TOPICS IN BIFURCATION THEORY

Thesis by
Dwight William Decker

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ABSTRACT

I. Existence and Structure of Bifurcation Branches

The problem of bifurcation is formulated as an operator equation in a Banach space, depending on relevant control parameters, say of the form $G(u,\lambda) = 0$. If $\dim N(G_u(u_o,\lambda_o)) = m$ the method of Lyapunov-Schmidt reduces the problem to the solution of m algebraic equations. The possible structure of these equations and the various types of solution behaviour are discussed. The equations are normally derived under the assumption that $G_\lambda^o \varepsilon R(G_u^o)$. It is shown, however, that if $G_\lambda^o \notin R(G_u^o)$ then bifurcation still may occur and the local structure of such branches is determined. A new and compact proof of the existence of multiple bifurcation is derived. The linearized stability near simple bifurcation and "normal" limit points is then indicated.

II. Constructive Techniques for the Generation of Solution Branches

A method is described in which the dependence of the solution arc on a naturally occurring parameter is replaced by the dependence on a form of pseudo-arclength. This results in continuation procedures through regular and "normal" limit points. In the neighborhood of bifurcation points, however, the associated linear operator is nearly singular causing difficulty in the convergence of continuation methods. A study of the approach to singularity of this operator yields convergence proofs for an iterative method for deter-

mining the solution arc in the neighborhood of a simple bifurcation point. As a result of these considerations, a new constructive proof of bifurcation is determined.

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Chapter I: Introduction to Bifurcation

(1) Introduction

The purpose of this Chapter is to provide a general introduction to bifurcation. Section (2) will describe, in heuristic terms, what bifurcation is, and will then indicate some of the varied bifurcation phenomena to be expected in particular problems. Several of the physical areas of application will be mentioned. Many of these diverse problems may be formulated in an abstract manner as a non-linear operator map between Banach spaces depending on relevant control parameters. It is this approach to bifurcation which will be pursued in Chapters II and III. As a preparation, Section (3) will present some of the concepts and results from basic functional analysis that will later be required.

(2) Bifurcation Phenomena

Bifurcation is a non-linear effect, intimately tied to the phenomenon of multiple solutions to non-linear equations. In this light it is a local theory, being concerned with a local change in the number of solutions to a particular problem. A point of intersection of two or more solution branches will be called a bifurcation point.

The simplest mathematical formulation exhibiting bifurcation is the linear eigenvalue problem.

$$Au = \lambda u \qquad (2.1)$$

$$u \in \mathbb{R}^n, \lambda \in \mathbb{R}$$

Here A is assumed to be a real nxn matrix. If we plot the solutions of equation (2.1) as a function of λ we arrive at a picture like that of Figure (2.1).

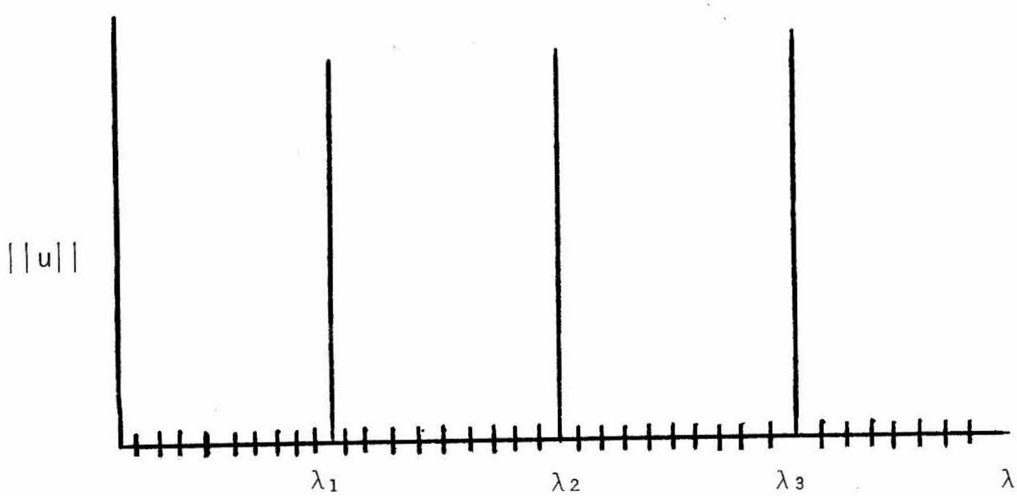


Figure (2.1)

The solution (o,λ) persists for all λ and is indicated by the hatched line. However, for particular values of λ (the real eigenvalues of A) non-trivial solutions to (2.1) exist as well. Thus we may call the points $\lambda_1,\lambda_2,\lambda_3,\ldots$ bifurcation points.

The above problem is actually non-linear due to the product term λu; but what behaviour may be expected if more bona fide non-linearities are allowed to enter? We indicate this non-linear equation schematically as

$$G(u,\lambda) = 0 \qquad (2.2)$$

and several of the possible situations are indicated in Figure (2.2). Here $N(u)$ is some scalar measure of the solution u which can take on both positive and negative values.

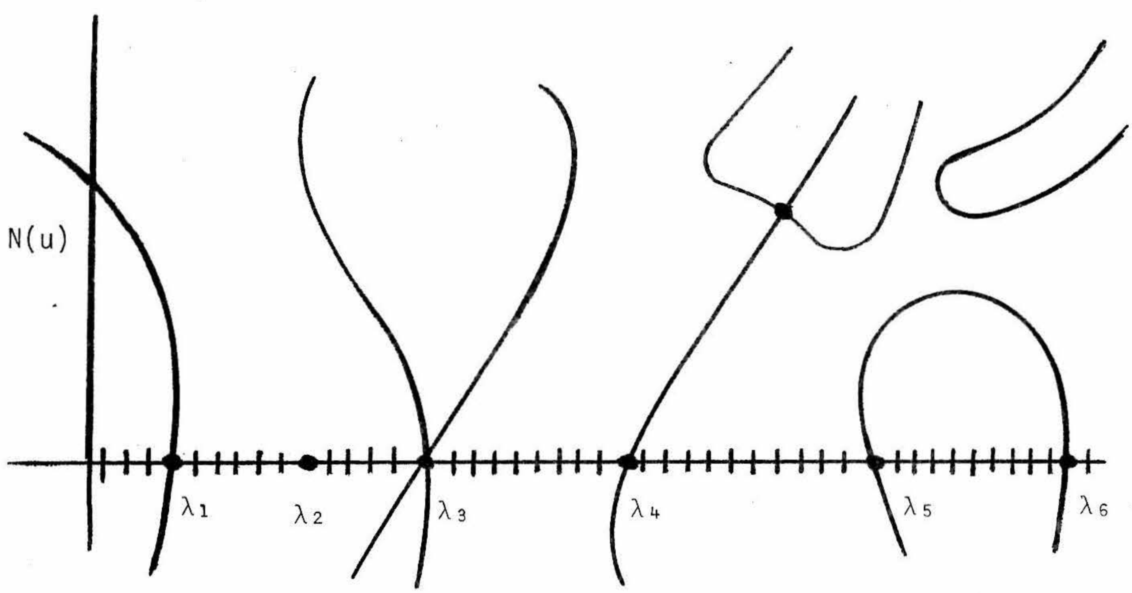


Figure (2.2)

We take particular note of the following:

(1) The eigenfunction branch emanating from λ_1 may become distorted. This situation falls under the category of simple bifurcation and has received considerable work. A comprehensive study may be found in Crandall and Rabinowitz (8). Simple bifurcation will be a special case of the equations studied in Chapter II

(2) There may be a point λ_2, which is an eigenvalue of the associated linear operator G_u, but from which no bifurcation occurs. This is in contrast to the linear eigenvalue problem, where a necessary and sufficient condition for bifurcation was λ to be a real eigenvalue of the linear operator. (Under rather general assumptions, however, topological arguments guarantee bifurcation at eigenvalues of odd multiplicity. (Krasnosel'sky (22))).

(3) Multiple bifurcation may occur, that is, more than two solution branches may intersect, as indicated at $\lambda=\lambda_3$. This may occur in the linear eigenvalue problem as well, but in that case the number of such branches is strictly limited to the geometric multiplicity of the eigenvalue. It will be seen in Chapter II that the general problem may have more bifurcating branches than the dimension of the null space of the linear operator. Sather (35), (36) has given considerable attention to multiple bifurcation in a Hilbert space setting.

(4) Secondary bifurcation may appear. This is the situation when at some point on a bifurcating branch, such as the one through λ_4, a second bifurcation takes place. (Pimbley (29)).

(5) We may find solution arcs from distinct bifurcation points λ_5 and λ_6 may join.

(6) Completely detached solution arcs may exist.

This already complicated picture can be made more intricate by the appearance in the physical problem of more than one relevant control parameter.

First, we may have a two parameter problem written as

$$G(u,\lambda,\tau) = 0 \tag{2.3}$$

with solutions indicated by Figures (2.3). Here τ is called an imperfection parameter, and measures the deviation of the physical problem from some idealized state. The solutions for $\tau=0$ are indicated by the hatched lines and for $\tau\neq0$ by the various dotted lines. We see that only for the particular value of $\tau=0$ does bifurcation occur. From this we may speak of the imperfection as breaking the bifurcation. This simple perturbed bifurcation has been studied quite generally by Keener and Keller (17) and more recently by Reiss (32) and Matkowsky and Reiss (26).

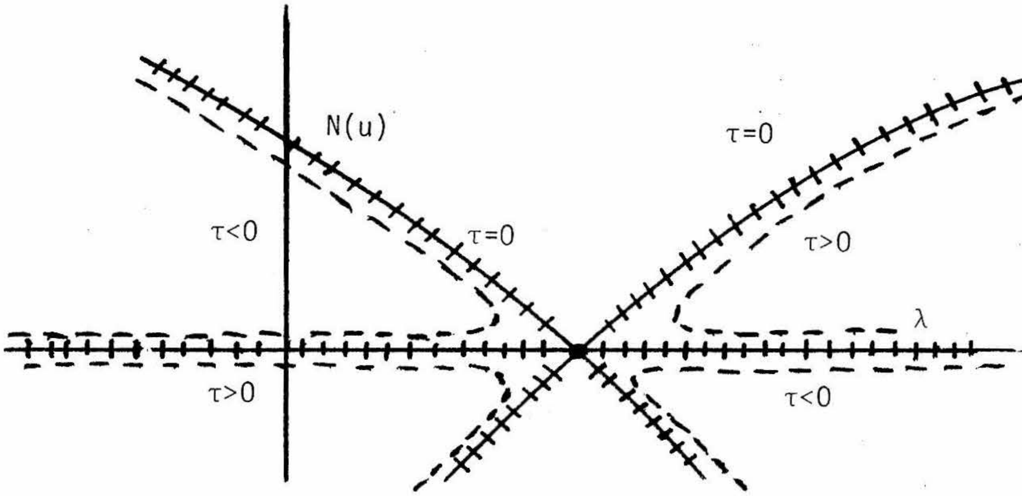


Figure (2.3)(a)

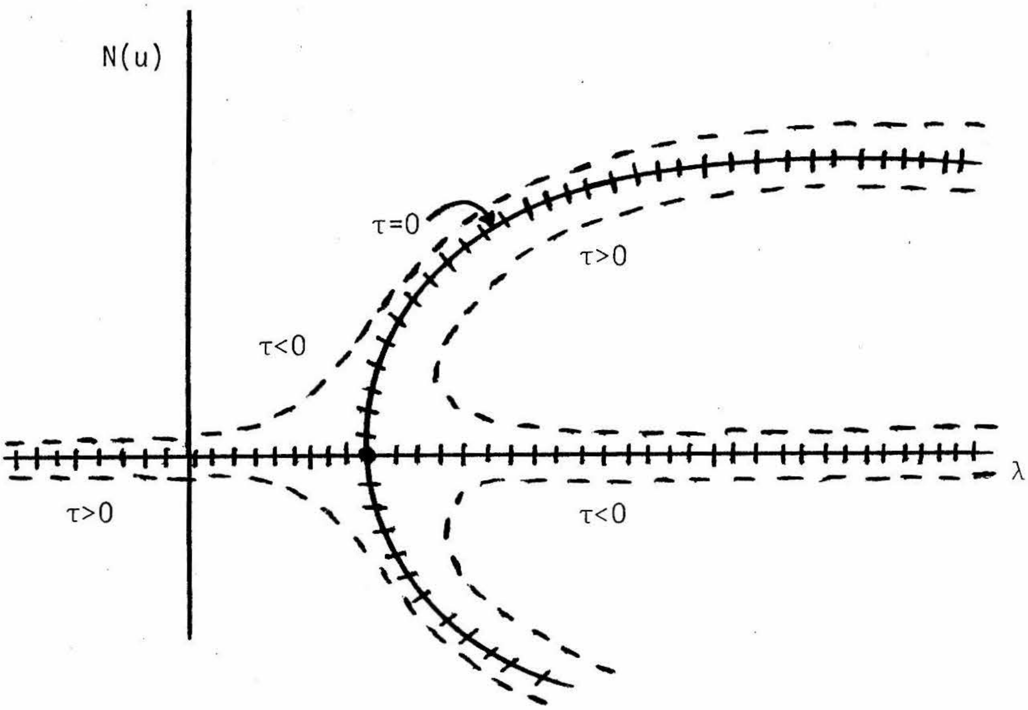


Figure (2.3)(b)

A different situation occurs when there are two bifurcation parameters present in the problem. We write this as

$$G(u,\lambda_1,\lambda_2) = 0 \qquad (2.4)$$

and exhibit some possible solution behaviour in Figures (2.4). Figure (2.4)(a) plots the solutions as a function of λ_1 for the particular value $\lambda_2=\lambda_2^*$. We see that multiple bifurcation occurs at λ_1^*. As λ_2 deviates from its critical value λ_2^* (indicated by Figure (2.4)(b)) the multiple bifurcation splits into two simple bifurcations but in addition a small closed loop of solutions appears. This small loop coalesces to a point as λ_2 returns to λ_2^*. It is this situation for which we say the appearance of an additional bifurcation parameter generates secondary bifurcation. This phenomenon has been considered by Bauer, Keller, and Reiss (2), Keener (15)-(16), Kreigsmann and Reiss (23), and Goldstein, Huerta, and Nearing (10).

We now mention just a few of the physical areas of application which exhibit some of these bifurcation phenomena.

(1) Perhaps the richest area of application is the study of non-linear elastic deformation. Revelant problems include the buckling of elastic plates, caps, shells, rods and beams. (Thompson and Hunt (44), Keller and Antman (21)).

(2) The general equations of chemical kinetics may be written as

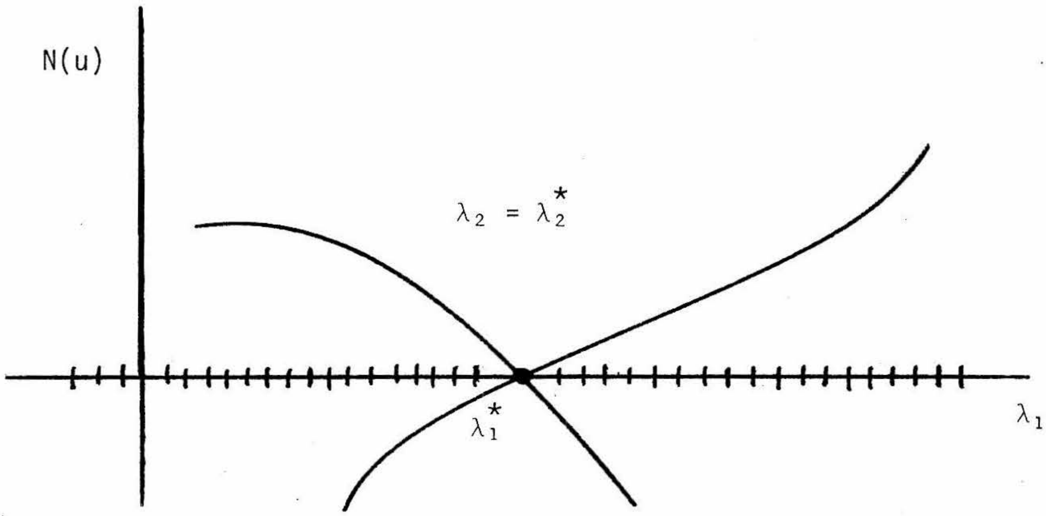


Figure (2.4)(a)

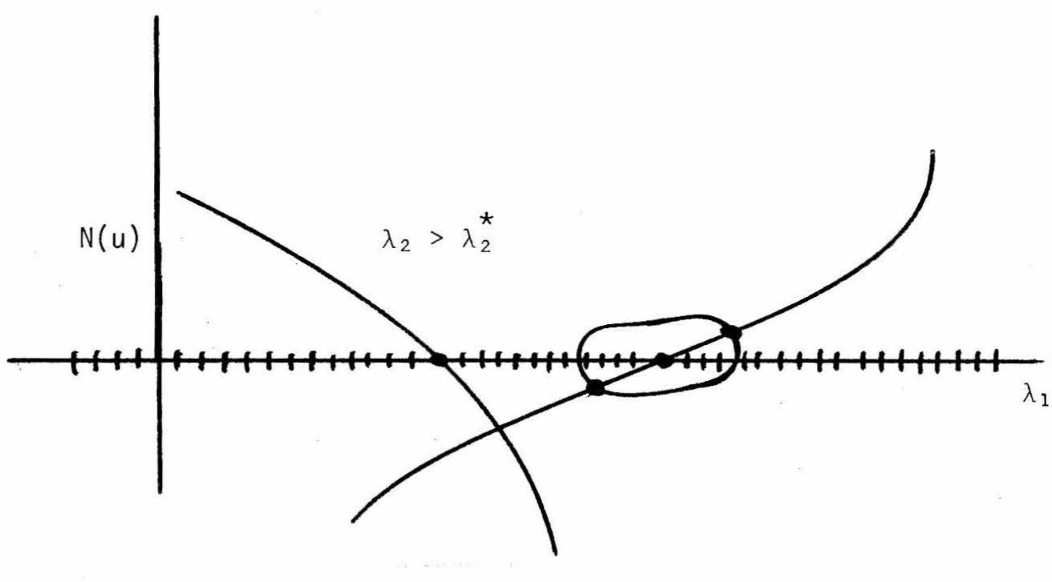


Figure (2.4)(b)

$$\frac{\partial C_i}{\partial t} = k_i \nabla^2 C_i + f_i(C_1, \ldots C_n, T)$$

$$\frac{\partial T}{\partial t} = K \nabla^2 T + g(C_1, \ldots C_n, T)$$

where $C_1, \ldots C_n$ are the concentrations of the various reactants and T is the temperature. This non-linear coupled parabolic system, supplemented with appropriate boundary conditions, admits solution sets with complicated bifurcation structure.
(Keener (16))

(3) Many of the equations governing fluid flow allow bifurcating solutions. For example :

(i) Thermal convection between heated flat plates
(Bénard problem)

(ii) Flow between rotating coaxial cylinders
(Taylor column)

(iii) Flow between rotating coaxial plates

(Sattinger (38), Joseph and Sattinger (12), Keller and Antman (21), Szeto (42))

The above list is essentially endless since virtually any physical theory which takes into account non-linear effects exhibits bifurcation. It is with this in mind that the work of Chapters II and III will not specify a particular set of governing equations or a particular physical model.

(3) Preliminary Definitions and Theorems

In the following chapters, use will be made of terms and concepts common to basic functional analysis. In the interests of completeness and consistency these terms will be briefly defined in this section. A more complete and precise description of the majority of these concepts can be found in either Rudin [34] or Schecter [41].

To begin, suppose we are given a vector space X, that is, a set whose elements are called vectors and for which addition and scalar multiplication are defined. The scalar field for most of our purposes is assumed to be the real numbers. The vector space X is called a normed space if there exists a functional from X into $\mathbb{R}$, denoted by $||\cdot||$, satisfying

(i) $||x+y|| \leq ||x|| + ||y|| \quad \forall\, x,y \in X$

(ii) $||\alpha x|| = |\alpha|\ ||x|| \quad \alpha$ a scalar

(iii) $||x|| > 0$ if $x \neq 0$

Every normed space may be regarded as a metric space in which the distance $d(x,y)$ between x and y is $||x-y||$. These most basic preliminaries lead to our first definition,

Definition: A Banach space is a normed space which is complete in the metric defined by its norm; that is, every Cauchy sequence is required to converge.

Our formulation of bifurcation will be in terms of an operator map between Banach spaces.

Suppose now we are given a mapping F between two normed vector spaces X and Y which is linear. A set $A \subseteq X$ will be called bounded if $\exists$ $m \ni ||a|| < m \; \forall a \in A$. The boundedness of our linear operator F is determined by its behaviour on these sets.

<u>Definition</u>: $F: X \to Y$ is a bounded linear operator iff F(A) is a bounded subset of Y for every bounded subset $A \in X$.

It is well known that if X is normed, boundedness and continuity of linear operators are equivalent. If X and Y are normed spaces we let B(X,Y) denote the set of all bounded linear mappings from X to Y. This set can be made into a normed space by defining

$$||F|| = \sup_{||x||=1} ||F(x)||$$

It can be shown that if Y is a Banach space the above norm makes B(X,Y) into a Banach space.

Using these ideas and the norm topologies of X and Y we can define the derivative of an operator map between X and Y.

<u>Definition</u>: Suppose X and Y are Banach spaces and F maps X into Y. F is Fréchet differentiable at x_0 iff there exists $DF(x_0) \in B(X,Y)$ such that

$$\lim_{u \to 0} \frac{||F(x_0+u) - F(x_0) - DF(x_0)u||}{||u||} = 0$$

It is clear that the Fréchet derivative (if it exists) is unique. If for some open set $A \varepsilon X$, the map $a \to DF(a)$ is continuous then F is said to be continuously differentiable in A.

In the following Chapters we study solutions of $G(u,\lambda) = 0$, and the structure of the Fréchet derivative of G with respect to u will be found to be of great importance.

Returning to the space $B(X,Y)$ we note the special case when $Y = \mathbb{R}$ (or $\mathbb{C}$). In this case the elements of $B(X,\mathbb{R})$ are called the continuous linear functionals on X and we have

Definition: The dual space of a Banach space X is the vector space X^* whose elements are all the continuous linear functionals $x^* \varepsilon B(X,\mathbb{R})$.

The norm in X^* is in this case

$$||x^*|| = \sup_{||x||=1} |x^*(x)|$$

and since $\mathbb{R}$ is complete X^* is a Banach space. We can now define an adjoint operator acting between dual spaces.

Definition: Let X and Y be Banach spaces and $F \varepsilon B(X,Y)$. Then define the adjoint F^* of F to be the operator for which

$$(F^*y^*)(x) = y^*(Fx)$$

It is easily seen that this defines F^* uniquely and that $||F^*||=||F||$.

We now may use the notions of a dual space and an adjoint operator to proceed toward a statement of a specialized form of the Fredholm Alternative. First, suppose M and N are subspaces of X and X^*, respectively. We define the annihilators $M^\perp$ and ${}^\perp N$ as

$$M^\perp = \{x^* | x^*(x) = 0 \ \forall \ x \varepsilon M\}$$

$${}^\perp N = \{x | x^*(x) = 0 \ \forall \ x^* \varepsilon N\}$$

Using these definitions we note the following sequence

$$\begin{array}{l} y^* \varepsilon N(F^*) \\ F^* y^* = 0 \\ (F^* y^*)(x) = 0 \quad \forall \ x \varepsilon X \\ y^*(Fx) = 0 \qquad \forall \ x \varepsilon X \\ y^* \varepsilon R(F)^\perp \end{array}$$

where each statement is equivalent to the one that follows or precedes it. That is, $N(F^*) = R(F)^\perp$ This means that to solve

$$Fx=y$$

it is necessary that $x^*(y)=0$ for all $x^* \varepsilon N(F^*)$. Under what conditions is this also sufficient? That is, when can we state $R(F)= {}^\perp N(F^*)$? This is answered by recalling that ${}^\perp M^\perp = \overline{M}$ for any subspace M. Thus, $\overline{R(F)} = {}^\perp N(F^*)$ and we see $R(F) = {}^\perp N(F^*)$ iff $R(F)$ is closed.

Now if X_1 is a closed subspace of X we may define the codim X_1 as the dimension of the quotient space X/X_1. If this dimension is finite then there exists a closed subspace X_2 complementing X_1 in X. That is we may write X as the direct sum $X = X_1 \oplus X_2$ with codim

$X_1 = \dim X_2$.

Using these ideas we may now state a modified Fredholm Alternative.

Theorem: Suppose $F:X \to X$ is a Fredholm operator of index zero.

That is;

$$\begin{array}{c} R(F) \text{ is closed} \\ \dim N(F)=d \\ \operatorname{codim} R(F)=d \end{array} \qquad d<\infty$$

Then with regard to the equation

$$Fx = y$$

we have either

(i) A unique solution for every $y \in Y$ (if $d = 0$)

or

(ii) an infinite number of solutions for some y and none for others. In the latter case we have a solution iff $y^*(y)=0$ for all $y^* \in N(F^*)$.

This is a considerably weaker statement than the usual Fredholm Alternative. Normally one considers $F=I-K$ with K compact and then proves our hypothesis that F is Fredholm of index zero. The above result is sufficient for our purposes however. We note that if F is Fredholm and codim $R(F)=d$ then A^* has exactly d null vectors $\psi_1^* \ldots \psi_d^*$ and $R(F) = \{x \in X | \psi_i^* x=0 \ i=1,..d\}$.

In finite dimensions, if an operator has an inverse then the inverse is bounded. In infinite dimensions this need not be the case since $R(F)$

may not be closed. We have however;

Theorem: (Bounded Inverse). Let $F: X \to Y$ be a continuous mapping from the Banach space X onto the Banach space Y. If $N(F)=\{\underline{0}\}$ then F^{-1} is bounded.

We shall use this result later in the following form. Suppose we have a bounded linear operator F with $N(F)=X_1$ being finite dimensional. Then there is a closed space X_2 complementing X_1 in X. Further suppose $R(F)=Y_1$ is closed. Then the restricted operator $\hat{F}=F|_{X_1} \to Y_1$ has a bounded inverse.

We now state a version of the Implicit Function Theorem which will be quoted in the existence theorems of Chapter II. The formulation follows that of Nirenberg [28].

Theorem: (Implicit Function)

Suppose X, Y and Z are Banach spaces and F a continuous mapping of an open set $U \subseteq X \times Y \to Z$. Assume that F has a Fréchet derivative with respect to x, denoted $F_x(x,y)$ which is continuous in U. Let $(x_0,y_0) \varepsilon U$ and suppose $F(x_0,y_0)=0$. Then if $DF(x_0,y_0)$ is an isomorphism from X onto Z we have,

(i) There exists a ball $B_r(y_0) = \{y \mid ||y-y_0||<r\}$ and a unique continuous mapping $u: B_r(y_0) \to X$ such that $u(y_0)=x_0$ and $F(u(y),y)=0$.

(ii) If F is a k-times continuously differentiable in U then u(y) is k times continuously differentiable.

We conclude this section with the statement of a basic lemma which is essential to many of the results of Chapter II. This lemma originated in Keller [19] and our presentation follows this work.

Lemma I: Let B be a Banach space and consider the linear operator $\hat{A}: B\times \mathbb{R}^{\gamma} \to B\times \mathbb{R}^{\gamma}$ of the form:

$$\hat{A} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

where

$A : B \to B$ $\qquad$ $B: \mathbb{R}^{\gamma} \to B$

$C^*: B \to \mathbb{R}^{\gamma}$ $\qquad$ $D: \mathbb{R}^{\gamma} \to \mathbb{R}^{\gamma}$

(1) If A is nonsingular then $\hat{A}$ is nonsingular iff

(a) $D - C^* A^{-1} B$ is non-singular

(2) If A is singular and

(b) $\dim N(A) = \operatorname{codim} R(A) = \gamma$

then $\hat{A}$ is non-singular iff:

(c_0) $\dim R(B) = \gamma$ $\qquad$ (c_1) $R(B) \cap R(A) = \{\underline{0}\}$

(c_2) $\dim R(C^*) = \gamma$ $\qquad$ (c_3) $N(A) \cap N(C^*) = \{\underline{0}\}$

(3) If A is singular and $\dim N(A) > \gamma$ then $\hat{A}$ is singular.

Proof: To prove $\hat{A}$ is nonsingular we must show it is both one to one and onto. We will do this by considering

$$\hat{A}\begin{pmatrix} x \\ \vec{\xi} \end{pmatrix} = \begin{pmatrix} y \\ \vec{\eta} \end{pmatrix}$$

where $x, y \in B$ and $\vec{\xi}, \vec{\eta} \in \mathbb{R}^{\nu}$. Considering

$$(\alpha) \qquad \begin{aligned} Ax + B\vec{\xi} &= y \\ C^{*}x + D\vec{\xi} &= \vec{\eta} \end{aligned} \qquad \text{(existence)}$$

if $\hat{A}$ is nonsingular, (α) must have a solution for any $(y, \vec{\eta})^{T}$. In addition the only solution of

$$(\beta) \qquad \begin{aligned} Ax + B\vec{\xi} &= 0 \\ C^{*}x + D\vec{\xi} &= 0 \end{aligned} \qquad \text{(uniqueness)}$$

must be $x = 0$, $\vec{\xi} = 0$. We now consider the various cases.

