ON THE DETERMINATION OF THE PROPERTIES OF A MEDIUM FROM ITS REFLECTION COEFFICIENT

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ABSTRACT

This thesis demonstrates how the parameters of a slightly non-homogeneous medium can be derived approximately from the reflection coefficient.

Two types of media are investigated. The first is described by the one-dimensional wave equation, the second by the more complex Timoshenko beam equation. In both cases, the media are assumed to be infinite in extent, with the media parameters becoming homogeneous as the space variable approaches positive or negative infinity.

Much effort is placed in deriving properties of the reflection coefficient for both cases. The wave equation is considered primarily to introduce the techniques used to investigate the more complex Timoshenko equation. In both cases, an approximation is derived for one of the medium parameters involving the reflection coefficient.

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CHAPTER I

INTRODUCTION

In this thesis we investigate some questions concerning wave propagation in various media. In Chapter II we shall be concerned with the familiar wave equation, while in Chapter III we will look at a medium which is described by the less familiar Timoshenko equation. The Timoshenko equation is a fourth order partial differential equation describing transverse vibrations of a beam. In both cases we shall consider only one space dimension, and the medium will extend from negative to positive infinity. The media to be considered will be non-homogeneous; that is, the parameters characterizing the media will depend on the space variable.

For media which are asymptotically homogeneous at infinity, there exist solutions which are sinusoidal waves as the space variable approaches infinity. We will be interested in solutions which can be interpreted as an incident sinusoidal wave which is partially transmitted and partially reflected. For this case, we can define reflection and transmission coefficients, corresponding to the reflected and transmitted parts of the wave.

Eventually, we will consider media which are only slightly non-homogeneous; i.e. those in which the parameters vary only slightly from their values at infinity. In this case we investigate the primary problem of this thesis, the inverse problem. In the inverse problem we will indicate how the reflection coefficient can be used to infer approximately the parameters of the medium.

CHAPTER II

THE WAVE EQUATION

1. Introduction.

In this chapter we consider the wave equation for a non-homogeneous medium:

$$\frac{\partial}{\partial x}\left[p(x)\,\frac{\partial y}{\partial x}\right] = \rho(x)\,\frac{\partial^2 y}{\partial t^2} \quad . \tag{1.1}$$

The nonhomogeneity of the medium manifests itself in the dependence of the parameters p and ρ on x.

In the first section, we will look briefly at some physical examples which lead to Eq.(1.1). In the second section, we transform Eq.(1.1) to a canonical form, separate out the time dependence and obtain an ordinary differential equation in x. In the last half of the section, we will illustrate what happens when the medium is homogeneous. In the third section we limit ourselves to media which become homogeneous as $|x|$ approaches infinity and are only slightly nonhomogeneous otherwise. In the fourth section we transform the ordinary differential equation derived in Section 2 into an integral equation. We prove the existence of a solution of the integral equation and, in the process, will derive bounds on the solution. In the fifth section we use the integral equation to define reflection and transmission coefficients for sinusoidal waves propagating in from infinity.

In Section 6 we derive an approximation for the reflection coefficient for a medium which is only slightly nonhomogeneous, and in

Section 7 we show how this approximation can be improved for high frequencies. Finally, in the last section, we will indicate how the reflection coefficient can be used in turn to obtain an approximation of the parameter describing the medium.

Many physical examples exist where the partial differential equation (1.1) appears. Several examples will be discussed below. References for the derivation of the wave equation for the various cases mentioned are readily available in the literature.

Eq. (1.1) arises in describing the transverse vibrations of a stretched string. The transverse displacement of the string from its equilibrium position at time t and position x is denoted by $y(x,t)$. The parameter $p(x)$ represents the tension at point x and $\rho(x)$ the mass per unit length at point x.

Eq. (1.1) also occurs in the theory of the loss-less transmission line, where $y(x,t)$ represents the electrical current in the line at the point x and time t. The quantity $\frac{1}{p(x)}$ is the capacitance per unit length of transmission line, while $\rho(x)$ is the inductance per unit length. An equivalent physical interpretation is one in which $y(x,t)$ is the voltage of the transmission line at point x and time t. In this case $\frac{1}{p(x)}$ is the inductance per unit length, $\rho(x)$ the capacitance per unit length.

In the case of longitudinal vibrations of a bar $y(x,t)$ represents the longitudinal particle displacement (that is, a displacement along the axis of the bar) from the equilibrium position, which is located at the point x along the bar. The parameter $\rho(x)$ is the density of the bar per unit length at the point x and $p(x)$ is Young's modulus. We assume that the parameters

$p(x)$ and $\rho(x)$ do not vary in a direction perpendicular to the longitudinal axis of the bar.

2. An Ordinary Differential Equation.

The partial differential equation (1.1) can be transformed into a standard or canonical form. In physical applications of interest the parameters $p(x)$ and $\rho(x)$ describing the medium are positive functions which are bounded away from both zero and infinity for all x. In addition $p(x)$ and $\rho(x)$ are assumed continuous and integrable in any finite x-interval.

We introduce a transformation of the independent variable. Let

$$\frac{d\tau}{dx} = \frac{1}{\beta(x)} \qquad \beta(x) = \sqrt{\frac{p(x)}{\rho(x)}} \, . \tag{2.1}$$

Then

$$\tau(x) = \int_0^x \frac{d\xi}{\beta(\xi)} \tag{2.2}$$

Note that $\beta(x)$ is positive, real, and bounded away from both zero and infinity. Thus we can state that $x \rightarrow \tau(x)$ is a real, continuous, one-to-one transformation. The new independent variable τ has an interesting interpretation which is independent of the type of physical medium under consideration. In the theory of hyperbolic partial differential equations, $\beta(x) = \sqrt{\frac{p(x)}{\rho(x)}}$ represents the slope of a characteristic curve in the x, t plane and has dimensions of velocity. This characteristic velocity represents the velocity at which a signal travels through the medium at the point x. The quantity τ has dimensions of time. The difference $\tau(x_1)-\tau(x_2)$ is the time required for a signal to propagate from point x_1 to point x_2. In terms of τ and t, (1.1) becomes

$$\frac{\partial}{\partial \tau}\left(\alpha^2(\tau)\frac{\partial y}{\partial \tau}\right) - \alpha^2(\tau)\frac{\partial^2 y}{\partial t^2} = 0 \tag{2.3}$$

where

$$\alpha(\tau) = \left[p(x)\ \rho(x)\right]^{\frac{1}{2}} . \tag{2.4}$$

We will be interested in solutions of the form

$$y(\tau, t) = \overline{y}(\tau, k)e^{-ikt} , \tag{2.5}$$

The ordinary differential equation satisfied by $\overline{y}(\tau, k)$ is

$$\frac{d}{d\tau}\left(\alpha^2(\tau)\frac{d\overline{y}}{d\tau}\right) + k^2\alpha^2(\tau)\overline{y} = 0 . \tag{2.6}$$

Let us assume that the medium is homogeneous, so that p(x) and $\rho(x)$ are constant. Thus in turn both $\alpha^2(\tau)$ and $\beta(x)$ are also constant. The variable τ becomes $\tau(k) = \frac{k}{\beta}$. The partial differential equation (2.3) reduces to

$$\frac{\partial^2 y}{2\tau^2} - \frac{\partial y}{\partial t^2} = 0 , \tag{2.7}$$

and the corresponding ordinary differential equation (2.6) reduces to

$$\frac{d^2\bar{y}}{d\tau^2} + k^2\bar{y} = 0 . \tag{2.8}$$

Eq. (2.8) has the following two independent solutions

$$\bar{y}^{(+)}(\tau, k) = e^{ik\tau} = e^{ik\frac{x}{\beta}} ,$$

$$\bar{y}^{(-)}(\tau, k) = e^{-ik\tau} = e^{-ik\frac{x}{\beta}} ,$$

so that we obtain the following two solutions to the partial differential equation (2.3):

$$y^{+}(\tau, t) = e^{-ik(t - x/\beta)} ,$$

$$y^{-}(\tau, t) = e^{-ik(t + x/\beta)} .$$

The above solutions can be interpreted as traveling waves.

3. The Slightly Nonhomogeneous Medium.

In this section we will be primarily concerned with a slightly nonhomogeneous medium. Briefly we will consider media which have the following properties.

1. The medium becomes homogeneous for large values of $|x|$.
2. The parameters describing the medium vary only slightly

from their values at infinity.

In terms of the parameters $p(x)$ and $\rho(x)$ describing the medium, our first assumption implies that as x approaches positive infinity, $p(x)$ and $\rho(x)$ approach constants p_∞ and ρ_∞. Likewise as x approaches negative infinity, $p(x)$ and $\rho(x)$ approach constants $p_{-\infty}$ and $\rho_{-\infty}$ which in general may be different from p_∞ and ρ_∞. We will restrict ourselves even further by requiring that the differences $p(x)-p_{\pm\infty}$ and $\rho(x)-\rho_{\pm\infty}$ be absolutely integrable in neighborhoods of $x = \pm\infty$. As x approaches ∞, $\beta(x)$ of (2.1) approaches the limit $\beta_\infty = \sqrt{p_\infty/\rho_\infty}$. Consider the difference

$$(3.1) \qquad \frac{1}{\beta(x)} - \frac{1}{\beta_\infty} = \frac{\rho^{1/2}(x)}{p^{1/2}(x)} - \frac{\rho_\infty^{1/2}}{p_\infty^{1/2}}$$

$$= \frac{1}{p^{1/2}(x)\,\rho^{1/2}(x)}\left\{ p_\infty^{1/2}\,\frac{\rho(x)-\rho_\infty}{\rho^{1/2}(x)+\rho_\infty^{1/2}} - \rho_\infty^{1/2}\,\frac{p(x)-p_\infty}{p^{1/2}(x)+p_\infty^{1/2}} \right\}.$$

The absolute integrability of the differences $p(x)-p_\infty$ and $\rho(x)-\rho_\infty$ in the neighborhood of $x=\infty$ implies that the difference $\beta^{-1}(x)-\beta_\infty^{-1}$ is also absolutely integrable in the neighborhood of $x=\infty$. We may now write $\tau(x)$ as follows:

$$(3.2) \qquad \tau(x) = \frac{x}{\beta_\infty} + T_\infty - \int_x^\infty \left[\frac{1}{\beta(\xi)} - \frac{1}{\beta_\infty}\right] d\xi$$

where

$$(3.3) \qquad T_\infty = \int_0^\infty \left[\frac{1}{\beta(\xi)} - \frac{1}{\beta_\infty}\right] d\xi.$$

Thus we have the following asymptotic behavior for $\tau(x)$:

$$(3.4) \qquad \tau(x) \sim \frac{x}{\beta_\infty} + T_\infty \quad \text{as} \quad x \to \infty .$$

In a similar manner we can show that

$$(3.5) \qquad \tau(x) = \frac{x}{\beta_{-\infty}} - T_{-\infty} + \int_{-\infty}^{x} \left[\frac{1}{\beta(\xi)} - \frac{1}{\beta_{-\infty}} \right] d\xi ,$$

$$(3.6) \qquad \tau(x) \sim \frac{x}{\beta_{-\infty}} - T_{-\infty} \quad \text{as} \quad x \to -\infty ;$$

where

$$(3.7) \qquad \begin{aligned} \beta_{-\infty} &= \sqrt{\frac{p_{-\infty}}{\rho_{-\infty}}} , \\ T_{-\infty} &= \int_{-\infty}^{0} \left[\frac{1}{\beta(\xi)} - \frac{1}{\beta_{-\infty}} \right] d\xi . \end{aligned}$$

The parameter $\alpha(\tau)$ of (2.4) has the following asymptotic properties:

$$(3.8) \qquad \begin{aligned} \alpha^2(\tau) &\sim \alpha_\infty^2 = \sqrt{p_\infty \rho_\infty} \quad \text{as} \quad \tau \to \infty , \\ \alpha^2(\tau) &\sim \alpha_{-\infty}^2 = \sqrt{p_{-\infty} \rho_{-\infty}} \quad \text{as} \quad \tau \to -\infty . \end{aligned}$$

In reference to the second property listed at the beginning of this section, we will assume that the parameters describing the medium vary only slightly from their values at infinity. In terms of $\alpha(\tau)$ we assume that the difference $\alpha(\tau)-\alpha_\infty$ remains small for all τ, so that

$$\frac{\max\limits_{-\infty<\tau<\infty} |\alpha(\tau)-\alpha_\infty|}{\alpha_\infty} << 1$$

We will go into a more thorough discussion of this point in the next section.

4. An Integral Equation.

In this section we will be interested in the solution $\overline{y}^+$ of

(4.1) $$\frac{d}{d\tau}\left(\alpha^2(\tau)\frac{d\overline{y}^+}{d\tau}\right) + k^2\alpha^2(\tau)\overline{y}^+ = 0$$

which has the property

(4.2) $$\overline{y}^+(\tau, k) \sim e^{ik\tau} \quad \text{as} \quad \tau \to \infty .$$

In order to express the problem in Eqs. (4.1) and (4.2) as an integral equation, we first write $\alpha^2(\tau) = \alpha_\infty^2 + r(\tau)$, where $r(\tau) \to 0$ as $\tau \to \infty$. Eq. (4.1) can be written as

(4.3) $$\frac{d^2\overline{y}^+}{d\tau^2} + k^2\overline{y}^+ = f(\tau) ,$$

where

(4.4) $$f(\tau) = -\frac{1}{\alpha_\infty^2}\left[\frac{d}{d\tau}\left(r(\tau)\frac{d\overline{y}^+}{d\tau}\right) + k^2 r(\tau)\overline{y}^+\right] .$$

Eq. (4.3) can be looked upon as a nonhomogeneous differential equation with the corresponding homogeneous differential equation

$$(4.5) \qquad \frac{d^2\bar{y}^+}{d\tau^2} + k^2\bar{y}^+ = 0.$$

Using variation of parameters and the two independent solutions $e^{ik\tau}$ and $e^{-ik\tau}$ of (4.5), we can derive the following particular solution of (4.3),

$$y_p(\tau,k) = \int_\tau^\infty \frac{\sin k(\xi-\tau)}{k} f(\xi)\, d\xi \quad .$$

If we add $e^{ik\tau}$ to the above particular solution of (4.3), the sum is also a solution of (4.3). We can write this sum as

$$(4.6) \qquad \bar{y}^+(\tau,k) = e^{ik\tau} - \frac{1}{\alpha_\infty^2}\int_\tau^\infty \frac{\sin k(\xi-\tau)}{k}\left[\frac{d}{d\xi}\left(r(\xi)\frac{d\bar{y}^+}{d\xi}\right) + k^2 r(\xi)\bar{y}^+(\xi,k)\right]d\xi,$$

where we have replaced $f(\tau)$ by its equivalent form (4.4).

Eq. (4.6) can be simplified. Assuming $r(\tau)$, $\frac{dr}{d\tau}$ approach zero as $\tau \to \infty$, we can show

$$(4.7) \qquad \int_\tau^\infty \frac{\sin k(\xi-\tau)}{k}\frac{d}{d\xi}\left[r(\xi)\frac{d\bar{y}^+}{d\xi}\right]d\xi = \int_\tau^\infty \cos k(\xi-\tau)\frac{dr(\xi)}{d\xi}\bar{y}^+(\xi,k)d\xi$$

$$-\int_\tau^\infty k\sin k(\xi-\tau) r(\xi)\bar{y}^+(\xi,k)d\xi + r(\tau)\bar{y}^+(\tau,k) \quad .$$

Combining (4.6) and (4.7) and simplifying, we obtain the following integral equation for $\bar{y}^+(\tau,k)$:

$$\alpha^2(\tau)\bar{y}^+(\tau,k) = \alpha_\infty^2\, ike^{ik\tau} - \int_\tau^\infty k\sin k(\xi-\tau)\,\frac{d\alpha^2(\xi)}{d\xi}\,\bar{y}^+(\xi,k)\,d\xi \quad . \tag{4.8}$$

We will now show that if the integral equation possesses a bounded continuous solution and $\frac{d\alpha^2(\xi)}{d\xi}$ is integrable, then that solution satifies the original differential equation (4.1) and the asymptotic condition (4.2).

We assume that $\bar{y}^+(\tau,k)$ is a bounded solution of the integral equation (4.8). We differentiate both sides of (4.8) twice. Differentiating once (the existence of the derivative of the right hand side insures the existence of the derivative of the left hand side) we obtain

$$\alpha^2(\tau)\,\frac{d\bar{y}^+}{d\tau} = \alpha_\infty^2\, ike^{ik\tau} - \int_\tau^\infty k\sin k(\xi-\tau)\,\frac{d\alpha^2(\xi)}{d\xi}\,\bar{y}^+(\xi,k)\,d\xi \quad . \tag{4.9}$$

Differentiating (4.9) (again the existence of the derivative of the right hand side insures the existence of the derivative of the left hand side) we obtain

$$\frac{d}{d\tau}\left(\alpha^2(\tau)\,\frac{d\bar{y}^+}{d\tau}\right) = -k^2\left\{\alpha_\infty^2 e^{ik\tau} - \int_\tau^\infty \cos k(\xi-\tau)\,\frac{d\alpha^2(\xi)}{d\xi}\,\bar{y}^+(\xi,k)\,d\xi\right\} . \tag{4.10}$$

Substituting (4.8) into (4.10), we retrieve the original differential eqn.(4.1).

We will now show that the bounded solution $\bar{y}^+(\tau,k)$ satisfies (4.2). Eq. (4.8) may be expressed as follows:

$$\bar{y}^+(\tau,k) - e^{ik\tau} = \frac{1}{\alpha^2(\tau)}\left\{\left[\alpha_\infty^2 - \alpha^2(\tau)\right]e^{ik\tau} - \int_\tau^\infty \cos k(\tau-\xi)\,\frac{d\alpha^2(\xi)}{d\xi}\,\bar{y}^+(\xi,k)\,d\xi\right\} . \tag{4.11}$$

Taking the absolute value in (4.11) gives

$$\left|\bar{y}^+(\tau,k) - e^{ik\tau}\right| \leq \frac{1}{\alpha^2(\tau)}\left|\alpha_\infty^2 - \alpha^2(\tau)\right| + C\int_\tau^\infty \left|\frac{d\alpha^2(\xi)}{d\xi}\right| d\xi \quad . \tag{4.12}$$

Since $\frac{d\alpha^2(\xi)}{d\xi}$ is integrable, the integral on the right hand side of (4.12) approaches zero as τ approaches infinity. The first term on the right hand side of (4.12) approaches zero. Thus we have

$$\lim_{\tau \to \infty} |\bar{y}^{+}(\tau, k) - e^{ik\tau}| = 0$$

which is equivalent to (4.2).

In passing we should note the following dual problem:

$$\frac{d}{d\tau}\left(\alpha^2(\tau)\frac{d\bar{y}^-}{d\tau}\right) + k^2\alpha^2(\tau)\,\bar{y}^- = 0 \tag{4.13}$$

$$\bar{y}^-(\tau, k) \sim e^{-ik\tau} \quad \text{as} \quad \tau \to -\infty \ . \tag{4.14}$$

Proceeding as we did for $\bar{y}^+(\tau, k)$, we can reduce the above problem to showing that the following integral equation has a bounded solution:

$$\alpha^2(\tau)\,\bar{y}^-(\tau, k) = \alpha^2_{-\infty} e^{-ik\tau} + \int_{-\infty}^{\tau} \cos k(\tau - \xi)\,\frac{d\alpha^2(\xi)}{d\xi}\,\bar{y}^-(\xi, k)\,d\xi \ . \tag{4.15}$$

Up to this point we have reduced the solution of the problem stated in (4.1) and (4.2) to showing that the integral equation (4.8) possesses a bounded solution. We need to show that the integral equation does indeed possess a bounded solution.

In the following lemma we prove that the integral equations (4.8) and (4.15) possess bounded solutions. In the process of proving this lemma, we will be able to construct bounds on the solutions $\bar{y}^+(\tau, k)$ and $\bar{y}^-(\tau, k)$.

Lemma: Let the derivative of $\alpha(\tau)$ be integrable on $-\infty < \tau < \infty$, $\alpha(\tau)$ be bounded away from zero and infinity. Then the integral equations (4.8) and (4.15) possess bounded solutions $\bar{y}^+(\tau, k)$ and $\bar{y}^-(\tau, k)$ with the following properties:

$$(4.16)\quad \begin{cases} |\overline{y}^{+}(\tau,k)| \leq \dfrac{\alpha_{\infty}^{2}}{\alpha^2(\tau)}\, e^{A_{+}(\tau)} \\[2ex] |\alpha^2(\tau)\overline{y}^{+}(\tau,k) - \alpha_{\infty}^{2}\, e^{ik\tau}| \leq \alpha_{\infty}^{2}\, A_{+}(\tau)\, e^{A_{+}(\tau)} \\[2ex] |\overline{y}^{-}(\tau,k)| \leq \dfrac{\alpha_{-\infty}^{2}}{\alpha^2(\tau)}\, e^{A_{-}(\tau)} \\[2ex] |\alpha^2(\tau)\overline{y}^{-}(\tau,k) - \alpha_{-\infty}^{2}\, e^{-ik\tau}| \leq \alpha_{-\infty}^{2}\, A_{-}(\tau)\, e^{A_{-}(\tau)} \end{cases}$$

where

$$(4.17)\quad \begin{cases} A_{+}(\tau) = \displaystyle\int_{\tau}^{\infty} \frac{2}{\alpha(\xi)} \left|\frac{d\alpha}{d\xi}\right| d\xi \\[3ex] A_{-}(\tau) = \displaystyle\int_{-\infty}^{\tau} \frac{2}{\alpha(\xi)} \left|\frac{d\alpha}{d\xi}\right| d\xi\, . \end{cases}$$

Proof: We shall prove the lemma for $\overline{y}^{+}(\tau,k)$. The proof for $\overline{y}^{-}(\tau,k)$ follows in the same manner as that for $\overline{y}^{+}(\tau,k)$.

