THE STABILITY OF TRAVELING WAVE SOLUTIONS OF PARABOLIC EQUATIONS

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ABSTRACT

A method for determining by inspection the stability or instability of any solution $u(t,x) = \phi(x-ct)$ of any smooth equation of the form

$$u_t = f(u_{xx}, u_x, u) \text{ where } \frac{\partial}{\partial a} f(a,b,c) > 0 \text{ for all arguments a, b, c,}$$

is developed. The connection between the mean wavespeed of solutions $u(t,x)$ and their initial conditions $u(0,x)$ is also explored. The mean wavespeed results and some of the stability results are then extended to include equations which contain integrals and also to include some special systems of equations. The results are applied to several physical examples.

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Chapter I

INTRODUCTION

We will investigate the class of equations

$$u_t = f(u_{xx}, u_x, u) \quad , \quad f_1 > 0 \quad , \tag{1.1}$$

and also some extensions of this class of equations. Specifically, we shall determine the stability of all traveling wave (and steady state) solutions of (1.1). We shall also study the dependence of the mean wavespeed of solutions of (1.1) on the initial conditions. Nearly all of these stability and wavespeed results will also be extended to certain generalizations of (1.1).

The reason we study (1.1) is its frequent occurrence in many different fields, such as biology, chemical reactions, and genetics. The importance of the traveling wave (and steady state) solutions comes from the fact that almost all solutions must evolve into traveling wave (and steady state) solutions as $t \to \infty$. We study the stability of these solutions since only stable solutions can occur naturally.

In order to demonstrate our stability results, we adopt the following typical procedure for finding traveling wave (and steady state) solutions of (1.1). This procedure involves introducing a moving coordinate system (to reduce traveling wave solutions to steady states) and introducing a phase-plane. Specifically, we switch to the moving coordinate system

$$t' = t \quad , \quad x' = x - ct \quad , \tag{1.2}$$

where the speed c is arbitrary but fixed. In terms of these new variables, equation (1.1) is

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad , \tag{1.3}$$

where the prime superscripts on the t's and x's have been dropped for convenience. All traveling wave (and steady state) solutions of (1.1) can now be treated as steady state solutions of equation (1.3) with the appropriate values of the parameter c. These steady state solutions $u(t, x) \equiv \phi(x)$ of (1.3) are then found by solving the first order system of equations

$$\begin{aligned} \phi_x &= v \\ f(v_x, v, \phi) + cv &= 0 \quad . \end{aligned} \tag{1.4}$$

The usual method of finding solutions of (1.4) is by studying the phase plane of system (1.4), where the vector (ϕ_x, v_x) is considered as a function of (ϕ, v).

Besides aiding in the search for solutions of (1.4), the phase plane provides a convenient way to classify the steady state solutions of equation (1.3). For example, non-constant monotonic steady state solutions can be classified as $N \to N$, $N \to S$, $S \to N$, or $S \to S$ depending on whether $(\phi(-\infty), v(-\infty))$ and $(\phi(+\infty), v(+\infty))$ are both nodes, a node and a saddle point, a saddle point and a node, or both saddle points, respectively. By using a maximum principle, we will find that the stability of any steady state solution of equation (1.3) depends only on its phase plane classification. Specifically, we will find that very nearly all non-monotonic steady states are unstable. We will find that a constant steady state $\phi(x) \equiv \phi_0$, $v \equiv 0$ is stable if $(\phi_0, 0)$ is a saddle point and unstable if it is a node, a spiral point, or a center. We will find that all non-constant monotonic steady states are stable to classes of perturbations which are determined by whether $\phi(x)$ is a $N \to N$, a $N \to S$, a $S \to N$, or a $S \to S$ type steady state. In particular, the $N \to N$ type steady states have the most limited stability classes (i.e. class of perturbations under which the steady state is stable), and the $S \to S$ type steady states have the largest stability classes. We will also find that our stability results are sharp.

These results are useful since now, for parabolic equations, one no longer has to solve an often difficult eigenvalue problem for each solution of each equation to determine stability. Note that in a degenerate sense the stability of all traveling wave (and steady state) solutions of elliptic equations is also known in advance, since all these solutions are unstable.

The presentation of these results begins in Chapter II, where an overview of the main results and their proofs is given. Only results pertaining to equation (1.1) are covered there. In this presentation of these results and proofs, complicating details are avoided. This hopefully clarifies the reasoning behind the results and thus shows why the results are true. Note however, that there are results not covered in Chapter II.

In Chapter III we do the necessary preliminary mathematical work. This work falls into three categories: modifying the original equation to prevent infinities from occurring, developing the needed mathematical tools (such as the maximum principle), and assembling the hypotheses under which we will work. In Chapter III, all this preliminary work is done for very general parabolic type systems of equations which include multiple independent spatial variables, multiple dependent variables, and even integrals. This is so every generalization of (1.1) that we will consider is included as a special case of the equations treated in Chapter III.

Before continuing, we briefly mention the idea behind the technical device of modifying the given equation, although this is discussed at length in Chapter III. Solutions u of the nonlinear equation $u_t = f(u_{xx}, u_x, u)$ may be able to develope infinities in u, u_x, or u_{xx} as time progresses. These infinities pose very serious mathematical problems. In order to avoid these, we select an arbitrarily large, but fixed, positive constant M and work with a modified equation, $u_t = f_M(u_{xx}, u_x, u)$. This new equation is identical to the original equation when $|u| \leq M$, $|u_x| \leq M$, and $|u_{xx}| \leq M$. However, there is a finite $N(M) > M$ such that if any of $|u_{xx}| > N$, $|u_x| > N$, or $|u| > N$ occurs, then the modified equation reduces to a heat equation. Moreover, the transition between the original equation and the heat equation is very smooth and the maximum principle holds for the modified equation as well as the original. Solutions to the modified equation, however, do not develope infinities as time progresses, which is the feature of the equations that we need. <u>All subsequent results are for the modified equation and hold for all M sufficiently large</u>. The physical reasons for - and consequences of - modifying the original nonlinear equation are discussed in detail in Chapter III. Briefly however, for any M as large as we please all our theorems will tell us how solutions $u(t,x)$ of the modified equations behave as a function of the initial conditions $u(0,x)$. Thus, for any solution $u(t,x)$ of the original equation which has $|u| < M$, $|u_x| < M$, and $|u_{xx}| < M$ for all $t \geq 0$, our theorems give the correct results. Also, for any solution $u(t,x)$ of the original equation, the results in our theorems are correct for all t until $|u| = M$, $|u_x| = M$, or $|u_{xx}| = M$ occurs.

In Chapter IV we obtain the stability results for monotonic traveling wave (and steady state) solutions of the basic equation

$$u_t = f(u_{xx}, u_x, u) \quad , \quad f_1 > 0 \quad . \tag{1.1}$$

We then derive the instability results for non-monotonic traveling waves (and steady states). Direct extensions of these results, such as to traveling plane waves in higher spatial dimensions and to boundary value problems on a finite spatial interval, are also considered.

In Chapter V we consider the connection between the mean wave-speed of a solution $u(t,x)$ of equation (1.1) and the initial condition $u(0,x)$. We first find when the existence of a monotone wave at speed $c = c_0$ implies the existence of nearby monotone waves with wavespeeds near c_0, and also when its existence implies the existence of other traveling waves at the same speed c_0. With these results, the techniques used in Chapter IV are used to find the dependence of the mean wavespeed of solutions $u(t,x)$ of (1.1) on the initial condition $u(0,x)$. Simple extensions of these results are also considered.

In Chapter VI we extend the results of Chapter IV and V to the more general class of equations

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} G(s,y,u(t-s, x-y))dyds) , f_1 > 0, f_4 \cdot G_3 \geq 0, \tag{1.5}$$

where T and Y are any positive constants. We consider this generalization because equation (1.5) occurs quite often in some fields, such as population dynamics. The stability for monotone waves and the mean wavespeed/initial condition results all hold for equation (1.5) as well as equation (1.1). In general, though, the proof of the instability of non-monotonic waves fails in two separate places.

In Chapter VII we extend the results of Chapters IV and V to the special class of parabolic systems

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}), \ f_1^{(i)} > 0, \ f_{3_j}^{(i)} \equiv \frac{\partial f^{(i)}(a,b,\tilde{c})}{\partial c^{(j)}} \geq 0$$

$$\text{all } j \neq i, \ i = 1,2,...,n \ . \tag{1.6}$$

These special parabolic systems occur in chemical reaction theory. Similar to Chapter VI, the stability and mean wavespeed results are established for system (1.6). Again, the proof of instability breaks down in the same two places as occurred in Chapter VI.

In Chapter VIII we use the results of the previous chapters on specific physical examples of equations (1.1), (1.5) and (1.6). We draw on the fields of genetics, biology, and chemical reactions for examples of (1.1). Population dynamics provides an example of equation (1.5). A reaction diffusion system is used as an example of system

(1.6) which occurs in chemical reaction theory.

In Chapter IX we briefly discuss our results in general terms. Specifically, we point out some short-comings of the results, make some reasonable conjectures, and discuss possible areas for further research.

Chapter II

OVERVIEW

In this chapter we present an overview of the main results (and their proofs) pertaining to the class of equations

$$u_t = f(u_{xx}, u_x, u) \quad , \quad f_1 > 0 \quad . \tag{2.1}$$

We will avoid most of the complicating details found in the more complete presentation contained in Chapters IV and V. This avoidance hopefully helps clarify the reasoning behind the results, showing why the results are true. Specifically, in section (2.1) we discuss the maximum principle. In section (2.2) the stability results for monotone waves are obtained. Section (2.3) deals with the instability of nonmonotonic waves. The last section, (2.4), is used to present some of the mean wavespeed/initial condition results.

2.1 The maximum principle. The maximum principle for the equation

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{2.2}$$

will be used in obtaining almost all our results. In essence this principle is the observation that if the inequality

$$u_t - f(u_{xx}, u_x, u) - cu_x \geq v_t - f(v_{xx}, v_x, v) - cv_x \tag{2.3}$$

holds for all $t > 0$ and if $u(0,x) \geq v(0,x)$ for all x, then

$u(t,x) \geq v(t,x)$ for all $t > 0$ as well. This principle motivates the definition of $\bar{u}(t,x)$ as an upper function of equation (2.2) and of $\underline{u}(t,x)$ as a lower function of (2.2) whenever

$$\bar{u}_t - f(\bar{u}_{xx}, \bar{u}_x, \bar{u}) - c\bar{u}_x \geq 0 \quad \text{for all} \quad t > 0, \quad \text{all} \quad x \qquad (2.4)$$

$$\underline{u}_t - f(\underline{u}_{xx}, \underline{u}_x, \underline{u}) - c\underline{u}_x \leq 0 \quad \text{for all} \quad t > 0, \quad \text{all} \quad x \qquad (2.5)$$

hold. This maximum principle and appropriate upper and lower functions can be employed to obtain stability proofs. For example, suppose $u(t,x) \equiv \phi(x)$ is a steady state solution of (2.2) and suppose that $\bar{u}(t,x)$ is an upper function of (2.2) such that $\bar{u}(0,x) \geq \phi(x)$ for all x and $\bar{u}(t,x)$ remains near to $\phi(x)$ for all x and all $t > 0$. Clearly when this occurs, the maximum principle implies a type of stability for $\phi(x)$. For in this case, the maximum principle shows that all solutions $u(t,x)$ of equation (2.2) whose initial conditions satisfy

$$\bar{u}(0, x) \geq u(0, x) \geq \phi(x) \quad \text{for all} \quad x \qquad (2.6)$$

also must satisfy

$$\bar{u}(t, x) \geq u(t, x) \geq \phi(x) \quad \text{for all} \quad t>0, \text{ all } x \,. \qquad (2.7)$$

Thus, employment of the maximum principle in this and similar manners reduces the question of stability to that of finding appropriate upper and lower functions.

We will now be specific. The maximum principle for equation (2.2) is the following:

<u>Maximum principle</u>: Suppose $f(\alpha,\beta,\gamma)$ is continuously differentiable in α,β, and γ and that $f_1(\alpha,\beta,\gamma) > 0$ for all arguments α,β, and γ. If $u(t,x)$ and $v(t,x)$ are any functions with u, u_x, u_{xx} and v, v_x, v_{xx} all bounded for all x and all $0 < t \leq T$, if

$$u_t - f(u_{xx}, u_x, u) - cu_x \geq v_t - f(v_{xx}, v_x, v) - cv_x \qquad (2.8)$$

holds for all (x,t) in $\mathbb{R} \times (0,T]$, and if $u(0,x) \geq v(0,x)$ for all x, then

$$u(t,x) \geq v(t,x) \quad \text{for all } x, \text{ all } t \text{ in } [0,T] . \qquad (2.9)$$

This maximum principle is a special case of the maximum principle presented in Chapter III. The following proof of this maximum principle is based on the material in Chapter III of reference [1].

<u>Proof</u>: We prove this principle by defining $h \equiv u-v$ and showing that h is positive. We start by defining a function of θ, $H([v],[h],\theta)$, by

$$H([v],[h],\theta) \equiv f(v_{xx} + \theta h_{xx}, v_x + \theta h_x, v + \theta h) - c(v_x + \theta h_x).$$

The derivative of H is

$$\frac{\partial H}{\partial \theta} = f_1 h_{xx} + (f_2 + c)h_x + f_3 h \quad ,$$

where the arguments of f_1, f_2, and f_3 are $v_{xx} + \theta h_{xx}$, $v_x + \theta h_x$, $v + \theta h$. Note that our assumptions imply that $\frac{\partial H}{\partial \theta}$ is bounded for all $\theta \in [0,1]$ and all $(x,t) \in \mathbb{R} \times [0,T]$. Thus, at any fixed x and t in $\mathbb{R} \times (0,T]$,

$$h_t = u_t - v_t \geq f(u_{xx}, u_x, u) + cu_x - f(v_{xx}, v_x, v) - cv_x \quad ,$$
$$= H([v], [h], 1) - H([v], [h], 0) \quad ,$$
$$= \frac{\partial H}{\partial \theta}([v], [h], \theta) \quad \text{for some } \theta \text{ in } [0,1] \quad .$$

This last step follows from the mean value theorem. Hence, at each (x,t) in $\mathbb{R} \times (0,T]$ there is a $\theta(t,x) \in [0,1]$ such that

$$h_t \geq f_1 h_{xx} + (f_2 + c)h_x + f_3 h \; , \tag{2.10}$$

where again the arguments of f_1, f_2, and f_3 are $v_{xx} + \theta(t,x)h_{xx}$, $v_x + \theta(t,x)h_x$, $v + \theta(t,x)h$. By our assumptions, f_1, $f_2 + c$, and f_3 are bounded for all (x,t) in $\mathbb{R} \times [0,T]$, and moreover $f_1 > 0$.

We will prove the maximum principle by showing that h must be positive for (x,t) in $\mathbb{R} \times [0,T]$ whenever the inequality

$$h_t \geq \alpha\, h_{xx} + \beta\, h_x + \gamma h \tag{2.11}$$

holds for all (x,t) in $\mathbb{R} \times (0,T]$. Here, α, β, and γ are arbitrary functions of (t,x) which are bounded over $(x,t) \in \mathbb{R} \times [0,T]$ and of which α is always positive. The maximum principle is then immediately established as (2.10) is a special case of (2.11).

