SOME PROBLEMS OF EDGE WAVES AND STANDING WAVES ON BEACHES

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ABSTRACT

Some problems of edge waves and standing waves on beaches are examined.

The nonlinear interaction of a wave normally incident on a sloping beach with a subharmonic edge wave is studied. A two-timing expansion is used in the full nonlinear theory to obtain the modulation equations which describe the evolution of the waves. It is shown how large amplitude edge waves are produced; and the results of the theory are compared with some recent laboratory experiments.

Traveling edge waves are considered in two situations. First, the full linear theory is examined to find the finite depth effect on the edge waves produced by a moving pressure disturbance. In the second situation, a Stokes' expansion is used to discuss the nonlinear effects in shallow water edge waves traveling over a bottom of arbitrary shape. The results are compared with the ones of the full theory for a uniformly sloping bottom.

The finite amplitude effects for waves incident on a sloping beach, with perfect reflection, are considered. A Stokes' expansion is used in the full nonlinear theory to find the corrections to the dispersion relation for the cases of normal and oblique incidence.

Finally, an abstract formulation of the linear water waves problem is given in terms of a self adjoint but nonlocal operator. The appropriate spectral representations are developed for two particular cases.

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INTRODUCTION

The purpose of this thesis is to examine some problems related to nonlinear waves on a sloping beach. To this end, perturbation expansions for the solutions to the equations of motion are found in various special cases. Both the full nonlinear theory and the shallow water approximation are examined in the various situations and their predictions compared. Since perturbation expansions are used, an appropriate understanding of the linear problem is needed. To this end, in the first chapter the relevant features of the linear problem are briefly described. The detailed discussion of some linear results that appear to be new is postponed to the last chapter.

The second chapter is concerned with the study of the nonlinear interactions of edge waves and incoming waves. It is shown, using the full nonlinear theory, that small edge waves become unstable when they interact with an incoming wave of twice their frequency. However, as the edge waves grow, higher order nonlinear effects become important stopping their growth and stabilizing the motion. To follow this process a two-timing expansion is developed and modulation equations for the edge wave are obtained. An expression for the final amplitude of the subharmonic edge wave in terms of the parameters of the incoming wave is found. The amplitude of the resulting standing edge wave was recently estimated by Guza and Inman in the shallow water case, and our results reduce to the shallow water ones in the appropriate limit. The results are compared with the available experimental data and good agreement is found.

Since edge waves are not only generated by instability mechanisms but also by moving pressure disturbances, the linear edge waves produced by a pressure disturbance moving parallel to the shore are found. The linear initial value problem is solved for the case of a uniformly sloping beach of finite depth. When the results obtained are calculated for small angles of the sloping beach, the known shallow water results obtained by Greenspan are recovered. This is the problem discussed in the third chapter.

The fourth chapter is devoted to the discussion of traveling shallow water edge waves. The problem of nonlinear traveling edge waves was recently studied by Whitham for the case of a uniformly sloping beach. In that case the shallow water theory gives an anomalous behavior for the nonlinear solution away from the shore. In this chapter the anomaly is shown to be associated with the invalidity of the shallow water theory away from the shore. When the shallow water theory is used, for depth distributions which remain finite and shallow, satisfactory results are obtained.

The work in the second chapter involved incoming waves with perfect reflection at the shoreline. Although not required to the order considered there, it is interesting to consider self-interaction finite amplitude effects for such standing waves. This problem was considered by Carrier and Greenspan, who found exact solutions for the nonlinear shallow water equations. However, as discussed previously, the shallow water approximation is not valid for a uniformly sloping beach. In the fifth chapter approximate solutions of the full nonlinear equations are found in the form of standing waves with finite amplitude.

The sixth chapter is concerned with an abstract formulation of the linear water waves problem. The linear problem is formulated in terms of a self adjoint but nonlocal operator. The spectral representation of the operator is then found in terms of the standing and edge wave solutions of the linear problem.

Finally, an appendix is added to provide detailed justification of some questions discussed in the main text.

CHAPTER 1

LINEAR WAVES ON SLOPING BEACHES

It is the purpose of this chapter to review some of the known features of linear waves on sloping beaches. Two new eigenfunction expansions that are used subsequently to solve nonhomogeneous problems arising from perturbation expansions are also briefly described; their detailed discussion is deferred to the last chapter. In the first section the full linear theory is discussed, while in the second section the shallow water theory is examined.

1.1 Full Linear Theory

For waves on a sloping beach, the linearized equations of motion for the velocity potential Φ are

$$\begin{aligned} &\frac{1}{g}\Phi_{tt} + \Phi_z = 0 \qquad \text{on } z=0,\ y \geq 0, \\ &\Phi_{yy} + \Phi_{zz} + \Phi_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ y \geq 0,\ -y\tan\beta \leq z \leq 0, \\ &\Phi_y \sin\beta + \Phi_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0. \end{aligned} \tag{1.1}$$

Here, x denotes the longshore coordinate, y the offshore coordinate, and z the vertical coordinate; the angle of the sloping beach is β. It is assumed throughout this work that $0 \leq \beta \leq \pi/2$. The surface elevation ζ is given in terms of Φ by

$$\zeta = -\frac{1}{g}\Phi_t\big|_{z=0}.$$

Consider now a solution of (1.1) in the form

$$\Phi(x,y,z,t) = \varphi(y,z)\cos(kx-\omega t). \tag{1.2}$$

Then substitution of (1.2) into (1.1) gives the equations for φ ; they are

$$\varphi_z - \frac{\omega^2}{g}\varphi = 0 \quad \text{on } z=0,\ y \geq 0,$$

$$\varphi_{yy} + \varphi_{zz} - k^2\varphi = 0 \quad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0, \tag{1.3}$$

$$\varphi_y \sin\beta + \varphi_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0.$$

We now discuss the solutions of (1.3) which will be of interest in the following chapters.

First of all, consider the case of no longshore variations: $k = 0$. A detailed discussion can be found in Stoker (1957) and we just summarize the results in a form suitable for our later purposes. For arbitrary angles β there is a continuous spectrum of solutions of (1.3) which represent incoming waves with perfect reflection. The general solution was found by Peters in 1952, but requires complicated integral representations which are difficult to use when combined with other effects. However, for the special angles $\beta = \pi/2n$, where n is integer, the desired solutions simplify to a finite sum of exponential and trigonometric functions (Stoker, 1957), and these are more manageable.

We note in detail the solutions for $\beta = \pi/4$ since these are the simplest of all; the behavior of the standing wave solutions for the other submultiples of $\pi/2$ is qualitatively the same.

The desired solutions for $\pi/4$ first found by Hanson (1926) are:

$$\bar{\Psi}_\ell(y,z) = \frac{1}{2}\left\{ e^{i\pi/4}\, e^{-\ell y - i\ell z} + e^{-i\pi/4}\, e^{\ell z - i\ell y} + c.c. \right\}, \quad \ell \geq 0, \tag{1.4}$$

where $\omega^2 = g\ell$. The surface value $\bar{\Psi}_\ell(y,0)$, which will play an essential role later, is given by:

$$\Psi_\ell(y,0) = \frac{1}{\sqrt{2}}\, e^{-\ell y} + \cos(\ell y + \pi/4). \tag{1.5}$$

The second term of (1.4) is the usual deep water wave, and the first term is needed in order to satisfy the boundary condition on the sloping bottom.

The solutions for the other submultiples of $\pi/2$ have a similar structure. For $\beta = \pi/2m$ they are (Stoker, 1957):

$$\Psi_\ell(y,z) = \sum_{k=1}^{m}{}' \; c_k \exp\big(\beta_k (y + iz)\big) + \text{c.c.} \tag{1.6}$$

where the complex constants β_k and c_k are given by:

$$\beta_k = \exp\left\{ i\pi\left(\frac{k}{m} + \frac{1}{2}\right)\right\}, \text{ for } k = 1, \ldots, m.$$

$$c_k = \exp\left\{ i\pi\left(\frac{m+1}{4} - \frac{k}{2}\right)\right\} \cot\frac{\pi}{2m} \cot\frac{2\pi}{2m} \cdots \cot\frac{k-1}{2m}\pi, \text{ for } k = 2, \ldots, m.$$

$$c_1 = \bar{c}_m.$$

Solutions (1.6) contain a deep water term and a finite number of terms exponentially small at infinity but important near the shore, which are needed to satisfy the boundary condition on the sloping bottom.

We will be interested in this work in solutions of nonhomogeneous problems in the form:

$$\begin{aligned} &\varphi_z - \ell_0 \varphi = f(y) && \text{on } z = 0,\ y \geq 0, \\ &\varphi_{yy} + \varphi_{zz} = 0 && \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0, \\ &\varphi_y \sin\beta + \varphi_z \cos\beta = 0 && \text{on } z = -y\tan\beta,\ y \geq 0. \end{aligned} \tag{1.7}$$

The function $f(y)$ will be assumed to be smooth and to tend to zero appropriately at infinity. To solve (1.7) the functions $\Psi_\ell(y,z)$ will be used. It can be shown that suitably normalized multiples $S_\ell'(y,z)$ of $\Psi_\ell(y,z)$ are complete, in the sense that every sufficiently integrable function can be represented as:

$$f(y) = \int_0^\infty S_\ell(y,0)\, d\ell \int_0^\infty S_\ell(t,0)\, f(t)\, dt \,. \tag{1.8}$$

In the last chapter it is proved that the functions $S_\ell(y,0)$ give the spectral representation of an appropriately defined self adjoint operator which describes the linear problem. Now we just give the formal solutions of (1.7), whose justification is provided in the Appendix.

First of all, consider the case $\ell_0 < 0$, then the homogeneous problem has no bounded solutions (i.e., ℓ_0 is not in the spectrum). In this case the square integrable solution for (1.7) is given by

$$\varphi(y,z) = \int_0^\infty \frac{S_\ell(y,z)}{\ell - \ell_0}\, d\ell \int_0^\infty S_\ell(y',0)\, f(y')\, dy' \,. \tag{1.9}$$

Consider now the case $\ell_0 > 0$. In this case ℓ_0 is in the continuous spectrum. A bounded solution, oscillatory as $y \to \infty$, is obtained. The solution is not unique since $S_{\ell_0}(y,z)$ solves the homogeneous problem. A convenient expression for the solution of (1.7) is:

$$\varphi(y,z) = \text{P.V.} \int_0^\infty \frac{S_\ell(y,z)}{\ell - \ell_0}\, d\ell \int_0^\infty S_\ell(t,0)\, f(t)\, dt + c\, S_{\ell_0}(y,z) \,; \tag{1.10}$$

where c is an arbitrary constant.

A particular forcing function in (1.7), which will be of interest later, is $f(y) = \dot{S}_{\ell_0}(y,z)$. The solution of (1.7) is obtained using the following procedure. Let $\omega^2 = g\ell$ in (1.3), differentiate (1.3) with respect to ℓ , and let $\ell = \ell_0$. Since the equation and the bottom boundary condition are independent of ℓ , we obtain for the derivative $\dot{S}_\ell(y,z)$:

$$\begin{aligned} &\dot{S}_\ell(y,z) - \ell\,\dot{S}_\ell(y,z) = S_\ell(y,z) \quad \text{on } z=0,\ y\geq 0, \\ &\dot{S}_{\ell yy} + \dot{S}_{\ell zz} = 0 \quad \text{for } y\geq 0,\ -y\tan\beta \leq z \leq 0, \\ &\dot{S}_{\ell y}\sin\beta + \dot{S}_{\ell z}\cos\beta = 0 \quad \text{on } z=-y\tan\beta\ ,\ y\geq 0. \end{aligned} \tag{1.11}$$

In (1.11) let $\ell = \ell_0$, then this gives $\dot{S}_{\ell_0}(y,z)$ as the solution of (1.7) when $f(y) = S_{\ell_0}(y,0)$.

We now consider (1.3) for the case $k \neq 0$, and take $k > 0$. It was shown by Hanson (1926), for angles $\beta = \pi/2m$, that a continuous spectrum of oblique waves with perfect reflection exists for $\omega^2 \geq gk$. In 1952 Peters obtained integral representations for the standing wave solutions valid for arbitrary angle β . There is, however, a very important qualitative difference between the cases $k=0$ and $k\neq 0$. This is the following:

For $k \neq 0$, besides the solutions in the continuous spectrum, there are solutions with finite energy which represent normal modes trapped at the shoreline. The first trapped mode solution was found by Stokes (1846), and it is known as Stokes' edge wave. In the notation of (1.3) it is given by

$$E(y,z) = e^{-ky\cos\beta + kz\sin\beta}\ , \quad \omega^2 = gk\sin\beta, \tag{1.12}$$

and is present for all $0 < \beta < \pi/2$.

It was shown by Ursell (1952) that as β decreases the number of trapped modes increases. The highest edge wave modes are linear combinations of exponentials, and they will be denoted by $E_m(y,z)$. The functions E_m are

$$E_m(y,z) = e^{-ky\cos\beta + kz\sin\beta} + \sum_{m=1}^{n} A_{mn}\left\{ e^{-ky\cos(2m-1)\beta - kz\sin(2m-1)\beta} + \right.$$
$$\left. + e^{-ky\cos(2m+1)\beta + kz\sin(2m+1)\beta} \right\}; \tag{1.13}$$

where the constants A_{mn} are given by:

$$A_{mn} = (-1)^m \prod_{r=1}^{m} \tan(n-r+1)\beta / \tan(n+r)\beta \, ,$$

and the dispersion relation is $\omega^2 = gk \sin(2n+1)\beta$. There is a restriction on the integer n since we want the exponentials in (1.13) to be decaying. This gives $(2n+1)\beta \leq \pi/2$. To summarize, there is a continuous spectrum for $\omega^2 \geq gk$, together with points in the spectrum, edge waves, whose number increases as β decreases.

We now consider the nonhomogeneous problem (1.3) when $k > 0$. This problem arises when edge waves produced by moving disturbances are considered. It also arises when the nonlinear theory for traveling edge waves is developed (Whitham, 1976). The nonhomogeneous problems of interest for our discussion are of the form:

$$\begin{aligned}
&\varphi_z - \lambda_0 \varphi = f(y) && \text{on } z = 0,\ y \geq 0, \\
&\varphi_{yy} + \varphi_{zz} - k^2 \varphi = 0 && \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0, \\
&\varphi_y \sin\beta + \varphi_z \cos\beta = 0 && \text{on } z = -y\tan\beta,\ y \geq 0.
\end{aligned} \tag{1.14}$$

Problem (1.14) was first solved by Whitham (1976), when $f(y)$ was a linear combination of exponentials, using a transform similar to (1.8). It will be shown in the last chapter how Whitham's transform provides the spectral representation of an appropriate self adjoint operator, analogous to the one encountered in (1.7).

Now we quote Whitham's results relevant to (1.14), and take for simplicity $\beta = \pi/4$. In this case, the oblique incoming waves with perfect reflection are given by

$$\Psi_\ell(y,z) = \frac{1}{2}\left\{ (\ell + i\lambda)\, e^{i\ell y + \lambda z} + (\ell - i\lambda)\, e^{-\lambda y + i\ell z} + \text{c.c.} \right\} \tag{1.15}$$

where $\lambda = (\ell^2 + k^2)^{1/2}$ and $\ell \geq 0$. In this case, there is only one edge wave solution for (1.14), the Stokes' solution, which is given by:

$$E(y,z) = e^{-\frac{k}{\sqrt{2}} y + \frac{k}{\sqrt{2}} z} .$$

Expansion (1.8) is now modified since the edge waves are also needed to represent an arbitrary pressure disturbance. The suitably normalized surface values $\bar{\Psi}_\ell(y, 0)$ are given by

$$S_\ell(y,0) = \sqrt{\frac{2}{\pi}}\, (\lambda^2 + \ell^2)^{1/2} \left\{ \frac{1}{2}(\ell + i\lambda)\, e^{i\ell y} + \text{c.c.} + \ell\, e^{-\lambda y} \right\},$$

and the normalized edge wave solution is:

$$E(y) = 2^{1/4} k^{1/2} e^{-\frac{k}{\sqrt{2}} y} .$$

In this notation Whitham's result reads as follows:

$$f(y) = \int_0^\infty f(t) E(t)\, dt \; E(y) + \int_0^\infty S_\ell(y,0)\, d\ell \int_0^\infty S_\ell(t,0) f(t)\, dt . \tag{1.16}$$

For smaller angles β more edge waves appear and they will be added to the expansion formula (1. 16).

We now use (1. 16) to solve (1. 14). First of all, assume that λ_0 is not in the spectrum. In this case there is a unique solution for (1. 14) and it is given by:

$$\varphi(y,z) = \int_0^\infty E(t) f(t)\, dt \; (k/\sqrt{2} - \lambda_0)^{-1} \, E(y,z) \; +$$

$$+ \int_0^\infty \frac{S_\ell(y,z)\; d\ell}{(\ell^2 + k^2)^{1/2} - \lambda_0} \int_0^\infty S_\ell(t,0)\, f(t)\, dt \,. \tag{1. 17}$$

When λ_0 is a point in the spectrum, a bounded solution is obtained provided f is orthogonal to E . In this case, the solution is not unique since an arbitrary multiple of E can be added to the particular solution. Finally, when λ_0 is in the continuous spectrum, the general solution which is oscillatory as $y \to \infty$ is conveniently expressed as:

$$\varphi(y,z) = \int_0^\infty E(t) f(t)\, dt \; (k/\sqrt{2} - \lambda_0)^{-1} \, E(y,z) \; +$$

$$+ \mathrm{P.V.} \int_0^\infty \frac{S_\ell(y,z)\; d\ell}{(\ell^2 + k^2)^{1/2} - \lambda_0} \int_0^\infty S_\ell(t,0)\, f(t)\, dt \; + \; c\, S_{\ell_0}(y,z) \,, \tag{1. 18}$$

where $\lambda_0 = (\ell_0^2 + k^2)^{1/2}$, and c is an arbitrary constant.

When nonlinear corrections are developed for oblique incident waves with perfect reflection, the problem analogous to (1. 11) arises. Then we need a solution of:

$$\varphi_z - \lambda_0 \varphi = \mathcal{S}_{\ell_0} \qquad \text{on } z=0,\ y \geq 0,$$

$$\varphi_{yy} + \varphi_{zz} - k^2 \varphi = 0 \qquad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0. \tag{1.19}$$

$$\varphi_y \sin\beta + \varphi_z \cos\beta = 0 \qquad \text{on } z = -y\tan\beta,\ y \geq 0.$$

in (1.19) ℓ_0 is given by $\ell_0^2 = \lambda_0^2 - k^2$. As in (1.11) differentiation with respect to λ of the solutions of the homogeneous problem (1.19) gives the desired solution of the nonhomogeneous problem.

1.2 The Linear Shallow Water Approximation

In the previous section the full linear theory was described and appropriate transforms were developed to solve boundary value problems. However, when β is small, expansions in the eigen functions (1.6) become increasingly complicated. It is therefore natural to turn to the shallow water approximation appropriate for small β.

The linearized shallow water equation for the surface elevation ζ (which is the same as the one for the velocity potential) is:

$$\frac{1}{g}\zeta_{tt} - \left\{ (h\,\zeta_y)_y + h\,\zeta_{xx} \right\} = 0, \text{ for } -\infty < x < \infty,\ y \geq 0. \tag{1.20}$$

In equation (1.20), due to the shallow water approximation, the z dependence is no longer present. The longshore coordinate is denoted by x, and the offshore coordinate is denoted by y. The function $h(y)$ represents the depth distribution, and it is taken to be a positive increasing function.

Consider now solutions of (1.20) in the form

$$\zeta(x,y,t) = f(y)\cos(kx - \omega t). \tag{1.21}$$

Substitution of (1.21) into (1.20) gives the equation for $f(y)$

$$(-hf')' + k^2 h f - \frac{\omega^2}{g} f = 0. \tag{1.22}$$

The appropriate boundary conditions are $f(0)$ finite and f bounded at ∞. As in the full linear theory, we consider first the case $k=0$. In this case, (1.22) becomes:

$$-(hf')' - \frac{\omega^2}{g} f = 0 \quad , \qquad f(0) \text{ finite}. \tag{1.23}$$

The usual example of depth distribution $h(y)$ used to describe shallow water waves on sloping beaches is $h(y) = \beta y$, where β is the angle of the sloping beach. For this case the appropriate wave solution of (1.23), with perfect reflection at the shore, is given by:

$$S_\ell(y) = \beta^{-1/2} J_0\left(2\sqrt{\frac{\ell y}{\beta}}\right), \quad \ell \geq 0 \tag{1.24}$$

and the dispersion relation is $\omega^2 = g\ell$. The expansion theorem (analogous to (1.8)) associated with the functions $S_\ell(y)$ is just the Fourier-Bessel expansion:

$$f(y) = \int_0^\infty \beta^{-1/2} J_0\left(2\sqrt{\frac{\ell}{\beta} y}\right) d\ell \int_0^\infty \beta^{-1/2} J_0\left(2\sqrt{\frac{\ell t}{\beta}}\right) f(t)\, dt. \tag{1.25}$$

The representation (1.25) is similar to (1.8) since there are no trapped modes for $k=0$.

Consider now (1.22) with $h(y) = \beta y$, $k > 0$. The equation for f takes the form:

$$y f'' + f' + \left(\frac{\omega^2}{g\beta} - k^2 y\right) f = 0, \qquad f(0) \text{ finite}. \tag{1.26}$$

Equation (1.26) is Laguerre's equation; hence, the eigenfunctions of

(1.26) are:

$$f_m(y) = e^{-ky} L_m(2ky), \tag{1.27}$$

and the natural frequencies are $\omega_m^2 = gk(2m+1)\beta$. There is an important difference between the shallow water theory and the full theory for $k \neq 0$. In the full theory the number of normal modes increases as $\beta \to 0$, but their number is finite and a continuous spectrum is always present. However, in (1.26) the trapped modes become infinite in number and the continuous spectrum disappears.

The difference just described is related to the fact that the shallow water theory is not appropriate to describe the waves in the offshore region. In fact, for a uniformly sloping beach the assumption of small depth is violated away from the shoreline. It will be shown later that when the depth distribution $h(y)$ is taken to remain shallow away from the shore, results similar to the ones of the full theory are obtained. There is a continous spectrum of incident waves with perfect reflection and a finite number of edge waves.

In linear problems it is appropriate to use the shallow water approximation on uniformly sloping beaches since the difference with the full theory arises when the amplitude is negligible. However, when nonlinear effects are examined the difference between the shallow water and full theory becomes important; and the shallow water theory will give the correct results at infinity only for beaches which remain shallow away from the shore.

CHAPTER 2

STANDING EDGE WAVES OF FINITE AMPLITUDE

This chapter concerns the excitation of standing edge waves by normally incident wavetrains. We study the evolution of a standing edge wave, which is a subharmonic of a wave normally incident on a sloping beach. A two-timing expansion is used in the full nonlinear theory to obtain the modulation equations for the amplitude and phase of the edge wave. The solution of the modulation equations for small amplitude of the edge wave recovers the instability results found by Guza and Davis (1974) using the shallow water approximation. Further study of the modulation equations shows that a periodic edge wave of finite amplitude is formed, since the nonlinear terms eventually stop the growth of the early stages. The amplitude of the final standing wave is calculated in terms of the known parameters of the incident wave. Finally, a comparison of the theory with available experimental data is made.

2.1 Introduction

The purpose is to examine the behavior of a standing edge wave in the presence of a wave normally incident on a sloping beach, whose frequency is twice the frequency of the edge wave concerned. Part of this problem has been studied by Guza and Davis (1974), using the nonlinear shallow water theory (to second order in the amplitude) for a uniformly sloping beach. They show that small edge waves become unstable, and compute numerically their growth rates. In this chapter, the full nonlinear theory (approximated to third order in the amplitude) is used to describe the interaction of the waves.

In the second and third sections, the problem is formulated using a two-timing expansion in order to obtain the nonlinear modulation equations which describe the edge wave. The interaction mechanism is found to be very close to the one which produces subharmonic resonance in simpler nonlinear oscillators. In the fourth section the modulation equations are examined to study the behavior of the edge waves. It is found that for all angles β small edge waves become unstable and grow; however, higher-order nonlinear effects become important and stop the growth of the edge wave. The amplitude of the edge wave in the final state is calculated in terms of the parameters of the incoming wave and the results are compared with the experiments. It is also shown that in the final state the motion consists of a large edge wave maintained by what now is the smaller incoming wave with its reflections.

