TWO NEW INTEGRAL TRANSFORMS AND THEIR APPLICATIONS

Thesis by

X X Newhall

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ABSTRACT

This thesis is in two parts. In Part I the independent variable θ in the trigonometric form of Legendre's equation is extended to the range $(-\infty, \infty)$. The associated spectral representation is an infinite integral transform whose kernel is the analytic continuation of the associated Legendre function of the second kind into the complex θ-plane. This new transform is applied to the problems of waves on a spherical shell, heat flow on a spherical shell, and the gravitational potential of a sphere. In each case the resulting alternative representation of the solution is more suited to direct physical interpretation than the standard forms.

In Part II separation of variables is applied to the initial-value problem of the propagation of acoustic waves in an underwater sound channel. The Epstein symmetric profile is taken to describe the variation of sound with depth. The spectral representation associated with the separated depth equation is found to contain an integral and a series. A point source is assumed to be located in the channel. The nature of the disturbance at a point in the vicinity of the channel far removed from the source is investigated.

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INTRODUCTION

Series and integral transforms are a powerful and widely used method of solving differential equations. The appropriate ones to apply to a given separable PDE are determined by the separated ODE's. Each of those transforms is called the spectral representation associated with the corresponding ODE.

Given a PDE in n variables we may apply successively any n-1 of the n associated spectral representations. The resulting ODE may be solved by any of the usual techniques. The ultimate form of the solution of the PDE is determined by which n-1 representations are applied. These different (but equivalent) forms are called alternative representations. The alternative representation most suitable for a given PDE is often not obvious beforehand.

The spectral representation associated with an ODE is found by a contour integration of the Green's function. We exploit that method in both parts of this thesis. In Part I we extend the domain of the angle θ in the trigonometric form of Legendre's equation to $-\infty < \theta < \infty$. The resulting spectral representation is a new integral transform that provides concise alternative representations for the solutions of a variety of well-known problems.

In Part II we investigate one aspect of the propagation of sound waves in the ocean. The PDE is in three independent variables. Two of the three associated spectral representations are the familiar Fourier and Hankel transforms, which lead to an alternative representation that can be analyzed only with difficulty. We derive the spectral representation associated with the third separated ODE and apply it to get a more useful form of the solution.

The numbers in square brackets [ ] throughout the text indicate references listed at the end.

PART I

An Infinite-Angle Generalized Mehler Transform and its Application.

Section 1. Introduction.

We shall need the spectral representation associated with the DE

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dw}{d\theta}\right)+\left[(\nu^2-\tfrac{1}{4})-\frac{\mu^2}{\sin^2\theta}\right]w=0,\ -\infty<\theta<\infty. \tag{1.1}$$

This equation arises from expressing the equation for electrostatic potential in spherical coordinates. The special case with $\mu = 0$ results from the equations for linear waves and heat flow on a spherical shell. Equation (1.1) is Legendre's equation of degree $\nu - \frac{1}{2}$ and order μ. The important feature here is that the independent variable θ in (1.1) is defined on the fully infinite interval $(-\infty, \infty)$; the usual range for θ is confined to $[0, \pi]$.

In the first three of the following sections we derive the spectral representation for (1.1). Section 2 interprets the infinite angle θ. We define the solutions of (1.1) in Section 3 and derive some of their properties. Finally in Section 4 we obtain two forms of the spectral representation along with the useful special case for $\mu = 0$.

The remainder of Part I is devoted to the application of these representations to three classical problems involving a sphere: waves on a spherical shell, the heat flow on a spherical shell, and the gravitational or electrostatic potential outside a sphere. Our transform provides useful new alternative representations for the solutions of these problems.

Section 2. Interpretation of the Infinite Range of θ.

Throughout nature a vast assortment of problems relate to the propagation of waves. One class of wave phenomena is that of long waves on the surface of a sphere. In the case of the earth itself seismic sea waves and extreme acoustical disturbances in the atmosphere are prominent examples.

For axisymmetric sphere surface waves initially confined close to one pole the qualitative behavior is easy to guess. The wave front will proceed symmetrically toward the opposite pole, reflect or run through itself there, and continue on toward the starting pole. It will again meet itself and continue on to the opposite pole, repeating this process indefinitely.

The fact that the wave front evidently proceeds forever suggests that we choose as our polar coordinate the total arc length or latitude θ over which the disturbance travels. For reasons of symmetry in the spectral representation, we define the physical surface of the sphere to be defined by the variables:

$$\text{physical latitude,} \qquad -\pi \leq \vartheta \leq \pi \tag{2.1}$$

$$\text{physical azimuth,} \qquad 0 \leq \phi < \pi \quad . \tag{2.2}$$

Figure 1 is a diagram of the sphere as seen by an observer looking along the polar axis down onto the equatorial plane.

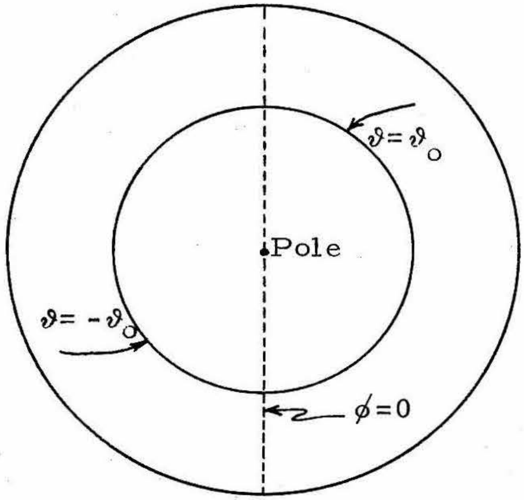

Figure 1. The modified coordinates of the surface of a sphere.

The inner circle is a parallel of physical latitude. The right half of the parallel is at $\vartheta = \vartheta_o$, the left half at $\vartheta = -\vartheta_o$. As the disturbance emanates from the pole, the portion of the wave front initially going in the direction of $\vartheta > 0$ is defined to travel over arc length range $0 \leq \theta < \infty$; the part initially going toward $\vartheta < 0$ travels over arc length $0 \geq \theta > -\infty$.

Using this convention, an observer situated at physical latitude $\vartheta > 0$ sees the first pass of the wave front when $\theta = \vartheta$. The other half of the wave passes him on its first return trip to the starting pole, at $\theta = -2\pi + \vartheta$. When the front passes the observer on its second journey toward the opposite pole, he sees the portion at total arc length $\theta = \vartheta + 2\pi$, and on the second return trip the other half meets him at $\theta = -4\pi + \vartheta$. Clearly, the total disturbance at physical latitude ϑ is the sum of the individual contributions at arc length $\vartheta + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. This fact is of great importance in the analysis of the problems in the following sections. It should be noted that $\vartheta \equiv \theta$ in the interval $[-\pi, \pi]$.

Section 3. The Analytic Continuation of Legendre Functions.

In this section we examine the nature of the solutions of Legendre's equation (1.1) defined on the infinite θ-interval. When this equation is restricted to the customary range $0 \leq \theta \leq \pi$, the two usual notations for the solutions are

$$w = P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta) \qquad \text{and} \qquad w = Q^{\mu}_{\nu-\frac{1}{2}}(\cos\theta) . \tag{3.1}$$

They are called the associated Legendre functions of the first and second kinds, respectively. We choose degree $\nu-\frac{1}{2}$ because $P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$ is an even function of ν, ([1], p. 140, (1)), and $Q^{\mu}_{-\nu-\frac{1}{2}}(\cos\theta)$ is a solution independent from $Q^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$.

To extend these solutions to an infinite θ interval, one might be tempted to observe that $\cos(\theta + 2n\pi) = \cos\theta$ and hence hold that

$$Q^{\mu}_{\nu-\frac{1}{2}}(\cos(\theta + 2n\pi)) = Q^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$$

$$\text{and} \quad P^{\mu}_{\nu-\frac{1}{2}}(\cos(\theta + 2n\pi)) = P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta) .$$

However, such a direct extension of the solutions is in error. The notation (3.1) for the solutions is intended only for the range $[0, \pi]$. It is well established (see [1], pp. 163-4) that the Legendre functions have branch points at $\theta = 0$ and $\theta = \pi$. We must use another device to bypass the branch points and continue onto the extended θ interval.

We let θ be an unrestricted complex variable and consider the mapping $z = \cos\theta$, shown in Figure 2.

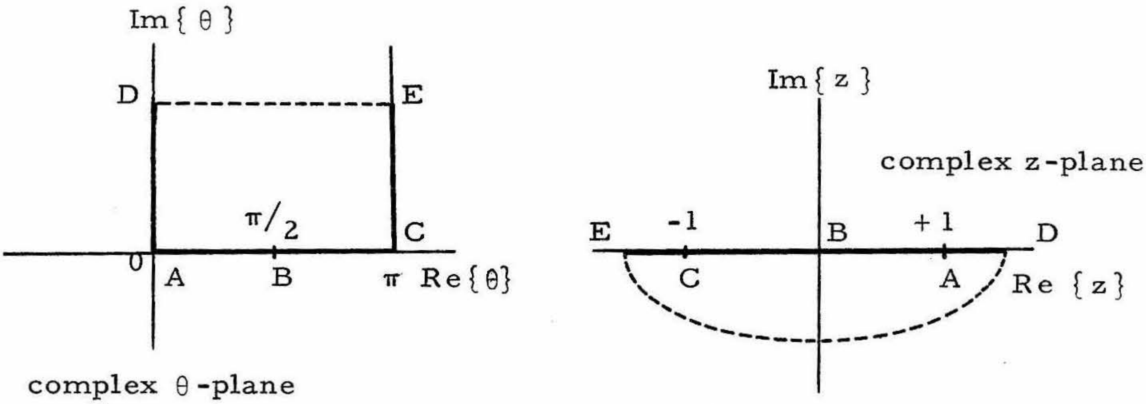

Figure 2. The mapping $z = \cos\theta$.

From this illustration it is apparent that the lower half of the z-plane $\mathscr{I}m\{z\} \leq 0$ maps onto the upper semi-infinite strip in the θ-plane: $0 \leq Re\{\theta\} \leq \pi$; $\mathscr{I}m\{\theta\} \geq 0$.

By making the substitution $z = \cos\theta$ in (1.1), we get the algebraic form of Legendre's equation:

$$\frac{d}{dz}\left[(1-z^2)\frac{dw}{dz}\right] + \left[(\nu^2 - \tfrac{1}{4}) - \frac{\mu^2}{(1-z^2)}\right]w = 0 \qquad (3.2)$$

This equation is of hypergeometric type. Of the many representations of the solutions we shall find it convenient to choose ([1], pp. 136-137):

$$w = Q^{\mu}_{\nu-\frac{1}{2}}(z) = e^{i\mu\pi}\sqrt{\pi}\,2^{\mu}\,\frac{\Gamma(\frac{1}{2}+\nu+\mu)}{\Gamma(1+\nu)}\,(z^2-1)^{\frac{1}{2}\mu}\,[z+(z^2-1)^{\frac{1}{2}}]^{-\frac{1}{2}-\nu-\mu}\,.$$

$$\cdot\, {}_2F_1\left[\tfrac{1}{2}+\mu,\ \tfrac{1}{2}+\nu+\mu;\ 1+\nu;\ \frac{z-(z^2-1)^{\frac{1}{2}}}{z+(z^2-1)^{\frac{1}{2}}}\right], \qquad (3.3)$$

where $\Gamma(\zeta)$ is the gamma function and

$${}_2F_1\,[a,b;c;\zeta] = 1 + \frac{a\cdot b}{c\cdot}\,\frac{\zeta}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\,\frac{\zeta^2}{2!} + \dots \qquad (3.4)$$

is the hypergeometric series.

Expression (3.3) has branch points at $z = \pm 1$. We must define the branches so that when $\mathcal{I}m\{z\} < 0$ (3.3) converges for $\mathcal{I}m\{\theta\} > 0$ under the substitution $z = \cos\theta$.

We write

$$(z^2-1)^{\frac{1}{2}} = (z-1)^{\frac{1}{2}}\,(z+1)^{\frac{1}{2}} = \left(R_1 e^{i\varphi_1}\right)^{\frac{1}{2}}\left(R_2 e^{i\varphi_2}\right)^{\frac{1}{2}}, \qquad (3.5)$$

where the magnitudes and arguments are shown in Figure 3.

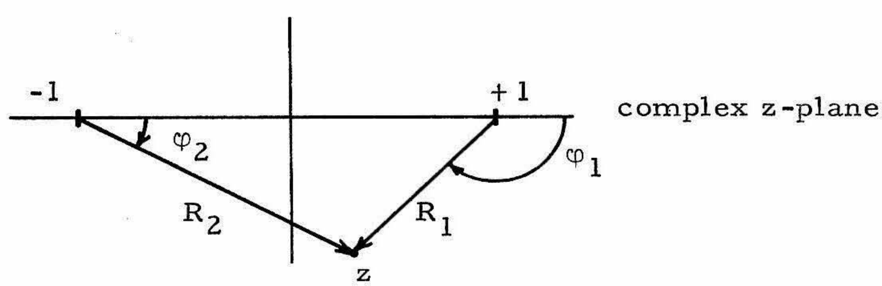

Figure 3. The numbers z-1 and z+1.

As $z \to 0$ from any point in the lower half plane, $(z^2-1) \to -1$, $R_1 \to 1$, $R_2 \to 1$, $\varphi_1 \to -\pi$, $\varphi_2 \to 0$. Therefore

$$-1 = R_1 R_2 \, e^{i(\varphi_1+\varphi_2)} = e^{-i\pi}, \tag{3.6}$$

$$\text{and} \quad (z^2-1)^{\frac{1}{2}} = e^{-\frac{\pi i}{2}} (1-z^2)^{\frac{1}{2}} . \tag{3.7}$$

Then if we choose the principal branch of the square root, the substitution $z = \cos\theta$ gives, from (3.7),

$$(z^2-1)^{\frac{1}{2}} = -i \sin\theta . \tag{3.8}$$

Making these substitutions in (3.3) we see that the entire expression will be a function of θ. Emphasizing that we wish to consider θ and not $\cos\theta$ as the independent variable, the result is

$$E^{\mu}_{\nu-\frac{1}{2}}(\theta) \equiv \sqrt{\pi}\, 2^{\mu} \frac{\Gamma(\frac{1}{2}+\nu+\mu)}{\Gamma(1+\nu)} e^{\frac{\pi i \mu}{2}} (\sin\theta)^{\mu} e^{i(\frac{1}{2}+\nu+\mu)\theta} \cdot {}_2F_1\left[\tfrac{1}{2}+\mu, \tfrac{1}{2}+\nu+\mu; 1+\nu; e^{2i\theta}\right] \tag{3.9}$$

The quantity $E^{\mu}_{\nu-\frac{1}{2}}(\theta)$ is of course equal to $Q^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$ for θ on the real axis between 0 and π.*

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*The notation $E^{\mu}_{\nu-\frac{1}{2}}(\theta)$ is an extension of that used by Clemmow [2], who derived a special case of the infinite-angle transform.

The hypergeometric series in (3.9) converges everywhere on and above the real axis in the θ-plane except at branch points located at $\theta = \pm n\pi$, $n = 0, 1, 2, \ldots$. If we put the cuts as shown in Figure 4 the expression (3.9) is defined for the entire upper half θ-plane.

