b}R_N n^{\alpha}\right)}{(n^{\alpha})^{N+2}} \quad (5.19)$$

where there is now rapid convergence of the sum.

A short computation was performed to find the non-dimensional quantity

$$\left[S\left(\tfrac{1}{b}\underline{r}\right)\right]_{\underline{r}=\underline{0}} \equiv -\frac{1}{B_1} - \frac{2\pi}{3}(B_1^2 + B_2^2 + \ldots + B_N^2)$$
$$+\frac{1}{\pi}\left(\frac{1}{2\pi B_1}\right)\cdots\left(\frac{1}{2\pi B_N}\right)\sum_{\alpha}{}' \frac{(\sin 2\pi B_1 n^{\alpha})\ldots(\sin 2\pi B_N n^{\alpha})}{(n^{\alpha})^{N+2}} \quad (5.20)$$

where

$$B_i = \frac{1}{b}R_i, \quad i = 1, \ldots, N.$$

The result was

$$S(\underline{0}) = -2.8373 \quad (5.21)$$

which agrees perfectly with Hasimoto's computed value. Thus, we have an expansion for $P(\underline{r})$ in the neighborhood of zero, namely

$$P(\underline{r}) = \frac{1}{b} S(\underline{0}) + \frac{2\pi}{3b^3} r^2 + \sum_{n=2}^{\infty} \sum_{m=0}^{m \le \frac{1}{2}n} a_{nm} Y_{2n}^{4m}(\underline{r}) \tag{5.22}$$

where the Y's are spherical harmonics*, and the cubic symmetry of the geometry has been used [Hasimoto (1959)]. The values of the constants a_{nm} are not needed for the order c term.

The second sum of (5.13), $R(\underline{r})$, satisfies the relation

$$\nabla^2 R(\underline{r}) = - P(\underline{r}) \tag{5.23}$$

and hence has the solution [Hasimoto (1959)]

$$R(\underline{r}) = \kappa - \frac{S(\underline{0})}{6b} r^2 - \frac{\pi}{30 b^3} r^4 - \sum_{n=2}^{\infty} \sum_{m \ne 0}^{m \le \frac{1}{2}n} \left(b_{nm} + \frac{a_{nm}}{2(4n+3)} \right) Y_{2n}^{4m}(\underline{r}) \tag{5.24}$$

where the actual values of the b_{nm}'s and κ are also not needed here. Hasimoto used the spherical harmonics to find the order c^2 in the expansion, but the only property of the expansion that is needed for the order c term in the simple cubic lattice is that the spherical harmonics are all of order equal or greater than 4. When this occurs, the order c term of the expansion is independent of the sum of spherical harmonics**.

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* $Y_n^m(x_1, x_2, x_3) = r^n P_n^m(\cos\theta) \cos m\phi$ where $\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta\cos\phi \\ x_3 = r\sin\theta\sin\phi . \end{cases}$

**In general, this means any lattice with the property that there are no spherical harmonics of order 2 in either (5.22) or (5.24) will have the same order c term in the expansion of the mean velocity of the array. The other two lattices examined by Hasimoto, namely the body-centered cubic and the face-centered cubic arrays, both have

Now that $P(\underline{r})$ and $R(\underline{r})$ are known, the value $v_i(\underline{r};0)$ can be found from (5.14). Looking forward to Faxen's law, we have

$$\left(1+\frac{a^2}{6}\nabla^2\right)v_i(\underline{r};0) = \frac{F_i}{4\pi\mu}\left(1+\frac{a^2}{6}\nabla^2\right)P(\underline{r}) + \frac{F_j}{4\pi\mu}\partial_i\partial_j\left(1+\frac{a^2}{6}\nabla^2\right)R(\underline{r})$$

$$= \frac{F_i}{4\pi\mu}\left[\frac{S(0)}{b} + \frac{2\pi}{3b^3}r^2 + \frac{2\pi a^2}{3b^3} + O(r^4)\right]$$

$$+\frac{F_j}{4\pi\mu}\left[-\frac{S(0)}{3b}\delta_{ij} - \frac{2\pi}{15b^3}r^2\delta_{ij} - \frac{4\pi}{15b^3}r_ir_j - \frac{2\pi a^2}{9b^3}\delta_{ij} - Y(\underline{r})\right] \qquad (5.25)$$

where $Y(\underline{r})$ has the properties $Y(\underline{0})=0$ and $\nabla^2 Y(\underline{0})=0$. $Y(\underline{r})$ comes from the double sum of (5.24). The expression (5.25) evaluated at $\underline{r}=\underline{0}$ gives the velocity of a finite sphere surrounded by point particles. In order to obtain the complete solution to order c of a finite sphere surrounded by finite particles, we return to (5.3) and estimate the higher multipole terms. This is easily done since the particles are widely separated. From the arguments in Chapter II we have

$$\begin{cases} F_{j\ell m} = F_{j\ell m}^{S.P.} = \dfrac{a^2}{6}\delta_{\ell m}F_j \\ \text{other multipoles} \quad 0 \end{cases},$$

which is correct to order c. Thus, from (5.3), the final solution is obtained by operating on (5.25) with $\left(1+\frac{a^2}{6}\nabla^2\right)$ and evaluating at $\underline{r}=\underline{0}$. The result is

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the property and the same order c term was found for these lattices. On the other hand, the coefficients of the respective $c^{1/3}$ terms are always dependent on the lattice.

$$\left(1+\frac{a^2}{6}\nabla^2\right)v_i(\underline{0};0) = \frac{F_i}{4\pi\mu}\left[\frac{2}{3}\frac{S(\underline{0})}{b} + \frac{8}{9}\pi\frac{a^2}{b^3}\right]$$

$$= \frac{F_i}{6\pi\mu a}\left[-1.76\,c^{1/3} + c\right] \qquad (5.26)$$

where the relation $c = \frac{4}{3}\pi\frac{a^3}{b^3}$ has been used. Then the settling velocity of the array is given by Faxen's law (2.16a) and the above equation as

$$\underline{V} = \underline{U}_0\left(1 - 1.76\,c^{1/3} + c + o(c)\right) \qquad (5.27)$$

where $U_{0i} = \frac{F_i}{6\pi\mu a}$.

While this seems to agree with the result derived by Hasimoto for the simple cubic lattice, some care is needed in the comparison of the two results. The result derived here is in the reference frame where the mean velocity of fluid plus solid is zero, whereas Hasimoto considers flow past a fixed array of spheres. Now the mean velocity of the fluid, $\underline{v}_f$, in our reference frame is given by

$$\underline{v}_f(1-c) + c\underline{V} = \underline{0} \qquad (5.28)$$

which expresses the fact that the net flow through any fixed surface is $\underline{0}$. The result (5.27) can be expressed in the fixed spheres reference frame by subtracting $\underline{V}$ from the velocities. Then the speed of the spheres is zero, and the mean velocity of the fluid $\underline{v}_f$ is given to order c by:

$$\underline{v}_f = -(1+c)\underline{V} \qquad (5.29)$$

The resulting relation between the mean velocity of the fluid and the force acting on a sphere is then

$$\frac{\underline{v}_f}{1+c} = \frac{\underline{F}}{6\pi\mu a}\left(1 - 1.76c^{1/3} + c + \ldots\right)$$

where (5.27) and (5.29) have been used. To order c, we thus have the relation

$$\underline{v}_f = \frac{\underline{F}}{6\pi\mu a}\left(1 - 1.76c^{1/3} + 2c + \ldots\right) \qquad (5.30)$$

between the mean fluid velocity $\underline{v}_f$ and the drag force $\underline{F}$ exerted on each sphere.

In contrast to this, Hasimoto considers the quantity $\mathcal{U}$ defined by

$$\mathcal{U} = \frac{1}{b^2}\iint_{-\frac{1}{2}b}^{+\frac{1}{2}b} v_1 \, dx_2 \, dx_3 \qquad (5.31)$$

where the region of integration is outside every sphere. The physical significance of $\mathcal{U}$ is that if fluid were flowing through an array of finite size, then the mean velocity of the fluid outside the array is $\mathcal{U}$, whereas inside the array the mean fluid velocity is $\underline{v}_f$. The relation between the two is simply

$$\underline{v}_f(1-c) = \mathcal{U} \qquad (5.32)$$

because there is less space inside the array, by volume fraction c, through which the fluid can move. Hasimoto expresses his answer in terms of F and $\mathcal{U}$ in the form

$$\mathcal{U} = \frac{F}{6\pi\mu a}\left(1 - 1.76c^{1/3} + c + \ldots\right) \qquad (5.33)$$

and by using (5.32) it can be seen that the two answers (5.30) and (5.33) are equivalent.

This concludes the discussion of the regular array, and it has been shown that the settling velocity of the regular array has the greatest dependence on concentration of the three cases considered. In the next section, we prove that this $c^{1/3}$ dependence is a property of the two-particle distribution function $G(\underset{\sim}{r}-\underset{\sim}{r}^{\beta})$ rather than exceptional kinematics of this particular array.

VI. RESULTS FOR THE GENERAL TWO-SPHERE DISTRIBUTION FUNCTION

The previous chapters considered only the conditional ensemble average (3.5),

$$\langle \sum_{\alpha}{}' f(\underline{r}^{\alpha}) \rangle_{\beta} = n \int_{|\underline{s}-\underline{r}^{\beta}| \geq 2a} f(\underline{s})\, d\underline{s} \tag{3.5}$$

which describes a completely random distribution where only mutual interpenetration of particles is forbidden. In order to include the effects of interparticle attraction or repulsion, other conditional averages are needed. The problem of actually finding the conditional average for given interparticle forces is a difficult one and is not attempted here. In this section we shall assume the conditional average is known and examine how it affects the previous results for both the random free and random fixed arrays. In particular, simple relations are shown between the results for the general conditional average and those found earlier using the ensemble average (3.5). Also, the relation between the cubic array and the other two arrays is made clear.

Using the notation of equations (3.3) and (3.4), a general two-sphere distribution is defined by:

$$\langle \sum_{\alpha}{}' f(\underline{r}^{\alpha}) \rangle_{\beta} = n \int f(\underline{s})[1 - G(\underline{s}-\underline{r}^{\beta})]\, d\underline{s} \tag{6.1}$$

where $G(\underline{s}-\underline{r}^{\beta}) \rightarrow 0$ as $|\underline{s}-\underline{r}^{\beta}| \rightarrow \infty$. We shall suppose that G has the form

$$G(\underline{s}-\underline{r}^{\beta}) = \begin{cases} 1, & |\underline{s}-\underline{r}^{\beta}| < 2a \\ g(\underline{s}-\underline{r}^{\beta}), & |\underline{s}-\underline{r}^{\beta}| \geq 2a \end{cases} \tag{6.2}$$

where g decays rapidly enough such that $\int_{|\underline{s}|\geq 2a} g(\underline{s})\,d\underline{s}$ is finite. The conditional ensemble average defined by (6.1) and (6.2) will be denoted $\langle \sum_{\alpha} f(\underline{r}^{\alpha}) | g \rangle_{\beta}$ to indicate the dependence of this average on g. In the special case where $g \equiv 0$, this ensemble average is exactly (3.5) and the argument $g = 0$ will be omitted.