In the last section, a separate point is discussed briefly. It is shown that free-standing periodic edge waves of finite amplitude are not possible since the energy is radiated to infinity due to the nonlinear self-interaction of the edge wave.

2.2 Formulation of the Problem

It is convenient in the equations of motion to eliminate the surface elevation in favor of the velocity potential. When the velocity potential is expressed as $\frac{g}{\omega}\Phi$, the equation for Φ becomes:

$$\frac{1}{g}\Phi_{tt} + \Phi_z = Q(\Phi) + C(\Phi) \quad \text{on } z=0, \quad y \geq 0,$$

$$\Phi_{yy} + \Phi_{zz} + \Phi_{xx} = 0 \quad \text{for } -\infty < x < \infty, \; y \geq 0, -y\tan\beta \leq z \leq 0 \qquad (2.1a)$$

$$\Phi_y \sin\beta + \Phi_z \cos\beta = 0, \quad \text{on } z = -y\tan\beta, \; y \geq 0, \tag{2.1b}$$

$\Phi \sim$ given incoming wave + outgoing wave as $y \to \infty$.

In (2.1) x denotes the longshore coordinate, y the offshore coordinate, and z the vertical coordinate. The expressions for the quadratic and cubic terms are conveniently taken in the form

$$Q(\Phi) = -\frac{1}{\omega}|\nabla\Phi|^2_t + \frac{1}{\omega}\left(\left(\Phi_z + \frac{1}{g}\Phi_{tt}\right)\Phi_t\right)_z ,$$

$$C(\Phi) = -\frac{g}{2\omega^2}\left(\Phi_x|\nabla\Phi|^2_x + \Phi_y|\nabla\Phi|^2_y + \Phi_z|\nabla\Phi|^2_z\right) + \frac{1}{\omega^2}\left(\Phi_t|\nabla\Phi|^2_t\right)_z +$$
$$+ \frac{g}{\omega^2}\left(\left(\Phi_z + \frac{1}{g}\Phi_{tt}\right)\left(|\nabla\Phi|^2 - \frac{1}{g}\Phi_t^2\right)\right)_z .$$

We now describe the mechanism of interaction between the incident wave and the edge wave. To this end, consider an incident wave solution (independent of x) of the linearized form of (2.1), with frequency ω and amplitude a_∞ . This is a solution in the form

$$a_\infty S(y,z)\, e^{i\omega t} + c.c,$$

which behaves as an incoming wave with perfect reflection. Denote the lowest-mode subharmonic edge wave solution of the linear part of (2.1) by

$$\chi\, e^{i\frac{\omega}{2}t} + c.c. \; ;$$

and assume that $|\chi| \ll a_\infty$. The nonlinear terms of (2.1) produce an interaction (S, χ^*) which resonates with χ ; this gives a growth in the amplitude of χ on a time scale proportional to a_∞ . Since $|\chi| \ll a_\infty$ in the early stages, the modification of S is neglected. However, as χ grows, the feedback interaction $(\chi,\chi) \to \varphi$ and the cubic self-interaction $(\chi^*, \chi, \chi) \to \chi$ become important, stopping the growth of χ and stabilizing the motion. This mechanism is analogous to the one responsible for the existence of steady subharmonic responses in simpler nonlinear oscillators.

To examine in more detail the situation just described, and to find the appropriate resonance conditions, we express the solution Φ in the form:

$$\Phi = \chi(x,y,z,t)\, e^{i\frac{\omega}{2}t} + \varphi(y,z,t)\, e^{i\omega t} + \text{c.c.} \; ; \tag{2.2}$$

where the time dependence of the functions φ and χ is slow compared with the period of oscillation. We will find the appropriate scale for this slow time later in the argument. For this reason, the derivatives $\varphi_t, \varphi_{tt}, \chi_{tt}$ can be neglected to the order considered.

To obtain the equations that govern the interaction, we substitute (2.2) in (2.1) and obtain

$$\begin{aligned}
&\chi_z - \frac{\omega^2}{4g}\chi + i\frac{\omega}{g}\chi_t = (\varphi, \chi^*) + (\chi,\chi,\chi^*) \quad \text{on } z=0,\ y \geq 0, \\
&\nabla^2\chi = 0 \quad \text{for } -\infty < x < \infty,\ y \geq 0,\ -y\tan\beta \leq z \leq 0, \\
&\chi_y \sin\beta + \chi_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0, \\
&\chi(y,z) \to 0 \text{ as } y \to \infty.
\end{aligned} \tag{2.3a}$$

$$\begin{aligned}
&\varphi_z - \frac{\omega^2}{g}\varphi = (\chi,\chi) \quad \text{on } z=0,\ y \geq 0, \\
&\varphi_{yy} + \varphi_{zz} = 0 \quad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0, \\
&\varphi_y \sin\beta + \varphi_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0, \\
&\varphi \sim \text{given incoming wave + outgoing wave as } y \to \infty.
\end{aligned} \tag{2.3b}$$

Since we are interested in the interaction of the lowest edge wave mode χ with the incident wave $a_\infty S$. We let χ be of the form

$$\chi(x,y,z,t) = \{ b(t)\chi^{(1)}(y,z) + \chi^{(2)} \} \cos kx ,$$

where

$$\chi^{(1)}(y,z) = E(y,z) = e^{-ky\cos\beta + kz\sin\beta} ,$$

and the resonance condition, $\omega^2/4g = k\sin\beta$, is satisfied. The incident wave is given by

$$\varphi = a_\infty S_{\ell_0}(y,z) ,$$

where the function S_{ℓ_0} is the solution of

$$\begin{aligned} &\varphi_z - \ell_0\varphi = 0 && \text{on } z=0,\ y \geq 0, \\ &\varphi_{yy} + \varphi_{zz} = 0 && \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0, \\ &\varphi_y\sin\beta + \varphi_z\cos\beta = 0 && \text{on } z = -y\tan\beta,\ y \geq 0. \end{aligned}$$

The solution S_{ℓ_0} behaves as an incoming wave with perfect reflection at the shore, and $\ell_0 = \omega^2/g = 4k\sin\beta$.

To study the early stages of the interaction, we assume that $|\chi| \ll |\varphi|$ and neglect the modifications of φ described by (2.3b). From (2.3a) we see that the resonant term (φ, χ^*) produces a growth for χ on a scale proportional to a_∞. More precisely, the equation for $\chi^{(2)}$ becomes

$$\begin{aligned} &\chi^{(2)}_z - k\sin\beta\,\chi^{(2)} = -i\frac{\omega}{g} b_t E + a_\infty b^*(E, S_{\ell_0}) && \text{on } z=0,\ y \geq 0, \\ &\chi^{(2)}_{yy} + \chi^{(2)}_{zz} - k^2\chi^{(2)} = 0 && \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0, \\ &\chi^{(2)}_y \sin\beta + \chi^{(2)}_z\cos\beta = 0 && \text{on } z = -y\tan\beta,\ y \geq 0, \\ &\chi^{(2)} \to 0 && \text{as } y \to \infty. \end{aligned} \tag{2.4}$$

Since the homogeneous form of (2.4) has a nontrivial bounded solution (the edge wave), there will be no bounded solution for (2.4) unless the forcing term satisfies an appropriate orthogonality condition. The use of the orthogonality condition, which is explained in detail in (2.9), gives the differential equation for the amplitude b in the form

$$i\frac{\omega}{g} b_t \int_0^\infty E^2 dy = a_\infty \int_0^\infty (\zeta_{l_0}, E) E \, dy \; b^* . \tag{2.5}$$

Equation (2.5) has exponentially-growing solutions which indicate instability of the edge wave.

When b^2 becomes of the same order of a_∞, the feedback interaction (χ, χ) becomes important, and a final steady state is suggested, with an edge wave amplitude $O(a_\infty^{1/2})$, and with φ modified by the interaction (χ, χ), but of course still $O(a_\infty)$. In the final state, the terms (φ, χ^*) and (χ, χ, χ^*) become of the same order. During the process, and in the final state, the terms $\varphi_t, \varphi_{tt}, \chi_{tt}$ are always of smaller order.

The arguments just described suggest a solution in the form of a slowly modulated edge wave. Therefore, in order to obtain the desired modulation equations, a two-timing expansion of the solution will be constructed in the next section.

2.3 The Modulation Equations

To implement the expansion in order to obtain the modulation equations, let

$$\begin{aligned} \varphi &= a_\infty \varphi^{(1)}(y, z, T) + a_\infty^2 \varphi^{(2)} + \dots + c.c. \\ \chi &= a_\infty^{1/2} \chi^{(1)}(y, z, T) \cos kx + a_\infty^{3/2} \chi^{(2)}(y, z, T) \cos kx + \dots + c.c. \end{aligned} \tag{2.6}$$

where the slow time T is $a_\infty t$. The expansion (2.6) can be easily made in terms of a nondimensional parameter proportional to a_∞ . However, we will use dimensional variables, and we will identify the "appropriate nondimensional parameter" later in the argument.

The expansion (2.6) is now substituted in (2.3a) and (2.3b). Then equation of like powers of a_∞ gives the equations for the successive orders as

$$
\begin{aligned}
&\chi^{(1)}_z - k\sin\beta\,\chi^{(1)} = 0\,, \quad \text{on } z=0,\ y \geq 0,\\
&\chi^{(1)}_{yy} + \chi^{(1)}_{zz} - k^2\chi^{(1)} = 0, \quad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0,\\
&\chi^{(1)}_y \sin\beta + \chi^{(1)}_z \cos\beta = 0 \quad \text{on } z = -\tan\beta,\ y \geq 0,\\
&\chi^{(1)} \to 0 \quad \text{as } y \to \infty.
\end{aligned}
\tag{2.7a}
$$

$$
\begin{aligned}
&\chi^{(2)}_z - k\sin\beta\,\chi^{(2)} = -\frac{\omega}{g}\chi^{(1)}_T + (\varphi^{(1)}, \chi^{(1)*}) + (\chi^{(1)}, \chi^{(1)}, \chi^{(1)*}) \ \text{on } z=0,\ y\geq 0,\\
&\chi^{(2)}_{yy} + \chi^{(2)}_{zz} - k^2\chi^{(2)} = 0\,, \quad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0,\\
&\chi^{(2)}_y \sin\beta + \chi^{(2)}_z \cos\beta = 0, \quad \text{on } z = -y\tan\beta,\ y \geq 0,\\
&\chi^{(2)} \to 0 \quad \text{as } y \to \infty.
\end{aligned}
\tag{2.7b}
$$

$$\varphi^{(1)}_z - l_0\varphi^{(1)} = (\chi^{(1)}, \chi^{(1)}) \quad \text{on } z = 0,\ y \geq 0,$$

$$\varphi^{(1)}_{yy} + \varphi^{(1)}_{zz} = 0 \quad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0, \qquad (2.7c)$$

$$\varphi^{(1)}_y \sin\beta + \varphi^{(1)}_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0,$$

$\varphi^{(1)} \sim$ given incoming wave + outgoing wave as $y \to \infty$.

The interaction of the waves is then completely described (to the order considered) by the solution of (2.7a) to (2.7c).

The edge wave solution of (2.7a), which is of interest for our purposes, is:

$$\chi^{(1)} = B(T)\, e^{-ky\cos\beta + kz\sin\beta}, \quad \text{with } \omega^2/4g = k\sin\beta. \qquad (2.8)$$

The solution for (2.7c) can be obtained, using the appropriate eigenfunction expansion, and will be discussed later.

As usual, the crucial step of the two-timing is the discussion of the resonant terms. In this case, (2.7b) is the important equation. The homogeneous form of equation (2.7b) has $E(y,z)$ as a nontrivial bounded solution; therefore, in order to obtain an acceptable solution for $\chi^{(2)}$, the forcing term must be orthogonal to E. The appropriate orthogonality condition is obtained using Green's theorem with $\chi^{(2)}$ and E. The orthogonality condition gives the nonlinear ordinary differential equation for B. This is

$$\int_0^\infty \left\{ -i\frac{\omega}{g}\dot{B}E + (\varphi^{(1)}, \chi^{(1)*}) + (\chi^{(1)}, \chi^{(1)}, \chi^{(1)*}) \right\} E\, dy = 0 \qquad (2.9)$$

and it describes the behavior of the edge wave amplitude.

To study the modulation equation (2.9) it is necessary to obtain an explicit expression for (2.9). We now indicate briefly the manipulations involved in the simplification of (2.9).

Substitution of (2.8) into (2.9) gives:

$$\int_0^\infty \Big\{ \frac{\omega}{g}\dot{B}E^2 + (\varphi^{(1)}_y E_y + \varphi^{(1)}_z E_z) EB^* + \frac{1}{2}\left(\left(\varphi^{(1)}_z - 4k\sin\beta\,\varphi^{(1)}\right)E\right)_z EB^* -$$

$$- i\,\frac{3\cos 2\beta}{4\sin\beta} k^3 E^4 B^2 B^* \Big\} dy = 0 . \qquad (2.10)$$

To simplify (2.10) the first-order z derivatives on $\varphi^{(1)}$ and E are expressed in terms of the boundary conditions from (2.7a) and (2.7c). The second-order z derivatives on $\varphi^{(1)}$ are replaced by y derivatives using Laplace's equation. Finally, the y derivatives are transferred from $\varphi^{(1)}$ to E by integration by parts. When the indicated calculations are performed, equation (2.10) becomes

$$\frac{\omega}{g}\dot{B}\int_0^\infty E^2 dy + B^*\Big\{ -4k^2\int_0^\infty \varphi^{(1)} E^2 dy + i\left(\frac{k^3\sin\beta}{2} - \frac{3}{4}\frac{\cos 2\beta}{\sin\beta}k^3\right)\int_0^\infty E^4 dy\, B^2 +$$

$$+ \frac{1}{2}\varphi^{(1)}_y + 2k\cos\beta\,\varphi^{(1)}\Big|_{y=0} \Big\} = 0 . \qquad (2.11)$$

The last term in (2.11) comes from the integrated part. This term can be expressed in a convenient way using the equation for $\varphi^{(1)}$. Using the boundary condition for (2.7c) on $z=0$, at $y=0$, we have:

$$\varphi_z^{(1)}(0,0) = 4k\sin\beta\, \varphi^{(1)}(0,0) - i k^2 B^2 ;$$

the bottom boundary condition gives at $y=0,\ z=0,$

$$\varphi_y^{(1)}(0,0) = -\varphi_z^{(1)}(0,0)\cot\beta .$$

Since $\varphi^{(1)}$ has continuous derivatives (see the Appendix), we have finally

$$\varphi_y^{(1)}(0,0) = -4k\cos\beta\,\varphi^{(1)}(0,0) + i k^2 \cot\beta\, B^2 . \tag{2.12}$$

Substitution of (2.12) in (2.11) and simplification gives the final form for the modulation equations. They are

$$\frac{\omega}{g}\dot{B} = 8\cos\beta\, k^3 B^* \int_0^\infty \varphi^{(1)} E^2 dy - \frac{5}{8} i \frac{k^3}{\sin\beta} B^2 B^* , \tag{2.13}$$

where the function $\varphi^{(1)}$ satisfies:

$$\begin{aligned}
&\varphi_z^{(1)} - l_0\varphi^{(1)} = -i k^2 B^2 E^2 && \text{on } z=0,\ y \geq 0, \\
&\varphi_{yy}^{(1)} + \varphi_{zz}^{(1)} = 0 && \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0, \\
&\varphi_y^{(1)}\sin\beta + \varphi_z^{(1)}\cos\beta = 0 && \text{on } z=-y\tan\beta,\ y \geq 0, \\
&\varphi^{(1)} \sim && \text{given incoming wave + outgoing wave as } y \to \infty.
\end{aligned} \tag{2.14}$$

To obtain the behavior of the waves it is necessary to solve (2.14) and then (2.13). This will be done in the next section.

2.4 The Solution of the Modulation Equations

The first part of this section is concerned with the study of the solution of (2. 13) and (2. 14) in the early stages of the motion when $|B| \ll 1$. Instability is shown, and an expression for the growth rate is obtained for general angle β. The second part deals with the derivation of the full nonlinear modulation solution for arbitrary angle β. Finally, in the third part, the solutions are examined in the phase plane, and the final steady state of finite amplitude is calculated.

(a) Instability of small edge waves. Assume $|B| \ll 1$; then the nonlinear terms in (2. 13) and (2. 14) can be neglected. For $\varphi^{(1)}$ we take the unmodified solution $S_{\ell_0}(y, z)$. In this case (2. 13) becomes

$$\frac{\omega}{g} \dot{B} = 8 \cos\beta \, k^3 \int_0^\infty S_{\ell_0}(y, 0) E^2(y) \, dy \; B^* . \tag{2. 15}$$

Equation (2. 15) has exponentially growing solutions; this shows instability of small edge waves for general angle of the sloping beach. The growth rate is

$$\frac{1}{2} \frac{\omega^2}{g} C(\beta) , \tag{2. 16}$$

where the function $C(\beta)$ is given by

$$C(\beta) = \frac{\cos\beta}{\sin^2\beta} a_\infty \int_0^\infty S_{\ell_0}(\eta/k, 0) \, e^{-2\eta\cos\beta} \, d\eta . \tag{2. 17}$$

In order to compare the growth rates for various angles, we choose a_∞ in such a way that the incoming wave has always the same amplitude a_0 at the shore for all β. The values of $C(\beta)$ can be calculated analytically for angles $\beta = \pi/2n$, since for that case the functions S'_{ℓ_0} can be expressed in terms of exponentials (Stoker 1957). However, the calculations become more involved as n increases; this is because the solution S'_{ℓ_0} contains n exponentials. For the case β very small, the shallow water approximation may be used. We now calculate some values of the function $c(\beta)$.

First of all, consider the shallow water region. In this case

$$a_\infty S'_{\ell_0}(\eta/k, 0) = a_0 J_0(4\sqrt{\eta})$$

where J_0 is the usual Bessel function. Then $c(\beta)$ is given by

$$c(\beta) = \frac{a_0}{\beta^2} \int_0^\infty J_0(4\sqrt{\eta})\, e^{-2\eta}\, d\eta . \tag{2.18}$$

Equation (2.18) is the result (after integration by parts) found by Guza and Davis (1974). The integral in (2.18) is standard, and its value is $1/2e^2$. (The value obtained numerically by Guza and Davis differs from the one just obtained. However, in a subsequent paper, Guza and Inman (1975) acknowledge the existence of numerical errors in their quantities related to $c(\beta)$.)

From equations (2.16) and (2.17) it is apparent that the non-dimensional parameter proportional to a_∞ is

$$a_0 \omega^2 \cos\beta / g \sin^2\beta$$

and our approximate theory will be valid for $a_0 \omega^2 \cos\beta / g \sin^2\beta \ll 1$. It was shown experimentally by Guza and Inman (1975) that the magnitude of the parameter $a_0 \omega^2 \cos\beta / g \sin^2\beta$ has a direct physical significance in the experiments. It was found that for values $a_0 \omega^2 \cos\beta / g \sin^2\beta > 1$ the incident waves did not produce edge waves.

To extend the shallow water results we calculate the values of $c(\pi/4)$, $c(\pi/6)$, and $c(\pi/8)$ using the full nonlinear theory. For the case $\beta = \pi/4$, the appropriate incident wave $a_\infty S'_{\ell_0}(\eta/k, 0)$ is given by

$$a_\infty S'_{\ell_0}(\eta/k, 0) = \frac{a_0}{\sqrt{2}} \left\{ \cos\left(\frac{4}{\sqrt{2}}\eta + \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} e^{-\frac{4}{\sqrt{2}}\eta} \right\},$$

and we find $c(\pi/4) = .066\, a_0$. When $\beta = \pi/6$ the corresponding incident wave is given by

$$a_\infty S'_{\ell_0}(\eta/k, 0) = -\frac{a_0}{\sqrt{3}} \left\{ \sin 2\eta + 2 e^{-\sqrt{3}\eta} \sin(\eta - \pi/3) \right\},$$

and $c(\pi/6) = .198\, a_0$. The value $c(\pi/8)$ involves four integrals and is $.430\, a_0$

The values just obtained show that the growth rate increases rapidly as $\beta \to 0$.

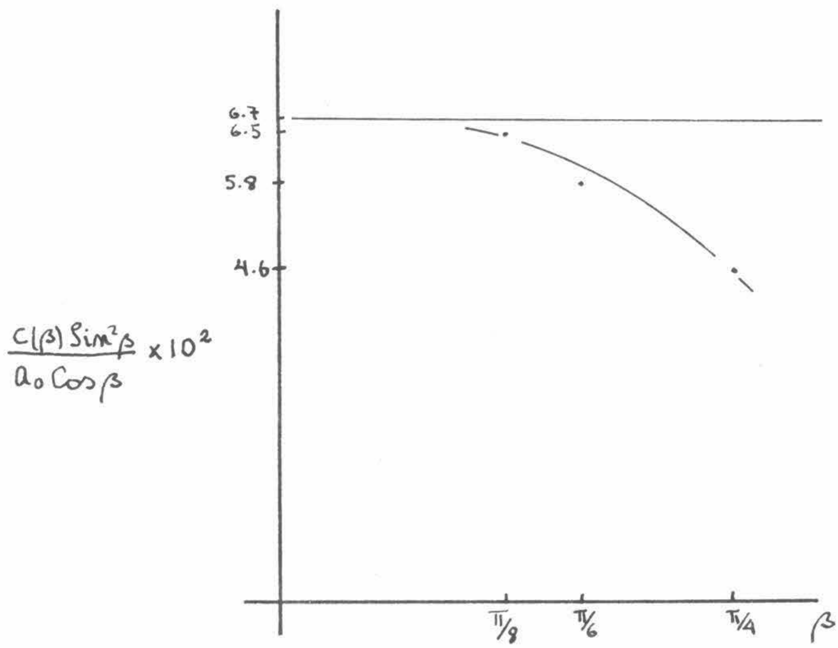
It is interesting to estimate the growth rates for finite depth using the shallow water result. The estimate is

$$c_s(\beta) = \frac{\cos\beta}{\sin^2\beta} \frac{e^{-2}}{2} a_0$$

The values of the growth rate are given by $c_s(\pi/4) = .10\, a_0$, $c_s(\pi/6) = .23\, a_0$, and $c_s(\pi/8) = .44\, a_0$.

To compare the shallow water estimate, $c_s(\beta)$, for the growth rate, with the values obtained using the finite depth theory, it is convenient to plot $c_s(\beta) \sin^2\beta / a_0 \cos\beta$ and $c(\beta) \sin^2\beta / a_0 \cos\beta$ as functions of the angle β.

Graph 1 shows the corresponding values.



Graph 1.

In Graph 1 the line is the shallow water result, and the points on the curve represent the finite depth results given by $c(\beta)\sin^2\beta/a_0\cos\beta$. From Graph 1 we conclude that the main angular dependence for the growth rate is given by the factor $\cot\beta/\sin\beta$.

