TOPICS IN LINEAR AND NONLINEAR DISPERSIVE WAVES

Thesis by

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ABSTRACT

The various singularities and instabilities which arise in the modulation theory of dispersive wavetrains are studied. Primary interest is in the theory of nonlinear waves,but a study of associated questions in linear theory provides background information and is of independent interest.

The full modulation theory is developed in general terms. In the first approximation for slow modulations, the modulation equations are solved. In both the linear and nonlinear theories, singularities and regions of multivalued modulations are predicted. Higher order effects are considered to evaluate this first order theory. An improved approximation is presented which gives the true behavior in the singular regions. For the linear case, the end result can be interpreted as the overlap of elementary wavetrains.In the nonlinear case, it is found that a sufficiently strong nonlinearity prevents this overlap. Transition zones with a predictable structure replace the singular regions.

For linear problems, exact solutions are found by Fourier integrals and other superposition techniques. These show the true behavior when breaking modulations are predicted.

A numerical study is made for the anharmonic lattice to assess the nonlinear theory. This confirms the theoretical predictions of nonlinear group velocities, group splitting, and wavetrain instability, as well as higher order effects in the singular regions.

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CHAPTER I. INTRODUCTION

This thesis is a study of the various singularities and instabilities that arise in the modulation theory of dispersive wavetrains. The main objective is the resolution of questions raised in the theory of nonlinear waves, but a detailed study of related questions in linear theory provides background information and is of independent interest.

Linear dispersive problems are characterized by the existence of elementary periodic solutions

$$A \cos \theta, \qquad \theta = kx - \omega t, \tag{1.1}$$

in which the frequency ω and the wave number k are related by a dispersion relation

$$\omega = W(k) \quad . \tag{1.2}$$

In corresponding nonlinear problems, solutions in the form of periodic wavetrains are also found but a basic new ingredient is the appearance of amplitude dependence in the dispersion relation. A typical variable, φ say, is given by

$$\varphi = \Phi(\theta), \qquad \theta = kx - \omega t , \tag{1.3}$$

where Φ is a periodic function of θ; Φ also includes an amplitude parameter A and the solution requires that (ω, k, A) satisfy a dispersion relation

$$\omega = W(k, A) \ . \tag{1.4}$$

The function Φ is no longer sinusoidal, in general, but the crucial change, leading to qualitatively new phenomena, is the inclusion of A in (1.4).

In the linear case, more general solutions can be obtained by Fourier superposition, but this is not possible in the nonlinear case. Instead, one approach has been to develop the theory of modulated wavetrains in which (ω, k, A) are slowly varying functions of x and t. In this extended form, ω and k are defined in terms of the phase $\theta(x, t)$ by

$$k = \theta_x , \qquad \omega = -\theta_t . \tag{1.5}$$

Even in linear theory, the modulation approach is useful to give a quick and informative derivation of the concepts of group velocity, and to extend results to nonuniform media where exact solutions cannot be found. In such cases it is equivalent to the W. K. B. approximation.

The simplest approach is to note that

$$k_t + \omega_x = 0$$

from (1.5). Then if one still assumes that (1.2) holds, k satisfies

$$k_t + C(k)k_x = 0 , \tag{1.6a}$$

where $C(k) = W'(k)$. This is a nonlinear hyperbolic equation with characteristic velocity $C(k)$. On the characteristic curves

$$\frac{dx}{dt} = C(k) ,$$

k is constant. Thus each wave number propagates with its own group velocity. It can also be shown that the amplitude satisfies a similar equation which may be written

$$(A^2)_t + (C(k)A^2)_x = 0 . \tag{1.6b}$$

This equation can be solved by integration along the same characteristics.

For nonlinear problems in which the amplitude is small, the dispersion relation (1.4) can be expanded in powers of A^2 ,

$$\omega = W_0(k) + A^2 W_2(k) + \ldots .$$

The corresponding modulation equations are

$$k_t + W_0'(k)k_x + 2AW_2 A_x = 0 , \tag{1.7a}$$

$$A_t + W_0'(k)A_x + \tfrac{1}{2} W_0'' Ak_x = 0 . \tag{1.7b}$$

A derivation will be given in Chapter V. These are two coupled equations for (k, A), which have characteristic velocities

$$C_{\pm} = W_0' \pm A\sqrt{W_0'' W_2} .$$

Accordingly, (1.7) are hyperbolic or elliptic if $W_0'' W_2$ is positive or negative, respectively.

In the hyperbolic case, the two velocities $C_{\pm}$ are the nonlinear analogs of the classical group velocity and both reduce to C(k) in the linear limit. If the modulations are confined to a finite region, the disturbance will split into two groups with speeds C_+ and C_-, a result quite different from linear theory.

For the elliptic case, small perturbations about constant solutions will grow. This implies that the periodic wavetrain (1.3) is unstable for $W_0''W_2 < 0$, a result which has no analog in linear problems.

In the linear problem, and in the nonlinear case when $W_0'' W_2 > 0$, the modulation equations are hyperbolic and nonlinear. Such equations always have breaking solutions. If the initial wave number distribution is such that C or $C_\pm$ decreases with increasing x, then some of the characteristics eventually cross. This produces singularities and multivalued solutions. Equations (1.6) or (1.7) are no longer valid at this stage since the modulations do not change slowly. This leaves the question of what actually happens near breaking. One possibility is that higher order effects become dominant and prevent breaking. There is also the intriguing possibility of some kind of shock structure being the end result. This would not be a shock in the usual sense, but a shock in the modulations; for instance, it might be the juxtaposition of two periodic wavetrain with some small transition region between.

In the linear case, however, even though (1.6a) is nonlinear, it approximates the behavior of a basic linear problem. So, in this case, superposition of wavetrains gives an acceptable interpretation of multivalued modulations.

In the following chapters we assess the validity of the modulation equations (1.6) and (1.7), and determine what happens in cases where this first order theory does not apply. First, in Chapter II, a concise and rigorous form of the full modulation theory is derived for linear problems. Equations (1.6) are found as a first approximation. In the next approximation an amplitude coupling term $\left(\frac{A_{xx}}{A}\right)_x$ appears in the wave number equation. Analysis of exact solutions in Chapters III and IV shows that this improved approximation is sufficient to predict the true behavior in all cases. This additional coupling effect prevents singularities and also produces an overlap region for large times.

In Chapter V, the full modulation theory is extended to nonlinear problems. Here it is found that a sufficiently strong nonlinearity changes the character of the solution near breaking. In the breaking region the wave number obeys a kind of Korteweg-de Vries equation. This predicts two kinds of transition region in which both the amplitude and wave number change abruptly; their speeds correspond to the two nonlinear group velocities. The improved modulation theory is also used to reassess the questions of instability in the elliptic case ($W_0'' W_2 < 0$ in (1.7)). It is found that only perturbations with a wavelength shorter than some critical value will grow. Also, solitary wave packets are shown to exist for this case.

In Chapter VI, numerical solutions are computed for an anharmonic lattice which is both nonlinear and dispersive. This problem is of physical interest in itself, but is presented here to assess the modulation theory and the questions of breaking. For the first time clear evidence of group splitting and instability is found, as well as verification of the predicted behavior near breaking.

CHAPTER II. LINEAR DISPERSIVE WAVES

We consider here the class of linear problems which exhibit dispersive wave behavior. In problems of one space dimension dispersive waves are recognized by the existence of elementary periodic solutions

$$\varphi(x,t) = A\, e^{i(kx-\omega t)} \quad , \tag{2.1}$$

where A, ω, and k are real constants: the amplitude, frequency, and wave number. Clearly A is arbitrary for linear problems, but typically it is found that ω and k must satisfy a dispersion relation

$$G(\omega, k) = 0 \quad , \tag{2.2a}$$

if (2.1) is to be a solution. We suppose that the dispersion relation can be solved for ω,

$$\omega = W(k) \quad . \tag{2.2b}$$

In general, there will be a number of such solutions which are referred to as different modes. As noted earlier, a significant quantity associated with each mode is the group velocity

$$C(k) = \frac{dW}{dk} \quad .$$

The variation of the group velocity with wave number is responsible for the dispersive behavior so we exclude the case where C is constant.

To see precisely how the dispersion is related to the governing equation, consider the general linear partial differential equation in x and t with constant coefficients:

$$P(\partial_t, \partial_x)\varphi(x,t) = 0 \quad , \tag{2.3}$$

where P is a polynomial. For periodic solutions (2.1), the derivative operators are replaced by multiplication

$$\partial_t \to -i\omega, \quad \partial_x \to ik \quad ,$$

to give

$$P(-i\omega, ik)A = 0 \quad .$$

If this has solutions for real ω and k, then (2.3) is dispersive, and the dispersion function must be

$$G(\omega, k) \equiv P(-i\omega, ik) \quad .$$

Interestingly, G contains all of the information about the problem. By reversing the above argument, the original equation can be reconstructed from the dispersion relation. Given the function G, the equation must have been

$$G(i\,\partial_t, -i\,\partial_x)\,\varphi(x,t) = 0 \quad . \tag{2.4}$$

We see that if G contains a mixture of odd and even powers, the differential equation involved is complex. For example the Schroedinger equation for a free particle,

$$i\,\varphi_t + \varphi_{xx} = 0$$

has $G \equiv \omega - k^2$.

Equation (2.3) will give rise to polynomial dispersion relations. However, there are other cases in which a more general dispersion arises, such as water waves, discrete problems, or integrodifferential equations. Also, if a single mode is being studied, the dispersion relation in the form (2.2b) may not be a polynomial. For these cases, an equivalent differential equation with an infinite series of terms can be constructed from (2.4), using the Taylor series expansion for G in powers of ω and k. For G in the form ω - W(k), this equivalent equation is

$$i\varphi_t - W(0)\varphi + i\,W'(0)\varphi_x + \frac{1}{2!}W''(0)\varphi_{xx} + \ldots\ldots = 0 \quad .$$

This is equivalent in that it has the same dispersion relation, and can be used in place of the original equation if desired.

The elementary solutions (2.1) can be superposed to give the general solution in terms of Fourier integrals. Considering just one mode, the solution is

$$\varphi(x, t) = \int_{-\infty}^{\infty} F(k) e^{ikx - i\,W(k)t}\, dk \quad ,$$

where F is determined from the initial conditions as

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x, o) e^{-ikx}\, dx \quad .$$

The Fourier integrals give the exact solution of the problem, but their content is hard to see. By formulating the solution in terms of modulations we can extract some of the main features which are common to all dispersive problems. The modulation theory can be obtained from the asymptotic behavior of the Fourier integrals, but it can also be derived directly in a way that lends itself to generalizations.

Modulation theory

We consider solutions which behave locally like the uniform periodic wave, but have a slowly varying amplitude, frequency, and wave number. We can write such a solution in terms of an amplitude $A(x, t)$ and phase $\theta(x, t)$ in the form

$$\varphi(x, t) = A(x, t)e^{i\theta(x, t)} \quad . \tag{2.5}$$

The proper generalization of frequency and wave number for a modulated wave is

$$\omega(x, t) = -\theta_t, \quad k(x, t) = \theta_x \quad .$$

The equations for A and θ are determined by substituting (2.5) into the differential equation (2.4). Each differentiation can be replaced by an operator acting on A,

$$\partial_t \varphi \to (-i\omega + \frac{\partial}{\partial t})A, \quad \partial_x \varphi \to (ik + \frac{\partial}{\partial x})A \ ,$$

so that (2.4) becomes

$$G(\omega + i\,\partial_t, \ k - i\,\partial_x)\, A(x, t) = 0 \quad . \tag{2.6}$$

It is convenient to work with the functions ω and k in place of θ, and supplement (2.6) by the compatibility relation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \ . \tag{2.7}$$

If (2.7) holds, then the operators $(\omega + i\,\partial_t)$ and $(k - i\,\partial_x)$ commute, however operations such as $k\partial_x$ and $\partial_x k$ cannot be interchanged. The real and imaginary parts of (2.6) along with (2.7) provide three equations for the variables (ω, k, A). These are the full (exact) modulation equations.

The particular form of G, and hence (2.6) is not unique when studying a single mode, and in some cases the proper choice will greatly simplify the equations. For instance, in the beam equation for which $\omega^2 = k^4$, one could take G to be either $\omega^2 - k^4$ or $\omega - k^2$ when studying the mode $\omega = k^2$. The latter choice is the simpler one for (2.6). In other cases, it is better not to solve for ω. In the Klein-Gordon equation for which $\omega^2 = k^2+1$, if G is chosen to be $\omega - \sqrt{k^2+1}$ it has to be interpreted as $\omega - (1 + \frac{1}{2}k^2 + \ldots)$. This results in an infinite series of terms in (2.6). It is better to take G to be $\omega^2 - (k^2+1)$ for that case.

A classical example of a modulated wave is obtained by superposing two elementary solutions of equal amplitude, say

$$\varphi = \exp[i(k+\Omega)x - iW(k-\Omega)t]$$
$$+ \exp[i(k-\Omega)x - i\,W(k-\Omega)t] \quad . \tag{2.8a}$$

This gives rise to a beating effect. By rearranging, (2.8a) can be put in terms of an amplitude and phase function

$$\varphi = A(x,t)e^{i(kx - \omega t)} \quad ,$$

where $$A = 2\cos(\Omega x - \tfrac{1}{2}[W(k+\Omega) - W(k-\Omega)]t) \quad , \tag{2.8b}$$

and $$\omega = \tfrac{1}{2}[W(k+\Omega) + W(k-\Omega)] \quad . \tag{2.8c}$$

The amplitude oscillations (or beats) move with a speed

$$V = \frac{W(k+\Omega) - W(k-\Omega)}{2\Omega} \quad . \tag{2.8d}$$

If Ω is small, then V is approximately the group velocity $\frac{dW}{dk}$. Yet this solution is exact, and the finite difference formula for this speed must be obtainable from the full modulation equations (2.6).

To see this we write (2.6) as

$$\tfrac{1}{2}\left\{G(\omega+i\frac{\partial}{\partial t},\ k-i\frac{\partial}{\partial x}) \pm G(\omega-i\frac{\partial}{\partial t},\ k+i\frac{\partial}{\partial x})A\right\} = 0\ ,$$

where the choice of (+) or (-) gives the real or imaginary part of (2.6). If ω and k are constant, and the amplitude is a sinusoidal steady profile, say

$$A = \cos\Omega(x-Vt) = R\ e^{i\Omega(x-Vt)}\ ,$$

then these equations require

$$\tfrac{1}{2}\left\{G(\omega+V\Omega,\ k+\Omega) \pm G(\omega-V\Omega,\ k-\Omega)\right\} = 0\ .$$

With $G(\omega,k) \equiv \omega - W(k)$, (+) and (-) give (2.8c) and (2.8d), respectively.

<u>Slow modulations</u>

There is not much advantage in replacing the original problem with the full modulation equations. Posed in this way the problem is much worse, being three-coupled nonlinear equations. However, we have not yet taken advantage of the fact that the solution of interest involves modulations of the uniform wavetrain. For such solutions the parameters change slowly relative to the local wavelength and period. Then, various levels of approximation become useful in understanding the nature of dispersive waves.

We assume that the modulation parameters are functions of slow variables

$$X = \epsilon x, \qquad T = \epsilon t, \qquad \epsilon \ll 1 ,$$

so that (2.6) becomes

$$G(\omega + i\epsilon\,\partial_T,\ k - i\epsilon\,\partial_X)\,A(X,T) = 0 . \tag{2.9}$$

This can be expanded in powers of ϵ by looking at the general term in the Taylor series for G. For instance, the first terms which arise from k^n are

$$\left.\begin{aligned}(k-i\epsilon\,\partial_X)^n A &= k^n A - i\epsilon(k^{n-1}A_X + k^{n-2}(kA)_X \\ &\quad + \dots (k^{n-1}A)_X) + O(\epsilon^2) \\ &= k^n A - i\epsilon(nk^{n-1}A_X + \tfrac{1}{2}A(nk^{n-1})_X) \\ &\quad + O(\epsilon^2) \\ &= k^n A - \frac{i\epsilon}{2A}\left(A^2 \frac{d}{dk}(k^n)\right)_X \\ &\quad + O(\epsilon^2) .\end{aligned}\right\}$$

The expansion of (2.9) in powers of $(i\epsilon)$ takes the form

$$\begin{aligned}G(\omega, k)A + \frac{-i\epsilon}{2A}\left[-(G_\omega A^2)_T + (G_k A^2)_X\right] \\ + (-i\epsilon)^2 E_2 + (-i\epsilon)^3 E_3 + O(\epsilon^4) = 0.\end{aligned} \tag{2.10a}$$

The higher order terms E_2, E_3 become quite complicated if the function G is left in general form. However, the interest here is in a single mode and we may take

$$G(\omega, k) \equiv \omega - W(k) \tag{2.11}$$

since the resulting equations must be equivalent. The corresponding expressions for E_2 and E_3 are

$$-E_2 = \frac{1}{2!} W''(k)A_{XX} + \frac{1}{3!} W'''(k)(3k_X A_X + k_{XX} A)$$
$$+ \frac{1}{4!} W^{(iv)}(k)\, 3k_X^2 A , \tag{2.10b}$$

$$-E_3 = \frac{1}{3!} \frac{d^3 W(k)}{dk^3} A_{XXX}$$
$$+ \frac{1}{4!} \frac{d^4 W(k)}{dk^4} (k_{XXX} A + 4k_{XX} A_X + 6k_X A_{XX}$$
$$+ \frac{1}{5!} \frac{d^5 W(k)}{dk^5} (10k_X k_{XX} A + 15k_X^2 A_X)$$
$$+ \frac{1}{6!} \frac{d^6 W}{dk^6}\, 15\, k_X^3 A . \tag{2.10c}$$

To simplify further we suppose that the <u>modulations</u> <u>are</u> <u>small</u> in the sense that ω and k are close to some constant values, i.e.

$$\omega = \omega_o + O(\epsilon), \; k = k_o + O(\epsilon) , \tag{2.12}$$

where ω_o and k_o can be chosen to satisfy the dispersion relation. The modulations ω and k will be seen to remain small so that (2.12) is valid uniformly in time. However this assumption cannot be made about the amplitude. With (2.11) and (2.12), E_2 is

$$E_2 = \frac{-1}{2!} W''(k_o)A_{XX} + O(\epsilon) . \tag{2.13}$$

The real part of (2.10) gives the corrected dispersion relation

$$\omega = W(k) - \tfrac{1}{2}\,\varepsilon^{2}\,W''(k_o)\,\frac{A_{XX}}{A} + O(\varepsilon^{3}) \ .$$

Using this in the compatibility relation (2.7) we obtain

$$k_T + C(k)k_X = \tfrac{1}{2}\,\varepsilon^{2}\,W''(k_o)\left(\frac{A_{XX}}{A}\right)_X + O(\varepsilon^{3}) \ .$$

Since k_T and k_X are $O(\varepsilon)$ according to (2.12), the correction to the wave number modulation is relatively of order ε. To be consistent with this we must neglect E_3 in the imaginary part of (2.10) which is a correction of order ε^2. Thus, at this level of approximation, the modulation equations are

$$k_T + C(k)k_X = \tfrac{1}{2}\,\varepsilon^{2}\,W''(k_o)\left(\frac{A_{XX}}{A}\right)_X \ , \qquad (2.14a)$$

$$(A^2)_T + (C(k)A^2)_X = 0 \ . \qquad (2.14b)$$

It would be consistent with assumption (2.12) to expand $C(k)$ as

$$C(k_o) + C'(k_o)(k-k_o)$$

in (2.14). But the more general form is kept, since even when $(k-k_o)$ is not small, the coupling term on the right of (2.14a) seems to be the most important one compared with other contributions.

For the special case $G \equiv \omega - k^2$, the full modulation equations (2.6) are

$$k_t + 2kk_x = \left(\frac{A_{xx}}{A}\right)_x \ ,$$

$$(A^2)_t + (2kA^2)_x = 0 \ .$$

In this case (2.14) are exact .

As mentioned, the modulation equations are valid no matter which form of $G(\omega,k)$ is chosen in (2.9) for a particular mode. However, in the study of nonlinear problems in Chapter V, it is not clear a priori that we have this choice. So it is of value to see how (2.14) is obtained from (2.10) using a general G. Within the approximation (2.12), E_2 is

$$\frac{1}{2!}\left[G_{\omega\omega}\Big|_o A_{TT} + G_{kk}\Big|_o A_{XX} - G_{\omega k}\Big|_o A_{XT}\right] + O(\epsilon) .$$

From (2.14b) we can make the approximation

$$A_T = -C(k_o)A_X + O(\epsilon) \quad ,$$

so that

$$E_2 = \frac{1}{2!}(G_{kk} + G_{\omega\omega}C^2 + G_{\omega k}C)\Big|_o A_{XX} + O(\epsilon) , \qquad (2.15a)$$

or

$$E_2 = \frac{-1}{2!} G_\omega\Big|_o W''(k_o) A_{XX} + O(\epsilon) \quad . \qquad (2.15b)$$

The general form of the corrected dispersion relation is

$$G(\omega,k)A + \tfrac{1}{2}\epsilon^2 G_\omega\Big|_o W''(k_o)A_{XX} + O(\epsilon^3) = 0 \quad .$$

When this is solved for ω and combined with the compatability relation we arrive at (2.14a). Using this ω in the amplitude equation

$$(G_\omega A^2)_T - (G_k A^2)_X + O(\epsilon^2) = 0$$

we arrive at (2.14b) since

$$\frac{G_k}{G_\omega} = -C(k) + O(\epsilon^2)$$

First order theory

If the ϵ^2 term in (2.14) is neglected, the modulation equations are

$$k_T + C(k)k_X = 0 \quad , \tag{2.16a}$$

$$(A^2)_T + (C(k)A^2)_X = 0 \quad , \tag{2.16b}$$

These equations can be solved in general, and their solution will be referred to as the first order modulation theory. Note that at this level of approximation the wave number can be computed from (2.16a) independently of the amplitude.

We solve (2.16) for the initial conditions

$$k(X,0) = k_0(X), \quad A(X,0) = A_0(X) \ .$$

From (2.16a), k is constant on the lines

$$\Gamma(\xi): \qquad X = C(k)T + \xi \quad ,$$

for any constant ξ.

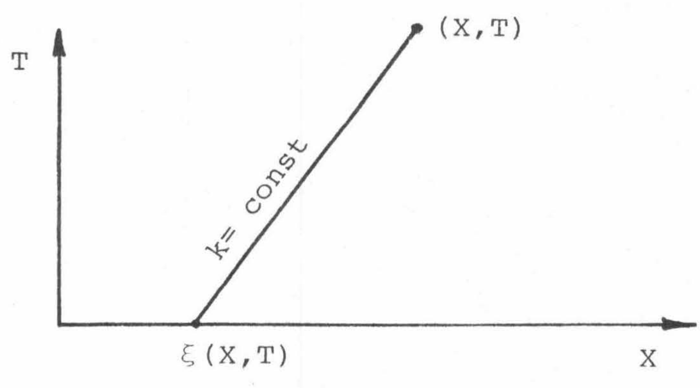


Fig. 2.1 Group Lines

Since the line $\Gamma(\xi)$ intersects the x axis at the point $X = \xi$, this constant must be $k_o(\xi)$ (see Fig. 2.1); thus we have

$$\Gamma(\xi): \qquad X = C(k_o(\xi))T + \xi \,. \tag{2.17}$$

These lines are the characteristics of equation (2.16a), and are called <u>group lines</u> since their slope is the group velocity. We see that different wave numbers propagate with the corresponding group velocity. If (2.17) can be solved uniquely for $\xi = \xi(X, T)$ the solution for k can be written in terms of the initial data as

$$k(X, T) = k_o(\xi(X, T)) \;. \tag{2.18}$$

With the solution for k known, the amplitude can be computed from (2.16b). We express this equation in terms of time derivatives along the group lines, i.e.

$$\frac{dA}{dT} + \tfrac{1}{2} W''(k)\, k_X\, A = 0 \;, \tag{2.19}$$

where $$\frac{d}{dT} \equiv \partial_T + C(k_o(\xi))\,\partial_X \;.$$

The factor $W''(k)k_X$ is now known and is computed from (2.17) and (2.18). We have

$$k_X = k_o'(\xi)\xi_X \;, \qquad \xi_X = \frac{1}{1+\rho(\xi)T} \;,$$

where ρ is the X-derivative of the group velocity for the initial conditions, i.e.,

$$\rho(X) = W''(k_o(X))k_o'(X) = \frac{d}{dX} C(k_o(X)) \;.$$

With this expression, the equation for A along the group lines is

$$\frac{dA}{dT} + \frac{1}{2} A \frac{\rho(\xi)}{1+\rho(\xi)T} = 0 \; .$$

Since $\rho(\xi)$ is constant on $\Gamma(\xi)$, it can be integrated at once to give

$$A(X,T) = \frac{A_0(\xi)}{(1+\rho(\xi)T)^{\frac{1}{2}}} , \qquad \xi = \xi(X,T) , \tag{2.20a}$$

which completes the solution of (2.16).

The "total amount of A^2" between any two group lines, say $\Gamma(\xi_1)$ and $\Gamma(\xi_2)$, can be defined as

$$Q(T) = \int_{X_1(T)}^{X_2(T)} A^2(X,T)dX = \int_{X_1(T)}^{X_2(T)} \frac{A_0^2(\xi)}{1+\rho(\xi)T} dX \quad ,$$

where $X_1(T)$ and $X_2(T)$ are the X-coordinates of $\Gamma(\xi_1)$ and $\Gamma(\xi_2)$ at time T. Since $\xi_X = (1+\rho T)^{-1}$, the expression for Q(T) can be converted to an integral with respect to ξ,

$$Q(T) = \int_{\xi_1}^{\xi_2} A_0^2(\xi)d\xi = Q(0) ,$$

which is constant in time. From this we see another role of the group velocity: the total A^2 between group lines remains constant, and in this sense A^2 is propagated with speed $C(k)$.

The X-derivatives of A will be needed later, so they are calculated here. For simplicity we choose the initial amplitude to be constant and take it to be unity;

$$A_0(\xi) = 1 \quad .$$

In (2.20a) A is expressed as a function of the independent variables

ξ and T, so $A_X = A_\xi \, \xi_X$ where

$$A_\xi = -\tfrac{1}{2} A^3 \rho' T, \quad \xi_X = A^2 \quad .$$

Then we can compute

$$\begin{aligned} A_X &= -\tfrac{1}{2} \rho' T A^5 \\ A_{XX} &= -\tfrac{1}{2} \rho'' T A^7 + 5(\tfrac{1}{2} \rho' T)^2 A^9 , \qquad (2.20b) \\ A_{XXX} &= -\tfrac{1}{2} \rho''' T A^9 + O(T^2) , \end{aligned}$$

etc.

Breaking modulations

According to first order theory, the values of k propagate with the group velocity C(k). If the initial modulation is such that C(k) decreases with X in some region, that is $\rho < 0$, then some of the group lines for different values of k will eventually cross, causing singularities and multivalued solutions. The appearance of singularities is clear from the formulas for A and k_X. As $T \to \frac{-1}{\rho(\xi)}$ both of these quantities become unbounded. After the singularity appears, the group lines cross leaving an overlap region where each point has more than one predicted value. This phenomenon is called breaking. Breaking solutions occur whenever $\rho < 0$, and the time at which a singularity first appears is given by

$$t_B = \operatorname*{Min}_x \left| \frac{1}{\rho(x)} \right| \quad .$$

The qualitative behavior of typical breaking modulations is sketched in Fig. 2.2.

