EXISTENCE, UNIQUENESS, AND STABILITY OF SOLUTIONS OF A CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this study we investigate the existence, uniqueness and asymptotic stability of solutions of a class of nonlinear integral equations which are representations for some time dependent nonlinear partial differential equations. Sufficient conditions are established which allow one to infer the stability of the nonlinear equations from the stability of the linearized equations. Improved estimates of the domain of stability are obtained using a Liapunov Functional approach. These results are applied to some nonlinear partial differential equations governing the behavior of nonlinear continuous dynamical systems.

TABLE OF CONTENTS

PART	TITLE	PAGE
	Acknowledgements	ii
	Abstract	iii
	Table of Contents	iv
	List of Symbols	vi
INTRODUCTION		1
CHAPTER 1.	GENERAL THEORY	3
	1.1. Discussion of an Integral Equation	3
	1.2. Existence and Uniqueness	4
	1.3. A Liapunov-Poincaré Type Theorem	9
	1.4. Liapunov's Direct Method for Stability	11
CHAPTER 2.	SPECIFIC EXAMPLES	15
	2.1. Some Comments on Existence	16
	2.2. Derivation of an Integral Equation	19
	2.3. Derivation of an Equality for a Liapunov Functional	22
	2.4. Examples	23
	Example A	24
	Example B	33
	Example C	38
	Example D	43
APPENDIX I.	A PROOF OF LEMMAS 1 AND 4	47
APPENDIX II.	SOBOLEV SPACE PROPERTIES	48
APPENDIX III.	PROPERTIES OF THE INTEGRAL KERNELS "A" AND "B"	50

PART TABLE OF CONTENTS (CONT'D) PAGE

APPENDIX IV. NONLINEARITY CONDITIONS 60

REFERENCES 66

LIST OF SYMBOLS

a. e.	almost everywhere
$\in$	belongs to
C^N	space of N times continuously differentiable functions
L_2	space of square integrable functions
R^N	Euclidean N-space
D	bounded subset of R^N
∂D	boundary of D
T	time
Ω	D X (0, T)
D^n	any n^{th} order spatial derivative
u_{z^m}	m^{th} order derivative of u with respect to z
M	mapping
K	constant
[a]	greatest integer $\leq$ a

INTRODUCTION

In this work we present a unified theory for treating the existence, uniqueness and asymptotic stability of solutions for a class of nonlinear partial differential equations governing the behavior of nonlinear continuous dynamical systems. From this class we treat the following initial boundary value problems in some detail;

$$u_{tt} - 2\alpha u_{xxt} - u_{xx} = f(u, u_x, u_t, u_{xx}, x, t) \qquad \text{(A)}$$

$$u_{tt} + 2\alpha u_t - u_{xx} = f(u, u_x, u_t, x, t) \qquad \text{(B)}$$

$$u_{tt} - 2\alpha \nabla^2 u_t - \nabla^2 u = f(u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{yy}, x, y, t) \qquad \text{(C)}$$

$$\left.\begin{aligned} u_{tt} - 2\alpha u_{xxt} - u_{xx} &= f_1(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, x, t) \\ v_{tt} - 2\alpha v_{xxt} - \sigma^2 v_{xx} &= f_2(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, x, t). \end{aligned}\right\} \qquad \text{(D)}$$

Some problems of this type have been treated before. Ficken and Fleishman [13] investigated the existence, uniqueness and stability of solutions for the initial value problem

$$u_{xx} - u_{tt} - 2\alpha_1 u_t - \alpha_2 u = \epsilon u^3 + b.$$

Greenberg, MacCamy and Mizel [8] have treated the initial boundary value problem $u_{tt} - u_{xxt} = \sigma'(u_x)\, u_{xx}$ (which is a special case of (A) above) using some results from the theory of parabolic equations.

Rabinowitz [10] and [11] has proven the existence of periodic solutions for $u_{tt} + 2\alpha u_t - u_{xx} = \epsilon f$ where ϵ is a small parameter and f is periodic in time. In [10] he treats $f = f(u, u_x, u_t, x, t)$ and in [11] he treats the fully nonlinear case $f = f(u, u_x, u_t, u_{xx}, u_{xt}, u_{xx}, x, t)$. To do this he uses methods from the theory of elliptic boundary value problems.

CHAPTER I

GENERAL THEORY

A number of time dependent nonlinear partial differential equations with appropriate initial data and boundary conditions can be written as Volterra integral equations. The solution of a particular integral equation can be viewed as a fixed point of some mapping M.

In this chapter we use a contraction mapping principle to prove the existence and uniqueness of a fixed point of such a mapping. Asymptotic stability of the fixed point follows from an application of the Gronwall lemma. We end this chapter by discussing the Liapunov Functional approach to stability; this approach is a generalization of Liapunov's Direct Method for ordinary differential equations.

1.1. Discussion of an Integral Equation

We consider those time dependent nonlinear partial differential equations which can be written as an integral equation of the form

$$\left.\begin{aligned} U(x,t) &= M\,U(x,t) \\ \text{where} \quad & \\ M\,U &= G[a(x)] + \int_0^t H[g(U)]\,d\tau\,. \end{aligned}\right\} \qquad (1.1)$$

U is a vector function defined for x belonging to some domain $D \subset R^N$ and $t \in [0, T]$ where T may be infinite. G is a linear integral operator on D which maps some initial data a(x) into a vector function of x and t. H is also a linear integral operator on D which maps a vector function g(U) into a vector function of x, t and τ. Notice that if $g(0) = 0$ then $U = 0$ is a fixed point of (1.1).

It is assumed that the components of U belong to $L_2(\Omega)$ where $\Omega = Dx(0,T)$. For the purpose of proving existence we define a Banach space $H(U,\Omega) \equiv H_T$ with elements U under the norm

$$\|U\|_2 = \left(\int_0^T \int_D [U_1^2 + U_2^2 + \cdots + U_n^2]\, dxdt \right)^{\frac{1}{2}} .$$

The subscript 2 is used because we also use an auxiliary norm

$$\|U\|_1 = \left(\int_D [U_1^2 + U_2^2 + \cdots + U_n^2]\, dx \right)^{\frac{1}{2}} .$$

A more appropriate norm, $\|U\|_2$, to use in what follows might be $\|U\|_2 = \max_{t \in [0,T]} \|U\|_1$; however we did not use this for the reason discussed in Section 2.1.

1.2. Existence and Uniqueness

We prove two existence theorems, one on a finite time interval and the other on an infinite time interval, by proving that M is a contraction mapping of a complete metric space into itself.

Let $B_{T_o}(\delta) \equiv B_{T_o}$ denote the set of elements U such that $U(x,t) \in H_{T_o}$, T_o finite, and $\|U\|_1 \le \delta$ a.e. for $t \in [0, T_o]$.

<u>Lemma 1:</u> B_{T_o} is a complete metric space.

Proof: See Appendix I.

Theorem 1. (Existence and Uniqueness for a Finite Time Interval):

If

a) $\|G[a]\|_1 \leq K_1 \eta$ where η depends only on a(x) and K_1 is independent of a(x),

b) $\|H[g(U)]\|_1 \leq K_2 \|g(U)\|_1$ for every $U \in B_{T_o}$, where K_2 is independent of g,

c) $\|g(U)\|_1 \leq L_1 \|U\|_1$ a.e. for every $U \in B_{T_o}$,

d) $\|g(U_1) - g(U_2)\|_1 \leq L_2 \|U_1 - U_2\|_1$ a.e. for every $U_1, U_2 \in B_{T_o}$,

e) $\eta \leq \frac{\gamma\delta}{K_1}$ for any $\gamma \in (0, 1)$,

f) $T_o < \min\left[\frac{1-\gamma}{K_2 L_1}, \frac{1}{K_2 L_2}\right]$,

then there exists a unique fixed point $U \in B_{T_o}$ of (1.1) on the interval $[0, T_o]$.

Proof: To prove this it is sufficient to prove that M is a contraction operator on B_{T_o} which maps B_{T_o} into itself.*

Lemma 2: M maps B_{T_o} into itself, that is if $\|U\|_1 \leq \delta$ a.e. then $\|MU\|_1 \leq \delta$ a.e. .

Proof: Conditions a) and b) imply that

$$\|MU\|_1 \leq K_1 \eta + K_2 \int_0^t \|g(U)\|_1 \, d\tau,$$

using c)

$$\|MU\|_1 \leq K_1 \eta + K_2 L_1 \int_0^t \|U\|_1 \, d\tau .$$

* See Korevaar [12], p. 213, for a discussion of contraction operators.

But $U \in B_{T_o}$, therefore

$$\|MU\|_1 \le K_1 \eta + K_2 L_1 \delta T_o.$$

Conditions e) and f) give

$K_1 \eta \le \gamma \delta$ and $K_2 L_1 \delta T_o < (1-\gamma)\delta$ which implies

$$\|MU\|_1 \le \delta.$$

Lemma 3: M is a contraction operator on B_{T_o}, that is $\|MU_1 - MU_2\|_2 \le r\|U_1 - U_2\|_2$ where r is a positive number < 1.

Proof: Condition b) implies

$$\|MU_1 - MU_2\|_1 \le K_2 \int_0^t \|g(U_1) - g(U_2)\|_1 \, d\tau,$$

using d)

$$\|MU_1 - MU_2\|_1 \le K_2 L_2 \int_0^t \|U_1 - U_2\|_1 \, d\tau \quad .$$

The Schwartz inequality yields

$$\|MU_1 - MU_2\|_1 \le K_2 L_2 \sqrt{T_o} \, \|U_1 - U_2\|_2 \quad .$$

Therefore

$$\|MU_1 - MU_2\|_2 \le K_2 L_2 T_o \|U_1 - U_2\|_2 \quad .$$

Condition f) implies $K_2 L_2 T_o < 1$, therefore M is a contraction mapping.

This concludes the proof of Theorem 1. Notice that the size of δ is restricted only by the size of the region where the Lipschitz conditions c) and d) hold.

Now let B_∞ be the set of elements $U(x,t)$ such that $U(x,t) \in H_\infty$ and $\|U\|_1 \le \delta e^{-\sigma t}$ a.e. for $t \in [0,\infty)$ and $\sigma > 0$.

Lemma 4: B_∞ is a complete metric space.

Proof: See Appendix I.

Theorem 2 (Existence and Uniqueness for an Infinite Time Interval):

If

a) $\|G[a]\|_1 \le K_1 e^{-\sigma t} \eta$ where $\sigma > 0$, η depends only on a(x) and K_1 is independent of a(x),

b) $\|H[g(U)]\|_1 \le K_2 e^{-\sigma(t-\tau)} \|g(U)\|_1$ for every $U \in B_\infty$, where K_2 is independent of g ,

c) $\|g(U)\|_1 \le L_1 \|U\|_1^2 \le L_1 \delta e^{-\sigma t} \|U\|_1$ a.e. for every $U \in B_\infty$,

d) $\|g(U_1) - g(U_2)\|_1 \le L_2 \delta e^{-\sigma t} \|U_1 - U_2\|_1$ a.e. for every $U_1, U_2 \in B_\infty$,

e) $\delta < \min\left[\frac{2\sigma}{K_2 L_2}, \frac{(1-\gamma)\sigma}{K_2 L_1}\right]$ where $\gamma \in (0, 1)$,

f) $\eta \le \frac{\gamma \delta}{K_1}$,

then there exists a unique fixed point of (1.1) on the interval $[0, \infty)$.

Proof: To prove this it is sufficient to prove that M is a contraction operator on B_∞ which maps B_∞ into itself.

Lemma 5: M maps B_∞ into itself, that is if $\|U\|_1 \le \delta e^{-\sigma t}$ a.e. then $\|MU\|_1 \le \delta e^{-\sigma t}$ a.e. .

Proof: Conditions a) and b) imply that

$$\|MU\|_1 \le K_1 e^{-\sigma t} \eta + K_2 \int_0^t e^{-\sigma(t-\tau)} \|g(U)\|_1 \, d\tau ,$$

using c)

$$\|MU\|_1 \leq K_1 \eta e^{-\sigma t} + K_2 L_1 \delta^2 e^{-\sigma t} \int_0^t e^{-\sigma \tau} d\tau$$

$$\leq K_1 \eta e^{-\sigma t} + \frac{K_2 L_1 \delta^2}{\sigma} e^{-\sigma t} .$$

Conditions e) and f) give $K_1 \eta \leq \gamma \delta$ and $\frac{K_2 L_1 \delta^2}{\sigma} < (1-\gamma)\delta$ which implies $\|MU\|_1 \leq \delta e^{-\sigma t}$.

Lemma 6: M is a contraction operator on B_∞, that is $\|MU_1 - MU_2\|_2 \leq r \|U_1 - U_2\|_2$ where r is a positive number < 1.

Proof: Condition b) implies

$$\|MU_1 - MU_2\|_1 \leq K_2 \int_0^t e^{-\sigma(t-\tau)} \|g(U_1) - g(U_2)\|_1 d\tau ,$$

using d)

$$\|MU_1 - MU_2\|_1 \leq K_2 L_2 \delta e^{-\sigma t} \int_0^t \|U_1 - U_2\|_1 d\tau .$$

The Schwartz inequality yields

$$\|MU_1 - MU_2\|_1 \leq K_2 L_2 \delta e^{-\sigma t} t^{\frac{1}{2}} \|U_1 - U_2\|_2 .$$

Squaring both sides, integrating and taking the square root yields

$$\|MU_1 - MU_2\|_2 \leq \frac{K_2 L_2 \delta}{2\sigma} \|U_1 - U_2\|_2 .$$

Condition e) implies $\frac{K_2 L_2 \delta}{2\sigma} < 1$,

therefore M is a contraction mapping.

This concludes the proof of Theorem 2. Notice that this theorem includes the result that the solution U is asymptotically stable, that is $\|U\|_1 \le \delta\, e^{-\sigma t}$.

1.3. A Liapunov-Poincaré Type Theorem

Since nonlinear problems cannot in general be solved, an interesting question is, "When does the solution of the linearized problem behave like the solution of the nonlinear one?".

In ordinary differential equations there are theorems attributed in various places to Liapunov, Poincaré and Perron, which say, in essence, that if the linearized equation is asymptotically stable and the nonlinearity is small then the nonlinear equation is asymptotically stable.