We shall use the method of successive approximations to construct the solution $\overline{y}^{+}(\tau,k)$ of the integral equation (4.8).

As a first approximation to the solution of the integral equation, we set

$$(4.18)\quad \overline{y}_0^{+}(\tau,k) = \frac{\alpha_{\infty}^{2}}{\alpha^2(\tau)}\, e^{ik\tau}\, .$$

To obtain a second approximation we can substitute the first approximation for $\overline{y}^{+}(\tau,k)$ into the right hand side of the integral equation. The second approximation will then be

$$\bar{y}_1^+(\tau,k) = \frac{\alpha_\infty^2}{\alpha^2(\tau)} e^{ik\tau} - \frac{1}{\alpha^2(\tau)} \int_\tau^\infty \cos k(\xi-\tau) \frac{d\alpha^2(\xi)}{d\xi} \bar{y}_0^+(\xi,k)\, d\xi \,.$$

Proceeding in this way we obtain the following for the $(n+1)^{st}$ approximation $\bar{y}_n^+(\tau,k)$:

$$(4.19) \quad \bar{y}_n^+(\tau,k) = \frac{\alpha_\infty^2}{\alpha^2(\tau)} e^{ik\tau} - \frac{1}{\alpha^2(\tau)} \int_\tau^\infty \cos k(\xi-\tau) \frac{d\alpha^2(\xi)}{d\xi} \bar{y}_{n-1}^+(\xi,k)\, d\xi \,.$$

We show that the sequence $\bar{y}_n^+(\tau,k)$ converges to a limit function and that this limit function satisfies the integral equation.

The iterates $\bar{y}_n(\tau,k)$ can be expressed in another form. Let

$$(4.20) \qquad \Delta_0(\tau,k) = \bar{y}_0^+(\tau,k) = \frac{\alpha_\infty^2}{\alpha^2(\tau)} e^{ik\tau}$$

$$(4.21) \qquad \Delta_n(\tau,k) = \bar{y}_n^+(\tau,k) - \bar{y}_{n-1}^+(\tau,k) \qquad n = 1, 2, \ldots$$

The iterates $\bar{y}_n(\tau,k)$ in turn can be expressed in terms of the differences $\Delta_n(\tau,k)$. Summing the differences $\Delta_m(\tau,k)$ from $m=0$ to $m=n$ and using (4.20) and (4.21) we find

$$(4.22) \qquad \bar{y}_n^+(\tau,k) = \sum_{m=0}^{n} \Delta_m(\tau,k) \,.$$

We can derive an expression for the $\Delta_n(\tau,k)$ similar to the expression in (4.19) for the $\bar{y}_n^+(\tau,k)$. Subtracting the expression for $\bar{y}_{n-1}^+(\tau,k)$ from the expression for $\bar{y}_n^+(\tau,k)$ we obtain the following:

$$\overline{y}_n^+(\tau,k)-\overline{y}_{n-1}^+(\tau,k) = -\frac{1}{\alpha^2(\tau)}\int_\tau^\infty \cos k(\xi-\tau)\,\frac{d\alpha^2(\xi)}{d\xi}\left[\overline{y}_{n-1}^+(\xi,k)-\overline{y}_{n-2}^+(\xi,k)\right]d\xi$$

or, in terms of the differences $\Delta_n(\tau,k)$,

$$\Delta_n(\tau,k) = -\frac{1}{\alpha^2(\tau)}\int_\tau^\infty \cos k(\xi-\tau)\,\frac{d\alpha^2(\xi)}{d\xi}\,\Delta_{n-1}(\xi,k)\,d\xi \quad . \tag{4.23}$$

Thus if the series

$$\sum_{m=0}^{\infty} \Delta_m(\tau,k) \tag{4.24}$$

converges, then so do the iterates $\overline{y}_n^+(\tau,k)$. First we obtain upper bounds on the absolute values of the differences $\Delta_m(\tau,k)$.

Taking absolute values of both sides of (4.20), we have the following bound on $\Delta_0(\tau,k)$:

$$\left|\Delta_0(\tau,k)\right| \leqslant \frac{\alpha_\infty^2}{\alpha^2(\tau)} \quad . \tag{4.25}$$

Using this bound for $\Delta_0(\tau,k)$ in the iteration equation (4.23), we can establish the following bound for $\Delta_1(\tau,k)$:

$$\left|\Delta_1(\tau,k)\right| \leqslant \frac{\alpha_\infty^2}{\alpha^2(\tau)}\; A_+(\tau) \tag{4.26}$$

where

$$A_+(\tau) = \int_\tau^\infty \frac{2}{\alpha(\xi)}\left|\frac{d\alpha}{d\xi}\right| d\xi \quad . \tag{4.27}$$

Using the conditions on $\alpha(\xi)$ that $\frac{d\alpha}{d\xi}$ be absolutely integrable on $-\infty<\tau<\infty$ and that $\alpha(\xi)$ be bounded away from zero, we have that the integral $A_+(\tau)$ exists and is bounded (in fact bounded by $A_+(-\infty)$). Again using the bound in (4. 26) on $\Delta_1(\tau,k)$ in (4. 23), we can show that

$$\left|\Delta_2(\tau,k)\right| = \frac{\alpha_\infty^2}{\alpha^2(\tau)} \int_\tau^\infty \frac{2}{\alpha(\xi)} \left|\frac{d\alpha}{d\xi}\right| A_+(\xi)\, d\xi \ . \tag{4.28}$$

However, note that the integral in (4. 28) can be integrated directly since

$$\frac{2}{\alpha(\xi)} \left|\frac{d\alpha}{d\xi}\right| d\xi = -dA_+(\xi) \ ,$$

$$A_+(\infty) = 0 \ .$$

Thus the bound on $\Delta_2(\tau,k)$ becomes the following

$$\begin{aligned} \left|\Delta_2(\tau,k)\right| &\leqslant -\frac{\alpha_\infty^2}{\alpha^2(\tau)} \int_\tau^\infty A_+(\xi)\, dA_+(\xi) \\ \left|\Delta_2(\tau,k)\right| &\leqslant \frac{\alpha_\infty^2}{\alpha^2(\tau)} \frac{[A_+(\tau)]^2}{2!} \ . \end{aligned} \tag{4.29}$$

Proceeding in this way, obtain the following bound on $\Delta_n(\tau,k)$:

$$\left|\Delta_n(\tau,k)\right| \leqslant \frac{\alpha_\infty^2}{\alpha^2(\tau)} \frac{[A_+(\tau)]^n}{n!} \ . \tag{4.30}$$

Note that the above bound is uniform in k. In addition we can derive the following bound on $\Delta_n(\tau,k)$ which is also uniform in τ:

$$\left|\Delta_n(\tau,k)\right| \leqslant \frac{\alpha_\infty^2}{\min\limits_{-\infty<\tau<\infty} \alpha^2(\tau)} \frac{[A_+(-\infty)]^n}{n!} \quad . \tag{4.31}$$

The series (4.24) can be bounded in absolute value by the sum of the absolute values of $\Delta_n(\tau,k)$. Using the bounds (4.31) gives

$$\left|\sum_{n=0}^{\infty} \Delta_n(\tau,k)\right| \leqslant \frac{\alpha_\infty^2}{\alpha^2(\tau)} \sum_{n=0}^{\infty} \frac{[A_+(\tau)]^n}{n!}$$

or

$$\left|\sum_{n=0}^{\infty} \Delta_n(\tau,k)\right| \leqslant \frac{\alpha_\infty^2}{\alpha^2(\tau)} e^{A_+(\tau)} \quad . \tag{4.32}$$

Thus the series (4.24) converges absolutely and uniformly with respect to τ and has the bound as is given in (4.32).

To show that the series (4.24) satisfies the integral equation (4.8) we return to the definition of $\Delta_n(\tau,k)$ in (4.23). Summing both sides of (4.23) we get

$$\sum_{n=1}^{\infty} \Delta_n(\tau,k) = -\sum_{n=1}^{\infty} \frac{1}{\alpha^2(\tau)} \int_{\tau}^{\infty} \cos k(\xi-\tau) \frac{d\alpha^2(\xi)}{d\xi} \Delta_{n-1}(\xi,k)\, d\xi . \tag{4.33}$$

We can interchange the summation and integration on the right hand side of the equation because of the uniform convergence of the series. Eq. (4.33) becomes

$$\begin{aligned}\sum_{n=1}^{\infty} \Delta_n(\tau,k) &= \sum_{n=0}^{\infty} \Delta_n(\tau,k) - \frac{\alpha_\infty^2}{\alpha^2(\tau)} e^{ik\tau} \\ &= -\frac{1}{\alpha^2(\tau)} \int_{\tau}^{\infty} \cos k(\xi-\tau) \frac{d\alpha^2(\xi)}{d\xi} \sum_{n=0}^{\infty} \Delta_n(\xi,k)\, d\xi\end{aligned}$$

which is our original integral equation. Thus $\bar{y}^+(\tau,k) = \sum_{n=0}^{\infty} \Delta_n(\tau,k)$ is a solution of the integral equation and from (4.32),

$$\left|\bar{y}^+(\tau,k)\right| \leqslant \frac{\alpha_\infty^2}{\alpha^2(\tau)} e^{A_+(\tau)} .$$

Let us return to the integral equation (4.8)

$$\alpha^2(\tau)\bar{y}^+(\tau,k) - \alpha_\infty^2 e^{ik\tau} = -\int_\tau^\infty \cos k(\xi-\tau)\, 2\alpha(\xi)\frac{d\alpha}{d\xi}\bar{y}^+(\xi,k)\, d\xi$$

Taking absolute values of both sides of the above equation and using the bound in (4.23), we can show

$$\left|\alpha^2(\tau)\bar{y}^+(\tau,K) - \alpha_\infty^2 e^{ik\tau}\right| \leqslant \alpha_\infty^2 \int_\tau^\infty \frac{2}{\alpha(\xi)} \left|\frac{d\alpha}{d\xi}\right| e^{A_+(\xi)} d\xi .$$

However, $A_+(\xi)$ is less than or equal to $A_+(\tau)$, if τ is less than or equal to ξ. Thus we can write the above as

$$\left|\alpha^2(\tau)\,\bar{y}^+(\tau,k) - \alpha_\infty^2 e^{ik\tau}\right| \leqslant \alpha_\infty^2 A_+(\tau)\, e^{A_+(\tau)} .$$

Note that if we rewrite the above as

$$\left|\alpha^2(\tau)\left|\bar{y}^+(\tau,k) - e^{ik\tau}\right| - \left|\alpha^2(\tau) - \alpha_\infty^2\right|\right| \leqslant \alpha_\infty^2 A_+(\tau) e^{A_+(\tau)}$$

$$\left|\bar{y}^+(\tau,k) - e^{ik\tau}\right| \leqslant \frac{\alpha_\infty^2}{\alpha^2(\tau)} A_+(\tau) e^{A_+(\tau)} + \frac{\left|\alpha^2(\tau) - \alpha_\infty^2\right|}{\alpha^2(\tau)}$$

then we can see that as τ approaches infinity

$$\left| \bar{y}^{+}(\tau, k) - e^{ik\tau} \right| \to 0 \quad .$$

The proof of the Lemma for $\bar{y}^{-}(\tau, k)$ proceeds in exactly the same way as the proof for $\bar{y}^{+}(\tau, k)$. The details of the proof for $\bar{y}^{-}(\tau, k)$ will be omitted.

5. Reflection and Transmission Coefficients.

In this section we define and interpret the reflection and transmission coefficients for a nonhomogeneous medium. We shall use the properties of $\bar{y}^{+}(\tau, k)$ which we derived in the previous section.

Let us return to investigating solutions of the partial differential equation (2.3). Using Eq. (2.5), we can combine the time factor e^{-ikt} with the function $\bar{y}^{+}(\tau, k)$ studied earlier to construct the following solution to the partial differential equation:

$$y^{+}(\tau, t) = e^{-ikt}\, \bar{y}^{+}(\tau, k) \; .$$

Let us now consider the asymptotic behavior of $y^{+}(\tau, t)$ for large τ. First using (4.2), we find that $y^{+}(\tau, t)$ has the following asymptotic behavior as τ approaches infinity:

$$\begin{aligned} y^{+}(\tau, t) &\sim e^{-ik(t-\tau)} \\ &\sim e^{-ik(t - \frac{x}{\beta_\infty})} \quad . \end{aligned}$$

Comparing the above with the results for the homogeneous medium, we see that the right hand side is just a simple harmonic wave propagating to the right with velocity β_∞.

Next we will derive the asymptotic expansion of $y^+(\tau,t)$ as τ approaches negative infinity. Using the integral equation (4.8) for $\bar{y}^+(\tau,k)$, we can write the function $y^+(\tau,t)$ as follows:

$$y^+(\tau,t) = \frac{\alpha_\infty^2}{\alpha^2(\tau)} e^{-ik(t-\tau)} - \frac{e^{-ikt}}{\alpha^2(\tau)} \int_\tau^\infty \cos(\tau-\xi) \frac{d\alpha^2(\xi)}{d\xi} \bar{y}^+(\xi,k)\, d\xi$$

$$= e^{-ik(t-\tau)} \left\{ \frac{\alpha_\infty^2}{\alpha^2(\tau)} - \frac{1}{\alpha^2(\tau)} \int_\tau^\infty e^{-ik\xi} \alpha(\xi) \frac{d\alpha(\xi)}{d\xi} \bar{y}^+(\xi,k) d\xi \right\}$$

$$- \frac{e^{-ik(t+\tau)}}{\alpha^2(\tau)} \int_\tau^\infty e^{ik\xi} \alpha(\xi) \frac{d\alpha(\xi)}{d\xi} \bar{y}^+(\xi,k)\, d\xi \;.$$

If we let τ approach minus infinity in the above representation for $y^+(\tau,t)$ we obtain the following asymptotic behavior for $y^+(\tau,t)$:

$$(5.1) \qquad \frac{\alpha_{-\infty}^2}{\alpha_\infty^2} y^+(\tau,t) \sim \left[1+P_1(k)\right] e^{-ik(t-\frac{x}{\beta_{-\infty}})} + P_2(k) e^{-ik(t+\frac{x}{\beta_{-\infty}})}$$

where

$$(5.2) \qquad P_1(k) = -\frac{1}{\alpha_\infty^2} \int_{-\infty}^{\infty} e^{-ik\xi} \alpha(\xi) \frac{d\alpha(\xi)}{d\xi} \bar{y}^+(\xi,k)\, d\xi$$

$$(5.3) \qquad P_2(k) = -\frac{1}{\alpha_\infty^2} \int_{-\infty}^{\infty} e^{ik\xi} \alpha(\xi) \frac{d\alpha(\xi)}{d\xi} \bar{y}^+(\xi,k)\, d\xi \;.$$

Note that the absolute integrability of $\frac{d\alpha}{d\xi}$ over the entire τ-axis and the boundedness of $\bar{y}^+(\tau,k)$ insures the existence of the above integrals.

Thus as τ approaches $-\infty$, the right hand side of (5.4) contains both a simple harmonic wave propagating to the right and one propagating to the left, both propagating with speed $\beta_{-\infty}$. If we divide (5.1) by $1+P_1(k)$, we can write the following for the asymptotic behavior of $v^+(\tau, t)$:

$$(5.4) \qquad v^+(\tau,t) \sim e^{-ik(t-\frac{x}{\beta_{-\infty}})} + \frac{P_2(k)}{1+P_1(k)} e^{-ik(t+\frac{x}{\beta_{-\infty}})} \quad \text{as } \tau \to -\infty \; (x \to -\infty)$$

$$(5.5) \qquad v^+(\tau,t) \sim \frac{e^{-ik(t-\frac{x}{\beta_{-\infty}})}}{1+P_1(k)} \quad \text{as } \tau \to \infty \; (x \to \infty) .$$

where $v^+(\tau, t)$ is defined as follows:

$$v^+(\tau,t) = \frac{\frac{\alpha^2_{-\infty}}{\alpha^2_{\infty}} y^+(\tau,t)}{1+P_1(k)} .$$

We can interprete the right hand side of the above as follows. A sinusoidal wave originates at $x = -\infty$ and propagates to the right with velocity $\beta_{-\infty}$. It is partially reflected and returns to $x = -\infty$ as a sinusoidal wave propagating to the left with velocity $\beta_{-\infty}$. In addition the original wave is partially transmitted and appears at $x = \infty$ as a sinusoidal wave propagating to the right with velocity β_{∞}. Thus as $x \to -\infty$ the solution $v^+(\tau, t)$ behaves like the sum of two simple harmonic waves, one wave with amplitude unity traveling to right and the other wave, a reflected wave, traveling to the left. While as $x \to \infty$ the solution $v^+(\tau,t)$ behaves like a transmitted simple harmonic wave.

We define the reflection coefficient $R^+(k)$ and the transmission coefficient $T^+(k)$ as the amplitude of the reflected and transmitted components of the solution $v^+(\tau, t)$.

$$(5.6)\qquad R^{+}(k) = \frac{P_2(k)}{1+P_1(k)} = -\frac{\int_{-\infty}^{\infty} e^{ik\xi}\alpha(\xi)\frac{d\alpha(\xi)}{d\xi}\,\bar{y}^{+}(\xi,k)d\xi}{\alpha_{\infty}^{2} - \int_{-\infty}^{\infty} e^{-ik\xi}\alpha(\xi)\frac{d\alpha(\xi)}{d\xi}\,\bar{y}^{+}(\xi,k)d\xi}$$

$$(5.7)\qquad T^{+}(k) = \frac{\alpha_{-\infty}^{2}/\alpha_{\infty}^{2}}{1+P_1(k)} = \frac{\alpha_{-\infty}^{2}}{\alpha_{\infty}^{2} - \int_{-\infty}^{\infty} e^{-ik\xi}\alpha(\xi)\frac{d\alpha(\xi)}{d\xi}\,\bar{y}^{+}(\xi,k)d\xi}\,.$$

Using this definition for $R^{+}(k)$, and $T^{+}(k)$, we can write (5.4) and (5.5) as follows:

$$(5.8)\qquad v^{+}(\tau,t) \sim e^{-ik(t-\frac{x}{\beta_{-\infty}})} + R^{+}(k)e^{-ik(t+\frac{x}{\beta_{-\infty}})}$$

$$\text{as } \tau \to -\infty \quad (x \to -\infty)$$

$$(5.9)\qquad v^{+}(\tau,t) \sim T^{+}(k)e^{-ik(t-\frac{x}{\beta_{\infty}})}$$

$$\text{as } \tau \to \infty \quad (x \to \infty)\,.$$

In a similar way we can manipulate the integral equation for $\bar{y}^{-}(\tau,k)$ to obtain the following transmission and reflection coefficients:

$$(5.10)\qquad R^{-}(k) = \frac{\int_{-\infty}^{\infty} e^{-ik\xi}\alpha(\xi)\frac{d\alpha}{d\xi}\,\bar{y}^{-}(\xi,k)d\xi}{\alpha_{-\infty}^{2} + \int_{-\infty}^{\infty} e^{ik\xi}\alpha(\xi)\frac{d\alpha}{d\xi}\,\bar{y}^{-}(\xi,k)\,d\xi}$$

$$(5.11)\qquad T^{-}(k) = \frac{\alpha_{-\infty}^{2}}{\alpha_{-\infty}^{2} + \int_{-\infty}^{\infty} e^{ik\xi}\alpha(\xi)\frac{d\alpha}{d\xi}\,\bar{y}^{-}(\xi,k)d\xi}\,.$$

If we define $v^-(\tau, k)$ as follows:

$$v^-(\tau, k) = T^-(k)\, y^-(\tau, t)$$

where

$$y^-(\tau, t) = e^{-ikt}\, \overline{y}^-(\tau, k)$$

then we can show that $v^-(\tau, t)$ has the following asymptotic behavior:

$$v^-(\tau,t) = T^-(k)y^-(\tau,t) \sim e^{-ik(t+\frac{x}{\beta_\infty})} + R^-(k)e^{-ik(t-\frac{x}{\beta_\infty})} \quad \text{as } \tau\to\infty\ (x\to\infty)$$

$$v^-(\tau,t) = T^-(k)y^-(\tau,t) \sim T^-(k)e^{-ik(t+\frac{x}{\beta_{-\infty}})} \quad \text{as } \tau\to-\infty\ (x\to -\infty) .$$

The above is just the opposite of the situation pertaining to $y^+(\tau,t)$. The above asymptotic behavior tells us that $v^-(\tau,t)$ starts out at $x=\infty$ as a simple harmonic wave propagating to the left and is partially reflected back to $x=\infty$ and partially transmitted to $x = -\infty$.