It is now simple to relate the general ensemble average to (3.5). Consider a function $h(\underline{r})$ defined by

$$h(\underline{r}) = \sum_{\alpha} f(\underline{r}^{\alpha}, \underline{r}) - n \int \bar{f}(\underline{s}, \underline{r})\,d\underline{s}$$

and assume there is a sphere at $\underline{r}^{\beta}$. Then, using the average (3.5), we have

$$\begin{aligned} \langle h(\underline{r}) \rangle_{\beta} &= f(\underline{r}^{\beta}, \underline{r}) + \langle \sum_{\alpha}{}' \langle f(\underline{r}^{\alpha}, \underline{r}) \rangle_{\beta} \rangle_{\beta} - n \int \bar{f}(\underline{s}, \underline{r})\,d\underline{s} \\ &= f(\underline{r}^{\beta}, \underline{r}) + n \int_{|\underline{s}-\underline{r}^{\beta}|\geq 2a} \langle f(\underline{s}, \underline{r}) \rangle_{\beta}\,d\underline{s} - n \int \bar{f}(\underline{s}, \underline{r})\,d\underline{s} \\ &= f(\underline{r}^{\beta}, \underline{r}) - n \int_{|\underline{s}-\underline{r}^{\beta}|< 2a} \bar{f}(\underline{s}, \underline{r})\,d\underline{s} + n \int_{|\underline{s}-\underline{r}^{\beta}|\geq 2a} \left[\langle f(\underline{s}, \underline{r}) \rangle_{\beta} - \bar{f}(\underline{s}, \underline{r}) \right] d\underline{s}, \end{aligned} \tag{6.3}$$

whereas when the general average is used, the corresponding result is

$$\begin{aligned} \langle h(\underline{r}) | g \rangle_{\beta} = {} & f(\underline{r}^{\beta}, \underline{r}) - n \int_{|\underline{s}-\underline{r}^{\beta}|< 2a} \bar{f}(\underline{s}, \underline{r})\,d\underline{s} + n \int_{|\underline{s}-\underline{r}^{\beta}|\geq 2a} \left[\langle f(\underline{s}, \underline{r}) \rangle_{\beta} - \bar{f}(\underline{s}, \underline{r}) \right] d\underline{s} \\ & - n \int_{|\underline{s}-\underline{r}^{\beta}|\geq 2a} \langle f(\underline{s}, \underline{r}) \rangle_{\beta}\, g(\underline{s}-\underline{r}^{\beta})\,d\underline{s} \,. \end{aligned} \tag{6.4}$$

Comparing (6.3) and (6.4) we have the simple relation

$$\langle h(\underline{r}) | g \rangle_\beta = \langle h(\underline{r}) \rangle_\beta - n \int\limits_{|\underline{s}-\underline{r}^\beta| \geq 2a} \langle f(\underline{s},\underline{r}) \rangle_\beta \, g(\underline{s}-\underline{r}^\beta) d\underline{s} \qquad (6.5)$$

between the conditional averages. This relation is exploited below to obtain results for the general two sphere distribution.

§1. Random Free Array

We begin with the equation for $\langle v_i(\underline{r};0) | g \rangle_0$ defined by (3.8).

$$\langle v_i(\underline{r};0) | g \rangle_0 = \langle \sum_\alpha{}' F_j S_{ij}(\underline{r}-\underline{r}^\alpha) | g \rangle_0 - n \int F_j S_{ij}(\underline{r}-\underline{s}) d\underline{s} + \dots \qquad (3.8)$$

The analysis is exactly the same as in Chapter III, except the extra term indicated in (6.5) is included in this calculation. In particular, the approximations $\langle F_{j\ell}(\underline{r}^\alpha) | g \rangle_0 = F_{j\ell}^{T.P.}(\underline{r}^\alpha)$, ... are used again. The analogous equation to (3.17) is then

$$\langle v_i(\underline{r};0) | g \rangle_0 = \langle v_i(\underline{r};0) \rangle_0 - n \int\limits_{|\underline{s}| \geq 2a} \left[F_j S_{ij}(\underline{r}-\underline{s}) + F_{j\ell}^{T.P.}(\underline{s}) \partial_\ell S_{ij}(\underline{r}-\underline{s}) + \dots \right] g(\underline{s}) d\underline{s} \qquad (6.6)$$

where (6.5) has been used on each term of the multipole expansion. Following Chapter III, the mean velocity, $\langle \underline{V} | g \rangle$, of the spheres is then found by operating on (6.6) with $\left(1+\frac{a^2}{6}\nabla^2\right)$ and applying Faxen's law. In particular, after using the relation (3.21),

$$V_i^{T.P.}(\underline{s}) - \mathcal{U}_{0i} = \left(1+\frac{a^2}{6}\nabla^2\right)\left[F_j S_{ij}(\underline{r}-\underline{s}) + F_{j\ell}^{T.P.}(\underline{s}) \partial_\ell S_{ij}(\underline{r}-\underline{s}) + \dots \right]_{\underline{r}=\underline{0}}, \qquad (3.21)$$

on the integrand, the result is simply

$$\langle \underline{V} | g \rangle = \langle \underline{V} \rangle - n \int\limits_{|\underline{s}| \geq 2a} \left[\underline{V}^{T.P.}(\underline{s}) - \underline{\mathcal{U}}_0 \right] g(\underline{s}) d\underline{s} \qquad (6.7)$$

where $\underline{V}^{T.P.}(\underline{s})$ is the terminal velocity of two spheres with relative position $\underline{s}$, and $\langle \underline{V} \rangle = \mathcal{U}_0(1-6.55c)$ from Chapter III. As previously mentioned, $\underline{V}^{T.P.}(\underline{s})$ is known numerically from the work of Goldman, Cox, and Brenner (1966).

Hence, for any given $g(\underline{s})$, the calculation of the mean velocity of the array can be obtained by the numerical integration of (6.7).

§2. Random Fixed Array

We will consider the effect of the general average (6.5) only on the leading term of the expansion in this case. The analysis is the same as used in Chapter IV, so only a few points will be highlighted here.

To obtain an integral equation for $\langle u_i(\underline{r}) | g \rangle_\beta$, the truncation

$$\langle v_j(\underline{r}^{\alpha};\alpha) | g \rangle_\beta - \langle u_j(\underline{r}^{\alpha}) | g \rangle_\beta = V_j - \frac{F_j}{6\pi\mu a}, \quad F_j \equiv \langle F_j \rangle \qquad (6.8)$$

is used, analogous to (4.8). This leads to an equation analogous to (4.9b), except the ensemble averages are now the general ensemble averages. Upon representing these ensemble averages by integrals, the resulting integral equation for $\langle u_i(\underline{r}) | g \rangle_\beta$ is

$$\langle u_i(\underline{r}) | g \rangle_\beta = F_j S_{ij}(\underline{r}-\underline{r}^{\beta}) - n \int_{|\underline{s}-\underline{r}^{\beta}|<2a} F_j S_{ij}(\underline{r}-\underline{s}) d\underline{s} - 6\pi\mu a n \int_{|\underline{s}-\underline{r}^{\beta}|\geq 2a} \langle u_j(\underline{s}) | g \rangle_\beta S_{ij}(\underline{r}-\underline{s}) d\underline{s}$$

$$+ \int_{|\underline{s}-\underline{r}^{\beta}|\geq 2a} [-nF_j + 6\pi\mu a n \langle u_j(\underline{s}) | g \rangle_\beta] S_{ij}(\underline{r}-\underline{s}) g(\underline{s}-\underline{r}^{\beta}) d\underline{s}, \qquad (6.9)$$

which replaces equation (4.10a). Using the definition (6.2) of G, equation (6.9) can be rewritten as

$$\langle u_i(\underline{r})|g\rangle_\beta = F_j S_{ij}(\underline{r}-\underline{r}^\beta) - 6\pi\mu a n\int\langle u_j(\underline{s})|g\rangle_\beta S_{ij}(\underline{r}-\underline{s})d\underline{s}$$

$$+\int\left[-nF_j + 6\pi\mu a n\langle u_j(\underline{s})|g\rangle_\beta\right]S_{ij}(\underline{r}-\underline{s})G(\underline{s}-\underline{r}^\beta)d\underline{s}\ . \qquad (6.\ 10)$$

Comparing (6. 10) with (4. 10b), it can be seen that the last two integrals of (4. 10b) are the effect of the two sphere distribution where $g \equiv 0$. Noting that the form of those two integrals had no effect in the analysis that led to (4. 18a), the solution for $\langle v_i(\underline{r};\beta)|g\rangle_\beta$ can be immediately written down [using (4. 18a)] as

$$\langle v_i(\underline{r};\beta)|g\rangle_\beta = -\frac{F_j}{6\pi\mu a}\cdot\frac{3}{2}a\left[\delta_{ij}A(\lambda r') - \frac{r_i r_j}{r^2}B(\lambda r')\right]$$
$$+\int\left[-nF_j + \mu\lambda^2\langle u_j(\underline{s})|g\rangle_\beta\right]G(\underline{s}-\underline{r}^\beta)C_{ij}(\underline{r}-\underline{s})\,d\underline{s} \qquad (6.\ 11)$$

where A , B , and C_{ij} are defined in (4. 18). Evaluating (6.11) at $\underline{r} = \underline{r}^\beta$, and using (4. 19) yields the following expression for the mean velocity $\langle V_i|g\rangle$:

$$\langle V_i|g\rangle = \frac{F_i}{6\pi\mu a}\left[1-\frac{3}{\sqrt{2}}c^{1/2}\right] + \int\left[-nF_j + \mu\lambda^2\langle u_j(\underline{s})|g\rangle_\beta\right]G(\underline{s}-\underline{r}^\beta)C_{ij}(\underline{r}-\underline{s})\,d\underline{s}\ . \qquad (6.\ 12)$$

This equation contains the function $\langle u_j(\underline{s})|g\rangle_\beta$ implicitly, and an estimate of it is needed to proceed further.

From (6. 10) we see that when $\underline{r}$ is in the neighborhood of $\underline{r}^\beta$, $\langle u_i(\underline{r})|g\rangle_\beta$ is approximated by the singular part of the right hand side, namely the Stokeslet term. Thus, when the function $G(\underline{s}-\underline{r}^\beta)$ is concentrated near $|\underline{s}-\underline{r}^\beta| = 0$, the approximation

$$\langle u_j(\underline{s})|g\rangle_\beta \approx F_j S_{ij}(\underline{s}-\underline{r}^\beta)$$

can be used in (6.12). Here, the term "concentrated" means G has a characteristic length on the order of the sphere radius a .

On the other hand, if G has a characteristic length on the order of the mean interparticle distance $n^{-1/3}$, then the (unknown) integral of $\langle u_j(\underline{s}) | g \rangle_\beta$ is of smaller order than the (known) integral of the nF_j term, and it can be ignored. To show this, we consider the function G^* defined by

$$G^*(\underline{s}-\underline{r}^\beta) = \begin{cases} 1, & |\underline{s}-\underline{r}^\beta| < n^{-1/3} \\ 0, & |\underline{s}-\underline{r}^\beta| \geq n^{-1/3} \end{cases}$$

and estimate the size of each integral. Using the relation

$$c = \frac{4}{3}\pi a^3 n$$

we see that the exponent $\lambda|\underline{r}^\beta-\underline{s}|$ of $C_{ij}(\underline{r}^\beta-\underline{s})$ is small in the region of integration since $\lambda n^{-1/3} \sim c^{1/6}$. Upon expanding $C_{ij}(\underline{r}^\beta-\underline{s})$ for a small argument, we find

$$C_{ij}(\underline{r}^\beta-\underline{s}) = S_{ij}(\underline{r}^\beta-\underline{s}) + \text{higher order terms} . \qquad (6.13)$$

Hence, for the nF_j integral we have (ignoring constants)

$$\int nF_j G^*(\underline{s}-\underline{r}^\beta) C_{ij}(\underline{r}^\beta-\underline{s}) d\underline{s} \approx nF_j \int_{|\underline{s}'|<n^{-1/3}} S_{ij}(\underline{s}') d\underline{s}' \approx \frac{nF_j}{\mu} \int_{|\underline{s}'|<n^{-1/3}} \frac{\delta_{ij}}{|\underline{s}'|} d\underline{s}' \approx U_{0_i} c^{1/3} \qquad (6.14)$$

On the other hand, $\langle u_j(\underline{s}) | g \rangle_\beta$ is bounded above by the Stokeslet $F_k S_{jk}(\underline{s}-\underline{r}^\beta)$ because of the shielding effect, so an estimate for the other integral is:

$$\int \mu\lambda^2 \langle u_j(\underline{s})|g\rangle_\beta G^*(\underline{s}-\underline{r}^\beta) C_{ij}(\underline{r}^\beta-\underline{s})\, d\underline{s} \approx \mu\lambda^2 F_k \int_{|\underline{s}'|<n^{-1/3}} S_{jk}(\underline{s}') S_{ij}(\underline{s}')\, d\underline{s}'$$

$$\approx \frac{F_k\lambda^2}{\mu} \int_{|\underline{s}'|<n^{-1/3}} \frac{\delta_{jk}\delta_{ij}}{|\underline{s}'|^2}\, d\underline{s}' \approx \mathcal{U}_0 c^{2/3}.$$

This shows the unknown function $\langle u|g\rangle_\beta$ may safely be ignored when G has the larger characteristic length $n^{-1/3}$.