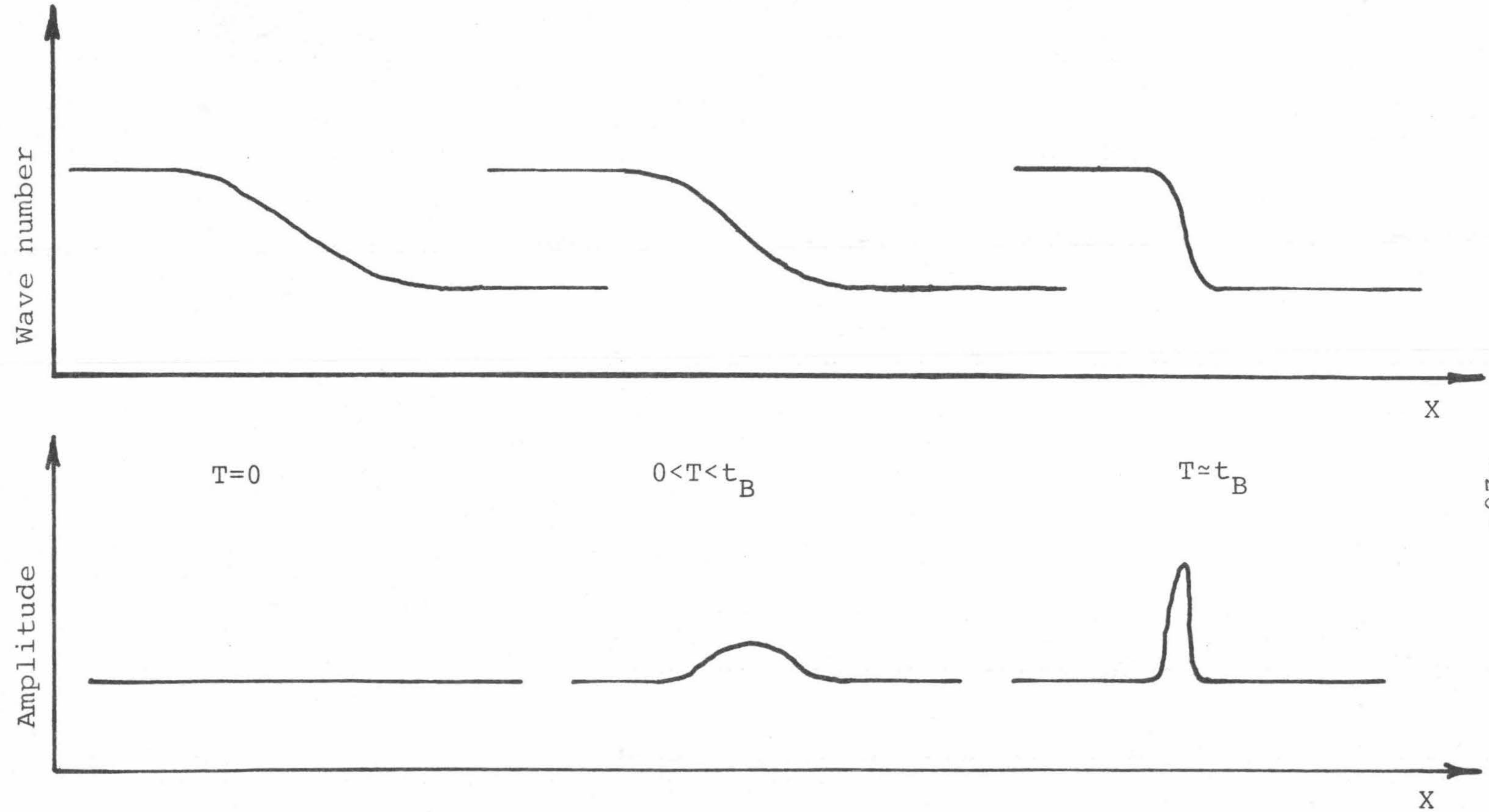


Fig. 2.2 . Typical breaking modulations predicted by the first order linear theory.

As breaking is approached, the assumption that the modulations are slow is certainly not valid. However, there is the possibility that the first order breaking effect dominates in the full theory so that singularities and regions of overlap actually develop, provided multivalued solutions can be given a meaning. Since the original problem is linear, one could interpret such solutions as the superposition of several wavetrains as in (2.8a).

Another possibility is that as breaking is approached, higher order effects become dominant and prevent the development of singularities. In this case there is the question of some kind of shock structure being the end result.

The existence of shocks in the modulations is an intriguing speculation, but doubtful for linear problems since such behavior is always associated with nonlinear effects in the original equations. In fact, it is shown in Chapter III that an overlap region does occur asymptotically for large times ($T \gg t_B$). This precludes the existence of a shock structure, and leads one to believe that development of singularities followed by overlap is the end result. This is not the case either, as we will see in the analysis of some exact solutions in Chapters III and IV.

Higher order effects

In the first order theory (2.16), the wave number is independent of the amplitude, but in the next approximation (2.14), there is an amplitude coupling term $\left(\frac{A_{XX}}{A}\right)_X$. This term becomes large as breaking is approached, so that it should have an important effect on the actual behavior.

To get a first impression of this effect we proceed by successive approximation. The first order solution for the amplitude given by (2.20a), call it $A^{(0)}$, can be used on the R.H.S. of (2.14a) since the error introduced in A will be $O(\epsilon^2)$,

$$A = A^{(0)} + O(\epsilon^2) .$$

The error is also proportional to T, so this is not valid for large times, though it does give the first tendencies. Using the derivatives (2.20b), the coupling term becomes

$$\left(\frac{A_{XX}}{A}\right)_X = -\tfrac{1}{2}\rho'''(\xi)T + O(T^2) .$$

For small times, neglecting $O(T^2)$, the wave number equation (2.14a) becomes

$$\frac{\partial \mathbf{k}}{\partial T} + W'(k)\frac{\partial k}{\partial X} = -\tfrac{1}{4}\epsilon^2 W''(k_0)\rho'''(\xi)T .$$

The characteristic curves for this equation, given by

$$\frac{dX}{dT} = W'(k) ,$$

are no longer straight lines since k is not constant on them. But for small T, they can be approximated by the straight group lines within the retained accuracy. Hence, ξ can be treated as a constant on the characteristics, and the equation can be integrated immediately;

$$k = k^{(0)} - \frac{1}{8}\epsilon^2 W''(k_0)\rho'''(\xi)T^2 \qquad (2.21)$$

on the group lines

$$X = W'(k(\xi, 0))T + \xi ,$$

where $k^{(0)}$ is the first order solution (2.18).

The solution in (2.21) gives an idea of the kind of error one can expect from using the first order modulation theory. In the case of breaking, the higher order effects tend to prevent the development of singularities. In Fig. 2.3 we have sketched a typical breaking profile for $W'' > 0$, and the corresponding ρ and ρ''' curves. The change in sign of ρ''' about the center of the disturbance causes the slope to flatten there, giving a reverse breaking effect, which grows initially as T^2.

In the next two chapters exact solutions are found by Fourier transforms and other superposition techniques to further evaluate the modulation theory and questions of breaking.

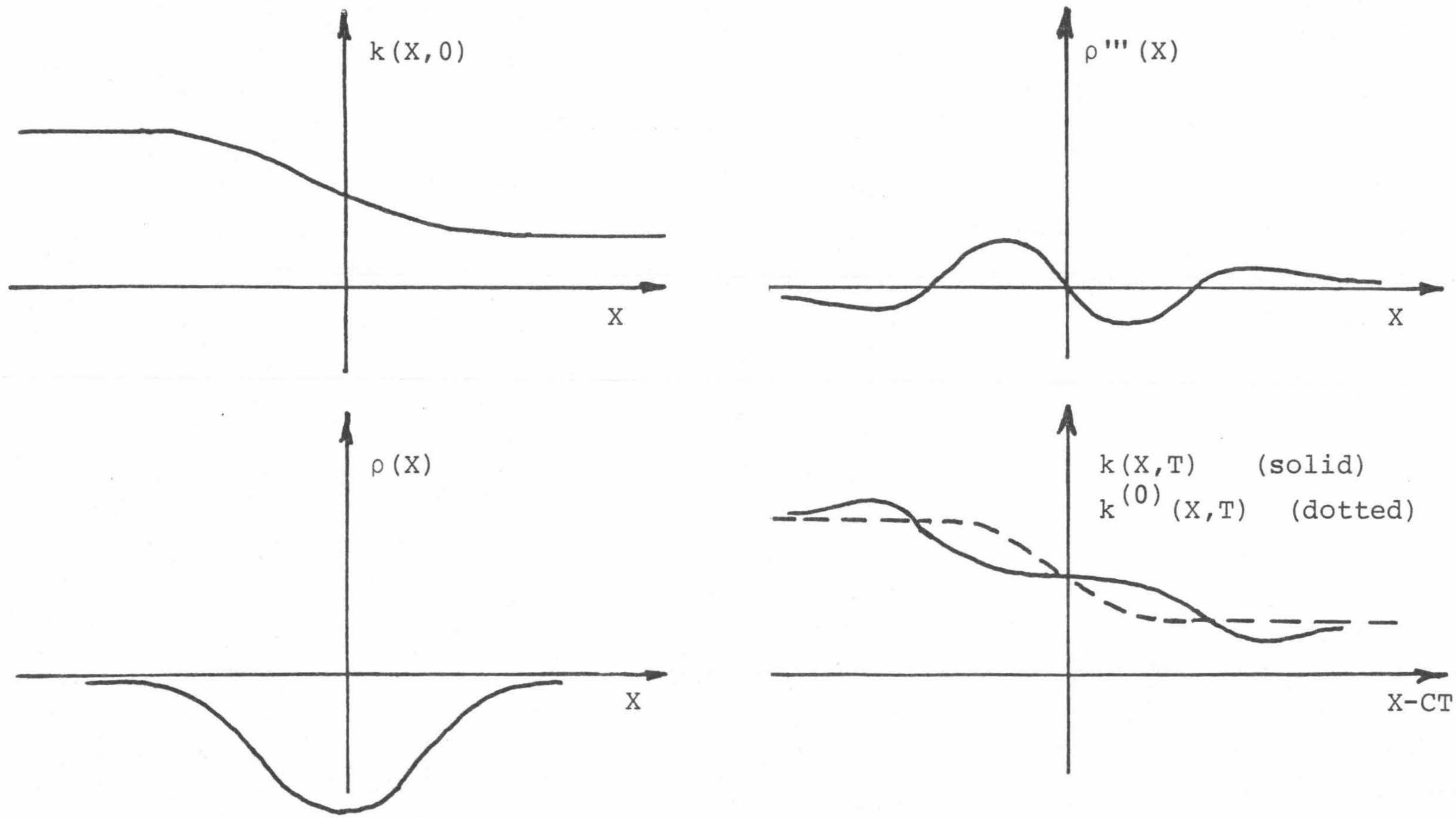


Fig. 2.3 . Higher order effects on breaking.

CHAPTER III. BREAKING MODULATIONS IN LINEAR PROBLEMS

For linear problems in uniform media we can obtain exact solutions in terms of Fourier integrals. In this chapter we use these exact solutions to study the true behavior when the first modulation theory predicts breaking.

Let us consider a solution $\varphi(x, t)$ which has the initial shape

$$\varphi(x, 0) = \begin{cases} e^{i k_1 x} & , \quad x < -L \\ e^{i(k_1 x - \frac{1}{2} R x^2)} & , \quad -L < x < L \\ e^{i k_2 x} & , \quad x > L \end{cases}$$

where $R = \dfrac{k_1 - k_2}{2L}$, and $C(k_1) > C(k_2)$. This represents a modulated wave with wave number

$$k(x, 0) = \begin{cases} k_1 , & x < -L \\ k_1 - Rx , & -L < x < L \\ k_2 , & x > L \end{cases} ,$$

and constant amplitude

$$A(x, 0) = 1 \quad .$$

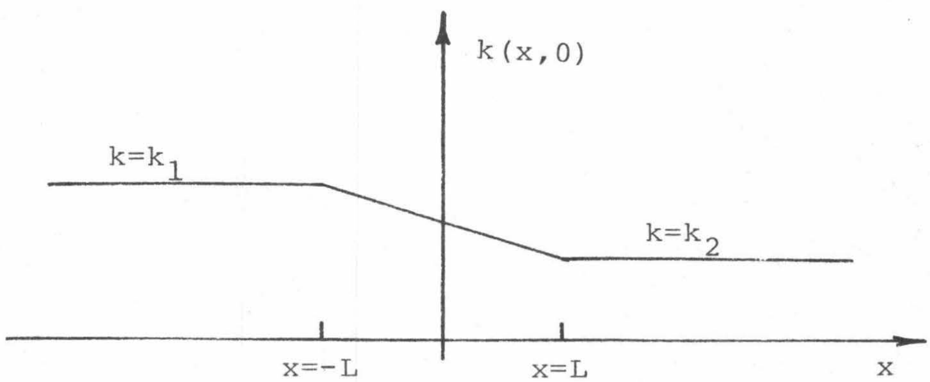


Fig. 3.1 Initial Modulations

The group velocity C(k) is decreasing toward the front of the disturbance. According to the first order modulation theory the group lines cross and breaking occurs. We now consider the exact solution.

Exact solution: $\omega = W(k)$

The initial conditions naturally separate into three regions, each of which can be analyzed separately and the results superposed to give the complete solution. We will compute three solutions $\varphi_1, \varphi_2, \varphi_3$ corresponding to the initial conditions

$$\varphi_1(x, 0) = \begin{cases} e^{ik_1 x} , & x < 0 \\ 0 , & x > 0 \end{cases} ,$$

$$\varphi_2(x, 0) = \begin{cases} 0 , & x < 0 \\ e^{ik_2 x} , & x > 0 \end{cases} ,$$

$$\varphi_3(x, 0) = \begin{cases} e^{i(k_1 x - \frac{1}{2}Rx^2)} . & -L < x < L \\ 0 , & \text{otherwise} \end{cases} .$$

The solution to the problem of interest is then

$$\varphi(x, t) = e^{i\psi_1} \varphi_1(x+L, t) + e^{i\psi_2} \varphi_2(x-L, t) + \varphi_3(x, t) ,$$

where ψ_1 and ψ_2 are chosen so that the three functions agree in phase at their junctions, i. e.

$$\psi_1 = -k_1 L - \tfrac{1}{4}(k_1 - k_2) L ,$$

$$\psi_2 = k_1 L - \tfrac{1}{4}(k_1 - k_2) L .$$

For a single mode with the dispersion relation

$$\omega = W(k) ,$$

a solution can be written

$$\varphi(x, t) = \int_{-\infty}^{\infty} F(k) e^{ikx - iW(k)t} \, dk \quad ,$$

where

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x, 0) e^{-ikx} \, dx \quad .$$

These formulas can be applied directly to find solution φ_3 ; we have

$$\varphi_3(x, t) = \int_{-L}^{L} e^{i(k_1 \eta - \frac{1}{2} R\eta^3)} \Phi(x - \eta, t) d\eta \quad , \tag{3.1a}$$

with

$$\Phi(\xi, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\xi - iW(k)t} \, dk \quad . \tag{3.1b}$$

Since $\varphi_1(x, 0)$ and $\varphi_2(x, 0)$ are not integrable on $(-\infty, \infty)$, a generalized Fourier representation must be used. These solutions are

$$\varphi_1(x, t) = \frac{1}{2\pi i} \int \frac{1}{k - k_1} e^{ikx - iW(k)t} \, dk \, , \tag{3.2}$$

$$\varphi_2(x, t) = \frac{1}{2\pi i} \int \frac{1}{k - k_2} e^{ikx - iW(k)t} \, dk \quad , \tag{3.3}$$

where the contours of integration are above and below the simple poles at k_1 and k_2, respectively. The initial conditions can be checked by evaluating the integrals at $t = 0$. For $x > 0$, the contours can be closed above to give

$$\varphi_1 = 0 \, , \qquad \varphi_2 = e^{ik_2 x} \quad .$$

For $x < 0$, they can be closed below to give

$$\varphi_I = e^{ik_1 x} \, , \qquad \varphi_2 = 0 \quad .$$

<u>Asymptotics-Overlap of elementary waves</u>

On the straight lines

$$x = Vt \tag{3.4}$$

the solution can be expressed as a function of **t** alone. Then, for large t, the integrals (3.1), (3.2), (3.3) can be evaluated by the usual methods of asymptotic analysis.

For integrals (3.2) and (3.3) the "saddle point method" is applicable. On the lines (3.4) integral (3.2) becomes

$$\varphi_1(t) = \frac{1}{2\pi i}\int_{C_1} \frac{1}{k-k_1} e^{i(kV-W(k))t}\, dk \ .$$

As $t \to \infty$, the major contribution comes from the stationary point (or saddle point) of the function

$$kV - W(k) \ .$$

Thus, the saddle point $k = k_s$, is determined from

$$\frac{dW}{dk}(k_s) \equiv C(k_s) = V \quad .$$

Line (3.4) can be interpreted as the group line corresponding to the wave number k_s. The path of steepest descent near $k = k_s$ is shown in Fig. 3.2. Integration along this path for $k_s \neq k_1$ gives terms of order $t^{-\frac{1}{2}}$ as $t \to \infty$. When the contour of integral (3.2) is deformed into the steepest descent path, there will be a contribution from the pole $k = k_1$ for $k_s < k_1$, and no contribution for $k_s > k_1$. Thus, provided $k_s \neq k_1$, we have as $t \to \infty$

$$\varphi_1 = \begin{Bmatrix} O(t^{-\frac{1}{2}}) & , & k_s > k_1 \\ e^{i\theta_1} + O(t^{-\frac{1}{2}}) , & & k_s < k_1 \end{Bmatrix} \ .$$

Similarly for φ_2 we have

$$\varphi_2 = \begin{Bmatrix} e^{i\theta_2} + O(t^{-\frac{1}{2}}) , & & k_s > k_2 \\ O(t^{-\frac{1}{2}}) & , & k_s < k_2 \end{Bmatrix} \ ,$$

where $\theta_j = k_j x - W(k_j)t$, $j = 1, 2$.

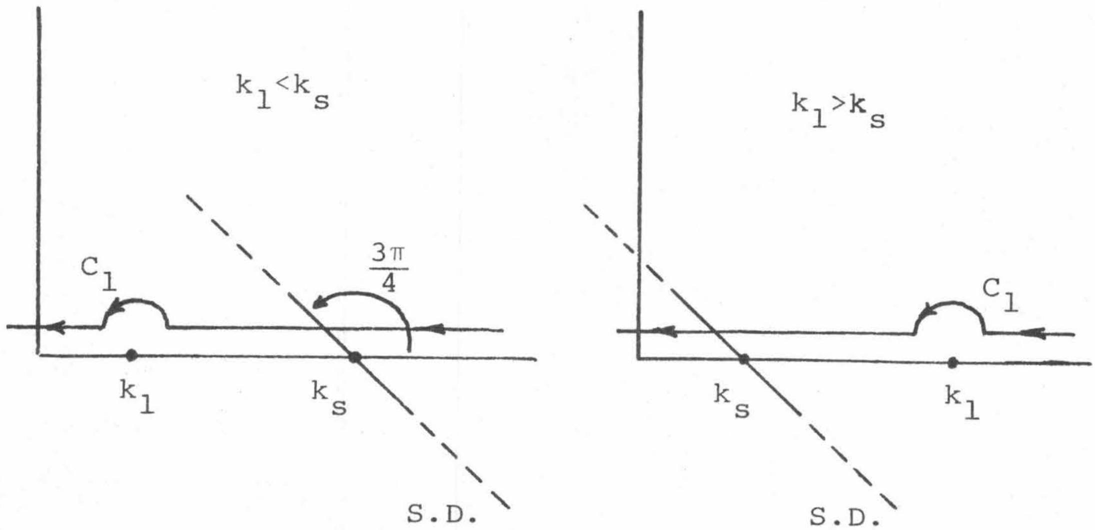


Fig 3.2 Steepest Descent Path Near Saddle Point for $\frac{dC}{dk} > 0$

Integral (3.1) can be found by a straightforward application of the "method of stationary phase" and is of the order $t^{-\frac{1}{2}}$ as t becomes large.

Thus, apart from the regions corresponding to $k_s = k_1$ or $k_s = k_2$, the asymptotic solution is the superposition of two semi-infinite uniform wavetrains moving with speeds $C(k_1)$ and $C(k_2)$. Since $C(k_1) > C(k_2)$, there is a region where the two wavetrains overlap (Fig. 3.3).

In terms of a single modulated wavetrain (Fig. 3.4), the region where there is just one elementary wave corresponds to constant wave number and amplitude. In the overlap region the sum of the two elementary waves produces a beating effect; the amplitude is harmonic

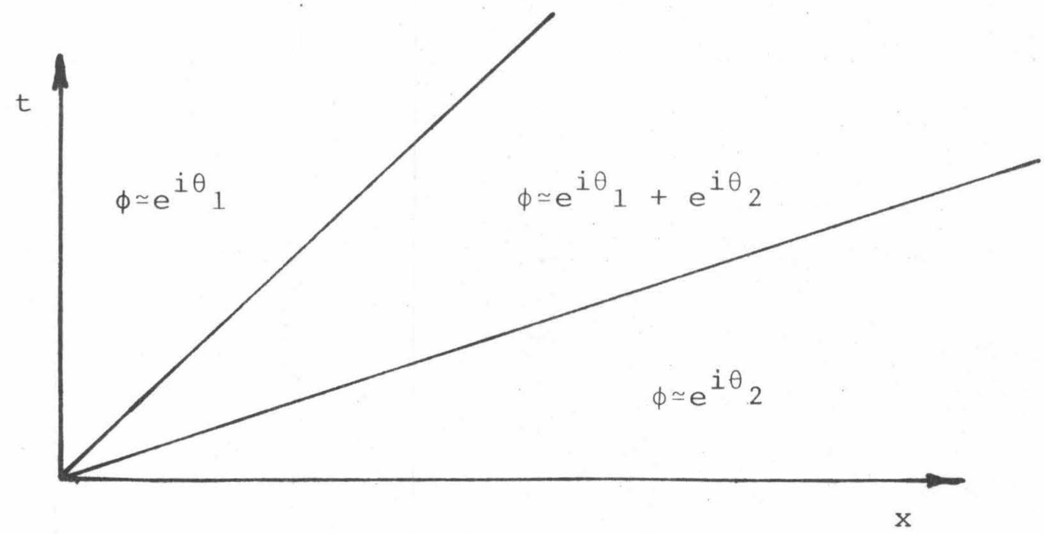


Fig. 3.3 Asymptotic solution($t \to \infty$).

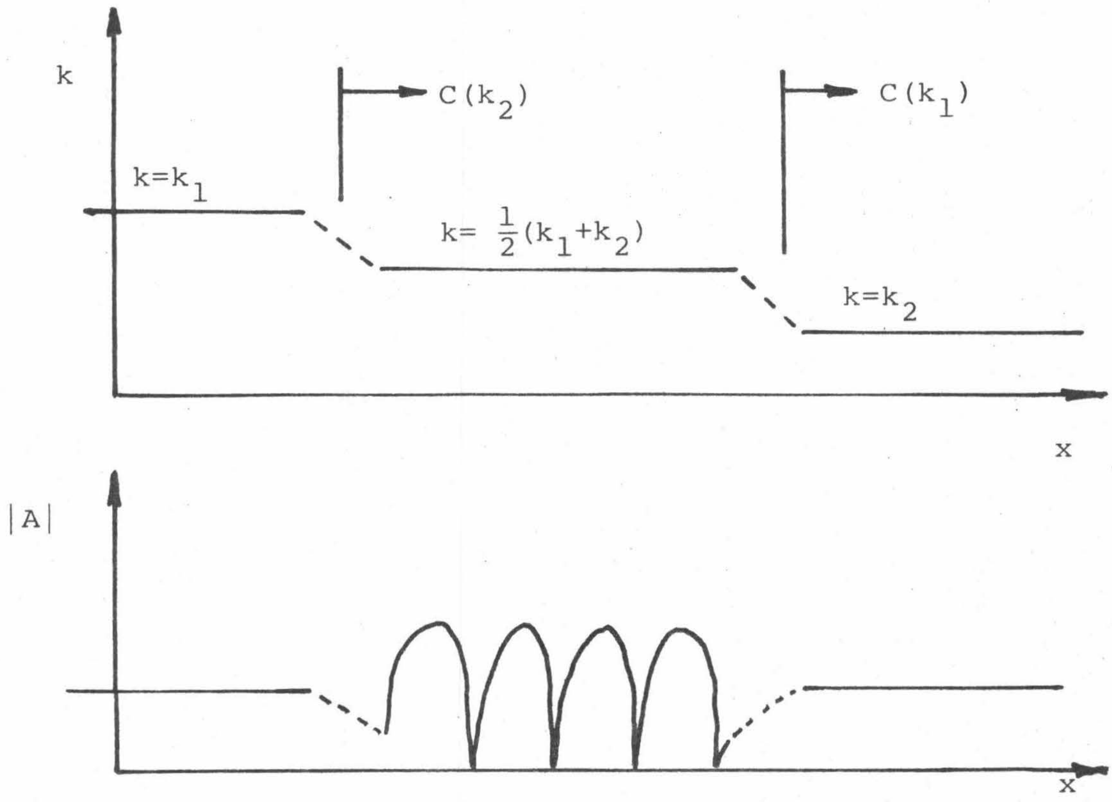


Fig. 3.4 Asymptotic modulations($t \to \infty$).

$$A = 2\cos\left[\frac{k_1 - k_2}{2}(x - Vt)\right] \tag{3.5}$$

with

$$V = \frac{W(k_1) - W(k_2)}{k_1 - k_2},$$

and the wave number and frequency are constant,

$$k = \tfrac{1}{2}(k_1 + k_2),$$

$$\omega = \tfrac{1}{2}[W(k_1) + W(k_2)].$$

It is interesting that this behavior for large time is compatible with the first order modulation theory. If the breaking solutions are interpreted as an overlap of two wavetrains then the first order solutions are valid as $t \to \infty$. One might speculate that the first order theory gives a fair picture of the solution for all times. We investigate this by looking at a special case.

Special case: $\omega = k^2$

As noted in Chapter II, the second order theory (2.14) is exact for the special case $\omega = k^2$. So the exact solution in this case also gives the typical behavior of the second order theory for a general dispersion relation. For $\omega = k^2$ the solution just obtained can be evaluated in terms of Fresnel integrals. For φ_1 we have

$$\varphi_1 = \frac{1}{2\pi i}\int \frac{1}{k - k_1} e^{ikx - ik^2 t}\, dk.$$

With the new variable of integration

$$\kappa = k - k_1,$$

this becomes

$$\varphi_1 = e^{ik_1 x - ik_1^2 t} F_1(x, t) ,$$

where

$$F_1 = \frac{1}{2\pi i} \int \frac{1}{\kappa} e^{i\kappa(x-2k_1 t) - i\kappa^2 t} d\kappa .$$

Making the further change of variables $\mu = \kappa t^{\frac{1}{2}}$ and evaluating the contribution from the semi-circle, we get

$$F_1 = \tfrac{1}{2} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\mu} e^{-i\mu^2} e^{i \frac{x-2k_1 t}{\sqrt{t}} \mu} d\mu .$$

Now, it is natural to take $\frac{x-2k_1 t}{\sqrt{t}}$ as a new independent variable. After more manipulation we find the solutions φ_1 and φ_2 in the form

$$\varphi_\alpha = e^{ik_\alpha x - ik_\alpha^2 t} F_\alpha(s_\alpha) , \qquad \alpha = 1, 2,$$

where
$$s_\alpha = \frac{x-2k_\alpha t}{\sqrt{2\pi t}} , \qquad t > 0 ,$$

$$F_\alpha(s) = \tfrac{1}{2} + \frac{(-1)^\alpha}{\sqrt{2}} e^{-i\frac{\pi}{4}} \mathcal{J}(s) ,$$

and $\mathcal{J}$ is the complex Fresnel integral function

$$\mathcal{J}(s) = \int_0^s e^{i\frac{\pi}{2} z^2} dz .$$

The function φ_3 can be standardized directly by completing the square in the exponential. For $t > 0$, we have

$$\varphi_3 = e^{i\theta_3} F_3 , \qquad \theta_3 = \frac{k_1 x - \frac{1}{2} R x^2 - k_1^2 t}{1 - 2Rt} ,$$

$$F_3 = e^{-i\frac{\pi}{4}} (2|1-2Rt|)^{-\frac{1}{2}} \int_{B(-L)}^{B(L)} \exp(i\tfrac{\pi}{2}\,\mathrm{sgn}(1-2Rt)u^2)du ,$$

$$B(L) = \left(\frac{|1-2Rt|}{2\pi t}\right)^{\frac{1}{2}} \left(L - \frac{x-2k_1 t}{1-2Rt}\right) .$$

Again, φ_3 is expressed in terms of $\mathcal{J}$ or its conjugate, depending upon the sign of $(1-2Rt)$.

This solution for $\omega = k^2$ is evaluated using the "τ-method of Lanczos for Fresnel integrals,"[6] which gives 5-6 significant figure accuracy. The modulations are calculated from the exact solution by measuring the peak values and the distance between zeroes. This gives an average value of amplitude and wave number which is a good approximation to the local values since the solution shows that the modulations do not change drastically over one wave length.