Theorem 2 of the last section includes a result of this type. Conditions a) and b) assert the asymptotic stability of the linearized equation and condition c) asserts that the nonlinearity is small. Since the space B_∞ contains only functions which are asymptotically stable (see the statement just preceding Lemma 4) the solution must be asymptotically stable. We now prove a result which gives a little sharper bound on the solution U. Recall that the bound we had from Theorem 2 was $\|U\|_1 \le \delta\, e^{-\sigma t}$.

Theorem 3 (A Liapunov-Poincaré Type Theorem):

Suppose all the conditions of Theorem 2 are satisfied, then

$$\|U\|_1 \le K_1\, \eta\, e^{\frac{K_2 L_1 \delta}{\sigma}(1-e^{-\sigma t})}\, e^{-\sigma t}, \tag{1.2}$$

where U is the fixed point of (1.1).

Proof: Conditions a) and b) of Theorem 2 imply that

$$\|U\|_1 \le K_1 \eta e^{-\sigma t} + K_2 \int_0^t e^{-\sigma(t-\tau)} \|g\|_1 \, d\tau \ .$$

But g being of second order, condition c), implies

$$e^{\sigma t}\|U\|_1 \le K_1 \eta + K_2 L_1 \delta \int_0^t \|U\|_1 \, d\tau \ .$$

Let $y(t) = e^{\sigma t} \|U\|_1$ which yields

$$y(t) \le K_1 \eta + K_2 L_1 \delta \int_0^t e^{-\sigma\tau} y(\tau) \, d\tau \ .$$

Let $R(t) = K_1 \eta + K_2 L_1 \delta \int_0^t e^{-\sigma\tau} y(\tau) \, d\tau$ which implies

$\dfrac{e^{\sigma t} R'(t)}{K_2 L_1 \delta} = y(t) \le R(t)$ which upon integration yields

$$R(t) \le R(0) \, e^{\frac{K_2 L_1 \delta}{\sigma}(1-e^{-\sigma t})}$$

Therefore $y(t) = e^{\sigma t} \|U\|_1 \le K_1 \eta \, e^{\frac{K_2 L_1 \delta}{\sigma}(1-e^{-\sigma t})}$.

This theorem not only says that the zero solution is asymptotically stable but it gives a lower bound on the region of asymptotic stability. Let us state this theorem another way.

If the system starts out so that $\eta \le \frac{\gamma}{K_1} \min\left[\frac{2\sigma}{K_2 L_2}, \frac{(1-\gamma)\sigma}{K_2 L_1}\right]$ (recall η is a measure of the initial data) then the system remains close to the zero solution in the sense of (1.2) and approaches zero as $t \to \infty$.

1.4. Liapunov's Direct Method for Stability

Stability by Liapunov's Direct Method has been applied extensively to ordinary differential equations and so it is natural to look for extensions of this method to partial differential equations. Several recent papers* treat stability for certain partial differential equations by such an extension. Greenberg, MacCamy and Mizel [8] and Rabinowitz [10] treat stability by a method which is essentially the same as the Direct Method.

In this section we prove a theorem on asymptotic stability which applies to any system with a state vector $U(x, t)$ for which a Liapunov Functional can be constructed. It is assumed that the system admits a zero (equilibrium) solution. The theorem and its proof are almost identical to a theorem proved by Kalman and Bertram [4] for ordinary differential equations.

Let $U(x, t, U_o)$ denote a possible state vector of this system where $U_o(x)$ denotes the initial state of the system, i.e. $U(x,0,U_o)=U_o(x)$. Let the measure of distance in this system be the norm, $\|U\|_1$, defined in Section 1.1. Before proceeding to the theorem we state three definitions.

Definition 1:

The zero solution is said to be stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\|U_o(x)\|_1 \leq \delta$ implies $\|U(x, t, U_o)\| \leq \epsilon$.

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* See for example Parks [1], Dickerson [2] and Infante and Plaut [3].

Definition 2:

The zero solution is said to be asymptotically stable in the large if

1) the zero solution is stable, and
2) all solutions which are bounded initially remain bounded for all time and approach zero as $t \to \infty$.

Definition 3:

V is a spatial integral operator which maps a vector function $U(x, t)$ into a scalar function of t denoted by $V[U(x, t), t] \equiv V(t)$.

Theorem 4:

Let $U(x, t, U_o)$ denote some state vector where $U(x, 0, U_o) = U_o(x)$ and $U(x, t, 0) \equiv 0$. Suppose there exists a scalar function $V[U, t] \equiv V(t)$ differentiable in t along every solution curve U such that $V[0, t] = 0$ and

a) $V[U, t]$ is positive definite, that is there exists a continuous nondecreasing scalar function β_1 such that $\beta_1(0) = 0$ and for all t and all $U \neq 0$, $0 \le \beta_1(\|U\|_1) \le V[U, t]$,

b) There exists a continuous scalar function γ such that $\gamma(0) = 0$ and the derivative $\dot{V}$ of V along the motion satisfies, for all $t > 0$ and $U \neq 0$, $\dot{V}[U, t] \le -\gamma(\|U\|_1) < 0$,

c) There exists a continuous, nondecreasing scalar function β_2 such that $\beta_2(0) = 0$ and, for all t, $V[U, t] \le \beta_2(\|U\|_1)$,

d) $\beta_1(\|U\|_1) \to \infty$ with $\|U\|_1 \to \infty$, then the zero solution is asymptotically stable in the large and $V[U,t]$ is called a Liapunov Functional.

Proof:

Lemma 7: The zero solution is stable.

Proof: Let $\|U_o\|_1 \le \delta$, where δ is as yet arbitrary, then by b) and c) $V[U,t] \le V[U_o, 0] \le \beta_2(\|U_o\|_1) \le \beta_2(\delta)$. Combining this with a) gives $\beta_1(\|U\|_1) \le V[U,t] \le \beta_2(\delta)$. But for every given $\epsilon > 0$ $\beta_1(\epsilon)$ is a fixed number > 0, since β_2 is continuous and $\beta_2(0) = 0$ there exists a δ such that $\beta_2(\delta) \le \beta_1(\epsilon)$. Therefore, for this δ, $\beta_1(\|U\|_1) \le \beta_1(\epsilon)$, which by the monotonicity of β_1 implies that $\|U\|_1 \le \epsilon$.

Lemma 8: All solutions bounded initially remain bounded for all time.

Proof: Take $\|U_o\|_1 \le r$ where r is arbitrary. We know from a), b) and c) that $\beta_1(\|U\|_1) \le V[U,t] \le V[U_o, 0] \le \beta_2(\|U_o\|_1) \le \beta_2(r)$. But by d) there exists a finite C, which depends on r, such that $\beta_2(r) \le \beta_1(C)$ which implies that $\beta_1(\|U\|_1) \le \beta_1(C)$. The monotonicity of β_1 then implies that $\|U\|_1 \le C$.

Lemma 9: All solutions bounded initially approach zero as $t \to \infty$.

Proof: By b) $V(t) = V[U,t]$ is a monotone nonincreasing function which is bounded below by zero. Therefore there exists a number V_∞ such that $\lim_{t \to \infty} V(t) \to V_\infty \ge 0$. Assume $V_\infty > 0$ and obtain a contradiction.

By assumption then

$\beta_2(\|U\|_1) \geq V(t) \geq V_\infty > 0$. Conditions c) and d) imply the existence of a number C_1 such that $V_\infty = \beta_2(C_1)$; combining this with Lemma 8 we have $C_1 \leq \|U\|_1 \leq C(r)$. Let $K = \min \gamma(y)$ for $y \in [C_1, C]$ which implies

$$V(t) \leq V(0) - \int_0^t \gamma(\|U\|_1) \leq V(0) - Kt .$$

This means that V becomes negative for large enough t which is a contradiction. Thus $V_\infty = 0$.

This proves Theorem 4.

Liapunov's Direct Method for stability has the advantage that it does not require any knowledge of the solution (except the knowledge that it satisfy a certain differential equation); however, it suffers in that there is no general way to find a Liapunov Functional.

Notice that Theorem 4 is more general than Theorem 3 in the sense that there is no restriction on the initial data (it can be arbitrarily large but finite in norm), and there is no explicit restriction on the nonlinearity.

CHAPTER 2

SPECIFIC EXAMPLES

In this chapter we show how the results of Chapter 1 can be applied to some specific nonlinear partial differential equations. In particular we show how these equations can be put in the form (1.1) such that $\|G[a]\|_1 \leq K e^{-\sigma t} \|a\|_1$ and $\|H[g]\|_1 \leq K e^{-\sigma(t-\tau)} \|g\|_1$. The existence and uniqueness theorems and the Liapunov-Poincaré theorem follow if g and η satisfy the appropriate conditions in Theorems 1 and 2. We also show how to construct a Liapunov Functional for these equations; stability then follows from Theorem 4.

The examples we treat are special cases of the following problem:

$$u_{tt} + L_1 u_t + L_2 u = f(u, u_t, \cdots, x, t) \tag{2.1}$$

where u is defined on some bounded spatial domain D for $t \in [0, T]$ and satisfies the homogeneous boundary condition

$$Bu(x, t) = 0 \quad \text{for} \quad x \in \partial D , \tag{2.2}$$

and the initial conditions

$$\left.\begin{aligned} u(x, 0) &= a_o(x) \\ u_t(x, 0) &= a_1(x) \end{aligned}\right\} \tag{2.3}$$

.

The function f may depend on u and some of its derivatives and is to be considered smooth in these variables. L_1 and L_2 are self-adjoint spatial operators such that the linear system (2.1)-(2.2) with $f \equiv 0$ exhibits classical normal modes.*

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* See Section 2.2 for a discussion of classical normal modes.

It should be noted that a partial differential equation does not have to be of the form (2.1)-(2.3) in order to apply the results of Chapter 1. The theory in Chapter 1 is quite general and can be applied to any partial differential equation which can somehow be written in the form of (1.1) such that G and H have the appropriate properties. Also, not every problem which can be written in the form of (2.1)-(2.3) can be treated by the methods of Chapter 1 because G and H may not have the appropriate properties.

Before going to the specific examples we discuss the meaning of existence of solutions to (2.1)-(2.3), derive an integral equation for (2.1)-(2.3) and derive an equality which is used for constructing a Liapunov Functional for (2.1)-(2.3).

2.1. Some Comments on Existence

To prove existence we replace (2.1)-(2.3) by an integral equation and then apply the theorems of Chapter 1. This, however, brings up the following questions:

1) In what sense is the integral equation equivalent to (2.1)-(2.3)?

2) How do we use the theory of Chapter 1 to prove existence of, for instance, classical solutions to (2.1)-(2.3), i.e., solutions such that u_{tt}, $L_1 u_t$, $L_2 u$, and $f(u, u_t, \cdots, x_1 t)$ are continuous functions of x and t?

The first question is answered simply by saying that if the integral equation has solutions which are sufficiently differentiable, then the two formulations will be equivalent. The second question can be answered in the framework of generalized derivatives and Sobolev spaces.* This is the approach taken here.

If we make the vector U of equation (1.1) contain u and some of its generalized derivatives then the Banach space H_T defined in Section 1.1 is the space of functions such that each function has these generalized derivatives belonging to $L_2(\Omega)$. For example we can take the components of U to be $u(x, y, t)$, u_t, $u_{xy_t^3}$ and u_{yy} then the norm of U is

$$\|U\|_2 = \left(\int_0^T \int_D [u^2 + u_t^2 + (u_{xy_t^3})^2 + u_{yy}^2] dx\, dy\, dt\right)^{\frac{1}{2}}$$

and H_T is the space of functions u whose generalized derivatives u_t, $u_{xy_t^3}$ and u_{yy} exist and are $L_2(\Omega)$ integrable.

If we make the vector U contain u and all its ℓ^{th} order derivatives then H_T is the ℓ^{th} order Sobolev space $W_2^{\ell}(\Omega)$ and we may apply the Sobolev embedding theorems. In essence the embedding theorems tell us that if the integer ℓ is large enough then every function in W_2^{ℓ} is equivalent to a function in C^k where k is some integer less than ℓ which is specified by the embedding theorem. For example, we can take the components of U to be $u(x, t)$, u_{xx}, u_{xt} and u_{tt} ; then H_T is the space $W_2^2(\Omega)$ and the embedding theorem tells us that u is equivalent to a function in C^o (class of continuous functions).

We use these ideas in the examples to prove existence, uniqueness and stability in two different spaces. In one space we use a norm, $\|U\|_2$, which assures the existence of the Liapunov Functional $V[U, t]$ and its derivative. In the other space we use a norm for a

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* See Appendix II for a definition of these along with a statement of a Sobolev Embedding Theorem.

Sobolev space which will assure the existence of classical solutions.

One more point in regard to these ideas. Recall that in Chapter 1 we mentioned that perhaps a more appropriate norm would be $\|U\|_2 = \max_{t \in [0, T]} \|U\|_1$. The reason we did not use this norm is because the properties of the functions in the Banach space generated by this norm are not known to us whereas the properties of generalized derivatives and Sobolev spaces are well known.*

2.2. Derivation of an Integral Equation

There are many representations one could use to derive an integral equation for (2.1)-(2.3); we want to use one that makes good use of the "damping term" $L_1 u_t$. Caughey and O'Kelly [7] in their paper on classical normal modes solve the linear problem

$$\left.\begin{array}{l} u_{tt} + L_1 u_t + L_2 u = f(x, t) \\ \text{with homogeneous boundary conditions:} \\ \qquad Bu(x, t) = 0 \quad \text{for } x \in \partial D \end{array}\right\} \tag{2.4}$$

where L_1 and L_2 are self-adjoint operators such that there exists a complete set of eigenfunctions $\varphi_n(x)$ which are eigenfunctions for both L_1 and L_2. This allows an expansion, for the solution of (2.4), of the form

$$u(x, t) = \sum u_n(t) \varphi_n(x) \quad . \tag{2.5}$$

Replacing $f(x, t)$ by $f(u, u_t, \cdots, t)$ in the solution of (2.4) generates an integral equation in the usual fashion.

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* See Sobolev [5] and Smirnov [6].

The general conditions on L_1 and L_2 such that an expansion of the form (2.5) exists for (2.4) can be summed up in a definition and theorem stated by Caughey and O'Kelly [7]. The theorem is proved in their paper; we state it here without proof.