Thus, the exact equation (6. 12) for $\langle \underline{V}|g\rangle$ may be approximated for two types of G's. When G is close to the completely random distribution (3. 5) in the sense of being concentrated near $|\underline{s}-\underline{r}^\beta| = 0$, then

$$\langle V_i|g\rangle \approx \mathcal{U}_{0_i}\left[1-\frac{3}{\sqrt{2}}c^{1/2}\right] + \int \left[-nF_j + \mu\lambda^2 F_k S_{jk}(\underline{s}-\underline{r}^\beta)\right] G(\underline{s}-\underline{r}^\beta) C_{ij}(\underline{r}^\beta-\underline{s})\, d\underline{s} \qquad (6.\,15)$$

is a good approximation to (6. 12). However, when G is not small over the mean interparticle distance $n^{-1/3}$, the leading order approximation to (6. 12) is obtained by neglecting the $\langle u|g\rangle_\beta$ integral altogether with the result

$$\langle V_i|g\rangle \approx \mathcal{U}_{0_i}\left[1-\frac{3}{\sqrt{2}}c^{1/2}\right] - nF_j \int G(\underline{s}-\underline{r}^\beta) C_{ij}(\underline{r}^\beta-\underline{s})\, d\underline{s}\,. \qquad (6.\,16)$$

As we have already shown in (6. 14), the integral term in the above equation is of order $\mathcal{U}_{0_i} c^{1/3}$ if $G(\underline{s}-\underline{r}^\beta)$ has a characteristic length of order $n^{-1/3}$. In this case, the integral becomes the leading term in the expansion for $\langle V|g\rangle$. If the length scale of G is somewhat less than $n^{-1/3}$, then the integral must be compared with the $c^{1/2}$ term to determine which is the leading order correction to the settling speed.

It can be seen from the above analysis that the leading correction to $\langle V|g\rangle$ is still the $\frac{3}{\sqrt{2}}c^{1/2}$ term derived previously unless the two sphere distribution G is far from being completely random. This point was made by Childress (1972).

§3. The Cubic Array as a Special Two-Sphere Distribution

It is easy to specify the cubic lattice in terms of the two sphere distribution by using delta functions at the lattice points and not allowing particle centers anywhere else. The G which is equivalent to the cubic lattice is then

$$G_{cubic}(\underline{s}-\underline{r}^{\beta}) = 1 - \frac{1}{n}\sum_{\alpha}{}' \delta(\underline{s}-\underline{r}^{\beta}-\underline{n}^{\alpha}) \tag{6.17}$$

where the notation of Chapter V is used.

A more interesting question is to find the property of G needed to create a $c^{1/3}$ dependence in the mean settling velocity of the random arrays. Suppose we impose the condition that no two particles may be closer than a distance of $n^{-1/3}$ (which is a property of the cubic array). Then G is defined by

$$G(\underline{s}-\underline{r}^{\beta}) = \begin{cases} 1, & |\underline{s}-\underline{r}^{\beta}|<n^{-1/3} \\ 0, & |\underline{s}-\underline{r}^{\beta}|\geq n^{-1/3} \end{cases} . \tag{6.18}$$

We wish to approximate the integral in (6.7) for the free array and the integral in (6.16) for the fixed array. Using (3.21) and (6.13), a first estimate to each integral is

$$-n\int F_j S_{ij}(\underline{s}-\underline{r}^{\beta})G(\underline{s}-\underline{r}^{\beta})d\underline{s} = -nF_j\int_{|\underline{s}-\underline{r}^{\beta}|<n^{-1/3}} S_{ij}(\underline{s}-\underline{r}^{\beta})d\underline{s} = 3.9\,\mathcal{U}_{0_i}c^{1/3} . \tag{6.19}$$ *

*The hindrance $3.9\,\mathcal{U}_0 c^{1/3}$ in (6.9) is much larger than the value

Thus, an order $c^{1/3}$ dependence in the settling speed for both the free and fixed arrays is caused by the absence of neighboring particles. This is an important result for the free array because it implies that strong interparticle repulsion will result in qualitative changes in the dependence of the settling speed on concentration.

$1.76\,U_0 c^{1/3}$ found for the cubic array. This is because particles near to $\underline{r}^{(1)}$ tend to increase its settling speed. In the cubic array, six particles are given at the distance $n^{-1/3}$, while in the random case these nearest neighbors are more distant. Thus, the cubic array settles more rapidly than the array defined by (6. 18).

VII. THE WALL EFFECT ON SEDIMENTATION

We considered in Chapter III the sedimentation of an infinite dispersion of freely moving spheres and found that the mean settling velocity was given by

$$\langle \underline{V} \rangle = \underline{U}_0 (1 - \beta c), \quad \beta = 6.55$$

This chapter examines the effect of the tube wall on the mean settling speed. It is found that to leading order the dependence on c is the same as for the infinite dispersion, but that the wall tends to reduce the dependence on c by order $\frac{a}{R_0}$, where R_0 is the tube radius. In addition, in the same manner as the settling of a single particle is hindered in a tube, we find that the settling of the dispersion is hindered by order $\frac{a}{R_0}$. The result is that the mean settling velocity is given by an equation of the form

$$\langle \underline{V} \rangle = \underline{U}_0 \left(1 - K\frac{a}{R_0} - c\left(6.55 - 8K\frac{a}{R_0}\right)\right)$$

where K is on the order of 5. An expression for K is given in the text [equation (7.22b)]. For the analysis we assume the dispersion is settling inside an infinitely long cylinder of radius R_0 , and that the dispersion is uniform throughout the tube. The method of solution is similar to that used for the unbounded dispersion, i.e., the suspension problem is reduced to a sum of one and two particle interactions. Here, the problem is reduced to the motion of a single particle in a tube, and to two particles in the presence of a wall. While the former problem has some approximate solutions (included below), the latter problem has not been studied. Consequently, only a simple bound for the effect of the two particle interactions can be given.

These estimates will be made clear in the analysis.

The representation (2.23) for $u_i(\underline{r})$ needs to be slightly modified for this problem in order to satisfy the no-slip boundary condition on the tube wall. Thus, we write

$$\begin{aligned} u_i(\underline{r}) = {} & \sum_{\alpha} F_j\left[S_{ij}(\underline{r}-\underline{r}^{\alpha}) - R_{ij}(\underline{r},\underline{r}^{\alpha})\right] - n\int F_j\left[S_{ij}(\underline{r}-\underline{s}) - R_{ij}(\underline{r},\underline{s})\right]d\underline{s} \\ & + \sum_{\alpha} F_{j\ell}^{\alpha}\left[\partial_\ell S_{ij}(\underline{r}-\underline{r}^{\alpha}) - R_{ij\ell}(\underline{r},\underline{r}^{\alpha})\right] - n\int \overline{F}_{j\ell}\left[\partial_\ell S_{ij}(\underline{r}-\underline{s}) - R_{ij\ell}(\underline{r},\underline{s})\right]d\underline{s} \\ & + \dots \end{aligned} \tag{7. 1}$$

where the functions R are the exact reflection of the Stokeslets from the wall. Thus, we have divided the fundamental solution for the motion of a particle in a tube into two parts -- one of them is the Stokeslet, S_{ij} , and the remainder is the reflection R . Then, by definition,

$$\left.\begin{aligned} S_{ij}(\underline{r}-\underline{r}^{\alpha}) - R_{ij}(\underline{r},\underline{r}^{\alpha}) &= 0 \\ \partial_\ell S_{ij}(\underline{r}-\underline{r}^{\alpha}) - R_{ij\ell}(\underline{r},\underline{r}^{\alpha}) &= 0 \\ &\vdots \end{aligned}\right\} \quad \text{for } \underline{r} \text{ on the cylinder wall.}$$

The domain of integration in (7. 1) consists of every point accessible to the sphere centers, which in this case means all points inside the cylinder farther than a distance a from the wall. This insures that the mean dispersion velocity is zero:

$$\langle u_i(\underline{r}) \rangle = 0 \, .$$

Note also that all the stresses exerted by the cylinder wall on the fluid are caused by the presence of the moving spheres, and that these stresses are included in (7. 1) by the presence of the reflection terms R .

As usual, the velocity of the spheres is found by considering the regular part of the velocity field in the neighborhood of a sphere. Thus, we consider

$$
\begin{aligned}
v_i(\underline{r};\beta) = & \sum_\alpha{}' F_j S_{ij}(\underline{r}-\underline{r}^\alpha) - \sum_\alpha F_j R_{ij}(\underline{r},\underline{r}^\alpha) - n\int F_j [S_{ij} - R_{ij}] d\underline{s} \\
& + \sum_\alpha{}' F_{j\ell}^\alpha \partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha) - \sum_\alpha F_{j\ell}^\alpha R_{ij\ell}(\underline{r},\underline{r}^\alpha) - n\int \overline{F}_{j\ell} [\partial_\ell S_{ij} - R_{ij\ell}] d\underline{s} \qquad (7.2) \\
& + \ldots \; .
\end{aligned}
$$

Note that the particle's own reflection in the wall contributes to the regular part of the velocity in the neighborhood of itself. This is the cause of the increased drag experienced by a single particle moving in the presence of a wall.

Once again, ensemble averages are used to find the mean settling speed. Here, these averages will be dependent on the distance of the particle from the axis of the cylinder: this dependence will be indicated by the non-dimensional parameter ρ where

$$
\rho \equiv \frac{\text{distance from axis}}{R_0} . \qquad (7.3)
$$

The mean value throughout the tube will then be a weighted average of the ensemble averages at different distances from the axis:

$$
\langle f \rangle = \frac{1}{\pi} \int_0^{1-a/R_0} \langle f | \rho \rangle \, 2\pi\rho \, d\rho \qquad (7.4)
$$

where

$$
\left\{ \begin{array}{l} \langle f|\rho\rangle = \text{mean value at a distance } \rho R_0 \text{ from the axis,} \\ \langle f \rangle = \text{mean value throughout the tube.} \end{array} \right.
$$

We will only use the "completely random" ensemble average (3.5) in this section.

Now averaging equation (7.2) at a fixed distance ρR_0 gives

$$\langle v_i(\underline{r};\beta)|\rho\rangle_\beta = \langle \sum_\alpha{}' F_j S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_\beta - n\int F_j S_{ij}(\underline{r}-\underline{s})d\underline{s} \tag{7.5a}$$

$$- F_j R_{ij}(\underline{r},\underline{r}^\beta) - \langle F_{j\ell}|\rho^\beta\rangle R_{ij\ell}(\underline{r},\underline{r}^\beta) \tag{7.5b}$$

$$- \langle \sum_\alpha{}' F_j R_{ij}(\underline{r},\underline{r}^\alpha)\rangle_\beta + n\int F_j R_{ij}(\underline{r},\underline{s})\,d\underline{s} \tag{7.5c}$$

$$+ \langle \sum_\alpha{}' \langle F_{j\ell}|\rho^\alpha\rangle_\beta \partial_\ell S_{ij}(\underline{r}-\underline{r}^\alpha)\rangle_\beta - n\int \langle F_{j\ell}|\rho^s\rangle \partial_\ell S_{ij}(\underline{r}-\underline{s})d\underline{s} \tag{7.5d}$$

$$- \langle \sum_\alpha{}' \langle F_{j\ell}|\rho^\alpha\rangle_\beta R_{ij\ell}(\underline{r},\underline{r}^\alpha)\rangle_\beta + n\int \langle F_{j\ell}|\rho^s\rangle R_{ij\ell}(\underline{r},\underline{s})d\underline{s} \tag{7.5e}$$

$$+ \ldots .$$

This equation is long, but it is not difficult to analyze line by line. Writing the ensemble average in integral form, we have [from (3.9)],

$$(7.5a) = -n\int_{|\underline{s}-\underline{r}^\beta|<2a} F_j S_{ij}(\underline{r}-\underline{s})d\underline{s} = \frac{F_i}{6\pi\mu a}\left[-6c + \frac{4}{5}\pi a n(\underline{r}-\underline{r}^\beta)^2 - \frac{2}{5}\pi a n(r_i - r_i^\beta)^2\right] \tag{7.6}$$

This is the same as the leading contribution to the unbounded dispersion problem, even though the sums and integrals are over different regions in the two problems. Note we have ignored the fact that the region of integration in (7.6) is somewhat reduced if the sphere center $\underline{r}^\beta$ is within $2a$ of the wall. This correction is small relative to other terms that will be found.

At $\underline{r} = \underline{r}^\beta$, it can be seen that (7.5b) expresses the fact that the particle at $\underline{r}^{\beta}$ is affected by its own reflection in the wall. As a first approximation to $\langle F_{j\ell} | \rho \rangle$ we will use $(F_{j\ell}|\rho)^{S.P.}$ as was done in Chapter III. This means the terms like (7.5b) can be approximated in terms of the known solutions for the motion of a single particle falling in a tube.