We continue by defining

$$w \equiv he^{-\eta t} \operatorname{sech} x \quad , \tag{2.12}$$

where $\eta > 0$ will be selected later. From (2.12) we find

$$
\begin{aligned}
h &= we^{\eta t} \cosh x \quad , \\
h_t &= w_t e^{\eta t} \cosh x + \eta w e^{\eta t} \cosh x \quad , \\
h_x &= w_x e^{\eta t} \cosh x + w e^{\eta t} \sinh x \quad , \\
h_{xx} &= w_{xx}\, e^{\eta t} \cosh x + 2\, w_x e^{\eta t} \sinh x + w e^{\eta t} \cosh x \quad .
\end{aligned} \tag{2.13}
$$

Substituting these expressions into the differential inequality (2.11) yields

$$w_t \geq \alpha\, w_{xx} + (\beta + 2\alpha \tanh x) w_x + (\gamma + \alpha + \beta \tanh x - \eta) w. \tag{2.14}$$

We now select η so large that $\gamma + \alpha + \beta\tanh x - \eta \leq -1$ for all $(x,t) \in \mathbb{R} \times [0,T]$. Thus (2.14) becomes

$$w_t \geq \alpha\, w_{xx} + \hat{\beta}\, w_x + \hat{\gamma}\, w \;,\; \alpha > 0 \;,\; \hat{\gamma} \leq -1 \;, \text{ for all } 0 < t \leq T \;. \tag{2.15}$$

We now come to the heart of the proof. Let $\varepsilon > 0$ be any positive constant. Suppose that $w(t,x) \leq -\varepsilon$ for some value of $(x,t) \in \mathbb{R} \times [0,T]$. From the boundedness of h and expression (2.12), we see that there is an $X > 0$ such that $|w| < \varepsilon/2$ for all $|x| \geq X$ and all $0 \leq t \leq T$. Thus there must be some point $\tilde{x},\tilde{t}$ in $|x| < X$, $0 \leq t \leq T$ where $w(\tilde{t},\tilde{x})$ is at a minimum and $w(\tilde{t},\tilde{x}) \leq -\varepsilon$. This minimum does not occur at $\tilde{t} \equiv 0$ since h (and thus w) is non-negative there. If it occurs at $\tilde{t} = T$ then $w_t(\tilde{t},\tilde{x}) \leq 0$, and if it occurs at $0 < \tilde{t} < T$ then $w_t(\tilde{t},\tilde{x}) = 0$. In either case $w_x(\tilde{t},\tilde{x}) = 0$ and $w_{xx}(\tilde{t},\tilde{x}) \geq 0$. Thus, (2.15) implies that

$$w_t(\tilde{t},\tilde{x}) \geq \hat{\gamma}\, w(\tilde{t},\tilde{x}) \geq \varepsilon > 0 \quad . \tag{2.16}$$

This contradicts $w_t(\tilde{t},\tilde{x}) \leq 0$. Hence $w(t,x) > -\varepsilon$ for all (x,t) in $\mathbb{R} \times [0,T]$. Since $\varepsilon > 0$ is arbitrary, $w(t,x) \geq 0$, and therefore

$h(t,x) \geq 0$ for all $(x,t) \in \mathbb{R} \times [0,T]$. Q.E.D.

In the next section we will use this maximum principle to establish sharp stability classes for the perturbations of constant and monotone traveling waves (and steady states).

2.2 Stability of monotone waves. In this section we find the stability of constant and non-constant monotonic traveling wave and steady state solutions of the equation

$$u_t = f(u_{xx}, u_x, u) \qquad f_1 > 0 \quad . \tag{2.1}$$

As before, we reduce any traveling wave to a steady state by switching to the moving coordinate system

$$t' = t \ , \ x' = x - ct \quad .$$

In terms of these new coordinates, equation (2.1) is

$$u_t = f(u_{xx}, u_x, u) + cu_x \ , \ f_1 > 0 \quad , \tag{2.2}$$

where the primes on the t's and x's have been conveniently dropped. Thus, all traveling wave solutions of equation (2.1) are now steady state solutions of equation (2.2) at the appropriate values of the parameter c. We will actually study the stability of the steady state solutions of equations (2.2).

As the first step we will define some types of stability which will enable us to state sharp stability results. Next, we will examine the stability of constant steady states. Then, the basic stability results for monotonic steady state solutions $u(t,x) \equiv \phi(x)$ of (2.2) will be given.

Finally, finding better upper and lower functions will enable us to improve this basic result whenever $\phi(-\infty)$ or $\phi(+\infty)$ (or both) are saddle points. The sharpness of these results will be established in section (2.4) as a by-product of the mean wavespeed/initial condition discussion.

We begin by making the needed stability definitions. Namely, we define C^w-stability and $\not{C}^w$-stability for any continuous function $w(x)$ with $w(x) \geq 1$ for all x. Given such a w, we define any steady state solution $u(t,x) \equiv \phi(x)$ of equation (2.2) to be C^w-stable if an only if for any given $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that every solution $u(t,x)$ of equation (2.2) satisfies

$$\left|\left[u(t,x) - \phi(x)\right]w(x)\right| \leq \varepsilon \quad \text{for all} \quad x \quad \text{and all} \quad t > 0 \;, \tag{2.17}$$

whenever the initial conditions $u(0,x)$ are smooth and satisfy

$$\left|\left[u(0,x) - \phi(x)\right] \cdot w(x)\right| \leq \delta(\varepsilon) \quad \text{for all} \quad x. \tag{2.18}$$

Similarly, $\phi(x)$ is defined to be $\not{C}^w$-stable if and only if for every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that every solution $u(t,x)$ of equation (2.2) satisfies

$$\left|u(t,x) - \phi(x)\right| \leq \varepsilon \quad \text{for all} \quad x \quad \text{and all} \quad t > 0 \;, \tag{2.19}$$

whenever the initial conditions $u(0,x)$ are smooth and satisfy (2.18).

In interpreting these definitions, we note that whenever $w(x)$ is bounded, then C^w-stability ($\not{C}^w$-stability) is equivalent to $C^{\bar{w}}$-stability ($\not{C}^{\bar{w}}$-stability) with $\bar{w}(x) \equiv 1$. We therefore turn our attention to the cases where $w(x) \to +\infty$ as $x \to -\infty$ and/or $x \to +\infty$. We see that, roughly speaking, $\phi(x)$ is C^w-stable ($\not{C}^w$-stable) if it is stable to perturbations applied at $t = 0$ which decay like $\frac{1}{w(x)}$ as $x \to -\infty$ and $x \to +\infty$. $\not{C}^w$-stability implies that the perturbations remain small and bounded as time increases. The stronger C^w-stability implies that the perturbations remain small and decay like $\frac{1}{w(x)}$ as $x \to -\infty$ and $x \to +\infty$ at all times $t > 0$. The definition of C^w-stability is very similar to the stability definitions used in reference [2].

The above stability definitions are adequate for our needs. We will therefore now examine the stability of an arbitrary constant steady state solution, $u(t,x) \equiv \phi_0$, of equation (2.2). The stability of this state will depend crucially on whether the point $\phi = \phi_0$, $v = 0$ is a node, a spiral point, or a saddle point of the first order system

$$\begin{aligned} \phi_x &= v \\ f(v_x, v, \phi) + cv &= 0 \quad . \end{aligned} \tag{2.20}$$

Note that $\phi = \phi_0$, $v = 0$ is a node, a spiral point, or a center when $f_3(0, 0, \phi_0) > 0$, and is a saddle point when $f_3(0,0,\phi_0) < 0$. (The case $f_3(0,0,\phi_0) = 0$ represents two or more singular points merged together in the phase plane). We observe that these signs of $f_3(0,0,\phi_0)$ imply that $u(t,x) \equiv \phi_0$ is stable to spatially <u>independent</u> perturbations if $\phi_0, 0$ is a saddle point, and is unstable to spatially <u>independent</u> perturbations when $\phi_0, 0$ is a node, a spiral point or a center. We expand this observation into the following:

<u>Theorem 2.1</u>: Suppose $u(t,x) \equiv \phi_0$ is a constant steady state solution of equation (2.2). Then

(1) $\phi(x)$ is C^w-stable with $w(x) \equiv 1$ if $\phi = \phi_0$, $v = 0$ is a saddle point of system (2.20), and

(2) $\phi(x)$ is $\not{C}^w$-unstable with $w(x) \equiv 1 + e^{-\kappa x} + e^{+\kappa x}$ for

$\kappa > 0$ sufficiently small if $\phi = \phi_0$, $v = 0$ is a node, a spiral point, or a center of system (2.20).

Proof: Part 1. Since $f_3(0,0,\phi_0) < 0$, we let $f_3(0,0,\phi_0) = -2\mu$. There exists an $h_0 > 0$ such that $f_3(0,0,\phi_0+h) < -\mu < 0$ for all $|h| < h_0$. Consider $u(h,t) \equiv \phi_0 + e^{-\mu t} h$. For all $0 < h < h_0$, we have

$$
\begin{aligned}
u_t(h,t) &- f(u_{xx}(h,t), u_x(h,t), u(h,t)) - cu_x(h,t) \\
&= -\mu e^{-\mu t} h - f(0,0,\phi_0 + he^{-\mu t}) \\
&\geq 0 \quad .
\end{aligned}
$$

Thus, $u(h,t)$ is an upper function of (2.2) for $0 < h < h_0$. Similarly, $u(h,t)$ is a lower function of (2.2) when $-h_0 < h < 0$. Let $\varepsilon > 0$ be given, and without loss we suppose that $\varepsilon < h_0$. Suppose $u(t,x)$ is any solution of (2.2) whose initial condition $u(0,x)$ satisfies $|u(0,x) - \phi(x)| < \varepsilon$ for all x. Then

$$u(\varepsilon,t) \geq u(t,x) \geq u(-\varepsilon,t) \text{ for all } x$$

is true at $t = 0$, and so the maximum principle implies that it is true for all $t \geq 0$ as well. I.e.,

$$\phi_0 + \varepsilon e^{-\mu t} \geq u(t,x) \geq \phi_0 - \varepsilon e^{-\mu t} \text{ for all } x, \text{ all } t \geq 0 \ . \quad (2.21)$$

Stability is thus established.

Part 2. Consider $u(h,t,x) \equiv \phi_0 + he^{\mu t} \operatorname{sech} \kappa x$, where $\mu \equiv \frac{1}{2} f_3(0,0,\phi_0) > 0$ and where $\kappa > 0$ will be defined later. We calculate

$$
\begin{aligned}
u_t(h,t,x) &- f(u_{xx}(h,t,x),\ u_x(h,t,x),\ u(h,t,x)) - cu_x(h,t,x) \\
&= \tfrac{1}{2} f_3\, he^{\mu t} \operatorname{sech} \kappa x \\
&- f(-\kappa^2 \cdot he^{\mu t} \operatorname{sech} \kappa x(1-2 \operatorname{sech}^2 \kappa x),\ -\kappa he^{\mu t} \operatorname{sech} \kappa x \tanh \kappa x, \\
&\qquad \phi_0 + he^{\mu t} \operatorname{sech} \kappa x) \\
&+ c\kappa he^{\mu t} \operatorname{sech} \kappa x \tanh \kappa x \\
&= he^{\mu t} \cdot (\operatorname{sech} \kappa x)\left[-\tfrac{1}{2} \cdot f_3 + (f_2+c)\kappa \tanh \kappa x + f_1 \kappa^2 (1-2 \operatorname{sech}^2 \kappa x)\right] \\
&+ O((he^{\mu t} \operatorname{sech} \kappa x)^2) \quad , \qquad (2.22)
\end{aligned}
$$

where f_1, f_2, and f_3 are evaluated at the arguments $0, 0, \phi_0$. Let κ be a fixed positive constant so small that the quantity in brackets in (2.22) is always less than $-\tfrac{1}{4}f_3$. Since the first term of (2.22) is then negative for $h > 0$, there exists an $h_0 > 0$ for which

$$u_t(h,t,x) - f(u_{xx}(h,t,x),\ u_x(h,t,x),\ u(h,t,x)) - cu_x(h,t,x) \le 0$$

for all t such that $0 < he^{\mu t} \le h_0$. Define $\tilde{u}(h,t,x)$ as the solution of (2.2) with the initial value $\tilde{u}(h,0,x) = \phi_0 + h \operatorname{sech} \kappa x$. Since $u(h,t,x)$ is a lower function of (2.2), the maximum principle implies

$$\tilde{u}(h,t,x) \ge \phi_0 + he^{\mu t} \operatorname{sech} \kappa x \text{ for all } t \text{ such that } 0 < he^{\mu t} \le h_0 \ .$$

This holds for all $0 < h < h_0$, and so part (2) is established.

Note that relation (2.21) shows that small perturbations about a saddle point die exponentially in time, and note that perturbations about a node, a spiral point, or a center can grow exponentially in time.

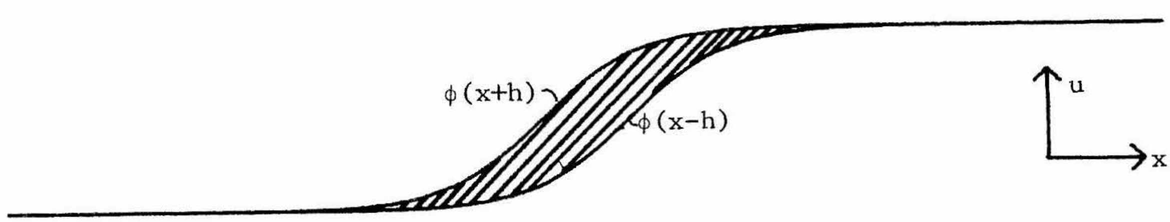
The stability of constant steady state solutions of (2.2) has been found, and so we turn our attention to the stability of non-constant monotonic steady state solutions. To see why monotonic waves are stable, let $u(t,x) \equiv \phi(x)$ be a monotonic steady state solution of the autonomous equation (2.2). In particular, let $\phi(x)$ be increasing. Since $\phi(x+h)$ also solves (2.2) for any h, it is both an upper and lower function of (2.2). Thus, for any $h > 0$ the maximum principle implies that all solutions $u(t,x)$ of (2.2) whose initial conditions $u(0,x)$ satisfy

$$\phi(x-h) \le u(0,x) \le \phi(x+h) \quad \text{for all } x$$

must satisfy

$$\phi(x-h) \le u(t,x) \le \phi(x+h) \quad \text{for all } x \text{ and all } t > 0$$

as well. That is, any solution of equation (2.2) which is initially in a region like the one shaded below will always remain in the region. Since $h > 0$ can be taken arbitrarily small, clearly monotonic states possess a type of stability. We make this precise in the following theorem.



Theorem 2.2: Let $u(t,x) \equiv \phi(x)$ be a bounded non-constant monotonic steady state solution of equation (2.2). Then it is C^w-stable with $w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$.

Proof: This proof precisely follows the above argument. Since $\phi(x)$ solves (2.2) and (2.2) is autonomous, $\phi(x+h)$ solves (2.2) for any h. Using the maximum principle twice shows that any solution $u(t,x)$ of (2.2) whose initial condition satisfies

$$\phi(x-h) \le u(0,x) \le \phi(x+h) \quad \text{for all } x$$

for some h, will also satisfy

$$\phi(x-h) \leq u(t,x) \leq \phi(x+h) \quad \text{for all} \quad x \quad \text{and all} \quad t \geq 0 \ . \qquad (2.23)$$

We now show that statement (2.23) implies C^w-stability with $w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$. From (2.23),

$$\begin{aligned}(1 + \frac{1}{|\phi'(x)|})[\phi(x-h) - \phi(x)] &\leq (1 + \frac{1}{|\phi'(x)|})[u(t,x) - \phi(x)] \\ &\leq (1 + \frac{1}{|\phi'(x)|})[\phi(x+h) - \phi(x)] \quad .\end{aligned}$$

As is discussed in Chapter IV, $\phi(x)$ being a bounded non-constant monotonic solution of

$$f(\phi_{xx}, \phi_x, \phi) + c\phi_x = 0$$

implies that $\phi'(x)$ can never be zero. Moreover, the functions $\phi''(x)$, $\phi'(x)$, and $|\phi''(x)/\phi'(x)|$ are all bounded for all x and $|\phi''(x)|$ is decreasing for all x with $|x|$ sufficiently large. Thus, there is a constant $B > 0$ such that

$$(1 + \frac{1}{|\phi'(x)|}) \ |\phi(x+h) - \phi(x)| \leq B|h| \quad \text{for all} \quad x \text{ and } h \quad .$$

Hence, given any $\varepsilon > 0$ we can conclude that

$$(1 + \frac{1}{|\phi'(x)|}) \ |u(t,x) - \phi(x)| < \varepsilon \text{ for all} \quad x \quad \text{and all} \quad t > 0$$

by taking $|h| < \varepsilon/B$. Moreover, there is also a $\delta(|h|) > 0$ such that

$$|u(0,x) - \phi(x)| \ (1 + \frac{1}{|\phi'(x)|}) \leq \delta(|h|) \quad \text{for all} \quad x$$

implies that

$$\phi(x-h) \leq u(0,x) \leq \phi(x+h) \quad \text{for all} \quad x$$

(where the sign of h is chosen so that $\phi(x+h) > \phi(x-h)$). Hence our theorem is established by taking $\delta = \delta(\varepsilon/B)$.

Theorem (2.2) states that monotonic steady state solutions $u(t,x) \equiv \phi(x)$ of

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{2.2}$$

are C^w-stable with $w(x) = 1 + \frac{1}{|\phi'(x)|}$. Roughly speaking, this means that a monotonic solution $u(t,x) = \phi(x)$ is stable to small perturbations which decay asymptotically like $|\phi'(x)|$ as x goes to $-\infty$ and to $+\infty$. Since these monotonic steady states $\phi(x)$ almost always decay exponentially as $x \to \pm\infty$, we can roughly state that a monotonic steady state solution of (2.2) is stable to small perturbations which decay at the same exponential rates as $\phi'(x)$ does, for $x \to -\infty$ and $x \to +\infty$. In Chapter IV we will examine the asymptotic decay (as $x \to \pm\infty$) of monotonic steady states $\phi(x)$ more carefully.