Case I: Suppose A is non-singular and (1) (a) holds. Then from (β)

$$x = A^{-1} B\vec{\xi}$$

and so

$$\hat{D}\vec{\xi} = 0$$

where $$\hat{D} \equiv D - C^{*} A^{-1} B \quad .$$

Thus if $\hat{D}$ is non-singular the only solution is $\vec{\xi} = 0$ forcing $x = 0$ and so we have uniqueness. Further, (α) implies

$$x = A^{-1}y - A^{-1} B\vec{\xi}$$

so
$$\hat{D}\vec{\xi} = \vec{\eta} - C^{*} A^{-1}y$$

hence
$$\vec{\xi} = \hat{D}^{-1}(\vec{\eta} - C^{*} A^{-1}y) \quad .$$

Thus for any $(y, \vec{\eta})^T$ we determine an $\vec{\xi}$ and consequently an x. Hence we have existence.

Case. II: Suppose A and $\hat{A}$ are non-singular, we wish to show $\hat{D}$ is non-singular.

As above we have

$$\hat{D}\vec{\xi} = \vec{\eta} - C^{*} A^{-1}y$$

and this has a solution for any $(y, \vec{\eta})^T$, in particular $y = 0$. Letting $\vec{\eta}$ vary we see $\hat{D}$ is non-singular.

Case III: Suppose (2) (b) holds and let (C_0) - (C_3) be satisfied.

We wish to show $\hat{A}$ is non-singular.

Considering (β) we see that if $B\vec{\xi} \neq 0$ then $Ax \neq 0$ violating (C_1). Thus $B\vec{\xi} = Ax = 0$ and (c_0) forces $\vec{\xi} = 0$. Hence $C^{*}x = 0$ so $x \in N(A) \cap N(C^{*})$ and thus $x = 0$. This give uniqueness.

From (2) (b) we can write

$$B = R(A) \oplus C$$

where $\dim C = \nu$. However since $\dim R(B) = \nu$ and $R(B) \cap R(A) = \{\underline{0}\}$ we may put

$$B = R(A) \oplus R(B) \quad .$$

Thus for any y there is a solution $(x_0, \vec{\xi})$ of

$$Ax_0 = y - B\vec{\xi} \quad .$$

The general solution to this equation is $x = x_0 + z_0$ where $z_0 \in N(A)$. Now to solve

$$C^* z_0 = \vec{\eta} - D\vec{\xi} - C^* x_0$$

we note $N(C^*) \cap N(A) = \{\underline{0}\}$ and $\dim R(C^*) = \nu$. Thus we have a solution z_0 for any $\vec{\eta}$ and existence is satisfied.

Case IV: Assume (2) (b) and let $\hat{A}$ be non-singular, we must show (C_0) - (C_3) are satisfied.

From existence, for any $y \in R(A)$ $\exists \vec{\xi} \in \mathbb{R}^\nu$ such that

$$Ax = y - B\vec{\xi} \in R(A).$$

Thus we must have $\dim R(B) = \nu$. Now if (C_3) does not hold then $\exists\, x_0 \neq 0$ such that $Ax_0 = C^* x_0 = 0$. Taking $(x, \vec{\xi})^T = (x_0, 0)^T$ we violate uniqueness. Writing

$$N(A) = sp\{\phi_1, \ldots \phi_\nu\}$$

then

$$\hat{A} \begin{pmatrix} \phi_j \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{\eta}_j \end{pmatrix}$$

where $\vec{\eta}_j = C^* \phi_j$. Since $\hat{A}$ is non-singular the $\vec{\eta}_j$ are independent giving $\dim R(C^*) = \nu$.

Further, if (C_1) did not hold, the equation

$$Ax + B\vec{\xi} = 0$$

would have a non-trivial solution violating uniqueness.

Case V: Suppose $\dim N(A) > \nu$. Then there exist independent $x_1, \ldots x_{\nu+1}$ for which $Ax_i = 0$. The $\nu + 1$ vectors $C^* x_i$ $i = 1, \ldots \nu+1$ are dependent and thus there is a non-zero $(\alpha_1, \ldots, \alpha_{\nu+1})$ for which

$$\Sigma \alpha_i C^* x_i = 0 \quad .$$

That is

$$A(\Sigma \alpha_i x_i) = 0$$

$$C^*(\Sigma \alpha_i x_i) = 0$$

violating uniqueness, so $\hat{A}$ is singular.

Chapter II Existence and Structure of Bifurcation Branches

(1) Introduction

This chapter deals with some of the theoretical considerations involved in bifurcation. Following the approach of Chapter I we study solutions to an equation of the form

$$G(u,\lambda) = 0 \tag{1.1}$$

Here u is an element in some Banach space B and G is a non-linear operator map from $B \times \mathbb{R}$ into B. Study is made of solutions in the neighborhood of a known solution point (u_0,λ_0).

It is shown in Section (2) that for the point (u_0,λ_0) to be a bifurcation point, certain requirements must be satisfied by the linearized operator $G_u(u_0,\lambda_0)$. Under the assumption that $G_u(u_0,\lambda_0)$ is Fredholm of index zero with an m-fold semi-simple eigenvalue, a necessary condition for bifurcation is seen to be the satisfaction of a set of algebraic equations. This approach is a slightly modified Lyapunov-Schmidt procedure (Vainberg and Trenogin (46)). The structure of higher order Algebraic Bifurcation Equations is also indicated.

Since isolation of the roots of these equations will play such an important role, Section (3) is concerned with some of the conditions under which isolation may occur.

Section (4) deals with the problem of constructing a formal perturbation series solution in the neighborhood of a bifurcation point. It is found that isolation of the root in question allows, in theory, the determination of the perturbation series to any order.

Most approaches to bifurcation assume that $G_\lambda(u_0,\lambda_0)$ is in the range of the operator $G_u(u_0,\lambda_0)$ Section (5) drops this assumption and derives a new set of algebraic equations, called the Limit Point Bifurcation Equations.

The root structure of either the Algebraic or the Limit Point Bifurcation Equations can be exceedingly complex. In general, they fall under the classification of singularities of vector fields and so aspects of catastrophe theory apply. (Thom (43), Chow, Hale, and Mallet-Paret (6)-(7), Thompson and Hunt (45)). Section (6), however, studies the root structure from a different viewpoint and indicates some of the various possibilities.

Section (7) contains the basic existence results. It is shown that for each isolated root of the Algebraic or Limit Point Bifurcation Equations, there exists a smooth solution branch bifurcating from (u_0,λ_0), with its local structure determined by this root.

The linearized stability of solution arcs in the neighborhood of simple normal limit points and simple bifurcation points is considered in Section (8).

(2) Algebraic Bifurcation Equations

In this section we study the problem of finding a formal solution to

$$G(u,\lambda) = 0 \tag{2.1}$$

in the neighborhood of a known solution point (u_o,λ_o). In what follows G will be a twice continuously Fréchet differentiable mapping from some Banach space $B_1 \equiv B \times \mathbb{R}$ into B. At the point (u_o,λ_o) we assume $G_u^o \equiv G_u(u_o,\lambda_o)$ is a bounded linear operator satisfying

$$N(G_u^o) \equiv X_1, \dim X_1 = d_1 < \infty$$

$$R(G_u^o) \equiv X_2 \text{ is a closed subspace of } B \tag{2.2}$$

$$\operatorname{codim} X_2 = d_2 < \infty$$

i.e., we are requiring G_u^o to be a Fredholm operator and define its index by

$$\operatorname{ind}(G_u^o) \equiv d_1 - d_2 \tag{2.3}$$

The simplest case with this structure is $d_1=d_2=0$; that is, G_u^o has a bounded inverse. Here the Implicit Function Theorem yields a unique solution arc $(u(\lambda),\lambda)$ through (u_o,λ_o). The next simplest possibility is $d_1 > 0$ but $d_2 = 0$. In this case we may define Y_1 as a closed complementing space to X_1 and $\hat{X}_1 \equiv X_1 \times \mathbb{R}$. Then applying the Implicit Function Theorem to

$$\hat{G}(Y_1,\hat{X}_1) \equiv G(Y_1+X_1,\lambda):Y_1 X \hat{X}_1 \to B$$

we see the solutions near (u_o,λ_o) form a d_1+1 dimensional manifold. In both these situations it is clear bifurcation does not occur. Hence, a necessary condition for bifurcation is that G_u^o be singular and that codim $R(G_u^o)$ be non-zero. (If B is finite dimensional, this is equivalent to dim $N(G_u^o) \neq 0$ since dim $N(G_u^o)$ = codim$R(G_u^o)$ in this case.) In this light we make the further assumption

$$d_1 = d_2 = m > 0$$

$$N(G_u^o) = \text{span } \{\phi_1 \ldots \phi_m\} \tag{2.4}$$

$$N(G_u^{o*}) = \text{span } \{\psi_1^* \ldots \psi_m^*\}$$

and since $R(G_u^o)$ is closed

$$R(G_u^o) = \{X \varepsilon B | \psi_i^* X = 0 \quad i=1,\ldots m\} \tag{2.5}$$

In addition we demand the structure of the zero eigenvalue of G_u^o to be such that

$$\psi_i^* \phi_j = \delta_{ij} \tag{2.6}$$

(All that is required is that the matrix $M = (\psi_i^* \phi_j)$ be non-singular, but the eigenfunctions can be then chosen for convenience so that $M = I$.) If we define the multiplicity μ of the eigenvalue by

$$\mu = \dim \bigcup_{k=1}^{\infty} N((G_u^o)^k) \qquad (2.7)$$

we see that (2.6) implies $\mu = m$. We call such an eigenvalue simple when $m = 1$ and semi-simple when $m > 1$ (Kato [13].) Such eigenvalues are often said to have Riesz index 1.

We now proceed with the study of solutions of (2.1) near (u_o,λ_o) by supposing the existence of a solution arc $(u(\varepsilon),\lambda(\varepsilon))$ depending smoothly on some parameter ε and finding the equations which must necessarily be satisfied by the arc. Hence, suppose we have

$$G(u(\varepsilon),\lambda(\varepsilon)) = 0 \quad |\varepsilon| \leq \varepsilon_o \qquad (2.8)$$

with $(u(o),\lambda(o)) = (u_o,\lambda_o)$, which depends as smoothly as desired on ε. Differentiating (2.8) with respect to ε and evaluating at $\varepsilon = 0$ we have

$$G_u^o\dot{u}(o) + G_\lambda^o\dot{\lambda}(o) = 0 \qquad (2.9)$$

$$G_u^o\ddot{u}(o) + G_\lambda^o\ddot{\lambda}(o) = -\left(G_{uu}^o\dot{u}(o)\dot{u}(o) + 2G_{u\lambda}^o\dot{u}(o)\dot{\lambda}(o) + G_{\lambda\lambda}^o\dot{\lambda}^2(o)\right) \qquad (2.10)$$

Here G_{uu}^o is the second Fréchet derivative of G with respect to u evaluated at (u_o,λ_o) etc. Since $R(G_u^o)$ is closed the Fredholm Alternative as stated in Chapter I specifies necessary and sufficient conditions for the existence of a solution $\dot{u}(o)$ of equation (2.9), i.e.,

$$\dot{\lambda}(o)\psi_i^* G_\lambda^o = 0 \qquad i = 1,\ldots.m \tag{2.11}$$

From (2.5) we see equation (2.9) presents two possible cases

(i) $G_\lambda^o \varepsilon R(G_u^o)$

(ii) $G_\lambda^o \notin R(G_u^o)$ But $\dot{\lambda}(o) = 0$

The second possibility is called a normal limit point and will be discussed in section (5). We assume case (i) for the present, and hence

$$\dot{u}(o) = \sum_{j=0}^{m} \xi_j \phi_j, \ \xi_o = \dot{\lambda}(o) \tag{2.12}$$

where the scalars ξ_j are as yet arbitrary and ϕ_o is the unique (from (2.6)) solution of,

$$\begin{aligned} &G_u^o \phi_o + G_\lambda^o = 0 \\ &\psi_j^* \phi_o = 0 \qquad j = 1,\ldots m \end{aligned} \tag{2.13}$$

Since $G^o_\lambda \varepsilon R(G^o_u)$ the necessary and sufficient condition for the existence of a solution $\ddot{u}(o)$ is that the righthand side of equation (2.10) be in $R(G^o_u)$. This requirement gives m equations for the m+1 unknowns $(\xi_o,\ldots,\xi_m)$

$$\sum_{j=1}^{m}\sum_{k=1}^{m} a_{ijk}\xi_j\xi_k + 2\sum_{j=1}^{m} b_{ij}\xi_j\xi_o + c_i\xi_o^2 = 0 \qquad (2.14)$$

$$i = 1\ldots.m$$

where we have defined, for i,j,k = 1,...m;

$$\left.\begin{aligned} a_{ijk} &= a_{ikj} = \psi_i^* G^o_{uu}\phi_j\phi_k \\ b_{ij} &= \psi_i^*(G^o_{uu}\phi_o + G^o_{u\lambda})\phi_j \\ c_i &= \psi_i^*(G^o_{uu}\phi_o\phi_o + 2G^o_{u\lambda}\phi_o + G^o_{\lambda\lambda}) \end{aligned}\right\} \qquad (2.15)$$

Clearly if $(\xi_o,\ldots\xi_m)$ is a solution of (2.14) then so is $(\beta\xi_o,\ldots\beta\xi_m)$ for any β; to avoid this non-uniqueness we add the equation

$$2\xi_o^2 + \xi_1^2 + \ldots\ldots \xi_m^2 = 1 \qquad (2.16)$$

We will find in the following section that this equation will allow a unique definition of the expansion parameter ε. We can rewrite (2.14)-(2.16) in a more compact vector-matrix notation as

$$\left.\begin{aligned} &A(\vec{\xi})\vec{\xi} + 2\xi_o B\vec{\xi} + \xi_o^2\vec{c} = 0 && \text{(a)} \\ &\vec{\xi}^T\vec{\xi} + 2\xi_o^2 = 1 && \text{(b)} \end{aligned}\right\} \qquad (2.17)$$

where

$$\vec{\xi} \equiv (\xi_1, \xi_2, \ldots \xi_m)^T \varepsilon\, \mathbb{R}^m \qquad A(\vec{\xi}) \equiv (A_{ij}) = \left(\sum_{k=1}^{m} a_{ijk}\xi_k \right)$$

$$B \equiv (b_{ij}) \qquad \vec{c} \equiv (c_1, c_2 \ldots c_m)^T \varepsilon\, \mathbb{R}^m \qquad (2.18)$$

The equations (2.17) are called the (quadratic) Algebraic Bifurcation Equations. Section (6) will consider several special cases of these equations and will indicate the varied solution structures to be expected. Clearly (2.17) (a) are m equations homogeneous of degree two in the m+1 variables $(\xi_o,\ldots\xi_m)$. In most applications the matrix B is non-zero, in fact, it may often be taken to be the identity. However, it often happens that

$$A(\vec{\xi})\equiv 0 \qquad a_{ijk}=0 \qquad i,j,k=1\ldots m \tag{2.19}$$

This can occur in two basic ways,

$$\begin{aligned} &\text{(i)} \quad G^o_{uu}:B\times B\to B \text{ is identically zero} \\ &\text{(ii)} \quad G^o_{uu}:X_1\times X_1\subseteq X_2 \end{aligned} \tag{2.20}$$

Since the second is less restrictive, we will assume (2.20)(ii). The assumptions (2.2) along with semi-simplicity of our eigenvalue allows a direct sum decomposition of B,

$$B = X_1 \oplus X_2 \tag{2.21}$$

and we see G^o_u maps X_2 in a one-one fashion onto X_2. Defining

$$\hat{G}_u \equiv G^o_u\Big|_{X_2} \tag{2.22}$$

and recalling that X_2 is closed, we have the existence of a bounded inverse for this restricted operator

$$\hat{G}_u^{-1}: X_2 \to X_2 \qquad (2.23)$$

As a preliminary to the construction of the higher order bifurcation equations we find the third and fourth order equations of (2.8).

$$G_u^o u^{(i)} + G_\lambda^o \lambda^{(i)} = r_i(u,\lambda,\ldots u^{(i-1)},\lambda^{(i-1)}) \qquad (2.24)$$

where

$$r_3 = -\Big(3\left[(G_{uu}^o\,\dot{u}+G_{u\lambda}^o\dot{\lambda})\ddot{u} + (G_{u\lambda}^o\dot{u}+G_{\lambda\lambda}^o\dot{\lambda})\ddot{\lambda}\right] + 3\,(G_{uu\lambda}^o\dot{u}\dot{u}\dot{\lambda}+G_{u\lambda\lambda}^o\dot{u}\dot{\lambda}^2)$$

$$+ G_{uuu}^o\dot{u}\dot{u}\dot{u} + G_{\lambda\lambda\lambda}^o\dot{\lambda}^3\Big) \qquad (2.25)$$

$$r_4 = -\Bigg(4\left[(G^o_{uu}\dot{u}+G^o_{u\lambda}\dot{\lambda})\dddot{u} + (G^o_{u\lambda}\dot{u}+G^o_{\lambda\lambda}\dot{\lambda})\dddot{\lambda}\right]$$

$$+3\left[(G^o_{uu}\ddot{u}+G^o_{u\lambda}\ddot{\lambda})\ddot{u} + (G^o_{u\lambda}\ddot{u}+G^o_{\lambda\lambda}\ddot{\lambda})\ddot{\lambda}\right]$$

$$+6\quad(G^o_{uuu}\dot{u}\dot{u}+2G^o_{uu\lambda}\dot{u}\dot{\lambda}+G^o_{u\lambda\lambda}\dot{\lambda}\dot{\lambda})\ddot{u} \tag{2.26}$$

$$+6\quad(G^o_{uu\lambda}\dot{u}\dot{u}+2G^o_{u\lambda\lambda}\dot{u}\dot{\lambda}+G^o_{\lambda\lambda\lambda}\dot{\lambda}^2)\ddot{\lambda}$$

$$+\ (G^o_{uuuu}\dot{u}\dot{u}\dot{u}\dot{u}+4G^o_{uuu\lambda}\dot{u}^3\dot{\lambda}+6G^o_{uu\lambda\lambda}\dot{u}^2\dot{\lambda}^2+4G^o_{u\lambda\lambda\lambda}\dot{u}\dot{\lambda}^3+G^o_{\lambda\lambda\lambda\lambda}\dot{\lambda}^4)\Bigg)$$

Now (2.17) with $A(\vec{\xi})\equiv 0$ becomes

$$\begin{aligned} \xi_o(2B\vec{\xi} + \xi_o\vec{c}) &= 0 \\ 2\xi_o^2 + \vec{\xi}^T\vec{\xi} &= 1 \end{aligned} \tag{2.27}$$

Equation (2.27) has two possible types of solutions

(i) $\xi_o = 0 \quad \vec{\xi}^T\vec{\xi} = 1$

(ii) $\vec{\xi} = -\alpha\vec{y}$ where $B\vec{y} = \vec{c}$

and $\xi_o = \alpha$

$$\alpha = \pm(8+||\vec{y}||^2)^{-\frac{1}{2}}$$

If B is non-singular case (ii) always gives one distinct solution, which in the next section will be shown to be isolated and hence (Section (7)) generates a bifurcating branch of solutions. If B is singular case (ii) may have one solution (ξ_o=0, $B\vec{\xi}$=0, $\vec{\xi}^T\vec{\xi}$=1, if $\vec{c}\notin R(B)$, dim N(B)=1) or a family of solutions (if $\vec{c}\in R(B)$ or dim N(B)>1) but in neither case can bifurcation be guaranteed. It is case (i) which requires the study of higher order bifurcation equations since all amplitudes of the eigenfunctions are left undetermined.

If we place $\dot{\lambda}(o) = \xi_o = 0$ in (2.10) we must solve

$$G_u^o\ddot{u} + G_\lambda^o\ddot{\lambda} = -G_{uu}^o\dot{u}(o)\dot{u}(o) \tag{2.28}$$

where

$$\dot{u}(o) = \xi_1\phi_1 + \ldots\ldots\xi_m\phi_m$$

and the ξ_j's are presently unknown. Equation (2.28) has a solution since the righthand side is in $R(G_u^o)$ and we make it unique by demanding

$$\psi_j^* \ddot{u} = 0 \qquad j = 1, \ldots m \tag{2.29}$$

Now from (2.19) we can solve uniquely for each (j,k)

$$\begin{aligned} G_u^o v_{jk} &= -G_{uu}^o \phi_j \phi_k \\ \psi_i^* v_{jk} &= 0 \quad i=1,\ldots m \end{aligned} \tag{2.30}$$

that is,

$$v_{jk} = \hat{G}_u^{-1} G_{uu}^o \phi_j \phi_k \tag{2.31}$$

and so we construct the unique $\ddot{u}(o)$ as

$$\ddot{u}(o) = \sum \xi_j \xi_k v_{jk} + \ddot{\lambda}(o)\phi_o \tag{2.32}$$

Now when $\dot{\lambda}(o) = 0$ (2.25) becomes

$$r_3 = -(G_{uuu}^o \dot{u}\dot{u}\dot{u} + 3G_{uu}^o \dot{u}\ddot{u} + 3G_{u\lambda}^o \dot{u}\ddot{\lambda}) \tag{2.33}$$

and the existence of a solution to (2.24) requires

$$A(\vec{\xi},\vec{\xi},\vec{\xi}) + 3\ddot{\lambda}(o)B\vec{\xi} = 0 \tag{2.34}$$

$$\dot{\lambda}(o) = 0 \quad \vec{\xi}^T\vec{\xi} = 1$$

where $A(\vec{\xi},\vec{\xi},\vec{\xi})$ is a trilinear operator on $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ defined by

$$A = (a_{ijk\ell}) = \psi_i^*[G^o_{uuu}\phi_j\phi_k\phi_\ell + 3G^o_{uu}\phi_j\hat{G}_u^{-1}G^o_{uu}\phi_k\phi_\ell) \tag{2.35}$$

We note that $A(\vec{x},\vec{y},\vec{z})$ is symmetric in its last two variables, and if (2.20)(i) holds it is symmetric in all three. In any case $A(\vec{\xi},\vec{\xi},\vec{\xi})$ is homogeneous of degree three. Equations (2.34) along with the normalization (2.17)(b) are called the cubic Algebraic Bifurcation Equations.

Although the pattern is clear, the quartic algebraic bifurcation equations are

$$A(\vec{\xi},\vec{\xi},\vec{\xi},\vec{\xi}) + 4B\vec{\xi}\dddot{\lambda}(o) = 0 \tag{2.36}$$

$$\dot{\lambda}(o) = \ddot{\lambda}(o) = 0 \qquad \vec{\xi}^T\vec{\xi} = 1$$

where $A \equiv (a_{ijk\ell m})$ and we have

$$a_{ijk\ell m} = \psi_i^* \Big(G^o_{uuuu}\phi_j\phi_k\phi_\ell\phi_m + 6G^o_{uuu}\phi_j\phi_k\hat{G}_u^{-1}G^o_{uu}\phi_\ell\phi_m$$

$$+ 3G^o_{uu}(\hat{G}_u^{-1}G^o_{uu}\phi_j\phi_k)(\hat{G}_u^{-1}G^o_{uu}\phi_\ell\phi_m) \qquad (2.37)$$

$$+ 4G^o_{uu}\phi_j\hat{G}_u^{-1}(G^o_{uuu}\phi_k\phi_\ell\phi_m + 3G^o_{uu}\phi_k\hat{G}_u^{-1}G^o_{uu}\phi_\ell\phi_m) \Big)$$

Once again we see $A(\vec{x},\vec{y},\vec{w},\vec{z})$ is symmetric in its last two arguments, its last three if $G^o_{uu}X_1X_1 \equiv 0$ and all arguments if $G^o_{uu}X_1X_1 = G^o_{uuu}X_1X_1X_1 \equiv 0$. In all cases $A(\vec{\xi},\vec{\xi},\vec{\xi},\vec{\xi})$ is homogeneous of degree four.

The preceding has shown that the existence of a smooth arc of solutions $(u(\varepsilon),\lambda(\varepsilon))$ of (2.8) through a singular point (u_o,λ_o) of the linearized operator forces a solution of a set of algebraic equations. That is, the satisfaction of the Algebraic Bifurcation Equations (of some order) is a necessary condition for bifurcation. It will be shown in section (7) that isolation of solutions of these equations is a sufficient condition to guarantee bifurcation. In preparation, the next section will define isolation and study circumstances under which isolation may occur.

(3) Isolation of Roots

Returning to the quadratic bifurcation equations, we define

$$i=1,\ldots m \qquad g_i(\xi_o,\ldots\xi_m)\equiv \sum_{j=1}^{m}\sum_{k=1}^{m} a_{ijk}\xi_j\xi_k+2\sum_{j=1}^{m} b_{ij}\xi_j\xi_o+c_i\xi_o^2$$

$$g_o(\xi_o,\ldots\xi_m)\equiv\xi_1^2+\ldots\xi_m^2+2\xi_o^2-1 \tag{3.1}$$

and

$$\vec{G}=\begin{bmatrix} g_1 \\ \vdots \\ g_m \\ g_o \end{bmatrix}$$

We now suppose that for $(\vec{\xi},\xi_o)=(\vec{\xi}^*,\xi_o^*)$ we have a root of

$$\vec{G}(\vec{\xi},\xi_o)=0 \tag{3.2}$$

Definition: Given H, a smooth mapping of a Banach space B into itself, and a solution x_o of

$$H(x_o)=0$$

then this root is isolated iff the Fréchet derivative $H_x(x_o)$ is non-singular.

This definition applied to our situation characterizes isolation in terms of the Jacobian of (3.2) at $(\vec{\xi}^*,\xi_0^*)$ If we define

$$2J = \left(\frac{\partial g_i}{\partial \xi_j}\right)\Bigg|_{(\vec{\xi}\ \xi_0)=(\vec{\xi}^*,\xi_0^*)} \tag{3.3}$$

and recall that $a_{ijk} = a_{ikj}$ we may write

$$J = \begin{bmatrix} A(\vec{\xi}^*) + \xi_0^* B & B\vec{\xi}^* + \xi_0^* \vec{c} \\ \vec{\xi}^{*T} & 2\xi_0^* \end{bmatrix} \tag{3.4}$$

Then if $\hat{J} \equiv |J|$ we find a root is isolated if $\hat{J} \neq 0$. This condition clearly demands non-singularity of an m+1 dimensional square matrix. This can be simplified slightly by an application of Lemma I. We state this in

<u>Lemma 3.1</u> Let $(\vec{\xi},\xi_0) = (\vec{\xi}^*,\xi_0^*)$ be a root of the Algebraic Bifurcation Equations (3.2). This root is isolated iff

(i) $A(\vec{\xi}^*) + \xi_0^* B$ is non-singular

or (ii) $\xi_0^* = 0$ is a simple eigenvalue of $A(\vec{\xi}^*)$ and (3.5)

$B\vec{\xi}^* \notin R(A(\vec{\xi}^*))$

<u>Proof</u>: We will suppress the * for convenience and will indicate the various requirements by referring to the notation of Lemma I in Chapter I.

First suppose $A(\vec{\xi}) + \xi_0 B$ is non-singular, i.e., case (1). To verify (a) we note

$$(A(\vec{\xi}) + \xi_0 B)\vec{\xi} = -\xi_0(B\vec{\xi} + \xi_0\vec{c})$$

$$\text{and so } \xi_0 \neq 0 \quad ,$$

since otherwise $A(\vec{\xi}) + \xi_0 B$ would be singular. Further

$$D - C^* A^{-1} B = 2\xi_0 + \frac{\vec{\xi}^T\vec{\xi}}{\xi_0} = \frac{1}{\xi_0} \neq 0 \text{ so } \hat{J} \neq 0 \text{ .}$$

Now suppose case (2) and (b) holds ($\gamma=1$ for our problem). Then from (3.6) (c_1) is violated unless $\xi_0=0$. Thus, $\vec{\xi}\neq 0$ is the unique eigenvector of $A(\vec{\xi})$ for the eigenvalue zero. If $B\vec{\xi}\notin R(A(\vec{\xi}))$ (c_1) and hence (c_0) are satisfied. Since $\vec{\xi}\neq 0$ (c_2) is satisfied and if zero is a simple eigenvalue (c_3) is fulfilled. This exhausts all possibilities since by case (3), if dim $N(A(\vec{\xi})+\xi_0 B)>1$ then $\hat{J} = 0$.