The next line, (7.5c), can be related to the motion of a single particle in a simple way. We have, using (3.5), that

$$(7.5c) = n \int_{|\underline{s}|<2a} F_j R_{ij}(\underline{r}, \underline{r}^{\beta} + \underline{s})\, d\underline{s} . \tag{7.7}$$

In addition, the reflections R are smooth enough that the integral in (7.7) is well approximated by the value of the integrand at the center of the region of integration multiplied by the volume of the region, i.e.,

$$n \int_{|\underline{s}|<2a} F_j R_{ij}(\underline{r}^{\beta}, \underline{r}^{\beta} + \underline{s})\, d\underline{s} \approx 8c\, F_j R_{ij}(\underline{r}^{\beta}, \underline{r}^{\beta}) . \tag{7.8}$$

Thus, terms similar to (7.5c) may also be related to the motion of a single particle in a tube.

Proceeding to the next terms of (7.5), we see that

$$(7.5d) = + n \int_{|\underline{s}-\underline{r}^{\beta}| \geq 2a} \left[\langle F_{j\ell} | \rho^{s} \rangle_{\beta} - \langle F_{j\ell} | \rho^{s} \rangle \right] \partial_{\ell} S_{ij}(\underline{r}, \underline{s})\, d\underline{s} - n \int_{|\underline{s}-\underline{r}^{\beta}|<2a} \langle F_{j\ell} | \rho^{s} \rangle \partial_{\ell} S_{ij}\, d\underline{s} \tag{7.9}$$

where the superscript s of ρ^{s} denotes that ρ^{s} relates to the variable of integration $\underline{s}$ rather than $\underline{r}$. In a similar fashion, (7.5e) is given by

$$(7.5e) = - n \int_{|\underline{s}-\underline{r}^{\beta}| \geq 2a} \left[\langle F_{j\ell} | \rho^{s} \rangle_{\beta} - \langle F_{j\ell} | \rho^{s} \rangle \right] R_{ij\ell}(\underline{r}, \underline{s})\, d\underline{s} + n \int_{|\underline{s}-\underline{r}^{\beta}|<2a} \langle F_{j\ell} | \rho^{s} \rangle R_{ij\ell}\, d\underline{s} . \tag{7.10}$$

These formulas can be reduced to solutions of one and two particle problems by making the approximations

$$\left\{ \begin{array}{l} <F_{j\ell}|\rho^s>_\beta - <F_{j\ell}|\rho^s> = (F_{j\ell}(\underline{s}-\underline{r}^\beta)|\rho^s)^{T.P.} - (F_{j\ell}|\rho^s)^{S.P.} \\ \qquad <F_{j\ell}|\rho^s> = (F_{j\ell}|\rho^s)^{S.P.} \end{array} \right. \qquad (7.11)$$

where $(F_{j\ell}(\underline{s}-\underline{r}^\beta)|\rho^S)^{T.P.}$ denotes the $F_{j\ell}$ of the particle at $\underline{s}$ in the presence of a particle at $\underline{r}^\beta$. This is the same approximation as used in (3.12) except the effect of the cylinder wall is included in the approximation (7.11). Further, the approximations

$$-n\int_{|\underline{s}-\underline{r}^\beta|<2a} <F_{j\ell}|\rho^s>\partial_\ell S_{ij}(\underline{r}-\underline{s})d\underline{s} \approx -n<F_{j\ell}|\rho^\beta>\int_{|\underline{s}-\underline{r}^\beta|<2a}\partial_\ell S_{ij}(\underline{r}-\underline{s})d\underline{s} \qquad (7.12)$$

and

$$+n\int_{|\underline{s}-\underline{r}^\beta|<2a} <F_{j\ell}|\rho^s>R_{ij\ell}(\underline{r},\underline{s})d\underline{s} \approx 8c<F_{j\ell}|\rho^\beta>R_{ij\ell}(\underline{r},\underline{r}^\beta) \qquad (7.13)$$

can be made in equations (7.9) and (7.10) because of the small region of integration.

Combining the results of (7.6) - (7.13), equation (7.5) can be written as:

$$\begin{aligned} <v_i(\underline{r};\beta)|\rho^r>_\beta = & -n\left[F_j + (F_{j\ell}|\rho^\beta)^{S.P.}\partial_\ell + \ldots\right]\int_{|\underline{s}-\underline{r}^\beta|<2a} S_{ij}(\underline{r}-\underline{s})d\underline{s} \\ & + n\int_{|\underline{s}-\underline{r}^\beta|\geq 2a}\left(\left[(F_{j\ell}(\underline{s}-\underline{r}^\beta)|\rho^s)^{T.P.} - (F_{j\ell}|\rho^s)^{S.P.}\right]\left[\partial_\ell S_{ij} - R_{ij\ell}\right] + \ldots\right)d\underline{s} \\ & -(1-8c)\left[F_j R_{ij}(\underline{r},\underline{r}^\beta) + (F_{j\ell}|\rho^\beta)^{S.P.} R_{ij\ell}(\underline{r},\underline{r}^\beta) + \ldots\right]. \end{aligned} \qquad (7.14)$$

Thus, the determination of $\langle v|\rho\rangle$ has been reduced to a sum of one and two particle problems in a tube. The mean settling velocity of the array can now be found by operating on (7. 14) with $\left(1+\frac{a^2}{6}\nabla^2\right)$ to apply Faxen's law, and averaging over all values of ρ using (7. 4). However, some of the required one and two particle results are not known, so the solution (7. 14) is not very useful at the present time.

In order to obtain a more concrete result, although a less accurate one, the following assumptions are made. In the first term on the right hand side of (7. 14) we assume the effect of the wall on the F 's can be neglected for most of the particles. This is certainly true for particles far from the wall. Thus, the approximations are

$$(F_{j\ell}|\rho^r)^{S.P.} = F_{j\ell}^{S.P.} \quad , \quad (F_{j\ell m}|\rho^r)^{S.P.} = F_{j\ell m}^{S.P.} \quad , \quad \ldots$$

where now the F 's are approximated by their values in an unbounded medium. The value of the resulting expression was computed in Chapter III, and was found to be

$$\left[\left(1+\frac{a^2}{6}\nabla^2\right)\left(-nF_j - nF_{j\ell}^{S.P.}\partial_\ell - \ldots\right)\int_{|\underline{s}-\underline{r}^\beta|<2a} S_{ij}(\underline{r}-\underline{s})\,d\underline{s}\,\right]_{\underline{r}=\underline{r}^\beta} = -\frac{F_i}{6\pi\mu a}\,5c\,. \tag{7. 15}$$

This is the estimate of the first term of (7. 14).

The second term on the right hand side of (7. 14) involves knowledge of the motion of two particles settling in a tube, and we estimate the magnitude of this term using the result for the unbounded dispersion. From (3. 23) and (3. 24), we recall for the unbounded dispersion that

$$n \int_{|\underline{s}-\underline{r}^\beta| \geq 2a} \left(1+\frac{a^2}{6}\nabla^2\right)\left[\left(F_{j\ell}^{T.P.}(\underline{s}-\underline{r}^\beta) - F_{j\ell}^{S.P.}\right)\partial_\ell S_{ij}(\underline{s}-\underline{r}^\beta) + \ldots \right] d\underline{s} = -1.55\, c\, \mathcal{U}_{0_i}$$

where the integral is over all space. Since the integrand is negative everywhere [Batchelor (1972)], the corresponding integral of the terms for two particles in a tube is less (in magnitude) than $1.55\, c\, \mathcal{U}_0$ because the range of integration is solely within the cylinder. This bound is actually approached when the point $\underline{r}^\beta$ is far from the cylinder wall because the integrand decays like $|\underline{s}-\underline{r}^\beta|^{-4}$ as $|\underline{s}-\underline{r}^\beta| \to \infty$.

The integral of the reflection terms R is more difficult to estimate, but it seems plausible that it will further reduce the magnitude of the two particle integral of (7. 14) because a particle at $\underline{r}^\beta$ sees both the particle at $\underline{r}^\alpha$ and the image of the particle at $\underline{r}^\alpha$, and this image acts on β in the opposite sense of the direct interaction* (see Figure 4). Furthermore, when the point $\underline{r}^\beta$ is far from the wall, the integral of the reflection terms R is of smaller order than the integral of the direct terms S_{ij} . Thus, an estimate of the second term of the right hand side of (7. 14) is

$$\left[\left(1+\frac{a^2}{6}\nabla^2\right) n \int_{|\underline{s}-\underline{r}^\beta| \geq 2a} \left[\left(F_{j\ell}^{T.P.}(\underline{s}-\underline{r}^\beta) - F_{j\ell}^{S.P.}\right)\left(\partial_\ell S_{ij} - R_{ij\ell}\right) + \ldots\right] d\underline{s}\right]_{\underline{r}=\underline{r}^\beta} \approx -1.55\, c\, \mathcal{U}_{0_i} \qquad (7.\ 16)$$

and we expect that $1.55\, c\, \mathcal{U}_0$ is an upper bound on the magnitude of the integral.

---

* An exact solution for the reflected velocity field due to a point particle in the presence of a plane wall was found by Lorentz (1907). It is given in Appendix C.

We now proceed to the third term of (7. 14), and consider the quantity

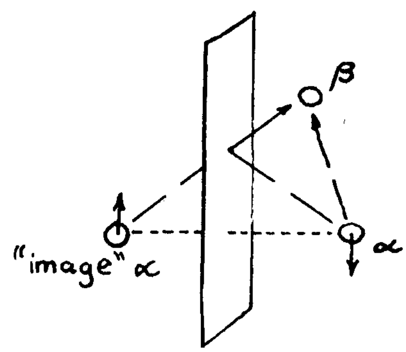
$$I = -\left[\left(1+\frac{a^2}{6}\nabla^2\right)\left(F_j R_{ij}(\underline{r},\underline{r}^\beta) + (F_{j\ell}|\rho_r)^{S.P.} R_{ij\ell}(\underline{r},\underline{r}^\beta) + \ldots\right)\right]_{\underline{r}=\underline{r}^\beta} \quad (7.17)$$

This expression is the change in the settling velocity of a single sphere in the presence of the cylindrical wall, and can be estimated numerically. There are different expressions for this quantity in different regions of the tube. When the sphere is away from the wall, the settling velocity is given by Cox and Mason (1971) as:

$$V = U_0\left[1-\frac{a}{R_0}f(\rho) + O\left(\frac{a^3}{R_0^3}\right)\right] \quad (7.18)$$

where $f(\rho)$ is given numerically by Greenstein and Happel (1968) for values of ρ between 0 and 0.9 (see Table 7-1). For ρ larger

Figure 4. The Form of a Two Particle Interaction in the Presence of a Wall.



than 0.9 and $\frac{1}{1-\rho}\frac{a}{R_0} \ll 1$, the settling velocity is given by Cox and Mason as:

$$V = U_0\left[1-\frac{a}{R_0}f(\rho)+\ldots\right] \text{ where } f(\rho) \sim \frac{9}{16}\frac{1}{1-\rho} \text{ as } \rho \to 1. \quad (7.19)$$

Finally, when the sphere is very near the wall, the cylinder wall can be approximated by a plane surface and the results of Goldman, Cox,

Table 7-1. Some Values of $f(\rho)$ Found by Greenstein and Happel.

ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$f(\rho)$	2.10	2.10	2.08	2.06	2.04	2.06	2.17	2.46	3.20	5.30

and Brenner (1967) may be used. They express their results in terms of a dimensionless drag force F_x^* on a sphere moving with velocity V parallel to a plane wall. Equating the excess weight, F, to the drag then gives

$$6\pi a\mu V F_x^* = F$$

or

$$V = \mathcal{U}_0 \frac{1}{F_x^*} \tag{7.20}$$

as the terminal velocity of a phere close to a plane wall. The dependence of V on the distance of the sphere center from the wall is given in Table 7-2. Note that near the wall there is an order 1 change in V, while for $\rho \leqslant 0.9$ there is only an order $\frac{a}{R_0}$ change in V. Between these two regions there is a qualitative change in behavior as described by (7.19). This behavior is not described sufficiently well by (7.19) since that equation does not match the solutions (7.18) and (7.20) at the endpoints (see Figure 5). Instead, a linear

Table 7-2. Dependence of V on the Distance from the Wall, where $V = \mathcal{U}_0(1 - g(\frac{h}{a}))$, h = distance of sphere center from the wall.

h/a	10.1	3.76	2.35	1.54	1.13	1.04
g(h/a)	0.06	0.15	0.24	0.36	0.54	0.62

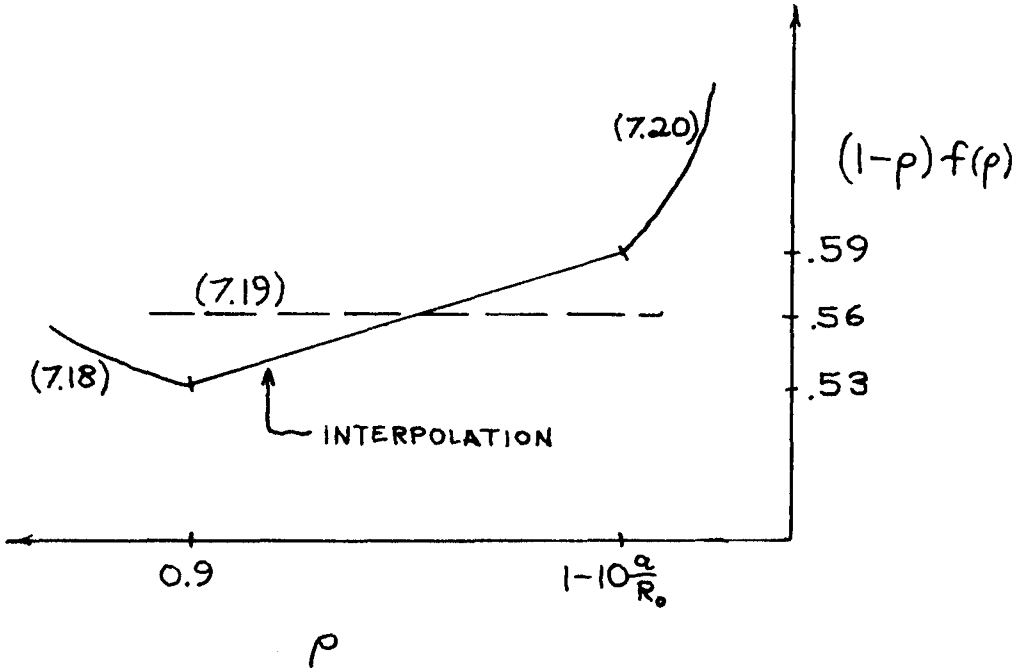


Figure 5. Wall hindrance for a single sphere settling in a cylinder.

interpolation for $f(\rho)(1-\rho)$ was used as shown in Figure 5. This maintained the behavior indicated by (7. 19) and continuously matched the values of V at the endpoints.

The average settling velocity of a single particle in the tube is given by equation (7. 4) as

$$\langle V \rangle = \int_0^{1-a/R_o} \langle V|\rho \rangle \, 2\rho \, d\rho \tag{7.21}$$

where $\langle V|\rho \rangle$ is defined for different values of ρ by (7. 18), (7. 20), and the correction of (7. 19). Using more values than are shown in the tables, the numerical result is

$$\langle V \rangle = U_o \left(1 - K\left(\frac{a}{R_o}\right)\frac{a}{R_o} \right) \tag{7.22a}$$

where

$$K\left(\frac{a}{R_0}\right) \equiv 5.0 + \left(1.06 + \frac{0.12}{1-100\frac{a}{R_0}}\right) \ln \frac{1}{100\,a/R_0} . \qquad (7.22b)$$

The above representation of $K\left(\frac{a}{R_0}\right)$ is only correct for $\frac{a}{R_0} < 0.01$. For $\frac{a}{R_0} > 0.01$ it can be seen that $0.9 > 1 - 10\frac{a}{R_0}$ so that Figure 5 is no longer valid.

The contribution of each region of integration to the total hindrance is given by

$$\rho \in [0, 0.9] \qquad -\frac{a}{R_0}\, 2.29$$

$$\rho \in \left[0.9, 1-10\frac{a}{R_0}\right] \qquad \frac{a}{R_0}\left[0.23 + \left(1.06 + \frac{.12}{1-100\,a/R_0}\right) \ln 100\frac{a}{R_0}\right]$$

$$\rho \in \left[1-10\frac{a}{R_0}, 1-\frac{a}{R_0}\right] \qquad -\frac{a}{R_0}\, 2.92 .$$

Notice the large effect of the boundary region. If the spheres tend to stay away from the walls, the overall wall effect will be greatly reduced.

Now we relate the suspension problem to the single particle result (7.22). The net decrease in settling speed indicated in (7.22) is equal to the mean of (7.17) throughout the tube. Hence,

$$\langle I \rangle = -\mathcal{U}_0 \frac{a}{R_0} K\left(\frac{a}{R_0}\right) . \qquad (7.23)$$

Now making use of the above results (7.15), (7.16), and (7.23), we see from (7.14) that

$$\left\langle \left(1 + \frac{a^2}{6}\nabla^2\right) v_i(\underline{r}^\beta; \beta) \right\rangle_\beta = -\mathcal{U}_0 c\, 6.55 - \mathcal{U}_0 (1-8c) \frac{a}{R_0} K\left(\frac{a}{R_0}\right)$$

which implies that the mean settling velocity of the spheres in the tube is given by

$$\langle V \rangle \approx \mathcal{U}_0\left[1 - \frac{a}{R_0}K\left(\frac{a}{R_0}\right) - c\left(6.55 - 8\frac{a}{R_0}K\left(\frac{a}{R_0}\right)\right)\right]. \quad (7.24)$$

Some typical values of the function $K\left(\frac{a}{R_0}\right)$ are given in Table 7-3.

Table 7-3. Some Values of the Function $K\left(\frac{a}{R_0}\right)$

$\frac{a}{R_o}$	10^{-4}	10^{-3}	10^{-2}	3×10^{-2}
$K(\frac{a}{R_o})$	10.1	7.5	5.2	4.1

(The last value was derived by considering the two regions $\rho \in [0, 0.7]$ and $\rho \in \left[0.7 = 1 - 10\frac{a}{R_0}, \ 1 - \frac{a}{R_0}\right]$.)

It is clear that, due to the lack of theoretical knowledge about the solutions of the relevant one and two particle problems, the above solution (7.24) is hardly exact. However, it does indicate the order of magnitude effect of the wall.

The solution (7.24) can be checked experimentally in the following manner. First, for fixed $\frac{a}{R_0}$, measure the normalized settling speed $V/\mathcal{U}_0$ for various values of c, and plot these points on a graph of $V/\mathcal{U}_0$ versus c. (See Figure 6.) Then the line through these points should intersect the axis $c = 0$ at the point $1 - \frac{a}{R_0}K\left(\frac{a}{R_0}\right)$. This experiment would check the theoretical value of $K\left(\frac{a}{R_0}\right)$ obtained above. The dependence of the settling speed on c could also be measured, and this provides an independent way of calculating $K\left(\frac{a}{R_0}\right)$ from the same set of experiments.

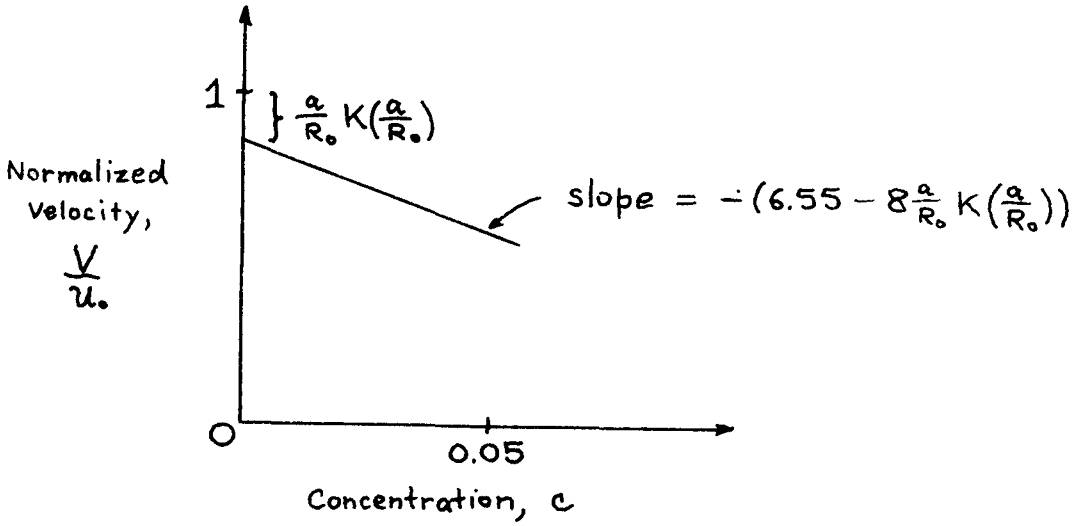


Figure 6. Predicted c dependence from (7-24).

Finally, even if the theoretical value of $K(\frac{a}{R_0})$ is incorrect, the form of the solution (7.24) could be checked because the dependence on $\frac{a}{R_0}$ of the slope of the line should be eight times as great as the dependence on $\frac{a}{R_0}$ of the intercept of the line with $c = 0$. Thus, the solution (7.24) can be checked quite thoroughly.

The next section summarizes some typical experimental results. Unfortunately, none of them are really suitable for obtaining the experimental information required above.

VIII. A SUMMARY OF EXPERIMENTAL RESULTS FOR SEDIMENTATION AND COMPARISON WITH THEORY

In Chapters III and VII results have been derived for the mean settling velocity of a dilute dispersion of spheres. The result for sedimentation in a tube was found to be

$$\frac{\mathcal{U}}{\mathcal{U}_0} = 1 - \frac{a}{R_0} K\left(\frac{a}{R_0}\right) - c\left(6.55 - 8\frac{a}{R_0} K\left(\frac{a}{R_0}\right)\right) \qquad (7.24)$$

where the value of $K\left(\frac{a}{R_0}\right)$ is given by (7.22b). Many experiments have been done to measure this hindered settling phenomenon, and some thorough summaries of these experiments are given in Happel and Brenner (1965) and Maude and Whitmore (1958). A representative selection of the experiments is given in Table 8-1. It can be seen from the last column of Table 8-1 that none of the experiments follow the theoretical prediction (7.24), but the reason for this is simply that the range of validity of the prediction (7.24) is not met by the experiments. The relation between theory and experiment is described below in more detail.

The most serious limitation of the theory is that it is only a linear theory, and as such, it predicts a settling speed of zero for $c = 16\%$, which is certainly not true. It is not expected that the theory would be valid for $c > 5\%$. This restriction leaves only the experiments by McNown and Lin, and Cheng and Schachman, and the results for $c = 5\%$ in Whitmore's experiments, to compare with the theory. The experimental results for higher concentrations are indicated in Figure 7. It can be seen that the straight lines predicted by theory widely diverge from experimental results for $c > 5\%$.

The experimental results for the mean settling velocity agree well with the theory

$$\frac{U}{U_0} = (1-c)^{\beta} \qquad \beta \approx 5 \tag{8.1}$$

derived by dimensional analysis by both Maude and Whitmore (1958) and Richardson and Zaki (1954). Using this experimental correlation, the size of the neglected c^2 term in the theory can be estimated by expanding (8.1) for small c with the result

$$\frac{U}{U_0} \sim 1 - 5c + 10c^2 - \ldots \quad . \tag{8.2}$$

This implies the theory will make a relative error on the order of

$$\frac{10c^2}{5c} = 2c$$

in calculating the hindrance to the mean settling speed.

The experiment by Cheng and Schachman (1955), while performed at sufficiently small values of c, was done in an ultracentrifuge, and the effect of rotation on the settling speed is not known. The results of their experiment are plotted in Figure 8.

The remaining experiment, done by McNown and Lin (1952), is interesting because of the unique $c^{1/3}$ dependence they found. The Reynolds number defined by

$$Re = \frac{2aU_0\rho}{\mu}$$

was about .75 in their experiment, and thus the theory we derived by assuming Stokes flow cannot predict this experimental result. Figure 8 shows the experimental points found by McNown and Lin, the straight line predicted by the theory (7.24), and a curve interpolated through the data points. The interpolation is based on the plot of

$\frac{\mathcal{U}}{\mathcal{U}_0}$ versus $c^{1/3}$ given in the experimenters' paper. A possible explanation for this $c^{1/3}$ dependence can be given on the basis of the results for general two sphere distributions described in Chapter VI. It is known experimentally [Happel and Brenner (1965)] that two particles settling at moderate Reynolds number (0.2 to 1.0) tend to separate from each other as long as one particle is not directly in the other particle's wake. If we suppose this complex two-particle interaction can be described by a hydrodynamic repulsion between particles, then it is likely that particles in a suspension will tend to separate from each other when settling at these moderate Reynolds numbers. In particular, the results of Chapter VI then indicate there will be a $c^{1/3}$ dependence in the settling speed when this repulsion is sufficiently strong. Thus, the $c^{1/3}$ dependence is probably caused by the magnitude of the Reynolds number, but a theory including inertia effects would be needed to verify this claim.