6. An Approximation to the Reflection Coefficient.

In this section we will assume that the medium is only slightly non-homogeneous. That is, we will assume that the parameters describing the medium are only a small perturbation from their values at infinity. With the assumption we will obtain an approximation for the reflection coefficient.

First we write $\alpha(\tau)$ as follows

$$\alpha(\tau) = \alpha_{\infty}\left[1 + \epsilon r(\tau)\right]$$

where we will assume that ϵ is small compared to 1.

We will further restrict $r(\tau)$ by assuming that its derivative is absolutely integrable on $-\infty < \tau < \infty$ and by assuming

(6.1) $$\epsilon \int_{-\infty}^{\infty} \left|\frac{dr}{d\xi}\right| d\xi = \frac{1}{\alpha_{\infty}} \int_{-\infty}^{\infty} \left|\frac{d\alpha}{d\xi}\right| d\xi << 1 .$$

Note that the above implies the following :

(6.2) $$\epsilon \left|r(\tau)\right| \leqslant \epsilon \int_{\tau}^{\infty} \left|\frac{dr}{d\tau}\right| d\xi << 1$$

$$\frac{\alpha(\tau)}{\alpha_{\infty}} = 1 + \epsilon\, r(\tau) = 1 + O(\epsilon) \text{ uniformly in } \tau .$$

The parameter ϵ is our perturbation parameter. We obtain a perturbation expansion for the reflection coefficient in terms of ϵ.

Let us consider the reflection coefficient

$$R^{+}(k) = \frac{P_2(k)}{1 + P_1(k)} = P_2(k) - \frac{P_1(k)\, P_2(k)}{1 + P_1(k)}$$

where $P_1(k)$ and $P_2(k)$ are given in (5.2) and (5.3). Using (4.23) we have

$$\left|\bar{y}^{+}(\tau, k)\right| \leqslant C$$

where C is independent of ϵ and k. Using this bound on $\bar{y}^+(\tau,k)$ we obtain the following bounds on $P_1(k)$ and $P_2(k)$:

$$|P_1(k)| \leq C_1(1+C_2\epsilon)\frac{1}{\alpha_\infty}\int_{-\infty}^{\infty}\left|\frac{d\alpha}{d\xi}\right| d\xi \leq C_3\epsilon$$

$$|P_2(k)| \leq C_1(1+C_2\epsilon)\frac{1}{\alpha_\infty}\int_{-\infty}^{\infty}\left|\frac{d\alpha}{d\xi}\right| d\xi \leq C_3\epsilon$$

where the C_i $(i=1,2,3)$ are independent of k and ϵ. Thus both $P_1(k)$ and $P_2(k)$ are $O(\epsilon)$ uniformly in k.

Let us take a closer look at $P_2(k)$. We can write $P_2(k)$ as follows:

$$P_2(k) = P_{21}(k) + P_{22}(k)$$

where

(6.3) $$P_{21}(k) = -\int_{-\infty}^{\infty} e^{2ik\xi}\frac{1}{\alpha(\xi)}\frac{d\alpha}{d\xi}\, d\xi$$

(6.4) $$P_{22}(k) = \frac{1}{\alpha_\infty^2}\int_{-\infty}^{\infty} e^{ik\xi}\frac{1}{\alpha(\xi)}\frac{d\alpha}{d\xi}\left[\alpha^2(\xi)\bar{y}^+(\xi,k)-\alpha_\infty^2 e^{ik\xi}\right] d\xi\ .$$

Using the integral equation (4.8), we can write the following:

$$|\alpha^2(\tau)\bar{y}^+(\tau,k)-\alpha_\infty^2 e^{ik\tau}| \leq 2\int_\tau^\infty \alpha(\xi)\left|\frac{d\alpha}{d\xi}\right|\, |\bar{y}^+(\xi,k)|\, d\xi$$

$$\alpha^2(\tau)\bar{y}^+(\tau,k)-\alpha_\infty^2 e^{ik\tau} = 2\alpha^2_\infty O(\epsilon)$$

uniformly in τ and k for all τ and k.

Using the above bound in (6.4), we can show that $P_{22}(k)$ satisfies

$$P_{22}(k) = O(\epsilon^2)$$

uniformly in k for all k.

Thus, combining the above results, we have the following for $R^+(k)$:

$$(6.5) \qquad R^+(k) = -\int_{-\infty}^{\infty} e^{2ik\xi} \frac{1}{\alpha(\xi)} \frac{d\alpha}{d\xi}\, d\xi + O(\epsilon^2)$$

uniformly in k for all k. Noting that

$$\frac{1}{\alpha(\xi)} \frac{d\alpha}{d\xi} = \frac{d \log \alpha}{d\xi}$$

we can write (6.5) in the following equivalent form:

$$(6.6) \qquad R^+(k) = -\int_{-\infty}^{\infty} e^{2ik\xi} \frac{d \log \alpha}{d\xi}\, d\xi + O(\epsilon^2) \,.$$

Thus we have that the first approximation of the reflection coefficient involves the Fourier transform of $\frac{d \log \alpha}{d\xi}$. We would like to be able to invert the Fourier transform and express $\frac{d \log \alpha}{d\xi}$ approximately in terms of the reflection coefficient $R^+(k)$. This will be the task of the next section.

7. The Approximation for High Frequency

We showed in the last section that in the expression for the reflection coefficient

$$R^+(k) = -\int_{-\infty}^{\infty} e^{2ik\xi} \frac{1}{\alpha\xi} \frac{d\alpha}{d\xi}\, d\xi + E(k,\epsilon) ;$$

$E(k,\epsilon)$ was $O(\epsilon^2)$ uniformly in k for all k. As $|k|$ approaches infinity, we can establish a better bound on $E(k,\epsilon)$. Obtaining this improved estimate will be the objective of this section.

We will first derive a different integral equation for the solution $\bar{y}^+(\tau,k)$. This new integral equation will be used to generate new bounds on $\bar{y}^+(\tau,k)$, bounds which will be useful as $|k|$ becomes large. In addition, the integral equation will be used to obtain a new expression for the reflection coefficient $R^+(k)$. Finally, applying the new bounds for $\bar{y}^+(\tau,k)$ in the new expression for $R^+(k)$, we obtain the desired bound on the error term $E(k,\epsilon)$.

First we will obtain bounds on $\bar{y}^+(\tau,k)$ for large $|k|$. We will do this in the following lemma.

Lemma. Let $\frac{d^2\alpha}{d\tau^2}$ along with $\frac{d\alpha}{d\tau}$ be absolutely integrable for τ in the interval $-\infty < \tau < \infty$. Then $\bar{y}^+(\tau,k)$ satisfies the following inequalities for $|k| > k_0 > 0$:

$$|\alpha(\tau)\bar{y}^+(\tau,k) - \alpha_\infty e^{ik\tau}| \leq \frac{e^{A_+(\tau)}}{|k|} \int_\tau^\infty \frac{\alpha_\infty^2}{\alpha^2(\xi)} \left| \frac{d^2\alpha}{d\xi^2} \right| d\xi \;.$$

Proof:

Let us define a new dependent variable as follows:

(7.1) $$w^+(\tau,k) = \frac{\alpha(\tau)}{\alpha_\infty}\, \bar{y}^+(\tau,k) .$$

The differential equation (4.1) for $\bar{y}^+(\tau,k)$ is transformed into the following differential equation for $w^+(\tau,k)$:

(7.2) $$\frac{d^2 w^+}{d\tau^2} + \left[k^2 - \frac{1}{\alpha(\tau)} \frac{d^2\alpha}{d\tau^2}\right] w^+ = 0 .$$

As τ approaches positive infinity, $\alpha(\tau)$ approaches α_∞ and $\bar{y}^+(\tau,k)$ approaches $e^{ik\tau}$. Thus

(7.3) $$w^+(\tau,k) \sim e^{ik\tau} \quad \text{as} \quad \tau \to \infty .$$

Rewriting (7.2) as follows

(7.4) $$\frac{d^2 w^+}{d\tau^2} + k^2 w^+ = f(\tau,k)$$

$$f(\tau,k) = \frac{1}{\alpha^2(\tau)} \frac{d^2\alpha}{d\xi^2} w^+(\tau,k)$$

we get a problem of the same form as that in (4.3), (4.4).

Proceeding as before we can generate an integral equation for $w^+(\tau,k)$. We can express $w^+(\tau,k)$ as follows:

$$w^+(\tau,k) = e^{ik\tau} + \int_\tau^\infty \frac{\sin k(\xi-\tau)}{k} f(\xi,k)d\xi$$

$$(7.5) \qquad = e^{ik\tau} + \int_\tau^\infty \frac{\sin k(\xi-\tau)}{k} \frac{1}{\alpha(\xi)} \frac{d^2\alpha}{d\xi^2} w^+(\xi,k)d\xi \cdot$$

In terms of $\bar{y}^+(\tau,k)$ we obtain

$$(7.6) \qquad \alpha(\tau)\bar{y}^+(\tau,k) = \alpha_\infty e^{ik\tau} + \int_\tau^\infty \frac{\sin k(\xi-\tau)}{k} \frac{d^2\alpha}{d\xi^2} \bar{y}^+(\xi,k)d\xi \cdot$$

We are now in a position to obtain the bound given in the lemma. First express (7.6) as follows:

$$\alpha(\tau)\bar{y}^+(\tau,k)-\alpha_\infty e^{ik\tau} = \int_\tau^\infty \frac{\sin k(\xi-\tau)}{k} \frac{d^2\alpha}{d\xi^2} \bar{y}^+(\xi,k)\,d\xi \,.$$

Next we take absolute values of both sides of the above equation and use (4.16)

$$(7.7) \qquad \begin{cases} \left|\alpha(\tau)\bar{y}^+(\tau,k)-\alpha_\infty e^{ik\tau}\right| \leqslant \displaystyle\int_\tau^\infty \frac{1}{|k|} \frac{\alpha_\infty^2}{\alpha^2(\xi)} \frac{d^2\alpha}{d\xi^2} e^{A_+(\xi)} d\xi \\[2ex] \left|\alpha(\tau)\bar{y}^+(\tau,k)-\alpha_\infty e^{ik\tau}\right| \leqslant \dfrac{e^{A_+(\tau)}}{|k|} \displaystyle\int_\tau^\infty \frac{\alpha_\infty^2}{\alpha^2(\tau)} \frac{d^2\alpha}{d\xi^2} d\xi \cdot \end{cases}$$

This establishes the lemma.

The integral equation (7.6) can be written as follows:

$$\bar{y}^+(\tau,k) = \frac{\alpha_\infty}{\alpha(\tau)} e^{ik\tau}\left[1- \frac{1}{2ik\alpha_\infty}\int_\tau^\infty e^{-ik\xi} \frac{d^2\alpha}{d\xi^2} \bar{y}^+(\xi,k)d\xi\right] + \frac{1}{2ik\alpha(\tau)}\int_\tau^\infty e^{ik\xi} \frac{d^2\alpha}{d\epsilon^2} \bar{y}^+(\xi,k)d\xi \,.$$

Using (7.7) we can establish the following asymptotic behavior:

(7.8) $$\frac{\alpha_{-\infty}}{\alpha_{\infty}}\,\bar{y}^{+}(\tau,k)\sim e^{ik\tau}\quad\text{as}\quad \tau\to\infty$$

(7.9) $$\frac{\alpha_{-\infty}}{\alpha_{\infty}}\,\bar{y}^{+}(\tau,k)\sim\left[1+Q_1(k)\right]e^{ik\tau}+Q_2(k)e^{-ik\tau}\quad\text{as}\quad \tau\to-\infty$$

where $Q_1(k)$ and $Q_2(k)$ are defined as follows:

(7.10) $$Q_1(k)=-\frac{1}{2ik\alpha_{\infty}}\int_{-\infty}^{\infty}e^{-ik\xi}\,\frac{d^2\alpha}{d\xi^2}\,\bar{y}^{+}(\xi,k)d\xi$$

(7.11) $$Q_2(k)=\frac{1}{2ik\alpha_{\infty}}\int_{-\infty}^{\infty}e^{ik\xi}\,\frac{d^2\alpha}{d\xi^2}\,\bar{y}^{+}(\xi,k)d\xi\ .$$

If we define the function $w^{+}(\tau,t)$ by

$$w^{+}(\tau,t)=\frac{\frac{\alpha_{-\infty}}{\alpha_{\infty}}\,\bar{y}^{+}(\tau,k)}{1+Q_1(k)}\,e^{-ikt}$$

then we can show that $w^{+}(\tau,t)$ has the following asymptotic behavior:

(7.12) $$w^{+}(\tau,t)\sim e^{-ik(t-\frac{x}{\beta_{-\infty}})}+\frac{Q_2(k)}{1+Q_1(k)}\,e^{-ik(t+\frac{x}{\beta_{-\infty}})}\quad\text{as}\ \tau\to-\infty\ (x\to-\infty)$$

(7.13) $$w^{+}(\tau,t)\sim\frac{e^{-ik(t-\frac{x}{\beta_{\infty}})}}{1+Q_1(k)}\quad\text{as}\ \tau\to\infty\ (x\to\infty)\ .$$

Comparing (7.12) and (7.13) with (5.7) and (5.9), we obtain an alternate representation for the reflection coefficient $R^{+}(k)$ and transmission coefficient $T^{+}(k)$.

(7.14)
$$R^{+}(k) = \frac{Q_2(k)}{1+Q_1(k)}$$

(7.15)
$$T^{+}(k) = \frac{1}{1+Q_1(k)} \cdot$$

We can write (7.14) as follows:

(7.16)
$$R^{+}(k) = Q_2(k) - \frac{Q_2(k)\,Q_1(k)}{1+Q_1(k)} .$$

If we replace $\alpha(\tau)\bar{y}^{+}(\tau,k)$ by $\alpha_{\infty}e^{ik\tau}$ in the term $Q_2(k)$, we have

$$Q_2(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{2ik\xi}\,\frac{1}{\alpha(\xi)}\frac{d^2\alpha}{d\xi^2}\,d\xi + \frac{1}{2ik\alpha_{\infty}}\int_{-\infty}^{\infty} e^{ik\xi}\,\frac{1}{\alpha(\xi)}\,\frac{d^2\alpha}{d\xi^2}\left[\alpha(\xi)\bar{y}^{+}(\xi,k)-\alpha_{\infty}e^{ik\xi}\right]d\xi .$$

If we assume $\frac{d^2\alpha}{d\xi^2}$ is absolutely integrable, then using lemma 2 we can obtain a bound on the second term and can write $Q_1(k)$ as follows:

(7.17)
$$Q_2(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{2ik\xi}\;\frac{1}{\alpha(\xi)}\,\frac{d^2\alpha}{d\xi^2}\;d\xi + O\left(\frac{\epsilon^2}{k^2}\right)$$

the order relation holding uniformly in k for $|k| \geqslant 1$.

The integral in (7.17) can be simplified. First,

$$\frac{1}{\alpha(\xi)}\,\frac{d^2\alpha}{d\xi^2} = \frac{d}{d\xi}\left[\frac{1}{\alpha(\xi)}\,\frac{d\alpha}{d\xi}\right] + \left[\frac{1}{\alpha(\xi)}\,\frac{d\alpha}{d\xi}\right]^2 .$$

Thus we can write $Q_2(k)$ as

(7.18)
$$Q_2(k) = I_1(k) + I_2(k) + O\left(\frac{\epsilon^2}{k^2}\right)$$

where

(7.19) $$I_1(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{2ik\xi} \frac{d}{d\xi}\left[\frac{1}{\alpha(\xi)} \frac{d\alpha}{d\xi}\right] d\xi$$

(7.20) $$I_2(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{2ik\xi} \left[\frac{1}{\alpha(\xi)} \frac{d\alpha}{d\xi}\right]^2 d\xi$$

the order relation in (7.18) holding uniformly for $|k| \geqslant 1$.

Consider next $I_2(k)$. We can integrate by parts once in $I_2(k)$; using the fact that $\frac{d\alpha}{d\xi}$ vanishes as $|\xi|$ approaches infinity, it follows that

$$I_2(k) = -\frac{1}{(2ik)^2} \int_{-\infty}^{\infty} e^{2ik\xi} \frac{2}{\alpha(\xi)} \frac{d\alpha}{d\xi} \left[\frac{1}{\alpha(\xi)} \frac{d^2\alpha}{d\xi^2} - \left(\frac{1}{\alpha} \frac{d\alpha}{d\xi}\right)^2\right] d\xi \, .$$

The absolute integrability of $\frac{d\alpha}{d\xi}$ and $\frac{d^2\alpha}{d\xi^2}$ yields

$$I_2(k) = O\left(\frac{\epsilon^2}{k^2}\right)$$

uniformly in k for $|k| \geqslant 1$. In the integral $I_1(k)$, we can integrate by parts again and show that

$$I_1(k) = -\int_{-\infty}^{\infty} e^{2ik\xi} \frac{1}{\alpha(\xi)} \frac{d\alpha}{d\xi} d\xi \, .$$

Thus

(7.21) $$Q_2(k) = -\int_{-\infty}^{\infty} e^{2ik\xi} \frac{1}{\alpha(\xi)} \frac{d\alpha}{d\xi} d\xi + O\left(\frac{\epsilon^2}{k^2}\right)$$

uniformly in k for $|k| \geqslant 1$.

We showed earlier that

$$\bar{y}^{+}(\tau, k) = O(1)$$

uniformly in τ and k. Using the above and the relationships, (7.10), and (7.11), we can establish the following for $Q_1(k)$ and $Q_2(k)$

$$Q_1(k) = O(\frac{\epsilon}{k})$$

$$Q_2(k) = O(\frac{\epsilon}{k})$$

uniformly in k for $|k| \geqslant 1$. Thus

$$\frac{Q_1(k)\, Q_2(k)}{1 + Q_1(k)} = O(\frac{\epsilon^2}{k^2}) \tag{7.22}$$

uniformly in k for $|k| \geqslant 1$.

Substituting (7.21) and (7.22) into (7.16), we derive the following result for the reflection coefficient $R^{+}(k)$:

$$R^{+}(k) = -\int_{-\infty}^{\infty} e^{2ik\xi} \frac{1}{\alpha(\xi)} \frac{d\alpha}{d\xi}\, d\xi + O(\frac{\epsilon^2}{k^2}) \tag{7.23}$$

the order relation holding uniformly in k for all k.

8. The Inverse Problem

Up to this point we have assumed that $\alpha(\tau)$ was known, and we worked toward approximating the reflection coefficient $R^+(k)$ associated with the medium described by the $\alpha(\tau)$. We will refer to this as a "forward problem." An equally interesting problem is one in which the reflection coefficent is known and an approximation to $\alpha(\tau)$ is desired. This second problem is referred to as an "inverse problem."

In the previous sections we have shown that the reflection coefficient to first approximation is equal to the Fourier transform of the derivative of $\log \alpha(\tau)$. That is, we have shown that

$$(8.1) \qquad R^+(k) = - \int_{-\infty}^{\infty} e^{2ik\xi} \frac{d \log \alpha}{d\xi} \, d\xi + E(k, \epsilon)$$

$$\frac{d \log \alpha}{d\xi} = \frac{1}{\alpha(\xi)} \frac{d\alpha}{d\xi} .$$

In addition, we have established the following bounds on the error term $E(k, \epsilon)$. In particular, in (6. 6) and (7. 23) we have shown that

$$E(k, \epsilon) = \begin{cases} O(\epsilon^2) , & \text{uniformly in } k, \ |k| < \infty \\ O\left(\frac{\epsilon^2}{k^2}\right) , & \text{uniformly in } k, \ |k| > 1 . \end{cases}$$

These two bounds can be consolidated into the following single bound:

$$(8.2) \qquad E(k, \epsilon) = O\left(\frac{\epsilon^2}{1+k^2}\right)$$

uniformly in k.

We are interested in inverting the Fourier transform to obtain a relation for $\frac{d \log \alpha}{d\xi}$ involving the reflection coefficient $R^{+}(k)$. Inverting the Fourier transform involves multiplying (8.1) by $\frac{1}{2\pi} e^{-2ik\xi}$ and integrating with respect to 2k over the interval $-\infty < 2k < \infty$. We must justify the existence of the integrals. The first term on the right of (8.1), as mentioned earlier, is the Fourier transform of a function. Conditions will be given on the function which will validate the inversion of the Fourier transform. As for the error term, we will show that it is absolutely integrable with respect to 2k on the interval $-\infty < 2k < \infty$, and thus we will be able to bound the inversion integral involving this term.

Consider first the error term $E(k,\epsilon)$. We can show that

$$\left| \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2ik\xi} E(k,\epsilon)\, dk \right| \leqslant \frac{C}{\pi} \epsilon^2 \int_{-\infty}^{\infty} \frac{dk}{1+k^2}$$

$$\leqslant C\epsilon^2$$

uniformly in ξ.

As for the first term on the right hand side of Eq. (8.1), we will need the following version of the

<u>Fourier Integral Theorem</u>: Let $f(\tau)$ be continuously differentiable and absolutely integrable on $-\infty < \tau < \infty$. Then the inversion integral of the Fourier transform exists and

$$f(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2ik\tau} \int_{-\infty}^{\infty} e^{2ik\xi} f(\xi)\, d\xi\, dk .$$

For the case we are considering, $f(\tau) = \frac{d \log \alpha}{d\tau}$. The absolute integrability of $f(\tau)$ follows immediately from previous assumptions on $\alpha(\tau)$. If $\frac{d^2\alpha}{d\tau^2}$ is assumed to be continuous, then $\frac{df}{d\tau}(\tau)$ is also continuous.