(b) <u>The solution for $\varphi^{(1)}$</u>. To describe the further growth of the edge wave and its interaction with the given incident wave, we need the full solution of (2.13) and (2.14). We begin by solving (2.14). The general solution for (2.14), regular at the origin, is given by:

$$\varphi^{(1)} = A(\tau)\, S_{\ell_0}(y,z) + i\, B^2(\tau)\, P(y,z), \tag{2.19}$$

where P satisfies:

$$P_z - 4k \sin\beta \, P = -k^2 E^2 \qquad \text{on } z = 0,\ y \geq 0,$$

$$P_{yy} + P_{zz} = 0 \qquad \text{for } y \geq 0,\ -y \tan\beta \leq z \leq 0, \tag{2.20}$$

$$P_y \sin\beta + P_z \cos\beta = 0 \qquad \text{on } z = -y\tan\beta,\ y \geq 0,$$

$$\varphi^{(1)} \sim \text{ given incoming wave + outgoing wave, as } y \to \infty.$$

To obtain the solution of (2.20) we use the appropriate eigenfunction expansions. For angles $\beta = \pi/2m$ suitably normalized multiples of the solutions of the homogeneous problem can be shown to be complete (see Chapter 6), in the sense that every sufficiently integrable function $g(y)$ can be expanded as

$$g(y) = \int_0^\infty S_\ell(y,0)\, d\ell \int_0^\infty g(\eta)\, S_\ell'(\eta,0)\, d\eta \,. \tag{2.21}$$

Using (2.21), the appropriate solution of (2.20) is

$$P(y,z) = -k^2\, \text{P.V.} \int_0^\infty \frac{S_\ell(y,z)}{\ell - \ell_0}\, d\ell \int_0^\infty S_\ell'(\eta,0)\, E^2(\eta)\, d\eta \,. \tag{2.22}$$

To find the asymptotic behavior of $P(y,0)$ as $y \to \infty$, for $\beta = \pi/2m$, we recall (Stoker 1957) that the eigenfunctions behave as $y \to \infty$ in the form

$$S_\ell(y,0) \sim N(\beta) \cos\left(\ell y + \tfrac{m-1}{4}\pi\right),$$

where the normalization constant $N(\beta)$ is independent of ℓ but depends on the angle β. The transform of the function E^2 is just a rational function which decays at infinity in the complex ℓ plane.

The path of integration in (2.22) can then be completed around the pole at $\ell = \ell_0$ (see the Appendix for details); and in this way it is possible to show that the dominant contribution comes from the pole $\ell = \ell_0$. It is found that

$$P(y,0) \sim k\, d(\beta)\, N(\beta) \sin\left(\ell_0 y + \frac{n-1}{4}\pi\right) \text{ as } y \to \infty, \tag{2.23}$$

where the nondimensional constant $d(\beta)$ is given by:

$$d(\beta) = \pi \int_0^\infty S_{\ell_0}(\eta/k, 0)\, e^{-2\eta \cos\beta}\, d\eta. \tag{2.24}$$

To complete the solution for $\varphi^{(1)}$ we need to determine the so far arbitrary function $A(T)$ in (2.19). The function $A(T)$ is determined using the boundary condition for $\varphi^{(1)}$ at infinity. The incoming wave component of $\bar{\Phi}$ remains unchanged during the growth of the edge wave. This condition implies that the incoming components of

$$a_\infty A S'_{\ell_0} e^{i\omega t} + a_\infty i \mathcal{B}^2(T) e^{i\omega t} P + \text{c.c.},$$

and

$$a_\infty S'_{\ell_0} e^{i\omega t} + \text{c.c.}$$

remain the same.

Therefore, in view of (2.23), we require

$$A(T) = 1 - k\, d(\beta)\, \mathcal{B}^2(T). \tag{2.25}$$

This completes the solution of (2.14).

As was remarked in the previous sections, the calculations for small β become very involved; and in order to obtain the desired results for small β we will use the shallow water approximation in evaluating certain integrals. However, before we outline the shallow water approximations, we provide the solution for $\beta = \pi/4$ and give the value $d(\pi/4)$. This will allow a check on whether or not the results vary considerably with β. The appropriate $S_\ell(y,z)$ for $\beta = \pi/4$ is given by

$$S_\ell(y,z) = \sqrt{\tfrac{2}{\pi}}\left\{ e^{\ell z}\cos\left(\ell y + \tfrac{\pi}{4}\right) + e^{-\ell y}\cos\left(\ell z - \pi/4\right)\right\}.$$

The solution to (2.14) for $\beta = \pi/4$ is

$$P(y,z) = -\frac{k2\sqrt{2}}{\pi}\ \mathrm{P.V.}\int_0^\infty \frac{e^{\eta k z}\cos(\eta k y + \pi/4) + e^{-\eta k y}\cos(\eta k z - \pi/4)}{(t-2\sqrt{2})(t^2+2)(t+\sqrt{2})}\, d\eta . \tag{2.26}$$

The asymptotic behavior of (2.26) on $z=0$ as $y \to \infty$ is given by:

$$P(y,0) \sim \frac{k}{15}\sqrt{\frac{2}{\pi}}\left\{\sqrt{\tfrac{2}{\pi}}\sin\left(2\sqrt{2}ky + \tfrac{\pi}{4}\right)\right\} ; \tag{2.27}$$

hence the constant $d(\pi/4)$ takes the value $\sqrt{2}/15\sqrt{\pi}$.

We now use the shallow water theory to obtain an approximation for P and $d(\beta)$ when β is small. The shallow water approximation of $P(y,0)$, now denoted by $P(y)$, satisfies the inhomogeneous shallow water equation:

$$\beta y P'' + \beta P' + 4k\beta P = k^2 e^{-2ky} ; \tag{2.28}$$

and the z dependence is absent in the shallow water approximation. The eigenfunctions of the homogeneous problem are the asymptotic values of $S_\ell(y,z)$ when $\beta \to 0$. They are

$$S_\ell(y) = \beta^{-1/2} J_0\left(2\sqrt{\frac{\ell}{\beta}y}\right), \text{ for } \ell \geq 0.$$

and the expansion theorem is just the familiar Fourier-Bessel expansion

$$f(y) = \int_0^\infty \beta^{-\frac{1}{2}} J_0\left(2\sqrt{\frac{ly}{\beta}}\right) dl \int_0^\infty \beta^{-\frac{1}{2}} J_0\left(2\sqrt{\frac{l\eta}{\beta}}\right) f(\eta)\, d\eta. \tag{2.29}$$

Using (2.29) the solution of (2.28) is given by

$$P(y) = -\frac{k^2}{\beta}\, \text{P.V.} \int_0^\infty \frac{J_0(2\sqrt{ly})}{l-4k}\, dl \int_0^\infty J_0(2\sqrt{l\eta})\, e^{-2k\eta}\, d\eta\,. \tag{2.30a}$$

The asymptotic behavior is again found from the contribution of the pole at $l = 4k$ and we have

$$P(y) \sim d(\beta)\, (\pi\beta)^{-1/2}\, (ky)^{-1/4} \sin\left(4\sqrt{ky} - \pi/4\right)$$

where

$$d(\beta) = \beta^{-1/2}\pi \int_0^\infty J_0(4\sqrt{\eta})\, e^{-2\eta}\, d\eta = \pi/2e^2\beta^{1/2}\,. \tag{2.30b}$$

The value of $d(\beta)$ is, of course, the asymptotic value, as $\beta \to 0$, of (2.24) when the shallow water approximation for the incident wave is used.

(c) <u>The amplitude equation.</u> We now consider equation (2.13) for the complete determination of the amplitude of the edge wave. To obtain the desired equation, the solution (2.19) of (2.14) is substituted in (2.13). We obtain

$$\frac{\omega}{g}\dot{B} = 8\cos\beta\, k^3 A B^* \int_0^\infty S_{l_0} E^2\, dy + i\frac{k^3}{\sin\beta}\mu(\beta)\, B^2 B^*\,, \tag{2.31}$$

where the function $\mu(\beta)$ is given by

$$\mu(\beta) = -\frac{5}{8} + 8 \sin\beta \cos\beta \int_0^\infty P(y,0)\, E^2(y)\, dy . \tag{2.32}$$

From (2.22) the nondimensional integral in (2.32) can be expressed conveniently as

$$\int_0^\infty P(y,0)\, E^2(y)\, dy = -k^2 \, \mathrm{P.V.} \int_0^\infty \frac{dl}{l - l_0} \left\{ \int_0^\infty S_l(\eta, 0)\, E^2(\eta)\, d\eta \right\}^2 \tag{2.33}$$

Again in formula (2.33), for angles $\beta = \pi/2m$, the integrals can be evaluated in closed form using partial fractions; however, the procedure is not practical for small β. In the case of small β, when the shallow water approximation (2.30a) is used, we obtain

$$\int_0^\infty P\, E^2\, dy = -\frac{1}{\beta}\, \mathrm{P.V.}\, \frac{1}{4} \int_0^\infty \frac{e^{-\eta}}{\eta - 4}\, d\eta = \frac{1}{4\beta}\, e^{-4}\, E_i(4) \tag{2.34}$$

where $E_i(x)$ is, in the notation of Abramowitz and Stegun (1965), the exponential integral.

To obtain the final expression for the amplitude equation, we substitute in (2.31) the value of A given by (2.25). Then

$$\frac{\omega}{g} \dot{B} = 8 \cos\beta\, k^3 \int_0^\infty S_{l_0} E^2\, dy\, B^* \left(1 - k d(\beta) B^2\right) + i \frac{k^3 \mu(\beta)}{\sin\beta} B^2 B^* . \tag{2.35}$$

Equation (2.35) describes the complete evolution of the standing wave in the presence of the given incoming wave.

(d) Solution of the amplitude equation. In this subsection we discuss the behavior of the solutions of (2.35) for general angle of the sloping beach. For this purpose it is convenient in (2.35) to change to nondimensional variables and functions in the following way.

Let

$$B(T) = k^{-1/2} \sin\beta^{1/2} R(k^2 g \omega^{-1} T) \tag{2.36}$$

then (2.35) gives the equation for R in the form

$$\dot{R} = \frac{8\cos\beta}{\pi} d(\beta) R^* \left(1 - d(\beta)\sin\beta R^2\right) + \mu(\beta) R^2 R^*, \tag{2.37}$$

since

$$8\cos\beta \int_0^\infty S_{\ell_0}(\eta/k, 0)\, e^{-2\eta\cos\beta}\, d\eta = 8\frac{\cos\beta}{\pi} d(\beta). \tag{2.38}$$

To examine equation (2.37) in the phase plane, let $R = e^{i\theta}$ then the equations for the amplitude and phase become:

$$\begin{aligned} \dot{r} &= \frac{8\cos\beta}{\pi} d(\beta)\left(\cos 2\theta - d(\beta)\sin\beta\, r^2\right) r, \\ \dot{\theta} &= -\frac{8\cos\beta}{\pi} d(\beta) \sin 2\theta + \mu(\beta) r^2. \end{aligned} \tag{2.39}$$

The system (2.39) is periodic as a function of θ with period π ; it is therefore sufficient to examine its behavior in the region $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The critical points of (2.39) represent the possible final steady states for amplitude and phase. The critical points of (2.39) in the (θ, r) plane are: $P_1 = (-\frac{\pi}{2}, 0)$, $P_2 = (0,0)$, $P_3 = (\frac{\pi}{2}, 0)$; the only critical point of finite amplitude is $P_4 = (\theta_0, r_0)$ where

$$\tan 2\theta_0 = \pi\mu(\beta) / 8\cos\beta \sin\beta\, d^2(\beta) \tag{2.40}$$

$$r_0^2 = \cos 2\theta_0 / d(\beta)\sin\beta$$

The critical points P_1, P_2 and P_3 are unstable (in the usual sense of ordinary differential equations). To examine the stability of P_4, let $r = r_0 + \rho$, $\theta = \theta_0 + \delta$, and linearize (2.39) around (θ_0, r_0) to obtain

$$\begin{pmatrix} \rho \\ \delta \end{pmatrix}' = \begin{pmatrix} -\frac{16}{\pi}\cos\beta \sin^2\beta \, d^2(\beta) r_0^2 & -\frac{16}{\pi}\cos\beta \, r_0 \sin 2\theta_0 \, d(\beta) \\ 2\mu(\beta) r_0 & -\frac{16}{\pi}\cos\beta \sin^2\beta \, d^2(\beta) r_0^2 \end{pmatrix} \begin{pmatrix} \rho \\ \delta \end{pmatrix}. \qquad (2.41)$$

The eigenvalues of the matrix of (2.41) are

$$\lambda_{1,2} = -\frac{16}{\pi}\cos\beta \sin^2\beta \, d^2(\beta) \pm 2 i r_0^2 |\mu(\beta)| ;$$

and therefore $\mathrm{Re}\,\lambda_{1,2} < 0$. Hence, the point (θ_0, r_0) is a stable spiral. Since there are no periodic orbits for (2.39) and no unbounded solutions, it follows that for all initial conditions the solutions of (2.39) will reach the steady state (θ_0, r_0) as $T \to \infty$.

The physical interpretation of the argument just described is clear. It means that small amplitude edge waves become unstable in the presence of incident waves of twice their frequency. They start growing until higher-order nonlinear effects become important and stabilize the motion. The final result is a steady periodic standing edge wave of amplitude $O(a_\infty^{1/2})$, determined by (2.40). The general behavior is the same for all angles β of the sloping beach; the difference will be only in the actual values for $d(\beta)$ and $\mu(\beta)$. The expression for the final edge wave amplitude is given, for arbitrary β, by

$$\sigma_0 T_i \sin\beta \left(\frac{A_\infty g}{N(\beta)\pi^2}\right)^{1/2}$$

where $A_\infty = a_\infty N(\beta)$ and T_i are the given offshore amplitude and period of the incident wave. The constant σ_0 is given by (2.40).

It is interesting to observe that the modulation equations (2.39) are very similar to the ones obtained for subharmonic responses of forced nonlinear one-dimensional oscillators. It is also possible to construct simpler examples of transfer of energy from an incoming wave to a trapped mode, using a two-dimensional membrane equation with an appropriate nonlinear restoring force. This is because the nonlinear mechanism is the same in all cases, and the only difference will be in the "correlation integrals" for the various modes.

(e) Expression of the run-up amplitude of the edge wave in terms of the amplitude of the incident wave at the shore. In order to compare the predictions of the present theory with laboratory experiments performed by Guza and Inman (1975), an expression for the final edge wave amplitude in terms of the parameters of the incoming wave at the shore is needed.

In the experiments, the measured run-up amplitude A_r, of the subharmonic edge wave, was taken from the distance between two antinodes measured along the beach: this is related to the amplitude of the surface elevation a_e by $A_r = 2a_e/\sin\beta$. We now calculate the run-up amplitude of the edge wave and the amplitude of the incident wave at shore for the case of small β.

Consider first of all the incident wave when no edge waves are present. The velocity potential Φ_i of the incident wave is given from (2.19) by

$$\Phi_i = \frac{a_\infty g}{\omega} S'_{l_0}(y)\, e^{i\omega t} + c.c. = \frac{a_\infty g}{\omega} \frac{2}{\beta^{1/2}} J_0(4\sqrt{ky}) \cos\omega t. \qquad (2.42)$$

Therefore, the surface elevation ζ_i is given by

$$\zeta_i = -\frac{1}{g}\Phi_{it} = \frac{2a_\infty}{\beta^{1/2}} J_0(4\sqrt{ky}) \sin\omega t, \qquad (2.43)$$

and the maximum amplitude of the incident wave at the shore, denoted by a_i, is

$$a_i = 2a_\infty/\beta^{1/2} \qquad (2.44)$$

The velocity potential Φ_e associated with the edge wave in the final state is given by the value of the first term in (2.6) when $T \to \infty$. This is

$$\Phi_e = a_\infty^{1/2} \frac{g}{\omega} \chi^{(1)}(y, z, \infty) \cos kx\, e^{i\frac{\omega}{2}t} + c.c. =$$

$$= 2a_\infty^{1/2} k^{-1/2} \beta^{\frac{1}{2}} r_0\, E(y,z) \cos kx \cos\left(\frac{\omega}{2}t + \theta_0\right); \qquad (2.45)$$

the surface elevation of the edge wave, ζ_e, is then given by

$$\zeta_e = \beta\left(a_\infty 4g/\omega^2\right)^{1/2} r_0\, e^{-ky\cos\beta} \cos kx \sin\left(\frac{\omega}{2}t + \theta_0\right) \qquad (2.46)$$

Therefore, the distance A_r between two antinodes (measured along the beach) is given by

$$A_r = 2r_0\left(a_\infty 4g/\omega^2\right)^{1/2} \qquad (2.47)$$

From (2.44) and (2.47) the run-up amplitude may be expressed in terms of the amplitude a_i and the period T_i of the incident wave as

$$A_r = 2T_i \left(a_i g \beta^{1/2} r_0^2 / 2\pi^2\right)^{1/2} \tag{2.48}$$

The value of r_0 is obtained from (2.40) where $d(\beta)$ and $\mu(\beta)$ are given by (2.30b) and (2.32). The final expression for A_r in terms of a_i to be compared with the experiments is

$$A_r = 2.5\, T_i \left(a_i g / \pi^2\right)^{1/2} . \tag{2.49}$$

It is interesting to compare the final edge wave amplitude for shallow water with a corresponding value for finite depth, say, for $\beta = \pi/4$. The calculation for the case $\beta = \pi/4$ is the same as just performed for the shallow water case, but using the values $d(\pi/4)$ and $\mu(\pi/4)$. The result is

$$A_r = 3.6\, T_i \left(a_i g / \pi^2\right)^{1/2}$$

Comparison with (2.49) (which is valid for small β) indicates that the equilibrium amplitude is not very sensitive to changes in the slope of the bottom.

2.5 Comparison with Experiment

In this section we compare the amplitude predicted by equation (2.49) with the experimental results of Guza and Inman (1975). However, before the actual comparison is made, it is interesting to recall some of the qualitative features observed in the experiments. It was observed that large edge wave run-up is produced for small incident wave amplitude. The amplitudes of the edge waves were larger for longer period of the incoming waves. Finally, when the parameter

$a_i\omega^2/g\beta^2$ exceeded one, no edge waves were produced.

In Table 1 the predictions of (2.49) are compared with the experimental results. The period of the incident wave is measured in seconds, and the amplitudes are measured in centimeters. When no experimental data are given, it is because no edge waves were produced since the condition $a_i\omega^2/g\beta^2 < 1$ was violated in the experiments.

T_i sec	a_i cm	A_r exp $\beta = 4^o$	A_r exp $\beta = 6^o$	A_r exp $\beta = 6.8^o$	A_r (2.49)
2.7	1.8	40	40	40	71
2.7	2	-	80	70	90
2.7	2.5	-	100	90	100
2.7	3	-	110	140	117
2.7	4	-	130	160	130
3.8	6.7	-	220	-	230

Table 1.

From Table 1 we conclude that good qualitative agreement is found in the sense that small incident waves produce large edge wave run-up. The present theory also agrees with the fact that incident waves with larger period produce larger edge waves. Unfortunately, the only data point for period different from 2.7 sec is the measurement for $T_i = 3.8$ sec, and it was not possible to compare other numerical values. Despite the fact that the parameter $a_i\omega^2/g\beta^2$ was close to one in the experiments, and was assumed small in our analysis, the numerical agreement of theory and experiment is quite good. The numerical agreement is better the larger the amplitude of the incident wave. A possible explanation is the following: it is observed in the experiments that the amplitude of the incident wave has to exceed a critical value in order to overcome viscous effects and excite edge waves. This suggests that viscous dissipation is important for small amplitudes of the incident wave, and the present theory does not include them.

2.6 Further Study of the Edge Wave Solution

In this section we examine the uniform validity of the expansion for χ to the order considered. It is expected, by analogy with the nonlinear traveling edge waves studied by Whitham (1976), that a non-uniformity as $y \to \infty$ leads to a modification of the rate of decay of the edge wave out to sea.

In order to discuss the appropriate modifications we consider again the problem for $\chi^{(2)}$, that is,

$$\begin{aligned}
&\chi^{(2)}_z - k\sin\beta\,\chi^{(2)} = -i\frac{\omega}{g}\dot{B}\,E + (\varphi^{(1)}, \chi^{(1)*}) + (\chi^{(1)}, \chi^{(1)}, \chi^{(1)*}) \quad \text{on } z=0,\ y \geq 0,\\
&\chi^{(2)}_{yy} + \chi^{(2)}_{zz} - k^2\chi^{(2)} = 0 \quad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0,\\
&\chi^{(2)}_y \sin\beta + \chi^{(2)}_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0,\\
&\chi^{(2)} \to 0 \quad \text{as } y \to \infty.
\end{aligned} \qquad (2.50)$$

Since the forcing term in (2.50) satisfies the orthogonality condition (2.9), there is a square integrable solution for $\chi^{(2)}$. However, it is not uniformly of order $\chi^{(1)}$ as $y \to \infty$. The situation in (2.50) is the same as the one examined by Whitham (1976), and we now use his results. The relevant one for this discussion is the asymptotic behavior of $\chi^{(2)}$ as $y \to \infty$. The leading order contribution as $y \to \infty$ comes from the first term of the forcing function in (2.50) and is given by

$$\chi^{(2)}(y, z) = -i \frac{\omega}{g} \dot{B} (y \tan\beta + z) E(y, z) + O(E(y, z)). \tag{2.51}$$

When (2.51) is substituted in (2.6) the expansion for χ takes the form

$$\chi(y,z) = a_\infty^{1/2} B(\tau) E(y,z) \cos kx \left\{ e^{i\omega t} - i a_\infty \frac{\omega}{g} \dot{B} B^{-1} (y \tan\beta + z) e^{i\omega t} \right\} + c.c. \tag{2.52}$$

We see that (2.52) does not provide a uniform expansion for the function χ. But it is recognized as the Taylor expansion of the function

$$\chi = a_\infty^{1/2} B E(y,z) \cos kx \ \exp i \left\{ \omega t - a_\infty \frac{\omega}{g} \dot{B} B^{-1} (y \tan\beta + z) \right\} + c.c. \tag{2.53}$$

which is the uniformly valid form. To justify this form the analysis can be recast starting from (2.6) using the method of strained coordinates to obtain (2.53). The modifications are minor but obscure the main steps, and will not be repeated here. In the revised form (2.53) the phase of the edge wave depends not only on the slow time but also on the offshore coordinate and the depth. The real part of the function $\dot{B} B^{-1}$ gives the offshore dependence of the phase while its imaginary part gives a modification in the rate of decay with y. This is in analogy with the traveling edge waves. In the final steady state, these small modifications are no longer present. When the shallow water theory is used to calculate the corrections to the phase, a logarithmic behavior as $y \to \infty$ is found. This is analogous to the situation found for traveling edge waves by Whitham (1976) and discussed in detail by Minzoni (1976). For this case, the same arguments apply. For the case of a more realistic beach with depth distribution $h(y) = \beta y$ for $0 \leq y \leq \ell_1$, $h(y) = h_1$ for $y \geq \ell_1$, $h_1 = \beta \ell_1$, the change in phase is given by

$$a_\infty \frac{\omega}{g} \dot{B} B^{-1} \beta (y - \ell_1) / h_1 . \tag{2.54}$$

It is linear in qualitative agreement with the behavior found using the full nonlinear theory in (2.53).

To conclude this section, we summarize the behavior of the solution for the edge wave. Incident waves of frequency ω and amplitude a_∞ produce growth of subharmonic edge waves. Also during the development of the edge wave, the rate of decay offshore is modified. When the amplitude of the edge wave becomes $O(a_\infty^{1/2})$ the feedback interaction modifies the incoming wave and the motion is stabilized, becoming a steady periodic oscillation. The final state involves a large edge wave together with a small incident wave and its reflections.

2.7 Free Standing Edge Waves

In this last section we discuss briefly a problem which is related to the previous discussion. It concerns the existence of free periodic edge waves of finite amplitude.

In order to find the periodic solution, if any, the boundary value problem to solve is (2.1) with a change in the boundary condition at infinity. In this case, since we assume that no energy goes into the system (except for that contained already in the edge wave), the appropriate boundary condition at infinity is that Φ behaves as an outgoing wave.

