I. EXACT SOLUTION OF SOME NONLINEAR EVOLUTION EQUATIONS

II. THE SIMILARITY SOLUTION FOR THE KORTEWEG-DE VRIES EQUATION AND THE RELATED PAINLEVÉ TRANSCENDENT

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Rodolfo Rubén Rosales

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ABSTRACT

In Part I, a method for finding solutions of certain diffusive-dispersive nonlinear evolution equations is introduced. The method consists of a straightforward iteration procedure, applied to the equation as it stands (in most cases), which can be carried out to all terms, followed by a summation of the resulting infinite series, sometimes directly and other times in terms of traces of inverses of operators in an appropriate space.

We first illustrate our method with Burgers' and Thomas' equations, and show how it quickly leads to the Cole-Hopft transformation, which is known to linearize these equations.

We also apply this method to the Korteweg and de Vries, nonlinear (cubic) Schrödinger, Sine-Gordon, modified KdV and Boussinesq equations. In all these cases the multisoliton solutions are easily obtained and new expressions for some of them follow. More generally we show that the Marčenko integral equations, together with the inverse problem that originates them, follow naturally from our expressions.

Only solutions that are small in some sense (i.e., they tend to zero as the independent variable goes to ∞) are covered by our methods. However, by the study of the effect of writing the initial iterate $u_1 = u_1(x,t)$ as a sum $u_1 = \tilde{u}_1 + \tilde{\tilde{u}}_1$, when we know the solution which results if $u_1 = \tilde{u}_1$, we are led to expressions that describe the interaction of two arbitrary solutions, only one of which is small. This should not be confused with Bäcklund transformations and is more in

the direction of performing the inverse scattering over an arbitrary "base" solution. Thus we are able to write expressions for the interaction of a cnoidal wave with a multisoliton in the case of the KdV equation; these expressions are somewhat different from the ones obtained by Wahlquist (1976). Similarly, we find multi-dark-pulse solutions and solutions describing the interaction of envelope-solitons with a uniform wave train in the case of the Schrödinger equation.

Other equations tractable by our method are presented. These include the following equations: Self-induced transparency, reduced Maxwell-Bloch, and a two-dimensional nonlinear Schrödinger. Higher order and matrix-valued equations with nonscalar dispersion functions are also presented.

In Part II, the second Painlevé transcendent is treated in conjunction with the similarity solutions of the Korteweg-de Vries equation and the modified Korteweg-de Vries equation.

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INTRODUCTION

In the last ten years there have been many developments in nonlinear wave theory, particularly in the aspect concerning exact solutions. Before the paper by Gardner, Greene, Kruskal and Miura (1967) very few instances of nonlinear equations exactly solvable were known outside the range of hyperbolic theory, and no systematic way of treating them was available. Gardner et al. were able to relate the Korteweg and de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

to an eigenvalue problem

$$\varphi_{xx} + u\varphi = \lambda\varphi ,$$

in which the solution of the KdV equation appeared as a potential, such that its spectrum remained invariant with time and the evolution of the scattering parameters could be computed explicitly. Thus, by the process of doing a scattering problem at t = 0 and an inverse scattering problem for t > 0, they were able to find a linear integral equation for the initial value problem of the KdV equation and derive a number of important results. These include the explicit solution for the interaction of any number of solitary waves; this problem corresponds to a vanishing reflection coefficient in the scattering problem.

Lax (1968) reformulated the method, opening the way for more equations to be solved by the inverse scattering transform, as the technique introduced by Gardner et al. has come to be known. In 1972, Zakharov and Shabat found an eigenvalue problem with which they were able to solve the nonlinear cubic Schrödinger equation

$$i\varphi_t + \varphi_{xx} \pm |\varphi|^2\varphi = 0.$$

Also in 1972, Wadati applied the same eigenvalue problem to solve the modified KdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0.$$

Several other examples of physical interest have been found since then, including not only partial differential equations, but difference equations and classical Hamiltonian systems as well.

The main difficulty with this method is in finding the appropriate eigenvalue problem for a given equation. In fact no a priori way of deciding whether a given equation is going to be solvable by this method is known, and certainly there is no systematic way of producing the eigenvalue problem. Most of the work done in the field so far seems to have gone in the other direction. That is, given an eigenvalue problem, find which interesting equations can be solved by it.

It is the purpose of the first part of this thesis to investigate the problem of finding alternative approaches which bypass some of the difficulties, and to learn how some of the more standard perturbation procedures, so successful in other areas, would fare on the particular equations solvable by the inverse scattering transform method. We are also interested in obtaining the linear integral equations for the initial value problem directly from the evolution equations, without the necessity of invoking a scattering problem.

We develop techniques by which perturbation expansions, valid in a limited region, can be summed to give the complete solution. In the process we obtain various exact solutions and the eigenvalue problems together with the linear integral equations associated with their inverse scattering problems. A general method of summing these perturbation expansions by means of operators is presented. The expressions thus obtained might have some usefulness in dealing with the asymptotic behavior of the solution for large time, but we have not explored this aspect as yet.

The approach presented is useful not only for the cases covered by the inverse scattering transform, but in others as well. We illustrate this in the first chapter, where we treat Burgers' equation

$$\eta_t + \eta_x \eta - \eta_{xx} = 0,$$

and Thomas' equation

$$\varphi_{xy} + \alpha \varphi_x + \beta \varphi_y + \gamma \varphi_x \varphi_y = 0 .$$

The Cole-Hopf (Hopf 1950, Cole 1951) and Thomas (1944) transformations that linearize these equations are shown to follow naturally from our expansions.

The second chapter is concerned with the study of the KdV equation. In the first section we find a non-uniform perturbation expansion for the solution. In the second section this expansion is summed in the particular case in which the initial iterate is a sum of exponentials. Thus we obtain the explicit solution for the interaction of any number of solitary waves. In the third section a linear integral equation for the solution is obtained from the perturbation expansion, from which it follows naturally. The eigenvalue problem associated with the KdV equation is also shown to follow from the expansion. In the fourth section the perturbation expansion is summed in a very general setting. The sum is expressed in terms of the trace of the inverse of an operator in a Hilbert space. An example is presented in section five in which the operator can be written as a composite of Fourier transforms. In the sixth section the problem of inversion is treated; that is, given initial values, find the corresponding parameters of the perturbation expansion. To do this we use the eigenvalue problem and show that the natural parameters of the expansion are precisely the scattering parameters. In the seventh section we study how the solution transforms when the parameters of the expansion are transformed. Formulas that describe the interaction of any number of solitary waves with a given arbitrary solution are found. The basic solution might be, for example, a cnoidal wave. The relationship of our expansions with the Bäcklund transformation for the KdV equation (Wahlquist and Estabrook 1973) is also presented in this section. Finally, in section eight, other equations that are solvable by the same type of perturbation expansion as the KdV are studied, including some nonlinear matrix partial differential equations.

The third chapter is concerned with the study of the cubic Schrödinger equation and the results follow the same lines of those of the KdV equation. In particular, we write explicit formulas for the interaction of envelope-wave solitons with a uniform wave train and the multisoliton solution of the vector-valued cubic Schrödinger.

In the remaining chapters of the first part the following equations are studied:

(Modified KdV) $\quad u_t \pm u^2 u_x + u_{xxx} = 0,$

(Sine-Gordon) $\quad u_{tt} - u_{xx} \pm \sin u = 0,$

and

(Boussinesq) $\quad u_{tt} \pm u_{xx} \pm u_{xxxx} \pm (u^2)_{xx} = 0.$

In all cases we find the multisoliton solutions, linear integral equations and eigenvalue problem. The Miura transformation between the modified KdV equation and the KdV equation (Miura 1968), is shown to follow naturally from our expansions.

Finally an appendix is added to provide detailed justification of some questions discussed in the main text.

The main difficulty of our approach lies in the algebraic manipulations needed. A lemma, which is presented in the Appendix, proves helpful in this, The method has some common points with the one presented by Hirota (1971, 1972 ab, 1973ab). However, we do not transform the equations previous to operating on them, and our expansions do not terminate (two main features of Hirota's work). From our approach, we obtain in some cases compact formulas for multisoliton solutions which are equivalent to Hirota's expansions.

The second part of the thesis is a short note dealing with the similarity solution of the KdV equation. It is shown that the ordinary differential equation for this similarity solution can be transformed into a second Painlevé transcendent equation

$$g'' = \eta g + 2g^3.$$

We then study a special class of solutions of the latter. These solutions do not seem to have been treated in the literature before.

Part I

EXACT SOLUTION OF SOME NONLINEAR EVOLUTION EQUATIONS

CHAPTER 1

BURGERS' AND THOMAS' EQUATIONS

1.1 Burgers' Equation

We consider Burgers' equation (Burgers 1948),

$$\eta_t + \eta_x \eta - \eta_{xx} = 0 \ , \ \eta = \eta(x,t), \tag{1.01}$$

and look for solutions in the form of perturbation expansions for small amplitude. We introduce an auxiliary "small" parameter ε and write

$$\eta(x,t) = \sum_{n=1}^{\infty} \varepsilon^n \eta_n(x,t) \tag{1.02}$$

We consider solutions such that $\eta_n \longmapsto 0$ as $x \longmapsto \infty$, for all n , and assume that (1.02) is valid for large x . Then we try to rewrite it so as to have a solution valid for all x .

Substituting (1.02) into (1.01) and collecting equal powers of ε , we have

$$\eta_{n,t} - \eta_{n,xx} = - \sum_{j=1}^{n-1} \eta_{j,x} \eta_{n-j} \ , \ (\forall n \geqslant 1). \tag{1.03}$$

For $n=1$, we have

$$\eta_{1,t} - \eta_{1,xx} = 0. \tag{1.04}$$

This is the linear, stable, heat equation and we choose to solve it by Fourier transforms in the form

$$\eta_1 = \int_C (aik) \exp(ikx - k^2 t) \, d\lambda(k). \tag{1.05a}$$

To allow flexibility in our notation we let $d\lambda(k)$ be an appropriate measure on the complex plane $\mathbb{C}$. For example, (1.05a) may be a superposition of real exponentials plus a usual Fourier transform

$$\eta_1 = \sum_m a_m \exp(-\kappa_m x + \kappa_m^2 t) + \int_{-\infty}^{\infty} (2ik)\exp(ikx - k^2 t)\beta(k)\,dk. \tag{1.05b}$$

The factor $(2ik)$ has been added in (1.05) for convenience in what follows.

At the second order we have

$$\eta_{2,t} - \eta_{2,xx} = -\eta_{1,x}\,\eta_1 = \int_{\mathbb{C}^2} (4ik_1^2 k_2)\exp[i(k_1+k_2)x - (k_1^2+k_2^2)t]\,d\lambda(k_1)\,d\lambda(k_2). \tag{1.06}$$

The right hand side of (1.06) suggests a solution of the form

$$\eta_2 = \int_{\mathbb{C}^2} \Phi_2(k_1,k_2)\exp[i(k_1+k_2)x - (k_1^2+k_2^2)t]\,d\lambda(k_1)\,d\lambda(k_2). \tag{1.07a}$$

Since $-(k_1^2+k_2^2)+(k_1+k_2)^2 = 2k_1k_2$ we have that (1.07a) solves (1.06) if

$$2k_1k_2\,\Phi_2(k_1,k_2) = 4ik_1^2k_2, \quad \Rightarrow \quad \Phi_2(k_1,k_2) = 2ik_1. \tag{1.07b}$$

We assume that any homogeneous solution of (1.06) that may be added to η_2 has been absorbed on η_1 .

For $n=3$, we have

$$\eta_{3,t} - \eta_{3,xx} = -\eta_{1,x}\,\eta_2 - \eta_{2,x}\,\eta_1 =$$

$$= \int_{\mathbb{C}^3} 4i[k_1^2k_2 + k_1(k_1+k_2)k_3]\,e^{i\Omega_3}\,[d\lambda(k)]^3, \tag{1.08}$$

where $\Omega_3 = (k_1+k_2+k_3)x + i(k_1^2+k_2^2+k_3^2)t$ and $[d\lambda(k)]^3 = d\lambda(k_1)\,d\lambda(k_2)\,d\lambda(k_3)$.

We take

$$\eta_3 = \int_{\mathbb{C}^3} \Phi_3(k_1, k_2, k_3)\, e^{i\Omega_3}\, [d\lambda(k)]^3. \tag{1.09a}$$

Since $-(k_1^2+k_2^2+k_3^2)+(k_1+k_2+k_3)^2 = 2k_1k_2+2k_1k_3+2k_2k_3 = 2k_1k_2+2(k_1+k_2)k_3$, we have

$$[2k_1k_2+2(k_1+k_2)k_3]\,\Phi_3(k_1, k_2, k_3) =$$

$$= 4ik_1^2k_2 + 4ik_1(k_1+k_2)k_3\,, \Rightarrow \Phi_3(k_1, k_2, k_3) = 2ik_1\,. \tag{1.09b}$$

Similarly for $n=4$ we find $\Phi_4 = 2ik_1$, and at this stage it seems likely that $\Phi_n(k_1,\dots,k_n) = 2ik_1$ for all n. To check this we take

$$\eta_n = \int_{\mathbb{C}^n} \Phi_n(k_1,\dots,k_n)\, e^{i\Omega_n}\, [d\lambda(k)]^n, \ (\forall n \geq 1), \tag{1.10}$$

where the meaning of Ω_n and $[d\lambda(k)]^n$ should be clear from (1.08). Substituting (1.10) into (1.03) we find

$$[-\sum_1^n k_j^2 + (\sum_1^n k_j)^2]\,\Phi_n(k_1,\dots,k_n) = -i\sum_1^{n-1}(k_1+\dots+k_j)\,\Phi_j(k_1,\dots,k_j)\,\Phi_{n-j}(k_{j+1},\dots,k_n). \tag{1.11}$$

Since

$$-\sum_1^n k_j^2 + (\sum_1^n k_j)^2 = 2\sum_{1\leq j<\ell\leq n} k_jk_\ell = 2\sum_2^n (k_1+\dots+k_{\ell-1})k_\ell =$$

$$= 2\sum_1^{n-1}(k_1+\dots+k_j)k_{j+1}\,, \tag{1.12}$$

we have

$$\Phi_n(k_1,\dots,k_n)\sum_1^{n-1}(k_1+\dots+k_j)k_{j+1} = -\frac{i}{2}\sum_1^{n-1}(k_1+\dots+k_j)\,\Phi_j(k_1,\dots,k_j)\,\Phi_{n-j}(k_{j+1},\dots,k_n).$$

This last equation is obviously satisfied by

$$\Phi_n(k_1,\ldots,k_n) = 2ik_1\ , \ (\forall n \geqslant 1). \tag{1.13}$$

The actual expressions for the Φ_n's are not unique, since different forms can lead to the same η_n's. For example $\Phi_n = ik_n$, $(i/n)(k_1+\ldots +k_n)$ or $(i/2)(k_1+k_2)$ could be used. The simplest one for subsequent manipulations is usually clear.

Formulas (1.02), (1.10) and (1.13) give

$$\eta = \sum_1^\infty \varepsilon^n \int_{\mathbb{C}^n} (2ik_1)\, e^{i\Omega_n} [d\lambda(k)]^n = \sum_1^\infty 2\, \zeta_x\, \zeta^{n-1}, \tag{1.14}$$

where

$$\zeta = \varepsilon \int_{\mathbb{C}} \exp(ikx - k^2 t)\, d\lambda(k). \tag{1.15}$$

We then have

$$\eta = 2\, \zeta_x (1-\zeta)^{-1} = -2\, \partial_x \ln(1-\zeta). \tag{1.16}$$

The function ζ, and therefore $\varphi = 1-\zeta$, is a general solution of the heat equation. Thus (1.15) and (1.16) provide us with the Cole-Hopf transformation (Hopf 1950, Cole 1951), $\eta = -2\,\partial_x \ln\varphi$ which linearizes (1.01). In the final form the solution is not limited to small amplitude and the parameter ε provides only a consistent ordering procedure.

We observe the key role played by the linear dispersion relation associated with (1.01), i.e. $G(\omega,k) = -i\omega + k^2 = 0$, which defines the basic (linear) harmonics $\exp(ikx - i\omega t)$ from whose interactions the full nonlinear solution is built up. In successive approximations $G(\omega_1+\dots+\omega_n, k_1+\dots+k_n)$ appears multiplying Φ_n, as in (1.11). The key steps concern the decomposition of this expression in a way compatible with the nonlinear terms, as is seen immediately after formulas (1.07a), (1.09a) and in (1.12).

It is possible to solve (1.01) when η is square matrix valued. The only change that has to be introduced is to take $d\lambda(k)$ matrix valued. Also, because of the noncommutativity of the product involved, (i) the Φ_n's are now uniquely determined, and (ii) the last equality in (1.16) does not hold.

1.2 Thomas' Equation

Consider now Thomas' equation, which describes certain chemical reactions. The equation (Thomas 1944) may be written

$$\varphi_{xy} + \alpha\varphi_x + \beta\varphi_y + \gamma\varphi_x\varphi_y = 0\,, \quad \varphi = \varphi(x,y)\,, \tag{1.17}$$

where α, β and γ are constants. Again we introduce an auxiliary "small" parameter ε and write

$$\varphi(x,y) = \sum_1^\infty \varepsilon^n \varphi_n(x,y)\,. \tag{1.18}$$

Then, if k and ℓ satisfy the linearized dispersion relation $G(k,\ell) = -k\ell + i\alpha k + i\beta\ell$ and $\Omega_n = \sum_1^n (k_j x + \ell_j y)$, we introduce[†]

[†]By analogy with (1.10).

$$\varphi_n = \int_{\mathbb{C}^n} \Phi_n(k_1,\dots,k_n)\, e^{i\Omega_n}\, [d\mu(k)]^n, \qquad (1.19)$$

where $d\mu(k)$ is an appropriate measure on $\mathbb{C}$. After some manipulation we find $\Phi_1 = 1$, $\Phi_2 = -\frac{\gamma}{2}$, $\Phi_3 = \frac{\gamma^2}{3}$ and we propose the general form

$$\Phi_n = (-1)^{n+1} \frac{\gamma^{n-1}}{n}, \ (\forall n \geq 1). \qquad (1.20)$$

This is equivalent to $\varphi_n = (-1)^{n+1} \frac{\gamma^{n-1}}{n} \varphi_1^n$, a formula that can be easily checked. Thus we have

$$\varphi = \sum_1^\infty (-1)^{n+1} \frac{\gamma^{n-1}}{n} (\varepsilon\varphi_1)^n = \frac{1}{\gamma} \ln(1+\varepsilon\gamma\varphi_1). \qquad (1.21)$$

Since φ_1 , and therefore $(1+\varepsilon\gamma\varphi_1)$, is a general solution of the linearized equation, (1.21) leads to the transformation $\gamma\varphi = \ln\psi$ used by Thomas to linearize (1.17).

CHAPTER 2

KORTEWEG AND DE VRIES EQUATION

The equation can be taken in the form

$$u_t + 3(u^2)_x + u_{xxx} = 0, \quad u = u(x,t). \tag{2.01}$$

This equation describes the development and propagation of long waves in shallow water (Korteweg and de Vries 1895) as well as many other important phenomena where a small quadratic nonlinearity is combined with a cubic dispersion relation.

2.1 Solution by Small Parameter Expansions

Substituting

$$u(x,t) = \sum_1^{\infty} \varepsilon^n u_n(x,t) \tag{2.02}$$

into (2.01), and collecting equal powers of the "small" parameter ε, we obtain

$$u_{n,t} + u_{n,xxx} = -3\,\partial_x \sum_1^{n-1} u_j u_{n-j} \,, \quad (\forall n \geq 1). \tag{2.03}$$

The function u_1 satisfies the linear Korteweg and de Vries (KdV) equation, so that we can take

$$u_1 = \int_{\mathbb{C}} (-k) \exp[i(kx + k^3 t)]\, d\lambda(k), \tag{2.04}$$

where, as before, $d\lambda(k)$ is an appropriate measure on the complex plane

$\mathbb{C}$ and the factor $(-k)$ has been introduced for convenience in what follows.

For $n=2$, we have

$$u_{2,t} + u_{2,xxx} = -3i\int_{\mathbb{C}^2} (k_1+k_2)\, k_1 k_2\, e^{i\Omega_2}\, [d\lambda(k)]^2 \tag{2.05}$$

where

$$\Omega_2 = (k_1+k_2)x + (k_1^3 + k_2^3)t \qquad \text{and} \quad [d\lambda(k)]^2 = d\lambda(k_1)\, d\lambda(k_2).$$

A solution in the form

$$u_2 = \int_{\mathbb{C}^2} \Phi_2(k_1,k_2)\, e^{i\Omega_2}\, [d\lambda(k)]^2, \tag{2.06a}$$

requires $i[(k_1^3+k_2^3)-(k_1+k_2)^3]\,\Phi_2(k_1,k_2) = -3i\,(k_1+k_2)k_1k_2$. Since $k_1^3+k_2^3-(k_1+k_2)^3 = -3k_1k_2(k_1+k_2)$ we have $\Phi_2 = 1$. Thus

$$u_2 = \int_{\mathbb{C}^2} e^{i\Omega_2}\, [d\lambda(k)]^2. \tag{2.06b}$$

For $n=3$, we have

$$u_{3,t} + u_{3,xxx} = -3\partial_x (u_1 u_2 + u_2 u_1) =$$

$$= 3i\int_{\mathbb{C}^3} (k_1+k_2+k_3)(k_1+k_3)\, e^{i\Omega_3}\, [d\lambda(k)]^3, \tag{2.07}$$

where

$$\Omega_3 = (k_1+k_2+k_3)x + (k_1^3+k_2^3+k_3^3)t \quad \text{and} \quad [d\lambda(k)]^3 = d\lambda(k_1)\, d\lambda(k_2)\, d\lambda(k_3).$$

A solution in the form

$$u_3 = \int_{\mathbb{C}^3} \Phi_3(k_1, k_2, k_3)\, e^{i\Omega_3}\, [d\lambda(k)]^3 \tag{2.08a}$$

requires $i[k_1^3 + k_2^3 + k_3^3 - (k_1+k_2+k_3)^3]\, \Phi_3(k_1, k_2, k_3) = 3i(k_1+k_2+k_3)(k_1+k_3)$.
Since $k_1^3 + k_2^3 + k_3^3 - (k_1+k_2+k_3)^3 = -3(k_1+k_2)(k_2+k_3)(k_3+k_1)$, we have
$\Phi_3 = -(k_1+k_2+k_3)(k_1+k_2)^{-1}(k_2+k_3)^{-1}$. Thus

$$u_3 = -\int_{\mathbb{C}^3} \frac{(k_1+k_2+k_3)}{(k_1+k_2)(k_2+k_3)}\, e^{i\Omega_3}\, [d\lambda(k)]^3. \tag{2.08b}$$

In a similar way we obtain

$$u_4 = \int_{\mathbb{C}^4} \frac{(k_1+k_2+k_3+k_4)}{(k_1+k_2)(k_2+k_3)(k_3+k_4)}\, e^{i\Omega_4}\, [d\lambda(k)]^4. \tag{2.09}$$

At this stage it seems natural to propose

$$u_n = (-1)^{n+1}\, i\partial_x \int_{\mathbb{C}^n} \frac{e^{i\Omega_n}}{\prod_1^{n-1}(k_j + k_{j+1})}\, [d\lambda(k)]^n, \quad (\forall n \geq 1). \tag{2.10}$$

In order to prove this we observe that if

$$u_n = \int_{\mathbb{C}^n} \Phi_n(k_1, \dots, k_n)\, e^{i\Omega_n}\, [d\lambda(k)]^n, \quad (\forall n \geq 1), \tag{2.11}$$

is substituted into (2.03) we obtain

$$G(-k_1^3 - \dots - k_n^3,\; k_1 + \dots + k_n)\, \Phi_n(k_1, \dots, k_n) =$$

$$= -3i(k_1 + \dots + k_n) \sum_1^{n-1} \Phi_j(k_1, \dots, k_j)\, \Phi_{n-j}(k_{j+1}, \dots, k_n), \quad (\forall n \geq 1), \tag{2.12}$$

where $G(\omega,k)=-i\omega-ik^3=0$ is the linear dispersion relation of (2.01). Introducing $\Phi_n\ (\forall n\geq 1)$, as given by (2.10), into (2.12) and multiplying through by Φ_n^{-1} we find we must prove that

$$G(-k_1^3-\dots-k_n^3,\ k_1+\dots+k_n)=-3i\sum_1^{n-1}(k_1+\dots+k_j)(k_j+k_{j+1})(k_{j+1}+\dots+k_n),\ (\forall n\geq 1). \tag{2.13}$$

This is a generalization for all n of the formulas found in (2.06) and (2.08) for the cases $n=2$ and $n=3$ respectively. To prove (2.13) we note that

$$-3i\sum_1^{n-1}(k_1+\dots+k_j)(k_j+k_{j+1})(k_{j+1}+\dots k_n)=-3i\Big[\sum_{1\leq i\leq j<l\leq n}k_ik_jk_l+\sum_{1\leq i<j\leq l\leq n}k_ik_jk_l\Big]=$$

$$=-6i\sum_{1\leq i<j<l\leq n}k_ik_jk_l-3i\sum_{1\leq j<l\leq n}(k_j^2k_l+k_jk_l^2)=$$

$$=-i\Big(\sum_1^n k_j\Big)^3+i\sum_1^n k_j^3=G\Big(-\sum_1^n k_j^3,\ \sum_1^n k_j\Big).$$

An alternative proof using (A.102) is also possible.

From (2.02) and (2.10) we now have

$$u=-i\partial_x\sum_1^\infty(-\epsilon)^n\int_{\mathbb{C}^n}\frac{e^{i\Omega_n}}{\prod_1^{n-1}(k_j+k_{j+1})}[d\lambda(k)]^n \tag{2.14}$$

As opposed to what happened in the case of Burgers' and Thomas' equations, it is not possible now to do a straightforward summation of the infinite series in formula (2.14). Different harmonics are now coupled together by the factors $(k_j+k_{j+1})^{-1}$ in the Φ_n's. No simple dependent variable transformation will linearize (2.01). However, it is still possible to perform the summation of (2.14) in a wide variety of cases. Its similarity with a geometric progression (which in fact it is, in the appropriate sense) will be exploited to do this. Moreover, the eigenvalue problem associated with (2.01) (Gardner et al. 1967), as well as the Marčenko linear integral equation of the corresponding inverse problem, is implicit in (2.14). Thus our perturbation method provides a simple and straightforward way of obtaining and solving the proper inverse scattering equations.

Introduce

$$b(x,t) = \frac{1}{2i}\int_{c} d\lambda(k)\, \exp i\left(\tfrac{1}{2}kx + k^3 t\right), \tag{2.15}$$

and assume that, as $x \longrightarrow \infty$, $b(x,t)$ as well as the n^{th} term of (2.14) tend to zero sufficiently fast. Then we can write (2.14) as

$$u = 2\partial_x \sum_{1}^{\infty} (-\varepsilon)^n \int_{[x,\infty)^{n-1}} dz_1 \ldots dz_{n-1}\, b(x+z_1,t)\, b(z_1+z_2,t) \ldots b(z_{n-1}+x,t). \tag{2.16}$$

We note that $b(2x,t)$ satisfies the linear KdV equation and that u is real if and only if b is real. The latter is equivalent to $d\lambda(k) = -d\lambda^*(-k^*)$.

2.2 Multisoliton Solutions

Assume now that $b(x,t)$ is a superposition of real exponentials, which we take in the form

$$b(z+y,t) = \frac{1}{2}\sum_{m} a_m^2 \exp\left[-\tfrac{1}{2}\kappa_m(z+y) + \kappa_m^3 t\right] = \frac{1}{2}\, p^T(z,t)\, p(y,t), \tag{2.17}$$

where the a_m's and κ_m's are positive real constants and $p(x,t)$ is the column real vector given by

$$p_m(x,t) = a_m \exp\left[\tfrac{1}{2}(-\kappa_m x + \kappa_m^3 t)\right], (\forall m). \tag{2.18}$$

The motivation for writing $b(z+y,t)$ as a product is obvious from (2.16) since then we will be able to perform each of the integrals over z_1, z_2, z_3, etc. separately, i.e., we have

$$u = \partial_x \sum_1^\infty (-\varepsilon)^n p^T(x,t)\left[\tfrac{1}{2}\int_x^\infty dz\, p(z,t)p^T(z,t)\right]^{n-1} p(x,t)\,. \tag{2.19}$$

This is a geometrical series, and if $B = B(x,t)$ is the square matrix given by

$$B(x,t) = \frac{1}{2}\int_x^\infty dz\, p(z,t)\, p^T(z,t) =$$

$$= \left\{ \frac{a_{m_1} a_{m_2}}{(\kappa_{m_1}+\kappa_{m_2})} \exp \tfrac{1}{2}\left[-(\kappa_{m_1}+\kappa_{m_2})x + (\kappa_{m_1}^3+\kappa_{m_2}^3)t\right]\right\}, \tag{2.20}$$

we can sum (2.19) to

$$u = -\varepsilon\, \partial_x\, p^T\cdot(I+\varepsilon B)^{-1}\cdot p = 2\varepsilon\, \partial_x \operatorname{Tr}\left[(I+\varepsilon B)^{-1}\cdot(-\tfrac{1}{2}pp^T)\right] =$$

$$= 2\varepsilon\, \partial_x \operatorname{Tr}\left[(I+\varepsilon B)^{-1}\partial_x B\right] = 2\,\partial_x^2 \operatorname{Tr} \ln (I+\varepsilon B). \tag{2.21}$$

We observe that the matrix B, as given by (2.20), is real, symmetric and positive definite. The positive definiteness of B is a consequence of the fact that, for any arbitrary column real vector $q \neq 0$, we have

$$q^T\cdot B(x,t)\cdot q = \frac{1}{2}\int_x^\infty dz\, [q^T\cdot p(z,t)]^2 > 0, \tag{2.22}$$

since q cannot be orthogonal to $p(z,t)$ for all $x \leq z < \infty$. Alternatively, (A.202) can be used to compute the principal minors of B, all of which are positive. It follows that for $\varepsilon > 0$ formula (2.21) is nonsingular. In particular, taking $\varepsilon = 1$ and making use of the identity $\exp \mathrm{Tr} = \det \exp$, which implies $\mathrm{Tr}\,\ln = \ln \det$, we have

$$u = -\partial_x p^T \cdot (I + B)^{-1} \cdot p = 2\partial_x^2 \,\mathrm{Tr}\,\ln(I + B) = 2\partial_x^2 \ln \det(I + B). \tag{2.23}$$

The last equality in this formula is the expression for the multisoliton solutions obtained by Gardner et al. (1967) and Hirota (1971).

2.3 Marčenko Integral Equation and Eigenvalue Problem

In an alternative manipulation of (2.16), we notice that if the following linear operator, defined on functions of two variables, is introduced

$$(\hat{b} \cdot f)(x,y) = \int_x^\infty f(x,z)\, b(z+y,t)\, dz \;, \quad (\forall f = f(x,y)), \tag{2.24}$$

then (2.16) can be written as

$$u = 2\partial_x \sum_1^\infty (-\varepsilon)^n (\hat{b}^{n-1} \cdot b)\Big|_{x=y}, \tag{2.25}$$

where b is interpreted as an argument for $\hat{b}$ in the form $b(x,z) = b(x+z,t)$, and the variable t participates only as a parameter. It is now natural to introduce the function $K = K(x,y,t)$ given by

$$K = \sum_1^\infty (-\varepsilon)^n \hat{b}^{n-1} \cdot b = -\varepsilon (I + \varepsilon \hat{b})^{-1} \cdot b\,. \tag{2.26}$$

Then the following equations ensue

$$u(x,t) = 2\,\partial_x K(x,x,t), \tag{2.27a}$$

$$0 = K(x,y,t) + \varepsilon\, b(x+y,t) + (\varepsilon \hat{b}.K)(x,y,t) =$$

$$= K(x,y,t) + \varepsilon\, b(x+y,t) + \varepsilon \int_x^\infty K(x,z,t)\, b(z+y,t)\,dz\,. \tag{2.27b}$$

We can think of these two equations as a way of summing (2.14), under the assumptions that led to (2.16). We recognize (2.27) as the Marčenko integral equation of the inverse scattering problem associated with (2.01) (Gardner et al. 1967).

From the definition of K in formula (2.26), and retracing the steps done to get (2.25) from (2.14), we can write

$$K(x,y,t) = \frac{1}{2i} \sum_1^\infty (-\varepsilon)^n \int_{\mathbb{C}^n} \frac{[d\lambda(k)]^n}{\prod_1^{n-1}(k_j + k_{j+1})} . \exp i\Big[\sum_1^{n-1}(k_j x + k_j^3 t) + \tfrac{1}{2}k_n(x+y) + k_n^3 t\Big]. \tag{2.28}$$

This formula provides an alternative definition of K which is independent of the assumption that $\lim\limits_{x \to \infty} b(x,t) = 0$.

Since u is now derived from the function K, it is natural to ask what equations K satisfies. The close relationship between the two functions is seen by comparing (2.14) and (2.28). There must be two equations for K, one involving the time dependence and the other characterizing the y dependence. The first one must be very closely related to (2.01) and is almost trivial to find, as we now proceed to show. From (2.28) we see that the effect of the operator $(\partial_x + \partial_y)$ on K is the same as that of the operator ∂_x on u. The operator ∂_t has the same effect on both K and u. Thus, by analogy with (2.01), we are led to study $[\partial_t+(\partial_x+\partial_y)^3]K$ and we immediately find that

$$[\partial_t+(\partial_x+\partial_y)^3]K(x,y,t)=-3u(x,t)(\partial_x+\partial_y)K(x,y,t), \tag{2.29}$$

where we have used (2.13), (2.14) and (2.28). The similarity of this formula with (2.01) is apparent.