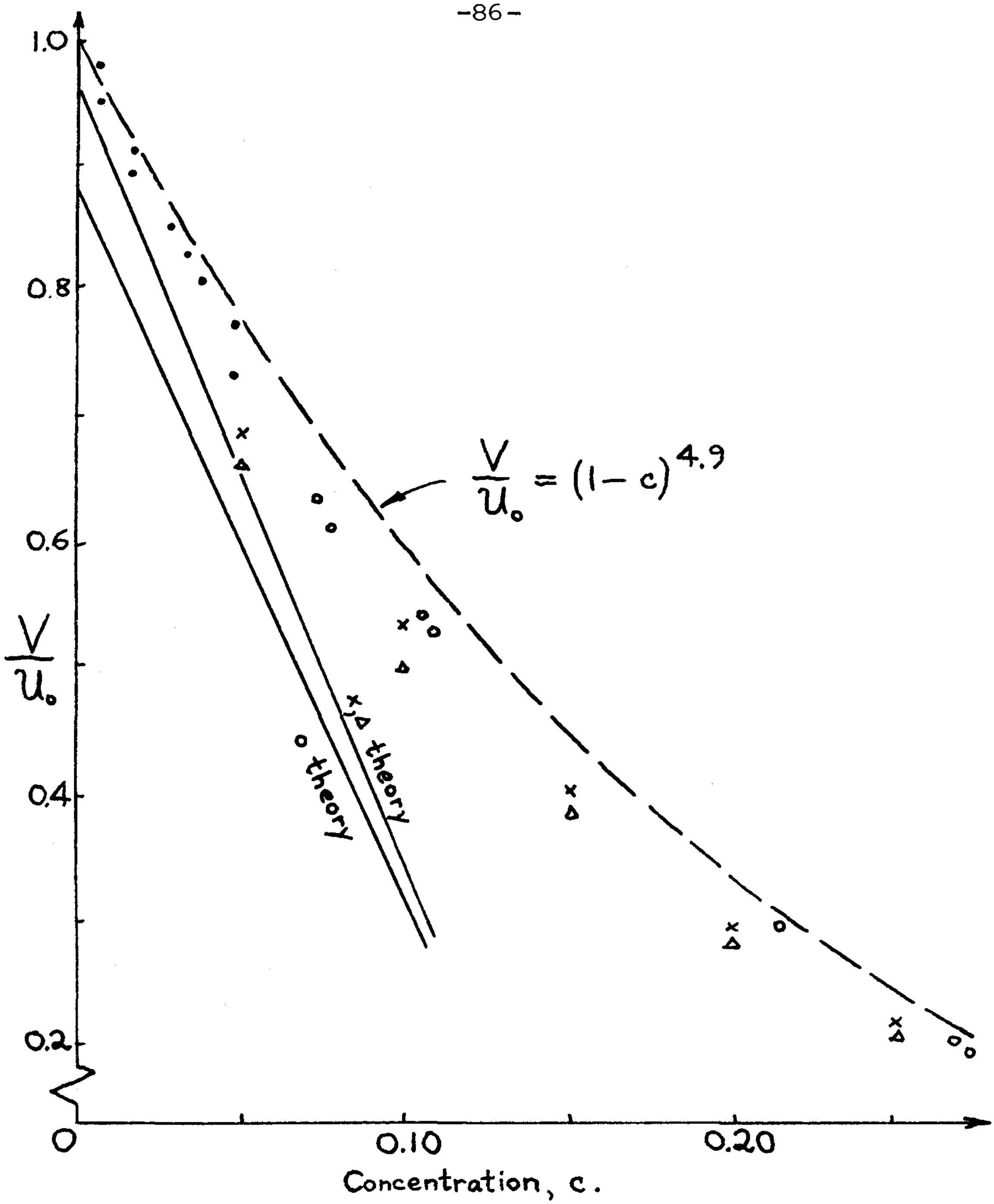
We conclude that at the present time there are no experiments that confirm or deny the theory (7.24) for the mean settling speed of a dilute suspension, and that other theories are needed to explain the experimental results given here.

TABLE 8-1. Some Typical Experimental Results

Experimenter	Physical Parameters: Spheres dia. (mm) $2a$, density (g/ml) ρ_s	Physical Parameters: Liquid viscosity (poise) μ, density (g/ml) ρ_f	Tube dia. (mm) $2R_o$	Non-dimensional Parameters: Range of volume concen- c	Reynolds number $Re = \frac{2aU}{\nu}$	$\frac{a}{R_o}$	$Re \frac{R_o}{a}$	Observed dependence
Whitmore (1955)	polystyrene 0.098 mm 1.0553 g/ml	lead nitrate 0.01024 p 1.0266 g/ml	30 mm	5%-25%	0.0143	.0033	4.3	$\frac{U}{U_o} = (1-c)^{5.0}$
	Kallodoc 0.096 mm 1.1881 g/ml	lead nitrate 0.01046 mm 1.0558 g/ml	30 mm	5%-25%	0.062	.0033	19	$\frac{U}{U_o} = (1-c)^{4.8}$
Steinour (1944)	tapioca 1.74 mm 1.38 g/ml	oil 7.13 mm 0.89 g/ml	62 mm	8%-50%	0.0026	.028	0.09	$\frac{U}{U_o} = (1-c)^{4.9}$
Cheng and Schachman (1955) (ultra-centrifuge)	polystyrene latex 2.6×10^{-4} mm 1.052 g/ml	water 0.00894 mm 0.997 g/ml	~10 mm	0.5%-6%	5.8×10^{-7}	$\sim10^{-5}$	0.02	$\frac{U}{U_o} = 1-5c$

TABLE 8-1. Some Typical Experimental Results (Cont'd.)

	Physical Parameters			Non-dimensional Parameters				
Experimenter	Spheres dia. (mm) $2a$ density (g/ml) ρ_s	Liquid viscosity (poise) μ density (g/ml) ρ_f	Tube dia. (mm) $2R_o$	Range of volume concen-	Reynolds number $Re = \frac{2aU}{\nu}$	a/R_o	$Re \frac{R_o}{a}$	Observed dependence
McNown and Lin (1952)	glass beads ~0.1 mm $U_o \cong .67$ cm/sec	water 0.00894 mm 0.997 g/ml	11 mm	0.1%-3%	~.75	.009	83	$\frac{U}{U_o} \cong .95 - 0.8c^{\frac{1}{3}}$
Richardson and Zaki (1954)	divinyl benzene 0.181 mm 1.058 g/ml	NaCl solution 0.0153 mm 1.001 g/ml	19 mm 28 mm 48 mm	$c > 10\%$	0.0789	.0095 .0065 .0038	~10	$\frac{U}{U_o} = \begin{cases} (1-c)^{4.9} \\ (1-c)^{4.79} \\ (1-c)^{4.76} \end{cases}$



o Steinour

x Whitmore (polystyrene)

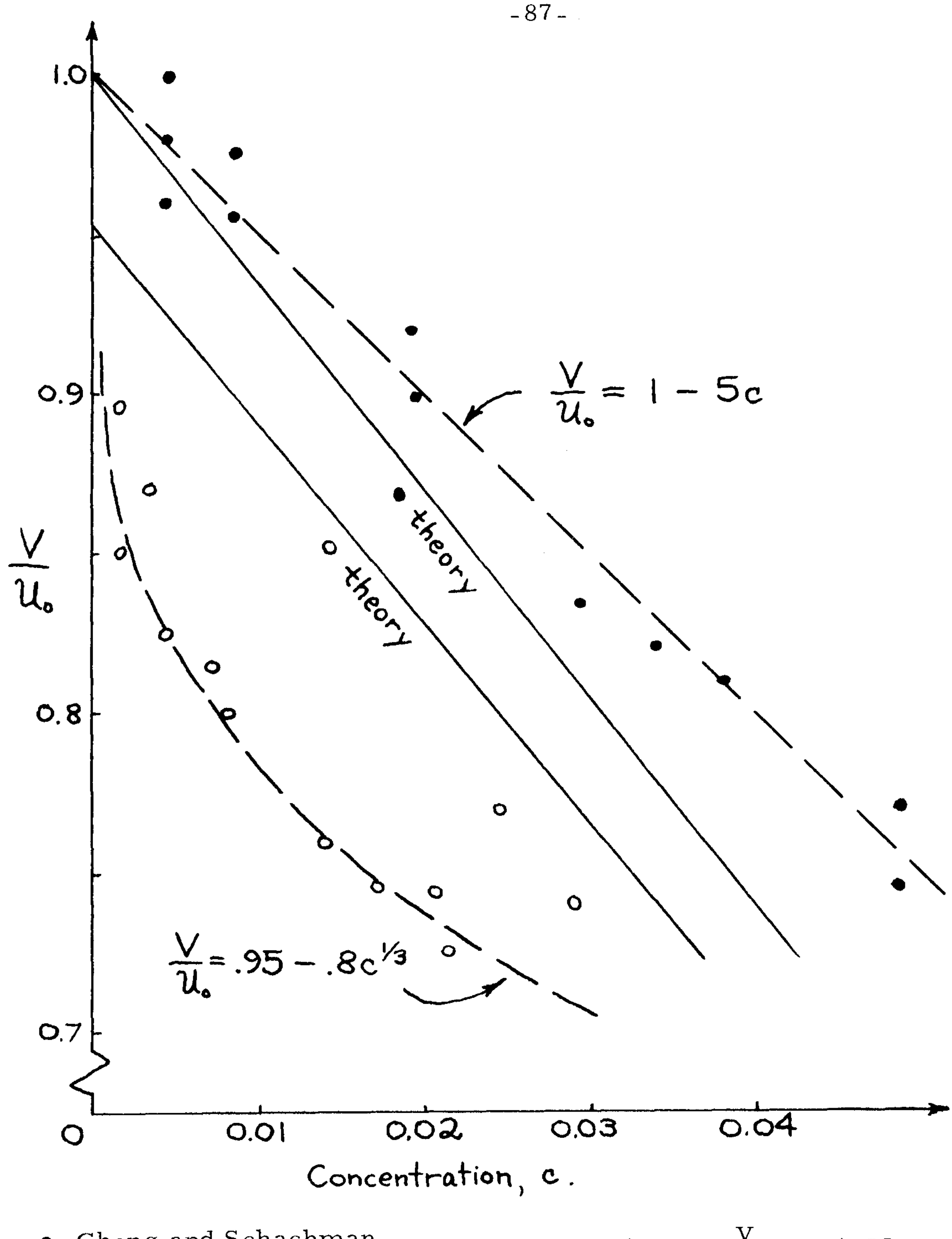
Δ Whitmore (Kallodoc)

• Cheng and Schachman

x,Δ theory $\frac{V}{U_o} = .975-6.35c$

o theory $\frac{V}{U_o} = .885-5.63c$

Figure 7. Experimental results for moderate concentrations.



● Cheng and Schachman

o McNown and Lin

● theory $\frac{V}{U_o} = 1 - 6.55c$

o theory $\frac{V}{U_o} = .955 - 6.19c$

Figure 8. Experimental results for low concentration.

APPENDIX A

A Comparison of the Random Free and Random Fixed Arrays

This section summarizes the results of Brinkman (1947) and Lundgren (1971). Brinkman first posed the porous medium model to find the settling velocity of spheres, while Lundgren directly related the equations for flow through a porous media to the equations for flow past a free suspension. The method of approach these authors use is to average the properties of the medium surrounding each sphere rather than to compute the individual hydrodynamic interactions as other authors do.

Brinkman's idea was to find an equation for flow through a suspension by combining the Stokes equation

$$\nabla p = \mu \nabla^2 \underline{v}$$

with Darcy's law

$$\nabla \overline{p} = -\frac{\mu}{k} \overline{\underline{v}}$$

for the mean flow through a porous medium. Thus, he postulated the equations of motion to be:

$$\begin{cases} \nabla p = -\frac{\mu}{k} \underline{v} + \mu' \nabla^2 \underline{v} \\ \nabla \cdot \underline{v} = 0 \end{cases} \tag{A-1}$$

where μ', the viscosity of the dispersion, was related in some way to μ, for instance by the Einstein viscosity relation

$$\mu' = \mu(1 + 2.5c).$$

His result was a change in the settling speed equal to $\frac{3}{\sqrt{2}} c^{1/2}$ for c small, which is the correct result for flow past the random fixed array. Saffman (1973) pointed out that Brinkman's postulated equations (A-1) are to leading order equivalent to (4.15) of this paper with $a = 0$.