Definition: The system (2.4) with $f \equiv 0$ is said to possess classical normal modes if there exists a complete set of orthonormal eigenfunctions φ_n, defined on D, such that

$$\left.\begin{aligned} L_1 \varphi_n(x) &= 2\lambda_n \varphi_n \\ L_2 \varphi_n(x) &= \mu_n \varphi_n \\ B \varphi_n(x) &= 0 \text{ for } x \in \partial D \ . \end{aligned}\right\} \qquad (2.6)$$

Theorem: The necessary and sufficient conditions for (2.4) with $f \equiv 0$ to possess classical normal modes are that

(a) the operators L_1 and L_2 commute i.e., $L_1 L_2 = L_2 L_1$

(b) the boundary conditions on the higher-order operator are derivable from a compatible set of boundary conditions on the lower order operator . (2.7)

If (2.5) is to be a solution of (2.4) the $u_n(t)$ must satisfy the differential equation

$$u_n''(t) + 2\lambda_n u_n'(t) + \mu_n u_n(t) = \int_D f(\xi, t)\, \varphi_n(\xi)\, d\xi \qquad (2.8)$$

with initial data

$$\left.\begin{aligned} u_n(0) &= \int_D a_o(\xi)\,\varphi_n(\xi)\,d\xi \\ u_n'(0) &= \int_D a_1(\xi)\,\varphi_n(\xi)\,d\xi \end{aligned}\right\} \tag{2.9}$$

where

$$u(x, 0) = a_o(x)$$

$$u_t(x, 0) = a_1(x) \quad .$$

The solution of (2.8) and (2.9) can be written in the standard form

$$u_n(t) = v_{1n}(t) \int_D a_o\varphi_n + v_{2n}(t) \int_D a_1\,\varphi_n + \int_o^t v_{2n}(t-\tau)\left(\int_D f(\xi,\tau)\varphi_n(\xi)d\xi\right)d\tau \ ,$$

where

$$v_{1n}(t) = e^{-\lambda_n t}\left[\cosh\sqrt{\lambda_n^2 - \mu_n}\;t + \frac{\lambda n}{\sqrt{}}\,\sinh\sqrt{}\;t\right]$$

and

$$v_{2n}(t) = e^{-\lambda_n t}\;\frac{\sinh\sqrt{}\;t}{\sqrt{}} \quad .$$

Substituting this result into (2.5) we find

$$\begin{aligned} u(x,t) = {} & \sum\varphi_n(x)\left(\int_D a_o\,\varphi_n\right)v_{1n}(t) + \sum\varphi_n(x)\left(\int_D a_1\varphi_n\right)v_{2n}(t) \\ & + \sum\varphi_n(x)\int_o^t\left(\int_D f(\xi,\tau)\,\varphi_n(\xi)d\xi\right)v_{2n}(t-\tau)\,d\tau \quad . \end{aligned} \tag{2.10}$$

If we make the identification

$$A(x,\xi;t) \equiv \sum\varphi_n(x)\,\varphi_n(\xi)\,v_{1n}(t)$$

and $\quad B(x,\xi;t) \equiv \sum\varphi_n(x)\,\varphi_n(\xi)\,v_{2n}(t) \qquad$ then

(2.10) can be written in a more compact form

$$u(x,t) = \int_D A(x,\xi;t)\, a_o(\xi) d\xi + \int_D B(x,\xi;t)\, a_1(\xi)$$

$$+ \int_o^t \int_D B(x,\xi;t-\tau)\, f(\xi,\tau)\, d\xi\, d\tau. \qquad (2.11)$$

Notice that $w \equiv u_{t^n}$ satisfies equation (2.1) with right hand side f_{t^n} and the same boundary conditions. The initial data become $w(x,0) = a_n(x) \equiv u_{t^n}(x,0)$ and $w_t(x,0) = a_{n+1}(x) \equiv u_{t^{n+1}}(x,0)$. Therefore we may write

$$u_{t^n}(x,t) = \int_D A\, a_n + \int_D B\, a_{n+1} + \int_o^t \int_D B\, f_{\tau^n} . \qquad (2.12)$$

The a_n are determined from the following relation

$$a_n = - L_1\, a_{n-1} - L_2\, a_{n-2} + f_{t^{n-2}} \qquad n \geq 2 . \qquad (2.13)$$

We will be dealing with u and some of its derivatives. Let D^n denote any n^{th} order spatial derivative, then

$$D^n u_{t^m} = \int_D D^n A\, a_m + \int_D D^n B\, a_{m+1} + \int_o^t \int_D D^n B\, f_{\tau^m} \qquad (2.14)$$

and

$$D^n u_{t^{m+1}} = \int_D D^n A_t\, a_m + \int_D D^n B_t\, a_{m+1} + \int_o^t \int_D D^n B_t\, f_{\tau^m} . \qquad (2.15)$$

The reason we choose this representation is that even though $\frac{\partial}{\partial t} \int_o^t \int_D Bf = \int_o^t \int_D B_t\, f$ (since $v_{2n}(0) = 0$), higher t derivatives bring contributions from the upper limit, for instance $\frac{\partial^2}{\partial t^2} \int_o^t \int_D Bf = \int^t \int_D B_{tt}\, f + f$. With the added contributions, derivatives of u cannot be written in the same form as equation (2.11) (compare equation (2.11) with equations (2.14) and (2.15)) which is a convenient form for putting U in the form of (1.1).

Define a vector $U(x, t)$ with components taken from $u(x, t)$ and $\{ D^n u_{t^m} \}$; U then satisfies the integral equation

$$U(x, t) = G[a_o, a_1] + \int_o^t H[g(U)] d\tau \quad , \tag{2.16}$$

where g depends on f and some of its derivatives. Therefore U is a fixed point of the mapping defined in (1.1).

2.3. Derivation of an Equality for a Liapunov Functional

Part of the Liapunov Functional can be constructed by multiplying equation (2.1) by $(u_t + \frac{1}{2} L_1 u)$ and integrating over the domain D. We assume in what follows that L_1 and L_2 are self-adjoint operators.

If the inner product of $f(x)$ and $g(x)$ is defined by

$$(f, g) = \int_D f\, g\, dx$$

then

$$(u_t + \tfrac{1}{2} L_1 u,\ u_{tt} + L_1 u_t + L_2 u) \equiv \tfrac{1}{2}(u_t, u_t)_t$$

$$+ (u_t, L_1 u_t) + (u_t, L_2 u) + (\tfrac{1}{2} L_1 u, u_{tt})$$

$$+ (\tfrac{1}{2} L_1 u,\ L_1 u_t) + (\tfrac{1}{2} L_1 u,\ L_2 u) = (u_t + \tfrac{1}{2} L_1 u,\ f) .$$

The relations

$$(u_t,\ L_2 u) = \tfrac{1}{2} (u,\ L_2 u)_t$$

$$(\tfrac{1}{2} L_1 u,\ u_{tt}) = (\tfrac{1}{2} L_1 u,\ u_t)_t - (\tfrac{1}{2} L_1 u_t, u_t)$$

$$(L_1 u,\ L_1 u_t) = (\tfrac{1}{2} L_1 u,\ L_1 u)$$

imply

$$\frac{1}{2}\frac{d}{dt}[(u_t,u_t) + (u, L_2u) + (L_1u, u_t) + \frac{1}{2}(L_1u, L_1u)]$$

$$= -\frac{1}{2}[(u_t, L_1u_t) + (L_1u, L_2u)] + (u_t + \frac{1}{2}L_1u, f) \quad . \tag{2.17}$$

Thus if we define

$$V_1(t) = \frac{1}{2}[(u, L_2u) + (u_t + \frac{1}{2}L_1u, u_t + \frac{1}{2}L_1u) + \frac{1}{4}(L_1u, L_1u)] \tag{2.18}$$

then (2.17) can be written

$$\dot{V}_1(t) = -\frac{1}{2}[(u_t, L_1u_t) + (L_1u, L_2u)] + (u_t + \frac{1}{2}L_1u, f) \quad . \tag{2.19}$$

2.4. Examples

In this section we deal with four specific examples. For each example we define the problem, state the integral equation and give some of the properties of the integral kernels A and B (see equation (2.10)).

We prove asymptotic stability in the large in terms of some norm, ρ, by constructing a functional V such that $0 \le \beta_1(\rho) \le V(t) \le \beta_2(\rho)$ and $\dot{V}(t) \le -\gamma(\rho)$ where β_1, β_2 and γ satisfy the conditions of Theorem 4. Stability follows from Theorem 4.

To prove existence and uniqueness we define an appropriate space H_T, construct the mapping M of (1.1) and show, for this mapping, that $\|G[a]\|_1 \le K e^{-\sigma t}\eta$ and $\|H[g(U)]\|_1 \le K e^{-\sigma(t-\tau)} \|g(U)\|_1$. Existence and uniqueness follow from Theorem 1 or 2 and asymptotic stability from Theorem 3 under the appropriate restrictions on η and g. As mentioned in Section 2.1 we use two different norms, one to prove existence of the Liapunov Functional, V, and its derivative and the other to prove existence of classical solutions. It should be noted however that the Liapunov Functional approach proves stability for a wider class of solutions than the class for which we prove existence.

Example A.

The differential equation is

$$\left.\begin{array}{l} u_{tt} - 2a\, u_{xxt} - u_{xx} = f(u, u_t, u_x, u_{xx}, x, t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = a_o(x) \qquad u_t(x,0) = a_1(x) \quad . \end{array}\right\} \qquad (A1)^{*}$$

The eigenfunctions and eigenvalues associated with L_1 and L_2 (see equation (2.6)) are

$$\varphi_n(x) = \sqrt{2}\ \sin n\pi x$$

$$\lambda_n = a\, n^2 \pi^2$$

$$\mu_n = n^2 \pi^2$$

so the integral equation (2.11) is

---

* A problem very similar to this has been studied by Greenberg, MacCamy and Mizel [8].

$$
\left.
\begin{aligned}
u(x, t) &= \int_0^1 A(x, \xi;t)a_o(\xi)d\xi + \int_0^1 B(x, \xi;t)a_1(\xi)d\xi \\
&\quad + \int_o^t \int_o^1 B(x, \xi;t-\tau)\, f(\xi, \tau)d\xi\, d\tau \\
&\text{where} \\
A(x, \xi;t) &= \sum_{n=1}^{\infty} 2 \sin n\pi x \sin n\pi\xi\, v_{1n}(t) \\
v_{1n}(t) &= e^{-a n^2 \pi^2 t}\Big[\cosh\sqrt{a^2 n^4 \pi^4 - n^2 \pi^2}\; t \\
&\qquad + \frac{a n^2 \pi^2}{\sqrt{\quad}} \sinh\sqrt{\quad}\; t\Big] \\
&\text{and} \\
B(x, \xi;t) &= \sum_{n=1}^{\infty} 2 \sin n\pi x \sin n\pi\xi\, v_{2n}(t) \\
v_{2n}(t) &= e^{-a n^2\pi^2 t}\, \frac{\sinh\sqrt{\quad}\; t}{\sqrt{\quad}} \quad .
\end{aligned}
\right\} \qquad \text{(A2)}
$$

A and B have the following properties which are needed for discussing existence and uniqueness; see Appendix III for the calculation of these results. It is assumed that $S(x)$ and its even order derivatives up to but not including order n vanish on the boundary.

$$
\left.
\begin{aligned}
&\Big\| \int_o^1 A_{x^n} S\Big\|_1 \le K e^{-a \pi^2 t} \|S_{x^n}\|_1 \\
&A_t = B_{xx} \\
&\Big\| \int_o^1 A_{t x^n} S\Big\|_1 = \Big\| \int_o^1 B_{x^{n+2}} S \Big\|_1 \le K e^{-a \pi^2 t} \| S_{x^n} \|_1 \\
&B_t = A + 2a B_{xx} \\
&\Big\| \int_o^1 B_t S \Big\|_1 \le K(1+2a)\, e^{-a \pi^2 t} \|S\|_1
\end{aligned}
\right\} \qquad \text{(A3)}
$$

These are the best bounds in the sense that they do not exist if S does not have the indicated differentiability.

In the method we use to prove existence, the structure of the kernel B puts a restriction on the contents of the nonlinear term f in (A1). One can argue that the only derivative higher than first order that f may contain is u_{xx} because otherwise the mapping $V = MU$ (M is defined in (1.1)) cannot be a mapping of the space H_T into itself. To see this consider the following: If f contains u_{xx}, u_{xt} and u_{tt} then the vector U (see equations (1.1) and (2.16)) must contain u_{xx}, u_{xt} and u_{tt}. If we assume that u_{xx}, u_{xt} and $u_{tt} \in L_2(\Omega)$, i.e. $U \in H_T$, but that higher order derivatives of u do not exist, then we find from equations (2.14), (2.15) and (A3) that $v_{xx} \in L_2(\Omega)$ and $v_{xt}, v_{tt}, \notin L_2(\Omega)$ where v_{xx}, v_{xt} and v_{tt} are components of V in the mapping $V = MU$. Therefore $V \notin H_T$ and M does not map H_T into itself. To be more specific v_{tt} is equal to the sum of three terms (see equation (2.15) with m = 1 and n = 0) one of which is $\int_0^t \int_0^1 B_t f_\tau \, d\xi d\tau$. This term exists only if $f_t(u, \cdots, u_{xx}, u_{xt}, u_{tt}, \cdots) \in L_2(\Omega)$ but this is not possible since u_{ttt} is assumed not to exist. The term v_{xx} exists because it depends on a term $\int_0^t \int_0^1 B_{xx} f \, d\xi d\tau$ which belongs to $L_2(\Omega)$ by (A3) since $f(u, \cdots, u_{xx}, u_{xt}, u_{tt}, \cdots) \in L_2(\Omega)$ by condition c) of Theorem 1 or 2.

Another way of looking at this is to examine B in (A2). Notice that each x derivative brings out an $n\pi$ whereas each t derivative brings out an order $n^2\pi^2$ term and so the convergence properties of B_{xx} and B_{tt} are different. In fact if S is an $L_2(\Omega)$ function then $\int_0^t \int_0^1 B_{xx} S \in L_2(\Omega)$ but $\int_0^t \int_0^1 B_{tt}$ S does not necessarily $\in L_2(\Omega)$.

Construction of a Liapunov Functional

We consider two types of f; one is $f = -g(u)$ and the other is $f = \frac{\partial}{\partial x} g(u_x)$.