We then assume an expansion for Φ in the form

$$\Phi = a_e B(T) E(y,z) \cos kx \, e^{i\frac{\omega}{2}t} + a_e^2 \varphi^{(1)} e^{i\omega t} + a_e^3 \chi^{(2)} e^{i\frac{\omega}{2}t} + \text{c.c.} \quad . \tag{2.55}$$

When (2.55) is substituted in (2.1), the equation obtained for $\varphi^{(1)}$ is again (2.7c), but the solution $\varphi^{(1)}$ now satisfies the radiation condition instead of (2.25). The interactions $(\varphi^{(1)}, \chi^{(1)*}) \to \chi^{(1)}$ and $(\chi^{(1)}, \chi^{(1)}, \chi^{(1)*}) \to \chi^{(1)}$ produce a change in the complex amplitude B on a time scale proportional to a_e^2 . The appropriate orthogonality condition necessary to obtain the solution for $\chi^{(2)}$ is again (2.9), and this leads to the modulation equation for $B(T)$. It is

$$\frac{\omega}{g} \dot{B} = -\frac{8\cos\beta}{\pi} d^2(\beta) k^3 B^2 B^* + i\frac{k^3 \mu(\beta)}{\sin\beta} B^2 B^* \, . \tag{2.56}$$

To express (2.56) in nondimensional form, let

$$B(T) = r(\tau) e^{i\theta(\tau)} \, , \tag{2.57}$$

where

$$\tau = \frac{k^3 g}{\omega} T \, .$$

Substitution of (2.57) in (2.56) gives the equations for the amplitude and phase. They are

$$\begin{aligned} \dot{r} &= -\frac{8\cos\beta}{\pi} \sin^2\beta \, d^2(\beta) r^3 \\ \dot{\theta} &= \mu(\beta) \sin\beta \, r^2 \end{aligned} \tag{2.58}$$

Since $d^2(\beta) \geq 0$ for all angles β , equations (2.58) show that the amplitude of a nonlinear standing edge wave decreases on a slow time scale $O(a_e k)^2$. This shows that a finite amplitude standing edge wave cannot exist as a periodic solution since all its energy is ultimately radiated at infinity due to the nonlinear effects. The amplitude decays as $\tau^{-1/2}$. This shows that the presence of incoming waves is indeed necessary to sustain finite amplitude standing edge waves.

It is interesting to note that the same situation (the radiation at infinity of the energy in a trapped mode of finite amplitude due to the nonlinear effects) will be present for all systems which allow a continuous spectrum in addition to the point spectrum. This situation is not usually encountered in simpler nonlinear standing waves problems, because the region considered is of bounded spatial dimensions and the continuous spectrum is absent in that case.

CHAPTER 3

EDGE WAVES PRODUCED BY MOVING PRESSURE DISTURBANCES

In this chapter we consider the edge waves produced by a pressure disturbance moving parallel to the shoreline. This problem has been considered by Greenspan (1956) using the linear shallow water approximation. We now discuss the same problem using the full linear theory. The same results found by Greenspan are shown to be valid for general angle of the sloping beach, and our results reduce to the shallow water ones in the appropriate limit. For our purposes, the pressure disturbance is taken in the form of an impulse concentrated a distance y_0 offshore, and moving parallel to the shore with velocity $-v$.

3.1 The Linear Initial Value Problem

The initial boundary value problem to solve for the velocity potential Φ and the surface elevation ζ is according to the linear theory

$$
\begin{aligned}
&\Phi_t + g\zeta = -\frac{F}{\rho}\,\delta(x+vt)\,\delta(y-y_0) && \text{on } z=0,\ y\geq 0,\\
&\zeta_t - \Phi_z = 0 && \text{on } -\infty<x<\infty,\ y\geq 0,\ z=0,\\
&\Phi_t(x,y,0,0) = \Phi(x,y,0,0) = 0 && \text{on } -\infty<x<\infty,\ y\geq 0, \qquad (3.1)\\
&\nabla^2\Phi = 0 && \text{for } -\infty<x<\infty,\ y\geq 0,\ -y\tan\beta\leq z\leq 0,\\
&\Phi_y \sin\beta + \Phi_z \cos\beta = 0 && \text{on } z=-y\tan\beta,\ y\geq 0.
\end{aligned}
$$

To solve (3.1) it is convenient to eliminate ζ , and to change variables in the form $x' = x + vt$, $t' = t$. After the appropriate manipulations (3.1) becomes:

$$\begin{aligned}
&v^2 \Phi_{x'x'} + 2v\,\Phi_{x't'} + \Phi_{t't'} + g\,\Phi_z = -\frac{v}{\rho} F\,\delta'(x')\,\delta(y-y_0), \text{ on } z=0,\ y \geq 0,\\
&\Phi_{t'}(x', y, 0, 0) = \Phi(x', y, 0, 0) = 0 \quad \text{for } -\infty < x' < \infty,\ y \geq 0,\\
&\Phi_{yy} + \Phi_{zz} + \Phi_{x'x'} = 0 \quad \text{for } -\infty < x' < \infty,\ y \geq 0,\ -y\tan\beta \leq z \leq 0,\\
&\Phi_y \sin\beta + \Phi_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0.
\end{aligned} \tag{3.2}$$

Equations (3.2) can be solved using a Laplace transform in time and a Fourier transform in x'. When the transforms are defined by

$$\bar{\Phi}(s) = \int_0^\infty e^{-st'}\,\Phi(t')\,dt', \qquad \hat{\Phi}(k) = \int_{-\infty}^\infty e^{-ikx'}\,\Phi(x')\,dx',$$

the equations for the transformed function are:

$$\begin{aligned}
&-k^2v^2\,\bar{\hat{\Phi}} + 2ikvs\,\bar{\hat{\Phi}} + s^2\bar{\hat{\Phi}} + g\bar{\hat{\Phi}}_z = -v\frac{ik}{\rho}F\,\delta(y-y_0), \text{ on } z=0,\ y \geq 0,\\
&\bar{\hat{\Phi}}_{yy} + \bar{\hat{\Phi}}_{zz} - k^2\,\bar{\hat{\Phi}} = 0 \quad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0,\\
&\bar{\hat{\Phi}}_y \sin\beta + \bar{\hat{\Phi}}_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0.
\end{aligned} \tag{3.3}$$

To solve (3.3) it is necessary to know the appropriate eigenfunction expansion associated with the homogeneous problem. This expansion was discussed briefly in (1.16) and will be examined in detail in the last chapter. In this case, we are only interested in the behavior of the lowest order edge wave mode since it is the one more easily excited by a moving storm. The higher order modes can be examined in the same way. The contribution from the continuous spectrum was examined by Hanson (1926), and the usual Kelvin's wave pattern is obtained.

Since we are only interested in the component of $\bar{\hat{\Phi}}$ along the lowest edge wave mode, it is only necessary to find the component of the forcing pressure along the desired mode and solve (3.3). When the desired solution is denoted by $\bar{\hat{\Phi}}_e$ we obtain

$$\bar{\hat{\Phi}}_e = -\frac{2i\, F v \cos\beta\, k|k|\, e^{-|k|\cos\beta\, y_0}}{\rho\, s\left[(s+ikv)^2 + g|k|\sin\beta\right]} \, .$$

In the original variables the solution is given by

$$\bar{\Phi}_e = -\frac{2i\, F v}{2\pi\rho}\cos\beta \int_{-\infty}^{\infty} e^{ikx}\, k|k|\, e^{-|k|(y+y_0)\cos\beta + |k| z \sin\beta}\, U(t',k)\, dk \, , \tag{3.4}$$

where the function $U(t',k)$ is given by

$$U(t',k) = \frac{1}{2\pi i}\int_{B} \frac{e^{st'}\, ds}{s\left[(s+ikv)^2 + g|k|\sin\beta\right]} \, .$$

The explicit form of $U(t',k)$ when substituted in (3.4) gives

$$\bar{\Phi}_e = \frac{4}{2\pi}\,\frac{Fv}{\rho}\cos\beta\, \mathrm{Im}\left\{ \int_0^{\infty}\left[\frac{1}{-k^2v^2 + gk\sin\beta} - \right.\right.$$

$$\left. - \frac{e^{i((gk\sin\beta)^{1/2} - kv)t'}}{2(gk\sin\beta)^{1/2}\left(-kv + (gk\sin\beta)^{1/2}\right)}\right] e^{-k(y+y_0)\cos\beta + kz\sin\beta}\, e^{ikx'}\, k^2\, dk -$$

$$\left. - \int_0^{\infty} \frac{e^{-i((gk\sin\beta)^{1/2} - kv)t'}}{2(gk\sin\beta)^{1/2}\left(+kv + (gk\sin\beta)^{1/2}\right)}\, e^{-k(y+y_0)\cos\beta + kz\sin\beta}\, e^{ikx'}\, k^2\, dk \right\} .$$

(3.5)

We are interested in the behavior of (3.5) as $t = t' \to \infty$. To obtain the dominant term in the expansion we consider each term in (3.5) in detail. First of all, in the second integral the phase of the exponent is never stationary; hence, it is $O(t'^{-1})$ as $t' \to \infty$. To find the asymptotic behavior of the first term it is convenient to deform the path of integration by adding an indentation in the lower half plane to avoid the point $k_0 = g \sin\beta / v^2$. Then we may consider each term in the integrand separately. In the second term the exponent $(gk\sin\beta)^{1/2} - kv$ has a positive imaginary part for $\operatorname{Img} k \leq 0$; therefore, the integral tends to zero as $t' \to \infty$. Hence, the long time behavior of (3.5) is given by

$$\bar{\Phi}_e \sim \frac{4\cos\beta\, F v}{2\pi\rho} \operatorname{Im} \int_C \frac{e^{-k(y+y_0)\cos\beta + kz\sin\beta}}{-k^2 v^2 + gk\sin\beta} e^{ikx'} k^2 \, dk , \tag{3.6}$$

where C is the indented contour. When $t' \to \infty$, $x' \to \infty$ as well; hence, the contributions from the real axis in (3.6) are $O(x'^{-1})$ by virtue of Rieman's lemma. The only contribution which remains to consider is the one from the semicircle in the lower half plane centered at $k_0 = g\sin\beta / v^2$. When $x' < 0$, and $x' \to -\infty$, we have $\Phi_e \to 0$ because $\operatorname{Re}(ikx') < 0$. For $x' > 0$ we deform the path in the upper plane; then the contribution from the semicircle is negligible, and the dominant term in $\bar{\Phi}_e$ is given by the residue at k_0. This is

$$\bar{\Phi}_e \sim \frac{4Fk_0}{\rho v} \cos\beta \cos(k_0 x + k_0 v t)\, e^{-k_0(y+y_0)\cos\beta + k_0 z \sin\beta} \tag{3.7}$$

for $x \geq -vt$. From (3.7) the surface elevation ζ_e is given by

$$\zeta_e = \frac{4F}{\rho g} \cos\beta \, k_0^2 \sin k_0(x+vt) \, e^{-k_0(y+y_0)\cos\beta} . \tag{3.8}$$

Equation (3.8) represents an edge wave traveling behind the disturbance, $(x \geq -vt)$ with phase velocity $-v$. The result (3.8) is valid for arbitrary angle β of the sloping beach. When $\beta \to 0$ in (3.8), and the Gaussian pressure distribution used by Greenspan is concentrated into an impulse, our result (3.8) agrees with Greenspan's. The only effect of the beach angle is the $\cos\beta$ factor.

CHAPTER 4

NONLINEAR EDGE WAVES AND SHALLOW WATER THEORY

In this chapter we consider the nonlinear effects for shallow water traveling edge waves on beaches with arbitrary depth distribution.

4.1 Introduction

In a recent paper (Whitham, 1976), both the shallow water approximation and the full water wave theory are used to discuss nonlinear effects in edge waves for the case of a uniformly sloping beach. In that case the shallow water approximation gives anomalous results for the amplitude decay away from the shoreline. This is attributed to the breakdown of the approximation as the depth increases. In this chapter, the shallow water theory is reconsidered for more general depth distributions which may be taken to remain finite and shallow at infinity. For finite depth the results are similar to those of the full theory for a constant slope. They differ in detail because the two cases now refer to different situations: in one, the depth offshore remains small compared to the wavelength; while in the other, it becomes large (in which case the precise depth distribution in the deep water is irrelevant since the waves are no longer influenced by the bottom).

Even in linear theory, the shallow water approximation has undesirable features for constant slope, since it predicts an infinite number of trapped modes at the shoreline (see equation (1.27)) and incoming waves with non-zero amplitude at infinity are not possible (see equation (1.24)). The full linear theory predicts just a finite number of edge waves and a continuous spectrum of incoming waves (see equations (1.13) and (1.15)). Again, the differences can be resolved by taking a depth distribution which becomes constant at large distances away from the shore. We discuss in some detail the spectrum of the operator associated with the linear theory and show that it has a finite number of isolated points (edge waves) and a continuous part, in agreement with the full linear theory. The nonlinear corrections for the lowest order mode are developed using a Stokes' expansion.

4.2 Linear Theory

The shallow water equations for a depth profile $h(y)$ are

$$\begin{aligned} &\varphi_t + \tfrac{1}{2}\varphi_x^2 + \tfrac{1}{2}\varphi_y^2 + g\zeta = 0, \\ &\zeta_t + \{(h(y)+\zeta)\varphi_x\}_x + \{(h(y)+\zeta)\varphi_y\}_y = 0, \end{aligned} \tag{4.1}$$

where ζ is the surface elevation, φ is a velocity potential for the horizontal velocity field, y denotes the offshore coordinate, and x the longshore coordinate. From the linearized form of (4.1) we obtain the following equation for ζ :

$$(h\zeta_y)_y + h\zeta_{xx} - \frac{1}{g}\zeta_{tt} = 0. \tag{4.2}$$

For a traveling wave solution of (4.2) of the form $\zeta = f(y)\sin(kx-\omega t)$, f satisfies

$$Lf + \lambda f \equiv (hk^{-1}f')' + (\lambda - kh)f = 0, \tag{4.3}$$

where $\lambda = \omega^2/gk$. In order to describe edge waves we need to discuss the spectrum of L .

For a constant beach angle β , $h(y) = \beta y$ and we have

Laguerre's equation. The spectrum is positive and discrete; the eigenvalues are $\omega^2 = gk(2m+1)/\beta$. However, this leads to the various discrepancies noted above. To model more realistic depth distributions, we choose $h(y)$ to be an increasing function such that $h(y) \sim \beta y$ as $y \to 0$ and $h(y) = h_1$ for $y \geq \ell_1$.

The domain of L is restricted to a class of functions which are finite as $y \to 0$. The operator L is self adjoint; therefore, the spectrum is confined to the real axis.

First, there are no points in the spectrum in the range $\lambda \leq 0$, for in that case any solution of (4.3) which is regular at the origin has $f(0)$ and $f'(0)$ of the same sign; it follows, writing (4.3) as

$$h f' = k v, \qquad v' = (kh - \lambda) f$$

that $|f|$ increases monotonically and cannot be bounded at infinity.

To find the spectrum of L in $\lambda \geq 0$, it is convenient to use the Liouville transformation. This is $f(y) = h^{-1/4}(y)\, u(s)$, where

$$s(y) = \int_0^y k^{-1/2} h^{-1/2}(t)\, dt .$$

The transformed equation for $u(s)$ is

$$(L_0 + \lambda) u \equiv u'' + (\lambda - q) u = 0, \tag{4.4}$$

where

$$q = kh - \frac{1}{16} \frac{h'^2}{kh} + \frac{h''}{4k} .$$

Since the Liouville transformation is in this case unitary, the spectrum of L_0 is the same as the spectrum of L. The general qualitative behavior of q is the following: since $h(y) \sim \beta y$ for $y \to 0$, $q(s) \sim -\frac{1}{4s^2}$ for $s \to 0$; $q(s)$ is an increasing function and is bounded by kh_1 ; $q(s) = kh_1$ for $s \geq s_1$; $q(s)$ has just one zero, which is smaller than s_1 . Thus, $q(s)$ is a potential well of infinite depth at the origin, width $s_1 = O\{\ell_1 k^{1/2} h_1^{-1/2}\}$ and height kh_1 .

For $\lambda \geq kh_1$, the solutions of (4.4) are oscillatory at infinity, and this range gives the continuous spectrum. (The relevant theorems are in Titchmarsh 1962, §§ 5.6, 5.7, 5.15.) For $0 \leq \lambda \leq kh_1$, there will be point eigenvalues (edge waves), whose number increases with the "size" of the well, which is measured by

$$s_1 (kh_1)^{1/2} = O(k\ell_1).$$

A natural choice for the depth distribution which incorporates the edge effect and remains shallow at infinity is

$$h(y) = \beta y \text{ for } 0 \leq y \leq \ell_1, \quad h(y) = h_1 \text{ for } y \geq \ell_1. \tag{4.5}$$

However, the discontinuity in h' would lead to singular functions in q , so as an example for (4.4) we take a smoothed version: $h(y) = \beta y$ for $0 \leq y \leq \ell_0$, $h(y)$ equal to a smooth increasing function for $\ell_0 \leq y \leq \ell_1$ and $h(y) = h_1$ for $y \geq \ell_1$. If $h_0 = \beta \ell_0$ is assumed fixed, we have

$$q(s) = \begin{cases} \frac{1}{4}\beta^2 s^2 - \frac{1}{4s^2} & \text{for } 0 \leq s \leq 2(kh_0)^{1/2}/\beta, \\ \text{smooth function for} & (4kh_0)^{1/2}/\beta \leq s \leq s_1(\beta) \\ kh_1 & \text{for } s_1(\beta) \leq s < \infty. \end{cases}$$

Here, the size of the well is measured by $2(kh_0)^{1/2}/\beta$. Comparison of the potential q in the interval $0 \leq s \leq (4kh_0)^{1/2}/\beta$ with the potential $\frac{1}{4}\beta^2 s^2$ for Hermite's equation confirms that for $0 \leq \lambda \leq k h_1$ there are points in the spectrum (for sufficiently small β) and that their number increases as β decreases.

The overall conclusion is that the nature of the spectrum for finite depth at infinity is the same as in the full linear theory for uniformly sloping beaches.

Finally, we return to (4.5) and work directly with (4.3) to find an explicit approximation to the linear dispersion relation for the lowest edge wave. We need the solution of

$$y f'' + f' + (\omega^2/g\beta - k^2 y) f = 0, \quad \text{for } 0 \leq y \leq \ell_1 \tag{4.6}$$

$$f'' + (\omega^2/gh_1 - k^2) f = 0, \quad \text{for } \ell_1 \leq y < \infty. \tag{4.7}$$

The interesting approximation for this discussion is for small β; this corresponds to large ℓ_1 if h_1 is kept fixed. For large ℓ_1 the solution of (4.6) is assumed to be close to e^{-ky}, and $\omega^2/g\beta$ is close to k. (These are the results for $\ell_1 = \infty$.) So we take

$$f(y) = e^{-ky} - \varepsilon \tilde{f}(y), \qquad \omega^2/g\beta = k(1-\varepsilon), \tag{4.8}$$

where ε will be related to ℓ_1 in the course of the argument. Then to first order in ε

$$y \tilde{f}'' + \tilde{f}' + (k - k^2 y)\tilde{f} = -k e^{-ky}.$$

The solution bounded at $y = 0$ is

$$\tilde{f}(y) = -e^{-ky} \int_0^y \frac{e^{2k\eta} - 1}{\eta} \, d\eta. \tag{4.9}$$

The appropriate solution of (4.7) is

$$f(y) = B\,e^{-\mu k y}\,, \quad \mu = \left(1 - \omega^2/gk^2 h_1\right)^{1/2}. \tag{4.10}$$

Since f and f' are continuous at $y = \ell_1$, the impedance $f'(\ell_1)/f(\ell_1)$ must be the same for (4.8) and (4.10). From (4.8) and (4.9),

$$f(\ell_1) \sim e^{-k\ell_1} + \varepsilon\, e^{k\ell_1}/4k\ell_1\,, \quad f'(\ell_1) \sim -k e^{-k\ell_1} + \varepsilon\, e^{k\ell_1}/4\ell_1\,,$$

$$f'(\ell_1)/f(\ell_1) \sim -\,k\left(1 - \varepsilon e^{2k\ell_1}/2k\ell_1\right); \quad k\ell_1 \gg 1.$$

(The second terms remain small since $\varepsilon\, e^{2k\ell_1}/k\ell_1$ is ultimately found to be small.) From (4.10),

$$f'(\ell_1)/f(\ell_1) = -\,k\mu = -k\left\{1 - (2k\ell_1)^{-1}\right\},$$

since $\omega^2 \sim gk\beta = gk\,h_1/\ell_1$ is a sufficient approximation in μ. Therefore, for the two values of $f'(\ell_1)/f(\ell_1)$ to agree,

$$\varepsilon = e^{-2k\ell_1}\,, \quad \omega^2 = g\beta k\left(1 - e^{-2k\ell_1} + O(e^{-4k\ell_1})\right). \tag{4.11}$$

4.3 Nonlinear Corrections

We now find the nonlinear corrections to the lowest edge wave mode. Following Whitham (1976), we consider Stokes' expansions for φ and ζ in the form of a traveling wave, and take

$$\varphi = a\,\varphi^{(1)}(y,\theta) + a^2\varphi^{(2)}(y,\theta) + a^3\varphi^{(3)}(y,\theta) + \cdots\,, \tag{4.12}$$

$$\zeta = a\,\zeta^{(1)}(y,\theta) + a^2\zeta^{(2)}(y,\theta) + a^3\zeta^{(3)}(y,\theta) + \cdots\,, \tag{4.13}$$

$$\omega = \omega_0 + a^2\omega_2 + \cdots\,, \tag{4.14}$$

where $\theta = kx - \omega t$. These are substituted in (4.1) to obtain the equations for the successive orders.

The first order problem is

$$(h\zeta^{(1)}_y)_y - (\omega_0^2/g - k^2 h)\zeta^{(1)}_{\theta\theta} = 0, \quad \zeta^{(1)}(0,\theta) \text{ finite.} \qquad (4.15)$$

Let $\zeta^{(1)} = f^{(1)}(y)\cos\theta$. Then $f^{(1)}$ satisfies

$$(h f^{(1)}_y)_y + (\omega_0^2/g - k^2 h) f^{(1)} = 0, \quad f^{(1)}(0) \text{ finite.} \qquad (4.16)$$

Choose ω^2/g to be the lowest eigenvalue of (4.16), and let $E(y)$ denote the corresponding edge wave solution for $f^{(1)}$. Notice that $E(y) \propto e^{-\mu k y}$ for $y \geq l_1$, where

$$\mu = (1 - \omega_0^2/gk^2h_1)^{1/2}.$$

The lowest order solution is

$$\zeta^{(1)} = E(y)\cos\theta, \quad \varphi^{(1)} = -g/\omega_0\, E(y)\sin\theta.$$

The second order problem takes the form

$$\begin{aligned} -\omega_0\varphi^{(2)}_\theta + g\zeta^{(2)} &= g^2k^2/\omega_0^2\left(M^{(2)}(y) + S^{(2)}(y)\cos 2\theta\right), \\ -\omega_0\zeta^{(2)}_\theta + (h\varphi^{(2)}_y)_y + k^2 h\,\varphi^{(2)}_{\theta\theta} &= -g k^2/\omega_0\, T^{(2)}(y)\sin 2\theta. \end{aligned} \qquad (4.17)$$

Let $\zeta^{(2)} = gk^2\omega_0^{-1} m^{(2)}(y) + f^{(2)}\cos 2\theta$. Then $f^{(2)}$ satisfies

$$(h f^{(2)}_y)_y + \left(4\frac{\omega_0^2}{g} - 4k^2h\right) f^{(2)} = k^2 R^{(2)}(y), \quad f^{(2)}(0) \text{ finite.} \qquad (4.18)$$

where $R^{(2)}$ is a quadratic in E and $R^{(2)}(y) = O(e^{-2\mu k y})$ as $y \to \infty$. We assume that the eigenvalues of the operator $(hf')' - (nk)^2 hf$ are not integer multiples of the lowest eigenvalue ω_0^2/g . Therefore, there is a solution of the second order problem (4.18) of the form

$$\zeta^{(2)}(y,\theta) = f^{(2)}(y)\cos 2\theta + g k^2\omega_0^{-2} m^{(2)}(y),$$

$$\varphi^{(2)}(y,\theta) = gk/\omega_0\, l^{(2)}(y)\sin 2\theta$$

where $f^{(2)}$, $m^{(2)}$ and $\ell^{(2)}$ are $O(e^{-2\mu k y})$ as $y \to \infty$.