Thus we have

(8.3) $$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2ik\tau} R^{+}(k)\, dk = - \frac{d \log \alpha(\tau)}{d\tau} + O(\epsilon^2)$$

uniformly in x for all x. Since the right hand side exists, we have the integral on the left hand side also exists.

Writing $\alpha(\tau) = \alpha_\infty \left[1 + \epsilon\, r(\tau)\right]$, we can express the right hand side of (8.3) as follows:

(8.4) $$F(\tau,\epsilon) = \epsilon \frac{dr}{d\tau} + O(\epsilon^2)$$

uniformly in τ for all τ, where

$$F(\tau,\epsilon) = -\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2ik\tau} R^{+}(k)\, dk .$$

Thus, we have shown that

(8.5) $$F(\tau,\epsilon) = \epsilon F_0(\tau) + O(\epsilon^2)$$

uniformly in τ for all τ and

$$F_0(\tau) = \frac{dr}{d\tau} .$$

Thus, if we are provided with a reflection coefficient, $R^+(k)$, corresponding to a medium described by $\alpha(\tau)$, then we have a means of recovering $\alpha(\tau)$ to order ϵ. We summarize this result more precisely in terms of the following theorem.

Theorem: Suppose

1. $\alpha(\tau)$ is twice continuously differentiable and both the first and second derivatives are absolutely integrable on the interval $-\infty < \tau < \infty$.
2. $\alpha(\tau)$ can be written as

$$\alpha(\tau) = \alpha_\infty\left[1 + \epsilon\, r(\tau)\right], \quad r(\tau) \to 0 \text{ as } \tau \to \infty.$$

Then

$$F(\tau,\epsilon) = -\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2ik\tau} R^+(k)\, dk$$

exists and can be expanded as follows:

$$F(\tau,\epsilon) = \epsilon\, F_0(\tau) + O(\epsilon^2)$$

uniformly in τ for all τ, and

$$\frac{dr}{d\tau} = F_0(\tau) \quad .$$

At this point it is interesting to note that if we wanted to approximate the original parameters $p(x)$ and $\rho(x)$ from the approximation to $\alpha(\tau)$, we could not do so uniquely.

CHAPTER III

THE TIMOSHENKO EQUATION

9. Introduction.

In this chapter, we will consider the Timoshenko equation describing the transverse vibrations of a non-homogeneous beam.

In the first section a brief discussion of the various theories relating to the transverse vibrations of a beam will be presented. Using the Timoshenko theory, we will derive the Timoshenko Beam equation for a non-homogeneous beam.

In the second section we will investigate the properties of solutions of the Timoshenko Beam equation for a homogeneous medium.

In the third section we shall return to the non-homogeneous beam. We shall assume that the material composition does not vary, although the beam may be tapered. Finally, as with the wave equation, we shall assume that the taper becomes uniform at infinity and that the taper is only slightly nonuniform otherwise.

In the fourth section, we separate the time dependence and derive a fourth order ordinary differential equation, which is expressed as an integral equation. Finally we prove the existence of solutions to the integral equation and, in the process derive bounds on the solution.

In the fifth section using the integral equation we define reflection and transmission coefficients for a sinusoidal wave incident from infinity.

In the sixth section we approximate the reflection coefficient for the case of a beam with a slightly non-uniform taper.

In the seventh section we improve the approximation to the reflection coefficient for high frequency incident waves.

In the final section we show how the medium parameters can be approximated from the reflection coefficient .

10. A Derivation of the Equation

In this section we shall consider transverse vibrations of a beam. This is but one of many ways that a beam can vibrate. Other types of vibrations of a beam are compressional vibrations, and torsional or twisting vibrations. One of the simplest examples of a flexure vibration is the motion of a struck tuning fork.

Problems involving transverse motions of a beam can be set up mathematically using the three-dimensional linear theory of elasticity. Although the partial differential equations and boundary conditions are strictly linear, the solution of these problems is extremely difficult. In fact, the only problems investigated up to the present time have been associated with beams which are composed of a homogeneous material and have a uniform circular cross-section (see Abramsom, [1]).

Because of the difficulty of solving the exact equations arising from the theory of elasticity, the exact theory must be abandoned in favor of an approximate one. In the exact theory we analyze the microscopic motions of the beam. In an approximate theory we cease trying to find out what happens in the small, but rather make some assumptions about the gross behavior of the beam. The partial differential equations and boundary conditions we obtained by making these assumptions are easier to handle than those arising from the exact theory.

There are three approximate theories for the flexure vibrations of a beam. In the elementary theory, a transverse slice of the beam is viewed as moving only transversely to the axis of the beam. In the Rayleigh theory, the slice is assumed to rotate as well. The Timoshenko theory includes both the assumptions of transverse and rotary motion of the slice of beam, and in addition takes account of distortion under the action of the shear force. These three theories can be ordered with the Timoshenko theory being the most complex and the elementary theory being the simplest.

Using the Timoshenko theory we proceed to the derivation of the Timoshenko beam equation. To do this we consider an increment of beam bounded by two plane cross-sectional faces, separated by a distance dx. When the beam is at rest, we align the centroidal axis so that it is horizontal and we choose the plane cross-sectional faces so that they are vertical. We now assume the beam is set in motion. We assume that the initial plane cross-sectional faces remain planes. This assumption is common to all three theories. We denote by y the distance that the centroidal axis is displaced from its equilibrium position and by φ the rotation angle of the plane cross-sectional face. We denote by z the direction perpendicular to x and y.

We next assume that y and φ do not vary much with position. That is, $\partial y/\partial x$ and $\partial\varphi/\partial x$ are small. Thus the angle that the centroidal axis makes with the horizontal is approximately $\partial y/\partial x$. We allow the various material and other parameters describing the beam to depend on x but to be independent of y and z. The cross-sectional faces are initially perpendicular to the centroidal axis. However, with the ensuing motion of the beam, we assume that the faces no longer remain perpendicular to the centroidal axis, but are bent from the perpendicular by an angle ψ due to the action of shear forces. Thus we set

$$\frac{\partial y}{\partial x} = \varphi + \psi . \tag{10.1}$$

Now the moment acting on the increment of the beam is just

$$M = E r^2 A \frac{\partial \varphi}{\partial x} \tag{10.2}$$

where E is Young's modulus and r is the radius of gyration of the cross-sectional area about the centroidal axis and A the cross-sectional area.

The shear stress at a point on one of the cross-sectional faces is equal to

$$\mu \gamma(y, z)$$

where $\gamma(y, z)$ is the local angle of shear, and μ is the modulus of rigidity. The total shear force Q acting on a face is given by

$$Q = \mu \int\int_{\overline{A}(x)} \gamma(y, z) \, dy \, dz$$

or

$$Q = \mu A \Gamma$$

where

$$\Gamma(x) = \frac{\iint_{\overline{A}(x)} \gamma(y,z)\, dy\, dz}{\iint_{\overline{A}(x)} dy\, dz}$$

is the average shear angle over the face and

$$A = \iint_{\overline{A}(x)} dy\, dz$$

is the area of the face. At this point the Timoshenko theory assumes $\Gamma(x)$ is proportional to $\psi = \frac{\partial y}{\partial x} - \varphi$. That is,

$$\Gamma = \lambda \psi \; .$$

It is further assumed that the constant of proportionality λ depends on the shape of the cross-section and not on its size. Putting this together, we obtain the following expression for the shear force.

$$Q = \lambda \mu A \psi \; . \tag{10.3}$$

Now we examine the momentum equations for the increment of the beam. The translational equation of motion is, (10.4) $$\frac{\partial Q}{\partial x} = \rho A \frac{\partial^2 y}{\partial t^2}$$

while the rotational equation of motion is

(10.5) $$\frac{\partial M}{\partial x} + Q = \rho r^2 \frac{\partial^2 \varphi}{\partial t^2} \,.$$

At this point we shall deviate from the standard derivation of the Timoshenko beam equation, in that we shall assume that the parameters appearing in the above equations are functions of position. The parameters may be classed into two groups. The parameters μ, E, and ρ depend on the material of which the beam is composed. If these parameters vary with the position, the beam is said to be nonhomogeneous. The remaining parameters λ, A, and r^2 are dependent only on the shape of the cross-sectional area. If these parameters vary, we shall speak of the beam being non-homogeneous in shape. Proceeding with this assumption, we combine (10.1), (10.2) and (10.4) to get

(10.6) $$\frac{\partial}{\partial x}\left(\lambda\mu A \frac{\partial y}{\partial x}\right) - \lambda\mu A \frac{\partial \varphi}{\partial x} - \varphi \frac{\partial}{\partial x}(\lambda\mu A) = \rho A \frac{\partial^2 y}{\partial t^2}$$

Combining (10.2), (10.3) and (10.5) we get

(10.7) $$\frac{\partial}{\partial x}\left(E r^2 A \frac{\partial \varphi}{\partial x}\right) - \lambda\mu A\left[\frac{\partial y}{\partial \varphi} - \varphi\right] = \rho r^2 A \frac{\partial^2 y}{\partial t^2} \,.$$

Now let

$$(10.8) \qquad \begin{cases} \alpha(x) = Er^2A & \beta(x) = \lambda\mu A \\ a(x) = \dfrac{\rho}{E} & b(x) = \dfrac{\rho}{\lambda\mu} \end{cases} .$$

We rewrite (10.6) and (10.7) using (10.8):

$$(10.9) \qquad \frac{\partial}{\partial x}\left(\alpha \frac{\partial \varphi}{\partial x}\right) + \beta \frac{\partial y}{\partial x} - \beta\varphi = a\alpha \frac{\partial^2 \varphi}{\partial t^2}$$

$$(10.10) \qquad \frac{\partial}{\partial x}\left(\beta \frac{\partial y}{\partial x}\right) - \beta \frac{\partial \varphi}{\partial x} - \varphi \frac{\partial \beta}{\partial x} = b\beta \frac{\partial^2 y}{\partial t^2} .$$

Thus for a nonhomogeneous beam of nonuniform shape we get a system of second order partial differential equations in the unknowns φ and y. The Timoshenko equation for a uniform beam can be easily retrieved from (10.9) and (10.10) by letting $\alpha(x) = \alpha_0$, $\beta(x) = \beta_0$, $a(x) = a_0$, and $b(x) = b_0$, where α_0, β_0, a_0 and b_0 are constants. Eqs. (10.9) and (10.10) then become

$$(10.11) \qquad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\beta_0}{\alpha_0}\frac{\partial y}{\partial x} - \frac{\beta_0}{\alpha_0}\varphi = a_0 \frac{\partial^2 \varphi}{\partial t^2}$$

$$(10.12) \qquad \frac{\partial^2 y}{\partial x^2} - \frac{\partial \varphi}{\partial x} = b_0 \frac{\partial^2 y}{\partial t^2}$$

Eliminating φ from these equations yields the following fourth order partial differential equation which is the Timoshenko beam equation for a homogeneous beam of uniform shape:

$$(10.13) \qquad \alpha_0 \frac{\partial^4 y}{\partial x^4} - \alpha_0(a_0 + b_0)\frac{\partial^4 y}{\partial t^2 \partial x^2} + \beta_0 b_0 \frac{\partial^2 y}{\partial t^2} + a_0 b_0 \alpha_0 \frac{\partial^4 y}{\partial t^4} = 0 .$$

If we eliminate y from (10.11) and (10.12), we obtain a partial differential equation for φ which has exactly the same form as the partial differential equation for y; that is

$$(10.14) \quad \alpha_0 \frac{\partial^4 \varphi}{\partial x^4} - \alpha_0 (a_0 + b_0) \frac{\partial^4 \varphi}{\partial x^2 \partial t^2} + \beta_0 b \frac{\partial^2 \varphi}{\partial t^2} + \alpha_0 a_0 b_0 \frac{\partial^4 \varphi}{\partial t^4} = 0 \quad .$$

11. The Timoshenko Equation for a Homogeneous Medium

In this section we investigate the partial differential equation describing lateral vibrations of a homogeneous uniform beam. We will study the properties of a particular set of solutions corresponding to traveling sinusoidal waves. Finally, we will derive an expression for the solution of the fourth order inhomogeneous ordinary differential associated with the partial differential equation (10.14).

We consider the following equation describing the transverse vibrations of a homogeneous beam:

$$(11.1) \qquad \frac{\partial^4 \varphi}{\partial x^4} - (a+b) \frac{\partial^4 \varphi}{\partial x^2 \partial t^2} + \frac{\beta b}{\alpha} \frac{\partial^2 \varphi}{\partial t^2} + ab \frac{\partial^4 \varphi}{\partial t^4} = 0 \quad .$$

The subscripts on the α, β, a, and b have been deleted in writing (11.1) since we shall only be considering the uniform beam in this section. The parameters α, β, a, and b are positive constants.

We look for solutions of (11.1) of the following form:

$$(11.2) \qquad \varphi(x,t) = e^{i\omega t}\, \overline{\varphi}(x,\omega) \quad .$$

Substituting into (11.1), we find that $\overline{\varphi}$ satisfies

$$\frac{d^4\overline{\varphi}}{dx^4} + (a+b)\omega^2 \frac{d^2\overline{\varphi}}{dx^2} - b\omega^2\left(\frac{\beta}{\alpha} - a\omega^2\right)\overline{\varphi} = 0 . \tag{11.3}$$

We now look for a solution $\overline{\varphi}(x,\omega)$ of (11.3) of the following form:

$$\overline{\varphi}(x,\omega) = e^{i\nu x} \tag{11.4}$$

Substituting into (11.3) we find that ν must satisfy the following characteristic equation:

$$\nu^4 - (a+b)\omega^2\nu^2 - b\omega^2\left(\frac{\beta}{\alpha} - a\omega^2\right) = 0 . \tag{11.5}$$

Eq. (11.5) provides an implicit relationship between ν and ω, and in fact is a fourth degree polynomial in ν. We can obtain the four roots

$$\begin{cases} \nu_1 = \omega\sqrt{\frac{1}{2}\left[b+a + \sqrt{(b-a)^2 + 4\frac{\beta}{\alpha}\frac{b}{\omega^2}}\right]} \\ \nu_2 = \omega\sqrt{\frac{1}{2}\left[b+a - \sqrt{(b-a)^2 + 4\frac{\beta}{\alpha}\frac{b}{\omega^2}}\right]} \\ \nu_3 = -\nu_1 \qquad \nu_4 = -\nu_2 . \end{cases} \tag{11.6}$$

The roots ν_1 and ν_3 are real and are respectively positive and negative for all real positive values of ω. The roots ν_2 and ν_4 are real and are respectively positive and negative for positive ω greater than ω_c, where

$$\omega_c = \beta/a\alpha > 0 . \tag{11.7}$$

For positive ω less than ω_c, ν_2 and ν_4 are pure imaginary. For negative ω we can deduce similar properties of ν_j by using

$$
(11.8) \qquad \begin{cases} \nu_1(-\omega) = -\nu_1(\omega) = \nu_3(\omega) \\ \\ \nu_2(-\omega) = -\nu_2(\omega) = \nu_4(\omega) \,. \end{cases}
$$

If ν is one of the roots ν_j, then

$$
(11.9) \qquad \varphi(x,t) = e^{i(\omega t + \nu_j x)} \qquad j = 1, 2, 3, 4
$$

is a solution of (11.1). For real values of ω and ν_j, the solution given by (11.9) can be interpreted as a simple harmonic wave. The frequency of vibration is $\frac{\omega}{2\pi}$, the wave-length is $\frac{2\pi}{\nu_j}$ and the velocity of propagation (or phase velocity) is $c_j = \frac{\omega}{\nu_j}$. If $|\omega| < \omega_c$, then ν_2 and ν_4, and thus the velocities c_2 and c_4, are imaginary, and (11.9) can no longer be interpreted as simple harmonic waves. For positive ω, c_1 is positive, while for positive ω greater than ω_c, c_2 is positive; and in each case (11.9) represents a wave propagating in the negative x-direction. The same holds true with negative ω. However, for positive ω, c_3 is negative, while for positive ω greater than ω_c, c_4 is negative; and in both cases (11.9) with j=3, 4 represents a wave propagating in the positive x-direction. Again with negative ω (11.9) with j=3, 4 represents a wave propagating in the positive x-direction. We will speak of the solution given by (11.9) for either ν_1 or ν_3 as the first mode of vibration and for either ν_2 or ν_4 as the second mode. Note that the magnitude of the velocity of the first and second modes varies with the frequency of vibration. This result is different than that for the wave equation of a homogeneous medium where simple harmonic waves propagate with a speed independent of the frequency.

The roots ν_1 and ν_2 have the following asymptotic behavior for certain limiting values of ω. As ω approaches zero, we can deduce that

$$\text{(11.10)} \qquad \begin{cases} \nu_1 \sim \sqrt{\dfrac{\beta b}{\alpha}}\ \omega^{1/2} \\ \nu_2 \sim 2\sqrt{\dfrac{\beta b}{\alpha}}\ \omega^{1/2} \quad . \end{cases}$$

Defining $\Delta\omega$ as $\omega - \omega_c$, we can show that as $\Delta\omega$ approaches zero,

$$\text{(11.11)} \qquad \begin{cases} \nu_1 \sim (a+b)^{1/2}\ \omega_c \\ \nu_2 \sim \left(\dfrac{2\omega_c \Delta\omega}{a+b}\right)^{1/2} \quad . \end{cases}$$

Finally for large ω

$$\text{(11.12)} \qquad \begin{cases} \nu_1 \sim \sqrt{\dfrac{a+b+|a-b|}{2}}\ \omega = \max(\sqrt{a}, \sqrt{b})\,\omega \\ \nu_2 \sim \sqrt{\dfrac{a+b-|a-b|}{2}}\ \omega = \min(\sqrt{a}, \sqrt{b})\,\omega \quad . \end{cases}$$

Using the definition of the propagation speeds c_j, $j = 1, 2$ and (11.6) we can show that as $|\omega|$ approaches infinity,

$$(11.13)\qquad \begin{cases} c_1 \sim \sqrt{\dfrac{2}{a+b+|b-a|}} = \dfrac{1}{\max(\sqrt{a},\ \sqrt{b})} \\[2ex] c_2 \sim \sqrt{\dfrac{2}{a+b-|b-a|}} = \dfrac{1}{\min(\sqrt{a},\ \sqrt{b})} \end{cases} .$$

Thus at high frequency the speed of propagation of the waves of the two modes becomes constant.

Consider the solutions

$$(11.14)\qquad \begin{cases} \overline{\varphi}_j(x,\omega) = e_j(x) , \\[1ex] e_j(x) = e^{i\nu_j x} \end{cases} \qquad j = 1, 2, 3, 4$$

where ν_j are as given by (11.6). Computing the Wronskian W of the solutions (11.14) we get

$$(11.15)\qquad W = -4\nu_1\nu_2(\nu_1{}^2-\nu_2{}^2)^2 .$$

For $\omega \neq 0,\ \omega_c$, the quantities ν_1, ν_2, and

$$(\nu_1{}^2-\nu_2{}^2)^2 = \omega^2\left[(b-a)\omega^2 + \frac{\beta}{\alpha}b\right]$$

are non-zero. Therefore, the Wronskian does not vanish. Thus the solutions given by (11.14) are linearly independent.

The independent solutions $\overline{\varphi}_j$ of the homogeneous differential equation (11.3) can be used to solve the inhomogeneous differential equation

$$\frac{d^4\bar{\varphi}}{dx^4} + (b+a)\omega^2 \frac{d^2\bar{\varphi}}{dx^2} - b\omega^2\left(\frac{\beta}{\alpha} - a\omega^2\right)\bar{\varphi} = f(x,\omega). \tag{11.16}$$

A particular solution of (11.16) is given by

$$\bar{\varphi}(x,\omega) = \sum_{k=1}^{4} \bar{\varphi}_k(x,\omega) \int_a^x \frac{W_k\{\bar{\varphi}_1 \cdots \bar{\varphi}_4\}(\xi)}{W\{\bar{\varphi}_1 \cdots \bar{\varphi}_4\}(\xi)} f(\xi)\, d\xi \tag{11.17}$$

where $W_k\{\bar{\varphi}_1 \cdots \bar{\varphi}_4\}(x)$ is derived from the determinant $W\{\bar{\varphi}_1 \cdots \bar{\varphi}_4\}(\xi)$ by replacing the kth column by

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

The constant a is arbitrary. We have shown that $e_k(x) = e^{i\nu_k x}$, k=1, 2, 3, 4, constitute a set of linearly independent solutions. Therefore, we can choose $\bar{\varphi}_k(x,\omega) = e_k(x)$ in (11.17). If in addition we let $a = -\infty$, then for each j

$$\hat{\varphi}_j(x,\omega) = e_j(x) + \sum_{k=1}^{4} e_k(x) \int_{-\infty}^x \frac{W_k\{e_1 \cdots e_4\}(\xi)}{W\{e_1 \cdots e_4\}} f(\xi)\, d\xi \tag{11.18}$$

is a solution of (11.16). The solution $\hat{\varphi}_j(x,\omega)$ behaves like $e_j(x) = e^{i\nu_j x}$ as x approaches $-\infty$. Evaluating $W_k(e_1 \cdots e_4)(\xi)$ we get

$$W_k(e_1 \cdots e_4)(\xi) = 2i\nu_{k-1}(-1)^k(\nu_k^2 - \nu_{k-2}^2)e_k(-\xi)$$

where for k = 1, 2 we let $\nu_0 = \nu_4$, $\nu_{-1} = \nu_3$. (11.18) becomes

$$\varphi_j(x,\omega) = e_j(x) + \sum_{k=1}^{4} \int_{-\infty}^{x} e_k(x-\xi) \frac{f(\xi)}{2i\nu_k(\nu_k^2-\nu_{k-2}^2)} d\xi \;. \tag{11.19}$$

Eq. (11.19) will be needed in the next section to transform certain differential equations into integral equations.