In the same manner one can find conditions guaranteeing isolation of a root of the higher order bifurcation equations. We indicate this for the cubic equations. Defining

$$J_3 = \left(\frac{\partial g_i}{\partial \xi_j}\right)\Bigg|_{(\vec{\xi}\ \xi_0)=(\vec{\xi}^*,\xi_0^*)} \tag{3.7}$$

where the $g_i(\vec{\xi},\xi_0)$ of (3.1) are replaced by their equivalents in (2.34). In this case we find

$$J_3 = \left[\begin{array}{c:c} A(\cdot,\vec{\xi}^*,\vec{\xi}^*) + 2A(\vec{\xi}^*,\vec{\xi}^*,\cdot) + 3B\ddot{\vec{\lambda}}(o) & 3B\vec{\xi}^* \\ \hdashline 2\vec{\xi}^{*T} & 0 \end{array}\right] \tag{3.8}$$

where $A(\cdot,\vec{\xi}^*,\vec{\xi}^*)$: $\mathbb{R}^m \to \mathbb{R}^m$ is the obvious linear operator.

If we are in the case where A is symmetric in all variables, we have $(\vec{\xi}^*,\xi_0^*)$ is isolated iff

(i) $A(\vec{\xi}^*,\vec{\xi}^*,\cdot) + \ddot{\lambda}(o)B$ is non-singular

or

(ii) $\ddot{\lambda}(o) = 0$ is a simple eigenvalue of $A(\vec{\xi}^*,\vec{\xi}^*,\cdot)$
and $B\vec{\xi}^* \notin R(A(\vec{\xi}^*,\vec{\xi}^*,\cdot))$.

This can be proved in the same manner as Lemma (3.1).

If we turn to the special case of a simple eigenvalue the Algebraic Bifurcation Equations reduce to

$$\begin{aligned} a_{111}\xi_1^2 + 2b_{11}\xi_1\xi_o + c_1\xi_o^2 &= 0 \\ 2\xi_o^2 + \xi_1^2 &= 1 \end{aligned} \tag{3.9}$$

and we see that Lemma (3.1) states that isolation is equivalent to

$$\begin{aligned} & a_{111}\xi_1 + \xi_o b_{11} \neq 0 \\ & \text{or } \xi_o = 0 \text{ and } b_{11} \neq 0 \end{aligned} \tag{3.10}$$

The second of equations (3.9) forces one of $(\xi_o\xi_1)$ not zero, say $\xi_o \neq 0$ and define $X = \xi_1/\xi_o$, then we need

$$a_{111}X^2 + 2b_{11}X + c_1 = 0 \tag{3.11}$$

which is a quadratic equation with real coefficients. Hence, if (3.11) has a real isolated solution, it must have a second real solution. Hence, if moving along a solution arc, one encounters a simple eigenvalue one need only check the known (from the assumed arc) solution of (3.9) for isolation to determine bifurcation. We will see in section (6) that for a semi-simple eigenvalue this is not in general correct.

We note that if (o,λ) is the known branch then $\xi_o = 1/\sqrt{2}$, $\xi_1 = 0$ is a root of (3.9) and this is isolated iff $b_{11} \neq 0$, i.e.,

$$\psi^* G^o_{u\lambda}\phi \neq 0 \tag{3.12}$$

which is the commonly quoted bifurcation condition.

(4) Bifurcation by Perturbation

Section (2) determines the solution of the Algebraic Bifurcation Equations as a necessary condition for the existence of a smooth solution of

$$\begin{aligned} G(u(\varepsilon),\ \lambda(\varepsilon)) &= 0 \\ (u(o),\lambda(o)) &= (u_o,\lambda_o) \end{aligned} \tag{4.1}$$

The parameter ε, however, was not explicitly defined. Section (3) is concerned with conditions under which a root may be isolated. In what follows we will describe a formal regular perturbation procedure for determining a solution of (4.1) and give an appropriate definition of ε. In addition, it will be shown that isolation of a root of the A.B.E.'s is sufficient to allow the determination of the perturbation series to any order.

Under the assumptions (2.2-6) define

$$\Phi_i = (\phi_i, 0) \quad i = 1,..m$$
$$\Phi_o = (\phi_o, 1) \tag{4.2}$$

and define the subspace $N_1 \subseteq B_1$ by

$$N_1 = \text{span}\,(\Phi_i, \quad i=0,...m) \tag{4.3}$$

We define P as the projection on B_1 with range N_1 and introduce an inner product on N_1 by

$$u = \alpha_o\Phi_o + \ldots + \alpha_m\Phi_m$$

$$v = \beta_o\Phi_o + \ldots + \beta_m\Phi_m$$

$$\langle u,v\rangle = \alpha_o\beta_o\langle(\phi_o,1),(\phi_o,1)\rangle + \ldots + \alpha_m\beta_m\langle(\phi_m,0),(\phi_m,0)\rangle$$

$$\equiv 2\alpha_o\beta_o + \alpha_1\beta_1 + \ldots \alpha_m\beta_m$$

Since the Φ_i are independent, this is a valid inner product which induces a norm equivalent to the original norm on B_1 when restricted to N_1. We use this to define ε as the norm of the projection of the solution in N_1. That is we attempt to solve

$$\left.\begin{array}{ll} G(u(\varepsilon),\lambda(\varepsilon)) = 0 & \text{(a)} \\ ||P((u(\varepsilon),\lambda(\varepsilon)) - (u_o,\lambda_o))||_{N_1} = \varepsilon & \text{(b)} \end{array}\right\} \quad (4.5)$$

where $||\ \ ||_{N_1}$ is the norm in N_1. We assume a solution of the form

$$\lambda(\varepsilon) = \lambda_o+\varepsilon\dot{\lambda}(o)+\frac{\varepsilon^2}{2}\ddot{\lambda}(o)+\ldots$$

$$u(\varepsilon) = u_o+\varepsilon\dot{u}(o)+\frac{\varepsilon^2}{2}\ddot{u}(o)+\ldots \quad (4.6)$$

Placing this into (4.5) we get equations (2.9), (2.10), (2.24), (2.25), (2.26) up to the fourth order in ε. As before

$$\dot{u}(o) \equiv \sum_{j=0}^{m} \xi_j\phi_j \qquad \dot{\lambda}(o) = \xi_o \quad (4.7)$$

and existence of a solution to (2.10) results in the Algebraic Bifurcation Equations. We suppose $(\vec{\xi},\xi_o)$ is an isolated root of (2.17). We see that to this order (4.5) is satisfied. Under these conditions the solution of

$$G_u^o\ddot{u}+G_\lambda^o\ddot{\lambda} = -(G_{uu}^o\dot{u}(o)\dot{u}(o)+2G_{u\lambda}^o\dot{u}(o)\dot{\lambda}(o)+G_{\lambda\lambda}\dot{\lambda}^2(o)) \quad (4.8)$$

may be written as

$$\ddot{u}(o) = \sum_{j=0}^{m} \xi_j^1\phi_j + w_1 \quad (4.9)$$

$$\xi_o^1 = \ddot{\lambda}(o)$$

where w_1 has no component in X_1 and is uniquely determined but the ξ_j^1 are arbitrary. If we use this expression in (2.25) then the existence of a solution at third-order forces

$$(A(\vec{\xi})+\xi_o B)\vec{\xi}_1 + (B\vec{\xi}+\xi_o\vec{c})\xi_o^1 = \vec{k}_1(\vec{\xi},\xi_o,w_1) \tag{4.10}$$

where $\vec{\xi}_1 \equiv (\xi_1^1,\ldots,\xi_m^1)^T$ and $\vec{k}_1$ is a known vector depending on derivatives of G and the indicated arguments. In addition we see, where $u_2(\varepsilon) = u_o+\varepsilon\dot{u}(o)+\frac{\varepsilon^2}{2}\ddot{u}(o)$

$$||P[(u_2(\varepsilon),\lambda_2(\varepsilon))-(u_o,\lambda_o)]||_{N_1} = \varepsilon(\vec{\xi}^T\vec{\xi}+2\xi_o^2)+\varepsilon^2(\vec{\xi}^T\vec{\xi}_1+2\xi_o\xi_o^1)$$

and so to satisfy (4.5)(b) to this order we demand

$$\vec{\xi}^T\vec{\xi}_1 + 2\xi_o\xi_o^1 = 0 \tag{4.11}$$

We can rewrite (4.10), (4.11) using the definition (3.3) as (with $k_1^o = 0$)

$$J\begin{bmatrix}\vec{\xi}_1\\ \xi_o^1\end{bmatrix} = \begin{bmatrix}\vec{k}_1\\ k_1^o\end{bmatrix} \tag{4.12}$$

Now since J is assumed non-singular we may uniquely determine $(\vec{\xi}_1,\xi_o^1)$. To show this process is the same at each step we consider

$$G_u(u(\varepsilon),\lambda(\varepsilon))\frac{du}{d\varepsilon}+G_\lambda(u(\varepsilon),\lambda(\varepsilon))\frac{d\lambda}{d\varepsilon} = 0 \qquad (4.13)$$

and differentiate w.r.t. ε (n-1) times to find

$$G_u(\varepsilon)u^{(n)} + G_\lambda(\varepsilon)\lambda^{(n)} = r_n(\varepsilon) \qquad \text{(a) (4.14)}$$

where
$$r_n(\varepsilon) = -\left[(n-1)\left(\frac{dG_u}{d\varepsilon}\, u^{(n-1)}+\frac{dG_\lambda}{d\varepsilon}\, \lambda^{(n-1)}\right) + H_o(u^{(1)},\ldots u^{(n-2)}, \lambda^{(1)},\ldots\lambda^{(n-2)}) + \frac{d^{(n-1)}G_u}{d\varepsilon^{(n-1)}}\, u^{(1)}+\frac{d^{(n-1)}G_\lambda}{d\varepsilon^{(n-1)}}\, \lambda^{(1)} \right] \qquad \text{(b)}$$

But

$$\frac{d^{(n-1)}G_u(\varepsilon)}{d\varepsilon^{(n-1)}} = G_{uu}u^{(n-1)}+G_{u\lambda}\lambda^{(n-1)}+H_1(u^{(1)},\ldots u^{(n-2)},\lambda^{(1)},\ldots\lambda^{(n-2)}) \qquad \text{(c)}$$

$$\frac{d^{(n-1)}G_\lambda(\varepsilon)}{d\varepsilon^{(n-1)}} = G_{u\lambda}u^{(n-1)}+G_{\lambda\lambda}\lambda^{(n-1)}+H_2(u^{(1)},\ldots u^{(n-2)},\lambda^{(1)},\ldots\lambda^{(n-2)}) \qquad \text{(d)}$$

$$\frac{dG_u}{d\varepsilon} = G_{uu}u^{(1)} + G_{u\lambda}\lambda^{(1)} \qquad \text{(e)}$$

$$\frac{dG_\lambda}{d\varepsilon} = G_{u\lambda}u^{(1)} + G_{\lambda\lambda}\lambda^{(1)} \qquad \text{(f)}$$

Collecting (4.14) (a)-(f) we find

$$G_u u^{(n)}+G_\lambda\lambda^{(n)}=-\Big[n\Big((G_{uu}u^{(1)}+G_{u\lambda}\lambda^{(1)})u^{(n-1)}+(G_{u\lambda}u^{(1)}+G_{\lambda\lambda}\lambda^{(1)})\lambda^{(n-1)}\Big) + H(u^{(1)},\dots u^{(n-2)},\ \lambda^{(1)},\dots\lambda^{(n-2)})\Big] \qquad (4.15)$$

From the structure of (4.15) we see that we can write

$$u^{(n+1)}(o) = \sum_{j=0}^{m} \xi_j^n\phi_j + w_n$$

$$\psi_j^* w_n = 0 \quad j = 1,\dots m \qquad (4.16)$$

where w_n is the unique solution of a compatible $(n+1)^{\underline{st}}$ equation and the ξ_j^n are determined by making the $(n+2)^{\underline{nd}}$ equation compatible, i.e.,

$$(A(\vec{\xi})+\xi_o B)\vec{\xi}_n + (B\vec{\xi}+\xi_o\vec{c})\xi_o^n = \vec{k}_n(\vec{\xi},\vec{\xi}_1,\dots\vec{\xi}_{(n-1)},w_1,\dots w_n) \qquad (4.17)$$

and we satisfy (4.5)(b) to this order by forcing

$$\vec{\xi}^T\vec{\xi}_n+2\xi_o\xi_o^n = k_o^n \equiv -\frac{n!}{2}\left(\sum_{j=1}^{n-1}(\vec{\xi}_j\vec{\xi}_{(n-j)}+2\xi_o^j\xi_o^{(n-j)})\Big/(j!)(n-j!)\right) \quad (4.18)$$

and once again we rewrite (4.17), (4.18) as

$$J\begin{bmatrix}\vec{\xi}_n \\ \xi_o^n\end{bmatrix} = \begin{bmatrix}\vec{k}_n(\vec{\xi},\ldots\vec{\xi}_{(n-1)}) \\ k_o^n(\vec{\xi},\ldots\vec{\xi}_{(n-1)})\end{bmatrix} \quad (4.19)$$

Thus we see that an isolated root of the Algebraic Bifurcation Equations is a sufficient condition for the existence of a unique perturbation expansion of (4.5) to any arbitrary order. The form of the solution is given by

$$u_n(\varepsilon) = \varepsilon\left(\sum_{j=0}^{m} \xi_j(\varepsilon)\phi_j\right) + \varepsilon^2 w(\varepsilon)$$

$$\lambda_n(\varepsilon) = \xi_o(\varepsilon) \quad (4.20)$$

where $\psi_j^* w(\varepsilon) = 0 \quad j = 1,\ldots m$

and $\xi_j(\varepsilon) = \xi_j + \frac{\varepsilon}{2}\xi_j^1 + \ldots$ with $|\,\vec{\xi}(\varepsilon)^T\vec{\xi}(\varepsilon) + 2\xi_o(\varepsilon)^2| = 1$ (4.21)

We note the especially simple case of bifurcation from the trivial state. Here N_1 = span $(\phi_1, 0)$ and $\|\ \|_{N_1}$ coincides with the norm in B_1

(if $||\phi_1|| = 1$) and our solution is

$$u_n(\varepsilon) = u_o+\varepsilon(\phi_1+\varepsilon w(\varepsilon))$$

$$\lambda_n(\varepsilon) = \lambda_o+\varepsilon\dot{\lambda}(o)+\ldots \tag{4.22}$$

$$\psi_1^* w(\varepsilon) = 0$$

and here ε is precisely the norm of the component of $u_n(\varepsilon)-u_o$ in the null space of G_u^o. Even in the semi-simple case if B_1 has a Hilbert space structure we may choose our Φ_j to be orthonormal and then once again $||P(\ )||_{N_1} = ||\ \ ||_{B_1}$ and so here also ε becomes the norm of solution in the eigenspace of $G_u^o + G_\lambda^o$.

The perturbation expansions derived in this section are in general not convergent to an actual solution. However, if $(u(\varepsilon),\lambda(\varepsilon))$ is the exact solution (guaranteed by section (7)), it is possible to show these expansions are asymptotic to $(u(\varepsilon),\lambda(\varepsilon))$. That is one may write

$$u(\varepsilon) = u_n(\varepsilon) + \varepsilon^{(n+1)}V_n(\varepsilon)$$

$$\lambda(\varepsilon) = \lambda_n(\varepsilon) + \varepsilon^{(n+1)}\mu(\varepsilon)$$

and by using isolation and contracting mapping techniques one may show $V_n(\varepsilon)$ and $\mu(\varepsilon)$ are uniformly bounded for $|\varepsilon|\leq\varepsilon_o$.

We conclude this section by quoting a Lemma which could be applied to give (4.15) and to indicate some of the structure of $H(u^{(1)},\ldots u^{(n-2)},\lambda^{(1)},\ldots\lambda^{(n-2)})$

Lemma (4.1) Let $G(u,\lambda)$ and $u(\varepsilon),\lambda(\varepsilon)$ have M continuous derivatives in some neighborhood of $G(u(o),\lambda(o))$ and $\varepsilon=0$, respectively. Then for some $\varepsilon_0>0$ and all $|\varepsilon|\leq\varepsilon_0$ and for $n = 1,2,\ldots M$

$$D^nG(u(\varepsilon),\lambda(\varepsilon)) = \sum_{k=1}^{n} \sum_{k_1+k_2=k} G_{k_1}^{k_2} h(k_1,k_2,n;\varepsilon)$$

$$h(k_1,k_2,n;\varepsilon)\equiv \sum_{|\vec{\gamma}(k_1,k_2,n)|=n} A(\vec{\gamma}(k_1,k_2,n))\prod_{i=1}^{k_1} D^{\alpha_i(k_1,n)}u(\varepsilon)\prod_{i=1}^{k_2} D^{\beta_i(k_2,n)}\lambda(\varepsilon)$$

where $$D\equiv\frac{d}{d\varepsilon}\ ,\quad G_{k_1}^{k_2}\equiv G_{\underbrace{u\ldots u}_{k_1}\ \underbrace{\lambda\ldots\lambda}_{k_2}}(u(\varepsilon),\lambda(\varepsilon))$$

$A(\vec{\gamma}(k_1,k_2,n)$ are positive integers and $\vec{\gamma}(k_1,k_2,n)$ is the multi-index with $k_1+k_2=k$ positive integer coefficients

$$\vec{\gamma}(k_1,k_2,n) = (\alpha_1(k_1,n),\ldots\alpha_{k_1}(k_1,n);\ \beta_1(k_2,n),\ldots\beta_{k_2}(k_2,n))$$

and $|\vec{\gamma}(k_1,k_2,n)| = \alpha_1 + \ldots\alpha_{k_1} + \beta_1 + \ldots\beta_{k_2}$

This Lemma is a generalization of Lemma (3.14) of Keller and Langford [20] and can be proved in a straightforward though laborious manner by induction.

(5) Normal Limit Points: Simple and Multiple

In this section we continue our study of solutions (2.1) in the neighborhood of a singular point of G_u of the type specified by (2.2)-(2.6). Early in section (2) we found that a solution of

$$G_u^o\dot{u}(o) + G_\lambda^o\dot{\lambda}(o) = 0 \qquad (5.1)$$

required

$$\dot{\lambda}(o)\vec{d} = 0$$
$$d_i \equiv \psi_i^* G_\lambda^o \quad i=1,\ldots m \qquad (5.2)$$

which presented two possible cases; (i) $\vec{d}=0$ or (ii) $\vec{d}\neq 0$ but $\dot{\lambda}(o)=0$. The first case was pursued in section (2) and led to the Algebraic Bifurcation Equations. In this section we shall consider case (ii). The point (u_o,λ_o) at which $\dot{\lambda}(o)=0$ but $G_\lambda^o \notin R(G_u^o)$ is called a normal limit point, simple if m=1, and multiple if m>1. Simple and multiple normal limit points exhibit qualitatively different behaviour. We shall see that there is a unique solution arc through any simple normal limit point. A multiple normal limit point, however, may exhibit bifurcation.

If we have (u_o,λ_o) a simple normal limit point and $\ddot{\lambda}(o)\neq 0$ then in a neighborhood of λ_o there are either two solutions or none. That is,

λ_o is the limiting value of λ for which a solution (on this branch) exists. The terminology is unfortunate, however, since the points A_o and C_o of Figure (5.1) are also normal limit points.

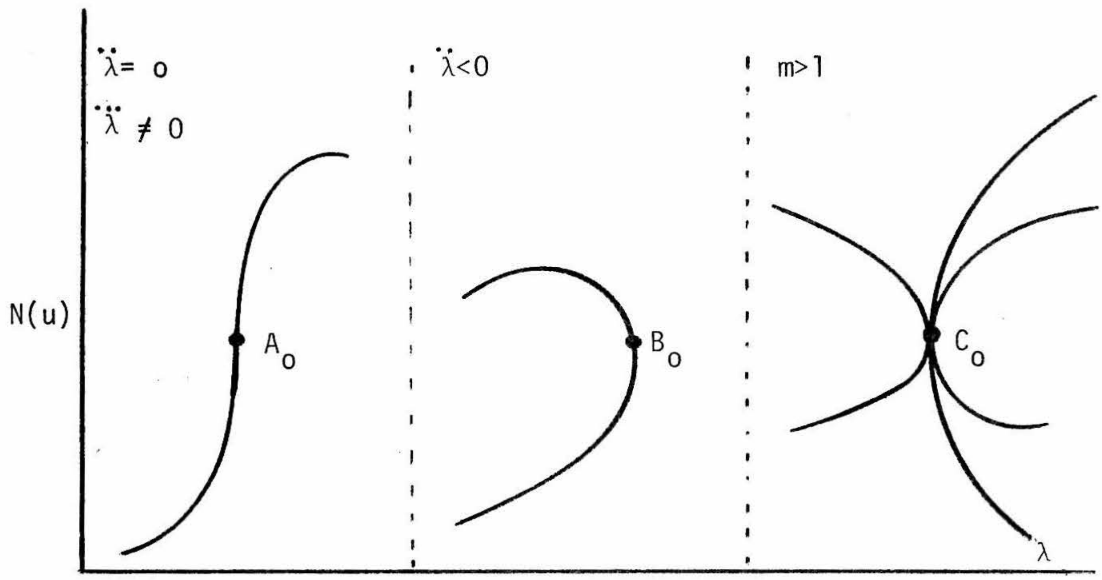


Figure 5.1

We now proceed to derive the equations which must be satisfied by any smooth solution arc through a normal limit point. With $\dot{\lambda}(o)=0$ Equations (2.9), (2.10) and (2.24) become

$$G_u^o\dot{u}(o) = 0 \tag{5.3}$$

$$G_u^o\ddot{u}(o)+G_\lambda^o\ddot{\lambda}(o) = -G_{uu}^o\dot{u}(o)\dot{u}(o) \tag{5.4}$$

$$G_u^o\dddot{u}(o)+G_\lambda^o\dddot{\lambda}(o) = -3(G_{uu}^o\ddot{u}\dot{u}+G_{u\lambda}^o\dot{u}\ddot{\lambda})-G_{uuu}^o\dot{u}(o)\dot{u}(o)\dot{u}(o) \tag{5.5}$$

The solution of (5.3) is given by

$$\dot{u}(o) = \sum_{j=1}^{m} \xi_j \phi_j \tag{5.6}$$

and existence of a solution $\ddot{u}(o)$ requires

$$A(\vec{\xi})\vec{\xi} + \ddot{\lambda}(o)\vec{d} = 0 \qquad \text{(a)} \quad (5.7)$$

In addition we may impose the normalization (4.5)(b) which forces

$$\xi_o = 0 \quad , \quad \vec{\xi}^T\vec{\xi} = 1 \qquad \text{(b)} \quad (5.7)$$

Each root of (5.7) (a),(b) gives a candidate for a solution arc. It will be shown in section (7) that each isolated root of (5.7) generates a solution branch through (u_o,λ_o). The Jacobian of the above system is given by

$$\hat{J} = |J| = \left| \begin{bmatrix} 2A(\vec{\xi}) & \vdots & \vec{d} \\ \cdots & \vdots & \cdot \\ 2\vec{\xi}^T & \vdots & 0 \end{bmatrix} \right| \tag{5.9}$$

Using this expression we can state conditions for the isolation of a root of the Limit Point Bifurcation Equations as;

<u>Lemma 5.1</u> Let $(\vec{\xi}^*,\ddot{\lambda}(o))$ be a root of (5.7)

This root is isolated iff

(i) $A(\vec{\xi}^*)$ is non-singular

or (ii) $\ddot{\lambda}(o) = 0$, zero is a simple eigenvalue of $A(\vec{\xi}^*)$, and $\vec{d} \notin R(A(\vec{\xi}^*))$.

Proof: Isolation is equivalent to non-singularity of J. We indicate the various requirements by referring to the notation of Lemma I in Chapter I.

Suppose we are in case (1), i.e., $A(\vec{\xi}^*)$ is non-singular, then $\ddot{\lambda}(o) \neq 0$ and hence

$$D - C^*A^{-1}B = \frac{\vec{\xi}^T\vec{\xi}}{\ddot{\lambda}(o)} = \frac{1}{\ddot{\lambda}(o)} \neq 0$$

and J is non-singular. Now suppose case (2) with $\gamma = 1$. Then from (5.7) (c_1) is violated unless $\ddot{\lambda}(o)=0$. Thus, $\vec{\xi}^*$ is the unique eigenvector of $A(\vec{\xi}^*)$ for the eigenvalue zero. Since $\vec{\xi}^* \neq 0$ (by 5.7) (c_2) is satisfied and if zero is a simple eigenvalue (c_3) is fulfilled. Thus, if $\vec{d} \notin R(A(\vec{\xi}^*))$ (c_1) is satisfied and J is non-singular. This exhausts the possibilities for isolation since by case (3) if $\dim N(A(\vec{\xi}^*))>1$ then $\hat{J}=0$.

Suppose we have a root $(\vec{\xi},\ddot{\lambda}(o))$ of (5.7), then the solution of (5.4) is

$$\ddot{u}(o) = \sum_{j=1}^{m} \gamma_j \phi_j + w \tag{5.10}$$

where w is the solution of (5.4) with $\psi_j^* w=0$ j=1,...m. Compatibility of equation (5.5) then forces

$$3A(\vec{\xi})\vec{\gamma} + \dddot{\lambda}(o)\vec{d} = \vec{k}(\vec{\xi},\ddot{\lambda}(o)) \qquad (a) \quad (5.11)$$

where $\vec{k}$ is a known vector. We add the corresponding normalization equation to this order

$$\vec{\xi}^T\vec{\gamma} = 0 \qquad (b) \quad (5.11)$$

and we see (5.11) can be written as

$$J \begin{bmatrix} \frac{3}{2}\vec{\gamma} \\ \\ \dddot{\lambda}(o) \end{bmatrix} = \begin{bmatrix} \vec{k} \\ \\ 0 \end{bmatrix} \qquad (5.12)$$

Hence, if J is non-singular we can uniquely determine $(\vec{\gamma},\dddot{\lambda}(o))$. It is a simple matter to show that isolation of a root $(\vec{\xi},\ddot{\lambda}(o))$ allows the determination of the terms in a perturbation series solution to arbitrary order.

It is also possible to arrive at higher order equations for a normal limit point. That is, if we assume

$$a_{ijk} = 0 \quad i,j,k = 1, \ldots m \qquad (5.13)$$

then the only solution of (5.7)(a) is $\ddot{\lambda}(o) = 0$, $\vec{\xi}$ arbitrary. This forces

$$\ddot{u}(o) = \sum_{i,j=1}^{m} \xi_i \xi_j v_{ij}$$

$$\text{where} \qquad G_u^o v_{ij} = G_{uu}^o \phi_i \phi_j \tag{5.14}$$

$$\psi_k^* v_{ij} = 0 \quad k = 1, \ldots m$$

and we place this in (5.5) to find

$$\begin{aligned} A(\vec{\xi},\vec{\xi},\vec{\xi}) + \dddot{\lambda}(o)\vec{d} = 0 \\ \dot{\lambda}(o) = \ddot{\lambda}(o) = 0 \qquad \vec{\xi}^T\vec{\xi}=1 \end{aligned} \tag{5.15}$$

Where the trilinear operator $A(\cdot,\cdot,\cdot)$ is given by (2.35). Isolation of roots of (5.15) can be handled in precisely the same manner as isolation of the roots of higher order bifurcation equations was considered in section (3).

We now consider the special case when the normal limit point is simple. Here equations (5.7) (a),(b) reduce to

$$\begin{aligned} a_{111}\xi_1^2 + \ddot{\lambda}(o)d_1 = 0 \\ \xi_1^2 = 1 \end{aligned} \tag{5.16}$$

Thus we have the solution

$$\ddot{\lambda}(o) = -a_{111}/d_1$$
$$\xi_1 = \pm 1 \tag{5.17}$$

This yields only one distinct solution since switching the sign of ξ_1 merely changes the sign of ε. This root is clearly isolated. Hence, if we apply the results of section (7) we find a unique solution arc through (u_o,λ_o). No bifurcation can occur at a simple normal limit point. In the case of higher order equations we find

$$\lambda^{(k)}(o) = -a_{\underbrace{11\ldots1}_{k+1}}/d_1 \tag{5.18}$$

to be the unique solution.

It will be found in section (6) that equations (5.7) may have as many as $2^m/2$ distinct isolated roots. For $m>1$ we may thus have several branches through (u_o,λ_o). All these branches, however, have $\frac{d\lambda}{d\varepsilon} = 0$ at (u_o,λ_o).

(6) Root Structure of the Algebraic Bifurcation Equations

From the work of the previous section it is clear that the solution of the Algebraic Bifurcation Equations is intimately tied to the solution of the full problem (2.1) in the neighborhood of a singular point. In this sense the reduction of an (in general) infinite dimensional problem to the solution of a finite set of polynomial equations is a tremendous simplification. Unfortunately these equations can exhibit a bewildering variety of possible solutions. This section will not attempt a comprehensive description of the possibilities. Rather several special examples will be considered which indicate the kind of results to be expected. An application of a fundamental theorem of algebraic geometry will be shown to give an upper limit to the number of possible isolated bifurcating branches.