In the search for the y-dependence equation we make use of (2.27), where time enters only as a parameter. The idea is to find an operator $\mathcal{L}$ that almost commutes with $\hat{b}$ and at the same time annihilates $b(x+y,t)$. Then applying $\mathcal{L}$ to (2.27b) and using (2.26) we should get an equation for K. To simplify our notation we will not display the time dependence in what follows. We first study the commutativity properties of $\hat{b}$ with respect to ∂_x and ∂_y. For any $f=f(x,y)$ we have

$$\partial_x(\hat{b}f)(x,y)=-f(x,x)\,b(x+y)+(\hat{b}\partial_x f)(x,y),$$

$$\partial_y(\hat{b}f)(x,y)=-f(x,x)\,b(x+y)-(\hat{b}\partial_y f)(x,y).$$

Therefore

$$(\partial_x-\partial_y)(\hat{b}f)(x,y)=[\hat{b}(\partial_x+\partial_y)f](x,y) \tag{2.30a}$$

$$(\partial_x+\partial_y)(\hat{b}f)(x,y)=[\hat{b}(\partial_x-\partial_y)f](x,y)-2f(x,x)\,b(x+y). \tag{2.30b}$$

It follows that $\mathcal{L}=(\partial_x^2-\partial_y^2)$ since we have $(\partial_x^2-\partial_y^2)b(x+y)=0$ and

$$[(\partial_x^2-\partial_y^2)\hat{b}f](x,y)=[\hat{b}(\partial_x^2-\partial_y^2)f](x,y)-2[f(x,x)]_x\,b(x+y). \tag{2.31}$$

Thus, applying $(\partial_x^2-\partial_y^2)$ to (2.27b), and using (2.27a), we have

$$(\partial_x^2-\partial_y^2)K(x,y,t)+\varepsilon\,\hat{b}(\partial_x^2-\partial_y^2)K(x,y,t)-\varepsilon u(x,t)\,b(x+y,t)=0.$$

Multiplying by $(I+\varepsilon\hat{b})^{-1}$ and using (2.26) it follows that

$$(\partial_x^2-\partial_y^2)K(x,y,t)+u(x,t)K(x,y,t)=0. \tag{2.32}$$

To prove (2.32) directly from (2.28) we simply observe that the effect of the operator $(\partial_x^2-\partial_y^2)=(\partial_x-\partial_y)(\partial_x+\partial_y)$ on K is multiplication of the integrands in (2.28) by

$$\begin{aligned}-(k_1+\dots+k_{n-1})(k_1+\dots+k_n) &= -(k_1+\dots+k_{n-1})^2-(k_1+\dots+k_{n-1})k_n = \\ &= -\sum_{1\le j\le \ell<n} k_jk_\ell-\sum_{1\le j<\ell\le n} k_jk_\ell = \\ &= -\sum_1^{n-1}(k_1+\dots+k_j)(k_j+k_{j+1}),\ (\forall n\ge 1),\end{aligned} \tag{2.33}$$

from which (2.32) immediately follows.

Since neither (2.29) nor (2.32) have coefficients depending on y, we can separate this variable. We do so by writing

$$K(x,y,t)=\int_{\mathbb{C}}\psi(x,t,\xi)\exp[i(\xi y+4\xi^3t)]\,d\mu(\xi), \tag{2.34}$$

for some measure $d\mu(\xi)$ on $\mathbb{C}$. The factor $\exp(4i\xi^3t)$ is introduced to simplify the time dependence of ψ. From (2.32) we find

$$\psi_{xx}+(\xi^2+u)\psi=0. \tag{2.35}$$

From (2.29) with the use of (2.35) we have

$$\psi_t = -4\,\psi_{xxx} - 6u\,\psi_x - 3u_x\,\psi = (4\xi^2 - 2u)\,\psi_x + u_x\,\psi. \tag{2.36}$$

We recognize in equation (2.35) the eigenvalue problem associated with (2.01) (Gardner et al. 1967), whose corresponding inverse scattering problem can be used to solve the initial value problem of (2.01) with (2.36) characterizing the time evolution of the scattering parameters. These two equations (2.35) and (2.36) will appear again in another context, when we study in Section 2.7 the effect of adding a set of Dirac δ's to $d\lambda(k)$ in (2.14).

We now proceed to write ψ and $d\gamma$ in (2.34) directly in terms of $d\lambda$. This is trivial if we look at formula (2.28), where y is practically separated and we only have to recognize $\xi = \frac{1}{2}k_n$. It follows that

$$\psi(x,t,\xi) = \left\{1 + \sum_1^\infty (-\epsilon)^n \int_{\mathbb{C}^n} \frac{e^{i\Omega_n}\,[d\lambda(k)]^n}{\prod_1^{n-1}(k_j+k_{j+1})\cdot(k_n+2\xi)}\right\} \exp[i(\xi x + 4\xi^3 t)], \tag{2.37a}$$

$$d\gamma(\xi) = -\frac{\epsilon}{2i}\, d\lambda(2\xi). \tag{2.37b}$$

It is also easy to check directly from (2.37a) formulas (2.35) and (2.36).

2.4 Operator Formalism

In this section we wish to sum (2.14) under circumstances more general than the ones that led to (2.23) or (2.27). In the process we will find formulas that generalize those in Section 2.2.

First we derive again the formulas for multisoliton solutions, starting now from (2.10). It is quite clear that (2.17) corresponds to

$$d\lambda(k) = \sum_m i\, a_m^2\, \delta(k - i\kappa_m)\, dk_R\, dk_I, \tag{2.38}$$

where $\delta(\cdot)$ is Dirac's δ-function on $\mathbb{C}$ thought as E^2, $k_R = \mathrm{Re}\, k$, $k_I = \mathrm{Im}\, k$ and the a_m 's and κ_m's are as in (2.17). Substitution of (2.38) in (2.10) gives

$$u_n = (-1)^n \partial_x \sum_{\substack{m_j \\ 1 \le j \le n}} a_{m_1}^2 \cdots a_{m_n}^2 \frac{e^{\sum_1^n (-\kappa_{m_j} x + \kappa_{m_j}^3 t)}}{\prod_1^{n-1} (\kappa_{m_j} + \kappa_{m_{j+1}})} =$$

$$= (-1)^n \partial_x \sum_{\substack{m_j \\ 1 \le j \le n}} p_{m_1} B_{m_1 m_2} \cdots B_{m_{n-1} m_n} p_{m_n} = (-1)^n \partial_x\, p^T B^{n-1} p, \tag{2.39}$$

where p and B are as in (2.18) and (2.20) respectively. From (2.39) all the other results in Section 2.2 follow easily. The factor $(\kappa_{m_1} + \kappa_{m_2})^{-1}$ in $B_{m_1 m_2}$ comes now naturally from the factor $(k_j + k_{j+1})^{-1}$ in Φ_n . These last manipulations can be generalized to a very general $d\lambda(k)$, not necessarily discrete as in (2.38). To do so we only need to replace the matrix B by an operator on a possibly infinite dimensional space, as shown in what follows.

By analogy with (2.38) assume now that we can write

$$d\lambda(k) = i\, a^2(k)\, d\rho(k), \tag{2.40}$$

where $d\rho(k)$ is a non-negative measure on $\mathbb{C}$ and $a = a(k)$ is a function defined on the domain $\mathcal{D}$ of $d\rho(k)$. Define now the operator B and the symmetric bilinear form $[\cdot,\cdot]$, both acting on functions defined on $\mathfrak{D}$, by means of the formulas

$$(\mathcal{B}f)(\ell) = \int_{\mathbb{C}} i\,p(\ell)\,(\ell+k)^{-1}\,p(k)\,f(k)\,d\rho(k)\,, \quad (\forall \ell \in \mathfrak{D}), \tag{2.41a}$$

$$[f,g] = \int_{\mathbb{C}} f(k)\,g(k)\,d\rho(k) = \langle \bar{f}, g\rangle, \tag{2.41b}$$

where

$$p(k) = a(k)\exp[\tfrac{i}{2}(kx+k^3t)], \quad (\forall k \in \mathfrak{D}), \tag{2.41c}$$

and $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathcal{L}^2(d\rho)$. The variables x and t enter as parameters in $\mathcal{B}$ and p.† We note that $\mathcal{B} = \mathcal{B}^T$ and that $\partial_x \mathcal{B} = -\frac{1}{2}pp^T$, where the transposes are defined with respect to $[\cdot,\cdot]$.†† In terms of $\mathcal{B}$ and p, u_n may be expressed as

$$u_n = (-1)^n\,\partial_x\, p^T\,\mathcal{B}^{n-1}\,p. \tag{2.42}$$

This formula follows from (2.10) in the same way as (2.39). If we assume that $p \in \mathcal{L}^2(d\rho)$ and that the singularities in the kernel of $\mathcal{B}$ can be taken care of in such a way that $\mathcal{B}$ is a bounded operator on $\mathcal{L}^2(d\rho)$, then we can sum (2.14), for $\|\epsilon\mathcal{B}\| < 1$, to

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†This dependence is displayed only when needed.

††For any operator A and functions f, g on $\mathfrak{D}$, transposes are defined by

$$[f, Ag] = [A^T f, g]\,, \qquad f^T g = [f,g]\,.$$

$$u = -\varepsilon \partial_x p^T (I+\varepsilon B)^{-1} p = 2\varepsilon \partial_x \operatorname{Tr}[(I+\varepsilon B)^{-1} \partial_x B]. \tag{2.43}$$

Whenever $(I+\varepsilon B)$ is not invertible, (2.43) is meaningless. However, at such points it is the actual solution of (2.01) that has a singularity, not the way we write it.

The functions b, K and ψ introduced earlier can be written in terms of p and B as follows:

$$b(x+y,t) = \tfrac{1}{2} p^T(x,t)\, p(y,t),$$

$$K(x,y,t) = -\tfrac{\varepsilon}{2} p^T(x,t)[I+\varepsilon B(x,t)]^{-1} p(y,t),$$

$$\psi(x,t,\xi) = \left\{1 + p^T(x,t)[I+\varepsilon B(x,t)]^{-1} q(x,t,\xi)\right\} \exp[i(\xi x + 4\xi^3 t)],$$

where q, as an element of $\mathcal{L}^2(d\rho)$, is given by $q(k) = 2i(k+2\xi)^{-1} p(k)$, $(\forall k \in \mathfrak{D})$.

As usual, the actual expressions for the Φ_n's, as introduced by formula (2.11), are not unique. Thus, for example, using that $(k_1+\dots+k_n) = (1/2)[(k_1+k_2)+(k_2+k_3)+\dots+(k_n+k_1)]$ we can rewrite equation (2.14) as

$$u = -2\partial_x^2 \sum_1^\infty \frac{(-\varepsilon)^n}{n} \int_{\mathbb{C}^n} \frac{e^{i\Omega_n}}{(k_1+k_2)\dots(k_{n-1}+k_n)(k_n+k_1)} [d\lambda(k)]^n. \tag{2.44}$$

Substitution of (2.38) in (2.44) gives directly $u = 2\partial_x^2 \operatorname{Tr} \ln(I+\varepsilon B)$, where B is as in (2.20). However, for more general $d\lambda(k)$'s (2.44) presents difficulties, since traces of infinite operators are hard to deal with, even when they exist.

2.5 Continuous Measures on the Real Line

It is the purpose of this section to give an example of the operator formalism developed in the preceding section, and at the same time to justify the boundedness assumptions made on the operator $\mathcal{B}$.

In the notation introduced in Section 2.4, let us take as $d\rho(k)$ the usual measure on the real line and assume that $a = a(k)$, defined for $-\infty < k < \infty$, is bounded and square integrable. Furthermore, assume that $a(k) = \pm a^*(-k)$ for all k , so that u is real. We must clarify the definition (2.41a) of $\mathcal{B}$ in this case. Several possibilities are open to us, depending on whether we integrate going over the singularity $k = -\ell$, under it or across it with a principal value. We choose to integrate going over the singularity. Then we have, for any $f \in \mathcal{L}^2(dk)$,

$$(\mathcal{B}f)(\ell) = \int_{-\infty+i0^+}^{\infty+i0^+} i\, p(\ell)(\ell+k)^{-1} p(k) f(k)\, dk = \int_{-\infty+i0^+}^{\infty+i0^+} p(\ell)\left[\int_0^\infty e^{i(\ell+k)\xi} d\xi\right] p(k) f(k)\, dk =$$

$$= p(\ell) \int_0^\infty e^{i\ell\xi}\left[\int_{-\infty}^{\infty} e^{ik\xi} p(k) f(k)\, dk\right] d\xi \;,\; -\infty < \ell < \infty . \tag{2.45}$$

It follows that if $\mathcal{P}$ is the operator multiplication by p, U is the operator multiplication by 1 on the positive real line and 0 on the negative real line, and F is the unitary Fourier transform

$$(Ff)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi k} f(k)\, dk \;,\; (\forall f \in \mathcal{L}^2(dk)),$$

We have

$$\mathcal{B} = 2\pi\, \mathcal{P} F^* U F^* \mathcal{P} \;,\; \|\mathcal{B}\| \leq 2\pi(\|a\|_\infty)^2. \tag{2.46}$$

The conditions that lead to formula (2.43) are thus satisfied if $|\varepsilon|(\|a\|_\infty)^2 < 1/2\pi$.

We now proceed to show that $(I+B)^{-1}$ exists for all x and t, even if $(\|a\|_\infty)^2 = 1/2\pi$, provided that

$$\limsup_{k\to 0} |\arg a(k) - \arg a(0)| < \pi/4 \quad \text{and} \quad \sup_{k\geq\delta} |a(k)| < 1/\sqrt{2\pi}, \ (\forall\delta>0).$$

For any fixed x and t these two conditions on $a=a(k)$ translate into $p(k)=a(k)\exp[\frac{i}{2}(kx+k^3t)]$. Thus there exists α independent of k (but not necessarily independent of x and t) such that $\|p^2-\alpha\|_\infty < 1/2\pi$. Since we have

$$(I+B)^{-1} = (I+2\pi PF^*UF^*P)^{-1} = I - 2\pi P(I+2\pi F^*UF^*P^2)^{-1}F^*UF^*P,$$

$$(I+2\pi F^*UF^*P^2)^{-1} = [I+2\alpha\pi F^*UF^* + 2\pi F^*UF^*(P^2-\alpha I)]^{-1} =$$

$$= [I+2\pi F^*UF^*(P^2-\alpha I)]^{-1}(I-2\alpha\pi F^*UF^*),$$

where we have used that $(F^*UF^*)^2=0$, the existence of $(I+B)^{-1}$ follows immediately from $\|2\pi F^*UF^*(P^2-\alpha I)\| < 1$.

Under suitable conditions u as given by formulas (2.43) and (2.46) vanishes as $|x|\to\infty$. For example, this is true if a is uniformly continuous and $\|a\|_\infty < 1/\sqrt{2\pi|\varepsilon|}$.

If we choose in (2.41a) to integrate going under the singularities or across them, we obtain

$$B = -2\pi PFUFP, \quad \|B\| \leq 2\pi(\|a\|_\infty)^2,$$

in the first case, and in the second case

$$B = i\pi PGP, \quad \|B\| \leq \pi(\|a\|_\infty)^2,$$

where G is the self-adjoint, unitary operator

$$(Gf)(\ell) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{f(k)}{\ell+k}\, dk \quad , -\infty < \ell < \infty, (\forall f \in \mathcal{L}^2(dk)).$$

2.6 The Problem of Inversion

From equation (2.37a) it follows that

$$\psi(x,t,\xi) = [1+o(1)] \exp[i(\xi x + 4\xi^3 t)] \text{ , as } x \to \infty, \tag{2.47}$$

provided that the same hypotheses that led to (2.16) hold. In this case we also have $u = o(1)$ as $x \to \infty$, and since ψ satisfies the equation

$$\psi_{xx} + (\xi^2 + u)\psi = 0, \tag{2.35}$$

it follows that, for $\operatorname{Im}\xi \geq 0$, $\psi(x,t,\xi) = \chi(x,t,\xi)\exp(4i\xi^3 t)$, where χ is the right Jost function corresponding to the scattering problem associated with (2.35). That is, χ is defined for $\operatorname{Im}\xi \geq 0$ by

$$\chi_{xx} + (\xi^2 + u)\chi = 0 \quad , \quad \chi \sim e^{i\xi x} \quad \text{as} \quad x \to \infty .$$

It is our purpose in this section to use this relationship between ψ and χ to write $d\lambda(k)$ in terms of the scattering parameters of (2.35), thus inverting (2.14) for $d\lambda$ in terms of u.

First, we write an integral equation for ψ , in which $d\lambda$ is in the kernel,

$$\psi(x,t,\xi) = \left[1 - \frac{\varepsilon}{2}\int_{\mathbb{C}} \frac{e^{i(\zeta x - 4\zeta^3 t)}}{(\xi+\zeta)} \psi(x,t,\zeta)\, dm(2\zeta)\right] e^{i(\xi x + 4\xi^3 t)} , \tag{2.48}$$

where $dm(k) = e^{ik^3 t} d\lambda(k)$. This equation follows easily from (2.37a) after the integration over k_n in each term of the summation is written separately from the others and the change $k_n = 2\zeta$ is made. Now $dm(k) \neq 0$ only for $\mathrm{Im}\, k \geq 0$ under the assumptions that led to (2.47). Thus we can write (2.48) as

$$\psi(x,t,\xi) = \left[1 - \frac{\varepsilon}{2} \int_{\mathbb{C}} \frac{e^{i\zeta x}}{(\xi+\zeta)} \chi(x,t,\zeta)\, dm(2\zeta)\right] e^{i(\xi x + 4\xi^3 t)}. \tag{2.49}$$

Assume now that $d\lambda$ is a combination of the cases treated in Sections 2.2 and 2.5. That is, dm is given by

$$\int_{\mathbb{C}} f(\zeta)\, dm(2\zeta) = \int_{-\infty+i0^+}^{\infty+i0^+} 2i\, f(\zeta)\, \alpha(2\zeta)\, d\zeta + 2i \sum_j \alpha_j f(i\kappa_j)\,, \ (\forall f), \tag{2.50}$$

where $\alpha = \alpha(k) = \alpha^*(-k)$, $-\infty < k < \infty$, is an appropriate function of the real variable k and the α_j and κ_j are positive real numbers. The α_j's and α evolve on time satisfying the equations $\alpha_t = ik^3\alpha$ and $\alpha_{j,t} = 8\kappa_j^3 \alpha_j$. In this case, under suitable assumptions on the function α, we have

$$u = O(1) \quad \text{as } |x| \longmapsto \infty,$$

and the left Jost function $\tilde{\chi}(x,t,\xi) \sim e^{-i\xi x}$ as $x \longmapsto -\infty$, $\mathrm{Im}\,\xi \geq 0$, corresponding to (2.35) can be defined. Then for $\mathrm{Im}\,\xi = 0$ we have

$$T(\xi)\tilde{\chi}(\xi) = \chi(-\xi) + \beta(\xi)\chi(\xi) \ , \ T(\xi)\chi(\xi) = \tilde{\chi}(-\xi) - \frac{T(\xi)}{T(-\xi)}\beta(-\xi)\tilde{\chi}(\xi), \tag{2.51}$$

where T and β are the transmission and reflection coefficients respectively.

We now seek to find a relationship between ψ and $\tilde{\chi}$. Substituting (2.50) into (2.49) we have

$$\psi(x,t,\xi)=\Big[\,1-\int_{-\infty+i0^+}^{\infty+i0^+}\frac{\varepsilon i\, e^{i\zeta x}\,\chi(x,t,\zeta)\,\alpha(2\zeta)\,d\zeta}{(\xi+\zeta)}\;-$$

$$-\,\varepsilon i\sum_j\frac{\chi(x,t,i\kappa_j)\,\alpha_j\,e^{-\kappa_j x}}{(\xi+i\kappa_j)}\Big]\,e^{i(\xi x+4\xi^3 t)}. \tag{2.52}$$

For ξ real, ψ is bounded as a function of x and the same is true for the integral term in (2.52). Thus, we must have that the summation term in (2.52) is bounded as a function of x for real ξ. The latter can only happen if the $\chi_j=\chi(x,t,i\kappa_j)$, $(\forall_j)$, are eigenfunctions of (2.35) with the κ_j^2, $(\forall_j)$, their corresponding eigenvalues. Without loss of generality, since we can always add new terms to the summation in (2.52) with $\alpha_j=0$, we can assume that all the eigenvalues of (2.35) are included in (2.52). Introduce now the time dependent constants $c_j=c_j(t)=\chi_j/\tilde{\chi}_j$, $(\forall_j)$, and substitute $\chi_j=c_j\tilde{\chi}_j$, $(\forall_j)$, and (2.51) into (2.52). Then, for $\operatorname{Im}\xi\neq 0$, we have

$$\psi(x,t,\xi)=\Big\{1-\int_{-\infty}^{\infty}\varepsilon i\big[T^{-1}(\zeta)\,\tilde{\chi}(-\zeta)-T^{-1}(-\zeta)\beta(-\zeta)\,\tilde{\chi}(\zeta)\big]\frac{\alpha(2\zeta)\,e^{i\zeta x}}{(\xi+\zeta)}\,d\zeta\;-$$

$$-\,\varepsilon i\sum_j\alpha_j c_j\,(\xi+i\kappa_j)^{-1}\,\tilde{\chi}_j\,e^{-\kappa_j x}\Big\}\,e^{i(\xi x+4\xi^3 t)}=$$

$$=\Big\{1+\int_{-\infty}^{\infty}\varepsilon i\,\frac{\beta(-\zeta)\,\alpha(2\zeta)}{T(-\zeta)(\xi+\zeta)}\,d\zeta-\varepsilon i\sum_j\frac{\alpha_j\,c_j}{(\xi+i\kappa_j)}+o(1)\Big\}\,e^{i(\xi x+4\xi^3 t)},\quad \text{as } x\to-\infty,$$

where the asymptotic behavior of $\tilde{\chi}$ at $-\infty$ has been used. It follows that, for $\operatorname{Im}\xi<0$,

$$\psi(x,t,\xi)=\left[1+\varepsilon i\int_{-\infty}^{\infty}\frac{\beta(-\zeta)\alpha(2\zeta)}{T(-\zeta)(\xi+\zeta)}d\zeta-\varepsilon i\sum_j\frac{\alpha_j c_j}{(\xi+i\kappa_j)}\right]e^{4i\xi^3 t}\tilde{\chi}(x,t,-\xi).$$

Taking the limit $\mathrm{Im}\,\xi\longrightarrow 0^-$ in this last formula, using the equation

$$\psi(\xi+i0^-)=2\pi\varepsilon\alpha(-2\xi)\chi(-\xi)e^{4i\xi^3 t}+\psi(\xi)=$$

$$=[2\pi\varepsilon\alpha(-2\xi)\chi(-\xi)+\chi(\xi)]e^{4i\xi^3 t},\quad \mathrm{Im}\,\xi=0,$$

a consequence of (2.52), and comparing with (2.51), we find for $\mathrm{Im}\,\xi=0$

$$\beta(\xi)=2\pi\varepsilon\alpha(2\xi),\quad T(\xi)=1+\int_{-\infty+i0^-}^{\infty+i0^-}\varepsilon i\frac{\beta(-\zeta)\alpha(2\zeta)}{T(-\zeta)(\zeta-\xi)}d\zeta+\varepsilon i\sum_j\frac{\alpha_j c_j}{(\xi-i\kappa_j)}. \tag{2.53}$$

The expression for $T(\xi)$ in (2.53) is obviously valid not only for $\mathrm{Im}\,\xi=0$, but for $\mathrm{Im}\,\xi\geq 0$. Using now the well known result

$$\mathrm{Res}\,T(\xi)\Big|_{\xi=i\kappa_j}=ic_j\gamma_j^{-1},\quad \gamma_j=\|\chi_j\|_2^2,\quad(\forall j),$$

and (2.53) we find $\alpha_j=1/\varepsilon\gamma_j$, $(\forall j)$. In particular, none of the α_j's vanishes, so that all the eigenvalues were originally considered in (2.52).

We are now ready to write $d\lambda(k)=e^{-ik^3t}dm(k)$ in terms of the scattering parameters of (2.35). From the expressions of the α_j's in terms of the normalization constants γ_j, from (2.50) and from (2.53), we have

$$\int_{\mathbb{C}} f(k)\,d\lambda(k)=\int_{\mathbb{C}} f(k)e^{-ik^3t}dm(k)=$$

$$=-\frac{1}{2\pi i\varepsilon}\int_{-\infty+i0^+}^{\infty+i0^+}f(k)\beta\left(\frac{k}{2}\right)e^{-ik^3t}dk+\frac{2i}{\varepsilon}\sum_j f(2i\kappa_j)\gamma_j^{-1}e^{-8\kappa_j^3 t},(\forall f). \tag{2.54}$$

2.7 Transformation Properties

From (2.14) it is rather obvious that any transformation between measures, $d\lambda \longrightarrow d\lambda'$, has a corresponding one $u \longrightarrow u'$, between solutions of (2.01) and vice versa. However, a simple transformation on one side need not correspond to a simple one on the other side, where by simple we mean that they can be explicitly displayed. We give in this section two examples in which the transformations on both sides are simple. The first one leads to an extension of the formulas in Section 2.4, which effectively corresponds to doing a small parameter expansion around an arbitrary solution of (2.01), instead of $u \equiv 0$ as in Section 2.1. The second example is the Bäcklund transformation for the KdV equation, first found by Wahlquist and Estabrook (1973).

(I) First we seek to find what is the effect on u of addition on the side of $d\lambda$. Let

$$d\lambda'(k) = d\lambda(k) + \sum_m i a_m^2 \, \delta(k - i\kappa_m) \, dk_R \, dk_I , \tag{2.55}$$

where $\delta(\cdot)$, k_R, k_I , the a_m 's and κ_m's are as in (2.38) and (2.17). Substituting (2.55) into (2.14) we find, after some manipulation, that

$$u' = u + \partial_x \sum_{n=1}^{\infty} (-\epsilon)^n \Big(\sum_0^{\infty} (-\epsilon)^\nu P_\nu\Big)^T \Big(\sum_1^{\infty} (-\epsilon)^{\nu-1} B_\nu\Big)^{n-1} \Big(\sum_0^{\infty} (-\epsilon)^\nu P_\nu\Big) , \tag{2.56}$$

where the P_ν 's and B_ν 's are column vectors and square matrices, respectively, given by

$$(P_0)_m = a_m \exp[\tfrac{1}{2}(-\kappa_m x + \kappa_m^3 t)] , (\forall m),$$

$$(P_\nu)_m = \int_{C^\nu} \frac{e^{i\Omega_\nu}\,[d\lambda(k)]^\nu}{\prod_1^{\nu-1}(k_j+k_{j+1})\cdot(k_\nu+iK_m)}\, a_m \exp\left[\tfrac{1}{2}(-K_m x + K_m^3 t)\right], \quad 1\leq\nu<\infty, (\forall m),$$
(2.57a)

$$(B_1)_{m_1 m_2} = \frac{1}{(K_{m_1}+K_{m_2})}(P_0)_{m_1}(P_0)_{m_2}, \quad (\forall m_1, m_2),$$

$$(B_{\nu+1})_{m_1 m_2} = i(P_0)_{m_1}\int_{C^\nu}\frac{e^{i\Omega_\nu}\,[d\lambda(k)]^\nu}{(iK_{m_1}+k_1)\prod_1^{\nu-1}(k_j+k_{j+1})(k_\nu+iK_{m_2})}(P_0)_{m_2}, \quad 1\leq\nu<\infty, (\forall m_1, m_2). \tag{2.57b}$$

Thus, if we define the column vector P and the matrix B by

$$P = \sum_0^\infty (-\varepsilon)^\nu P_\nu \quad , \quad B = \sum_0^\infty (-\varepsilon)^\nu B_{\nu+1}, \tag{2.58}$$

we have

$$u' = u - \varepsilon\,\partial_x\, P^T(I+\varepsilon B)^{-1}P\,. \tag{2.59}$$

From (2.37a), (2.57a) and (2.58) it is easy to see that

$$(P)_m(x,t) = a_m\,\psi_m(x,t,iK_m/2)\,, (\forall m). \tag{2.60}$$

Thus from (2.35) and (2.36) we see that P satisfies the equations

$$\partial_x^2 P = -(u+\Lambda)P\,, \quad \partial_t P = u_x P - (2u - 4\Lambda)\,\partial_x P\,, \tag{2.61}$$

where Λ is the diagonal matrix with diagonal elements $-K_m^2/4$. The following equations are also easy to verify, using (2.57b) and (2.58),

$$\partial_x B = -\tfrac{1}{2}PP^T \quad , \quad \partial_t B = -uPP^T - 2(\Lambda PP^T + PP^T\Lambda + P_x P_x^T)\,, \tag{2.62a}$$

$$2(B\Lambda - \Lambda B) = PP_x^T - P_x P^T\,, \quad B = B^T. \tag{2.62b}$$

Now let u be any solution of (2.01), not necessarily representable in terms of a measure $d\lambda$ as in (2.14). Then we can check directly that any solution of (2.61) and (2.62) gives, through (2.59), another solution u' of (2.01). p can be allowed to take values in a Hilbert space with a conjugation (see A.3), with B and Λ symmetric operators there.

Equations (2.61) and (2.62a) are consistent, provided that u satisfies (2.01), and they imply ∂_x and ∂_t of (2.62b), which is then true up to a constant. If a solution p of (2.61) is such that p and p_x vanish fast enough as $x \longmapsto \infty$, then $B=(1/2)\int_x^\infty pp^T$ solves (2.62).

When B is a matrix, using that $B_x=-(1/2)pp^T$ (2.59) can also be written as follows

$$u' = u + 2\partial_x^2 \, \mathrm{Tr} \, \ln (I+\varepsilon B) = u + 2\partial_x^2 \ln \det (I+\varepsilon B). \tag{2.63}$$

Examples

(i) Take $u = x/6t$ for $t>0$ and Λ diagonal, with real diagonal elements λ_j; then a solution of (2.61), (2.62) is

$$p_j = i a_j t^{1/6} \mathrm{Ai}\left[-x/(6t)^{1/3} - \lambda_j (6t)^{2/3}\right], (\forall j) \quad \text{and} \quad B = -\tfrac{1}{2}\int_{-\infty}^{x} pp^T,$$

where the a_j's are arbitrary real constants and $\mathrm{Ai}(\cdot)$ is the Airy function. We note that B is a real, symmetric and positive definite matrix, so that, for $\varepsilon>0$ and $t>0$, u' as given by (2.59) or (2.63) is nonsingular.

(ii) Take $u\equiv 0$ and $\Lambda = 0$ (scalar). Then $p=rx$ and $B=-\frac{1}{6}r^2x^3-ar^2t$, where r is an arbitrary real number, solve (2.61) and (2.62), then after some manipulation we have, from (2.63),

$$u' = -6x(x^3 - 24t')(x^3 + 12t')^{-2}, \text{ where } t' = t + 1/2\varepsilon r^2.$$

We observe that $u'(x=0) = u'(x=\infty) = 0$ and that u' is nonsingular for $x, t' > 0$. This solution is the same as the one presented by Moses (1976).

(iii) When u is a cnoidal wave the first equation in (2.61) is a Lamé equation of index $n = 1$. Its solution can then be expressed explicitly in terms of σ-functions and ζ-functions (Ince 1956). We then can write explicit formulas for the interaction of a cnoidal wave with a multisoliton solution. Similar results have been obtained by Wahlquist (1976) using Bäcklund transformation techniques.

(iv) Floquet theory (Ince 1956) and (2.59) through (2.63) can be used to prove that the result of the interaction of a soliton with a periodic solution of (2.01) is another periodic solution of the same period as the original one. We do not know whether these two periodic solutions are actually the same one, save for a (possibly complex) phase shift, as in the case of interaction of cnoidal waves with solitons, or if a more complicated relationship is involved.

(II) Introduce now the potential function v defined by $u = -iv_x$. The expression for v in terms of $d\lambda$ is obvious from (2.14). Let us now assume that the transformation $d\lambda \mapsto d\lambda'$ is such that, for some $\tau \in \mathbb{C}$, we have

$$(k + i\tau)\, d\lambda(k) + (k - i\tau)\, d\lambda'(k) = 0. \tag{2.64}$$

It is then easy to prove by induction that

$$-\sum_1^{n-1}(k_j+k_{j+1})[d\lambda'(k)]^j[d\lambda(k)]^{n-j}=(k_1+i\tau)[d\lambda(k)]^n+(k_n-i\tau)[d\lambda'(k)]^n,$$

$$-\sum_1^{n-1}(k_j+k_{j+1})[d\lambda(k)]^j[d\lambda'(k)]^{n-j}=(k_1-i\tau)[d\lambda'(k)]^n+(k_n+i\tau)[d\lambda(k)]^n,$$

from which it follows that, for all $n=1,2,\ldots$

$$\sum_1^{n-1}(k_j+k_{j+1})\left\{[d\lambda'(k)]^j-[d\lambda(k)]^j\right\}\left\{[d\lambda'(k)]^{n-j}-[d\lambda(k)]^{n-j}\right\}=$$

$$=2(k_1+\ldots+k_n+i\tau)[d\lambda(k)]^n+2(k_1+\ldots+k_n-i\tau)[d\lambda'(k)]^n.$$

This last formula has an immediate counterpart in terms of v and v'

$$(v'-v)^2=-2i(v'+v)_x-2i\tau(v'-v),$$

or

$$(v'+i\tau-v)^2+\tau^2=2(u'+u). \tag{2.65}$$

This is the time independent part of the Bäcklund transformation for (2.01).

2.8 Higher Order KdV Equations. Lax's Sequence

We might ask what other equations for u could be solved in the form

$$u=-i\partial_x\sum_1^\infty(-\varepsilon)^n\int_{\mathbb{C}^n}\frac{e^{i\theta_n}}{\prod_1^{n-1}(k_j+k_{j+1})}[dm(k,t)]^n, \tag{2.66a}$$

$$\partial_t dm(k,t)=-i\omega(k)\,dm(k,t), \tag{2.66b}$$

where $\theta_n = (k_1 + \cdots + k_n)x$, when the dispersion function $\omega = \omega(k)$ is no longer $-k^3$ as in (2.14). For example, we might ask what nonlinear equation corresponds to the linear equation $u_t + \partial_x^5 u = 0$, i.e., to $\omega(k) = k^5$.

It is clear that if we define, for any function $\nu = \nu(k)$,

$$\mathcal{M}_\nu = \sum_1^\infty (-\varepsilon)^n \int_{\mathbb{C}^n} \frac{(\nu_1 + \cdots + \nu_n)}{\prod_1^{n-1}(k_j + k_{j+1})} e^{i\theta_n} [dm(k,t)]^n, \tag{2.67}$$

then u satisfies the equation

$$u_t + \partial_x \mathcal{M}_\omega = 0. \tag{2.68}$$

Now the question is whether $\mathcal{M}_\nu$ can be written explicitly in terms of u and its derivatives. Obviously $\mathcal{M}_k = u$ and $\mathcal{M}_{k^3} = -3u^2 - u_{xx}$, where the latter follows from the results in Section 2.1. We now search for an inductive argument to get $\mathcal{M}_{k^{2n+1}}$, for all $n = 0, 1, 2, \ldots$. In doing so, the following identity is useful

$$\begin{aligned}
(\sum_1^n b_j)^3 (\sum_1^n a_j) &= (\sum_1^n b_j)(\sum_1^n b_j^2 a_j) + \sum_1^{n-1} [(b_1 + \cdots + b_j)^2 (b_j + b_{j+1})(a_{j+1} + \cdots + a_n) + \\
&\quad + (a_1 + \cdots + a_j)(b_j + b_{j+1})(b_{j+1} + \cdots + b_n)^2 + 2(b_1 + \cdots + b_j)(b_j + b_{j+1})(b_{j+1} + \cdots + b_n)(a_{j+1} + \cdots + a_n) + \\
&\quad + 2(b_1 + \cdots + b_j)(a_1 + \cdots + a_j)(b_j + b_{j+1})(b_{j+1} + \cdots + b_n)] + \sum_{1 \le j < l < m \le n} [\\
&\quad (b_1 + \cdots + b_j)(b_j + b_{j+1})(a_l b_m - b_l a_m) + (b_j a_l - a_j b_l)(b_{m-1} + b_m)(b_m + \cdots + b_n)],
\end{aligned} \tag{2.69}$$

where a_j and b_j $(1 \le j \le n = 1, 2, 3, \ldots)$ are arbitrary numbers. To prove (2.69) we first verify it for $n=4$ and then use (A.102). With $b_j = k_j$ and $a_j = \nu_j$, (2.69) implies

$$\partial_x^3 \mathcal{M}_\nu = -\partial_x \mathcal{M}_{k^2\nu} - 2u_x \mathcal{M}_\nu - 4u\partial_x \mathcal{M}_\nu, \tag{2.70}$$

where the last Σ in (2.69) does not contribute because of its commutator nature. Equation (2.70) gives $\mathcal{M}_{k^2\nu}$ in terms of $\mathcal{M}_\nu$, save for an arbitrary constant of integration, which is determined by the boundary condition $\mathcal{M}_\nu = 0$ if $u \equiv 0$. We can now write an inductive process that gives $K_n(u) = \mathcal{M}_{k^{2n+1}}, (\forall n \geq 0)$, starting from $K_0(u) = u$,

$$\partial_x K_{n+1}(u) = (-\partial_x^3 - 2u_x - 4u\partial_x) K_n(u), \quad K_{n+1}(0) = 0. \tag{2.71}$$

The K_n's are polynomials in u and its partial derivatives. The first few are $K_1 = -3u^2 - u_{xx}$, $K_2 = 10u^3 + 5u_x^2 + 10uu_{xx} + u_{xxxx}$, etc. The sequence of equations $u_t + \partial_x K_n(u) = 0$, which by construction is solved by (2.66) with $\omega(k) = k^{2n+1}$, is the Lax sequence of generalized KdV equations (Lax 1968). The particular case $n=1$ gives back the KdV equation.