From equations (2.18) and (2.19)

$$\left.\begin{aligned} V_1(t) &= \tfrac{1}{2}\int_0^1 [u_x^2 + (u_t - a\,u_{xx})^2 + a^2 u_{xx}^2]\,dx \\ \mathring{V}_1(t) &= -a\int_0^1 [u_{xt}^2 + u_{xx}^2 + u_{xx} f]dx + \int_0^1 u_t f\,dx \end{aligned}\right\} \qquad (A4)$$

<u>Case I.</u> Take $f = -g(u)$ where $u\,g(u) > 0$ for $u \neq 0$ and $g'(u) > 0$ for $u \neq 0$. Define $V(t) = V_1(t) + V_2(t)$ where

$$V_2(t) = \int_0^1 \left(\int_0^u g(\xi)d\xi \right) dx$$

which upon differentiation yields

$$\dot{V}_2(t) = \int_0^1 u_t\, g(u)dx \quad .$$

This leads to

$$\left.\begin{aligned} V(t) &= \tfrac{1}{2}\int_0^1 [u_x^2 + (u_t - a u_{xx})^2 + a^2 u_{xx}^2 + 2\int_0^u g(\xi)d\xi]dx \\ \mathring{V}(t) &= -a\int_0^1 [u_{xt}^2 + u_{xx}^2 + u_x^2\, g'(u)]dx \quad . \end{aligned}\right\} \qquad (A5)$$

To prove asymptotic stability in the large we show that V satisfies the conditions of Theorem 4 with

$$\rho^2 = \int_0^1 [u^2 + u_x^2 + u_t^2 + u_{xx}^2]\,dx \quad . \qquad (A6)$$

The following inequalities will be useful in our proof. If u is such that $u(0)=u(1)=0$ and u_{xx} exists, then

$$|u| \le \int_0^1 |u_x|dx \Rightarrow u^2 \le \int_0^1 u_x^2\,dx \le \rho^2 \quad , \qquad (A7)$$

and

$$|u_x| \le \int_0^1 |u_{xx}|dx \Rightarrow u_x^2 \le \int_0^1 u_{xx}^2 \, dx \le \rho^2 \quad . \tag{A8}$$

The second inequality is based on the fact that u_x has a zero since $\int_0^1 u_x = 0$, and u_x is continuous.

Conditions a) and d) of Theorem 4.

Making use of (A5), (A6) and the inequality

$$|u_t| \le |u_t - a\, u_{xx}| + |\, a\, u_{xx}|$$

we find that

$$\rho^2 \le \int_0^1 [2\, u_x^2 + 2(u_t - a\, u_{xx})^2 + 2\, a^2\, u_{xx}^2 + u_{xx}^2\,]$$

$$\le 2(2 + \frac{1}{a^2})\ V_1(t) = K_1\ V_1(t) \le K_1\ V(t) \quad .$$

Therefore $V(U, t) \ge \frac{1}{K_1}\ \rho^2 = \beta_1(\rho)$ where it is clear that β_1 satisfies conditions a) and d) of Theorem 4.

Condition c) of Theorem 4.

From (A4) we notice

$$V_1(t) \le \tfrac{1}{2} \int_0^1 [u_x^2 + 2u_t^2 + 3\, a^2\, u_{xx}^2\,] \le K_2\, \rho^2 \, .$$

Now consider $V_2(t) = \int_0^1 (\int_0^u g(\xi)d\xi)dx$ and let $h(|u|) \equiv \max(\int_0^u g, \int_0^{-u} g)$ which implies that $h(0) = 0$, h is a nondecreasing function of $|u|$ and $\int_0^u g \le h(|u|)$. By (A7) $|u| \le \rho$, therefore

$$V_2(t) = \int_0^1 \int_0^u g \le \int_0^1 h(|u|) \le \int_0^1 h(\rho) = h(\rho).$$

Combining this with the result for V_1 yields $V(t) \le K_2\rho^2 + h(\rho) \equiv \beta_2(\rho)$ where we observe that β_2 satisfies condition c).

Condition b) of Theorem 4.

Condition b) will be satisfied if we find a γ such that

$\gamma(\rho) \leq \alpha \int_0^1 [u_{xt}^2 + u_{xx}^2 + u_x^2 g'(u)]dx.$ Using (A7) and (A8) we find

$$\rho^2 = \int_0^1 u^2 + u_x^2 + u_t^2 + u_{xx}^2 \leq \int_0^1 3 u_{xx}^2 + u_{xt}^2 ,$$

but $g'(u)$ is positive; therefore

$$\rho^2 \leq 3 \int_0^1 [u_{xx}^2 + u_{xt}^2 + u_x^2 g'(u)] .$$

If we take $\gamma(\rho) = \frac{\alpha \rho^2}{3}$, condition b) is satisfied.

We have therefore proved the following result: If $\beta_1(\rho) = \frac{1}{K_1} \rho^2$, $\beta_2(\rho) = K_2 \rho^2 + h(\rho)$ and $\gamma(\rho) = \frac{\alpha}{3} \rho^2$, then $\beta_1(\rho) \leq V(t) \leq \beta_2(\rho)$ and $\dot{V}(t) \leq -\gamma(\rho)$. Asymptotic stability in the large for the norm ρ follows.

Case II. Take $f = \frac{\partial}{\partial x} g(u_x) \equiv u_{xx} g'(u_x)$ where $u_x g(u_x) > 0$ for $u_x \neq 0$ and $g'(u_x) \geq 0$.

Define $V(t) = V_1(t) + V_2(t)$ where $V_2(t) = \int_0^1 (\int_0^{u_x} g(\xi)d\xi)dx$ which upon differentiation yields

$$\dot{V}_2(t) = \int_0^1 u_{xt} g(u_x)\, dx = -\int_0^1 u_t \frac{\partial g}{\partial x} dx.$$

Therefore

$$V(t) = V_1(t) + \int_0^1 (\int_0^{u_x} g(\xi)\, d\xi)\, dx$$

$$\dot{V}(t) = -\alpha \int_0^1 [u_{xt}^2 + u_{xx}^2 + u_{xx}^2 g'(u_x)]\, dx.$$

The construction of β_1 and γ is the same as in Case I, the construction of β_2 goes through the same except we use $|u_x| \leq \rho$ instead of $|u| \leq \rho$.

Conditions for Existence

For the existence of V and $\mathring{V}$ we need the existence in $L_2(\Omega)$ of u and its generalized derivatives u_{xx}, u_{xxt} and u_{tt} (the existence of u_{xx} and u_{tt} imply the existence of u_x and u_t). We therefore define $H_T = \{u(x,t) \| u, u_{xx}, u_{xxt}, u_{tt} \in L_2(\Omega), u|_{\partial D} = 0, u(x,0) = a_o(x), u_t(x,0) = a_1(x)\}$ with the norm $\|U\|_1^2 = \int_0^1 [u^2 + u_{xx}^2 + u_{xxt}^2 + u_{tt}^2]dx$.

By equations (2.14) and (2.15) we have

$$u = \int_o^1 A\, a_o + \int_o^1 B\, a_1 + \int_o^t \int_o^1 B\, f$$

$$u_{xx} = \int_o^1 A_{xx} a_o + \int_o^1 B_{xx} a_1 + \int_o^t \int_o^1 B_{xx}\, f$$

$$u_{xxt} = \int_o^1 A_{xx} a_1 + \int_o^1 B_{xx} a_2 + \int_o^t \int_o^1 B_{xx}\, f_T$$

$$u_{tt} = \int_o^1 A_t\, a_1 + \int_o^1 B_t\, a_2 + \int_o^t \int_o^1 B_t\, f_T \quad .$$

By (A3) there exists a constant K such that

$$\|G[a]\|_1 \le K\, e^{-a\pi^2 t} [\|a_o\|_1 + \|a_1\|_1 + \|a_{oxx}\|_1 + \|a_{1xx}\|_1 + \|a_2\|_1] \equiv K\, e^{-a\pi^2 t}\, \eta \quad ,$$

as long as $a_o = a_1 = 0$ on ∂D. Also from (A3) we find

$$\|H[g(U)]\|_1^2 \le K^2 e^{-2a\pi^2 t} [2\, \|f\|_1^2 + 2\, \|f_T\|_1^2] \equiv K^2 e^{-2a\pi^2 t} \|g(U)\|_1^2 .$$

This assures the existence of V and $\dot{V}$ as long as

1) $a_o = a_1 = 0$ on ∂D

2) η and g satisfy the conditions of Theorem 1 or 2.

See Appendix IV where we show that condition 2) above is satisfied for $f = h(u_x)\, u_{xx}$ and $f = h(u)$.

Now we prove that if a_o and a_1 are smooth enough then there exists a solution $u(x,t)$ of (A1) such that $u \in C^3(\Omega)$.

The Sobolev space $W_2^5(\Omega)$ in the two dimensional case, i.e. x, t, is embedded in $C^3(\Omega)$ and the norm associated with this space is

$$\|u\|^2_{W_2^5(\Omega)} = \int_0^T \int_0^1 [u^2 + \sum_{n+m=5} (D^n u_{t^m})^2]\, dx\, dt .$$

However the existence of u_{t^5} and u_{xt^4} requires the existence of $u_{x^2t^4}$, as we shall see below, therefore the vector U is made to contain u, all fifth order derivatives of u and $u_{x^2t^4}$. Since the inequalities in (A3) hold under the assumption that S and its even order derivatives vanish on ∂D we define

$$H_T = \{u(x,t) \| u \in W_2^5(\Omega),\ u_{x^2t^4} \in L_2(\Omega),\ u|_{\partial D} = u_{xx}|_{\partial D} = u_{x^4}|_{\partial D} = 0,\ u_{t^n}(x,0) = a_n(x),\ n = 0, 4\}$$

with the norm $\|U\|_2^2 = \|u\|^2_{W_2^5(\Omega)} + \int_0^T \int_0^1 (u_{x^2t^4})^2\, dx\, dt$. Notice that H_T is a subspace of $W_2^5(\Omega)$.

By (2.13) and (2.14) we have

$$\left.\begin{aligned}
u &= \int_0^1 A\, a_0 + \int_0^1 B\, a_1 + \int_0^t \int_0^1 B\, f \\
u_{x^5} &= \int_0^1 A_{x^5}\, a_0 + \int_0^1 B_{x^5}\, a_1 + \int_0^t \int_0^1 B_{x^5}\, f \\
u_{x^4t} &= \int_0^1 A_{x^4}\, a_1 + \int_0^1 B_{x^4}\, a_2 + \int_0^t \int_0^1 B_{x^4}\, f_T \\
u_{x^3t^2} &= \int_0^1 A_{x^3}\, a_2 + \int_0^1 B_{x^3}\, a_3 + \int_0^t \int_0^1 B_{x^3} f_{T^2} \\
u_{x^2t^3} &= \int_0^1 A_{x^2}\, a_3 + \int_0^1 B_{x^2}\, a_4 + \int_0^t \int_0^1 B_{x^2}\, f_{T^3} \\
u_{xt^4} &= \int_0^1 A_x\, a_4 + \int_0^1 B_x\, a_5 + \int_0^t \int_0^1 B_x\, f_{T^4}, \text{ but if } f_{T^4} \text{ is to exist so must } u_{x^2t^4} \\
u_{x^2t^4} &= \int_0^1 A_{xx}\, a_4 + \int_0^1 B_{xx}\, a_5 + \int_0^t \int_0^1 B_{xx}\, f_{T^4} \\
u_{t^5} &= \int_0^1 A_t\, a_4 + \int_0^1 B_t\, a_5 + \int_0^t \int_0^1 B_t\, f_{T^4} .
\end{aligned}\right\} = G[a] + \int_0^t H[g(U)]$$

By (A3) there exists a constant K such that

$$\|G(a)\|_1 \le K e^{-\alpha \pi^2 t} (\|a_o\|_1 + \|a_1\|_1 + \|a_{o_x 5}\|_1 + \|a_{1_x 3}\|_1$$

$$+ \|a_{1_x 4}\|_1 + \|a_{2_x 2}\|_1 + \|a_{2_x 3}\|_1 + \|a_{3x}\|_1 + \|a_{3_x 2}\|_1 + \|a_4\|_1$$

$$+ \|a_{4x}\|_1 + \|a_5\|_1 + \|a_{4_x 2}\|_1) = K e^{-\alpha \pi^2 t} \eta ,$$

under the following conditions:

1) $a_o = a_{oxx} = a_{o_x 4} = a_{o_x 6} = a_1 = a_{1xx} = a_{1_x 4} = a_{1_x 6} = 0$ on ∂D,

2) $f(x, 0) = f_{xx}(x, 0) = f_{x^4}(x, 0) = f_t(x, 0) = f_{tx^2}(x, 0) = 0$ on ∂D.

A little calculation based on equation (2.13) shows that η exists if $a_{o_x 8}$ and $a_{1_x 8} \in L_2(D)$. Also by (A3) we see that

$$\|H[g(U)]\|_1^2 \le K^2 e^{-2\alpha \pi^2 (t-\tau)} (\|f\|_1^2 + \|f_{x^3}\|_1^2$$

$$+ \|f_{x^2 \tau}\|_1^2 + \|f_{x \tau^2}\|_1^2 + \|f_{\tau^3}\|_1^2 + 3 \|f_{\tau^4}\|_1^2)$$

$$= K^2 e^{-2\alpha\pi^2(t-\tau)} \|g(U)\|_1^2 ,$$

under the following condition:

$$f(x, t) = f_{xx}(x, t) = 0 \quad \text{on} \quad \partial D .$$

We have therefore proved the following result:

If

a) $a_{o_x 8}, a_{1_x 8} \in L_2(D)$

b) $a_{o_x 2n} = a_{1_x 2n} = 0$ on ∂D for $n = 1, 2, 3$

c) $f(x, t) = f_{xx}(x, t) = 0$ on ∂D

d) $f_{x^4}(x, 0) = 0$ on ∂D

e) g and η satisfy the conditions of Theorem 1 or 2

then there exists a three times continuously differentiable solution of (A1) on either $[0, T_o]$ or $[0, \infty)$.

Conditions c), d) and e) could be satisfied by $f = h(u)$ or $f = h(u_x)u_{xx}$ as long as h is suitably smooth and satisfies some condition at zero. See Appendix IV for some discussion of this.

Example B.