By using $\phi(x-h)$ and $\phi(x+h)$ as lower and upper functions we have established the stability of all monotonic steady state solutions to perturbations which decay asymptotically at least as fast as the solution does as $x \to \pm\infty$. However, $u(t,x) \equiv \phi(x+h)$ and $u(t,x) \equiv \phi(x-h)$ are actually solutions of equation (2.2). By using these as our upper and lower functions we have not taken advantage of the generality allowed by the inequalities in the definitions of upper and lower functions. (See definitions in (2.4) and (2.5)). In the next two lemmas we will utilize this generality to find better upper and lower functions.

Lemma 2.3: Suppose that $u(t,x) \equiv \phi(x)$ is a bounded non-constant monotonic steady state solution of (2.2) at some c. Let $\phi(-\infty) \equiv \phi_-$ and $\phi(+\infty) \equiv \phi_+$. Then

(1) If $\phi = \phi_-$, $v = 0$ is a node and $\phi = \phi_+$, $v = 0$ is a saddle point of system (2.20), then

$$\begin{aligned} \overline{u}(t,x) &\equiv \phi(x+h(t)) + q(t)[\phi(x+h(t)) - \phi_-] \quad \text{and} \\ \underline{u}(t,x) &\equiv \phi(x-h(t)) - q(t)[\phi(x-h(t)) - \phi_-] \quad , \end{aligned} \tag{2.24}$$

where $h(t)$ and $q(t)$ are given by

$$h(t) = \alpha\kappa(1 - e^{-st}) + h_0 \quad , \quad q(t) = \alpha e^{-st} \quad , \tag{2.25}$$

are an upper and lower function of (2.2), respectively, In (2.25) κ and s are some positive constants, h_0 is arbitrary, and α is any constant with sufficiently small magnitude and with sign the same as $\phi'(x)$.

(2) If $\phi = \phi_-$, $v = 0$ is a saddle point and $\phi = \phi_+$, $v = 0$ is a node of system (2.20), then

$$\begin{aligned} \overline{u}(t,x) &\equiv \phi(x+h(t)) + q(t)\cdot[\phi_+ - \phi(x+h(t))] \quad \text{and} \\ \underline{u}(t,x) &\equiv \phi(x-h(t)) - q(t)[\phi_+ - \phi(x-h(t))] \quad , \end{aligned} \tag{2.26}$$

where $h(t)$ and $q(t)$ are given by (2.25), are an upper and lower function of (2.2), respectively.

We are mainly interested in these upper and lower functions at $t = 0$ and $t \to \infty$. We have sketched in figures (1) and (2) the upper and lower functions at $t = 0$ and $t = +\infty$ from both (2.24) and (2.26), assuming for illustrative purposes that $\phi(x)$ is monotonically increasing. Note that in all sketches the value of h_0 used for the upper functions is Δh larger than that used for the lower functions.

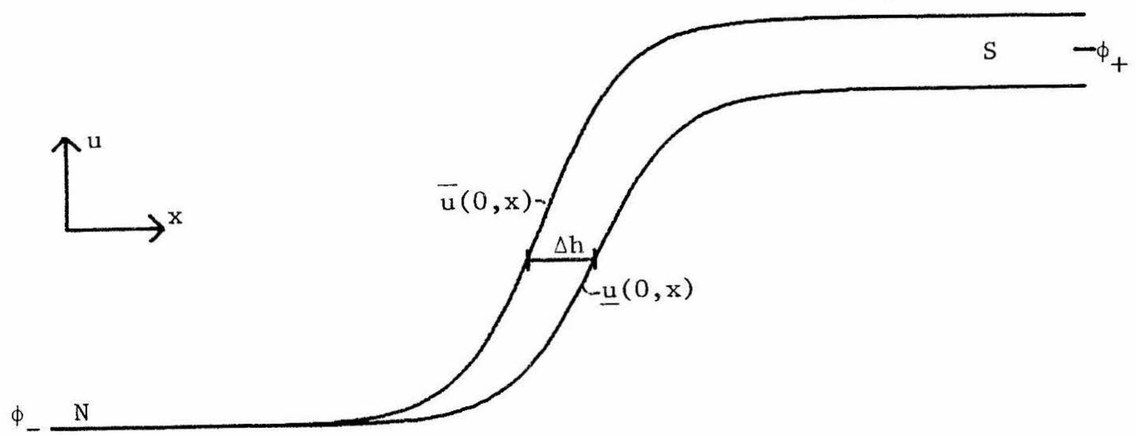


Figure (1a): The functions $\bar{u}(0,x)$ and $\underline{u}(0,x)$ when ϕ_- is a node (N) and ϕ_+ is a saddle point (S), from (2.24).

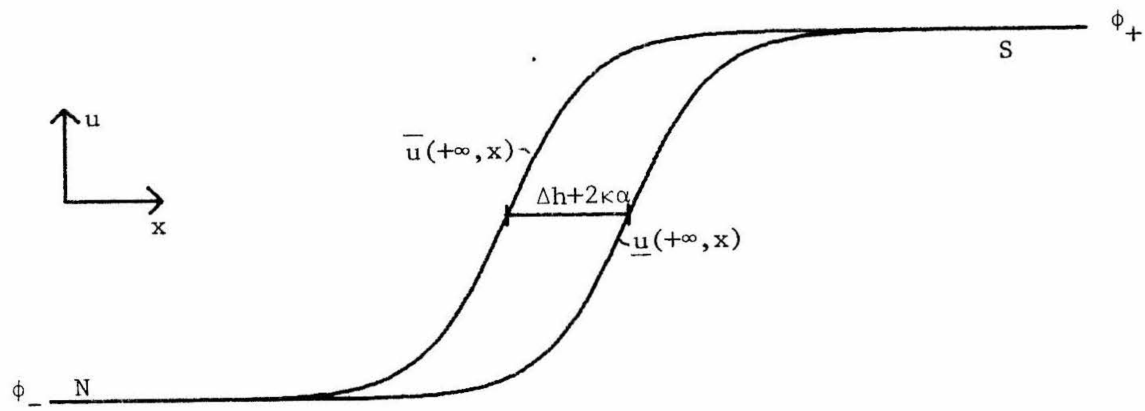


Figure (1b): The functions $\bar{u}(+\infty,x)$ and $\underline{u}(+\infty,x)$ when ϕ_- is a node (N) and ϕ_+ is a saddle point (S), from (2.24).

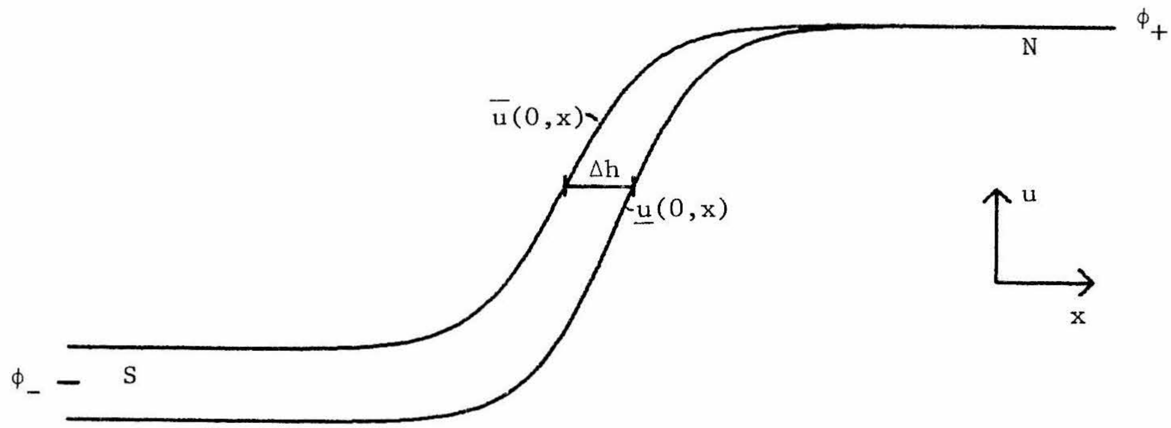


Figure (2a): The functions $\overline{u}(0,x)$ and $\underline{u}(0,x)$ when ϕ_- is a saddle point (S) and ϕ_+ is a node (N), from (2.26).

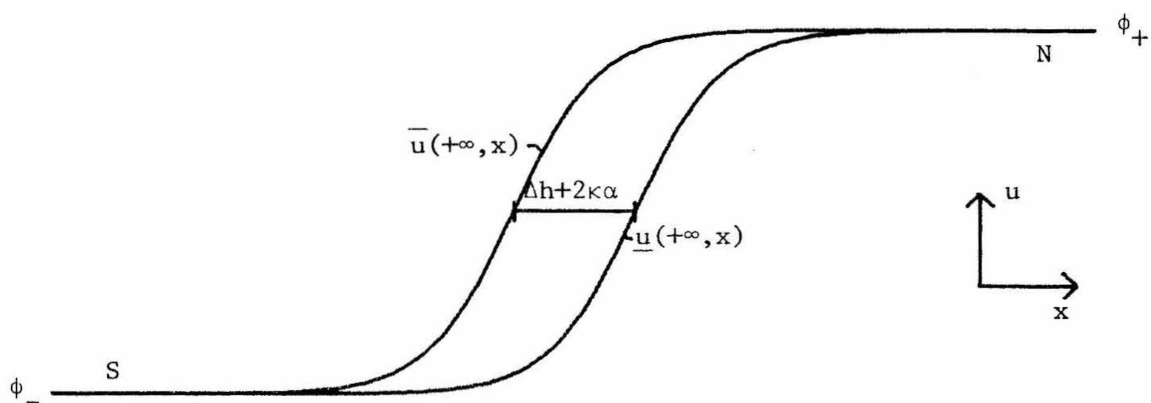


Figure (2b): The functions $\overline{u}(+\infty,x)$ and $\underline{u}(+\infty,x)$ when ϕ_- is a saddle point (S) and ϕ_+ is a node (N), from (2.26).

Proof of lemma 2.3: We prove only that $\overline{u}(t,x)$ in (2.24) is an upper function of (2.2) when $\phi(x)$ is increasing. The proof when $\phi(x)$ is decreasing and the proofs of the other parts of the lemma follow from very similar calculations.

We will prove that $\overline{u}(t,x)$ in (2.22) is an upper function by showing that $\overline{u}_t - f(\overline{u}_{xx}, \overline{u}_x, \overline{u}) - c\overline{u}_x \geq 0$ for all α sufficiently small.

We have

$$\bar{u} = \phi + q(\phi-\phi_-) \quad ,$$
$$\bar{u}_t = \phi' \ h_t(1+q) + q_t(\phi-\phi_-) \quad ,$$
$$\bar{u}_x = \phi'(1+q) \quad ,$$
$$\bar{u}_{xx} = \phi''(1+q) \quad .$$

For $x + h(t) < - x_0$ ($x_0 > 0$ very large), we expand about ϕ, ϕ', and ϕ''. We find

$$\begin{aligned} &\bar{u}_t - f(\bar{u}_{xx}, \bar{u}_x, \bar{u}) - c\bar{u}_x \\ &\quad \geq \phi' h_t + q_t(\phi-\phi_-) - f_1\phi'' q - (f_2+c)\phi' q - f_3(\phi-\phi_-)q \qquad (2.27) \\ &\qquad + h.o.(\phi'' q, \phi' q, (\phi-\phi_-)q) \quad , \end{aligned}$$

where the arguments of f_1, f_2, f_3 are ϕ'', ϕ', ϕ, and the argument of ϕ'', ϕ', and ϕ is $x + h(t)$. Here h.o.(a,b,c) stands for terms which are of at least quadratic order in a,b, and c. For $x + h(t) < - x_0$, ϕ''/ϕ' and $(\phi-\phi_-)/\phi'$ are both bounded (as will be shown in Chapter IV). Thus (2.27) shows that there exists positive constants M^-, N^-, and q^- such that

$$\bar{u}_t - f(\bar{u}_{xx}, \bar{u}_x, \bar{u}) - c\bar{u}_x \geq 0 \quad \text{for all} \quad x + h(t) < - x_0$$

whenever

$$h_t + M^- q_t \geq N^- q \quad \text{and} \quad 0 \leq q \leq q^- \quad .$$

For $x + h(t) > x_0$ ($x_0 > 0$ very large), we again expand about ϕ'', ϕ', ϕ. As before, we find

$$\begin{aligned} &\bar{u}_t - f(\bar{u}_{xx}, \bar{u}_x, \bar{u}) - c\bar{u}_x \\ &\quad \geq \phi' h_t + q_t(\phi-\phi_-) - f_1\phi'' q - (f_2+c)\phi' q - f_3(\phi-\phi_-)q \qquad (2.27) \\ &\quad + h.o.(\phi'' q, \phi' q, (\phi-\phi_-)q) \end{aligned}$$

where the arguments of f_1, f_2, f_3 and ϕ'', ϕ', ϕ are the same as before. Since $f_3(0,0,\phi_+) < 0$, by taking x_0 sufficiently large we ensure that

$$- f_3(\phi'', \phi', \phi) \geq 2s \quad \text{for all} \quad x + h(t) \geq x_0 \quad ,$$

for some positive constant s. Since $\phi(x+h(t))$ is increasing in x, we have

$$0 < \phi(x_0) - \phi_- \leq \phi(x+h(t)) - \phi_- \leq \phi_+ - \phi_- \quad \text{for all} \quad x+h(t) \geq x_0 \ ;$$

i.e., $\phi - \phi_-$ is bounded away from zero and is also bounded. Noting that ϕ''/ϕ' is bounded for $x+h(t) \geq x_0$, we see that (2.27) implies that there exists an $N^+ > 0$ and a $q^+ > 0$ such that

$$\bar{u}_t - f(\bar{u}_{xx}, \bar{u}_x, \bar{u}) - c\bar{u}_x \geq 0 \quad \text{for all} \quad x+h(t) \geq x_0$$

whenever

$$0 \leq - q_t \leq sq \ , \ h_t \geq N^+ q \ , \text{ and } 0 \leq q \leq q^+ \quad .$$

We now consider the middle region. Linearizing about ϕ'', ϕ', and ϕ as before again yields (2.27). Since $\phi'(x) \neq 0$ for all x (this is shown in Chapter IV), for any $x_0 > 0$ there is a $\delta > 0$ such that $\phi'(x) > \delta$ for all x in $[-x_0, x_0]$. Since ϕ'' and $\phi-\phi_-$ are bounded, (2.27) shows that there are constants $M^0 > 0$, $N^0 > 0$, and $q^0 > 0$ for which

$$\bar{u}_t - f(\bar{u}_{xx}, \bar{u}_x, \bar{u}) - c\bar{u}_x \geq 0 \quad \text{all} \quad x+h(t) \in [-x_0, x_0]$$

is satisfied whenever

$$h_t + M^0 q_t \geq N^0 q \qquad 0 \leq q \leq q^0 \quad .$$

To summarize this calculation, we can conclude

$$\bar{u}_t - f(\bar{u}_{xx}, \bar{u}_x, \bar{u}) - c\bar{u}_x \geq 0 \quad \text{for all} \quad x \quad , \tag{2.28}$$

when

$$\begin{aligned} &h_t \geq \max\{M^-, M^0\} \cdot (-q_t) + \max\{N^-, N^0, N^+\}q \quad , \\ &0 \leq - q_t \leq sq \quad , \quad \text{and} \\ &0 \leq q \leq \min\{q^-, q^0, q^+\} \quad , \end{aligned} \tag{2.29}$$

hold. Hence, we take

$$q = \alpha e^{-st} \quad , \; h = \alpha\kappa(1-e^{-st}) + h_0 \tag{2.25}$$

where $\kappa = \max\{M^-, M^0\}s + \max\{N^-, N^0, N^+\}$, and note that (2.29) is satisfied for $0 \leq \alpha \leq \min\{q^-, q^0, q^+\}$. Thus (2.28) is true, and hence $\bar{u}$ is an upper function.

Lemma (2.3) provides good upper and lower functions when the monotone wave goes from a node to a saddle. These functions still decay asymptotically like $\phi'(x)$ as x goes to the node (at either $-\infty$ or $+\infty$). However, as x goes to the saddle point at, say, $+\infty$, the upper and lower functions asymptote to $\phi_+ + \Delta$ and $\phi_+ - \Delta$ for some $\Delta > 0$. When used for stability proofs, this translates into a larger stability class than that in theorem (2.2). Namely, these upper and lower functions will prove the monotonic wave stable to perturbations which decay like $\phi'(x)$ as x goes to the node (at $x = -\infty$ or $x = +\infty$), but which only need be bounded as x goes to the saddle point (at $x = -\infty$ or $x = +\infty$).