From (2.67) it is obvious that $\mathcal{M}_\nu$ is linear in ν. Thus we can write $\mathcal{M}_\nu$ for any ν of the form $\nu(k) = k f(k^2)$, with f an entire function, i.e., if $f(\zeta) = \sum_0^\infty f_n \zeta^n$ then $\mathcal{M}_\nu = \sum_0^\infty f_n K_n(u)$. Moreover, if we assume that dm is such that $u, \mathcal{M}_\nu$ and all the necessary partial derivatives vanish sufficiently rapidly as $x \to \infty$, then we can write $\mathcal{M}_\nu$ for any ν such that $\nu(k) = k f(k^2)/g(k^2)$, with f and g entire functions. To prove this we observe that in this case (2.70) may be written as

$$\partial_x \mathcal{M}_{k^2\nu} = \mathcal{L}\, \partial_x \mathcal{M}_\nu \qquad \text{with} \qquad \mathcal{L} = -\partial_x^2 - 4u + 2u_x \int_x^\infty . \tag{2.72}$$

so that $\partial_x \mathcal{M}_{kf(k^2)/g(k^2)} = g^{-1}(\mathcal{L})\, f(\mathcal{L})\, \partial_x \mathcal{M}_k = g^{-1}(\mathcal{L}) f(\mathcal{L})\, u_x$. Thus (2.66), with $\omega(k) = k f(k^2)/g(k^2)$, solves

$$g(\mathcal{L})\, u_t + f(\mathcal{L})\, u_x = 0 , \tag{2.73}$$

with the boundary condition that u vanishes fast enough as $x \mapsto \infty$. Equation (2.73) is the same equation found by Ablowitz et al.(1974a) as solvable by the inverse scattering transform associated with equation (2.35).

We give now another example of a class of ν's for which $\mathcal{M}_\nu$ can be written in terms of u. Assume that $dm(k,t)$ vanishes for $\mathrm{Im}\, k < 0$ and that u and $\mathcal{M}_\nu$ decay fast enough as $x \mapsto \infty$. Assume also that ν is given by

$$\nu(k) = \int_{-\infty}^{\infty} \frac{k\, h(\xi)}{(k^2 - 4\xi^2)}\, d\xi \quad (\mathrm{Im}\, k > 0) \quad \text{and} \quad \nu(k) = \nu(k + i0^+) \ (\mathrm{Im}\, k = 0), \tag{2.74}$$

where h is some given function of ξ . Then we have

$$\mathcal{M}_\nu = \int_{-\infty}^{\infty} h(\xi)\, d\xi \left\{ \sum_1^\infty (-\epsilon)^n \int_{\mathbb{C}^n} \frac{1}{2} \frac{\sum_1^n (k_j + 2\xi)^{-1} + (k_j - 2\xi)^{-1}}{\prod_1^{n-1} (k_j + k_{j+1})} \cdot e^{i\theta_n} [dm(k,t)]^n \right\} .$$

But $\sum_1^n (k_j + 2\xi)^{-1} + (k_j - 2\xi)^{-1} = (k_1 - 2\xi)^{-1} + \sum_1^{n-1} \frac{(k_j + k_{j+1})}{(k_j + 2\xi)(k_{j+1} - 2\xi)} + (k_n + 2\xi)^{-1}$, so that if

$$\varphi(x,t,\xi) = \left\{ 1 + \sum_1^\infty (-\epsilon)^n \int_{\mathbb{C}^n} \frac{e^{i\theta_n} [dm(k,t)]^n}{\prod_1^{n-1} (k_j + k_{j+1}) \cdot (k_n + 2\xi)} \right\} e^{i\xi x} , \tag{2.75}$$

we have

$$\Omega_0 = (1/2)\int_{-\infty}^{\infty} h(\xi)\,[\varphi(\xi)\varphi(-\xi) - 1]\, d\xi. \qquad (2.76)$$

Therefore (2.68) implies that, for $[dm(-k^*,t)]^* = -dm(k,t)$, $h(\xi)$ real and

$$\omega(k) = \int_{-\infty}^{\infty} \frac{k\,h(\xi)}{(k^2-4\xi^2)}\, d\xi = -\omega^*(-k^*) \quad (\mathrm{Im}\,k>0)\,,\ \omega(k) = \omega(k+i0^+) \quad (\mathrm{Im}\,k=0),$$

equations (2.66) and (2.75) will provide a real solution of the system

$$u_t = (1/2)\,\partial_x \int_{-\infty}^{\infty} h(\xi)\,(1-|\varphi(\xi)|^2)\, d\xi \quad , \quad \varphi_{xx} + (\xi^2+u)\varphi = 0\,, \qquad (2.77).$$

with the boundary conditions $u(x=\infty)=0$ and $|\varphi^2(x=\infty)|=1$. In writing (2.77) from (2.76) we have used that $\varphi(-\xi)=\varphi^*(\xi)$ $(\forall -\infty<\xi<\infty)$ for u real. The equation for φ is the same as (2.35), and can be proved in the same way, since φ and ψ have the same functional form.

Results similar to, or the same as, those found in the previous sections apply to the class of equations found in this section.

A final remark is that equation (2.01), for u square-matrix valued, is also solvable by small parameter expansions. In fact all the formulas of Sections 2.1, 2.3 and the second half of Section 2.7 remain valid for this case if $d\lambda$ is taken as a matrix valued measure. Results similar to those in Sections 2.2, 2.4, 2.5, 2.6 and the first part of Section 2.7 apply in this case. It is also possible to generalize the results of this section to the matrix case. For example, the matrix version of (2.70) is

$$\partial_x^3 \mathcal{M}_\nu = -\partial_x \mathcal{M}_{k^2\nu} - \{u_x, \mathcal{M}_\nu\} - 2\{u, \partial_x \mathcal{M}_\nu\} - [u, \mathcal{N}_\nu],$$

$$\partial_x \mathcal{N}_\nu = [u, \mathcal{M}_\nu] \quad , \quad \mathcal{M}_\nu\big|_{u\equiv 0} = \mathcal{N}_\nu\big|_{u\equiv 0} = 0, \tag{2.78}$$

where $\{X,Y\}=XY+YX$ and $[X,Y]=XY-YX$ for any matrices X and Y. Formula (2.78) follows from (2.69) upon using the following identity

$$(b_1+\dots+b_n)\sum_{1\le\ell<m\le n}(a_\ell b_m - b_\ell a_m) = \sum_1^{n-1}(a_1+\dots+a_j)(b_j+b_{j+1})(b_{j+1}+\dots+b_n) - \\ - \sum_1^{n-1}(b_1+\dots+b_j)(b_j+b_{j+1})(a_{j+1}+\dots+a_n), \tag{2.79}$$

valid for all numbers a_j, b_j $(1\le j\le n)$ and all $n=1,2,3,\dots$. This identity can be easily proved using (A.102). Furthermore, since equations (2.69) and (2.79) remain valid if the a_j's are matrix valued, it is possible to treat the case of nonscalar dispersion functions. More precisely, equations solved by (2.66a) with (2.66b) replaced by

$$\partial_t\, dm(k,t) = -i\,[\omega(k)\,dm(k,t) - dm(k,t)\,\omega(-k)], \tag{2.80}$$

where $\omega=\omega(k)$ is now a matrix valued function of k, can be written.

CHAPTER 3

NONLINEAR CUBIC SCHRÖDINGER EQUATION

The methods developed for the KdV equation are now applied to the nonlinear cubic Schrödinger equation

$$i\varphi_t + \varphi_{xx} = -2\sigma\varphi\varphi^*\varphi = -2\sigma|\varphi|^2\varphi\ , \quad \varphi = \varphi(x,t), \tag{3.01}$$

where φ is complex valued and $\sigma = \pm 1$. This equation describes the modulation of a plane stationary light beam in a medium with nonlinear refractive index (Yariv 1975), as well as many other important phenomena involving time-dependent dispersive waves in a nonlinear medium (Whitham 1974).

To avoid having to work with complex conjugates, we replace (3.01) by the system

$$i\varphi_t + \varphi_{xx} + 2\sigma\varphi\psi\varphi = 0\ , \quad -i\psi_t + \psi_{xx} + 2\sigma\psi\varphi\psi = 0. \tag{3.02}$$

Then (3.01) is recovered if we require $\psi = \varphi^*$.

3.1 Solution by Small Parameter Expansions

Substituting

$$\varphi = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \varphi_m \quad , \qquad \psi = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \psi_m\ , \tag{3.03}$$

into (3.02), and collecting equal powers of the "small" parameter ε , we obtain

$$i\varphi_{m,t}+\varphi_{m,xx}=-2\sigma\sum_{j+\ell+s=m\ (j,\ell,s\ \text{odd})}\varphi_j\psi_\ell\varphi_s\ ,\quad(\forall m), \tag{3.04a}$$

$$-i\psi_{m,t}+\psi_{m,xx}=-2\sigma\sum_{j+\ell+s=m\ (j,\ell,s\ \text{odd})}\psi_j\varphi_\ell\psi_s\ ,\quad(\forall m), \tag{3.04b}$$

The functions φ_1 and ψ_1 satisfy the linearized equations, so that we can take

$$\varphi_1=\int_{\mathbb{C}}\exp[i(kx-k^2t)]\,d\lambda(k)\quad,\quad\psi_1=\int_{\mathbb{C}}\exp[i(kx+k^2t)]\,d\mu(k), \tag{3.05}$$

where, as usual, $d\lambda(k)$ and $d\mu(k)$ are appropriate measures on the complex plane $\mathbb{C}$.

The condition $\varphi_1^*=\psi_1$ will be satisfied if

$$d\lambda^*(-k^*)=d\mu(k). \tag{3.06}$$

For $n=1,\ m=3$ we have

$$i\varphi_{3,t}+\varphi_{3,xx}=-2\sigma\varphi_1\psi_1\varphi_1=-2\sigma\int_{\mathbb{C}^3}e^{i\Omega_3}\,d\lambda(k_1)d\mu(k_2)d\lambda(k_3), \tag{3.07}$$

where $\Omega_3=(k_1+\dots+k_3)x-(k_1^2-k_2^2+k_3^2)t$, and a similar equation for ψ_3 . A solution in the form

$$\varphi_3=\int_{\mathbb{C}^3}\Phi_3(k_1,k_2,k_3)\,e^{i\Omega_3}\,d\lambda(k_1)d\mu(k_2)d\lambda(k_3), \tag{3.08a}$$

requires

$$-2(k_1+k_2)(k_2+k_3)\,\Phi_3(k_1,k_2,k_3)=\{k_1^2-k_2^2+k_3^2-(k_1+k_2+k_3)^2\}\Phi_3(k_1,k_2,k_3)=-2\sigma.$$

Therefore we have

$$\varphi_3 = \sigma \int_{\mathbb{C}^3} \frac{e^{i\Omega_3}}{(k_1+k_2)(k_2+k_3)} d\lambda(k_1) d\psi(k_2) d\lambda(k_3), \tag{3.08b}$$

and similarly

$$\psi_3 = \sigma \int_{\mathbb{C}^3} \frac{e^{i\Lambda_3}}{(k_1+k_2)(k_2+k_3)} d\psi(k_1) d\lambda(k_2) d\psi(k_3), \tag{3.08c}$$

where $\Lambda_3 = (k_1+k_2+k_3)x + (k_1^2 - k_2^2 + k_3^2)t$.

For $n = 2$, $m = 5$ we find

$$\varphi_5 = \sigma^2 \int_{\mathbb{C}^5} \frac{e^{i\Omega_5}}{(k_1+k_2)(k_2+k_3)(k_3+k_4)(k_4+k_5)} d\lambda_1 d\psi_2 d\lambda_3 d\psi_4 d\lambda_5 , \tag{3.09a}$$

$$\psi_5 = \sigma^2 \int_{\mathbb{C}^5} \frac{e^{i\Lambda_5}}{(k_1+k_2)(k_2+k_3)(k_3+k_4)(k_4+k_5)} d\psi_1 d\lambda_2 d\psi_3 d\lambda_4 d\psi_5 \tag{3.09b}$$

where $d\lambda_j = d\lambda(k_j)$, $d\psi_j = d\psi(k_j)$, $(\forall 1 \leq j \leq 5)$, and Ω_5, Λ_5 are the obvious generalizations of Ω_3, Λ_3 above. We now propose that in general

$$\varphi_m = \sigma^n \int_{\mathbb{C}^m} \frac{e^{i\Omega_m}}{(k_1+k_2)\dots(k_{2n}+k_m)} d\lambda_1 d\psi_2 \dots d\lambda_m \tag{3.10a}$$

$$\psi_m = \sigma^n \int_{\mathbb{C}^m} \frac{e^{i\Lambda_m}}{(k_1+k_2)\dots(k_{2n}+k_m)} d\psi_1 d\lambda_2 \dots d\psi_m \tag{3.10b}$$

for every $m = 2n+1$, $0 \leq n < \infty$. Substituting (3.10) into (3.04) we find that we must have

$$\left\{ k_1^2 - k_2^2 + \dots - k_{2n}^2 + k_m^2 - (k_1 + \dots + k_m)^2 \right\} \frac{\sigma^n}{(k_1+k_2)\dots(k_{2n}+k_m)} = -2\sigma \sum_{\substack{j+l+s=m \\ j,l,s \text{ odd.}}} \Big\{$$

$$\frac{\sigma^{(j-1)/2}}{(k_1+k_2)\dots(k_{j-1}+k_j)}\cdot\frac{\sigma^{(\ell-1)/2}}{(k_{j+1}+k_{j+2})\dots(k_{j+\ell-1}+k_{j+\ell})}\cdot\frac{\sigma^{(s-1)/2}}{(k_{j+\ell+1}+k_{j+\ell+2})\dots(k_{2n}+k_m)}\Bigg\} =$$

$$= -\frac{2\sigma^n}{(k_1+k_2)\dots(k_{2n}+k_m)} \sum_{\substack{1\le j<r<m \\ j \text{ odd},\ r \text{ even.}}} (k_j+k_{j+1})(k_r+k_{r+1}) \,. \tag{3.11}$$

That is, we need the following factorization of the linearized dispersion relation

$$\left\{k_1^2-k_2^2+\dots-k_{2n}^2+k_m^2-(k_1+\dots+k_m)^2\right\}=-2\sum_{\substack{1\le j<r<m \\ j \text{ odd},\ r \text{ even.}}}(k_j+k_{j+1})(k_r+k_{r+1}) \,. \tag{3.12}$$

Since

$$-2\sum_{\substack{1\le j<r<m \\ j \text{ odd},\ r \text{ even.}}}(k_j+k_{j+1})(k_r+k_{r+1}) = -2\sum_{\substack{1\le j<\ell\le m \\ j \text{ odd.}}} k_j k_\ell \; -2\sum_{\substack{1< j\le \ell\le m \\ j \text{ even.}}} k_j k_\ell =$$

$$= -2\sum_{1\le j<\ell\le m} k_j k_\ell \; -2\sum_{j \text{ even.}} k_j^2 \quad ,$$

(3.12) follows immediately.

From (3.03) and (3.10) we now have

$$\varphi = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \sigma^n \int_{\mathbb{C}^m} \frac{e^{i\Omega_m}}{\prod_{j=1}^{m-1}(k_j+k_{j+1})}\, d\lambda_1\, d\lambda_2 \dots d\lambda_m \quad , \tag{3.13a}$$

$$\psi = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \sigma^n \int_{\mathbb{C}^m} \frac{e^{i\Lambda_m}}{\prod_{j=1}^{m-1}(k_j+k_{j+1})} d\mu_1 d\lambda_2 \dots d\mu_m , \tag{3.13b}$$

and it turns out that (3.06) is enough to make not only $\varphi_1^* = \psi_1$, but $\varphi^* = \psi$. These last two expressions for φ and ψ imply

$$\varphi\psi = -i\partial_x \sum_{n=1}^{\infty} \varepsilon^{2n}\sigma^{n-1} \int_{\mathbb{C}^{2n}} \frac{e^{i\sum_1^{2n}(k_j x + (-1)^j k_j^2 t)}}{\prod_{j=1}^{2n-1}(k_j+k_{j+1})} d\lambda_1 d\mu_2 \dots d\lambda_{2n-1} d\mu_{2n} , \tag{3.14a}$$

$$\psi\varphi = -i\partial_x \sum_{n=1}^{\infty} \varepsilon^{2n}\sigma^{n-1} \int_{\mathbb{C}^{2n}} \frac{e^{i\sum_1^{2n}(k_j x - (-1)^j k_j^2 t)}}{\prod_{j=1}^{2n-1}(k_j+k_{j+1})} d\mu_1 d\lambda_2 \dots d\mu_{2n-1} d\lambda_{2n} . \tag{3.14b}$$ †

Formulas (3.13) and (3.14) are very similar to (2.14). The same methods used to sum (2.14) will work with (3.13) and (3.14). Again the inverse scattering problem associated with (3.01)-(3.02) (Zakharov and Shabat 1972), as well as its corresponding Marčenko integral equations, is implicit in (3.13) and (3.14). Thus perturbation expansions provide again a simple and straightforward way of arriving at the proper inverse scattering transform.

Introduce

$$b(x,t) = \frac{1}{2ix} \int_{\mathbb{C}} d\lambda(k) \exp i\left(\frac{k}{2}x - k^2 t\right) , \quad d(x,t) = \frac{1}{2ix} \int_{\mathbb{C}} d\mu(k) \exp i\left(\frac{k}{2}x + k^2 t\right), \tag{3.15}$$

---

† The reason for making a difference between $\varphi\psi$ and $\psi\varphi$ is that, as far as this section and Section 3.3 are concerned, all the formulas remain valid when φ and ψ are matrix valued (see Section 3.6).

where $\sigma = \chi^{-2}$. Assume that, as $x \to \infty$, $b(x,t)$ as well as $d(x,t)$ and each of the terms of the summations in (3.13) and (3.14), vanish sufficiently fast. Then we can write (3.13) and (3.14) as

$$\varphi = 2i\chi \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \int_{[x,\infty)^{2n}} dz_1 \ldots dz_{2n}\, b(x+z_1,t)\, d(z_1+z_2,t) \ldots b(z_{2n}+x,t)\,, \tag{3.16a}$$

$$\psi = 2i\chi \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \int_{[x,\infty)^{2n}} dz_1 \ldots dz_{2n}\, d(x+z_1,t)\, b(z_1+z_2,t) \ldots d(z_{2n}+x,t)\,, \tag{3.16b}$$

$$\varphi\psi = \frac{2}{\sigma}\partial_x \sum_{n=1}^{\infty} \varepsilon^{2n} \int_{[x,\infty)^{2n-1}} dz_1 \ldots dz_{2n-1}\, b(x+z_1,t) d(z_1+z_2,t) \ldots b(z_{2n-2}+z_{2n-1},t) d(z_{2n-1}+x,t)\,, \tag{3.16c}$$

$$\psi\varphi = \frac{2}{\sigma}\partial_x \sum_{n=1}^{\infty} \varepsilon^{2n} \int_{[x,\infty)^{2n-1}} dz_1 \ldots dz_{2n-1}\, d(x+z_1,t) b(z_1+z_2,t) \ldots d(z_{2n-2}+z_{2n-1},t) b(z_{2n-1}+x,t)\,. \tag{3.16d}$$

We note that $b(2x,t)$ and $d(2x,t)$ satisfy the linear Schrödinger equation, and that equation (3.06) is equivalent to $b^* = -(\mathrm{sg}\,\sigma)\, d$.

3.2 Multiple Envelope-Soliton Solutions

Assume now that b and d are a superposition of exponentials in the form

$$b(x+y,t) = \frac{1}{2i\chi} \sum_{j=1}^{N} \lambda_j^2 \exp i\left[z_j \frac{(x+y)}{2} - z_j^2 t\right] = \frac{1}{2i\chi}\, p^T(x,t)\, p(y,t)\,, \tag{3.17a}$$

$$d(x+y,t) = \frac{1}{2i\chi} \sum_{\ell=1}^{M} \mu_\ell^2 \exp i\left[w_\ell \frac{(x+y)}{2} + w_\ell^2 t\right] = \frac{1}{2i\chi}\, q^T(x,t)\, q(y,t)\,, \tag{3.17b}$$

where $\mathrm{Im}\, z_j > 0$, $\mathrm{Im}\, w_\ell > 0$ and $p(x,t)$ and $q(x,t)$ are the column vectors given by

$$p_j(x,t) = \lambda_j \exp \frac{i}{2}(z_j x - z_j^2 t) \quad , \; (1 \leq j \leq N), \tag{3.18a}$$

$$q_\ell(x,t) = \psi_\ell \exp \frac{i}{2}(w_\ell x + w_\ell^2 t) \; , \; (1 \leq \ell \leq M). \tag{3.18b}$$

Therefore, just as in Section 2.2,

$$\varphi = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \sigma^n p^T (BD)^n p \quad , \quad \psi = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \sigma^n q^T (DB)^n q \; , \tag{3.19a}$$

$$\varphi\psi = -i\partial_x \sum_{n=1}^{\infty} \varepsilon^{2n} \sigma^{n-1} p^T (BD)^{n-1} B q \; , \; \psi\varphi = -i\partial_x \sum_{n=1}^{\infty} \varepsilon^{2n} \sigma^{n-1} q^T D (BD)^{n-1} p \; , \tag{3.19b}$$

where B and D are the $N \times M$ and $M \times N$ matrices given by

$$B(x,t) = \frac{1}{2i} \int_x^\infty dz \; p(z,t) q^T(z,t) = \left\{ \lambda_j \psi_\ell (z_j + w_\ell)^{-1} \exp \frac{i}{2}\left[(z_j + w_\ell)x - (z_j^2 - w_\ell^2)t\right] \right\}, \tag{3.20a}$$

$$D(x,t) = \frac{1}{2i} \int_x^\infty dz \; q(z,t) p^T(z,t) = B^T(x,t). \tag{3.20b}$$

It follows immediately from (3.19) that

$$\varphi = \varepsilon p^T (I - \sigma\varepsilon^2 BD)^{-1} p \quad , \quad \psi = \varepsilon q^T (I - \sigma\varepsilon^2 DB)^{-1} q \; , \tag{3.21a}$$

$$\varphi\psi = -i\varepsilon^2 \partial_x p^T (I - \sigma\varepsilon^2 BD)^{-1} B q \; , \; \psi\varphi = -i\varepsilon^2 \partial_x q^T D (I - \sigma\varepsilon^2 BD)^{-1} p. \tag{3.21b}$$

Since $\varphi\psi = \psi\varphi$ from (3.21b), we have

$$\begin{aligned} \varphi\psi = \frac{1}{2}(\varphi\psi + \psi\varphi) &= \frac{\varepsilon^2}{2i} \partial_x \left\{ p^T (I - \sigma\varepsilon^2 BD)^{-1} B q + q^T D (I - \sigma\varepsilon^2 BD)^{-1} p \right\} = \\ &= \frac{\varepsilon^2}{2i} \partial_x \operatorname{Tr} \left\{ (I - \sigma\varepsilon^2 BD)^{-1} (B q p^T + p q^T D) \right\} = -\varepsilon^2 \partial_x \operatorname{Tr} \left\{ (I - \sigma\varepsilon^2 BD)^{-1} (BD)_x \right\}. \end{aligned} \tag{3.21c}$$

This can also be expressed as

$$\varphi\psi = \sigma^{-1}\partial_x^2 \operatorname{Tr} \ln(I - \sigma\varepsilon^2 BD) = \sigma^{-1}\partial_x^2 \ln\det(I - \sigma\varepsilon^2 BD). \tag{3.22}$$

When $\mathcal{N} = \mathcal{M}$, $\lambda_j = \mu_j^*$ and $z_j = -w_j^*$ we have

$$p = \bar{q}\ , \quad D = -\bar{B}\ , \quad B^* = -B \quad \text{and} \quad \varphi = \psi^*. \tag{3.23}$$

Then (3.21a) and (3.22) give

$$\varphi = \varepsilon p^T (I + \sigma\varepsilon^2 B\bar{B})^{-1} p\ , \quad |\varphi|^2 = \sigma^{-1}\partial_x^2 \ln\det(I + \sigma\varepsilon^2 B\bar{B}). \tag{3.24}$$

In this case (iB) is hermitian and positive definite, as can be seen either by computing its principal minors using (A.202), or from the formula $iB = (1/2)\int_x^\infty (pp^*)\,dz$ which follows from the definition of B and (3.23). Thus from (A.3) the eigenvalues of $B\bar{B} = (iB)\overline{(iB)}$ are all real and positive. Moreover, they tend to zero as $x \mapsto \infty$, and to ∞ as $x \mapsto -\infty$. It follows that $(I + \sigma\varepsilon^2 B\bar{B})^{-1}$ exists for all $-\infty < x, t < \infty$ when $\sigma > 0$, and that it has singularities at some x's for all $-\infty < t < \infty$ when $\sigma < 0$, i.e., in one case formulas (3.24) are nonsingular (and thus represent physically meaningful solutions) and in the other they are not. This corresponds to the fact that in the case $\sigma > 0$ uniform wavetrains in equation (3.01) are linearly unstable, and presumably break into a sequence of wave packages (i.e., envelope-solitons such as those given by (3.24), while in the case $\sigma < 0$ they are stable.

That the eigenvalues of $B\bar{B}$ tend to zero as $x \mapsto \infty$, and to ∞ as $x \mapsto -\infty$, follows from the inequalities $\|B^{-1}\|_2^{-2} \leqslant \|(B\bar{B})^{-1}\|_2^{-1} \leqslant \text{spectrum}(B\bar{B}) \leqslant$

$\leqslant \| B\bar{B} \|_2 \leqslant \| B \|_2^2$, where $\|B\|_2 \longmapsto 0$ as $x \longmapsto \infty$, and $\|B^{-1}\|_2 \longmapsto 0$ as $x \longmapsto -\infty$ (since from (3.20a) B is a product of two diagonal matrices with exponentials on the diagonal, and a third matrix independent of x .

The expressions (3.21) for the multiple envelope-soliton solutions are new in the literature. The second formula of (3.24) was first obtained by Zakharov and Shabat (1972) and Hirota (1973a).† Hirota also obtains expressions for φ . These can be obtained from (3.21a) as follows. Let $\hat{B}$ and $\hat{D}$ be the $N \times N$ and $M \times M$ matrices given by

$$\hat{B} = pp^T \qquad D = qq^T . \tag{3.25}$$

Then (3.21a) can also be written as

$$\varphi = \varepsilon \,\mathrm{Tr} \{ (I - \sigma\varepsilon^2 BD)^{-1} \hat{B} \} , \quad \psi = \varepsilon \,\mathrm{Tr} \{ (I - \sigma\varepsilon^2 DB)^{-1} \hat{D} \} . \tag{3.26}$$

But if H is the complex valued function of square matrices defined by $H(A) = \mathrm{Tr}\, \ln (I + A) = \ln \det (I + A)$, we have $(\partial_A H).T = \mathrm{Tr} \{ (I + A)^{-1} T \} = \{ \det (I + A) \}^{-1} (\partial_{I+A} \det).T$, where ∂ denotes the Jacobian. Thus if we define

$$\begin{aligned} G_\varphi &= \varepsilon (\partial_{(I - \sigma\varepsilon^2 BD)} \det).\hat{B} \quad , \quad G_\psi = \varepsilon (\partial_{(I - \sigma\varepsilon^2 DB)} \det).\hat{D} , \\ F &= \det (I - \sigma\varepsilon^2 BD) = \det (I - \sigma\varepsilon^2 DB) , \end{aligned} \tag{3.27}$$

we have

$$\varphi = G_\varphi / F \quad , \qquad \psi = G_\psi / F . \tag{3.28}$$

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†Both references have misprint errors. The factor σ^{-1} has been replaced by $\sqrt{4\sigma}$ in the former and by σ in the latter.

When (3.23) holds, $G_\varphi = G^*_\psi$, F is real and (upon expansion of the determinants involved) (3.27) and (3.28) give Hirota's formulas.

3.3 Marčenko Integral Equations and Eigenvalue Problem

In an alternative manipulation of (3.16), we notice that if the following linear operators, defined on functions of two variables, are introduced

$$(\hat{b}f)(x,y)=\int_x^\infty f(x,z)\,b(z+y,t)\,dz\ ,\ (\hat{d}f)(x,y)=\int_x^\infty f(x,z)\,d(z+y,t)\,dz\ ,\quad (\forall f=f(x,y)), \tag{3.29}$$

then we have

$$\varphi = 2i\,x\sum_{\substack{n=0\\ m=2n+1}}^{\infty} \varepsilon^m\left[(\hat{b}\hat{d})^n b\right]_{y=x}\ ,\quad \psi = 2i\,x\sum_{\substack{n=0\\ m=2n+1}}^{\infty}\varepsilon^m\left[(\hat{d}\hat{b})^n d\right]_{y=x}\ , \tag{3.30a}$$

$$\varphi\psi = 2\sigma^{-1}\partial_x\sum_{n=1}^{\infty}\varepsilon^{2n}\left[\hat{d}(\hat{b}\hat{d})^{n-1}b\right]_{y=x}\ ,\quad \psi\varphi = 2\sigma^{-1}\partial_x\sum_{n=1}^{\infty}\varepsilon^{2n}\left[\hat{b}(\hat{d}\hat{b})d\right]_{y=x}\ , \tag{3.30b}$$

where b and d are interpreted as arguments for $\hat{d}$ and $\hat{b}$, respectively, in the form $b(x,z)=b(x+z,t)$ and $d(x,z)=d(x+z,t)$, and the variable t participates only as a parameter. Introduce now the functions of the variables x, y and t

$$K_1 = \sum_{n=0}^{\infty}\varepsilon^{2n+1}(\hat{b}\hat{d})^n b = \varepsilon(I-\varepsilon^2\hat{b}\hat{d})^{-1}b\ , \tag{3.31a}$$

$$K_2 = \sum_{n=0}^{\infty}\varepsilon^{2n+1}(\hat{d}\hat{b})d = \varepsilon(I-\varepsilon^2\hat{d}\hat{b})^{-1}d\ , \tag{3.31b}$$

$$K_3 = \sum_{n=1}^{\infty}\varepsilon^{2n}\,\hat{d}(\hat{b}\hat{d})\,b = \varepsilon^2\hat{d}(I-\varepsilon^2\hat{b}\hat{d})^{-1}b\ , \tag{3.31c}$$

$$K_4 = \sum_{n=1}^{\infty} \epsilon^{2n}\, \hat{b}\,(\hat{d}\,\hat{b})^{n-1} d = \epsilon^2\, \hat{b}\,(I - \epsilon^2 \hat{d}\,\hat{b})^{-1} d. \tag{3.31d}$$

Then we have the following equations

$$0 = K_1(x,y,t) - \epsilon\, b(x+y,t) - \epsilon \int_x^{\infty} K_3(x,z,t)\, b(z+y,t)\, dz\ , \tag{3.32a}$$

$$0 = K_2(x,y,t) - \epsilon\, d(x+y,t) - \epsilon \int_x^{\infty} K_4(x,z,t)\, d(z+y,t)\, dz\ , \tag{3.32b}$$

$$0 = K_3(x,y,t) - \epsilon \int_x^{\infty} K_1(x,z,t)\, d(z+y,t)\, dz\ , \tag{3.32c}$$

$$0 = K_4(x,y,t) - \epsilon \int_x^{\infty} K_2(x,z,t)\, b(z+y,t)\, dz\ , \tag{3.32d}$$

$$\varphi(x,t) = 2i\chi\, K_1(x,x,t) \quad , \quad \psi(x,t) = 2i\chi\, K_2(x,x,t)\ , \tag{3.32e}$$

$$(\varphi\psi)(x,t) = 2\sigma^{-1} \partial_x K_3(x,x,t) \quad , \quad (\psi\varphi)(x,t) = 2\sigma^{-1} \partial_x K_4(x,x,t)\ . \tag{3.32f}$$

We can think of the equations as the result of summing (3.13) and (3.14). They are the Marčenko integral equations of the inverse scattering problem corresponding to (3.01)-(3.02) (Zakharov and Shabat 1972). We now write formulas for the K_j's independent of the assumption that $b,d \longmapsto 0$ as $x \longmapsto \infty$. We do this from their definition, retracing in reverse the steps used to obtain (3.16) from (3.13) and (3.14).

$$K_1(x,y,t) = \sum_{\substack{n=0\\ m=2n+1}}^{\infty} \left(\frac{\epsilon^m \sigma^n}{2i\chi}\right) \int_{\mathbb{C}^m} \frac{e^{i\left\{\sum_1^{2n}(k_j x + (-1)^j k_j^2 t) + \frac{1}{2} k_m (x+y) - k_m^2 t\right\}}}{\prod_1^{2n}(k_j + k_{j+1})}\, d\lambda_1 d\mu_2 \dots d\lambda_m\ , \tag{3.33a}$$

$$K_2(x,y,t) = \sum_{\substack{n=0\\ m=2n+1}}^{\infty} \left(\frac{\epsilon^m \sigma^n}{2i\chi}\right) \int_{\mathbb{C}^m} \frac{e^{i\left\{\sum_1^{2n}(k_j x - (-1)^j k_j^2 t) + \frac{1}{2} k_m (x+y) + k_m^2 t\right\}}}{\prod_1^{2n}(k_j + k_{j+1})}\, d\mu_1 d\lambda_2 \dots d\mu_m\ , \tag{3.33b}$$

$$K_3(x,y,t) = \sum_{n=1}^{\infty} \left(\frac{\epsilon^{2n} \sigma^n}{2i}\right) \int_{\mathbb{C}^{2n}} \frac{e^{i\left\{\sum_1^{2n-1}(k_j x + (-1)^j k_j^2 t) + \frac{1}{2} k_{2n} (x+y) + k_{2n}^2 t\right\}}}{\prod_1^{2n-1}(k_j + k_{j+1})}\, d\lambda_1 d\mu_2 \dots d\lambda_{2n-1} d\mu_{2n}\ , \tag{3.33c}$$

$$K_4(x,y,t)=\sum_{n=1}^{\infty}\left(\frac{\varepsilon^{2n}\sigma^n}{2i}\right)\int_{\mathbb{C}^{2n}}\frac{e^{i\left\{\sum_1^{2n-1}(k_jx-(-1)^jk_j^2t)+\frac{1}{2}k_{2n}(x+y)-k_{2n}^2t\right\}}}{\prod_1^{2n-1}(k_j+k_{j+1})}d\mu_1 d\lambda_2\cdots d\mu_{2n-1}d\lambda_{2n}\,. \quad (3.33d)$$

Just as we did for the KdV equation, we now look for equations satisfied by the K_j's. Again the time dependent equations are closely related to the equations satisfied by φ for K_1, ψ (for K_2), and the potentials of $\varphi\psi$ (for K_3) and $\psi\varphi$ (for K_4). In fact, it is just a matter of replacing, whenever they appear as last factors in each term of the mentioned equations, φ by $2iK_1$, ψ by $2iK_2$, etc., and ∂_x by $(\partial_x+\partial_y)$. These time dependent equations are

$$[i\partial_t+(\partial_x+\partial_y)^2]K_1(x,y,t)+2\sigma\,\varphi(x,t)\psi(x,t)K_1(x,y,t)=0\,, \quad (3.34a)$$

$$[-i\partial_t+(\partial_x+\partial_y)^2]K_2(x,y,t)+2\sigma\,\psi(x,t)\varphi(x,t)K_2(x,y,t)=0\,, \quad (3.34b)$$

$$i\partial_t K_3(x,y,t)-i\chi^{-1}[\varphi(x,t)(\partial_x+\partial_y)-\varphi_x(x,t)]K_2(x,y,t)=0\,, \quad (3.34c)$$

$$-i\partial_t K_4(x,y,t)-i\chi^{-1}[\psi(x,t)(\partial_x+\partial_y)-\psi_x(x,t)]K_1(x,y,t)=0\,. \quad (3.34d)$$

Equations (3.34a,b) are a consequence of (3.12), (3.13) and (3.33a,b). Equations (3.34c,d) follow from (3.13), (3.33) and the following identity, valid for all $n=1,2,3,\ldots$:

$$k_1^2-k_2^2+\cdots-k_{2n}^2=\sum_{\substack{1\le j<2n\\ j\ \text{odd.}}}(k_1+\cdots+k_j)(k_j+k_{j+1})-(k_j+k_{j+1})(k_{j+2}+\cdots+k_{2n}). \quad (3.35)$$

This identity is motivated by the equations $i(\varphi\psi)_t=\partial_x(\varphi\psi_x-\varphi_x\psi)$ and $-i(\psi\varphi)_t=\partial_x(\psi\varphi_x-\psi_x\varphi)$, in the same way that (3.12) is motivated by (3.02).