The first order modulations, say $A^{(0)}$ and $k^{(0)}$, can also be found explicitly for this case, giving a comparison between the exact solution and the first approximation. The time of breaking is

$$t_B = \frac{1}{2R} .$$

Outside the disturbed regions, i.e.

$$\text{I:} \qquad x < 2k_1 t - L , \qquad t < t_B ,$$

$$\text{II:} \qquad x > 2k_2 t + L , \qquad t < t_B ,$$

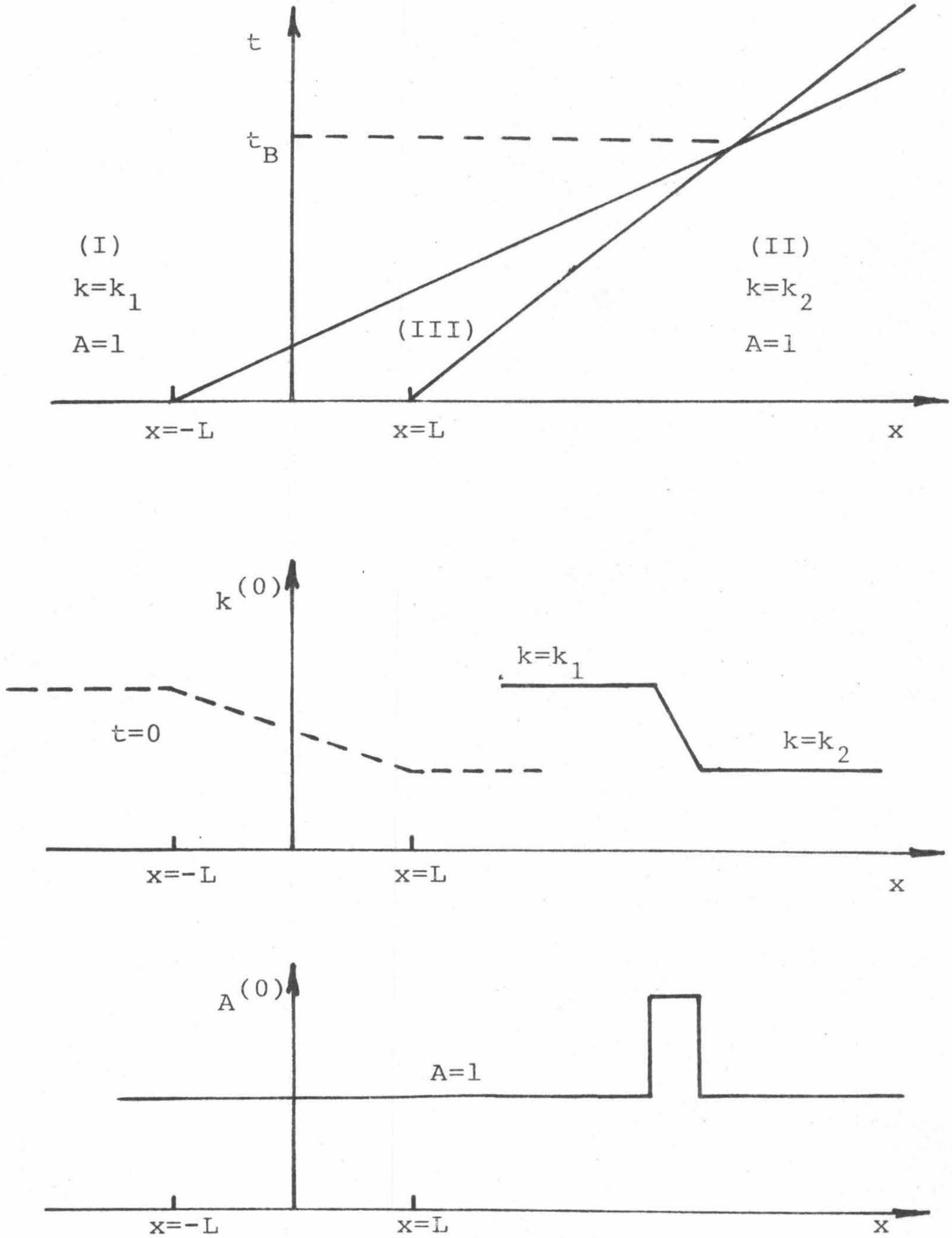


Fig. 3.5 Solution of the first order modulation equations for the case $\omega=k^2$.

the values of $A^{(o)}$ and $k^{(o)}$ are constant (see Fig. 3.5). In the range of influence of the disturbance

$$\text{III:} \qquad 2k_1 t - L < x < 2k_2 t + L, \quad t < t_B \quad ,$$

we have

$$k^{(o)} = k_1 + \tfrac{1}{2} \frac{x - 2k_1 t + L}{t - t_B} \quad ,$$

$$A^{(o)} = \left(1 - \frac{t}{t_B}\right)^{-\frac{1}{2}} \quad .$$

Results for the case

$$k_1 = 2.3 \, , \qquad k_2 = 2.0, \qquad R = .025 \, ,$$

are shown in Fig. 3.6. The first order modulations are plotted along with the calculated solution. The time of breaking is $t_B = 20$. For small times (e.g. $t/t_B = .3$) the behavior agrees well with the first order theory. As the solution progresses further (e.g. $t/t_B = .77$), the higher order effects shown in Fig. 2.3 become apparent; the breaking of k at the center has stopped, and oscillations are beginning to appear at the front and back. As t/t_B increases past unity the slope of k flattens at the center until a region of constant k is left.

In the last picture ($t/t_B = 5.13$), the asymptotic form of the solution is beginning to develop. There is an interval of approximately 50 units in length for which k is constant, aside from the two spikes which appear at the near zeros of the amplitude. This constant value is $\frac{1}{2}(k_1 + k_2)$. The amplitude is beginning to show the beating effect in (3.5). The appearance of the spikes in k is a spurious result of the algorithm used to calculate the modulations. The wave number is computed from the distance between zeros of the solution. At points where the amplitude vanishes there is an extra zero; this gives a high value for the computed wave number near these points.

Figs. 3.6.1 - 3.6.6

Modulations from the exact solution (solid)

and from first order theory (dotted)

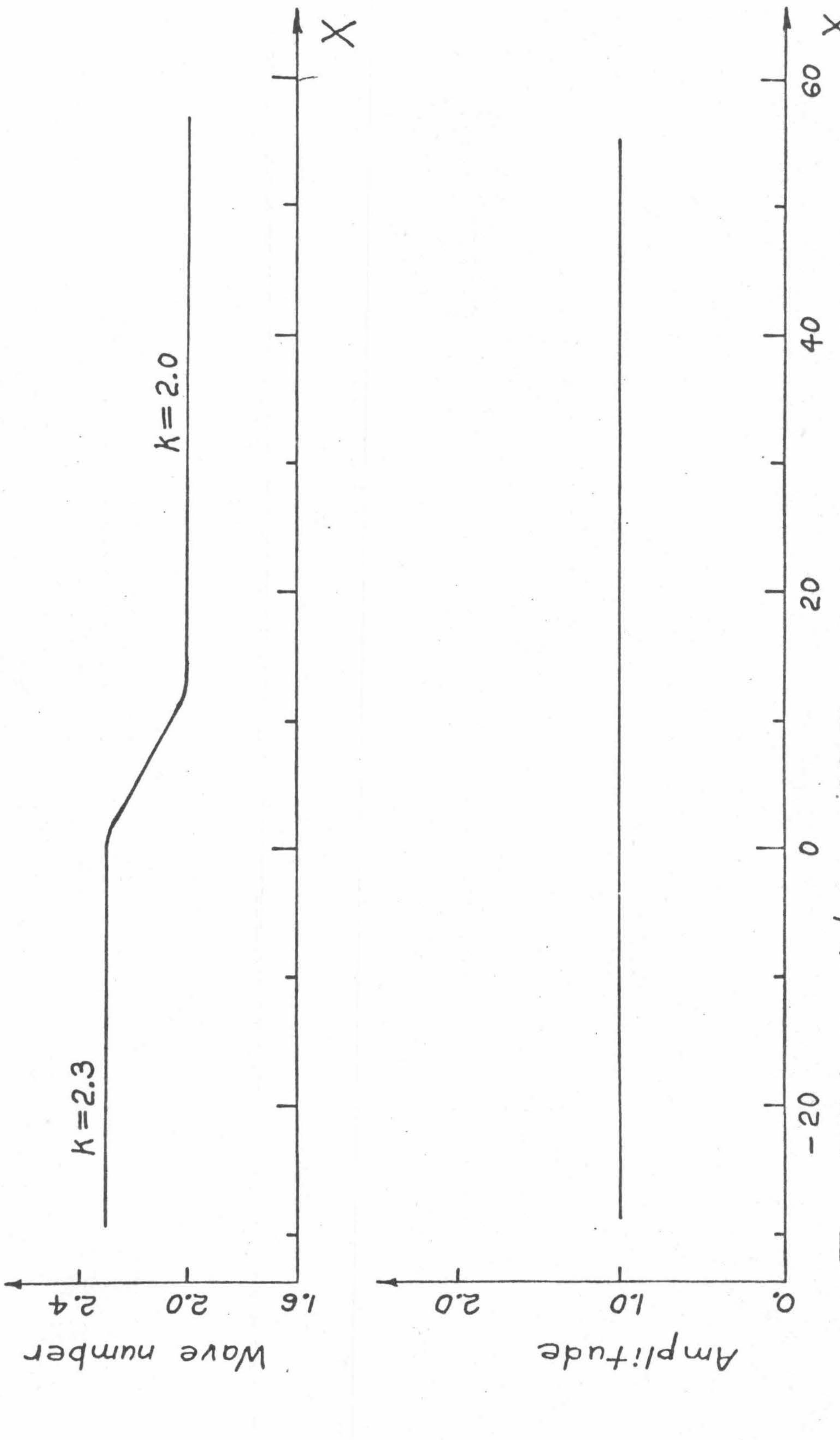


Fig. 3.6.1 $t/t_B = .005$

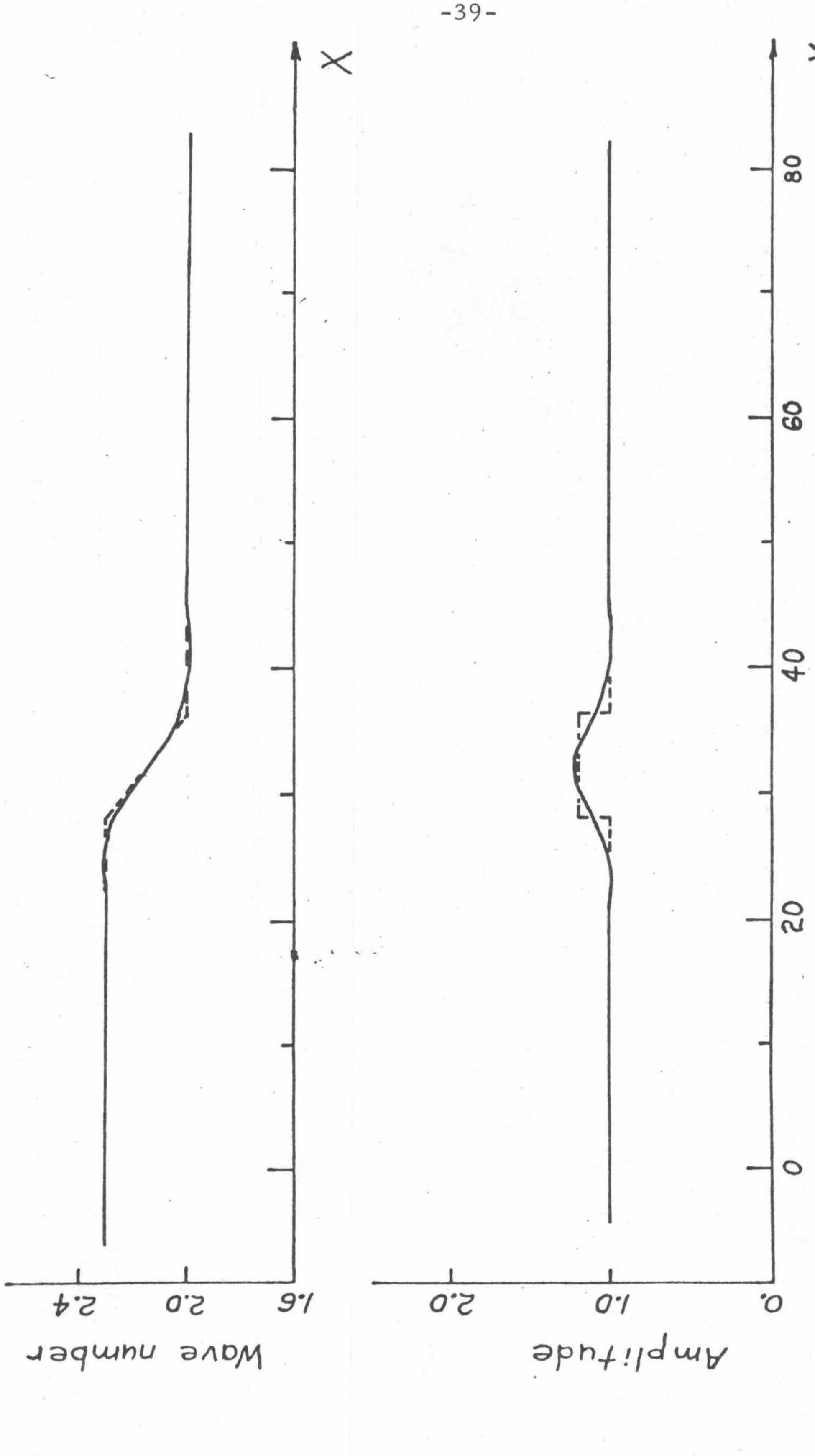


Fig. 3.6.2 $t/t_B = .308$

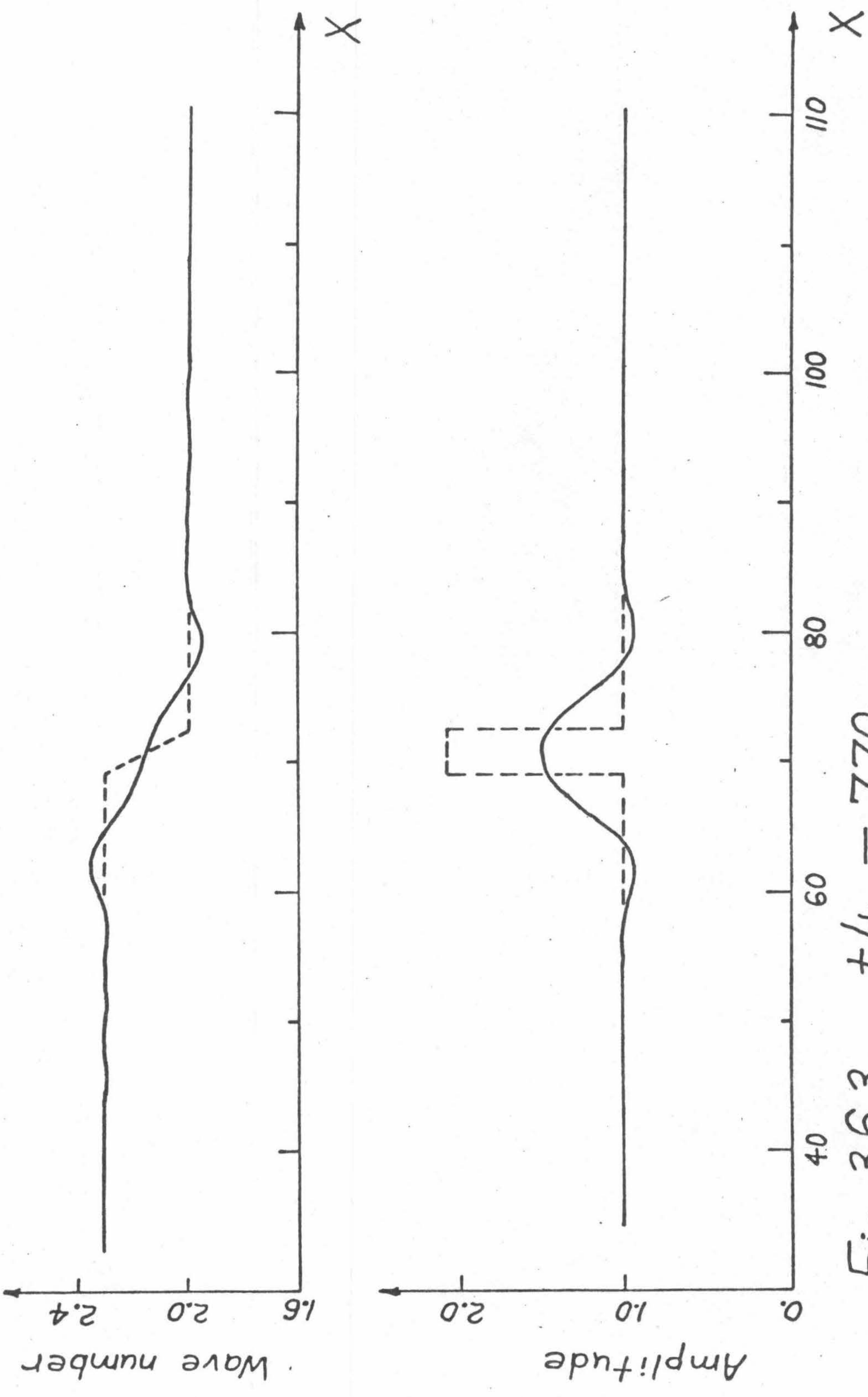


Fig. 3.6.3 $t/t_B = .770$

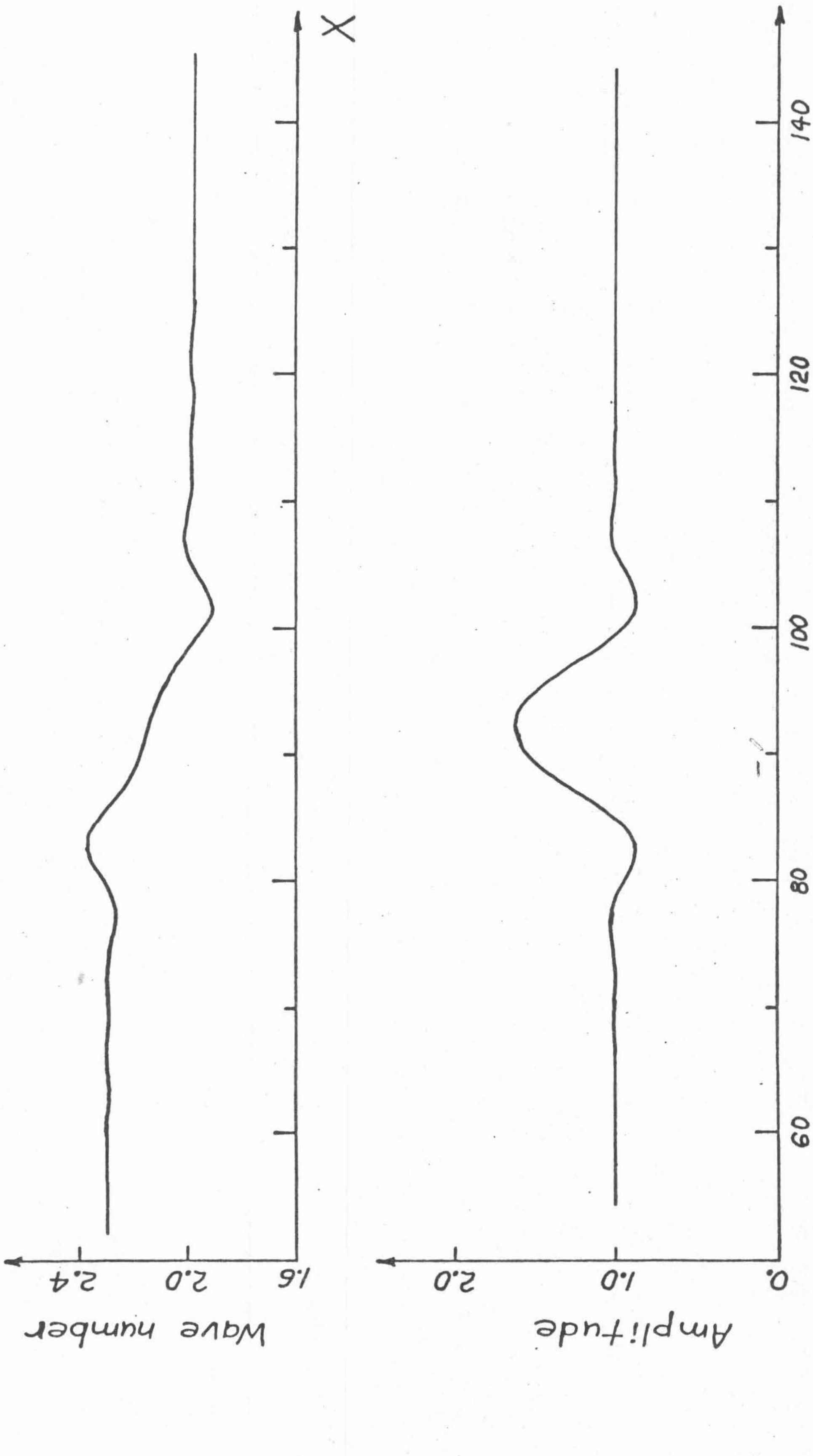


Fig. 3.6.4 $t/t_B = 1.025$

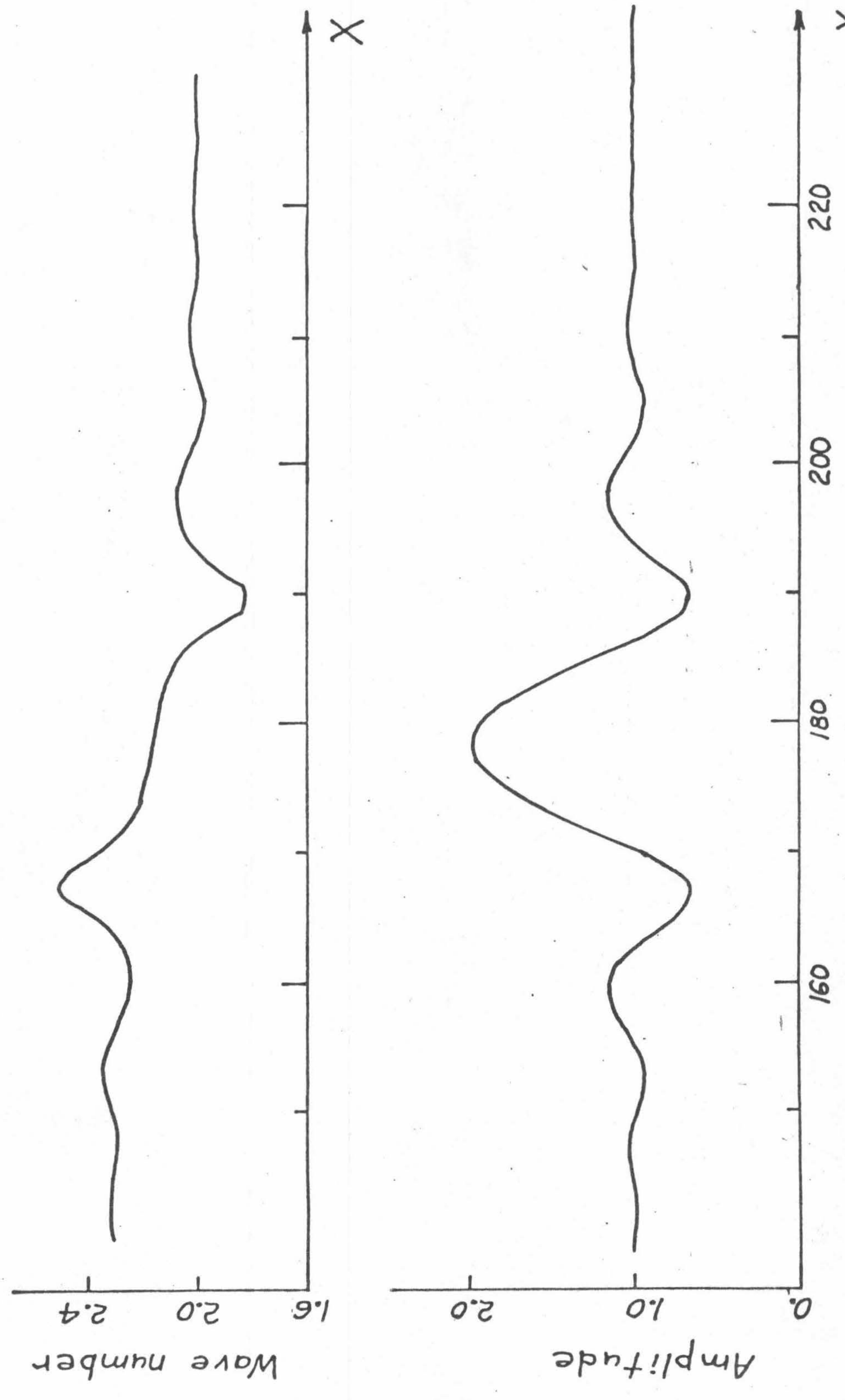


Fig. 3.6.5 $t/t_B = 2.050$

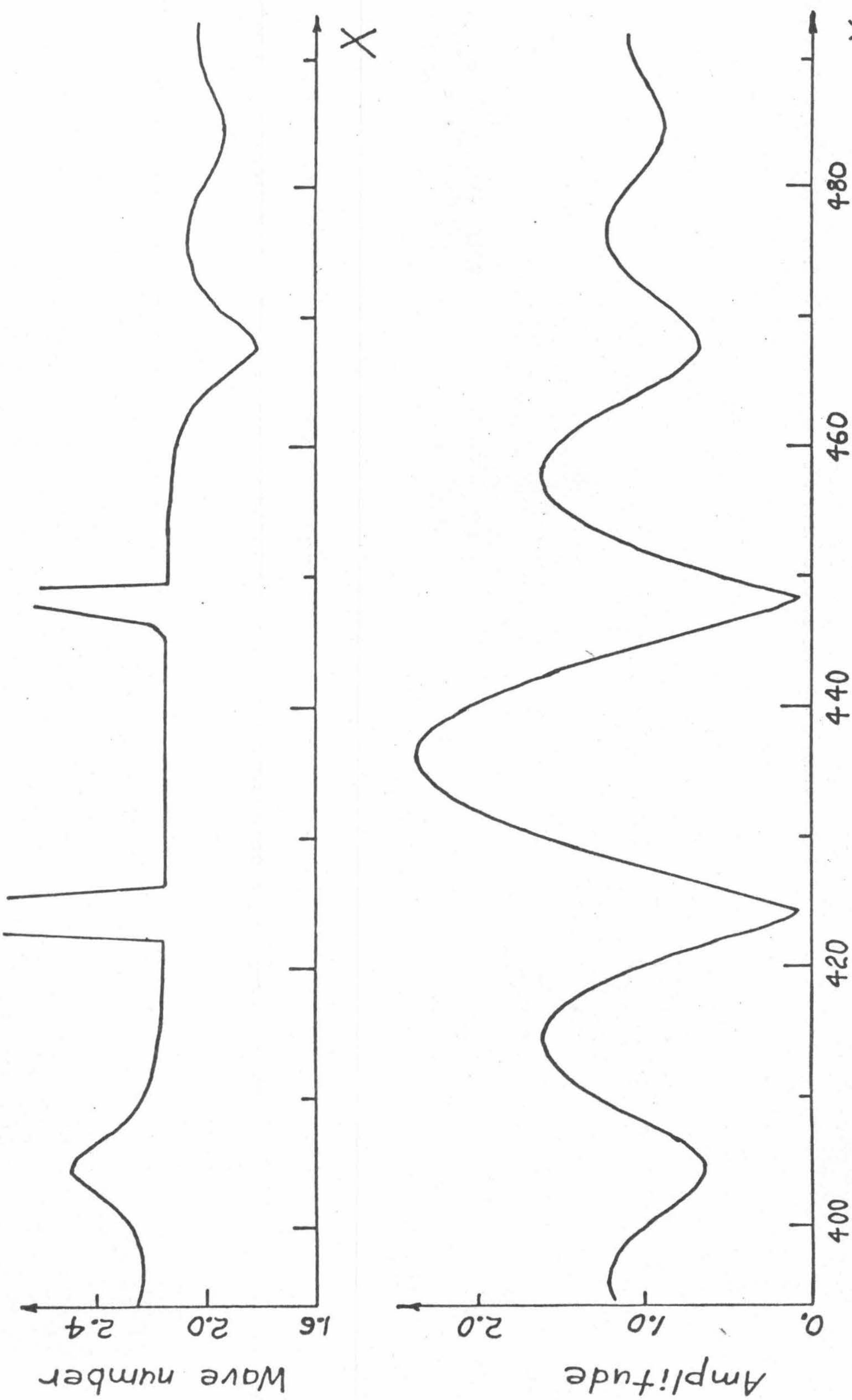


Fig. 3.6.6 $t/t_B = 5.130$

Conclusions

In the first approximation for slow modulations, the wave number is governed by a nonlinear equation which is independent of the amplitude. Higher order effects enter in a complicated way, but if the modulations are also small, i.e. (2.12), the only significant contribution in the next approximation is an amplitude coupling term $\left(\frac{A_{xx}}{A}\right)_x$ in the wave number equation. As long as this term remains negligible, the modulations agree well with the first order theory. However, in breaking problems this coupling term becomes large near the singularity and its effect must be considered.

The exact solution shown in Fig. 3.6 for the special case $\omega = k^2$ also gives the typical behavior predicted by the next higher order theory (2.14) for the general problem. In this solution the breaking is stopped short of a singularity, and is reversed; the final result is a region of constant k at the center, and an amplitude behavior like $|\sin \Omega(x-Vt)|$ for some Ω and V. This can be interpreted as the beating phenomenon which results from the superposition of two elementary waves. Given this interpretation, the solution agrees with the asymptotic analysis of the exact solution obtained by Fourier integrals. That is, the end result is an overlap of elementary wavetrains which appear as beats.

Since the second approximation (2.14) resolves the singularity and agrees with the asymptotic analysis of the exact solution, there is no need to look beyond (2.14). Even higher order effects will add only small corrections to the predicted behavior. This is borne out by further exact solutions discussed in Chapter IV. There, the Schroedinger equation, Klein-Gordon equation, harmonic lattice, and the linearized Korteweg-de Vries equation are all found to exhibit the behavior predicted by (2.14). This is true even when the modulations $(k-k_0)$ are not small. Therefore, it appears that the effects considered in (2.14) are the most important, even when (2.12) does not hold.