In lemma (2.3) we were able to improve our upper and lower functions when a single saddle point is present, and so one expects that still better functions can be obtained when both $x = -\infty$ and $x = +\infty$ are saddle points. The following lemma shows this to be so.

<u>Lemma 2.4</u>: Suppose $u(t,x) \equiv \phi(x)$ is a bounded, monotonic, steady state

solution of (2.2) at some c. Let $\phi(-\infty) \equiv \phi_-$ and $\phi(+\infty) \equiv \phi_+$. If $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are both saddle points of system (2.20), then

$$\begin{aligned} \overline{u}(t,x) &\equiv \phi(x+h(t)) + |q(t)| \quad \text{and} \\ \underline{u}(t,x) &\equiv \phi(x-h(t)) - |q(t)| \quad , \end{aligned} \tag{2.30}$$

with

$$h(t) \equiv \alpha\kappa(1-e^{-st}) + h_0 \ , \ q(t) \equiv \alpha e^{-st} \quad , \tag{2.31}$$

are upper and lower functions, respectively. Here κ and s are fixed positive constants, h_0 is arbitrary, and α is any constant with sufficiently small magnitude and with sign like that of $\phi'(x)$.

Proof: Lemma 2.4 follows from calculations very similar to those in the proof of lemma (2.3).

It should be noted that the upper and lower functions in lemma (2.4) were devised in reference [3] for the class of equations $u_t = u_{xx} + h(u)$. These upper and lower functions are sketched in figures (3a) and (3b) at $t = 0$ and $t = +\infty$, respectively. For illustrative purposes we have taken $\phi(x)$ to be increasing. Also in sketching these functions, we have used a value of h_0 for the upper functions which is Δh larger than the value used for the lower functions.

We now use the upper and lower functions provided by lemmas (2.3) and (2.4) in conjunction with the maximum principle. This will yield our main stability result. In order to state this result succinctly, we introduce the notation

$$r^{+}(\phi'(x)) \equiv \begin{cases} \phi'(x) & x \geq 0 \\ \phi'(0) & x \leq 0 \end{cases} \quad , \text{ and}$$

$$r^{-}(\phi'(x)) \equiv \begin{cases} \phi'(0) & x \geq 0 \\ \phi'(x) & x \leq 0 \end{cases} \quad .$$

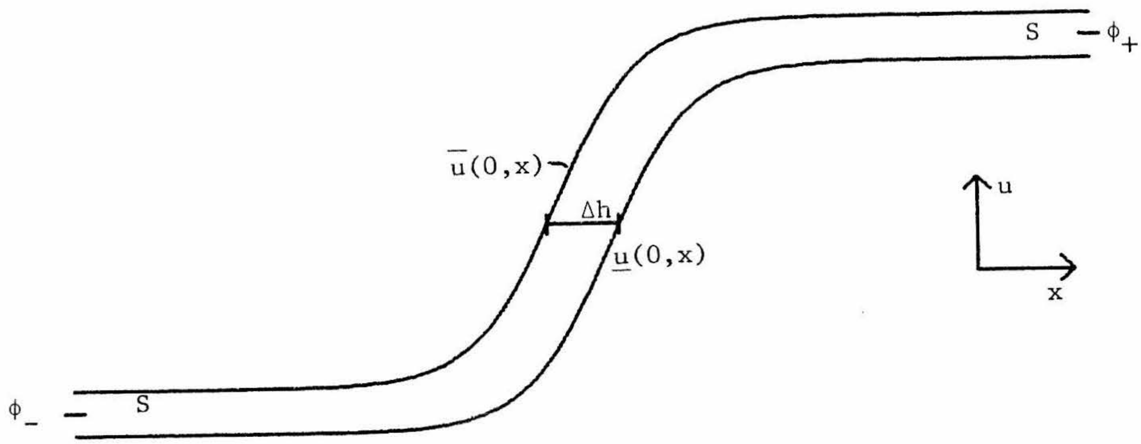


Figure (3a): The functions $\overline{u}(0,x)$ and $\underline{u}(0,x)$ when ϕ_- and ϕ_+ are both saddle points (S), from (2.30).

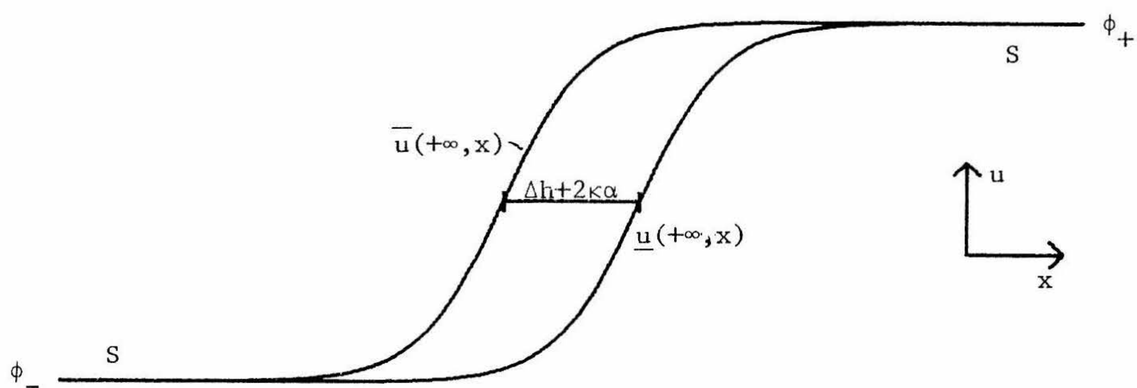


Figure (3b): The functions $\overline{u}(+\infty,x)$ and $\underline{u}(+\infty,x)$ when ϕ_- and ϕ_+ are both saddle points (S), from (2.30).

<u>Theorem 2.5</u>: Suppose $u(t,x) \equiv \phi(x)$ is a bounded, monotonic steady state

solution of equation (2.2) for some value of c. Then $u(t,x) \equiv \phi(x)$ is C^w-stable where

(1) if $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are both saddle points of system (2.20), then $w(x) \equiv 1$;

(2) if $\phi = \phi_-$, $v = 0$ is a node and $\phi = \phi_+$, $v = 0$ is a saddle point of system (2.20), then $w(x) \equiv 1 + \frac{1}{|r^-(\phi'(x))|}$;

(3) if $\phi = \phi_-$, $v = 0$ is a saddle point and $\phi = \phi_+$, $v = 0$ is a node of system (2.20), then $w(x) \equiv 1 + \frac{1}{|r^+(\phi'(x))|}$; and

(4) if $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are both nodes of system (2.20), then $w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$.

Proof: To prove part (1), we use the upper and lower functions contained in lemma (2.4). To prove parts (2) and (3) we use the upper and lower functions in lemma (2.3). The maximum principle shows that any solution $u(t,x)$ of equations (2.2) which is initially between an upper and a lower function will always stay between those functions. This immediately implies that $\phi(x)$ is stable, because the size of α (see equations (2.24), (2.26), and (2.30)) can be taken as small as one wishes. Inspection of the formulas for the upper and lower function shows that the classes of perturbations bounded by the upper and lower functions are the same as those allowed in the definition of C^w-stability, with the functions $w(x)$ as given by the theorem. Part (4) is a special case of theorem (2.2).

As a rough summary, we have shown that monotone steady state solutions of equation (2.2) are stable with respect to perturbations which are small and

(1) bounded as $x \to +\infty$ [as $x \to -\infty$] when $\phi(+\infty)$ [$\phi(-\infty)$] is a saddle point, and

(2) decay asymptotically like $\phi'(x)$ as $x \to +\infty$ [as $x \to -\infty$] when $\phi(+\infty)$ [$\phi(-\infty)$] is a node. This is equivalent to showing that the traveling wave solutions $u(t,x) = \phi(x-ct)$ of equation (2.1) ($u_t = f(u_{xx}, u_x, u)$) are C^{w^*}-stable with $w^*(t,x) = w(x-ct)$. That is, our results measure the deviations of the perturbed traveling wave relative to a function $\frac{1}{w}$ which moves with the wave. This seems physically appropriate.

The stability results in theorem (2.5) are shown to be sharp in section (2.4). There it is seen that in the $N \to S$, $S \to N$, and $N \to N$ cases, some perturbations which slightly violate the asymptotic decay conditions of parts (2), (3), and (4) of theorem (2.5) lead to solutions which travel at velocities slightly different than c. Since these perturbed waves will gradually drift away from the unperturbed wave, the traveling wave is unstable to these perturbations.

In the next section we show that very nearly all non-monotonic waves are unstable. This will complete the stability picture for steady state solutions of equation (2.2).

2.3 Instability of non-monotonic waves. In this section we show that very nearly all non-monotonic steady states are unstable. Specifically, we will show that if $u(t,x) \equiv \phi(x)$ is a non-monotonic solution of (2.2), then

(1) if $\phi(x)$ has a relative extrema at at least two distinct points x, then there is an x_0 and an x_1 such that $u(t,x) = \phi(x)$ is unstable to all smooth initial perturbations $p(x)$ which are non-negative

for $x \notin [x_0, x_1]$ and which are positive for all x in $[x_0, x_1]$;

(2) if $\phi(x)$ has only a single relative extremum then

(a) if $\phi(-\infty)$ or $\phi(+\infty)$ is a saddle point then $u(t,x) = \phi(x)$ is unstable to perturbations which decay like $|\phi'(x)|$ as x goes to $-\infty$ and goes to $+\infty$;

(b) if both $\phi(-\infty)$ and $\phi(+\infty)$ are nodes then $u(t,x) = \phi(x)$ may be stable or unstable to perturbations which decay asymptotically like $|\phi'(x)|$ as x goes to $-\infty$ and goes to $+\infty$.

The result in (1) for non-monotonic waves with at least two relative extrema is as strong a result as one can hope for, since it shows that most non-monotonic waves are unstable even to arbitrarily small perturbations of finite extent. The weaker result in (2a) leaves open the question of whether the non-monotonic waves it treats can be stable to perturbations which decay at an asymptotically faster rate than $\phi'(x)$ as x goes to $-\infty$ and to $+\infty$. We will not treat case (2b) in this chapter. In Chapter IV we will be able to characterize when a wave $u(t,x) = \phi(x)$ in case (2b) is stable or unstable to perturbations which decay asymptotically like $\phi'(x)$ as $x \to -\infty$ and as $x \to +\infty$. However, we will be unable to determine whether all non-monotonic waves in case (2b) are unstable or whether some are stable and some are unstable.

We now state these results precisely in the theorem below.

Theorem 2.6: Suppose that $u(t,x) \equiv \phi(x)$ is a non-monotonic bounded steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad . \tag{2.2}$$

Then:

(1) If there are at least two finite values of x at which $\phi(x)$ has relative extrema, then there is a finite interval $[x_0, x_1]$ and a $\Delta > 0$ such that for any $\varepsilon > 0$ there is a $p(x)$ satisfying

$$0 \leq p(x) \leq \varepsilon \quad \text{for all} \quad x \in (x_0, x_1)$$

$$p(x) \equiv 0 \quad \text{for all} \quad x \notin (x_0, x_1) \quad ,$$

for which the solution $u(t,x)$ of equation (2.2) with initial condition

$$u(0,x) = \phi(x) + p(x)$$

satisfies

$$|u(t,x) - \phi(x)| > \Delta$$

for some x and some $t > 0$.

(2) If there is only a single finite value of x, $x = x_e$, where $\phi(x)$ has a relative extremum and if either $\phi(-\infty)$ or $\phi(+\infty)$ is a saddle point, then $\phi(x)$ is $\mathfrak{C}^w$-unstable with

$$w(x) \equiv \begin{cases} 1 + \dfrac{1}{|\phi'(x)|} + \dfrac{1}{|\phi'(x_e+1)|} & x \leq x_e - 1 \\ 1 + \dfrac{1}{|\phi'(x_e-1)|} + \dfrac{1}{|\phi'(x_e+1)|} & x_e - 1 \leq x \leq x_e + 1 \\ 1 + \dfrac{1}{|\phi'(x_e-1)|} + \dfrac{1}{|\phi'(x)|} & x \geq x_e + 1 \end{cases} \quad . \tag{2.32}$$

When we prove theorem (2.6), we will actually prove for part (1) much more than is claimed by the theorem. We will actually show that the non-monotonic steady states $u(t,x) = \phi(x)$ with at least two relative extrema are unstable to <u>all</u> smooth initial perturbations $p(x)$ which are

(a) non-negative outside the interval $[x_0, x_1]$

(b) positive inside the interval $[x_0, x_1]$.

Note also that the $w(x)$ of part (2) of this theorem is essentially $1 + \frac{1}{|\phi'(x)|}$ modified so that it remains finite at $x = x_e$ where $\phi'(x)$ is zero. The constants $\frac{1}{|\phi'(x_e-1)|}$ and $\frac{1}{|\phi'(x_e+1)|}$ were included in $w(x)$ because formally we have only defined $\mathcal{C}^w$-instability for continuous w, and these constants can be ignored without loss.

The proof of theorem (2.6) will be based on a so-called "hair-trigger" effect. Before proving this theorem, we will illustrate this effect with the example where it was apparently first discovered in reference [4].

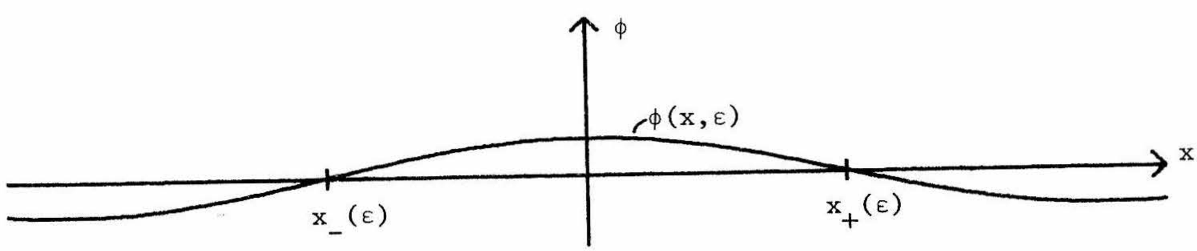
For this example we consider Fischer's equation

$$u_t = u_{xx} + u(1-u) \tag{2.33}$$

and note that $u(t,x) \equiv 0$ and $u(t,x) \equiv 1$ are its <u>only</u> bounded non-negative steady states. We define the steady state solution $u(t,x) \equiv \phi(x, \varepsilon)$ of (2.33) by

$$\phi_{xx} + \phi(1-\phi) = 0 \qquad \phi(0,\varepsilon) = \varepsilon \qquad \phi'(0,\varepsilon) = 0 \quad ,$$

where $\varepsilon > 0$ is small. This steady state is illustrated below, and we see that $\phi(x,\varepsilon) = 0$ at $x = x_-(\varepsilon) < 0$ and at $x = x_+(\varepsilon) > 0$.