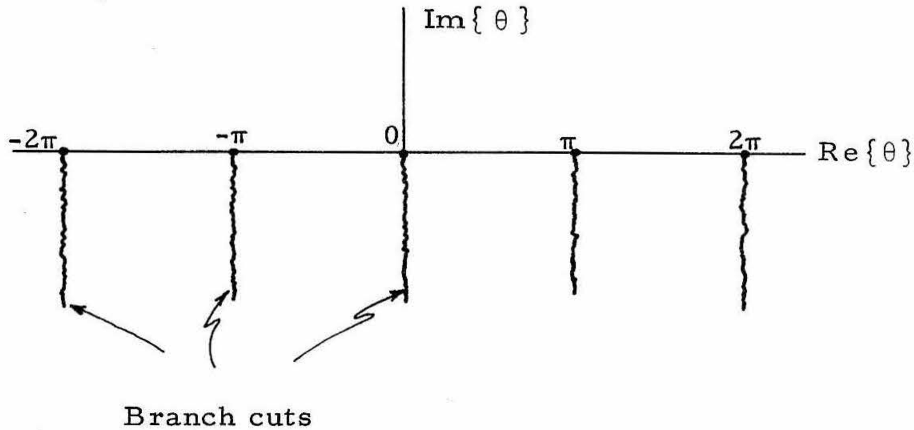

Figure 4. $E^{\mu}_{\nu-\frac{1}{2}}(\theta)$ is defined on and above the real axis except at branch points.

If we replace θ by $\theta + n\pi$ in (3.9) we get

$$E^{\mu}_{\nu-\frac{1}{2}}(\theta+n\pi) = \sqrt{\pi}\, 2^{\mu} \frac{\Gamma(\frac{1}{2}+\nu+\mu)}{\Gamma(1+\nu)} e^{\frac{\pi i\mu}{2}} e^{-in\pi\mu} (\sin\theta)^{\mu} e^{i(\frac{1}{2}+\nu+\mu)\theta} e^{i(\frac{1}{2}+\nu+\mu)n\pi} .$$

$$\cdot {}_2F_1\left[\tfrac{1}{2}+\mu,\ \tfrac{1}{2}+\nu+\mu;\ 1+\nu;\ e^{2i(\theta+n\pi)}\right], \qquad (3.10)$$

from which we get the important result

$$E^{\mu}_{\nu-\frac{1}{2}}(\theta+n\pi) = i^n e^{in\pi\nu} E^{\mu}_{\nu-\frac{1}{2}}(\theta), \; n = 0, \pm 1, \pm 2, \ldots . \tag{3.11}$$

Equation (3.11) is the addition formula. It shows directly that the analytic continuation of $Q^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$ in the θ-plane is non-periodic. Equation (3.11) is crucial in the evaluation of integrals occurring later.

One may ask why we chose to extend $Q^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$ rather than the more common solution $P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$. The reason is that the hypergeometric representations for $P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$ do not yield the convenient form of the addition formula (3.11).

However, it is of interest to examine some form of the continuation of $P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$. There are a number of identities relating the two standard solutions. In particular, from [1], p. 140, we have

$$Q^{\mu}_{\nu}(z)\sin[\pi(\nu+\mu)] - Q^{\mu}_{-\nu-1}(z)\sin[\pi(\nu-\mu)]$$

$$= \pi e^{i\mu\pi}\cos(\nu\pi)\, P^{\mu}_{\nu}(z).$$

The substitution $z = \cos\theta$, $\mathcal{I}m\{z\} \leq 0$, and $\nu \to \nu - \frac{1}{2}$ in the above form gives the continuation of the left-hand side and consequently the right side. Thus, we can define

$$\mathcal{P}^{\mu}_{\nu-\frac{1}{2}}(\theta) \equiv -\frac{e^{-i\mu\pi}}{\pi\sin\pi\nu}\left[\cos\pi(\nu+\mu)E^{\mu}_{\nu-\frac{1}{2}}(\theta) - \cos\pi(\nu-\mu)E^{\mu}_{-\nu-\frac{1}{2}}(\theta)\right] \tag{3.12}$$

to be the continuation of $P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$. Similarly, using

$$Q^{\mu}_{-\nu-1}(z) - Q^{\mu}_{\nu}(z) = e^{i\mu\pi}\cos(\nu\pi)\,\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)\,P^{-\mu}_{\nu}(z),$$

we get

$$\mathcal{P}^{-\mu}_{\nu-\frac{1}{2}}(\theta) = -\frac{e^{-i\mu\pi}}{\sin\pi\nu}\,\frac{\left[E^{\mu}_{\nu-\frac{1}{2}}(\theta) - E^{\mu}_{-\nu-\frac{1}{2}}(\theta)\right]}{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}\,. \qquad (3.13)$$

Another fundamental definition is

$$E^{\mu}_{\nu-\frac{1}{2}}(\theta) = \tfrac{1}{2}\pi\,\frac{e^{i\mu\pi}}{\cos\pi(\nu+\mu)}\left[i\,e^{i\pi\nu}P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta) + P^{\mu}_{\nu-\frac{1}{2}}(-\cos\theta)\right] \qquad (3.14)$$

for $0 \leq \mathrm{Re}\{\theta\} \leq \pi$. From this expression and the addition formula (3.11) we get the identity

$$E^{\mu}_{\nu-\frac{1}{2}}(\theta) - E^{\mu}_{\nu-\frac{1}{2}}(-\theta) = i\pi\,e^{i\mu\pi}\,\frac{\cos\nu\pi}{\cos\pi(\nu+\mu)}\,P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta), \qquad (3.15)$$

$0 \leq \mathrm{Re}\{\theta\} \leq \pi$.

These relationships are useful for the evaluation of certain integrals. It should be mentioned that while (3.12) does give the analytic continuation of $P^{\mu}_{\nu-\frac{1}{2}}(\cos\theta)$, it is not especially useful outside the range $[-\pi,\pi]$.

Section 4. Derivation of the Transform.

In this section we derive two forms of the spectral representation associated with (1.1). Appendix B outlines the theory of constructing the spectral representation associated with an ODE. (For a full treatment of the subject, see Titchmarsh [3].)

Starting from (B.7), we define $\lambda \equiv \nu^2 - \frac{1}{4}$. Then our fundamental formula is the contour integral expansion theorem for Dirac's δ-function:

$$\delta(\theta - \theta_o) = \frac{1}{\pi i} \int_{C_1} \mathcal{G}(\theta, \theta_o, \nu)\, \nu \, d\nu \tag{4.1}$$

where $\mathcal{G}(\theta, \theta_o, \nu)$ is the Green's function for (1.1) and C_1 is the infinite semicircle shown in Figure 5.

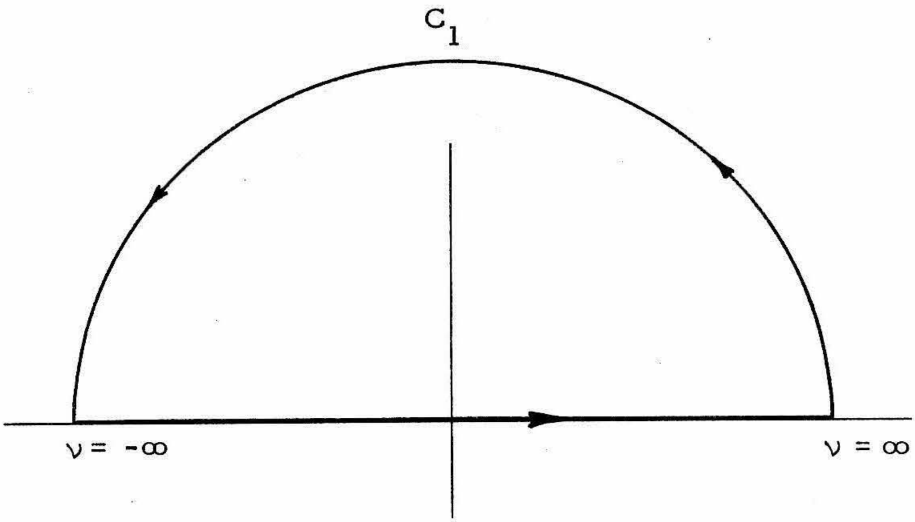

Figure 5. Integration path for (4.1) and (4.5) to derive the spectral representation associated with (1.1).

In Appendix C one form of the Green's function for (1.1) is shown to be

$$\mathscr{G}(\theta, \theta_o, \nu) = - \frac{E^{\mu}_{\nu-\frac{1}{2}}(\theta_>) \, E^{\mu}_{-\nu-\frac{1}{2}}(\theta_<) \, \sin\theta_o \, e^{-2i\mu\pi}}{\sin\pi\nu \; \Gamma(\frac{1}{2}+\nu+\mu) \; \Gamma(\frac{1}{2}-\nu+\mu)}, \qquad (4.2)$$

where $\theta_>$, $\theta_<$ are the greater and lesser of θ and θ_o, respectively.

We note that $\delta(\theta-\theta_o)$ is the value of the integral in (4.1) taken along the arc C_1 itself. To get an equivalent representation for the right-hand side of (4.1), we close the contour C_1 along the real axis (Figure 5) and apply Cauchy's integral theorem.

The spectral expansion (4.1) becomes

$$\delta(\theta-\theta_o) = -\frac{1}{\pi i}\int_{-\infty}^{\infty} \mathscr{G}(\theta, \theta_o, \nu) \, \nu \, d\nu + \text{Residues}. \qquad (4.3)$$

By comparing (4.2) with (3.9), we observe that the Γ-functions in the denominator of (4.2) cancel those contributed to the numerator by the E's. The factor $\Gamma(1+\nu)\,\Gamma(1-\nu)$ which will appear in the denominator obeys the identity

$$\Gamma(1+\nu)\,\Gamma(1-\nu) = \pi\nu \csc \pi\nu$$

and is canceled by the remaining factors in $\nu\mathscr{G}(\theta, \theta_o, \nu)$. Therefore the integrand in (4.3) has no singularities, and we get the δ-function expansion:

$$\delta(\theta - \theta_o) = -\frac{i}{\pi} e^{-2i\mu\pi} \int_{-\infty}^{\infty} \frac{E^{\mu}_{\nu-\frac{1}{2}}(\theta_>)\ E^{\mu}_{-\nu-\frac{1}{2}}(\theta_<)\ \sin\theta_o\ \nu}{\sin\pi\nu\ \Gamma(\frac{1}{2}+\nu+\mu)\ \Gamma(\frac{1}{2}-\nu+\mu)}\, d\nu\ . \tag{4.4}$$

Each side is symmetric in θ and θ_o. Therefore, we can set $\theta_< = \theta_o$ and $\theta_> = \theta$. Multiply each side by $f(\theta_o)$ and integrate from $-\infty$ to ∞ to get the transform pair:

$$f(\theta) = \int_{-\infty}^{\infty} \tilde{f}(\nu)\ \frac{E^{\mu}_{\nu-\frac{1}{2}}(\theta)\ \nu}{\sin\pi\nu\ \Gamma(\frac{1}{2}+\nu+\mu)\ \Gamma(\frac{1}{2}-\nu+\mu)}\ d\nu\ , \tag{4.5}$$

$$\tilde{f}(\nu) = -\frac{i}{\pi}\ e^{-2i\mu\pi} \int_{-\infty}^{\infty} f(\theta)\ E^{\mu}_{-\nu-\frac{1}{2}}(\theta)\ \sin\theta\ d\theta\ . \tag{4.6}$$

These last two expressions are called the infinite-angle generalized Mehler transform, after F. Mehler [4] who constructed a somewhat similar transform in connection with the conical functions $P_{i\nu-\frac{1}{2}}(\cos\theta)$.

We can set the order $\mu = 0$ in (4.5) and (4.6). If we define $E^{o}_{\nu-\frac{1}{2}}(\theta) \equiv E_{\nu-\frac{1}{2}}(\theta)$ as the continuation of $Q_{\nu-\frac{1}{2}}(\cos\theta)$, the transform pair reduces to

$$f(\theta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{f}(\nu)\ E_{\nu-\frac{1}{2}}(\theta)\ \nu \cot\pi\nu\ \ d\nu, \tag{4.7}$$

$$\tilde{f}(\nu) = -\frac{i}{\pi} \int_{-\infty}^{\infty} f(\theta) E_{-\nu-\frac{1}{2}}(\theta)\ \ \sin\theta\ d\theta. \tag{4.8}$$

This special case was derived by Clemmow [2]. In some of the examples that follow it will turn out that (4.7)-(4.8) is the appropriate transform pair to use.

In the special case of the reduced transform, the expansion of the δ-function becomes

$$\delta(\theta-\theta_o) = -\frac{i}{\pi^2}\int_{-\infty}^{\infty} E_{-\nu-\frac{1}{2}}(\theta_o)\, E_{\nu-\frac{1}{2}}(\theta)\sin\theta_o\, \nu\cot\pi\nu\, d\nu \tag{4.9}$$

We will offer an alternate form of (4.4)-(4.6). We can choose $E^{\mu}_{\nu-\frac{1}{2}}(\theta)$ and $E^{-\mu}_{-\nu-\frac{1}{2}}(\theta)$ as our independent solutions to (1.1). Appendix C shows that with this pair the Green's function is

$$\mathscr{G}(\theta, \theta_o, \nu) = -\frac{1}{\pi}\, \frac{E^{\mu}_{\nu-\frac{1}{2}}(\theta_>)\, E^{-\mu}_{-\nu-\frac{1}{2}}(\theta_<)\sin\theta_o \cos\pi(\nu+\mu)}{\sin\pi\nu}\,. \tag{4.10}$$

Definition (3.9) implies that the numerator of (4.10) contains the factor

$$\Gamma(\tfrac{1}{2}+\nu+\mu)\,\Gamma(\tfrac{1}{2}-\nu-\mu) = \pi\sec\pi(\nu+\mu), \tag{4.11}$$

and the demoninator contains

$$\Gamma(1+\nu)\,\Gamma(1-\nu) = \pi\nu\csc\pi\nu \tag{4.12}$$

These factors cancel with others in (4.10), so there will be no residue terms. By integrating over the contour in Figure 5 we get the expansion of the δ-function:

$$\delta(\theta-\theta_o) = -\frac{i}{\pi^2}\int_{-\infty}^{\infty} E^{\mu}_{\nu-\frac{1}{2}}(\theta)\, E^{-\mu}_{-\nu-\frac{1}{2}}(\theta_o) \sin\theta_o\, \nu \frac{\cos\pi(\nu+\mu)}{\sin\pi\nu}\, d\nu. \tag{4.13}$$

The resulting transform pair is

$$f(\theta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \tilde{f}(\nu)\, E^{\mu}_{\nu-\frac{1}{2}}(\theta)\, \nu \frac{\cos\pi(\nu+\mu)}{\sin\pi\nu}\, d\nu\,, \tag{4.14}$$

$$\tilde{f}(\nu) = -\frac{i}{\pi}\int_{-\infty}^{\infty} f(\theta)\, E^{-\mu}_{-\nu-\frac{1}{2}}(\theta) \sin\theta\, d\theta\,. \tag{4.15}$$

Section 5. Application to Waves on a Sphere.

An elegant use of the infinite-angle transform is its application to axisymmetric waves on a sphere. The physical problem selected for study is long free-surface water waves on a shallow spherical ocean. In Appendix A we show that the linearized equation for such waves is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi(\theta,t)\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\theta,t) = 0 , \tag{5.1}$$

where (θ, t) is the elevation of the free surface above the equilibrium position. As explained in Section 2, the variable θ in (5.1) can be regarded as the total arc length covered by the wave front, so $-\infty < \theta < \infty$. The radius of the sphere has been normalized to 1.