The first order theory predicts that the lines of constant k converge and eventually cross. This leaves a region of multivalued modulations. However, near $t/t_B = 1$, the next approximation (2.14) must be used. The lines of constant k actually diverge near breaking (see Fig. 3.7). This divergence eventually leads to beats, which can be interpreted as overlap of elementary wavetrains. This overlap corresponds to the region of multivalued modulations in the first order theory.

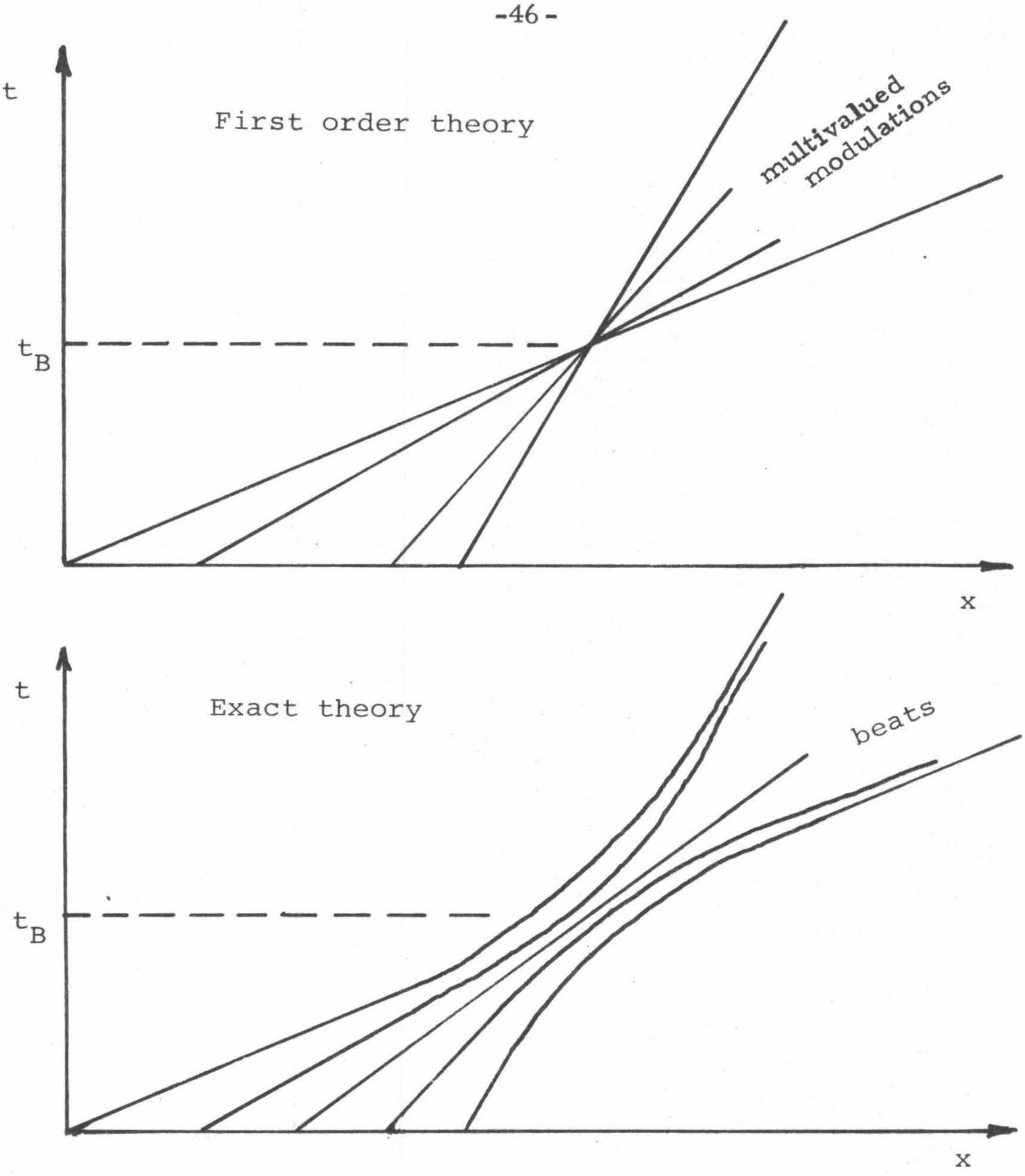


Fig. 3.7 . Lines of constant k, according to the first order theory, and the exact theory.

CHAPTER IV. SINUSOIDAL FREQUENCY MODULATION IN A DISPERSIVE MEDIUM

Exact solutions for initial sinusoidal modulations in the wave number can be found in the form of a rapidly convergent series for any linear dispersive wave problem. These provide additional examples of the true behavior when breaking is predicted.

The phase function

$$\theta(x) = k_o x + \frac{\Delta k}{\nu} \sin \nu x$$

has the wave number

$$k(x) \equiv \theta_x = k_o + \Delta k \cos \nu x \quad .$$

The complex periodic wave for this modulation is

$$\varphi(x) = e^{i\theta} = e^{ik_o x} e^{i\frac{\Delta k}{\nu} \sin \nu x} \quad .$$

Since the second exponential factor is itself periodic, it has a Fourier expansion. Thus

$$\varphi(x) = \sum_{n=-\infty}^{\infty} C_n e^{ik_o x + in\nu x}$$

where the Fourier coefficients C_n are

$$\frac{\nu}{2\pi} \int_0^{\frac{2\pi}{\nu}} \exp\left(i \frac{\Delta k}{\nu} \sin \nu x - i \nu n x\right) dx \quad .$$

Taking $\psi = \nu x$, the C_n's can be expressed in terms of Bessel functions, i.e.

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} e\left(i \frac{\Delta k}{\nu} \sin\psi - in\psi\right) d\psi$$

$$= J_n\left(\frac{\Delta k}{\nu}\right) \quad .$$

Using the property of Bessel functions,

$$J_n(y) = (-1)^n \; J_{-n}(y) ,$$

the real part of φ gives the series representation

$$\cos(k_o x + \frac{\Delta k}{\nu} \sin \nu x) = C_o \cos k_o x$$

$$+ \sum_{n=1}^{\infty} C_n (\cos k_n x + (-1)^n \cos k_{-n} x) , \qquad (4.1)$$

where $k_n = k_o + n\nu$.

For a dispersive wave problem with the dispersion relation

$$\omega = W(k) ,$$

we have elementary solutions

$$\cos[k_n x - W(k_n)t]$$

for each n. Since these can be superposed, a dispersive wave solution can be constructed from the R.H.S. of (4.1) by adding the appropriate time dependence in each phase; i.e.,

$$\varphi(x,t) = C_o \cos \theta_o$$

$$+ \sum_{n=1}^{\infty} C_n(\cos \theta_n + (-1)^n \cos \theta_{-n}) \qquad (4.2)$$

where $\theta_n = k_n x - W(k_n)t$. This solution can be cast in the form of a modulated wave

$$\varphi(x,t) = R \; A(x,t)e^{i\,\theta(x,t)} ,$$

and from (4.1) the initial modulations are

$$k(x, 0) = k_o + \Delta k \cos \nu x ,$$

$$A(x, 0) = 1 .$$

The series (4.2) converges very rapidly, so that a numerical plot of the solution using the first several terms gives accurate results, e.g., when $\frac{\Delta k}{\nu} = 1$, $C_7 = 1.5 \times 10^{-6}$.

An example

As a graphic example we take the case of the Schroedinger equation,

$$W(k) = k^2,$$

for the conditions

$$\Delta k = .1 , \quad \nu = .1 ,$$

$$k_o = 2 , \quad t_B = \frac{1}{2\Delta k \nu} = 50 .$$

The solution is evaluated using the first seven terms of the series (4.2). The amplitude and wave number are then estimated from the solution by the same technique used in Chapter III. The results are shown in Fig. 4.1.

The results substantiate the conclusions drawn in the previous chapters. As the amplitude derivatives become large, the higher order effects discussed in Chapter II and shown in Fig. 2.3 begin to appear. Eventually the breaking is reversed, leaving a region of constant k. This is similar to the results shown in Fig. 3.5. Again, although this is computed for a special case, it is also the typical behavior predicted by the second order theory (2.14) for

Figs. 4.1.1 - 4.1.5

Modulations from solution (4.2) (solid)

and from first order theory (dotted)

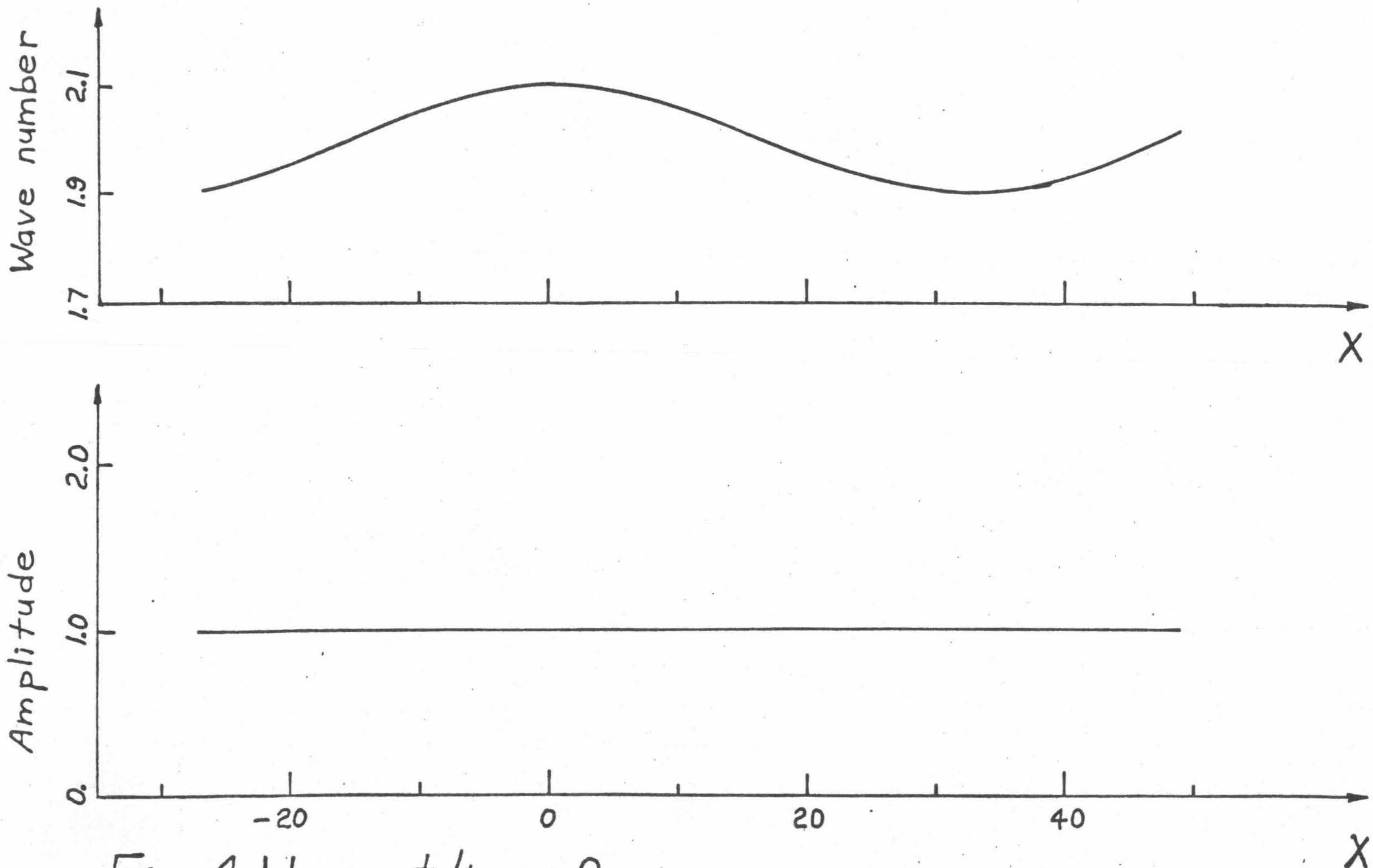


Fig. 4.1.1 $t/t_B = 0$

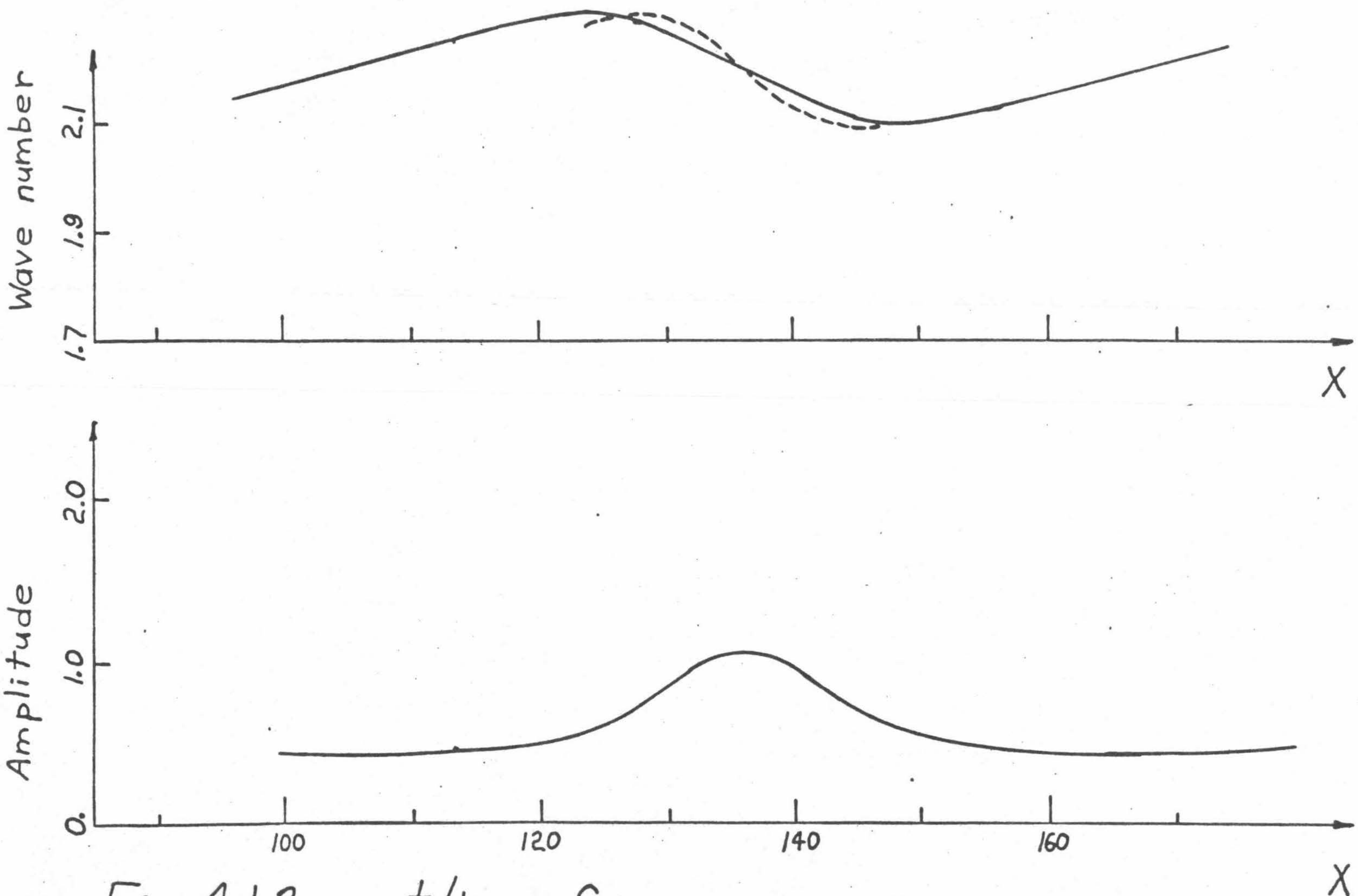


Fig. 4.1.2 $t/t_B = .6$

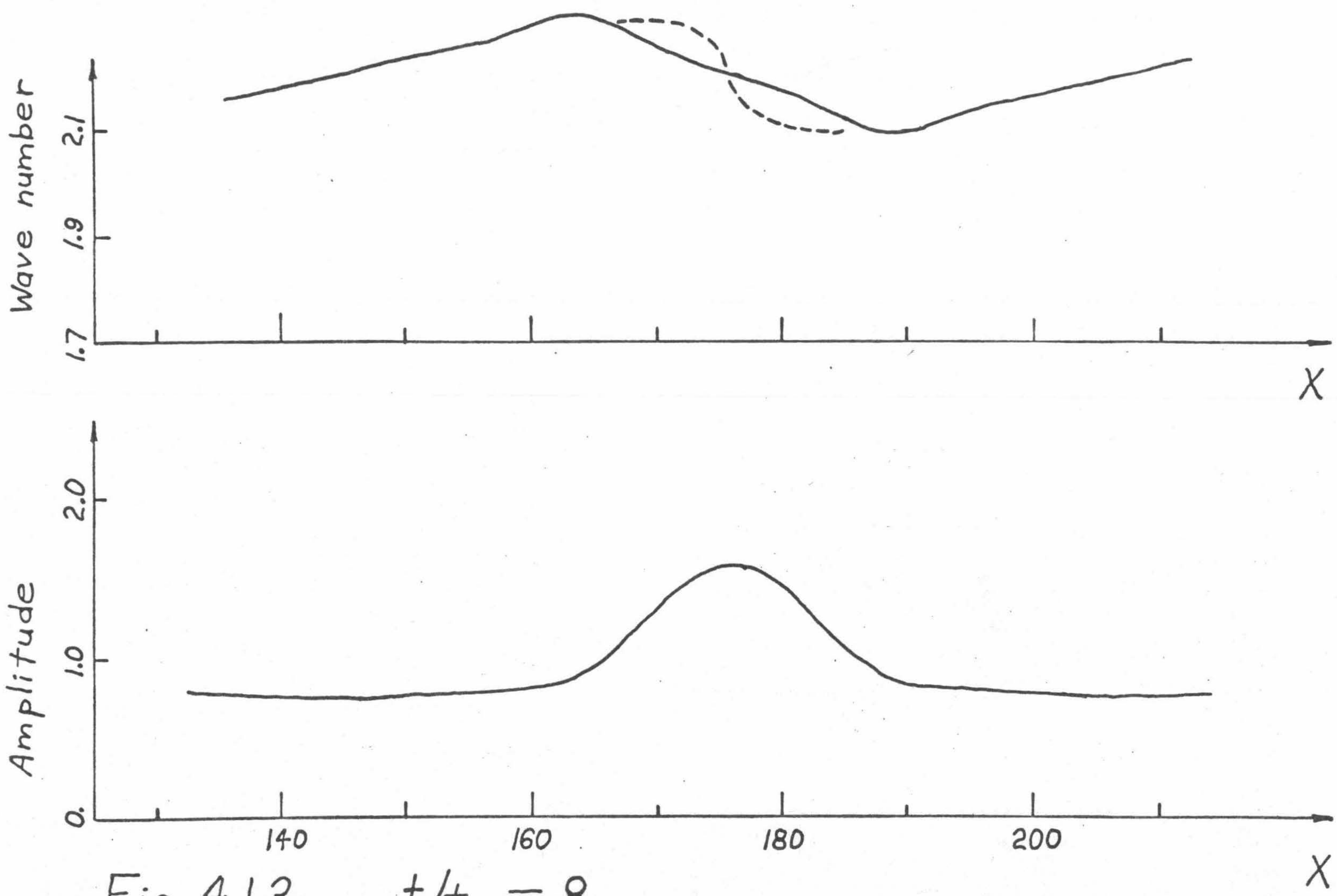


Fig. 4.1.3 $t/t_B = .8$

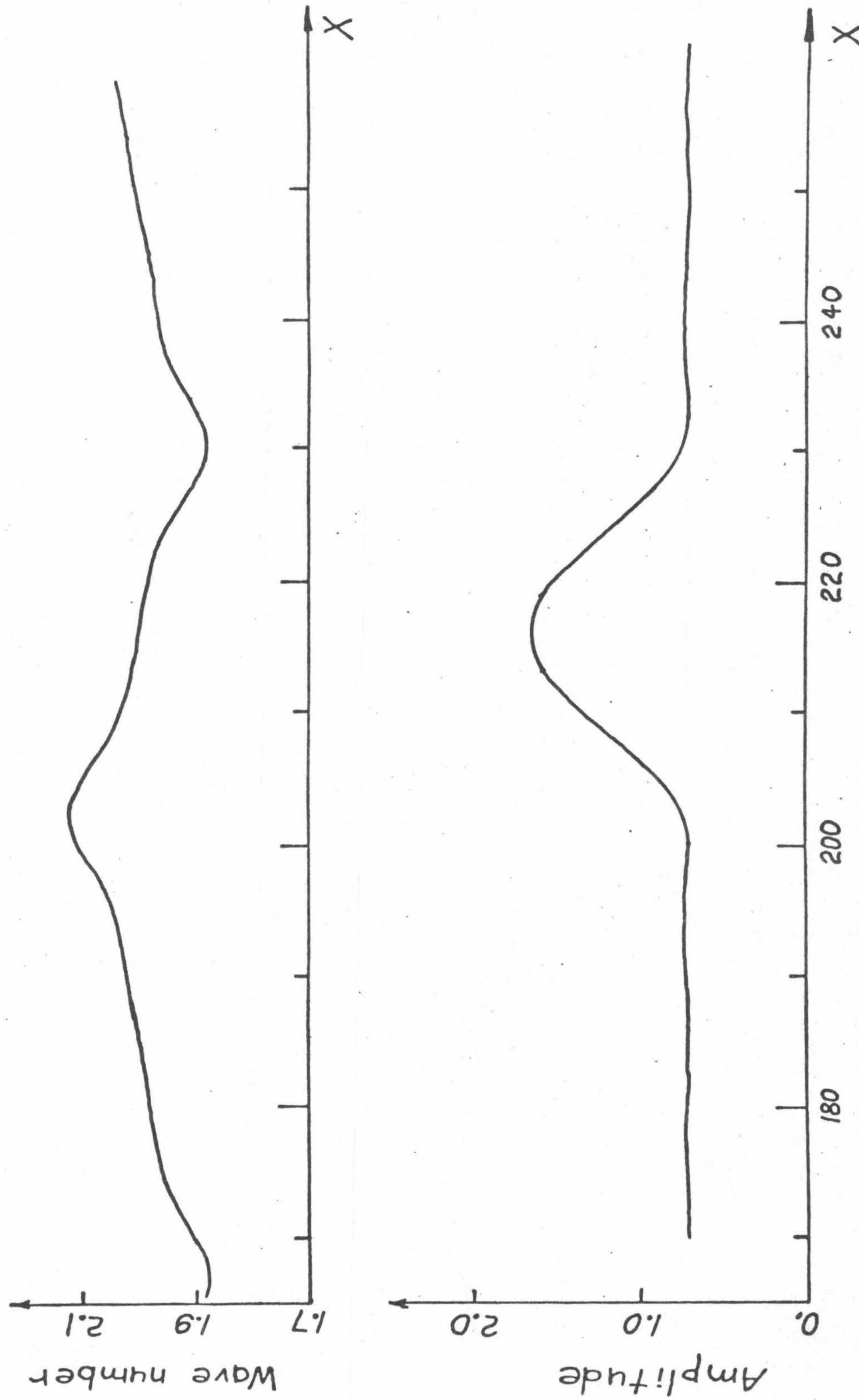


Fig. 4.1.4 $t/t_B = 1.0$

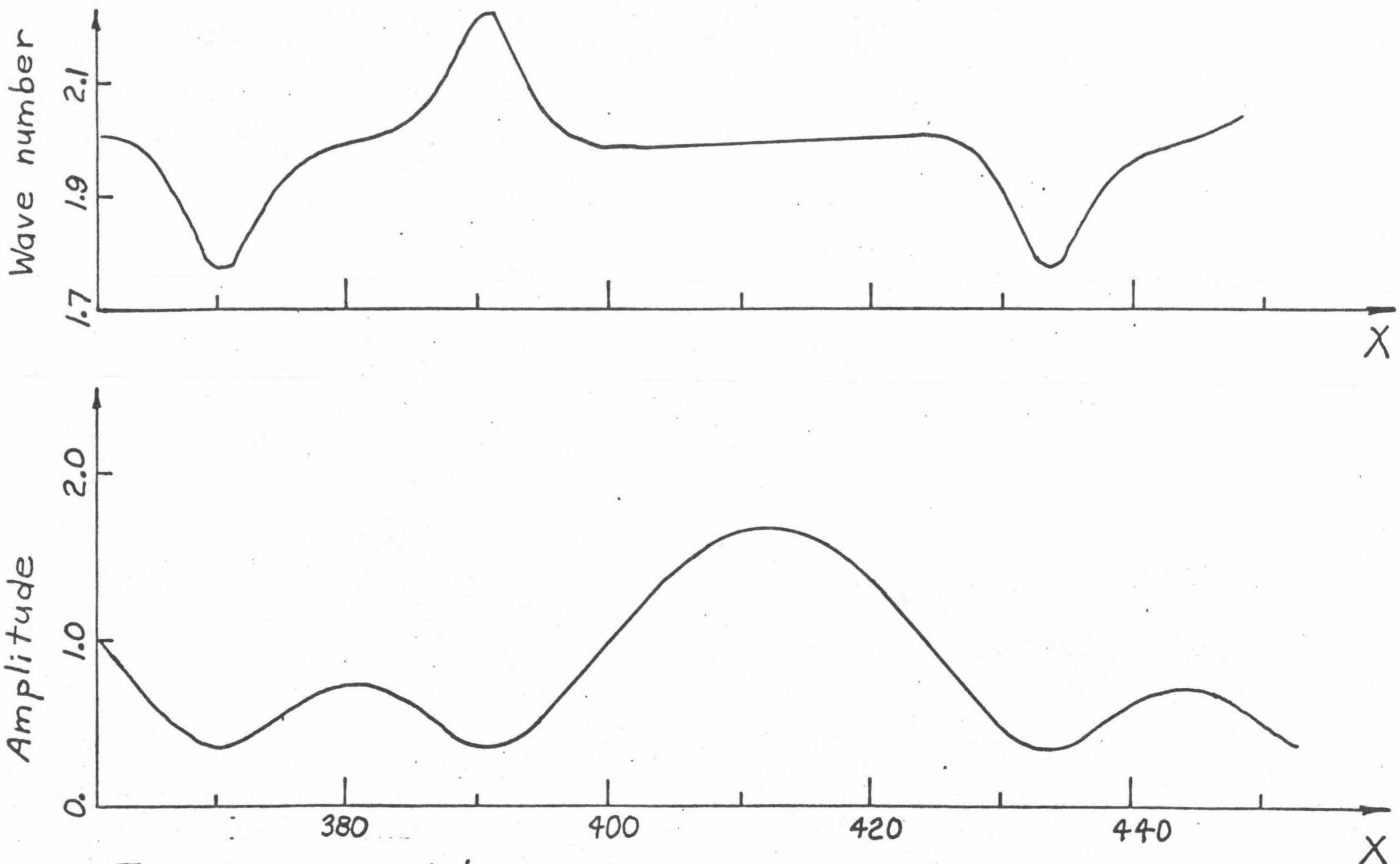


Fig. 4.1.5 $t/t_B = 2.0$

the general case.

This technique can be used for any dispersion relation. Solutions for several other problems have been calculated, including

$$\omega = (k^2+1)^{\frac{1}{2}} \quad ,$$

$$\omega = 2 \sin \frac{k}{2} \quad ,$$

and

$$\omega = k - k^3 \quad ,$$

which are the dispersion relations for the Klein-Gordon equation, the harmonic lattice, and the linearized Korteweg-deVries equation, respectively. All of them give similar results. This is strong evidence that the behavior for breaking modulations is qualitatively the same in all problems, and is predicted by the second order approximation (2.14).

Modulations

Although (4.2) is useful for the numerical evaluation, it is not the best form for analytic discussion of the modulations. However, using the usual trigonometric identities, we can rewrite the solution as

$$\varphi = \alpha \cos \theta_0 - \beta \sin\theta_0 \quad ,$$

where

$$\theta_0 = k_0 x - W(k_0)t$$

$$\alpha = C_0 + 2 \sum_{n \text{ even}} C_n \cos(\Delta_n t)\cos(n\nu N_n)$$

$$+ 2 \sum_{n \text{ odd}} C_n \sin(\Delta_n t)\sin(n \nu N_n) \; ,$$

$$\beta = -2 \sum_{n \text{ even}} C_n \sin(\Delta_n t)\cos(n\nu N_n)$$

$$+ 2 \sum_{n \text{ odd}} C_n \cos(\Delta_n t)\sin(n\nu N_n) ,$$

$$\Delta_n = \tfrac{1}{2}[W(k_n) + W(k_{-n}) - 2\,W(k_o)] ,$$

and

$$N_n = x - \frac{W(k_n) - W(k_{-n})}{n\nu} t .$$

The amplitude and wave number can now be written explicitly

$$A(x, t) = (\alpha^2 + \beta^2)^{\frac{1}{2}} \tag{4.3a}$$

$$k(x, t) = k_o + \frac{\alpha\beta_x - \beta\alpha_x}{A^2} \tag{4.3b}$$

If information about the modulations is desired, it can be computed directly from (4.3). One sees immediately that there can be no singularities in the wave number. For the possible exception of points where the amplitude vanishes, (4.3b) has a finite limit as $A \to 0$.