We thus define $\tilde{u}(t,x,\varepsilon)$ as the solution of (2.33) with the initial condition

$$\tilde{u}(0,x,\varepsilon) \equiv \begin{Bmatrix} \phi(x,\varepsilon) & x_-(\varepsilon) \le x \le x_+(\varepsilon) \\ 0 & x \notin [x_-(\varepsilon),\ x_+(\varepsilon)] \end{Bmatrix}.$$

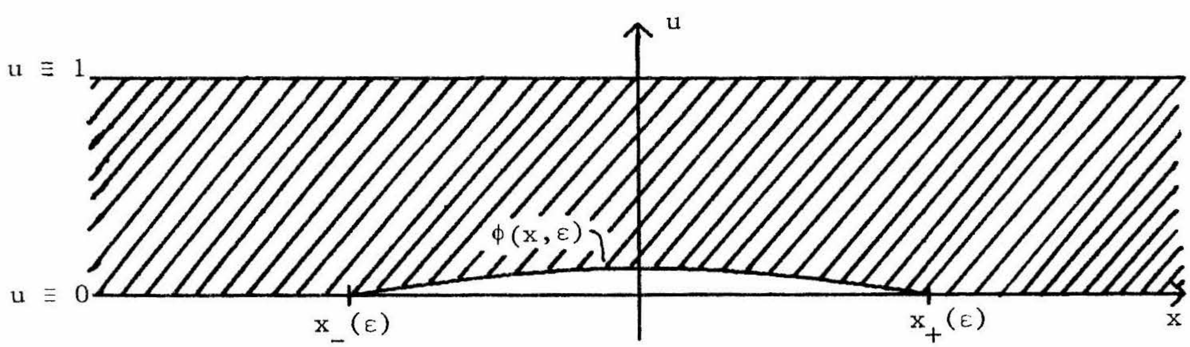
By using the maximum principle repeatedly, one can show that $\tilde{u}(t,x,\varepsilon) \to \phi_\infty(x,\varepsilon)$ as $t \to +\infty$, where $\phi_\infty(x,\varepsilon)$ is the least steady state solution of (2.33) larger than $\tilde{u}(0,x,\varepsilon)$ for all x. Thus, $\phi_\infty(x,\varepsilon)$ must be the constant steady state $\phi_\infty(x,\varepsilon) \equiv 1$. Therefore $u(t,x,\varepsilon) \to 1$ as $t \to \infty$ no matter how small $\varepsilon > 0$ is. This is the hair-trigger effect, in which a slight positive bulge in the initial condition caused $\tilde{u}(t,x,\varepsilon)$ to increase to the next steady state. This effect shows that $u(t,x) \equiv 0$ is an unstable steady state of Fischer's equation. Moreover, since $\tilde{u}(t,x,\varepsilon)$ and $u(t,x) \equiv 1$ are both solutions of Fischer's equation, the maximum principle implies that all solutions $u(t,x)$ of Fischer's equation whose initial conditions $u(0,x)$ satisfy

$$\tilde{u}(0,x,\varepsilon) \le u(0,x) \le 1 \quad \text{for all } x$$

must also satisfy

$$\tilde{u}(t,x,\varepsilon) \le u(t,x) \le 1 \quad \text{for all } x, \text{ all } t \ge 0 .$$

In particular this implies that $u(t,x) \to 1$ as $t \to +\infty$ for all x. This is illustrated in the following figure.



If $u(t,x)$ is any solution of Fischer's equation whose initial condition $u(0,x)$ lies in the shaded region, then $u(t,x) \to 1$ as $t \to +\infty$ for all x.

We will use the hair-trigger effect to prove theorem (2.6). In analogy with the preceding example, the instability of a non-monotonic steady state $\phi(x)$ will be established in three steps. For the first step, appropriate initial conditions $\tilde{u}(0,x,\varepsilon)$ for $\varepsilon > 0$ are defined with the properties

$$\tilde{u}(0,x,\varepsilon) \equiv \phi(x) \qquad \text{for } x \notin (x_-(\varepsilon), x_+(\varepsilon)) ,$$

$$\tilde{u}(0,x,\varepsilon) > \phi(x) \qquad \text{for } x \in (x_-(\varepsilon), x_+(\varepsilon)) , \text{ and}$$

$$\tilde{u}(0,x,\varepsilon) - \phi(x) = 0(\varepsilon) \quad \text{for } x \in (x_-(\varepsilon), x_+(\varepsilon)) .$$

In addition, for case (1) of theorem (2.6) we are able to take $x_-(\varepsilon)$ and $x_+(\varepsilon)$ to be bounded as $\varepsilon \to 0$, but for case (2) they are unbounded. For the second step, we use the maximum principle to show that $\tilde{u}(t,x,\varepsilon)$ increases in t to the least steady state $\phi_\infty(x,\varepsilon) \geq \tilde{u}(0,x,\varepsilon)$ for all x; i.e. $\tilde{u}(t,x,\varepsilon)$ is increasing in t for all x and $\tilde{u}(+\infty,x,\varepsilon) = \phi_\infty(x,\varepsilon)$. For the third step, we show that the least steady state $\phi_\infty(x,\varepsilon) \geq \tilde{u}(0,x,\varepsilon)$ for all x is always the least <u>constant</u> steady state solution $u(t,x) \equiv \phi_0$ satisfying

$$\phi(x) < \phi_0 \quad \text{for all } x,$$

whenever $\varepsilon > 0$ is sufficiently small. This third step establishes the instability, since $\phi_\infty(x,\varepsilon) \equiv \phi_0$ and $\phi(x)$ remain a finite distance apart as ε goes to zero.

It should be noted that hair-trigger effects were previously used to show instability of non-monotonic <u>steady state</u> solutions of the equation

$$u_t = u_{xx} + f(u)$$

in reference [5]. There C^w-stability with $w(x) \equiv 1$ was considered, and it was shown all non-monotonic steady state solutions are C^w-unstable (with $w(x) \equiv 1$), although the arguments used in [5] imply stronger instabilities. Theorem (2.6) extends these results to include traveling wave as well as steady state solutions of the general class of equations

$$u_t = f(u_{xx}, u_x, u) \qquad f_1 > 0 \quad . \tag{2.1}$$

Proof of theorem 2.6: We prove the instability of the non-monotonic steady state solution $\phi(x)$ of (2.2). As indicated above, the proof is in three parts. The first part of the proof is in finding appropriate $\phi(x,\varepsilon)$ with which to construct the bulges in the initial conditions. The second step of the proof is establishing the hair-trigger effect. This step constitutes the heart of the proof. The third and last step is showing that the final steady states $\phi_\infty(x,\varepsilon)$ of the perturbed solutions remain a finite distance from $\phi(x)$ as ε goes to zero.

We will actually only carry out the second step of the proof here. The calculations and estimates involved in the first and third steps are somewhat lengthy and tedious, and so for these steps we will use the results obtained in Chapter IV (namely, lemmas (4.7) and (4.8)). Moreover, the proof of the second step shows the main principle behind the instability of $\phi(x)$.

For the first step, we use the functions $\phi(x,\varepsilon)$ constructed in detail in Chapter IV. The results of this construction are contained in the following lemma from Chapter IV.

Lemma 4.7: Suppose that $u(t,x) \equiv \phi(x)$ is a bounded non-monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad . \tag{2.34}$$

(1) If $\phi(x)$ has relative extrema at at least two distinct finite points x, then there are functions $\phi(x,\varepsilon)$, $x_-(\varepsilon)$, $x_+(\varepsilon)$ (with $\phi(x,\varepsilon)$

in C_x^3) such that for all ε in $(0,\varepsilon_0)$ (for some $\varepsilon_0 > 0$) the following conditions are satisfied:

(a) $x_-(\varepsilon) < x_e < x_+(\varepsilon)$

(b) $f(\phi_{xx}, \phi_x, \phi) + c\phi_x = 0$ for $\phi = \phi(x,\varepsilon)$ when x is in $[x_-(\varepsilon), x_+(\varepsilon)]$,

(c) $\phi(x,\varepsilon) > \phi(x)$ for all x in $(x_-(\varepsilon), x_+(\varepsilon))$

(d) $\phi(x_-(\varepsilon),\varepsilon) = \phi(x_-(\varepsilon))$, $\phi(x_+(\varepsilon),\varepsilon) = \phi(x_+(\varepsilon))$,

(e) $\displaystyle\max_{x_-(\varepsilon)\le x\le x_+(\varepsilon)} |\phi(x,\varepsilon) - \phi(x)| \to 0$ as $\varepsilon \to 0$

(f) $x_0 \le x_-(\varepsilon) < x_+(\varepsilon) \le x_1$ for some $-\infty < x_0 < x_1 < +\infty$.

Here, in condition (a) the point $x = x_e$ is any point where $\phi(x)$ has a relative extremum.

(2) If $\phi(x)$ has a relative extremum only at a single finite value $x = x_e$ and if $\phi = \phi(-\infty)$, $v = 0$ or $\phi = \phi(+\infty)$, $v = 0$ is a saddle point of

$$\phi_x = v$$

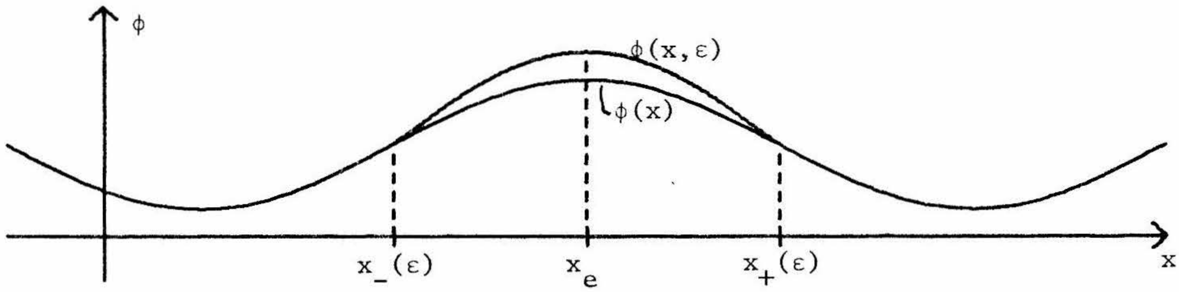
$$f(v_x, v, \phi) + cv = 0 \quad ,$$

then there are functions $\phi(x,\varepsilon)$, $x_-(\varepsilon)$, $x_+(\varepsilon)$ (with $\phi(x,\varepsilon)$ in C_x^3) such that for all ε in $(0,\varepsilon_0)$ (for some $\varepsilon_0 > 0$) conditions (a), (b), (c), (d), and (e) are satisfied. Now however, $x_-(\varepsilon) \to -\infty$ or $x_+(\varepsilon) \to +\infty$ as $\varepsilon \to 0$ and

(f') $\displaystyle\max_{\substack{|x|>x_e+1 \\ x_-(\varepsilon)<x<x_+(\varepsilon)}} \{|\phi(x,\varepsilon) - \phi(x)|\cdot(1 + \frac{1}{|\phi'(x)|})\} \to 0$ as $\varepsilon \to 0$.

Here $x = x_e$ is the single point where $\phi(x)$ has a relative extremum.

The establishment of this result in Chapter IV is the first step in our proof. Typical representatives of $\phi(x,\varepsilon)$, $x_-(\varepsilon)$, $x_+(\varepsilon)$, and $\phi(x)$ are illustrated on the next page.



We now show that if the function $\phi(x,\varepsilon)$ satisfies conditions (a), (b), (c), and (d), then the solution $\tilde{u}(t,x,\varepsilon)$ of

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{2.34}$$

with initial condition

$$\tilde{u}(0,x,\varepsilon) = \left\{ \begin{array}{ll} \phi(x,\varepsilon) & x \in [x_-(\varepsilon), x_+(\varepsilon)] \\ \phi(x) & x \notin [x_-(\varepsilon), x_+(\varepsilon)] \end{array} \right\}, \tag{2.35}$$

is increasing in t. In fact we shall show that

$$\tilde{u}(t,x,\varepsilon) \to \phi_\infty(x,\varepsilon) \quad \text{as} \quad t \to \infty \quad ,$$

where $\phi_\infty(x,\varepsilon)$ is the smallest steady state solution of (2.34) which is larger than $\tilde{u}(0,x,\varepsilon)$ for all x. This second step constitutes the heart of the proof, and we now establish it.

Let $\phi(x,\varepsilon)$ satisfy conditions (a), (b), (c), and (d), and define $\tilde{u}(t,x,\varepsilon)$ as the solution of (2.34) with its initial condition given by (2.35). We first show that $\tilde{u}(t,x,\varepsilon) \geq \tilde{u}(0,x,\varepsilon)$ for all x and all $t \geq 0$. Since $\tilde{u}(0,x,\varepsilon) \geq \phi(x)$ for all x, and since $\tilde{u}(t,x,\varepsilon)$ and $\phi(x)$ are both solutions of (2.34), the maximum principle implies that $\tilde{u}(t,x,\varepsilon) \geq \phi(x)$ for all x and all $t \geq 0$. In particular,

$$\tilde{u}(t,x_-(\varepsilon),\varepsilon) \geq \phi(x_-(\varepsilon)) = \phi(x_-(\varepsilon),\varepsilon) \quad \text{and}$$

$$\tilde{u}(t,x_+(\varepsilon),\varepsilon) \geq \phi(x_+(\varepsilon)) = \phi(x_+(\varepsilon),\varepsilon)$$

for all $t \geq 0$. Also, from (2.35) we see that

$$\tilde{u}(0,x,\varepsilon) \geq \phi(x,\varepsilon) \qquad \text{for all} \quad x \quad \text{in} \quad [x_-(\varepsilon),\ x_+(\varepsilon)] \quad .$$

This allows us to apply the maximum principle for finite domains to $\tilde{u}$, since $\tilde{u}(t,x,\varepsilon)$ and $\phi(x,\varepsilon)$ both satisfy equation (2.34) on the interval $x_-(\varepsilon) \leq x \leq x_+(\varepsilon)$. (Note that the maximum principle presented in section (2.1) applies only to infinite domains. However, the maximum principle for equation (2.34) over finite domains is a special case of the general maximum principle in Chapter III). Hence,

$$\tilde{u}(t,x,\varepsilon) \geq \phi(x,\varepsilon) \quad \text{for all} \quad x \in [x_-(\varepsilon),\ x_+(\varepsilon)] \ , \ \text{all } t \geq 0 \quad .$$

Since we have already established that $\tilde{u}(t,x,\varepsilon) \geq \phi(x)$ for all x and all $t > 0$,

$$\tilde{u}(t,x,\varepsilon) \geq \tilde{u}(0,x,\varepsilon) \quad \text{for all} \quad x, \quad \text{all} \quad t \geq 0 \quad . \qquad (2.36)$$

We now use (2.36) to show that $\tilde{u}(t,x,\varepsilon)$ is increasing in t. From (2.36), $\tilde{u}(h,x,\varepsilon) \geq \tilde{u}(0,x,\varepsilon)$ for any $h > 0$ and for all x. Since $\tilde{u}(t+h,x,\varepsilon)$ and $\tilde{u}(t,x,\varepsilon)$ both solve equation (2.34), the maximum principle shows that

$$\tilde{u}(t+h,x,\varepsilon) \geq \tilde{u}(t,x,\varepsilon) \ \text{for all } x, \ \text{all} \quad t \geq 0, \ \text{and all} \quad h \geq 0 \quad .$$

Further, let $\tilde{\phi}(x,\varepsilon)$ be any steady state solution of (2.34) with

$$\tilde{u}(0,x,\varepsilon) \leq \tilde{\phi}(x,\varepsilon) \ \text{for all } x \quad .$$

(It is shown in Chapter IV that we can assume such a $\tilde{\phi}$ exists without loss). From the maximum principle we conclude that

$$\tilde{u}(t,x,\varepsilon) \leq \tilde{\phi}(x,\varepsilon) \quad \text{for all} \quad x, \quad \text{all} \quad t \geq 0 \quad .$$

Thus at any given x and ε, $\tilde{u}(t,x,\varepsilon)$ is increasing and bounded in t. Hence, $\lim_{t\to\infty} \tilde{u}(t,x,\varepsilon) \equiv \phi_\infty(x,\varepsilon)$ exists. Since $\tilde{u}(t,x,\varepsilon)$ evolves into this time-independent function, $\phi_\infty(x,\varepsilon)$ must be a steady state solution of (2.34). Moreover from the maximum principle it is easily seen that $\phi_\infty(x,\varepsilon)$ must be the smallest steady state larger than $\tilde{u}(0,x,\varepsilon)$ for all x. That is, if $\tilde{\tilde{\phi}}(x,\varepsilon)$ is any other steady state solution of (2.34) with

$$\tilde{u}(0,x,\varepsilon) \leq \tilde{\tilde{\phi}}(x,\varepsilon) \quad \text{for all} \quad x, \tag{2.37}$$

then

$$\phi_\infty(x,\varepsilon) \leq \tilde{\tilde{\phi}}(x,\varepsilon) \quad \text{for all} \quad x \quad . \tag{2.38}$$

This establishes the hair-trigger effect for equation (2.34).

For the third and final step of the proof, we need to establish that the steady states $\phi_\infty(x,\varepsilon)$ remain a finite distance from $\phi(x)$ as $\varepsilon \to 0$. We establish this by using the following result from Chapter IV:

<ins>Lemma 4.8</ins>: Assume that $\phi(x)$ is any bounded non-monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad . \tag{2.34}$$

In addition, if $\phi(x)$ has only a single relative extremum then assume that at least one of $\phi = \phi(-\infty)$, $v = 0$ and $\phi = \phi(+\infty)$, $v = 0$ is a saddle point of the system

$$\phi_x = v$$
$$f(v_x,v,\phi) + cv = 0 \quad .$$

Then if $\tilde{\phi}(x)$ is <ins>any other</ins> steady state solution of (2.34) and if $\tilde{\phi}(x)$ satisfies

$$\phi(x) \leq \tilde{\phi}(x) \quad \text{for all} \quad x \quad ,$$

then $\tilde{\phi}(x) \geq \phi_0$ for all x. Here ϕ_0 is the <ins>least constant</ins> steady state solution of (2.34) with

$$\phi(x) < \phi_0 \quad \text{for all} \quad x \quad .$$

Thus $\qquad \phi(x) < \phi_0 \leq \tilde{\phi}(x) \quad \text{for all} \quad x$

where ϕ_0 is the least solution of $f(0,0,\phi_0) = 0$ satisfying

$$\phi(x) < \phi_0 \quad \text{for all} \quad x.$$

Note that we can always assume such a constant steady state ϕ_0 exists.