Equation (5.1) with $0 \leqslant \theta \leqslant \pi$ has no doubt been solved innumerable times using the standard Legendre polynomial expansion. However, this form of the solution is difficult to analyze or interpret physically. We seek a more transparent expression. We shall examine (5.1) with initial conditions

$$\psi(\theta, 0) = 0 , \tag{5.2}$$

$$\psi_t(\theta,0) = \begin{cases} v_0 , & |\theta| < \theta_0 & (5.3a) \\ 0 , & \theta_0 < |\theta| < \pi - \theta_0 & (5.3b) \\ -v_0 , & \pi - \theta_0 < |\theta| < \pi . & (5.3c) \end{cases}$$

These conditions correspond to the sphere's being given an initial outward velocity throughout a zone at one pole and an identical inward velocity around the other pole. (This balanced initial condition is required by the continuity equation.)

From the principle of superposition we can treat (5.3a) and (5.3c) separately. To illustrate the use of our transform, we will solve the problem with condition (5.3a) only, stressing that the case with (5.3c) follows in the identical manner. The conventional form of the solution of (5.1)-(5.3a) is

$$\psi(\theta, t) = \tfrac{1}{2} v_0 t(1-\cos\theta_0) + \frac{v_0}{c} \sum_{n=1}^{\infty} \frac{n+\frac{1}{2}}{\sqrt{n(n+1)}} \left\{ \int_0^{\theta_0} P_n(\cos\xi) \sin\xi \, d\xi \right\} \cdot$$

$$\cdot \; P_n(\cos\theta) \sin\sqrt{n(n+1)}\, ct \; . \tag{5.5}$$

It is difficult to interpret (5.5) directly. For example, it is not at all clear that, for sufficiently small t, $\psi(\theta, t) = 0$ for any $\theta > \theta_0$. The difficulty is that (5.5) represents the entire sphere by its physical latitude range. The first term in (5.5) is the average disturbance over the whole domain and the remaining terms are the "correction" terms.

Peters [5] was able to get a more meaningful representation of (5.5) by clever and intricate manipulation of the special functions involved, but his result is valid only at the poles of the sphere. (There is one interesting case where (5.5) is useful. If $\theta_0 = \pi$, then (5. 3) implies that the entire sphere is given an initial outward velocity. It is easily shown that

$$\int_0^{\pi} P_n(\cos\xi) \sin\xi \, d\xi = 0$$

for n = 1, 2, ... In such a case (5. 5) shows that

$$\Psi(\theta, t) = v_o t. \tag{5.6}$$

The entire sphere expands uniformly into space forever. Such behavior may be at odds with one's intuition, but the result is correct for the very simple physical system which equation (5. 1) describes.)

We will now extract the solution of (5. 1)-(5. 3a) using the infinite-angle transform. From (4. 9) and (4. 10), define

$$\Psi(\theta, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \widetilde{\Psi}(\nu, t) \, E_{\nu-\frac{1}{2}}(\theta) \, \nu \cot \pi\nu \, d\nu \tag{5.7}$$

and

$$\widetilde{\Psi}(\nu, t) = -\frac{i}{\pi}\int_{-\infty}^{\infty} \Psi(\theta, t) \, E_{-\nu-\frac{1}{2}}(\theta) \sin\theta \, d\theta. \tag{5.8}$$

Multiply (5. 1) by $-\frac{i}{\pi} E_{-\nu-\frac{1}{2}}(\theta) \sin\theta$ and integrate from $-\infty$ to ∞:

$$-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)E_{-\nu-\frac{1}{2}}(\theta)\,d\theta+\frac{i}{\pi}\;\frac{1}{c^2}\int_{-\infty}^{\infty}\Psi_{tt}E_{-\nu-\frac{1}{2}}(\theta)\sin\theta\,d\theta = 0. \tag{5.9}$$

The first integral in (5.9) can be integrated by parts:

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)E_{-\nu-\frac{1}{2}}(\theta)\,d\theta=\sin\theta\frac{\partial\Psi}{\partial\theta}E_{-\nu-\frac{1}{2}}(\theta)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial\Psi}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}E_{-\nu-\frac{1}{2}}(\theta)\,d\theta. \tag{5.10}$$

We assume $\Psi(\theta,t)$ and $\Psi_{\theta}(\theta,t)\rightarrow 0$ as $\theta\rightarrow\pm\infty$. Therefore, the boundary term in (5.10) vanishes and we get

$$-\int_{-\infty}^{\infty}\frac{\partial\Psi}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}E_{-\nu-\frac{1}{2}}(\theta)\,d\theta=-\Psi\sin\theta\frac{\partial}{\partial\theta}E_{-\nu-\frac{1}{2}}(\theta)\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\Psi\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}E_{-\nu-\frac{1}{2}}(\theta)\right)d\theta. \tag{5.11}$$

The boundary term in (5.11) vanishes as before, so (5.9) can be written as

$$-\frac{i}{\pi}\int_{-\infty}^{\infty}\Psi\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}E_{-\nu-\frac{1}{2}}(\theta)\right)d\theta+\frac{i}{\pi}\;\frac{1}{c^2}\int_{-\infty}^{\infty}\Psi_{tt}E_{-\nu-\frac{1}{2}}(\theta)\sin\theta\,d\theta = 0. \tag{5.12}$$

Now $E_{-\nu-\frac{1}{2}}(\theta)$ is a solution to Legendre's equation (1.1) with $\mu=0$, so (5.12) becomes

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\Psi(\theta,t)\,(\nu^2-\tfrac{1}{4})\,E_{-\nu-\frac{1}{2}}(\theta)\sin\theta\,d\theta+\frac{i}{\pi}\,\frac{1}{c^2}\int_{-\infty}^{\infty}\Psi_{tt}(\theta,t)E_{-\nu-\frac{1}{2}}(\theta)\sin\theta d\theta=0. \tag{5.13}$$

Appealing to (5.8), this becomes

$$\widetilde{\Psi}_{tt}(\nu,t)+(\nu^2-\tfrac{1}{4})\;c^2\;\widetilde{\Psi}(\nu,t)=0. \tag{5.14}$$

Transforming the initial conditions (5.2) and (5.3) gives

$$\widetilde{\Psi}(\nu,0)=0\quad;\quad\widetilde{\Psi}_t(\nu,0)=-\frac{i}{\pi}\,v_o\int_{-\theta_o}^{\theta_o}E_{-\nu-\frac{1}{2}}(\xi)\,\sin\xi\,d\xi.$$

It follows that

$$\widetilde{\Psi}(\nu,t)=-\frac{i}{\pi}\;\frac{v_o}{c}\int_{-\theta_o}^{\theta_o}E_{-\nu-\frac{1}{2}}(\xi)\,\sin\xi\,d\xi\cdot\;\frac{\sin\sqrt{\nu^2-\frac{1}{4}}\;ct}{\sqrt{\nu^2-\frac{1}{4}}}, \tag{5.15}$$

and the inverse transform (4.7) yields the solution

$$\widetilde{\Psi}(\theta,t)=-\frac{i}{\pi^2}\;\frac{v_o}{c}\int_{-\infty}^{\infty}\left[\int_{-\theta_o}^{\theta_o}E_{-\nu-\frac{1}{2}}(\xi)\,\sin\xi\,d\xi\right]\cdot\frac{\sin\sqrt{\nu^2-\frac{1}{4}}ct}{\sqrt{\nu^2-\frac{1}{4}}}E_{\nu-\frac{1}{2}}(\theta)\;\nu\cot\pi\nu\;d\nu. \tag{5.16}$$

This solves (5.1) - (5.3) for $-\infty < \theta < \infty$. Interchange the order of integration and rearrange terms to get

$$\Psi(\theta, t) = \frac{v_o}{c} \int_{-\theta_o}^{\theta_o} d\xi \cdot \left(-\frac{i}{\pi^2}\right) \int_{-\infty}^{\infty} \frac{\sin\sqrt{\nu^2 - \frac{1}{4}}\, ct}{\sqrt{\nu^2 - \frac{1}{4}}}$$

$$E_{-\nu-\frac{1}{2}}(\xi)\, E_{\nu-\frac{1}{2}}(\theta) \sin \xi\, \nu \cot \pi\nu\, d\nu. \tag{5.17}$$

As stated before, the variable θ in (5.17) represents total arc length, measured from the starting pole, covered by the wave. In Section 2 we showed that the disturbance $\Phi(\vartheta, t)$ observed at physical latitude ϑ is the sum of the individual disturbances at arc lengths $\vartheta + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Thus we get

$$\Phi(\vartheta, t) = \sum_{n=-\infty}^{\infty} \Psi(\vartheta + 2n\pi, t)$$

or

$$\Phi(\vartheta, t) = \frac{v_o}{c} \sum_{n=-\infty}^{\infty} \int_{-\theta_o}^{\theta_o} d\xi \int_{-\infty}^{\infty} \frac{\sin\sqrt{\nu^2 - \frac{1}{4}}\, ct}{\sqrt{\nu^2 - \frac{1}{4}}}$$

$$E_{-\nu-\frac{1}{2}}(\xi)\, E_{\nu-\frac{1}{2}}(\vartheta + 2n\pi) \sin \xi\, \nu \cot \pi\nu\, d\nu. \tag{5.18}$$

From the addition formula (3.11) we write (5.18) as

$$\Phi(\vartheta, t) = \frac{v_o}{c}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\theta_o}^{\theta_o} d\xi \left(-\frac{i}{\pi^2}\right)\int_{-\infty}^{\infty} \frac{\sin\sqrt{\nu^2-\frac{1}{4}}\, ct}{\sqrt{\nu^2-\frac{1}{4}}}\, e^{2in\pi\nu}.$$

$$\cdot\; E_{-\nu-\frac{1}{2}}(\xi)\, E_{\nu-\frac{1}{2}}(\vartheta)\, \sin\xi\, \nu \cot\pi\nu\, d\nu. \tag{5.19}$$

The ν-integral in (5.19) has the structure of a Fourier transform. We will evaluate it using the convolution theorem for Fourier integrals. Let $F(\nu)$ be the Fourier transform of $f(\alpha)$:

$$F(\nu) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\, e^{-i\nu\alpha}\, d\alpha.$$

If $G(\nu)$ is the Fourier transform of $g(\alpha)$, then the convolution theorem states that

$$\int_{-\infty}^{\infty} e^{i\nu\alpha}\, F(\nu)\, G(\nu)\, d\nu = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\zeta)\, g(\alpha-\zeta)\, d\zeta. \tag{5.20}$$

Define $$F(\nu) \equiv \frac{\sin\sqrt{\nu^2-\frac{1}{4}}\, ct.}{\sqrt{\nu^2-\frac{1}{4}}}. \tag{5.21}$$

Then, from [6], p. 26, (30), we find that

$$f(\alpha) = \pi\, I_o\left[\frac{c}{2}\sqrt{t^2-\frac{\alpha^2}{c^2}}\right]\, H\left(t-\left|\frac{\alpha}{c}\right|\right) \tag{5.22}$$

where $I_o(z)$ is the modified Bessel function of the first kind, order zero, and H(x) is the Heaviside unit function: $H(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$.

Define $G(\nu) \equiv -\left(\frac{i}{\pi^2}\right) E_{-\nu-\frac{1}{2}}(\xi)\, E_{\nu-\frac{1}{2}}(\vartheta)\, \sin\xi\, \nu \cot\pi\nu\, d\nu$ (5.23)

From the δ-function expansion (4.13) and the addition formula (3.11) we get the representation

$$\delta(\vartheta + 4n\pi - \xi) = -\frac{i}{\pi^2}\int_{-\infty}^{\infty} e^{i4n\pi\nu}\, E_{-\nu-\frac{1}{2}}(\xi)\, E_{\nu-\frac{1}{2}}(\vartheta)\, \sin\xi\, \nu \cot\pi\nu\, d\nu. \tag{5.24}$$

Let $\alpha \equiv 4n\pi$ and each side of (5.24) is a function of α. We get the Fourier formula

$$\int_{-\infty}^{\infty} e^{i\alpha\nu} \cdot \left(-\frac{i}{\pi^2}\right) E_{-\nu-\frac{1}{2}}(\xi)\, E_{\nu-\frac{1}{2}}(\vartheta)\, \sin\xi\, \nu\cot\pi\nu\, d\nu = \delta(\vartheta+\alpha-\xi). \tag{5.25}$$

From the results in (5.22) and (5.25) we will use the convolution formula (5.20) to determine the ν-integral in (5.19).

$$\int_{-\infty}^{\infty} e^{i2n\pi\nu}\, \frac{\sin\sqrt{\nu^2-\frac{1}{4}}\, ct}{\sqrt{\nu^2-\frac{1}{4}}} \left(-\frac{i}{\pi^2}\right) E_{-\nu-\frac{1}{2}}(\xi)\, E_{\nu-\frac{1}{2}}(\vartheta)\, \sin\xi\, \nu\cot\pi\nu\, d\nu =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} I_0\left[\frac{c}{2}\sqrt{t^2-\frac{\zeta^2}{c^2}}\right] H\left(t-\frac{|\zeta|}{c}\right)\, \delta(\vartheta+2n\pi-\zeta-\xi)\, d\zeta$$

Evaluating the integrand at the zero of the δ-function argument yields the result for the total physical disturbance:

$$\Phi(\vartheta, t) = \frac{v_o}{2c}\sum_{n=-\infty}^{\infty}(-1)^n \int_{-\theta_o}^{\theta_o} I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\vartheta+2n\pi-\xi)^2}{c^2}}\right] H\left(ct - |\vartheta+2n\pi-\xi|\right) d\xi. \tag{5.26}$$

For any finite t this expression contains only a finite number of terms, each one representing a separate passage of the wave front around the sphere. We shall analyze the behavior of the wave.

From the symmetry of the physical problem the solution must be symmetric in ϑ. Therefore, we will consider the behavior seen by an observer at latitude $\vartheta > 0$.

Case 1. $\vartheta > \theta_o$.

In this situation the observer is located outside the zone of initial velocity. We examine $H\left(ct - |\vartheta + 2n\pi - \xi|\right)$ in (5.26). The argument first becomes positive in the $n = 0$ term, $\Phi_o(\vartheta, t)$. The integration variable ξ is confined to $(-\theta_o, \theta_o)$, and hence

$$\Phi(\vartheta, t) \equiv 0 \quad \text{for} \quad t < \frac{\vartheta - \theta_o}{c} .$$

This fact shows directly that the initial wavefront travels with speed c into a quiescent medium and that there is no advance disturbance.

After the wave front first arrives the expression (5.26) becomes

$$\Phi(\vartheta, t) = \frac{v_o}{2c}\int_{A_o}^{\theta_o} I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\vartheta-\xi)^2}{c^2}}\right] d\xi, \quad \frac{\vartheta - \theta_o}{c} < t < \frac{2\pi - \vartheta - \theta_o}{c} , \tag{5.27}$$

where A_o is the algebraically lesser of $-\theta_o$ and $\vartheta - ct$. This expression represents the first passage of the initial wavefront. Now, $I_o(z)$ is bounded near the origin. Its appearance in the integrand of (5.27) implies that the surface level rises continuously and monotonically from zero for the time interval specified in (5.27).

This behavior shows that the wave propagation on the sphere is not "clean-cut" but leaves an infinitely long wake.