12. The Timoshenko Equation for a Non-homogeneous Medium

In the present chapter we shall restrict attention to certain types of non-uniform beams. With this restriction we will be able to simplify the system of partial differential equations (10.6) and (10.7) to a single fourth order partial differential equation. We will reduce the fourth order partial differential equation further to a fourth order ordinary differential equation. Finally, we will transform the differential equation to an integral equation for solutions with particular asymptotic properties for large $|x|$. We solve these integral equations by successive approximations.

First we assume that the beam is composed of homogeneous material throughout. This assumption implies that the material parameters such as the density ρ, Young's modulus E, and the modulus of rigidity μ do not vary with position.

Secondly we shall assume that the shapes of the cross-sections remain geometrically similar along the beam, although we allow their sizes to vary. As an example for a rectangular-cross-section, we would limit ourselves to the case where the ratio of the lengths and the ratio of the widths of the sides of two cross-sections are the same. The variations of sizes of the cross-sectional areas will result in a taper of the beam. We shall refer to such beams as nonuniform.

The second assumption implies that λ is constant, as we mentioned in Chapter 1. The parameter λ depends on the shape and not the size of the cross-section.

The second assumption also has certain implications for the cross-sectional area A(x) and the radius of gyration r(x).

We have the following expressions for A(x) and r(x) :

$$(12.1) \quad \begin{cases} A(x) = \iint_{\overline{A}(x)} dz\, dy \\ \\ r^2(x)\, A(x) = \iint_{\overline{A}(x)} y^2\, dy\, dz \end{cases}$$

where $\overline{A}(x)$ is the region in the y, z plane at the point x whose cross-sectional area is A(x). The value of A(x') and $r^2(x')$ at another position x' will be given by similar integrals.

$$(12.2) \quad \begin{cases} A(x') = \iint_{\overline{A}(x')} dy'\, dz' \\ \\ r^2(x')\, A(x') = \iint_{\overline{A}(x')} (y')^2\, dy'\, dz' \end{cases}$$

However, by our second assumption, the dimensions of $\overline{A}(x')$ are just some constant multiple m of the dimensions of $\overline{A}(x)$. We make the following change of variables

$$y' = my$$
$$z' = mz$$

in the integrals of (12.2). The area in terms of y and z is just $\overline{A}(x)$ since the shape is the same, and the ratio of the corresponding dimensions of the two areas is m. Eqs. (12.2) become

$$A(x') = \int\int_{\overline{A(x)}} dy\, dz = m^2 A(x)$$

$$r^2(x')\, A(x') = \int\int_{\overline{A(x)}} y^2\, dy\, dz = m^4 r^2(x)\, A(x)$$

Thus

(12.3) $$\frac{r^2(x)}{A(x)} = \frac{r^2(x')}{A(x')}$$

which implies α/β is constant. Thus we can show

(12.4) $$\frac{\alpha'}{\alpha} \quad \frac{\beta'}{\beta} \quad .$$

With these assumptions on a, b, α, and β we shall now reduce the system of second order partial differential equations to one fourth order differential equation. First (10.9), (10.10) can be written as

(12.5) $$\beta(\varphi - \frac{\partial y}{\partial x}) = \frac{\partial}{\partial x}\left(\alpha \frac{\partial \varphi}{\partial x}\right) - a\alpha \frac{\partial^2 \varphi}{\partial t^2} \;, \qquad \alpha = \alpha(x)$$

(12.6) $$\frac{\partial}{\partial x}\left[\beta(\varphi - \frac{\partial y}{\partial x})\right] + b\beta \frac{\partial^2 y}{\partial t^2} = 0 \;, \qquad \beta = \beta(x) \;.$$

Divide (12.6) by β and differentiate once with respect to x to obtain

(12.7) $$\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\frac{\partial}{\partial x}\left[\beta(\varphi - \frac{\partial y}{\partial x})\right]\right\} + b\,\frac{\partial^3 y}{\partial t^2 \partial x} = 0 \;.$$

After some manipulating, (12.7) can be written in the following form:

$$(12.8) \qquad \frac{\partial}{\partial x}\left\{\frac{1}{\beta}\frac{\partial}{\partial x}\left[\beta\left(\varphi-\frac{\partial y}{\partial x}\right)\right]\right\} - b\frac{\partial^2}{\partial t^2}\left(\varphi-\frac{\partial y}{\partial x}\right)+b\frac{\partial^2\varphi}{\partial t^2} = 0 \ .$$

Now we substitute the right hand side of (12.5) into (12.8), obtaining

$$(12.9) \quad \frac{\partial}{\partial x}\left\{\frac{1}{\beta}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\alpha\frac{\partial\varphi}{\partial x}\right)-a\alpha\frac{\partial^2\varphi}{\partial t^2}\right]\right\} - b\frac{\partial^2}{\partial t^2}\left\{\left[\frac{\partial}{\partial x}\left(\alpha\frac{\partial\varphi}{\partial x}\right)-a\alpha\frac{\partial^2\varphi}{\partial t^2}\right]\right\}+b\frac{\partial^2\varphi}{\partial t^2} = 0 \ .$$

Expanding (12.9) and using (12.4), we derive the following partial differential equation for $\varphi(x,t)$:

$$(12.10) \quad \frac{\partial^4\varphi}{\partial x^4} - (a+b)\frac{\partial^4\varphi}{\partial x^2\partial t^2} + ab\frac{\partial^4\varphi}{\partial t^4} + \frac{\beta}{\alpha}\frac{\partial^2\varphi}{\partial t^2} =$$

$$-\left\{2\frac{\alpha'}{\alpha}\frac{\partial^3\varphi}{\partial x^3} + \left[3\frac{\alpha''}{\alpha} - 2\left(\frac{\alpha'}{\alpha}\right)^2\right]\frac{\partial^2\varphi}{\partial x^2} - (a+b)\frac{\alpha'}{\alpha}\frac{\partial^3\varphi}{\partial x\partial t^2}\right.$$

$$\left. + \left[\frac{\alpha'''}{\alpha} - \frac{\alpha''\alpha'}{\alpha^2}\right]\frac{\partial\varphi}{\partial x} - a\left(\frac{\alpha'}{\alpha}\right)^2\frac{\partial^2\varphi}{\partial t^2}\right\}$$

where we have written α' for $d\alpha/dx$.

13. An Integral Equation

In this section we will begin by separating out the time from (12.10), thus reducing the partial differential equation to an ordinary differential equation. We then transform the ordinary differential equation into an integral equation. In the remainder of this section, we will show the existence of solutions of the integral equation and, in the process, derive bounds on the solution.

We ask if (12.10) has any solutions of the following form:

$$\varphi(x,t) = e^{i\omega t}\,\overline{\varphi}(x,\omega) \quad . \tag{13.1}$$

Substituting this into (12.10), we arrive at the following ordinary differential equation for $\overline{\varphi}(x,\omega)$:

$$\frac{d^4\overline{\varphi}}{dx^4} + (a+b)\omega^2 \frac{d^2\overline{\varphi}}{dx^2} + b\omega^2\left(a\omega^2 - \frac{\beta}{\alpha}\right)\overline{\varphi} = L\overline{\varphi} \tag{13.2}$$

where $L\overline{\varphi}$ is given by

$$L\overline{\varphi}(x,\omega) = -\left\{2\frac{\alpha'}{\alpha}\frac{d^3\overline{\varphi}}{dx^3} + \left[3\frac{\alpha''}{\alpha} - 2\left(\frac{\alpha'}{\alpha}\right)^2\right]\frac{d^2\overline{\varphi}}{dx^2}\right. \tag{13.3}$$

$$\left. + \left[\frac{\alpha'''}{\alpha} - \frac{\alpha''\alpha'}{\alpha^2}\right]\frac{d\overline{\varphi}}{dx} + (a+b)\frac{\alpha'}{\alpha}\omega^2\frac{d\overline{\varphi}}{dx} + a\left(\frac{\alpha'}{\alpha}\right)^2\omega^2\overline{\varphi}\right\} \quad .$$

The operator on the left hand side of (13.2) has constant coefficients and is identical to that of (11.3) for a uniform beam.

We shall consider the case where the beam becomes uniform as x approaches $\pm\infty$. That is, we shall assume that $\alpha(x)$ becomes constant and all derivatives of α vanish, as $|x|$ approaches ∞. In addition, further assumptions about the derivatives of $\alpha(x)$ at $\pm\infty$ will be made later in connection with the requirement that certain integrals converge.

The above assumptions have implications on the solutions $\overline{\varphi}(x,\omega)$ of the differential equation (13.2). Suppose we look for a set of solutions of this differential equation which are bounded and have bounded derivatives of various orders as x approaches $-\infty$. Then applying the assumptions in the previous paragraph, the differential form $L\overline{\varphi}(x,\omega)$ will vanish as x approaches $-\infty$, and, we would thus expect that bounded solutions of (13.2) would approach the solutions (11.14) of the differential equation (11.3) as x approaches $-\infty$. That is, we would expect that there are solutions $\overline{\varphi}_j(x,\omega)$, $j=1,2,3,4$ of the differential equation (13.2) which behave like

$$e_j(x) = e^{i\nu_j x}$$

as x approaches $-\infty$.

In the following work, we shall derive integral equations for the solutions $\overline{\varphi}_j(x,\omega)$ which have the above asymptotic behavior as x approaches $-\infty$. In addition, we shall derive conditions on $\alpha(x)$ for the existence of such solutions and obtain bounds on those solutions.

If we compare (13.2) with (11.16) we see that the two equations are the same with $f(x) = L\overline{\varphi}(x,\omega)$. We may therefore write

(13.4) $$\overline{\varphi}(x,\omega) = e_j(x) + \sum_{k=1}^{4} \int_{-\infty}^{x} e_k(x-\xi) \frac{L\overline{\varphi}_j(\xi,\omega)}{2i\nu_k(\nu_k^2 - \nu_{k-1}^2)} \, d\xi \quad .$$

The differential form $L\overline{\varphi}_j(x,\omega)$ contains derivatives of $\overline{\varphi}_j(x,\omega)$. Performing suitable integrations by parts in (13.4) yields

$$\overline{\varphi}_j(x,\omega) = e_j(x) - \int_{-\infty}^{x} k(x,\xi;\omega)\,\overline{\varphi}_j(\xi,\omega)\, d\xi$$

where

$$k(x,\xi;\omega) = \sum_{k=1}^{4} e_k(x-\xi)\, M_k(\xi) \tag{13.5}$$

and M_k is given by

$$M_k(\xi) = -\frac{1}{2(\nu_k^2-\nu_{k-1}^2)}\left\{ \left[-2\nu_k^2+(a+b)\omega^2\right]\frac{\alpha'}{\alpha} - i\nu_k\left[3\,\frac{\alpha''}{\alpha} - 4\left(\frac{\alpha'}{\alpha}\right)^2\right] \right.$$
$$+ \left[\frac{\alpha'''}{\alpha} - 5\,\frac{\alpha''\alpha'}{\alpha^2} + 4\left(\frac{\alpha'}{\alpha}\right)^3\right]$$
$$\left. - \frac{\omega^2}{i\nu_k}\left[(a+b)\,\frac{\alpha''}{\alpha} - (2a+b)\left(\frac{\alpha'}{\alpha}\right)^2\right]\right\} . \tag{13.6}$$

The boundary terms appearing after the integration by parts leading to (13.4) vanish. This follows at the lower limit since the various derivatives of $\alpha(x)$ vanish at the lower limit, $\xi = -\infty$. The vanishing of such terms at the upper limit $\xi = x$ is due to the following identity:

$$\sum_{k=1}^{4} \frac{\nu_k^n}{\nu_k^2 - \nu_{k-1}^2} = 0 \qquad n = -1,\ 0,\ 1 \ . \tag{13.7}$$

Eq. (13.4) is an integral equation for $\bar{\varphi}_j(x,\omega)$. The solution of the integral equation satisfies the differential equation (13.2) and behaves like $e_j(x)$ as x approaches $-\infty$.

We can expand (13.5) and obtain an explicit relation for $k(x,\xi;\omega)$. The nature of the kernel will depend on whether $|\omega| > \omega_c$ or $|\omega| < \omega_c$. For $|\omega| > \omega_c$,

$$(13.8)\quad k(x,\xi;\omega) = -\frac{2\alpha'/\alpha}{\nu_1^2-\nu_2^2}\left[\nu_1^2\cos\nu_1(x-\xi)-\nu_2^2\cos\nu_2(x-\xi)\right]$$

$$+\frac{\left[(a+b)\omega^2\,\alpha'/\alpha+\alpha''/\alpha-5\,\alpha''\alpha'/\alpha^2+4(\alpha'/\alpha)^3\right]}{\nu_1^2-\nu_2^2}\left[\cos\nu_1(x-\xi)-\cos\nu_2(x-\xi)\right]$$

$$+\frac{\left[3\,\alpha''/\alpha-4(\alpha'/\alpha)^2\right]}{\nu_1^2-\nu_2^2}\left[\nu_1\sin\nu_1(x-\xi)-\nu_2\sin\nu_2(x-\xi)\right]$$

$$-\frac{\omega^2\left[(a+b)\alpha''/\alpha-(2a+b)(\alpha'/\alpha)^2\right]}{\nu_1^2-\nu_2^2}\left[\frac{\sin\nu_1(x-\xi)}{\nu_1}-\frac{\sin\nu_2(x-\xi)}{\nu_2}\right].$$

For $0<|\omega|<\omega_c$

$$(13.9)\quad k(x,\xi;\omega) = -\frac{2\,\alpha'/\alpha}{|\nu_1|^2-|\nu_2|^2}\left[|\nu_1|^2\cosh|\nu_1|(x-\xi)-|\nu_2|^2\cosh|\nu_2|(x-\xi)\right]$$

$$-\frac{\left[(a+b)\omega^2\,\alpha'/\alpha+\alpha'''/\alpha-5\,\alpha''\alpha'/\alpha^2+4(\alpha'/\alpha)^3\right]}{|\nu_1|^2-|\nu_2|^2}\left[\cosh|\nu_1|(x-\xi)-\cosh|\nu_2|(x-\xi)\right]$$

$$-\frac{\left[3\,\alpha''/\alpha-4(\alpha'/\alpha)^2\right]}{|\nu_1|^2-|\nu_2|^2}-\left[|\nu_1|\sinh|\nu_1|(x-\xi)-|\nu_2|\sinh|\nu_2|(x-\xi)\right]$$

$$+\frac{\omega^2\left[(a+b)\alpha''/\alpha-(2a+b)(\alpha'/\alpha)^2\right]}{|\nu_1|^2-|\nu_2|^2}\left[\frac{\sinh|\nu_1|(x-\xi)}{|\nu_1|}-\frac{\sinh|\nu_2|(x-\xi)}{|\nu_2|}\right].$$

In later work we will be considering $\alpha(x)$ of the following form

$$\alpha(x) = [1+\epsilon\, r(x)]\,\alpha_\infty$$

where $r(x)$ approaches zero as x approaches ∞ and ϵ is small. To facilitate future work, we will use this form for $\alpha(x)$ in the remainder of this section. In the next section we will consider the case of small ϵ.

In this section the only restriction we need place on ϵ is that $\alpha(x)$ be bounded away from zero for all x.

In the following we will show that the integral equation has a solution and we will obtain bounds on the solution. For the work done in the following sections we need only bounds on the solution $\overline{\varphi}_j(x,\omega)$ for $|\omega| > \omega_c$. Therefore, we will restrict ourselves to $|\omega| > \omega_c$.

First consider x less than zero. Let

$$\overline{\varphi}_j(x,\omega) = \sum_{n=0}^{\infty} \Delta_{jn}(x,\omega) \tag{13.10}$$

where

$$\begin{cases} \Delta_{j0}(x,\omega) = e_j(x) \\ \Delta_{jn}(x,\omega) = -\int_{-\infty}^{x} k(x,\xi;\omega)\, \Delta_{jn-1}(\xi,\omega)\, d\xi \ . \end{cases} \tag{13.11}$$

We will show that (13.10), along with (13.11), provides a solution to the integral equation (13.4) for x less than zero.

For $|\omega| > \omega_c$

$$|\Delta_{j0}(x,\omega)| < 1 \ .$$

From (13.8) we obtain the following bound on $k(x,\xi;\omega)$ for $|\omega| > \omega_c$:

$$|k(x,\xi;\omega)| \leqslant \epsilon\, C\, E(\xi) \tag{13.12}$$

where

$$(13.13) \qquad E(\xi) = \max\{|r'|, |r''|, |r'|^2, |r'''|, |r'r''|, |\xi r''|, |\xi(r')^2|\} .$$

The constant C is independent of ω and ϵ. In the derivations which follow we will use the symbol C, occasionally with a subscript, to denote any constant independent of ω and ϵ.

Using (13.11), (13.12) we get the following bound on $\Delta_{j1}(x,\omega)$ for $x < 0$:

$$|\Delta_{j1}(x,\omega)| \leq \epsilon C \int_{-\infty}^{x} E(\xi)\, d\xi .$$

Substituting the above into (13.11), we get the following bound on $\Delta_{j2}(x,\omega)$:

$$|\Delta_{j2}(x,\omega)| \leq \epsilon\, C^2 \int_{-\infty}^{x} E(\xi) \int_{-\infty}^{\xi} E(\xi_1)\, d\xi_1$$

$$|\Delta_{j2}(x,\omega)| \leq \epsilon^2 C^2 \frac{1}{2!} \left[\int_{-\infty}^{x} E(\xi)\, d\xi\right]^2 .$$

Continuing in this way, we can show that

$$(13.14) \qquad |\Delta_{jn}(x,\omega)| \leq \frac{1}{n!} \left[\epsilon C \int_{-\infty}^{x} E(\xi)\, d\xi\right]^n .$$

Thus, using (13.10), we see that

$$(13.15)\qquad \begin{aligned} |\overline{\varphi}_j(x,\omega)| &\leq \sum_{n=0}^{\infty} \frac{1}{n!}\left[C\epsilon \int_{-\infty}^{x} E(\xi)d\xi\right]^n \\ |\overline{\varphi}_j(x,\omega)| &\leq \exp\left[C\epsilon \int_{-\infty}^{x} E(\xi)d\xi\right] . \end{aligned}$$

In addition we can show,

$$(13.16)\qquad |\overline{\varphi}_j(x,\omega)-e_j(x)| \leq C\epsilon \int_{-\infty}^{x} E(\xi)d\xi \,\exp\left[C\epsilon \int_{-\infty}^{x} E(\xi)d\xi\right] .$$

Thus we have some bounds on $\overline{\varphi}_j(x,\omega)$ for x less than zero.

To show that for x less than zero $\overline{\varphi}_j(x,\omega)$ as given by (13.10) does satisfy the integral equation, sum both sides of (13.11). Thus

$$(13.17)\qquad \sum_{n=0}^{\infty} \Delta_{jn}(x,\omega) = e_j(x) - \sum_{n=0}^{\infty} \int_{-\infty}^{x} k(x,\xi,\omega)\,\Delta_{jn}(\xi,\omega)d\xi .$$

Using (13.15), we see that the series given in (13.10) converges absolutely and uniformly for $|\omega| > \omega_c$. Thus we can interchange the summation and integration in (13.17) and we retrieve the integral equation.