To begin we recall the Quadratic Algebraic Bifurcation Equations as

$$i=1,\dots m \qquad f_i(\xi_o,\dots\xi_m) = \sum_{j,k=1}^{m} a_{ijk}\xi_j\xi_k + 2\sum_{j=1}^{m} b_{ij}\xi_j\xi_o + c_i\xi_o^2 \qquad \text{(a)}$$

$$f_o(\xi_o,\dots\xi_m) = 2\xi_o^2 + \dots\xi_m^2 - 1 \qquad \text{(b)}$$

$$(6.1)$$

As noted before f_i, $i=1, \dots m$ are m homogeneous equations in m+1 scalar variables and as such exhibit lines of roots in $\mathbb{R}^{m+1}$. That is if $(\xi_o,\dots\xi_m)$ is a root then $(\alpha\xi_o,\dots\alpha\xi_m)$ is a root for any $\alpha\in\mathbb{R}$. Equation (6.1) (b) excludes the identically zero root and removes most of the non-uniqueness. However, if $(\xi_o,\dots\xi_m)$ is a root $(-\xi_o,\dots-\xi_m)$ is still a root. This corresponds to the two tangent vectors $\vec{t}_\pm = \pm(\dot{u}(o),\dot{\lambda}(o))$, determined by the direction of approach to (u_o,λ_o). For our purposes these roots will <u>not</u> be considered distinct.

As a first case we consider the equations for a simple eigenvalue mentioned at the end of section (3). They are

$$a_{111}X^2+2b_{11}X+c_1 = 0 \quad X=\frac{\xi_1}{\xi_o} \quad \xi_o\neq 0$$

$$\text{or} \quad c_1X^2+2b_{11}X+a_{111} = 0 \quad X=\frac{\xi_o}{\xi_1} \quad \xi_1\neq 0 \qquad (6.2)$$

In either case we clearly have a quadratic equation in one variable giving as possible real solutions either 2 isolated roots, one double root or no roots. Viewed as an equation over $\mathbb{C}$ we always have two roots (counting multiplicities). A theorem of Bézout indicates that this behaviour is carried over to the case where $m>1$. For any point $(\xi_o,\ldots\xi_m)\in\mathbb{C}^{m+1}$ we associate the line through the origin $(\beta\xi_o,\ldots.\beta\xi_m)$ $\forall\beta\in\mathbb{C}$, with a point in the m-dimensional complex projective plane $\mathbb{C}P^m$. Then one of the first results of algebraic geometry states (Abhyankar [1]).

<u>Theorem</u>: (Bézout-1770): Let $f_i(\xi_o,\ldots\xi_m) = 0$ $i = 1,2,\ldots m$ be m algebraic equations in the m+1 complex variables $\xi_o,\ldots\xi_m$, with the i^{th} equation homogeneous of degree m_i. Suppose the f_i have no common intersection component. Then these equations have

$$M = \prod_{i=1}^{m} m_i$$

roots in $\mathbb{C}P^m$. (counting multiplicities!)

The multiplicity of a root is a delicate concept. But for our purposes it is sufficient to note that a root has multiplicity one iff it is isolated. A common intersection component is loosely a one parameter family of roots. As an example, suppose

$$f_i(\xi_o,\ldots\xi_m) = \left(\sum_{o}^{m}\alpha_j\xi_j\right)\left(\sum_{o}^{m}\beta_j^{(i)}\xi_j\right)$$

that is, each quadratic can be factored into two linear terms. The plane $0=g(\xi_o,\ldots\xi_m) = \left(\sum_{o}^{m}\alpha_j\xi_j\right)$ is a common intersection component If we apply this result to the equations (6.1)(a) we immediately deduce

<u>Lemma (6.1)</u> The quadratic algebraic bifurcation equations arising from a multiplicity m semi-simple eigenvalue can have at most 2^m isolated real roots.

For m=1 this clearly reduces to our known result for a simple eigenvalue. We can use the structure of (6.1)(a) to say a bit more in the quadratic case.

<u>Lemma (6.2)</u> The number of real roots (counting multiplicities and <u>assuming</u> no roots from common intersection components) for the quadratic bifurcation equations is even.

<u>Proof</u>: The coefficients of equations (6.1)(a) are real. If there is no common intersection component, then there are 2^m roots in $\mathbb{C}P^m$. Hence, if P_o is a root $\overline{P}_o$ (complex conjugate) is a root. Since the total number of roots is even and real roots are fixed by conjugation, we see the number of real roots is even (counting multiplicities).

For the case of a simple eigenvalue these two lemmas show that existence of 1 real isolated root forces the existence of a second isolated real root and hence, bifurcation. For m>1 this is not so simple. For example, if m=2, we have 4 roots in $\mathbb{C}P^2$, if we know the existence of 1 isolated real root (from a known branch) we could either have a second isolated real root or a real root of multiplicity 3. In the second instance, the existence of bifurcation could not be claimed.

If the problem in question requires the consideration of the cubic bifurcation equations (2.34), then we may alternately write $\ddot{\lambda}(o)=\pm\gamma^2$ to get 2 sets of m equations homogeneous of degree 3.

$$A(\vec{\xi},\vec{\xi},\vec{\xi}) + 3\gamma^2 B\vec{\xi} = 0 \qquad \text{(a)}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.3)$$
$$A(\vec{\xi},\vec{\xi},\vec{\xi}) \; -3\gamma^2 B\vec{\xi} = 0 \qquad \text{(b)}$$

We see that if $(\vec{\xi},\gamma)$ is a root of (6.3)(a) then $(\vec{\xi},i\gamma)$ is a root of (6.3)(b) Thus, all real roots of (2.34) are given by the real roots of (6.3)(a) plus roots $(\vec{\xi},i\gamma)$ of (6.3)(a) with $(\vec{\xi},\gamma)$ real. Here Bézouts Theorem would allow at most 3^m real isolated roots. Since 3^m is odd, it is possible to have a single real root. In the same way we write the quadratic limit point equations (5.7)(a) as

$$A(\vec{\xi})\vec{\xi} + \gamma^2\vec{d} = 0 \qquad (6.4)$$

and the real roots of (5.7)(a) are all roots of (6.4) of the form

$(\vec{\xi},\gamma)$ or $(\vec{\xi},i\gamma)$ for $(\vec{\xi},\gamma)$ real. The number of such roots can at most be 2^m. However, suppose we have a real root $(\vec{\xi},\ddot{\lambda}(o) = \gamma^2)$, then clearly $(-\vec{\xi},\ddot{\lambda}(o) = (-\gamma)^2)$ is also a root which appears to be distinct. As before these two roots correspond to the two tangent vectors $\vec{t}_{\pm}=\pm(\dot{u}(o),\dot{\lambda}(o))$ determined by the direction of approach to (u_o,λ_o). (Equivalently they correspond to different choices of the sign of ε.) They will not be considered distinct for our purposes and so the maximum number of isolated roots would be $2^m/2$.

All the previous analysis gives an upper bound on the number of isolated roots but does not guarantee the existence of any real roots of equations (6.1), (6.3) or (6.4). We now mention a technique which in special cases can prove existence of real roots of the Algebraic Bifurcation Equations.

Suppose our problem has $(u,\lambda) = (o,\lambda)$ as a solution for all λ. Then it is easy to see the bifurcation equations (6.1) become

$$A(\vec{\xi})\vec{\xi} + 2\xi_o B\vec{\xi} = 0 \qquad \text{(a)}$$

$$2\xi_o^2 + \vec{\xi}^T\vec{\xi} = 1 \qquad \text{(b)} \qquad (6.5)$$

and we see one root is given by $\xi_o = 1/\sqrt{2}$, $\vec{\xi} = 0$. (This is the root corresponding to the trivial branch). This root is isolated if B is non-singular. Assuming this, we can write (6.5)(a) as

$$T(\vec{\xi}) = \xi_o\vec{\xi} \qquad (6.6)$$

where

$$T(\vec{\xi}) = -\frac{1}{2}B^{-1}A(\vec{\xi})\vec{\xi} \tag{6.7}$$

Problems of the type (6.6) have been studied by Birkoff and Kellog [4] and Berger and Berger [3] under the name invariant direction problems. Keller [19], using the Birkhoff-Kellog theorem, has shown,

Lemma (6.3) Let B be non-singular and m-odd then (6.6) (and hence 6.5) has at least one root with $\vec{\xi}\neq 0$.

We now extend this result to bifurcation from the non-trivial state.

Lemma (6.4) Let $(\gamma_0,\vec{\gamma})$ be an isolated root of (6.1) with $\gamma_0 \neq 0$. Then if m is odd (6.1) has at least one root distinct from $(\gamma_0,\vec{\gamma})$.

Proof: We reduce our problem to that of Lemma (6.3).

If we consider

$$A(\vec{\gamma})\vec{\gamma} + 2\gamma_0 B\vec{\gamma} + \vec{c}\gamma_0^2 = 0$$

and

$$A(\vec{\eta})\vec{\eta} + 2\eta_0(A(\vec{\gamma}) + \gamma_0 B)\vec{\eta} = 0 \tag{6.8}$$

then it is easily seen that $\vec{\xi} = \vec{\eta} + \eta_0\vec{\gamma}$, $\xi_0 = \eta_0\gamma_0$ is a root of (6.1)(a) Thus, we may reduce the problem of finding a root of (6.1)(a) to finding a root of (6.8), which is in the form of bifurcation from the trivial state. By isolation and $\gamma_0 \neq 0$ we may write (6.8) (since then $A(\vec{\gamma})+\gamma_0 B$ is invertible),

$$T_1(\vec{\eta}) = \eta_o\vec{\eta}$$
$$T_1 \equiv -\frac{1}{2}(A(\vec{\gamma})+\gamma_o B)^{-1}A(\vec{\eta})\vec{\eta} \tag{6.9}$$

Applying Lemma (6.3) to (6.9) we find a root of (6.8) with $\vec{\eta}\neq 0$. All that remains is to show $(\xi_o,\vec{\xi})$ is distinct from $(\gamma_o,\vec{\gamma})$. Suppose $(\beta\gamma_o,\beta\vec{\gamma}) = (\xi_o,\vec{\xi})$ then $\beta\gamma_o = \xi_o = \eta_o\gamma_o$ so $\beta = \eta_o$ and then $\vec{\eta} = \vec{\xi} - \eta_o\vec{\gamma}$ $= (\beta-\eta_o)\vec{\gamma} = 0$, a contradiction. ///

We note that there is no guarantee that the roots provided by these lemmas are isolated. Also we see that if we assume no common intersection components, then Lemma (6.2) gives the above results for arbitrary m.

Further results based on invariant direction properties are possible if $T(\vec{\xi})$ of (6.6) is a gradient system. That is, when

$$T_i(\vec{\xi}) = \frac{\partial}{\partial\xi_i}\psi(\vec{\xi}) \qquad i=1,\ldots m \tag{6.10}$$

Langford [24] discusses several possibilities and notes that scalar self-adjoint two-point boundary value problems generally lead to gradient system bifurcation equations.

We conclude this section with two simple algebraic examples indicating some of the situations described above. First, we consider

$$G(u,\lambda) = \begin{bmatrix} \dfrac{u_1^2+u_2^2}{2} - \lambda \\ u_1u_2 \end{bmatrix} \tag{6.11}$$

Here $G_u^o = G_u(o,o) = \begin{pmatrix} o & o \\ o & o \end{pmatrix}$ and $G_\lambda^o = \begin{bmatrix} -1 \\ o \end{bmatrix} \notin R(G_u^o)$.

This is a multiple limit point situation and the bifurcation equations are easily seen to be

$$\begin{aligned} \xi_1^2 + \xi_2^2 - \ddot{\lambda} &= 0 \\ \xi_1\xi_2 &= 0 \qquad\qquad (6.12) \\ \xi_1^2 + \xi_2^2 &= 1 \end{aligned}$$

We have the maximum four isolated roots $(\xi_1,\xi_2,\lambda) = (\pm 1,0,1), (0,\pm 1,1)$ but we see this gives only two distinct tangents. These roots correspond to the two bifurcating solutions given by

$$(u^{(1)},\lambda^{(1)}) = (0,\sqrt{2\lambda},\lambda)$$

$$(u^{(2)},\lambda^{(2)}) = (\sqrt{2\lambda},0,\lambda)$$

and indicated in Figure (6.1) .

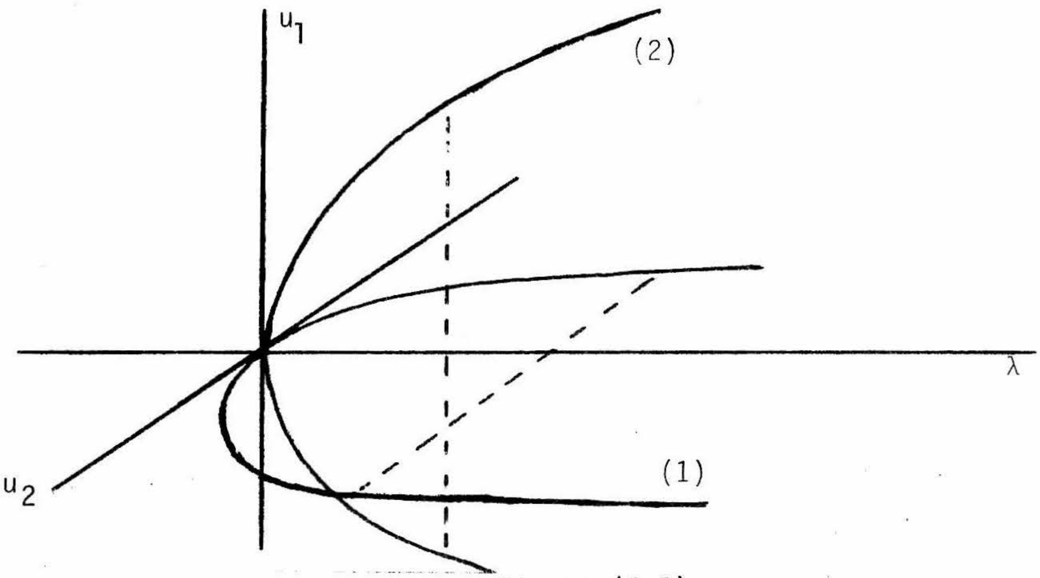


Figure (6.1)

Second, we consider a special case of a set of equations described by Stakgold [40].

$$G(u,\lambda) = \begin{bmatrix} u_1(\lambda-\gamma_1(u_1^2+u_2^2)) \\ \\ u_2(\lambda-\gamma_2(u_1^2+u_2^2)) \end{bmatrix} \gamma_i>0 \qquad (6.13)$$

Once again $G_u^o \equiv 0$ so we have a double eigenvalue. Here, however, the algebraic bifurcation equations are cubic;

$$\xi_1(2\gamma_1(\xi_1^2+\xi_2^2)-\ddot{\lambda}(o)) = 0 \qquad (a)$$

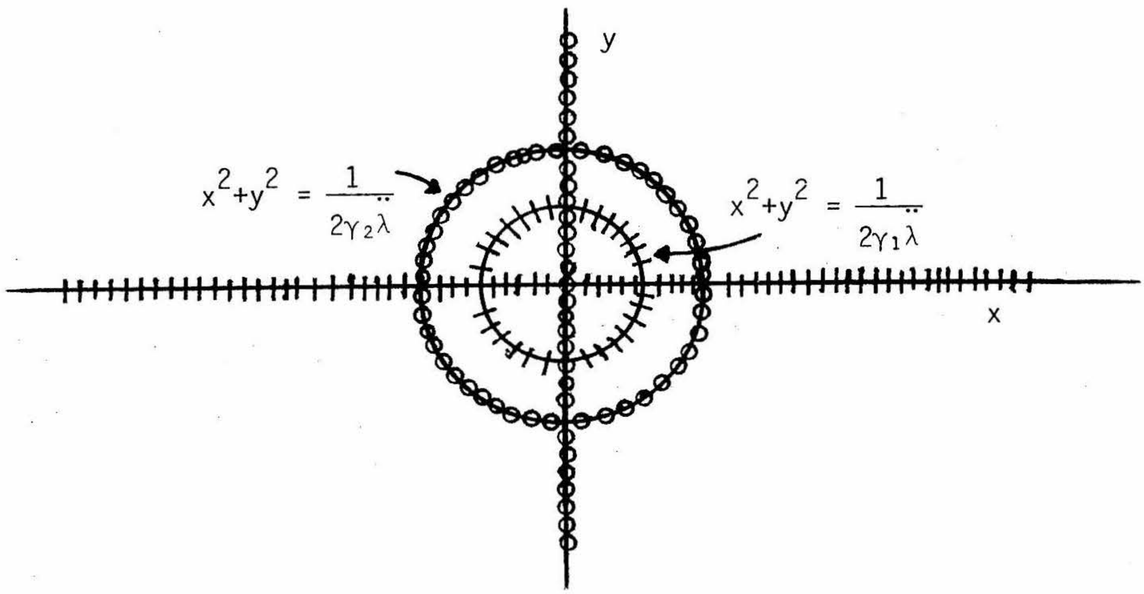
$$\xi_2(2\gamma_2(\xi_1^2+\xi_2^2)-\ddot{\lambda}(o)) = 0 \qquad (b) \qquad (6.14)$$

$$\xi_1^2+\xi_2^2 = 1 \qquad (c)$$

These equations have four roots if $\gamma_1 \neq \gamma_2$.

(i) $\xi_1=0$, $\xi_2=\pm 1$, $\ddot{\lambda}=2\gamma_2$ or (ii) $\xi_2=0, \xi_1=\pm 1$, $\ddot{\lambda}=2\gamma_1$.

If we define $X = \frac{\xi_1}{\ddot{\lambda}}$, $Y = \frac{\xi_2}{\ddot{\lambda}}$ then (6.14) (a),(b) are represented in Figure (6.2).



(a) = 0 ooo
(b) = 0 1111

Figure (6.2)

If $\gamma_1 \neq \gamma_2$ these four intersections are isolated and yield two bifurcating branches similar to Fig. (6.1). However, if $\gamma_1=\gamma_2$ the equations have a common intersection component. It turns out that each point on the co-incident circles yields a branch and hence, we have a sheet of bifurcating solutions indicated by Fig. (6.3).

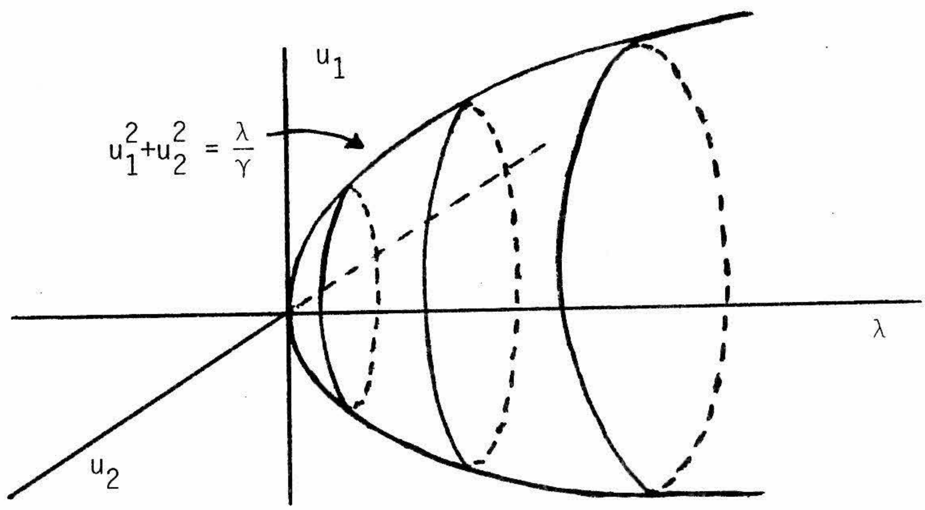


Figure (6.3)

In general neither multiple roots nor intersection components can be extended to bifurcating solutions. Often, however, the original equation may be invariant under some group action. In this case Sattinger [37] has shown the Algebraic Bifurcation Equations also exhibit this invariance. Indeed all the higher order equations are invariant. This invariance can easily be seen to generate a common intersection component. For this situation it is often possible that any point of the component will yield a bifurcating branch, and the group action will then generate a sheet of solutions. Whether this is always true is at present not known. This problem has received some consideration by Ruelle [33] and recently by Büchner, Marsden and Schecter [5] .

Although bifurcation in the presence of a symmetry group is made difficult because the roots of the bifurcation equations are not isolated, this situation may also introduce simplifications. The Algebraic

Bifurcation Equations of order m in general involve $O(m^3)$ independent coefficients (a_{ijk}, etc.). Sattinger [37] and [39] has shown that consideration of the irreducible representations of the group under study can lead to a dramatic reduction in the number of these independent coefficients.

A different situation occurs when the lowest order Algebraic Bifurcation Equations have an intersection component due to some invariance which is not shared by the higher order equations. In this case one cannot expect each point of the component to yield a branch.

(7) Existence of Multiple Bifurcation Branches

Most of the work of the previous sections considered formal solutions of

$$G(u(\varepsilon),\lambda(\varepsilon)) = 0 \qquad (7.1)$$

depending smoothly on a parameter ε. If a smooth solution $(u(\varepsilon),\lambda(\varepsilon))$ of (7.1) was assumed to exist then its local structure was determined by our analysis of sections (2), (4), and (5). This approach will be justified in this section in the following manner. For each isolated root $(\vec{\xi},\xi_o)$ of the Algebraic Bifurcation Equations (2.17) we will show the existence of a smooth branch of solutions bifurcating from (u_o,λ_o) with its local structure determined by the root $(\vec{\xi},\xi_o)$. In addition, if $G^o_\lambda \notin R(G^o_u)$ we will show that each isolated root of the Limit Point Bifurcation Equations (5.7) generates a smooth solution arc $(u(\varepsilon),\lambda(\varepsilon))$ through (u_o,λ_o). These results will be presented separately in Theorems (7.1) and (7.2). The existence proof in both cases will rely upon applications of the Basic Lemma I.

(H1): We suppose $G(u,\lambda)$ is twice continuously differentiable with respect to both u and λ in some neighborhood of (u_o,λ_o). We assume that at (u_o,λ_o) the linear operator $G_u(u_o,\lambda_o)$ satisfies (2.2) - (2.6). That is, G^o_u is a Fredholm operator of index zero with an m-fold semi-simple zero eigenvalue. Also assume $G^o_\lambda \varepsilon R(G^o_u)$.

(H2) Suppose $(\vec{\xi}^*, \xi_0^*)$ is an isolated root of the quadratic Algebraic Bifurcation Equations. That is

$$A(\vec{\xi}^*)\vec{\xi}^* + 2\xi_0^* B\vec{\xi}^* + \vec{C}\xi_0^{*2} = 0$$

$$2\xi_0^{*2} + \vec{\xi}^{*T}\vec{\xi}^* = 1 \qquad (7.2)$$

and

$$\hat{J} = |J| = \left| \begin{bmatrix} A(\xi^*) + \xi_0^* B & \vdots & B\vec{\xi}^* + \xi_0^* \vec{C} \\ \cdots & \vdots & \cdots \\ \vec{\xi}^{*T} & \vdots & 2\xi_0^* \end{bmatrix} \right| \neq 0 \qquad (7.3)$$

Under the assumptions (H1),(H2) we construct the unique solution v_0 of

$$\left.\begin{aligned} &G_u^o v_o + \tfrac{1}{2}(G_{uu}^o \phi\phi + 2G_{u\lambda}^o \phi\xi_o^* + G_{\lambda\lambda}^o \xi_o^{*2}) = 0 && \text{(a)} \\ &\psi_i^* v_o = 0 \quad i=1,\dots m && \text{(b)} \\ &\phi \equiv \sum_{j=0}^{m} \xi_j^* \phi_j && \text{(c)} \end{aligned}\right\} \quad (7.4)$$

From this root $(\vec{\xi}^*, \xi_0^*)$ and solution v_0 we construct a branch emanating from (u_0, λ_0).

Theorem (7.1) Let (H1),(H2) hold and let v_0 be the unique solution of (7.4). Then $\exists \varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ there exists a unique $\phi(\varepsilon)$, $v(\varepsilon)$ for which

$$G(u_0 + \varepsilon(\phi(\varepsilon) + \varepsilon v(\varepsilon)), \lambda_0 + \varepsilon \xi_0(\varepsilon)) = 0$$

$$\phi(\varepsilon) = \sum_{j=0}^{m} \xi_j(\varepsilon)\phi_j \qquad (7.5)$$

$$\psi_i^* v(\varepsilon) = 0 \quad i=1,\ldots m$$

The $\xi_j(\varepsilon)$, $v(\varepsilon)$ are continuous functions of ε with

$$\xi_j(o) = \xi_j^* \quad j=o,\ldots m$$

$$v(o) = v_o \qquad (7.6)$$

If in addition $G(u,\lambda)$ is three times continuously differentiable w.r.t. u and λ then $\phi(\varepsilon)$ and $v(\varepsilon)$ are continuously differentiable.

Proof: In order to apply the Implicit Function Theorem we define the following functions.

$$g(\xi_o,\vec{\xi},v,\varepsilon) = \begin{cases} \frac{1}{\varepsilon^2}G(u_o+\varepsilon(\phi+\varepsilon v),\lambda_o+\varepsilon\xi_o) & \varepsilon\neq 0 \\ G_u^o v+\frac{1}{2}(G_{uu}^o\phi\phi+2G_{u\lambda}^o\phi\xi_o+G_{\lambda\lambda}^o\xi_o^2) & \varepsilon=0 \end{cases}$$

$$h_o(\xi_o,\vec{\xi},v,\varepsilon) = 2\xi_o^2 + \vec{\xi}^T\vec{\xi} - 1 \tag{7.7}$$

$$h_i(\xi_o,\vec{\xi},v,\varepsilon) = \psi_i^* v \qquad i=1,\ldots m$$

We see immediately that $g(\xi_o^*,\vec{\xi}^*,v_o,o)=\mathbf{0}$, $h_i(\xi_o^*,\vec{\xi}^*,v_o,o) = 0$ $i = 0, \ldots m$. To apply the Implicit Function Theorem we must consider

$$J_1 \equiv \left.\frac{\partial(g,\vec{h},h_o)}{\partial(v,\vec{\xi},\xi_o)}\right|_{\substack{\xi_o=\xi_o^* \\ \vec{\xi}=\vec{\xi}^* \\ v=v_o \\ \varepsilon=0}}$$

and we find

$$J_1=\begin{bmatrix} G_u^o(G_{uu}^o\phi\phi_1+G_{u\lambda}^o\phi_1\xi_o^*) & \cdots & (G_{uu}^o\phi\phi_m+G_{u\lambda}^o\phi_m\xi_o^*) & (G_{u\lambda}^o\phi+\xi_o^*G_{\lambda\lambda}^o) \\ \psi_1^* & 0 \quad \cdots & 0 & 0 \\ \vdots & \vdots \quad \ddots & \vdots & \vdots \\ \psi_m^* & 0 \quad \cdots & 0 & 0 \\ 0 & 2\xi_1^* \quad \cdots & 2\xi_m^* & 4\xi_o^* \end{bmatrix}$$

To apply Lemma I we relabel

$$J_1 \equiv \left[\begin{array}{cc:c} \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{\hat{A}}} & \\ A_o & B_o & \hat{B} \\ C_o^* & D_o & \\ \hdashline \multicolumn{2}{c:}{\hat{C}^*} & \hat{D} \end{array}\right] \tag{7.9}$$

We consider $\xi_o^*=0$ and $\xi_o^*\neq 0$ separately. First, suppose $\xi_o^*\neq 0$. Using the notation of Lemma I applied to the operator $\hat{A}$ we see we are in case (2) with $\dim N(A_o) = \gamma = m$. Condition C_2 is satisfied since $\psi_1^*\ldots\psi_m^*$ are linearly independent. $N(A_o)\cap N(C_o^*) = \{\underline{0}\}$ since if $\phi_k \varepsilon N(C_o^*)$ then $\psi_j^*\phi_k=0$ $\forall_j$ contradicting the assumed semi-simplicity. Thus, C_3 is verified. $R(A_o)\cap R(B_o) = \{\underline{0}\}$ since if $B_o\vec{\gamma}\varepsilon R(A_o)$ then $\psi_i^* B_o\vec{\gamma} = 0$ $\forall_i$, that is $(A(\vec{\xi}^*)+\xi_o^* B)\vec{\gamma} = 0$, and by (7.3) with $\xi_o^*\neq 0, A(\vec{\xi}^*)+\xi_o^* B$ is non-singular forcing $\vec{\gamma}=0$. In the same manner $\dim R(B_o)=m$ since $B_o\vec{\gamma}=0$ implies $\vec{\gamma}=0$. Hence C_o and C_1 are satisfied and we conclude $\hat{A}$ is non-singular. Thus, we are in case (1) when applied to J_1 and so we must calculate

$$\hat{S} = \hat{D} - \hat{C}^*\hat{A}^{-1}\hat{B} \tag{7.10}$$

To do this we solve $\hat{A}X = \hat{B}$ with $X = \binom{v}{\vec{\gamma}} \varepsilon B \times \mathbb{R}^m$, i.e.,

$$A_o v + B_o\vec{\gamma} = G^o_{u\lambda}\phi + \xi_o^* G^o_{\lambda\lambda}$$

$$C_o^* v + D_o\vec{\gamma} = 0$$

$$G_u^o v = -(G_{uu}^o\phi(\gamma_j\phi_j)+\xi_o^* G_{u\lambda}^o\phi_j\gamma_j-(G_{u\lambda}^o\phi+\xi_o^* G_{\lambda\lambda}^o)) \quad \text{(a)}$$

$$\text{(sum over } j) \qquad (7.11)$$

$$\psi_i^* v = 0 \quad i = 1,\ldots m \qquad \text{(b)}$$

Compatibility of (7.11)(a) requires $(A(\vec{\xi}^*) + \xi_o^* B)\vec{\gamma} = (B\vec{\xi}^* + \xi_o^*\vec{C})$ and the unique solution of this equation is $\vec{\gamma} = -\vec{\xi}^*/\xi_o^*$. Using this in (7.11) determines the unique $v = -2v_o/\xi_o^*$. Hence (7.10) becomes

$$\hat{S} = 4\xi_o^* + \frac{2\vec{\xi}^{*T}\vec{\xi}^*}{\xi_o^*} = \frac{2}{\xi_o^*} \neq 0$$

and so J_1 is non-singular.