At time

$$t = \frac{2\pi - \vartheta - \theta_o}{c}$$

the n=-1 term $\Phi_{-1}(\vartheta, t)$ in (5.26) appears. It represents the first return trip of the wavefront to the starting pole. In the interval

$$\frac{2\pi - \vartheta - \theta_o}{c} < t < \frac{2\pi + \vartheta - \theta_o}{c}$$

the total disturbance is

$$\Phi(\vartheta, t) = \frac{v_o}{2c}\left\{\int_{A_o}^{\theta_o} I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\vartheta - \xi)^2}{c^2}}\right]d\xi - \int_{-\theta_o}^{A_{-1}} I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\vartheta - 2\pi - \xi)^2}{c^2}}\right]d\xi\right\}. \tag{5.28}$$

Here A_o is defined as for (5.27), and A_{-1} is the algebraically smaller of θ_o and $\vartheta + ct - 2\pi$. The disturbance obeys this formulation until

$t = \dfrac{2\pi + \vartheta - \theta_o}{c}$, at which time the n=1 term $\Phi_1(\vartheta, t)$ appears and is to be included in the total expression for $\Phi(\vartheta, t)$. It is:

$$\Phi_1(\vartheta, t) = -\frac{v_o}{2c}\int_{A_1}^{\theta_o} I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\vartheta+2\pi-\xi)^2}{c^2}}\right] d\xi, \tag{5.29}$$

where A_1 is the algebraically lesser of $-\theta_o$ and $\vartheta + 2\pi - ct$.

The next two terms to enter are the $n = -2$ and $n = 2$ terms, respectively. From the factor $(-1)^n$ in (5.26) it is evident that successive pairs of terms are alternately positive and negative. This fact has the remarkable interpretation of a phase reversal occurring at the far pole but not at the starting pole. Φ_o, the initial wave coming from the starting pole, is positive. The contribution from Φ_{-1}, the first return from the far pole, is negative. The term Φ_1, representing the second pass from the starting pole, is also negative, indicating no change of sign. The second return wave Φ_{-2} is positive, exhibiting another change of sign at the far pole.

<u>Case 2. $\vartheta < \theta_o$.</u>

The observer here is situated within the zone of initial velocity. Rather than evaluate (5.26) directly we make the following observation. The single initial condition (5.3) can be written as the sum of the two conditions

$$\Psi_t(\theta, 0) = f_1(\theta) + f_2(\theta), \tag{5.30}$$

where $f_1(\theta) = v_o, \quad |\theta| \leq \pi$ (5.31) and $\quad f_2(\theta) = -v_o\,, \quad \theta_o < |\theta| \leq \pi.$ (5.32)

This corresponds physically to our combining the solutions to separate initial-value problems: one in which the entire sphere receives an initial outward velocity, and one where the zone now under consideration is initially quiet while the remainder of the sphere is given an inward velocity $-v_o$.

It follows either from (5.5) or directly from (5.1) and (5.2) that, with condition (5.31) as the initial velocity,

$$\Phi(\vartheta, t) = v_o t. \tag{5.33}$$

When we include (5.32) as our initial velocity distribution, then, except for a change in coordinate limits, we have precisely the physical problem of Case 1 above. Making the necessary adjustments in the notation of (5.26), we can combine the separate solutions to the problems using (5.31) and (5.32) individually to get

$$\Phi(\vartheta, t) = v_o t - \frac{v_o}{2c} \sum_{n=-\infty}^{\infty} (-1)^n \int_{\theta_o}^{2\pi-\theta_o} I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\vartheta+2n\pi-\xi)^2}{c^2}}\right] H\left(ct - |\vartheta+2n\pi-\xi|\right) \tag{5.34}$$

as the solution to our original problem (5.1)-(5.3) with $\vartheta < \theta_o$. We shall analyze (5.34) for the first few moments of wave travel. For

$$t < \frac{\theta_o - \vartheta}{c}$$

the series contributes nothing, and

$$\Phi(\vartheta, t) = v_o t,$$

a uniform outward motion. At

$$t = \frac{\theta_o - \vartheta}{c}$$

the $n = 0$ term $\Phi_o(\vartheta, t)$ enters and we get

$$\Phi(\vartheta, t) = v_o t - \frac{v_o}{2c}\int_{\theta_o}^{A_o} I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\xi-\vartheta)^2}{c^2}}\right] d\xi\,, \quad \frac{\theta_o - \vartheta}{c} < t < \frac{\theta_o + \vartheta}{c} \tag{5.35}$$

where A_o is the lesser of $\vartheta + ct$ and $2\pi - \theta_o$. The integral in (5.35) represents the propagation of the effects of initial quiescence from the nearer boundary of the initial disturbance inward to the observer's latitude. At

$$t = \frac{\theta_o + \vartheta}{c}$$

the $n = 1$ term appears. It can be interpreted as the arrival of quiescence effects from the far boundary of the initial disturbance.

As in Case 1, (5.34) admits more and more terms as t increases. However, we must make the distinction that (5.34) describes the disturbance in terms of the propagation of quiescence effects, whereas (5.26) represents the acutal disturbance itself. Thus, we find that while the physical wave undergoes a phase reversal at the far pole, the quiescence terms encounter such a change only at the starting pole.

Propagation Of An Explosion.

Of particular interest is a special case of the above problem representing the behavior of the sphere following a point explosion at one pole. We reduce the size of the zone $|\theta| < \theta_o$ in (5.3) and increase the velocity v_o in such a way that the total initial energy remains constant as $\theta_o \to 0$.

We define the initial energy per unit area to be $\frac{1}{2} v_o^2$. Then the total energy ϵ_o exhibited by initial condition (5.3) is

$$\epsilon_o = \pi v_o^2 (1 - \cos \theta),$$

so,

$$v_o = \sqrt{\frac{\epsilon_o}{\pi}} \frac{1}{\sqrt{1 - \cos \theta_o}} , \qquad (5.36)$$

Substitute (5.36) into the solution (5.26) to get

$$\Phi(\vartheta, t) = \frac{1}{2c} \sqrt{\frac{\epsilon_o}{\pi}} \frac{1}{\sqrt{1 - \cos \theta_o}} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\theta_o}^{\theta_o} I_o\left[\frac{c}{2} \sqrt{t^2 - \frac{(\vartheta + 2n\pi - \xi)^2}{c^2}}\right] \cdot H\left(ct - |\vartheta + 2n\pi - \xi|\right) d\xi . \qquad (5.37)$$

Applying the mean value theorem for integrals and expanding $\sqrt{1 - \cos \theta_o}$ in a power series for θ_o gives

$$\Phi(\vartheta, t) = \frac{1}{2c}\sqrt{\frac{\varepsilon_o}{\pi}} \frac{1}{\theta_o\sqrt{1+O(\theta_o)^2}} \sum_{n=-\infty}^{\infty} (-1)^n I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\vartheta+2n\pi-\theta_1)}{c^2}}\right] \cdot 2\theta_o \cdot H\left(ct - |\vartheta+2n\pi-\theta_1|\right), \tag{5.38}$$

where $-\theta_o < \theta_1 < \theta_o$.

Take the limit of (5.38) as $\theta_o \to 0$ to yield

$$\Phi(\vartheta, t) = \frac{1}{c}\sqrt{\frac{\varepsilon_o}{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(\vartheta+2n\pi)^2}{c^2}}\right] H\left(ct - |\vartheta+2n\pi|\right) \tag{5.39}$$

as the representation for the disturbance at any physical latitude ϑ resulting from an explosion at the starting pole at $t=0$. It reveals quite clearly the behavior of the wave.

The first contribution again comes from the $n = 0$ term. The factor $H(ct-\vartheta)$ shows that there is no disturbance at ϑ before $t = \frac{\vartheta}{c}$, the first arrival time of the wave front.

Now $I_o(0) = 1$, so when the front passes the observer the surface elevation jumps abruptly to height

$$\Phi(\vartheta, \frac{\vartheta}{c}) = \frac{1}{c}\sqrt{\frac{\varepsilon_o}{\pi}} .$$

As the wave proceeds, the surface at ϑ continues to rise monotonically until $t = \frac{2\pi-\vartheta}{c}$, the arrival time of the return wave.

In the interval $\frac{2\pi - \vartheta}{c} < t < \frac{2\pi + \vartheta}{c}$, the total disturbance is

$$\Phi(\vartheta, t) = \frac{1}{c}\sqrt{\frac{\epsilon_o}{\pi}} \left\{ I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{\vartheta^2}{c^2}}\right] - I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{(2\pi - \vartheta)^2}{c^2}}\right]\right\}$$

The n = 1 term appears at $t = \frac{\vartheta + 2\pi}{c}$ and represents the passage of the wave on its second trip toward the far pole.

Because of the infinite initial velocity at one pole it is worthwhile to examine the wave for possible subsequent singularities at either pole. From (5. 39), there is no disturbance for $t < \frac{\pi}{c}$. The first wave contributes

$$\Phi(\pi, t) = \frac{1}{c}\sqrt{\frac{\epsilon_o}{\pi}}\; I_o\left[\frac{c}{2}\sqrt{t^2 - \frac{\pi^2}{c^2}}\right], \quad \frac{\pi}{c} < t < \frac{3\pi}{c}.$$

We see that, although the radial velocity of the wave front changes abruptly with the arrival of the wave, the amplitude remains bounded and has no physical singularity. From (5..39) we deduce that similar behavior is true at the starting pole.

Section 6. The Flow of Heat on a Spherical Shell.

In this section we derive an alternative representation for an axisymmetric initial-value heat-flow problem on a spherical shell. The standard form of the heat diffusion equation is

$$k \nabla^2 u - u_t = 0 \tag{6.1}$$

We consider an initial distribution of temperature over a zone. Equation (6.1) becomes

$$\frac{k}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) - \frac{\partial u}{\partial t} = 0, \tag{6.2}$$

with initial condition $u(\theta, 0) = u_o, \ |\theta| < \theta_o$. (6.3)

Classically, the eigenfunction expression in terms of Legendre polynomials yields

$$u(\theta, t) = \sum_{n=0}^{\infty} c_n(t) \, P_n(\cos\theta)$$

with the result

$$u(\theta, t) = \frac{u_o}{2}(1 - \cos\theta_o) + \sum_{n=1}^{\infty} (n + \tfrac{1}{2}) \left[\int_0^{\theta_o} P_n(\cos\xi) \sin\xi \, d\xi\right] P_n(\cos\theta) \cdot e^{-n(n+1)kt} . \tag{6.4}$$

This equation is useful for large kt. The total heat remains constant, and the temperature tends to a uniform distribution over the entire sphere.

For small kt the series in (6.4) is slowly convergent. To get a more rapidly convergent form, we define θ to be arc length on the sphere and hence $-\infty < \theta < \infty$ in (6.2). The flow of heat in an initial-value problem resembles an infinitely fast, exponentially weak wave. Therefore, we follow Section 2 and note that the temperature $U(\vartheta, t)$ at physical latitude ϑ is

$$U(\vartheta, t) = \sum_{n=-\infty}^{\infty} u(\vartheta + 2n\pi, t) . \tag{6.5}$$

Now we shall solve (6.2)-(6.3). Multiply (6.2) by $-\frac{i}{\pi} E_{-\nu-\frac{1}{2}}(\theta) \sin\theta$ and integrate from $-\infty$ to ∞.

$$-\frac{i}{\pi} k \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) E_{-\nu-\frac{1}{2}}(\theta)\, d\theta + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} E_{-\nu-\frac{1}{2}}(\theta) \sin\theta\, d\theta = 0 \tag{6.6}$$

From the exponential decay of the solution to the heat equation in an infinite domain we assume $u(\theta, t) \to 0$ for $\theta \to \pm\infty$. Therefore when we integrate the first term in (6.6) by parts twice the boundary terms vanish, leaving

$$-\frac{i}{\pi}k\int_{-\infty}^{\infty} u(\theta,t)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\;E_{-\nu-\frac{1}{2}}(\theta)\right)d\theta + \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}E_{-\nu-\frac{1}{2}}(\theta)\sin\theta\,d\theta = 0. \tag{6.7}$$

$E_{-\nu-\frac{1}{2}}(\theta)$ satisfies (1. 1) with $\mu = 0$, so (6. 7) becomes

$$\frac{i}{\pi}k\int_{-\infty}^{\infty} u(\theta,t)\,(\nu^2-\tfrac{1}{4})\,E_{-\nu-\frac{1}{2}}(\theta)\sin\theta\,d\theta + \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}E_{-\nu-\frac{1}{2}}(\theta)\sin\theta\,d\theta = 0. \tag{6.8}$$

Each term here has the form of the infinite-angle transform (4. 15), so (6. 8) becomes

$$\tilde{u}_t(\nu,t) + k\,(\nu^2-\tfrac{1}{4})\,\tilde{u}(\nu,t) = 0. \tag{6.9}$$

Transforming the initial condition (6. 3) solves (6. 9):

$$\tilde{u}(\nu,t) = -\frac{i}{\pi}u_o e^{\frac{1}{4}kt}\left\{\int_{-\theta_o}^{\theta_o} E_{-\nu-\frac{1}{2}}(\xi)\sin\xi\,d\xi\right\}e^{-\nu^2 kt}.$$

The inverse transform (4. 14) gives

$$u(\theta,t) = -\frac{i}{\pi^2}u_o e^{\frac{1}{4}kt}\int_{-\infty}^{\infty}\left\{\int_{-\theta_o}^{\theta_o}E_{-\nu-\frac{1}{2}}(\xi)\sin\xi\,d\xi\right\}\cdot e^{-\nu^2 kt}E_{\nu-\frac{1}{2}}(\theta)\,\nu\cot\pi\nu\,d\nu \tag{6.9}$$

Equation (6.9) gives the temperature at any arc length θ. To calculate the result at physical latitude ϑ, we appeal to (6.5); this expression, together with the addition formula (3.11) and interchanging the order of integration, gives

$$U(\vartheta, t) = u_o e^{\frac{1}{4}kt} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\theta_o}^{\theta_o} d\xi \left(-\frac{i}{\pi^2}\right) \int_{-\infty}^{\infty} e^{2in\pi\nu} e^{-\nu^2 kt} E_{-\nu-\frac{1}{2}}(\xi)$$

$$E_{\nu-\frac{1}{2}}(\theta) \sin \xi \nu \cot \pi \nu \, d\nu \, . \qquad (6.10)$$

It is well known that

$$\int_{-\infty}^{\infty} e^{i\alpha\nu} e^{-\nu^2 kt} d\nu = \frac{\sqrt{\pi}}{\sqrt{kt}} e^{-\frac{\alpha^2}{4kt}} . \qquad (6.11)$$

Incorporating (6.11) and (5.25) into the convolution theorem, we get

$$-\frac{i}{\pi^2} \int_{-\infty}^{\infty} e^{i2n\pi\nu} e^{-\nu^2 kt} E_{-\nu-\frac{1}{2}}(\xi) E_{\nu-\frac{1}{2}}(\vartheta) \sin \xi \nu \cot \pi\nu \, d\nu =$$

$$= \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{\zeta^2}{4kt}} \delta(\vartheta + 2n\pi - \zeta - \xi) \, d\zeta \, .$$

From this expression (6.10) becomes

$$U(\vartheta, t) = \frac{u_o e^{\frac{1}{4}kt}}{2\sqrt{\pi kt}} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\theta_o}^{\theta_o} e^{-\frac{(\vartheta+2n\pi-\xi)^2}{4kt}} d\xi . \tag{6.12}$$

We shall analyze the behavior of the temperature $U(\vartheta, t)$.

Case 1. $\vartheta > \theta_o$.