The differential equation is

$$\left.\begin{aligned} u_{tt} + 2\alpha u_t - u_{xx} &= f(u, u_x, u_t, x, t) \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= a_o(x) \qquad u_t(x,0) = a_1(x) . \end{aligned}\right\} \quad (B1)^*$$

The eigenfunctions and eigenvalues given by equation (2.6) are

$$\varphi_n(x) = \sqrt{2} \sin n\pi x$$

$$\lambda_n = \alpha$$

$$\mu_n = n^2 \pi^2 .$$

A and B have the following properties which are calculated in

* This problem has been treated for the existence of periodic solutions by Rabinowitz [10]. In a more recent paper [11] he uses a technique developed by Möser [9] to treat the case where f contains u_{xx}, u_{xt} and u_{tt}.

Appendix III. It is assumed that S and its even order derivatives up to but not including order n vanish at x = 0, 1.

$$\left.\begin{aligned}
&A_t = B_{xx} \\
&B_t = A - 2\alpha B \\
&\| \int_o^1 A_{x^n} S \|_1 \le K e^{-\alpha t} \| S_{x^n} \|_1 \\
&\| \int_o^1 A_{tx^{n-1}} S \|_1 = \| \int_o^1 B_{x^{n+1}} S \|_1 \le K e^{-\alpha t} \| S_{x^n} \|_1 \\
&\| \int_o^1 B_{tx^n} S \|_1 \le K e^{-\alpha t} \| S_{x^n} \|_1
\end{aligned}\right\} \quad \text{(B2)}$$

These are the best bounds in the sense that they do not exist if $S_{x^n} \notin L_2(D)$.

The structure of B again puts a limit on the contents of f but this time f can contain nothing higher than first order derivatives. The reasoning is the same as in Example A, here suffice it to say that if $S \in L_2(D)$ then $\int_o^1 B_{xx} S$, $\int_o^1 B_{xt} S$ and $\int_o^1 B_{tt} S$ do not necessarily belong to $L_2(D)$.

Construction of a Liapunov Functional

We consider the same two types of f as in Example A, but let

$$\rho^2 = \int_o^1 [u^2 + u_x^2 + u_t^2] \, dx \quad .$$

Case BI. f = -g(u)

Using (2.18) and (2.19) we determine

$$V(t) = \frac{1}{2}\int_0^1 [u_x^2 + (u_t^2 + au)^2 + a^2u^2 + 2\int_0^u g(\xi)d\xi]dx$$

$$\mathring{V}(t) = -a\int_0^1 [u_t^2 + u_x^2 + ug(u)]\,dx \quad .$$

The following result can be established just as in Example A:

If $\beta_1(\rho) = \frac{1}{4}\rho^2$, $\beta_2(\rho) = K\rho^2 + h(\rho)$ where $h(\rho)$ is the same as in Example A and $\gamma(\rho) = \frac{a}{2}\rho^2$ then $\beta_1(\rho) \le V(t) \le \beta_2(\rho)$ and $\dot{V}(t) \le -\gamma(\rho)$. We observe that β_1, β_2 and γ satisfy the conditions of Theorem 4.

Case BII.

$f = \frac{\partial}{\partial x} g(u_x)$ (Notice the existence theorem does not apply in this case)

Using (2.18) and (2.19) we determine

$$V(t) = \frac{1}{2}\int_0^1 [u_x^2 + (u_t+au)^2 + a^2 u^2 + 2\int_0^{u_x} g(\xi)d\xi]dx$$

$$\mathring{V}(t) = -a\int_0^1 [u_t^2 + u_x^2 + u_x\, g(u_x)]\,dx \quad .$$

If β_1, β_2 and γ are the same as in Case BI, the conditions of the Theorem 4 will be satisfied.

Conditions for Existence

For the existence of V and $\mathring{V}$ we need the existence in $L_2(\Omega)$ of u and its generalized derivatives u_{xx} and u_{tt} (the existence of u_{xx} and u_{tt} imply the existence of u_x and u_t). Since the inequalities in (B2) for $n \le 1$ hold for $S = 0$ on ∂D we define $H_T = \{u \| u \in W_2^2(\Omega),\ u|_{\partial D} = 0,$ $u(x, 0) = a_0(x),\ u_t(x, 0) = a_1(x)\}$ with the norm

$$\|U\|_1^2 = \int_0^1 [u^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2]\,dx \quad .$$

By equations (2.14) and (2.15) we have

$$
\left.\begin{aligned}
u &= \int_o^1 A\, a_o + \int_o^1 B\, a_1 + \int_o^t \int_o^1 B\, f \\
u_{xx} &= \int_o^1 A_{xx}\, a_o + \int B_{xx}\, a_1 + \int_o^t \int_o^1 B_{xx}\, f \\
u_{xt} &= \int_o^1 A_x\, a_1 + \int_o^1 B_x\, a_2 + \int_o^t \int_o^1 B_x\, f_\tau \\
u_{tt} &= \int_o^1 A_t\, a_1 + \int_o^1 B_t\, a_2 + \int_o^t \int_o^1 B_t\, f_\tau
\end{aligned}\right\} = G[a] + \int_o^t H[g(U)]
$$

By (B2) there exists a constant K such that

$$\|G[a]\|_1 \le K e^{-\alpha t}[\|a_o\|_1 + \|a_1\|_1 + \|a_{oxx}\|_1 + \|a_{1x}\|_1 + \|a_2\|_1] = K e^{-\alpha t}\,\eta ,$$

as long as $a_o = a_1 = 0$ on ∂D. Also from (B2) we find

$$\|H[g]\|_1^2 \le K^2 e^{-2\alpha(t-\tau)} [\|f\|_1^2 + \|f_x\|_1^2 + 2\|f_\tau\|_1^2] = K^2 e^{-2\alpha(t-\tau)} \|g\|_1 ,$$

as long as $f(x, t) = 0$ on ∂D. This assures the existence of V and $\dot{V}$ if

1) $a_o = a_1 = 0$ on ∂D

2) $f(x, t) = 0$ on ∂D

3) η and g satisfy the conditions of Theorem 1 or 2.

Now we prove that if a_o and a_1 are smooth, providing η and g satisfy the appropriate conditions, then there exists a solution $u(x, t)$ of (B1) such that $u(x, t) \in C^2(\Omega)$.

The Sobolev space $W_2^4(\Omega)$ in the two dimensional case, i.e. x, t, is embedded in $C^2(\Omega)$ and the norm associated with this space is

$$\|u\|^2_{W_2^4(\Omega)} = \int_o^T \int_o^1 [u^2 + \sum_{m+n=4} (D^n u_{t^m})^2]\, dxdt \quad .$$

Since the inequalities in (B2) hold under the assumption that S and its even order derivatives vanish on ∂D, we define

$$H_T = \{u(x,t) \| u \in W_2^4(\Omega),\ u|_{\partial D} = u_{xx}|_{\partial D} = 0,\ u_{t^n}(x,0) = a_n(x), n=0,3\}$$

with the norm $\|U\|_2^2 = \|u\|^2_{W_2^4(\Omega)}$.

By (2.14) and (2.15) we have

$$u = \int_0^1 A\, a_o + \int_0^1 B\, a_1 + \int_0^t \int_0^1 B\, f$$

$$u_{x^4} = \int_0^1 A_{x^4} a_o + \int_0^1 B_{x^4} a_1 + \int_0^t \int_0^1 B_{x^4} f$$

$$u_{x^3 t} = \int_0^1 A_{x^3}\, a_1 + \int_0^1 B_{x^3} a_2 + \int_0^t \int_0^1 B_{x^3} f_\tau$$

$$= G[a] + \int_0^t H[g]\, d\tau$$

$$u_{x^2 t^2} = \int_0^1 A_{x^2}\, a_2 + \int_0^1 B_{x^2}\, a_3 + \int_0^t \int_0^1 B_{x^2} f_{\tau^2}$$

$$u_{xt^3} = \int_0^1 A_x\, a_3 + \int_0^1 B_x\, a_4 + \int_0^t \int_0^1 B_x\, f_{\tau^3}$$

$$u_{t^4} = \int_0^1 A_t\, a_3 + \int_0^1 B_t\, a_4 + \int_0^t \int_0^1 B_t\, f_{\tau^3} \ .$$

By (B2) there exists a constant K such that

$$\|G(a)\|_1 \le K e^{-\alpha t} [\|a_o\|_1 + \|a_1\|_1 + \|a_{o_{x^4}}\|_1 + \|a_{1_{x^3}}\|_1$$

$$+ \|a_{2_{x^2}}\|_1 + \|a_{3x}\|_1 + \|a_4\|_1] = K e^{-\alpha t} \eta ,$$ under the following

conditions:

1) $a_o = a_{oxx} = a_1 = a_{1xx}$ on ∂D ,

2) $f(x,0) = f_\tau(x,0) = 0$ on ∂D .

A little calculation based on equation (2. 13) shows that η exists if $a_{o_{x^4}}$ and $a_{1_{x^4}} \in L_2(D)$. Also by (B2) we see that

$$\|H[g(U)]\|_1^2 \le K^2 e^{-2a(t-\tau)} [\|f\|_1^2 + \|f_{x^3}\|_1^2 + \|f_{\tau xx}\|_1^2 + \|f_{\tau^2 x}\|_1^2 + 2\|f_{\tau^3}\|_1^2]$$

$$\equiv K^2 e^{-2a(t-\tau)} \|g(U)\|_1^2 ,$$

under the following condition:

$$f(x,t) = f_{xx}(x,t) = 0 \quad \text{on} \quad \partial D.$$

We have therefore proved that if

a) $a_{o_{x^4}},\ a_{1_{x^4}} \in L_2(D)$

b) $a_o = a_{oxx} = a_1 = a_{1xx} = 0$ on ∂D

c) $f(x,t) = f_{xx}(x,t) = 0$ on ∂D

d) η and g satisfy the conditions of Theorem 1 or 2

then there exists a two times continuously differentiable solution of (B1) on either $[0, T_o]$ or $[0, \infty)$.

Example C.

The differential equation is

$$\left.\begin{array}{l} u_{tt} - 2a\,\nabla^2 u_t - \nabla^2 u = f(u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{yy}, x, y, t) \\ D = \{\, 0 < x < 1,\ 0 < y < 1 \} \\ u(x,y,t) = 0 \quad \text{for} \quad x, y \in \partial D \\ u(x,y,0) = a_o(x,y) \quad u_t(x,y,0) = a_1(x,y) \end{array}\right\} \qquad \text{(C1)}$$

The eigenfunctions and eigenvalues (see equation (2.6)) associated with L_1 and L_2 are

$$\varphi_{mn}(x) = \sqrt{2}\ \sin m\pi x\ \sqrt{2}\ \sin n\pi y$$

$$\lambda_{mn} = a\,\pi^2\,(m^2+n^2)$$

$$\mu_{mn} = \pi^2\,(m^2+n^2) .$$

A and B have the following properties where ∂D_x denotes the boundary at $x = 0, 1$ and ∂D_y denotes the boundary at $y = 0, 1$. See Appendix III for details.

$$\left.\begin{array}{l}
A_t = \nabla^2 B \\
B_t = A + 2a\,\nabla^2 B \\
\| \int_D \nabla^2 A\, g \|_1 \le K\, e^{-2a\,\pi^2 t} \|\nabla^2 g\|_1 \quad \text{if } g|_{\partial D} = 0 \\
\| \int_D A_t\, g \|_1 = \| \int_D \nabla^2 B\, g\|_1 \le K\, e^{-2a\,\pi^2 t} \|g\|_1 \\
\| \int_D B_t\, g \|_1 \le K\, e^{-2a\,\pi^t} \|g\|_1 \\
\| \int_D \begin{pmatrix} A_{x^n y^m} \\ B_{x^{n+1} y^{m+1}} \end{pmatrix} g \|_1 \le K\, e^{-2a\,\pi^2 t} \|g_{x^n y^m}\|_1 \\
\text{if} \quad g_{x^{2k}}\big|_{\partial D_x} = g_{y^{2j}}\big|_{\partial D_y} = 0 \quad k = 0, [\frac{n-1}{2}];\ j = 0, [\frac{m-1}{2}] \\
\| \int_D B_{x^{n+2}}\, g \|_1 \le K\, e^{-2\,a\,\pi^2 t} \|g_{x^n}\|_1 \\
\qquad \text{if } g_{x^{2k}}\big|_{\partial D_x} = 0 \quad k = 0, [\frac{n-1}{2}] \\
\| \int_D B_{y^{n+2}}\, g\|_1 \le K\, e^{-2a\,\pi^2 t} \|g_{y^n}\|_1 \\
\qquad \text{if } g_{y^{2k}}\big|_{\partial D_y} = 0 \quad k = 0, [\frac{n-1}{2}].
\end{array}\right\} \quad \text{(C2)}$$

The structure of B puts limits on the contents of f just as in Example A.

Construction of a Liapunov Functional

We consider the case for $f = -g(u)$ where $ug(u) > 0$ for $u \neq 0$ and $g'(u) \geq 0$.

Using (2.18) and (2.19) we determine

$$V(t) = \tfrac{1}{2} \int_D [u_x^2 + u_y^2 + (u_t - a\nabla^2 u)^2 + a^2(\nabla^2 u)^2 + \int_0^u g(\xi)d\xi]dxdy$$

$$\overset{\circ}{V}(t) = -a \int_D [u_{xt}^2 + u_{yt}^2 + (\nabla^2 u)^2 + g'(u)(u_x^2 + u_y^2)]dxdy \quad .$$

If we let $\rho^2 = \int_D [u^2 + u_x^2 + u_y^2 + u_t^2 + (\nabla^2 u)^2]\,dxdy$, then the following result can be proved just as in Example A. If $\beta_1(\rho) = \frac{1}{4}\rho^2$, $\beta_2(\rho) = K\rho^2 + h(\rho)$ and $\gamma(\rho) = \frac{a}{4}\rho^2$, then $\beta_1(\rho) \leq V(t) \leq \beta_2(\rho)$ and $\dot{V}(t) \leq -\gamma(\rho)$. We observe that β_1, β_2 and γ satisfy the conditions of Theorem 4 so asymptotic stability in the large follows.