Now suppose that $\xi_o^* = 0$, then if we solve $\hat{A}X = 0$, i.e.,

$$G_u^o v + G_{uu}^o\phi(\sum_{j=1}^{m}\gamma_j\phi_j) = 0 \qquad \text{(a)}$$

$$(7.12)$$

$$\psi_i^* v = 0 \quad i=1,\ldots m \qquad \text{(b)}$$

Compatibility of (7.12)(a) forces $A(\vec{\xi}^*)\vec{\gamma} = 0$ and so by isolation $\vec{\gamma} = \vec{\xi}^*$, and then from (7.12)(b) $v = 2v_o$. Hence $\dim N(\hat{A})=1$. Now if we solve $y^*\hat{A} \equiv 0$ where $y^* = (v^*,\vec{\gamma}^T)$

$$G_u^{o*} v^* + \sum_{j=1}^{m}\gamma_j\psi_j^* = 0 \qquad \text{(a)}$$

$$(7.13)$$

$$v^* G_{uu}^o\phi\phi_j = 0 \quad j = 1,\ldots m \qquad \text{(b)}$$

Then $\psi_i^*\phi_j = \delta_{ij}$ forces $\vec{\gamma} \equiv 0$ hence $v^* = \sum_{j=1}^{m} \alpha_j\psi_j^*$ and (7.13)(b) demands $A(\vec{\xi}^*)\vec{\alpha} = 0$ so $\vec{\alpha} = \vec{\xi}^*$. Thus codim $R(\hat{A}) = 1$ and we are in case (2) of Lemma I. Now $\hat{B} \neq \underline{0}$ since $B\vec{\xi}^* \neq 0$ and so C_0 is satisfied. Since $\vec{\xi}^{*T}\vec{\xi}^* = 1$, $\hat{C}^* \neq \underline{0}$ and C_2 is verified. $N(\hat{A}) \cap N(\hat{C}^*) = \{\underline{0}\}$ since $N(\hat{A})$ span $\{(2v_0, \vec{\xi}^*)^T\}$ and $\hat{C}^* \begin{pmatrix} 2v_0 \\ \vec{\xi}^* \end{pmatrix} = \vec{\xi}^{*T}\vec{\xi}^* = 1$. Thus C_3 is satisfied.

Finally $R(\hat{B}) \cap R(\hat{A}) = \{\underline{0}\}$ since if $\hat{A}X = \hat{B}$ with $X = (v, \vec{\gamma})^T$ we see $A(\vec{\xi}^*)\vec{\gamma} = B\vec{\xi}^*$, i.e., $B\vec{\xi}^* \in R(A(\vec{\xi}^*))$ a contradiction of the assumed isolation of $(\vec{\xi}^*, o)$. Thus in this case as well J_1 is non-singular.

We see from the definitions of $g, \vec{h}, h_0$ that these functions are continuous in all arguments and continuously differentiable with respect to $\vec{\xi}, \xi_0$, and v. Thus, we can apply the Implicit Function Theorem to the system (7.7) to claim the existence of continuous $\vec{\xi}(\varepsilon), \xi_0(\varepsilon)$, $v(\varepsilon)$ satisfying (7.5) with initial values given by (7.4). We note that if G is in $C^3(B_\rho(u_0,\lambda_0))$ then $g(\varepsilon)$ is $C^1(|\varepsilon| \leq \varepsilon_0)$, and hence our solution $(\phi(\varepsilon), v(\varepsilon))$ is continuously differentiable.

The uniqueness given by the Implicit Function Theorem only applies to solutions nearby the solution with initial structure given by (7.4). Each isolated root of (7.2) generates a smooth solution arc through (u_0, λ_0).

We now turn to the case of limit point bifurcation. We shall assume

(H3) Let H1 hold but $G^o_\lambda \notin R(G^o_u)$

(H4) Let $(\vec{\xi}^*, \eta^*)$ be an isolated root of the quadratic Limit Point Bifurcation Equations. That is

$$A(\vec{\xi}^*)\vec{\xi}^* + 2\eta^*\vec{d} = 0$$
$$\vec{\xi}^{*T}\vec{\xi}^* = 1 \tag{7.14}$$

and

$$\hat{J} = |J| = \left| \begin{bmatrix} A(\vec{\xi}^*) & \vdots & \vec{d} \\ \cdots & \vdots & \cdot \\ \vec{\xi}^{*T} & \vdots & 0 \end{bmatrix} \right| \neq 0 \tag{7.15}$$

Under the assumptions (H3) and (H4) we construct the unique solution v_o of

$$\left.\begin{aligned} &G^o_u v_o + \tfrac{1}{2}(G^o_{uu}\phi\phi + \eta^* G^o_\lambda) = 0 && \text{(a)} \\ &\psi^*_j v_o = 0 \quad j=1,\dots m && \text{(b)} \\ &\phi = \sum_{j=1}^{m} \xi^*_j \phi_j && \text{(c)} \end{aligned}\right\} \tag{7.16}$$

From this root $(\vec{\xi}^*,\eta^*)$ and solution v_o we construct a branch emanating from (u_o,λ_o).

Theorem 7.2 Let (H3) and (H4) hold and let v_o be the unique solution of (7.16). Then $\exists\varepsilon_o>0$ such that for all $|\varepsilon|\leq\varepsilon_o$ there exists a unique $(\phi(\varepsilon),v(\varepsilon),\eta(\varepsilon))$ for which

$$G(u_o+\varepsilon(\phi(\varepsilon)+\varepsilon v(\varepsilon)),\lambda_o+\varepsilon^2\eta(\varepsilon)) = 0$$

$$\phi(\varepsilon) = \sum_{j=1}^{m}\xi_j(\varepsilon)\phi_j \tag{7.17}$$

$$\psi_i^* v(\varepsilon) = 0 \quad i=1,\ldots m$$

where the $\xi_j(\varepsilon),\eta(\varepsilon)$, $v(\varepsilon)$ are continuous functions of ε with

$$\xi_j(o) = \xi_j^* \quad j=1,\ldots m$$

$$\eta(o) = \eta^* \tag{7.18}$$

$$v(o) = v_o$$

If in addition $G(u,\lambda)\varepsilon\ C^3(B_\rho(u_o,\lambda_o))$ then $\phi(\varepsilon)$ and $v(\varepsilon)$ are continuously differentiable with respect to ε.

<u>Proof:</u> The proof proceeds in the same manner as in Theorem (7.1).

Define

$$g(\eta,\vec{\xi},v,\varepsilon) = \begin{cases} \frac{1}{\varepsilon^2}G(u_0+\varepsilon(\phi+\varepsilon v),\lambda_0+\varepsilon^2\eta) & \varepsilon\neq 0 \\ G_u^0 v+\frac{1}{2}(G_{uu}^0\phi\phi+2\eta G_\lambda^0) & \varepsilon=0 \end{cases} \tag{7.19}$$

$$h_0(\eta,\vec{\xi},v,\varepsilon) = \vec{\xi}^T\vec{\xi} - 1$$

$$h_i(\eta,\vec{\xi},v,\varepsilon) = \psi_i^* v \qquad i=1,\ldots m$$

We clearly have a root of this system $\eta = \eta^*$, $\vec{\xi} = \vec{\xi}^*$, $v = v_0$, $\varepsilon = 0$. To generate a smooth family of roots we study

$$J_1 \equiv \left.\frac{\partial(g,h,h_0)}{\partial(v,\vec{\xi},\eta)}\right|_{\substack{v = v_0 \\ \eta = \eta^* \\ \vec{\xi} = \vec{\xi}^* \\ \varepsilon = 0}}$$

where

$$
J_1 = \begin{bmatrix}
G_u^o & G^o_{uu}\phi\phi_1 & \cdots & G^o_{uu}\phi\phi_m & G^o_\lambda \\
\psi_1^* & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots \\
\psi_m^* & 0 & \cdots & \cdots & 0 \\
0 & 2\xi_1^* & \cdots & 2\xi_m^* & 0
\end{bmatrix} \qquad (7.20)
$$

Relabelling

$$
J_1 \equiv \left[\begin{array}{cc:c}
\multicolumn{2}{c}{\overbrace{\qquad\qquad}^{\hat{A}}} & \\
A_o & B_o & \\
 & & \hat{B} \\
C_o^* & D_o & \\
\hdashline
\multicolumn{2}{c:}{\hat{C}^*} & \hat{D}
\end{array}\right] \qquad (7.21)
$$

We first suppose $\eta^* \neq 0$. Then clearly $\dim N(A_o) = \operatorname{codim} R(A_o) = m$ and we are in case (2) as applied to $\hat{A}$. Since $\psi_1^* \ldots \psi_m^*$ are linearly independent C_2 is satisfied. $N(A_o) \cap N(C_o^*) = \{\underline{0}\}$ since if $\phi_k \varepsilon N(C_o^*)$ then $\psi_j^* \phi_k = 0$ $j=1,\ldots m$ a contradiction of semi-simplicity. Thus (C_3) is verified. Now suppose $B_o\vec{\gamma} \varepsilon R(A_o)$, then $\psi_i^* B\vec{\gamma} = 0 \;\forall_i$ so $A(\vec{\xi}^*)\vec{\gamma} = 0$ forcing $\vec{\gamma} = 0$ since $A(\vec{\xi}^*)$ is non-singular by isolation. Thus, (C_1) is satisfied. Finally $\dim R(B_o) = m$ since $B_o\vec{\gamma} = 0$ forces $A(\vec{\xi}^*)\vec{\gamma} = 0$ requiring $\vec{\gamma} = 0$. Thus (C_o) is verified and Lemma I states that $\hat{A}$ is non-singular.

To deduce the same of J_1 we must consider

$$\hat{S} = \hat{D} - \hat{C}^{*}\hat{A}^{-1}\hat{B} \tag{7.22}$$

To calculate $\hat{S}$ we solve $\hat{A}X = \hat{B}$ for $X = \begin{pmatrix} v \\ \vec{\gamma} \end{pmatrix} \varepsilon BXR^{m}$, i.e.,

$$G_u^o v + G_{uu}^o \phi\left(\sum_{j=1}^{m} \gamma_j \phi_j\right) = G_\lambda^o \qquad \text{(a)}$$

$$\psi_i v = 0 \quad i=1,\ldots m \qquad \text{(b)} \tag{7.23}$$

Compatibility of (7.23)(a) requires

$$A(\vec{\xi}^{*})\vec{\gamma} = \vec{d}$$

which has the unique solution $\vec{\gamma} = -\vec{\xi}^{\,*}/\eta^{*}$. Then (7.23)(b) forces $v = -2v_o/\eta^{*}$.

Thus $\quad \hat{S} = 0 + \dfrac{2\vec{\xi}^{*T}\vec{\xi}^{*}}{\eta^{*}} = 2/\eta^{*} \neq 0$ and J_1 is non-singular.

Now suppose $\eta^{*} = 0$. If we solve $\hat{A}X=\underline{0}$ we see

$$G_u^o v + G_{uu}^o \phi\left(\sum_{j=1}^{m} \gamma_j \phi_j\right) = 0 \qquad \text{(a)}$$

$$\psi_i^{*} v=0 \quad i=1,\ldots m \qquad \text{(b)} \tag{7.24}$$

Compatibility of (7.24) requires $A(\vec{\xi}^{*})\vec{\gamma} = 0$ which has the unique solution

$\vec{\gamma} = \vec{\xi}^*$, and then from (7.24)(b) we find $v = 2v_o$. Thus $\dim N(\hat{A}) = 1$. To solve $y^*\hat{A}=0$, we find

$$G_u^{o*}v^* + \sum_{j=1}^{m} \gamma_j \psi_j^* = 0 \qquad \text{(a)}$$

$$v^* G_{uu}^{o} \phi\phi_j = 0 \quad j=1,\ldots m \qquad \text{(b)} \tag{7.25}$$

Since $\psi_j^* \notin R(G_u^{o*})$ we see $\vec{\gamma} = 0$ and so $v^* = \sum_{j=1}^{m} \alpha_j \psi_j^*$ and then (7.25)(b) forces $\vec{\alpha} = \vec{\xi}^*$. Thus codim $R(\hat{A})=1$ and we are in case (2) applied to J_1. $\dim R(\hat{B})=1$ since $\vec{d}\neq 0$ implies $G_\lambda^o \neq 0$, verifying (C_o). To check (c_2) we see $\vec{\xi}^{*T}\vec{\xi}=1$ so $\dim R(\hat{C}^*)=1$. $N(\hat{A})\cap N(\hat{C}^*)=\{\underline{0}\}$ since $N(\hat{A})=\text{span}\{(2v_o,\vec{\xi}^*)^T\}$ and $\hat{C}^*\begin{pmatrix}2v_o\\ \vec{\xi}^*\end{pmatrix} = \vec{\xi}^{*T}\vec{\xi}^* = 1$. Finally $R(\hat{B})\cap R(\hat{A}) = \{\underline{0}\}$ for if $\hat{A}X = \hat{B}$ we would have $A(\vec{\xi}^*)\vec{\gamma} = \vec{d}$ a contradiction since $\vec{d}\notin R(A(\vec{\xi}^*))$ by the assumed isolation. So in this case as well J_1 is non-singular.

Thus we can apply the Implicit Function Theorem to generate the solution arc given by (7.17) - (7.18).

---

8. Stability of Simple Bifurcation and Normal Limit Points

In this section we will study the stability behaviour of solution arcs containing simple bifurcation points as well as simple normal limit points. This will be done within the context of linear stability, that is, attention will be paid only to the eigenvalues of the appropriate linearized operator.

Suppose we have a solution point (u_o,λ_o) of $G(u_o,\lambda_o)$. We consider the linear eigenvalue problem

$$G_u(u_o,\lambda_o)\phi = \alpha\phi \tag{8.1}$$

The point (u_o,λ_o) will be considered stable if all eigenvalues α of (8.1) have $Re\alpha<0$. If at least one eigenvalue has $Re\alpha>0$ then (u_o,λ_o) will be called linearly unstable. The point (u_o,λ_o) is neutrally stable if the largest real part of the eigenvalues of G_u^o is zero. We will assume this situation when studying the zero eigenvalue of G_u^o at simple bifurcation and normal limit points. Similar stability results for bifurcation points are determined in Crandall and Rabinowitz [9] and are indicated in Sattinger [38].

To begin suppose G_u^o is Fredholm of index zero with a simple zero eigenvalue. In addition suppose $(u(s),\lambda(s))$ is a smooth solution arc through $(u(s_o),\lambda(s_o)) = (u_o,\lambda_o)$ depending on some parameter s. Then for s near s_o, $G_u(u(s),\lambda(s))$ is also Fredholm of index zero, and the results of Section (4) (Chapter III) allow us to decompose B in a natural way. These results can be summarized in the following fashion.

Suppose G_u is continuously differentiable with respect to u, λ and

s. Assuming $\alpha(s_o)=0$ is a simple eigenvalue then for s near s_o there exists a pair $(\alpha(s),\phi(s))$, continuously differentiable with respect to s for which

$$G_u(u(s),\lambda(s))\phi(s) = \alpha(s)\phi(s) \tag{8.2}$$

In addition, this small real eigenvalue remains simple. That is, we can define

$$\begin{aligned} N(G_u(s) - \alpha(s)I) &= N_1(s) \\ R(G_u(s) - \alpha(s)I) &= X_1(s) \end{aligned} \tag{8.3}$$

and these subspaces decompose B, i.e.,

$$B = N_1(s) \oplus X_1(s) \tag{8.4}$$

from this decomposition we have the adjoint eigenfunction $\psi^*(s)$ satisfying

$$\begin{aligned} &(G_u^*(s) - \alpha(s)I)\psi^*(s) = 0 \\ \text{and} \quad &X_1(s) = \{x\varepsilon B|\psi^*(s)x = 0\} \end{aligned} \tag{8.5}$$

We normalize to $\psi^*(s)\phi(s) = 1$ and observe

$$\psi^*(s)G_u(s)w(s) = 0 \qquad w(s)\varepsilon X_1(s) \tag{8.6}$$

Now from

$$G(u(s),\lambda(s)) = 0 \tag{8.7}$$

we see $$G_u(u(s),\lambda(s))\frac{du}{ds}+G_\lambda(u(s),\lambda(s))\frac{d\lambda}{ds} = 0 \tag{8.8}$$

If we decompose

$$\frac{du}{ds} = \beta(s)\phi(s) + w(s) \tag{8.9}$$

where $$w(s)\varepsilon X_1(s)$$

then

$$G_u(s)w(s)+\alpha(s)\beta(s)\phi(s)+G_\lambda(s)\frac{d\lambda}{ds} = 0 \tag{8.10}$$

Acting upon (8.10) with $\psi^*(s)$ we find

$$\alpha(s)\beta(s) = -\psi^*(s)G_\lambda(u(s),\lambda(s))\frac{d\lambda}{ds} \tag{8.11}$$

This expression is exact as a function of s and will be used to relate the behaviour of $\alpha(s)$ to that of $\frac{d\lambda}{ds}$ as s approaches s_o.

First we consider the case when (u_o,λ_o) is a simple normal limit point. Here

$$\frac{du}{ds}(s_o) = \phi(s_o) = \phi_1$$

since satisfaction of the simple normal limit point equations forces $\beta(s_o)=1$ (Section 5). Further $\psi^*(s_o)G_\lambda^o=d_1\neq 0$ and so we find

$$\alpha(s) = -d_1\frac{d\lambda}{ds} + O(|(s-s_o)\frac{d\lambda}{ds}|) \tag{8.12}$$

That is, suppose for some m>0 we have $\lambda(s) = \lambda_o+a(s-s_o)^m+\ldots$, then $\alpha(s) = b(s-s_o)^{m-1}+\ldots$ This result forces a change in stability in the situation where $\frac{d\lambda}{ds}$ changes sign through s_o. (This is the case when the solution arc turns back on itself (i.e., m is even) and is indicated in Figure (8.1).)

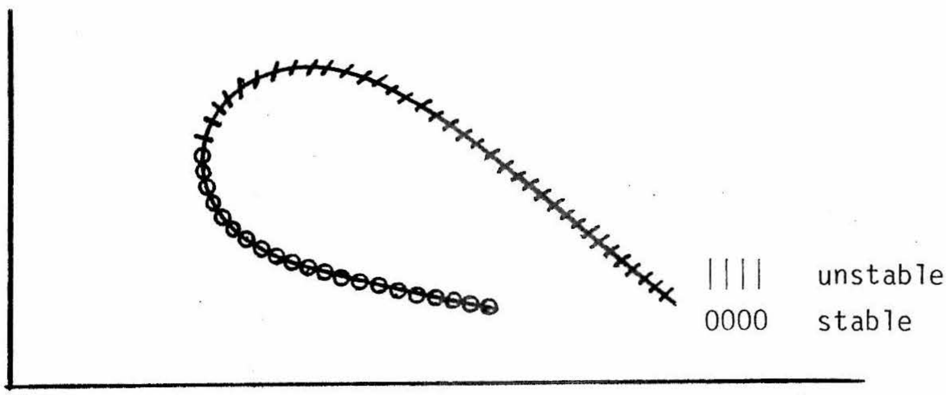


Figure (8.1)

This stability change can be shown directly by an application of Leray-Schauder degree theory but here the particularly simple calculation results in an explicit determination of the behaviour of $\alpha(s)$ near the singular point.

We now consider the expression (8.10) when (u_o,λ_o) is a simple bifurcation point. To do this we write

$$\psi^*(s)G_\lambda(s)=\psi^*(s_o)G_\lambda^o+(s-s_o)\left(\frac{d\psi^*}{ds}(s_o)G_\lambda^o+\psi^*(s_o)\frac{dG_\lambda^o}{ds}\right)+O(|s-s_o|^2) \qquad (8.13)$$

Now at a bifurcation point $\psi^*(s_o)G_\lambda^o=0$, so we must calculate the linear term.

$$\psi^*(s_o)\frac{dG_\lambda^o}{ds} = \psi^*(s_o)\left(G_{u\lambda}^o\frac{du^o}{ds}+G_{\lambda\lambda}^o\frac{d\lambda^o}{ds}\right) \qquad (8.14)$$

where $\frac{du^o}{ds} = \xi_o\phi_o + \xi_1\phi_1, \frac{d\lambda^o}{ds} = \xi_o$.

To calculate $\frac{d\psi^*}{ds}(s_o)G^o_\lambda$ we consider the decomposition

$$G_\lambda(s) = \gamma(s)\phi(s) + v(s) \tag{8.15}$$

where $v(s)\varepsilon\ X_1(s)$, and solve

$$G_u\phi_o(s) + v(s) = 0 \tag{8.16}$$

where $\psi^*(s)\phi_o(s) = 0$. We see $\phi_o(s_o)= \phi_o$ and $v(s_o) = G^o_\lambda$. Since both terms in (8.16) are in $X_1(s)$ we have

$$\psi^*(s)G_u(s)\phi_o(s) \equiv 0 \tag{8.17}$$

and so

$$\frac{d\psi^*}{ds}(s_o)G^o_u\phi_o + \psi^*(s_o)\frac{dG^o_u}{ds}\phi_o + \psi^*(o)G^o_u\frac{d\phi_o}{ds} = 0$$

Now $\psi^*(s_o)G^o_u \equiv 0$ and from (8.16) at $s = s_o$, $G^o_u\phi_o = -G^o_\lambda$.

Hence

$$\frac{d\psi^*}{ds}(s_o)G^o_\lambda = \psi^*(s_o)\frac{dG^o_u}{ds}\phi_o \tag{8.18}$$

Placing the expression (8.18) in (8.11) we find

$$\alpha(s)\beta(s) = -\psi^*(s_o)(\frac{dG^o_u}{ds}\phi_o+\frac{dG^o_\lambda}{ds})\frac{d\lambda}{ds}(s-s_o)+0(|(s-s_o)^2\ \frac{d\lambda}{ds}|) \tag{8.19}$$

Expanding the linear term results in

$$\alpha(s)\beta(s)= -(b_{11}\xi_1+c_1\xi_o)\frac{d\lambda}{ds}(s-s_o)+0(|(s-s_o)^2\ \frac{d\lambda}{ds}|) \tag{8.20}$$

This expression will be used to study the behaviour of $\alpha(s)$ on a solution arc with $\frac{d\lambda^o}{ds} = 0$. First, however, we will derive an expression to be used when $\frac{d\lambda^o}{ds} \neq 0$. From our definition

$$G_u(s)\phi(s) = \alpha(s)\phi(s) \tag{8.21}$$

we find

$$G_u\frac{d\phi}{ds} + \frac{dG_u}{ds}\phi = \frac{d\alpha}{ds}\phi + \alpha\frac{d\phi}{ds} \tag{8.22}$$

Evaluating (8.22) at $s=s_o$ and acting upon the result with $\psi^*(s_o)=\psi_1^*$ we have, (using $\alpha(s_o)=0$, $\psi_1^* G_u^o \equiv 0$ and $\psi_1^* \phi_1 = 1$)

$$\frac{d\alpha^o}{ds} = \psi_1^*(G^o_{uu}\frac{du}{ds} + G^o_{u\lambda}\frac{d\lambda}{ds})\phi_1 \tag{8.23}$$

That is

$$\frac{d\alpha^o}{ds} = (a_{111}\xi_1 + b_{11}\xi_o) \tag{8.24}$$

Recalling the Algebraic Bifurcation Equations and the isolation requirement given in Section (3) we see either

$$a_{111}\xi_1 + b_{11}\xi_o \neq 0 \quad \text{and} \quad \xi_o \neq 0$$

or

$$a_{111}\xi_1 + b_{11}\xi_o = 0, \quad \xi_o = 0 \quad \text{But } b_{11} \neq 0 \tag{8.25}$$

The first case allows us to state

(i) Along any simple bifurcating arc for which $\frac{d\lambda^o}{ds} \neq 0$ then $\frac{d\alpha^o}{ds} \neq 0$ and hence

there is a stability change through s_o.

For the second case we use (8.20) and note that if $\xi_o=0$ then $\beta(s_o)=1$ and so $(b_{11}\neq 0)$

$$\alpha(s) = -b_{11}(s-s_o)\frac{d\lambda}{ds} + 0(|(s-s_o)^2\frac{d\lambda}{ds}|) \qquad (8.26)$$

This allows the statement

(ii) Along a simple bifurcation arc for which $\frac{d\lambda}{ds}$ changes sign the eigenvalue

$\alpha(s)$ remains of one sign and there is no change in stability.

Particular examples of these two cases are indicated in Figures (8.2) and (8.3). The situations depicted actually indicate an exchange of stability, that is as one solution arc loses stability the other arc becomes stable. We now indicate why this is to be expected.

First consider the case of Figure (8.2). The two roots (ξ_o^1,ξ_1^1), (ξ_o^2,ξ_1^2) have $\xi_o^1\xi_o^2\neq 0$ and so defining $x=\xi_1/\xi_o$ they are the two distinct roots x_1,x_2 of

$$f(x) \equiv a_{111}x^2 + 2b_{11}x + c_1 = 0 \qquad (8.27)$$

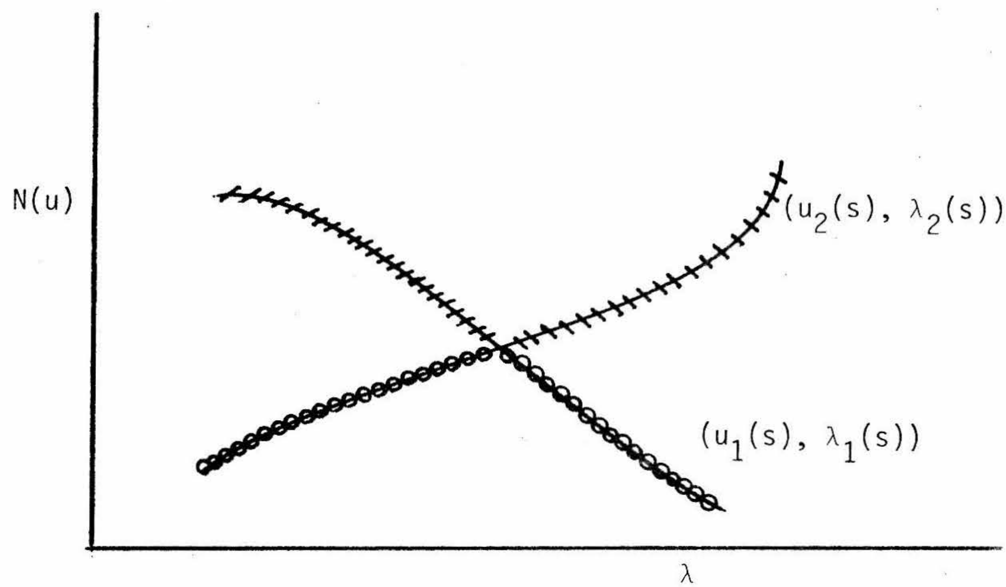


Figure (8.2)

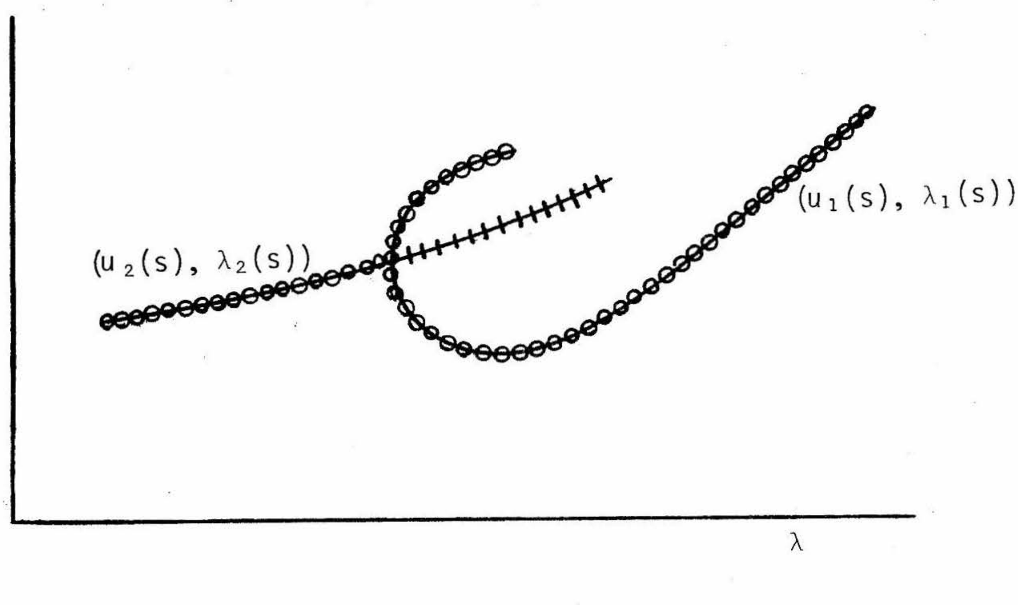


Figure (8.3)

It is easily shown that

$$f'(x_1)f'(x_2) = a_{111}c_1 - b_{11}^2 \tag{8.28}$$

and hence this product is negative since the quadratic has two real roots. Now in order to move along the two branches in the same direction we choose $\frac{d\lambda_i^o}{ds}$, i=1,2 to be of the same sign. With this choice and (8.24) we see

$$\alpha_1'(s_o)\alpha_2'(s_o) = \frac{f'(x_1)f'(x_2)}{\xi_o^1\ \xi_o^2} < 0 \tag{8.29}$$

Hence the eigenvalues cross the origin in opposite directions and we have an exchange of stability.