Clearly the stability of a steady state solution $u(t,x) \equiv \phi(x)$ of (2.34) cannot depend on the behavior of the equation at values of ϕ larger than

$$\phi_{max} + 1 \equiv \sup_x\{\phi(x)\} + 1 \quad .$$

Thus if no constant steady state ϕ_0 exists, we can change the function $f(\phi_{xx},\phi_x,\phi)$ for values of $\phi > \phi_{max} + 1$ so that $f(0,0,\phi)$ has a zero at, say, $\phi_{max} + 2$.

The proof is now easily completed. Let ϕ_0 be as in the above lemma, and define $\Delta > 0$ by

$$\phi_0 - \phi(0) \equiv 2\Delta \quad .$$

Now, for any $\varepsilon > 0$ no matter how small,

$$\tilde{u}(t,x,\varepsilon) \to \phi_\infty(x,\varepsilon) \quad \text{as} \quad t \to +\infty$$

and so at $x = 0$,

$$u(t,0,\varepsilon) \to \phi_\infty(0,\varepsilon) \geq \phi_0 = \phi(0) + 2\Delta \quad .$$

Thus at some x (for example, $x = 0$), $\tilde{u}(t,x,\varepsilon) - \phi(x) > \Delta$ for all t sufficiently large. We now only need to note that since $\tilde{u}(0,x,\varepsilon)$ is given by (2.35) and since $\phi(x,\varepsilon)$ satisfies conditions (e) and (f) (in case (1)) or conditions (e) and (f') (in case (2)), the perturbations

$$p(x,\varepsilon) \equiv \tilde{u}(0,x,\varepsilon) - \phi(x)$$

satisfy all requirements posed by theorem (2.6). Thus, theorem (2.6) is established.

The above proof shows much more than instability of $\phi(x)$. Specifically note that because of conditions (e) and (f) or (e) and (f'),

$$\tilde{u}(0,x,\varepsilon) < \phi_0 \quad \text{for all} \quad x$$

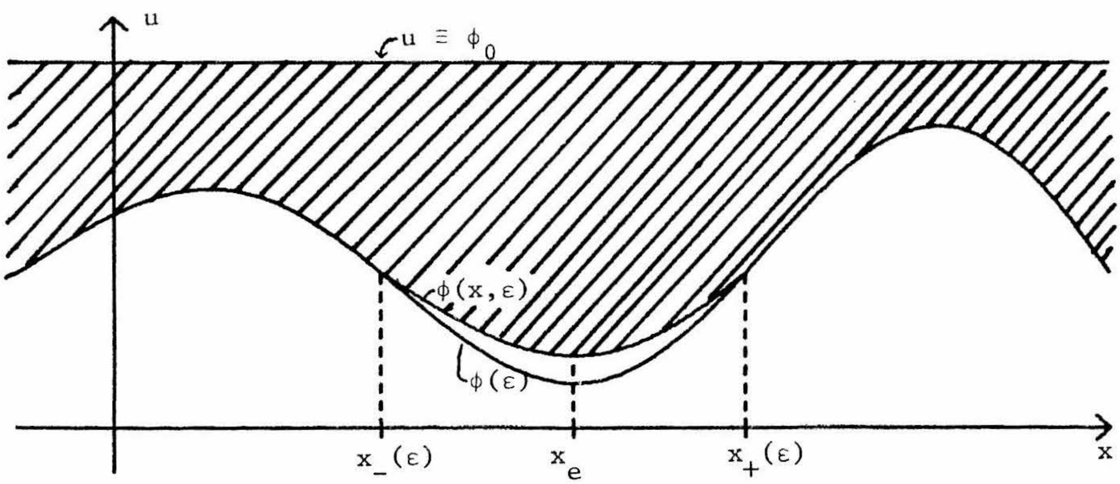
must hold for all $\varepsilon > 0$ sufficiently small. Note also that $\tilde{u}(t,x,\varepsilon)$ and ϕ_0 are both solutions of (2.34). Thus if $u(t,x)$ is <u>any</u> solution of (2.34) whose initial condition $u(0,x)$ satisfies

$$\tilde{u}(0,x,\varepsilon) \leq u(0,x) \leq \phi_0 \quad \text{for all } x \quad ,$$

then the maximum principle shows that $u(t,x)$ must satisfy

$$\tilde{u}(t,x,\varepsilon) \leq u(t,x) \leq \phi_0 \text{ for all } x, \text{ all } t > 0 \quad .$$

In particular, since $u(t,x,\varepsilon) \to \phi_0$ as $t \to +\infty$ for all x, then $u(t,x) \to \phi_0$ as $t \to +\infty$ for all x as well. Pictorially, any solution $u(t,x)$ of equation (2.34) which initially is in a region like the shaded region in the figure below, must have $u(+\infty,x) \equiv \phi_0$.



All solutions $u(t,x)$ whose initial values $u(0,x)$ are in the shaded region must have $u(+\infty,x) \equiv \phi_0$.

Theorem (2.6) very nearly completes the stability picture for steady state solutions of

$$u_t = f(u_{xx},u_x,u) + cu_x \quad . \qquad (2.34)$$

Roughly speaking, steady states $\phi(x)$ with at least two relative extrema are unstable, even to arbitrarily small perturbations of finite extent. Steady states $\phi(x)$ with exactly one relative extrema and with either $\phi(-\infty)$ or $\phi(+\infty)$ being a saddle point are unstable to arbitrarily small perturbations which decay asymptotically like $\phi'(x)$ as $x \to -\infty$ and $x \to +\infty$. Steady states $\phi(x)$ with exactly one relative extrema and with both $\phi(-\infty)$ and $\phi(+\infty)$ being nodes apparently may be stable or unstable (although perhaps all of these steady states are unstable) to arbitrarily small perturbations which decay asymptotically like $\phi'(x)$ as $x \to -\infty$ and as $x \to +\infty$. (A stability criterion for these steady state solutions is developed in Chapter IV). However, steady states $\phi(x)$ with no relative extrema (i.e. non-constant monotonic steady states) are stable, at least to small perturbations which decay asymptotically at least as fast as $\phi'(x)$ does as $x \to -\infty$ and as $x \to +\infty$. The precise stability of these monotonic steady states depends on their phase plane classification as a $N \to N$, a $N \to S$, a $S \to N$, or a $S \to S$ type steady state, and is given in theorem (2.5). In summary we see that <u>the stability of any steady state solution $\phi(x)$ of</u>

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{2.34}$$

<u>is generic</u>: the stability of $\phi(x)$ depends only on a few easily determined characteristics of ϕ and f, and is independent of the detailed natures of both ϕ and f.

In this section and the previous section we dealt with the stability of steady state solutions of (2.34) over an unbounded spatial domain. One of the extensions we will make in Chapter IV is the extension of the stability/instability results to boundary-initial value problems over finite spatial domains.

We now focus our attention on another topic. In the next section we consider the connection between the mean wavespeed of a solution $u(t,x)$ of

$$u_t = f(u_{xx}, u_x, u) \tag{2.39}$$

and its initial condition $u(0,x)$. As a by-product of this analysis, we will find that the stability results contained in theorem (2.5) for bounded monotonic steady state solutions of (2.34) are sharp in most cases.

<u>2.4 Mean wavespeeds and the initial conditions</u>. In this section, we present some results from Chapter V. As in section (2.2), we consider equations

$$u_t = f(u_{xx}, u_x, u) \tag{2.39}$$

which admit non-constant <u>monotonic</u> solutions

$$u(t,x) = \phi(x-ct,c) \tag{2.40}$$

for some values of c (which may be zero), since these are the non-trivial <u>stable</u> traveling wave solutions of (2.39). We first determine when the existence of a <u>monotonic</u> solution $\phi(x-ct,c)$ of (2.39) at a particular wavespeed c implies the existence of other nearby monotonic traveling wave solutions, both at the same and slightly different wavespeeds. We then use these results and the maximum principle to establish the connection between the mean wavespeed of solutions $u(t,x)$ of (2.39) and their initial conditions $u(0,x)$.

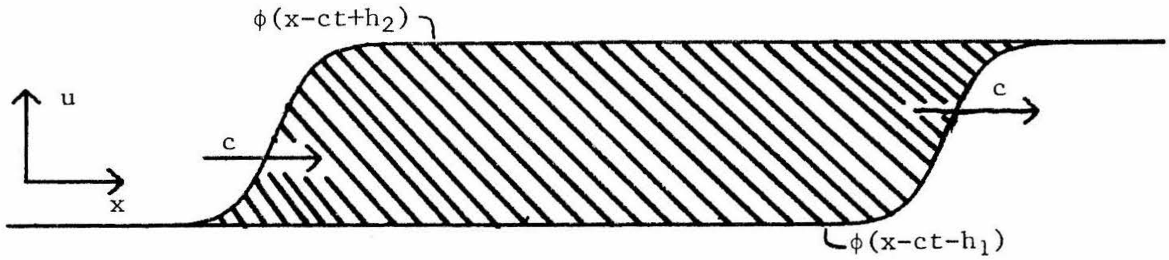
To see how such results can be obtained, let $u(t,x) \equiv \phi(x-ct)$ be an <u>increasing</u> traveling wave solution of (2.39). Then for all h_1, $h_2 > 0$ (no matter how large) $\phi(x+h_2-ct)$ and $\phi(x-h_1-ct)$ also solve (2.39). The maximum principle therefore implies that all solutions $u(t,x)$ of (2.39) with initial conditions $u(0,x)$ satisfying

$$\phi(x-h_1) \leq u(0,x) \leq \phi(x+h_2) \quad \text{for all } x \quad , \tag{2.41}$$

must satisfy

$$\phi(x-ct-h_1) \leq u(t,x) \leq \phi(x-ct+h_2) \quad \text{for all} \quad x \quad \text{and all} \quad t \geq 0 \quad . \tag{2.42}$$

This is illustrated in the following figure, where the implication of the maximum principle is that all solutions of (2.39) which are initially in the shaded region will remain in the shaded region for all $t \geq 0$.



It is apparent that these solutions $u(t,x)$ travel with mean wavespeed c in an appropriate sense. Moreover, h_1 and h_2 can be arbitrarily large. Thus the main restrictions on which initial conditions $u(0,x)$ can be bounded as in (2.41) are asymptotic in nature. It is also clear that stronger results can be obtained by using the upper and lower functions found in section (2.2).

In this section we will consider the four main types of monotonic waves, $S \to S$, $N \to S$, $S \to N$, and $N \to N$, separately. For each case, assuming a single monotonic traveling wave solution exists, we will determine the existence or non-existence of nearby monotonic waves traveling with both the same and nearly the same velocities. We will then use the maximum principle and the upper and lower functions constructed in section (2.2) to obtain the mean wavespeed/initial conditions results. We begin with the simplest case, namely the case where $u(t,x) \equiv \phi(x-c_0t, c_0)$ is a monotonic $S \to S$ type traveling wave solution of (2.39).

Case I: $S \to S$. Suppose $u(t,x) \equiv \phi(x,c_0)$ is a monotonic bounded steady state solution of

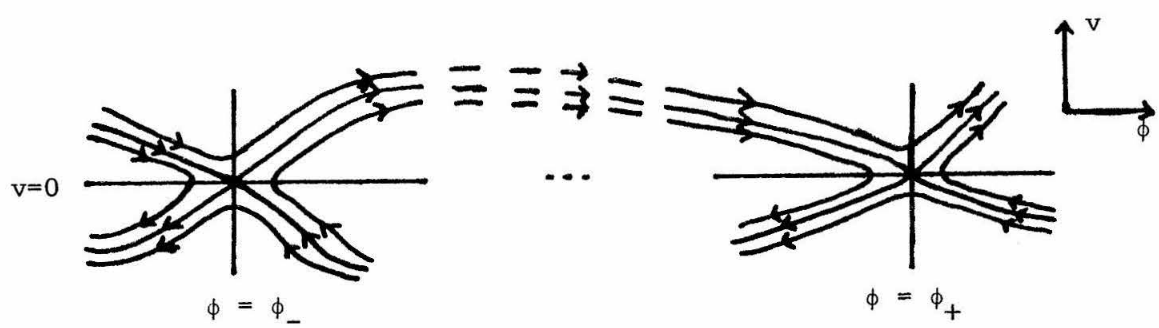
$$u_t = f(u_{xx},u_x,u) + cu_x \ , \ f_1 > 0 \tag{2.43}$$

at the wavespeed $c = c_0$, and suppose also that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v=0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v=0$ are both saddle points of

$$\begin{aligned} \phi_x &= v \\ f(v_x,v,\phi) + cv &= 0 \end{aligned} \tag{2.44}$$

at $c = c_0$. In addition we assume that $\phi(x,c_0)$ is increasing, since the analysis for $\phi(x,c_0)$ decreasing is essentially the same.

With these assumptions, the phase plane of (2.44) at $c=c_0$ looks like



Since $\phi = \phi_-$, $v=0$ and $\phi = \phi_+$, $v=0$ are saddle points at $c = c_0$, they are saddle points for all values of c. Thus, for each c there exists a function $\Psi_-(x,c)$ and $\Psi_+(x,c)$ such that every steady state solution $u(t,x) = \phi(x,c)$ of equation (2.43) with $\phi(-\infty,c) = \phi_-$ and with $\phi_x(x,c) > 0$ for all x sufficiently small must be

$$\phi(x,c) \equiv \Psi_-(x+h,c) \text{ for all } x \text{ and some constant } h \ .$$

Similarly, if $u(t,x) = \phi(x,c)$ solves equation (2.43), if $\phi(+\infty,c) = \phi_+$, and if $\phi_x(x,c) > 0$ for all x sufficiently large, then

$$\phi(x,c) \equiv \Psi_{+}(x+h,c) \quad \text{for all } x \text{ and some constant } h.$$

Thus, for any c there is at most one steady state solution $u(t,x) = \phi(x,c)$ of (2.43) (modulo translations in x) which is both monotonic and goes from $\phi(-\infty,c) = \phi_{-}$ to $\phi(+\infty,c) = \phi_{+}$. One sees that finding a value c_0 for c at which such a solution exists is equivalent to finding a c_0 at which

$$\Psi_{-}(x+h,c_0) \equiv \Psi_{+}(x,c_0) \quad \text{for all } x \text{ for some } h \ .$$

This bears a resemblance to an eigenvalue problem.

We now establish the wavespeed/initial condition result for this case. Consider equation (2.43) at $c = 0$, namely

$$u_t = f(u_{xx},u_x,u) \quad . \tag{2.45}$$

This is the given equation (2.1) in terms of the original stationary coordinate system. Let $\phi(x,c_0)$ be the monotonic steady state solution of (2.43) at $c = c_0$ with $\phi(-\infty,c_0) = \phi_{-}$, $\phi(+\infty,c_0) = \phi_{+}$, and with $\phi = \phi_{-}$, $v = 0$ and $\phi = \phi_{+}$, $v = 0$ being saddle points of system (2.44) at $c = c_0$. Then $u(t,x) \equiv \phi(x-c_0t,c_0)$ solves equation (2.45). We now utilize the upper and lower functions of lemma (2.4) and the maximum principle. This immediately shows that if $\overline{u}(t,x)$ and $\underline{u}(t,x)$ are any of the upper and lower functions given in lemma (2.4), then

$$\underline{u}(0,x) \le u(0,x) \le \overline{u}(0,x) \quad \text{for all } x$$

implies that

$$\underline{u}(t,x-c_0t) \le u(t,x) \le \overline{u}(t,x-c_0t) \text{ for all } x, \text{ all } t \ge 0$$

for any solution $u(t,x)$ of equation (2.45). Substituting for $\underline{u}$ and $\overline{u}$ from (2.30), we find that for any $q(0) > 0$ sufficiently small (and for any h_1 and h_2), all solutions $u(t,x)$ of (2.45) satisfying

$$\phi(x-h_1,c_0) - q(0) \le u(0,x) \le \phi(x+h_2,c_0) + q(0) \text{ for all } x \tag{2.46}$$

must also satisfy

$$\phi(x-c_0t-h_1-\kappa q(0),c_0) - q(t) \leq u(t,x) \leq \phi(x-c_0t+h_2+\kappa q(0),c_0) + q(t)$$
$$\text{for all } x, \text{ all } t \geq 0 \ . \qquad (2.47)$$

Here, $q(t)$ is given by (2.31) and thus $q(t) \to 0$ monotonically as $t \to \infty$. We illustrate in figure (4) the bounds on $u(t,x)$ given in (2.47). From this illustration it is clear that whenever $u(0,x)$ satisfies (2.46) for a small enough $q(0) > 0$ and any h_1 and h_2, the resulting bounds (2.47) on the solution $u(t,x)$ imply that $u(t,x)$ travels with mean wavespeed c_0 in the appropriate sense.