For $x > 0$ we proceed as follows. Let

$$(13.18)\qquad \overline{\varphi}_j(x,\omega) = \sum_{n=0}^{\infty} \Delta^+_{jn}(x,\omega)$$

where now

$$(13.19)\qquad \begin{aligned} \Delta^+_{j0}(x,\omega) &= e_j(x) - \int_{-\infty}^{0} k(x,\xi);\omega)\,\overline{\varphi}_j(\xi,\omega)d\xi \\ \Delta^+_{jn}(x,\omega) &= - \int_{0}^{x} k(x,\xi;\omega)\,\Delta_{jn-1}(\xi,\omega)d\xi . \end{aligned}$$

For $x > 0$ we have the following bound on $k(x,\xi;\omega)$:

(13.20) $$|k(x,\xi;\omega)| \leq x\epsilon\, C_1 E_1(\xi) + \epsilon\, C E(\xi)$$

where $E(\xi)$ is given in (13.13), C and C_1 are constants independent of ω and ϵ, and

(13.21) $$E_1(\xi) = \max\left[|r'|,\ |r''|,\ |r'''|,\ |r'|^2,\ |r'r''|\right] .$$

Using the bound in (13.16) for $\overline{\varphi}_j(x,\omega)$ or $x < 0$, we get

$$|\Delta^+_{j0}(x,\omega)| \leq 1 + x\epsilon\, C_1 \int_{-\infty}^{x} E_1(\xi)d\xi + \epsilon\, C \int_{-\infty}^{x} E(\xi)d\xi$$

$$|\Delta^+_{j0}(x,\omega)| \leq C_2(1 + \epsilon x) .$$

Using the above in (13.19), we get the following bound on $\Delta_{j1}(x,\omega)$:

$$|\Delta^+_{j1}(x,\omega)| \leq C_2(1 + \epsilon x)\epsilon \int_0^x \left\{C_1 \xi E_1(\xi) + C E(\xi)\right\} d\xi .$$

Substituting this into (13.19) we obtain

$$|\Delta^+_{j2}(x,\omega)| \leq C_2(1+\epsilon x)\epsilon^2 \int_0^x \left[C_1 \xi E_1(\xi) + CE(\xi)\right] \int_0^{\xi} \left[C_1 \eta E_1(\eta) + CE(\eta)\right] d\eta\, d\xi$$

$$|\Delta^+_{j2}(x,\omega)| \leq C_2(1+\epsilon x)\, \frac{\left\{\epsilon \int_0^x [C_1 \xi E_1(\xi) + CE(\xi)]\, d\xi\right\}^2}{2!} .$$

Continuing in this way, we can show that

$$|\Delta^{+}_{jn}(x,\omega)| \leqslant C_2\,(1+\epsilon x)\,\frac{\{\epsilon \int_0^x [C_1 \xi E_1(\xi) + CE(\xi)]\}^n}{n!} \quad .$$

Thus, using (13.19) and the above, we can show

$$(13.22) \qquad |\overline{\varphi}_j(x,\omega)| \leqslant C_2\,(1+\epsilon x) \sum_{n=0}^{\infty} \frac{\{\epsilon \int_0^x [C_1 \xi E_1(\xi) + CE(\xi)]\, d\xi\}^n}{n!}$$

$$|\overline{\varphi}_j(x,\omega)| \leqslant C_2\,(1+\epsilon x) \exp\left\{\epsilon \int_0^x \left[C_1 \xi E_1(\xi) + CE(\xi)\right] d\xi\right\} \quad .$$

We can verify that the series in (13.18) satisfies the integral equation (13.4) for x greater than zero in exactly the same way as we did with the series in (13.10).

If we restrict ourselves to $|\omega| \geqslant \omega_0 > \omega_c$, the analysis becomes much simpler, as we do not have to contend with the singular behavior of $1/\nu_k$. Again we express the solution in terms of a series

$$\overline{\varphi}_j(x,\omega) = \sum_{n=0}^{\infty} \Delta_{jn}(x,\omega)$$

where $\Delta_{jn}(x,\omega)$ satisfies some iteration formula. If we were to successively derive bounds for $\Delta_{jn}(x,\omega)$, we would find that $\Delta_{jn}(x)$ is bounded as follows for all x:

$$|\Delta_{jn}(x,\omega)| \leqslant \frac{[\, C\epsilon \int_{-\infty}^{x} E_3(\xi) d\xi \,]^n}{n!}$$

where

$$E_3(\xi) = \max\left[\, |r'|,\ |r''|,\ |r'''| \,\right] \quad .$$

Thus, we would obtain the following bounds on $\overline{\varphi}_j(x,\omega)$ valid for $|\omega| \geqslant \omega_0 > \omega_c$:

$$|\varphi_j(x,\omega)| \leqslant \exp\left[C\epsilon \int_{-\infty}^{x} E_3(\xi)d\xi\right]$$

$$|\varphi_j(x,\omega) - e_j(x)| \leqslant C\epsilon \int_{-\infty}^{x} E_3(\xi) \exp\left[C\epsilon \int_{-\infty}^{x} E_3(\xi)d\xi\right] .$$

With this, we have completed the investigation of the solutions $\overline{\varphi}_j(x,\omega)$.

14. Reflection and Transmission Coefficents

We now move on to defining and deriving expressions for reflection and transmission coefficients for the beam. We will find that the integral equation (13.4) is quite useful to this end. It can be written as follows:

(14.1) $$\overline{\varphi}_j(x,\omega) = e_j(x) - \sum_{k=1}^{4} e_k(x) \int_{-\infty}^{x} e_k(-\xi)M_k(\xi)\overline{\varphi}_j(\xi,\omega)d\xi$$

where $M_k(\xi)$ is as given in (13.6). The solution $\overline{\varphi}_j(x,\omega)$ has the following asymptotic behavior:

(14.2) $$\begin{cases} \overline{\varphi}_j(x,\omega) \sim e_j(x) & \text{as} \quad x \to -\infty \\ \overline{\varphi}_j(x,\omega) \sim e_j(x) + \sum_{k=1}^{4} P_{jk}(\omega)\, e_k(x) & \text{as } x \to \infty \end{cases}$$

where

(14.3) $$P_{jk}(\omega) = \int_{-\infty}^{\infty} \frac{e_k(-\xi)M_k(\xi)}{\nu_k^2 - \nu_{k-1}^2} \varphi_j(\xi,\omega)\, d\xi .$$

We will first consider $|\omega| > \omega_c$. For these values of ω, the roots ν_k are real and $e_k(x) = e^{i\nu_k x}$, $k = 1, 2, 3, 4$, is a bounded function whose real and imaginary parts are cosines and sines.

For the wave equation in a non-homogeneous medium, we saw that we could interpret certain solutions as an incoming wave which was partially reflected and partially transmitted. In such a case we could define quantities called reflection and transmission coefficients which characterized the reflected and transmitted waves and depended only on the wavelength of the incoming wave.

In this section we will also define reflection and transmission coefficients for reflected and transmitted waves in a Timoshenko beam of nonuniform cross-section. We will find that there are two reflection and two transmission coefficients, instead of one each in the case of the wave equation. Finally, we will derive explicit expressions for the transmission and reflection coefficients.

We will look for solutions which look like incoming waves which have been partially transmitted and reflected. That is, as x approaches $-\infty$, we seek a solution which behaves like a sinusoidal wave, part of which is travelling to the right and part to the left. However, as x approaches ∞, we want that same solution to appear only as a sinusoidal wave propagating to the right.

The solutions $\overline{\varphi}_j(x,\omega)$ are independent. Therefore, the general solution of the differential equation (13.2) can be written as a linear combination of the solutions $\overline{\varphi}_j(x,\omega)$. Consider the following linear combination:

$$(14.4) \qquad v_j(x,\omega) = \overline{\varphi}_j(x,\omega) + \sum_{k=3}^{4} A_{jk}\, \overline{\varphi}_k(x,\omega) .$$

The function $v_j(x,\omega)e^{i\omega t}$ is a solution of the partial differential equation (11.28). As x approaches $-\infty$

$$v_j(x,\omega)e^{i\omega t} \sim e_j(x-c_1 t) + \sum_{k=3}^{4} A_{jk}\, e_k(x-c_k t)$$

where c_j and c_k are the phase velocities defined earlier. Thus, as x approaches $-\infty$ for $j=1,2$, $v_j(x,\omega)e^{i\omega t}$ looks like a sinusoidal wave propagating in the positive x-direction with speed $c_j > 0$ and two reflected sinusoidal waves propagating to the left with speed c_1 and c_2 (note $c_1 = -c_3$, $c_2 = -c_4$). Sinusoidal waves propagating with speed c_1 associated with the first mode of vibration will be referred to as "type I" waves. Sinusoidal waves propagating with speed c_2 associated with the second mode of vibration will be referred to as "type II" waves.

As x approaches ∞ for $j = 1, 2$, we want $v_j(x,\omega)$ to be composed only of transmitted sinusoidal waves, i.e. waves propagating to the right only. Therefore, we will choose A_{j3}, A_{j4} so that any waves travelling to the left will not be present. Using (14.4), the asymptotic approximation (14.2) of $\overline{\varphi}_j(x,\omega)$ as x approaches ∞, and the requirement that no waves be propagating to the left as x approaches ∞, we find that A_{j3} and A_{j4} must satisfy the following:

$$A_{jk} + \sum_{m=3}^{4} A_{jm} P_{mk} = -P_{jk} \qquad \begin{array}{l} j = 1, 2 \\ k = 3, 4 \end{array} \quad . \tag{14.5}$$

For each fixed j we have two simultaneous linear inhomogeneous equations in the unknowns A_{j3} and A_{j4}. Solving the system of equations in (14.5), we get

$$\begin{aligned} A_{j3} &= -\frac{P_{j3}(1+P_{44}) - P_{j4}P_{43}}{(1+P_{33})(1+P_{44})-P_{34}P_{43}} \\ A_{j4} &= -\frac{P_{j4}(1+P_{33})-P_{j3}P_{34}}{(1+P_{33})(1+P_{44})-P_{34}P_{43}} \end{aligned} \tag{14.6}$$

as long as the denominator does not vanish (we will investigate the possibility of the denominator vanishing later).

With the above choice for A_{jk} we find that $v_j(x)$ has the following asymptotic behavior

$$\begin{cases} v_j(x,\omega) \sim e_j(x) + \sum_{k=1}^{2} R_{jk}(\omega) e_{k+2}(x) & \text{as } x \to -\infty \\ v_j(x,\omega) \sim \sum_{k=1}^{2} T_{jk}(\omega)\, e_{k+2}(x) & \text{as } x \to \infty \end{cases} \tag{14.7}$$

where

$$R_{jk}(\omega) = A_{jk+2} \quad , \tag{14.8}$$

$$T_{jk}(\omega) = \delta_{jk} + P_{jk} + \sum_{m=3}^{4} A_{jm} P_{mk} \tag{14.9}$$

where again j = 1, 2 and k = 1, 2. We shall refer to R_{jk} as reflection coefficients and to T_{jk} as transmission coefficients.

Note that we have a total of two reflection and two transmission coefficients. Compare this with the wave equation of a nonhomogeneous medium for which we have only one reflection and one transmission coefficient. For the Timoshenko beam equation we see the possibility that a sinusoidal wave propagating from $x = -\infty$ can generate two reflected waves and two transmitted waves of different wavelengths.

15. An Approximation to the Reflection Coefficient

In this section we examine one particular reflection coefficient, namely $R_{22}(\omega)$. This reflection coefficient is associated with the reflected type II wave generated by an incident or incoming type II wave from $x = -\infty$. We restrict attention to the case in which the non-uniformity in the beam is small, so that with

$$\alpha(x) = \left[1 + \epsilon\, r(x)\right] \alpha_{\infty} .$$

We shall require $\epsilon << 1$. The quantity ϵ is our perturbation parameter. In this section we will derive a perturbation expansion for $R_{22}(\omega)$ in terms of ϵ.

From (14.6) and (14.8) we can derive the following expression for $R_{22}(\omega)$,

$$(15.1) \qquad R_{22}(\omega) = - \frac{P_{24}(\omega)\left[1+P_{33}(\omega)\right] - P_{23}(\omega)\, P_{34}(\omega)}{\left[1+P_{33}(\omega)\right]\left[1+P_{44}(\omega)\right] - P_{34}(\omega)\, P_{43}(\omega)}$$

where from (14.3)

$$(15.2) \qquad P_{jk}(\omega) = -\int_{-\infty}^{\infty} e_k(-\xi)\, M_k(\xi)\, \overline{\varphi}_j(x,\omega)\, d\xi$$

and

$$(15.3) \quad M_k(\xi) = \frac{1}{2(\nu_k^2 - \nu_{k-1}^2)} \Bigg\{ \left[-2\nu_k^2 + (a+b)\omega^2\right] \frac{\alpha'}{\alpha} - i\nu_k \left[3\,\frac{\alpha''}{\alpha} - 4\left(\frac{\alpha'}{\alpha}\right)^2\right]$$

$$+ \frac{\alpha'''}{\alpha} - 5\,\frac{\alpha''\alpha'}{\alpha'} + 4\left(\frac{\alpha'}{\alpha}\right)^3 - \frac{\omega^2}{i\nu_k}\left[(a+b)\,\frac{\alpha''}{\alpha} - (2a+b)\left(\frac{\alpha'}{\alpha}\right)^2\right] \Bigg\} .$$

Writing $\alpha(x) = \left[1 + \epsilon\, r(x)\right]\alpha_\infty$, we have the following equivalent forms

$$\frac{\alpha'(x)}{\alpha(x)} = \frac{\epsilon\, r'(x)}{1+\epsilon\, r(x)} = \epsilon\, r'(x)\,[1 + O(\epsilon)]$$

$$\frac{\alpha''(x)}{\alpha(x)} = \frac{\epsilon\, r''(x)}{1+\epsilon\, r(x)} = \epsilon\, r''(x)\,[1 + O(\epsilon)]$$

$$\frac{\alpha'''(x)}{\alpha(x)} = \frac{\epsilon\, r'''(x)}{1+\epsilon\, r(x)} = \epsilon\, r'''(x)\,[1 + O(\epsilon)]$$

as $\epsilon \to 0$, uniformly in x for all x. In addition, in the last section we showed that $\overline{\varphi}_j(x,\omega) = O(1)$ as $\epsilon \to 0$ uniformly in x and ω.

We are interested in approximating the reflection coefficient $R_{22}(\omega)$ for $|\omega| \geq \omega_c$. Note that for $\omega = \omega_c$ the roots ν_2 and ν_4 vanish. The function $M_k(\xi)$ contains terms which have factors $1/\nu_k$. For $k = 2, 4$ these terms become infinitely large as $|\omega|$ approaches ω_c. As a first step, we will work toward approximating $R_{22}(\omega)$ for $|\omega|$ bounded away from ω_c; that is

$$|\omega| \geq \omega_0 > \omega_c \quad .$$

In this case we can show that ν_k and $\nu_k^2 - \nu_{k-1}^2$ are bounded away from zero.

Substituting the above into the expression (15. 2), we can show the following for $P_{jk}(\omega)$:

(15. 4) $$P_{jk}(\omega) = O(\epsilon)$$

as $\epsilon \to 0$, uniformly in ω for $|\omega| \geq \omega_0 > \omega_c$. Thus, using (15.1), we can show that

$$R_{22}(\omega) = \frac{-P_{24}(\omega) + O(\epsilon^2)}{1 + O(\epsilon)}$$

or

(15. 5) $$R_{22}(\omega) = -P_{24}(\omega) + O(\epsilon^2) \quad .$$

as $\epsilon \to 0$ uniformly in ω. Thus, we are left with the problem of approximating $P_{24}(\omega)$.

Instead of approximating $P_{24}(\omega)$, we will work toward approximating the general term $P_{jk}(\omega)$. We will find that $M_k(\xi)$ can be expressed in a form which will simplify later work. First we note that

$$\frac{\alpha''}{\alpha} = \left(\frac{\alpha'}{\alpha}\right)' + \left(\frac{\alpha'}{\alpha}\right)^2 ,$$

$$\frac{\alpha'''}{\alpha} = \left(\frac{\alpha'}{\alpha}\right)'' + 3\,\frac{\alpha''\alpha'}{\alpha^2} - 2\left(\frac{\alpha'}{\alpha}\right)^3 .$$

Using the above expressions, we write $M_k(\xi)$ as follows:

(15.6)
$$M_k(\xi) = \frac{1}{2(\nu_k^2-\nu_{k-1}^2)}\left\{ \left[-2\nu_k^2 + (a+b)\omega^2\right]\frac{\alpha'}{\alpha} - i\nu_k\left[3\left(\frac{\alpha'}{\alpha}\right)' - \left(\frac{\alpha'}{\alpha}\right)^2\right] + \left(\frac{\alpha'}{\alpha}\right)'' - 2\,\frac{\alpha''\alpha'}{\alpha^2} + 2\left(\frac{\alpha'}{\alpha}\right)^3 - \frac{\omega^2}{i\nu_k}\left[(a+b)\left(\frac{\alpha'}{\alpha}\right)' - a\left(\frac{\alpha'}{\alpha}\right)^2\right]\right\} .$$

Using (15.6) and replacing $\overline{\varphi}_j(x,\omega)$ by $e_j(x)$, we can divide up the expression (15.5) for $P_{jk}(\omega)$ as follows:

(15.7)
$$P_{jk}(\omega) = \overset{1}{I}_{jk}(\omega) + \overset{2}{I}_{jk}(\omega) + \overset{3}{I}_{jk}(\omega)$$

where

(15.8) $$I^{1}_{jk}(\omega) = -\int_{-\infty}^{\infty} e_k(-\xi)\, M_k(\xi)\, e_j(\xi)\, d\xi \ ,$$

(15.9) $$I^{2}_{jk}(\omega) = -\int_{-\infty}^{\infty} e_k(-\xi)\, M_{k1}(\xi) \left[\overline{\varphi}_j(\xi,\omega) - e_j(\xi)\right] d\xi \ ,$$

(15.10) $$I^{3}_{jk}(\omega) = -\int_{-\infty}^{\infty} e_k(-\xi)\, M_{k2}(\xi)\, \overline{\varphi}_j(\xi,\omega)\, d\xi \ ,$$

(15.11) $$M_{k1}(\xi) = \frac{1}{2(\nu_k^2 - \nu_{k-1}^2)} \left\{ \left[-2\nu_k^2 + (a+b)\omega^2\right] \frac{\alpha'}{\alpha} - 3i\nu_k \left(\frac{\alpha'}{\alpha}\right)' + \left(\frac{\alpha'}{\alpha}\right)'' - \frac{\omega^2}{i\nu_k}(a+b)\left(\frac{\alpha'}{\alpha}\right)' \right\}$$

(15.12) $$M_{k2}(\xi) = \frac{1}{2(\nu_k^2 - \nu_{k-1}^2)} \left\{ i\nu_k \left(\frac{\alpha'}{\alpha}\right)^2 - 2\left[\frac{\alpha''\alpha'}{\alpha^2} - \left(\frac{\alpha'}{\alpha}\right)^3\right] + \frac{a\omega^2}{i\nu_k}\left(\frac{\alpha'}{\alpha}\right)^2 \right] \right\}$$

(15.13) $$M_k(\xi) = M_{k1}(\xi) + M_{k2}(\xi) \ .$$

First we will consider $I^{1}_{jk}(\omega)$. The order of the derivatives of $\frac{\alpha'}{\alpha}$ can be reduced by integrating by parts. As $|\xi|$ approaches infinity, the terms $\frac{\alpha'}{\alpha}$ and $\left(\frac{\alpha'}{\alpha}\right)'$ vanish so that the integrated terms vanish. Thus, the expression (15.8) can be simplified to the following after the integration by parts.

(15.14) $$I^{1}_{jk}(\omega) = \frac{\nu_j(\nu_k + \nu_j) - \omega^2 (a+b)\, \nu_j/\nu_k}{2(\nu_k^2 - \nu_{k-1}^2)} \int_{-\infty}^{\infty} e_j(\xi)\, e_k(-\xi)\, \frac{\alpha'}{\alpha}\, d\xi$$

We showed in the last section that

$$\varphi_j(x,\omega) - e_j(x) = O(\epsilon) \qquad \text{as } \epsilon \to 0 \, ,$$

uniformly in x and ω .

Thus we can show that

$$(15.15) \qquad \begin{cases} R_{22}(\omega) = -I^{1}_{24}(\omega) + O(\epsilon^2) \\ R_{22}(\omega) = \dfrac{\omega^2 (a+b)}{2(\nu_1^2 - \nu_2^2)} \displaystyle\int_{-\infty}^{\infty} e_4(-\xi) e_2(\xi) \frac{\alpha'}{\alpha} d\xi + O(\epsilon^2) \\ R_{22}(\omega) = \dfrac{\omega^2 (a+b)}{2(\nu_1^2 - \nu_2^2)} \displaystyle\int_{-\infty}^{\infty} e^{2i\nu_2 \xi} \frac{\alpha'}{\alpha} d\xi + O(\epsilon^2) \end{cases}$$

as $\epsilon \to 0$ uniformly in ω for $|\omega| \geq \omega_0 > \omega_c$.

As $|\omega|$ approaches ω_c, we must be more careful in our approximation of $R_{22}(\omega)$. We see that for $|\omega| = \omega_c$ the roots ν_2 and ν_4 vanish. Thus any term in $R_{22}(\omega)$ which contains a factor $1/\nu_2$ or $1/\nu_4$ will become infinite. However, the terms ν_1, ν_3 , and $\nu_k^2 - \nu_{k-1}^2$ for all k remain bounded away from zero.