Now consider the case when one branch has $\xi_o^1=0$. From the Algebraic Bifurcation Equations this forces $a_{111}=0$. Applying (8.24) to the branch with $\xi_o^2\neq 0$

$$\alpha_2'(s_o) = b_{11}\frac{d\lambda_2}{ds}(s_o) \tag{8.30}$$

However (8.20) yields

$$\alpha_1(s) = -b_{11}(s-s_o)\frac{d\lambda_1}{ds}(s)+0(|s-s_o|^2) \tag{8.31}$$

We also note that $(s-s_o)\frac{d\lambda_1}{ds}(s) \geq 0$ regardless of the choice for the direction of increasing s. (See Figure (8.3))

The two expressions (8.30)-(8.31) show that if $(u_2(s),\lambda_2(s))$ loses stability (i.e., $b_{11}>0$, $\frac{d\lambda_2}{ds}(s_o)>0$) then $(u_1(s),\lambda_1(s))$ is stable and so that here also we have an exchange of stability.

The situation for multiple eigenvalue bifurcation is considerably more complicated. The main problem rests on guaranteeing the existence of m continuously differentiable eigenvalue-eigenfunction pairs $(\alpha_i(s),\phi_i(s))$. At present it appears that analyticity of $G_u(u(s),\lambda(s))$ with respect to a complex parameter s is required. However, if existence is assumed, similar formulas for the determination of the behaviour near zero of the various eigenvalues are possible.

Chapter III Construction of Bifurcation Branches

(1) Introduction

In this chapter we continue our study of solutions to problems of the general form:

$$G(u,\lambda) = 0 \tag{1.1}$$

where G is a nonlinear operator mapping the Banach space $B \times \mathbb{R}$ into B. Once again λ has been explicitly separated from u to indicate its importance as a state or control parameter for the particular problem.

Suppose now we have a point (u_0,λ_0) which satisfies (1.1). Several natural questions arise.

(1) What is the structure of the solutions of (1.1) near (u_0,λ_0)? In particular, can we give sufficient conditions for the existence of a smooth solution branch $\Gamma(s)$

$$\Gamma(s) \equiv \{(u(s),\lambda(s)) \mid G(u(s),\lambda(s)) = 0 \quad (u(s_0),\lambda(s_0)) = (u_0,\lambda_0)\} \tag{1.2}$$

through (u_0,λ_0) depending smoothly on some parameter s?

(2) What explicit methods can we derive which will allow the construction of these solution branches?

The first question was studied in Chapter II where the point (u_0,λ_0)

was a singular point of G_u^o of various special types. The second question will be considered in this chapter.

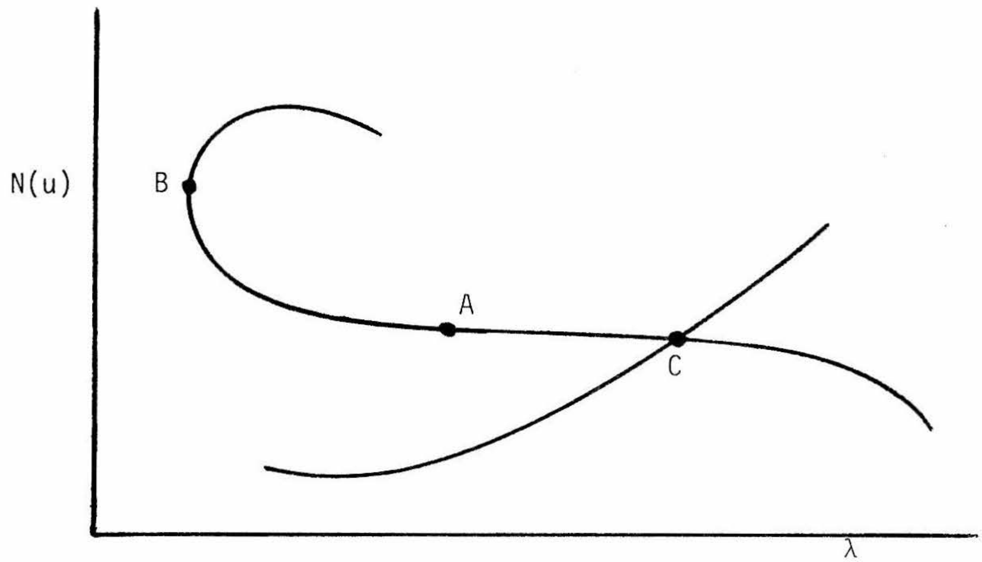


Figure (1.1)

In the specification of the solution arc $\Gamma(s)$ of (1.2) the parameter s is left undefined. Referring to Figure (1.1) we see that λ itself would seem to be an acceptable choice of parameter in the neighborhood of point A. Such a point will be defined in Section (2) as a regular point. Intuitively it is clear that this choice would be unacceptable in the neighborhood of point C. If the point B is a simple normal limit point we shall see that choosing our parameter to be an approximation to arc-length in $B \times \mathbb{R}$ will allow continuation past B without difficulty. This choice of parametrization leads to what will be called the inflated problem.

Sections (3) through (7) are concerned with the continuation problem in the neighborhood of a simple bifurcation point (Point C). In Section (3) the structure of the inflated problem linearized about the solution point $\{(u(s_0),\lambda(s_0)),(\dot{u}(s_0),\dot{\lambda}(s_0))\}$ is studied. It is seen that in general the inflated system inherits the structure of $G_u(u_0,\lambda_0)$. The linearized inflated operator, called $A(s_0)$, will be Fredholm of index zero, and provided $\dot{\lambda}(s_0)\neq 0$, will have a simple zero eigenvalue. Section (4) introduces results on the perturbation of such eigenvalues. The linear operator $A(s)$ is non-singular for s in some deleted neighborhood about s_0, and it is the size of its smallest eigenvalue $\alpha(s)$ which will allow us to determine the manner in which $A(s)$ approaches singularity. With this idea in mind, Section (5) calculates the rate at which the simple eigenvalue $\alpha(s)$ travels to zero. This rate is seen to be related to the Algebraic Bifurcation Equations. When $\dot{\lambda}(s_0)=0$ the operator $A(s_0)$ has a single null vector but the eigenvalue is of multiplicity two. Section (6) is concerned with the perturbation of this eigenvalue. Conditions are determined under which the eigenvalue splits into two simple eigenvalues which approach zero like $\sqrt{s-s_0}$.

With these preparations Section (7) considers the convergence of an iterative method in the neighborhood of a simple bifurcation point. The scheme presented will be Newton's method, with the initial guess being a point on the tangent ray through the bifurcation point. Assuming the existence of a smooth solution branch $(u(s),\lambda(s))$ through (u_0,λ_0), it will be shown that the Newton iterates can be guaranteed to converge at least geometrically. A second result, resting on a finer estimate for the distance between the first two iterates, will allow the statement of quadratic convergence. This result will not require the assumption of a solution branch through (u_0,λ_0), and will thus furnish a constructive proof for the existence of bifurcation. The application of this scheme <u>at</u> the bifurcation point is avoided since it is known (Rall (30), Reddien (31)) that Newton's method will not generally converge quadratically in such a situation.

(2) The Inflated Problem

In order to construct a solution arc $\Gamma(s)$ through (u_o,λ_o) given by (1.2) an immediate problem is the choice of an appropriate parametrization of the arc. This choice is quite arbitrary, but since λ has already been distinguished as an important physical parameter, it is a natural choice. In the neighborhood of a solution point isolated with respect to λ this choice is usually acceptable. More precisely, if $G(u,\lambda)$ is in $C^1(B_\rho(u_o,\lambda_o))$ and $G_u^o \equiv G_u(u_o,\lambda_o)$ is non-singular then the Implicit Function Theorem guarantees the existence of a unique smooth arc of solutions $(u(\lambda),\lambda)$ where $|\lambda-\lambda_o|<\rho_o$ for some $\rho_o>0$. (Points (u_o,λ_o) where G_u^o is non-singular will be called regular points.) However, if (u_o,λ_o) were a simple normal limit point this choice would be unsatisfactory. To avoid this difficulty and to give greater flexibility in the choice of a parameter we use a technique developed by Keller [18].

We specify the parameter s by imposing a normalization on the solution, that is we replace (1.1) by

$$
\begin{aligned}
G(u,\lambda) &= 0 \qquad &\text{(a)}\\
N(u,\lambda,s) &= 0 \qquad &\text{(b)}
\end{aligned}
\qquad (2.1)
$$

Here N is a functional on $B \times \mathbb{R}^2$. This approach makes $(u(s),\lambda(s))$ the solution of an inflated problem. Introducing $X \varepsilon B_1 \equiv B \times \mathbb{R}$ and $P: B_1 \times \mathbb{R} \to B_1$ as

$$P(X,s) = \begin{bmatrix} G(u,\lambda) \\ \\ N(u,\lambda,s) \end{bmatrix} \qquad \text{(a)}$$

$$X = (u,\lambda) \qquad \text{(b)} \qquad (2.2)$$

then our solution arc $\Gamma(s)$ satisfies

$$P(X(s),s) = 0 \qquad (2.3)$$

For a given s_o our solution $X(s_o)$ is isolated with respect to s if

$$A(s_o) \equiv P_X(X(s_o),s_o) = \begin{bmatrix} G_u(u(s_o),\lambda(s_o)) & G_\lambda(u(s_o),\lambda(s_o)) \\ \\ N_u^*(u(s_o),\lambda(s_o)) & N_\lambda(u(s_o),\lambda(s_o)) \end{bmatrix} \qquad (2.4)$$

is non-singular.

We write $N_u^* \equiv N_u(u(s_o),\lambda(s_o))$ since N_u is a linear functional on B and since all normalizations we consider will make N_u continuous. A major advantage in considering this inflated system is that $A(s_o)$ may be non-singular when G_u^o is singular. The conditions under which this occurs are specified by an application of the Basic Lemma I. To satisfy these conditions one must choose a proper normalization. The normalization used here will be an approximation to arc-length.

If we define s to be arc-length in the Banach space B_1, then we require,

$$N_1(u,\lambda,s) = \left|\left|\frac{du}{ds}\right|\right|^2 + \left(\frac{d\lambda}{ds}\right)^2 - 1 \qquad (2.5)$$

This expression is not the most useful since there is no explicit dependence upon s, hence an approximation to (2.5) is derived. If we know a solution (u_o,λ_o) of (2.1)(a) at $s=s_o$, and if in addition we solve for $(\dot{u}(s_o),\dot{\lambda}(s_o))$ satisfying

$$G_u^o\dot{u}(s_o) + G_\lambda^o\dot{\lambda}(s_o) = 0 \qquad (2.6)$$

then we define

$$N_2(u,\lambda,s) = \dot{u}(s_o)^*(u(s)-u_o)+\dot{\lambda}(s_o)(\lambda-\lambda_o)-(s-s_o) \qquad (2.7)$$

Here $\dot{u}(s_o)^*\varepsilon B^*$ and satisfies $\dot{u}(s_o)^*\dot{u}(s_o) = ||\dot{u}(s_o)||^2$. This expression is an $O(s-s_o)$ approximation to arclength in B_1 and is the normalization that will be used in all of what follows unless otherwise stated.

Differentiating (2.3) with respect to s we see

$$A(s)\dot{X}(s) = r(s) \qquad (2.8)$$

where

$$r(s)=\begin{pmatrix} 0 \\ -N_s \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (2.9)$$

From (2.8) we note that

$$||\dot{X}(s)||^2 \equiv ||\dot{u}(s)||^2+||\dot{\lambda}(s\ )||^2 > 0 \qquad (2.10)$$

We now show that with our normalization a simple normal limit point with respect to λ becomes a regular point with respect to s. Suppose $s = s_o$ is a simple normal limit point and hence by assumption $G^o_\lambda \notin R(G^o_u)$ and thus $\dot{\lambda}(s_o) = 0$. From (2.6) $\dot{u} = \phi_1 \varepsilon N(G^o_u)$, and $||\phi_1|| \neq 0$ by (2.10). Thus $N^*_u \phi_1 = \dot{u}^* \phi_1 = ||\phi_1|| \neq 0$ and so $N(N^*_u) \bigcap N(G^o_u) = \{\underline{0}\}$. Lemma I now states that $A(s_o)$ is non-singular. We note that simplicity is not strictly required. Suppose ψ^*_1 is the unique solution of

$$G^{o*}_u \psi^*_1 = 0 \qquad (2.11)$$

Then one could have $\psi^*_1 \phi_1 = 0$, so $\phi_1 \varepsilon R(G^o_u)$ without affecting the result. This means that G^o_u could have ϕ_1 as its only null vector and the zero eigenvalue could have multiplicity $\mu \geq 2$, yet $A(S_o)$ would remain non-singular. In all previous applications of Lemma I $m = \dim N(G^o_u) = \mu$ was assumed.

In this manner the difficulty with normal limit points disappears. An obvious question is can $A(s_o)$ be singular when G^o_u is not? That is, can a regular point with respect to λ become a singular point with respect to s? This would occur if

$$S = N_\lambda - N^*_u G^{-1}_u G_\lambda = 0 \qquad (2.12)$$

Now if we suppose $N_s \neq 0$ then

$$S = N_\lambda + N^*_u \dot{u}/\dot{\lambda} = -N_s/\dot{\lambda} \neq 0$$

($\dot{\lambda} \neq 0$ since $\dot{\lambda} = 0$ implies $\dot{u} = 0$ from (2.6) which is a contradiction of (2.10).) Hence if any normalization is chosen with $N_s \neq 0$, regular points remain regular points.

The preceding analysis indicates the structure of the inflated system near regular and normal limit points $(u_0,\lambda_0) \varepsilon B \times \mathbb{R}$. In the next section we will consider the inflated system in the neighborhood of a singular point of $G(u,\lambda)$ of the type leading to simple bifurcation.

(3) The Inflated System at a Simple Bifurcation Point

In the previous section we considered situations where the structure of $G(u,\lambda)$ was modified by embedding this equation in an inflated system. Here we want to find conditions under which $P(X(s),s)$ inherits the structure of $G(u,\lambda)$. In particular we wish to study $A(s_0)$ at a point $(u(s_0),\lambda(s_0))$ of the type yielding simple bifurcation. We do this by applying the following lemma

Lemma II. Let B be a Banach space, $B_1 \equiv B \times \mathbb{R}$, and consider the linear operator $\hat{A}:B_1 \to B_1$ of the form

$$\hat{A} = \begin{bmatrix} A & B \\ C^* & D \end{bmatrix} \qquad \begin{matrix} A: B \to B \\ B: \mathbb{R} \to B \end{matrix} \qquad \begin{matrix} C^*: B \to \mathbb{R} \\ D: \mathbb{R} \to \mathbb{R} \end{matrix}$$

where C^* and D are continuous linear functionals. Suppose A is Fredholm of index zero with a simple zero eigenvalue. That is

$$N(A) = \text{span}\ \{\phi_1\}$$

$$N(A^*) = \text{span}\ \{\psi_1^*\}$$

$$R(A) = \{X \varepsilon B | \psi_1^* X = 0\} \tag{3.1}$$

$$\psi_1^* \phi_1 = 1$$

Then we may conclude $\hat{A}$ is Fredholm of index zero with

$$N(\hat{A}) = \text{span}\ (\Phi\ )$$

$$N(\hat{A}^*) = \text{span}\ (\Psi^*)$$

iff (i) or (ii) are satisfied.

(i) $B \notin R(A)$ and $\phi_1 \varepsilon N(C^*)$

Then we may take

$$\Phi = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}, \quad \Psi^* = (y_o^* + \alpha_1 \psi_1^*, 1) \tag{3.3}$$

where
$$-\alpha_1 \equiv \frac{y_o^* B + D}{\psi_1^* B}$$

and y_o^* is the unique solution of

$$A^* y_o^* + C^* = 0 \qquad y_o^* \phi_1 = 0$$

Here $$\Psi^* \Phi = \alpha_1 = -\left(\frac{y_o^* B + D}{\psi_1^* B}\right) \tag{3.4}$$

(ii) $B \varepsilon R(A)$ and $|C^* \phi_o + D| + |C^* \phi_1| \neq 0$

where ϕ_o is the unique solution of

$$A\phi_o + B = 0$$

$$\psi_1^* \phi_o = 0$$

Then we may take

(a) if $C^* \phi_1 \neq 0$

$$\Phi = \begin{pmatrix} \phi_o + \alpha_2 \phi_1 \\ 1 \end{pmatrix} \quad \Psi^* = (\psi_1^*, o) \tag{3.5}$$

where $$\alpha_2 = -\left(\frac{C^* \phi_o + D}{C^* \phi_1}\right)$$

So $$\Psi^* \Phi = \alpha_2$$

(b) if $C^* \phi_1 = 0 \quad (C^* \phi_o + D \neq 0)$

$$\Phi = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} \qquad \Psi^* = (\psi_1^*, o) \qquad \Psi^* \Phi = 1 \tag{3.6}$$

The zero eigenvalue of $\hat{A}$ is simple if $\alpha_1 \neq 0$ in case (i) or $\alpha_2 \neq 0$ in case (ii).

<u>Proof:</u> We consider separately the two cases $B \in R(A)$, $B \notin R(A)$.

(i) $B \notin R(A)$. Writing

$$\begin{bmatrix} A & B \\ C^* & D \end{bmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix} = 0 \quad \text{we see } \alpha = 0$$

Then $X = \gamma\phi_1$, $\gamma \in \mathbb{R}$ and we see that there is a unique (up to a multiple) eigenvector iff $C^*\phi_1 = 0$, and we may take $\Phi = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$. Now solving

$$(y^*, \beta) \begin{bmatrix} A & B \\ C^* & D \end{bmatrix} = 0 \quad \text{with } B \notin R(A) \text{ and } \phi_1 \in N(C^*)$$

we see (since $y^*A \equiv A^*y^*$) ,

$$A^*y^* + \beta C^* = 0$$

$$y^*B + \beta D = 0$$

Since $C^*\phi_1 = 0$, $C^* \in R(A^*)$ so $\exists y_0^*$ such that $A^*y_0^* + C^* = 0$ and y_0^* is unique if we demand $y_0^*\phi_1 = 0$. Thus $y^* = \beta y_0^* + \gamma\psi_1^*, \gamma \in \mathbb{R}$ and so $\beta(y_0^*B + D) + \gamma\psi_1^*B = 0$. Since $\psi_1^*B \neq 0$ we get $\gamma = -\beta\left(\frac{y_0^*B+D}{\psi_1^*B}\right)$ and we may take

$$\Psi^* = \left(y_0^* - \left(\frac{y_0^*B+D}{\psi_1^*B}\right)\psi_1^*, 1\right)$$

as our unique eigenvector.

(ii) $B \in R(A)$ then

$$\begin{bmatrix} A & B \\ C^* & D \end{bmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix} = 0$$

implies $X = \alpha\phi_o + \gamma\phi_1$, $\gamma \varepsilon \mathbb{R}$ where ϕ_o is the unique solution of

$A^*\phi_o + B = 0$, $\psi_1^*\phi_o = 0$. Then we have,

$$(C^*\phi_o + D)\alpha + (C^*\phi_1)\gamma = 0$$ and we can solve uniquely for

α/γ or γ/α iff $C^*\phi_o + D$ and $C^*\phi_1$ are not both zero. If this is the case we may take

$$\Phi = \begin{pmatrix} \phi_o + \alpha_2\phi_1 \\ 1 \end{pmatrix} \quad , \quad \alpha_2 = \frac{-(C^*\phi_o + D)}{C^*\phi_1}$$

and if $C^*\phi_1 = 0$ we may take $\Phi = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$

To solve for Ψ^* we set

$$A^*y^* + \beta C^* = 0$$

$$y^*B + D\beta = 0$$

First suppose $C^*\phi_1 \neq 0$, then $\beta = 0$ and $y^* = \gamma\psi_1^*$ and so we need $\gamma\psi_1^*B = 0$ which is true for any γ. Here we may take

$\Psi^* = (\psi_1^*, o)$. Now suppose $C^*\phi_1 = 0$ but $C^*\phi_o + D \neq 0$

Then $C^* \varepsilon R(A^*)$ so $y^* = \beta y_o^* + \gamma\psi_1^*$ and we require

$$\beta(y_o^*B + D) + \gamma\psi_1^*B = 0$$

Thus $\beta(y_o^*B + D) = 0$ since $\psi_1^*B = 0$

Since $B = -A\phi_o$ we see $y_o^*B = -y_o^* A\phi_o = -A^*y_o^*\phi_o = C^*\phi_o$

Thus $y_o^* B+D = C^* \phi_o + D \neq 0$ and so $\beta = 0$. Here we may take our eigen-vector to be

$$\Psi^* = (\psi_1^*, 0)$$

To conclude the proof it is necessary to show $R(\hat{A})$ is closed. Suppose we have a sequence $(y_n, \beta_n)^T \varepsilon R(\hat{A})$ for which $(y_n, \beta_n)^T \to (y, \beta)^T$. Then there exist $(X_n, \alpha_n)^T$ such that

$$\begin{aligned} AX_n + B\alpha_n &= y_n \\ & \qquad n=1,\ldots \\ C^* X_n + D\alpha_n &= \beta_n \end{aligned} \tag{3.7}$$

Now we may write y_n as the direct sum $y_n = \gamma_n B + Z_n$ where Z_n has no component of B. Then (γ_n, Z_n) converges to some (γ, Z). From (3.7) we may take $\alpha_n = \gamma_n$ and so $AX_n = Z_n$. Since $R(A)$ is closed and $Z_n \to Z$ there is an X for which $AX = Z$. Thus we have $AX+B\alpha=y$ and since C^* and D are continuous $C^* X + D\alpha = \beta$. Hence $R(\hat{A})$ is closed.

__________ _____ __________

To apply this lemma we suppose we have an arc of solutions satis-fying (using the notation of section (2))

$$P(X(s), s) = 0 \tag{3.8}$$

and

$$A(s)\dot{X}(s) = r(s) \tag{3.9}$$

where

$$A(s) = \begin{bmatrix} G_u(s) & G_\lambda(s) \\ \\ N_u^*(s) & N_\lambda(s) \end{bmatrix} \tag{3.10}$$

Suppose further that at $s = s_o, G_u^o$ is a Fredholm operator of index zero with a simple zero eigenvalue. Finally suppose $G_\lambda^o \varepsilon R(G_u^o)$ so that we are in the case where bifurcation may occur. Define ϕ_o as the unique solution of

$$\begin{aligned} G_u^o\phi_o + G_\lambda^o &= 0 \\ \psi_1^*\phi_o &= 0 \end{aligned} \tag{3.11}$$

From (3.9) with our normalization (2.7) we must have

$$\begin{aligned} G_u^o\dot{u}(s_o) + G_\lambda^o\dot{\lambda}(s_o) &= 0 \qquad \text{(a)} \\ \dot{u}(s_o)^*\dot{u}(s_o) + \dot{\lambda}(s_o)^2 &= 1 \qquad \text{(b)} \end{aligned} \tag{3.12}$$

We see that (3.12)(a) has the solution $\dot{u}(s_o) = \xi_o\phi_o + \xi_1\phi_1$ with $\dot{\lambda}(s_o) = \xi_o$. To determine $\dot{u}(s_o)^*$ we construct the two linear functionals ϕ_o^*, ϕ_1^* satisfying

$$\begin{aligned} \phi_o^*\phi_1 &= 0 \qquad & \phi_o^*\phi_o &= 1 \\ \phi_1^*\phi_o &= 0 \qquad & \phi_1^*\phi_1 &= 1 \end{aligned} \tag{3.13}$$

Then if we place $\dot{u}(s_o)^* = \xi_o\phi_o^* + \xi_1\phi_1^*$ we see $\dot{u}(s_o)^* u(s_o) = \xi_o^2 + \xi_1^2$, which which is never zero unless $\dot{u}(s_o) = 0$. Thus $\dot{u}(s_o)^* \dot{u}(s_o) = \beta||\dot{u}(s_o)||^2$ and our constructed linear functional is the desired one to within a non-zero factor. We use this expression since if (ξ_o, ξ_1) satisfies the simple Algebraic Bifurcation Equations

$$a_{111}\xi_1^2 + 2b_{11}\xi_1\xi_o + c_1\xi_o^2 = 0 \qquad \text{(a)}$$

$$2\xi_o^2 + \xi_1^2 - 1 = 0 \qquad \text{(b)} \tag{3.14}$$

we see that (3.14)(b) forces the satisfaction of (3.12)(b).

Now since $G_\lambda^o \epsilon R(G_u^o)$ we are in case (ii) of Lemma II and we find

$$C^*\phi_o + D = 2\xi_o$$

$$C^*\phi_1 = \xi_1 \tag{3.15}$$

From (3.14)(b) $|C^*\phi_o + D| + |C^*\phi_1| = 2|\xi_o| + |\xi_1| \neq 0$. Thus $A(s_o)$ has a zero eigenvalue and unique eigenvector Φ inherited from G_u^o. This zero eigenvalue is simple if $\Psi^*\Phi = \alpha_2 \neq 0$ where

$$\alpha_2 = \frac{-(C^*\phi_o + D)}{C^*\phi_1} = -\frac{2\xi_o}{\xi_1} \tag{3.16}$$

Hence we have simplicity unless $\dot{\lambda}(s_o) = \xi_o = 0$. Provided $\dot{\lambda}(s_o) \neq 0$ we redefine our eigenvectors as

$$\Phi = \frac{1}{\alpha_2}\left[\begin{pmatrix}\phi_o\\ 1\end{pmatrix} + \alpha_2\begin{pmatrix}\phi_1\\ 0\end{pmatrix}\right] \qquad \Psi^* = (\psi_1^*, o) \tag{3.17}$$

so that $\Psi^* \Phi = 1$.

The above work has shown that at a solution point (u_0, λ_0) leading to the simple Bifurcation Equations the inflated problem inherits the structure of $G(u,\lambda) = 0$. In particular $A(s_0)$ is usually (unless $\dot{\lambda}(s_0) = 0$) a Fredholm operator of index zero with a simple zero eigenvalue. The next section will present some results on the perturbation of simple eigenvalues. These results will then be applied to the simple eigenvalue of $A(s)$ resulting in a bound on the growth of $||A(s)^{-1}||$ as s approaches s_0. A bound of this type will be found to be necessary in section (7) to guarantee contraction of an iteration scheme designed to construct solutions near a bifurcation point.

(4) Perturbation of Simple Eigenvalues

In the previous section we found that at a simple bifurcation point, the linearized operator of the inflated system has a simple zero eigenvalue provided $\xi_0 = \dot{\lambda}(o) \neq 0$. We will now show that for s near s_0, $A(s)$ has a small simple real eigenvalue $\alpha(s)$ and corresponding eigenfunction $\phi(s)$. If $A(s)$ is continuously differentiable with respect to s then so are $\alpha(s)$ and $\phi(s)$. These conclusions are not valid if the zero eigenvalue of $A(s_0)$ is not simple; in this case, the eigenvalue $\alpha(s)$ may not remain real and may not be continuously differentiable at zero.

Most of the results of this section were derived by Crandall and Rabinowitz [9] in the context of a discussion of linearized stability. Here the conclusions will be shown to follow from particularly simple applications of Lemma I.

We recall the formulation of the inflated problem from section (2) as

$$P(X(s),s) = 0 \tag{4.1}$$

and then

$$A(s)\dot{X}(s) = r(s) \tag{4.2}$$

where

$$A(s) = \begin{bmatrix} G_u(X(s)) & G_\lambda(X(s)) \\ \\ N_u^*(X(s)) & N_\lambda(X(s)) \end{bmatrix} \tag{4.3}$$

With this description in mind we state the following:

<u>Lemma (4.1)</u> Suppose $A(s):B_1 \to B_1$ and $\psi^*(s)\varepsilon B_1^*$ are given continuously differentiable mappings for $|s-s_o|<\delta_o$. In addition, at $s=s_o$ suppose $A(s_o)$ is a Fredholm operator of index zero with a simple zero eigenvalue and $\psi^*(s_o) = \Psi^*$ where $A^*(s_o)\Psi^* = 0$. Then $\exists\,\delta_1>0$, $\rho_o>0$ and unique continuously differentiable functions $(\phi(s),\alpha(s))\varepsilon B_{\rho_o}((\Phi,0))$ satisfying

$$\begin{aligned} A(s)\ \phi(s) &= \alpha(s)\ \phi(s) \\ \psi^*(s)\ \phi(s) &= 1 \end{aligned} \tag{4.4}$$

for $|s-s_o|<\delta_1$. At $s=s_o$ we have

$$\phi(s_o) = \Phi$$

$$\alpha(s_o) = 0$$

where $A(s_o)\Phi = 0$.