Lundgren was more rigorous in his derivation of the relevant equations of motion. He extended a statistical formulation used by Saffman (1971) to average the properties of the medium surrounding each sphere. He found that the proper resistance term that should be added to the Stokes equations differed depending on whether the arrays were free or fixed. For the free array, his result is

$$\nabla p = \mu \nabla^2 \underline{v} + B \nabla^2 \langle \underline{v} \rangle + C g \qquad \text{(A-2)}$$

where he finds the constants B and C, and $\langle \underline{v} \rangle$ is the velocity of the composite material. Thus, the resistance term added to Stokes law was not found to be proportional to $\langle \underline{v} \rangle$ as Brinkman had supposed. For flow past a fixed array, Lundgren's result is

$$\nabla p = \mu \nabla^2 \underline{v} + A \langle \underline{v} \rangle + B \nabla^2 \langle \underline{v} \rangle \qquad \text{(A-3)}$$

where now $\langle \underline{v} \rangle$ is the seepage velocity, and he finds the constants A and B. Comparing (A-2) and (A-3), it is seen that there is more resistance to the motion of the fluid for the fixed array, and so the c-dependence is larger in this case.

The resulting equation he derived for the mean settling speed of a suspension is interesting in view of Faxen's law. He found that $\underline{V}$, the mean sphere velocity, and $\langle \underline{v} \rangle$, the composite material velocity, were related by

$$\underline{V} = \frac{\underline{\tilde{F}}}{6\pi \tilde{\mu} a} + \langle \underline{v} \rangle + \frac{a^2}{6} \nabla^2 \langle \underline{v} \rangle \qquad \text{(A-4)}$$

where $\tilde{\mu}$ = effective viscosity of the suspension,
$\underline{\tilde{F}}$ = excess weight of a sphere relative to the suspension.

Equation (A-4) is just Faxen's law (2.16a) with the mean properties of the suspension replacing the properties of the fluid alone. His result for the mean settling velocity of suspensions is

$$\underline{V} = \underline{U}_0 (1 - \beta c + \ldots), \quad \beta = \frac{7}{2}$$

which is as near to experimental values $\beta \approx 5$ as other methods (e.g. $\beta = 6.55$ derived in Chapter III).

The idea of replacing the fluid viscosity μ with the suspension viscosity $\tilde{\mu}$ was tested experimentally by Whitmore (1955). He compared the settling speeds of particles falling through pure fluid to those falling through a dilute suspension of neutrally buoyant spheres. The result was the falling spheres were hindered more when settling through the suspension of neutrally buoyant spheres <u>if</u> the concentration of falling spheres was less than 10 per cent. This indicates that the suspension viscosity $\tilde{\mu}$ may be the correct parameter in the equations of motion. However, at greater concentrations there were streaming effects and the spheres settled more rapidly through the suspension than through the fluid alone.

APPENDIX B

Evaluation of $\int\limits_{r\geq 2a}\left(\frac{rD}{1+dD}+\frac{rF}{1+dF}\right)dr$

In Chapter IV, the following equation for $\langle v_1(\underline{0};0)\rangle_0$ is derived:

$$\langle v_1(\underline{0};0)\rangle_0 = -\frac{4\pi}{3}nd^2V_1\int\limits_{r\geq 2a}\left(\frac{rD}{1+dD}+\frac{rF}{1+dF}\right)dr\,. \tag{4.38}$$

This section derives the leading terms in the expansion for small c of the right hand side of (4.38). The method used involves dividing the region of integration into three parts, and finding separate expansions for each region.

First, we make the change of variables $\lambda r = x$ in the integral. This gives

$$\langle V_1\rangle_0 = -\frac{4\pi}{3}n\frac{d^2}{\lambda}V_1\int_{2a\lambda}^{\infty}\left(\frac{\frac{1}{x^2}D'}{1+\frac{d\lambda}{x^3}D'}+\frac{\frac{1}{x^2}F'}{1+\frac{d\lambda}{x^3}F'}\right)dx \tag{B-1}$$

where

$$\begin{cases} D' \equiv 2\left[1-e^{-x}(1+x)\right] \sim x^2\left(1-\frac{2}{3}x+\ldots\right) \text{ as } x\to 0 \\ F' \equiv -1+e^{-x}(1+x+x^2) \sim \frac{1}{2}x^2\left(1-\frac{4}{3}x+\ldots\right) \text{ as } x\to 0 \end{cases} \tag{B-2}$$

The coefficient of the integral in (B-1) is order $V_1c^{1/2}$ since

$$n\frac{d^2}{\lambda}V_1 = n\left(\frac{3}{2}\frac{a}{1-\lambda a}\right)^2\frac{V_1}{(6\pi a n)^{1/2}} \sim V_1 n^{1/2}a^{3/2} \sim V_1 c^{1/2}.$$

Thus, we may neglect the terms of order $c^{1/2}$ in the integral itself since these produce only order c effects in $\langle v_1\rangle_0$. It is convenient to define the small parameter

$$\mu = 2a\lambda = 3\sqrt{2}\, c^{1/2}$$

to simplify the notation.

Now divide the interval of integration into three parts:

$$\int_{\mu}^{\infty} (\cdot)\, dx = \int_{\mu}^{\mu^{1/2}} (\cdot)\, dx + \int_{\mu^{1/2}}^{1} (\cdot)\, dx + \int_{1}^{\infty} (\cdot)\, dx . \qquad \text{(B-3)}$$

Consider first the integral involving D'. In the region $\mu \le x \le \mu^{1/2}$ the integrand may be written

$$\frac{\frac{1}{x^2} D'}{1 + \frac{d\lambda}{x^3} D'} = \frac{\frac{1}{x} D'}{x + d\lambda\left(1 - \frac{2}{3}x + \ldots\right)} = \frac{\left(x - \frac{2}{3}x^2 + \ldots\right)}{x + d\lambda}\left[1 + d\lambda \frac{\left(\frac{2}{3}x + \ldots\right)}{x + d\lambda}\right] \qquad \text{(B-4)}$$

where D' has been expanded for x small using (B-2). Only the first term of (B-4) gives a contribution greater than $c^{1/2}$ since the other terms are bounded by:

$$\left| \int_{\mu}^{\mu^{1/2}} \frac{x^2}{x + d\lambda}\, dx \right| = \int_{\mu}^{\mu^{1/2}} |x| \left| \frac{x}{x + d\lambda} \right| |dx| \le \mu^{1/2} \cdot 1 \cdot \mu^{1/2} = \mu \approx c^{1/2}$$

and

$$\left| \int_{\mu}^{\mu^{1/2}} \frac{x}{x + d\lambda}\, d\lambda \frac{\left(\frac{2}{3}x + \ldots\right)}{x + d\lambda}\, dx \right| \le 1 \cdot d\lambda \cdot 1 \cdot \mu^{1/2} \approx c^{3/4} .$$

The first term gives a contribution of

$$\int_{\mu}^{\mu^{1/2}} \frac{x}{x + d\lambda}\, dx = \int_{\mu}^{\mu^{1/2}} \left(1 - \frac{d\lambda}{x + d\lambda}\right) dx = \mu^{1/2} - d\lambda\left[\ln(\mu^{1/2} + d\lambda) - \ln(\mu + d\lambda)\right] + O(\mu) \qquad \text{(B-5)}$$

which will later be put in terms of c . This is the leading order contribution from the interval $[\mu, \mu^{1/2}]$.

In the other two regions of integration, the denominator of the integrand may be expanded directly as

$$\frac{\frac{1}{x^2}D'}{1+\frac{d\lambda}{x^3}D'} = \frac{1}{x^2}D'\left[1 - \frac{d\lambda}{x^3}D' + \left(\frac{d\lambda}{x^3}D'\right)^2 - \ldots\right] \qquad \text{(B-6)}$$

In the region $x \geqslant 1$, the terms with $d\lambda$ are order μ , so for $x \geqslant 1$ we need only keep the term

$$\int_1^\infty \frac{1}{x^2}D'\,dx \ . \qquad \text{(B-7)}$$

In the region $\mu^{1/2} \leqslant x \leqslant 1$, the term involving $\left(\frac{d\lambda}{x^3}D'\right)^2$ may be neglected since

$$\left(\frac{d\lambda}{x^3}D'\right)^2 \sim \frac{(d\lambda)^2}{x^2}\left(1-\frac{4}{3}x+\ldots\right) \leqslant M\mu \quad \text{for} \quad x \geqslant \mu^{1/2} .$$

Then expanding the first two terms on the right hand side of (B-6) gives

$$\frac{1}{x^2}D'\left(1-\frac{d\lambda}{x^3}D'\right) = \frac{1}{x^2}D' - \left(1-\frac{2}{3}x+\ldots\right)\frac{d\lambda}{x}\left(1-\frac{2}{3}x+\ldots\right)$$
$$= \frac{1}{x^2}D' - \frac{d\lambda}{x} + O(\mu).$$

Thus, in the region $[\mu^{1/2}, 1]$, we need only keep the contribution

$$\int_{\mu^{1/2}}^{1}\left(\frac{1}{x^2}D' - \frac{d\lambda}{x}\right)dx \qquad \text{(B-8)}$$

Combining the results (B-5), (B-7), and (B-8) we see that the integral of the D' term is given by

$$\int_{\mu}^{\infty} \frac{\frac{1}{x^2}D'}{1+\frac{d\lambda}{x^3}D'}\,dx = \int_{\mu}^{\mu^{1/2}} \frac{x}{x+d\lambda}\,dx - \int_{\mu^{1/2}}^{1} \frac{d\lambda}{x}\,dx + \int_{\mu^{1/2}}^{\infty} \frac{1}{x^2}D'\,dx + O(c^{1/2}). \quad \text{(B-9)}$$

In a similar manner, the integral involving F' can be expressed as

$$\int_{\mu}^{\infty} \frac{\frac{1}{x^2}F'}{1+\frac{d\lambda}{x^3}F'}\,dx = \frac{1}{2}\int_{\mu}^{\mu^{1/2}} \frac{x}{x+\frac{d\lambda}{2}}\,dx - \frac{1}{4}\int_{\mu^{1/2}}^{1} \frac{d\lambda}{x}\,dx + \int_{\mu^{1/2}}^{\infty} \frac{1}{x^2}F'\,dx + O(c^{1/2}). \quad \text{(B-10)}$$

The integrals in (B-9) and (B-10) are easily evaluated. We have

$$\int_{\mu^{1/2}}^{\infty} \frac{1}{x^2}(D'+F')\,dx = \int_{\mu^{1/2}}^{\infty} \left(\frac{1}{x^2} + e^{-x} - \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2}\right) dx$$

$$= \frac{1}{\mu^{1/2}} - e^{-\mu^{1/2}} - \frac{e^{-\mu^{1/2}}}{\mu^{1/2}}$$

$$= 2 - \frac{3}{2}\mu^{1/2} + O(\mu) \quad \text{as } \mu \to 0,$$

$$-\frac{5}{4}\int_{\mu^{1/2}}^{1} \frac{d\lambda}{x}\,dx = \frac{5}{4}\,d\lambda \ln \mu^{1/2} = \frac{5}{8}\,d\lambda \ln \mu,$$

and

$$\int_{\mu}^{\mu^{1/2}} \left(\frac{x}{x+d\lambda} + \frac{1}{2}\frac{x}{x+\frac{d\lambda}{2}}\right) dx = \frac{3}{2}\mu^{1/2} - d\lambda \ln(\mu^{1/2}+d\lambda) + d\lambda \ln(\mu+d\lambda)$$
$$- \frac{d\lambda}{4}\ln\left(\mu^{1/2}+\frac{d\lambda}{2}\right) + \frac{d\lambda}{4}\ln\left(\mu+\frac{d\lambda}{2}\right) + O(\mu).$$

Then using the relations

$$\mu = 3\sqrt{2}\,c^{1/2} \qquad d\lambda = \frac{9}{4}\sqrt{2}\,c^{1/2} + O(c)$$

to express the above results in terms of the volume concentration c, we have

$$\int_{\mu}^{\infty}\left(\frac{\frac{1}{x^2}D'}{1+\frac{d\lambda}{x^3}D'}+\frac{\frac{1}{x^2}F'}{1+\frac{d\lambda}{x^3}F'}\right)dx = 2+\frac{45}{32}\sqrt{2}\,c^{1/2}\ln c + O(c^{1/2}).$$

Therefore, since the coefficient of the integral is given by

$$-\frac{4\pi}{3}n\frac{d^2}{\lambda}V_1 = -V_1\frac{3}{4}\sqrt{2}\,c^{1/2}$$

the expansion of $\langle v_1(\underline{0};0)\rangle_0$ is

$$\langle v_1(\underline{0};0)\rangle_0 = V_1\left(-\frac{3}{\sqrt{2}}c^{1/2}-\frac{135}{64}c\ln c + O(c)\right).$$

This is the result cited in Chapter IV.