Using (15.14), (15.10) and (15.9), we can show that

$$P_{23}(\omega) = \frac{\nu_2[(\nu_2-\nu_1)+\omega^2(a+b)\frac{1}{\nu_1}]}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e^{i(\nu_1+\nu_2)\xi} \frac{\alpha'}{\alpha} d\xi + O(\epsilon^2)$$

$$P_{43}(\omega) = \frac{\nu_2[(\nu_2+\nu_1)-\omega^2(a+b)\frac{1}{\nu_1}]}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e^{i(\nu_1-\nu_2)\xi} \frac{\alpha'}{\alpha} d\xi + O(\epsilon^2)$$

(15.16)

$$P_{34}(\omega) = -\frac{\nu_1[(\nu_1+\nu_2)-\omega^2(a+b)\frac{1}{\nu_2}]}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e^{i(\nu_2-\nu_1)\xi} \frac{\alpha'}{\alpha} d\xi + O(\epsilon^2)$$

$$P_{33}(\omega) = \frac{2\nu_1^2-\omega^2(a+b)}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} \frac{\alpha'}{\alpha} d\xi + O(\epsilon^2)$$

as $\epsilon \to 0$ uniformly in ω for $|\omega| \geqslant \omega_c$. Combining the above, we have

(15.17) $$P_{23}(\omega)\, P_{34}(\omega) = O(\epsilon^2) + O(\frac{\epsilon^3}{\nu_2})$$

(15.18) $$P_{43}(\omega)\, P_{34}(\omega) = O(\epsilon^2) + O(\frac{\epsilon^3}{\nu_2}) .$$

Again the order relations hold as $\epsilon \to 0$ uniformly in ω for $|\omega| > \omega_c$.

Let us look more closely at $P_{24}(\omega)$. First, let us consider $I_{24}^{3}(\omega)$. We can write this term as follows,

$$I_{24}^{3}(\omega) = -\int_{-\infty}^{\infty} e_4(-\xi)\, M_{k2}(\xi)\, e_2(\xi)\, d\xi$$

$$- \int_{-\infty}^{\infty} e_4(-\xi)\, M_{k2}(\xi) \left[\overline{\varphi}_2(\xi),\omega)-e_2(\xi)\right] d\xi$$

$$I_{24}^{3}(\omega) = -\int_{-\infty}^{\infty} M_{k2}(\xi)\, d\xi + O(\frac{\epsilon^3}{\nu_2})$$

as $\epsilon \to 0$, uniformly in ω for $|\omega| \geq \omega_c$. Now isolating those terms which become infinite in $I_{24}^3(\omega)$ we have

$$(15.19) \qquad I_{24}^3(\omega) = \frac{a\omega^2}{2i\nu_2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} \left(\frac{\alpha'}{\alpha}\right)^2 d\xi + O(\epsilon^2) + O\left(\frac{\epsilon^3}{\nu_2}\right)$$

as $\epsilon \to 0$, uniformly in ω for $|\omega| \geq \omega_c$.

Next we look at the term $I_{24}^2(\omega)$. Isolating the term which becomes unbounded, we can write $I_{24}^2(\omega)$ as follows,

$$(15.20) \; I_{24}^2(\omega) = \frac{(a+b)\omega^2}{2i\nu_2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e_4(-\xi)\, \frac{\alpha'}{\alpha}\left[\overline{\varphi}_2(\xi,\omega)-e_2(\xi)\right] d\xi + O(\epsilon^2)$$

uniformly in ω for $|\omega| \geq \omega_c$. To continue further, we will have to take a closer look at the expression $\overline{\varphi}_2(x,\omega)-e_2(x)$.

Using the integral equation (13.4), we can write the following:

$$\overline{\varphi}_2(x,\omega)-e_2(x) = -\int_{-\infty}^{x} \sum_{k=1}^{4} M_k(\xi)\, e_k(x-\xi)\, \overline{\varphi}_2(\xi,\omega)\,d\xi$$

$$\overline{\varphi}_2(x,\omega)-e_2(x) = J_1(x,\omega) + J_2(x,\omega)$$

where

$$J_i(x,\omega) = -\int_{-\infty}^{x} \sum_{k=1}^{\infty} M_{ki}(\xi)\, e_k(x-\xi)\, \overline{\varphi}_2(\xi,\omega)\, d\xi \qquad i = 1, 2$$

and $M_{ki}(\xi)$ is given in (15.11) and (15.12).

Let us consider first the term $J_2(x,\omega)$. Using (15.12) we can write $J_2(x,\omega)$ as follows:

$$(15.21)\quad J_2(x,\omega) = -\int_{-\infty}^{x} \sum_{k=1}^{4} \frac{e_k(x-\xi)}{2(\nu_k^2-\nu_{k-1}^2)} \left\{ i\nu_k\left(\frac{\alpha'}{\alpha}\right)^2 - 2\left[\frac{\alpha''\alpha'}{\alpha^2} - \left(\frac{\alpha'}{\alpha}\right)^3\right] \right\} \bar{\varphi}_2(\xi,\omega)\,d\xi$$

$$- \int_{-\infty}^{x} \frac{i a\omega^2 \left(\frac{\alpha'}{\alpha}\right)^2}{(\nu_1^2-\nu_2^2)} \frac{\sin\nu_1(x-\xi)}{\nu_1} \bar{\varphi}_2(\xi,\omega)\,d\xi$$

$$+ \int_{-\infty}^{x} \frac{i a\omega^2 \left(\frac{\alpha'}{\alpha}\right)^2}{(\nu_1^2-\nu_2^2)} \frac{\sin\nu_2(x-\xi)}{\nu_2} \bar{\varphi}_2(\xi,\omega)\,d\xi \;.$$

Since ν_1 is bounded away from zero for $|\omega| \geq \omega_c$, we can show that the first two terms on the right hand side of (15.21) are $O(\epsilon^2)$ uniformly in x and ω for $|\omega| \geq \omega_c$. As for the third term, using the bound

$$\left| \frac{\sin\nu_2(x-\xi)}{\nu_2} \right| \leq (x-\xi)$$

we have

$$J_2(x,\omega) = O(\epsilon^2)$$

for $\epsilon << 1$ uniformly in x and ω for $|\omega| \geq \omega_c$.

Continuing to $J_1(x,\omega)$, we can write this as follows:

$$J_1(x,\omega) = -\int_{-\infty}^{x} \sum_{k=1}^{4} M_{k1}(\xi)\, e_k(x-\xi)\, e_2(\xi)\, d\xi$$

$$-\int_{-\infty}^{x} \sum_{k=1}^{4} M_{k1}(\xi)\, e_k(x-\xi) \left[\bar{\varphi}_2(\xi,\omega) - e_2(\xi)\right] d\xi \;.$$

The second term on the right hand side can be approximated in much the same way that the term $J_2(x,\omega)$ was approximated. In fact, we can show that the second term is $(|x|+1)\,O(\epsilon^2)$. The first term on the right is integrated by parts once, giving us the following for $J_1(x,\omega)$:

$$J_1(x,\omega) = -\sum_{k=1}^{4} e_2(x)\left\{\frac{[-3i\nu_k - i(\nu_2-\nu_k) - \frac{\omega^2}{i\nu_k}(a+b)]\frac{\alpha'}{\alpha}}{2(\nu_k^2-\nu_{k-1}^2)} - \frac{(\frac{\alpha'}{\alpha})'}{2(\nu_k^2-\nu_{k-1}^2)}\right\}$$

$$-\int_{-\infty}^{x}\sum_{k=1}^{4}\frac{\nu_2(\nu_k+\nu_2)-\omega^2(a+b)\frac{\nu_2}{\nu_k}}{\nu_k^2-\nu_{k-1}^2}\, e_k(x-\xi)e_2(\xi)d\xi$$

$$+(|x|+1)\,O(\epsilon^2) \qquad \text{as } \epsilon\to 0,$$

uniformly in x and ω for $|\omega| \geq \omega_c$. If we sum the first term, we find that it vanishes. In the second term we isolate the parts of the term which do not vanish. Thus we find that $J_1(x,\omega)$, and thus $\overline{\varphi}_2(x,\omega)-e_2(x)$, can be expressed as follows:

$$(15.22)\quad J_1(x,\omega) = \frac{(a+b)\omega^2}{2(\nu_1^2-\nu_2^2)}\left[e_2(-x)\int_{-\infty}^{x}\frac{\alpha'}{\alpha}e_2(2\xi)d\xi - e_2(x)\int_{-\infty}^{x}\frac{\alpha'}{\alpha}d\xi\right]$$

$$+\nu_2\,O(\epsilon) + (|x|+1)\,O(\epsilon^2)$$

$$(15.23)\qquad \overline{\varphi}_2(x,\omega)-e_2(\xi) = J_1(x,\omega) + (|x|\,'1)\,O(\epsilon^2)$$

as $\epsilon\to 0$, uniformly in x and ω for $|\omega| \geq \omega_c$.

Substituting (15.22) and (15.23) into (15.20), we find that

$$(15.24)\quad I_{24}^{2}(\omega) = -\frac{\omega^4 (a+b)^2}{4i\nu_2(\nu_1^2-\nu_2^2)}\left[\int_{-\infty}^{\infty}\left(\frac{\alpha'}{\alpha}\right)' e_2(2\xi)\int_{-\infty}^{\xi}\frac{\alpha'}{\alpha}\,d\eta\,d\xi - \int_{-\infty}^{\infty}\left(\frac{\alpha'}{\alpha}\right)'\int_{-\infty}^{\xi}\frac{\alpha'}{\alpha}\,e_2(2\eta)d\eta\,d\xi\right] + O(\epsilon^2) + \frac{1}{\nu_2}O(\epsilon)$$

uniformly in ω for $|\omega| > \omega_c$. We can simplify (15.24) further by integration by parts. In fact, we can show

$$\int_{-\infty}^{\infty}\left(\frac{\alpha'}{\alpha}\right)' e_2(2\xi)\int_{-\infty}^{\xi}\frac{\alpha'}{\alpha}\,d\eta\,d\xi = -\int_{-\infty}^{\infty}\frac{\alpha'}{\alpha}\left[2i\nu_2\int_{-\infty}^{\xi}\frac{\alpha'}{\alpha}\,d\eta + \frac{\alpha'}{\alpha}\right]e_2(2\xi)d\xi$$

$$\int_{-\infty}^{\infty}\left(\frac{\alpha'}{\alpha}\right)'\int_{-\infty}^{\xi}\frac{\alpha'}{\alpha}\,e_2(2\eta)d\eta\,d\xi = -\int_{-\infty}^{\infty}\left(\frac{\alpha'}{\alpha}\right)^2 e_2(2\xi)d\xi\ .$$

Combining the above equations with (15.24), we can establish the following:

$$(15.25)\qquad I_{24}^{2}(\omega) = O(\epsilon^2) + \frac{1}{\nu_2}O(\epsilon^3)$$

uniformly in ω for $|\omega| \geq \omega_c$.

Finally, using (15.14), we obtain the following for $I_{24}^{1}(\omega)$:

$$(15.26)\qquad I_{24}^{1}(\omega) = -\frac{(a+b)\omega^2}{(2\nu_1^2-\nu_2^2)}\int_{-\infty}^{\infty} e^{2i\nu_2\xi}\,\frac{\alpha'}{\alpha}\,d\xi.$$

Combining Eqs. (15.26), (15.25), and (15.20), we have the following for $P_{24}(\omega)$:

$$(15.27) \quad P_{24}(\omega) = -\frac{(a+b)\omega^2}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e^{2i\nu_2\xi} \frac{\alpha'}{\alpha} d\xi$$

$$+ \frac{a\omega^2}{2i\nu_2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} \left(\frac{\alpha'}{\alpha}\right)^2 d\xi + O(\epsilon^2) + O\left(\frac{\epsilon^3}{\nu_2}\right) .$$

The analysis of $P_{44}(\omega)$ proceeds in the same way as the preceding analysis of $P_{24}(\omega)$, and the details will be omitted. We obtain the following for $P_{44}(\omega)$:

$$(15.28) \quad P_{44}(\omega) = -\frac{[2\nu_2^2-(a+b)\omega^2]}{2(\nu_1^2-\nu_2^2)} \log\frac{\alpha_{\infty}}{\alpha_{-\infty}}$$

$$+ \frac{a\omega^2}{2i\nu_2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} \left(\frac{\alpha'}{\alpha}\right)^2 d\xi + O(\epsilon^2) + O\left(\frac{\epsilon^3}{\nu_2}\right) .$$

Using the various approximations which we have obtained for the terms $P_{jk}(\omega)$, we derive the following expression for $R_{22}(\omega)$:

$$(15.29) \qquad R_{22}(\omega) = \frac{A(\omega)+B_1(\omega)}{1+O(\epsilon)+B_2(\omega)}$$

$$A(\omega) = -\frac{(a+b)\omega^2}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e^{2i\nu_2\xi} \frac{\alpha'}{\alpha} d\xi + O(\epsilon^2)$$

$$B_1(\omega) = \frac{a\omega^2}{2i\nu_2(\nu_1^2-\nu_2^2)} \left[\int_{-\infty}^{\infty} \left(\frac{\alpha'}{\alpha}\right)^2 d\xi + O(\epsilon^3)\right]$$

$$B_2(\omega) = B_1(\omega) + O\left(\frac{\epsilon^3}{\nu_2}\right)$$

the order relations holding uniformly in ω for $|\omega| \geq \omega_c$. Note that

$$\int_{-\infty}^{\infty} \left(\frac{\alpha'}{\alpha}\right)^2 d\xi = O(\epsilon^2) \ .$$

With a little modification, we can write (15.29) as follows:

(15.30) $$R_{22}(\omega) = -\frac{(a+b)\omega^2}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e^{2i\nu_2\xi} \frac{\alpha'}{\alpha} d\xi + E(\omega,\epsilon)$$

(15.31) $$E(\omega,\epsilon) = \frac{i\frac{\epsilon^2}{\nu_2} B_3(\omega)[1+O(\epsilon)]}{1+O(\epsilon)+i\frac{\epsilon^2}{\nu_2} B_3(\omega)[1+O(\epsilon)]}$$

$$= \frac{i\frac{\epsilon^2}{\nu_2} B_3(\omega)}{1+i\frac{\epsilon^2}{\nu_2} B_3(\omega)} \Big[1+O(\epsilon)\Big]$$

$$= \frac{i\epsilon^2 B_3(\omega)}{\nu_2+i\epsilon^2 B_3(\omega)} \Big[1+O(\epsilon)\Big]$$

where

(15.32) $$B_3(\omega) = \frac{-a\omega^2}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} (r')^2 \, d\xi \ .$$

Note that the denominator of $E(\omega,\epsilon)$ in (15.31) is a complex quantity. The real part of the denominator for $|\omega| \geq \omega_c$ is ν_2, while the imaginary part is $\epsilon^2 B_3(\omega)$. The magnitude of the denominator can be bounded below by either the magnitude of its real part or the magnitude of its imaginary part. Since $B_3(\omega)$ is bounded away from zero for finite $|\omega| \geq \omega_c$, we have that the denominator of $E(\omega,\epsilon)$ can never vanish for finite $|\omega| \geq \omega_c$. In addition, we have the following two bounds on $E(\omega,\epsilon)$:

(15.33) $$E(\omega,\epsilon) = 1 + O(\epsilon)$$

(15.34) $$E(\omega,\epsilon) = O\left(\frac{\epsilon^2}{\nu_2}\right)$$

uniformly in ω for $|\omega| \geq \omega_c$.

16. The Approximation for High Frequency

We shall need bounds on the solution $\varphi_j(x,\omega)$ for large values of $|\omega|$. To obtain the bounds we return to the original fourth order differential equation (13.2) and make a change from the dependent variable $\varphi_j(x,\omega)$ to a new dependent variable $\psi_j(x,\omega)$. The differential equation in terms of the new unknown will be transformed into an integral equation. This integral equation will be used to obtain a bound on $\varphi_j(x,\omega)$.

Let $\psi_j = \sqrt{\alpha}\,\varphi_j$. Substituting this into Eq. (13.2), the original differential equation becomes

$$(16.1)\quad \frac{d^4\psi_j}{dx^4} + (a+b)\omega^2 \frac{d^2\psi_j}{dx^2} + b\,\omega^2\left(a\,\omega^2 - \frac{\beta}{\alpha}\right)\psi_j = L_1\,\psi_j$$

where

$$(16.2)\quad \begin{aligned} L_1\psi = -\Big\{ &\frac{1}{2}\left(\frac{\alpha'}{\alpha}\right)^2 \frac{d^2\psi}{dx^2} + \left[-\frac{\alpha'''}{\alpha} + 2\,\frac{\alpha''\alpha'}{\alpha^2} - \left(\frac{\alpha'}{\alpha}\right)^3\right]\frac{d\psi}{dx} \\ &+ \left[-\frac{1}{2}\,\frac{\alpha''''}{\alpha} + \frac{3}{2}\,\frac{\alpha'''\alpha'}{\alpha^2} - \frac{3}{4}\left(\frac{\alpha''}{\alpha}\right)^2 - 3\,\frac{\alpha''(\alpha')^2}{\alpha^3} + \frac{21}{16}\left(\frac{\alpha'}{\alpha}\right)^4\right]\psi \\ &+ \omega^2\left[-\frac{1}{2}(a+b)\,\frac{\alpha''}{\alpha} - \frac{(5a+b)}{4}\left(\frac{\alpha'}{\alpha}\right)^2\right]\psi\Big\}\,. \end{aligned}$$

Compare the differential equation (16.1) with the differential equation (13.2). The left hand sides of the two equations have exactly the same form. We can transform the differential equation (16.1) into an integral equation in exactly the same way that we did the differential equation in (13.2). We get

$$(16.3)\quad \psi_j(x,\omega) = \sqrt{\alpha_{-\infty}}\; e_j(x) + \sum_{k=1}^{4} \int_{-\infty}^{x} e_k(x-\xi)\, \frac{L_1\,\psi_j(\xi,\omega)}{2_i\,\nu_k(\nu_k^2 - \nu_{k-2}^2)}\, d\xi\ .$$

Note that as x approaches $-\infty$

$$\psi_j(x,\omega) = \sqrt{\alpha(x)}\,\varphi_j(x,\omega) \sim \sqrt{\alpha_{-\infty}}\; e_j(x)$$

where $\alpha_{-\infty}$ is the limit of $\alpha(x)$ as x approaches $-\infty$.

Next we remove the derivatives of the $\psi_j(x,\omega)$ in $L_1\,\psi_j(x,\omega)$ appearing in the integral in the integral equation (16.3). We accomplish this by integrating by parts in the appropriate way. Carrying out the integration by parts and re-expressing in terms of $\varphi_j(x,\omega)$, we get

$$\text{(16.4)} \qquad \sqrt{\alpha(x)}\,\bar{\varphi}_j(x,\omega) = \sqrt{\alpha_{-\infty}}\; e_j(x) - \int_{-\infty}^{x} K_1(x,\xi;\omega)\bar{\varphi}_j(\xi,\omega)d\xi$$

where

$$K_1(x,\xi,\omega) = \sum_{k=1}^{4} e_k(x-\xi)\sqrt{\alpha(\xi)}\; N_k(\xi)$$

$$\text{(16.5)} \qquad N_k(\xi) = \frac{-1}{2i\nu_k(\nu_k^2-\nu_{k-1}^2)}\left\{\frac{1}{2}\frac{\alpha''''}{\alpha} - \frac{5}{2}\frac{\alpha'''\alpha'}{\alpha^2} - \frac{9}{4}\left(\frac{\alpha''}{\alpha}\right) + 10\frac{\alpha''(\alpha')^2}{\alpha^3}\right.$$
$$- \frac{75}{16}\left(\frac{\alpha'}{\alpha}\right)^4 + 2\nu_k\left[-\frac{\alpha'''}{\alpha} + 4\frac{\alpha''\alpha'}{\alpha^2} - 3\left(\frac{\alpha'}{\alpha}\right)^3\right] + \nu_k^2\left(\frac{\alpha'}{\alpha}\right)^2$$
$$\left. - \frac{1}{2}\omega^2\left[(a+b)\frac{\alpha''}{\alpha} + \frac{(5a+b)}{4}\left(\frac{\alpha'}{\alpha}\right)^2\right]\right\}.$$

For $|\omega|$ sufficiently large and bounded away from zero and ω_c we have, using earlier asymptotic estimates,

$$|\nu_k| \le C\,|\omega|$$

$$\left|\frac{1}{2i\nu_k(\nu_k^2 - \nu_{k-1}^2)}\right| \le \frac{C_1}{|\omega|^3}$$

Using these estimates, we can obtain the following bound for $K_1(x, \xi;\omega)$:

$$(16.6) \qquad |K_1(x, \xi;\omega)| \le \frac{C}{|\omega|} E_2(\xi)$$

where

$$E_2(\xi) = \max(|\alpha''''|, |\alpha'''|, |\alpha''|, |\alpha'|) .$$

The constants C and C_1 are independent of x, ξ, and ω.

We have shown that the function $\bar{\varphi}_j(x, \omega)$ is bounded for all values of ω. Thus we can show, using (16.4) and (16.6) that

$$(16.7) \qquad \left|\sqrt{\alpha(x)}\, \bar{\varphi}_j(x, \omega) - \sqrt{\alpha_{-\infty}}\, e_j(x)\right| \le \epsilon \frac{C_2}{|\omega|} \int_{-\infty}^{x} E_2(\xi)d\xi$$

$$\sqrt{\alpha(x)}\, \bar{\varphi}_j(x, \omega) - \sqrt{\alpha_{-\infty}}\, e_j(x) = \frac{1}{|\omega|} O(\epsilon^2)$$

uniformly in x and ω for all x and large $|\omega|$.