We have exact expressions for the modulations (4.3), and these should agree to some extent with the first order solutions (2.18) and (2.20). If we again consider the special case $W(k) = k^2$, α and β simplify somewhat since we have

$$\Delta_n t = (n\nu)^2\, t = \frac{n^2\nu}{2\Delta k}\left(\frac{t}{t_B}\right)$$

and

$$N_n = N(x, t) \equiv x - 2\,k_o\, t ,$$

where $t_B = (2\Delta k\nu)^{-1}$.

The quantities α, β, A, k are all functions of t/t_B and N. If the expressions for α and β are expanded in powers of t/t_B, the infinite series can be summed using formulas similar to (4.1)! After some work one obtains

$$A^2 = 1 + \frac{t}{t_B} \sin \nu\xi + \left(\frac{t}{t_B}\right)^2 \sin^2 \nu\xi + O\left(\frac{t}{t_B}\right)^3 ,$$

and

$$k = k(\xi, 0) + O\left(\frac{t}{t_B}\right)^2$$

where $\xi = \xi(x, t)$ is given by

$$x = 2k(\xi, 0)t + \xi .$$

The first order solutions are

$$A^{(0)^2} = \left(1 - \frac{t}{t_B} \sin \nu\xi\right)^{-1}$$

$$k^{(0)} = k(\xi, 0) .$$

Thus we have a significant solution for which the exact modulations can be found. These agree with the first order solutions for $t/t_B \ll 1$, and the error is of the order $(t/t_B)^3$ in the amplitude and $(t/t_B)^2$ in the wave number.

CHAPTER V. NONLINEAR DISPERSIVE WAVES

Nonlinear problems can also exhibit dispersive wave behavior. The concepts of periodic solutions, dispersion, modulations, and group velocity can be generalized to include the nonlinear case. Problems of breaking occur, but the results are qualitatively different from the linear problem, even for small nonlinearities!

In nonlinear problems, dispersive behavior is characterized by the existence of a family of periodic solutions, which are analogous to the elementary solutions

$$Ae^{i\theta}, \qquad \theta = kx - \omega t$$

of linear theory. These solutions are not harmonic in general, and their shape usually depends upon the amplitude and wave number. In the case of a single dependent variable $\varphi(x,t)$

$$\varphi(x,t) = \Phi(\theta, a, k), \quad \theta = kx - \omega t, \tag{5.1}$$

which are 2π periodic in θ. The constant a determines the amplitude of Φ. Typically it is found that for Φ to be a solution, a relation between ω, k, and the amplitude parameter a must be satisfied, i.e.

$$G(\omega, k, a) = 0 . \tag{5.2}$$

This is the <u>nonlinear dispersion relation.</u>

As an example, we take the cubic Schroedinger equation,

$$i\varphi_t + \varphi_{xx} + \sigma|\varphi|^2\varphi = 0 .$$

This has periodic solutions

$$\varphi = a e^{ikx - i\omega t} ,$$

provided (ω, k, a) satisfy

$$\omega - k^2 + \sigma a^2 = 0 .$$

Another example, in which φ is not sinusoidal, is a nonlinear version of the Klein-Gordon equation:

$$\varphi_{tt} - \varphi_{xx} + V'(\varphi) = 0 .$$

Periodic solutions are found by taking

$$\varphi = \Phi(\theta) , \qquad \theta = kx - \omega t .$$

Upon substitution and one integration, the solution is found to be

$$\theta = [\tfrac{1}{2}(\omega^2 - k^2)]^{\frac{1}{2}} \int \frac{d\Phi}{[a - V(\Phi)]^{\frac{1}{2}}} .$$

The constant a is introduced by the integration and is related to the amplitude of Φ. For the special case where $V(\Phi)$ is either cubic, quartic, or trigonometric, $\Phi(\theta)$ can be expressed in terms of standard elliptic functions. If we require that Φ be 2π periodic in θ, then

$$\tfrac{1}{2}[\omega^2 - k^2]^{\frac{1}{2}} \oint \frac{d\Phi}{[a - V(\Phi)]^{\frac{1}{2}}} = 2\pi$$

where the circular integral is an integration over one period of Φ. This relation between ω, k, and a is the amplitude dependent dispersion relation.

Near-Linear Theory

If the nonlinearity is small ($a \ll 1$), more can be said about the form of the periodic wave (5.1) and the dispersion relation (5.2). In this case the solution can be found by perturbation methods. The dependent variable φ is expressed in the form of a Stokes' expansion,

$$\varphi = a \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + O(a^4), \qquad (5.3)$$

and G, which is determined by the supression of secular terms, is expressed in powers of the amplitude,

$$G(\omega, k, a) \equiv G_0(\omega, k) + a^2 G_2(\omega, k) + O(a^4) \quad . \qquad (5.4a)$$

This can be solved for ω to give the alternative form

$$\omega = W_0(k) + a^2 W_2(k) + O(a^4) \quad . \qquad (5.4b)$$

To see specifically how (5.3) and (5.4) are calculated, let us consider the special case where the first nonlinear effect is cubic in φ and its derivatives, i.e.

$$P(\partial_t, \partial_x)\varphi + \sigma \frac{\partial^p \varphi}{\partial x^p} \frac{\partial^q \varphi}{\partial x^q} \frac{\partial^r \varphi}{\partial x^r} + O(\varphi^4) = 0 \, . \qquad (5.5)$$

Only x-derivatives are considered here for simplicity. Other problems can be analyzed in the same manner by adding more nonlinear terms.

The function $G_0(\omega, k)$ in (5.4) must correspond with the dispersion relation for the linearized problem

$$P(\partial_t, \partial_x)\varphi = 0 \quad .$$

From the discussion in Chapter II we know

$$G_0(\omega, k) = P(-i\omega, ik) .$$

Substituting the Stokes' expansion (5. 3) into (5. 5), and using

$$P(\partial_t, \partial_x)e^{in\theta} = G_0(n\omega, nk)e^{in\theta} ,$$

we have

$$\mathcal{R}\left[G_0(\omega, k)ae^{i\theta} + a_2 G_0(2\omega, 2k)e^{i2\theta} + a_3 G_0(3\omega, 3k)e^{i3\theta}\right]$$

$$+ \sigma a^3 \mathcal{R}\left[(ik)^p e^{i\theta}\right] \cdot \mathcal{R}\left[(ik)^q e^{i\theta}\right] \cdot \mathcal{R}\left[(ik)^r e^{i\theta}\right] + O(a^4) = 0 ,$$

where $\mathcal{R}[f]$ is the real part of f. The expansion of the first nonlinear term gives

$$\mathcal{R}\left[\tfrac{1}{4}\sigma(ik)^N a^3 (\beta e^{i\theta} + e^{i3\theta})\right]$$

where $N = p + q + r, \quad \beta = (-1)^p + (-1)^q + (-1)^r.$

By equating the coefficient of $e^{i\theta}$ to zero we find the amplitude dependent dispersion relation

$$G_0(\omega, k) + \tfrac{1}{4}\sigma\beta(ik)^N a^2 + O(a^3) = 0 . \qquad (5.6)$$

The coefficients of $e^{i2\theta}$ and $e^{i3\theta}$ give

$$a_2 = 0, \quad a_3 = \frac{\sigma(ik)^N}{4G_0(3\omega, 3k)} a^3 .$$

The nonlinear equation (5. 5) is dispersive if (5. 6) has solutions for real (ω, k, a). This requires that the number of derivatives in each term be even so that (5.6) is real, or that the number of derivatives in each term is odd so that (5.6) is purely imaginary and a factor i cancels throughout.

Given a G, one can always construct a corresponding equation, although it will not be unique. One such construction with a cubic nonlinearity is

$$G_0(i\partial_t, -i\partial_x)\varphi + |\varphi|^2\, G_2(i\partial_t, -i\partial_x)\varphi = 0. \tag{5.7}$$

This has complex periodic solutions

$$\varphi = a\, e^{ikx - i\omega t}$$

provided $\quad G_0(\omega, k) + a^2\, G_2(\omega, k) = 0\,.$

Other equations can be found by comparing (5.5) and (5.6).

Modulation theory

In linear problems we saw that the modulation viewpoint was useful in extracting information about dispersive waves. In nonlinear problems, the periodic solutions cannot be superposed to give general results. Here the modulation theory is not only informative, it provides a method of treating the general nonlinear problem which cannot be handled in any other way.

As in the linear case, we suppose there are solutions which are locally of the form (5.1), with the amplitude, wave number, and frequency varying in space and time. We will restrict ourselves to the near linear case (5.5). However, the full nonlinearity can be treated concisely and rigorously by the method of the averaged Lagrangian.[1]

The modulated wave can be cast in the form of a Stokes' expansion,

$$\varphi(x, t) = a(x, t)\cos\theta(x, t) + a_3(x, t)\cos 3\theta + O(a^5) ,$$

where a_3 is to be determined. Upon substitution of this into equation (5.5) we have

$$\mathcal{R}\Big[e^{i\theta}\, G_0(\omega + i\,\partial_t,\ k - i\,\partial_x)\, a$$

$$+ e^{i3\theta}\, G_0(3\omega + i\,\partial_t,\ 3k - i\,\partial_x)\, a_3\Big]$$

$$+\sigma\,\mathcal{R}\Big[e^{i\theta}(ik+\partial_x)^p a\Big]\cdot\mathcal{R}\Big[e^{i\theta}(ik+\partial_x)^q a\Big]\cdot\mathcal{R}\Big[e^{i\theta}(ik+\partial_x)^r a\Big]$$

$$+ O(a^5) = 0 . \qquad (5.8)$$

The nonlinear term of (5.8) can be expanded if desired, giving a mixture of $e^{i\theta}$ and $e^{i3\theta}$ terms. As before, we obtain two equations by setting the coefficients of $e^{i\theta}$ and $e^{i3\theta}$ to zero separately; one of them determines the function a_3, the other gives the modulation equations for (ω, k, a). These are supplemented by the compatibility relation

$$k_t + \omega_x = 0 .$$

<u>Slow modulations</u>

As in the linear problem, we treat the modulation parameters (ω, k, a) as functions of slow variables

$$X = \epsilon x , \qquad T = \epsilon t , \qquad \epsilon \ll 1 .$$

Equation (5.8) becomes

$$\mathcal{R}\Big\{e^{i\theta}\, G_0(\omega + i\epsilon\partial_T,\ k - i\epsilon\,\partial_X)a$$

$$+\, e^{i3\theta}\, [G_0(3\omega, 3k)a^3 + O(\epsilon)]a^3\Big\}$$

$$+\, \sigma a^3\, \mathcal{R}[e^{i\theta}(ik)^p] \cdot \mathcal{R}[e^{i\theta}(ik)^q] \cdot R(e^{i\theta}(ik)^r]$$

$$+\, O(\epsilon a^3,\ a^5) = 0.$$

This is a double expansion in which ϵ and a are both small. The coefficient of $e^{i\theta}$ gives

$$G_0(\omega + i\epsilon\,\partial_T,\ k - i\epsilon\,\partial_X)a + a^3\, G_2(\omega, k) + O(\epsilon a^3, a^5) = 0. \qquad (5.9)$$

Equation (5.9) is the nonlinear analog of equation (2.9), and is simply the full modulation form of the linearized problem augmented by the cubic term $a^3 G_2$. Although (5.9) was derived for a special case, the result can be shown by other techniques [(1)] to be true in general.

Following the expansion (2.10), we can expand the term $G_0(\omega + i\epsilon\,\partial_T,\ k - i\epsilon\,\partial_X)a$ in powers of $(i\epsilon)$ as

$$G_0(\omega, k)a + \frac{i\epsilon}{2a}\Big[(G_{o,\omega}\, a^2)_T - (G_{o,k}\, a^2)_X\Big]$$

$$(-i\epsilon)^2\, E_2 + (-i\epsilon)^3\, E_3 + O(\epsilon^4 a)\,. \qquad (5.10)$$

If we suppose that ω and k are close to some constant values,

$$\omega = \omega_0 + O(\epsilon), \quad k = k_o + O(\epsilon), \qquad (5.11)$$

then E_2 is given by (2.15), that is

$$E_2 = \frac{-1}{2!}\, G_{o,\omega}(\omega_o, k_o) W_o''(k_o) a_{XX} + O(\epsilon\, a)\,.$$

With this, the real part of (5.9) gives

$$[G_0(\omega, k) + a^2 G_2(\omega, k)]a + \frac{\epsilon^2}{2!} G_{o,\omega}(\omega_o, k_o) W_o''(k_o) a_{XX}$$

$$+ O(a^5, \epsilon a^3, \epsilon^3 a) = 0. \qquad (5.12a)$$

This can be solved for ω,

$$\omega = W_o(k) + a^2 W_2(k) - \frac{\epsilon^2}{2!} W_o''(k_o) \frac{a_{XX}}{a}$$

$$+ O(a^4, \epsilon a^2, \epsilon^3), \qquad (5.12b)$$

where W_0 and W_2 are the functions which arise in the alternative form of the dispersion relation (5.4b). The imaginary part of (5.9) gives the amplitude equation

$$\frac{1}{2a}\left[(-G_{o,\omega}\, a^2)_T + (G_{o,k}\, a^2)_X\right] + O(\epsilon^2 a,\ a^3,\ \frac{a^5}{\epsilon}) = 0. \qquad (5.13)$$

As in linear theory, we must neglect the correction $\epsilon^2 a$ as well as a^3 to be consistent with the retained accuracy in (5.12). So to this approximation the modulation equations are

$$k_T + \left[W_o(k) + a^2 W_2(k)\right]_X = \tfrac{1}{2}\epsilon^2 W_o''(k_o)\left(\frac{a_{XX}}{a}\right)_X, \qquad (5.14a)$$

$$(a^2)_T + [W_o'(k) a^2]_X = 0. \qquad (5.14b)$$

These equations correspond to (2.14) in the linear theory. To be consistent with (5.11) we should expand the terms

$$W_o + a^2 \mathbf{W}_2 = W_o(k_o) + W_o'(k_o)(k - k_o)$$

$$+ a^2 W_2(k_o),$$

and

$$W_0' = W_0'(k_0) + W_0''(k_0)(k-k_0)$$

in (5.14). But the more general form is kept since the term on the right of (5.14a) seems to be the most important contribution from E_2 even when $(k-k_0)$ is not small.

If the reduced form

$$G \equiv \omega - W_0(k) - a^2 W_2(k) + O(a^4)$$

were used in (5.9) instead of the general G, the result would still be equations (5.14). Thus when studying a single mode, the modulation equations may be taken as

$$\omega a + i\epsilon a_T = W_0(k - i\epsilon \frac{\partial}{\partial x})a + a^3 W_2(k)$$

within the accuracy of (5.14). Further, the modulation equations (5.9) involve only the dispersion functions G_0 and G_2, so that the modulations are independent of which particular equation led to these functions. Thus, for the mode

$$\omega = W_0(k) + a^2 W_2(k) + O(a^4),$$

the equation

$$i\varphi_t = W_0\left(-i\frac{\partial}{\partial x}\right)\varphi + |\varphi|^2 W_2\left(-i\frac{\partial}{\partial x}\right)\varphi \qquad (5.15)$$

has the same modulations as the original problem. When it is a sufficient approximation to take W_0 as quadratic, and W_2 as constant, this can be transformed into the cubic Schroedinger equation

$$i\varphi_t + \varphi_{xx} = \sigma |\varphi|^2 \varphi .$$

First order theory

In the first approximation to equations (5.14) we have

$$k_T + (W_0(k) + a^2 W_2(k))_X = 0 ,$$

$$(a^2)_T + (W_0'(k)a^2)_X = 0 .$$

In determining the characteristics of these equations, it turns out that it is sufficient to work with the pair

$$k_T + W_0'(k)k_X + 2\, a\, W_2(k)a_X = 0 , \qquad (5.16a)$$

$$a_T + W_0'(k)a_X + \tfrac{1}{2} W_0''(k)a\, k_X = 0 . \qquad (5.16b)$$

The term omitted in (5.16a) leads to corrections of smaller order. These provide the nonlinear version of the first order modulation theory (2.16). The significant difference introduced by the nonlinearity is that the wave number is coupled to the amplitude at the lowest order through the term $2aW_2\, a_X$ in (5.16a). This is responsible for qualitatively new results.

To solve these equations by the method of characteristics we multiply (5.16b) by $\pm 2\left(\frac{W_2}{W_0''}\right)^{\frac{1}{2}}$, and add to (5.16a). This gives

$$(k_T + C_\pm k_X) \pm 2\left(\frac{W_2}{W_0''}\right)^{\frac{1}{2}} \operatorname{sgn} W_0''(a_T + C_\pm a_X) = 0 ,$$

where $$C_\pm = W_0' \pm a\sqrt{W_0'' W_2}$$

are the characteristic velocities of this system. Thus, the modulation equations (5.16) are hyperbolic for $W_0'' W_2 > 0$, and elliptic for $W_0'' W_2 < 0$. We consider first the hyperbolic case in which

$C_\pm$ are real. On the characteristic curves

$$\Gamma^\pm : \frac{dX}{dT} = C_\pm$$

the equations become

$$F'(k) \frac{dk}{dT} \pm \frac{da}{dT} = 0 , \tag{5.17}$$

where $$F = \frac{1}{2} \int_0^k \left(\frac{W_0''}{W_2} \right)^{\frac{1}{2}} \operatorname{sgn} W_0'' \, dk.$$

These are known as the characteristic equations. They can be integrated once to give expressions which are constant along $\Gamma^\pm$, i.e.

$$F(k) \pm a = \gamma^\pm . \tag{5.18}$$

The quantities $\gamma^\pm$ are the Riemann invariants of equations (5.16).

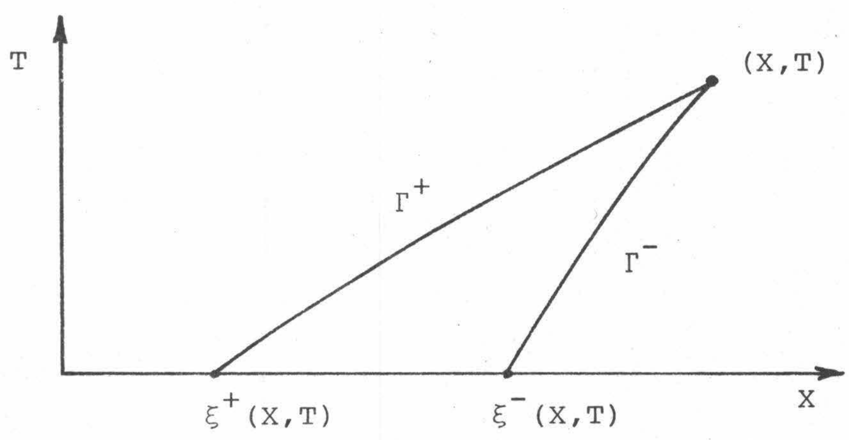


Fig. 5.1 Characteristics of modulation equations (5.16)

If there is a unique Γ^+ and a unique Γ^- from the X-axis to the point (X, T) (see Fig. 5.1), we can associate with this point the coordinates $\xi^+(X, T)$ and $\xi^-(X, T)$ where the $\Gamma^\pm$ cross the X-axis. In terms of the initial conditions

$$a(X, 0) = a_o(X), \quad k(X, 0) = k_o(X),$$

the Riemann invariants are

$$\gamma^\pm = F(k_o(\xi^\pm)) \pm a_o(\xi^\pm) \quad . \tag{5.19}$$

At the point (X, T) where Γ^+ and Γ^- intersect, the values of k and a are the same for each curve, so that equations (5.18) can be added and subtracted to give

$$F(k) = \tfrac{1}{2}(\gamma^+ + \gamma^-) , \tag{5.20a}$$

$$a = \tfrac{1}{2}(\gamma^+ - \gamma^-) . \tag{5.20b}$$

In principle, to calculate the solution at (X, T), the two characteristic curves must be followed back to the X-axis to locate ξ^+ and ξ^-. The Riemann invariants are computed from the initial data using (5.19). The solution is then given by (5.20). However, since the characteristics cannot be determined without knowing the solution at each point, the procedure must be carried out numerically, in general.

Splitting

When the initial modulations are confined to a segment of the X-axis, say

$$[k(X,0),\ a(X,0)] = \begin{bmatrix} [k_1,\ a_1] & , & X < 0 \\ [k_0(X),\ a_0(X)], & & 0 \le X \le L \\ [k_2,\ a_2] & , & X > L \end{bmatrix} ,$$

more can be said about the solution. The two families of characteristics from the disturbance on $[0, L]$ will eventually split at some time, say $T = T_s$. This time is estimated by

$$T_s \doteq \frac{L}{C_+(k_1, a_1) - C_-(k_2, a_2)} .$$

It is assumed that the nonlinearity is strong enough to cause this splitting before there is any crossing of characteristics of the same family. This splitting naturally divides the X-T plane into three

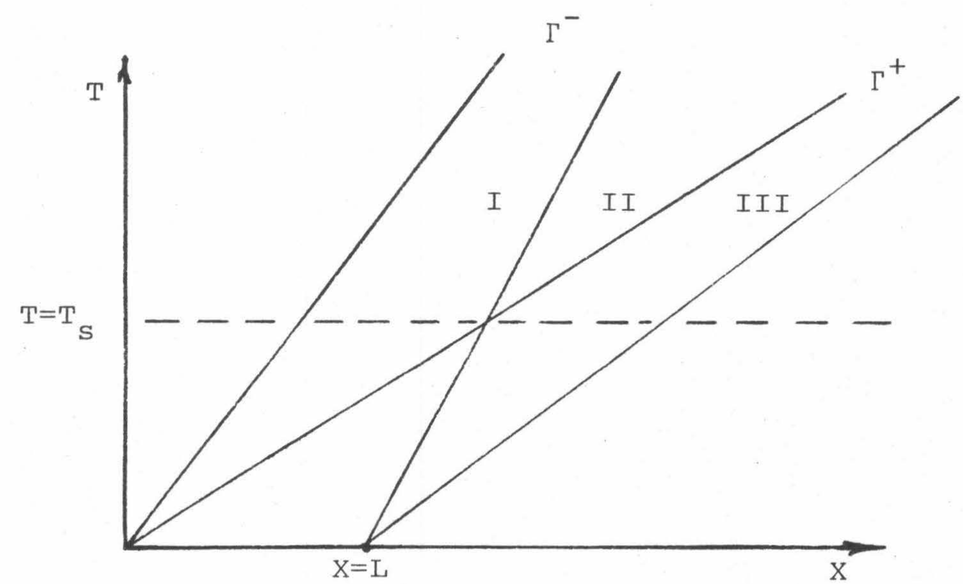


Fig. 5.2 Splitting of the Modulations

distinct regions as shown in Fig. 5.2. In region I, the Riemann invariant γ^+ is constant everywhere. It has the value

$$\gamma^+ = F(k_1) + a_1 \quad .$$

The other invariant γ^- is constant on each Γ^-, so that k and a are individually constant on each Γ^- curve in I, and therefore these curves are straight lines. Similarly in region III, k and a are constant in each Γ^+, which are also straight lines. For the center region II, both Γ^+ and Γ^- eminate from constant initial conditions so that γ^+ and γ^- are constant,

$$\gamma^+ = F(k_1) + a_1 \quad , \qquad \gamma^- = F(k_2) - a_2 \ ,$$

everywhere in II. Thus, k and a are constant throughout II, with the values

$$F(k) = \tfrac{1}{2}[F(k_1) + F(k_2)] + \tfrac{1}{2}[a_1 - a_2] \ , \tag{5.21a}$$

$$a = \tfrac{1}{2}[F(k_1) - F(k_2)] + \tfrac{1}{2}[a_1 + a_2] \ . \tag{5.21b}$$

So an isolated disturbance is eventually split into two groups which move with speeds C_+ and C_-. These two characteristic velocities are the nonlinear analogs of group velocity, and both reduce to the classical group velocity in the linear limit. Both the wave number and the amplitude propagate with the corresponding group velocity in each group. Since one of the Riemann invariants is constant throughout a group, we can eliminate one of the dependent variables from the equation. For instance, in the C_- group, we have

$$\gamma^+ = F(k_1) + a_1 = F(k) + a \ .$$

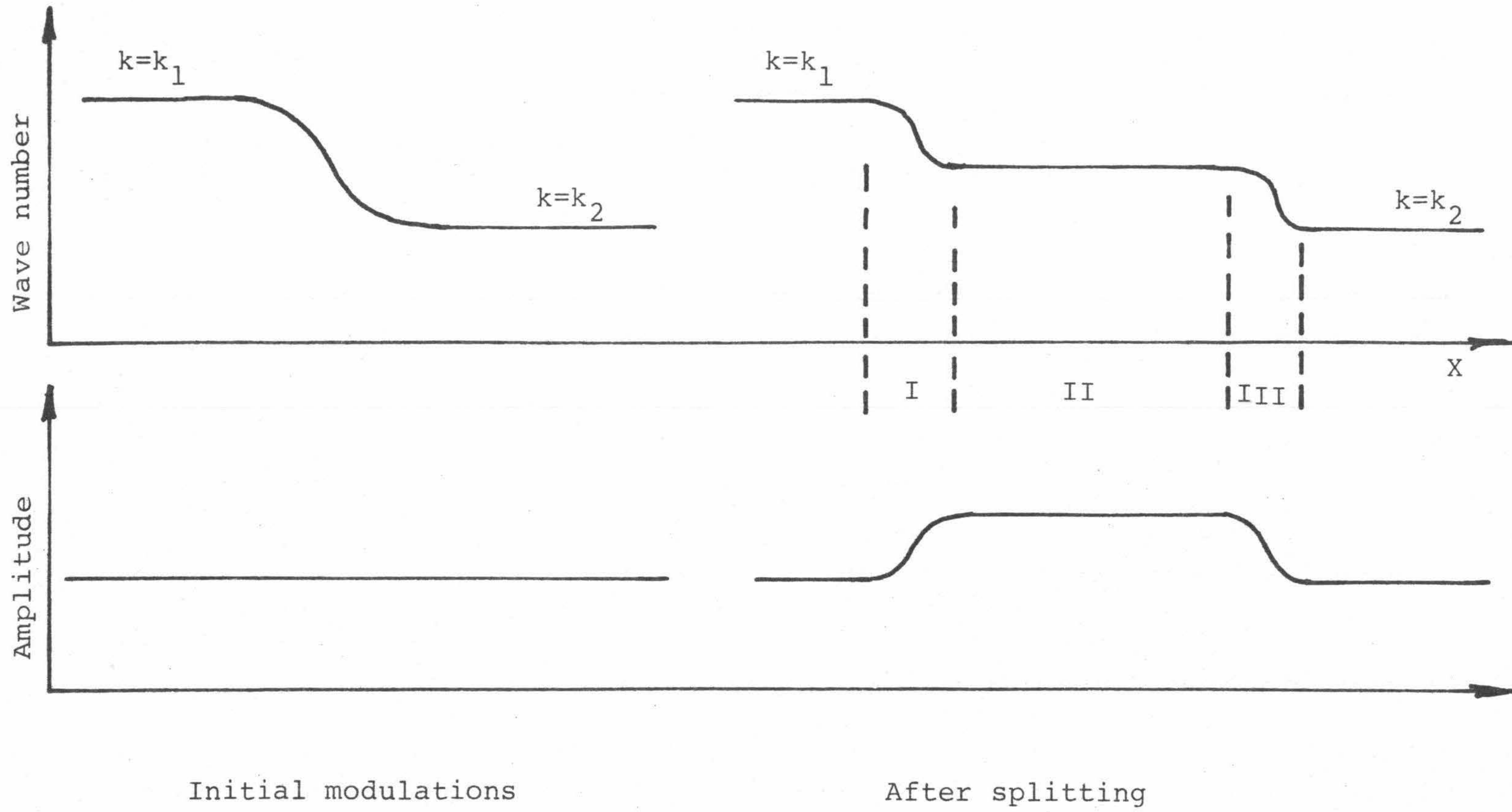


Fig. 5.3 Typical breaking modulations predicted by the first order nonlinear theory.

This can be used to express a in terms of k alone,

$$a = a_1 - [F(k) - F(k_1)] \tag{5.22a}$$

valid in region I. In region III we have

$$a = a_2 + [F(k) - F(k_1)]. \tag{5.22b}$$

The group velocities depend on k and a, so that characteristics of the same family can eventually cross, leading to singularities and multivalued solutions. The solution of (5.16) for typical compressive modulations is sketched in Fig. 5.3. This is to be compared with the solution for the linear problem in Fig. 2.2. The nonlinearity has split the disturbance into two groups, each resulting in breaking.