The third order problem is

$$(h\mathcal{J}^{(3)}_y)_y - (\omega_0^2/g - k^2 h)\mathcal{J}^{(3)}_{\theta\theta} = \frac{\omega_0}{g}\Big(-2\omega_2 E(y)\cos\theta +$$

$$+\omega_0 k^2 R^{(3)}(y)\cos\theta + \omega_0 k^2 S^{(3)}(y)\cos 3\theta\Big),\ \mathcal{J}^{(3)}(0,\theta) \text{ finite}. \tag{4.19}$$

The forcing term in (4.19) proportional to $\cos 3\theta$ does not resonate; hence, it gives a contribution $O(e^{-3\mu k y})$ as $y \to \infty$. The crucial part of the discussion of the nonlinear problem concerns the resonant terms in (4.19) proportional to $\cos\theta$. Let $\mathcal{J}^{(3)} = f^{(3)}\cos\theta$. Then $f^{(3)}$ satisfies

$$(hf^{(3)}_y)_y + \left(\frac{\omega_0^2}{g} - k^2 h\right) f^{(3)} = \frac{\omega_0}{g}\left(-2\omega_2 E(y) + \omega_0 k^2 R^{(3)}(y)\right). \tag{4.20}$$

In order to have a square integrable solution which satisfies the boundary condition $f^{(3)}(0) =$ finite, the right hand side of (4.20) must be orthogonal to the function $E(y)$. This orthogonality condition determines ω_2:

$$\omega_2 = \tfrac{1}{2}\gamma\omega_0 k^2, \text{ where } \gamma = \int_0^\infty R^{(3)}(y)E(y)\,dy \Big/ \int_0^\infty E^2(y)\,dy. \tag{4.21}$$

The expression for $R^{(3)}$ in terms of E is complicated for general $h(y)$ and in any case $R^{(3)}$ is not known explicitly. But $E(y) = O(e^{-\mu k y})$ and $R^{(3)}(y) = O(e^{-3\mu k y})$ as $y \to \infty$, so γ will differ little from the value $\frac{1}{2}$ obtained for the case $h(y) = \beta y$. More precisely, if $h(y) = \beta y$ for $0 \le y \le \ell_1$, then, as shown in (4.11), the correction is $O(e^{-2k\ell_1})$.

However, finding small changes in the dispersion relation is not the object of this chapter. The questions concern the interpretation of the behavior of $f^{(3)}(y)$ as $y \to \infty$ and the uniform validity of the expansions.

To study the behavior of $f^{(3)}(y)$ as $y \to \infty$, we solve (4.20) by variation of parameters. The solution is

$$f^{(3)}(y) = \frac{\omega_0^2}{g} k^2 E(y) W(y) ,$$

where

$$W(y) = -\int_0^y \frac{1}{h(\eta) E^2(\eta)} \int_\infty^\eta \{\gamma E^2(\xi) - R^{(3)}(\xi) E(\xi)\, d\xi\, d\eta . \qquad (4.22)$$

In all cases $W(y) \to \infty$ as $y \to \infty$, so the third order terms in (4.12) and (4.13) become large compared with the first order terms, which are proportional to $E(y)$, and the expansions are not uniformly valid as $y \to \infty$. We have

$$\zeta = a\left(E(y) + \frac{\omega_0^2}{g} k^2 a^2 E(y) W(y)\right) \cos\theta + \cdots .$$

For large y, $E(y) \propto e^{-\mu k y}$, so this becomes

$$\zeta \sim a\left(e^{-\mu k y} + \frac{\omega_0^2}{g} k^2 a^2 W(y) e^{-\mu k y}\right) \cos\theta + \cdots .$$

The method of strained coordinates suggests that this is the Taylor expansion of

$$\zeta \sim a \exp\left\{-\mu k y + \frac{\omega_0^2}{g} a^2 k^2 W(y)\right\} \cos\theta , \qquad (4.23)$$

and that this modified form is the correct, uniformly valid one. For the beach of constant slope discussed by Whitham, $h(y) = \beta y$, $E(y) = e^{-ky}$, $R^{(3)}(y) = e^{-3ky}$, and

$$W(y) \sim (4k\beta)^{-1} \log (ky) \text{ as } y \to \infty.$$

The logarithmic behavior seemed unnatural and is attributed to the inadequacy of the shallow water theory for this case. This view was confirmed, since the full water wave theory gave $W(y) \propto y$ and could be interpreted satisfactorily as yielding an amplitude dependence in the rate of decay. We are now in a position to discuss the behavior for more general distributions $h(y)$ which do not violate the shallow water approximations.

The asymptotic behavior of $W(y)$ is given by the first term in (4.22), i.e.,

$$W(y) \sim \int_{\text{Const}}^{y} \frac{1}{h E^2(\eta)} \int_{\eta}^{\infty} \gamma E^2(\xi)\, d\xi\, d\eta. \tag{4.24}$$

When $h \to h_1$ as $y \to \infty$, we have $E(y) \propto e^{-\mu k y}$,

and

$$W(y) \propto \gamma (2\mu k h_1)^{-1} y .$$

This is the same type of behavior as in the full theory, and again we have a clear interpretation of the result as a nonlinear modification to the rate of the exponential decay. According to (4.23) the appropriate rate of decay is now

$$k\left(1 - \frac{\omega_0^2}{gk^2h_1}\right)^{1/2} + \frac{\omega_0^2 \gamma}{2gkh_1}\left(1 - \frac{\omega_0^2}{gk^2h_1}\right)^{-\frac{1}{2}} (ak)^2 .$$

It is interesting that the term (4.24) originates from the frequency correction ω_2 , introduced in (4.14) to eliminate secular terms in the Stokes' expansion but then leads to nonuniformities in γ ! It is unusual in nonlinear vibration problems that terms needed to construct a uniform expansion in one variable produce nonuniformities in other variables. However, in simpler examples the region considered is finite in space, and then all nonuniformities appear in the time variable. When Stokes' expansions are used to discuss periodic solutions which represent trapped modes in infinite regions, we may expect the behavior found here.

CHAPTER 5

NONLINEAR CORRECTIONS FOR STANDING WAVES ON A UNIFORMLY SLOPING BEACH

The problem of nonlinear standing waves in an infinitely deep fluid was first discussed by Lord Rayleigh (1915) who found solutions to the third order of approximation in the amplitude. In 1952 Penney and Price discussed the same problem but to a much higher order of approximation. Finally in 1960 I. Tadjbakhsh and J. B. Keller found approximate solutions to the equations of motion which represent standing waves in a fluid of finite constant depth. It is the purpose of this chapter to discuss the corresponding nonlinear corrections for standing waves on a sloping beach. Here there is no longshore dependence; we are concerned, except for the last section, with a normally incident wavetrain and its reflection. The problem has been previously discussed by Carrier and Greenspan (1958) on the basis of the nonlinear shallow water approximation. Carrier and Greenspan found exact solutions (as implicitly defined functions) for the nonlinear shallow water equations. However, as remarked in Chapter 1, the shallow water approximation for a uniformly sloping beach is invalid away from the shore; therefore, differences between the full theory and the shallow water approximation can be expected in analogy with the situation discussed in Chapter 4.

This chapter consists of four sections. In the first section the problem is formulated and the approximation scheme is described. The second section is devoted to a detailed discussion of the case $\beta = \pi/4$. Since the calculations for $\beta = \pi/4$ are less involved, this provides an understanding of the self interactions of the wave. In the third section, the case of arbitrary angle β is examined and the results are compared with those for the shallow water. In the last section, the nonlinear correction to the dispersion relation for an oblique wave is found for a beach slope $\beta = \pi/4$.

5.1 Formulation of the Problem

The problem of developing the nonlinear corrections for a standing wave on a uniformly sloping beach is different mathematically from the more familiar problems of standing waves in a bounded region. In the case of a bounded region the normal modes of the linear problem are used as a first approximation and the nonlinear corrections for the natural frequencies are found using an appropriate orthogonality condition. However, in the present case, the first approximation is not a "proper" eigenfunction; but it is a function in the continuous spectrum, and the orthogonality condition is no longer meaningful. Therefore, a different criterion must be used to determine the corrections to the dispersion relation. This criterion is the requirement of uniform validity of the expansions away from the shoreline.

To discuss the problem for arbitrary angle β of the sloping beach, we consider the full nonlinear equations of motion approximated to third order in the amplitude. In this case, there is no advantage in the elimination of the surface elevation in terms of the velocity potential. When nondimensional variables and functions are introduced for the velocity potential Φ and for the surface elevation ζ by the relations

$$\Phi(y,z,t) = \frac{ag}{\omega}\,\varphi(y',z',t')$$
$$\zeta(y,t) = a\,\zeta'(y',t') \tag{5.1}$$

where $y' = ly$, $z' = lz$, $t' = \omega t$, $\omega^2 = gl$, the equations of motion, after the primes are dropped, become

$$\begin{aligned}
&\varphi_t + \zeta + \varepsilon\{\varphi_{tz}\zeta + \tfrac{1}{2}(\varphi_y^2 + \varphi_z^2)\} + \varepsilon^2\{(\varphi_y\varphi_{yz} + \varphi_z\varphi_{zz})\zeta + \tfrac{1}{2}\varphi_{tzz}\zeta^2\}, \text{ on } z=0,\ y\geq 0,\\
&\zeta_t - \varphi_z + \varepsilon\{\varphi_y\zeta_y - \varphi_{zz}\zeta\} + \varepsilon^2\{\varphi_{yz}\zeta\zeta_y - \tfrac{1}{2}\varphi_{zzz}\zeta^2\} = 0 \text{ on } z=0,\ y\geq 0,\\
&\varphi_{yy} + \varphi_{zz} = 0 \quad \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0,\\
&\varphi_y \sin\beta + \varphi_z \cos\beta = 0 \quad \text{on } z = -y\tan\beta,\ y \geq 0.
\end{aligned} \tag{5.2}$$

where the small parameter $\varepsilon = al$ is a measure of the wave slope.

We are interested in standing wave solutions of (5.2), that is in bounded periodic solutions of (5.2), whose period will depend on the amplitude of the oscillation. To obtain an approximation to the solution, we consider Stokes' expansions for φ and ζ in the form

$$\varphi \sim \sum_{m=0}^{\infty} \varepsilon^m \varphi^{(m)}(y,z,\theta), \tag{5.3}$$

$$\zeta \sim \sum_{m=0}^{\infty} \varepsilon^m \zeta^{(m)}(y,\theta), \tag{5.4}$$

$$\theta = \omega(\varepsilon)t = \sum_{m=0}^{\infty} \varepsilon^m \omega_m\, t, \tag{5.5}$$

where the functions $\varphi^{(m)}$ and $\zeta^{(m)}$ are 2π periodic functions of θ. When (5.3) to (5.5) are substituted in (5.2) the equations for the successive orders are obtained and their solution gives the desired approximation to the finite amplitude standing wave solution of (5.2). This perturbation procedure differs from the one used by Penney and Price and Tadjbakhsh and Keller. They expand the solutions φ and $\mathcal{J}$ in terms of the normal modes of the system, and then obtain nonlinear ordinary differential equations for the time dependent coefficients in their expansions. However, this method relies on the fact that products of trigonometric functions (the normal modes in their case) can be easily expanded in terms of the normal modes of the linear problem. This is not the case for the normal modes for a uniformly sloping beach, and the expansions (5.3) to (5.5) are preferred. The scheme obtained using (5.3) to (5.5) of course recovers the results for the case of infinte depth when used in that situation.

5.2 The Special Case $\beta = \pi/4$

In this section we describe in detail how the approximate solutions of (5.2) are obtained for the simplest case $\beta = \pi/4$. When (5.3) to (5.5) are substituted in (5.2) the lowest order equations take the form

$$\begin{aligned}
&\omega_0 \varphi^{(1)}_\theta + \mathcal{J}^{(1)} = 0 && \text{on } z = 0,\ y \geq 0, \\
&\omega_0 \mathcal{J}^{(1)}_\theta - \varphi^{(1)}_z = 0 && \text{on } z = 0,\ y \geq 0, \\
&\varphi^{(1)}_{yy} + \varphi^{(1)}_{zz} = 0 && \text{for } y \geq 0,\ -y\tan\beta \leq z \leq 0. \\
&\varphi^{(1)}_y + \varphi^{(1)}_z = 0 && \text{on } z = -y,\ y \geq 0.
\end{aligned} \tag{5.6}$$

For a standing wave solution of (5.6), we let $\varphi^{(1)}(y,z,\theta) = f^{(1)}(y,z)\cos\theta$, eliminate $\mathcal{J}^{(1)}$ from (5.6) and obtain the following equations for $f^{(1)}$:

$$f_z^{(1)} - \omega_0^2 f^{(1)} = 0 \qquad \text{on } z=0,\ y \geq 0.$$

$$f_{yy}^{(1)} + f_{zz}^{(1)} = 0 \qquad \text{for } y \geq 0,\ -y \leq z \leq 0. \tag{5.7}$$

$$f_y^{(1)} + f_z^{(1)} = 0 \qquad \text{on } z = -y,\ y \geq 0.$$

The appropriate standing wave solutions of (5.7) are the "eigenfunctions of the continuous spectrum" discussed in section (1.1). The appropriate solution of (5.7) is the function $\bar{\Psi}_\ell(y,z)$ defined in equation (1.4). Therefore the required solution of (5.6) is

$$\varphi^{(1)} = f^{(1)}(y,z)\cos\theta = \left\{e^{z}\cos(y+\pi/4) + e^{-y}\cos(z-\pi/4)\right\}\cos\theta,$$

$$\mathcal{J}^{(1)} = f^{(1)}(y,0)\sin\theta = \left\{\cos(y+\pi/4) + 1/\sqrt{2}\ e^{-y}\right\}\sin\theta, \tag{5.8}$$

$$\omega_0 = 1.$$

The second order problem is

$$\varphi_\theta^{(2)} + \mathcal{J}^{(2)} + \omega_1\varphi_\theta^{(1)} + \varphi_{\theta z}^{(1)}\mathcal{J}^{(1)} + \frac{1}{2}\left(\varphi_y^{(1)2} + \varphi_z^{(1)2}\right) = 0, \qquad \text{on } z=0,\ y\geq 0,$$

$$\mathcal{J}_\theta^{(2)} - \varphi_z^{(2)} + \omega_1\mathcal{J}_\theta^{(1)} + \varphi_y^{(1)}\mathcal{J}_y^{(1)} - \varphi_{zz}^{(1)}\mathcal{J}^{(1)} = 0, \qquad \text{on } z=0,\ y\geq 0,$$

$$\varphi_{yy}^{(2)} + \varphi_{zz}^{(2)} = 0 \qquad \text{for } y\geq 0,\ -y\leq z\leq 0, \tag{5.9}$$

$$\varphi_y^{(2)} + \varphi_z^{(2)} = 0 \qquad \text{on } z=-y,\ y\geq 0.$$

When the expressions (5.8) are substituted in (5.9) and $\mathcal{J}^{(2)}$ is eliminated, (5.9) becomes

$$\varphi_{\theta\theta}^{(2)} + \varphi_z^{(2)} = 2\omega_1\bar{\Psi}_1(y,0)\cos\theta + \Big\{\left(\sin(y+\pi/4) + 2^{-\frac{1}{2}}e^{-y}\right)^2 + \frac{1}{2}e^{-2y} +$$

$$+ \frac{3}{2}\left(\cos(y+\pi/4) + \frac{1}{\sqrt{2}}e^{-y}\right)^2 - \frac{1}{2}\cos^2(y+\pi/4)\Big\}\sin 2\theta,$$

$$\varphi_{yy}^{(2)} + \varphi_{zz}^{(2)} = 0 \qquad \text{for } y\geq 0,\ -y\leq z\leq 0. \tag{5.10}$$

$$\varphi_y^{(2)} + \varphi_z^{(2)} = 0 \qquad \text{on } z=-y,\ y\geq 0.$$

Since the appropriate solution for $\varphi^{(2)}$ must be periodic we take

$$\varphi^{(2)} = A_1^{(2)} \cos\theta + A_2^{(2)} \sin 2\theta . \tag{5.11}$$

Substitution of (5.11) into (5.10) gives equations for $A_1^{(2)}$ and $A_2^{(2)}$ in the form

$$\begin{aligned} &A_{1z}^{(2)} - A_1^{(2)} = 2\omega_1 \bar{\Psi}_1(y,0) \quad \text{on } z=0,\ y \geq 0, \\ &A_{1yy}^{(2)} + A_{1zz}^{(2)} = 0 \quad \text{for } y \geq 0,\ -y \leq z \leq 0, \\ &A_{1y}^{(2)} + A_{1z}^{(2)} = 0 \quad \text{on } z=-y,\ y \geq 0. \end{aligned} \tag{5.12}$$

$$\begin{aligned} &A_{2z}^{(2)} - 4A_2^{(2)} = 1 + R_2^{(2)}(y) \quad \text{on } z=0,\ y \geq 0, \\ &A_{2yy}^{(2)} + A_{2zz}^{(2)} = 0 \quad \text{for } y \geq 0,\ -y \leq z \leq 0, \\ &A_{2y}^{(2)} + A_{2z}^{(2)} = 0 \quad \text{on } z=-y,\ y \geq 0. \end{aligned} \tag{5.13}$$

where the forcing function $R_2^{(2)}(y)$ is given by

$$R_2^{(2)}(y) = \frac{7}{4} e^{-2y} + \sqrt{2}\, e^{-y} \sin\left(y + \frac{\pi}{4}\right) + \frac{3}{\sqrt{2}} e^{-y} \cos\left(y + \pi/4\right) .$$

Now consider (5.12). Equation (5.12) is the one discussed in the first chapter, and its solution is given in (1.11). Therefore the solution of (5.12) is given by

$$A_1^{(2)} = 2\omega_1 \frac{\partial}{\partial \ell} \Psi_\ell(y,z) \Big|_{\ell=1} .$$

The constant ω_1 is so far arbitrary. To determine it, the requirement that (5.3) provides a uniformly valid expansion offshore is used. To examine the uniform validity of (5.3) we consider the expression

$$A_1^{(2)}(y,z) = 2\omega_1 \left\{ z\left(e^{z} \cos(y+\tfrac{\pi}{4}) - e^{-y} \sin(z-\tfrac{\pi}{4})\right) - y\left(e^{z} \sin(y+\tfrac{\pi}{4}) - e^{-y} \cos(z-\pi/4)\right) \right\} . \tag{5.14}$$

When (5. 14) is used to find the asymptotic expansion, away from the shore, we obtain

$$\varphi(y,z) \sim e^{z} \cos\left(y+\frac{\pi}{4}\right)\cos\theta + \varepsilon\left\{-2\omega_1 y e^{z} \sin\left(y+\frac{\pi}{4}\right)\sin 2\theta\right\}. \tag{5.15}$$

However (5. 15) is not uniformly valid as $y \to \infty$, unles $\omega_1 = 0$. Hence,to obtain the desired uniform expansion we take $\omega_1 = 0$, which is usual in water waves problems. The general solution of (5. 12) is obtained by adding to the particular integral $A_1^{(2)}(y,z)$ the general solution of the homogeneous problem. This amounts to a redefinition of the amplitude and phase of the solution for the linear problem; therefore there is no loss in taking it to be identically zero.

To find the solution of (5. 13), we use the expansion discussed in the first chapter. Using equation (1. 10) we obtain a particular integral for (5. 13) in the form

$$A_2^{(2)}(y,z) = -\frac{1}{4} + \text{P.V.}\int_0^\infty \frac{S_\ell(y,z)}{\ell-4}\,d\ell \int_0^\infty S_\ell(t,0)\,R_2^{(2)}(t)\,dt. \tag{5.16}$$

As before the general solution of (5. 13) is obtained by adding to the particular integral (5. 16) the general solution of the homogeneous problem. In order to find the appropriate solution of the homogeneous problem we again examine the behavior of $\varphi^{(2)}$ as $y \to \infty$. To this end we need the asymptotic expansion of $A_2^{(2)}(y,z)$ as $y \to \infty$. This expansion is discussed in detail in the Appendix, and for the present discussion it is sufficient to observe that the main contribution as $y \to \infty$ comes from the pole $\ell = 4$ and is given by:

$$A_2^{(2)}(y,z) = -\frac{1}{4} - \sqrt{\frac{2}{\pi}} \int_0^\infty R_2^{(2)}(t)\, S_4'(t,0)\,dt \; e^{4z} \sin\left(4y + \frac{\pi}{4}\right) + a_2^{(2)}(y,z) , \tag{5.17}$$

where $a_2^{(2)}(y,z) = O(y^{-2})$ as $y \to \infty$. From (5.17) we deduce that $\varphi^{(2)}$ behaves as a standing wave at infinity, hence the particular integral (5.16) is an appropriate solution for (5.13) since in order to find a periodic solution we need energy balance at infinity. However, a comment is necessary on the solutions of the homogeneous problem. First of all consider the solution $c\, S_4(y,z) \cos 2\theta$, where c is an arbitrary constant. When this solution is added to (5.17) the resulting solution behaves as an incoming or outgoing wave depending on the sign of c. This is not desirable since at the next order an increase or decay in amplitude is obtained, and we are interested in periodic solutions; hence we take c = 0. The second solution of the homogeneous problem $D\, S_4(y,z) \sin 2\theta$ will not upset the periodicity of the solution φ, however we take D = 0 since this corresponds to a choice of the incoming wave at infinity. The same situation is encountered in the work of Todjbakhsh and Keller and the same choice is made there.

The above arguments give the solution for the second order problem in the form:

$$\begin{aligned} \varphi^{(2)} &= -\frac{1}{4} \sin 2\theta - \sqrt{\frac{2}{\pi}} \int_0^\infty R_2^{(2)}(t)\, S_4(t,0)\,dt \; e^{4z} \sin\left(4y + \frac{\pi}{4}\right) \sin 2\theta + \\ &\quad + a_2^{(2)}(y,z) \sin 2\theta . \\ \mathcal{J}^{(2)} &= \frac{1}{4} \cos 2\left(y + \frac{\pi}{4}\right) - \frac{1}{4} \cos 2(y + \pi/4) \cos 2\theta + \\ &\quad + 2\sqrt{\frac{2}{\pi}} \int_0^\infty R_2^{(2)}(t)\, S_4'(t,0)\,dt \, \sin\left(4y + \frac{\pi}{4}\right) \cos 2\theta + b_2^{(2)}(y,z) \cos 2\theta , \end{aligned} \tag{5.18}$$

where $a_2^{(2)}(y,z)$ and $b_2^{(2)}(y,z)$ are $O(y^{-2})$ as $y \to \infty$.

The third order equations take the form:

$$\varphi_\theta^{(3)} + \zeta^{(3)} = -\omega_2 \varphi_\theta^{(1)} - \varphi_{\theta z}^{(1)} \zeta^{(2)} - \varphi_{\theta z}^{(2)} \zeta^{(1)} - \frac{1}{2} \varphi_{\theta zz}^{(1)} \zeta^{(1)2} - \varphi_y^{(1)} \varphi_y^{(2)} -$$
$$- \varphi_z^{(1)} \varphi_z^{(2)} - \varphi_y^{(1)} \varphi_{yz}^{(1)} \zeta^{(1)} - \varphi_z^{(1)} \varphi_{zz}^{(1)} \zeta^{(1)}, \quad \text{on } z=0,\ y \geq 0;$$

$$\zeta_\theta^{(3)} - \varphi_z^{(3)} = -\omega_2 \zeta_\theta^{(1)} + \varphi_{zz}^{(1)} \zeta^{(2)} + \varphi_{zz}^{(2)} \zeta^{(1)} + \frac{1}{2} \varphi_{zzz}^{(1)} \zeta^{(1)2} -$$
$$- \varphi_{yz}^{(1)} \zeta^{(1)} \zeta_y^{(1)} - \varphi_y^{(1)} \zeta_y^{(2)} - \varphi_y^{(2)} \zeta_y^{(2)}, \quad \text{on } z=0,\ y \geq 0, \tag{5.19}$$

$$\varphi_{yy}^{(3)} + \varphi_{zz}^{(3)} = 0 \quad \text{for } y \geq 0,\ -y \leq z \leq 0,$$

$$\varphi_y^{(3)} + \varphi_z^{(3)} = 0 \quad \text{on } z=-y,\ y \geq 0.$$

Substitution of (5.8) and (5.18) in (5.19) and elimination of the surface elevation $\zeta^{(3)}$ gives the equation for $\varphi^{(3)}$ in the form

$$\varphi_{\theta\theta}^{(3)} + \varphi_z^{(3)} = 2\omega_2 \varphi^{(1)}(y,0) + R_1^{(3)}(y) \cos\theta + R_3^{(3)}(y) \cos 3\theta, \quad \text{on } z=0,\ y \geq 0,$$
$$\varphi_{yy}^{(3)} + \varphi_{zz}^{(3)} = 0 \quad \text{for } y \geq 0, \quad -y \leq z \leq 0. \tag{5.20}$$
$$\varphi_y^{(3)} + \varphi_z^{(3)} = 0 \quad \text{on } z=-y,\ y \geq 0.$$

To solve (5.20) let $\varphi^{(3)} = A_1^{(3)} \cos\theta + A_3^{(3)} \cos 3\theta$. Then the equation for $A_3^{(3)}(y,z)$ is:

$$A_{3z}^{(3)} - 9 A_3^{(3)} = R_3^{(3)}(y) \quad \text{on } z=0,\ y \geq 0,$$
$$A_{3yy}^{(3)} + A_{3zz}^{(3)} = 0 \quad \text{for } y \geq 0,\ -y \leq z \leq 0, \tag{5.21}$$
$$A_{3y}^{(3)} + A_{3z}^{(3)} = 0 \quad \text{on } z=-y,\ y \geq 0.$$

where the function $R_3^{(3)}(y)$ is a linear combination of trigonometric functions with wave number different from nine and functions which are $O(y^{-2})$ as $y \to \infty$. An acceptable solution for (5.21) is readily obtained and has the same features discussed in (5.16). It will be shown in detail in the Appendix that the trigonometric terms give no problem since all have wave numbers different from nine.