Conditions for Existence

For the existence of V and $\dot{V}$ we need the existence in $L_2(\Omega)$ of u and its generalized derivatives $\nabla^2 u$, $\nabla^2 u_t$ and u_{tt}. By (2.14) and (2.15) we have

$$\begin{aligned} u &= \int_D A a_o + \int_D B a_1 + \int_o^t \int_D Bf \\ \nabla^2 u &= \int_D (\nabla^2 A)a_o + \int_D (\nabla^2 B)a_1 + \int_o^t \int_D (\nabla^2 B)f \\ \nabla^2 u_t &= \int_D (\nabla^2 A)a_1 + \int_D (\nabla^2 B)a_2 + \int_o^t \int_D (\nabla^2 B)f_\tau \\ u_{tt} &= \int_D A_t a_1 + \int_D B_t a_2 + \int_o^t \int_D B_t f_\tau \end{aligned} \qquad = G[a] + \int_o^t H[g]\,d\tau \quad .$$

Take $H_T = \{u(x,y,t) \| \; u, \; \nabla^2 u, \; \nabla^2 u_t, \; u_{tt} \in L_2(\Omega), \; u|_{\partial D} = 0,$

$$u(x, y, 0) = a_o, \; u_t(x, y, 0) = a_1\}$$

with norm $\|U\|_1^2 = \int_D [u^2 + (\nabla^2 u)^2 + (\nabla^2 u_t)^2 + u_{tt}^2] \, dxdy$.

By (C2) there exists a constant, K, such that

$$\|G[a]\|_1 \leq K e^{-2\alpha\pi^2 t} [\|a_o\|_1 + \|a_1\|_1 + \|\nabla^2 a_o\|_1 + \|\nabla^2 a_1\|_1 + \|a_2\|_1] = K e^{-2\alpha\pi^2 t} \eta$$

and such that

$$\|H[g]\|_1^2 \leq K^2 e^{-4\alpha\pi^2(t-\tau)} [2\|f\|_1^2 + 2\|f_\tau\|_1^2] \equiv K^2 e^{-4\alpha\pi^2(t-\tau)} \|g\|_1^2 .$$

This assures the existence of V and $\dot{V}$ as long as

1) $a_o = a_1 = 0$ on ∂D

2) η and g satisfy the conditions of Theorem 1 or 2.

Now we prove that if a_o and a_1 are smooth, providing η and g satisfy the appropriate conditions, then there exists a solution u(x, y, t) of (C1) such that $u(x, y, t) \in C^3(\Omega)$.

The Sobolev space in the three dimensional case, i.e. x, y, t, is embedded in $C^3(\Omega)$, so in a manner analogous to Example A we define

$$H_T = \{u(x, y, t) \| \; u \in W_2^5(\Omega), \; D^2 u_{t^4} \in L_2(\Omega),$$
$$u|_{\partial D} = u_{xx}|_{\partial D_x} = u_{yy}|_{\partial D_y} = u_{x^4}|_{\partial D_x} = u_{y^4}|_{\partial D_y} = 0,$$
$$u_{t^n}(x, y, 0) = a_n, \quad n = 0, 4\}$$

with norm

$$\|U\|_2^2 = \|u\|^2_{W_2^5(\Omega)} + \int_0^T \int_D [\Sigma (D^2 u_{t^4})^2] \, dx\,dy\,dt.$$

By (2.14) and (2.15) we have

$$u = \int A a_o + \int B a_1 + \int_o^t \int B f$$

$$D^5 u = \int (D^5 A) a_o + \int (D^5 B) a_1 + \int_o^t \int (D^5 B) f$$

$$D^4 u_t = \int (D^4 A) a_1 + \int (D^4 B) a_2 + \int_o^t \int (D^4 B) f_\tau$$

$$D^3 u_{tt} = \int (D^3 A) a_2 + \int (D^3 B) a_3 + \int_o^t \int (D^3 B) f_{\tau^2}$$

$$D^2 u_{t^3} = \int (D^2 A) a_3 + \int (D^2 B) a_4 + \int_o^t \int (D^2 B) f_{\tau^3}$$

$$D u_{t^4} = \int (DA) a_4 + \int (DB) a_5 + \int_o^t \int (DB) f_{\tau^4}$$

but if f_{τ^4} is to exist so must $D^2 u_{t^4}$

$$D^2 u_{t^4} = \int (D^2 A) a_4 + \int (D^2 B) a_5 + \int_o^t \int (D^2 B) f_{\tau^4}$$

$$u_{t^5} = \int A_t a_4 + \int B_t a_5 + \int_o^t \int B_t f_{\tau^4} .$$

In the same manner as in the previous examples we can state the following result.

If

a) $D^8 a_o, D^8 a_1 \in L_2(D)$

b) $a_o, a_1, f(x,y,0)$ and $f(x,y,t)$ satisfy the appropriate conditions on ∂D

c) g and η satisfy the conditions of Theorem 1 or 2

then there exists a three times continuously differentiable solution of (C1) on either $[0, T_o]$ or $[0, \infty)$.

Example D.

The differential equations are

$$u_{tt} - 2a\, u_{xxt} - u_{xx} = f_1(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, x, t)$$

$$v_{tt} - 2a\, v_{xxt} - \sigma^2 v_{xx} = f_2(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, x, t)$$

$$u(0, t) = v(0, t) = u(1, t) = v(1, t) = 0$$

$$u(x, 0) = a_o(x), \ v(x, 0) = b_o(x), \ u_t(x, 0) = a_1(x), \ v_t(x, 0) = b_1(x)$$

The eigenfunctions and eigenvalues (see equation (2.6)) are

$$\varphi_n(x) = \sqrt{2}\ \sin n\pi x, \ \lambda_n(u) = a\, n^2\pi^2, \ \mu_n(u) = n^2\pi^2$$

$$\psi_n(x) = \sqrt{2}\ \sin n\pi x, \ \lambda_n(v) = a\, n^2\pi^2, \ \mu_n(v) = \sigma^2 n^2\pi^2$$

so the integral equations are

$$u(x, t) = \int_o^1 A\, a_o + \int_o^1 B\, a_1 + \int_o^t \int_o^1 B\, f_1$$

$$v(x, t) = \int_o^1 C\, b_o + \int_o^1 E\, b_1 + \int_o^t \int_o^1 E\, f_2$$

where A and B are the same as in Example A and

$$C = \sum \sqrt{2}\ \sin n\pi x \sqrt{2}\ \sin n\pi\xi\ v_{1n}(t)$$

$$v_{1n}(t) = e^{-an^2\pi^2 t}\left(\cosh\sqrt{a^2n^4\pi^4 - \sigma^2 n^2\pi^2}\ t + \frac{a n^2\pi^2}{\sqrt{}}\sinh\sqrt{}\ t\right)$$

$$E = \sum \sqrt{2}\ \sin n\pi x \sqrt{2}\ \sin n\pi\xi\ v_{2n}(t)$$

$$v_{2n}(t) = e^{-a\, n^2\pi^2 t}\ \frac{\sinh\sqrt{}\ t}{\sqrt{}} .$$

From Appendix III we see that the form of the bounds for C and E do not depend on σ so we may use the bounds given in equation (A3).

The structure of B and E puts a limit on the contents of f_1 and f_2 as before.

Construction of a Liapunov Functional

We consider the case for

$$f_1 = \frac{\partial}{\partial x} \frac{\partial}{\partial u_x} \psi(u_x, v_x) \quad \text{and} \quad f_2 = \frac{\partial}{\partial x} \frac{\partial \psi}{\partial v_x}$$

where ψ has the following properties:

a) $\psi(u_x, v_x) > 0$ unless $u_x = v_x = 0$

b) $\psi(0, 0) = 0$

c) $u_{xx} \frac{\partial}{\partial x} \frac{\partial \psi}{\partial u_x} + v_{xx} \frac{\partial}{\partial x} \frac{\partial \psi}{\partial v_x}$

$$= u_{xx}^2 \psi_{u_x u_x} + 2 v_{xx} u_{xx} \psi_{u_x v_x} + v_{xx}^2 \psi_{v_x v_x} \geq 0$$

d) $h(K) = \max_{u_x^2 + v_x^2 = K^2} \psi(u_x, v_x)$ is a nondecreasing function of K.

Using (2.18) and (2.19) we determine

$$V(t) = \tfrac{1}{2} \int_0^1 [u_x^2 + (u_t - a u_{xx})^2 + a^2 u_{xx}^2 + \sigma^2 v_x^2$$

$$+ (v_t - a v_{xx})^2 + a^2 v_{xx}^2 + 2 \psi] dx$$

$$\dot{V}(t) = -a \int_0^1 [u_{xt}^2 + u_{xx}^2 + v_{xt}^2 + \sigma^2 v_{xx}$$

$$+ u_{xx} \frac{\partial}{\partial x} \frac{\partial \psi}{\partial u_x} + v_{xx} \frac{\partial}{\partial x} \frac{\partial \psi}{\partial v_x}] dx \quad .$$

If we let

$$\rho^2 = \int_0^1 [u^2 + v^2 + u_x^2 + v_x^2 + u_t^2 + v_t^2 + u_{xx}^2 + v_{xx}^2]\,dx$$

then the following result can be proven just as in Example A. If $\beta_1(\rho) = K_1\rho^2$, $\beta_2 = K_2\rho^2 + h(\rho)$ and $\gamma(\rho) = K_3\rho^2$ for some constants K_1, K_2 and K_3 then $\beta_1(\rho) \leq V(t) \leq \beta_2(\rho)$ and $\mathring{V}(t) \leq -\gamma(\rho)$. Thus β_1, β_2 and γ satisfy the conditions of Theorem 4.

Conditions for Existence

For the existence of V and $\mathring{V}$ we need the existence in $L_2(\Omega)$ of u and v and their generalized derivatives u_{xx}, v_{xx}, u_{xxt}, v_{xxt}, u_{tt} and v_{tt}. By (2.14) and (2.15) we have

$$u = \int_0^1 A\,a_0 + \int_0^1 B\,a_1 + \int_0^t\int_0^1 B\,f_1$$

$$v = \int_0^1 C\,b_0 + \int_0^1 E\,b_1 + \int_0^t\int_0^1 E\,f_2$$

$$u_{xx} = \int_0^1 A_{xx}a_0 + \int_0^1 B_{xx}a_1 + \int_0^t\int_0^1 B_{xx}\,f_1$$

$$v_{xx} = \int_0^1 C_{xx}b_0 + \int_0^1 E_{xx}b_1 + \int_0^t\int_0^1 E_{xx}\,f_2$$

$$u_{xxt} = \int_0^1 A_{xx}a_1 + \int_0^1 B_{xx}a_2 + \int_0^t\int_0^1 B_{xx}f_{1\tau}$$

$$v_{xxt} = \int_0^1 C_{xx}b_1 + \int_0^1 E_{xx}b_2 + \int_0^t\int_0^1 E_{xx}f_{2\tau}$$

$$u_{tt} = \int_0^1 A_t\,a_1 + \int_0^1 B_t\,a_2 + \int_0^t\int_0^1 B_t\,f_{1\tau}$$

$$v_{tt} = \int_0^1 C_t\,b_1 + \int_0^1 E_t\,b2 + \int_0^t\int_0^1 E_t\,f_{2\tau}$$

Take $H_T = \{u(x,t) \text{ and } v(x,t) \| \, u, v, u_{xx}, v_{xx}, u_{xxt}, v_{xxt},$
$u_{tt}, v_{tt} \in L_2(\Omega), \; u|_{\partial D} = v|_{\partial D} = 0, \; u(x,0) = a_o(x) \;,$
$u_t(x,0) = a_1(x), \; v(x,0) = b_o(x), \; v_t(x,0) = b_1(x) \}$

with norm $\|U\|_1^2 = \int_o^1 [u^2 + v^2 + u_{xx}^2 + v_{xx}^2 + u_{xxt}^2$

$$+ v_{xxt}^2 + u_{tt}^2 + v_{tt}^2]\, dx \;.$$

By (A3) there exists a constant K such that

$$\|G(a)\|_1 \leq K e^{-\alpha \pi^2 t} [\|a_o\|_1 + \|b_o\|_1 + \|a_1\|_1 + \|b_1\|_1$$

$$+ \|a_{oxx}\|_1 + \|b_{oxx}\|_1 + \|a_{1xx}\|_1 + \|b_{1xx}\|_1 + \|a_2\|_1 + \|b_2\|_1]$$

$$\equiv K e^{-\alpha \pi^2 t} \eta$$

and such that

$$\|H(g)\|_1^2 \leq K^2 e^{-2\alpha \pi^2 (t-\tau)} [2\|f_1\|_1^2 + 2\|f_2\|_1^2 + 2\|f_{1\tau}\|_1^2 + 2\|f_{2\tau}\|_1^2]$$

$$\equiv K^2 e^{-2\alpha\pi^2(t-\tau)} \|g\|_1^2 \;.$$

This assures the existence of V and $\dot{V}$ as long as

1) $a_o = a_1 = b_o = b_1 = 0$ on ∂D

2) η and g satisfy the conditions of Theorem 1 or 2.

The proof of existence of solutions to (DI) belonging to $C^3(\Omega)$ is a direct extension of the work in the previous examples.

APPENDIX I

A PROOF OF LEMMAS 1 AND 4

Lemmas 1 and 4 can be proved in the same manner; here we prove Lemma 4. The following statement, which we prove, is equivalent to Lemma 4: Every Cauchy sequence in B_∞ converges to a limit function in B_∞.

Proof:

Let U_n be an arbitrary Cauchy sequence in H_∞ such that

$$\|U_n\|_1 \leq \delta e^{-\sigma t} \quad \text{a.e.}, \quad \text{i.e. } U_n \in B_\infty .$$

Since H_∞ is a Banach space there exists a $U \in H_\infty$ such that

$$\int_0^\infty f_n(t)dt \to 0 , \quad f_n(t) = \int_D (U_n - U)^2 \, dx \quad . \tag{I 1}$$

But (I 1) implies the existence of a subsequence* n_K such that

$$f_{n_K} \to 0 \quad \text{a.e.} \quad ,$$

therefore

$$\|U_{n_K} - U\|_1 \to 0 \quad \text{a.e.} \quad .$$

By the triangle inequality

$$\|U\|_1 \leq \|U_{n_K} - U\|_1 + \|U_{n_K}\|_1 \leq \epsilon + \delta e^{-\sigma t} \quad \text{a.e.}$$

where $\epsilon \to 0$ as $n_K \to \infty$. Since U is independent of ϵ we have

$$\|U\|_1 \leq \delta e^{-\sigma t} \quad \text{a.e.} \quad ,$$

therefore $U \in B_\infty$.

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* For a discussion of this see Korevaar [12] .

APPENDIX II

SOBOLEV SPACE PROPERTIES

In this appendix we define the generalized derivative and the Sobolev space $W_2^{\ell}(\Omega)$ and state an embedding theorem for $W_2^{\ell}(\Omega)$. For a complete discussion of these concepts see Sobolev [5] and Smirnov [6].