Linearized modulation equations

If equations (5.16) are linearized about some constant values, say

$$k = k_0, \quad a = a_0, \quad \omega_0 = W_0(k_0) + a_0^2 W_2(k_0),$$

the general solution can be easily found, and in some cases this is more illuminating than the full solution just discussed. The linearized equations have solutions

$$a - a_0 = \exp[i\,\mu(\lambda t - x)]$$

$$k - k_0 = \alpha \exp[i\,\mu(\lambda t - x)]$$

where α and λ are found upon substitution to be

$$\alpha = \pm \frac{2\,W_2}{\sqrt{W_o''\,W_2}} \;, \quad \lambda = W_o' \pm a_o\sqrt{W_o''\,W_2} \;.$$

Superposition of these solutions for all values of μ will give the general solution.

An interesting application of the linearized solution when $W_o''\,W_2 > 0$ is the signaling problem. For a small sinusoidal amplitude modulation at $X = 0$,

$$k(0, T) = k_o$$

$$a(0, T) = a_o + \delta \cos\nu T, \quad \delta \ll 1 \;,$$

the solution is

$$k = k_o - \delta \frac{2\,W_2}{\sqrt{W_o''\,W_2}} \sin \nu\left(T - \frac{X}{W_o'}\right) \sin\gamma X \;,$$

$$a = a_o + \delta \cos\nu\left(T - \frac{X}{W_o'}\right) \cos \gamma X \;,$$

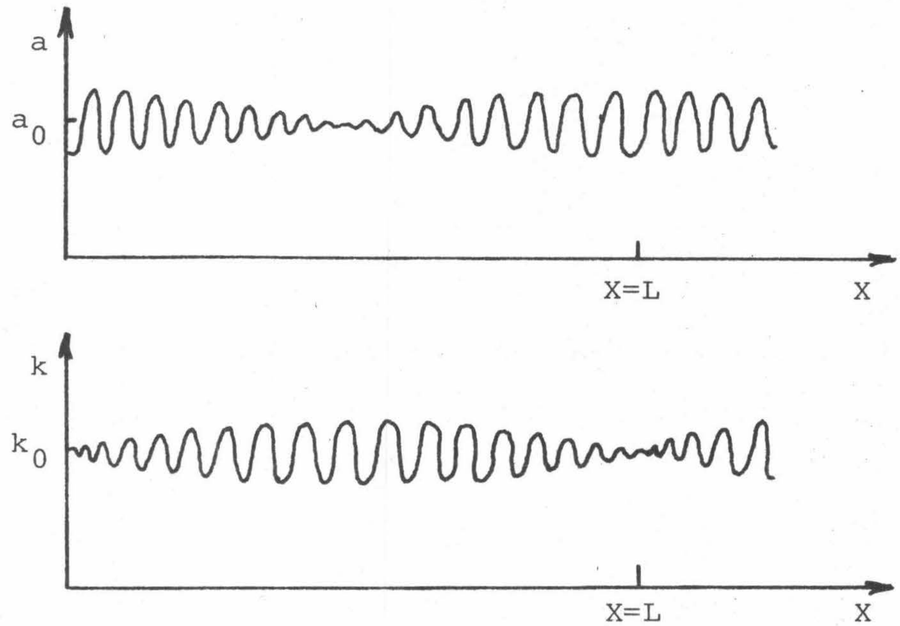
where $\gamma = \nu\, a_o \dfrac{\sqrt{W_o''\,W_2}}{W_o'^2}$.

We see a beating effect in both the amplitude and wave number, Fig. 5.4. This is due to the superposition of solutions which move with the two nonlinear group velocities C_+ and C_-. The distance between modes of the beats is

$$L = \frac{\pi\, W_o'^2}{\nu\, a_o\sqrt{W_o''\,W_2}} \;.$$

This beating phenomenon could provide a good test when searching for experimental evidence of two nonlinear group velocities. Such an experiment should produce steady time behavior

Fig. 5.4 Signaling problem



similar to standing waves; it should be easier to measure than a propagating disturbance.

Higher order effects

The presence of both the nonlinear term and the higher order dispersion effect in (5.14) provides interesting phenomena which do not occur when these are considered separately. One is a structured transition region for breaking modulations. Another is the existence of solitary wave packets in the elliptic case.

Breaking

The first order modulation equations (5.16) have breaking solutions in some cases. As breaking is approached, these equations are no longer valid. As in linear theory the higher order dispersive effects neglected in the first approximation must be considered.

From (5.14) we see that at the next level of approximation these effects enter the equations through the term $(\frac{a_{XX}}{a})_X$. This is the same amplitude term which introduces higher order effects in the linear theory (see equation (2.14)). However, in the linear problem the amplitude is given approximately by (2.20), whereas in the nonlinear problem it is given approximately by (5.22). The higher order effects are quite different in the two cases; we expect qualitatively different behavior near breaking for the nonlinear problem.

To study this behavior we return to equations (5.14),

$$k_T + [W_0(k) + a^2 W_2(k)]_X = \tfrac{1}{2}\epsilon^2 W_0''(k_0)\left(\frac{a_{XX}}{a}\right)_X , \qquad (5.23a)$$

$$(a^2)_T + [W_0'(k)a^2]_X = 0 . \qquad (5.23b)$$

The two terms in (5.23a) which involve the amplitude are both of higher order. Thus after splitting is completed, the first order result (5.22) can be used to eliminate the amplitude from (5.23a). This will give an equation that involves only k and is valid in the region of breaking.

Relation (5.22) gives the amplitude as a function of wave number, which we denote by

$$a = a^*(k) . \qquad (5.23)$$

We have

$$a_{XX} = \frac{\partial a^*}{\partial k}(k_0)\, k_{XX}(1 + O(\epsilon)) ,$$

since k_X^2 is of smaller order according to (5.11). The function $\frac{\partial a^*}{\partial k}$ can be found from (5.17) and (5.22),

$$\frac{\partial a^*}{\partial k} = \pm \frac{1}{2}\left(\frac{W_0''}{W_2}\right)^{\frac{1}{2}} \operatorname{sgn} W_0'' \;,$$

where + or - is chosen for the C_+ or C_- group respectively. With the assumptions in (5.11), (5.24) implies that a is also close to some constant value

$$a = a_0 + O(\epsilon) \;.$$

We would like this deviation in the amplitude to be small compared to the background value a_0. For instance we may take $a_0 = O(\epsilon^{\frac{3}{4}})$. This allows the simplification

$$\frac{a_{XX}}{a} = \frac{a_{XX}}{a_0}\;(1 + o(\epsilon))\;.$$

With this, (5.23a) becomes

$$k_T + (W_0'(k) \pm a^* \sqrt{W_0''\,W_2} + a^{*2}\,W_2')\,k_X = \beta\, k_{XXX} \qquad (5.25)$$

with the constant $\beta = \pm \dfrac{\epsilon^2\,|W_0''(k_0)|}{4\,a_0}\left(\dfrac{W_0''}{W_2}\right)^{\frac{1}{2}}$.

This equation gives the next approximation to the true behavior in the regions of breaking. The choice of (+) or (-) gives solutions which apply in the C_+ or C_- groups respectively. Again, we do not expand the nonlinear term about k_0 and a_0, since the R. H. S. seems to give the important contribution even when $(k-k_0)$ is not small.

The higher order term βk_{XXX} adds dispersion to the modulation equations. In fact, if we retain only the linear part of the coefficient of k_X, we have the Korteweg-de Vries equation for the variable $K = k-k_0$. That is

$$K_T + (c_0 + \gamma K)K_X = \beta K_{XXX} ,$$

where
$$c_0 \triangleq W_0'(k_0) + a_0\sqrt{W_0'' W_2} , \qquad (5.26)$$

$$\gamma \triangleq W_0''(k_0) .$$

This equation is well known as a model for nonlinear dispersive problems. Therefore we digress here to study (5.26); we wish to determine the effects of dispersion on solutions for which the nonlinearity alone predicts breaking.

First we consider the possibility of a shock-like solution to (5.26). A shock is a transition from one equilibrium state to another which is steady in some frame of reference. One looks for steady profile solutions

$$K(X, T) = f(X-VT) ,$$

such that

$$K \to K_1 \quad \text{as} \quad (X-VT) \to -\infty$$

and
$$K \to K_2 \quad \text{as} \quad (X-VT) \to +\infty .$$

However, it is known that the only steady solutions to the Korteweg-deVries equation are periodic, or solitary waves of the form

$$a_0 + a \operatorname{sech}^2 \Omega (X - VT) .$$

This precludes the existence of a steady shock structure.

There is still the possibility of unsteady solutions which have some kind of structure in the region of breaking. Although no such exact solutions are known, their qualitative aspects can be inferred by studying the nonlinear and dispersive effects separately.

The linearized problem ($\gamma = 0$),

$$K_T + C_o K_X = \beta K_{XXX},$$

can be solved by looking for similarity solutions of the form

$$K(X, T) = f\left[\frac{X - C_o T}{(3|\beta|T)^n}\right]. \qquad (5.28)$$

Upon substitution one finds that $n = \frac{1}{3}$, and for $\beta > 0$, $f(\xi)$ must satisfy

$$f''' + \xi f'(\xi) = 0 .$$

This is Airy's equation for $f'(\xi)$, and f can be expressed in terms of the Airy function:

$$f(\xi) = -\int_{-\infty}^{\xi} A_i(\eta)d\eta ,$$

where $$A_i(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta s + i\frac{1}{3}s^3} ds .$$

For $\beta < 0$, $$f(\xi) = \int_{-\infty}^{-\xi} A_i(\eta)d\eta .$$

These solutions are shown in Fig. (5.5).

As $T \to 0$, the solution in (5.28) approaches a unit step function at $X = C_o T$. There is a definite transition region around the point $X = C_o T$ where the character of the solution changes abruptly. Oscillations appear at the front or back depending upon the sign of β. The interpretation here is that the transition creates oscillations which propagate with their own group velocity. Since

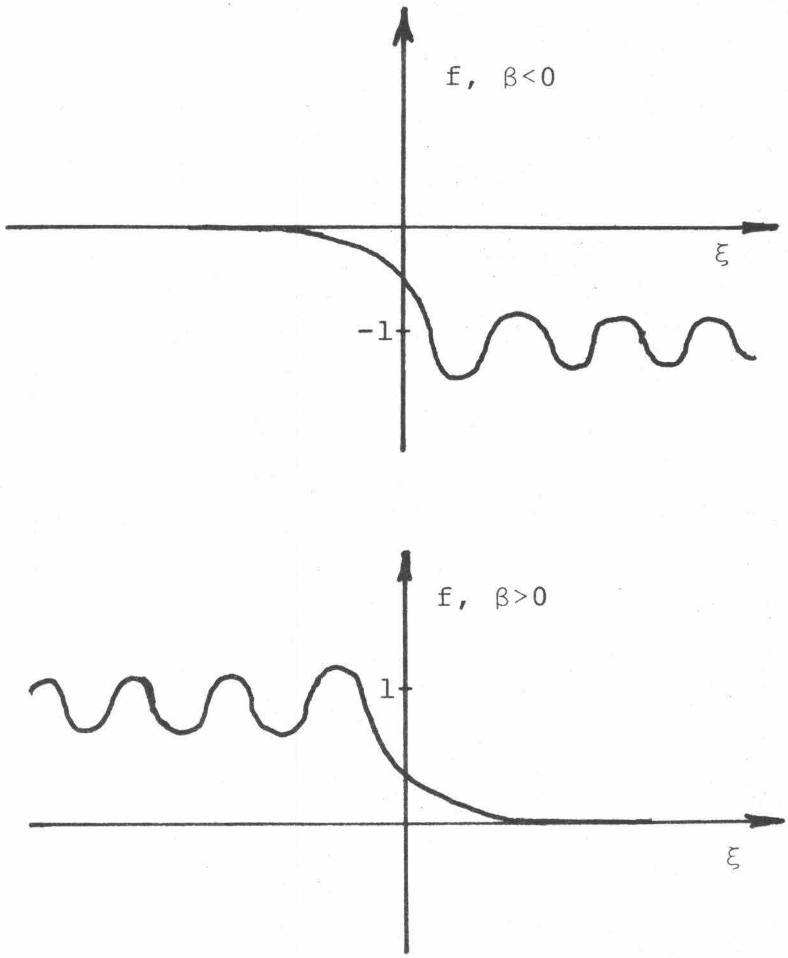


Fig. 5.5 Similarity solutions of the linearized K.dV. equation.

$$C(k) = C_0 + 3\beta k^2$$

the oscillations move faster than C_0 for $\beta > 0$, and slower for $\beta < 0$. The slope at $X = C_0 T$ is

$$K_X = \left(\frac{1}{3|\beta|T}\right)^{\frac{1}{3}} f'(0) ,$$

showing that the transition flattens as the solution progresses.

This solution of the linear problem demonstrates that the dispersion tends to cancel breaking in the transition. So it is plausible that a balance is reached between these two effects leaving a nearly steady profile aside from the oscillations. The speed of this disturbance should be $V = C_0 + \gamma K$ evaluated at some intermediate point.

These deductions are speculative but they have been borne out by numerical studies detailed in Chapter VI for a similar problem (Fig. 6.3). There seems to be a somewhat permanent structure to the solution in the region where nonlinear effects alone would cause breaking and overlap. There is a definite transition region where the character of the solution changes abruptly. The solution in the immediate neighborhood of the transition is almost a steady profile which appears to change slowly; the final behavior of the solution on the oscillatory side is unknown.

Since the wave number is governed by the nonlinear dispersive equation (5.25), these transition regions can occur in the modulations. Both types are possible since the third derivative in (5.25) can have either sign (see Fig. 5.6). The propagation speeds are the nonlinear group velocities

$$V = W_0' \pm a\sqrt{W_0'' W_2} \qquad (5.29a)$$

at some typical values of (k, a). The unsteady oscillations will always be ahead for the C_+ group, and behind for the C_- group. The amplitude and the wave number both change across these transitions. Their values, (k_1, a_1) at $-\infty$, and (k_2, a_2) at $+\infty$, are related by

$$a_2 - a_1 = \pm [F(k_2) - F(k_1)] \ . \qquad (5.29b)$$

For these results it was assumed that the amplitude is large enough to cause splitting before breaking. This requires

$$a > \frac{C_\pm(k_1) - C_\pm(k_2)}{2(W_0'' W_2)^{\frac{1}{2}}} \ . \qquad (5.29c)$$

The nonlinearity must be sufficient to overcome the overlapping tendency $[C(k_1) - C(k_2)]$ of the group velocity. Otherwise the behavior will follow linear theory. Therefore, only transitions up to a certain size, as measured by $(a_1 - a_2)$ or $(k_1 - k_2)$, can exist for a given nonlinear effect.

In general, both types of transition will emerge from an initial disturbance, leaving between them an undisturbed periodic wave. For the breaking problem considered earlier (Fig. 5.3), the result is the sausage shaped wave packet shown in Fig. 5.7. This packet grows in length and has unsteady oscillations at the front and rear.

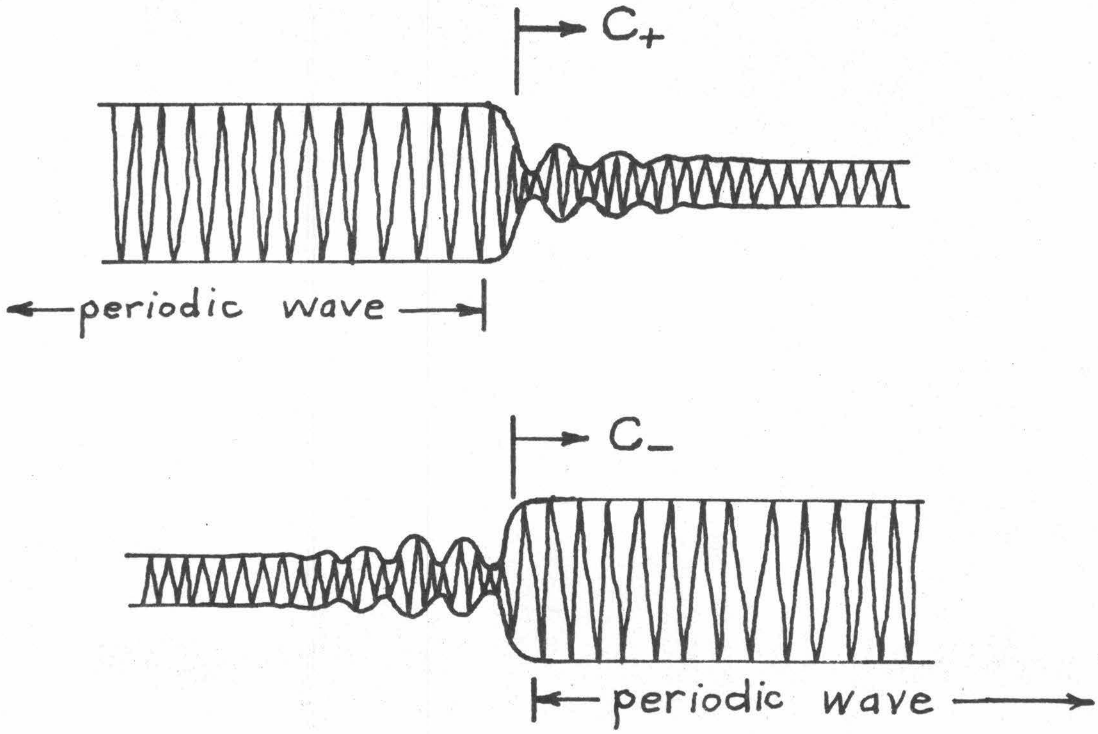


Fig. 5.6 Two possible transition regions for breaking modulations

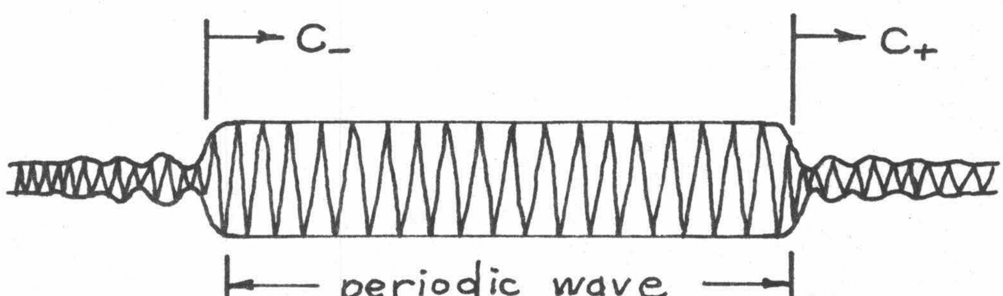


Fig. 5.7 Both transitions emerging from a disturbance

In linear theory, the end result of breaking is a region of modulations which look like beats. This can be interpreted as the overlap of the two semi-infinite wavetrains on either side of the breaking region. Here we see that a sufficiently strong nonlinearity can stop this overlap, leaving a transition zone separating the two wavetrains. However, this transition lacks the permanence of a true shock structure; the long time behavior of the unsteady oscillations on one side of the transition is unknown.

Solitary Wave Packets

An important class of solutions to (5.14) consists of steady profile waves which propagate without change in shape. With (ω, k, a) as functions of $\xi = X - VT$, (5.12b) and (5.14b) are

$$[\omega - W_0(k) - a^2 W_2(k)]a + \lambda a_{\xi\xi} = 0, \tag{5.30a}$$

$$- V(a^2)_\xi + (W_0'(k)a^2)_\xi = 0, \tag{5.30b}$$

where $\lambda = \frac{1}{2}\epsilon^2 W_0''(k_0)$. The second of these can be integrated immediately to give

$$[W_0'(k) - V]a^2 = \text{constant} . \tag{5.31}$$

An interesting special case has (ω, k) constant within the retained accuracy of (5.14), and the amplitude vanishing as $\xi \to \pm\infty$. By choosing

$$V = W_0'(k) , \tag{5.32}$$

(5.30b) is satisfied identically and (5.30a) can be integrated to give

$$\frac{da}{d\xi} = \pm a\left(\frac{W_2}{2\lambda}(a^2 - a_M{}^2)\right)^{\frac{1}{2}} \tag{5.33}$$

where $a_M = +\left(2\,\dfrac{\omega - W_0(k)}{W_2(k)}\right)^{\frac{1}{2}}$

or $$\omega = W_0(k) + \tfrac{1}{2} a_M{}^2 W_2(k) . \tag{5.34}$$

The only positive bounded solutions of (5.33) occur for

$$0 \le a \le a_M ,$$

which further requires that $\frac{W_2}{2\lambda} < 0$, or equivalently

$$W_o'' W_2 < 0 \quad .$$

The solution of (5. 33) is a solitary wave with a maximum value a_M .

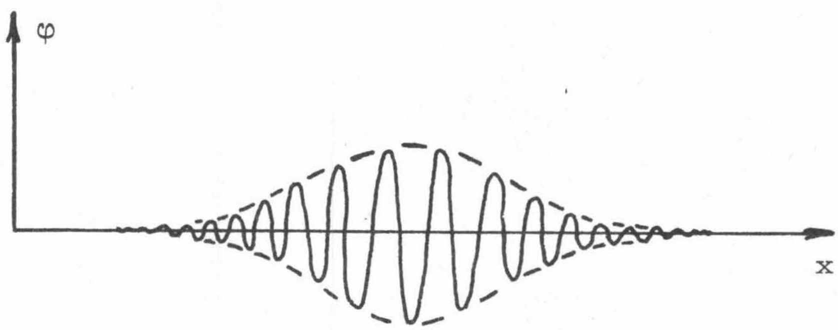


Fig. 5. 9 Solitary Wave Packet

Thus, for the case when the first order modulation equations are elliptic, there are solitary wave packets as shown in Fig. 5. 9. They have constant (ω, k), and propagate undistorted. Their speed is the linear group velocity (5. 32) (within $O(a^2)$), and (ω, k) satisfy the amplitude dependent dispersion relation at $\frac{1}{\sqrt{2}}$ of the maximum value.

Instability

The first order modulation equations are either hyperbolic or elliptic depending upon the sign of $W_o'' W_2$. We will show here that this is associated with the stability of the uniform periodic wave.

The periodic wave is denoted by constant solutions of the modulation equations, say ω_o, k_o, a_o. To investigate the behavior of small perturbations of the periodic wave, we study the modulation equations (5.14) linearized about this constant solution, i.e.

$$k_T + W_o'(k_o)k_X + 2a_oW_2(k_o)a_X - b\,a_{XXX} = 0, \qquad (5.35a)$$

$$a_T + W_o'(k_o)a_X + \tfrac{1}{2} W_o''(k_o)a_o\, k_X = 0, \qquad (5.35b)$$

with $b = \frac{1}{2}\epsilon^2 \dfrac{W_o''(k_o)}{a_o}$. The term $a_o^2 W_2'(k_o)k_X$ has been dropped in (5.35a) since it is small compared with the term $2a_o W_2(k_o)a_X$. These equations have the solution

$$k = e^{i\mu(\lambda T - X)}, \qquad a = \alpha e^{i\mu(\lambda T - X)}$$

where α and λ are determined upon substitution to be

$$\alpha = \pm\frac{1}{2}\left(\frac{W_o''}{W_2}\right)^{\frac{1}{2}}\left(1 - \left(\frac{\epsilon\mu}{2a_o}\right)^2 \frac{W_o''}{W_2}\right),$$

$$\lambda = W_o' \pm a_o\left(W_o''W_2 + \left(\frac{\epsilon\mu W_o''}{2a_o}\right)^2\right)^{\frac{1}{2}}.$$

For λ complex, the solution will grow exponentially in time implying instability of the periodic wave. Thus a necessary condition for stability is that

$$W_o'' W_2 > -\left(\frac{1}{2}\epsilon\mu\frac{W_o''}{a_o}\right)^2 .$$

For the case $W_o'' W_2 < 0$, this requires that

$$\epsilon \mu \geq 2 a_o \left(- \frac{W_2}{W_o''} \right)^{\frac{1}{2}} \quad .$$

There is a critical value for which the perturbations become unstable. In terms of the unscaled space coordinate x, perturbations $e^{i\mu x}$ are unstable for

$$\mu < 2 a_o \left(- \frac{W_2}{W_o''} \right)^{\frac{1}{2}} \quad . \tag{5.36}$$

According to the first order theory ($\epsilon = 0$), the stability criterion is

$$W_o'' W_2 > 0 \quad .$$

So, in the first approximation the periodic wave is stable or unstable depending on whether the modulation equations (5.16) are hyperbolic or elliptic. The higher order terms have a stabilizing effect.

The elliptic case also implies the existence of solitary wave packets. Thus, the speculation that the end result of instability is not a chaotic disturbance, but a sequence of such wave packets.

CHAPTER VI. NUMERICAL STUDY

In Chapter V, several results for nonlinear dispersive problems are predicted from modulation theory. These include the existence of two group velocities which produce splitting, a structured transition region for breaking modulations, and the possible instability of the periodic wave. The derivation of exact solutions for nonlinear problems exhibiting such behavior is unlikely, and experimental evidence of splitting and breaking modulations is not yet available. We present here numerical calculations for a model of physical interest; these verify splitting, the predicted behavior near breaking, and the stability criterion. A calculation showing breaking solutions in the usual sense is also presented for a nonlinear dispersive problem.

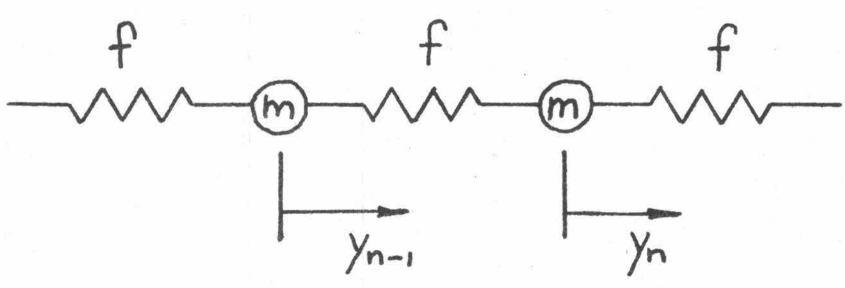


Fig. 6.1. Mass-spring lattice

A discrete problem

The numerical solution of partial differential equations is a difficult task, especially for the large ranges of space and time needed to observe the effects discussed. However, there are some discrete problems of physical interest which exhibit dispersive behavior, and offer a relatively easy numerical solution since they involve coupled ordinary differential equations. Such a system is the infinite mass-spring chain shown in Fig. 6.1. If the extension of the nth spring is $r_n(t) = y_n - y_{n-1}$, the equations are

$$m\,\ddot{r}_n(t) = f(r_{n+1}) + f(r_{n-1}) - 2f(r_n), \qquad (6.1)$$

where f is the restoring force of the spring. This is used by Lowell[3] to model lattice vibrations in crystals. Equation (6.1) has other applications, such as the study of lumped transmission lines as discussed by Hirota and Suzuki.[4] These authors provide experimental evidence of the predicted behavior for solitary waves. Others[5] have used lumped transmission lines to verify the concepts of dispersion relation and group velocity for linear problems. If we assume that the displacements are small but finite, and that f is an odd function, the force law can be approximated by

$$f(r) = \alpha r + \beta r^3 \quad .$$

The scales of r_n and t can be chosen so that the problem takes the normalized form

$$\ddot{r}_n(t) = g(r_{n+1}) + g(r_{n-1}) - 2g(r_n) , \qquad (6.2a)$$

$$g(r) = r + \sigma r^3 \qquad . \qquad (6.2b)$$

We will use problem (6.2) as a model to study numerically the behavior of dispersive systems, but first we show that (6.2) is indeed dispersive, and calculate its periodic wave and dispersion relation.

In the linearized limit of (6.2) we have

$$\ddot{r}_n(t) = r_{n+1} + r_{n-1} - 2 r_n \qquad . \qquad (6.3)$$

For this to have periodic solutions of the form

$$r_n(t) = A e^{i\theta_n(t)} , \qquad \theta_n = pn - \omega t , \qquad (6.4)$$

one finds upon substitution that p and ω must satisfy

$$-\omega^2 + 2(1 - \cos p) = 0 , \qquad (6.5a)$$

$$\omega = 2 \sin(\frac{p}{2}) \qquad . \qquad (6.5b)$$

If p is interpreted as the spatially discrete analog of the wave number then (6.5) is the dispersion relation. In modulation theory, the variables (ω, k, A) are functions of continuous x as well as t. We can extend this problem so that these concepts apply by defining r(x, t) over all space such that

$$\frac{\partial^2 r}{\partial t^2}(x, t) = r(x+1, t) + r(x-1, t) - 2r(x, t) \qquad . \qquad (6.6a)$$

Equation (6.3) is then the special case where x = n, and $r_n(t) = r(n, t)$. Moreover, we can formulate the problem in terms of a partial differential equation by expanding the R.H.S. of (6.6a) in a Taylor series

$$\frac{\partial^2 r}{\partial t^2} = \frac{\partial^2 r}{\partial x^2} + \frac{1}{12}\frac{\partial^4 r}{\partial x^4} + \frac{2}{6!}\frac{\partial^6 r}{\partial x^6} + \dots \quad . \qquad (6.6b)$$

Whichever view is taken, the modulation theory of Chapter II can be applied; i.e. solutions of (6.6) are expressed as

$$r(x,t) = A(x,t)e^{i\theta(x,t)} \quad ,$$

where A and θ must satisfy the modulation equations (2.6) in which G is given by either of (6.5).