Proof: We define the operators

$$F_i(s;u,t): B_1 \times \mathbb{R} \to B \quad i=1,2$$

by

$$\begin{aligned} F_1(s;u,t) &= A(s)u-tu \\ F_2(s;u,t) &= \psi^*(s)u-1 \end{aligned} \tag{4.5}$$

We see immediately that $F_i(s_o;\Phi,o) = 0$ and the Fréchet derivative of these equations at $s=s_o$ is given by

$$DF_o \equiv \left.\frac{\partial(F_1,F_2)}{\partial(u,t)}\right|_{s=s_o} = \begin{bmatrix} A(s_o) & \Phi \\ \Psi^* & 0 \end{bmatrix} \tag{4.6}$$

Since $\Psi^*\Phi \neq 0$ we see DF_o is non-singular. In addition the $F_i(s;u,t)$ are continuously differentiable with respect to all arguments. The Implicit Function Theorem thus gives the desired results.

With very little more work we can see that the eigenvalue $\alpha(s)$ remains simple. This is contained in

<u>Lemma (4.2)</u> Given $(\phi(s),\alpha(s))$ satisfying (4.4) for $|s-s_0|<\delta_1$ then there is no solution of $(A(s)-\alpha(s)I)w = \phi(s)$ for $|s-s_0|<\delta_2, \delta_2>0$.

<u>Proof</u>: We define the <u>linear</u> operator $F(s)$ by

$$F(s)\begin{pmatrix} u \\ t \end{pmatrix} \equiv \begin{pmatrix} A(s)-\alpha(s)I & \phi(s) \\ \psi^*(s) & 0 \end{pmatrix}\begin{pmatrix} u \\ t \end{pmatrix} \tag{4.7}$$

Simplicity is equivalent to requiring that $F(s)\begin{pmatrix} u \\ t \end{pmatrix} = 0$ have only the trivial solution $(u,t)^T = (o,o)^T$. We see that $F(o)\begin{pmatrix} o \\ o \end{pmatrix} = 0$

and $$DF_o \equiv \left.\frac{\partial(F)}{\partial(t,u)}\right|_{s=s_o} = \begin{bmatrix} A(s_o) & \Phi \\ \Psi^* & 0 \end{bmatrix}$$ is non-singular.

Thus we have a unique solution $(u(s),t(s))^T$ with $(u(s_o),t(s_o))^T=(o,o)^T$. Since our operator is linear $(o,o)^T$ remains a solution and by uniqueness is the only one.

It is necessary to strengthen the result of Lemma (4.1) for later use. In general the Implicit Function Theorem guarantees uniqueness of the constructed solution only about the known solution point. In

Lemma (4.1) this would be about $(u,t) = (\Phi,o)$. Lemma (4.2) does not suffer from this defect since in the linear case local uniqueness is equivalent to global uniqueness. We now show that all eigenfunctions (u,t) of (4.5), where t is sufficiently small, are of the form given by Lemma (4.1). We know $\phi(s) = \Phi+w(s)$ where $||w(s)||\to 0$ as $s\to s_0$. To find a distinct eigenfunction we must solve

$$A(s)u = tu$$

$$\text{with} \qquad \Psi^{*}u = 0$$

Then we have

$$\begin{aligned} A(s_0)u &= \left[(A(s_0)-A(s))+tI\right] u \qquad &(a)\\ \Psi^{*}u &= 0 \qquad &(b) \end{aligned} \qquad (4.7)$$

From (4.7)(b) we see $u\varepsilon R(A(s_0))$ and since $A(s_0)$ is non-singular when restricted to this subspace we find

$$||u|| \leq K(||A(s)-A(s_0)|| + |t|)\ ||u||$$

Thus for s near s_0 and t sufficiently small the only solution of (4.7) is $u = 0$. From this we may state that for s sufficiently close to s_0, the smallest eigenvalue of $A(s)$ is $\alpha(s)$ with eigenfunction $\phi(s)$.

One assumption of Lemma (4.1) is that $A(s_0)$ is a Fredholm operator of index zero. Since $A(s) - \alpha(s)I$ is near in norm to $A(s_0)$ it also has this property (Schecter [41]). The previous lemmas show $A(s)-\alpha(s)I$ has a simple zero eigenvalue. These conclusions allow us to decompose B_1 naturally as

$$B_1 = N_1 \oplus X_1 \tag{4.8}$$

where
$$N_1(s) \equiv N(A(s)-\alpha(s)I)$$
$$X_1(s) \equiv R(A(s)-\alpha(s)I) \tag{4.9}$$

It is the subspace X_1 that $\psi^*(s)$ of (4.4) is meant to characterize. This linear functional is assumed known for the purpose of Lemma (4.1) but in practice one really only knows $\psi^*(s_0)$. To implement the lemma one could take $\psi^*(s)$ to be the smooth mapping $\psi^*(s) \equiv \Psi^*$. However once the existence of $(\phi(s),\alpha(s))$ is guaranteed (and since $A(s)-\alpha(s)I$ is Fredholm) we have the existence of $\hat{\psi}^*(s)$ satisfying

$$A^*(s)\hat{\psi}^*(s\ ) = \alpha(s)\hat{\psi}^*(s) \tag{4.10}$$

This linear functional will be used in later sections and the ^ will be dropped. For simplicity we normalize so that $\psi^*(s)\phi(s) = 1$. We may now characterize X_1 as

$$X_1(s) = \{X \varepsilon B_1 | \psi^*(s)X = 0\} \tag{4.11}$$

Corresponding to these subspaces we define the projections

$$Q_1(s)B_1 = N_1$$
$$Q_2(s)B_1 = X_1 \tag{4.12}$$
$$Q_1(s)+Q_2(s) = I$$

These projectors will be used in later work to bound the growth of

$||A(s)^{-1}||$ as $s \to s_0$.

(5) Calculation of $\alpha'(s_0)$

In this section we study the manner in which the small eigenvalue $\alpha(s)$ approaches zero with $s-s_0$. This eigenvalue and the eigenfunction $\phi(s)$ possess the same differentiability properties as the operator $A(s)$. We assume the required smoothness of $A(s)$.

Writing

$$A(s)\phi(s) = \alpha(s)\phi(s) \tag{5.1}$$

and differentiating with respect to s we have

$$A(s)\phi'(s)+A'(s)\phi(s) = \alpha'(s)\phi(s)+\alpha(s)\phi'(s) \tag{5.2}$$

We now apply the adjoint eigenfunction and evaluate at $s = s_0$

$$\psi^*(s_0)A(s_0)\phi'(s_0)+\psi^*(s_0)A'(s_0)\phi(s)=\alpha'(s_0)\psi^*(s_0)\phi(s_0)+\alpha(s_0)\psi^*(s_0)\psi'(s_0) \tag{5.3}$$

Using $\alpha(s_0) = 0$ and $\psi^*(s_0)A(s_0) = 0$

$$\alpha'(s_0)\psi^*(s_0)\phi(s_0) = \psi^*(s_0)A'(s_0)\phi(s_0) \tag{5.4}$$

This allows us to calculate $\alpha'(s_0)$ provided $\psi^*(s_0)\phi(s_0) \neq 0$, i.e., if the eigenvalue is simple.

We now apply this result when $A(s)$ is given by (2.4), i.e.

$$A(s) = \begin{pmatrix} G_u(u(s),\lambda(s)) & N_u(u(s),\lambda(s)) \\ N_u^*(u(s),\lambda(s),s) & N_\lambda(u(s),\lambda(s),s) \end{pmatrix} \tag{5.5}$$

Here we assume $G(u,\lambda)$ and $N(u,\lambda,s)$ are the operators defining the inflated system, but we do not assume $(u(s),\lambda(s))$ is the solution arc, rather we presently assume they are smooth functions of s for which $(u(s_0),\lambda(s_0)) = (u_0,\lambda_0)$, the known solution point. This will prove useful later when they will only be the linear approximations to the solution arc.

Now we have

$$A'(s) = \begin{bmatrix} G_{uu}\frac{du}{ds} + G_{u\lambda}\frac{d\lambda}{ds} & G_{u\lambda}\frac{du}{ds} + G_{\lambda\lambda}\frac{d\lambda}{ds} \\ N^*_{uu}\frac{du}{ds} + N^*_{u\lambda}\frac{d\lambda}{ds} & N^*_{u\lambda}\frac{du}{ds} + N_{\lambda\lambda}\frac{d\lambda}{ds} \end{bmatrix} \tag{5.6}$$

We find that using the normalization defined by (2.8) forces $N^*_{uu} = N^*_{u\lambda} = N_{\lambda\lambda} \equiv 0$. To evaluate (5.4) we use $\phi(s_0)=\Phi, \psi^*(s_0) = \Psi^*$, the eigenfunctions defined by (3.17). Then

$$\Psi^* A'(s_0)\Phi = \psi_1^*(G^o_{uu}\frac{du^o}{ds} + G^o_{u\lambda}\frac{d\lambda^o}{ds})\phi_1 + \frac{1}{\alpha_2}\psi_1^*(G^o_{uu}\frac{du^o}{ds} + G^o_{u\lambda}\frac{d\lambda^o}{ds})\phi_o$$

$$+ \frac{1}{\alpha_2}\psi_1^*(G^o_{u\lambda}\frac{du^o}{ds} + G^o_{\lambda\lambda}\frac{d\lambda^o}{ds}) \tag{5.7}$$

To evaluate further we need expressions for $\frac{du^o}{ds}$, $\frac{d\lambda^o}{ds}$.

We now assume $(u(s),\lambda(s))$ are such that

$$\frac{du^o}{ds} = \xi_o\phi_o + \xi_1\phi_1$$
$$\frac{d\lambda^o}{ds} = \xi_o \tag{5.8}$$

Where (ξ_o,ξ_1) is a solution of the one dimensional Algebraic Bifurcation Equations,

$$f_1(\xi_o,\xi_1) = a_{111}\xi_1^2 + 2b_{11}\xi_1\xi_o + c_1\xi_o^2 = 0$$
$$f_2(\xi_o,\xi_1) = \xi_1^2 + 2\xi_o^2 - 1 = 0 \tag{5.9}$$

We evaluate the Jacobian of (5.9) at (ξ_o,ξ_1) and find

$$4\Delta \equiv \left| \left. \frac{\partial(f_1 f_2)}{\partial(\xi_o,\xi_1)} \right|_{(\xi_o,\xi_1)} \right|$$

$$\Delta = 2\xi_o(a_{111}\xi_1 + \xi_o b_{11}) - \xi_1(\xi_1 b_{11} + \xi_o c_1) \tag{5.10}$$

Now placing our expressions (5.8) into (5.7)

$$\Psi^* A'(s_o)\Phi = \psi_1^*(G^o_{uu}(\xi_o\phi_o+\xi_1\phi_1)+G^o_{u\lambda}\xi_o)\phi_1 + \frac{1}{\alpha_2}\psi_1^*(G^o_{uu}(\xi_o\phi_o+\xi_1\phi_1)+G^o_{u\lambda}\xi_o)\phi_o$$
$$+ \frac{1}{\alpha_2}\psi_1^*(G^o_{u\lambda}(\xi_o\phi_o+\xi_1\phi_1) + G^o_{\lambda\lambda}\xi_o)$$

and thus

$$\Psi^* A'(s_o)\Phi = (\xi_1 a_{111}+\xi_o b_{11}) + \frac{1}{\alpha_2}(\xi_1 b_{11}+\xi_o C_1) \tag{5.11}$$

Now recalling from (3.16) that $\alpha_2 = -\frac{2\xi_o}{\xi_1}$, and that $\Psi^*\Phi = 1$ we have

$$\alpha'(o) = \Psi^* A'(s_o)\Phi = \Delta/2\xi_o \tag{5.12}$$

Hence we may state that if (ξ_o,ξ_1) is an isolated root of the Algebraic

Bifurcation Equations (forcing $\Delta \neq 0$) then $\alpha(s)$ goes to zero like $s-s_o$ provided $\frac{d\lambda^o}{ds} = \xi_o \neq 0$.

This may be restated in the following manner. Suppose we have a point $G(u_o,\lambda_o) = 0$ for which G_u^o is a Fredholm operator of index zero with a simple zero eigenvalue. If the Algebraic Bifurcation Equations at this point have an isolated solution (guaranteeing two distinct branches through (u_o,λ_o)) then along any solution arc for which $\frac{d\lambda^o}{ds} \neq 0$ the linearized operator $A(s)$ of the inflated system (2.3) has an eigenvalue $\alpha(s)$ for which $\frac{d\alpha^o}{ds} \neq 0$. In this case $\alpha(s) \to \alpha(s_o) = 0$ in the same manner as $\lambda(s) \to \lambda(s_o)$. This is the case in Fig. (5.1)(a).

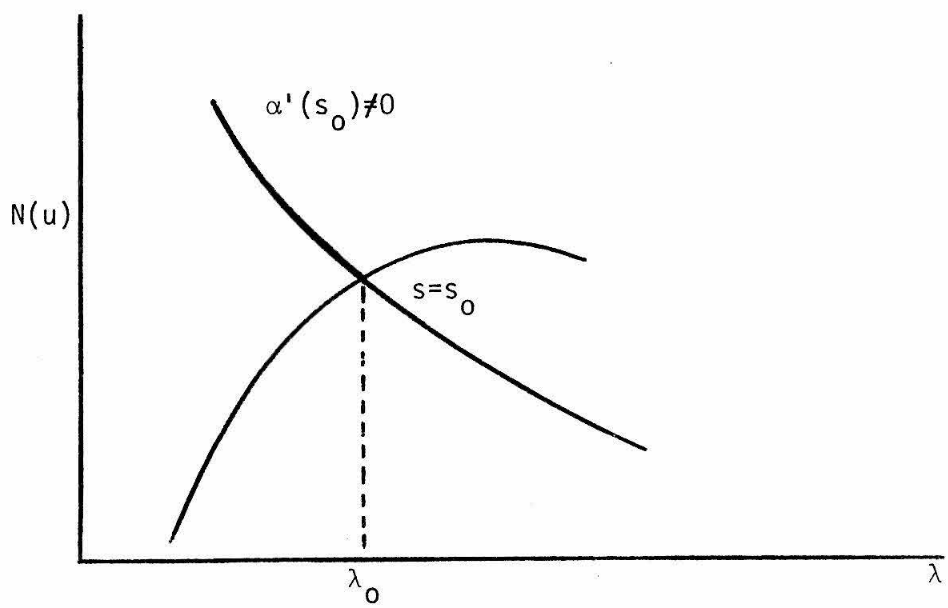


Figure (5.1) (a)

The situation of Fig. (5.1)(b) is more complicated because $A(s_0)$ does not have zero as a simple eigenvalue. Thus the existence of smoothly differentiable $(\alpha(s),\phi(s))$ is not guaranteed and formula (5.4) is not necessarily valid. However, we now show that isolation of the root $(\xi_0,\xi_1) = (0,1)$ of case (b) gives two simple eigenvalues $\alpha_{\pm}(s)$ which approach zero like $\sqrt{s-s_0}$.

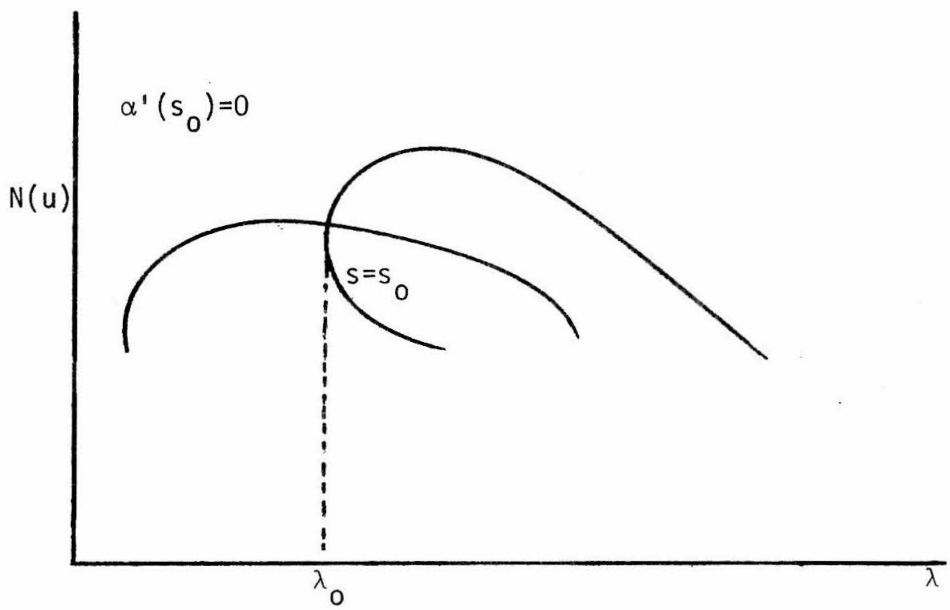


Figure (5.1) (b)

(6) Perturbation of Multiplicity Two Eigenvalue

In this section we shall study the case in which the inflated linearized operator has a non-simple zero eigenvalue. This occurs at a simple bifurcation point with $\xi_0 = 0$. We will find that in this case the eigenvalue is of multiplicity two. To indicate the possible behaviour to be expected we consider the very simple algebraic example

$$A(s) = \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix} \tag{6.1}$$

Here zero is a non-simple multiplicity two eigenvalue for s=0 but for $s \neq 0$ the eigenvalue splits into two simple eigenvalues $\pm\sqrt{s}$ with corresponding eigenvectors

$$\phi_\pm = \begin{pmatrix} \pm\sqrt{s} \\ 1 \end{pmatrix} \tag{6.2}$$

We see neither the eigenvalues nor eigenvectors are continuously differentiable at s=0 and that the eigenvalues are not always real for $|s-s_0|<\delta$. We will find that if the root $(\xi_0,\xi_1)=(0,1)$ is isolated then our inflated system will behave in this manner as well.

Recalling the work of Section (3) we can write $A(s_0)$ for our case as

$$A(s_0) = \begin{pmatrix} G_u^0 & G_\lambda^0 \\ \phi_1^* & 0 \end{pmatrix} \tag{6.3}$$

We see $\Phi_1=(\phi_0,1)^T$ is the unique eigenvector (to within normalization) for the eigenvalue zero. However solving

$$\begin{pmatrix} G_u^o & G_\lambda^o \\ \phi_1^* & 0 \end{pmatrix} \begin{pmatrix} v \\ \alpha \end{pmatrix} = \begin{pmatrix} \phi_o \\ 1 \end{pmatrix} \tag{6.4}$$

we get a solution (unique up to a multiple of ϕ_1)

$$\Phi_2 = \begin{pmatrix} \phi_1 + \phi_2 \\ 0 \end{pmatrix} \tag{6.5}$$

where $G_u^o \phi_2 = \phi_o$, $\psi_1^* \phi_2 = 0$ (This has a solution since $\psi_1^* \phi_o = 0$ making $\phi_o \varepsilon R(G_u^o)$). However trying to solve

$$A(s_o) \begin{pmatrix} v \\ \alpha \end{pmatrix} = \Phi_2 \tag{6.6}$$

we get

$$G_u^o v + G_\lambda^o \alpha = \phi_1 + \phi_2 \tag{6.7}$$

Acting on (6.7) by ψ_1^* we find no solution is possible. Hence the zero eigenvalue has multiplicity $\mu=2$. (Chapter II, Section (2)).

From Lemma II of Section (3) we know $R(A(s_o))$ has codimension 1. Hence we may take.

$$B_1 = sp\{\Phi_2\} \oplus R(A(s_o)) \tag{6.8}$$

and if we define

$$\begin{aligned} R(A(s_o)) &= sp\{\Phi_1\} + Y_2^o \\ Y_1^o &= sp\{\Phi_1, \Phi_2\} \end{aligned} \tag{6.9}$$

then we may write

$$B_1 = Y_1^o \oplus Y_2^o \tag{6.10}$$

and the operator $A(s_o)$ maps Y_2 in a 1-1 fashion onto itself and hence is non-singular on this subspace. We now define projections P_1^o and P_2^o whose ranges are these subspaces

$$\begin{gathered} P_1^o B_1 = Y_1^o \qquad P_2^o B_1 = Y_2^o \\ P_1^o + P_2^o = I \end{gathered} \tag{6.11}$$

We see that for $x \varepsilon B_1$ that $P_1^o x = 0$ is equivalent to

$$\Psi_1^* x = \Psi_2^* x = 0 \tag{6.12}$$

where $\Psi_1^* = (\psi_1^*, 0)$, $\Psi_2^* = (0,1)$.

Now we may write

$$\begin{bmatrix} A(s_o) & 0 \\ 0 & A(s_o) \end{bmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = B^o \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{6.13}$$

where

$$B^o = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.14}$$

We shall prove later in this Section that an equation of the form (6.13) is satisfied for s near s_o. More precisely we have

$$\begin{bmatrix} A(s) & 0 \\ 0 & A(s) \end{bmatrix} \begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \end{pmatrix} = B(s) \begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \end{pmatrix} \tag{6.15}$$

where all quantities depend continuously differentiably on s and

$$B(s_o) \equiv B^o$$
$$\Phi_i(s_o) \equiv \Phi_i \tag{6.16}$$

For the present we will assume this result to be true. It then allows the definition of smooth projections and invariant subspaces

$$Y_1(s) = \text{span}\{\Phi_1(s), \Phi_2(s)\}$$
$$P_1(s)B_1 = Y_1(s)$$
$$Y_1(s) \oplus Y_2(s) = B_1 \tag{6.17}$$
$$P_1(s) + P_2(s) = I$$

with $P_i(s_o) = P_i^o$, $Y_i(s_o) = Y_i^o$ $i = 1,2$. It is the action of $A(s)$ on the two dimensional subspaces $Y_1(s)$ which will allow a determination of $||A(s)^{-1}||$. To study this behaviour we wish to determine the eigenvalues of $A(s)$ when restricted to $Y_1(s)$. These are clearly the eigenvalues of $B(s)$. From the form of $B(s_o)$ given by (6.14) it is easy to see that the element $b_{12}(s)$ plays the most important role. That is, if $b_{12}(s) = b_{12}'(s_o)(s-s_o) + O((s-s_o)^2)$ with $b_{12}'(s_o) \neq 0$ then the eigenvalues of $B(s)$ are distinct and approach zero like $\sqrt{s-s_o}$. Assuming the expression (6.15) we now determine $b_{12}'(s_o)$.

Differentiating (6.15) we find

$$\begin{bmatrix} A(s) & 0 \\ 0 & A(s) \end{bmatrix} \begin{pmatrix} \Phi_1'(s) \\ \Phi_2'(s) \end{pmatrix} + \begin{bmatrix} A'(s) & 0 \\ 0 & A'(s) \end{bmatrix} \begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \end{pmatrix} =$$

$$B'(s) \begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \end{pmatrix} + B(s) \begin{pmatrix} \Phi_1'(s) \\ \Phi_2'(s) \end{pmatrix} \tag{6.18}$$

Evaluating this at $s=s_o$ and acting upon the result with $\Psi_1^*, (\Psi_1^* A(s_o) \equiv 0)$ we find

$$\Psi_1^* A'(s_o)\Phi_1 = b_{11}'(s_o)\Psi_1^*\Phi_1 + b_{12}'(s_o)\Psi_1^*\Phi_2 \tag{6.19}$$

Now $\Psi_1^*\Phi_1 = 0$, $\Psi_1^*\Phi_2 = 1$, and

$$A'(s_o) = \begin{pmatrix} G_{uu}^o\phi_1 & G_{u\lambda}^o\phi_1 \\ 0 & 0 \end{pmatrix} \tag{6.20}$$

Hence

$$\Psi_1^* A'(s_o)\Phi_1 = \psi_1^* G_{uu}^o \phi_1 \phi_0 + \psi_1^* G_{u\lambda}\phi_1 = b_{11} = -\Delta$$

where we have evaluated the Jacobian Δ given by (5.10) at the root $(\xi_o, \xi_1) = (0,1)$.

Thus we see that if our root is isolated then

$$b_{12}'(s_o) = -\Delta \neq 0 \tag{6.21}$$

and so A(s) has two distinct eigenvalues approaching zero like $\sqrt{s-s_o}$.

This estimate was derived assuming the expression (6.15) and we now proceed to justify this assumption. The proof is a modification of a technique employed by McLeod and Sattinger [25] for a simple double eigenvalue.

Without loss of generality we may take

$$\Phi_i(s) = \Phi_i + \chi_i(s) \tag{6.22}$$

$$\text{with} \qquad \Psi_i^* \chi_j(s) = 0 \qquad i,j=1,2$$

and assuming A(s) is at least twice continuously differentiable with respect to s we attempt to solve

$$\begin{pmatrix} A(s_o) & 0 \\ 0 & A(s_o) \end{pmatrix} \begin{pmatrix} \Phi_1 + \chi_1 \\ \Phi_2 + \chi_2 \end{pmatrix} + (s-s_o) \begin{bmatrix} A'(s_o) & 0 \\ 0 & A'(s_o) \end{bmatrix} \begin{pmatrix} \Phi_1 + \chi_1 \\ \Phi_2 + \chi_2 \end{pmatrix} =$$

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (s-s_o) B_1 \right] \begin{pmatrix} \Phi_1 + \chi_1 \\ \Phi_2 + \chi_2 \end{pmatrix} + \begin{pmatrix} R_1(s,\Phi_i,\chi_i) \\ R_2(s,\Phi_i,\chi_i) \end{pmatrix} \tag{6.23}$$

where $R_i(s,\Phi_i,\chi_i) = 0(|s-s_o|^2)$

We may consider (6.23) as an equation for the six quantities χ_i and b_{ij}^1 $i,j = 1,2$. We solve this by forcing both P_1^o and P_2^o acting on (6.23) to be zero. This results in the equations

$$\begin{bmatrix} A(s_o) & 0 \\ 0 & A(s_o) \end{bmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + (s-s_o) \begin{bmatrix} P_2^o A'(s_o) & 0 \\ 0 & P_2^o A'(s_o) \end{bmatrix} \begin{pmatrix} \Phi_1 + \chi_1 \\ \Phi_2 + \chi_2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + (s-s_o)B_1\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} P_2^o R_1(s) \\ P_2^o R_2(s) \end{pmatrix} \tag{6.24}$$

and action by P_1^o is equivalent to forcing

$$(s-s_o)\ \Psi_i^* \left\{ \begin{bmatrix} A'(s_o) & 0 \\ 0 & A'(s_o) \end{bmatrix} \begin{pmatrix} \Phi_1+\chi_1 \\ \Phi_2+\chi_2 \end{pmatrix} - B_1 \begin{pmatrix} \Phi_1+\chi_2 \\ \Phi_2+\chi_2 \end{pmatrix} \right\} = \psi_i^* \begin{pmatrix} R_1(s) \\ R_2(s) \end{pmatrix} \tag{6.25}$$

Now evaluating (6.24) at $s=s_o$ forces

$$\begin{aligned} A(s_o)\chi_1 &= 0 \\ A(s_o)\chi_2 &= \chi_1 \end{aligned} \tag{6.26}$$

Since $\chi_i \varepsilon Y_2^o$ the only solution of (6.26) is $\chi_1 = \chi_2 = 0$. Placing this in (6.25) and using

$$\lim_{s \to s_o} R_i(s)(s-s_o)^{-1} = 0 \tag{6.27}$$

and $\Psi_1^* \Phi_1 = 0 \quad \Psi_1^* \Phi_2 = 1\ , \quad \Psi_2^* \Phi_1 = 1\ , \quad \Psi_2^* \Phi_2 = 0$

we find:

$$\begin{aligned} \psi_2^* A'(s_o)\phi_1 &= b_{11}^1 \\ \psi_1^* A'(s_o)\phi_1 &= b_{12}^1 \\ \psi_2^* A'(s_o)\phi_2 &= b_{21}^1 \\ \psi_1^* A'(s_o)\phi_2 &= b_{22}^1 \end{aligned} \tag{6.28}$$

Thus we can solve (6.26), (6.28) uniquely for χ_i, b^1_{ij} and the Fréchet derivative of these equations is given by

$$\begin{bmatrix} A(s_0) & 0 & 0 & 0 & 0 & 0 \\ 0 & A(s_0) & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad (6.29)$$

which is non-singular as a mapping from $Y^o_2 \times Y^o_2 \times \mathbb{R}^4$ into $Y^o_2 \times Y^o_2 \times \mathbb{R}^4$. Hence the Implicit Function Theorem yields the desired results.