As usual the crucial equation of the analysis is the one that contains the resonant terms. In this case it is the equation for $A_1^{(3)}$, namely:

$$A_{1z}^{(3)} - A_1^{(3)} = 2\omega_2 \bar{\Psi}_1(y,0) + \frac{8}{32}\cos(y+\tfrac{\pi}{4}) + T_1^{(3)}(y), \text{ on } z=0, y \geq 0,$$
$$A_{1yy}^{(3)} + A_{1zz}^{(3)} = 0 \quad \text{for } y \geq 0, \ -y \leq z \leq 0, \tag{5.22}$$
$$A_{1y}^{(3)} + A_{1z}^{(3)} = 0 \quad \text{on } z=-y, \ y \geq 0.$$

where the function $T_1^{(3)}$ has terms which are $O(y^{-2})$ and contains trigonometric terms with wave number different from one.

We now discuss the solution of (5.22). First of all consider the second term in the forcing function of (5.22). This is the result of the cubic self interaction of the deep water component of the linear solution, and of the deep water component of the mean elevation with the lowest order solution. It is in fact the same term obtained in the case of infinite depth. To find the correction to the dispersion relation ω_2, add and subtract from the forcing term in (5.22) the function $(4\sqrt{2})^{-1} e^{-y}$. This gives

$$A_{1z}^{(3)} - A_1^{(3)} = (2\omega_2 + \tfrac{1}{4})\,\bar{\Psi}_1(y,0) + T_1^{(3)}(y) - \frac{1}{4\sqrt{2}} e^{-y} \text{ on } z=0, y \geq 0,$$
$$A_{1yy}^{(3)} + A_{1zz}^{(3)} = 0 \quad \text{for } y \geq 0, \ -y \leq z \leq 0, \tag{5.23}$$
$$A_{1y}^{(3)} + A_{1z}^{(3)} = 0 \quad \text{on } z=-y, \ y \geq 0.$$

The second and third terms in the forcing function of (5.23) produce acceptable solutions for $\varphi^{(3)}$. However, the first term resonates on the continuous spectrum, and as was shown in (5.15) it does not give an acceptable solution. Therefore, we must take $\omega_2 = -1/8$. This determines the nonlinear correction to the dispersion relation. The final solution, in dimensional variables, is given to lowest order by

$$\Phi = \frac{ag}{\omega}\left\{ e^{-ly}\cos(lz-\pi/4) + e^{lz}\cos(ly+\pi/4)\right\}\cos\theta + O(al)^2 \tag{5.24}$$

where $\theta = \omega t = (gl)^{1/2}\left(1-\frac{1}{8}(al)^2+(al)^4\right)t$.

The perturbation analysis just described indicates that nonlinear waves with perfect reflection are possible on a uniformly sloping beach.

5.3 The Solution for Arbitrary β

The analysis just described can be applied with more labor to the case of an arbitrary angle β. However, it was noted in (5.22) that the interactions which produce the resonant term in (5.23) are: the self-interaction of the deep water component of the linear solution, and the interaction between the mean elevation and the linear solution. This interaction is readily calculated, for arbitrary β, if we recall from equation (1.6) that the deep water component of the linear solution $S_l(y,z)$ is independent of the angle β. Therefore, since the correction ω_2 depends only on the deep water behavior we have $\omega_2 = -1/8$ for all β. The actual form of the waves will be different but to the order considered the behavior of the dispersion relation is the same as the one obtained for infinite depth. In particular for small values of β the correction is - 1/8. However, the exact solutions of Carrier and Greenspan (1958) are finite amplitude standing waves satisfying the linear dispersion relation. This discrepancy is due to the fact that the shallow water theory, for a uniformly sloping beach, is not valid at large distance offshore. In fact the linear theory does not allow solutions with nonzero amplitude at infinity. When the nonlinear shallow water approximation is studied using (5.3) to (5.5), the expansion of the solution obtained by Carrier and Greenspan is found. However, in this case no change in frequency is needed since the self-interactions of the wave never produce the oscillatory part of the linear solution which gives the change in the dispersion relation. When a more realistic depth distribution which remains shallow at infinity is considered, the self-interactions of the wave produce a change in the dispersion relation, in analogy with the full theory.

5.4 Nonlinear Corrections for Oblique Incident Waves on a Sloping Beach

The nonlinear effects in the dispersion relation for a monochromatic wave obliquely incident on a sloping beach are found using a Stokes' expansion. The arguments involved in this section are the same ones used to discuss the normally incident waves. The difference is in the basic linear problem, and also in the calculations since now the longshore dependence is included. As in the case of normal incidence the correction to the dispersion relation depends only on the deep water terms; the only edge effect is the introduction of extra terms which fall off exponentially away from the shoreline.

To find the desired approximate solutions consider the full nonlinear problem

$$\begin{aligned}
&\Phi_t + g\zeta' + \tfrac{1}{2}\{\Phi_x^2 + \Phi_y^2 + \Phi_z^2\} = 0, \quad \text{on } z = \zeta'(x,y,t),\ -\infty<x<\infty,\ y\geq 0,\\
&\zeta'_t - \Phi_z + \Phi_x\zeta'_x + \Phi_y\zeta'_y = 0, \quad \text{on } z = \zeta'(x,y,t), -\infty<x<\infty,\ y\geq 0,\\
&\nabla^2\Phi = 0 \quad \text{for } -\infty<x<\infty,\ y\geq 0,\ -y\leq z\leq \zeta'(x,y,t).\\
&\Phi_y + \Phi_z = 0 \quad \text{on } z=-y,\ y\geq 0.
\end{aligned} \tag{5.25}$$

For (5.25) we want a solution in the form of a travelling wave, and we let

$$\Phi(y,z,x,t) = \varphi(y,z,\theta), \tag{5.26}$$

$$\zeta'(y,x,t) = \zeta(y,\theta), \tag{5.27}$$

where $\theta = k_0x+\omega t$, $k_0>0$, $\omega>0$. When (5.26) and (5.27) are substituted in (5.25), and the nonlinear boundary conditions are approximated to third order we obtain for φ and ζ the following equations

$$\begin{aligned}
&\omega\varphi_\theta + g\zeta + \omega\varphi_{\theta z}\zeta + \tfrac{\omega}{2}\varphi_{\theta zz}\zeta^2 + \tfrac{1}{2}\left(k_0^2\varphi_\theta^2 + \varphi_y^2 + \varphi_z^2\right) +\\
&\quad + \left(k_0^2\varphi_\theta\varphi_{\theta z} + \varphi_y\varphi_{yz} + \varphi_z\varphi_{zz}\right)\zeta = 0, \quad \text{on } z=0,\ y\geq 0,\\
&\omega\zeta_\theta - \varphi_z - \varphi_{zz}\zeta - \tfrac{1}{2}\varphi_{zzz}\zeta^2 + k_0^2\varphi_\theta\zeta_\theta + \varphi_y\zeta_y +\\
&\quad + k_0^2\varphi_{\theta z}\zeta\zeta_\theta + \varphi_{yz}\zeta\zeta_y = 0, \quad \text{on } z=0,\ y\geq 0,\\
&\varphi_{yy} + \varphi_{zz} + k_0^2\varphi_{\theta\theta} = 0 \quad \text{for } -\infty<\theta<\infty,\ y\geq 0,\ -y\leq z\leq 0,\\
&\varphi_y + \varphi_z = 0 \quad \text{on } z=-y,\ y\geq 0.
\end{aligned} \tag{5.28}$$

Now we find approximate solutions of (5.28) which represent oblique incident waves with perfect reflection at the shoreline. We take

$$\varphi \sim \sum_{n=1}^{\infty} a^n \varphi^{(n)}(y, z, \theta) \tag{5.29}$$

$$\mathcal{J} \sim \sum_{n=1}^{\infty} a^n \mathcal{J}^{(n)}(y, \theta) \tag{5.30}$$

$$\omega \sim \sum_{n=0}^{\infty} a^n \omega_n \tag{5.31}$$

where $\varphi^{(n)}$ and $\mathcal{J}^{(n)}$ are 2π periodic functions of θ .

When expressions (5.29) to (5.31) are substituted in (5.28), the equations for the successive orders are obtained.

The lowest order problem is just the linear one, i.e.

$$\begin{aligned} &\varphi^{(1)}_z + \frac{\omega_0^2}{g}\varphi^{(1)} = 0 \qquad \text{on } z=0,\ y \geq 0, \\ &\varphi^{(1)}_{yy} + \varphi^{(1)}_{zz} + k_0^2 \varphi^{(1)}_{\theta\theta} = 0 \qquad \text{for } -\infty < \theta < \infty,\ y \geq 0,\ -y \leq z \leq 0, \\ &\varphi^{(1)}_y + \varphi^{(1)}_z = 0, \qquad \text{on } z = -y,\ y \geq 0. \end{aligned} \tag{5.32}$$

An appropriate solution of (5.32) which represents a wave incident in the direction $-(k_0, l_0)$, $l_0 > 0$, with perfect reflection at the shore, is given by Hanson's solution. That is

$$\varphi^{(1)} = \frac{g}{\omega_0}\left\{ e^{-y(k_0^2+l_0^2)^{1/2}} \cos(l_0 z + \varepsilon) + e^{z(k_0^2+l_0^2)^{1/2}} \cos(l_0 y + \varepsilon)\right\} \cos\theta, \tag{5.33}$$

$$\mathcal{J}^{(1)} = \left\{ e^{-y(k_0^2+l_0^2)^{1/2}} \cos\varepsilon + \cos(l_0 y + \varepsilon)\right\} \sin\theta, \tag{5.34}$$

where $\omega_0^2 = g(l_0^2 + k_0^2)^{1/2}$, and $\cos\varepsilon = l_0/(l_0^2+\lambda_0^2)^{1/2}$ where $\lambda_0^2 = l_0^2 + k_0^2$.

The second order problem is

$$\omega_0 \varphi_\theta^{(2)} + g\zeta^{(2)} = -\omega_0 \varphi_{\theta z}^{(1)} \zeta^{(1)} - \frac{1}{2}\left(k_0^2 \varphi_\theta^{(1)2} + \varphi_y^{(1)2} + \varphi_z^{(1)2}\right), \quad \text{on } z=0,\ y \geq 0,$$

$$\omega_0 \zeta_\theta^{(2)} - \varphi_z^{(2)} = \varphi_{zz}^{(1)} \zeta^{(1)} - k_0^2 \varphi_\theta^{(1)} \zeta_\theta^{(1)} - \varphi_y^{(1)} \zeta_y^{(1)}, \quad \text{on } z=0,\ y \geq 0,$$

$$\varphi_{yy}^{(2)} + \varphi_{zz}^{(2)} + k_0^2 \varphi_{\theta\theta}^{(2)} = 0, \quad \text{for } -\infty < \theta < \infty,\ y \geq 0,\ -y \leq z \leq 0, \qquad (5.35)$$

$$\varphi_y^{(2)} + \varphi_z^{(2)} = 0 \quad \text{on } z = -y,\ y \geq 0.$$

The term proportional to ω_1 does not appear since the correction to the frequency only comes in third order. When the expressions for $\varphi^{(1)}$ and $\zeta^{(1)}$ are substituted in (5.35), $\zeta^{(2)}$ is eliminated from the equations and $\varphi^{(2)}$ is taken in the form $\varphi^{(2)}(y,z,\theta) = f^{(2)}(y,z)\sin 2\theta$, we obtain the equations for $f^{(2)}$:

$$f_z^{(2)} - 4\left(l_0^2 + k_0^2\right)^{1/2} f^{(2)} = R_2^{(2)}(y) \quad \text{on } z=0,\ y \geq 0,$$

$$f_{yy}^{(2)} + f_{zz}^{(2)} - 4k_0^2 f^{(2)} = 0, \quad \text{for } y \geq 0,\ -y \leq z \leq 0, \qquad (5.36)$$

$$f_y^{(2)} + f_z^{(2)} = 0, \quad \text{on } z = -y,\ y \geq 0.$$

where $R_2^{(2)}(y)$ is given by:

$$R_2^{(2)}(y) = \frac{g l_0^2}{\omega_0} + \alpha e^{-2y(l_0^2+k_0^2)^{1/2}} + \beta e^{-y(l_0^2+k_0^2)^{1/2}} \cos(l_0 y + \varepsilon) + $$

$$+ \gamma e^{-y(l_0^2+k_0^2)^{1/2}} \sin(l_0 y + \varepsilon), \qquad (5.37)$$

and the constants α, β, γ, depend on k_0 and l_0.

The solutions of (5.37) may be discussed in detail, and their behavior is similar to the one already described for normally incident waves. In this case there is a standing wave with $2l_0$ as wave number in the y direction. There is no contribution from the point spectrum, the edge wave, and this shows that an edge wave with the appropriate decay for the wave number $2k_0$ is not produced. There is a contribution which is estimated as $O(e^{-2k_0 y})$; as described in the Appendix. The most important term, for our discussion, is the one which comes from the constant forcing term; this is a contribution which does not vanish nor oscillate as $y \to \infty$. Hence the expression for $\varphi^{(2)}$ takes the form

$$\varphi^{(2)}(y,z,\theta) = \{(g\ell_0)^2/\omega_0(2gk-4\omega_0^2)\}\, e^{2k_0 z}\, \sin 2\theta +$$

$$+ O(\sin 2\ell_0 y, \cos 2\ell_0 y) + O(e^{-2k_0 y}). \qquad (5.38)$$

The surface elevation is given by

$$\zeta^{(2)}(y,\theta) = -\frac{\omega_0}{g}\varphi_\theta^{(2)} - \frac{\omega_0}{g}\varphi_{\theta z}^{(1)}\zeta^{(1)} - \frac{k_0^2}{2g}\varphi_\theta^{(1)2} - \varphi_y^{(1)2} - \varphi_z^{(1)2} \;; \qquad (5.39)$$

the nonlinear effect is the introduction of a mean level (independent of time) which is:

$$\zeta^{(2)}_{mean} = -\frac{g}{4\omega_0^2}\Big\{ \ell_0^2 + (\ell_0^2+k_0^2)\cos^2(\ell_0 y+\varepsilon) + 2\ell_0(\ell_0^2+k_0^2)^{1/2} e^{-y(\ell_0^2+k_0^2)^{1/2}} \times$$

$$\times \sin(\ell_0 y+\varepsilon 2) + 2k_0^2\cos\varepsilon\, e^{-y(\ell_0^2+k_0^2)^{1/2}}\cos(\ell_0 y+\varepsilon) + (\ell_0^2+2k_0^2\cos^2\varepsilon)\, e^{-2y(\ell_0^2+k_0^2)^{1/2}}\Big\}.$$

The exponential terms are due to the sloping bottom; they are not present when the depth is infinite.

The third order problem is

$$\omega_0\varphi_\theta^{(3)} + g\zeta^{(3)} = -\omega_0\varphi_{\theta z}^{(2)}\zeta^{(1)} - \omega_0\zeta^{(2)}\varphi_{\theta z}^{(1)} - \frac{\omega_0}{2}\varphi_{\theta zz}^{(1)}\zeta^{(1)2} -$$

$$-(k_0^2\varphi_\theta^{(1)}\varphi_\theta^{(2)} + \varphi_y^{(1)}\varphi_y^{(2)} + \varphi_z^{(1)}\varphi_z^{(2)}) - (k_0^2\varphi_\theta^{(1)}\varphi_{\theta z}^{(1)} + \varphi_y^{(1)}\varphi_{yz}^{(1)} + \varphi_z^{(1)}\varphi_{zz}^{(1)})\zeta^{(1)} -$$

$$-\omega_2\varphi_\theta^{(1)}, \quad \text{on } z=0,\; y \geq 0,$$

$$\omega_0\zeta_\theta^{(3)} - \varphi_z^{(3)} = \varphi_{zz}^{(1)}\zeta^{(2)} + \varphi_{zz}^{(2)}\zeta^{(1)} + \frac{1}{2}\varphi_{zzz}^{(1)}\zeta^{(1)2} - k_0^2(\varphi_\theta^{(1)}\zeta_\theta^{(1)} +$$

$$+ \mathcal{J}^{(1)}_{\theta}\varphi^{(2)}_{\theta}) - \varphi^{(1)}_{y}\mathcal{J}^{(2)}_{y} - \varphi^{(2)}_{y}\mathcal{J}^{(1)}_{y} - k_0^2\varphi^{(1)}_{\theta z}\mathcal{J}^{(1)}\mathcal{J}^{(1)}_{\theta} - \varphi^{(1)}_{yz}\mathcal{J}^{(1)}\mathcal{J}^{(1)}_{y} -$$

$$\omega_2\mathcal{J}^{(1)}_{\theta}, \quad \text{on } z=0,\ y \geq 0,$$

$$\varphi^{(3)}_{yy} + \varphi^{(3)}_{zz} + \varphi^{(3)}_{\theta\theta} = 0, \quad \text{for } -\infty < \theta < \infty,\ y \geq 0,\ -y \leq z \leq 0,$$

$$\varphi^{(3)}_{y} + \varphi^{(3)}_{z} = 0, \quad \text{on } z=-y,\ y \geq 0. \tag{5.40}$$

Substitution of (5.33), (5.34), (5.38) and (5.39) in (5.40) gives the equations which determine $\varphi^{(3)}$ and $\mathcal{J}^{(3)}$. However, the terms in the right hand side of (5.40) which determine the nonlinear correction to the dispersion relation are just the ones which resonate with the basic frequency, i.e., terms proportional to $\cos(l_0 y+\varepsilon)\cos\theta$, $\cos(l_0 y+\varepsilon)\sin\theta$. When the appropriate calculations are performed in (5.40), and the resonant terms $f^{(3)}(y,z)\cos\theta$ for $\varphi^{(3)}$ are considered, $f^{(3)}$ satisfies

$$f^{(3)}_{z} - (l_0^2+k_0^2)^{1/2} f^{(3)} = R^{(3)}_{1}(y), \quad \text{on } z=0,\ y \geq 0,$$

$$f^{(3)}_{yy} + f^{(3)}_{zz} - k_0^2 f^{(3)} = 0, \quad \text{for } y \geq 0,\ -y \leq z \leq 0, \tag{5.41}$$

$$f^{(3)}_{y} + f^{(3)}_{z} = 0 \quad \text{on } z=-y,\ y \geq 0.$$

where $R^{(3)}_{1}$ is given by

$$R^{(3)}_{1}(y) = 2\omega_2\left(e^{-y(l_0^2+k_0^2)^{1/2}}\cos\varepsilon + \cos(l_0 y+\varepsilon)\right) - \delta\cos(l_0 y+\varepsilon),$$

δ is given by

$$\delta = 2\omega_0\left\{-\frac{3}{8}l_0^2 + \frac{9}{32}k_0^2 + \frac{1}{4}\frac{l_0^2k_0^2}{l_0^2+k_0^2} - \frac{3}{2}k_0(l_0^2+k_0^2)^{1/2}\frac{A}{\omega_0} - (l_0^2-k_0^2)\frac{A}{\omega_0}\right\}$$

and $\displaystyle \frac{A}{\omega_0} = -\frac{\omega_0^4 - g^2k_0^2}{2\omega_0^2(2\omega_0^2-gk_0)}$.

Again, as previously discussed, we want a uniformly valid expansion as $y \to \infty$, therefore it is necessary to avoid terms of the form $y \sin(l_o y + \varepsilon)$ in the solution. These are produced by the terms proportional to $\cos(l_o y + \varepsilon)$; hence we must take $\omega_2 = \delta/2$. This determines the nonlinear correction to the dispersion relation.

The final form for the surface elevation is to lowest order

$$\zeta(y,\theta) = a \left\{ e^{-y(l_o^2 + k_o^2)^{1/2}} \cos\varepsilon + \cos(l_o y + \varepsilon) \right\} \sin(k_o x + \omega t) ,$$

where $\omega = \omega_o \left(1 + \frac{\delta}{2} a^2\right)$. The Stokes' expansion has two small parameters ak_o and al_o, since the problem is two-dimensional. The dispersion relation $\omega = \omega_o\left(1 + \frac{\delta}{2} a^2\right)$ reduces to the one for normal incidence when $k_o = 0$.

It was shown that no second order edge waves are generated by a monochromatic wave. A packet of obliquely incident waves will excite edge waves provided its spectrum is appropriate to produce a resonance on the edge wave modes. The situation just described was studied by Gallagher in 1971.

CHAPTER 6

THE LINEAR OPERATOR FORMULATION OF THE LINEAR WATER WAVES PROBLEM

In this chapter we consider some general features of the linear description of water waves. In the first section it is shown how the linear problem can be formulated in terms of a non-local but self-adjoint linear operator, as may be expected since the dispersion relation for water waves is not a polynomial. Using the standard argument it is shown how self-adjointness implies conservation of energy. In the second section two special cases are discussed in detail, and the eigenfunction expansions (1.8) and (1.16) are shown to provide the spectral representation of the linear operator associated with (1.7) and (1.14).

6.1 Formulation of the Linear Problem

Before we begin the discussion it is necessary to introduce some notation. Assume that the fluid is contained in a bounded or unbounded region $\Omega \subset \mathbb{R}^3$ with smooth boundaries and that Green's theorem may be applied to it. Denote the longshore and offshore coordinates by x and y respectively and let z be the vertical coordinate. Let the upper part of the boundary of Ω, which is denoted by Δ lie in the plane z = 0; $\partial(\Omega)$ denotes the remaining part of the boundary.

With the above conventions the linearized equations of motion for the velocity potential φ take the form

$$\frac{1}{g}\varphi_{tt} + \varphi_z = 0\ , \quad \text{on } z=0,\ (x,y)\in\Delta,$$

$$\varphi(x,y,0,0) = f(x,y)\ ; \quad \varphi_t(x,y,0,0) = g(x,y)\ , \quad \text{for } (x,y)\in\Delta,$$

$$\varphi_{yy} + \varphi_{zz} + \varphi_{xx} = 0 \quad \text{in } \Omega, \tag{6.1}$$

$\varphi_n = 0$ on $\partial(\Omega)$ if the region Ω is finite; or $\varphi \to 0$ and $\varphi_n \to 0$ sufficiently fast at infinity when Ω is not bounded.

To reinterpret (6.1) in an abstract fashion we consider the homogeneous problem obtained from (6.1) after separation of variables. This is

$$\varphi_z - \lambda\varphi = 0 \qquad \text{on } z=0,\ (x,y)\in\Delta,$$

$$\varphi_{yy} + \varphi_{zz} + \varphi_{xx} = 0 \quad \text{in } \Omega,$$

$$\varphi_n = 0 \ \text{ on } \partial(\Omega)\ ; \text{ or } \varphi \text{ and } \varphi_n \to 0 \text{ at infinity} \tag{6.2}$$

when Ω is not finite.