For kt sufficiently small the $n=0$ term dominates all others because of the rapid decay of the exponential in the integrand. (It also dominates the factor $\frac{1}{\sqrt{\pi kt}}$.) We have

$$U(\vartheta, t) \approx \frac{u_o e^{\frac{1}{4}kt}}{2\sqrt{\pi kt}} \int_{-\theta_o}^{\theta_o} e^{-\frac{(\vartheta-\xi)^2}{4kt}} d\xi ,$$

or $$U(\vartheta, t) \approx \frac{u_o}{\sqrt{\pi}} e^{\frac{1}{4}kt} \int_{\frac{\vartheta-\theta_o}{2\sqrt{kt}}}^{\frac{\vartheta+\theta_o}{2\sqrt{kt}}} e^{-\xi^2} d\xi .$$

We can write this as

$$U(\vartheta, t) \approx \frac{u_o}{2} e^{\frac{1}{4}kt} \left(\mathrm{E\,rf}\left(\frac{\vartheta+\theta_o}{2\sqrt{kt}}\right) - \mathrm{E\,rf}\left(\frac{\vartheta-\theta_o}{2\sqrt{kt}}\right)\right) . \tag{6.13}$$

Case 2. $\vartheta < \theta_o$.

In this case the observer is situated within the zone of initial temperature u_o. As with the wave problem of the previous section we treat this case as the superposition of separate problems, one with initial condition

$$u_1(\theta, 0) = u_o \quad |\theta| \leq \pi, \tag{6.14}$$

and the other with initial condition

$$u_2(\theta, 0) = -u_o, \quad \theta_o < |\theta| \leq \pi. \tag{6.15}$$

The solution to (6.2) and (6.14) is easily seen to be

$$U_1(\vartheta, t) \equiv u_o \quad . \tag{6.16}$$

We can solve (6.2) and (6.15) by a change in the integral limits in (6.12), and so our complete solution for this case is

$$U(\vartheta, t) = u_o - \frac{u_o e^{\frac{1}{4}kt}}{2\sqrt{\pi kt}} \sum_{n=-\infty}^{\infty} (-1)^n \int_{\theta_o}^{2\pi-\theta_o} e^{-\frac{(\vartheta + 2n\pi - \xi)^2}{4kt}} d\xi$$

Here again, for sufficiently small kt, the n=0 term dominates, and

$$U(\vartheta, t) \approx u_o\left(1 - \frac{e^{\frac{1}{4}kt}}{2\sqrt{\pi kt}} \int_{\theta_o}^{2\pi-\theta_o} e^{-\frac{(\xi - \vartheta)^2}{4kt}} d\xi\right), \quad \vartheta < \theta_o.$$

Section 7. The Potential of a Sphere.

In this section we apply the generalized transform to obtain an alternative representation for the electrostatic or gravitational potential of a sphere.

We solve Laplace's equation in spherical coordinates:

$$r^2 V_{rr} + 2r V_r + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta} V_{\varphi\varphi} = 0, \tag{7.1}$$

with boundary conditions

$$V(a, \theta, \varphi) = f(\theta, \varphi), \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi \;, \tag{7.2}$$

$$\lim_{r\to\infty} r\, V\, (r, \theta, \varphi) < \infty \;. \tag{7.3}$$

We specify that $-\infty < \theta, \varphi < \infty$ in (7.1). Laplace's equation is one of static equilibrium, so we cannot interpret the solution as the contribution from a traveling disturbance. Because φ as well as θ occupies an infinite range, we must now distinguish between the extended azimuth φ and physical azimuth ϕ exactly as between θ and ϑ in Section 2. ϕ is the physical azimuth of an observer. The total potential at physical point (r, ϑ, ϕ) is the algebraic sum of the individual contributions at the extended coordinates $(r, \vartheta + 2n\pi, \phi + 2m\pi)$, where $n, m = 0, \pm 1, \pm 2, \ldots$. (Note that $0 \le \vartheta \le \pi$.)

The usual form of the solution to (7.1) expressed as a series of associated Legendre functions is

$$V(r, \theta, \varphi) = \frac{1}{2\pi}\left(\frac{a}{r}\right)\int_0^{2\pi} d\xi \int_0^{\pi} d\zeta \, \sin\zeta f(\zeta, \xi) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m \left(\frac{a}{r}\right)^n \cdot$$

$$\cdot \, (n+\tfrac{1}{2}) \, P_n^m(\cos\theta) \, P_n^{-m}(\cos\zeta) \, e^{im(\varphi - \xi)}. \qquad (7.4)$$

For $r >> a$ the factor $\left(\frac{a}{r}\right)^n$ implies that the series is rapidly convergent. The first few terms give a good approximation to the actual potential.

For $r \approx a$ we would require an inconvenient number of terms for accurate description of the potential, since $\left(\frac{a}{r}\right)^n$ is no longer dominant. This fact, for instance, hampers concise calculation of the gravity force on a spacecraft orbiting a planet. We will use the Mehler transform to derive a representation useful for $r \approx a$.

We define the Fourier transform $\widetilde{g}(\mu)$ of $g(\varphi)$ as

$$\widetilde{g}(\mu) \equiv \int_{-\infty}^{\infty} g(\varphi) \, e^{i\mu\varphi} \, d\varphi \qquad (7.5)$$

Take the Fourier transform of (7.1). The boundary terms arising from the integration by parts of the last term vanish, leaving

$$r^2 \widetilde{V}_{rr}(r, \theta, \mu) + 2r \widetilde{V}_r(r, \theta, \mu) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \widetilde{V}}{\partial\theta}(r, \theta, \mu)\right) - \frac{\mu^2}{\sin^2\theta} \widetilde{V}(r, \theta, \mu) = 0. \qquad (7.6)$$

Multiply (7.6) by $-\frac{i}{\pi} E^{-\mu}_{-\nu-\frac{1}{2}}(\theta) \sin\theta$ and integrate over θ from $-\infty$ to ∞. We integrate the third term by parts. The boundary terms again vanish and we get

$$- \frac{i}{\pi} r^2 \int_{-\infty}^{\infty} \tilde{V}_{rr}(r, \theta, \mu) E^{-\mu}_{-\nu-\frac{1}{2}}(\theta) \sin\theta d\theta - \frac{i}{\pi} 2r \int_{-\infty}^{\infty} \tilde{V}_r(r, \theta, \mu) \cdot$$

$$\cdot E^{-\mu}_{-\nu-\frac{1}{2}}(\theta) \sin\theta d\theta - \frac{i}{\pi} \int_{-\infty}^{\infty} \tilde{V}(r, \theta, \mu) \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} E^{-\mu}_{-\nu-\frac{1}{2}}(\theta) \right)$$

$$+ \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\mu^2}{\sin\theta} \tilde{V}(r, \theta, \mu) E^{-\mu}_{-\nu-\frac{1}{2}}(\theta) d\theta = 0 . \tag{7.7}$$

With the aid of Legendre's equation (1.1) we can simplify the third and fourth integrals; combining terms in the resulting expression gives

$$- \frac{i}{\pi} \int_{-\infty}^{\infty} \left[r^2 \tilde{V}_{rr}(r, \theta, \mu) + 2r \tilde{V}_r(r, \theta, \mu) - (\nu^2 - \tfrac{1}{4}) \tilde{V}(r, \theta, \mu) \right] E^{-\mu}_{-\nu-\frac{1}{2}}(\theta) \sin\theta = 0 . \tag{7.8}$$

From the Mehler transform formula (4.15) we define

$$\Psi(r, \nu, \mu) \equiv - \frac{i}{\pi} \int_{-\infty}^{\infty} \tilde{V}(r, \theta, \mu) E^{-\mu}_{-\nu-\frac{1}{2}}(\theta) \sin\theta d\theta. \tag{7.9}$$

Equation (7.8) becomes

$$r^2\, \Psi_{rr} + 2r\, \Psi_r - (\nu^2 - \tfrac{1}{4})\, \Psi = 0 \ , \tag{7.10}$$

whose solution is

$$\Psi = A(\nu,\mu)\; r^{\nu-\frac{1}{2}} + B(\nu,\mu)\, r^{-\nu-\frac{1}{2}} \ . \tag{7.11}$$

From (7.5) we have

$$\widetilde{V}(r,\theta,\mu) = \int_{-\infty}^{\infty} V(r,\theta,\varphi)\, e^{i\mu\varphi}\, d\varphi \ . \tag{7.12}$$

Boundary condition (7.3) implies

$$\lim_{r\to\infty} r^{\alpha}\, V(r,\theta,\varphi) = 0 \quad \text{for} \quad \alpha < 1 \ ,$$

which together with (7.12) gives

$$\lim_{r\to\infty} r^{\alpha}\, \widetilde{V}(r,\theta,\mu) = 0 \quad \text{for} \quad \alpha < 1 \ . \tag{7.13}$$

Imposing (7.13) on (7.9), we get

$$\lim_{r\to\infty} r^{\alpha}\Psi(r,\nu,\mu) = 0 \quad \text{for} \quad \alpha < 1 \ .$$

Therefore, the solution (7.11) becomes

$$\Psi(r,\nu,\mu) = \begin{cases} A(\nu,\mu)\; r^{\nu-\frac{1}{2}} \ , & \nu \leq -\frac{1}{2} \\ 0 & -\frac{1}{2} < \nu < \frac{1}{2} \\ B(\nu,\mu)\; r^{-\nu-\frac{1}{2}} & \nu \geq \frac{1}{2} \end{cases} \tag{7.14}$$

By transforming initial condition (7.3) and inverting (7.12) and (7.9) we find the result

$$V(r,\theta,\varphi) = \frac{1}{2\pi}\sqrt{\frac{a}{r}}\int_0^{2\pi} d\xi \int_0^{\pi} d\zeta \cdot f(\zeta,\xi) \cdot \left(-\frac{i}{\pi^2}\right)\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} d\nu \left\{ \left(\frac{a}{r}\right)^{|\nu|} e^{-i\mu(\varphi-\xi)} \cdot \right.$$

$$\left. \cdot\, E^{-\mu}_{-\nu-\frac{1}{2}}(\zeta)\, E^{\mu}_{\nu-\frac{1}{2}}(\theta) \sin\zeta\, \nu \frac{\cos\pi(\nu+\mu)}{\sin\pi\nu} H\left(|\nu|-\tfrac{1}{2}\right)\right\}, \tag{7.15}$$

where H(x) is the Heaviside unit function.

This expression is the potential at the extended coordinates θ and φ. The total physical potential $\Phi(r,\vartheta,\phi)$ is given by

$$\Phi(r,\vartheta,\phi) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} V(r,\vartheta+2n\pi,\phi+2m\pi).$$

From the addition formula (3.11) we get

$$\Phi(r,\vartheta,\phi) = \frac{1}{2\pi}\sqrt{\frac{a}{r}}\sum_{n=-\infty}^{\infty}(-1)^n \sum_{m=-\infty}^{\infty}\int_0^{\pi} d\zeta \int_0^{2\pi} d\xi\; f(\zeta,\xi)\int_{-\infty}^{\infty} d\mu\, e^{-i\xi(\phi+2m\pi-\xi)} \cdot$$

$$\cdot\left\{\left(-\frac{i}{\pi^2}\right)\int_{-\infty}^{\infty} d\nu \left(\frac{a}{r}\right)^{|\nu|} e^{i2n\pi\nu} E^{-\mu}_{-\nu-\frac{1}{2}}(\zeta)\, E^{\mu}_{\nu-\frac{1}{2}}(\vartheta)\sin\zeta\,\nu\frac{\cos\pi(\nu+\mu)}{\sin\pi\nu} H\left(|\nu|-\tfrac{1}{2}\right)\right\} \tag{7.16}$$

We first evaluate the ν-integral in (7.16). Recalling the hypergeometric series definition (3.9) of $E^{\mu}_{\nu-\frac{1}{2}}(\theta)$, we find that

$$- \frac{i}{\pi^2} \; E^{-\mu}_{-\nu-\frac{1}{2}}(\zeta) \, E^{\mu}_{\nu-\frac{1}{2}}(\vartheta) \sin\zeta \, \nu \, \frac{\cos\pi(\nu+\mu)}{\sin\pi\nu} \; =$$

$$= e^{i\nu(\vartheta-\zeta)} \left(-\frac{i}{\pi}\right) e^{\frac{i}{2}(\theta+\zeta)} \, e^{i\mu(\vartheta-\zeta)} \sin\zeta \; {}_2F_1[\ldots e^{2i\zeta}]$$

$${}_2F_1[\ldots e^{2i\vartheta}]. \tag{7.17}$$

Integrate both sides of (7.17) over ν from $-\infty$ to ∞. The left-hand side is just the δ-function expansion (4.13). Therefore

$$\int_{-\infty}^{\infty} e^{i\nu(\vartheta-\zeta)} \, C(\vartheta, \zeta, \mu, \nu) \; d\nu = \delta(\vartheta-\zeta) \tag{7.18}$$

where $C(\theta, \zeta, \mu, \nu)$ represents the remaining factors on the right-hand side of (7.17).

Another well-known expansion is

$$\delta(\vartheta-\zeta) = \int_{-\infty}^{\infty} e^{i\nu(\vartheta-\zeta)} \frac{1}{2\pi} \, d\nu \tag{7.19}$$

Subtracting (7.19) from (7.18) we obtain that

$$C(\vartheta, \zeta, \mu, \nu) = \frac{1}{2\pi}$$

and hence

$$- \frac{i}{\pi^2} E^{-\mu}_{-\nu-\frac{1}{2}}(\zeta) E^{\mu}_{\nu-\frac{1}{2}}(\vartheta) \sin \zeta\nu \frac{\cos\pi(\nu+\mu)}{\sin\pi\nu} = \frac{1}{2\pi} e^{i\nu(\vartheta-\zeta)} \tag{7.20}$$

Let N stand for the ν- integral in (7.16). We can write N as

$$N = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\frac{a}{r})^{|\nu|} e^{i\nu(\vartheta + 2n\pi - \zeta)} H[|\nu| - \tfrac{1}{2}] d\nu$$

or

$$N = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} (\frac{a}{r})^{\nu} \cos \nu(\vartheta + 2n\pi - \zeta) d\nu .$$

The result is

$$N = \frac{1}{\pi} \sqrt{\frac{a}{r}} (-1)^n \left\{ \frac{\log(\frac{r}{a}) \cos \frac{1}{2}(\vartheta - \zeta) - (\vartheta + 2n\pi - \zeta) \sin \frac{1}{2}(\vartheta - \zeta)}{[\log(\frac{r}{a})]^2 + [\vartheta + 2n\pi - \zeta]^2} \right\}$$

There is no μ-dependence in this integral, so we can do the μ-integral in (7.16) to give

$$\int_{-\infty}^{\infty} d\mu\, e^{-i\mu(\phi + 2m\pi - \xi)} = 2\pi\delta(\phi + 2m\pi - \xi) \quad . \tag{7.21}$$

Now $0 < \xi < 2\pi$, so the only contribution to (7.16) from (7.21) is for $m = 0$. Therefore our final result is

$$\Phi(r, \vartheta, \phi) = \frac{1}{\pi} (\frac{a}{r}) \sum_{n=-\infty}^{\infty} \int_0^{\pi} f(\zeta, \phi) \left\{ \frac{\log(\frac{r}{a}) \cos \frac{1}{2}(\vartheta - \zeta) - (\vartheta + 2n\pi - \zeta) \sin \frac{1}{2}(\vartheta - \zeta)}{[\log(\frac{r}{a})]^2 + [\vartheta + 2n\pi - \zeta]^2} \right\} d\zeta \tag{7.22}$$

This representation for the potential is superior to the usual expression (7.4). In modern applications (7.4) has been used extensively in the calculation of the gravitational force of a celestial body on a nearby spacecraft. Not only do we have to calculate a double integral numerically, but also for $r \approx a$ several dozen terms in the double series are required for a good approximation to the potential. Unfortunately, the calculation of such a large number of associated Legendre functions poses serious numerical problems.