Definition: (Generalized Derivative)

Let Ω be an open subset of R^{N+1} and let $f(x,t)$ and $g(x,t) \in L_2(\Omega)$ such that

$$\int_{\Omega} f(x,t)\, D^n_{t^m} \varphi(x,t)\, dx\, dt = (-1)^{\ell} \int_{\Omega} g(x,t)\varphi(x,t) dx\, dt$$

for any $\ell = m+n$ times continuously differentiable function φ with compact support in Ω.

In this case, $g(x,t)$ is called the generalized derivative of the type $D^n_{t^m} f$ of $f(x,t)$ in Ω.

Definition: (Sobolev Space $W_2^{\ell}(\Omega)$)

Let Ω be an open subset of R^{N+1}. Given a positive integer ℓ, define $W_2^{\ell}(\Omega)$ as the linear space of those $f \in L_2(\Omega)$ having all their generalized derivatives of order ℓ belonging to $L_2(\Omega)$. The norm on $W_2^{\ell}(\Omega)$ is given by

$$\|f\|^2_{W_2^{\ell}(\Omega)} = \int_{\Omega} \Big[f^2 + \sum_{n+m=\ell} (D^n_{t^m} f)^2 \Big] dx\, dt\ .$$

Theorem:

The space $W_2^{\ell}(\Omega)$ is a Banach space.

Theorem: (Embedding Theorem)

If $\ell > \frac{N+1}{2}$ and the integer m satisfies

$$0 < m < \ell - \frac{N+1}{2}$$

then every function $f(x, t) \in W_2^{\ell}(\Omega)$ is equivalent to a function which is continuously differentiable in Ω up to and including order m.

APPENDIX III

PROPERTIES OF THE INTEGRAL KERNELS "A" AND "B"

The properties of A and B depend on the fundamental solutions of the equation (see Section 2.2 for some discussion of A and B)

$$v_n'' + 2\lambda_n v_n' + \mu_n v_n = 0$$

$$v_{1n}(0) = 1 \qquad v_{2n}(0) = 0$$

$$v_{1n}'(0) = 0 \qquad v_{2n}'(0) = 1 \quad .$$

The fundamental solutions are

$$v_{1n}(t) = e^{-\lambda_n t}\left[\cosh\sqrt{\lambda_n^2 - \mu_n}\; t + \frac{\lambda_n}{\sqrt{}} \sinh\sqrt{}\; t\right]$$

$$v_{2n}(t) = e^{-\lambda_n t} \frac{\sinh\sqrt{}\; t}{\sqrt{}} \quad .$$

It follows that

$$v_{1n}'(t) = -\mu_n v_{2n}(t)$$

$$v_{2n}'(t) = v_{1n}(t) - 2\lambda_n v_{2n}(t) \, .$$

Since

$$A = \sum \varphi_n(x)\, \varphi_n(\xi)\, v_{1n}(t)$$

$$B = \sum \varphi_n(x)\, \varphi_n(\xi)\, v_{2n}(t)$$

$$L_1 \varphi_n = 2\lambda_n \varphi_n$$

$$L_2 \varphi_n = \mu_n \varphi_n$$

we find

$$A_t = -\sum \varphi_n(x)\, \varphi_n(\xi)\, \mu_n\, v_{2n}(t)$$

$$B_t = \sum \varphi_n(x)\, \varphi_n(\xi)\, v_{1n}(t) - \sum \varphi_n(x)\, \varphi_n(\xi)\, 2\lambda_n\, v_{2n}(t) \; .$$

Therefore

$$A_t = -L_2 B$$

$$B_t = A - L_1 B \; .$$

To obtain bounds for the norms of the integral operators we need bounds for $v_{1n}(t)$ and $v_{2n}(t)$. In examples A, C and D the eigenvalues of the operators L_1 and L_2 differ only by constants, hence for these examples $v_{1n}(t)$ and $v_{2n}(t)$ can be written in the form

$$v_{1n}(t) = e^{-a\omega t}\left(\cosh\sqrt{a^2\omega^2 - \sigma^2\omega}\; t + \frac{a\omega}{\sqrt{}}\sinh\sqrt{}\; t\right)$$

$$v_{2n}(t) = e^{-a\omega t}\;\frac{\sinh\sqrt{}\; t}{\sqrt{}} \quad , \qquad \alpha > 0 \; ,$$

where $\omega = \omega(n)$ or as in Example C, $\omega = \omega(m, n)$. We condider the three cases

$$\omega(n) \geq \omega_1 > \frac{\sigma^2}{a^2} \; , \qquad \omega(n) = \frac{\sigma^2}{a^2} \text{ and}$$

$$\omega_{min} \leq \omega \leq \omega_o < \frac{\sigma^2}{a^2} \; .$$

<u>Case I.</u> Consider $\omega \geq \omega_1 > \dfrac{\sigma^2}{a^2}$.

Lemma: If $f(\omega) = -a\,\omega + \sqrt{a^2\omega^2 - \sigma^2\omega}$

then $e^{f(\omega)t} \leq e^{-\frac{\sigma^2}{2\alpha}t}$ for $\omega \geq \omega_1$.

Proof: Since

$$f'(\omega) = -a + \tfrac{1}{2}\,\frac{2a^2\omega - \sigma^2}{\sqrt{a^2\omega^2 - \sigma^2\omega}} \geq 0 \quad ,$$

f is a nondecreasing function of ω such that $\lim_{\omega\to\infty} f(\omega) = -\frac{\sigma^2}{2a}$. It follows that $e^{f(\omega)t} \leq e^{-\frac{\sigma^2}{2\alpha}t}$.

Notice that

$$0 \leq v_{1n}(t) = e^{-a\omega t}\left[\tfrac{1}{2}\left(e^{\sqrt{}\,t} + e^{-\sqrt{}\,t}\right) + \frac{a\omega}{2\sqrt{}}\left(e^{\sqrt{}\,t} - e^{-\sqrt{}\,t}\right)\right]$$

$$= \tfrac{1}{2}e^{f(\omega)t}\left(1 + \frac{a\omega}{\sqrt{}}\right) - e^{-(a\omega + \sqrt{})t}\left(\frac{a\omega}{\sqrt{}} - 1\right) ,$$

but $\left(\frac{a\omega}{\sqrt{}} - 1\right) > 0$, therefore

$$v_{1n}(t) \leq \tfrac{1}{2}e^{f(\omega)t}\left(1 + \frac{a\omega}{\sqrt{}}\right) .$$

Let $g(\omega) = \dfrac{a\omega}{\sqrt{a^2\omega^2 - \sigma^2\omega}} > 0$, then

$$2gg' = \frac{d}{d\omega}g^2(\omega) = -\frac{a^2\sigma^2}{(a^2\omega - \sigma^2)^2} < 0.$$ But

$g > 0$ therefore $g' < 0$ which implies g has its maximum at $\omega = \omega_1$.

Therefore

$$|v_{1n}(t)| \leq \tfrac{1}{2}e^{-\frac{\sigma^2}{2a}t}\left(1 + \frac{a\omega_1}{\sqrt{a^2\omega_1^2 - \sigma^2\omega_1}}\right) .$$

We also have

$$0 \leq v_{2n}(t) = e^{-a\omega t}\,\frac{e^{\sqrt{}\,t} - e^{-\sqrt{}\,t}}{\sqrt{}}$$

$$\leq \tfrac{1}{2}\,\frac{e^{f(\omega)t}}{\sqrt{}} \quad ,$$

therefore

$$|v_{2n}(t)| \leq \tfrac{1}{2} \frac{e^{-\frac{\sigma^2}{2a} t}}{\sqrt{a^2 \omega_1^2 - \sigma^2 \omega_1}} ,$$

and

$$0 \leq \omega \, v_{2n}(t) \leq \tfrac{1}{2} \frac{\omega}{\sqrt{}} \; e^{f(\omega)t} ,$$

therefore

$$|\omega \, v_{2n}(t)| \leq \tfrac{1}{2} \frac{\omega_1}{\sqrt{a^2 \omega_1^2 - \sigma^2 \omega_1}} \; e^{-\frac{\sigma^2}{2a} t} .$$

Collecting these results we see there exists a constant K such that

$$\left.\begin{aligned} |v_{1n}(t)| &\leq K e^{-\frac{\sigma^2}{2a} t} \\ |v_{2n}(t)| &\leq K e^{-\frac{\sigma^2}{2a} t} \qquad \text{for } \omega \leq \omega_1 \\ |\omega v_{2n}(t)| &\leq K e^{-\frac{\sigma^2}{2a} t} , \end{aligned}\right\}$$

Case II. Consider $\omega = \frac{\sigma^2}{a^2}$. In this case

$$v_{1n} = e^{-\frac{\sigma^2}{a} t} + \frac{\sigma^2}{a} t \; e^{-\frac{\sigma^2}{a} t}$$

$$v_{2n} = t \; e^{-\frac{\sigma^2}{a} t} .$$

Therefore there exists a constant K such that

$$\left.\begin{aligned} &|v_{1n}| \\ &|v_{2n}| \\ &\left|\frac{\sigma^2}{a^2} v_{2n}\right| \end{aligned}\right\} \leq K e^{-\frac{a^2}{2a} t} .$$

Case III. Consider $\omega_{min} \le \omega \le \omega_o < \frac{\sigma^2}{a^2}$.

In this case we have sin and cos instead of sinh and cosh therefore

$$|v_{1n}(t)| \le e^{-a\omega t}\left(1 + \frac{a^2\omega^2}{\sigma^2\omega - a^2\omega^2}\right)^{\frac{1}{2}}$$

$$= e^{-a\omega t}\left(\frac{1}{1 - \frac{a^2}{\sigma^2}\omega}\right)^{\frac{1}{2}} \le e^{-a\omega_{min}t}\left(\frac{1}{1 - \frac{a^2}{\sigma^2}\omega_o}\right)^{\frac{1}{2}}$$

$$|v_{2n}(t)| \le e^{-a\omega_{min}t}\frac{1}{\sqrt{\sigma^2\omega_o - a^2\omega_o^2}}$$

$$|\omega v_{2n}(t)| \le e^{-a\omega_{min}t}\frac{\omega_o}{\sqrt{\sigma^2\omega_o - a^2\omega_o^2}} .$$

If we consider the damping to be small then the bounds in Case III are the largest and we may write for some constant K that

$$\left.\begin{array}{l} |v_{1n}(t)| \\ |v_{2n}(t)| \\ |\omega v_{2n}(t)| \end{array}\right\} \le K e^{-a\omega_{min}t} \quad \text{for every } \omega . \tag{III-1}$$

Taking a to be small is a matter of convenience and makes no difference in the results of this paper because no matter what a is we still have bounds of the form $K e^{-at}$, $a > 0$. Notice that the exponent in (III-1) is independent of σ .

With this background let us examine each example.

Example A. $\omega = n^2\pi^2$, $\omega_{min} = \pi^2$

Examine $\int_o^1 A_{x^k}\, g$.

$$\text{Let } f_k(nx) = \begin{cases} \sqrt{2}\ \sin n\pi x & k \text{ even} \\ \sqrt{2}\ \cos n\pi x & k \text{ odd.} \end{cases}$$

Then

$$\int_o^1 A_{x^k}\, g = \sum_1^\infty \pm (n\pi)^k f_k(nx)\left(\int_0^1 \sqrt{2}\sin n\pi\xi\, g(\xi)\, d\xi\right) v_{1n}(t)$$

and

$$\int_o^1 \sqrt{2}\ \sin n\pi\xi\ g(\xi)d\xi = \pm \int_o^1 \frac{1}{(n\pi)^k}\ f_k(n\xi)\, g_{\xi^k}\, d\xi ,$$

under the assumption that g and all its even order derivatives up to but not including order k vanish at zero and one.

Therefore

$$\int_o^1 \left(\int_o^1 A_{x^k}\ g\, d\,\xi\right)^2 dx = \sum_{n=1}^\infty \left(\int_o^1 f_k(n\xi)\, g_{\xi^k}\, d\xi\right)^2 v_{1n}^2(t)$$

$$\leq K^2 e^{-2a\pi^2 t} \int_o^1 g_{x^k}^2\ dx \qquad ,$$

where we have made use of (III-1) and Bessel's inequality. But this is equivalent to

$$\left\| \int_o^1 A_{x^k}\, g \right\|_1 \leq K e^{-a\pi^2 t} \| g_{x^k} \|_1 \quad .$$

Examine $\int_o^1 B_{x^n}\, g$.

In a similar fashion we show

$$\int_o^1 \left(\int_o^1 B_{x^{k+2}}\, g\right)^2 dx = \sum_{n=1}^\infty \left(\int_o^1 f_k(nx)\, g_{x^k}\right)^2 (n^2\pi^2 v_{2n}(t))^2$$

$$\leq K^2 e^{-2\,a\,\pi^2\, t} \int_o^1 g_{x^k}^2\ dx \quad .$$

Therefore,

$$\| \int_o^1 B_{x^{k+2}} g \|_1 \leq K e^{-a\pi^2 t} \| g_{x^k} \|_1 \quad .$$

Example B.

Take a small enough so that $v_{1n}(t)$ and $v_{2n}(t)$ are sinusoidal; then

$$v_{1n}(t) = e^{-at} \left(\cos\sqrt{n^2\pi^2 - a^2}\ t + \frac{a}{\sqrt{}} \sin\sqrt{}\ t\right)$$

$$v_{2n}(t) = e^{-at} \frac{\sin\sqrt{}\ t}{\sqrt{}}$$

which implies

$$\left.\begin{array}{r} |v_{1n}(t)| \\ |v_{2n}(t)| \\ |n\pi\ v_{2n}(t)| \end{array}\right\} \leq K e^{-at}$$

for every n and for some constant K.

Examine $\int A_{x^k} g$.

In exactly the same manner as in Example A we find

$$\| \int A_{x^k} g \|_1 \leq K e^{-at} \| g_{x^k} \|_1 \quad .$$

Examine $\int_o^1 B_{x^{k+1}} g$.

$$\int_o^1 \left(\int_o^1 B_{x^{k+1}} g\right)^2 dx = \Sigma \left(\int_o^1 f_k(nx)\, g_{x^k}\right)^2 (n\pi\ v_{2n}(t))^2$$

$$\leq K^2 e^{-2at} \int_o^1 g^2_{x^k} \quad .$$

Therefore

$$\|\int B_{x^{k+1}}\, g\|_1 \leq K e^{-\alpha t} \|g_{x^k}\|_1.$$

Example C. $\omega = \pi^2 (n^2 + m^2)$, $\omega_{min} = 2\pi^2$

Let ∂D_x denote the boundary at $x = 0, 1$ and ∂D_y denote the boundary at $y = 0, 1$.