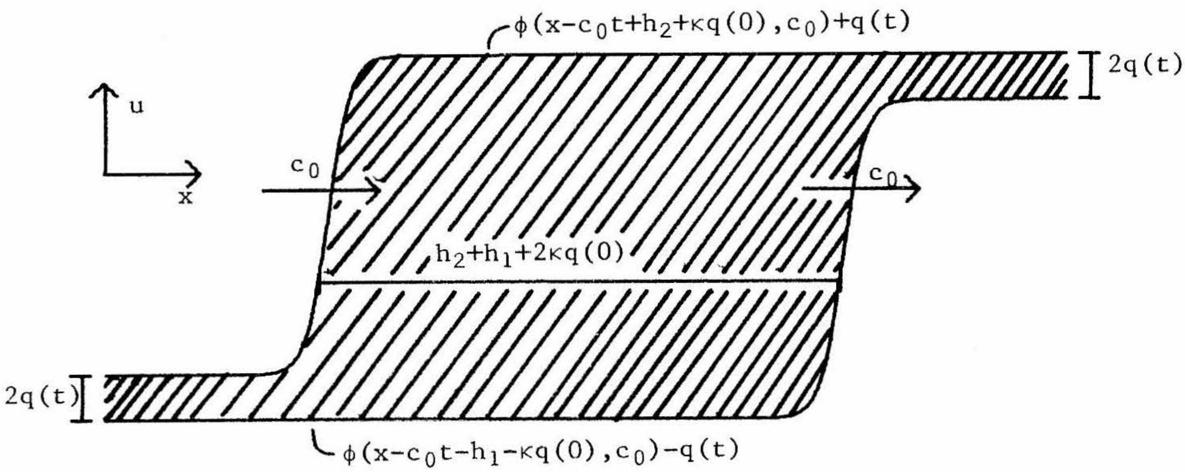


Figure (4): Since both of the functions bounding the shaded region move with speed c_0, since $q(t)\to 0$ as $t\to+\infty$, and since $u(t,x)$ must be in the shaded region for all $t\geq 0$, $u(t,x)$ must propagate with mean wavespeed c_0.

Thus when $u(0,x)$ can be bounded as in (2.46), we found that the resulting solution must travel with speed c_0. To obtain the mean wavespeed/initial condition result, we need only identify the class of initial conditions which can be bounded by (2.46). Therefore, we note that (2.46) is satisfied for a given $q(0) > 0$ and some h_1 and h_2 sufficiently large whenever the conditions

$$\phi_- - \alpha' < u(0,x) < \phi_+ + \alpha' \quad \text{for all} \quad x$$
$$\phi_- - \alpha' < u(0,x) < \phi_- + \alpha' \quad \text{for all} \quad x < -x_0 \qquad (2.48)$$
$$\phi_+ - \alpha' < u(0,x) < \phi_+ + \alpha' \quad \text{for all} \quad x > +x_0$$

are satisfied for any $x_0 > 0$ and any α' in $(0,q(0))$. Hence, whenever conditions (2.48) are fulfilled for any $\alpha' > 0$ small enough and for any $x_0 > 0$, then the solution $u(t,x)$ of

$$u_t = f(u_{xx}, u_x, u) \qquad (2.45)$$

travels with mean wavespeed c_0. As an immediate corollary, we see that there is at most one speed c_0 for which a monotone solution $\phi(x-c_0t,c_0)$ (with $\phi(-\infty,c_0) = \phi_-$ and $\phi(+\infty,c_0) = \phi_+$) exists.

In summary, we have discovered:

Theorem 2.7 ($S \to S$): Suppose that $u(t,x) \equiv \phi(x-c_0t,c_0)$ is a bounded monotonic traveling wave (or steady state) solution of

$$u_t = f(u_{xx}, u_x, u) \quad , \qquad (2.45)$$

and also suppose that $\phi = \phi(-\infty,c_0)$, $v = 0$ and $\phi = \phi(+\infty,c_0)$, $v = 0$ are both saddle points of system (2.44) at $c = c_0$. Then if $u(t,x) \equiv \tilde{\phi}(x-\tilde{c}t,\tilde{c})$ is any other monotonic traveling wave solution of (2.45) with $\tilde{\phi}(-\infty,\tilde{c}) = \phi(-\infty,c_0)$ and $\tilde{\phi}(+\infty,\tilde{c}) = \phi(+\infty,c_0)$, then

$$\tilde{\phi}(x-\tilde{c}t,\tilde{c}) \equiv \phi(x-c_0t+h,c_0) \quad \text{for all} \quad x, \quad \text{all} \quad t \geq 0$$

for some h. In particular $\tilde{c} = c_0$.

Theorem 2.8 ($S \to S$): Suppose that $u(t,x) \equiv \phi(x-c_0t,c_0)$ is a monotonic bounded traveling wave (or steady state) solution of (2.45), and also suppose that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ are both saddle points of system (2.44) at $c = c_0$. Then if $u(t,x)$ is any solution of (2.45) with initial conditions $u(0,x)$ satisfying

$$\phi_- - \alpha' < u(0,x) < \phi_- + \alpha' \quad \text{for all} \quad x < -x_0 ,$$

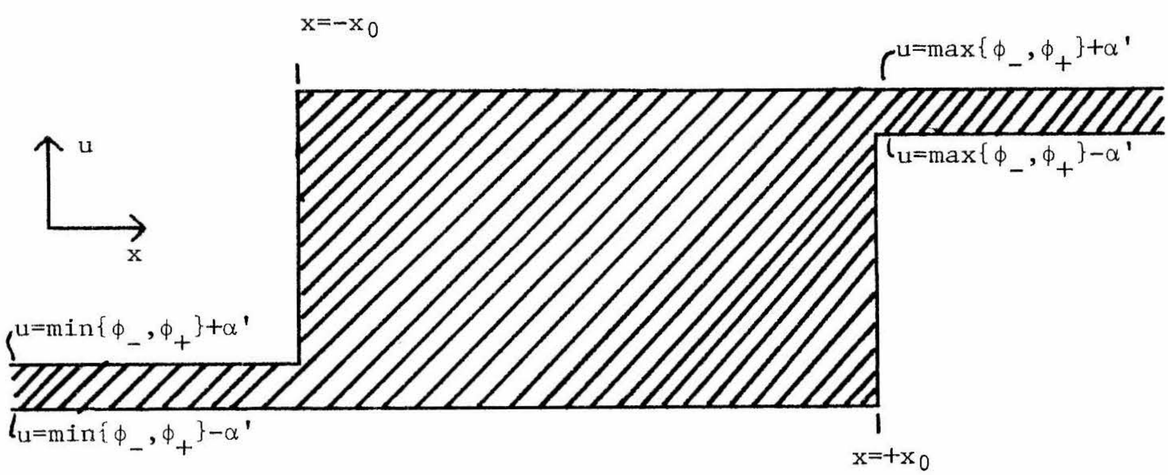
$$\phi_+ - \alpha' < u(0,x) < \phi_+ + \alpha' \quad \text{for all} \quad x > +x_0 ,$$

$$\min\{\phi_-,\phi_+\} - \alpha' < u(0,x) < \max\{\phi_-,\phi_+\} + \alpha' \quad \text{for all} \quad x ,$$

<u>for any $\alpha' > 0$ sufficiently small and any $x_0 > 0$, then $u(t,x)$ travels with mean wavespeed c_0.</u>

Note that we have established the above theorems only in the case where $\phi(x,c_0)$ is increasing in x. However, a similar analysis easily establishes the theorems for the case of $\phi(x,c_0)$ decreasing.

Roughly speaking, theorem 2.7 (S→S) shows that if ϕ_- and ϕ_+ are both saddle points, then given that $\phi(-\infty,c_0) = \phi_-$ and $\phi(+\infty,c_0) = \phi_+$ there is at most one traveling wave $u(t,x) = \phi(x-c_0t,c_0)$ modulo translations in x, and it travels with a unique wavespeed c_0. Moreover, from theorem 2.8 (S→S), we see that any solutions $u(t,x)$ whose initial conditions $u(0,x)$ remotely resemble this unique traveling wave must travel with mean wavespeed c_0. This is illustrated in the following figure.



If $u(0,x)$ is contained in any region like the one shaded above (where x_0 can be arbitrarily large), then the solution $u(t,x)$ must travel with mean wavespeed c_0.

This completes the $S \to S$ case. We continue now by analyzing the $N \to S$ case.

Case II: $N \to S$. Suppose that $u(t,x) \equiv \phi(x,c_0)$ is a bounded monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{2.49}$$

at $c = c_0$. We also assume that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ is a node and that $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ is a saddle point of the system

$$\begin{aligned} \phi_x &= v \\ f(v_x,v,\phi) + cv &= 0 \end{aligned} \tag{2.50}$$

at $c = c_0$. Finally, we will assume that $\phi(x,c_0)$ is increasing, since the analysis for $\phi(x,c_0)$ decreasing proceeds similarly.

We will first use a continuity argument to show that if $\phi(x,c_0)$ exists and has the properties assumed above, then usually for each wavespeed c' in at least a small range (c_1,c_2) about c_0 there is a monotonic steady state solution $\phi(x,c')$ of (2.49) at $c = c'$. Furthermore, we will find that $\phi(-\infty,c') = \phi(-\infty,c_0) \equiv \phi_-$ and that $\phi(+\infty,c') = \phi(+\infty,c_0) \equiv \phi_+$. As a by-product of this analysis we will find that at any given wavespeed c there is at most one such solution of (2.49) (modulo translations in x). We will then summarize these results in a theorem. Finally, we will quote and prove the mean wavespeed/initial value results for this case. We now proceed.

Let $\phi(x,c_0)$ be the monotonic steady state solution of (2.49) at $c = c_0$ with all of the properties assumed above. Then the phase plane of system (2.50) at $c = c_0$ must look like the phase plane sketched in figure (5) below. Since $\phi = \phi_+$, $v = 0$ is a saddle point at $c = c_0$, it is a saddle point at each value of c. Similarly, since $\phi = \phi_-$, $v=0$

is an unstable node at $c = c_0$, it is an unstable node at each value of

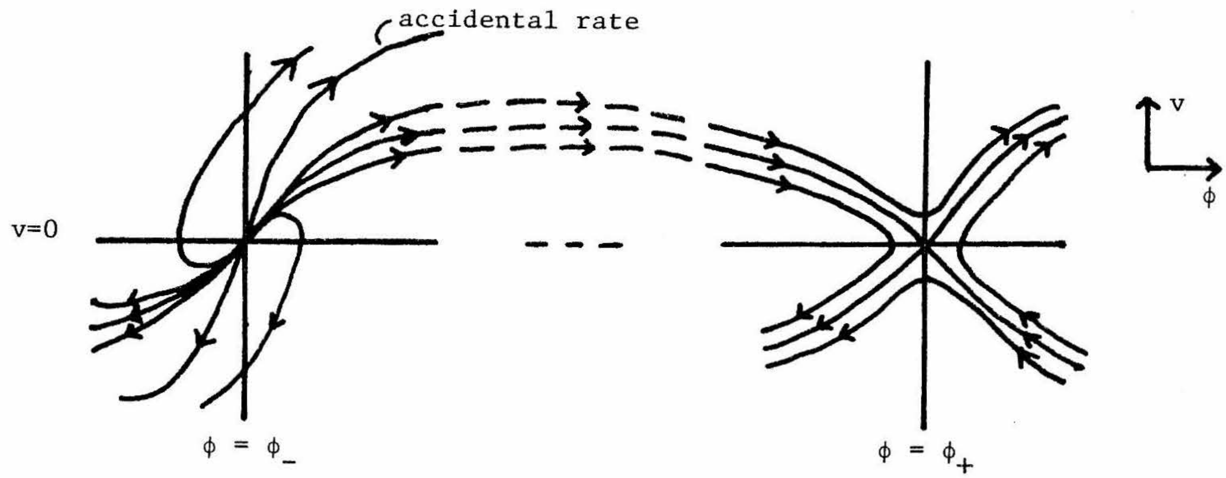


Figure (5)

$c \le c_{max}$, where c_{max} is

$$c_{max} \equiv -2\sqrt{f_1(0,0,\phi_-)\, f_3(0,0,\phi_-)} - f_2(0,0,\phi_-) \quad .$$

We now show that the existence of the monotonic steady state solution $\phi(x,c_0)$ of (2.49) at $c = c_0$ usually implies the existence of similar monotonic steady state solutions of (2.49) for all $c \le c_{max}$ sufficiently near c_0. Since $\phi = \phi_+$, $v = 0$ is a saddle point of system (2.50) for each c, at each c there is a solution $\phi = \Psi(x,c)$, $v = \frac{\partial}{\partial x}\Psi(x,c)$ of system (2.50) such that $\Psi(x,c) \to \phi_+$ as $x \to +\infty$ and such that $\Psi(x,c)$ is increasing in x for all x sufficiently large. Moreover there can be only one such solution (modulo translations in x) since $\phi = \phi_+$, $v = 0$ is a saddle point. Thus, for each $c \le c_{max}$ there is at most one steady state solution $\phi(x,c)$ of (2.49) (modulo translations in x) which is monotone and which goes from $\phi(-\infty,c) = \phi_-$ to $\phi(+\infty,c) = \phi_+$. (Of course for $c > c_{max}$ the point $\phi = \phi_-$, $v = 0$ is no longer an unstable node, and so no such solutions can exist for $c > c_{max}$).

By using the translational freedom in x for each c in the definition of $\Psi(x,c)$, we can make $\Psi(x,c)$ and $v(x,c) \equiv \frac{\partial}{\partial x}\Psi(x,c)$ both be continuously differentiable in c. (This is an implication of Chapter 13 of reference [6], for example). Moreover, by further translation of $\Psi(x,c)$ we can in addition set $\Psi(x,c_0) \equiv \phi(x,c_0)$.

Let $\tilde{\phi}_+ < \phi_+$ be defined such that $f_3(0,0,\phi) < 0$ for all ϕ in $[\tilde{\phi}_+, \phi_+]$, and let $x_+(c)$ be defined by

$$\Psi(x,c) \geq \tilde{\phi}_+ \quad \text{for all} \quad x \geq x_+(c) \quad .$$

From the phase plane of system (2.50), one realizes that

$$v(x,c) \equiv \frac{\partial}{\partial x}\Psi(x,c) > 0 \quad \text{for all} \quad x \geq x_+(c) \quad .$$

Let $\tilde{c}_1$, $\tilde{c}_2$ with $\tilde{c}_1 < c_0 < \tilde{c}_2$ be selected, and let x_+ be defined by

$$x_+ \equiv \max_{\tilde{c}_1 \leq c \leq \tilde{c}_2} \{x_+(c)\} \quad .$$

We have observed from the phase plane that $\Psi(x,c)$ is monotone for $x \geq x_+$ when c is in $(\tilde{c}_1,\tilde{c}_2)$. Suppose a constant x_- with $x_- < x_+$ is selected No matter how small x_- is, the uniform continuity of $v(x,c)$ in c when x is restricted to the interval $[x_-,x_+]$ shows that for some $\tilde{\tilde{c}}_1$ in $[\tilde{c}_1,c_0)$ and some $\tilde{\tilde{c}}_2$ in $(c_0,\tilde{c}_2]$, the function $v(x,c) > 0$ for all x in $[x_-,x_+]$ when c is in $(\tilde{\tilde{c}}_1,\tilde{\tilde{c}}_2)$. Hence, we now know that for any x_- (no matter how small) there is a $\tilde{\tilde{c}}_1 < c_0$ and a $\tilde{\tilde{c}}_2 > c_0$ such that $\Psi(x,c)$ is monotonic for all $x \geq x_-$ when c is in $(\tilde{\tilde{c}}_1,\tilde{\tilde{c}}_2)$.