In (7.22) we have the advantage of having to evaluate only a single integral and series. Even more significantly, we need only sines and cosines and avoid completely the computation of any Legendre functions.

PART II

Sound Waves in the Ocean

Section 8. Introduction.

We shall derive the spectral representation associated with the DE

$$w'' + \left[\frac{\omega^2}{c_\infty^2} \left(1 + \frac{M}{\cosh^2(\frac{1}{2}mz)} \right) - \lambda \right] w = 0 \qquad (8.1)$$

and apply it to solve the PDE modeling the propagation of sound waves in the ocean. The spectrum is found to be mixed, requiring both an integral and a series to constitute the complete representation.

Equation (8.1) arises from the separation of variables method applied to the linear wave equation with variable sound speed. In Section 9 we start with the PDE and derive the spectral representation associated with (8.1). We solve the PDE in Section 10 and analyze certain aspects of the initial-value wave problem it represents.

Section 9. Derivation of the Spectral Representation.

A problem of interest is the propagation of small-amplitude sound waves in the ocean. It has been widely determined experimentally that the speed of sound in the oceans of the earth is non-uniform. There is a decided minimum speed at a depth of roughly 1200 meters, with a sharp increase above and below it.

The depth of minimum speed is called the sound channel. The far field in the vicinity of the channel shows prominently the trapping and focusing effects caused by the non-uniform sound speed when a point source is located in the channel.

There has been substantial investigation of the steady-state case of this problem, using a variety of models for the variation of sound speed with depth. (For an extensive bibliography, see the paper by Deavenport [9].) Blum and Cohen [8] investigated the initial-value problem, where the source begins emitting sound at time $t = 0$. Their analysis converts the Fourier and Hankel representation to an asymptotically more convenient form by a Watson contour integration. One of our aims is to derive the exact expression for this form using a new spectral representation.

For simplicity we consider an ocean of infinite extent in all directions with a vertical variation of sound speed. Because of the horizontal uniformity we express our problem in cylindrical coordinates, with the origin in the channel and the z-axis directed along the vertical. We pick the symmetric Epstein profile for sound speed c(z):

$$c(z) = c_{\infty}\left[1+\frac{M}{\cosh^2(\frac{1}{2}mz)}\right]^{-1/2} . \qquad (9.1)$$

Its variation with depth is shown in Figure 6. c_{∞} is the limiting speed for $z \to \pm\infty$, M is a measure of the minimum sound speed (in the channel, where $c(0) = \frac{c_{\infty}}{\sqrt{1+M}}$), and m is inversely related to the width of the profile.

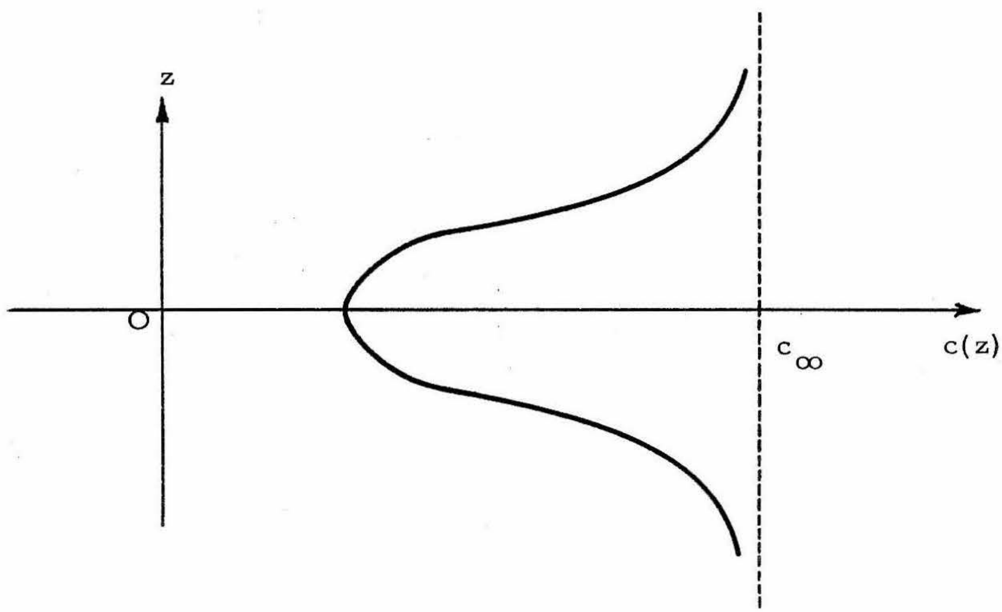

Figure 6. The symmetric Epstein profile (9.1).

The initial-value problem we study using speed (9.1) is

$$p_{rr}(r, z, t) + \frac{1}{r} p_r(r, z, t) + p_{zz}(r, z, t) - \frac{1}{c_\infty^2} \left[1 + \frac{M}{\cosh^2(\frac{1}{2}mz)} \right] p_{tt} =$$

$$= - \frac{1}{2\pi} \frac{\delta(r)}{r} \delta(z - z_0) f(t) H(t) , \qquad (9.2)$$

$$-\infty < z < \infty , \quad 0 \le r < \infty , \quad -\infty < t < \infty$$

$$p(r, z, 0) = 0 \qquad (9.3)$$

$$p_t(r, z, 0) = 0 \qquad (9.4)$$

$$p(r, z, t) \to 0 \text{ and outgoing waves} \qquad (9.5)$$

$$\text{as } r \to \infty, \ z \to \pm \infty .$$

The quantity $p(r, z, t)$ is the observed deviation at point (r, z) from the ambient pressure. This problem represents a point source at radius $r = 0$ and height $z = z_0$ that emits an acoustical disturbance according to $f(t)$ beginning at time $t = 0$.

We separate variables in the homogeneous form of (9.2):

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} - \frac{1}{c_\infty^2} \left[1 + \frac{M}{\cosh^2(\frac{1}{2}mz)} \right] \frac{T''}{T} = 0 . \qquad (9.6)$$

Define $\frac{T''}{T} \equiv -\omega^2$, and we see that the t-representation is the Fourier transform. Equation (9.6) becomes

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{Z''}{Z}+\frac{\omega^2}{c_\infty^2}\left[1+\frac{M}{\cosh^2(\frac{1}{2}mz)}\right] = 0 . \tag{9.7}$$

The terms are separately functions of r and z. Define

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R} \equiv -\lambda$$

and the equation for Z yields

$$Z'' + \left\{\frac{\omega^2}{c_\infty^2}\left[1+\frac{M}{\cosh^2(\frac{1}{2}mz)}\right] - \lambda\right\} Z = 0 . \tag{9.8}$$

It can be easily shown that two solutions of (9.8) are

$$Z = P^{-\mu}_{\nu-1/2}(\tanh \tfrac{1}{2} mz) \text{ and } Z = P^{-\mu}_{\nu-1/2}(-\tanh \tfrac{1}{2} mz), \tag{9.9}$$

where

$$\nu \equiv \left(\frac{1}{4} + 4\,\frac{\omega^2}{c_\infty^2 m^2}\, M\right)^{1/2} \tag{9.10}$$

and

$$\mu \equiv \frac{2}{m}\left(\lambda - \frac{\omega^2}{c_\infty^2}\right)^{1/2} . \quad \text{(Principal Branch)} \tag{9.11}$$

By imposing the outgoing wave condition (9.5) we construct the Green's function $\mathcal{G}(z, z_0, \mu(\lambda))$ in Appendix D, to get

$$\mathcal{G}(z, z_0, \mu(\lambda)) = \frac{1}{m} P^{-\mu}_{\nu-1/2}(\tanh\tfrac{1}{2}mz_>) P^{-\mu}_{\nu-1/2}(-\tanh\tfrac{1}{2}mz_<) \cdot$$

$$\cdot\, \Gamma(\tfrac{1}{2}+\nu+\mu)\,\Gamma(\tfrac{1}{2}-\nu+\mu) \qquad (9.12)$$

where $z_>$ and $z_<$ are the greater and lesser of z and z_0, respectively.

Now we can apply the fundamental spectral formula (B.7):

$$\delta(z - z_0) = \frac{1}{2\pi i} \int_{C_B} \mathcal{G}(z, z_0, \mu(\lambda))\, d\lambda + \sum (\text{Residues in } \lambda\text{-plane}). \qquad (9.13)$$

From (9.11) and (9.12) we see that the point $\mu = 0$ is a branch point of $\mathcal{G}(z, z_0, \mu(\lambda))$ in the λ-plane. In Appendix D it is shown that convergence of the solutions at $z = \pm\infty$ demands that $\mathrm{Re}\{\mu\} \geq 0$. Therefore we make the change of variable $\lambda = \frac{m^2}{4}\mu^2 + \frac{\omega^2}{c_\infty^2}$ and arrange the cut in the λ-plane so that $\mathrm{Re}\{\mu\} \geq 0$. The resulting contours are shown in Figure 7.

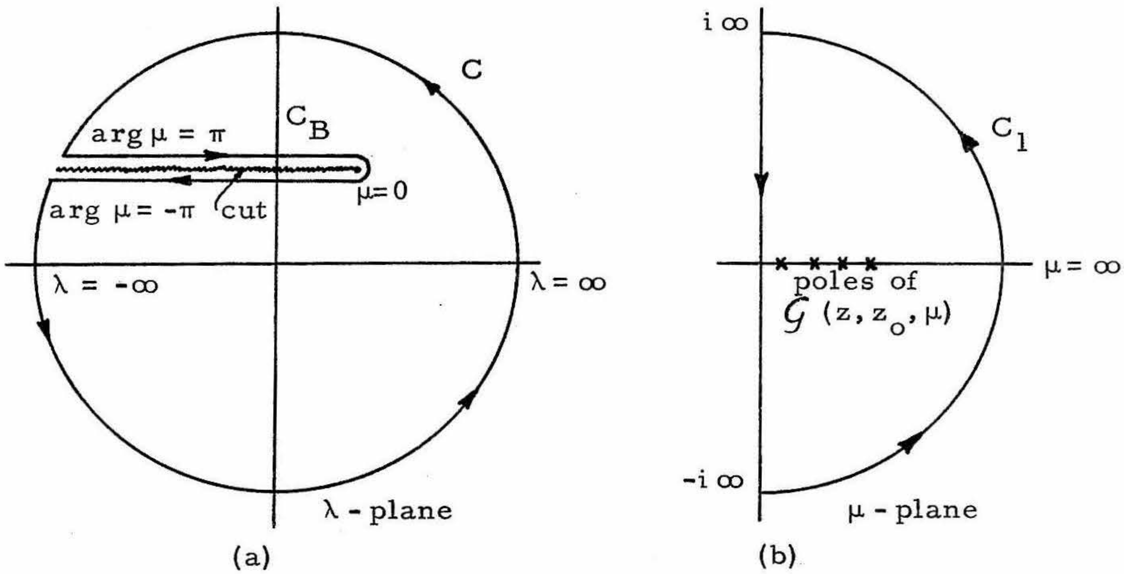

Figure 7. Contours in λ- and μ- planes for the integrals (9.13) and (9.14).

The spectral formula in the μ-plane is

$$\delta(z-z_0) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \mathcal{G}(z, z_0, \mu) \frac{m^2}{2} \mu \, d\mu + \sum (\text{Residues in right half of } \mu\text{-plane}) . \qquad (9.14)$$

Now

$$\mathcal{G}(z, z_0, \mu) \frac{m^2}{2} \mu = \tfrac{1}{2} m\mu \, P^{-\mu}_{\nu-1/2}(\tanh \tfrac{1}{2} mz) P^{-\mu}_{\nu-1/2}(-\tanh \tfrac{1}{2} mz_0) \Gamma(\tfrac{1}{2}+\nu+\mu) \cdot \Gamma(\tfrac{1}{2} - \nu + \mu) . \qquad (9.15)$$

The function $P^{-\mu}_{\nu-\frac{1}{2}}(\zeta)$ is an entire function of μ. In Appendix E we show that ν is real in the ω-range of interest, and that there are simple poles from the Γ-functions at $\mu = |\nu| - \frac{1}{2} - n$, $n = 0, 1, \ldots, N$, where N is the largest integer less than $|\nu| - \frac{1}{2}$.

The δ-function expansion (9.14) becomes

$$\delta(z-z_0) = \frac{m}{4\pi i}\int_{-i\infty}^{i\infty} P^{-\mu}_{\nu-\frac{1}{2}}\left(\tanh\frac{mz}{2}\right) P^{-\mu}_{\nu-\frac{1}{2}}\left(-\tanh\frac{mz_0}{2}\right)\mu\Gamma\left(\tfrac{1}{2}+\nu+\mu\right)\Gamma\left(\tfrac{1}{2}-\nu+\mu\right)d\mu$$

$$+\sum_{n=0}^{n<|\nu|-\frac{1}{2}} \frac{(-1)^n}{n!} P^{-(|\nu|-\frac{1}{2})+n}_{\nu-\frac{1}{2}}\left(\tanh\frac{mz}{2}\right) P^{-(|\nu|-\frac{1}{2})+n}_{\nu-\frac{1}{2}}\left(-\tanh\frac{mz_0}{2}\right)\left(|\nu|-\tfrac{1}{2}-n\right)\cdot$$

$$\cdot\,\Gamma(2|\nu|-n)\,. \tag{9.16}$$

Multiply each side of (9.16) by $f(z_0)$ and integrate over z_0 from $-\infty$ to ∞. The resulting transform pair is

$$f(z) = \int_{-i\infty}^{i\infty} \tilde{f}(\mu) P^{-\mu}_{\nu-\frac{1}{2}}\left(\tanh\frac{mz}{2}\right)\mu\Gamma\left(\tfrac{1}{2}+\nu+\mu\right)\Gamma\left(\tfrac{1}{2}-\nu+\mu\right)d\mu$$

$$+\sum_{n=0}^{n<|\nu|-\frac{1}{2}} \tilde{f}\,\frac{(-1)^n}{n!} P^{-(|\nu|-\frac{1}{2})+n}_{\nu-\frac{1}{2}}\left(\tanh\frac{mz}{2}\right)\left(|\nu|-\tfrac{1}{2}-n\right)\cdot$$

$$\cdot\,\Gamma(2|\nu|-n) \tag{9.17}$$

where

$$\tilde{f}(\mu) = \frac{m}{4\pi i} \int_{-\infty}^{\infty} f(z) P_{\nu-1/2}^{-\mu} \left(-\tanh \frac{mz}{2}\right) dz \tag{9.18}$$

and

$$\tilde{f}_n = \frac{m}{2} \int_{-\infty}^{\infty} f(z)\, P_{\nu-1/2}^{-(|\nu|-1/2)+n} \left(-\tanh \frac{mz}{2}\right) dz \;. \tag{9.19}$$

For any fixed ν this spectral representation contains an integral plus a finite number of discrete terms.

Section 10. Application to Sound Waves in the Ocean.

We apply the spectral representation (9.17) to obtain a new form of the solution of the initial-value problem (9.2)–(9.5). First, we employ the complex Fourier transform (with frequency ω). The integration path lies in the upper half ω-plane above all singularities of the integrand. Define

$$p(r, z, t) = \int_{-\infty+i\gamma}^{\infty+i\gamma} \tilde{p}(r, z, \omega) e^{-i\omega t} \, d\omega \tag{10.1}$$

$$\tilde{p}(r, z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(r, z, t) e^{i\omega t} \, dt . \tag{10.2}$$

The contour for (10.1) is shown in Figure 8.