APPENDIX C

A Point Particle in the Presence of a Plane Wall

A result for the velocity field due to a point particle in the presence of a plane wall was given by Lorentz (1907). It is presented here because the result is not easily found in the literature.

Let the fluid velocity due to the particle in an unbounded fluid be denoted by $(u^{(0)}, v^{(0)}, w^{(0)})$, and let the wall be the plane $y=0$. Denote the image motion by $(u^{(1)}, v^{(1)}, w^{(1)})$, where, in the image, the components of motion parallel to the wall remain the same, while the component perpendicular to the wall is reversed. Then, on $y=0$, we have $u^{(1)} = u^{(0)}$, $v^{(1)} = -v^{(0)}$, $w^{(1)} = w^{(0)}$. It was shown by Lorentz that the velocity field $(u^{(0)}+u^{(2)}, v^{(0)}+v^{(2)}, w^{(0)}+w^{(2)})$ vanishes on $y=0$, where

$$\left\{\begin{aligned} u^{(2)} &= -u^{(1)} - 2y\frac{\partial v^{(1)}}{\partial x} + \frac{y^2}{\mu}\frac{\partial p^{(1)}}{\partial x} \\ v^{(2)} &= v^{(1)} - 2y\frac{\partial v^{(1)}}{\partial y} + \frac{y^2}{\mu}\frac{\partial p^{(1)}}{\partial y} \\ w^{(2)} &= -w^{(1)} - 2y\frac{\partial v^{(1)}}{\partial z} + \frac{y^2}{\mu}\frac{\partial p^{(1)}}{\partial z} \\ p^{(2)} &= p^{(1)} + 2y\frac{\partial p^{(1)}}{\partial y} - 4\mu\frac{\partial v^{(1)}}{\partial y} \end{aligned}\right. \qquad \text{(C-1)}$$

Here, $p^{(1)}$ is the pressure corresponding to the velocity $(u^{(1)}, v^{(1)}, w^{(1)})$ and $p^{(2)}$ is similarly defined. The velocity field $(u^{(2)}, v^{(2)}, w^{(2)})$ satisfies the homogeneous Stokes flow equations in the fluid, and thus is the "wall reflection" of the velocity field $(u^{(0)}, v^{(0)}, w^{(0)})$. Of course, if the flow $(u^{(0)}, v^{(0)}, w^{(0)})$ were due to a finite particle, the boundary conditions on the surface of the particle would not be satisfied by the reflected field $(u^{(2)}, v^{(2)}, w^{(2)})$, and equations (C-1) would then describe only an approximate solution to the problem.

APPENDIX D

A Summary of the Multiple Fourier Transforms Used

As few books have tables of multiple integrals, it was felt worthwhile to summarize the transforms used in this thesis. A sample calculation is provided at the end of this section. The range of integration in each case is unbounded three-dimensional space.

$$\int e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = 8\pi^3 \delta(\underline{r})$$

$$\int \frac{1}{k^2} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = 2\pi^2 \frac{1}{r}$$

$$\int \frac{k_i k_j}{k^2} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = -2\pi^2 \partial_i \partial_j \left(\frac{1}{r}\right) = \frac{2\pi^2}{r^3}\left(\delta_{ij} - 3\frac{r_i r_j}{r^2}\right)$$

$$\int \frac{1}{k^4} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = -\pi^2 r$$

$$\int \frac{k_i k_j}{k^4} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = -\pi^2 \partial_i \partial_j r = \frac{\pi^2}{r}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)$$

$$\int \frac{1}{k^2+\lambda^2} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = \frac{2\pi^2}{r} e^{-\lambda r}$$

$$\int \frac{1}{k^2(k^2+\lambda^2)} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = \frac{2\pi^2}{r\lambda^2}\left(1 - e^{-\lambda r}\right) \qquad \text{(D-0)}$$

$$\int \frac{k_i k_j}{k^2(k^2+\lambda^2)} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = -\partial_i \partial_j \left[\frac{2\pi^2}{r\lambda^2}\left(1 - e^{-\lambda r}\right)\right]$$

$$\int \frac{k_i k_j}{k^4(k^2+\lambda^2)} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k} = -\frac{2\pi^2 \delta_{ij}}{r^3\lambda^4}\left[1 - \tfrac{1}{2}\lambda^2 r^2 - e^{-\lambda r}(1+\lambda r)\right]$$

$$+ \frac{6\pi^2}{r^3\lambda^4}\frac{r_i r_j}{r^2}\left[1 - \tfrac{1}{6}\lambda^2 r^2 - e^{-\lambda r}\left(1 + \lambda r + \tfrac{1}{3}\lambda^2 r^2\right)\right].$$

Using the above results, and the notation $P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$, we also have

$$\frac{1}{8\pi^3\mu}\int \frac{P_{ij}}{k^2} e^{i\underline{k}\cdot\underline{r}} d\underline{k} = S_{ij}(\underline{r}) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right)$$

$$\int \frac{P_{ij}}{k^2+\lambda^2} e^{i\underline{k}\cdot\underline{r}} d\underline{k} = 2\pi^2\left[\frac{e^{-\lambda r}}{r}\delta_{ij} + \partial_i\partial_j\left(\frac{1-e^{-\lambda r}}{r\lambda^2}\right)\right]$$

$$\int \frac{P_{ij}}{k^2(k^2+\lambda^2)} e^{i\underline{k}\cdot\underline{r}} d\underline{k} = \frac{2\pi^2}{r^3\lambda^4}\delta_{ij}\left[1 + \tfrac{1}{2}\lambda^2 r^2 - e^{-\lambda r}(1+\lambda r + \lambda^2 r^2)\right]$$
$$- \frac{6\pi^2}{r^3\lambda^4}\frac{r_i r_j}{r^2}\left[1 - \tfrac{1}{6}\lambda^2 r^2 - e^{-\lambda r}(1+\lambda r + \tfrac{1}{3}\lambda^2 r^2)\right].$$

As an example of the method of integration, consider the integral

$$I = \int \frac{k_i k_j}{k^4(k^2+\lambda^2)} e^{i\underline{k}\cdot\underline{r}} d\underline{k} .$$

Making use of polar coordinates with the axis $\varphi = 0$ parallel to $\underline{r}$, we have

$$I = \int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{k_i k_j}{k^4(k^2+\lambda^2)} e^{ikr\cos\varphi} k^2 \sin\varphi \, d\theta \, d\varphi \, dk \qquad \text{(D-1)}$$

Next, write the product $k_i k_j$ in new Cartesian coordinates where one axis is parallel to $\underline{r}$. The transformation is defined by

$$k_i = (\underline{e}_i \cdot \underline{\ell}_n)\, k'_n \qquad \text{(D-2)}$$

where $\begin{cases} \underline{e}_i & \text{are the unit vectors in the unprimed coordinates,} \\ \underline{\ell}_n & \text{are the unit vectors in the primed coordinates,} \end{cases}$

and the primed coordinates are related to the polar coordinates by

$$\begin{cases} k_1' = k \cos\varphi \\ k_2' = k \sin\varphi \cos\theta \\ k_3' = k \sin\varphi \sin\theta \end{cases}$$

This transformation enables us to use the relation

$$k_i k_j = (\underline{e}_i \cdot \underline{\ell}_m)(\underline{e}_j \cdot \underline{\ell}_n)\, k_m' k_n'$$

to divide the integral in (D-1) into several parts. Upon integration with respect to θ, it is seen that the integrals involving $k_1' k_2'$, $k_2' k_3'$, and $k_3' k_1'$ are zero, and that I is given by

$$I = (\underline{e}_i \cdot \underline{\ell}_1)(\underline{e}_j \cdot \underline{\ell}_1) I_1 + \left[(\underline{e}_i \cdot \underline{\ell}_2)(\underline{e}_j \cdot \underline{\ell}_2) + (\underline{e}_i \cdot \underline{\ell}_3)(\underline{e}_j \cdot \underline{\ell}_3)\right] I_2 \qquad \text{(D-3)}$$

where

$$I_1 = 2\pi \int_0^\infty \int_0^\pi \frac{e^{ikr\cos\varphi}}{k^2+\lambda^2} \cos^2\varphi \sin\varphi \, d\varphi \, dk$$

$$I_2 = \pi \int_0^\infty \int_0^\pi \frac{e^{ikr\cos\varphi}}{k^2+\lambda^2} \sin^2\varphi \sin\varphi \, d\varphi \, dk$$

Using the integral (D-0), it can be seen that

$$2I_2 + I_1 = \frac{2\pi^2}{r\lambda^2}\left(1 - e^{-\lambda r}\right) \qquad \text{(D-4)}$$

and so only the evaluation of I_1 is necessary. In order to find I_1, we note that

$$\begin{aligned} \int_0^\pi e^{ikr\cos\varphi} \cos^2\varphi \sin\varphi \, d\varphi &= \frac{1}{(ik)^2} \frac{\partial^2}{\partial r^2} \int_0^\pi e^{ikr\cos\varphi} \sin\varphi \, d\varphi \\ &= -\frac{2}{k^2} \frac{\partial^2}{\partial r^2}\left(\frac{\sin kr}{kr}\right) \\ &= -\frac{2}{r^2} \frac{\partial^2}{\partial k^2}\left(\frac{\sin kr}{kr}\right). \end{aligned} \qquad \text{(D-5)}$$

The last step is obvious from the general relation

$$\frac{\partial^2}{\partial k^2} f(kr) = r^2 f''(kr) = \frac{r^2}{k^2}\frac{\partial^2}{\partial r^2} f(kr) .$$

Using (D-5) we have

$$I_1 = -\frac{4\pi}{r^2}\int_0^\infty \frac{\partial^2}{\partial k^2}\left(\frac{\sin kr}{kr}\right)\frac{dk}{k^2+\lambda^2}$$

and when integrated by parts this gives

$$I_1 = \frac{8\pi}{r^3}\left[\int_0^\infty \frac{\sin kr}{k(k^2+\lambda^2)^2}dk - 4\int_0^\infty \frac{k \sin kr}{(k^2+\lambda^2)^3}dk\right]. \qquad \text{(D-6)}$$

These two integrals can be found in standard tables, e.g. Gradshteyn and Ryzhiz (1965). The result for I_1 is then

$$I_1 = \frac{4\pi^2}{r^3\lambda^4}\left[+1 - e^{-\lambda r}(1+\lambda r + \tfrac{1}{2}\lambda^2 r^2)\right]$$

and using (D-4) the result for I_2 is

$$I_2 = \frac{2\pi^2}{r^3\lambda^4}\left[-1 + \tfrac{1}{2}\lambda^2 r^2 + e^{-\lambda r}(1+\lambda r)\right].$$

The expression for I in (D-3) may then be simplified somewhat by noting that $(\underline{e}_i\cdot\underline{\ell}_1)$ is the normalized component of $\underline{r}$ in the direction i, so that

$$(\underline{e}_i\cdot\underline{\ell}_1)(\underline{e}_j\cdot\underline{\ell}_1) = \frac{r_i r_j}{r^2} . \qquad \text{(D-7)}$$

Also, by the orthogonality conditions for the transformation [Courant and Hilbert (1953)], we have

$$(\underline{e}_i\cdot\underline{\ell}_2)(\underline{e}_j\cdot\underline{\ell}_2) + (\underline{e}_i\cdot\underline{\ell}_3)(\underline{e}_j\cdot\underline{\ell}_3) = \delta_{ij} - \frac{r_i r_j}{r^2} . \qquad \text{(D-8)}$$

Making use of these last two relations in (D-3), we find

$$I = \delta_{ij} I_2 + \frac{r_i r_j}{r^2}(I_1 - I_2)$$

$$= -\frac{2\pi^2}{r^3\lambda^4}\delta_{ij}\left[1 - \tfrac{1}{2}\lambda^2 r^2 - e^{-\lambda r}(1+\lambda r)\right]$$

$$+ \frac{6\pi^2}{r^3\lambda^4}\frac{r_i r_j}{r^2}\left[1 - \tfrac{1}{6}\lambda^2 r^2 - e^{-\lambda r}(1 + \lambda r + \tfrac{1}{3}\lambda^2 r^2)\right],$$

which is exactly the result cited above.

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