Examine $\int_D A_{x^k y^\ell}\, g(\xi, \eta)\, d\xi\, d\eta$

where $A = A(x, \xi; y, \eta; t)$.

Assume that

$$g_{x^{2i}}\Big|_{\partial D_x} = g_{y^{2j}}\Big|_{\partial D_y} = 0 \quad i = 0, [\tfrac{k-1}{2}]\,; \quad j = 0, [\tfrac{\ell-1}{2}]\,.$$

Then

$$\int_D \Big(\int_D A_{x^k y^\ell}\, g\Big)^2 dxdy = \Sigma\,\Sigma\,\Big(2\int_D \begin{Bmatrix} \sin m\pi x \sin n\pi y \\ \sin m\pi x \cos n\pi y \\ \cos m\pi x \sin n\pi y \\ \cos m\pi x \cos n\pi y \end{Bmatrix} g_{x^k y^\ell}\Big)^2 v^2_{1\,mn}(t)$$

$$\leq K^2 e^{-4\alpha\pi^2 t} \int_D g^2_{x^k y^\ell}\, dxdy\,,$$

therefore

$$\|\int_D A_{x^k y^\ell}\, g\|_1 \leq K e^{-2\alpha\pi^2 t} \|g_{x^k y^\ell}\|_1\,.$$

Examine $\int_D [\nabla^2 A(x, \xi; y, \eta; t)]\, g(\xi, \eta)\, d\xi d\eta$.

Assume that $g|_{\partial D} = 0$, then

$$\int_D \Big(\int_D \nabla^2 A\, g\Big)^2 = \Sigma\,\Sigma\,\Big(\int_D 2 \sin m\pi x \sin n\pi y\, \nabla^2 g\Big)^2 v^2_{1\,m\,n}(t)$$

$$\leq K^2 e^{-4\alpha\pi^2 t} \int_D (\nabla^2 g)^2.$$

Therefore

$$\| \int_D \nabla^2 A g \|_1 \le K e^{-2 a \pi^2 t} \| \nabla^2 g \|_1 \quad .$$

Examine $\int_D B_{x^{k+2}}\, g(\xi, \eta)\, d\xi d\eta$ where $B = B(x, \xi; y, \eta; t)$.

Assume that $g_{x^{2i}} \big|_{\partial D_x} = 0 \qquad i = 0, [\frac{k-1}{2}]$, then

$$\int_D \left(\int_D B_{x^{k+2}}\, g \right)^2 = \sum \sum \left(\int_D \begin{Bmatrix} 2 \sin \sin \\ 2 \cos \sin \end{Bmatrix} g_{x^k} \right)^2 (m^2 \pi^2 v_{2mn})^2$$

where

$$|m^2 \pi^2 v_{2mn}| \le |(m^2 + n^2) \pi^2 v_{2mn}| = |\omega v_{2mn}| \le K e^{-2 a \pi^2 t} .$$

Therefore

$$\| \int_D B_{x^{k+2}}\, g \|_1 \le K e^{-2 a \pi^2 t} \| g_{x^k} \|_1 \quad .$$

Examine $\int_D B_{x^{k+1} y^{\ell+1}}\, g(\xi, \eta)\, d\xi d\eta$.

Assume that

$$g_{x^{2i}} \big|_{\partial D_x} = g_{y^{2j}} \big|_{\partial D_y} = 0 \qquad i = 0, [\frac{k-1}{2}];\ j = 0, [\frac{\ell - 1}{2}]$$

then

$$\int_D \left(\int_D B_{x^{k+1} y^{\ell+1}}\, g \right)^2 = \sum \sum \left(2 \int_D \begin{Bmatrix} \sin \sin \\ \sin \cos \\ \cos \sin \\ \cos \cos \end{Bmatrix} g_{x^k y^\ell} \right)^2 (\pi^2 mn\, v_{2mn})^2$$

where

$$|\pi^2 mn\, v_{2mn}| \le |\omega v_{2mn}| \le K e^{-2 a \pi^2 t} \quad .$$

Therefore

$$\| \int_D B_{x^{k+1} y^{\ell+1}} g \|_1 \leq K e^{-2 \alpha \pi^2 t} \| g_{x^k y^\ell} \|_1 .$$

Examine $\int_D \nabla^2 B g$.

$$\int_D \left(\int_D \nabla^2 B g \right)^2 = \sum \sum \left(2 \int_D \sin m\pi x \sin n\pi y \ g \right)^2 [(m^2+n^2)\pi^2 v_{2mn}]^2$$

$$\leq K^2 e^{-4 \alpha \pi^2 t} \int_D g^2 .$$

Therefore

$$\| \int_D \nabla^2 B g \|_1 \leq K e^{-2 \alpha \pi^2 t} \| g \|_1 .$$

Example D.

Since σ does not enter the exponent in (III-1) the bounds for C and E are the same as for A and B of Example A.

APPENDIX IV

NONLINEARITY CONDITIONS

The purpose of this appendix is to show the conditions under which the nonlinearities, $f = h(u)$ and $f = h(u_x)\, u_{xx}$, mentioned in Example A (see pages 30 and 33), satisfy conditions c) and d) of Theorem 2. We do this for the case where

$$H_T = \{ u(x,t) \,\|\, u, u_{xx}, u_{xxt}, u_{tt} \in L_2(\Omega),\ u|_{\partial D} = 0 \},$$

which is the case for the existence of V and $\mathring{V}$ in Example A (see page 30).

The space of functions considered in Theorem 2 is B_∞, so in what follows we assume that $u \in B_\infty \subset H_T$, where H_T is defined above. Since $U \in B_\infty$ we have the following inequalities (see pages 27 and 28 for some discussion of these inequalities):

$$|u|^2 \le \int_0^1 u_x^2 \le \int_0^1 u_{xx}^2 \le \|U\|_1^2 \le \delta^2 e^{-2\sigma t} \qquad \text{a. e.}$$

$$|u_x|^2 \le \int_0^1 u_{xx}^2 \qquad \text{a. e.}$$

$$|u_t|^2 \le \int_0^1 u_{xt}^2 \le \int_0^1 u_{xxt}^2 \le \|U\|_1^2 \le \delta^2 e^{-2\sigma t} \qquad \text{a. e.} \tag{IV-1}$$

$$|u_{xt}|^2 \le \int_0^1 u_{xxt}^2 \qquad \text{a. e. .}$$

<u>Lemma 1 $(f = h(u))$:</u>

If

a) $h(u)$ is twice continuously differentiable for $|u| \le \delta$,

$$\left.\begin{array}{l} |h(u)| \leq k_1(\delta)\ |u|^2 \\ \\ |h'(u)| \leq k_2(\delta)\ |u| \\ \\ |h''(u)| \leq k_3(\delta) \end{array}\right\} |u| \leq \delta,$$

b)

then

$$\|g(U)\|_1 = (2\|f\|_1^2 + 2\|f_t\|_1^2)^{\frac{1}{2}} \leq L_1 \|U\|_1^2 \qquad \text{a. e.}$$

and

$$\|g(U) - g(V)\|_1 = (2\|f(U) - f(V)\|_1^2 + 2\|f_t(U) - f_t(V)\|_1^2)^{\frac{1}{2}}$$
$$\leq L_2 \delta\, e^{-\sigma t} \|U-V\|_1 \ \text{a. e.}\ .$$

The inequalities in the following proof hold a. e. in t.

<u>Proof:</u> The inequalities (IV-1) and condition b) imply that

$$|f| = |h(u)| \leq k_1 |u|^2 \leq k_1 \|U\|_1^2,$$

therefore

$$\|f\|_1 \leq k_1 \|U\|_1^2 .$$

In a similar fashion

$$|f_t| = |h'(u)\, u_t| \leq k_2 |u|\ |u_t| \leq k_2 \|U\|_1^2 ,$$

which implies

$$\|f_t\|_1 \leq k_2 \|U\|_1^2 .$$

Therefore

$$\|g(U)\|_1 \leq \|U\|_1^2 (2k_1^2 + 2k_2^2)^{\frac{1}{2}} = L_1 \|U\|_1^2 .$$

By the mean value theorem

$$|f(U) - f(V)| = |h(u) - h(v)| \leq |h'(\eta)| \, |u-v|$$

for some $\eta \in [u, v]$. This implies that

$$|f(U) - f(V)| \leq k_2 \, |\eta| \, \|U-V\|_1 \leq k_2 \, \delta \, e^{-\sigma t} \|U-V\|_1$$

since $|\eta| \leq \delta e^{-\sigma t}$. Therefore

$$\|f(U) - f(V)\| \leq k_2 \delta \, e^{-\sigma t} \|U-V\|_1.$$

By the same reasoning as above we find

$$|f_t(U) - f_t(V)| = |h'(u) \, u_t - h'(v) \, v_t|$$

$$= |h'(v) \, (u_t - v_t) + h''(v) \, (u-v) \, u_t|$$

$$\leq (k_2 + k_3) \; \delta \, e^{-\sigma t} \|U-V\|_1,$$

which implies that

$$\|f_t(U) - f_t(V)\|_1 \leq (k_2 + k_3) \; \delta \, e^{-\sigma t} \; \|U-V\|_1.$$

Therefore

$$\|g(U) - g(V)\|_1 \leq \delta \, e^{-\sigma t} \; \|U-V\|_1 \; (2 \, k_2^2 + 2 \, (k_2 + k_3)^2)^{\frac{1}{2}}$$

$$= L_2 \; \delta \, e^{-\sigma t} \; \|U-V\|_1.$$

Lemma 2 ($f = h(u_x)\, u_{xx}$):

If

a) h is twice continuously differentiable for $|u_x| \leq \delta$,

$$\left.\begin{aligned} &|h(u_x)| \leq k_1(\delta)\, |u_x| \\ \text{b)}\quad &|h'(u_x)| \leq k_2(\delta) \\ &|h''(u_x)| \leq k_3(\delta) \end{aligned}\right\} \quad |u_x| \leq \delta,$$

then

$$\| g(U) \|_1 = (2\| f \|_1^2 + 2\| f_t \|_1^2)^{\frac{1}{2}} \leq L_1 \| U \|_1^2 \qquad \text{a. e.}$$

and

$$\| g(U) - g(V) \|_1 = (2\| f(U) - f(V) \|_1^2 + 2\, \| f_t(U) - f_t(V) \|_1^2)^{\frac{1}{2}}$$

$$\leq L_2 \delta\, e^{-\sigma t} \| U - V \|_1 \qquad \text{a. e.} \quad .$$

The inequalities in the following proof hold a. e. in t.

Proof: The inequalities (IV-1) and condition b) imply that

$$|f| = | h(u_x)\, u_{xx} | \leq k_1 \, | u_x | \, | u_{xx} | \leq k_1 \| U \|_1 \, | u_{xx} |,$$

therefore

$$\| f \|_1 \leq k_1 \| U \|_1^2.$$

In a similar fashion

$$|f_t| = |h'(u_x)\, u_{xt} u_{xx} + h(u_x)\, u_{xxt}|$$

$$\leq k_2 \|U\|_1 \, |u_{xx}| + k_1 \|U\|_1 \, |u_{xxt}|,$$

which implies

$$\|f_t\|_1 \leq (k_1 + k_2) \, \|U\|_1^2.$$

Therefore

$$\|g(U)\|_1 \leq (2k_1^2 + 2(k_1 + k_2)^2)^{\frac{1}{2}} \, \|U\|_1^2$$

$$= L_1 \|U\|_1^2.$$

By the mean value theorem

$$|f(U) - f(V)| = |h(v_x)\,(u_{xx} - v_{xx}) + h'(\eta)\, u_{xx}(u_x - v_x)|$$

$$\leq k_1 \|U\|_1 \, |u_{xx} - v_{xx}| + k_2 \|U-V\|_1 \, |u_{xx}|,$$

therefore

$$\|f(U) - f(V)\|_1 \leq (k_1 + k_2)\, \delta\, e^{-\sigma t} \, \|U-V\|_1.$$

Notice that

$$f_t(U) - f_t(V) = (h'(u_x)\, u_{xt}\, u_{xx} - h'(v_x)\, v_{xt}\, v_{xx})$$

$$+ (h(u_x)\, u_{xxt} - h(v_x)\, v_{xxt}).$$

By a calculation similar to the previous one we find

$$\|h(u_x)\, u_{xxt} - h(v_x)\, v_{xxt}\|_1 \leq (k_1 + k_2)\, \delta\, e^{-\sigma t} \, \|U-V\|_1.$$

Using the mean value theorem, we find

$$|h'(u_x)\, u_{xt}\, u_{xx} - h'(v_x)\, v_{xt}\, v_{xx}|$$

$$= |h'(v_x)\, [u_{xt}(u_{xx} - v_{xx}) + v_{xx}(u_{xt} - v_{xt})]$$

$$+ h''(\eta)\,(u_x - v_x)\, u_{xt}\, u_{xx}|$$

$$\leq k_2 \|U\|_1\, |u_{xx} - v_{xx}| + k_2\, |v_{xx}|\, \|U - V\|_1$$

$$+ k_3\, \|U - V\|_1\, \|U\|_1\, |u_{xx}|\,,$$

which implies

$$\|h'(u_x)\, u_{xt}\, u_{xx} - h'(v_x)\, v_{xt}\, v_{xx}\|_1$$

$$\leq (2k_2 + k_3\,\delta)\;\delta\, e^{-\sigma t} \|U - V\|_1.$$

Therefore

$$\|f_t(U) - f_t(V)\|_1 \leq (k_1 + 3k_2 + k_3\delta)\;\delta\, e^{-\sigma t}\, \|U - V\|_1,$$

and finally

$$\|g(U) - g(V)\|_1 \leq (2(k_1 + k_2)^2 + 2(k_1 + 3K_2 + k_3\delta)^2)^{\frac{1}{2}}\;\delta\, e^{-\sigma t} \|U - V\|_1$$

$$= L_2\,\delta\, e^{-\sigma t} \|U - V\|_1.$$

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