We now formulate (6.2) as the eigenvalue problem for an appropriate linear operator. To this end some more notation is needed.

Denote by $L^2(\Delta)$ the Hilbert space of square integrable functions of two variables defined on Δ. It is necessary to introduce the subspace $\mathcal{D} \subset L^2(\Delta)$ defined as

$$\mathcal{D} = \left\{ f\in L^2(\Delta) \;\middle|\; f(x,y) = \varphi(x,y,0) \text{ in } \Delta, \text{ where } \varphi \text{ satisfies (i), (ii), (iii)} \right\} ;$$

(i) $\nabla^2 \varphi = 0$ in Ω .

(ii) $\varphi_n = 0$ on $\partial(\Omega)$ or $\varphi, \varphi_n \to 0$ at infinity if Ω is unbounded.

(iii) $\int\limits_{\Delta} \left| \frac{\partial \varphi}{\partial z}(x,y,0) \right|^2 dx\,dy < \infty .$

The set $\mathcal{D}$ is not empty and in fact is dense in $L^2(\Delta)$. This is due to the fact that functions of two variables with compact support, say, generate solutions of Laplace's equations satisfying (i), (ii), (iii).

Now it is possible to define the operator that describes the linear problem. The definition of L is as follows:

$$L : \mathcal{D} \to L^2(\Delta)$$

$$(Lf)(x,y) = \frac{\partial}{\partial z} \varphi(x,y,z)\Big|_{z=0} \quad \text{for } (x,y) \in \Delta . \tag{6.3}$$

The operator L is well defined since the definition of $\mathcal{D}$ was made to accomplish this. Notice that the operator L is not local since the value of the transformed function Lf at a point depends on all the values that f takes in Δ .

We now use definition (6.3) to express in an abstract form the initial boundary value problem (6.1) in the following way. Solving (6.1) is equivalent to finding an $L^2(\Delta)$ valued function of a real variable, u(t), which satisfies

$$\frac{1}{g} u_{tt} = \bar{L} u$$

$$u(0) = f \; ; \; u_t(0) = g. \tag{6.4}$$

where the derivatives in (6.4) are taken in the strong sense and $\bar{L}$ is an extension of L to an appropriate domain. For a solution of

(6.4) to exist; the initial data f and g have to be restricted to an appropriate subspace of $L^2(\Delta)$.

To obtain the actual solution of (6.4) the spectral representation for $\bar{L}$ is needed; hence, the problem is to show that $\bar{L}$ admits a spectral representation. We will prove that L has a self-adjoint extension $\bar{L}$. This implies that a spectral representation for $\bar{L}$ exists. It will be also proved that the spectrum of $\bar{L}$ is positive, as one may expect since (6.4) describes oscillations. To prove that L admits a self-adjoint extension some more notation and another definition are needed.

Denote the usual scalar product in $L^2(\Delta)$ of two functions f and g by (f,g). It is also necessary to recall the definitions of symmetric, positive operators, and also the extension theorem of Friedrichs.

An operator A with domain $\mathfrak{D}$ (dense in $L^2(\Delta)$) is called positive if $(Af,f) \geqslant 0$, for all $f \in \mathfrak{D}$, $f \neq 0$.

An operator A with domain $\mathfrak{D}$ (dense in $L^2(\Delta)$) is called symmetric if $(Af,g) = (f,Ag)$, for all $f,g \in \mathfrak{D}$.

The extension theorem of Friedrichs (Friedrichs 1973) is: Let A be a densely defined positive symmetric operator. Then A admits a self-adjoint extension with positive spectrum.

Therefore to prove the existence of $\bar{L}$ it is only necessary to show that L is symmetric and positive. These two properties are direct consequences of the construction of L and $\mathfrak{D}$.

To prove the symmetry let $f(x,y) = F(x,y,0)$ and $g(x,y) = G(x,y,0)$ be elements of $\mathfrak{D}$. The use of Green's theorem on F and G gives

$$(Lf,g) - (f,Lg) = \int_\Delta gF_z - \int_\Delta fG_z = \int_\Omega F\nabla^2 G - G\nabla^2 F = 0, \tag{6.5}$$

since the other terms vanish by the construction of $\mathfrak{D}$.

The positivity of L also follows from Green's theorem since

$$(Lf,f) = \int_\Delta fF_z = \int_\Omega |\nabla F|^2 > 0 \quad \text{for } f \neq 0. \tag{6.6}$$

Formulas (6.5) and (6.6) show that a self-adjoint extension of L with positive spectrum exists; this is denoted by $\bar{L}$.

To conclude this section we prove that conservation of energy for solutions of (6.4) is a consequence of the self-adjointness of $\bar{L}$. Assume a solution u(t) of (6.4) with two continuous derivatives, and define the quadratic functional E(t) by

$$E(t) = \frac{\rho}{2g}(u_t, u_t) + \frac{\rho}{2}(u, \bar{L}u). \tag{6.7}$$

We now prove that E(t) is constant. In fact the differentiation of (6.7) gives

$$\dot{E}(t) = \frac{\rho}{g}(u_t, u_{tt}) + \frac{\rho}{2}(u_t, L\bar{u}) + \frac{\rho}{2}(u, Lu_t).$$

Since $\bar{L}$ is self-adjoint the last two terms can be combined and we obtain

$$\dot{E}(t) = \frac{\rho}{g}\left(u_t, \frac{1}{g}u_{tt} + \bar{L}u\right) = 0$$

since u satisfies (6.4). Hence the self-adjointness of $\bar{L}$ implies the conservation of E(t). To identify E(t) with the energy of the waves, recall that the surface elevation ζ is expressed in terms of the velocity potential as

$$\zeta(x,y,t) = -\frac{1}{g}\varphi_t(x,y,0) = -\frac{1}{g}u_t. \tag{6.8}$$

Therefore using (6.8) in (6.7) we obtain

$$E(t) = \rho g \int_\Delta \int_0^\zeta z\, dz\,dx\,dy + \frac{1}{2}\rho \int_\Omega |\nabla\varphi|^2 dx\,dy\,dz, \tag{6.9}$$

which is the total energy for a solution of (6.4).

6.2 The Spectral Representation of $\bar{L}$ in Two Particular Cases

In this section we show that the eigenfunction expansions (1.8) and (1.16) provide the spectral representation for $\bar{L}$ for two special domains Ω. In the first part we consider the case when no longshore variation is present, and in the second part the edge wave contribution to the expansion formula is examined.

(a) The case of no longshore variation. In this case the appropriate problem to consider is (1.7), and for simplicity we take $\beta = \pi/4$. To find the desired self-adjoint extension of L we need to prove the completeness of the eigenfunctions $S_\ell(y,0)$.

In this case the functions $S_\ell(y,z)$ take the form:

$$S_\ell(y,z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\left\{e^{i\pi/4}e^{-\ell y - i\ell z} + e^{-i\pi/4}e^{\ell z - i\ell y} + \text{c.c.}\right\}; \tag{6.10}$$

and we now prove the completeness of the functions $S_\ell(y,0)$ for a special class of functions $f(y)$. More precisely let f be a C^∞ function which vanishes at infinity faster than any power with an integrable transform $\hat{f}(\ell)$ defined by: $\hat{f}(\ell) = \int_0^\infty S_\ell(y,0)\, f(y)\, dy.$

Then the inversion formula

$$f(y) = \int_0^\infty \hat{f}(\ell)\, S_\ell(y,0)\, d\ell \tag{6.11}$$

holds.

To prove (6.11) consider its explicit form. That is

$$\int_0^\infty S_\ell(y,0)\,\hat{f}(\ell)\,d\ell = \lim_{M\to\infty} \frac{1}{2\pi}\Big\{ e^{i\frac{\pi}{2}}\int_0^M d\ell \int_0^\infty e^{-\ell(y+y')} f(y')\,dy' +$$

$$+\int_0^M d\ell \int_0^\infty \Big\{ e^{-\ell(y'+iy)} + e^{-\ell(y+y')} + e^{i\pi/2} e^{-\ell(y'-iy)} + e^{i\pi/2} e^{-\ell(y+iy')}\Big\} f(y')\,dy' +$$

$$+\int_0^M d\ell \int_0^\infty e^{-\ell(y+y')} f(y')\,dy' + e^{-i\pi/2}\int_0^M d\ell \int_0^\infty e^{i\ell(y+y')} f(y')\,dy' +$$

$$+\int_0^M d\ell \int_0^\infty e^{i\ell(y-y')} f(y')\,dy' + \text{c. c.}\Big\} \tag{6.12}$$

which holds because $\hat{f}(\ell)$ is integrable.

We now show that for $y > 0$ each term in (6.12) has a limit as $M \to \infty$ and we calculate their values. For the first term we have

$$\lim_{M\to\infty} i\int_0^M d\ell \int_0^\infty e^{-\ell(y+y')} f(y')\,dy' =$$

$$\lim_{M\to\infty} i\int_0^\infty \Big\{\frac{1}{y+y'} - \frac{e^{-M(y+y')}}{y+y'}\Big\} f(y')\,dy' ; \tag{6.13}$$

since the interchange of the order of integration is legitimate because f vanishes at infinity faster than any power and the domain of integration is bounded in the ℓ variable. To prove the existence of the limit in the right hand side of (6.13) notice that the integrand is dominated by the function $2|f(y')|(y+y')^{-1}$ which is integrable since $y > 0$. Hence, by the dominated convergence theorem the limit can be interchanged with the integration, and the limit of (6.13) is

$$i\int_0^\infty \frac{f(y')}{y+y'}\,dy'$$

This is purely imaginary and therefore does not contribute to (6.12) since the complex conjugate is also included. Exactly the same argument is used with the second term and its complex conjugate, and it is found that they do not contribute to (6.12). Now consider the third and fourth terms with the complex conjugates. Observe that

$$\int_0^\infty e^{-\ell y'} f(y')\,dy' = O(|\ell|^{-1}) \quad \text{as } |\ell| \to \infty$$

for ℓ in the first quadrant of the complex plane. Therefore the path of integration in the third term can be deformed into the imaginary axis since the contribution from the large semicircle tends to zero. Combining the third and fourth terms we obtain

$$\lim_{M\to\infty} -\frac{i}{2\pi}\left\{\int_{-M}^{M} e^{i\ell(y+y')}\,d\ell \int_0^\infty f(y')\,dy'\right\} = 0$$

by Fourier's theorem. Finally when the last term is combined with its complex conjugate we obtain

$$\int_0^\infty S_\ell(y,0)\hat{f}(\ell)\,d\ell = \lim_{M\to\infty} \frac{1}{2\pi}\int_{-M}^{M} e^{i\ell(y-y')}\,d\ell \int_0^\infty f(y')\,dy' = f(y), \text{ for } y>0. \quad (6.14)$$

Since f is continuous from the right as $y \to 0$ and $S_\ell(y,0)$ is bounded it follows, using again the dominated convergence theorem, that (6.14) is valid for $y = 0$. This completes the proof of the desired result.

To construct the desired spectral representation we need an extension of (6.11) to the whole space $L^2[0,\infty)$. More precisely, consider the operator V whose domain $\mathcal{B}$ is a subspace of differentiable functions in $L^1 \cap L^2$ defined by:

$$V : \mathcal{B} \to L^2[0,\infty)$$

$$(Vf)(\ell) = \int_0^\infty S_\ell(y,0)\, f(y)\, dy. \tag{6.15}$$

We will show that V can be extended to an isomorphism of L^2. First of all we will show that V admits a closed extension $\bar{V}$. (That is, there exists an operator $\bar{V}$ such that: whenever a sequence $\{x_n\} \subset \text{Domain of } \bar{V}$ converges to x, and $\bar{V}x_n \to y$, then $x \in \text{Domain of } \bar{V}$ and $\bar{V}x = y$. Also $\bar{V}x = Vx$ for all $x \in \mathcal{B}$.) To prove the existence of $\bar{V}$ observe that the kernel $S_\ell(y,0)$ is proportional to $\cos(\ell y + \pi/4) + 2^{-1/2} e^{-\ell y}$, then as a consequence of Plancherel's theorem the first term generates a bounded, and therefore closed operator. With this observation it is now only necessary to prove that the operator with kernel $e^{-\ell y}$ admits a closed extension. To show that the operator $V_2 f$ defined by

$$(V_2 f)(\ell) = \int_0^\infty e^{-\ell y} f(y)\, dy, \quad \text{for } f \in \mathcal{B} \tag{6.16}$$

admits a closed extension, it is sufficient to prove (Yosida 1968) that for every sequence $\{f_n\} \subset \mathcal{B}$ such that $f_n \to 0$ and $V_2 f_n \to g$, we have g = 0. To verify this statement in (6.16) consider the sequence $\{f_n\} \subset \mathcal{B}$ such that $f_n \to 0$ in $L^2[0,\infty)$

and $V_2 f_n \to g$ in L^2. In this case there exists a subsequence $\{f_{n_k}\}$ such that

$$\int_0^\infty e^{-\ell y} f_{n_k}(y)\,dy \to g(\ell)$$

almost everywhere. Also from (6.16) we have

$$|g(\ell)|^2 \leq \int_0^\infty |f_{n_k}(t)|^2 dt \int_0^\infty e^{-2\ell t} dt . \tag{6.17}$$

Equation (6.17) implies $g(\ell) = 0$ a.e. and therefore g = 0 in L^2. This shows that V_2 and therefore that V admits a closed extension $\bar{V}$.

The extension of V to an isomorphism U of L^2 is obtained using the orthogonality and the completeness relations given in (6.11). To find the desired extension it is convenient to express (6.11) using $\bar{V}$ and its adjoint $(\bar{V})^*$.

Since $\bar{V}$ is densely defined $(\bar{V})^*$ exists; also $\mathcal{B} \subset \mathrm{Domain}\,(\bar{V})^*$ and the operator $(\bar{V})^*$ is given by

$$((\bar{V})^* g)(y) = \int_0^\infty S_\ell(y,0)\, g(\ell)\, d\ell$$

for $g \in \mathcal{B}$, since

$$(\bar{V} f, g) = \int_0^\infty g(\ell)\, d\ell \int_0^\infty S_\ell(y,0)\, f(y)\,dy = \int_0^\infty f(y)\,dy \int_0^\infty S_\ell(y,0)\, g(\ell)\, d\ell =$$

$$= (f, (\bar{V})^* g)$$

for all f and g in $\mathcal{B}$. Using the operators $\bar{V}$ and $(\bar{V})^*$ (6.11) implies

$$(\bar{V})^* \bar{V} \ell_m = \ell_m \tag{6.18}$$

$$\mathfrak{D} = \left\{ f \in L^1 \cap L^2 \;\middle|\; \hat{f}(\ell) = O(|\ell|^{-4}) \text{ as } |\ell| \to \infty \right\} .$$

When $f \in \mathfrak{D}$, then $f(y) = \varphi(y, 0)$, where φ is a solution of Laplace's equation which satisfies conditions (i), (ii) and (iii) of the previous section. Therefore, L is defined as:

$$(Lf)(y) = \frac{\partial}{\partial z} \varphi(y,z)\Big|_{z=0} , \quad \text{for } y \geq 0. \tag{6.22}$$

To find a suitable self-adjoint extension of L for this case, consider the subspace

$$\hat{\mathfrak{D}} = \left\{ f \in L^2 \;\middle|\; \int_0^\infty \ell^2 |f(\ell)|^2 \, d\ell < \infty \right\},$$

and define the operator

$$\hat{L} : \hat{\mathfrak{D}} \longrightarrow L^2[0,\infty)$$

$$(\hat{L} f)(\ell) = \ell f(\ell) \quad \text{for } \ell \geq 0.$$

The operator $\hat{L}$ is self-adjoint, has no point spectrum, and its continuous spectrum is the positive real axis. Using the operator $\hat{L}$ a self-adjoint extension of L is constructed as follows:

Let

$$\begin{aligned} &\bar{L} : U^*(\hat{\mathfrak{D}}) \to L^2[0,\infty) \\ &\bar{L} = U^* \hat{L} U . \end{aligned} \tag{6.23}$$

The operator $\bar{L}$ is self-adjoint since U is an isomorphism. We now need to prove that $\bar{L}$ is an extension of L. The fact $\mathfrak{D} \subset U^*(\hat{\mathfrak{D}})$ follows, since by construction $U(\mathfrak{D}) \subset \hat{\mathfrak{D}}$ and U is an isomorphism. To prove that $\bar{L} f = L f$ for $f \in \mathfrak{D}$ observe that for

for all the Laguerre functions ℓ_m. Since $\bar{V}$ is closed we have $(\bar{V})^{**} = \bar{V}$, and from (6.18) we obtain:

$$\left((\bar{V})^* \bar{V} \ell_m, \ell_m\right) = \left(\ell_m, \ell_m\right) = \left(\bar{V} \ell_m, \bar{V} \ell_m\right) \tag{6.19}$$

Consider now the operator V' which is defined on the Laguerre functions by $V' \ell_m = \bar{V} \ell_m$. Then (6.19) and the fact that the Laguerre functions are dense in $L^2[0, \infty)$ implies that V' can be extended to an isometry U defined for all $x \varepsilon L^2[0, \infty)$. Also, since $\bar{V}$ is closed we have $Ux = \bar{V}x$ for $x \varepsilon \mathcal{D}$omain of $\bar{V}$. To prove that the isometry U is an isomorphism we need to show that the range of U is the space L^2. To prove that U is onto, observe that from (6.11) the orthogonality relation

$$f(\ell) = \int_0^\infty S_\ell(y, 0)\, dy \int_0^\infty S_\ell(y, 0)\, f(\ell)\, d\ell \tag{6.20}$$

holds for all the Laguerre functions; because the kernel $S_\ell(y, 0)$ is symmetric in y and ℓ. In terms of $\bar{V}$ and $(\bar{V})^*$ (6.20) takes the form

$$\bar{V}(\bar{V})^* \ell_m = \ell_m = U(\bar{V})^* \ell_m . \tag{6.21}$$

Equation (6.21) shows that the range of U is dense in L^2. To prove that the range of U is all L^2, let $y \varepsilon L^2$, then there exists a sequence $Ux_m \rightarrow y$; by (6.19) the sequence x_m has a limit x, and since U is bounded $Ux = y$.

Now it is possible to construct the operator L associated with (1.7). To define L let

$f \in \mathfrak{D}$ the operator $\bar{L}$ is represented as

$$(Lf)(y) = \int_0^\infty S_\ell(y,0)\, \ell \, d\ell \int_0^\infty S_\ell(t,0)\, f(t)\, dt \tag{6.24}$$

and that the function

$$\varphi(y,z) = \int_0^\infty S_\ell(y,z)\, d\ell \int_0^\infty S_\ell(t,0)\, f(t)\, dt$$

is a function that satisfies (i), (ii) and (iii). Therefore calculation of Lf using (6.22), and interchange of differentiation and integration (which is shown to be permissible in (A.7) gives

$$(Lf)(y) = \frac{\partial}{\partial z}\varphi(y,z)\Big|_{z=0} = \int_0^\infty S_\ell(y,0)\, \ell \, d\ell \int_0^\infty S_\ell(t,0)\, f(t)\, dt = (\bar{L} f)(y),$$

and this proves that $\bar{L}$ is the desired self-adjoint extension of L.

To conclude this section,we summarize the results obtained. The "incoming waves with perfect reflection $S_\ell(y,0)$ " were shown to be complete in the sense that they are the kernel for an isomorphism $\mathcal{U}$ of L^2. Also, the construction (6.23) shows that the functions $S_\ell(y,0)$ provide the spectral representation of an appropriate self-adjoint extension of L.

(b) Case of longshore variation. In this subsection it is shown how the result of Whitham mentioned in (1.16) is used to provide the spectral representation for the operator $\bar{L}$ when it has one point in the spectrum. The arguments involved are essentially the same ones discussed in the previous subsection.

We now consider in detail the case $\beta = \pi/4$. First of all the functions $S_\ell(y,0)$ and the edge wave solution are given by:

$$E(y) = 2^{-1/4} k^{1/2} e^{-\frac{k}{\sqrt{2}} y}$$

$$\mathcal{S}_\ell(y,0) = \sqrt{\tfrac{2}{\pi}} \left(\ell^2 + \lambda^2\right)^{1/2} \left\{ \tfrac{1}{2}(\ell + i\lambda)\, e^{i\ell y} + \text{c.c.} + \ell e^{-\ell y} \right\},$$

where $\lambda = \left(\ell^2 + k^2\right)^{1/2}$ and $\ell \geqslant 0$. The variable k appears after the separation of the x dependence as shown in (1.14).

The functions $\mathcal{S}_\ell(y,0)$ and $E(y)$ satisfy the following relations due to Whitham (1976):

The orthogonality relations

$$\int_0^\infty \mathcal{S}_\ell(y,0)\, E(y)\, dy = 0, \quad \text{for } \ell \geqslant 0, \tag{6.24}$$

$$\int_0^\infty \mathcal{S}_\ell(y,0)\, dy \int_0^\infty \mathcal{S}_\ell(y,0)\, f(\ell)\, d\ell = f(\ell) \tag{6.25}$$

for all $f \in C^\infty$ which vanish at the origin, and vanish at infinity faster than any power.

The completeness of the functions $\mathcal{S}_\ell(y,0)$ and $E(y)$ for the Laguerre functions is expressed as:

$$\int_0^\infty \ell_m(t) E(t)\, dt\; E(y) + \int_0^\infty \mathcal{S}_\ell(y,0)\, d\ell \int_0^\infty \mathcal{S}_\ell(t,0)\, \ell_m(t)\, dt = \ell_m(y) \tag{6.26}$$

In order to find the operators $\mathcal{V}$ and $\mathcal{U}$ appropriate for this case it is convenient to introduce a new Hilbert space $\mathcal{H}^{(1)}$ defined as

$$\mathcal{H}^{(1)} = \left\{ (a, f) \;\middle|\; a \in \mathbb{R},\ f \in L^2[0,\infty) \right\}$$

with scalar product given by:

$$\left\langle (a,f), (b,g) \right\rangle = ab + \int_0^\infty fg\, d\ell$$

The space $\mathcal{H}^{(1)}$ may be identified with the space $L^2_\mu(-\infty,\infty)$, where the measure μ is the Lebesgue measure on $[0,\infty)$ and has its mass in $(-\infty,0)$ concentrated at one point. This space is nothing but the space of the "representers" (Friedrichs 1973) which will provide a representation of $\bar{L}$ as a multiplication.

With the definition of $\mathcal{H}^{(1)}$ it is possible to define the operator $\mathcal{V}$ appropriate for this case. First of all, consider the operator $\mathcal{W}$, whose domain $\mathcal{B}$ is a subspace of differentiable functions in $L^1 \cap L^2$, defined by

$$(\mathcal{W} f)(\ell) = \int_0^\infty S_\ell(y,0)\, f(y)\, dy ,$$

which is densely defined and whose adjoint $\mathcal{W}^*$ is given by

$$(\mathcal{W}^* g)(y) = \int_0^\infty S_\ell(y,0)\, g(\ell)\, d\ell$$

for all $g \in \mathcal{B}$. Define now the operator $\mathcal{V}$ by:

$$\mathcal{V} : \mathcal{B} \to \mathcal{H}^{(1)}$$

$$\mathcal{V} f = \left(\int_0^\infty f E ,\ \mathcal{W} f \right).$$

The same arguments used in the previous section show that $\mathcal{V}$ admits a densely defined closed extension $\bar{\mathcal{V}}$ whose adjoint $(\bar{\mathcal{V}})^*$ is given by

$$(\bar{\mathcal{V}})^*(a,h) = aE + \mathcal{W}^* h$$

for $a \in \mathbb{R}$, $h \in \mathcal{B}$; and satisfies

$$\langle \bar{\mathcal{V}} f, g \rangle = (f, (\bar{\mathcal{V}})^* g)$$

The completeness relation can be written in terms of $\bar{V}$ and $(\bar{V})^*$ as

$$(\bar{V})^* \bar{V} \ell_m = \ell_m,$$

and since $\bar{V}$ is closed we have:

$$\langle \bar{V} \ell_m, \bar{V} \ell_m \rangle = (\ell_m, \ell_m). \tag{6.27}$$

Equation (6.27 is essentially the same as (6.19). The same arguments used in (6.19) (since they do not depend on the fact that V maps L^2 into a subspace of L^2) show that the operator $\bar{V}$ can be extended to an isometry of L^2 into $\mathcal{H}^{(1)}$. To prove that U is an isomorphism it is sufficient to show, as in the previous section, that the range of U is dense in $\mathcal{H}^{(1)}$. This fact follows, as before, from the orthogonality relations (6.24), (6.25). In fact, when (6.24) and (6.25) are expressed in terms of $\bar{V}$ and $(\bar{V})^*$ we have:

$$U(\bar{V})^*(a,f) = \bar{V}(\bar{V})^*(a,f) = \bar{V}\left(aE + \int_0^\infty s_\ell(\cdot,0) f(\ell)\,d\ell\right) = (a,f) \tag{6.28}$$

for all real numbers a, and C^∞ functions f which vanish at the origin and at infinity faster than any power. The vectors (a,f) are dense in $\mathcal{H}^{(1)}$; therefore, it follows from (6.28) that the range of U is dense in $\mathcal{H}^{(1)}$. The argument used in the previous section shows that U is onto $\mathcal{H}^{(1)}$ and therefore U is the desired isomorphism between $L^2[0,\infty)$ and $\mathcal{H}^{(1)}$.