In the nonlinear problem we use the continuous extension of (6.2):

$$\frac{\partial^2 r}{\partial t^2}(x,t) = g(r(x+1,t)) + g(r(x-1,t)) - 2g(r(x,t)).$$

If the R.H.S. is expanded in a Taylor series,

$$r_{tt} = r_{xx} + \frac{1}{12} r_{xxxx} + \dots$$
$$+ \sigma[3r^2 r_{xx} + 6rr_x^2 + \dots]$$

we see that only terms which are even in the number of x-derivatives appear. Therefore, the nonlinear problem is also dispersive. The periodic wave can be found in terms of a Stokes' expansion for small amplitudes,

$$r = \varphi(\theta, a, k) = a\cos\theta + \mu_1 a^3 \cos 3\theta$$
$$+ a^5(\mu_2 \cos 3\theta + \nu_2 \cos 5\theta) + \dots \quad , \qquad (6.7a)$$

where

$$\mu_1 = \frac{\frac{1}{2}\sigma(\cos 3k-1)}{G_0(3W_0, 3k)}, \quad \mu_2 = \frac{3\sigma(\cos 3k-1)}{G_0(3W_0, 3k)}\mu_1 ,$$

$$\nu_2 = \frac{\frac{3}{2}\sigma(\cos 5k-1)}{G_0(5W_0, 5k)}\mu_1$$

$$G_0(\omega, k) \equiv -\omega^2 + 2(1 - \cos k) ,$$

and $$W_0 = 2\sin\frac{k}{2} .$$

The supression of secular terms in the expansion gives the amplitude dependent dispersion relation

$$\omega = 2\sin\left(\frac{k}{2}\right)\left(1 + \frac{3}{8}\sigma a^2 + \left(\frac{3}{8}\sigma\mu_1 - \frac{9}{128}\sigma^2\right)a^4 + \ldots\right) . \qquad (6.7b)$$

Moreover, the exact periodic wave can be found for the special case $k = \frac{2\pi}{3}$, i.e.

$$r = a\cos\left(\frac{2\pi}{3}x - \omega t\right) , \qquad (6.8)$$

where $$\omega = 2\sin\left(\frac{\pi}{3}\right)\left(1 + \frac{3}{4}\sigma a^2\right)^{\frac{1}{2}} ,$$

is a solution for arbitrary a. This gives credibility to the belief that a periodic solution exists in general, and to the validity of the above Stokes' expansion for small amplitudes.

According to modulation theory, the stability of the periodic wave is related to $\mathrm{sgn}(W_0'' W_2)$. For this example we have $W_0'' = -\frac{1}{2}\sin\left(\frac{k}{2}\right)$ and $W_2 = \frac{3}{4}\sigma\sin\left(\frac{k}{2}\right)$ so that (6.7a) is stable for the case of a soft spring $\sigma < 0$. For a hard spring $\sigma > 0$ the periodic wave should exhibit instability.

Numerical solution

In the numerical solution of (6.2), a finite chain must be used, say from n = 0, to n = N as shown in Fig. 6.2.

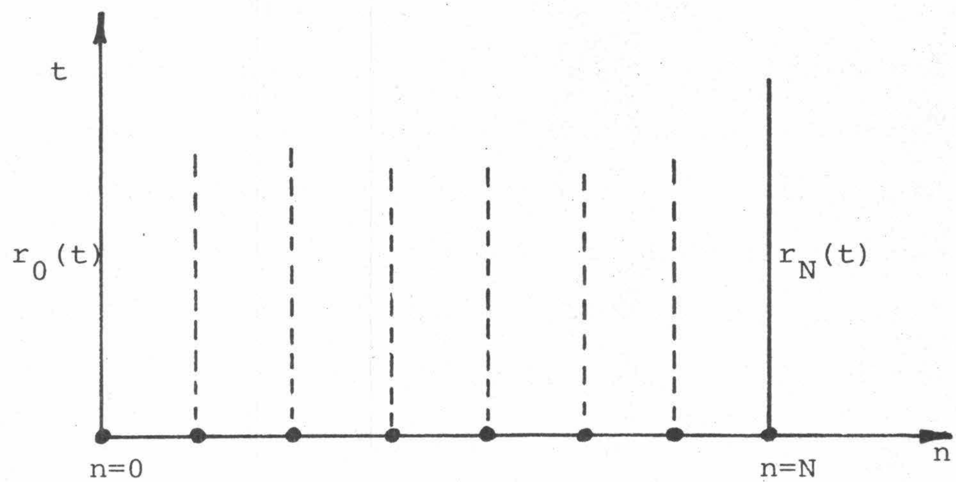


Fig. 6.2 Numerical Solution

A signal $r_o(t)$ can be applied at n = 0, and the disturbance observed as it propagates down the lattice. This line should behave as a semi-infinite chain until the signal nears n = N. The number of node points must be large enough for the dispersive phenomena to develop before the signal reaches the terminal. It has been found that lines of the order of 100 elements are sufficient, and also computationally practical.

The equations are integrated using a pre-written program (MODDEQ) available on the Caltech Fortran library. This routine uses the Adams-Moulton predictor-corrector, with the method of

Runge-Kutta-Gill to start the integration process.

Breaking in a nonlinear dispersive medium

First we consider the question of breaking in the detailed profile of a wave (as opposed to breaking in the modulations) when dispersion is present. An initial profile is introduced into (6. 2) as shown in Fig. 6. 3. The nonlinear effect alone would cause such a profile to break. As this profile steepens, the dispersive effect due to the discreteness becomes important. This effect retards breaking in the main transition, while producing oscillations behind.

In Chapter V, speculation about such solutions were made for the Korteweg-de Vries equation (5. 25). We can see the correspondence between (6. 2) for a single mode, and the Korteweg-de Vries equation by expanding the dispersion relation (6. 7b) in powers of k,

$$\omega = \left(k - \frac{1}{24} k^3 + O(k^5)\right) \left(1 + \frac{3}{8} \sigma a^2 + O(a^4)\right) .$$

By comparing (5. 5) and (5. 6), we see that an equation which also has this dispersion relation is

$$r_t + (1 + \tfrac{3}{2} \sigma r^2) r_x + O(\partial_x^{\,5})$$
$$= - \frac{1}{24} (1 + \tfrac{3}{2} \sigma r^2) r_{xxx} + O(r^5) .$$

This has a structure similar to (5. 25) for $\beta < 0$. The analysis of Chapter V predicts that oscillations appear behind the transition for this sign of β, in agreement with the computations presented here.

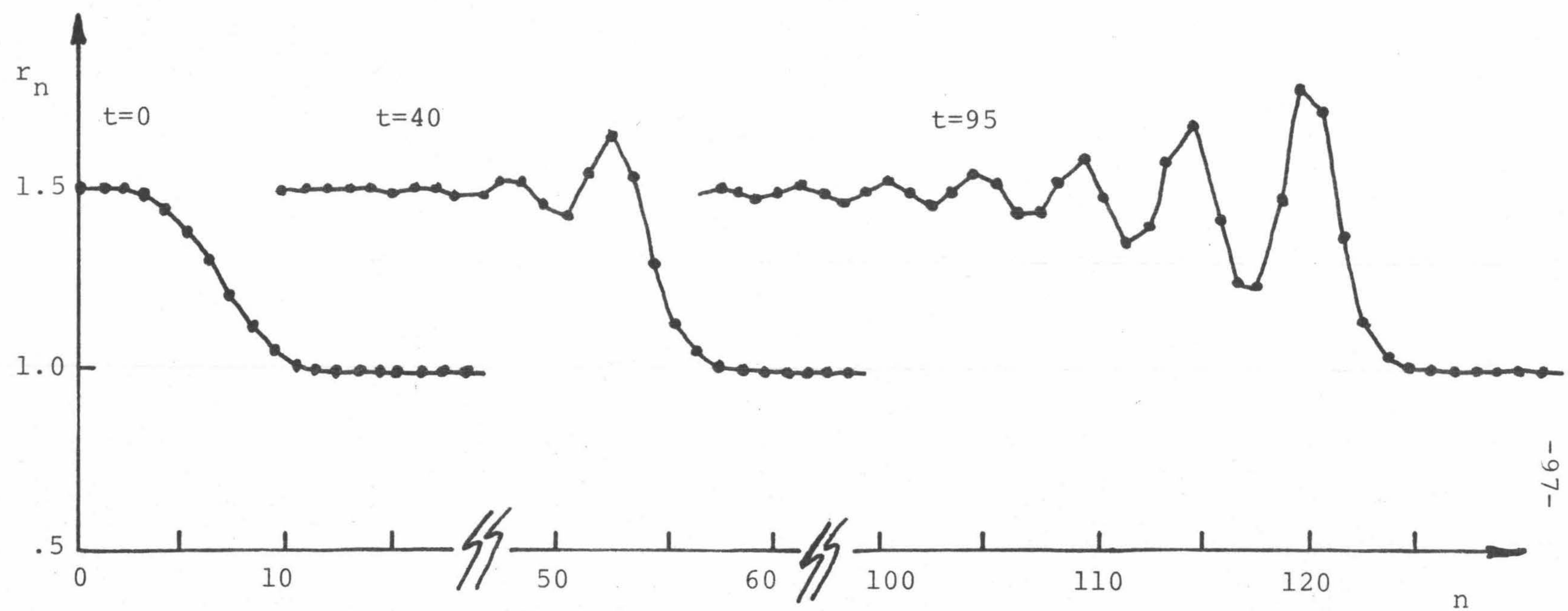


Fig. 6.3 The effect of dispersion on breaking in nonlinear problems.

Modulated waves

To study the propagation of modulations, the periodic solution $\varphi(\theta, a, k)$ given by (6.7) is imposed initially for some constant wave number and amplitude, i.e.

$$r_n(0) = \varphi(\theta_n(0), a, k), \quad \theta_n(t) = kn - \omega t ,$$

$$\dot{r}_n(0) = -\omega \varphi_\theta(\theta_n(0), a, k) ,$$

for n = 0, 1, ... N. The boundary condition at n = 0 is varied to send a modulated wave down the lattice. For a signal with an amplitude $a_o(t)$ and frequency $\omega_o(t)$, the phase function is computed as

$$\theta_o(t) = -\int_o^t \omega_o(t)\,dt ,$$

If the amplitude and frequency vary slowly, the appropriate boundary condition is

$$r_o(t) = \varphi(\theta_o(t), a_o(t), k_o(t)) , \qquad (6.9)$$

where k_o is determined by ω_o and a_o through the dispersion relation. The boundary condition at n = N is set to correspond to the initial periodic wave,

$$r_N(t) = \varphi(\theta_N(t), a, k) , \; \theta_N(t) = kN - \omega t ,$$

so that no reflections occur at this boundary due to the unmodulated signal.

When the functions $r_n(t)$ are computed numerically, they must be interpreted in terms of modulations $\omega_n(t)$, $k_n(t)$, n=0,N. The amplitude is determined at the nth node by measuring the local maxima of $r_n(t)$ and interpolating linearly between crests as shown in Fig. 6.4. The frequency is determined similarly by measuring the time between zeroes of $r_n(t)$.

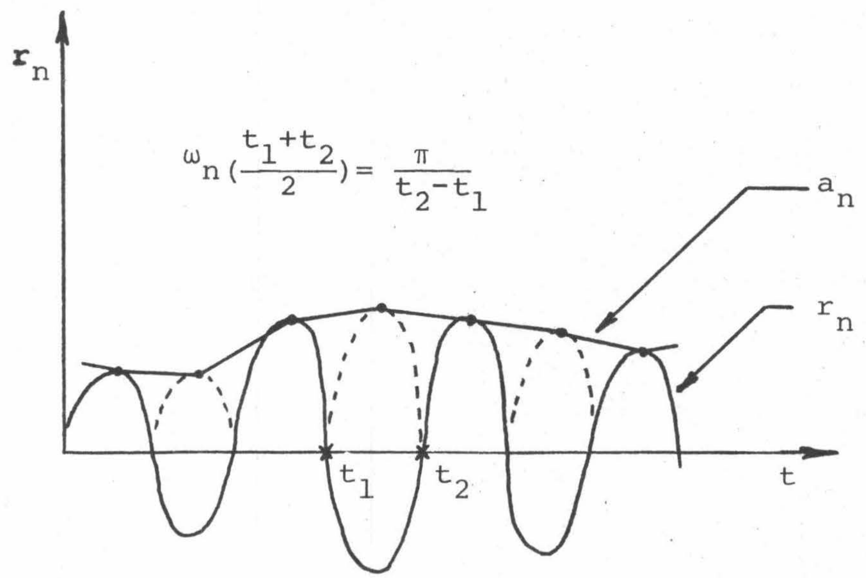


Fig. 6.4. Numerically computing the modulations

As mentioned in Chapter III, this is an approximation which computes the average values over one period.

To check the accuracy and feasability of the scheme, the solution is computed for the linear case ($\sigma = 0$) and no modulations at the boundary (6.9). The exact solution is then known,

$$r_n(t) = a \cos(kn - \omega t)$$

and can be compared with the computed values. For this calculation we take a lattice of sixty elements, and

$$k = \frac{2\pi}{3}, \quad \omega = 1.732, \; a = 1 \; .$$

The integration is performed over $t \in (0, 50)$ for time step sizes $h = .1$ and $h = .05$. The results:

N = 60, t = 50		h = .1	h = .05
Max. error	r_n	.3%	.01%
"	a	.3%	.1%
"	ω	<.05%	<.05%

(6.10)

The total computing time on the IBM 370/155 is about 80 sec for $k = .05$.

Nonlinear Case-Splitting and Breaking

A signal is sent down the lattice by decreasing the frequency at $n = 0$, while the amplitude is held constant. This will cause a disturbance along both families of characteristics for which breaking is predicted by the first order theory.

The initial parameters are

$$k = \frac{2\pi}{3}, \quad a = 1 ,$$

$$\omega = W(\tfrac{2\pi}{3}, 1) = 1.665 \quad .$$

For this choice of wave number the exact formula (6.8) can be used for the initial periodic wave. Since the amplitude is of the order 1, the parameter σ is chosen to have a small value,

$$\sigma = -.1 ,$$

so that the near-linear theory will still be reasonable.

A lattice of 100 elements is used with an integration step size $h = .05$. Table (6.10) shows an error of .01% in the computed $r_n(t)$ at time $t = 50$. The results should be accurate for times on the order of $t = 100$.

The frequency is decreased linearly over the interval $t \in (2,8)$ to a final value $\omega = 1.465$. The time and distance down the lattice when splitting of the characteristics occurs can be estimated as

$$x_s \approx \frac{C_+ C_-}{C_+ - C_-} \Delta t, \quad t_s \approx t_0 + \frac{C_+}{C_+ - C_-} \Delta t ,$$

where the signal at the source begins at time t_0 and lasts for a time Δt. Using near-linear theory, the group velocities in terms of ω and a are

$$C_{\pm} = W_0' \pm \frac{\frac{1}{2}\omega}{1 + \frac{3}{8}\sigma a^2} \left(\frac{3}{8} \sigma a^2\right)^{\frac{1}{2}} ,$$

$$\text{where} \quad W_0' = \cos \sin^{-1} \left(\frac{\frac{1}{2}\omega}{1 + \frac{3}{8}\sigma a^2} \right) .$$

For an intermediate value of the frequency, $\omega = 1.56$, and $a = 1$, we calculate

$$C_+ = .743 , \qquad C_- = .429 ,$$

$$x_s \approx 8 \quad , \qquad t_s \approx 21 \quad .$$

The position and time of breaking can also be estimated. For the C_+ family, the group lines cross at

$$x_B = C^2\left(\frac{dC_+}{dt}\right)^{-1} , \quad t_B = C_+\left(\frac{dC_+}{dt}\right)^{-1} + t_0$$

where $\frac{dC_+}{dt}$ is the rate of change of the group velocity at the source. For $\omega = 1.66$, $\frac{dC_+}{dt} = .025$, $C_+ = .673$ we have

$$x_B \approx 18, \quad t_B \approx 29 \quad .$$

The modulations are separated completely before any breaking occurs, and according to condition (5.29c), the nonlinearity is strong enough to produce the transition regions instead of overlap after breaking.

The solution shown in Fig. 6.5 agrees very well with the predicted behavior, Fig. 5.7. The two transitions have clearly separated from the disturbance leaving a region between which is precisely an unmodulated periodic wave. Using the background values at $\pm\infty$.

$$\omega_1 = 1.465, \ a_1 = 1, \ k_1 = 1.730,$$

$$\omega_2 = 1.665, \ a_2 = 1, \ k_2 = 2.090 ,$$

we can calculate the parameters for this center region from (5.21). Here we have

$$F(k) = \frac{-1}{\sqrt{6|\sigma|}} \ k \ .$$

The predicted values are

$$a_{TH} = 1.232 , \quad \omega_{TH} = 1.540 \ ;$$

these compare well with the values from the computation,

$$a_{NUM} = 1.23 , \quad \omega_{NUM} = 1.52 \quad .$$

Looking at the disturbance in front, we see an almost steady transition, preceded by a growing oscillation in both the frequency and the amplitude. The speed of this steady region is

$$V = .67 \quad .$$

Behind this, the values are

$$\omega = 1.52, \quad a = 1.23, \quad k = 1.87$$

which give $C_+ = .78$. In front, as $x \to +\infty$, the values are

$$\omega = 1.67, \quad a = 1.00, \quad k = 2.09 ,$$

which give $C_+ = .67$. The suggestion in (5.29a) that V be close to a typical C_+ is verified. The change in wave number across the disturbance is $\Delta k = .22$. According to (5.29b), the change in amplitude should be

$$\Delta a_{TH} = \frac{-1}{\sqrt{6|\sigma|}} \Delta k = .28 ,$$

and the computed value is

$$\Delta a_{NUM} = .23 \quad .$$

Thus, we have confirmed the theoretical predictions concerning nonlinear group velocities and group splitting as well as the higher order effects near breaking.

In comparing Fig. 6.5.4 with the first order solution Fig. 5.3, we see that they are very similar, even though the numerical solution has progressed well past breaking ($t/t_B \approx 4$). Apparently we can extend the first order theory (5.16) past breaking by replacing the singular regions with the transition zones shown in Fig. 5.6. The result should agree well with the true behavior for times up to t_B and well beyond. This is in contrast to the linear problem, for which the first order theory cannot be extended to give the behavior near t_B, and only in a limited sense for $t \gg t_B$.