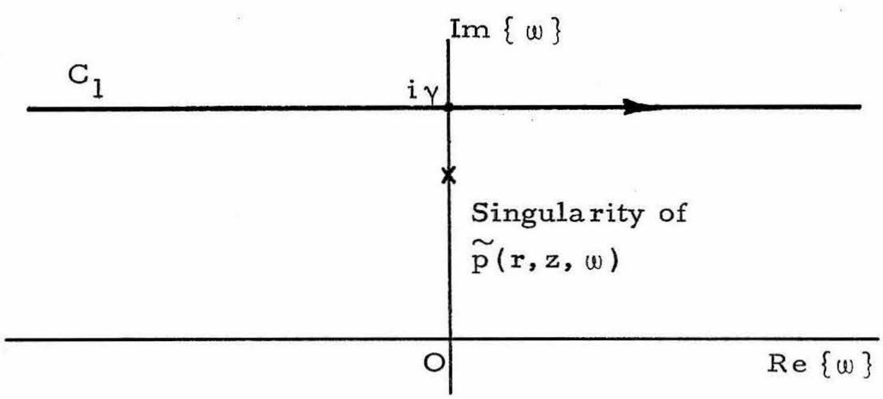

Figure 8. Integration path for (10.1).

We define $F(\omega) \equiv -\frac{1}{2\pi}\int_0^\infty f(t)\, e^{i\omega t}\, dt$. Then in the usual way (9.2)

transforms to

$$\tilde{p}_{rr} + \frac{1}{r}\tilde{p}_r + \tilde{p}_{zz} + \frac{\omega^2}{c_\infty^2}\left[1 + \frac{M}{\cosh^2(\frac{1}{2}mz)}\right]\tilde{p} = \frac{\delta(r)}{2\pi r}\delta(z-z_0)F(\omega). \tag{10.3}$$

Now we apply the transform (9.17)–(9.19). Multiply (10.3) by $P^{-\mu}_{\nu-1/2}\left(-\tanh\frac{mz}{2}\right)$ and integrate over z from $-\infty$ to ∞.

Equation (10.3) becomes

$$\int_{-\infty}^{\infty}\left(\tilde{p}_{rr} + \frac{1}{r}\tilde{p}_r + \tilde{p}_{zz} + \frac{\omega^2}{c_\infty^2}\left[1 + \frac{M}{\cosh^2(\frac{1}{2}mz)}\right]\tilde{p}\right)P^{-\mu}_{\nu-1/2}\left(-\tanh\frac{mz}{2}\right)dz$$

$$= \frac{\delta(r)}{2\pi r}F(\omega)\,P^{-\mu}_{\nu-1/2}\left(-\tanh\frac{mz_0}{2}\right). \tag{10.4}$$

From (9.5) it follows that $\tilde{p}$, $\tilde{p}_z$, $\to 0$ as $z \to \pm\infty$.

Therefore, we can integrate the third term on the left in (10.4) by parts twice and the boundary terms vanish. Combining this result with the fact that $P^{-\mu}_{\nu-1/2}\left(-\tanh\frac{mz}{2}\right)$ satisfies (9.8), we can simplify (10.4) to

$$\int_{-\infty}^{\infty}\left(\tilde{p}_{rr} + \frac{1}{r}\tilde{p}_r + \left(\frac{m^2}{4}\mu^2 + \frac{\omega^2}{c_\infty^2}\right)\tilde{p}\right)P^{-\mu}_{\nu-1/2}\left(-\tanh\frac{mz}{2}\right)dz =$$

$$= \frac{\delta(r)}{2\pi r} \; F(\omega) \; P^{-\mu}_{\nu-\frac{1}{2}} \left(- \tanh \frac{mz_0}{2} \right) \; . \tag{10.5}$$

From (9.18) we define

$$\frac{m}{4\pi i} \int_{-\infty}^{\infty} \tilde{p} \; (r, z, \omega) P^{-\mu}_{\nu-\frac{1}{2}} \left(-\tanh \frac{mz}{2} \right) dz \equiv \psi (r, \mu, \omega) \tag{10.6}$$

which yields from (10.5)

$$\psi_{rr} + \frac{1}{r} \psi_r \left(\frac{m^2}{4} \mu^2 + \frac{\omega^2}{c_\infty^2} \right) \psi = \frac{im}{8\pi^2} F(\omega) P^{-\mu}_{\nu-\frac{1}{2}} \left(-\tanh \frac{mz_0}{2} \right) \frac{\delta(r)}{r} \; . \tag{10.7}$$

We recognize (10.7) as the zero-order Bessel equation. Its solution obeying the outgoing wave condition with time factor $e^{-i\omega t}$ is

$$\psi (r, \mu, \omega) = \frac{m}{16\pi} H_0^{(1)} \left[\operatorname{sgn}(\mathrm{Re}\{\omega\}) r \left(\frac{m^2}{4} \mu^2 + \frac{\omega^2}{c_\infty^2} \right)^{\frac{1}{2}} \right] F(\omega)$$

$$P^{-\mu}_{\nu-\frac{1}{2}} \left(-\tanh \frac{mz_0}{2} \right) \; .$$

$H_0^{(1)}(z)$ is the Hankel function of the first kind, and

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} .$$

To get the full representation for $\tilde{p}(r, z, \omega)$ (and consequently for $p(r, z, t)$), (9.17) implies that we must have a contribution $\psi_n(r, \omega)$ from the discrete spectrum. It is derived precisely as $\psi(r, \mu, \omega)$. It is shown in Appendix E that we can translate the ω-contour down to the real axis in the discrete spectrum. The variable ν is real for real ω, so the inverse transforms (9.17) and (10.1) give the final result

$$p(r, z, t) = \int_{-\infty+i\gamma}^{\infty+i\gamma} d\omega\, e^{-i\omega t} F(\omega) \left\{ \frac{m}{16\pi} \int_{-i\infty}^{i\infty} H_0^{(1)}\left[\operatorname{sgn}(\operatorname{Re}\{\omega\})\, r \left(\frac{m^2}{4}\mu^2 + \frac{\omega^2}{c_\infty^2} \right)^{1/2} \right] \cdot \right.$$

$$\left. \cdot\, P_{\nu-1/2}^{-\mu}\left(-\tanh\frac{mz_0}{2}\right) P_{\nu-1/2}^{-\mu}\left(\tanh\frac{mz}{2}\right) \mu\Gamma\left(\tfrac{1}{2}+\nu+\mu\right)\Gamma\left(\tfrac{1}{2}-\nu+\mu\right) d\mu \right\}$$

$$+ \frac{im}{8} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} F(\omega) \sum_{n=0}^{n<|\nu|-1/2} \frac{(-1)^n}{n!} H_0^{(1)}\left[\operatorname{sgn}(\omega)\, r \left| \left(\frac{m^2}{4}\left(|\nu|-\tfrac{1}{2}-n\right)^2 + \frac{\omega^2}{c_\infty^2} \right)^{1/2} \right| \right] \cdot$$

$$\cdot P_{\nu-1/2}^{-(|\nu|-1/2)+n}\left(-\tanh\frac{mz_0}{2}\right) P_{\nu-1/2}^{-(|\nu|-1/2)+n}\left(\tanh\frac{mz}{2}\right)(|\nu|-\tfrac{1}{2}-n)\cdot$$

$$\cdot\,\Gamma(2|\nu|-n) \quad . \qquad (10.8)$$

This expression represents the complete disturbance observed at any point (r, z) with the source located at $(0, z_0)$. We shall analyze certain aspects of this result.

When the source and receiver are both far above the channel it is apparent from Figure 6 that the surrounding speed is nearly uniform. Therefore, the corresponding solution will not differ significantly from a pure spherical wave.

The behavior of the solution in the vicinity of the channel is more complex. (The variation of sound speed throughout the actual oceans of the earth compares quite favorably with the central portion of the Epstein profile.) By performing a Watson transformation on the Hankel representation of $p(r, z, t)$, Blum and Cohen [8] obtained an expression for the case $z = z_0 = 0$ (source and receiver in the channel) asymptotically valid for $r \rightarrow \infty$. We can reduce expression (10.8) to theirs if we ignore the continuous spectrum, set $z = z_0 = 0$, and replace the Hankel function by its asymptotic form for $r \rightarrow \infty$. It was found that the strongest signals arriving at a receiver in the channel are not "clean-cut" — they have tails or wakes that tend to obscure subsequent signals close behind.

It is known [8] that the continuous spectrum in (10.8) is $O\left(\frac{1}{r}\right)$ as $r \to \infty$ for all z, and that the discrete part is $O\left(\frac{1}{\sqrt{r}}\right)$ for $z = z_0 = 0$. When the source is in the channel (10.8) becomes

$$p(r,z,t) \sim \frac{im}{8} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} F(\omega) \sum_{n=0}^{n<|\nu|-\frac{1}{2}} \Bigg\{ H_0^{(1)}[\cdots\cdot] \cdot P_{\nu-\frac{1}{2}}^{-(|\nu|-\frac{1}{2})+n}(0) \cdot$$

$$\cdot\, P_{\nu-\frac{1}{2}}^{-(|\nu|-\frac{1}{2})+n}\left(\tanh \frac{mz}{2}\right) \cdot$$

$$\cdot\, (|\nu| - \tfrac{1}{2} - n)\, \Gamma\,(2|\nu| - n) \Bigg\} \quad , \quad r \to \infty . \qquad (10.9)$$

From [1], p. 143, (6), we obtain

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} {}_2F_1(1+\nu, -\nu; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}x) ,$$

from which (10.9) becomes, after interchanging summation and integration,

$$p(r,z,t) \sim \frac{im}{8} \sum_{n=0}^{\infty} \int_{\frac{mc_\infty}{2\sqrt{M}}(n^2+n)^{\frac{1}{2}}}^{\infty} e^{-i\omega t}\, F(\omega) \Bigg\{ H_0^{(1)}[\cdots\cdot]\, P_{\nu-\frac{1}{2}}^{-(|\nu|-\frac{1}{2})+n}(0) \cdot$$

$$\cdot \frac{1}{\Gamma(\frac{1}{2} + |\nu| - n)} \quad e^{-\frac{m}{2}(|\nu| - \frac{1}{2} - n)z} \cdot F\left(\frac{1}{2} + \nu, \frac{1}{2} - \nu ; \frac{1}{2} + |\nu| - n ; \frac{1}{1+e^{mz}}\right) \Bigg\} \quad (10.10)$$

For $mz \ll 1$, each of the integrals in the series is reduced by a factor $e^{-\frac{m}{2}(|\nu|-\frac{1}{2}-n)z}(1-O(mz))$. Because of the exponential the high-frequency components of $F(\omega)$ will be drastically attenuated, even at a small distance from the channel. The implication is that if the forcing function $f(t)$ at the source is a sharp pulse, the dominant behavior just outside the channel will be a smooth rise and long decay. Thus, long distance, high frequency communication is even more restricted than it is with the receiver in the channel.

APPENDIX A

Derivation of the Equation for Waves on a Sphere

We shall derive the equation for the axisymmetric free-surface water waves on a shallow ocean covering a solid sphere. For inviscid, incompressible, irrotational fluids the fundamental equations are

Continuity: $$\nabla \cdot \underline{u} = 0 \tag{A.1}$$

Momentum: $$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = \underline{F} - \frac{1}{\rho} \nabla p \tag{A.2}$$

where $\underline{u} = u_r \hat{r} + u_\theta \hat{\theta} + u_\phi \hat{\phi}$ is the instantaneous velocity of a particle of fluid, $\underline{F}$ is the vector sum of external forces, ρ is the constant density, and p is the local pressure. The symbol ∇ is the gradient operator, and $\nabla \underline{u}$ is a dyadic appropriate to spherical coordinates.

In the simplest approximation we ignore the non-linear term $\underline{u} \cdot \nabla \underline{u}$. For axisymmetric waves there is no azimuth dependence and $\frac{\partial}{\partial \phi} \equiv 0$. The spherical form of the continuity equation (A.1) is

$$\frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} (r^2 \sin\theta \, u_r) + \frac{\partial}{\partial \theta} (r \sin\theta \, u_\theta) \right] = 0 , \tag{A.3}$$

which can be written as

$$\frac{1}{r} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, u_\theta) = 0 \tag{A.4}$$

In the momentum equation we assume that the only external force is gravity and that it is constant throughout the depth of the ocean. Then

$$\underline{F} = - g \hat{r} \tag{A.5}$$

and (A.2) reduces to the pair of equations

$$\frac{\partial u_r}{\partial t} = - g - \frac{1}{\rho}\frac{\partial p}{\partial r} \quad , \tag{A.6}$$

$$\frac{\partial u_\theta}{\partial t} = - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} \ . \tag{A.7}$$

As we are interested in free surface waves, we define $\psi(\theta,t)$ to be the instantaneous elevation of the surface above the equilibrium position. Figure A-1 shows the relevant quantities.

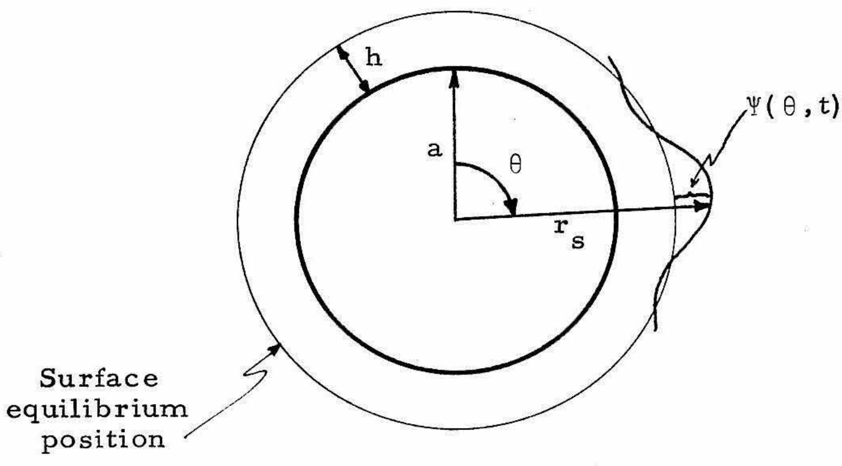

Figure A-1. Definition of parameters describing the free-surface waves.

If r_s is the distance from the center to the free surface, then

$$\psi(\theta,t) = r_s - a - h \ . \tag{A.8}$$

The boundary conditions on (A.4), (A.6), and (A.7) are

(i) The radial velocity vanishes at the bottom:

$$u_r(a,\theta) = 0 \tag{A.9}$$

(ii) The pressure at the surface vanishes:

$$p(a + h + \psi) = 0 \tag{A.10}$$

(iii) The kinematic condition relating u_r and ψ:

$$u_r(a + h + \psi) = \frac{\partial\psi}{\partial t} . \tag{A.11}$$

In the lowest order theory we ignore radial acceleration; i.e., assume

$$\frac{\partial u_r}{\partial t} = 0$$

and (A.6) can be integrated to

$$p = -\rho g r + C(\theta,t) .$$

Using (A.10) we evaluate $C(\theta,t)$ to get

$$p = \rho g(a + h + \psi(\theta,t) - r) . \tag{A.12}$$

Therefore

$$\frac{\partial p}{\partial\theta} = \rho g \frac{\partial\psi}{\partial\theta} \tag{A.13}$$

Differentiate (A.4) with respect to time and use (A.7), (A.12), and (A.13):

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_r}{\partial t}\right) - g \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) = 0 . \tag{A.14}$$

Integrate (A.13) with respect to r from $r = a$ to the free surface:

$$\left[(a+h+\psi)^2\right]\frac{\partial u_r}{\partial t}(a+h+\psi) - \frac{g(h+\psi)}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = 0 . \qquad (A.15)$$

We assume $a >> h >> \psi$. Using relation (A.11) and keeping only the dominant terms, (A.15) reduces to

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{a^2}{gh}\frac{\partial^2\psi}{\partial t^2} = 0. \qquad (A.16)$$

Let $\frac{a^2}{gh} = c^2$, and we get the final result

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 . \qquad (A.17)$$

Equation (A.17) was derived using the usual definition of polar latitude $0 \leq \theta \leq \pi$. In Section 2 we employ the modified polar latitude $-\pi \leq \vartheta \leq \pi$. In the left-hand half of the sphere as shown in Figure 1 the disturbance propagates in the $-\theta$ direction. We note, however, that (A.17) is invariant under the transformation $\theta \rightarrow -\theta$ and hence is valid in the entire range $-\infty < \theta < \infty$.