In order to find the y-dependent equations we follow the same procedure used in Section 2.3 for the KdV equation. First we write (3.32) as

$$\begin{bmatrix} 1 & -\varepsilon\hat{b} \\ -\varepsilon\hat{d} & 1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_3 \end{bmatrix} = \begin{bmatrix} \varepsilon b(x+y) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -\varepsilon\hat{b} \\ -\varepsilon\hat{d} & 1 \end{bmatrix}\begin{bmatrix} K_4 \\ K_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon d(x+y) \end{bmatrix}, \qquad (3.36)$$

where the time dependence is not displayed to simplify the notation. Next we apply the operators

$$\begin{bmatrix} (\partial_x - \partial_y) & 0 \\ 0 & (\partial_x + \partial_y) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (\partial_x + \partial_y) & 0 \\ 0 & (\partial_x - \partial_y) \end{bmatrix}$$

to (3.36), and use (2.30) to obtain

$$\begin{bmatrix} 1 & -\varepsilon\hat{b} \\ -\varepsilon\hat{d} & 1 \end{bmatrix}\begin{bmatrix} (\partial_x - \partial_y)K_1 \\ (\partial_x + \partial_y)K_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\varepsilon K_1(x,x)\, d(x+y) \end{bmatrix} = 0 ,$$

$$\begin{bmatrix} 1 & -\varepsilon\hat{b} \\ -\varepsilon\hat{d} & 1 \end{bmatrix}\begin{bmatrix} (\partial_x + \partial_y)K_4 \\ (\partial_x - \partial_y)K_2 \end{bmatrix} + \begin{bmatrix} 2\varepsilon K_2(x,x)\, b(x+y) \\ 0 \end{bmatrix} = 0 .$$

Thus multiplying both of these last two equations by $\begin{bmatrix} 1 & -\varepsilon\hat{b} \\ -\varepsilon\hat{d} & 1 \end{bmatrix}^{-1}$ and using (3.32e) and (3.36), we have

$$(\partial_x - \partial_y)K_1 - i\chi^{-1}\varphi(x,t)K_4 = 0 , \quad (\partial_x + \partial_y)K_4 - i\chi^{-1}\psi(x,t)K_1 = 0, \qquad (3.37a)$$

$$(\partial_x - \partial_y)K_2 - i\chi^{-1}\psi(x,t)K_3 = 0 , \quad (\partial_x + \partial_y)K_3 - i\chi^{-1}\varphi(x,t)K_2 = 0. \qquad (3.37b)$$

An alternative, and perhaps simpler, way to obtain (3.37) is directly from (3.33). In fact, comparison of (3.33) with (3.13) and (3.14) shows that K_1 is "almost" φ , K_2 is "almost" ψ , K_3 is "almost" a potential for $\varphi\psi$ and, finally, K_4 is "almost" a potential for $\psi\varphi$. Thus $\psi K_1 \approx \psi\varphi$ must have a close relationship with some derivative of K_4 , etc. The study of the products ψK_1 , φK_2 , ψK_3 and φK_4 with use of (3.13), (3.14) and (3.33) leads immediately to (3.37).

Since neither (3.34) nor (3.37) have coefficients dependent on y , we can separate this variable and write ($j=1,2,3$ & 4)

$$K_j(x,y,t) = \int_{\mathbb{C}} \Delta_j(x,t,\xi)\, \exp i(\xi y + 2\nu_j \xi^2 t)\, d\eta_j(\xi) \;, \tag{3.38}$$

for some measures $d\eta_1 = d\eta_4$ and $d\eta_2 = d\eta_3$, where $\nu_1 = \nu_4 = -1$ and $\nu_2 = \nu_3 = 1$. The factor $\exp(2\nu_j i\xi^2 t)$ is added because it simplifies the formulas. Then we have from (3.37)

$$\Delta_{1,x} - i\xi\Delta_1 = i\chi^{-1}\varphi\Delta_4 \quad , \qquad \Delta_{4,x} + i\xi\Delta_4 = i\chi^{-1}\psi\Delta_1 \;, \tag{3.39a}$$

$$\Delta_{2,x} - i\xi\Delta_2 = i\chi^{-1}\psi\Delta_3 \quad , \qquad \Delta_{3,x} + i\xi\Delta_3 = i\chi^{-1}\varphi\Delta_2 \;, \tag{3.39b}$$

and from (3.34), after some manipulations in which we use (3.39) to eliminate space derivatives,

$$i\Delta_{1,t} - 2\xi^2\Delta_1 - 2\xi\chi^{-1}\varphi\Delta_4 + i\chi^{-1}\varphi_x\Delta_4 + \sigma\varphi\psi\Delta_1 = 0 \;, \tag{3.40a}$$

$$-i\Delta_{4,t} - 2\xi^2\Delta_4 + 2\xi\chi^{-1}\psi\Delta_1 + i\chi^{-1}\psi_x\Delta_1 + \sigma\psi\varphi\Delta_4 = 0 \;, \tag{3.40b}$$

$$-i\Delta_{2,t} - 2\xi^2\Delta_2 - 2\xi\chi^{-1}\psi\Delta_3 + i\chi^{-1}\psi_x\Delta_3 + \sigma\psi\varphi\Delta_2 = 0 \;, \tag{3.40c}$$

$$i\Delta_{3,t} - 2\xi^2\Delta_3 + 2\xi\chi^{-1}\varphi\Delta_2 + i\chi^{-1}\varphi_x\Delta_2 + \sigma\varphi\psi\Delta_3 = 0 \,. \tag{3.40d}$$

We note that Δ_1 and Δ_4 satisfy the same system of equations as Δ_3 and Δ_2, save for the change $\xi \longmapsto -\xi$. Equations (3.39a) or (3.39b) constitute the eigenvalue problem whose inverse scattering problem was used by Zakharov and Shabat (1972) to solve (3.01) and find many interesting properties of its solutions. Then (3.40) provides the time evolution of the scattering parameters. Equations (3.39) and (3.40) are also important in the study of the effect of adding to $d\lambda$ and dy a set of δ-functions of Dirac (see Section 3.5).

To write the Δ_j's and $d\eta_j$'s directly in terms of $d\lambda$ and dy is easy. In fact just by looking at (3.33) it is obvious that

$$d\eta_1(\xi) = d\eta_4(\xi) = \frac{\varepsilon}{2i\chi}\, d\lambda(2\xi) \quad , \quad d\eta_2(\xi) = d\eta_3(\xi) = \frac{\varepsilon}{2i\chi}\, dy(2\xi) \tag{3.41a}$$

$$\Delta_1 = \left\{ 1 + \sum_{n=1}^{\infty} \varepsilon^{2n}\sigma^n \int_{\mathbb{C}^{2n}} \frac{e^{i\left\{\sum_1^{2n}(k_j x + (-1)^j k_j^2 t)\right\}}}{\prod_1^{2n-1}(k_j + k_{j+1})\cdot(k_{2n}+2\xi)}\, d\lambda_1 dy_2 \dots d\lambda_{2n-1} dy_{2n} \right\} e^{i(\xi x - 2\xi^2 t)}, \tag{3.41b}$$

$$\Delta_2 = \left\{ 1 + \sum_{n=1}^{\infty} \varepsilon^{2n}\sigma^n \int_{\mathbb{C}^{2n}} \frac{e^{i\left\{\sum_1^{2n}(k_j x - (-1)^j k_j^2 t)\right\}}}{\prod_1^{2n-1}(k_j + k_{j+1})\cdot(k_{2n}+2\xi)}\, dy_1 d\lambda_2 \dots dy_{2n-1} d\lambda_{2n} \right\} e^{i(\xi x + 2\xi^2 t)}, \tag{3.41c}$$

$$\Delta_3 = \left\{ \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} (\varepsilon/\chi)^m \int_{\mathbb{C}^m} \frac{e^{i\left\{\sum_1^{m}(k_j x + (-1)^j k_j^2 t)\right\}}}{\prod_1^{2n}(k_j + k_{j+1})\cdot(k_m+2\xi)}\, d\lambda_1 dy_2 \dots d\lambda_m \right\} e^{i(\xi x + 2\xi^2 t)} \quad , \tag{3.41d}$$

$$\Delta_4 = \left\{ \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} (\varepsilon/\chi)^m \int_{\mathbb{C}^m} \frac{e^{i\left\{\sum_1^{m}(k_j x - (-1)^j k_j^2 t)\right\}}}{\prod_1^{2n}(k_j + k_{j+1})\cdot(k_m+2\xi)}\, dy_1 d\lambda_2 \dots dy_m \right\} e^{i(\xi x - 2\xi^2 t)} \quad . \tag{3.41e}$$

Formulas (3.39) and (3.40) can be proved directly from (3.41).

3.4 Operator Formalism

It is our purpose in this section to sum (3.13) and (3.14) under circumstances more general than the ones in Section 3.2. The same ideas that proved useful in the case of the KdV equation (Section 2.4) should work here too, since (3.13) and (3.14) have the same nature as (2.14).

Assume that we can write

$$d\lambda(k) = a_1^2(k)\, d\rho_1(k) \quad , \quad d\gamma(k) = a_2^2(k)\, d\rho_2(k) \ , \tag{3.42}$$

where $d\rho_1(k)$ and $d\rho_2(k)$ are positive measures on $\mathbb{C}$, and $a_1(k)$ and $a_2(k)$ are functions defined on the domains of $d\rho_1(k)$ and $d\rho_2(k)$ respectively. Let us now define, by analogy with Section 3.2, the following operators and symmetric forms (closely related to the inner products in $\mathcal{L}^2(d\rho_1)$ and $\mathcal{L}^2(d\rho_2)$)

$$(\mathcal{B}f)(k_1) = \int_{\mathbb{C}} p(k_1)(k_1+k_2)^{-1} q(k_2) f(k_2) d\rho_2(k_2) \ , \ (\forall f = f(k_2)) \ , \tag{3.43a}$$

$$(\mathcal{D}g)(k_2) = \int_{\mathbb{C}} q(k_2)(k_2+k_1)^{-1} p(k_1) g(k_1) d\rho_1(k_1) \ , \ (\forall g = g(k_1)) \ , \tag{3.43b}$$

$$[g_1, g_2]_1 = \int_{\mathbb{C}} g_1(k_1) g_2(k_1)\, d\rho_1(k_1) \ , \ [f_1, f_2]_2 = \int_{\mathbb{C}} f_1(k_2) f_2(k_2)\, d\rho_2(k_2) \ , \ (\forall g_1, g_2, f_1, f_2), \tag{3.43c}$$

where k_1 and k_2 range over the domains of $d\rho_1$ and $d\rho_2$, respectively, and p and q are given by

$$p(k_1) = a_1(k_1) \exp \tfrac{i}{2}(k_1 x - k_1^2 t) \qquad \text{and} \qquad q(k_2) = a_2(k_2) \exp \tfrac{i}{2}(k_2 x + k_2^2 t) \ . \tag{3.44}$$

We note the following properties of the operators $\mathcal{B}$ and $\mathcal{D}$

(i) $\mathcal{B}^T = \mathcal{D}$ i.e. $[g, \mathcal{B}f]_1 = [\mathcal{D}g, f]$, $(\forall g = g(k_1), f = f(k_2))$.

(ii) $\mathcal{B}_x = -\frac{1}{2i}\, p q^T = -\frac{1}{2i}\, p [q, \cdot]_2$.

(iii) $\mathcal{D}_x = -\frac{1}{2i}\, q p^T = -\frac{1}{2i}\, q [p, \cdot]_1$.

If we assume that $p \in \mathcal{L}^2(d\rho_1)$, $q \in \mathcal{L}^2(d\rho_2)$, and that $B : \mathcal{L}^2(d\rho_2) \longmapsto \mathcal{L}^2(d\rho_1)$ and $D : \mathcal{L}^2(d\rho_1) \longmapsto \mathcal{L}^2(d\rho_2)$ are bounded operators, then we can sum (3.13) and (3.14) to

$$\varphi = \varepsilon[p,(I-\sigma\varepsilon^2 BD)^{-1}p]_1 \quad , \quad \psi = \varepsilon[q,(I-\sigma\varepsilon^2 DB)^{-1}q]_2 , \tag{3.45a}$$

$$\psi\varphi = -i\varepsilon^2\partial_x [q, D(I-\sigma\varepsilon^2 BD)^{-1}p]_2 , \quad \varphi\psi = -i\varepsilon^2\partial_x [p,(I-\sigma\varepsilon^2 BD)^{-1}Bq]_1 . \tag{3.45b}$$

Also, since $\varphi\psi = \psi\varphi$, we have

$$\varphi\psi = -\varepsilon^2\partial_x \operatorname{Tr}\{(I-\sigma\varepsilon^2 BD)^{-1}(BD)_x\} . \tag{3.46}$$

We do not have, in general, a formula analogous to (3.22) because of the difficulties of dealing with traces and determinants of infinite dimensional operators. Formulas (3.45) and (3.46) constitute an actual summation of (3.13) and (3.14). It is also possible to sum the expressions (3.33) and (3.41) for the K_j's and Δ_j's to formulas similar to (3.45). Whenever $(I-\sigma\varepsilon^2 BD)$ or $(I-\sigma\varepsilon^2 DB)$ is not invertible, these formulas are meaningless. However, at this point it is the actual solution of (3.01)-(3.02) that has a singularity, not the way we write it.

Let us now assume that (3.06) holds, with $d\rho_1(k) = d\rho_2(-k^*)$ and $a_1(k) = a_2^*(-k^*)$. Then if $S : \mathcal{L}^2(d\rho_1) \longleftrightarrow \mathcal{L}^2(d\rho_2)$ is the isomorphism given by

$$(Sh)(k) = h(-k^*) , \; (\forall\, h = h(k_1) \text{ or } h = h(k_2)),$$

we have $Sp = \bar{q}$, $Sq = \bar{p}$ and $\overline{BS} = -SD$. Thus, since $S^2 = I$, from (3.45) we have

$$\varphi = \psi^* = \varepsilon[p,(I+\sigma\varepsilon^2 BS\overline{BS})^{-1}p]_1 . \tag{3.47}$$

Now (iBS) is self-adjoint, and for any g we have $2\partial_x\langle g, iBSg\rangle_1 =$
$= 2\partial_x \int_{\mathbb{C}^2} i\, d\rho_1(k_1)\, d\rho_2(k_2)\, g^*(k_1)\, p(k_1)(k_1+k_2)^{-1} g(k_2)\, g(-k_2^*) = -\int_{\mathbb{C}} d\rho_1(k_1)\, p(k_1)\, g^*(k_1) \int_{\mathbb{C}} d\rho_2(k_2)\, q(k_2)\, g(-k_2^*) =$
$= -\int_{\mathbb{C}} d\rho_1(k_1)\, p(k_1)\, g^*(k_1) \int_{\mathbb{C}} d\rho_1(k_1)\, p^*(k_1)\, g(k_1) = -\left|\int_{\mathbb{C}} d\rho_1(k_1)\, p(k_1)\, g^*(k_1)\right|^2$. Thus if a_1 and $d\rho_1$ are such that we can write $\langle g, iBSg\rangle_1 = -\int_x^{\infty} \partial_x \langle g, iBSg\rangle_1 dz$ or $\langle g, iBSg\rangle_1 = \int_{\infty}^{x} \partial_x \langle g, iBSg\rangle dz$, (iBS) is semidefinite (positive or negative) and from (A.3) the spectrum of $(iBS)\overline{(iBS)}$ is real and nonnegative. It follows that, for $\sigma > 0$, (3.47) will not have any singularities. For $\sigma < 0$ uniform bounds, independent of x and t on $\|B\|$ are necessary to guarantee this. As we saw in Section 3.2, this is not always possible.

The particular case $d\rho_1(k) = d\rho_2(k) = dk$, $-\infty < k < \infty$, i.e., $d\rho_1$ and $d\rho_2$ are the usual measure on the real line, can be treated in a way completely analogous to the equivalent case for the KdV equation in Section 2.5. Then $\mathcal{B}$ and $\mathcal{D}$ can be expressed in terms of Fourier transform operators on the real line, both are uniformly bounded for all $-\infty < x, t < \infty$, and if (3.06) holds (iBS) is semidefinite (when integrating over or under the singularity $(k_1+k_2)^{-1}$. Moreover, it is also possible to treat a combination of this last case with the one in Section 3.2, and then relate the parameters of the measures $d\lambda$ and $d\gamma$ † to the scattering parameters of (3.39).

3.5 Transformation Properties

The purpose of this section is to investigate the effect on φ and ψ of adding to $d\lambda$ and $d\gamma$ δ-functions of Dirac. More precisely, let

$$d\lambda'(k) = d\lambda(k) + \sum_{j=1}^{N} \xi_j^2\, \delta(k - z_j)\, dk_R\, dk_I \quad , \tag{3.48a}$$

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†The same type of arguments used in Section 2.6 are useful for this.

$$d\psi'(k) = d\psi(k) + \sum_{j=1}^{\mathcal{M}} \eta_j^2 \, \delta(k-w_j)\, dk_R\, dk_I \ , \tag{3.48b}$$

where $k_R = \mathrm{Re}\,k$, $k_I = \mathrm{Im}\,k$, $\delta(\cdot)$ is Dirac's δ-function on the complex plane thought of as a two-dimensional Euclidean space, and the ξ_j's, η_j's, z_j's and w_j's are constants. Call φ' and ψ' the solutions of (3.02) that result when $d\lambda'$ and $d\psi'$ are substituted in (3.13) for $d\lambda$ and $d\psi$. Introduce the $(\mathcal{N}+\mathcal{M})\times(\mathcal{N}+\mathcal{M})$ matrix

$$C = \begin{bmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{bmatrix} ,$$

where C^{11}, C^{12}, C^{21} and C^{22} are the matrices whose components are given by

$$(C^{11})_{\upsilon\ell} = \Big\{ \sum_{\substack{n=0\\ m=2n+1}}^{\infty} (\varepsilon/\chi)^m \int_{\mathbb{C}^m} \frac{e^{i\Omega_m}\, d\lambda_1 d\psi_2 \ldots d\lambda_m}{(w_\upsilon+k_1)\prod_j (k_j+k_{j+1})\,(k_m+w_\ell)} \Big\} \eta_\upsilon \eta_\ell \, e^{\frac{i}{2}[(w_\upsilon+w_\ell)x+(w_\upsilon^2+w_\ell^2)t]} , (\forall\, 1\leq \upsilon,\ell \leq \mathcal{M}),$$

$$(C^{22})_{\upsilon\ell} = \Big\{ \sum_{\substack{n=0\\ m=2n+1}}^{\infty} (\varepsilon/\chi)^m \int_{\mathbb{C}^m} \frac{e^{i\Lambda_m}\, d\psi_1 d\lambda_2 \ldots d\psi_m}{(z_\upsilon+k_1)\prod_j (k_j+k_{j+1})\,(k_m+z_\ell)} \Big\} \xi_\upsilon \xi_\ell \, e^{\frac{i}{2}[(z_\upsilon+z_\ell)x-(z_\upsilon^2+z_\ell^2)t]} , (\forall\, 1\leq \upsilon,\ell \leq \mathcal{N}),$$

$$(C^{12})_{\upsilon\ell} = (C^{21})_{\ell\upsilon} = \eta_\upsilon \xi_\ell \, e^{\frac{i}{2}[(w_\upsilon+z_\ell)x+(w_\upsilon^2-z_\ell^2)t]} \Big\{ \frac{1}{w_\upsilon+z_\ell} +$$

$$+ \sum_{n=1}^{\infty} (\varepsilon/\chi)^{2n} \int_{\mathbb{C}^{2n}} \frac{e^{i[\sum_j (k_j x + (-1)^j k_j^2 t)]}\, d\lambda_1 d\psi_2 \ldots d\lambda_{2n-1} d\psi_{2n}}{(w_\upsilon+k_1)\prod_j (k_j+k_{j+1})(k_{2n}+z_\ell)} \Big\} , (\forall\, 1\leq \upsilon\leq \mathcal{M}, 1\leq \ell\leq \mathcal{N}).$$

Then, after some manipulations, and if r and s are the $(\mathcal{N}+\mathcal{M})$ vectors given by

$$r_j = \eta_j\, \Delta_3(x,t,\tfrac{1}{2}w_j) \ , \quad r_{\mathcal{M}+\ell} = \xi_\ell\, \Delta_1(x,t,\tfrac{1}{2}z_\ell) \ ,$$

$$s_j = \eta_j\, \Delta_2(x,t,\tfrac{1}{2}w_j) \ , \quad s_{\mathcal{M}+\ell} = \xi_\ell\, \Delta_4(x,t,\tfrac{1}{2}z_\ell) \ ,$$

$(\forall\, 1\leq j\leq \mathcal{M},\ 1\leq \ell\leq \mathcal{N})$, we find

$$\varphi' = \varphi + \varepsilon r^T\{I-(\varepsilon/\chi)C\}^{-1}r \quad , \quad \psi' = \psi + \varepsilon s^T\{I-(\varepsilon/\chi)C\}^{-1}s . \tag{3.49}$$

Similarly, from (3.14), we have

$$\varphi'\psi' - \varphi\psi = -i\varepsilon\chi\partial_x r^T\{I-(\varepsilon/\chi)C\}^{-1}s = -i\varepsilon\chi\partial_x s^T\{I-(\varepsilon/\chi)C\}^{-1}r . \tag{3.50}$$

If Λ is the $(N+M)$ diagonal matrix with diagonal elements $\Lambda_{jj}=(1/2)w_j$, $\Lambda_{\ell\ell}=-(1/2)z_{\ell-M}$, $(\forall\ 1\leqslant j\leqslant M,\ M<\ell\leqslant M+N)$, then r, s and C satisfy the following equations:

$$r_x + i\Lambda r = i\chi^{-1}\varphi s \quad , \quad s_x - i\Lambda s = i\chi^{-1}\psi r , \tag{3.51a}$$

$$i r_t - 2\Lambda^2 r + 2\chi^{-1}\varphi\Lambda s + i\chi^{-1}\varphi_x s + \sigma\varphi\psi r = 0 ,$$

$$-i s_t - 2\Lambda^2 s - 2\chi^{-1}\psi\Lambda r + i\chi^{-1}\psi_x r + \sigma\psi\varphi r = 0 , \tag{3.51b}$$

$$C_x = -(1/2i)(rs^T + sr^T) \quad , \quad C_t = 2(\Lambda C_x + C_x\Lambda) - i\chi^{-1}(\varphi ss^T - \psi rr^T) , \tag{3.51c}$$

$$C\Lambda - \Lambda C = (1/2)(rs^T - sr^T) \quad , \quad C^T = C . \tag{3.51d}$$

It can now be checked directly that for any given symmetric operator Λ , in a Hilbert space with a conjugation (see A.3), any solution of (3.51) gives through (3.49) and (3.50) a new solution of (3.02). It is not necessary that φ and ψ be representable by formulas like (3.13) and (3.14). Equations (3.51a,b,c) are consistent, provided that φ and ψ satisfy (3.02), and they imply ∂_x and ∂_t of (3.51d). If r, r_x, s and s_x vanish (as $x \mapsto \infty$) fast enough, then $C=(1/2i)\int_x^\infty (rs^T+sr^T)dz$ solves (3.51c,d).

If the vectors r and s, and the operators Λ and C are in a finite dimensional space, then (3.50) has the alternative form

$$\varphi'\psi' - \varphi\psi = \sigma^{-1}\partial_x^2 \operatorname{Tr} \ln\{I - (\varepsilon/\chi)C\} = \sigma^{-1}\partial_x^2 \ln \det\{I - (\varepsilon/\chi)C\} , \tag{3.52}$$

where we have used (3.51c).

If $\varphi = \psi^*$ then a condition that will ensure that $(\varphi')^* = \psi'$ is that there exists an operator U such that

$$r^* = s^T U^T , \quad C^* = (sg\sigma)\, U C U^T \quad \text{and} \quad U U^T = U^T U = I . \tag{3.53}$$

Then $(\varphi')^* = \varphi^* + \varepsilon r^* \{I - (\varepsilon/\chi^*) C^*\}^{-1} \bar{r} = \psi + \varepsilon s^T U^T \{U U^T - (\varepsilon/\chi) U C U^T\}^{-1} U s = \psi'$.

Examples

(i) Assume $d\lambda = d\mu = 0$. Then $\varphi = \psi = 0$ and the formulas for φ' and ψ' reduce to the ones in Section 3.4, since (using the notation of (3.43) and (3.44))

$$r = \begin{bmatrix} 0 \\ p \end{bmatrix} , \quad s = \begin{bmatrix} q \\ 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & D \\ B & 0 \end{bmatrix}$$

is a solution of (3.51).

(ii) Take now $\varphi = \psi^* = \rho \exp i[\kappa x - (\kappa^2 - 2\sigma\rho^2)t + \theta_0]$, where $\rho > 0$, κ and θ_0 are real constants. Then for $\Lambda = \lambda \in \mathbb{C}$, a solution of (3.51a,b) is given by

$$r = \alpha \exp i[Tx - (T\kappa + \lambda\kappa - 2T\lambda - \sigma\rho^2)t] = \alpha E_\lambda^1 , \tag{3.54a}$$

$$s = \alpha\chi(T+\lambda)\rho^{-1}\exp i[(T-\kappa)x-(T\kappa+\lambda\kappa-2T\lambda-\kappa^2+\sigma\rho^2)t-\theta_0] =$$

$$= \alpha\chi(T+\lambda)\varphi^{-1}E^1_\lambda = \alpha E^2_\lambda, \tag{3.54b}$$

where $\alpha\in\mathbb{C}$ and $T=(1/2)(\kappa\pm\sqrt{(\kappa+2\lambda)^2+4\sigma\rho^2})$, i.e. $(T+\lambda)(T-\kappa-\lambda)=\sigma\rho^2$. Let λ be such that $\mathrm{Im}T\neq 0$ and take the sign on the square root always so that $\mathrm{Im}T>0$. Then we have

$$T^*(\lambda)+T(\lambda^*)=\kappa \qquad \text{and } (E^1_\lambda)^* = \rho\frac{e^{i\theta_0}}{\chi(T(\lambda^*)+\lambda^*)}E^2_{\lambda^*}. \tag{3.54c}$$

Assume now $\sigma<0$. Then we can take λ real, satisfying $(\kappa+2\lambda)^2<-4\sigma\rho^2$. Thus $r^* = (\alpha/\alpha^*)\rho\dfrac{e^{i\theta_0}}{\chi(T+\lambda)}s$, and since $\left|\rho\dfrac{e^{i\theta_0}}{\chi(T+\lambda)}\right|=1$, we can choose $\alpha=\rho\dfrac{e^{i\theta_0}}{\chi(T+\lambda)}\alpha^*$. Then $r^*=s$ and $C^*=(-i\int_x^\infty rs)^* = -C = (\mathrm{sg}\,\sigma)C$. Condition (3.53) is thus satisfied with $\upsilon=1$, and from (3.49) and (3.52) we have the following solution for (3.01)

$$\varphi' = \varphi+\varepsilon\alpha^2(E^1_\lambda)^2\left(1-\frac{\varepsilon\alpha^2}{\chi(2T-\kappa)}E^1_\lambda E^2_\lambda\right)^{-1}, \tag{3.54d}$$

$$|\varphi'|^2 = \rho^2+\sigma^{-1}\partial_x^2\ln\left(1-\frac{\varepsilon\alpha^2}{\chi(2T-\kappa)}E^1_\lambda E^2_\lambda\right). \tag{3.54e}$$

Call $\xi=+\sqrt{-(\kappa+2\lambda)^2-4\sigma\rho^2} = -i(2T-\kappa)$, $\delta = -\dfrac{\varepsilon\alpha^2}{\chi(2T-\kappa)}\cdot\dfrac{\chi(T+\lambda)}{\rho}e^{-i\theta_0} = -\dfrac{\varepsilon\alpha^2}{i\chi\xi}\dfrac{\alpha^*}{\alpha} = -\varepsilon|\alpha|^2/i\chi\xi$ and $\eta=\kappa+2\lambda$. Then ξ is a positive real number with $\xi^2<-4\sigma\rho^2$, δ is an arbitrary real number, $\eta=\pm\sqrt{-4\sigma\rho^2-\xi^2}$ and we have

$$r = \beta\exp\{\tfrac{i}{2}[\kappa x-(\kappa^2-2\sigma\rho^2)t+\theta_0]\}\exp\{-\tfrac{1}{2}\xi[x+(\eta-2\kappa)t]\}, \quad s=r^*,$$

$$\varphi' = \varphi\left\{1 + (\epsilon\beta^2/\rho)\,\frac{e^{-\xi[x+(\eta-2\kappa)t]}}{1+\delta\exp\{-\xi[x+(\eta-2\kappa)t]\}}\right\} ,$$

$$|\varphi'|^2 = \rho^2 + \sigma^{-1}\partial_x^2 \ln\left\{1+\delta\, e^{-\xi[x+(\eta-2\kappa)t]}\right\} , \qquad (3.54f)$$

where $\beta = \alpha e^{-i\theta_0/2} = \sqrt{-i\chi\delta\xi/\epsilon}\,\exp\{-\frac{i}{2}[\arg\chi + \arctan(\xi/\eta)]\}$, and $\epsilon\beta^2 = -i|\chi|\delta\xi\exp[-i\arctan(\xi/\eta)] = \delta\xi(\xi+i\eta)/2\sigma\rho$. In this new formulation we have replaced the "free" parameters λ and $\epsilon\alpha^2$ of (3.54d,e) by ξ and δ. If we take $\delta>0$ we see that φ' is a nonsingular solution, asymptotic to φ as $x\mapsto\infty$ and to $(1+\frac{\epsilon\beta^2}{\rho\delta})\varphi$ as $x\mapsto-\infty$.

We recognize in (3.54f) the dark pulse solution (envelope-hole solution) of (3.01). This solution was first found by Hasegawa and Tappert (1973). See also Hirota (1974) and Zakharov and Shabat (1973). The solution corresponding to the interaction of several of these dark pulse solutions can be easily found by taking Λ a real diagonal matrix, with diagonal elements λ_j satisfying $(\kappa+2\lambda_j)^2<-4\sigma\rho^2$ for all j's. Then we take $r_j = \alpha_j E^1_{\lambda_j}$, $s_j = \alpha_j E^2_{\lambda_j}$ and $C = (1/2i)\int_x^\infty (rs^T + sr^T)$, where $\alpha_j = \rho\chi^{-1}(T_j+\lambda_j)^{-1}e^{i\theta_0}\alpha_j^*$ for all j's.

Take now φ as in example (ii) and $\Lambda = \Lambda_\lambda$, where Λ_λ is the (2×2) diagonal matrix with entries λ and λ^*. Then for some $\alpha_1,\alpha_2\in\mathbb{C}$ we can take

$$r^T = [\alpha_1 E^1_\lambda, \alpha_2 E^1_{\lambda^*}] , \quad s^T = [\alpha_1 E^2_\lambda, \alpha_2 E^2_{\lambda^*}]$$

and

$$C = \frac{1}{2i}\int_x^\infty (rs^T + sr^T) = \begin{bmatrix} \frac{\alpha_1^2}{(2T-\kappa)} E^1_\lambda E^2_\lambda & \frac{\alpha_1\alpha_2}{2(T-T^*)}(E^1_\lambda E^2_{\lambda^*} + E^1_{\lambda^*}E^2_\lambda) \\ \frac{\alpha_1\alpha_2}{2(T-T^*)}(E^1_\lambda E^2_{\lambda^*} + E^1_{\lambda^*}E^2_\lambda) & \frac{\alpha_2^2}{(\kappa-2T^*)} E^1_{\lambda^*}E^2_{\lambda^*} \end{bmatrix}$$

where $T = T(\lambda)$ is as in example (ii). Let U be the (2×2) matrix defined by $r^* = s^T U^T$, i.e., $U_{11} = U_{22} = 0$, $U_{12} = \frac{\alpha_1^*}{\alpha_2} \rho \frac{e^{i\theta_0}}{\chi(\kappa - T^* + \lambda^*)}$ and $U_{21} = \frac{\alpha_2^*}{\alpha_1} \rho \frac{e^{i\theta_0}}{\chi(T+\lambda)}$ from (3.54c). Then, since $U_{21} U_{12}^* =$ $= -\frac{1}{|\chi|^2 (T+\lambda)(T-\kappa-\lambda)} \rho^2 = -\frac{1}{|\chi|^2 \sigma \rho^2} \rho^2 = -(sg\sigma)$, we have $U\bar{U} = \bar{U}U =$ $= -(sg\sigma) U$. Using this it follows that $C^* = (sg\sigma) U C U^T$. Moreover, if we choose $\alpha_2 = \alpha_1^* \rho e^{i\theta_0} / (\kappa - T^* + \lambda^*)\chi$, we have $U_{12} = -(sg\sigma) U_{21} = 1$ so that $UU^T = U^T U = I$. It follows then from (3.53) that $(\varphi')^* = \psi'$, so that we have a solution of (3.01).