Figs. 6.5.1 - 6.5.4

Breaking modulations for the nonlinear case

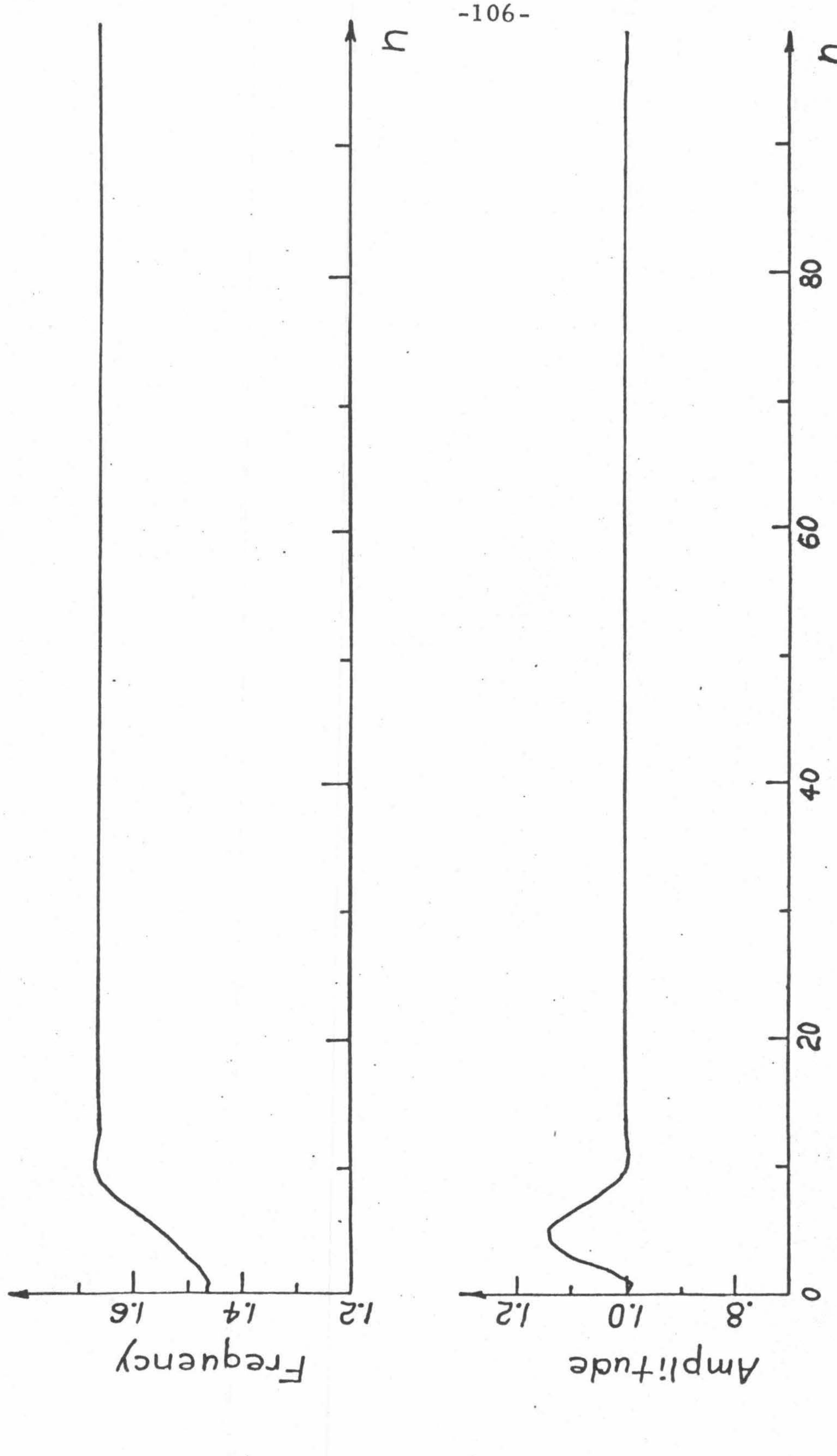


Fig. 6.5.1 $t = 15$

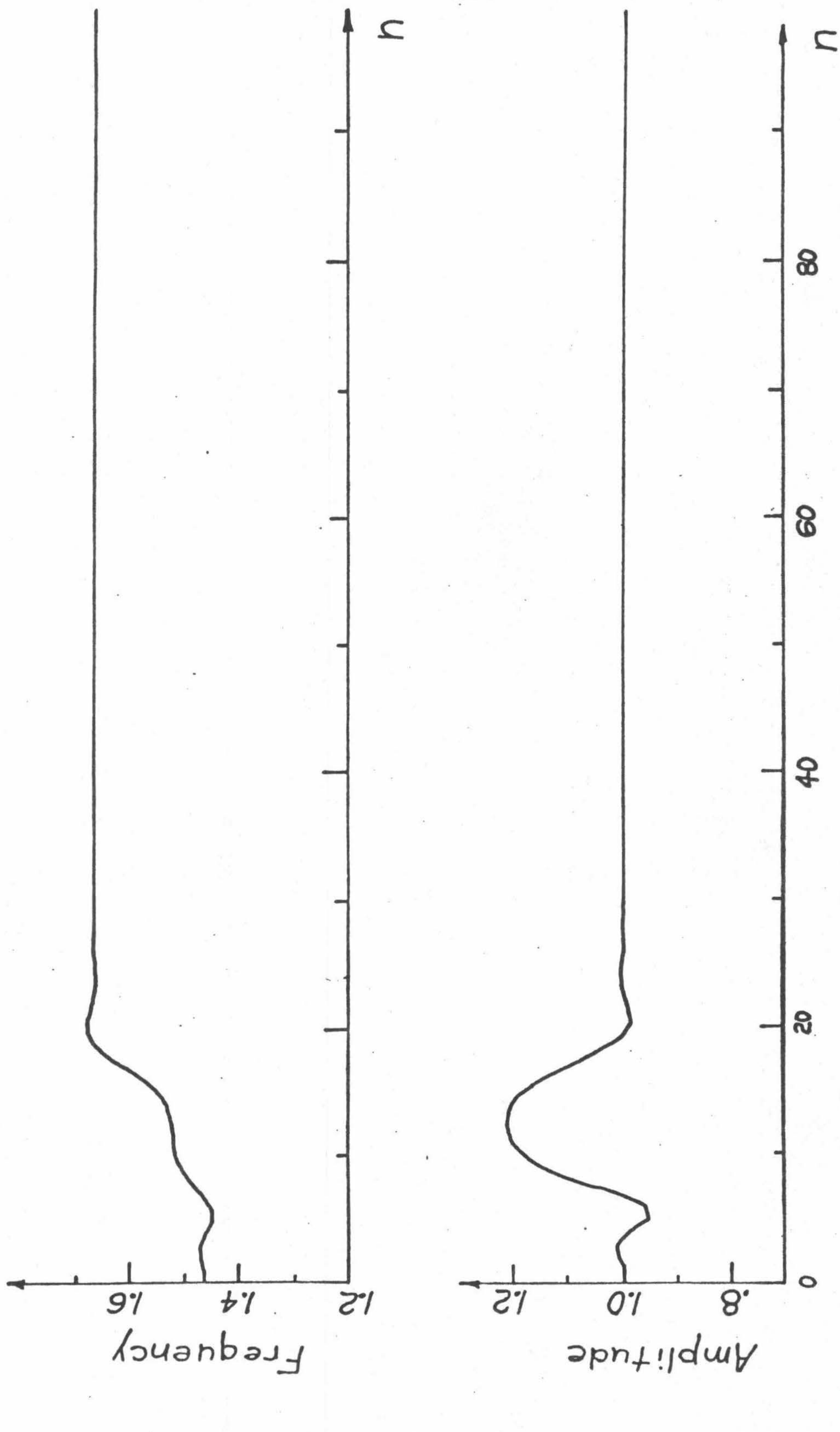


Fig. 6.5.2 $t = 30$

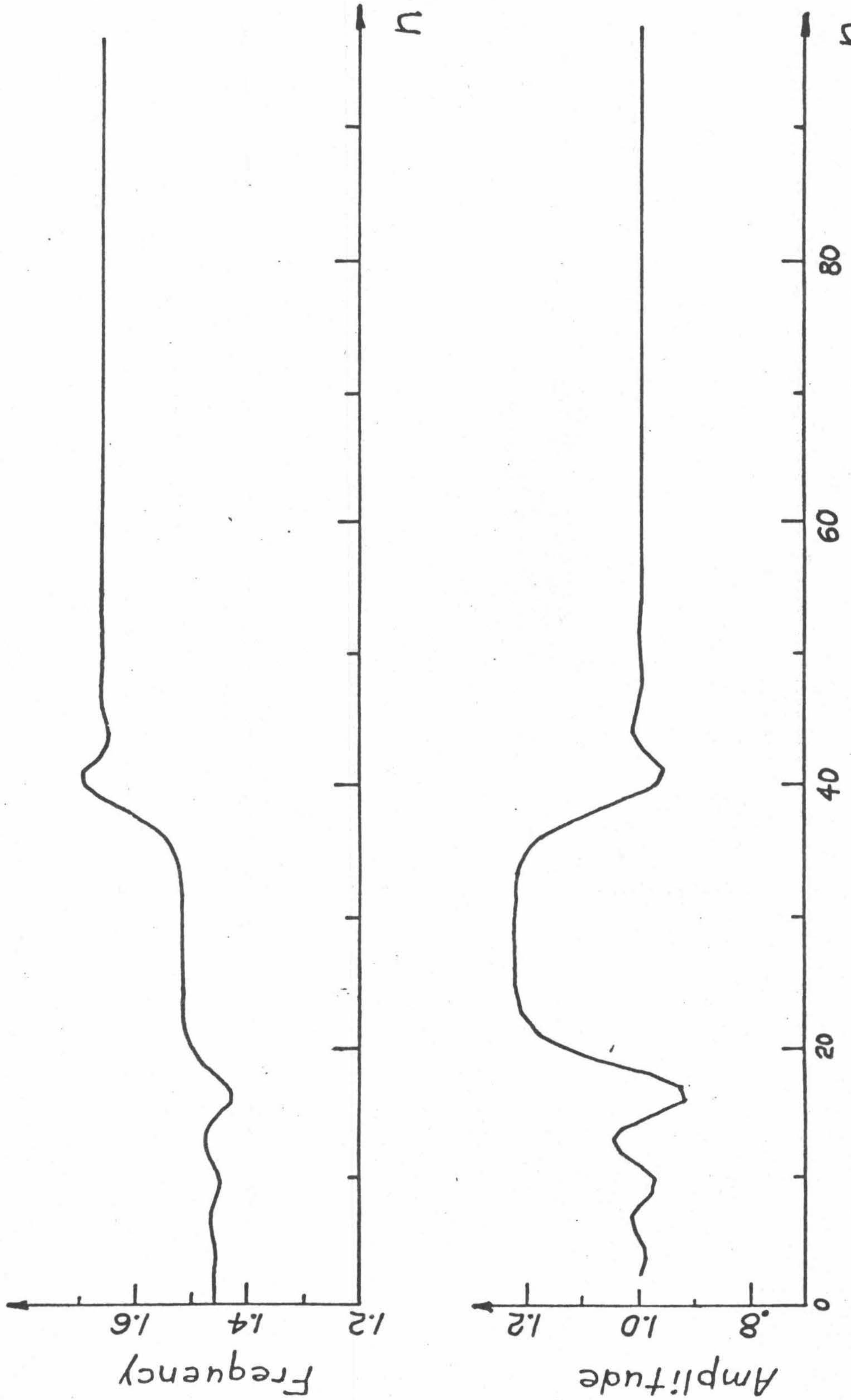


Fig. 6.5.3 $t = 60$

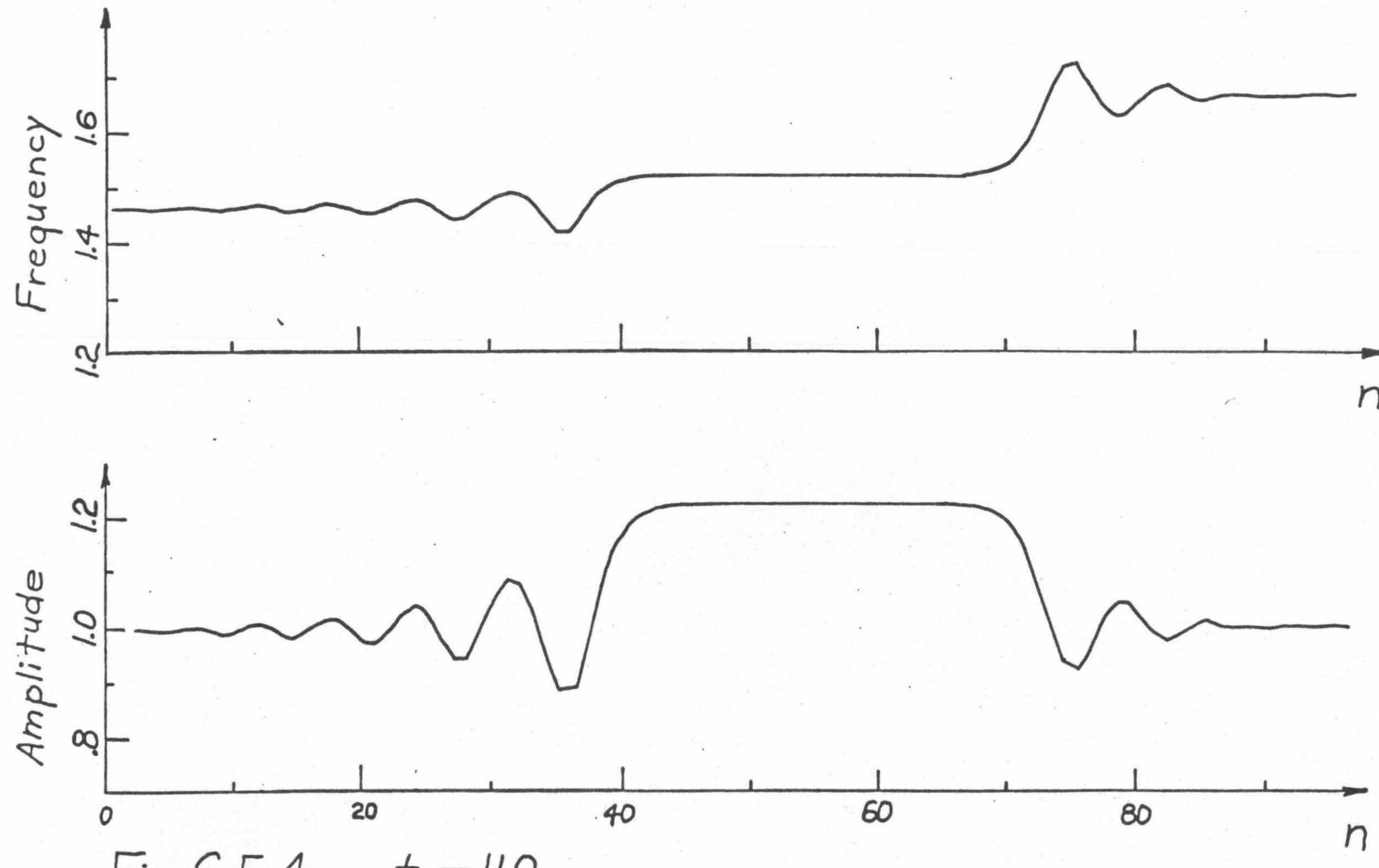


Fig. 6.5.4 $t = 110$

Linear case

The previous case is solved again for a linear force law $\sigma = 0$, $f(r) = r$, as another example of the breaking problem analyzed in Chapter III. The numerical solution, Fig. 6.6, agrees with the exact solution, Fig. 3.5, demonstrating that the numerical methods used are reliable.

The time of breaking for this case is

$$t_B = 24 \quad .$$

The speed of the amplitude peak near breaking is

$$V = .56 \quad ,$$

which agrees with the group velocity calculated at the center ($\omega = 1.64$),

$$C = .57 \quad .$$

Figs. 6.6.1 - 6.6.3

Breaking modulations for the linear case

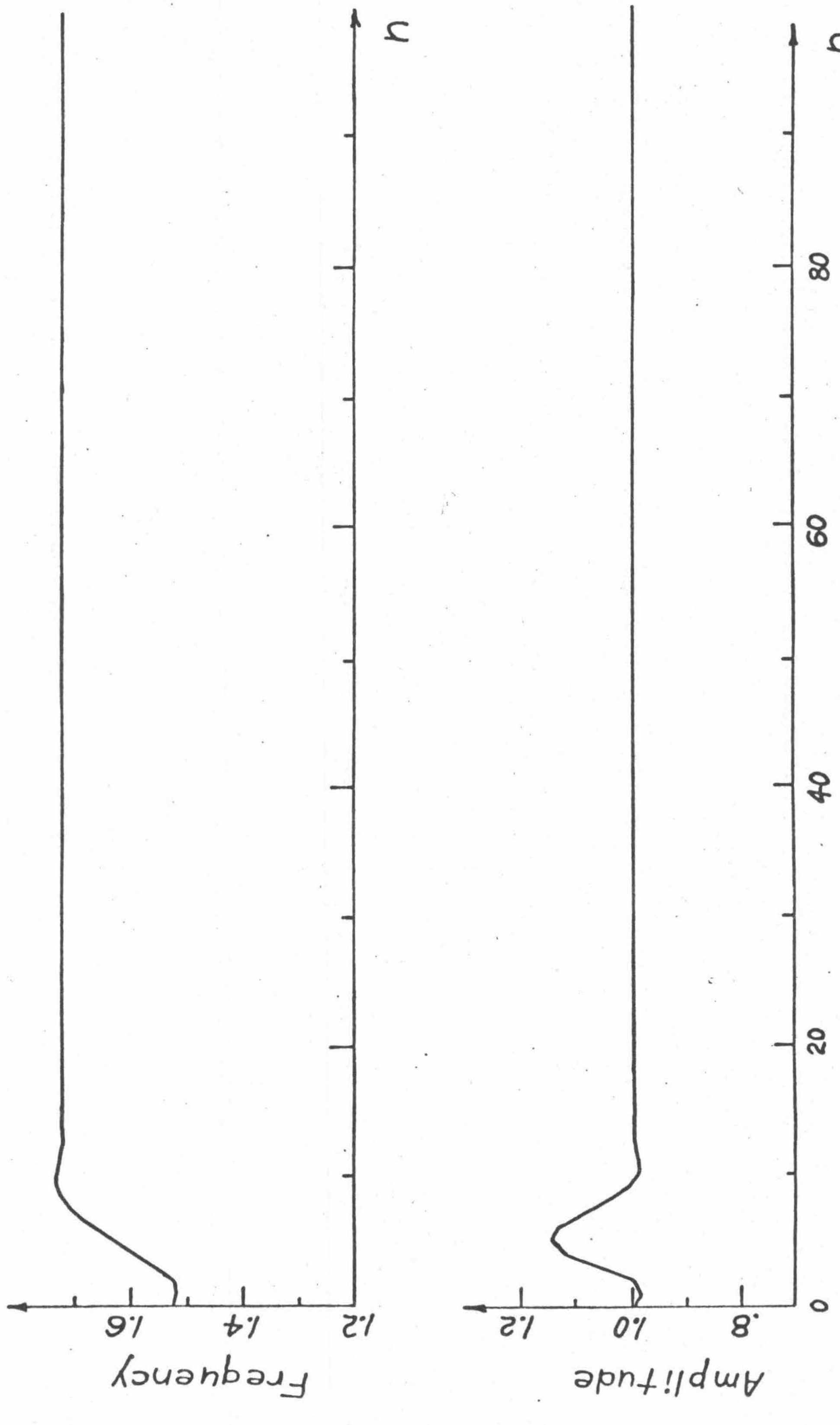


Fig. 6.6.1 t = 15

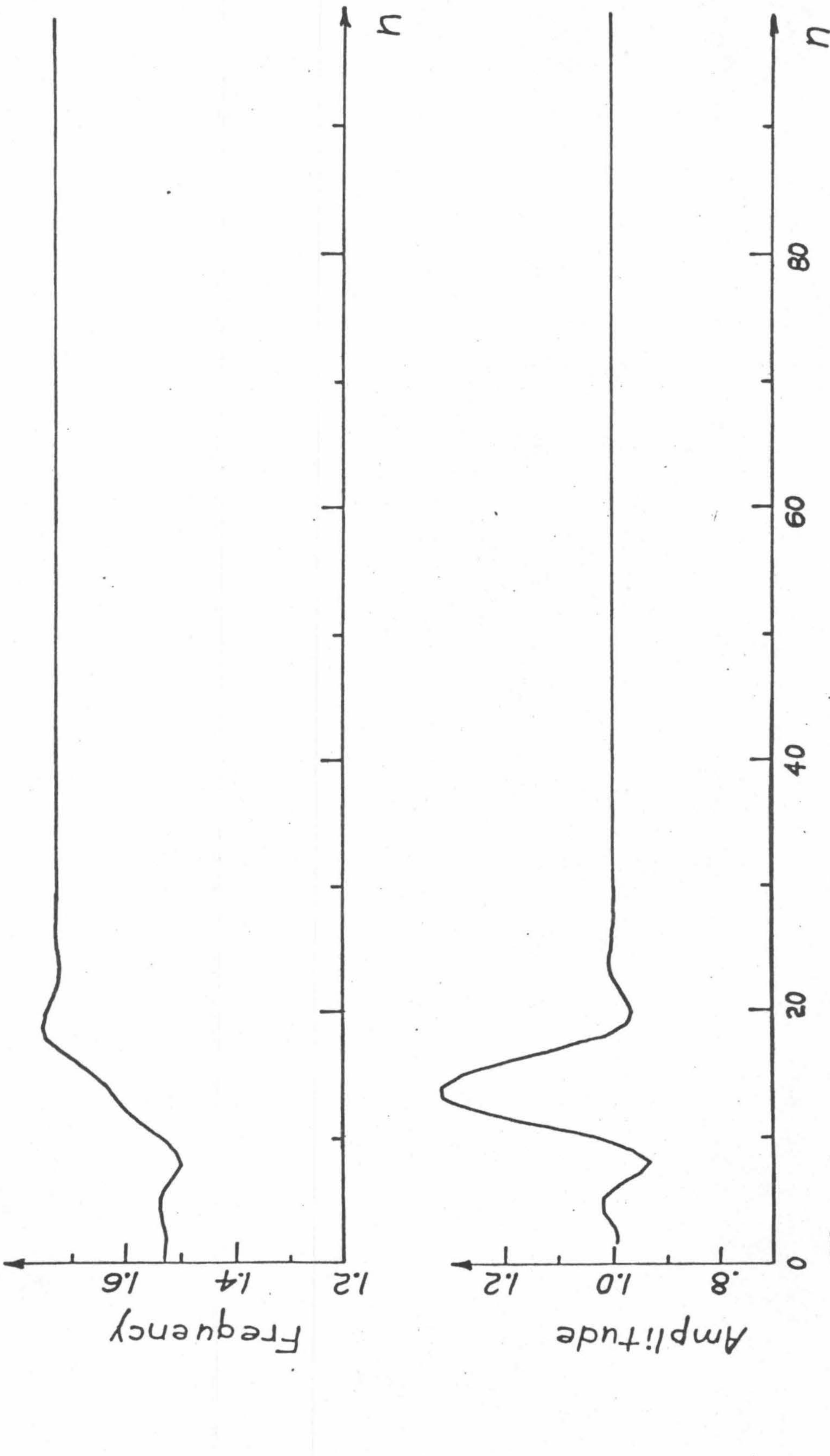


Fig. 6.6.2 $t = 30$

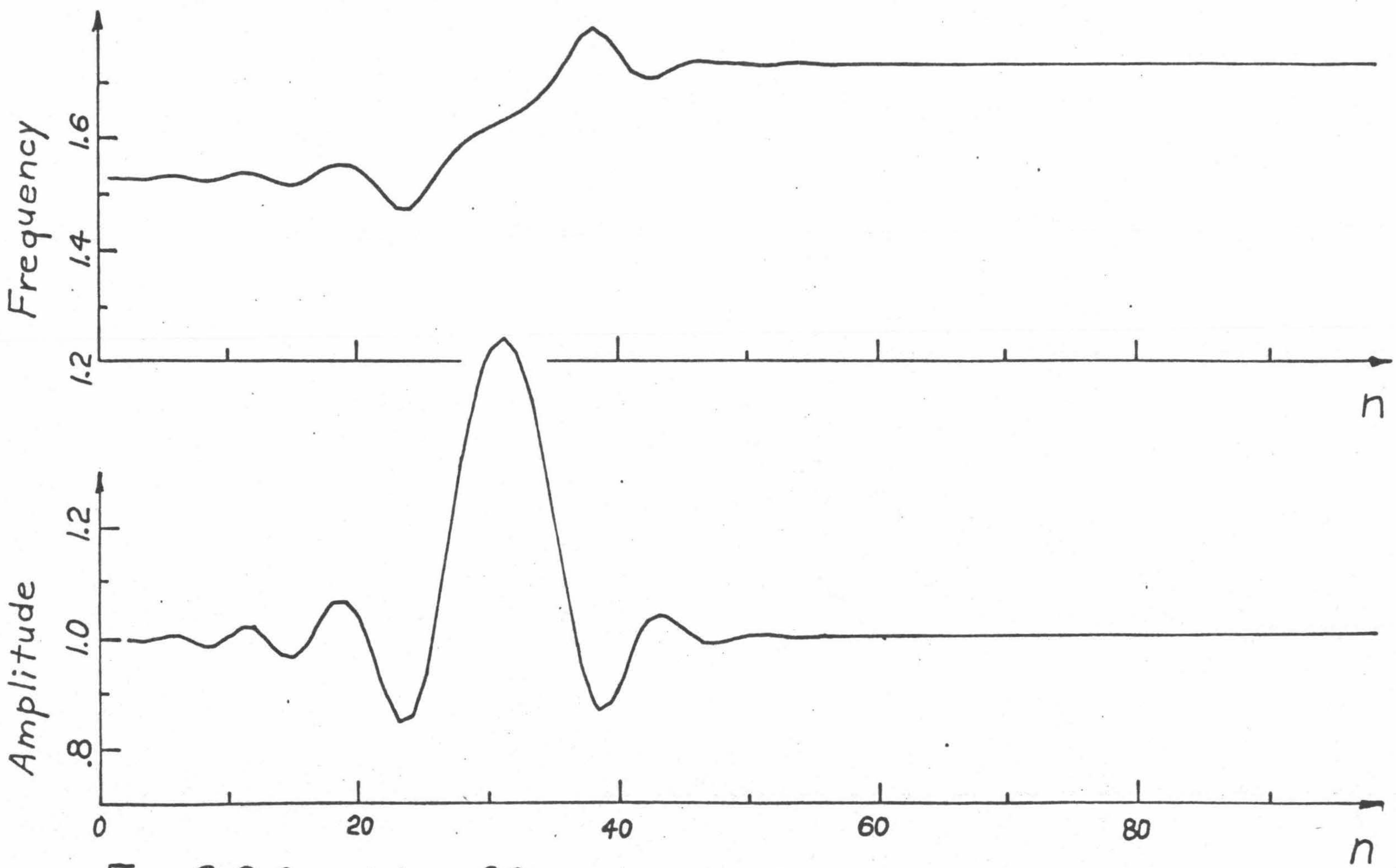


Fig. 6.6.3 $t = 60$

Instability

The modulation theory predicts that the periodic wave (6.7) is unstable for the case of a hard spring $\sigma > 0$. For the value

$$\sigma = +.1 \ ,$$

a small sinusoidal perturbation given by

$$\omega_0(t) = 1.667 + .05 \sin(.07t) \ ,$$

is introduced at the boundary. The rapid growth of this disturbance, shown in Fig. 6.7, demonstrates that this instability actually occurs.

Due to computational limitations, the solution cannot be followed to its end result. Therefore, we cannot say from these calculations whether or not instability results in a sequence of solitary wave packets.

Figs. 6.7.1 - 6.7.3

Instability of the periodic wave

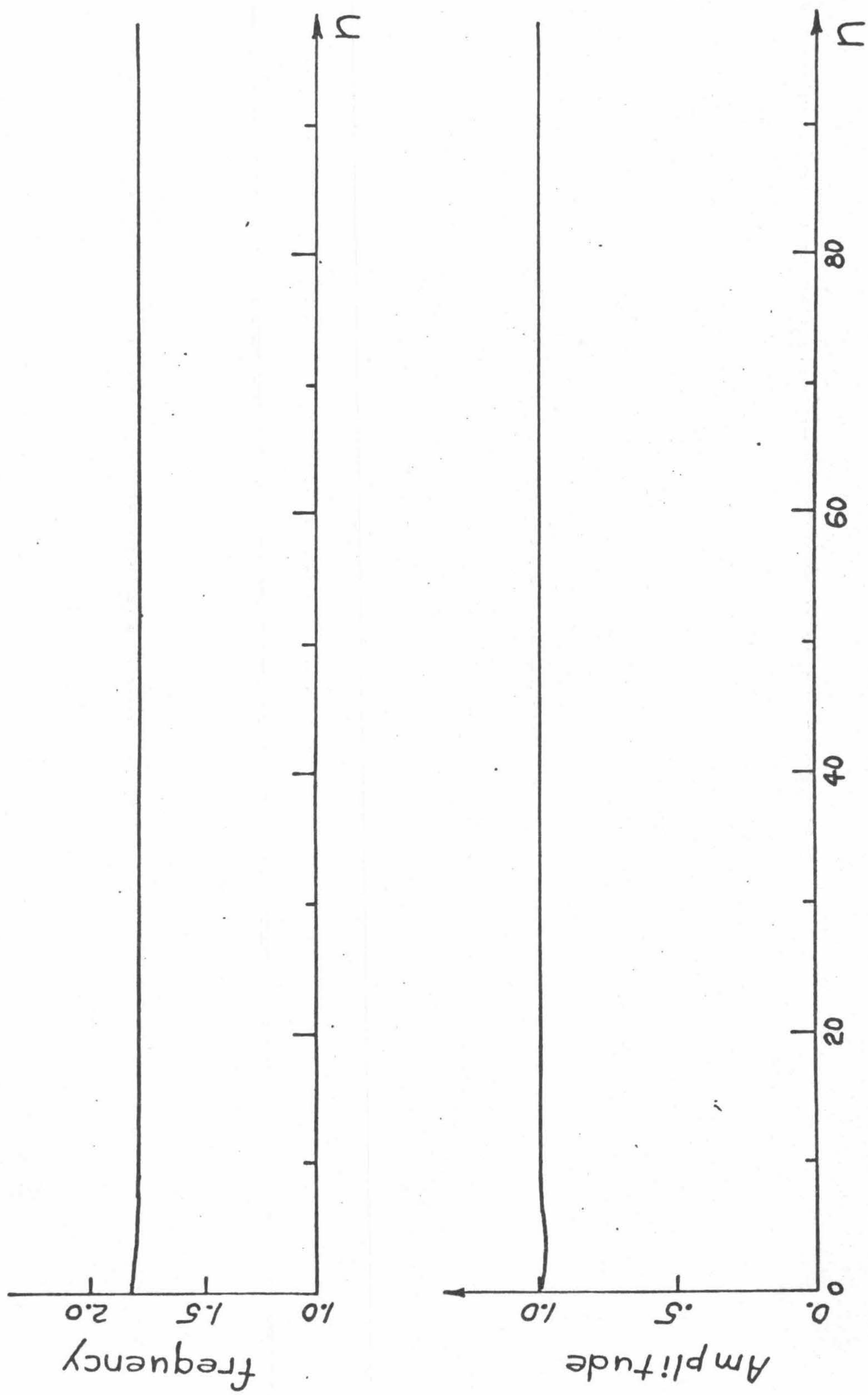


Fig. 6.7.1 t = 10

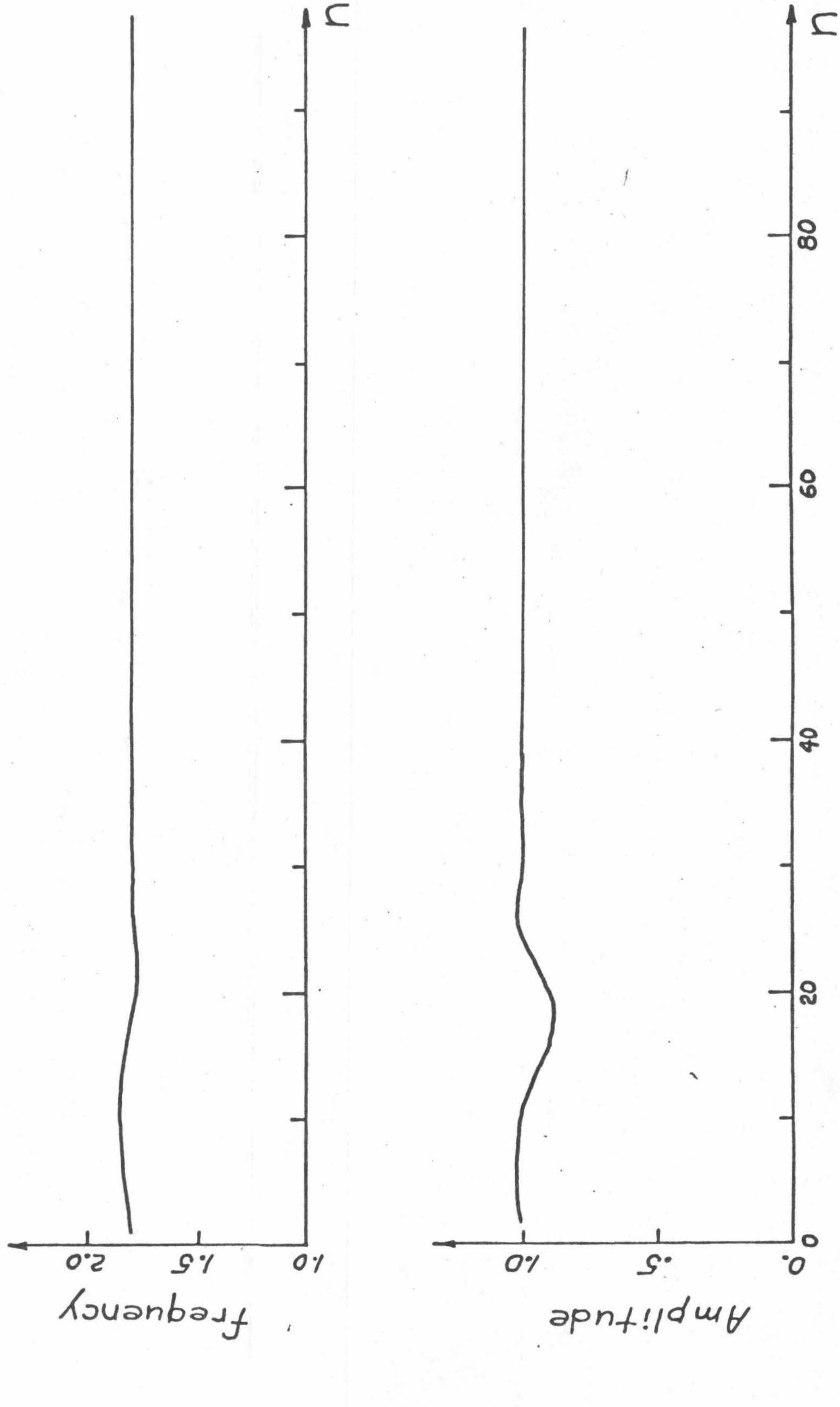


Fig. 6.7.2 $t = 40$

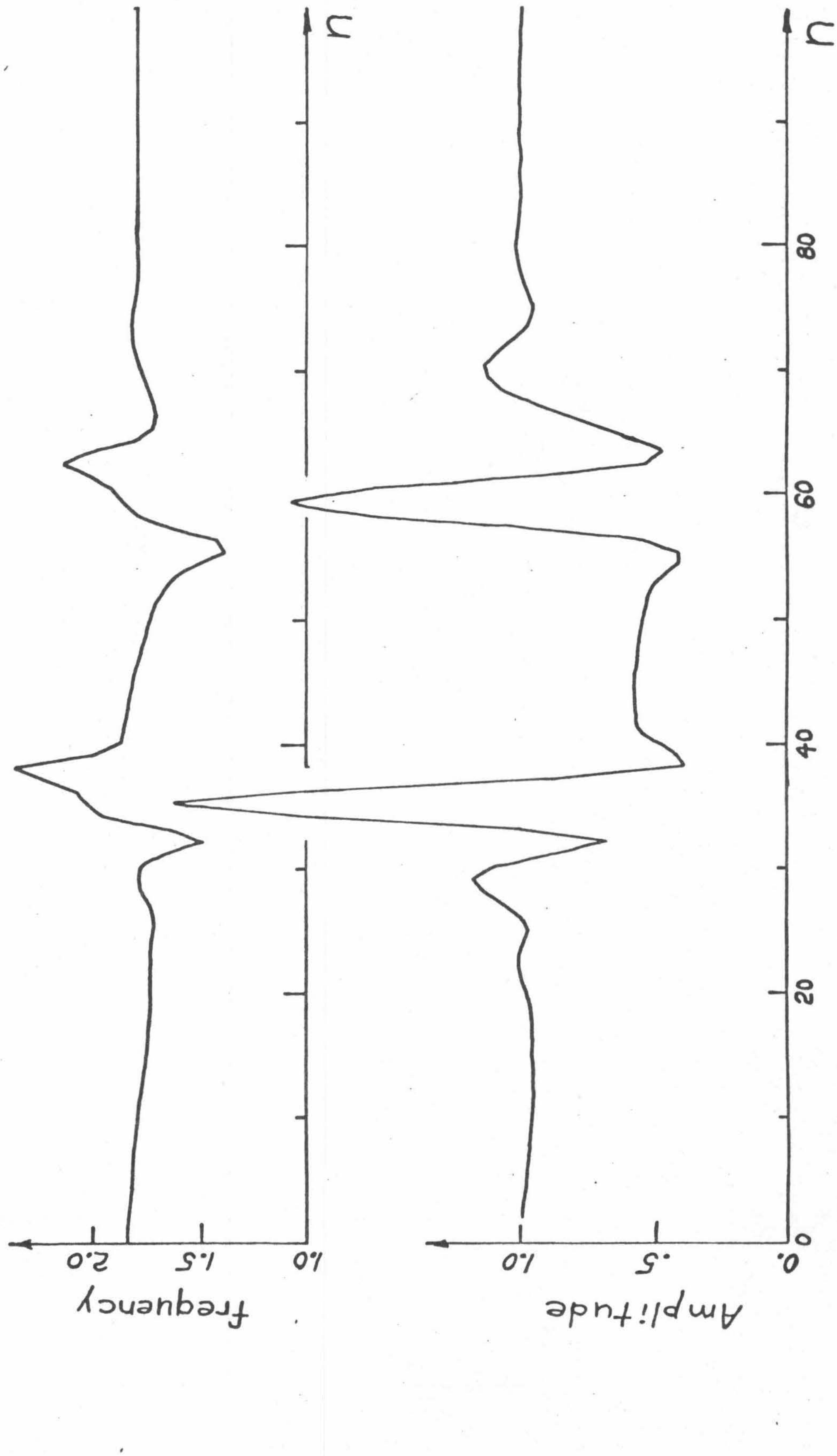


Fig. 6.7.3 $t = 100$

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