APPENDIX B

Finding the Spectral Representation Associated with an ODE

The method for deriving systematically the spectral representation associated with any linear ODE and its boundary conditions is most thoroughly covered in Titchmarsh [3]. It can be motivated as follows:

Let L be a linear differential operator, and consider the inhomogeneous equation:

$$(\lambda - L)u(x,\lambda) = f(x). \tag{B.1}$$

where $f(x)$ is a known function of x. From the theory of eigenfunction expansions we can expand $u(x,\lambda)$ and $f(x)$ in the eigenfunctions $u_n(x)$ of the operator L:

$$u(x,\lambda) = \sum_n \alpha_n(\lambda)\, u_n(x) \; ; \tag{B.2}$$

$$f(x) = \sum_n \beta_n u_n(x). \tag{B.3}$$

Let $Lu_n(x) = \lambda_n u_n(x)$, where λ_n is the nth eigenvalue of L. Substituting (B.2) and (B.3) into (B.1) we get

$$\sum_n \alpha_n(\lambda)(\lambda - \lambda_n)\, u_n(x) = \sum_n \beta_n u_n(x) \; ,$$

which implies

$$\sum_n \alpha_n(\lambda)\, u_n(x) \equiv u(x,\lambda) = \sum_n \frac{\beta_n}{\lambda - \lambda_n} u_n(x) \; . \tag{B.4}$$

Integrate (B.4) around an infinite circle C in the λ-plane:

$$\frac{1}{2\pi i} \oint_C u(x,\lambda)\, d\lambda = \sum_n \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda - \lambda_n} \cdot \beta_n u_n(x) \tag{B.5}$$

From Cauchy's integral formula we have

$$\frac{1}{2\pi i} \oint_C \frac{d\lambda}{\lambda - \lambda_n} \equiv 1 ,$$

since $|\lambda_n| < \infty$. Then (B.5) becomes

$$\frac{1}{2\pi i} \oint_C u(x,\lambda)\, d\lambda = \sum_n \beta_n u_n(x) \equiv f(x) . \tag{B.6}$$

In (B.1) let $f(x) = \delta(x - x_o)$. Then $u(x,\lambda)$ is by definition the Green's function $\mathcal{G}(x, x_o, \lambda)$, and (B.6) gives us the expansion theorem for the δ-function:

$$\delta(x - x_o) = \frac{1}{2\pi i} \oint_C \mathcal{G}(x, x_o, \lambda)\, d\lambda$$

where C is a circle of infinite radius in the λ-plane.

In general the Green's function may not be single-valued over one complete cycle around C. We can get an expression equivalent to the above integral by placing branch cuts in the λ-plane and using Cauchy's integral theorem. Our fundamental identity for the δ-function becomes

$$\delta(x - x_o) = \frac{1}{2\pi i} \int_{C_B} \mathcal{G}(x, x_o, \lambda)\, d\lambda + \sum (\text{Residues in whole } \lambda\text{-plane}), \tag{B.7}$$

where C_B represents contours around all branch cuts.

APPENDIX C

Construction of the Green's Function for (1.1).

The Green's function $\mathcal{G}(\theta, \theta_o, \nu)$ solves the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathcal{G}}{\partial\theta}\right) + \left[(\nu^2 - \tfrac{1}{4}) - \frac{\mu^2}{\sin^2\theta}\right]\mathcal{G} = \delta(\theta - \theta_o). \quad \text{(C.1)}$$

From the form of (C.1), we see that $\mathcal{G}$ is the product of two independent solutions of the homogeneous equation divided by their Wronskian evaluated at $\theta = \theta_o$. Sturm-Liouville theory shows that the Wronskian W must be of the form

$$W = \frac{C(\nu)}{\sin\theta},$$

where $C(\nu)$ depends on the individual solutions selected. If we specify that $\mathcal{G}$ must behave like an outgoing wave at $\theta = \pm\infty$, then by selecting $E^{\mu}_{\nu-\frac{1}{2}}(\theta)$ and $E^{\mu}_{-\nu-\frac{1}{2}}$ as our independent solutions we find that

$$\mathcal{G} = \frac{E^{\mu}_{\nu-\frac{1}{2}}(\theta_>)E^{\mu}_{-\nu-\frac{1}{2}}(\theta_<)}{W},$$

where $\theta_>$ and $\theta_<$ are the greater and lesser of θ and θ_o, respectively.

From [1], p. 123, (13), and p. 140, (3), we see that

$$W = -\frac{\sin\pi\nu e^{2i\pi\mu}}{\sin\theta_o}\Gamma(\tfrac{1}{2} + \nu + \mu)\Gamma(\tfrac{1}{2} - \nu + \mu).$$

The Green's function is

$$\mathcal{G}(\theta,\theta_o,\nu) = -\frac{E^{\mu}_{\nu-\frac{1}{2}}(\theta_>)\,E^{\mu}_{-\nu-\frac{1}{2}}(\theta_<)\sin\theta_o\, e^{-2\pi i\mu}}{\sin\pi\nu\,\Gamma(\frac{1}{2}+\nu+\mu)\Gamma(\frac{1}{2}-\nu+\mu)} \quad . \qquad (C.2)$$

We may also choose $E^{-\mu}_{-\nu-\frac{1}{2}}(\theta)$ as the second solution to (1.1). We get a more compact form of the Wronskian

$$W = -\pi\frac{\sin\pi\nu}{\cos\pi(\nu+\mu)}\;\frac{1}{\sin\theta} \quad ,$$

and the resulting Green's function from (C.1) is

$$\mathcal{G}(\theta,\theta_o,\nu) = -\frac{1}{\pi}\,E^{\mu}_{\nu-\frac{1}{2}}(\theta_>)E^{-\mu}_{-\nu-\frac{1}{2}}(\theta_<)\sin\theta\,\frac{\cos\pi(\nu+\mu)}{\sin\pi\nu} \quad . \qquad (C.3)$$

APPENDIX D

Construction of the Green's Function for the Epstein Equation (9.8).

In this appendix we derive the Green's function satisfying

$$\frac{d^2}{dz^2}\mathcal{G}(z,z_o,\lambda) + \left\{\frac{\omega^2}{c_\infty^2}\left[1 + \frac{M}{\cosh^2(\frac{1}{2}mz)}\right] - \lambda\right\}\mathcal{G}(z,z_o,\lambda) = \delta(z - z_o) \tag{D.1}$$

$$\mathcal{G}(z,z_o,\lambda) \to 0 \quad \text{as} \quad z \to \pm\infty, \quad \text{Re}\,\{\mu\} > 0\,. \tag{D.2}$$

We define ν and μ in (9.10) and (9.11). With this notation we know that one solution w to (9.8) is

$$Z = P^{-\mu}_{\nu-\frac{1}{2}}\left(\tanh\frac{mz}{2}\right), \tag{D.3}$$

where $\mu = \mu(\lambda)$. From [1], p. 163, (8), we find that, for $z \to \infty$

$$P^{-\mu}_{\nu-\frac{1}{2}}\left(\tanh\frac{mz}{2}\right) \sim \frac{e^{-\frac{1}{2}m\mu z}}{\Gamma(1+\mu)}\,. \tag{D.4}$$

By choosing the cut in the λ-plane as shown in Figure 7, it follows that $\text{Re}\,\{\mu\} \geq 0$ if we use the principal branch of the square root in (9.11). Then (D.4) obeys condition (D.2).

If we replace z by $-z$ in (9.8), the equation is unchanged. Therefore, a second solution is

$$Z = P^{-\mu}_{\nu-\frac{1}{2}}\left(-\tanh\frac{mz}{2}\right). \tag{D.5}$$

It is clear from (D.4) that $P^{-\mu}_{\nu-\frac{1}{2}}(-\tanh \frac{mz}{2}) \to 0$ as $z \to -\infty$ for $\mathrm{Re}\,\{\mu\} > 0$.

From [1] the Wronskian W of the two solutions is

$$W = \frac{m}{\Gamma(\frac{1}{2}+\nu+\mu)\Gamma(\frac{1}{2}-\nu+\mu)} ,$$

and the Green's function is immediately seen to be

$$\mathcal{G}(z, z_o, \lambda) = \frac{1}{m} P^{-\mu}_{\nu-\frac{1}{2}}\left(\tanh \frac{mz_>}{2}\right) P^{-\mu}_{\nu-\frac{1}{2}}\left(-\tanh \frac{mz_<}{2}\right) \Gamma(\tfrac{1}{2}+\nu+\mu)\Gamma(\tfrac{1}{2}-\nu+\mu) \tag{D.6}$$

where $z_>$ and $z_<$ are the greater and lesser of z and z_o, respectively.

APPENDIX E

Translation of the ω-contour to the Real Axis.

The general theory states that the ω-contour in the inverse Fourier transform must be above all singularities of the integrand in the upper half plane. In this appendix we show that the contour can be taken to run along the real axis for the discrete part of (10.8).

On the original ω-contour (Figure 8) we can write the argument D of the discrete part of (10.8) as

$$D = H_o^{(1)}\left[\operatorname{sgn}(\operatorname{Re}\{\omega\})r\left|\left(\frac{m^2}{4}(\nu-\tfrac{1}{2}-n)^2+\frac{\omega^2}{c_\infty^2}\right)^{\frac{1}{2}}\right|\right]\cdot$$

$$\cdot\, P_{\nu-\frac{1}{2}}^{-(\nu-\frac{1}{2})+n}\left(-\tanh\frac{mz_o}{2}\right)P_{\nu-\frac{1}{2}}^{-(\nu-\frac{1}{2})+n}\left(\tanh\frac{mz}{2}\right)(\nu-\tfrac{1}{2}-n)\Gamma(2\nu-n). \quad (E.1)$$

The point $\nu = 0$ in the ω-plane is a branch point of D. There are no other singularities in the upper half ω-plane. Therefore, we deform the contour C_1 of Figure 8 to the contour C_2 as shown in Figure E-1.

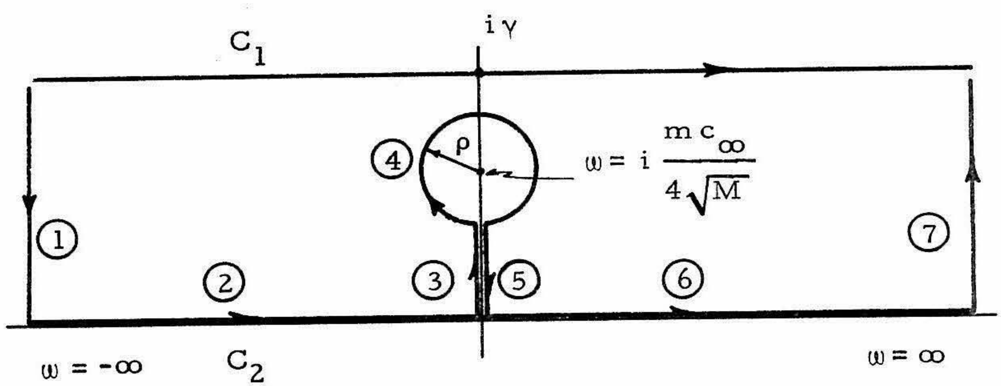

Figure E-1. The alternate contour C_2 for (E.1).

The theory of Fourier transforms implies that, for $|F(\omega)| = O(\frac{1}{\omega^{\epsilon}})$, $\epsilon > 0$, the integrals along ① and ⑦ vanish for $t > 0$.

ν is defined as

$$\nu = \frac{2\sqrt{M}}{mc_{\infty}} \left(\omega - i\frac{mc_{\infty}}{4\sqrt{M}}\right)^{\frac{1}{2}} \left(\omega + i\frac{mc_{\infty}}{4\sqrt{M}}\right)^{\frac{1}{2}} = \sqrt{R_1}\, e^{\frac{1}{2}i\varphi_1} \sqrt{R_2}\, e^{\frac{1}{2}i\varphi_2},$$

where we choose the principal branches of the square roots. The magnitudes and arguments are shown in Figure E-2.

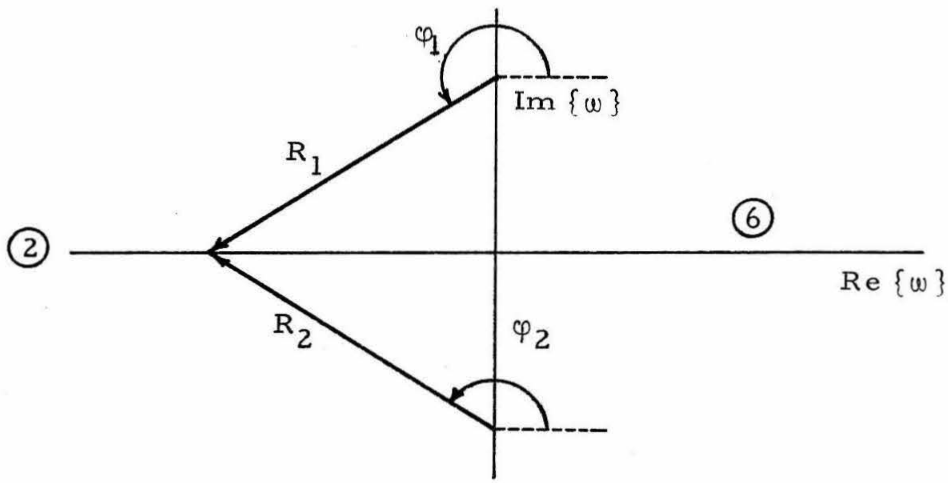

Figure E-2. Definition of arguments to determine the sign of ν.

From the symmetry we see that $\varphi_1 + \varphi_2 = 2\pi$ for ω on ② and ③, so $\nu = -\sqrt{R_1 R_2}$ when ω is on those two segments. Similarly, $\varphi_1 + \varphi_2 = 0$ for ω on ⑤ and ⑥ and $\nu = +\sqrt{R_1 R_2}$. It follows that $|\nu| < \frac{1}{2}$ for ω on ③ and ⑤, and $|\nu| < \frac{1}{2}$ on ④ for sufficiently small ρ. Therefore, in light of the restriction $0 \leq n \leq \mathrm{Re}\,\{\nu\} - \frac{1}{2}$ for the Green's function integrand (9.15) to contribute to the residue series, we see that ω and hence ν are purely real, and

$$\nu = \mathrm{sgn}\,(\omega) \left| \left(\frac{1}{4} + 4 \frac{\omega^2}{m^2 c_\infty^2} M \right)^{\frac{1}{2}} \right| . \tag{E.2}$$

There is no residue contribution from the apparent pole of $\Gamma(\frac{1}{2} - \nu + \mu)$ in (9.16) because of the factor μ. The range of n in the summation over residues is thus restricted to $n < |\nu| - \frac{1}{2}$, and (E.2) requires that we use the absolute value symbols $|\nu|$ in the notation for the discrete spectrum.

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