We now introduce again the quantities $\xi = -i(2T - \kappa)$, $\eta = \kappa + 2\lambda$ and $\beta = \alpha_1 e^{-i\theta_0/2}$. Then ξ and β are arbitrary complex numbers with $\mathrm{Re}\,\xi > 0$, $\eta^2 + \xi^2 + 4\sigma\rho^2 = 0$ and we have

$$r_1 = \beta \exp \tfrac{1}{2}(-\Theta + i\theta) = s_2^*, \quad r_2 = -(sg\sigma)\left[\frac{\beta\chi(\eta + i\xi)}{2\rho}\right] \exp \tfrac{1}{2}(-\Theta^* + i\theta) = -(sg\sigma)\, s_1^*,$$

$$C_{11} = (Sg\sigma)\, C_{22}^* = (\beta^2 \chi(\eta + i\xi)/2i\rho\xi) \exp(-\Theta),$$

$$C_{12} = C_{21} = (|\beta|^2 / 2i\mathrm{Re}\,\xi)(1 - (sg\sigma)|\chi(\eta + i\xi)|^2 / 4\rho^2) \exp(-\mathrm{Re}\,\Theta), \tag{3.55a}$$

where

$$\theta = \kappa x - (\kappa^2 - 2\sigma\rho^2)t + \theta_0 \quad \text{and} \quad \Theta = \xi[x + (\eta - 2\kappa)t]. \tag{3.55b}$$

It is easy to see now that $C_{12} = C_{21}$ are purely imaginary and that, for $\sigma < 0$, $|C_{12}| \geq |C_{11}|$. It follows that for $\sigma < 0$, the eigenvalues of C are purely imaginary and that for $\sigma > 0$, they are complex conjugates with nonvanishing imaginary parts. Thus we see that the solution just found is nonsingular for $\sigma > 0$. For $\sigma \leq 0$ the solution is singular at all times, since not only are the eigenvalues of C purely imaginary;

but they move from $\pm i\infty$ to o as x goes from $-\infty$ to ∞ ; unless ξ and η are real, in which case $|x(\eta+i\xi)| = 2\rho$ so that $|C_{12}| = |C_{11}|$. Then we get back the solution of example (ii), with $\delta = -(\varepsilon/\rho\xi)\,\mathrm{Im}[\beta^2(\eta+i\xi)]$. This nonsingularity of the solution mainly for $\sigma>0$, might again be related to the instability of the uniform wave trains in this case.

Assume now $\sigma>0$, and let F and G be the functions

$$F = \det[I-(\varepsilon/x)C] = 1-2(\varepsilon/x)\mathrm{Re}\,C_{11} + \varepsilon^2\sigma(|C_{11}|^2+|C_{12}|^2) =$$

$$= 1-(\varepsilon/\rho)\mathrm{Re}\left[\beta^2(\eta+i\xi)e^{-\Theta}/i\xi\right] + (\varepsilon^2|\beta|^4/4\rho^2)\left[\frac{|\eta+i\xi|^2}{|\xi|^2} + \frac{(4\sigma\rho^2-|\eta+i\xi|^2)^2}{16\sigma\rho^2(\mathrm{Re}\,\xi)^2}\right]e^{-2\mathrm{Re}\,\Theta},$$

$$G = \varepsilon\left\{\left[1-(\varepsilon/x)C_{22}\right]r_1^2 + 2(\varepsilon/x)C_{12}r_1r_2 + \left[1-(\varepsilon/x)C_{11}\right]r_2^2\right\}.$$

Then from (3.49) and (3.52) we have

$$\varphi' = \varphi + G/F \quad , \qquad |\varphi'|^2 = \rho^2 + \sigma^{-1}\partial_x^2 \ln F \ . \tag{3.55c}$$

Now the leading term of G for $x \longmapsto -\infty$ is

$$(\varepsilon^2/x)\left[2C_{12}r_1r_2 - C_{22}r_1^2 - C_{11}r_1^2\right] =$$

$$=(\varepsilon^2|\beta|^4(\eta+i\xi)^*/2\rho\,\mathrm{Re}\,\xi)(\mathrm{Im}\,\xi)\left[\frac{1}{\xi^*} + \frac{|\eta+i\xi|^2}{4\sigma\rho^2\xi}\right]e^{i\theta-2\mathrm{Re}\,\Theta}.$$

Thus, from (3.55c) and the definitions of F and G , we see that

$$\varphi' \sim \varphi \quad \text{as} \quad x \longmapsto \infty \qquad \text{and} \qquad \varphi' \sim z\varphi \quad \text{as} \quad x \longmapsto -\infty$$

where z is a unitary complex number given by

$$z = 1 + \frac{-2\,\mathrm{Re}(i\xi^2)}{[(|\eta|^2+|\xi|^2)|\xi|^2 + 4\sigma\rho^2\,\mathrm{Re}\,\xi^2]}[(\eta+i\xi)^*\xi^* - (\eta-i\xi)\xi] ,$$

since $(z-1)$ is ρ times the coefficient of the leading term of G as $x \mapsto -\infty$, divided by the coefficient of the leading term of F.

The solution we have just obtained can be thought of as the solution representing the result of the interaction of an envelope-soliton with the traveling, spatially homogeneous wave solution φ. We see that the final outcome is a phase shift in φ of magnitude $\arg z$. Note that the phase shifts of the solutions (ξ,η) and $(\xi,-\eta)$ are opposite, while if ξ is real $z=1$. The solution representing the interaction of φ with several envelope-solitons can be easily found simply by taking Λ as a direct sum of Λ_λ's in (3.51).

3.6 Vector Valued Schrödinger Equation

As we pointed out before, the formulas in Sections 3.1 and 3.3 are valid also in the case in which φ and ψ are matrix valued, with matrices of the appropriate sizes so that the products $\varphi\psi\varphi$ and $\psi\varphi\psi$ make sense. In this context $*$ means hermitian adjoint in (3.01). Of course $d\lambda$ and $d\gamma$ are then matrix valued measures of the same sizes as φ and ψ respectively. It is possible to extend the results of Sections 3.2, 3.4 and 3.5 to this case. As an example we give now expressions for the multisoliton solution, in the vector valued case, of equation (3.01).

Let $d\lambda(k) = d\gamma^*(-k^*)$ be given by

$$d\lambda(k) = \sum_1^N \lambda^\ell\, \delta(k-z_\ell)\, dk_R\, dk_I \; , \tag{3.56}$$

where the z_ρ's are complex numbers of positive imaginary parts, the λ^ℓ's are arbitrary column complex vectors of dimension s, $k_R = \mathrm{Re}\,k$, $k_I = \mathrm{Im}\,k$, and $\delta(\cdot)$ is Dirac's δ-function on the complex plane. For any $1 \leq \nu \leq s$, define the column vector p and the matrices B_ν, Λ_ν by

$$P_j = \exp\left[\frac{i}{2}(z_j x - z_j^2 t)\right], \quad (\Lambda_\nu)_{j\rho} = (\lambda^\ell)_\nu \, \delta_{j\ell}, \quad (1 \leq j, \ell \leq N),$$

$$B_\nu = (1/2i)\int_x^\infty p\,(\Lambda_\nu p)^* = \left\{ (z_j - z_\rho^*)^{-1} (\lambda^\ell)_\nu^* \, P_j P_\rho^* \right\}. \tag{3.57}$$

Then we have from (3.13) and (3.14)

$$(\varphi)_j = \varepsilon p^T \Lambda_j (I + \sigma\varepsilon^2 \sum_1^s B_\rho \bar{B}_\rho)^{-1} p, \quad (1 \leq j \leq s) \tag{3.58a}$$

$$(\varphi\varphi^*)_{\ell\nu} = -i\varepsilon^2 \partial_x \, p^T \Lambda_\ell (I + \sigma\varepsilon^2 \sum_1^s B_j \bar{B}_j)^{-1} B_\nu \bar{p} = \tag{3.58b}$$

$$= 2\partial_x \,\mathrm{Tr}\left\{ (I + \sigma\varepsilon^2 \sum_1^s B_j \bar{B}_j)^{-1} B_\nu (\bar{B}_\rho)_x \right\},$$

$$|\varphi|^2 = i\varepsilon^2 \partial_x p^* (\sum_1^s \Lambda_\rho^* \bar{B}_\rho)(I + \sigma\varepsilon^2 \sum_1^s B_j \bar{B}_j)^{-1} p = 2\partial_x \mathrm{Tr}\left\{ (I + \sigma\varepsilon^2 \sum_1^s B_j \bar{B}_j)^{-1} \sum_1^s (B_\rho)_x \bar{B}_\rho \right\}. \tag{3.58c}$$

Now since $|\varphi|^2 = \varphi^* \varphi = \mathrm{Tr}(\varphi\varphi^*) = (1/2)\{ |\varphi|^2 + \mathrm{Tr}(\varphi\varphi^*) \}$, (3.58b,c) give

$$|\varphi|^2 = \sigma^{-1} \partial_x^2 \,\mathrm{Tr} \ln (I + \sigma\varepsilon^2 \sum_1^s B_\rho \bar{B}_\rho) = \sigma^{-1} \partial_x^2 \ln \det (I + \sigma\varepsilon^2 \sum_1^s B_\rho \bar{B}_\rho). \tag{3.59}$$

Again, as in the scalar case, there are no problems with the existence of $(I + \sigma\varepsilon^2 \sum_1^s B_\rho \bar{B}_\rho)^{-1}$ when $\sigma > 0$. This since $\sum_1^s B_\rho \bar{B}_\rho = \sum_1^s R \Lambda_\rho^* R^T \Lambda_\rho$ is similar to $\sum_1^s R^{1/2} \Lambda_\rho^* R^T \Lambda_\rho R^{1/2} = \sum_1^s (R^{1/2\,T} \Lambda_\rho R^{1/2})^* (R^{1/2\,T} \Lambda_\rho R^{1/2})$, which is self-adjoint and nonnegative. Here R stands for the self-adjoint and positive definite N-square matrix $R=(1/2)\int_x^\infty PP^*$. There are no multisoliton solutions for the case $\sigma<0$.

We note that if φ is a solution of the vector valued Schrödinger equation and A is a self-adjoint matrix, then

$$\Phi(x,t) = e^{iAt}\varphi(x,t) , \tag{3.60}$$

solves the equation

$$i\Phi_t + A\Phi + \Phi_{xx} + 2\sigma \|\Phi\|^2 \Phi = 0 . \tag{3.61}$$

In particular Φ can take values in $\mathcal{L}^2(du)$, and if we take $A=\pm\partial_u^2$, we see that (3.60) will solve the two-dimensional nonlinear Schrödinger equation

$$i\Phi_t + \Phi_{xx} \pm \Phi_{uu} + 2\sigma\left\{\int(\Phi^*\Phi)du\right\}\Phi = 0 , \quad \Phi = \Phi(x,u,t) . \tag{3.62}$$

As a final remark, we point out that if φ and ψ satisfy (3.02), then the following equations can also be solved by our perturbation approach

$$iu_t + u_{xx} = 2i\chi^{-1}\varphi v_x , \qquad -iv_t + v_{xx} = 2i\chi^{-1}\psi u_x . \tag{3.63}$$

If $\sigma<0$, so that $(i\chi^{-1})$ is real, and $\varphi=\psi^*$ then we can impose the condition $u=v^*$ and obtain the single equation

$$iu_t + u_{xx} = 2i\chi^{-1}\varphi u_x^* . \tag{3.64}$$

Other equations closely related to (3.01) can also be solved by our approach.

CHAPTER 4

MODIFIED KORTEWEG AND DE VRIES EQUATION

The equation can be taken in the form

$$v_t + 3\rho v_x v^2 + 3\rho v^2 v_x + v_{xxx} = 0 , \quad v = v(x,t) , \tag{4.01}$$

where ρ is a constant and v is real valued. This equation is the simplest modification of the KdV equation treated in Chapter 1. The particular way in which the nonlinear term is written in (4.01) is due to the fact that when v is taken matrix valued this is the right generalization solvable by our expansions.

4.1 Solution by Small Parameter Expansions

Substituting

$$v = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m v_m \tag{4.02}$$

into (4.01) and collecting equal powers of ε, we obtain for all $m=2n+1$ $(n=0,1,2,\dots)$

$$v_{m,t} + v_{m,xxx} = -3\rho \sum_{\substack{j+l+s=m \\ j,l,s \text{ odd.}}} (v_j v_l v_{s,x} + v_{j,x} v_l v_s) . \tag{4.03}$$

Since the linear dispersion relation of (4.01) is $G(\omega,k) = -i\omega - ik^3 = 0$, introduce

$$\Omega_n = (k_1 + \dots + k_n)x + (k_1^3 + \dots + k_n^3)t , \quad (\forall n = 1,2,3,\dots), \tag{4.04}$$

and write, as usual,

$$v_m = \int_{\mathbb{C}^m} \Phi_m \, e^{i\Omega_m} \, (d\gamma(k))^m \quad , \quad (\forall m = 2n+1,\ n = 0,1,2,\dots), \tag{4.05}$$

where $d\gamma(k)$ is an appropriate measure on the complex plane $\mathbb{C}$ and $\Phi_m = \Phi_m(k_1,\dots,k_m)$. For $m=1$ (4.05) satisfies (4.03). For $m = 2n+1$, $n = 1,2,3,\dots$ we must have

$$G(-k_1^3 - \dots - k_m^3,\ k_1+\dots+k_m)\,\Phi_m = -3i\rho \sum [(k_1+\dots+k_j) + (k_{j+\ell+1}+\dots+k_m)]\,\Phi_j\,\Phi_\ell\,\Phi_s\ ;$$
$$j+\ell+s = m,\ (j,\ell,s \text{ odd}). \tag{4.06}$$

where Φ_1 is arbitrary and the variables $k_1,\dots,k_m$ in the products $\Phi_j\,\Phi_\ell\,\Phi_s$ are evaluated sequentially. We note that G is the same as in Chapter 2.

Taking $\Phi_1 = 1$ in (4.06), and since $G(-k_1^3-k_2^3-k_3^3,\ k_1+k_2+k_3) = -3i(k_1+k_2)(k_2+k_3)(k_3+k_1)$, we find $\Phi_3 = \rho\,(k_1+k_2)^{-1}(k_2+k_3)^{-1}$. Similarly $\Phi_5 = \rho^2(k_1+k_2)^{-1}(k_2+k_3)^{-1}(k_3+k_4)^{-1}(k_4+k_5)^{-1}$ and we postulate that for all $m = 2n+1$, $n = 0,1,2,\dots$

$$\Phi_m = \rho^n \prod_{j=1}^{m-1} (k_j + k_{j+1})^{-1}. \tag{4.07}$$

Substituting this expression into (4.06) and multiplying through by Φ_m^{-1} we find that the following identity must hold for $m = 2n+1,\ n=0,1,2,\dots$:

$$G(-k_1^3 - \dots - k_m^3,\ k_1+\dots+k_m) = -3i\sum_{\ell=1}^{n}\sum_{j=1}^{\ell} (k_{2j-1}+k_{2j})(k_{2\ell}+k_{2\ell+1})(k_{2\ell+1}+\dots+k_m) -$$
$$-3i\sum_{\ell=1}^{n}\sum_{j=\ell}^{n} (k_1+\dots+k_{2\ell-1})(k_{2\ell-1}+k_{2\ell})(k_{2j}+k_{2j+1}). \tag{4.08}$$

This equality follows easily from (2.13). Alternatively (A.102) can be used to prove it.

Thus we have

$$v = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \rho^n \int_{\mathbb{C}^m} \frac{e^{i\Omega_m}}{\prod_{j=1}^{m-1} (k_j + k_{j+1})} (d\psi(k))^m . \tag{4.09}$$

It follows that

$$v^2 = -i\partial_x \sum_{n=1}^{\infty} \varepsilon^{2n} \rho^{n-1} \int_{\mathbb{C}^{2n}} \frac{e^{i\Omega_{2n}}}{\prod_{j=1}^{2n-1} (k_j + k_{j+1})} (d\psi(k))^{2n} . \tag{4.10}$$

The two last formulas are very similar to the ones in Chapters 2 and 3, and the same techniques we used to sum them can be used here. The eigenvalue problem associated with (4.01) (Wadati 1972), together with the corresponding Marčenko integral equations of its inverse scattering problem, follows easily from (4.09) and (4.10).

Introduce

$$b(x,t) = (1/2) \int_{\mathbb{C}} \exp[i(kx/2 + k^3 t)]\, d\psi(k) \tag{4.11}$$

and assume that as $x \to \infty$, $b(x,t)$ as well as each of the terms of the summations in (4.09) and (4.10) tends to zero sufficiently fast. Then we have

$$v = 2 \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m (-\rho)^n \int_{[x,\infty)^{m-1}} dz_1 \dots dz_{m-1}\, b(x+z_1,t)\, b(z_1+z_2,t) \dots b(z_{m-1}+x,t) , \tag{4.12a}$$

$$v^2 = \frac{2}{\rho} \partial_x \sum_{n=1}^{\infty} \varepsilon^{2n} (-\rho)^n \int_{[x,\infty)^{2n-1}} dz_1 \dots dz_{2n-1}\, b(x+z_1,t)\, b(z_1+z_2,t) \dots b(z_{2n-1}+x,t) . \tag{4.12b}$$

We note that $b(2x,t)$ satisfies the linearized equation and that b must be real for v to be real. The latter is equivalent to $d\psi(k) =$

$=d\psi^*(-k^*)$.

4.2 Multisoliton Solutions

Assume now that $b(x,t)$ is a sum of exponentials, which we take in the form

$$b(x+y,t)=\frac{1}{2}\sum_j \psi_j \exp i[z_j\frac{(x+y)}{2}+z_j^3 t] = \frac{1}{2}p^T(x,t)p(y,t)\ , \tag{4.13}$$

where $\{\psi_j, z_j\}_j$ are arbitrary complex numbers such that $\operatorname{Im} z_j > 0$ for all j's , and p is the column vector given by

$$p_j(x,t) = \psi_j \exp\frac{i}{2}(z_j x + z_j^3 t)\ ,\ (\forall j)\ . \tag{4.14}$$

Moreover, we assume that for every j there exists a unique ℓ_j such that $z_j^* = -z_{\ell_j}$ and $\psi_j^* = \psi_{\ell_j}$. This condition guarantees that b is real. Obviously $\ell_{\ell_j} = j$ so that $j \mapsto \ell_j$ is a permutation.

Substituting (4.13) into (4.12) and introducing the square matrix

$$B(x,t)=\frac{1}{2}\int_x^\infty pp^T = \{ i(z_j+z_\ell)^{-1} p_j(x,t)\, p_\ell(x,t)\}\ , \tag{4.15}$$

we have

$$v = \epsilon p^T(I+\epsilon^2\rho B^2)^{-1}p = -2\epsilon\, \mathrm{Tr}\{(I+\epsilon^2\rho B^2)^{-1}B_x\} = -2\kappa^{-1}\partial_x \mathrm{Tr}\arctan(\epsilon\kappa B), \tag{4.16a}$$

$$v^2 = -\epsilon^2\partial_x p^T(I+\epsilon^2\rho B^2)^{-1}Bp = 2\epsilon^2\partial_x \mathrm{Tr}\{(I+\epsilon^2\rho B^2)^{-1}BB_x\} =$$

$$= \rho^{-1}\partial_x^2\, \mathrm{Tr}\ln(I+\epsilon^2\rho B^2) = \rho^{-1}\partial_x^2 \ln\det(I+\epsilon^2\rho B^2)\ , \tag{4.16b}$$

where $\kappa^2=\rho$. Since $B^2 = D\bar{D}$, where D is the positive definite self-adjoint matrix $\mathcal{D}=(1/2)\int_x^\infty \underline{p}\,\underline{p}^{*}$ † we have from (A.3) that all the eigenvalues of $\mathcal{B}^2$ are positive. It follows that for $\rho>0$, formulas (4.16) are nonsingular. For $\rho<0$ they are singular, since the eigenvalues of $\mathcal{B}^2$ take all the values from ∞ to 0 as x varies from $-\infty$ to ∞. As in the case of the Schrödinger equation, the nonexistence of solitons for $\rho<0$ is related to the stability of uniform wave trains. Let $\rho>0$ in what follows.

Take $\epsilon=-K^{-1}$ in (4.16a). Then we can write

$$v=-iK^{-1}\partial_x \operatorname{Tr}\ln (I+iB)(I-iB)^{-1}=-iK^{-1}\partial_x \ln\frac{\det(I+iB)}{\det(I-iB)}=2K^{-1}\partial_x \arctan\left(\frac{g}{f}\right) \tag{4.17}$$

where

$$2ig=\det(I+iB)-\det(I-iB)\quad,\quad 2f=\det(I+iB)+\det(I-iB) \tag{4.18}$$

and we have used: $\operatorname{artan} z=\frac{1}{2i}\ln\frac{(1+iz)}{(1-iz)}$ and $\tan\left(\frac{1}{2i}\ln z\right)=-i\frac{(z-1)}{(z+1)}$ for any z. The speed of each soliton component in (4.16) is given by $(\operatorname{Im} z_j^3)/(\operatorname{Im} z_j)$. Since the z_j's need not be purely imaginary (as in the case of the KdV equation), we can have several solitons moving at the same speed. In fact this is going to happen (whenever for some j, z_j is not purely imaginary) with the pair $z_j\,,\,-z_j^*=z_{\ell_j}$. The most elementary solution of this type can be written, using (4.17) and (4.18) as

$$v=2K^{-1}\partial_x \arctan\left\{\frac{\beta}{2\alpha}\sin\left[\alpha(x+(\alpha^2-3\beta^2)t)+\xi_0\right]\operatorname{sech}\left[\beta(x+(\beta^2-3\alpha^2)t)+\eta_0\right]\right\}, \tag{4.19}$$

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† $\mathcal{D}=\mathcal{B}\Gamma^T$ where Γ is the permutation matrix that gives $\bar{p}=\Gamma p$ so that $\mathcal{D}\bar{\mathcal{D}}=\mathcal{D}\mathcal{D}^T=\mathcal{B}\Gamma^T\Gamma\mathcal{B}=\mathcal{B}^2$.

where α, β, ξ_0 and η_0 are real constants. This solution is called a "breather".

The formula $v^2 = \rho^{-1}\partial_x^2 \ln \det(I + B^2)$ where we have taken $\varepsilon = -\kappa^{-1}$ in (4.16b) was first obtained by Wadati (1972), using the inverse scattering transform. The formula $v = 2\kappa^{-1}\partial_x \arctan(\frac{g}{f})$, when the determinants in (4.18) are expanded using (A.203), was first obtained by Hirota (1972a).

4.3 Marčenko Integral Equations and Eigenvalue Problem

Introduce the operator $\hat{b}$ as in (2.24). Then from (4.12) we have

$$v(x,t) = 2\,K_1(x,x,t) \quad , \qquad v^2(x,t) = 2\rho^{-1}\partial_x K_2(x,x,t) \quad , \tag{4.20}$$

$$K_1(x,y,t) - \varepsilon\, b(x+y,t) - \varepsilon \int_x^\infty K_2(x,z,t)\, b(z+y,t)\, dz = 0 \,, \tag{4.21a}$$

$$K_2(x,y,t) + \varepsilon\rho \int_x^\infty K_1(x,z,t)\, b(z+y,t)\, dz = 0 \,, \tag{4.21b}$$

where

$$K_1 = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m (-\rho)^n\, \hat{b}^{m-1} b = \varepsilon\,(I + \varepsilon^2\rho\,\hat{b}^2)^{-1} b, \tag{4.22a}$$

$$K_2 = \sum_{n=1}^{\infty} \varepsilon^{2n} (-\rho)^n\, \hat{b}^{2n-1} b = -\varepsilon^2\rho\,\hat{b}\,(I + \varepsilon^2\rho\,\hat{b}^2)^{-1} b, \tag{4.22b}$$

and b is interpreted as an argument for $\hat{b}$ in the usual way: $b(x,y) = b(x+y,t)$. Expressions for K_1 and K_2 directly in terms of d_y can be easily be obtained. We recognize in (4.20) and (4.21) the Marčenko integral equations of the inverse scattering problem associated with (4.01) (Wadati 1972). These equations can be thought of as a way of summing (4.09) and (4.10). An alternative way following the lines set in Sections 2.4 and 3.4 can also be pursued. We then obtain formulas that generalize (4.16) with $\mathcal{B}$ an infinite dimensional operator instead of a finite matrix.

Partial differential equations for K_1 and K_2 can be easily obtained following the same procedure used in the preceding chapters. The equations are the following:

$$[\partial_t + 3\rho v^2(x,t)(\partial_x+\partial_y) + 3\rho v_x(x,t)v(x,t) + (\partial_x+\partial_y)^3]K_1(x,y,t) = 0, \qquad (4.23a)$$

$$[\partial_t + (\partial_x+\partial_y)^3]K_2(x,y,t) = 3\rho[v_x(x,t)(\partial_x+\partial_y) - \rho v^3(x,t)]K_1(x,y,t), \qquad (4.23b)$$

$$(\partial_x - \partial_y)K_1(x,y,t) = -v(x,t)K_2(x,y,t), \qquad (4.23c)$$

$$(\partial_x + \partial_y)K_2(x,y,t) = \rho v(x,t)K_1(x,y,t), \qquad (4.23d)$$

Equation (4.23a) is the "K-version" of (4.01) and (4.23b) is the "K-version" of the equation satisfied by $\gamma = v^2$, i.e., $\gamma_t + \gamma_{xxx} = 3(v_x^2 - \rho v^4)_x$. Equations (4.20c,d) follow from (4.21), although they can also be proved directly using the formulas for K_1 and K_2 in terms of $d\gamma$. We now separate variables and write

$$K_j(x,y,t) = \int_{\mathbb{C}} \Delta_j(x,t,\xi)\exp[i(\xi y + 4\xi^3 t)]\,d\tilde{\gamma}(\xi), \ (j=1,2). \qquad (4.24)$$

Then we have from (4.23c,d)

$$\Delta_{1,x} - i\xi\Delta_1 = -v\Delta_2 \quad , \qquad \Delta_{2,x} + i\xi\Delta_2 = \rho v\Delta_1 \quad , \qquad (4.25)$$

and from (4.23a,b), upon using (4.25) to eliminate derivatives with respect to x,

$$\Delta_{1,t} - 4i\xi^3\Delta_1 + 2i\xi\rho v^2\Delta_1 + 4\xi^2 v\Delta_2 - v_{xx}\Delta_2 - 2i\xi v_x\Delta_2 - 2\rho v^3\Delta_2 =$$

$$= \rho\,(v v_x - v_x v)\,\Delta_1 , \qquad (4.26a)$$

$$\Delta_{2,t} + 4i\xi^3\Delta_2 - 2i\xi\rho v^2\Delta_2 - 4\xi^2\rho v\Delta_1 + \rho v_{xx}\Delta_1 - 2i\xi\rho v_x\Delta_1 + 2\rho^2 v^3\Delta_1 =$$

$$= \rho\,(v v_x - v_x v)\,\Delta_2 . \qquad (4.26b)$$

The reason for keeping the right hand sides of these last two formulas is that all the formulas in this section, as well as those in Sections 4.1 and 4.4, remain valid if v is matrix valued. We recognize in (4.25) the eigenvalue problem associated with (4.01). The time evolution of the scattering parameters is then characterized by (4.26). It is possible to write Δ_1 , Δ_2 and $d\tilde{\gamma}$ directly in terms of $d\gamma$ and then prove (4.25) and (4.26) using these expressions. If Δ_1 is normalized so that its first term in the ε-dependent expansion is $\exp[i(\xi x + 4\xi^3 t)]$, then $d\tilde{\gamma}(\xi) = (\varepsilon/2)\,d\gamma(2\xi)$ and $\Delta_2 = O(\varepsilon)$.

As a final remark, we point out that it is possible to relate $d\gamma$ with the scattering parameters of (4.25), when $d\gamma$ is a continuous measure on the real line plus a sum of Dirac deltas on the complex plane. This is done following the same procedure used in Section 2.6 for the KdV equation.

4.4 Miura Transformation

Comparing formulas (4.09) and (4.10) with formula (2.14) we immediately see that

$$u = -i\kappa v_x + \rho v^2 , \quad \rho = \kappa^2 , \qquad (4.27)$$

is a solution of the KdV equation (2.01), given by (2.14) with

$$d\lambda(k) = \kappa\, d\gamma(k). \tag{4.28}$$

We recognize in (4.27) the transformation studied by Miura (1968). This transformation relates solutions of (2.01) and (4.01), independent of whether or not they can be written in terms of a measure.

4.5 Transformation Properties

By studying how formulas (4.09) and (4.10) transform, when a set of δ-functions of Dirac in the complex plane is added to $d\gamma$, we arrive at the following result. If v is a solution of (4.01), then so is

$$v' = v + \varepsilon r^T (I - \varepsilon\kappa C)^{-1} r = v + \varepsilon s^T (I - \varepsilon\kappa C)^{-1} s\,, \tag{4.29}$$

where $r = r(x,t)$, $s = s(x,t)$ and $C = C(x,t)$ are two column vectors and a matrix, respectively, that satisfy the following equations:

$$r_x + \Lambda r = i\kappa v s \qquad , \qquad s_x - \Lambda s = i\kappa v r\,, \tag{4.30a}$$

$$r_t - 4\Lambda^3 r - 2\rho v^2 \Lambda r = -4i\kappa v \Lambda^2 s - i\kappa v_{xx} s + 2i\kappa v_x \Lambda s - 2i\kappa\rho v^3 s,$$

$$s_t + 4\Lambda^3 s + 2\rho v^2 \Lambda s = -4i\kappa v \Lambda^2 r - i\kappa v_{xx} r - 2i\kappa v_x \Lambda r - 2i\kappa\rho v^3 r, \tag{4.30b}$$

$$C_x = -\frac{1}{2i}(r s^T + s r^T)$$

$$C_t = -4(\Lambda^2 C_x + \Lambda C_x \Lambda + C_x \Lambda^2) - 2\rho v^2 C_x + \kappa v_x (s s^T + r r^T) +$$
$$+ 2\kappa v \left[\Lambda (r r^T - s s^T) + (r r^T - s s^T)\Lambda\right], \tag{4.30c}$$

$$\Lambda C - C\Lambda = \frac{1}{2i}(rs^T - sr^T) \quad , \qquad C = C^T \, , \tag{4.30d}$$

$$r = U^T s \quad , \quad U C U^T = C \quad , \quad U^{-1} = U^T \, , \tag{4.30e}$$

where Λ and U are arbitrary constant square matrices with Λ symmetric. The conditions $U = U^T$ and $U\Lambda = -\Lambda U$ make (4.30e) compatible with the other equations. Moreover, we have

$$(v')^2 = v^2 - \varepsilon i \kappa^{-1} \partial_x r^T (I - \varepsilon \kappa C)^{-1} s = v^2 - \varepsilon \kappa^{-1} \partial_x \operatorname{Tr}\{(I - \varepsilon \kappa C)^{-1} C_x\} =$$

$$= v^2 + \rho^{-1} \partial_x^2 \operatorname{Tr} \ln (I - \varepsilon \kappa C) = v^2 + \rho^{-1} \partial_x^2 \ln \det (I - \varepsilon \kappa C) . \tag{4.31}$$

If v is real, then a condition that will insure that v' is real is the existence of a matrix T such that

$$r^* = s^T T^T \quad , \quad C^* = (\operatorname{sg}\rho)\, T C T^T \text{ and } T^{-1} = T^T . \tag{4.32}$$

To make (4.32) compatible with the other equations we require $T\bar{T} = -(\operatorname{sg}\rho) I$ and $T\Lambda = -\bar{\Lambda} T$.

Equations (4.30a,b,c) are consistent, and they imply ∂_x and ∂_t of (4.30d). If r and s decay to zero, as $x \mapsto \infty$, fast enough, then $C = (1/2i)\int_x^\infty (rs^T + sr^T)$ solves (4.30c,d), and (4.30e) and (4.32) if $U = U^T$, $U\Lambda = -\Lambda U$, $T\bar{T} = -(\operatorname{sg}\rho) I$ and $T\Lambda = -\bar{\Lambda} T$.

Except for the last two equalities in (4.31), all the equations in this section can be generalized for C, Λ, etc., operators in a Hilbert space with a conjugation (see A.3) with respect to which transposes are defined.

(i) By taking $v \equiv \alpha$ in (4.30), we can generate solutions of the equation

$$w_t + w_{xxx} + 12\alpha\rho\, w w_x + 6\rho\, w^2 w_x = 0 ,$$

by means of the transformation

$$w(x,t) + \alpha = v(x + 6\rho\alpha^2 t, t) .$$

(ii) The interaction of periodic solutions of (4.01) with solitary waves can be studied using the formulas in this section.

4.6 A Related Equation

The equation

$$v_t + 6\rho\, |v|^2 v_x + v_{xxx} = 0 , \tag{4.33}$$

which for real v reduces to (4.01), can also be readily solved using our method. In fact introducing $w = v^*$, (4.33) can be written as the system

$$v_t + 3\rho\, v w v_x + 3\rho\, v_x w v + v_{xxx} = 0 , \tag{4.34a}$$

$$w_t + 3\rho\, w v w_x + 3\rho\, w_x v w + w_{xxx} = 0 , \tag{4.34b}$$

whose solution follows the same lines of the solution of (3.02), with (4.08) replacing (3.12). The formulas for v and w end up looking like (3.13), with σ replaced by ρ and the linear dispersion $\omega = \pm k^2$ replaced by $\omega = -k^3$.

As (3.01), (4.33) presents multi-envelope soliton solutions for $\rho > 0$, multi-dark pulse solutions for $\rho < 0$ and solutions representing the interaction of many envelope-soliton solutions with the traveling, spatially homogeneous wave solution of (4.33) for $\rho > 0$. All this can be obtained by an analysis similar to that in Sections 3.5 and 4.5. Formulas for the multi-envelope soliton solutions for this equation were first presented by Hirota (1973a).