(7) Convergence in the Neighborhood of a Bifurcation Point

In the previous sections we have collected the results needed to allow a solution of the main problem. Suppose through some continuation procedure or otherwise we have located a simple bifurcation point (u_0,λ_0). Can we devise an iteration procedure, based on information available at (u_0,λ_0) which will converge to a point of the desired branch emanating from (u_0,λ_0)? This procedure would allow continuation of solution arcs through bifurcation points.

The procedure to be presented will be Newton's method applied to the inflated problem, where the initial guess will be a point of the ray tangent to $(u(s),\lambda(s))$ at (u_0,λ_0). Two basic results will be stated. The first will show that assuming the existence of a smooth solution branch $x(s)$ through a simple bifurcation point $x(s_0)$, then Newton's method, started with a tangent approximation, will converge to this branch for s sufficiently close to s_0. This method may initially only converge geometrically. The second result, which uses a special property of the initial guess to determine a refined bound on the difference between the first two iterates will prove quadratic convergence of Newton's method. In addition the assumption of the existence of a solution arc $x(s)$ will be dropped. Hence this result will constitute an explicit constructive proof for the existence of a bifurcating branch through $x(s_0)$.

To begin we assume a known solution to the following problem

$$P(x(s_o),s_o) = 0 \quad \text{(a)}$$
$$P_x(x(s_o),s_o)\dot{x}(s_o) = r(s_o) \quad \text{(b)} \qquad (7.1)$$

and we assume $s=s_o$ is a simple bifurcation point. That is G_u^o is a Fredholm operator of index zero with a simple zero eigenvalue (Section (3)). Further the solution $\dot{x}(s_o)$ of (7.1)(b) is forced to satisfy the Algebraic Bifurcation Equations. That is, $\dot{x}(s_o)=(\dot{u}(s_o),\dot{\lambda}(s_o))^T = (\xi_o\phi_o+\xi_1\phi_1,\xi_o)^T$

$$f_1(\xi_o,\xi_1)=a_{111}\xi_1^2+2b_{11}\xi_1\xi_o+c_1\xi_o^2 = 0$$
$$f_2(\xi_o,\xi_1)=2\xi_o^2+\xi_1^2-1 = 0 \qquad (7.2)$$

We assume this root to be isolated and, therefore, Δ defined by (5.10) is non-zero.

The initial guess to the solution point $x(s)$ is defined by

$$x_o(s)=x(s_o)+(s-s_o)\dot{x}(s_o) \qquad (7.3)$$

and we define the linear operator

$$A_o(s)\equiv A(x_o(s)) = P_x(x_o(s),s) \qquad (7.4)$$

Now since $A(s_o)$ is assumed assumed Fredholm of index zero, then so is $A_o(s)$ for s near s_o. From the results of Sections (4)-(6) we see the smallest eigenvalue of $A_o(s)$ is non-zero and thus $A_o(s)$ is non-singular for sufficiently small $|s-s_o|>0$. We define

$$||A_0(s)^{-1}|| = M(s) \tag{7.5}$$

and will find expressions for M(s) later in this section.

The sequence of Newton iterates for this problem is defined by

$$A_\nu(s) = P_x(x_\nu(s),s)$$

$$\text{and} \qquad A_\nu(s)(x_{\nu+1}(s)-x_\nu(s)) = -P(x_\nu(s),s) \tag{7.6}$$

To prove convergence we must show $x_0(s)$ is in an appropriate domain of attraction to x(s) and that all the $A_\nu(s)$ are non-singular. Conditions under which this occurs are stated in: (for the Chord method see [18])

Lemma (7.1) Let x(s) be a twice continuously differentiable arc of solutions of

$$P(x(s),s) = 0 \quad \text{for} \quad |s-s_0|<\delta, \text{ some } \delta>0$$

Suppose we have a known solution of (7.1)-(7.2) and define the initial iterate by (7.3). Then for $|s-s_0|<\delta$ we assume:

$$\max_{s_0\le t\le s} ||\ddot{x}(t)|| \le \kappa(s)$$

$$||A(y,s)-A(x(s),s)|| \le K(s)||y-x(s)|| \tag{7.7}$$

$$\text{for } ||y-x(s)|| \le r(s)$$

$$\text{where} \qquad r(s) \equiv \tfrac{1}{2}(s-s_0)^2\kappa(s)$$

Then if

$$2M(s)K(s)r(s) \equiv \Theta(s) < \frac{1}{2} \tag{7.8}$$

the Newton iterates defined by (7.6) converge at least geometrically with factor

$$\frac{\Theta(s)}{1-\Theta(s)} \tag{7.9}$$

Proof: The proof proceeds by induction. First we note $||x_o(s)-x(s)|| \leq \frac{1}{2}K(s)(s-s_o)^2 = r(s)$. By definition

$$||x_1(s)-x(s)|| = ||x_o(s)-x(s)-A_o^{-1}(s)(P(x_o(s),s)-P(x(s),s))||$$

$$= ||A_o^{-1}(s)\left[A_o(s)(x_o(s)-x(s))-(P(x_o(s),s)-P(x(s),s))\right]||$$

$$= ||A_o^{-1}(s)(A_o(s)-A(y(s),s))(x_o(s)-x(s))||$$

where $y(s) = tx(s)+(1-t)x(s)$ for some $t\varepsilon[0,1]$ and thus

$$||x_1(s)-x(s)|| \leq ||A_o^{-1}(s)||\ ||x_o-x(s)||\ ||A_o(s)-A(y(s),s)||$$

Writing

$$||A_o(s)-A(y(s),s)|| \leq ||A(x_o(s))-A(x(s),s)||+||A(y(s),s)-A(x(s),s)||$$

and since $||x_o(s)-x(s)|| \leq r(s), ||y(s)-x(s)|| \leq r(s)$ we have

$$||x_1(s)-x(s)|| \leq 2M(s)K(s)r(s)||x_o(s)-x(s)||$$

$$\leq \theta||x_o(s)-x(s)||$$

We now suppose $||x_{k+1}(s)-x(s)|| \leq \frac{\theta}{1-\theta}\ ||x_k(s)-x(s)||$ and that

$A_k(s)$ is invertible for $k=0,1,\ldots\nu-1$.

Now

$$A_\nu(s) = A_o(s)(I+A_o^{-1}(s)(A_\nu(s)-A_o(s)))$$

and

$$||A_o^{-1}(s)(A_\nu(s)-A_o(s))|| \leq 2K(s)M(s)r(s) = \theta$$

Thus by the Banach Lemma $A_\nu(s)$ is invertible and

$$||A_\nu(s)^{-1}|| \leq \frac{M(s)}{1-\theta}$$

But then

$$||x_{\nu+1}(s)-x(s)|| = ||x_\nu(s)-x(s)-A_\nu^{-1}(s)(P(x_\nu(s),s)-P(x(s),s))||$$

$$\leq ||A_\nu^{-1}(s)||\ ||x_\nu(s)-x(s)||\ ||A_\nu(s)-A(y,s)||$$

with $y(s) = tx_\nu(s)+(1-t)x(s)$ for some $t\varepsilon[0,1]$

Thus

$$||x_{\nu+1}(s)-x(s)|| \leq \frac{2M(s)K(s)r(s)}{1-\theta}||x_\nu(s)-x(s)||$$

$$= \frac{\theta}{1-\theta}\ ||x_\nu(s)-x(s)||\ .$$

This completes the induction and we see

$$||x_\nu(s)-x(s)|| \leq \left(\frac{\theta}{1-\theta}\right)^\nu ||x_o(s)-x(s)||.$$

———— ———— ————

To apply this lemma it is necessary to determine the magnitude of

$$\theta=2M(s)K(s)r(s)$$

An immediate conflict is apparent. For most problems one would expect $K(s)$ and $\kappa(s)$ to be well behaved, so from (7.7) $r(s) = O((s-s_0)^2)$ which can be made as small as desired by choosing s near s_0. However, $A_0(s)$ becomes singular as s approaches the bifurcation point and thus $M(s)\to\infty$. Hence, it is important to determine the rate at which $M(s)$ becomes unbounded.

First, we consider the case for which $A(s_0)=P_x(x(s_0),s_0)$ has zero as a simple eigenvalue. In this situation we recall the projections Q_1,Q_2 and subspaces N_1,X_1 defined in Section (4). These quantities were originally defined for $A(s) = P_x(x(s),s)$ but the corresponding definitions for $A_0(s)$ are immediate. We see

$$\begin{aligned} ||A_0^{-1}(s)|| &= ||A_0^{-1}(s)(Q_1+Q_2)|| \\ &\leq ||A_0^{-1}(s)Q_1(s)|| + ||A_0^{-1}(s)Q_2(s)|| \end{aligned} \tag{7.10}$$

Since $Q_1(s)B_1=sp\{\phi_1(s)\}$ and $A_0(s)\phi(s)=\alpha_0(s)\phi(s)$ it is clear we may take

$$||A_0^{-1}(s)Q_1(s)|| \leq \frac{M_1(s)}{|\alpha_0(s)|} \tag{7.11}$$

where $M_1(s)$ is well behaved as $s\to s_0$.

Since

$$(A_o(s)-\alpha_o(s)I)X_1 = X_1$$

we see $A_o(s)X_1=X_1$. Now $A_o(s)$ has no null vector in X_1 because from Section (4) $\alpha_o(s)$ is the smallest eigenvalue of $A_o(s)$ for s sufficiently close to s_o. Hence $A_o(s)$ is one-one and onto X_1 and so

$$||A_o^{-1}(s)Q_2(s)|| \leq M_2(s) \tag{7.12}$$

where $M_2(s)$ is bounded as $s \to s_o$.

Hence we may write

$$||A_o^{-1}(s)|| \equiv M(s) \leq \frac{M_o(s)}{|\alpha_o(s)|} \quad \text{as } s \to s_o \tag{7.13}$$

and $M_o(s)$ is uniformly bounded for $|s-s_o|<\delta$. Now we apply the results of Section (5) on the approach to zero of $\alpha_o(s)$. It was found that provided $A(s_o)$ had a simple zero eigenvalue (forcing $\xi_o \neq 0$) then

$$\alpha_o'(o) = \frac{\Delta}{2\xi_o} \tag{7.14}$$

and from our assumed isolation $\Delta \neq 0$.

Hence

$$\text{(simple)} \qquad M(s) \leq \frac{K_o}{|s-s_o|} \quad \text{for } |s-s_o|<\delta \quad . \tag{7.15}$$

This bound will be shown to guarantee iterative convergence.

Now consider the case when $A_o(s_o)$ has a non-simple eigenvalue. Under the assumptions of Section (6) we recall the projections P_1, P_2

and subspaces Y_1, Y_2. Then

$$||A_0^{-1}(s)|| \leq ||A_0^{-1}(s)P_1|| + ||A_0^{-1}(s)P_2|| \tag{7.16}$$

Since $A_0^{-1}(s)P_1$ restricts $A_0^{-1}(s)$ to the two dimensional subspace Y_1 we may take

$$||A_0^{-1}(s)P_1|| \leq \frac{N_1(s)}{|\alpha_1(s)|} \tag{7.17}$$

where $\alpha_1(s)$ is either of the eigenvalues of $A_0(s)$ which approach zero. (Since our root (ξ_0, ξ_1) is isolated both these eigenvalues approach zero at the same rate.) As before we find $A_0^{-1}(s)$ maps Y_2 in a one-one fashion onto itself and hence

$$||A_0^{-1}(s)P_2(s)|| \leq N_2(s) \tag{7.18}$$

Placing (7.17)-(7.18) into (7.16) we find

$$||A_0^{-1}(s)|| \leq \frac{N_0(s)}{|\alpha_1(s)|}$$

where $N_0(s)$ is bounded as s approaches s_0. In Section (6) it was found that isolation determined the behaviour of $\alpha_1(s)$ near zero and we may state:

$$\text{(non-simple)} \qquad ||A_0^{-1}(s)|| \leq \frac{K_0}{|s-s_0|^{\frac{1}{2}}} \qquad \text{for } |s-s_0|<\delta \tag{7.19}$$

These bounds now allow a guarantee of convergence for s sufficiently close to s_0.

Theorem (7.1) Assume the hypotheses of Lemma (7.1).

Then we have the two cases:

(i) If $A(s_0)$ has a simple zero eigenvalue, then $\exists\delta<0$ such that for $|s-s_0|<\delta$

$$\theta = 2M(s)K(s)r(s) \leq K_0K(s)\kappa(s)|s-s_0| < \frac{1}{2} \tag{7.20}$$

and the Newton iterates defined by (7.6) converge.

(ii) Under the assumptions of Section (6) if $A(s_0)$ has zero as a multiplicity 2 eigenvalue then $\exists\delta>0$ such that for $|s-s_0|<\delta$

$$\theta \leq K_0K(s)\kappa(s)|s-s_0|^{3/2} < \frac{1}{2} \tag{7.21}$$

and the Newton iterates converge.

Proof Placing (7.15) and (7.19) into (7.8) of Lemma (7.1) the result follows immediately.

The above result only guarantees geometric convergence, to force the standard quadratic convergence of Newton's method we make use of the following theorem.

Theorem (7.2) Let $F(x)$ be a differentiable mapping from a Banach space B into itself. Let $x_0\varepsilon B$ be such that the Frechet derivative of F at x_0, $DF(x_0)$, is invertible and we have

$$\begin{aligned} &\text{(i)} \quad ||DF^{-1}(x_0)|| \le a \\ &\text{(ii)} \quad ||DF^{-1}(x_0)F(x_0)|| \le b \\ &\text{(iii)} \quad ||DF(x)-DF(y)|| \le c||x-y|| \\ &\qquad\qquad \text{for} \quad ||x-x_0|| \le 2b \\ &\qquad\qquad\qquad\quad ||y-y_0|| \le 2b \end{aligned} \tag{7.22}$$

Then if $abc<\frac{1}{2}$ the Newton iterates

$$x_{\nu+1} = x_\nu - DF^{-1}(x_\nu)\,F(x_\nu)$$

are defined and converge to a unique element x^* for which

$$F(x^*) = 0 \qquad ||x_0-x^*|| \le 2b$$

and the error converges quadratically, i.e.,

$$||x_\nu - x^*|| \le \frac{2b}{2^{2^\nu}}$$

This result is a slight modification of Theorems in Isaacson and Keller [11] or Kantorovich [14].

Before attempting to apply this result with our previous estimates we assume a bound on the linear operator $P_x(y,s)$ near the solution arc $x(s)$.

$$||P_x(y,s)|| \le L(s) \quad \text{for} \quad ||y-x(s)|| \le r(s) \quad \ldots \tag{7.23}$$

We now find expressions for the bounds a,b,c as applied to the ν-th

Newton iterate. From Lemma (7.1)

$$||P_x(x\ (s),s)|| \leq \frac{K_o}{|s-s_o|^{\gamma}} \left(\frac{1}{1-\theta}\right) \tag{7.24}$$

$$c \leq K(s)$$

where $\gamma=1$ (simple) or $\gamma=\frac{1}{2}$ (non-simple).

Further

$$||P_x^{-1}(x_\nu(s),s)P(x_\nu(s),s)|| \leq ||P_x^{-1}(x_\nu(s),s)||\ ||P(x_\nu(s),s)-P(x(s),s)||$$

and

$$||P(x_\nu(s),s)-P(x(s),s)|| \leq ||L(s)||x_\nu(s)-x(s)||$$

since $\quad ||x_\nu(s) - x(s)|| \leq r(s)$

Hence

$$abc \leq \left[\frac{K_o}{|s-s_o|^{\gamma}} \left(\frac{1}{1-\theta}\right)\right]^2 K(s)L(s)||x_\nu(s)-x(s)||$$

But

$$\begin{aligned} ||x_\nu(s)-x(s)|| &\leq \left(\frac{\theta}{1-\theta}\right)^{\nu} ||x_o(s)-x(s)|| \\ &\leq \left(\frac{\theta}{1-\theta}\right)^{\nu} \frac{1}{2}\kappa(s)(s-s_o)^2 \end{aligned}$$

and so we find

$$abc \leq \frac{N(s)}{|s-s_o|^{2\gamma}} (s-s_o)^2 \left(\frac{\theta}{1-\theta}\right)^{\nu} \tag{7.25}$$

where under general smoothness assumptions on $P(x,s)$ and $x(s)$ the function $N(s)$ is well behaved as s approaches s_0.

If we consider (7.25) in the case where $P(x(s_0),s_0)$ has a simple zero eigenvalue, then $abc<\frac{1}{2}$ cannot be guaranteed for $\nu=0$. However, a point in the iteration will be reached for which quadratic convergence can be assured. The situation when $\gamma=\frac{1}{2}$ is different and may be stated as

<u>Theorem (7.3)</u> Assume the requirements of Lemma (7.1). Under the assumptions of Section (6) suppose $P_x(x(s_0),s_0)$ has zero as a multiplicity two eigenvalue . Then $\exists\,\delta<0$ such that for $|s-s_0|<\delta$

$$abc\leq N(s)|s-s_0|<\frac{1}{2}$$

and the Newton iterates (7.6) with initial guess $x_0(s)$ defined by (7.3) converge quadratically to the solution $x(s)$ of

$$P(x(s),s) = 0$$

We now return to the case where $A(s_0)$ has a simple zero eigenvalue and show that quadratic convergence can be salvaged by determining a finer estimate for $||x_1-x_0||$. This calculation will rest strongly on the fact that $x_0(s)$ is determined from a solution of the simple Algebraic Bifurcation Equations. Moore and Spence [27] use a similar approach to prove convergence of a finite dimensional difference approximation for a two-point boundary value problem.

Once again the required estimates result from a consideration of the projections

$$Q_1(s)B_1 = \phi(s)$$
$$Q_1(s)+Q_2(s) = I \tag{7.26}$$

where
$$A_o(s)\phi(s) = \alpha_o(s)\phi(s) \tag{7.27}$$

We write

$$\begin{aligned}||x_1(s)-x_o(s)|| &= ||A_o^{-1}(s)P(x_o(s),s)||\\ &\le ||A_o^{-1}(s)Q_1(s)P(x_o(s),s)||+||A_o^{-1}(s)Q_2(s)P(x_o(s),s)||\\ &\le ||A_o^{-1}(s)Q_1(s)||\ ||Q_1(s)P(x_o(s),s)||+||A_o^{-1}(s)Q_2(s)||\ ||Q_2(s)P(x_o(s),s)||\end{aligned}$$

and so we wish to evaluate $||Q_i(s)P(x_o(s),s)||$. To do this we expand about $s=s_o$ to find

$$\begin{aligned}P(x_o(s),s)&=P(x_o(s_o),s_o)+(s-s_o)\left[P_x(x_o(s_o),s_o)\frac{dx_o(s_o)}{ds}+P_s(x_o(s_o),s_o)\right]\\ &+\frac{(s-s_o)^2}{2}\left[P_x(x_o(s_o),s_o)\frac{d^2x_o(s_o)}{ds^2}+P_{xx}(x_o(s_o),s_o)\left(\frac{dx_o(s_o)}{ds}\right)^2\right.\\ &\left.\quad+2P_{xs}(x_o(s_o),s_o)\frac{dx_o(s_o)}{ds}+P_{ss}(x_o(s_o),s_o)\right]\\ &+O((s-s_o)^3)\end{aligned} \tag{7.28}$$

However $P(x(s_o),s_o) = 0$ and $x_o(s) = x(s_o) + (s-s_o)\dot{x}(s_o)$.

Hence

$$x_o(s_o) = x(s_o)$$

$$\frac{dx_o(s_o)}{ds} = \dot{x}(s_o)$$

$$\frac{d^2x_o(s_o)}{ds^2} = 0$$

From our definition of P(x,s) (Section (2)) we find

$P_{xs} = P_{ss} \equiv 0$. Further

$$P_x(x(s_o),s_o)\dot{x}(s_o) = -P_s(x(s_o),s_o)$$

and so

$$P(x_o(s),s) = \frac{(s-s_o)^2}{2} P_{xx}(x(s_o),s_o)\dot{x}(s_o)\dot{x}(s_o) + O((s-s_o)^3) \qquad (7.29)$$

In addition

$$P_{xx}(x(s_o),s_o)\dot{x}(s_o) = \begin{bmatrix} G^o_{uu}\dot{u}+G^o_{u\lambda}\dot{\lambda} & G^o_{u\lambda}\dot{u}+G^o_{\lambda\lambda}\dot{\lambda} \\ 0 & 0 \end{bmatrix} \qquad (7.30)$$

where $\quad \dot{x}(s_o) = (\dot{u}(s_o),\dot{\lambda}(s_o))^T$

Now since $\psi^*(s)\phi(s) = 1$

$$Q_1(s)P(x_o(s),s) = (\psi^*(s)P(x_o(s),s))\phi(s) \qquad (7.31)$$

Evaluating this coefficient with $\dot{u}(s_o)=\xi_o\phi_o+\xi_1\phi_1$, $\dot{\lambda}(s_o)=\xi_o$ and $\psi^*(s_o) = (\psi_1^*,0)$ we find

$$\psi^{*}(s)P(x(s_0),s_0)=(a_{111}\xi_1^2+2b_{11}\xi_1\xi_0+c_1\xi_0^2)\frac{(s-s_0)^2}{2}+R_1(s)(s-s_0)^3 \qquad (7.32)$$

Since $(\dot{u},\dot{\lambda})^T$ satisfies the Algebraic Bifurcation Equations the term quadratic in $(s-s_0)$ is zero and hence

$$||Q_1(s)P(x_0(s),s)||=R_1(s)|s-s_0|^3 \qquad (7.33)$$

Now unless $P_{xx}(x(s_0),s_0)\dot{x}(s_0)\dot{x}(s_0)\equiv 0$ we will have

$$||Q_2(s)P(x_0(s),s)||=R_2(s)(s-s_0)^2 \qquad (7.34)$$

Placing (7.33)-(7.34) as well as the expressions (7.11)-(7.12), (7.15) into the formula for the difference between the first two Newton iterates we find

$$||x_1(s)-x_0(s)||\leq\frac{K_0R_1(s)}{|s-s_0|}|s-s_0|^3+M_2(s)R_2(s)(s-s_0)^2\equiv R_0(s)(s-s_0)^2 \qquad (7.35)$$

Our previous estimate only allowed $||x_1(s)-x_0(s)||=O(|s-s_0|)$.

This new estimate rests on the fact that the linear operator $A_0(s)$ becomes singular only on a one-dimensional subspace of B_1, and it is precisely in this subspace that the initial guess $x_0(s)$ is a power of Δs more accurate due to the satisfaction of the Algebraic Bifurcation Equations. Our calculation above allows these two factors to balance one another.

With this preparation we can state

<u>Theorem (7.4)</u> Let $P(x,s)$ be three times continuously differentiable. Assume a solution $(x(s_0),\dot{x}(s_0))$ of (7.1) resulting from an isolated root of the Algebraic Bifurcation Equations (7.2), and let $A(s_0)$ have a

simple zero eigenvalue. Define the initial guess as

$$x_0(s) = x(s_0)+(s-s_0)\dot{x}(s_0)$$
$$A_0(s) = P_x(x_0(s),s) \tag{7.36}$$

and the sequence of Newton iterates by

$$A_\nu(s)(x_{\nu+1}(s)-x_\nu(s)) = -P(x_\nu(s),s) \tag{7.37}$$

Suppose for some $\rho>0$ and all $|s-s_0|<\delta_0, \delta_0>0$

$$||A(y,s)-A(z,s)|| \leq K(s)||y-z||$$
$$||y-x_0(s)|| \leq 2\rho$$
$$||z-x_0(s)|| \leq 2\rho$$

Then $\exists\ \delta>0$ such that for $|s-s_0|<\delta$ the Newton iterates (7.37) are all well defined and converge quadratically to a solution $x^*(s)$ of

$$P(x^*(s),s) = 0$$

This solution is unique in the ball $||x_0(s)-x||<2\rho$.

<u>Proof</u>: We must satisfy the hypotheses of Theorem (7.2). First, from (7.15) $\exists\ \delta_1>0$ for which

$$||P_x^{-1}(x_0(s),s)|| \leq \frac{K_0}{|s-s_0|} \quad \text{for } 0<|s-s_0|<\delta_1$$

and from the previous work $\exists\ \delta_2>0$ for which

$$||P_x^{-1}(x_0(s),s)P(x_0(s),s)|| \leq R_0(s)(s-s_0)^2 \qquad |s-s_0|<\delta_2$$

Now $\exists\ \delta_3>0$ such that $R_0(s)(s-s_0)^2<2\rho$ for $|s-s_0|<\delta_3$.
Taking $\delta=\min(\delta_1,\delta_2,\delta_3)$ we find

$$abc \leq \frac{K_0}{|s-s_0|} R_0(s)(s-s_0)^2 K(s)$$

$$= K_0 K(s) R_0(s)|s-s_0| \qquad \text{for } |s-s_0|<\delta$$

and for δ sufficiently small $abc<\frac{1}{2}$ guaranteeing convergence.

This iteration procedure allows the prescription of a method for switching branches at simple bifurcation points. The procedure is indicated schematically in Figure (7.1). Suppose we employ regular Euler-Newton continuation to calculate a solution arc $y(t)$ and suppose we locate a bifurcation point $y(t_0)$. At this point we solve the Algebraic Bifurcation Equations to find two distinct tangents (provided the roots are isolated). Using the tangent associated with the second arc $x(s)$ we construct the initial guess $x_0(s)$ defined by (7.36). Theorem (7.4) then guarantees that for $|s-s_0|$ sufficiently small the Newton iterates will converge to a point on the new solution arc. (The cones indicate the region in which geometric contraction is guaranteed by Lemma (7.1), i.e., $2M(s)K(s)r(s)\leq\frac{1}{2}$.)

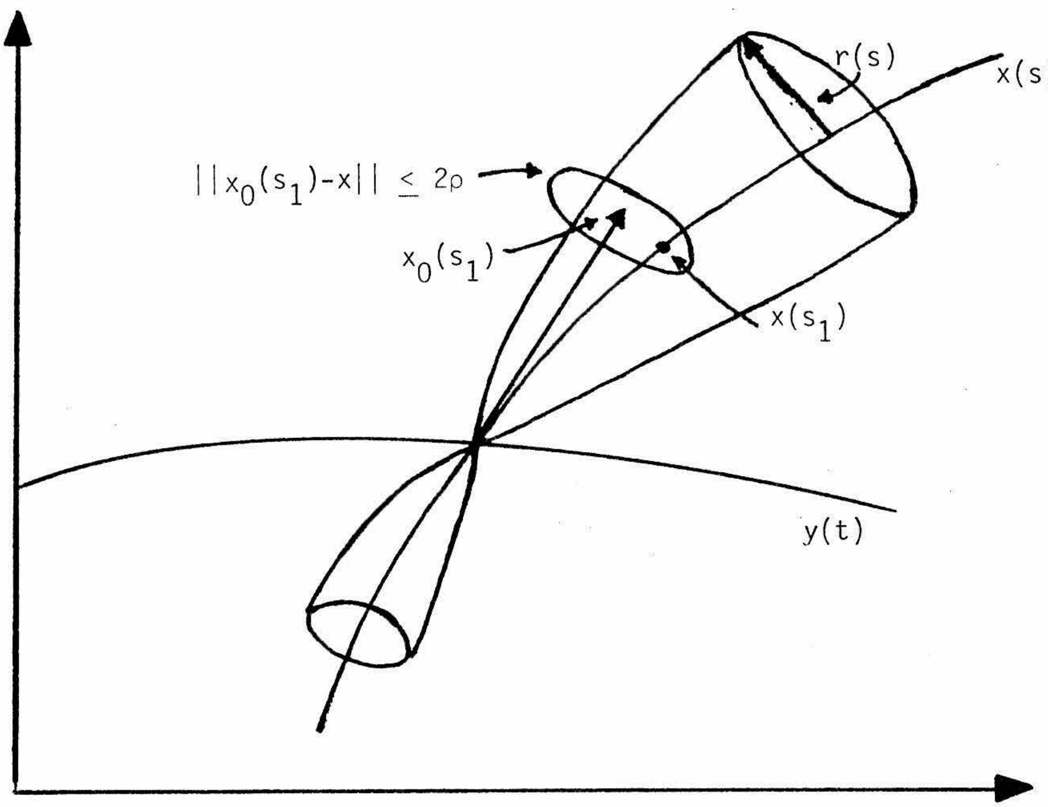


Figure (7.1)

Theorem (7.4) does not assume the existence of a solution arc x(s) through $x(s_0)$. Hence, the guarantee of convergence constitutes a constructive existence proof for a bifurcating solution arc. This result is compatible with the work of Chapter II, since to obtain $\dot{x}(s_0)$ we must solve the Algebraic Bifurcation Equations, and to get the desired behaviour of $\alpha_0(s)$ (and hence $||A_0^{-1}(s)||$) we required this root to be isolated. It may be noted that the existence results of Chapter II are in a sense constructive since their proofs rely on the

Implicit Function Theorem and hence on a Contraction Mapping Theorem. The above approach is, however, more explicit and yields quadratic convergence; whereas an application of a contractive mapping technique could only guarantee linear convergence.

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