Eq. (16.4) can be rewritten as follows:

$$(16.8) \quad \sqrt{\frac{\alpha(x)}{\alpha_{-\infty}}}\, \bar{\varphi}_j(x, \omega) = e_j(x) - \sum_{k=1}^{4} e_k(x) \int_{-\infty}^{x} e_k(-\xi)\sqrt{\frac{\alpha(\xi)}{\alpha_{-\infty}}}\, N_k(\xi)\bar{\varphi}_j(\xi, \omega)d\xi .$$

As x approaches minus infinity, (16.8) has the following asymptotic behavior:

$$(16.9) \qquad \sqrt{\frac{\alpha(x)}{\alpha_{-\infty}}}\, \bar{\varphi}_j(x, \omega) \sim e_j(x) .$$

As x approaches plus infinity, (16.8) behaves as follows:

$$(16.10) \qquad \sqrt{\frac{\alpha(x)}{\alpha_{-\infty}}}\, \bar{\varphi}_j(x, \omega) = e_j(x) + \sum_{k=1}^{4} e_k(x)\, Q_{jk}(\omega)$$

$$(16.11) \qquad Q_{jk}(\omega) = \frac{1}{\sqrt{\alpha_{-\infty}}} \int_{-\infty}^{x} e_k(-\xi)\sqrt{\alpha(\xi)}\, N_k(\xi)\bar{\varphi}_j(\xi, \omega)d\omega .$$

We compare the asymptotic forms (16.9) and (16.10) with the asymptotic forms (14.2). As we did earlier, we can multiply $\sqrt{\alpha(x)/\alpha_{-\infty}}\ \bar{\varphi}_j(x,\omega)$ by $e^{i\omega t}$ and, for large $|x|$, we can interpret incident, reflected, and transmitted sinusoidal waves. We consider the linear combination

(16.12) $$w_j(x) = \sqrt{\frac{\alpha(x)}{\alpha_{-\infty}}}\ \bar{\varphi}_j(x,\omega) + \sum_{k=3}^{4} B_{jk}(\omega)\sqrt{\frac{\alpha(x)}{\alpha_{-\infty}}}\ \bar{\varphi}_j(x,\omega)\ .$$

As x approaches minus infinity, $e^{i\omega t}\ w_j(x)$ appears as an outgoing type I sinusoidal wave propagating to the right and a type I and type II sinusoidal wave returning to the left.

As x approaches +x, we choose $B_{jk}(\omega)$ so that $e^{i\omega t}\ w_j(x,t)$ is composed of sinusoidal waves propagating to the right only. This condition on $B_{jk}(\omega)$ gives us two simultaneous equations which B_{j3} and B_{j4} must satisfy. Solving these equations, we obtain the following:

(16.13) $$B_{j3}(\omega) = -\frac{Q_{j3}(\omega)[1+Q_{44}] - Q_{j4}(\omega)\, Q_{43}(\omega)}{[1+Q_{33}(\omega)][1+Q_{44}(\omega)] - Q_{43}(\omega)\, Q_{34}(\omega)}$$

(16.14) $$B_{j4}(\omega) = -\frac{Q_{j4}(\omega)\ [1+Q_{33}] - Q_{j3}(\omega)\, Q_{34}(\omega)}{[1+Q_{33}(\omega)][1+Q_{44}(\omega)] - Q_{43}(\omega)\, Q_{34}(\omega)}\ .$$

$$j = 1, 2$$

Again, as we did in (14.8) and (14.9), we can define reflection and transmission coefficients as follows:

(16.15) $$R_{jk}(\omega) = B_{jk+2}(\omega)$$

(16.16) $$T_{jk} = \delta_{jk+2} + Q_{jk}(\omega) + \sum_{m=3}^{4} B_{jm}\, Q_{mk+2}$$

$$j = 1, 2$$

$$k = 1, 2 \ .$$

Thus, we have another representation for the reflection and transmission coefficients. This representation will be useful for large $|\omega|$.

Returning to the reflection coefficient R_{22}, we have the following:

(16.17) $$R_{22}(\omega) = B_{24} = - \frac{Q_{24}(\omega)[1+Q_{33}(\omega)]-Q_{23}(\omega)\, Q_{34}(\omega)}{[1+Q_{33}(\omega)][1+Q_{44}(\omega)]-Q_{34}(\omega)\, Q_{43}(\omega)} \ .$$

If we separate out the term Q_{24} in the expression (16.17) we have

(16.18) $$R_{22}(\omega) = - Q_{24} - \frac{Q_{24}\, Q_{44}(1+Q_{33})+Q_{34}(Q_{23}+Q_{24}\, Q_{43})}{(1+Q_{33})\,(1+Q_{44}) - Q_{34}\, Q_{43}} \ .$$

Let us take a closer look at $Q_{jk}(\omega)$. First we can establish the following bound on part of the integrand of $Q_{jk}(\omega)$:

(16.19) $$\left| e_k(-\xi)\, N_k(\xi) \sqrt{\alpha(\xi)} \right| \le \frac{C}{|\omega|} E_2(\xi) \ .$$

Thus, since $\overline{\varphi}_j(x, \omega)$ is uniformly bounded for large $|\omega|$, we have

(16.20) $$Q_{jk}(\omega) = O(\frac{\varepsilon}{\omega})$$

the order relation holding uniformly in ω for $|\omega| \gg \omega_c$. Thus we can establish that

(16.21) $$R_{22}(\omega) = - Q_{24}(\omega) + O\left(\frac{\varepsilon^2}{\omega^2}\right)$$

uniformly in ω for $|\omega| >> \omega_c$.

Let us consider $Q_{24}(\omega)$. First we can write $Q_{24}(\omega)$ as follows:

(16.22) $$Q_{24}(\omega) = \int_{-\infty}^{\infty} e_4(-\xi) N_k(\xi) e_2(\xi)$$

$$+ \frac{1}{\sqrt{\alpha_{-\infty}}} \int_{-\infty}^{\infty} e_4(-\xi) N_k(\xi)[\sqrt{\alpha(\xi)}\, \overline{\varphi}_j(\xi;\omega) - \sqrt{\alpha_{-\infty}}\, e_j(\xi)]d\xi \ .$$

Thus, using (16.7) and (16.19) we can bound the second term on the right hand side of (16.22) and obtain the following:

(16.23) $$Q_{24}(\omega) = \int_{-\infty}^{\infty} e_2(2\xi) N_k(\xi) d\xi + O\left(\frac{\epsilon^2}{\omega^2}\right)$$

uniformly in ω for $|\omega| >> \omega_c$.

We now look at the terms which are $O(\epsilon)$ in $N_k(\xi)$. We find that each of the terms α''''/α, α'''/α, α''/α, α'/α are $O(\epsilon)$. We will be interested in integrating the integrals. Rewriting $N_k(\xi)$, we obtain the following:

$$N_k(\xi) = \frac{-1}{2i\nu_k(\nu_k^2 - \nu_{k-1}^2)} \left\{ \frac{1}{2}\left(\frac{\alpha'}{\alpha}\right)''' - \frac{1}{2}\frac{\alpha'''\alpha'}{\alpha^2} - \frac{3}{4}\left(\frac{\alpha''}{\alpha}\right)^2 \right.$$

$$+ 4\,\frac{\alpha''(\alpha')^2}{\alpha^3} - \frac{123}{16}\left(\frac{\alpha'}{\alpha}\right)^4 + 2\nu_k\left[-\left(\frac{\alpha'}{\alpha}\right)'' + \frac{\alpha''\alpha'}{\alpha^2} - \left(\frac{\alpha'}{\alpha}\right)^3\right]$$

$$\left. - \frac{1}{2}\,\omega^2\left[(a+b)\left(\frac{\alpha'}{\alpha}\right)' + \frac{9a+5b}{4}\left(\frac{\alpha'}{\alpha}\right)^2\right]\right\}$$

Substituting the above into (16.23) and retaining the order ϵ terms and terms which are $O(\epsilon^2/\omega)$, we can simplify the expression for $Q_{24}(\omega)$ to the following:

(16.24)
$$Q_{24}(\omega) = -\int_{-\infty}^{\infty} \frac{e_2(2\xi)}{2i\nu_2(\nu_1^2-\nu_2^2)} \left\{\frac{1}{2}\left(\frac{\alpha'}{\alpha}\right)''' - i\nu_k\left(\frac{\alpha'}{\alpha}\right)'' - \frac{1}{2}\omega^2\left[(a+b)\left(\frac{\alpha'}{\alpha}\right)' + \frac{9a+5b}{4}\left(\frac{\alpha'}{\alpha}\right)^2\right]\right\} d\xi + O\left(\frac{\epsilon^2}{\omega^2}\right)$$

the order relation holding uniformly in ω for $|\omega| >> \omega_c$.

Consider the following term in (16.24):

$$I(\omega) = \int_{-\infty}^{\infty} \frac{e_2(2\xi)}{2i\nu_2(\nu_1^2-\nu_2^2)} \frac{9a+5b}{8}\omega^2\left(\frac{\alpha'}{\alpha}\right)^2 d\xi .$$

If we integrate by parts in $I(\omega)$, we can show

$$I(\omega) = \frac{\omega^2}{4\nu^2(\nu_1^2-\nu_2^2} \int_{-\infty}^{\infty} e_2(2\xi)\frac{9a+5b}{4}\left[\frac{\alpha'\alpha''}{\alpha^2} - \left(\frac{\alpha'}{\alpha}\right)^3\right] d\xi$$

and thus

$$I(\omega) = O\left(\frac{\epsilon^2}{\omega^2}\right)$$

uniformly in ω for $|\omega| >> \omega_c$. Thus we have the following for $Q_{24}(\omega)$:

$$Q_{24}(\omega) = -\int_{-\infty}^{\infty} \frac{e_2(2\xi)}{2i\nu_2(\nu_1^2-\nu_2^2)} \left\{\frac{1}{2}\left(\frac{\alpha'}{\alpha}\right)''' - i\nu_k\left(\frac{\alpha'}{\alpha}\right)'' - \frac{1}{2}\omega^2(a+b)\left(\frac{\alpha'}{\alpha}\right)'\right\} d\xi + O\left(\frac{\epsilon^2}{\omega^2}\right) .$$

Integrating by parts in the above, we can finally simplify the expression for $Q_{24}(\omega)$ to the following:

$$Q_{24}(\omega) = \frac{1}{2}\frac{\omega^2(a+b)}{\nu_1^2-\nu_2^2}\int_{-\infty}^{\infty} e^{2i\nu_2\xi}\frac{\alpha'}{\alpha} d\xi + O\left(\frac{\epsilon^2}{\omega^2}\right) .$$

Substituting the above results for $Q_{24}(\omega)$ into (16.21), we derive the following for $R_{22}(\omega)$:

$$R_{22}(\omega) = -\frac{(a+b)\omega^2}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e^{2i\nu_2\xi} \frac{\alpha'}{\alpha} d\xi + O\left(\frac{\epsilon^2}{\omega^2}\right) \tag{16.25}$$

17. The Inverse Problem

Up to this point we have solved the "forward problem" of approximating the reflection coefficient $R_{22}(\omega)$ in terms of the parameter $\alpha(x)$. In this section, we will concentrate on solving the "inverse problem": given the reflection coefficient $R_{22}(\omega)$, approximate the parameter $\alpha(x)$.

In previous sections, we have shown that the reflection coefficient to a first approximation is proportional to the Fourier transform of the derivative of log $\alpha(x)$. That is, we have shown that

$$R_{22}(\omega) = -\frac{(a+b)\omega^2}{2(\nu_1^2-\nu_2^2)} \int_{-\infty}^{\infty} e^{2i\nu_2} \frac{d\log\alpha}{d\xi} d\xi + E(\nu^2,\xi) \tag{17.1}$$

In addition, we have established the following uniform bounds on $E(\nu_2,\xi)$ in (15.33). (15.34), and (16.25):

$$\begin{cases} E(\nu_2,\epsilon) = 1+O(\epsilon) & \text{for } |\nu_2| \text{ bounded away from zero} \\ E(\nu_2,\epsilon) = O\left(\frac{\epsilon^2}{\nu_2}\right) & \text{for all } \nu_2 \\ E(\nu_2,\epsilon) = O\left(\frac{\epsilon^2}{\nu_2^2}\right) & \text{for all } \nu_2 \end{cases} \tag{17.2}$$

The last two bounds can be consolidated into the following single bound:

$$(17.3) \qquad E(\nu_2, \varepsilon) = O\left(\frac{\varepsilon^2}{|\nu_2| + \nu_2^2}\right)$$

uniformly in ν_2 .

As in the case of the wave equation discussed in Part I, we want to invert the Fourier transform in (17.1) and obtain a relation for the derivative of log $\alpha(x)$ involving the reflection coefficient $R_{22}(\omega)$. To do this we multiply (17.1) by the factor

$$-\frac{1}{\pi} \frac{\nu_1^2 - \nu_2^2}{(a+b)\,\omega^2} e^{-2i\,\nu_2\,\xi}$$

and then integrate over the interval $-\infty < 2\,\nu_2 < \infty$. A few words should be said about the parameters ν_1 and ν_2 as functions of ω. From (11.6), we can show that ν_1 and ν_2 are monotonic functions of ω. Thus, the functions $\nu_1(\omega)$ and $\nu_2(\omega)$ are one-to-one mappings. Therefore, ν_1 and ω can be expressed as functions of ν_2.

As for the integral arising from the first term on the right hand side of (17.1), we can apply the Fourier integral theorem given in the last section of Part I. The conditions which we assumed earlier on the parameter $\alpha(x)$ are enough to insure the existence of inversion of the Fourier transform of the derivative of log $\alpha(x)$.

As for the integral containing the error term, we will be able to bound it in terms of the integral of the absolute value of $E(\nu_2, \varepsilon)$. First we will bound $E(\nu_2, \varepsilon)$ as follows:

$$E(\nu_2, \epsilon) \leq \begin{cases} 1 + O(\epsilon) & |\nu_2| \leq \epsilon^2 \\ 0\left(\frac{\epsilon^2}{|\nu^2| + \nu_2^2}\right) & |\nu_2| > \epsilon^2 \end{cases} .$$

Splitting the range of integration into two parts, we can obtain the following:

$$\left| \int_{-\infty}^{\infty} \frac{\nu_1^2 - \nu_2^2}{\pi \omega^2 (a+b)} e^{-2i\nu_2 x} E(\nu_2, \epsilon) d\nu_2 \right| \leq C \int_0^{\epsilon^2} d\nu_2$$

$$+ C_1 \epsilon^2 \int_{\epsilon^2}^{\infty} \frac{d\nu_2}{\nu_2 + \nu_2^2}$$

$$\leq C \epsilon^2 + C_1 \epsilon^2 \operatorname{Log} \epsilon^2 .$$

Thus, we can show that the integral involving the error term is $O(\epsilon^2 \log \epsilon)$, uniformly in x for all x.

Thus we have

$$(17.4) \qquad \frac{d \log \alpha(x)}{d x} = - \frac{1}{\pi(a+b)} \int_{-\infty}^{\infty} \frac{(\nu_1^2 - \nu_2^2)}{\omega^2} e^{-2i\nu_2 x} R_{22}(\omega) d\nu_2 + O(\epsilon^2 \log \epsilon)$$

uniformly in x for all x. Since we showed that the integral of the right hand side of (17.1) exists, the integral on the right hand side of (17.4) must also exist.

If we express $\alpha(x)$ as

$$\alpha(x) = \alpha_{-\infty} [1 + \epsilon r(x)]$$

then we can write (17.4) as follows:

$$F(x, \varepsilon) = \varepsilon \frac{dr}{dx} + O(\varepsilon^2 \log \varepsilon) \tag{17.5}$$

uniformly in x for all x, where

$$F(x, \varepsilon) = -\frac{1}{\pi(a+b)} \int_{-\infty}^{\infty} \frac{\nu_1^2 - \nu_2^2}{\omega^2} e^{-2i\nu_2 x} R_{22}(\omega) d\nu_2 \quad .$$

Eq. (17.5) says that the function $F(x, \varepsilon)$ can be approximated as follows:

$$F(x, \varepsilon) = \varepsilon F_0(x) + O(\varepsilon^2 \log \varepsilon) \quad ,$$

where $F_0(x)$ equals the derivative of $r(x)$. Thus, if we have the reflection coefficient $R_{22}(\omega)$, we can construct the function $F(x, \varepsilon)$ and, in turn, approximate the derivative of $r(x)$. All of the above results can be stated more precisely in the following theorem.

Theorem: Let $\alpha(x)$ satisfy the following:

1. $\alpha(x)$ possesses a fourth order partial derivative for all x.
2. The following integrals exist:

$$\int_{-\infty}^{\infty} \left| \frac{d^n \alpha}{dx^n} \right| dx \,, \qquad n = 1, 2, 3, 4 \,,$$

$$\int_{-\infty}^{\infty} \left| x \frac{d^m \alpha}{dx^m} \right| dx \,, \qquad m = 1, 2,$$

3. $\alpha(x)$ can be written as

$$\alpha(x) = \alpha_{-\infty}[1 + \varepsilon r(x)]$$

Then the function

$$F(x, \epsilon) = - \frac{1}{\pi (a+b)} \int_{-\infty}^{\infty} \frac{\nu_1^2 - \nu_2^2}{\omega^2} e^{-2i \nu_2 x} R_{22}(\omega) d\nu_2$$

exists and can be expanded as follows:

$$F(x, \epsilon) = \epsilon F_o(x) + O(\epsilon^2 \log \epsilon)$$

uniformly in x for all x, and

$$F_o(x) = \frac{dr}{dx} .$$

The proof of the theorem follows from the previous work.

CHAPTER IV

CONCLUSION

The Inverse problem has wide physical interest. Two situations where the inverse problem is physically applicable are discussed below.

In the first situation, suppose we were required to determine the parameters characterizing a non-homogeneous medium. However, suppose we were limited by the fact that the medium were inaccessible, so we were unable to measure the parameters directly. Yet, if we could stand off at infinity and measure the reflection coefficient associated with sinusoidal waves of various frequencies, our work here provides us with a means of approximating the parameters. Thus, we have a means of determining properties of a certain medium from a remote location.

A second situation is essentially a synthesis problem. In this situation, we would be required to synthesize or construct a medium which would have a desired reflection coefficient or whose reflection coefficient would have certain desired properties. Again, our work would provide a means of approximating the medium parameters, and thus provide us with a means of approximately designing and synthesizing certain systems or media.

The primary goal in our investigation of both the wave equation and the Timoshenko equation was to provide an approximate method of solving the inverse problem for a slightly non-homogeneous medium. In both cases, we found that the approximation to the derivative of the parameter α was simply a Fourier integral involving the reflection coefficient.

In the case of the wave equation, there are alternative approaches for obtaining an approximation which differ from the one taken here. We will discuss briefly these approaches in the paragraphs which follow.

Eq. (7.2) of Chapter II is recognized as the one dimensional analog of the Schroedinger equation from quantum mechanics, with the potential $V(\tau)$ being given by

$$V(\tau) = \frac{1}{\alpha(\tau)} \frac{d^2 \alpha}{d\tau^2} .$$

If we had transformed our differential equation into the Schroedinger form, and then tried working with this for all values of k, we would have run into difficulties for small k. If we had worked with the integral equation corresponding to the Schroedinger equation for a slightly non-homogeneous medium, we would have obtained the familiar Born approximation for the reflection coefficient. The Born approximation is unsatisfactory in our case, because as k approaches zero, the best we can do is show that the reflection coefficient is $O(1)$. However, we showed that the reflection coefficient is $O(\epsilon)$ uniformly in k for all k. Thus, by avoiding the transformation to the Schroedinger form of the differential equation, we were able to obtain a better bound on the reflection coefficient for small k. We were content in using the Schroedinger type equation to approximate the reflection coefficient for large k.

In addition, if we had used the Born approximation to retrieve the parameter $\alpha(\tau)$, the best we could have done would have been to approximate $\frac{d\alpha}{d\tau}$ with an error which was $O(\epsilon^2 \log \epsilon)$ uniformly in x for all x. As it was, we were able to approximate $\frac{d\alpha}{d\tau}$ with an error which was $O(\epsilon^2)$ uniformly in x for all x.

A second approach involves solving the inverse problem exactly. Kay [6] demonstrated how the inverse problem for the wave equation could be solved exactly in terms of the reflection coefficient. Their method relies on work done by Gelfand and Levitan [5]. The exact solution does not give a closed form for the solution, but merely reduces the problem to solving an integral equation. The solution of the integral equation is related to the parameter

$$V(\tau) = \frac{1}{\alpha(\tau)} \frac{d^2\alpha}{d\tau^2} \quad .$$

In the case of a slightly non-homogeneous medium, by using successive approximations, an approximate solution to the integral equation can be obtained (Faddeyev, [4] pp. 81 and 90), and thus the parameter $\frac{d\alpha}{d\tau}$ can be approximated. There is no problem in obtaining a uniform approximation on the semi-infinite interval (as done in Faddeyev [4]); However, the uniformity of the approximation cannot be extended over the entire infinite interval.

The Gelfand-Levitan method has been extended (see Agranovich and Marchenko [2]) to cover the inverse problem associated with certain systems of second order differential equations. In turn, this extension can be used to solve the inverse problem for certain fourth order differential equations.

One objection to the exact approach is that it can be applied to only certain fourth order differential equations, and unfortunately, the Timoshenko equation cannot be transformed into a form which can be covered by this method. Thus, we are left with our direct approach in solving the inverse problem approximately.

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