As a final remark we point out that the matrix-valued case of (4.34), and in particular the vector-valued case, can also be treated.

CHAPTER 5

SINE-GORDON EQUATION

The methods developed in previous chapters are now applied to the Sine-Gordon equation, which can be taken in the form

$$u_{XX} - u_{TT} = u_{xt} = \sigma \sin u \,, \tag{5.01}$$

where σ is a real constant and $u = u(X,T) = u(x,t)$ is real valued, with $x = \frac{1}{2}(X+T)$ and $t = \frac{1}{2}(X-T)$. This equation arises in many branches of mathematics and physics (Scott 1970, Rubinstein 1970, Barone et al. 1971). In order to avoid the complications of having to deal with a transcendental nonlinearity (i.e., $\sin u$) when performing our expansions, we introduce the new variables $\varphi = u_x$ and $\eta = -1 + \cos u$. Then we have the equations, which we write in the most symmetric form possible,

$$\varphi_{xt} - \sigma\varphi = \frac{\sigma}{2}(\varphi\eta + \eta\varphi) \quad , \quad 2\sigma^2\eta + (\sigma\eta)^2 + \varphi_t^2 = 0 \,. \tag{5.02}$$

We can easily recover (5.01) from (5.02), since we have $\eta \neq -1$, by means of the transformation $\sigma \sin u = \varphi_t$, $\cos u = \eta + 1$, consistent because of the second equation in (5.02). The first of these equations then readily gives $u_x = \varphi$, which implies (5.01).

5.1 Solution by Small Parameter Expansions

An expansion of the form $u = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m u_m$ corresponds to the following expansions for φ and η :

$$\varphi = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \varphi_m \qquad \text{and} \qquad \eta = \sum_{\substack{n=1 \\ s=2n}}^{\infty} \varepsilon^s \eta_s \,. \tag{5.03}$$

Let $G(Q,P)=G(\omega,k)=-P^2+Q^2-\sigma=k\omega-\sigma=0$ be the linearized dispersion relation of (4.01), where $k=P-Q$ and $\omega=-P-Q=\sigma k^{-1}$. Define

$$\Omega_n=(P_1+\dots+P_n)X-(Q_1+\dots+Q_n)T=(k_1+\dots+k_n)x-(\omega_1+\dots+\omega_n)t, \quad (\forall\, 1\leq n<\infty), \tag{5.04}$$

and write for m odd and s even,

$$\varphi_m=\int_{\mathbb{C}^m}\Phi_m e^{i\Omega_m}[d\lambda(k)]^m \qquad \eta_s=\int_{\mathbb{C}^s}H_s e^{i\Omega_s}[d\lambda(k)]^s \tag{5.05}$$

where $d\lambda(k)$ is an appropriate measure on $\mathbb{C}$, $\omega_j=\sigma k_j^{-1}$ for all j's and the Φ_m's and H_s's are functions of the k_j's. Then, substituting (5.03), (5.04) and (5.05) into (5.02), we have

$$G(\omega_1+\dots+\omega_m, k_1+\dots+k_m)\,\Phi_m=\frac{\sigma}{2}\sum_{j=1}^{n}\Phi_{2(n-j)+1}H_{2j}+H_{2j}\Phi_{2(n-j)+1}, \tag{5.06a}$$

$$2\sigma^2 H_s+\sigma^2\sum_{j=1}^{n-1}H_{2j}H_{2(n-j)}-$$
$$-\sum_{j=0}^{n-1}(\omega_1+\dots+\omega_{2j+1})(\omega_{2j+2}+\dots+\omega_s)\,\Phi_{2j+1}\Phi_{2(n-j)-1}=0, \tag{5.06b}$$

where $m=2n+1$, $s=2n$, $n=1,2,3,\dots$, Φ_1 is arbitrary and the variables in the products $\Phi_{m-2j}H_{2j}$, etc., are evaluated sequentially.

For $n=1$ we have from (5.06b), $2\sigma^2 H_2-\omega_1\omega_2\,\Phi_1(k_1)\Phi_1(k_2)=0$. Therefore, taking $\Phi_1\equiv 1$, we get $H_2=\frac{1}{2\sigma^2}\omega_1\omega_2=\frac{1}{2\sigma}\frac{(\omega_1+\omega_2)}{(k_1+k_2)}$. Then, from (5.06a), it follows that

$$[(k_1+k_2+k_3)(\omega_1+\omega_2+\omega_3)-\sigma]\,\Phi_3=\frac{\sigma}{2}[H_2(k_2,k_3)+H_2(k_1,k_2)]=$$

$$=\frac{\sigma}{2}\left[\frac{1}{2\sigma^2}\omega_2\omega_3+\frac{1}{2\sigma^2}\omega_1\omega_2\right]=\frac{1}{4\sigma}\omega_2(\omega_1+\omega_3)=\frac{\sigma}{4}\frac{(k_1+k_3)}{k_1k_2k_3}.$$

Now $(\omega_1+\omega_2+\omega_3)(k_1+k_2+k_3)-\sigma=\sigma(k_1+k_2)(k_2+k_3)(k_3+k_1)/k_1k_2k_3$ as it is easy to see from the formulas $\omega_j=\sigma/k_j$. It follows that

$\Phi_3 = \{4(k_1+k_2)(k_2+k_3)\}^{-1}$.

For $n=2$ we find similarly that $H_4 = \frac{1}{8\sigma}\frac{(\omega_1+\omega_2+\omega_3+\omega_4)}{(k_1+k_2)(k_2+k_3)(k_3+k_4)}$ and $\Phi_5 = \{16(k_1+k_2)(k_2+k_3)(k_3+k_4)(k_4+k_5)\}^{-1}$. Thus, it is natural to propose

$$\Phi_m = \{4^n \prod_{j=1}^{m-1}(k_j+k_{j+1})\}^{-1} = \{4^n \prod_{j=1}^{m-1}(P_j-Q_j+P_{j+1}-Q_{j+1})\}^{-1}, \tag{5.07a}$$

$$H_s = \frac{2(\omega_1+\cdots+\omega_s)}{4^n\sigma\prod_1^{s-1}(k_j+k_{j+1})} = -\frac{2(P_1+\cdots+P_s+Q_1+\cdots+Q_s)}{4^n\sigma\prod_1^{s-1}(P_j-Q_j+P_{j+1}-Q_{j+1})} \ . \tag{5.07b}$$

Substitution of (5.07) into (5.06) shows that the following identities must be satisfied:

$$G(\omega_1+\cdots+\omega_m, k_1+\cdots+k_m) =$$
$$= \sum_1^n\{(k_{2j-1}+k_{2j})(\omega_{2j}+\cdots+\omega_m)+(\omega_1+\cdots+\omega_{2j})(k_{2j}+k_{2j+1})\} \ , \tag{5.08a}$$

$$\sigma(\omega_1+\cdots+\omega_s) + \sum_1^{n-1}(\omega_1+\cdots+\omega_{2j})(k_{2j}+k_{2j+1})(\omega_{2j+1}+\cdots+\omega_s) -$$
$$-\sum_0^{n-1}(\omega_1+\cdots+\omega_{2j+1})(k_{2j+1}+k_{2j+2})(\omega_{2j+2}+\cdots+\omega_s) = 0 \ . \tag{5.08b}$$

Directly, or using (A.102), it can be proved that $G(\omega_1+\cdots+\omega_m, k_1+\cdots+k_m) = \sum_1^m(-1)^{j+1}G(\omega_j,k_j) + \sum_1^n\{(k_{2j-1}+k_{2j})(\omega_1+\cdots+\omega_m)+(\omega_1+\cdots+\omega_{2j})(k_{2j}+k_{2j+1})\}$ which, since $G(\omega_j,k_j)=0 \ (\forall j)$, reduces to (5.08a). (5.08b) follows easily upon expansion of the Σ's and use of the identities $\omega_j k_j = \sigma, (\forall j)$.

Finally, from (5.05) and (5.07), we have

$$\varphi = \sum_{\substack{n=0\\m=2n+1}}^{\infty} 2(\varepsilon/2)^m \int_{C^m} \frac{e^{i\Omega_m}}{\prod_{j=1}^{m-1}(k_j+k_{j+1})}[d\lambda(k)]^m \tag{5.09a}$$

$$\eta = \frac{2}{\sigma} i \partial_t \sum_{\substack{n=1 \\ s=2n}}^{\infty} (\varepsilon/2)^s \int_{\mathbb{C}^s} \frac{e^{i\Omega_s}}{\prod_{j=1}^{s-1} (k_j + k_{j+1})} [d\lambda(k)]^s . \tag{5.09b}$$

Real solutions are obtained when $d\lambda(k) = d\lambda^*(-k^*)$. The following is an immediate consequence of (5.09a):

$$\varphi^2 = -4i \partial_x \sum_{\substack{n=1 \\ s=2n}}^{\infty} (\varepsilon/2)^s \int_{\mathbb{C}^s} \frac{e^{i\Omega_s}}{\prod_{j=1}^{s-1} (k_j + k_{j+1})} [d\lambda(k)]^s . \tag{5.10}$$

The similarity of these formulas with the ones in Chapters 2, 3 and (especially) 4 is obvious, and we can sum (5.09) and (5.10) using the same techniques used there. Again the inverse scattering problem associated with (5.01) and (5.03) (Ablowitz et al. 1973), together with the corresponding Marčenko linear integral equations, follows from (5.09) and (5.10).

Introduce now

$$b(x,t) = \frac{1}{2} \int_{\mathbb{C}} \exp\left[\frac{i}{2}(kx - 2\sigma k^{-1} t)\right] d\lambda(k), \tag{5.11}$$

and assume that, as $x \mapsto \infty$, b, as well as each of the terms in the summations of (5.09) and (5.10), vanishes. Then we can write

$$\varphi = 2 \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^m \left(-\frac{1}{4}\right)^n \int_{[x,\infty)^{m-1}} dz_1 \ldots dz_{m-1}\, b(x+z_1,t)\, b(z_1+z_2,t) \ldots b(z_{m-1}+x,t) , \tag{5.12a}$$

$$\eta = -\frac{4}{\sigma} \partial_t \sum_{\substack{n=1 \\ s=2n}}^{\infty} \varepsilon^s \left(-\frac{1}{4}\right)^n \int_{[x,\infty)^{s-1}} dz_1 \ldots dz_{s-1}\, b(x+z_1,t)\, b(z_1+z_2,t) \ldots b(z_{s-1}+x,t) , \tag{5.12b}$$

and

$$\varphi^2 = 8\,\partial_x \sum_{\substack{n=1\\ s=2n}}^{\infty} \varepsilon^s\left(-\tfrac{1}{4}\right)^n \int_{[x,\infty)^{s-1}} dz_1 \dots dz_{s-1}\, b(x+z_1,t)\,b(z_1+z_2,t)\dots b(z_{s-1}+x,t). \tag{5.12c}$$

As usual $b(2x,t)$ satisfies the linearized equations. It must be real for u to be real.

5.2 Multisoliton Solution

Assume now that b is given by a sum of exponentials. That is, take

$$b(x+y,t) = \frac{1}{2}\sum_{j=1}^{N} \lambda_j^2 \exp\left\{\frac{i}{2}\left[z_j(x+y) - 2\sigma z_j^{-1} t\right]\right\} = \frac{1}{2}\, p^T(x,t)\, p(y,t), \tag{5.13}$$

where $\{\lambda_j, z_j\}_{j=1}^{N}$ are arbitrary complex numbers such that $\operatorname{Im} z_j > 0$ for all j's, and we assume that for every $1 \leqslant j \leqslant N$ there exists a unique $1 \leqslant \ell_j \leqslant N$ such that $z_j = -z_j^*$ and $\lambda_j^* = \lambda_j$. This last condition guarantees that b is real. The column vector p is defined by

$$p_j(x,t) = \lambda_j \exp\left\{\frac{i}{2}\left(z_j x - \sigma z_j^{-1} t\right)\right\}, \quad (1 \leqslant j \leqslant N). \tag{5.14}$$

Substituting (5.13) into (5.12), introducing the matrix

$$B(x,t) = (1/2)\int_x^{\infty} p(y,t)\, p^T(y,t)\, dy = \left\{ i(z_j + z_\ell)\, p_j(x,t)\, p_\ell(x,t) \right\}, \tag{5.15}$$

and summing, we have

$$\varphi = \varepsilon p^T\left[I + (\varepsilon/2)^2 B^2\right]^{-1} p = -2\varepsilon \operatorname{Tr}\left\{\left[I + (\varepsilon/2) B^2\right]^{-1} B_x\right\} = -4\partial_x \operatorname{Tr} \arctan\left(\tfrac{\varepsilon}{2} B\right), \tag{5.16a}$$

$$\eta = (\varepsilon^2/2\sigma)\,\partial_t\, p^T\left[I + (\varepsilon/2)^2 B^2\right]^{-1} B p = -(\varepsilon^2/\sigma)\,\partial_t \operatorname{Tr}\left\{\left[I + (\varepsilon/2)^2 B^2\right]^{-1} B B_x\right\} =$$

$$= -(2/\sigma)\,\partial_x\partial_t \operatorname{Tr} \ln\left[I + (\varepsilon/2)^2 B^2\right] = -(2/\sigma)\,\partial_x\partial_t \ln\det\left[I + (\varepsilon/2)^2 B^2\right], \tag{5.16b}$$

$$\varphi^2 = 4\,\partial_x^2\, \mathrm{Tr}\,\ln[I + (\varepsilon/2)^2 B^2] = 4\,\partial_x^2 \ln\det[I + (\varepsilon/2)^2 B^2]\,. \tag{5.16c}$$

All these formulas are nonsingular, since the eigenvalues of B are all real. This follows from the fact that $B^2 = D\bar{D}$, where D is the positive definite, self-adjoint matrix $D = \frac{1}{2}\int_x^\infty pp^*$; therefore, from (A.3), all the eigenvalues of B^2 are positive. D is related to B by the formula $D = B\Gamma^T$, where Γ is the permutation matrix that gives $\bar{p} = \Gamma p$.

We now use the relationship of u with φ and η, to write (Take $\varepsilon = 2$)

$$u_x^2 = 4\,\partial_x^2 \ln\det(I + B^2) \tag{5.17a}$$

$$\cos u = 1 - 2\sigma^{-1}\partial_x\partial_t \ln\det(I + B^2) \tag{5.17b}$$

$$\begin{aligned} u &= 4\,\mathrm{Tr}\arctan B = -2i\,\mathrm{Tr}\ln(I + iB)(I - iB)^{-1} = \\ &= -2i\ln\frac{\det(I+iB)}{\det(I-iB)} = 4\arctan\left(\frac{g}{f}\right), \end{aligned} \tag{5.17c}$$

where

$$2ig = \det(I + iB) - \det(I - iB)\,, \quad 2f = \det(I + iB) + \det(I - iB)\,. \tag{5.18}$$

The speed of each soliton component is given by $\frac{\mathrm{Im}\,\sigma z_j^{-1}}{\mathrm{Im}\, z_j} = -\sigma|z_j|^{-1}$. Since the z_j's need not be purely imaginary, we can have bound states composed of several solitons moving at the same speed. The simplest of these bound states is the one produced by a pair $z_j, z_{\ell_j} = -z_j^*$ and is known by the name "breather" or "0π-pulse". A formula similar to (4.19) can be obtained for it. The solutions produced by one purely imaginary

z_j are known by the name "kinks" or "2π-pulses".

Formula (5.17a) was first obtained by Ablowitz et al. (1973), using the inverse scattering transform. The formula $u = 4\arctan(\frac{g}{f})$, for purely imaginary z_j's, when g and f are expanded using (A.203), was first obtained by Hirota (1972b). The particular case corresponding to $N=2$ was first reported by Perring and Skyrme in 1962. Other alternative formulas have been obtained by Caudrey et al. (1973a).

5.3 Marčenko Integral Equations and Eigenvalue Problem

Introduce the operator $(\hat{b}f)(x,y) = \int_x^\infty f(x,z)b(z+y,t)\,dz$. Then from (5.12) we have

$$\varphi(x,t) = u_x(x,t) = 4\,K_1(x,x,t) , \tag{5.19a}$$

$$-\eta(x,t) = 1 - \cos u(x,t) = 4\sigma^{-1}\partial_t K_2(x,x,t) , \tag{5.19b}$$

$$\varphi^2(x,t) = u_x^2(x,t) = 8\,\partial_x K_2(x,x,t) \tag{5.19c}$$

$$K_1(x,y,t) - \frac{\varepsilon}{2}b(x+y,t) - \frac{\varepsilon}{2}\int_x^\infty K_2(x,z,t)\,b(z+y,t)\,dz = 0, \tag{5.20a}$$

$$K_2(x,y,t) + \frac{\varepsilon}{2}\int_x^\infty K_1(x,z,t)\,b(z+y,t)\,dz = 0, \tag{5.20b}$$

where

$$K_1(x,y,t) = \sum_{\substack{n=0\\ m=2n+1}}^{\infty}(-1)^n(\varepsilon/2)^m\,\hat{b}^{m-1}b = \frac{\varepsilon}{2}[I+(\varepsilon/2)^2B^2]^{-1}b , \tag{5.21a}$$

and

$$K_2(x,y,t) = \sum_{\substack{n=1\\ s=2n}}^{\infty}(-1)^n(\varepsilon/2)^s\,\hat{b}^{s-1}b = -\frac{\varepsilon^2}{4}\hat{b}[I+(\varepsilon/2)^2B^2]^{-1}b. \tag{5.21b}$$

In (5.21) b is interpreted as an argument for $\hat{b}$ in the usual way: $b(x,z) = b(x+z,t)$. Expressions for K_1 and K_2 directly in terms of $d\lambda$ can easily be written. As in the preceding cases, we recognize in (5.19) and (5.20) the Marčenko integral equations of the inverse scattering problem associated with (5.01) (Ablowitz et al. 1973).

Following the same procedure used in the preceding chapters, partial differential equations for K_1 and K_2 can immediately be obtained. The equations satisfied by K_1 and K_2 are the following:

$$\left\{(\partial_x+\partial_y)\partial_t - \sigma[1+\tfrac{1}{2}\eta(x,t)]\right\}K_1(x,y,t) = -\tfrac{1}{2}\varphi(x,t)\partial_t K_2(x,y,t)\,, \tag{5.22a}$$

$$(\partial_x+\partial_y)K_2(x,y,t) = \tfrac{1}{2}\varphi(x,t)\,K_1(x,y,t)\,, \tag{5.22b}$$

$$(\partial_x-\partial_y)K_1(x,y,t) = -\tfrac{1}{2}\varphi(x,t)K_2(x,y,t)\,. \tag{5.22c}$$

(5.22a) is the "K-version" of the first equation in (5.02) and (5.22b,c) follow from (5.20), although they can also be proved directly using the formulas for K_1 and K_2 in terms of $d\lambda(k)$. We now separate variables, writing

$$K_j(x,y,t) = \int_{\mathbb{C}} \Delta_j(x,y,\xi)\exp[i(\xi y - \tfrac{\sigma}{4}\xi^{-1}t)]\,d\gamma(\xi)\,,\ (j=1,2). \tag{5.23}$$

Then we have from (5.22b,c)

$$\Delta_{2,x} + i\xi\,\Delta_2 = \tfrac{1}{2}\varphi\,\Delta_1 \quad \text{and} \quad \Delta_{1,x} - i\xi\,\Delta_1 = -\tfrac{1}{2}\varphi\,\Delta_2\,. \tag{5.24}$$

From (5.22a) we have, upon using the second equation of (5.24) to eliminate derivatives with respect to x of Δ_1 ,

$$2i\xi\,\Delta_{1,t} = \frac{\sigma}{2}\,(\Delta_1 \cos u + \Delta_2 \sin u)\,. \tag{5.25a}$$

Taking now ∂_x of this last equation, using (5.24) and (5.25a) to eliminate all derivatives of Δ_1 , dividing by φ and using $\eta_x = -\frac{1}{2\sigma}(\varphi^2)_t$, which follows from (5.02), we have

$$2i\xi\,\Delta_{2,t} = \frac{\sigma}{2}\,(\Delta_1 \sin u - \Delta_2 \cos u)\,. \tag{5.25b}$$

(5.24) is the eigenvalue problem associated with (5.01)-(5.03) and (5.25) gives the time evolution of the scattering parameters.

It is possible to write Δ_1, Δ_2 and $d\psi$ directly in terms of $d\lambda$ and then prove (5.24) and (5.25) using these expressions. If Δ_1 is normalized so that its first term in the ε-dependent expansion is $\exp[i(\xi x - \frac{\sigma}{4}\xi^{-1}t)]$, then $d\psi(\xi) = \frac{\varepsilon}{4}\,d\lambda(2\xi)$ and $\Delta_2 = O(\varepsilon)$.

Equations (5.19) and (5.20) can be thought of as a way of summing (5.09) and (5.10). An alternative way, following the lines set in Sections 2.4 and 3.4 can also be pursued. We then obtain formulas that generalize the results of 5.2 with $\mathcal{B}$ an infinite dimensional operator, instead of a matrix.

Transformation relations similar to those in Sections 2.7, 3.5 and 4.5 can also be obtained for (5.01)-(5.02). Finally, it is possible to relate $d\lambda$ with the scattering parameters of (5.24) by a procedure similar to the one used in Section 2.6.

5.4 Miura Transformation

Comparison of formulas (5.09a) and (5.10) with formula (2.66) shows that

$$u = -\frac{i}{2}\varphi_x + \frac{1}{4}\varphi^2 \tag{5.26}$$

solves equations (2.68) and (2.73) with $\omega(k)=\sigma k^{-1}$, $f(k^2)=\sigma$ and $g(k^2)=k^2$. That is, we have

$$-u_{xxt} - 4uu_t + 2u_x\int_x^\infty u_t + \sigma u_x = 0. \tag{5.27}$$

This equation can be checked directly from (5.02) and (5.26), using the fact that $\eta = \frac{2}{\sigma}\int_x^\infty u_t + \sigma^{-1}\varphi_t$, as implied by $\eta_x = -\frac{1}{2\sigma}(\varphi^2)_t$.

5.5 Higher Order Equations

The equations treated in this and the preceding two chapters are all included in the general class of equations solvable by expressions of the form

$$\varphi = \sum_{\substack{n=0\\ m=2n+1}}^{\infty} \varepsilon^m\rho^n\int_{\mathbb{C}^m} \frac{e^{i\theta_m}}{\prod_1^{m-1}(k_j+k_{j+1})}\, dr(k_1,t)\,ds(k_2,t)\ldots dr(k_m,t)\ , \tag{5.28a}$$

$$\psi = \sum_{\substack{n=0\\ m=2n+1}}^{\infty} \varepsilon^m\rho^n\int_{\mathbb{C}^m} \frac{e^{i\theta_m}}{\prod_1^{m-1}(k_j+k_{j+1})}\, ds(k_1,t)\,dr(k_2,t)\ldots ds(k_m,t)\ , \tag{5.28b}$$

$$\partial_t dr(k,t) = -i\omega(k)\,dr(k,t)\ ,\quad \partial_t ds(k,t) = -i\nu(k)\,ds(k,t)\ , \tag{5.28c}$$

where $\theta_m = (k_1+\cdots+k_m)x, (\forall m)$ and ρ is a constant. To find equations in this class we follow the same procedure used in Section 2.8. Instead of equation (2.69) we now use the identity

$$-\Big(\sum_1^m b_j\Big)\Big(\sum_1^m a_j\Big)+\sum_1^m(-1)^j b_j a_j=\sum_0^{n-1}(b_{2j+1}+b_{2j+2})(a_{2j+2}+\cdots+a_m)+(a_1+\cdots+a_{2j+2})(b_{2j+2}+b_{2j+3})=$$

$$=\sum_{0\le j<\ell\le n}(b_{2j+1}+b_{2j+2})\frac{(a_{2j+2}+\cdots+a_{2\ell})(b_{2\ell}+b_{2\ell+1})+(b_{2\ell}+b_{2\ell+1})(a_{2\ell+1}+\cdots+a_m)}{(b_{2j+2}+\cdots+b_m)}+$$

$$+\sum_{0\le\ell<j\le n}\frac{(a_1+\cdots+a_{2\ell+1})(b_{2\ell+1}+b_{2\ell+2})+(b_{2\ell+1}+b_{2\ell+2})(a_{2\ell+2}+\cdots+a_{2j})}{(b_1+\cdots+b_{2j})}(b_{2j}+b_{2j+1})\,, \tag{5.29}$$

valid for all arbitrary numbers a_j and b_j $(1\le j\le m=2n+1,\ n=0,1,2,\dots)$. To prove the first equality in (5.29) we first verify it for $n=2$ and then we use (A.102). The second equality is trivial.

The simplest equation solved by (5.28) is

$$i\varphi_t=\varphi\,,\qquad\qquad -i\psi_t=\psi\,, \tag{5.30}$$

which corresponds to $\omega\equiv 1$ and $\nu\equiv -1$. Then using (5.29) we can construct the equation solved by (5.28) for $\nu(k)=-\omega(-k)$ and $\omega(k)$ a quotient of entire functions, say f and g . This equation is

$$g(\mathcal{L})\begin{bmatrix} i\varphi_t\\ -i\psi_t\end{bmatrix}=f(\mathcal{L})\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\quad,\ \omega(k)=f(k)/g(k)\,, \tag{5.31a}$$

where $\mathcal{L}$ is the operator given by

$$(\mathcal{L}r)_1=-i\,r_{1,x}+i\rho\,\varphi.\int^x(r_2\varphi-\psi r_1)\;-i\rho\int^x(r_1\psi-\varphi r_2)\cdot\varphi\quad,$$

$$(\mathcal{L}r)_2=i\,r_{2,x}-i\rho\,\psi.\int^x(r_1\psi-\varphi r_2)+i\rho\int^x(r_2\varphi-\psi r_1)\cdot\psi\quad, \tag{5.31b}$$

for any 2-vector valued function $r=r(x)$. In the context of (5.31a) the indefinite integrals $\int^x$ present in the definition of $\mathcal{L}$ are evaluated in such a way that the boundary condition $f(\mathcal{L})[\varphi,\psi]^T \equiv 0$ when $\varphi \equiv 0$ and $\psi \equiv 0$, is satisfied.† When g is not a constant it is necessary to assume that φ_t and ψ_t vanish sufficiently rapidly as $x \mapsto \infty$ (or $x \mapsto -\infty$), so that ∞ (or $-\infty$) may be taken as the lower limit of integration. Each application of the operator $\mathcal{L}$ produces the transformation $\omega(k) \longmapsto k\omega(k)$ and $\nu(k) \longmapsto -k\nu(k)$ in the dispersion functions, just the same as the operator $\mathcal{L}$ of Section 2.8 produced the transformation $\omega(k) \longmapsto k^2\omega(k)$. Equation (5.31) is the same equation found by Ablowitz et al. (1974a) as solvable by the inverse scattering transform associated with the eigenvalue problem

$$p_x - i\xi p = -\varphi q \quad , \quad q_x + i\xi q = p\psi p \, , \tag{5.32}$$

of which (3.39), (4.25) and (5.24) are particular cases. The cubic-Schrödinger equation, the modified KdV equation, the Sine-Gordon equation and equation (4.33) are all particular cases of (5.31).

When $\omega(k) = \omega(k+io^+)$ and $\nu(k) = \nu(k+io^+)$ for $\mathrm{Im}\,k = 0$ with

$$\omega(k) = \int_{-\infty}^{\infty} \frac{h(\xi)}{(k+2\xi)}\, d\xi \quad , \quad \nu(k) = \int_{-\infty}^{\infty} \frac{h(\xi)}{(k-2\xi)}\, d\xi \quad , \quad (\mathrm{Im}\,k > 0) \, , \tag{5.33}$$

for some function $h = h(\xi)$, it is also possible to write the equation solved by (5.28). Assume that φ and ψ vanish fast enough as $x \mapsto \infty$ and that $dr(k,t) \equiv ds(k,t) \equiv 0$ for $\mathrm{Im}\,k < 0$. Then the same kind of

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†Under these conditions it is easy to check that $\mathcal{L}^n[\varphi,\psi]^T$, $(\forall n = 0, 1, 2, \ldots)$ is a polynomial in φ, ψ and their first n partial derivatives with respect to x.

argument used to obtain (2.76) can be applied here to see that

$$\varphi_t = \int_{-\infty}^{\infty} h(\xi)\, \Delta_1(-\xi)\, \Delta_3(\xi)\, d\xi \quad , \quad \psi_t = \int_{-\infty}^{\infty} h(\xi)\, \Delta_2(\xi)\, \Delta_4(-\xi)\, d\xi \, , \tag{5.34}$$

where

$$\Delta_1 = \left\{ 1 + \sum_{n=1}^{\infty} \varepsilon^{2n} \rho^n \int_{\mathbb{C}^{2n}} \frac{e^{i\theta_{2n}}}{\prod_1^{2n-1} (k_j + k_{j+1}) \cdot (k_{2n} + 2\xi)} dr_1 ds_2 \dots dr_{2n-1} ds_{2n} \right\} e^{i\xi x} \, , \tag{5.35a}$$

$$\Delta_2 = \left\{ 1 + \sum_{n=1}^{\infty} \varepsilon^{2n} \rho^n \int_{\mathbb{C}^{2n}} \frac{e^{i\theta_{2n}}}{\prod_1^{2n-1} (k_j + k_{j+1}) \cdot (k_{2n} + 2\xi)} ds_1 dr_2 \dots ds_{2n-1} dr_{2n} \right\} e^{i\xi x} \, , \tag{5.35b}$$

$$\Delta_3 = -i \left\{ \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^{m} \rho^n \int_{\mathbb{C}^{m}} \frac{e^{i\theta_{m}}}{\prod_1^{m-1} (k_j + k_{j+1}) \cdot (k_{m} + 2\xi)} dr_1 ds_2 \dots dr_{m} \right\} e^{i\xi x} \, , \tag{5.35c}$$

$$\Delta_4 = -i \left\{ \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \varepsilon^{m} \rho^n \int_{\mathbb{C}^{m}} \frac{e^{i\theta_{m}}}{\prod_1^{m-1} (k_j + k_{j+1}) \cdot (k_{m} + 2\xi)} ds_1 dr_2 \dots ds_{m} \right\} e^{i\xi x} \, . \tag{5.35d}$$

The functions $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 satisfy

$$\Delta_{1,x} - i\xi\, \Delta_1 = -\rho\, \varphi\, \Delta_4 \quad , \quad \Delta_{4,x} + i\xi\, \Delta_4 = \psi\, \Delta_1 \, , \tag{5.36a}$$

$$\Delta_{2,x} - i\xi\, \Delta_2 = -\rho\, \psi\, \Delta_3 \quad , \quad \Delta_{3,x} + i\xi\, \Delta_3 = \varphi\, \Delta_2 \, . \tag{5.36b}$$

Introduce

$$\Phi(\xi) = \Delta_1(-\xi)\, \Delta_3(\xi) \quad , \quad \Psi(\xi) = \Delta_2(\xi)\, \Delta_4(-\xi) \, ,$$

$$\mathcal{N}(\xi) = \Delta_1(-\xi)\, \Delta_2(\xi) - \rho\, \Delta_3(\xi)\, \Delta_4(-\xi) \, . \tag{5.37}$$

Then using (5.34) and (5.36) we see that the equation satisfied by (5.28), with the choice of $\omega(k)$ given by (5.33), is

$$\varphi_t = \int_{-\infty}^{\infty} h(\xi)\, \Phi(\xi)\, d\xi \quad , \quad \psi_t = \int_{-\infty}^{\infty} h(\xi)\, \Psi(\xi)\, d\xi \quad , \tag{5.38a}$$

$$\Phi_x = -2i\xi\,\Phi + \varphi\mathcal{N} \quad , \quad \Psi_x = 2i\xi\,\Psi + \psi\mathcal{N} \, , \tag{5.38b}$$

$$\mathcal{N}_x = -2\rho(\psi\Phi + \varphi\Psi) \, , \tag{5.38c}$$

with φ, ψ, Φ and Ψ small as $x \mapsto \infty$ and $\mathcal{N} \sim 1$. If h is real then $\upsilon(k) = -\omega^*(-k^*)$ and solutions satisfying $\varphi = \psi^*$ can be found. Then $\Phi = \Psi^*$ and $\mathcal{N} = \mathcal{N}^*$ for ξ real. If h is even, then $\upsilon = \omega$ and we can ask that $\varphi = \psi$. Then $\Phi(\xi) = \Psi(-\xi)$ and $\mathcal{N}(\xi) = \mathcal{N}(-\xi)$ for ξ real. Assume now that h is real and even. Introduce the real functions

$$\mathcal{E}(x,t) = -\varphi(t-x, x) = -\psi(t-x, x) \, , \tag{5.39a}$$

$$\mathcal{P}(x,t,2\xi) = \mathrm{Re}\,\Phi(t-x, x, \xi) \quad , \quad \mathcal{Q}(x,t,2\xi) = -\mathrm{Im}\,\Phi(t-x, x, \xi) \, , \tag{5.39b}$$

$$\mathcal{N}(x,t,2\xi) = -\mathcal{N}(t-x, x, \xi) \, , \tag{5.39c}$$

$$h(\xi) = -[g(2\xi) + g(-2\xi)] \, , \tag{5.39d}$$

where $-\infty < \xi < \infty$. We note that $\mathcal{P}$ and $\mathcal{N}$ are even in ξ and $\mathcal{Q}$ is odd. Thus we have

$$\mathcal{E}_t + \mathcal{E}_x = \int_{-\infty}^{\infty} g(\zeta)\, \mathcal{P}(x,t,\zeta)\, d\zeta \, , \tag{5.40a}$$

$$\mathcal{P}_t = -\zeta\,\mathcal{Q} + \mathcal{E}\,\mathcal{N} \quad , \qquad \mathcal{Q}_t = \zeta\,\mathcal{P} \, , \tag{5.40b}$$

$$\mathcal{N}_t = -4\rho\,\mathcal{E}\,\mathcal{P}. \tag{5.40c}$$

These are the self-induced transparency equations (SIT) in dimensionless variables. For $g(\xi)=\alpha\delta(\xi-\xi_0)$ † they are the reduced Maxwell-Bloch equations (RMB). The SIT and RMB equations have been studied using various techniques, including inverse scattering, by Caudrey et al. (1973ab, 1974); Lamb (1973); and Ablowitz et al. (1974b).

Equations solvable by expressions of the form (5.28) when φ and ψ are matrix valued†† can also be found. In fact in (5.31) φ and ψ can be taken matrix valued, since when defining $\mathcal{L}$ in (5.31b) the possibility of having noncommutative products was considered. Furthermore, since (5.29) remains valid if the a_j's are matrices, nonscalar dispersion functions are also possible in this context. More precisely, (5.28c) can be replaced by

$$\partial_t \, dr(k,t) = -i\,[\,\omega(k)\,dr(k,t) + dr(k,t)\,\nu(k)\,] \ , \tag{5.41a}$$

$$\partial_t \, ds(k,t) = i\,[\,\nu(-k)\,ds(k,t) + ds(k,t)\,\omega(-k)\,] \ , \tag{5.41b}$$

where $\omega=\omega(k)$ and $\nu=\nu(k)$ are now square-matrix valued functions of of appropriate sizes. The simplest equation furnished by (5.41) is

$$i\varphi_t = \omega_0\varphi + \varphi\nu_0 \ , \qquad -i\psi_t = \nu_0\psi + \psi\omega_0 \ , \tag{5.42}$$

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†There is no harm in taking g (and thus h) a δ-function. It is easy to see that the requirement in (5.33), that the integration be with a proper function kernel, is unduly restrictive. In fact the integration need not even be over ξ real.