Now we construct the appropriate self-adjoint extension of L when one edge wave is present. To study the operator L it is convenient to introduce the subspace $\mathcal{D} \subset L^2[0,\infty)$ defined as:

$\mathcal{D} = \{ f \in L^2[0,\infty) \mid f$ is a finite linear combination of Laguerre functions $\}$.

If $f \in \mathcal{D}$, it may be verified that there exists a unique φ such that $f(y) = \varphi(y, 0)$, where the function φ satisfies:

(i) $\varphi_{yy} + \varphi_{zz} - k^2 \varphi = 0$, for $y \geq 0$, $-y \leq z \leq 0$,

(ii) $\varphi_y + \varphi_z = 0$ on $z = -y$, $y \geq 0$,

(iii) $\varphi = O(r^{-1})$, $\varphi_r = O(r^{-2})$, as $r = (x^2 + y^2)^{1/2} \to \infty$,

(iv) $\int_0^\infty \left| \frac{\partial}{\partial z} \varphi(y,z) \Big|_{z=0} \right|^2 dy < \infty$.

Using the subspace $\mathcal{D}$ define the operator L by

$$L : \mathcal{D} \to L^2[0, \infty)$$

$$(Lf)(y) = \frac{\partial}{\partial z} \varphi(y, z)\Big|_{z=0} \quad \text{for } y \geq 0.$$

Now the construction of a self-adjoint extension $\bar{L}$ of L is analogous to the one of the preceding section. In more detail consider the subspace

$$\hat{\mathcal{D}} = \left\{ (a, f) \in \mathcal{H}^{(1)} \,\Big|\, \int_0^\infty (\ell^2 + k^2) |f(\ell)|^2 \, d\ell < \infty. \right\}$$

and define the operator $\hat{L}$ by:

$$\hat{L} : \hat{\mathcal{D}} \to \mathcal{H}^{(1)}$$

$$\hat{L}(a, f) = \left(\frac{k}{\sqrt{2}} a, \; ((\cdot)^2 + k^2)^{1/2} f(\cdot) \right).$$

The operator $\hat{L}$ is self-adjoint, and its spectrum consists of the point $\frac{k}{\sqrt{2}}$, and the half line $\ell \geq k$. We now define, using $\hat{L}$ the desired self-adjoint extension $\bar{L}$ of L as follows:

$$\bar{L} : U^{-1}(\hat{\mathfrak{D}}) \to L^2[0,\infty) \quad ; \quad \bar{L} = U^{-1}\hat{L}U \tag{6.29}$$

The operator $\bar{L}$ is self-adjoint since U is an isomorphism. To prove that $\bar{L}$ is an extension of L we need to prove that $\mathfrak{D} \subset U^{-1}(\hat{\mathfrak{D}})$, and that $\bar{L}f = Lf$ for all $f \varepsilon \mathfrak{D}$. That $\mathfrak{D} \subset U^{-1}(\hat{\mathfrak{D}})$ follows from $U(\mathfrak{D}) \subset \hat{\mathfrak{D}}$ and the fact that U is an isomorphism. The statement $\bar{L}f = Lf$ for $f \varepsilon \mathfrak{D}$ follows, as in the previous section, by direct calculation and the fact that for $f \varepsilon \mathfrak{D}$ (6.29) gives:

$$(\bar{L}f)(y) = \frac{k}{\sqrt{2}} \int_0^\infty f(t)E(t)\,dt\ E(y) + \int_0^\infty (\ell^2+k^2)^{1/2} S_\ell(y,0)\,d\ell \int_0^\infty S_\ell(t,0)f(t)\,dt \ .$$

APPENDIX

In this appendix we prove that the formal solutions for the linear problems (1.7) and (1.14) given by (1.8) and (1.18) are indeed solutions of the desired problems. This justifies their use in obtaining approximate solutions for the nonlinear problems discussed in Chapters 2, 4 and 5.

The asymptotic expansions for the solutions (1.8) and (1.18) are found for large distances away from the shore. It is also shown how the dominant term in the expansions can be found (in a nonrigorous way) directly from the equations.

In the first section the case of no edge waves present is discussed, and in the second section the contribution from the edge wave is examined.

A.1 The Solution of Equations (1.7)

In this section we consider the problem

$$\begin{aligned} &\varphi_z - \ell_0 \varphi = R(y) && \text{on } z=0,\ y \geq 0, \\ &\varphi_{yy} + \varphi_{zz} = 0, && \text{for } y \geq 0,\ -y \leq z \leq 0, \\ &\varphi_y + \varphi_z = 0 && \text{on } z=-y,\ y \geq 0. \end{aligned} \tag{A.1}$$

where $\ell_0 > 0$ and $R(y)$ is a function which satisfies:

$$\hat{R}(\ell) = \int_0^\infty S_\ell(y,0)\, R(y)\, dy = o(|\ell|^{-3}) \quad \text{as } |\ell| \to \infty,$$

for all ℓ's in the first quadrant of the complex plane. This is indeed the case of interest in the second chapter since $R(y) = \alpha\, e^{-\nu y}$, and therefore $\hat{R}(\ell) = \alpha \pi^{1/2} \nu^2 (\ell^2 + \nu^2)^{-1} (\ell + \nu)^{-1}$. The functions that appear as forcing terms in equations (5.21) and (5.22) are functions whose transforms are $O(|\ell|^{-3})$ and $O(|\ell|^{-2})$; and trigonometric functions. We will show that in all cases (1.8) provides the desired solution to the problem.

The formal solution for (A.1) is:

$$\varphi(y,z) = \text{P.V.}\sqrt{\tfrac{2}{\pi}}\int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0}\left\{e^{\ell z}\cos\left(\ell y+\tfrac{\pi}{4}\right)+e^{-\ell y}\cos\left(\ell z-\tfrac{\pi}{4}\right)\right\}d\ell\,, \tag{A.3}$$

but for our present purposes (A.3) is more conveniently written as:

$$\varphi(y,z) = \text{P.V.}\sqrt{\tfrac{2}{\pi}}\,\text{Re}\left\{\int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0}\,e^{\ell z}\,e^{i(\ell y+\frac{\pi}{4})}\,d\ell+\int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0}\,e^{-\ell y}\,e^{i(\ell z-\frac{\pi}{4})}\,d\ell\right\}. \tag{A.4}$$

To prove that (A.4) provides a solution of (A.1) it is necessary to show that (A.4) is well defined, has the desired derivatives, and satisfies (A.1). To prove that (A.4) is well defined and sufficiently differentiable it is sufficient to show that the functions

$$\text{P.V.}\sqrt{\tfrac{2}{\pi}}\int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0}\,e^{\ell z}\,e^{i(\ell y+\pi/4)}\,d\ell \tag{A.5}$$

and

$$\text{P.V.}\sqrt{\tfrac{2}{\pi}}\int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0}\,e^{-\ell y}\,e^{i(\ell z-\frac{\pi}{4})}\,d\ell \tag{A.6}$$

exist for y and z in the region of interest, can be differentiated twice with respect to y and z, and that differentiation can be interchanged with the principal value.

We now examine in detail (A.5) and (A.6). Integration of the analytic function

$$\hat{R}(\ell)(\ell-\ell_0)^{-1}\, e^{\ell z}\, e^{i(\ell y+\frac{\pi}{4})}$$

on the contour formed by the real axis, a small semicircle C_1 in the upper half plane centered at $\ell=\ell_0$, a large semicircle at infinity and the line L_1 given by $\ell=\zeta e^{i\pi/4}$, $\zeta \geq 0$, shows (when the small semicircle is collapsed to ℓ_0 and the large at ∞) that the desired principal value exists and may be expressed by

$$\text{P.V.}\int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0} e^{\ell z} e^{i(\ell y+\frac{\pi}{4})} d\ell = \int_{L_1} \frac{\hat{R}(\ell)}{(\ell-\ell_0)} e^{\ell z} e^{i(\ell y+\frac{\pi}{4})} d\ell + \pi i \hat{R}(\ell_0) e^{\ell_0 z} e^{i(\ell_0 y+\pi/4)} . \quad \text{(A. 7)}$$

The contribution from the semicircle at infinity is zero because $\hat{R}(\ell)=o(|\ell|^{-3})$ or $O(|\ell|^{-2})$ as $|\ell|\to\infty$.

The same argument, but now used in the lower half plane, with the line L_2 parametrized by $\ell=\zeta e^{-i\frac{\pi}{4}}$, $\zeta\geq 0$, gives for (A. 6)

$$\text{P.V.}\int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0} e^{-\ell y} e^{i(\ell z-\frac{\pi}{4})} d\ell = \int_{L_2} \frac{\hat{R}(\ell)\, e^{-\ell y}}{\ell-\ell_0} e^{i(\ell z-\pi/4)} d\ell + \pi i \hat{R}(\ell_0) e^{-\ell_0 y} e^{i(\ell_0 z-\frac{\pi}{4})} . \quad \text{(A. 8)}$$

Also,formulas (A. 6) and (A. 7) are valid for $0\leq y<\infty$, $-y\leq z\leq 0$; this shows that φ is defined in the region of interest.

To prove that (A. 5) is differentiable, consider (A. 7). In the first term of (A. 7) differentiation can be interchanged with integration for all y and z not zero, and when $\hat{R}(\ell)=o(|\ell|^{-3})$ two derivatives can be calculated in this way when y and z are both zero. When $\hat{R}(\ell)=O(|\ell|^{-2})$ the second derivatives of φ may not exist at the origin. This is the case for $\varphi^{(3)}$ in (5. 19) since the interaction $\varphi^{(2)}_{zz}\mathcal{J}^{(1)}$ produces a term whose transform is $O(|\ell|^{-2})$; however, the solution is still acceptable because it has finite velocity at the shore. This shows that φ has the desired number of derivatives.

We now prove that the derivatives can be interchanged with the principal value; for example, we now show that

$$\frac{\partial}{\partial z} \text{P.V.} \int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0} e^{\ell z} e^{i(\ell y+\frac{\pi}{4})} d\ell = \text{P.V.} \frac{\partial}{\partial z} \int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0} e^{\ell z} e^{i(\ell y+\frac{\pi}{4})} d\ell, \tag{A.9}$$

the argument being the same in the other cases. To verify (A. 9), compute the left term using (A. 7); this gives:

$$\frac{\partial}{\partial z} \text{P.V.} \int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0} e^{\ell z} e^{i(\ell y+\frac{\pi}{4})} d\ell = \int_0^\infty \frac{\hat{R}(\ell)\,\ell}{\ell-\ell_0} e^{\ell z} e^{i(\ell y+\frac{\pi}{4})} d\ell +$$

$$+ \pi i \ell_0 \hat{R}(\ell_0) e^{\ell_0 z} e^{i(\ell_0 y+\frac{\pi}{4})}. \tag{A.10}$$

Now compute the right hand side of (A. 9); this gives

$$\text{P.V.} \frac{\partial}{\partial z} \int_0^\infty \frac{\hat{R}(\ell)}{\ell-\ell_0} e^{\ell z} e^{i(\ell y+\frac{\pi}{4})} d\ell = \text{P.V.} \int_0^\infty \frac{\hat{R}(\ell)\,\ell}{\ell-\ell_0} e^{\ell z} e^{i(\ell y+\pi/4)} d\ell. \tag{A.11}$$

The second term in (A. 11) is now calculated using (A. 7) and the right hand term of (A. 10) is obtained. This proves (A. 9). The existence of second derivatives is proved in the same way. Therefore, in (A. 4), first and second derivatives can be interchanged with the principal value.

Finally, to prove that (A. 3) satisfies (A. 1), observe that the function

$$e^{\ell z} \cos(\ell y+\tfrac{\pi}{4}) + e^{-\ell y} \cos(\ell z-\pi/4)$$

satisfies Laplace's equation and the bottom boundary condition; hence, interchanging the derivatives with the P. V. we see that (A. 3) satisfies Laplace's equation and the bottom boundary condition. To show that the surface condition is also satisfied observe that:

$$\frac{\partial}{\partial z}\varphi(y,z)\Big|_{z=0} = \text{P.V.}\int_0^\infty \ell\, S_\ell'(y,0)\,\frac{\hat{R}(\ell)}{\ell-\ell_0}\,d\ell\,, \quad \text{for } y \geq 0.$$

hence

$$\frac{\partial}{\partial z}\varphi(y,z) - \ell_0\,\varphi(y,z)\Big|_{z=0} = \text{P.V.}\int_0^\infty \frac{(\ell-\ell_0)}{(\ell-\ell_0)}\,\hat{R}(\ell)\,S_\ell'(y,0)\,d\ell = R(y), \text{ for } y \geq 0. \tag{A. 12}$$

This shows that (A. 3) provides a solution of (A. 1).

The solution of (A. 1) is not unique since the homogeneous problem has a solution; however, the appropriate solutions for each time dependent problem have been already discussed in the previous chapters.

We now consider the asymptotic behavior of $\varphi(y,0)$ as $y \to \infty$. To find the desired expansion we express (A. 3) in terms of (A. 7) and (A. 8); this gives

$$\begin{aligned}\varphi(y,0) = \text{Re}\sqrt{\frac{2}{\pi}}\Big\{ &\int_0^\infty \frac{\hat{R}(\zeta e^{i\pi/4})}{\zeta e^{i\pi/4}-\ell_0}\, e^{i\pi/2}\, e^{-(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}})\zeta y}\, d\zeta \,+ \\ &+ \int_0^\infty \frac{\hat{R}(\zeta e^{-i\pi/4})}{\zeta e^{i\pi/4}-\ell_0}\, e^{-i\frac{\pi}{2}}\, e^{-(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}})\zeta y}\, d\zeta - \hat{R}(\ell_0)\sin(\ell_0 y+\pi/4) \,+ \\ &+ \hat{R}(\ell_0)\, e^{-\ell_0 y}\sin\frac{\pi}{4}\Big\}.\end{aligned} \tag{A. 13}$$

To find the asymptotic expansion of the first two terms in (A. 13) we use Watson's lemma and we find that their sum is $O(y^{-2})$ since both integrands have the same value with opposite signs at $\zeta = 0$. Therefore the dominant contribution in (A. 13) is given by the third term which represents a standing wave.

We now consider the solution of (A. 1) when $R(y) = \sin m_0 y$ with $m_0 \neq l_0$, since the case $m_0 = l_0$ was discussed in the first chapter. We will show that (A. 1) has a bounded solution as $y \to \infty$.

To prove the desired result consider the problem:

$$\begin{aligned} &\varphi_z - l_0 \varphi = e^{-\mu y} \sin m_0 y \qquad \text{on } z = 0,\ y \geq 0, \\ &\varphi_{yy} + \varphi_{zz} = 0 \qquad \text{for } y \geq 0,\ -y \leq z \leq 0, \\ &\varphi_y + \varphi_z = 0 \qquad \text{on } z = -y,\ y \geq 0. \end{aligned} \tag{A. 14}$$

where μ is a positive parameter. Let $\varphi_\mu(y,z)$ be a solution of (A. 14); we will show that $\lim_{\mu \to 0} \varphi_\mu(y,z)$ exists, and that provides a solution of (A. 14) when $\mu = 0$. To prove the desired result it is sufficient to verify that the derivatives can be interchanged with the limiting process.

We now show that the function

$$\varphi(y,z) = \lim_{\mu \to 0} \frac{2}{\pi} \text{ P.V.} \int_0^\infty \frac{1}{\sqrt{2}} \frac{\operatorname{Img}(\mu - i m_0)^2 (l - l_0)^{-1}}{(l + \mu - i m_0)(l^2 + (\mu - i m_0)^2)} \Big\{ e^{lz} \times$$

$$\times \cos(ly + \pi/4) + e^{-ly} \cos(lz - \tfrac{\pi}{4}) \Big\} \, dl \tag{A. 15}$$

is well defined, and that the derivatives up to the second order exist for φ , and that differentiation can be interchanged with the limiting processes. The arguments involved are the same used in the discussion of (A. 7), and now we just outline them. Consider for example the term

$$\frac{2}{\pi} \lim_{\mu \to 0} \text{P.V.} \int_0^\infty \operatorname{Img} \frac{1}{\sqrt{2}} \left\{ \frac{(\mu - i m_o)^2}{(\ell + \mu - i m_o)(\ell^2 + (\mu - i m_o)^2)(\ell - \ell_o)} \right\} e^{\ell z} e^{i(\ell y + \pi/4)} \, d\ell \, . \tag{A. 16}$$

The function

$$\operatorname{Img} \frac{1}{\sqrt{2}} \frac{(\mu - i m_o)}{(\ell + \mu - i m_o)(\ell^2 + (\mu - i m_o)^2)} = \frac{1}{2\sqrt{2}} \frac{(\ell + m_o) + \mu}{(\ell + m_o)^2 + \mu^2} -$$

$$- \frac{1}{2\sqrt{2}} \frac{(\ell - m_o) + \mu}{(\ell - m_o)^2 + \mu^2} + \frac{1}{2\sqrt{2}} \frac{(\ell + m_o) + \mu}{(\ell + m_o)^2 + \mu^2} ,$$

which is the transform of $e^{-\mu y} \sin m_o y$, is an analytic function of ℓ , with poles at $\ell = m_o \pm i\mu$ in the first quadrant and $O(|\ell|^{-3})$ uniformly in μ as $|\ell| \to \infty$. This implies as in (A. 7) that the P. V. exists, and that the path of integration can be deformed to the line L_1 , however now besides the contribution of the pole $\ell = \ell_o$, there is a contribution from the pole at $\ell = m_o + i\mu$ which is

$$- i \frac{e^{i m_o y - \mu y} \, e^{(m_o + i\mu) z}}{m_o - \ell_o + i\mu} \, . \tag{A. 17}$$

The term (A. 17) has as limit as $\mu \to \infty$ a differentiable function of y and z. Since the contribution from the path L_1 is of the form

$$\int_{L_1} f(\ell, \mu) \, e^{\ell z} e^{i(\ell y + \pi/4)} \, d\ell \tag{A. 18}$$

it follows, because $f(\ell, \mu) = O(|\ell|^{-4})$ uniformly in μ , that (A. 18) can be differentiated under the integral sign, and that the derivatives up to the second order can be interchanged with the limit. As in

(A. 12) it follows that (A. 15) satisfies Laplace's equation and the bottom boundary condition. Finally, to verify that (A. 15) satisfies the boundary condition at the surface, we observe that

$$\lim_{\mu\to\infty}\left\{\frac{\partial}{\partial z}\varphi_\mu(y,z)-l_0\varphi_\mu(y,z)\right\}\Big|_{z=0}=\frac{\partial}{\partial z}\lim_{\mu\to 0}\varphi_\mu(y,0)-l_0\lim_{\mu\to 0}\varphi_\mu(y,0)=$$

$$=\lim_{\mu\to 0}\quad e^{-\mu y}\sin m_0 y\ =\ \sin m_0 y\qquad \text{for } y\geq 0.$$

To conclude this section, we find the asymptotic expansion of (A. 15) as $y\to\infty$ on the line z = 0. The dominant contribution is now given by two standing waves, one obtained from the residue at $l=l_0$, and the other which is (A. 17) when $\mu=0$. Observe that the value of (A. 17), when $\mu=0$, is the solution of (A. 14) with $\mu=0$ obtained by separation of variables when the bottom boundary condition is neglected in the deep water region.

A. 2 The Solution of Equations (1. 14).

In this section we examine briefly the solution for the problem

$$\begin{aligned}
&\varphi_z-\lambda_0\varphi=e^{-\beta y} && \text{on } z=0,\ y\geq 0,\\
&\varphi_{yy}+\varphi_{zz}-k^2\varphi=0 && \text{for } y\geq 0,\ -y\leq z\leq 0, \qquad\qquad (A.19)\\
&\varphi_y+\varphi_z=0 && \text{on } z=-y,\ y\geq 0.
\end{aligned}$$

where $\lambda_0>k>0$, i.e. λ_0 is a point in the continuous spectrum. The case $\lambda_0=k/\sqrt{2}$ was discussed by Whitham (1976) in his work on nonlinear effects in traveling edge waves.

The proposed solution for (A. 19) is:

$$\varphi(y,z) = \frac{2k\, e^{-\frac{k}{\sqrt{2}}y + \frac{k}{\sqrt{2}}z}}{(k/\sqrt{2} - \lambda_0)(\sqrt{2}\,\beta + k)} + \text{P.V.}\,\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(2\beta^2 - k^2)\ell^2\, e^{-\lambda y - i\ell z}}{(\ell^2+\beta^2)(\lambda+\beta)(\lambda^2+\ell^2)(\lambda-\lambda_0)}\,d\ell \; +$$

$$+ \text{P.V.}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(2\beta^2-k^2)\ell^2\quad e^{i\ell y + \lambda z}\ d\ell}{(\ell^2+\beta^2)(\lambda+\beta)(\ell+i\frac{k}{\sqrt{2}})(\ell-i\frac{k}{\sqrt{2}})(\lambda-\lambda_0)}\;; \tag{A. 20}$$

where $\lambda = (\ell^2 + k^2)^{1/2}$ and the branch is chosen so that $\operatorname{Re}\lambda \geq 0$ for $\operatorname{Im}\ell \geq 0$.

To show that (A. 20) is well defined and provides a solution for (A. 19) we proceed as in (A. 7). To deform the contours in the appropriate way we note the singularities of the integrands. There are simple poles at $\lambda = \lambda_0$, $\ell = \pm i\beta$, $\ell = \pm ik/\sqrt{2}$ and branch points at $\ell = \pm ik$. The integrands are at most $O(|\ell|^{-4})$ as $|\ell| \to \infty$.

Consider now the first integral in (A. 20). Deform the path of integration to a line L_3 parallel to the real axis without crossing singularities. The contribution from the pole $\lambda = \lambda_0$ is proportional to $e^{-\lambda_0 y}$. The contribution from the integral along L_3 is $O(e^{-ky})$ since $|e^{-\lambda y}| \leq e^{-ky}$ and the factor in front of the exponential is $O(|\ell|^{-4})$ as $|\ell| \to \infty$.

The second integral is evaluated deforming the path of integration to a contour around the branch cut along the imaginary axis. The contribution from the poles at $\lambda = \lambda_0$ gives a standing wave. In this case the residues at the poles $\ell = ik/\sqrt{2}$ and $\ell = i\beta$ must be added. The residue at $\ell = ik/\sqrt{2}$ cancels the first term in (A. 20). The contribution of the pole $\ell = i\beta$ only appears for $\beta < k$ and is given by

$$\left\{ (k^2-\beta^2)^{1/2} - \lambda_0 \right\} e^{(k^2-\beta^2)^{1/2} z} \, e^{-\beta y} . \tag{A.21}$$

The integral along the branch cut is $O(e^{-ky})$. In the case $\beta > k$, all the terms, except the standing wave, are $O(e^{-ky})$.

Finally, we observe that the term (A. 21) is the solution of (A. 19) when the bottom boundary condition is neglected for large y.

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