††The dimensions of φ and ψ must be such that the products $\varphi\psi$ and $\psi\varphi$ make sense.

for $\omega=\omega_0$ and $\nu=\nu_0$ independent of k. The others are constructed from this one using the operator $\mathcal{L}$, just as (5.31) followed from (5.30).

As a final remark we point out that the subclass of equations for which we can require $\varphi=\bar{\psi}$ is related to the class of equations treated in Section 2.8 by means of the transformation

$$u = -i\sqrt{\rho}\,\varphi_x + \rho\varphi^2. \tag{5.43}$$

This generalizes the results of Sections 4.4 and 5.4.

CHAPTER 6

BOUSSINESQ EQUATION

As a final example we consider in this chapter the equation

$$u_{tt} - \gamma u_{xx} - \frac{\nu^2}{3} u_{xxxx} + (\sigma u^2)_{xx} = 0 , \tag{6.01}$$

where γ, ν^2 and σ are real constants and $u = u(x,t)$ is real valued. This equation is a nonlinear Boussinesq equation. It occurs in one-dimensional nonlinear lattices (Zabusky 1967), water waves (Ursell 1953), etc.

6.1 Solution by Small Parameter Expansions

Following the same procedure we used in the preceding chapter, we arrive at the formulas

$$u = \frac{i\nu}{\sigma} \partial_x \sum_{n=1}^{\infty} (-\varepsilon)^n \int_{\mathbb{C}^{2n}} \frac{e^{i\Omega_n}}{\prod_1^{n-1} (\alpha_j + \beta_{j+1})} [d\lambda(k,\omega)]^n =$$

$$= \frac{2\nu^2}{\sigma} \partial_x^2 \sum_{n=1}^{\infty} (-\varepsilon)^n n^{-1} \int_{\mathbb{C}^{2n}} \frac{e^{i\Omega_n}}{(\alpha_n + \beta_1) \prod_1^{n-1} (\alpha_j + \beta_{j+1})} [d\lambda(k,\omega)]^n , \tag{6.02}$$

where $d\lambda(k,\omega)$ is an appropriate measure on $\mathbb{C}^2$ such that

$$d\lambda(k,\omega) \neq 0 \quad \text{only if} \quad G(\omega,k) = -\omega^2 + \gamma k^2 - \frac{\nu^2}{3} k^4 = 0 , \tag{6.03}$$

$$\Omega_n = (k_1 + \cdots + k_n)x - (\omega_1 + \cdots + \omega_n)t , \ (\forall n = 1,2,3,\ldots) \tag{6.04}$$

and

$$\alpha_j = \alpha(k_j,\omega_j) = \omega_j k_j^{-1} + \nu k_j \ , \ \beta_j = \beta(k_j,\omega_j) = -\omega_j k_j^{-1} + \nu k_j \ , \qquad (6.05)$$
$$(\forall\, 1 \leq j \leq n,\ n = 1,2,3,\ldots)\ .$$

The equality of the two expressions for u in (6.02), follows from noticing that

$$\left\{ \sum_{j=1}^{n-1} (\alpha_j + \beta_{j+1}) \right\} + (\alpha_n + \beta_1) = 2\nu \sum_{j=1}^{n} k_j \ , \qquad (6.06)$$

as is evident from (6.05). Therefore $-2\nu i n^{-1}\partial_x$ is equivalent to eliminating one of the factors in the denominator of the integrand of the last expression of (6.02). The condition for u to be real is

$$\left\{ \nu\, d\lambda(-k^*, -\omega^*) \right\}^* = -\nu\, d\lambda(k,\omega)\ . \qquad (6.07)$$

To prove that (6.02) is a solution of (6.01) we use the identity

$$G(a_1+\cdots+a_n,\ b_1+\cdots+b_n) - (b_1+\cdots+b_n)\sum_{j=1}^{n} b_j^{-1}\, G(a_j, b_j) =$$

$$= -(1/2)\sum_{l=1}^{n-1} \sum_{\pi\in\mathscr{C}_n} (b_{\pi_1}+\cdots+b_{\pi_l})(z_{\pi_l} + w_{\pi_{l+1}})(b_{\pi_{l+1}}+\cdots+b_{\pi_n})(z_{\pi_n} + z_{\pi_1})\ , \qquad (6.08)$$

where $n = 1,2,3,\ldots$, the b_j 's and a_j 's are arbitrary complex numbers such that $b_j \neq 0$ $(\forall 1 \leq j \leq n)$, $z_j = a_j b_j^{-1} + \nu b_j$, $w_j = -a_j b_j^{-1} + \nu b_j$, $(\forall 1 \leq j \leq n)$ and $\mathscr{C}_n$ is the set of cyclic permutations of $\{1,\ldots,n\}$. For $n = 4$ (6.08) can be verified directly, then (A.102) proves it for all n . Substituting (6.02) into (6.01) and using (6.08) we then have

$$(\partial_t^2 - 4\partial_x^2 - \frac{\nu^2}{3}\partial_x^4)u = \frac{2\nu^2}{\sigma}\partial_x^2 \sum_{n=1}^{\infty}(-\varepsilon)^n n^{-1}\int_{\mathbb{C}^{2n}} \frac{G(\omega_1+\dots+\omega_n, k_1+\dots+k_n)}{(\alpha_n+\beta_1)\prod_1^{n-1}(\alpha_j+\beta_{j+1})} e^{i\Omega_n}[d\lambda(k,\omega)]^n =$$

$$= -\frac{\nu^2}{\sigma}\partial_x^2 \sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{\pi\in\mathcal{G}_n}(-\varepsilon)^n n^{-1}\int_{\mathbb{C}^{2n}} \frac{(k_{\pi_1}+\dots+k_{\pi_l})(k_{\pi_{l+1}}+\dots+k_{\pi_n})}{\prod_{j=1}^{l-1}(\alpha_{\pi_j}+\beta_{\pi_{j+1}})\cdot\prod_{j=l+1}^{n-1}(\alpha_{\pi_j}+\beta_{\pi_{j+1}})} e^{i\Omega_n}[d\lambda(k,\omega)]^n =$$

$$= -\sigma\partial_x^2 \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{\nu}{\sigma}(-\varepsilon)^l \int_{\mathbb{C}^{2l}} \frac{(k_1+\dots+k_l)}{\prod_{j=1}^{l-1}(\alpha_j+\beta_{j+1})} e^{i\Omega_l}(d\lambda)^l \cdot \frac{\nu}{\sigma}(-\varepsilon)^{n-l}\int_{\mathbb{C}^{2(n-l)}} \frac{(k_1+\dots+k_{n-l})}{\prod_{j=1}^{n-l-1}(\alpha_j+\beta_{j+1})} e^{i\Omega_{n-l}}(d\lambda)^{n-l} =$$

$$= -\sigma\partial_x^2 u^2 . \tag{6.09}$$

This proves that (6.02) is a solution of (6.01).

Formula (6.02) is quite similar to the expressions found for the solutions of the equations in the preceding chapters, and the same techniques used before apply here for its summation. Introduce the function

$$b(x,y,t) = \frac{1}{2\nu i}\int_{\mathbb{C}^2} d\lambda(k,\omega)\exp\{\frac{i}{2\nu}(\beta x + \alpha y) - i\omega t\} , \tag{6.10}$$

and assume that, as $x \to \infty$, $(\partial_\varepsilon^n u)|_{\varepsilon=0} \longrightarrow 0$ $(\forall n)$ and $b \to 0$, the latter being true also for $y \to \infty$. Then we have, from (6.02), upon using $\alpha+\beta = 2\nu k$

$$u = -\frac{2\nu^2}{\sigma}\partial_x \sum_{n=1}^{\infty}(-\varepsilon)^n \int_{[x,\infty)^{n-1}} dz_1\dots dz_{n-1}\, b(x,z_1,t)\, b(z_1,z_2,t)\dots b(z_{n-1},x,t). \tag{6.11}$$

Equation (6.07) is equivalent to b being real. It is clear that $b(x,x,t)$ satisfies the linearized version of (6.01).

6.2 Multisoliton Solutions

Take now

$$b(x,y,t) = \frac{1}{2}\sum_m a_m^2 \exp\{-\tfrac{1}{2}(\xi_m x + \eta_m y) + \Lambda_m t\} = \frac{1}{2} p^T(x,t)\, q(y,t), \tag{6.12}$$

where p and q are the column vectors whose components are given by

$$p_m(x,t) = a_m \exp(-\tfrac{1}{2}\xi_m x - \tfrac{i\nu}{4}\xi_m^2 t)\ ,\quad q_m(x,t) = a_m \exp(-\tfrac{1}{2}\eta_m x + \tfrac{i\nu}{4}\eta_m^2 t), \tag{6.13}$$

the a_m's are real numbers and

$$\xi_m = \kappa_m - \frac{1}{i\nu}\frac{\Lambda_m}{\kappa_m}\ ,\qquad \eta_m = \kappa_m + \frac{1}{i\nu}\frac{\Lambda_m}{\kappa_m}\ , \tag{6.14}$$

where the κ_m's and Λ_m's are real numbers, the κ_m's positive, such that

$$0 = G(i\Lambda_m, i\kappa_m) = \Lambda_m^2 - 4\kappa_m^2 - \frac{\nu^2}{3}\kappa_m^4 . \tag{6.15}$$

In writing the second equality in (6.12) we have used that $\eta_m^2 - \xi_m^2 = \frac{4}{i\nu}\Lambda_m$.

Introduce the square matrix

$$B(x,t) = \frac{1}{2}\int_x^\infty q(z,t)\, p^T(z,t)\, dz = \{(\eta_{m_1} + \xi_{m_2})^{-1} q_{m_1}(x,t)\, p_{m_2}(x,t)\} . \tag{6.16}$$

Then from (6.11) we have

$$u = \frac{\nu^2}{\sigma}\varepsilon\partial_x \underline{p}^T(I+\varepsilon B)^{-1}\underline{q} = -\frac{2\nu^2}{\sigma}\varepsilon\partial_x \mathrm{Tr}\{(I+\varepsilon B)^{-1}B_x\} =$$

$$= -\frac{2\nu^2}{\sigma}\partial_x^2 \mathrm{Tr}\,\ln(I+\varepsilon B) = -\frac{2\nu^2}{\sigma}\partial_x^2 \ln\det(I+\varepsilon B). \tag{6.17}$$

For ν real, i.e., $\nu^2>0$, $\underline{p}^T = \underline{q}^*$ and B is positive definite. Thus for $\varepsilon>0$, formula (6.17) is nonsingular. For ν purely imaginary, i.e., $\nu^2<0$, further conditions on the κ_m's are necessary to guarantee the nonsingularity of (6.17). A sufficient condition is that for all pairs $m_1 \neq m_2$

$$(\xi_{m_1}-\xi_{m_2})(\eta_{m_1}-\eta_{m_2}) = (\kappa_{m_1}-\kappa_{m_2})^2 + \nu^{-2}\left(\frac{\Lambda_{m_1}}{\kappa_{m_1}} - \frac{\Lambda_{m_2}}{\kappa_{m_2}}\right)^2 \geq 0, \tag{6.18a}$$

or

$$(\xi_{m_1}+\xi_{m_2})(\eta_{m_1}+\eta_{m_2}) = (\kappa_{m_1}+\kappa_{m_2})^2 + \nu^{-2}\left(\frac{\Lambda_{m_1}}{\kappa_{m_1}} - \frac{\Lambda_{m_2}}{\kappa_{m_2}}\right) \leq 0. \tag{6.18b}$$

Then (A.203) shows that all the minors of B are nonnegative, so that no eigenvalue is negative (since the polynomial $\det(\lambda I+B)$ will have all its coefficients positive). It follows that, again for $\varepsilon>0$, (6.17) is nonsingular. For two soliton solutions (6.18) is also a necessary condition. After some manipulation, (6.18) can be reduced to the following single condition:

$$(2\Lambda_{m_1}/\kappa_{m_1})^2 + (2\Lambda_{m_2}/\kappa_{m_2})^2 + 4\Lambda_{m_1}\Lambda_{m_2}/\kappa_{m_1}\kappa_{m_2} \geq 34, \ (\forall m_1 \neq m_2). \tag{6.19}$$

Multisoliton solutions for equation (6.01) were first presented by Hirota (1973b).

6.3 Marčenko Integral Equation and Eigenvalue Problem

Introduce the operator $\hat{b}$ defined by

$$(\hat{b}f)(x,y) = \int_x^\infty f(x,z)\, b(z,y,t)\, dz \ , \ (\forall f = f(x,y)) . \tag{6.20}$$

Then, from (6.11),

$$u = -2\nu^2 \sigma^{-1} \partial_x K(x,x,t) \ , \tag{6.21a}$$

$$K(x,y,t) + \varepsilon b(x,y,t) + \varepsilon \int_x^\infty K(x, \ ,t)\, b(z,y,t)\, dz \ , \tag{6.21b}$$

where $K = \sum_1^\infty (-\varepsilon)^n \hat{b}^{n-1} b = -\varepsilon(I + \varepsilon\hat{b})^{-1} b$. This equation provides, in effect, a summation of (6.11). We recognize in it the Marčenko integral equation used by Zakharov and Shabat (1974) to solve (6.01) by a variant of the inverse scattering method.

We now look for equations satisfied by K . This time we do not expect the t-dependent equation to follow directly from (6.01). This is so because the proof of (6.02), in (6.09), uses the highly symmetrical dependence of u on x . Indeed (6.08) involves summation over $\mathscr{P}_n$. On the other hand, K , whose expression in terms of $d\lambda$ is

$$K = \frac{1}{2\nu i} \sum_{n=1}^\infty (-\varepsilon)^n \int_{\mathbb{C}^{2n}} \frac{e^{i\Omega_{n-1} + \frac{i}{2\nu}(\beta_n x + \alpha_n y) - i\omega t}}{\prod_1^{n-1} (\alpha_j + \beta_{j+1})} [d\lambda(k,\omega)]^n \ , \tag{6.22}$$

has this symmetry destroyed by the presence of the variable y . However, whatever the equation satisfied by K is, we expect its linear part to be determined by the first term in the expansion (6.22), i.e., $\mathbf{b}$. Writing ω in (6.10) in terms of α and β, $\omega = \frac{1}{4\nu}(\alpha^2 - \beta^2)$, we see that b satisfies

$$b_t + i\nu\,(b_{xx} - b_{yy}) = 0. \tag{6.23}$$

Thus we are led to study the effect of the operator $\partial_t + i\nu(\partial_x^2 - \partial_y^2)$ on $\mathcal{K}$. Now we have

$$-i(\omega_1 + \dots + \omega_n) - \frac{i}{4\nu}\left\{(\beta_1 + \dots + \beta_n + \alpha_1 + \dots + \alpha_{n-1})^2 - \alpha_n^2\right\} =$$

$$= -\frac{i}{4\nu}\left\{\alpha_1^2 - \beta_1^2 + \dots + \alpha_n^2 - \beta_n^2 + (\beta_1 + \dots + \beta_n + \alpha_1 + \dots + \alpha_{n-1})^2 - \alpha_n^2\right\} =$$

$$= -\frac{i}{4\nu}\left\{2\sum_{1 \le l < j \le n} \beta_l\beta_j + 2\sum_{1 \le l < j < n} \beta_l\alpha_j + 2\sum_{1 \le l < j \le n} \alpha_l\beta_j + 2\sum_{1 \le l \le j < n} \alpha_l\alpha_j\right\} =$$

$$= -\frac{i}{2\nu}\sum_{j=1}^{n-1}(\beta_1 + \dots + \beta_j + \alpha_1 + \dots + \alpha_j)(\alpha_j + \beta_{j+1}).$$

Therefore

$$\left\{\partial_t + i\nu(\partial_x^2 - \partial_y^2)\right\}\mathcal{K}(x,y,t) = (i\sigma/\nu)\,u(x,t)\,\mathcal{K}(x,y,t). \tag{6.24}$$

To find another equation we proceed from (6.21). First we write the dispersion relation in terms of $\alpha = \nu k + \omega k^{-1}$ and $\beta = \nu k - \omega k^{-1}$:

$$0 = \alpha^3 + \beta^3 - 3\gamma(\alpha + \beta). \tag{6.25}$$

It follows that b satisfies the equation

$$0 = \mathcal{L}b = \left\{4\nu^2(\partial_x^3 + \partial_y^3) + 3\gamma(\partial_x + \partial_y)\right\} b. \tag{6.26}$$

This equation and (6.20) imply that

$$(\mathcal{L}\hat{b}f)(x,y) = (\hat{b}\mathcal{L}f)(x,y) - 12\nu^2[f(x,x)]_x\, b_x(x,y,t) -$$

$$-6\nu^2\left\{[f(x,x)]_{xx} + f_{xx}(x,x) - f_{yy}(x,x)\right\} b(x,y,t)\,,\quad (\forall f = f(x,y)). \tag{6.27}$$

Thus, multiplying (6.21b) through by $(I+\varepsilon\hat{b})^{-1}\mathcal{L}$, using

$$\varepsilon(I+\varepsilon\hat{b})^{-1} b_x = -K_x(x,y,t) - K(x,x,t)\,K(x,y,t)\,, \tag{6.28}$$

and (6.21a), we obtain

$$0 = (\mathcal{L}K)(x,y,t) - 6\sigma u(x,t)K_x(x,y,t) - 6\sigma u(x,t)K(x,x,t)K(x,y,t) -$$

$$-3\sigma u_x(x,t)K(x,y,t) + 6\nu^2[K_{xx}(x,x,t) - K_{yy}(x,x,t)]K(x,y,t)\,.$$

Introduce

$$v = -2\nu^2\sigma^{-1}\partial_t K(x,x,t) = \frac{2i\nu^3}{\sigma}[K_{xx}(x,x,t) - K_{yy}(x,x,t)] - -2\nu i\,u(x,t)K(x,x,t). \tag{6.29}$$

Then

$$\left\{4\nu^2(\partial_x^3+\partial_y^3) + 3\gamma(\partial_x+\partial_y) - 6\sigma u\,\partial_x - 3\sigma u_x - 3i\frac{\sigma}{\nu}v\right\}K = 0. \tag{6.30}$$

This formula can also be proved directly from (6.22). Finally, an eigenvalue problem for (6.01) can be found separating the variable y in (6.24) and (6.30).

We note that introducing v into (6.01) we can write it as the system

$$v_t = \gamma u_x + \frac{\nu^2}{3}u_{xxx} - (\sigma u^2)_x \quad,\qquad u_t = v_x\,. \tag{6.31}$$

CHAPTER 7

AN EQUATION PRESENTING MULTISOLITON BEHAVIOR

7.1 Introduction

Recently Caudrey et al. (1976) introduced an equation presenting multisoliton solutions. The equation is a generalization to fifth order of the KdV equation, and can be taken in the form

$$u_t + (u_{xxxx} + 3uu_{xx} + u^3)_x = 0. \tag{7.01}$$

Following Hirota's approach they introduce

$$u = \partial_x^2 \ln f . \tag{7.02}$$

This transformation reduces (7.01) to a homogeneous equation of degree two in the variable f, which can be written in the following form

$$\left\{(\partial_x - \partial_{x'})\, G(-(\partial_t - \partial_{t'}), \partial_x - \partial_{x'})\, f(x,t) f(x',t')\right\}_{\substack{x=x'\\ t=t'}} = 0 , \tag{7.03}$$

where $0 = G(\Omega, P) = -\Omega + P^5$ is the linear dispersion relation of (7.01). Then they prove that a solution of (7.03) is given by

$$f = \sum_{\nu=0}^{n} \sum_{\binom{n}{\nu}} a(j_1, \ldots, j_\nu) \exp(\eta_{j_1} + \cdots + \eta_{j_\nu}) , \tag{7.04}$$

where $\sum_{\binom{n}{\nu}}$ indicates summation over all possible combinations of ν indices $j_1, \ldots, j_\nu$ out of $1, 2, 3, \ldots n$,

$$\eta_j = P_j x - \Omega_j t + \eta_{j0} \quad , \quad G(\Omega_j, P_j) = 0 \; , \; (\forall\, 1 \leq j \leq n) ,$$

$$a(\ell,j) = -\frac{(P_\ell-P_j)\,G(\Omega_\ell-\Omega_j, P_\ell-P_j)}{(P_\ell+P_j)\,G(\Omega_\ell+\Omega_j, P_\ell+P_j)} = \frac{(P_\ell-P_j)^2(P_\ell^2-P_\ell P_j+P_j^2)}{(P_\ell+P_j)^2(P_\ell^2+P_\ell P_j+P_j^2)}, \quad (\forall\ 1\leq j,\ell\leq n),$$

$$a(j_1,\ldots,j_\nu) = \prod_{1\leq s<\ell\leq\nu} a(j_s,j_\ell),$$

and the P_j 's, Ω_j 's and η_{jo} 's are constants. The solution of (7.01) provided by (7.02) and (7.04) presents all the characteristics of a multisoliton solution.

No eigenvalue problem that would render (7.01) solvable by an inverse scattering transform is known. However, the fact that the equation supports multisoliton solutions is an encouraging sign that there might be one. Motivated by this, we tried to apply to (7.01) our small parameter expansion technique, without success. In view of this we started a search for a lower order equation that would also support multisoliton solutions, of the same functional form (7.04) but with a different dispersion function Ω. Presumably it would be easier to find a perturbation expansion for a lower order equation. Then we would only have to take a different dispersion function in it to obtain a perturbation expansion valid for (7.01). Although our search for a lower order equation was successful, we have not been able to write a small parameter expansion for it. The equation is

$$r - r_{xt} = -6r\int_x^\infty r_t\,dy \qquad \text{or} \qquad v_x - v_{xxt} = 6v_x v_t, \tag{7.05}$$

where $r = v_x$.

7.2 Multisoliton Solution

Introduce

$$v = \partial_x \ln f = f_x f^{-1} \tag{7.06}$$

into (7.05). Then f satisfies the equation

$$\{(\partial_x-\partial_{x'})\, G(-(\partial_t-\partial_{t'}),\, \partial_x-\partial_{x'})\, f(x,t) f(x',t')\}_{\substack{x=x'\\ t=t'}} = 0\,, \tag{7.07}$$

where now $0 = G(\Omega,P) = P + P^2\Omega$ is the linear dispersion relation of (7.05). We assert that, with use of this latter form of G, (7.04) solves (7.07). Instead of $\Omega_j = P_j^5, (\forall\, 1 \leq j \leq n)$ we now have $\Omega_j = -P_j^{-1}, (\forall\, 1 \leq j \leq n)$, but the $a(\ell,j)$'s, which characterize the functional form of the solution, remain the same. To prove our assertion we substitute (7.04) into (7.07). Then the coefficient of

$$\exp(2\eta_{i_1}+\cdots+2\eta_{i_p}+\eta_{j_1}+\cdots+\eta_{j_q})\,, \quad \{i_p\}_1^p \cap \{j_q\}_1^q = \phi\,,$$

is

$$\sum_{m=0}^{q} \sum_{\binom{q}{m}} a(i_1,\ldots,i_p,\ell_1,\ldots,\ell_m)\, a(i_1,\ldots,i_p,\ell_{m+1},\ldots,\ell_q)\, H(\Omega^{\ell_m}, P^{\ell_m})\,,$$

where $\{\ell_s\}_1^q = \{j_s\}_1^q$, $H(\Omega,P) = PG(\Omega,P)$, $\sum_{\binom{q}{m}}$ means summation over all possible combinations $\ell_1,\ldots,\ell_m$ of m indices out of the q indices $j_1,\ldots,j_q$, $P^{\ell_m} = P_{\ell_1}+\cdots+P_{\ell_m} - P_{\ell_{m+1}} - \cdots - P_{\ell_q}$ and similarly for Ω^{ℓ_m}.
This coefficient must vanish. Taking out the common factor $a^2(i_1,\ldots,i_p)\prod_{r,s} a(i_r,\ell_s)$ we see that what we want to prove is

$$\sum_{m=0}^{q} \sum_{\binom{q}{m}} a(\ell_1,\ldots,\ell_m)\, a(\ell_{m+1},\ldots,\ell_q)\, H(\Omega^{\ell_m}, P^{\ell_m}) = 0\,.$$

Introduce $\alpha(P_\ell,P_s) = (P_\ell+P_s)^2(P_\ell^2+P_\ell P_s+P_s^2)$. Then this last formula is equivalent to

$$0 = \sum_{m=0}^{q} \sum_{\binom{q}{m}} \prod_{1\leq r<s<m} \alpha(P_{\ell_r},-P_{\ell_s}) \prod_{\substack{1\leq r\leq m\\ m<s\leq q}} \alpha(P_{\ell_r},P_{\ell_s})\, H(\Omega^{\ell_m}, P^{\ell_m}) = C_q(P_{j_1},\ldots,P_{j_q})\,.$$

Now for any $1 \leq s \leq q$, $P_{j_s} \longmapsto -P_{j_s}$ leaves C_q invariant. Thus C_q is a function of $P_{j_s}^2$, $(1 \leq s \leq q)$, only. Since the singularities of C_q come from the $\Omega_{j_s} = -P_{j_s}^{-1}$, which only occur linearly, they must cancel out. It follows that C_q is a polynomial of degree at most $2q^2-2q+3$. Furthermore, C_q must be of even degree. Thus degree $C_q \leq 2q^2-2q+2$.

An alternative way of writing C_q is

$$C_q(P_{j_1},\dots,P_{j_q}) = \sum_{\substack{\epsilon_\rho = \pm 1 \\ 1 \leq \ell \leq q}} \prod_{1 \leq T < s \leq q} \alpha(\epsilon_T P_{j_T}, -\epsilon_s P_{j_s})\, H\Big(\sum_1^q \epsilon_\rho \Omega_{j_\rho}, \sum_1^q \epsilon_\rho P_{j_\rho}\Big).$$

Thus we see that C_q is symmetric. Let us now denote

$$P_\epsilon = \sum_2^q \epsilon_\rho P_{j_\rho}, \quad \Omega_\epsilon = \sum_2^q \epsilon_\rho \Omega_\rho, \quad A_\epsilon(x) = \sum_{1 < s \leq q} \alpha(x, \epsilon_s P_{j_s}).$$

Then

$$C_q(P_{j_1},\dots,P_{j_q}) = \sum_{\substack{\epsilon_\rho = \pm 1 \\ 1 \leq \ell \leq q}} \prod_{1 < T < s \leq q} \alpha(\epsilon_T P_{j_T}, -\epsilon_s P_{j_s}) \Big\{ A_\epsilon(P_{j_1})\big[-2P_\epsilon^2 + P_\epsilon^3 \Omega_\epsilon -$$

$$- P_\epsilon^3 P_{j_1}^{-1} + O(P_{j_1})\big] + A_\epsilon(-P_{j_1})\big[-2P_\epsilon^2 + P_\epsilon^3 \Omega_\epsilon + P_\epsilon^3 P_{j_1} + O(P_{j_1})\big] \Big\}.$$

Thus, since $A_\epsilon'(0) = -3\Omega_\epsilon A_\epsilon(0)$, we have

$$C_q(0, P_{j_2},\dots,P_{j_q}) = -4 A_\epsilon(0)\, C_{q-1}(P_{j_2},\dots,P_{j_q}).$$

Moreover,

$$C_q(P_{j_1}, -P_{j_1}, P_{j_3},\dots,P_{j_q}) = 24 P_{j_1}^4 \Big\{ \prod_{3 \leq s \leq q} \alpha(P_{j_1}, P_{j_s})\, \alpha(P_{j_1}, -P_{j_s}) \Big\} C_{q-2}(P_{j_3},\dots,P_{j_q}).$$

Because of these last two formulas and the symmetry of C_q it follows that, if $C_{q-1} \equiv C_{q-2} \equiv 0$, $\Big\{ \prod_{s=1}^q P_{j_s}^2 \cdot \prod_{1 \leq T < s \leq q} (P_{j_s}^2 - P_{j_T}^2) \Big\}$ is a factor of C_q. The degree of this factor is $2q^2 > 2q^2 - 2q + 2$ for $q > 1$. Thus $C_q \equiv 0$. Since $C_0 \equiv C_1 \equiv 0$, the proof follows by induction.

7.3 Conservation Laws

We have found the following two conservation laws for (7.05):

$$(2r^3 - r_x^2)_t + \left(6r^2 \int_x^\infty r_t \, dy + r^2\right)_x = 0 , \tag{7.08a}$$

$$(3r^4 - 6rr_x^2 + r_{xx}^2)_t +$$

$$+\left(12r^3 \int_x^\infty r_t \, dy - 6r_x^2 \int_x^\infty r_t \, dy + 6rr_x r_t + 2r^3 - 2r_x^2\right)_x = 0. \tag{7.08b}$$

These conservation laws match the ones found by Caudrey et al. (1976) for (7.01) and give more reason to expect the existence of a whole class of equations, similar to those found in Sections 2.8 and 5.5, supporting multisoliton solutions of the form (7.04). The particular one corresponding to a dispersion function $\omega(k) = k^3$ could be of interest, since it would constitute a variant of the KdV equation.

APPENDIX

In this appendix we prove several formulas and results quoted in the main text of this thesis.

A.1 Some Results and Identities for Polynomials

Let $\mathcal{V}$ and $\mathcal{W}$ denote two finite dimensional vector spaces over a field .

Lemma. Let $\mathcal{P}=\mathcal{P}(\tilde{x}_1,\ldots,\tilde{x}_n):\mathcal{V}^n \longrightarrow \mathcal{W}$ be a polynomial function such that degree $\mathcal{P} < n$. Then

$$0 = \sum_{\sigma\in T_n} (sg\sigma)\,\mathcal{P}(\sigma_1\tilde{x}_1,\ldots,\sigma_n\tilde{x}_n) = (-1)^n\,\mathcal{P}(\tilde{x}_1,\ldots,\tilde{x}_n) +$$

$$+ (-1)^{n-1}[\mathcal{P}(\tilde{0},\tilde{x}_2,\ldots,\tilde{x}_n)+\ldots+\mathcal{P}(\tilde{x}_1,\ldots,\tilde{x}_{n-1},0)]+\ldots+\mathcal{P}(\tilde{0},\ldots,\tilde{0}) , \qquad (A.101)$$

where the summation is extended over the set T_n of all functions

$$\sigma : \{1,\ldots,n\} \longmapsto \{0,1\} ,$$

and

$$sg\sigma = (-1)^{\sum_j \sigma_j} .$$

Proof: By linearity it is enough to consider the case $\mathcal{W}=F$ and $\mathcal{P}$ a monic monomial in some base $\{\tilde{e}_j\}_1^m$ of $\mathcal{V}$. Then if $\tilde{x}_\ell = \sum_j^m x_{\ell j}\tilde{e}_j$ $(1\leq \ell\leq n)$, we have

$$\mathcal{P}(\tilde{x}_1,\ldots,\tilde{x}_n) = \prod_{\kappa_{\ell j}\neq 0} x_{\ell j}^{\kappa_{\ell j}}$$

where $\kappa=(\kappa_{\ell j})$ is an $n\times m$ matrix of natural numbers with $\sum_{\ell,j}\kappa_{\ell j} < n$. Then

$$\sum_{T_n}(sg\sigma)\,\mathcal{P}(\sigma_1\tilde{x}_1,\ldots,\sigma_n\tilde{x}_n) = \sum_{T_n}(sg\sigma)\prod_{\kappa_{\ell j}\neq 0}(\sigma_\ell x_{\ell j}^{\kappa_{\ell j}}) =$$

$$= \mathcal{P}(\tilde{x}_1,\ldots,\tilde{x}_n).\Big(\sum_{T_n}(sg\sigma)\prod_{\kappa_{\ell j}\neq 0}\sigma_\ell\Big) = 0.$$

To see this let $1 \leq \ell_1 < \dots < \ell_s \leq n$ be such that $x_{\ell j} \neq 0$ for some $1 \leq j \leq m$ if and only if $\ell = \ell_r$ for some $1 \leq r \leq s$, and let T_n' be the subset of T_n of all σ's such that $\sigma_{\ell_r} = 1$ $(\forall 1 \leq r \leq s)$. Then

$$\sum_{T_n} (\operatorname{sg}\sigma) \prod_{x_{\ell j} \neq 0} \sigma_\ell = \sum_{T_n'} (\operatorname{sg}\sigma) = (-1)^s \sum_{r=1}^{n-s} (-1)^r \binom{n-s}{r} = 0 \,,$$

since $s < n$.

<u>Lemma</u>. Let $\{P_n : V^n \longrightarrow W\}_{n=1}^{\infty}$ and $\{Q_n : V^n \longrightarrow W\}_{n=1}^{\infty}$ be two sequences of polynomial functions such that, for some $1 \leq r < \infty$,

(i) degree $P_n \leq r$ and degree $Q_n \leq r$ $(\forall 1 \leq n < \infty)$,

(ii) $P_n(\tilde{x}_1, \dots, \tilde{x}_n) = P_{n-1}(\tilde{x}_1, \dots, \tilde{x}_{j-1}, \tilde{x}_{j+1}, \dots, \tilde{x}_n)$,

$Q_n(\tilde{x}_1, \dots, \tilde{x}_n) = Q_{n-1}(\tilde{x}_1, \dots, \tilde{x}_{j-1}, \tilde{x}_{j+1}, \dots, \tilde{x}_n)$,

$(\forall 1 < n < \infty, 1 \leq j \leq n)$, whenever $\tilde{x}_j = \tilde{0}$,

(iii) $P_r = Q_r$

Then

$$P_n = Q_n \,, \quad (\forall 1 \leq n < \infty) . \tag{A.102}$$

Proof: (ii) and (iii) imply $P_n = Q_n$ for $1 \leq n \leq r$. From (ii) and the preceding lemma we see that P_n can be written in terms of $P_1, \dots, P_{n-1}$ and Q_n in terms of $Q_1, \dots, Q_{n-1}$ $(\forall r < n < \infty)$. Thus (A.102) follows by induction.

A.2 Expansions of Certain Determinants

<u>Lemma</u>. Let $A = (a_{\ell j})$ be an $n \times n$ matrix over a field F, and let $\{\lambda_j\}_1^n$ be indeterminates. Then

$$\det(\delta_{\ell j} + a_{\ell j}\lambda_j) = 1 + \sum_{m=1}^{n} \sum_{\binom{n}{m}} (\det A_{\ell_1 \dots \ell_m}) . \prod_1^m \lambda_{\ell_j} \,, \tag{A.201}$$

where

(i) $\sum_{\binom{n}{m}}$ indicates summation over all possible combinations of m indices $1 \leq \ell_1 < \dots < \ell_m \leq n$ out of $\{1, 2, \dots, n\}$, and

(ii) $A_{\ell_1 \dots \ell_m}$ denotes the $m \times m$ principal submatrix of A formed by the intersection of the columns and rows $\ell_1, \dots, \ell_m$.

Lemma. Let $\tilde{z}, \tilde{w} \in F^n$ be such that $z_\ell + w_j \neq 0 \ (\forall 1 \leq \ell \neq j \leq n)$, where F is as before. Then if we define $(\forall 1 \leq \ell \neq j \leq n)$,

$$a_{\ell j} = \frac{(z_\ell - z_j)(w_\ell - w_j)}{(z_\ell + w_j)(z_j + w_\ell)} = a_{j\ell} \quad \text{and} \quad s_{\ell j} = \frac{z_\ell + w_\ell}{z_\ell + w_j}, \ s_{\ell\ell} = 1,$$

we have

$$\det S = \prod_{1 \leq \ell < j \leq n} a_{\ell j} = a_{1 \dots n}. \tag{A.202}$$

Proof: Whenever $z_\ell = z_j$ or $w_\ell = w_j$ $(\forall 1 \leq \ell < j \leq n)$, the determinant vanishes, since then the rows ℓ and j, or the columns ℓ and j respectively, are linearly dependent. Thus for some $\alpha \in F$

$$\prod_{1 \leq \ell \neq j \leq n} (z_\ell + w_j) \det S = \det\left\{ \prod_{\substack{r=1 \\ r \neq j}}^{n} (z_\ell + w_r) \right\} = \alpha \prod_{1 \leq \ell < j \leq n} (z_\ell - z_j)(w_\ell - w_j)$$

since we are dealing with polynomials of the same degree $n(n-1)$ with the same zeros. Thus $\det S = \alpha\, a_{1 \dots n}$. Evaluating at $z_j + w_j = 0 \ (\forall 1 \leq j \leq n)$, we see that $\alpha = 1$, since then $S = I$ and $a_{\ell j} = 1 \ (\forall 1 \leq \ell \neq j \leq n)$.

Corollary. Let S be as in the preceding lemma and $\{\lambda_j\}_1^n$ as in (A.201). Then

$$\det(\delta_{\ell j} + s_{\ell j} \lambda_j) = 1 + \sum_{m=1}^{n} \sum_{\binom{n}{m}} a_{\ell_1 \dots \ell_m} \prod_{1}^{m} \lambda_{\ell_j}. \tag{A.203}$$

A.3 Conjugations and Positivity of Spectrum

Let H be a separable Hilbert space over the complex field $\mathbb{C}$. A conjugation over H is an antilinear, idempotent operator J such that $\langle Ju,v\rangle = \langle Jv,u\rangle$ $(\forall u,v\in H)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product on H. For any bounded linear operator A on H we define its conjugate A^J and transpose A^T operators by

$$A^J = JAJ \qquad \text{and} \qquad A^T = JA^*J = (A^*)^J = (A^J)^*,$$

where $*$ denotes the hermitian adjoint operation.

If H is a space of square integrable functions with the usual inner product, then the pointwise conjugation is a conjugation over H in the sense just defined, and we denote it with a bar, i.e., $\bar{u}$ for Ju and $\bar{A}$ for A^J. If $H=\mathbb{C}^n$ with the standard scalar product, and we consider the componentwise conjugation on H, then the transpose takes its usual meaning.

Lemma. Let B be a self-adjoint, nonnegative, bounded linear operator on H. Then the spectrum of $BB^J = BB^T$ is contained in the nonnegative real numbers. Moreover, if B is invertible, then $BB^J = BB^T$ is similar to a self-adjoint, positive definite operator.

Proof: Let $B=K^2$ with $K=K^*\geq 0$ bounded. Then $B^J=(K^J)^2$ with $(K^J)=(K^J)^*\geq 0$ also bounded, and we have

$$(I-\alpha BB^J)^{-1} = [I-\alpha K^2(K^J)^2]^{-1} = I+\alpha K[I-\alpha K(K^J)^2K]^{-1}K(K^J)^2,$$

and

$$[I-\alpha K(K^J)^2K]^{-1} = I+\alpha K(K^J)^2(I-\alpha BB^J)^{-1}K.$$

It follows that the spectrum of BB^J is the same, with the possible exception of the origin, as that of $K(K^J)^2K = (KK^J)(KK^J)^* \geqslant 0$. In fact $(BB^J)^{-1}$ exists if and only if $[(KK^J)(KK^J)^*]^{-1}$ exists, so that the origin is in either both or none of the spectra.

For the second statement we observe that if B^{-1} exists, so does K^{-1} and we have $BB^J = K(KK^J)(KK^J)^*K^{-1}$ with $(KK^J)(KK^J)^* > 0$.

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II. THE SIMILARITY SOLUTION FOR THE KORTEWEG-DE VRIES EQUATION AND THE RELATED PAINLEVÉ TRANSCENDENT

1. Introduction

The Korteweg-de Vries (KdV) equation describes the development and propagation of moderately small amplitude shallow water waves (Korteweg and de Vries 1895), and many other important phenomena where a small nonlinearity is combined with a cubic dispersion relation. In various contexts its similarity solutions become important. The equation can be normalized to

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1}$$

and the similarity solutions can be taken in the form

$$u(x,t) = (3t)^{-\frac{2}{3}} f(\eta), \qquad \eta = x/(3t)^{\frac{1}{3}}. \tag{1.2}$$

Substitution of (1.2) into (1.1) gives the following ordinary differential equation for f:

$$f''' + 6ff' - 2f - \eta f' = 0. \tag{1.3}$$

The solutions of primary interest decay exponentially as $\eta \to \infty$. In this limit, they approach solutions of the linearized equation

$$f''' - 2f - \eta f' = 0, \tag{1.4}$$

and the derivative of the Airy function, $Ai'(\eta)$, is the relevant solution of this linearized version. Therefore, we take the boundary condition

$$f \sim aAi'(\eta), \qquad \eta \to \infty, \tag{1.5}$$

where a is an amplitude parameter.

Preliminary numerical computations by Berezin and Karpman (1964) show that when a is small enough f becomes oscillatory as $\eta \to -\infty$, but otherwise f may develop singularities. We will show that there is a critical value a_1 of a which separates the oscillatory from the singular solutions. For

$|a| = a_1$, $f(\eta) \sim \frac{\eta}{2}$ as $\eta \to -\infty$. For $|a| < a_1$, $f(\eta)$ becomes oscillatory as $\eta \to -\infty$. For $a > a_1$, $f(\eta)$ develops a singularity at a finite η. We have not made the analysis for $a < -a_1$, since the original interest (discussed next) was in solutions with $f \to 0^-$ as $\eta \to \infty$. Numerically we compute $a_1 = 1 + O(10^{-13})$.

Ablowitz and Newell (1973) studied the solution of (1.1) when the initial data decay sufficiently rapidly as $|x| \to \infty$ and no solitons are generated (Scott, Chu and McLaughlin 1973). The similarity solution was proposed for the long time structure in the region $x/t^{\frac{1}{3}} = O(1)$. In this context the matching condition with the region $(x/t^{\frac{1}{3}}) \gg 1$ gives $a = -\beta_0(k)$, where $\beta_0 = \beta_0(k)$ is the reflection coefficient of the scattering problem associated with (1.1) (Gardner, Greene, Kruskal and Miura 1967). But for most reasonable initial conditions $\beta_0(0) = -1$, and this corresponds to the critical value $a = 1$. Therefore the matching with the oscillatory region $(x/t^{\frac{1}{3}}) \ll -1$ of (1.1) is not possible. A revised discussion of this question has been presented recently by Ablowitz and Segur (1977).

2. Second Painlevé Transcendent

In order to study (1.3) it is convenient to make the following transformation due to G. B. Whitham:

$$f = g' - g^2 \quad . \tag{2.1}$$

This transformation was suggested by the relationship between the KdV equation and the Modified (MKdV) equation (Miura 1968). Then g satisfies the equation

$$(g'' - \eta g - 2g^3)'' - 2g(g'' - \eta g - 2g^3)' = 0 \quad . \tag{2.2}$$

This equation can be integrated once to

$$(g'' - \eta g - 2g^3)' = \alpha \exp\{-2\int_\eta^\infty g(\xi)d\xi\}$$

where α is an arbitrary constant. Since we are interested only in solutions for which g decays exponentially as $\eta \to \infty$, α must be 0. Another integration then gives

$$g'' - \eta g - 2g^3 = 0. \tag{2.3}$$

This equation is the ordinary differential equation corresponding to the similarity solution of the MKdV equation

$$v_t - 2v^2 v_x + v_{xxx} = 0. \tag{2.4}$$

The MKdV equation with positive nonlinear term gives (2.3) with the opposite sign for $2g^3$.

From (2.3) we see that $g = g(\eta)$ is a Second Painlevé Transcendent (Ince 1956). This form simplifies the discussion of the solution, and the relation to (1.1) and (2.4) stimulates renewed interest in the Painlevé equation. In particular g can only have first order poles as singularities; the question of the various singularities is a basic feature of Painlevé's classification. Near such a singularity, at $\eta = \eta_0$ say, $g(\eta)$ has one or other of the expansions

$$g(\eta) = \pm\{\frac{1}{\eta - \eta_0} - \frac{\eta_0}{6}(\eta - \eta_0) - \frac{1}{4}(\eta - \eta_0)^2 + \ldots\}. \tag{2.5}$$

It is interesting to observe that when the minus sign is chosen, (2.1) will lead to a function f regular at η_0. The plus sign will produce a double pole in f at η_0. This last case is the one that developes for $a > a_1$, as we will see in what follows.

In terms of g, (1.5) can be written

$$g(\eta, a) \sim a\mathrm{Ai}(\eta), \quad \text{as} \quad \eta \to \infty. \tag{2.6}$$

We consider now the structure of this class of solutions, as a varies from 0 to ∞. As far as the equation for g is concerned it is not necessary to consider $a < 0$, since $g(\eta_1, -a) = -g(\eta_1, a)$.

First the equation (2.3) is written

$$g'' = (\eta + 2g^2)g. \tag{2.7}$$

We see then that we can divide the (η, g) plane into four regions; (I) $g > 0$, $\eta + 2g^2 > 0$, (II) $g > 0$, $\eta + 2g^2 < 0$, (III) $g < 0$, $\eta + 2g^2 < 0$, and (IV) $g < 0$, $\eta + 2g^2 > 0$. Any solution $g = g(\eta)$ of (2.7) will be strictly concave in regions (I) and (III), and strictly convex in the other two regions. Going back to (2.6) we see that as η decreases from ∞, $g(\eta, a) > 0$ increases, while $g'(\eta, a) < 0$ decreases and we have the following cases (see Fig. 1).

(i) If a is large enough $g = g(\eta, a)$ will completely avoid the parabola $\eta + 2g^2 = 0$, remaining always in region (I). In fact $g = g(\eta, a)$ will develop a singularity at a finite $\eta = s(a)$: a simple pole with residue equal to one. Solutions in this range are "nested", i.e. $g(\eta, a') > g(\eta, a'')$ and $g'(\eta, a') < g'(\eta, a'')$ if $a' > a''$. Moreover as $a \to \infty$, $s(a) \to \infty$, strictly monotonically.

(ii) Call the infimum of the a's for which (i) is true, a_2. Then everything said in (i) is valid for $g = g(\eta, a_2)$; except for the existence of a point of tangency with the parabola $\eta + 2g^2 = 0$, at $\eta = \eta_T$ say. As $a \to a_2$, $s(a)$ decreases monotonically to the finite limit $s(a_2)$.

(iii) Now let a be such that $0 < a < a_2$. In this case as η moves from ∞ to the left, there is going to be a point

$\eta = \eta_c(a)$ at which $g = g(\eta, a)$ crosses the parabola $\eta + 2g^2 = 0$, going from region I into region II. Then the solution becomes convex. However, if a is close enough to a_2, the crossing of the parabola will be almost tangential and the solution will not separate much from the parabola as η continues to decrease. Thus the factor $\eta + 2g^2$ will be small, and so will the curvature of $g = g(\eta, a)$, in fact it will be less than the curvature of $\eta + 2g^2 = 0$. A second crossing at some $\eta = \eta_e(a)$ then occurs, back to region I. From then on $g = g(\eta, a)$ remains in region I and, as in cases (i) and (ii), develops a singularity. If a_1 is the infimum of the a's for which all of this happens, then for $a_1 < a < a_2$, we have $\eta_e(a) < \eta_T < \eta_c(a)$; η_e, s and η_c are monotonic in a; when $a \to a_2^-$, $\eta_e \to \eta_T^-$, $\eta_c \to \eta_T^+$ and $s \to s(a_2)^-$; when $a \to a_1^+$, $\eta_e \to -\infty$ and $s \to -\infty$ strictly monotonically.

(iv) For $a = a_1$, $g = g(\eta, a)$ crosses the parabola at only one point, $\eta = \eta_c(a_1)$, it remains in region II for all $\eta < \eta_c(a_1)$, and asymptotes to $\eta + 2g^2 = 0$ from below as $\eta \to -\infty$. Solutions in the range $a \geqslant a_1$ are nested. This extends the result in (i).

(v) For $0 < a < a_1$, the convexity is large enough to make $g'(\eta, a) = 0$ at a point $\eta = \eta_1(a) < \eta_c(a)$. Then $g = g(\eta, a)$ will turn down, cross the line $g = 0$, enter region III and have a minimum. Then it turns back, crosses $g = 0$ again, has a maximum and so on. In other words $g = g(\eta, a)$ becomes oscillatory. As $\eta \to -\infty$ the amplitude and wavelength of the oscillations decrease, due to the fact that $\eta + 2g^2 \to -\infty$. Solutions in this range look very much like Airy functions, shifted to the left, the shift being larger the closer a is to a_1. As $a \to a_1^-$, all the zeros, maxima and minima of $g = g(\eta, a)$ move towards $-\infty$. Also, the size of the oscillations increases and, in particular, the first "hump" sticks very close to the parabola $\eta + 2g^2 = 0$ for an ever increasing range of η's. As $a \to 0$, $g(\eta, a)$ approaches $a\mathrm{Ai}(\eta)$.

All these statements can be proved quite rigorously, one of the key elements in the proof being the analyticity of $g(\eta, a)$ in both its arguments in the range $-\infty < a < \infty$, $\eta > s(a)$. (We define $s(a) = -\infty$ for $-a_1 \leqslant a \leqslant a_1$).

3. Asymptotic Expansions

The complete asymptotic expansion as $\eta \to \infty$ is

$$g(\eta, a) \sim \sum_{n=0}^{\infty} a^{2n+1}\psi_n(\eta), \tag{3.1}$$

where $\psi_0(\eta) = \mathrm{Ai}(\eta)$ and $\psi_n(\eta)$ are the unique solutions of

$$\psi_n'' - \eta\psi_n = 2 \sum_{i+j+k=n-1} \psi_i\psi_j\psi_k \ , \quad (n \geqslant 1),$$

$$\psi_n(\eta) = o(\mathrm{Ai}(\eta)) \quad \text{as} \quad \eta \to \infty. \tag{3.2}$$

The ψ_n's can be written explicitly in terms of multiple integrals. They have asymptotic, non-convergent expansions of the form,

$$\psi_n(\eta) \sim \{\frac{1}{2\sqrt{\pi}} \exp\left(-\tfrac{2}{3}\eta^{3/2}\right)\}^{2n+1}\eta^{-\frac{1}{4}} \sum_{j=n}^{\infty} \alpha_{jn}\left(\tfrac{2}{3}\eta^{3/2}\right)^{-j}, \tag{3.3}$$

$$\text{as} \quad \eta \to \infty, (n = 0, 1, 2 \ldots).$$

The α_{jn}'s, $j \geqslant n \geqslant 0$ are constants. A few of them are,

$$\alpha_{j0} = (-1)^j\Gamma(3j + \tfrac{1}{2})/54^j\Gamma(j + \tfrac{1}{2})\Gamma(j + 1) =$$

$$(-1)^j(2j + 1)(2j + 3) \ldots (6j - 1)/216^j . j!, (j \geqslant 0), \tag{3.4}$$

$$\alpha_{nn} = (1/6)^n, (n \geqslant 0), \tag{3.5}$$

$$\alpha_{n+1\,n} = -(1/6)^n\left(\frac{35}{36}n - \frac{1}{72}\right) - \frac{1}{12}\delta_{n,0}, \quad (n \geqslant 0). \tag{3.6}$$

The asymptotic series (3.3) are all alternating. From (3.3) and (3.5) we see that,

$$\psi_n(\eta) \sim \eta^{-\frac{1}{4}}\left\{\frac{1}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}\eta^{3/2}\right)\right\}^{2n+1}\{4\eta^{3/2}\}^{-n}, (n \geqslant 0). \tag{3.7}$$

Thus the series (3.1) is not only asymptotic, but convergent, provided that a is small enough or η large enough. More precisely,

$$\frac{a^2}{16\pi\eta^{3/2}} \exp\left(-\frac{4}{3}\eta^{3/2}\right) << 1.$$

Moreover, we can write

$$g(\eta, a) = aAi(\eta)\{1 + \epsilon(\eta, a)\},$$

$$\epsilon(\eta, a) \sim \frac{a^2}{16\pi\eta^{3/2}} \exp\left(-\frac{4}{3}\eta^{3/2}\right), \quad \text{as} \quad \eta \to \infty. \tag{3.8}$$

We compute several values of ϵ, to get an idea of their sizes:

$$\epsilon(4, a) \cong 5.8 \times 10^{-8}a^2, \quad \epsilon(5, a) \cong 6.0 \times 10^{-10}a^2,$$

$$\epsilon(6, a) \cong 4.2 \times 10^{-12}a^2, \quad \epsilon(10, a) \cong 3.1 \times 10^{-22}a^2. \tag{3.9}$$

It is seen that $g(\eta, a) = aAi(\eta)$ is a very good approximation, even for moderately sized η. Similarly, $g'(\eta, a) = aAi'(\eta)$ is also a good approximation.

In the case $|a| < a_1$, the asymptotic expansion for $\eta \to -\infty$ is

$$g(\eta, a) \sim (-\eta)^{-\frac{1}{4}}\left\{\sum_{n=0}^{\infty} \phi_n(r) \exp i(2n+1)\theta\right\} + (c.c.), \quad \eta \to -\infty, \quad |a| < a_1, \tag{3.10}$$

where

$$\theta = \frac{2}{3}(-\eta)^{3/2} - 2d^2 \ln\{\frac{2}{3}(-\eta)^{3/2}\}, \qquad r = \frac{3}{2}(-\eta)^{-3/2}$$

and d is a constant. The ϕ_n's have expansions of the form,

$$\phi_n \sim \sum_{j=n}^{\infty} \beta_{jn} r^j, \quad \text{as} \quad r \to 0 \ , \quad (n = 0, 1, 2, \ldots) \tag{3.11}$$

and are the solutions of certain singular equations, with appropriate boundary conditions. The only free parameter in (3.10) - (3.11) is β_{00}, and d is related to it by

$$d = |\beta_{00}|. \tag{3.12}$$

We do not know the connection formula $\beta_{00} = \beta_{00}(a)$, $|a| < a_1$. In a recent paper, Ablowitz and Segur (1977) propose that

$$d^2 = -\frac{1}{4\pi}\ln(1 - a^2).$$

To first order (3.10) gives,

$$g(\eta, a) \sim 2d(-\eta)^{-\frac{1}{4}} \cos\{\frac{2}{3}(-\eta)^{3/2} - 2d^2\ln\{\frac{2}{3}(-\eta)^{3/2}\} + \theta_0\}, \tag{3.13}$$

where $\theta_0 = \arg \beta_{00}$.

For the critical solution $a = a_1$ which asymptotes to the parabola, it is readily checked that the asymptotic expansion is

$$g(\eta, a_1) \sim (-\frac{\eta}{2})^{\frac{1}{2}} \sum_0^{\infty} r_n (2\eta)^{-3n}, \quad \text{as} \quad \eta \to \infty, \tag{3.14}$$

where $r_0 = 1$, $r_1 = 1$, $r_2 = -\frac{73}{2}$, $r_3 = \frac{10657}{2}$, etc.

4. Numerical Computations

In this section we describe the numerical computations for the values of a_1 and a_2. Equation (2.7) was integrated with the initial conditions $g(10) = aAi(10)$, $g'(10) = aAi'(10)$, for various values of a. According to (3.8) and (3.9) these initial conditions are accurate up to 20 or more significant digits for $a = O(1)$. We computed Ai and Ai' using the expansions (Abramowitz and Stegun 1965)

$$Ai(\eta) \sim \frac{1}{2\sqrt{\pi}} \eta^{-\frac{1}{4}} \exp\left(-\frac{2}{3}\eta^{3/2}\right) \sum_{0}^{\infty} (-1)^k c_k \left(\frac{2}{3}\eta^{3/2}\right)^{-k}, \quad \eta \to \infty; \tag{4.1}$$

$$Ai'(\eta) \sim -\frac{1}{2\sqrt{\pi}} \eta^{\frac{1}{4}} \exp\left(-\frac{2}{3}\eta^{3/2}\right) \sum_{0}^{\infty} (-1)^k d_k \left(\frac{2}{3}\eta^{3/2}\right)^{-k}, \quad \eta \to \infty; \tag{4.2}$$

up to and including the fifteenth term. Here we have

$$c_0 = 1, \quad c_k = \Gamma(3k + \tfrac{1}{2})/54^k k!\,\Gamma(k + \tfrac{1}{2}), \quad (k = 1, 2, 3, \ldots)$$

and

$$d_k = -\frac{6k + 1}{6k - 1} c_k, \quad (k = 0, 1, 2, \ldots). \tag{4.3}$$

The error committed is of the order of the first deleted term. Since $c_{16} \cong 3.15 \times 10^6$ and $d_{16} \cong -3.21 \times 10^6$, and

$$c_{16}\left(\frac{2}{3}10^{3/2}\right)^{-16} \cong 2.06 \times 10^{-15}, \quad -d_{16}\left(\frac{2}{3}10^{3/2}\right)^{-16} \cong 2.11 \times 10^{-15}, \tag{4.4}$$

we see that we had at least fourteen significant digits in our initial conditions.

To integrate (2.7) we used a fourth order Runge-Kutta scheme (Abramowitz and Stegun 1965) on an IBM 370/155 computer, with double-precision. The step size was set at $h = 0.001$, $h = 0.002$ and $h = 0.004$. The value $h = 0.001$ is about the optimum for a truncation error of $O(10^{-15})$.

A check on the integration procedure was made at $\eta = 6$.

There we compared the values g and g' resulting from the numerical integration with $aAi(6)$ and $aAi'(6)$, which by (3.8), (3.9) coincide with the true solution in up to eleven digits. These comparison values of $aAi(6)$ and $aAi'(6)$ were computed using (4.1) and (4.2) up to and including the fifteenth term. Since

$$c_{16}(\tfrac{2}{3}6^{3/2})^{-16} \cong 4.4 \times 10^{-10}, \qquad -d_{16}(\tfrac{2}{3}6^{3/2})^{-16} \cong 4.4 \times 10^{-10}, \tag{4.5}$$

nine significant digits were obtained. The relative discrepancies $\Delta_1 = |g(6) - aAi(6)|/|g(6)|$, $\Delta_2 = |g'(6) - aAi'(6)|/|g'(6)|$ turned out to be $\Delta_1 \cong 2 \times 10^{-10}$, $\Delta_2 \cong 3 \times 10^{-10}$, which fit with (4.5) perfectly. Changes in the step size did not affect this last result.

For values of $a \leq 1 - 10^{-9}$ the solutions became oscillatory as $\eta \to -\infty$, as shown in figures 1, 3 and 4. For $a \geq 1 + 10^{-9}$ the solutions had unbounded growth as $\eta \to -\infty$ as in Fig. 1. This was independent of the particular value chosen for h. The solution for $a = 1$ was consistently oscillatory for the Runge-Kutta scheme, but when other integration schemes, of the predictor corrector type, were used its behaviour was erratic. This was probably due to the nature of the truncation error for the Runge-Kutta scheme, which in the particular equation we were solving was of constant sign and tended to make the computed solution consistently smaller. For $a \geq 1.02$ we found that the solution completely avoided the parabola $\eta + 2g^2 = 0$, and for $a \leq 1.0175$ it did not. From these results we conclude that

$$a_1 = 1 + O(10^{-9}), \quad 1.0175 < a_2 < 1.02. \tag{4.6}$$

At the same time as these calculations were being made, J.M. Greene was also computing the critical curve, and he also predicts $a_1 = 1$.

Another check of the result $a_1 = 1$ was made using the boundary value problem solver PASVAR (Lentini and Pereyra 1977). We solved equation (2.7) in the interval $T \leqslant \eta \leqslant 10$ with the boundary conditions

$$g(10)Ai'(10) = g'(10)Ai(10) = 0, \quad g(T) \text{ given},$$

where the value of g at $\eta = T$ was computed using (3.14). T was taken to be -5.5, -6.0 and -6.5. The solution was computed to a relative error of $O(10^{-10})$ for $T = -5.5$, $O(10^{-12})$ for $T = -6.0$ and $O(10^{-13})$ for $T = -6.5$. To within these errors then the value of a_1 was

$$a_1 = g(10)/Ai(10) = g'(10)/Ai'(10).$$

In all cases we obtained $a_1 = 1$, to all significant digits.

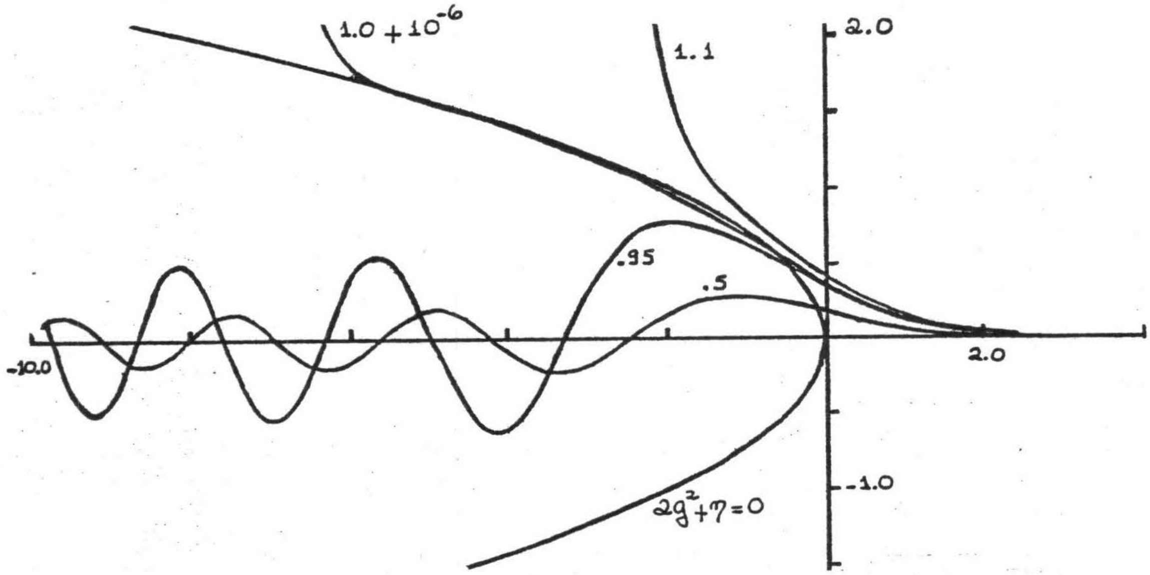

Fig. 1. Solutions of (2.6)-(2.7) for a = 1.1, a = $1.0 + 10^{-6}$, a = .95 and a = .5 .

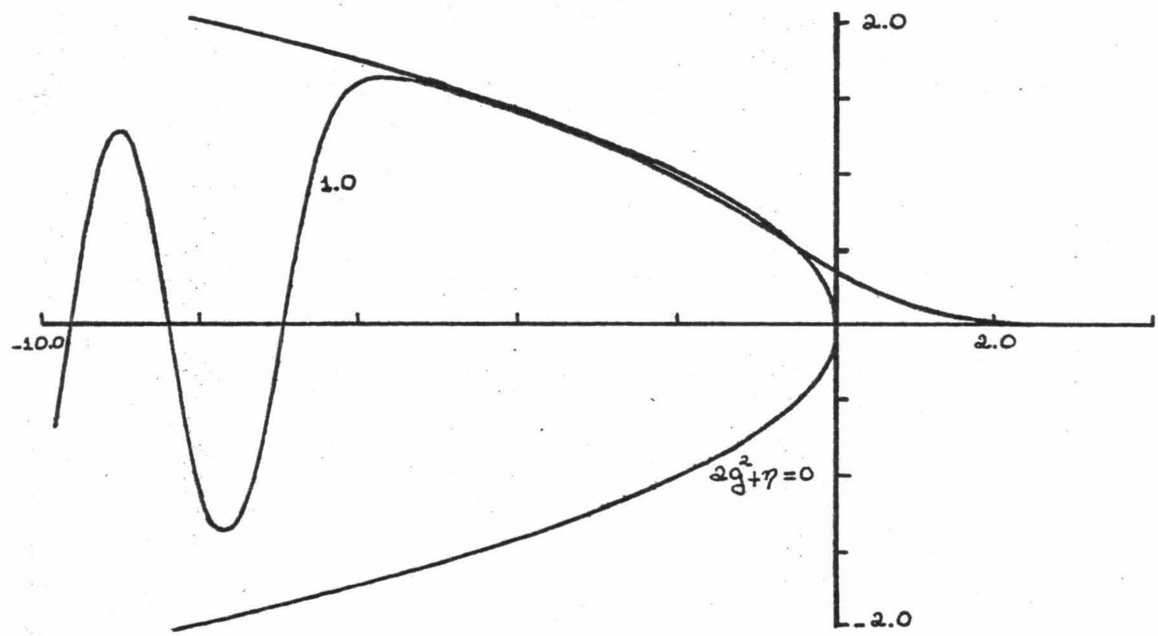

Fig. 2. Solution of (2.6)-(2.7) for a = 1.0. Other numerical schemes or different step sizes gave nonoscillatory solutions.

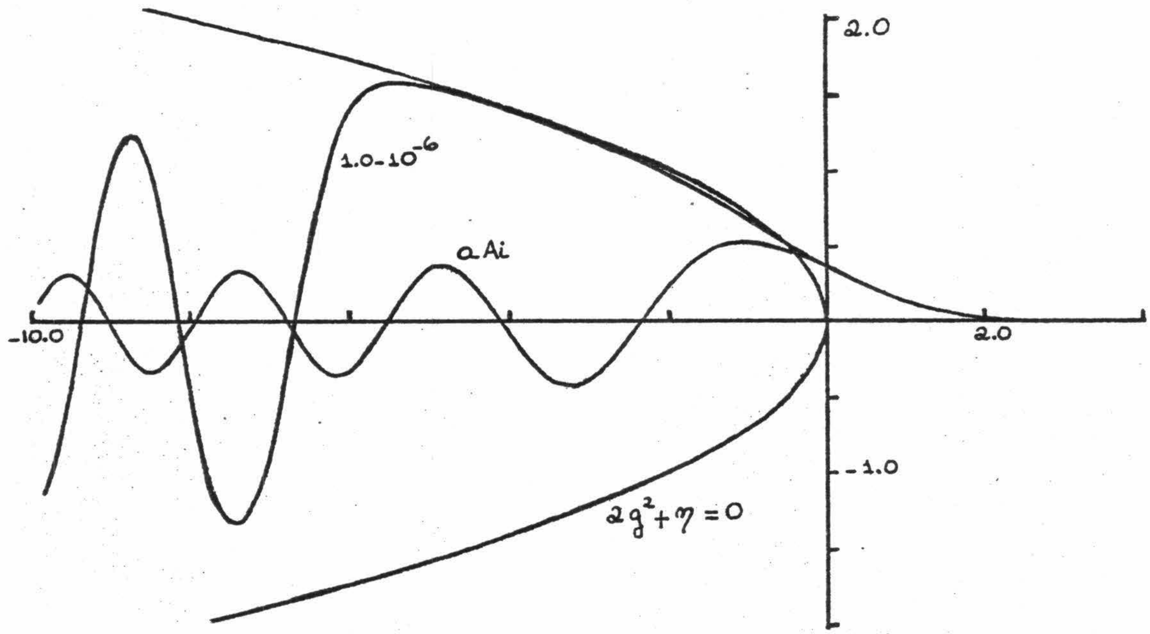

Fig. 3. Solution of (2.6)-(2.7) for $a = 1.0 - 10^{-6}$ and comparison with aAi

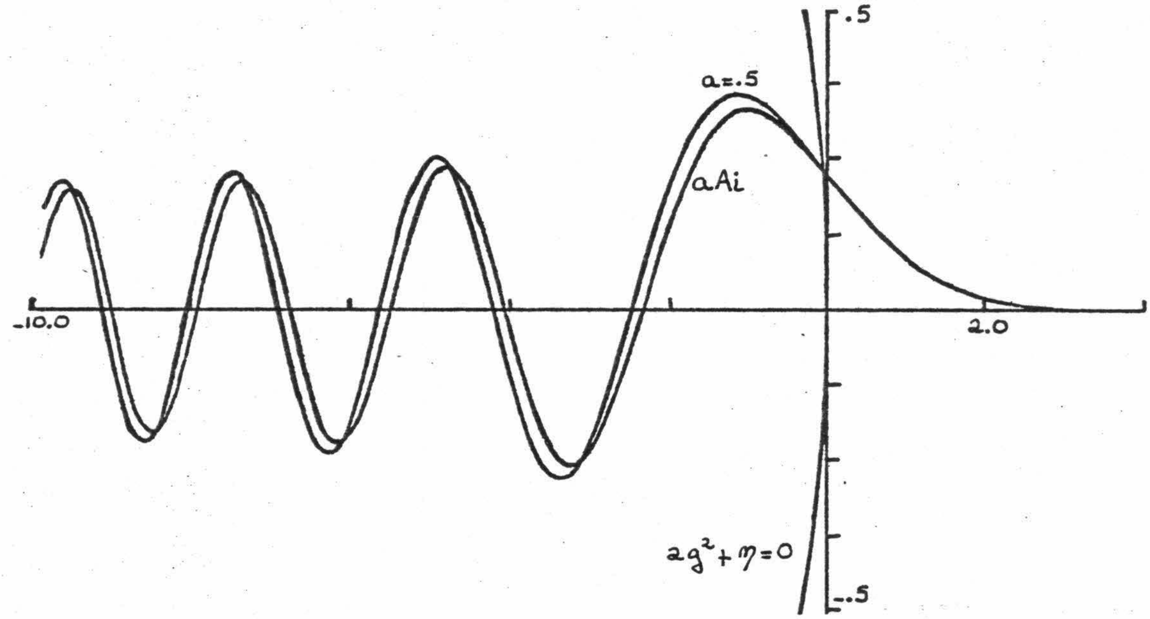

Fig. 4. Comparison of the solution of (2.6)-(2.7) for a = .5 with aAi.

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