Now x_- can be taken arbitrarily small, and so $\Psi(x_-,c_0)$ and $v(x_-,c_0)$ can be made to be very near $\Psi(-\infty,c_0) \equiv \phi(-\infty,c_0) = \phi_-$ and $v(-\infty,c_0) = 0$. Moreover, by restricting the interval $(\tilde{\tilde{c}}_1,\tilde{\tilde{c}}_2)$ about c_0 sufficiently, we can make $\Psi(x_-,c)$ and $v(x_-,c)$ very near $\Psi(x_-,c_0)$ and

$v(x_-,c_0)$ for all c in $(\tilde{\tilde{c}}_1,\tilde{\tilde{c}}_2)$. Therefore, we need to examine the behavior of solutions of system (2.50) near the node at $\phi = \phi_-$, $v = 0$. In particular, let $\tilde{\phi}(x,c)$ be any solution of (2.50) with $\tilde{\phi}(-\infty,c) = \phi_-$ and for which $\tilde{v}(x,c) \equiv \frac{\partial}{\partial x}\tilde{\phi}(x,c)$ is positive for all x sufficiently small. As in reference [6], solving the asymptotic equation

$$f_1(0,0,\phi_-)\tilde{\phi}_{xx} + (f_2(0,0,\phi_-)+c)\tilde{\phi}_x + f_3(0,0,\phi_-)(\tilde{\phi}-\phi_-) = 0 \qquad (2.51)$$

shows that for $c < c_{max}$ either

$$\begin{aligned} &\tilde{\phi}(x,c) \sim \phi_- + ae^{k_1(c)x} \text{ as } x\to-\infty \quad \text{(usual asymptotic decay rate) or} \\ &\tilde{\phi}(x,c) \sim \phi_- + ae^{k_2(c)x} \text{ as } x\to-\infty \quad \text{(accidental asymptotic decay rate)} \end{aligned} \qquad (2.52)$$

for some positive constant a. Here,

$$k_1(c),k_2(c) \equiv \frac{-(f_2(0,0,\phi_-)+c) \pm \sqrt{(f_2(0,0,\phi_-)+c)^2 - 4f_1(0,0,\phi_-)f_3(0,0,\phi_-)}}{2f_1(0,0,\phi_-)} \qquad (2.53)$$

where $0 < k_1(c) < k_2(c)$. Similarly in the limiting case of $c = c_{max}$ either

$$\begin{aligned} &\tilde{\phi}(x,c) \sim \phi_- - axe^{k_1(c)x} \text{ as } x\to-\infty \quad \text{(usual asymptotic decay rate) or} \\ &\tilde{\phi}(x,c) \sim \phi_- + ae^{k_1(c)x} \text{ as } x\to-\infty \quad \text{(accidental asymptotic decay rate)} \end{aligned} \qquad (2.54)$$

for some positive constant a. Let us note that at any $c \leq c_{max}$ all solutions $\tilde{\phi}(x,c)$ of system (2.50) which are increasing in x for all x sufficiently small and which also decay to ϕ_- at the <u>accidental</u> rate as $x \to -\infty$ are represented by a single phase plane trajectory. Thus all these solutions are translates of each other.

<u>Suppose now that $\phi(x,c_0)$ (which is also $\Psi(x,c_0)$) decays to ϕ_- at the usual rate as $x \to -\infty$.</u> Define $\tilde{\phi} > \phi_-$ such that $f_3(0,0,\phi)>0$ for all ϕ in $[\phi_-,\tilde{\phi}_-]$. By selecting x_- sufficiently small and selecting $\tilde{\tilde{c}}_1 < c_0$ and $\tilde{\tilde{c}}_2 > c_0$ sufficiently near c_0, the uniform continuity of $\Psi(x,c)$ in c for x in $[x_-,x_+]$ shows that $\Psi(x,c) \equiv \tilde{\phi}_-$ at exactly

one point $x = x_-(c)$ in $[x_-, x_+]$. We will now show that for all $c \leq c_{max}$, c sufficiently near c_0, the steady state solution $\Psi(x,c)$ decays to ϕ_- monotonically for $x < x_-(c)$ and decays to ϕ_- at the usual rate as $x \to -\infty$.

Consider the phase plane of system (2.50) near $\phi = \phi_-$, $v = 0$ at any value of $c \leq c_{max}$, as illustrated in figure (6) below. Let us

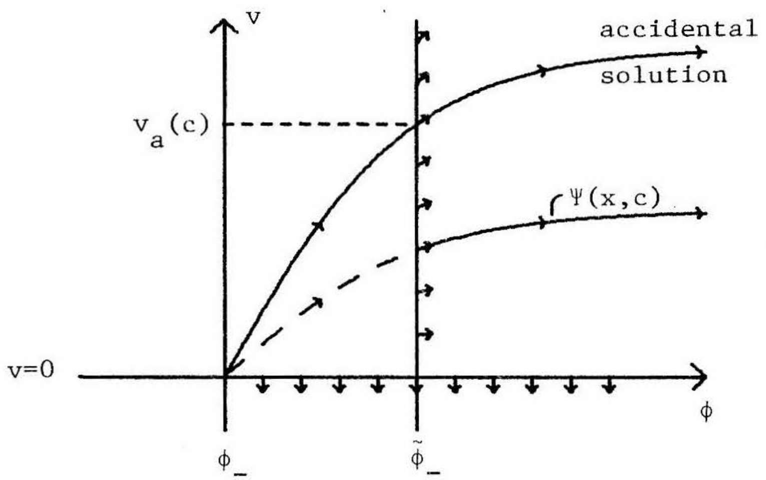


Figure (6): Phase plane of system (2.50) near $\phi=\phi_-$, $v=0$ at any $c \leq c_{max}$. If the phase plane trajectory of $\Psi(x,c)$ intersects the $\phi=\tilde{\phi}_-$ line at any positive point v below the crossing point $v_a(c)$ of the trajectory of the accidentally decaying solution, then $\Psi(x,c)$ must decay monotonically to ϕ_- at the usual rate as $x \to -\infty$. This is because the phase plane directors point downward on the v=0 line between ϕ_- and $\tilde{\phi}_-$ and because the horizontal components of the phase plane directors on the $\phi=\tilde{\phi}_-$ line are positive for $v>0$.

examine the phase plane trajectories of all solutions of system (2.50) which decrease from $\phi = \tilde{\phi}_-$ to $\phi = \phi_-$ at the usual rate as $x \to -\infty$. We see that all these trajectories must cross the $\phi = \tilde{\phi}_-$ line at a positive point v which is smaller than the point $v = v_a(c)$ at which the accidental solution (i.e. the solution which decays to $\phi = \phi_-$ at the accidental rate as $x \to -\infty$) crosses the $\phi = \tilde{\phi}_-$ line. Conversely, as illustrated in figure (6), <u>any solution of system (2.50) which crosses the $\phi = \tilde{\phi}_-$ line at a positive point $v < v_a(c)$ must decrease monotonically from $\phi = \tilde{\phi}_-$ to $\phi = \phi_-$ at the usual rate as x decreases to $-\infty$.</u>

Now we have already shown that whenever c is in $(\tilde{\tilde{c}}_1, \tilde{\tilde{c}}_2)$ then $\Psi(x,c)$ is monotonic for $x \geq x_-(c)$ and $\Psi(+\infty, c) = \phi_+$, where $x_-(c)$ has been defined as the point x at which

$$\Psi(x_-(c),c) = \tilde{\phi}_- \quad .$$

Thus to conclude that $u(t,x) = \Psi(x,c')$ is a monotonic steady state solution of (2.49) at $c = c'$ with $\Psi(-\infty,c') = \phi_-$, with $\Psi(+\infty,c') = \phi_+$, and with $\Psi(x,c')$ decaying to ϕ_- at the usual rate as $x \to -\infty$ for any c' in $(\tilde{\tilde{c}}_1,\tilde{\tilde{c}}_2) \cap (-\infty,c_{max}]$, we now need only to show that

$$v(x_-(c'),c') \equiv \frac{\partial}{\partial x} \Psi(x,c')|_{x=x_-(c')} < v_a(c') \quad .$$

However, since $\Psi(x,c_0)$ decays to ϕ_- at the usual rate,

$$v(x_-(c_0), c_0) < v_a(c_0) \quad \text{for} \quad c = c_0 \quad .$$

Moreover, $v(x_-(c),c)$ and $v_a(c)$ are continuous in c for $c \leq c_{max}$. Thus for some c_1 in $[\tilde{\tilde{c}}_1,c_0)$ and some c_2 in $(c_0,\tilde{\tilde{c}}_2]$, both sufficiently near c_0, we can conclude that $v(x_-(c'),c') < v_a(c')$ for all c' in $(c_1,c_2) \cap (-\infty,c_{max}]$ as is needed.

Thus, for some $c_1 < c_0$ and some $c_2 > c_0$ we have now shown by a continuity argument that for each c in $(c_1,c_2) \cap (-\infty,c_{max}]$, there is a traveling wave solution $u(t,x) = \Psi(x-ct,c)$ of the equation

$$u_t = f(u_{xx},u_x,u) \tag{2.55}$$

which is monotone, which decays to ϕ_- at the usual rate as $x \to -\infty$, and which has $\Psi(+\infty,c) = \phi_+$. Furthermore, we have shown that at each wave-speed c there is at most one such solution (modulo translations in x). In Chapter V we will analyze this continuity argument further, and this analysis will enable us to characterize the extremal wavespeeds c_1 and $\min\{c_2, c_{max}\}$ in that chapter. We now summarize the results of this present analysis in the following theorem.

Theorem 2.7 $(N \to S)$: Suppose that $u(t,x) \equiv \phi(x-c_0t, c_0)$ is a bounded monotonic traveling wave (or steady state if $c_0 = 0$) solution of

$$u_t = f(u_{xx}, u_x, u) \quad . \tag{2.55}$$

Suppose further that $\phi = \phi(-\infty, c_0) \equiv \phi_-$, $v = 0$ is a node and $\phi = \phi(+\infty, c_0) \equiv \phi_+$, $v = 0$ is a saddle point of the system

$$\begin{aligned} \phi_x &= v \\ f(v_x, v, \phi) + cv &= 0 \end{aligned} \tag{2.50}$$

at $c = c_0$. Finally suppose that $\phi(x,c_0)$ decays to ϕ_- at the usual rate as $x \to -\infty$. Then there is a c_1 and a c_2 with

$$-\infty \le c_1 < c_0 < c_2$$

such that for each c' in $(c_1, c_2) \cap (-\infty, c_{max}]$ there exists a $\phi(x,c')$ satisfying the following conditions:

(1) $\phi(x,c')$, $\phi_x(x,c')$ are continuously differentable in c',

(2) $\phi(x,c')$ is monotonic in x,

(3) $u(t,x) \equiv \phi(x-c't, c')$ solves equation (2.55),

(4) $\phi(-\infty, c') = \phi_-$ and $\phi(+\infty, c') = \phi_+$, and

(5) $\phi(x,c')$ decays to ϕ_- at the usual asymptotic rate as $x \to -\infty$.

Also, if $\phi_1(x,c')$ and $\phi_2(x,c')$ are any functions satisfying (2), (3), and (4) at some c', then $\phi_1(x+h,c') \equiv \phi_2(x,c')$ for all x and for some h at that c'.

The above theorem states that if $u(t,x) = \phi(x-c_0t, c_0)$ is a monotonic $N \to S$ type traveling wave solution of (2.55) which decays to

$\phi(-\infty,c_0)$ at the <u>usual</u> rate, then for at least a limited range of speeds c there is a similar monotonic $N \to S$ type traveling wave $u(t,x) = \phi(x-ct,c)$ at each wavespeed c. It also shows that for any wavespeed c these solutions are unique to within translations in x.

In Chapter V we will find a characterization of the extremal wavespeeds c_1 and c_2, and this characterization will lead to stronger results than those contained in the above theorem in some cases. For example, in Chapter V we will be able to show that if there are no constant steady state solutions $u(t,x) \equiv \phi_0$ of (2.55) between ϕ_- and ϕ_+ (i.e. if $f(0,0,\phi_0) \neq 0$ for all ϕ_0 with $\min\{\phi_-,\phi_+\} < \phi_0 < \max\{\phi_-,\phi_+\}$), then $c_1 = -\infty$. That is, monotonic $N \to S$ type traveling wave solutions exist at all wavespeeds $c \leq \min\{c_2,c_{max}\}$.

In section (2.2) we obtained stability results for monotonic waves in theorem (2.5). The above theorem, theorem 2.7 ($N \to S$), shows that the stability results contained in theorem (2.5) for the $N \to S$ case are <u>sharp</u> whenever the monotonic wave $u(t,x) = \phi(x-c_0t,c_0)$ decays to ϕ_- at the <u>usual</u> rate as $x \to -\infty$. In particular, theorem (2.5) says that a bounded monotonic $N \to S$ type solution $u(t,x) = \phi(x,c_0)$ of (2.49) at $c = c_0$ is stable to small perturbations which are bounded as $x \to +\infty$ and which decay asymptotically like $\phi_x(x,c_0)$ as $x \to -\infty$. Suppose that $\phi(x,c_0)$ decays at the <u>usual</u> rate as $x \to -\infty$. Then theorem 2.7 ($N \to S$) shows that for each $\tilde{c}$ near c_0 there exists a monotonic solution $u(t,x,\tilde{c}) \equiv \phi(x-(\tilde{c}-c_0)t,\tilde{c})$ of

$$u_t = f(u_{xx},u_x,u) + c_0u_x \quad .$$

Since for $\tilde{c} \neq c_0$ the solutions $\phi(x,c_0)$ and $\phi(x-(\tilde{c}-c_0)t,\tilde{c})$ drift apart as time increases, the solution $u(t,x) = \phi(x,c_0)$ of

$$u_t = f(u_{xx}, u_x, u) + c_0 u_x$$

is unstable to the initial perturbations $p(x,\tilde{c})$ given by

$$u(0,x) = \phi(x,c_0) + p(x,\tilde{c}) \equiv \phi(x,c_0) + [\phi(x,\tilde{c}) - \phi(x,c_0)]$$

for all $\tilde{c} \neq c_0$. Since $\phi(x,c)$ is continuous in c, since

$$\phi_x(x,c) \sim a(c)e^{k_1(c)x} \text{ as } x \to -\infty \quad ,$$

and since $k_1(c)$ is continuous in c for $c \leq c_{max}$, we see that the solution $\phi(x,c_0)$ of (2.49) at $c = c_0$ is <u>not</u> stable to all arbitrarily small perturbations which decay exponentially (as $x \to -\infty$) at any slightly slower rate than $\phi_x(x,c_0)$. Thus the asymptotic decay restriction on the perturbations allowed by theorem (2.5) cannot be significantly relaxed. Hence theorem (2.5) is <u>sharp</u> for the $N \to S$ case whenever the asymptotic decay as $x \to -\infty$ is at the <u>usual</u> rate.

We now establish the mean wavespeed/initial condition result for this case. Consider equation (2.49) at $c = 0$. This is

$$u_t = f(u_{xx}, u_x, u) \quad , \tag{2.55}$$

and is the given equation in terms of the original <u>stationary</u> coordinate system. Suppose for some c_0 that $u(t,x) = \phi(x-c_0t, c_0)$ is a bounded monotonic solution of (2.55), that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ is a node, that $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ is a saddle point, and that $\phi(x,c_0)$ decays to ϕ_- at the usual rate as $x \to -\infty$. We can now apply theorem 2.7 $(N \to S)$, and we conclude that for each c in $(c_1,\tilde{c}_2)$ there is a bounded monotonic solution $u(t,x) = \phi(x-ct,c)$ which has $\phi(+\infty,c) = \phi_+$ and which decays to ϕ_- at the usual rate as $x \to -\infty$. Here $\tilde{c}_2$ has been defined as

$$\tilde{c}_2 \equiv \min\{c_2, c_{max}\} \quad .$$

Let us assume that $\phi(x,c_0)$ is increasing in x, since the analysis for $\phi(x,c_0)$ decreasing is very similar. Now for each solution

$u(t,x) = \phi(x-ct,c)$ lemma (2.3) yields corresponding upper and lower functions $\bar{u}(t,x-ct,c)$ and $\underline{u}(t,x-ct,c)$. From equation (2.24) and (2.25) of lemma (2.3), these functions are

$$\begin{aligned}\bar{u}(t,x,c) &\equiv \phi(x+h_1+h(t,c),c) + q(t,c)\cdot\left[\phi(x+h_1+h(t,c),c) - \phi_-\right]\\ \underline{u}(t,x,c) &\equiv \phi(x-h_2-h(t,c),c) - q(t,c)\cdot\left[\phi(x-h_2-h(t,c),c) - \phi_-\right]\end{aligned} \tag{2.60}$$

where

$$h(t,c) \equiv q(0)\cdot\kappa(c)(1 - e^{-s(c)t}),q(t,c) \equiv q(0)e^{-s(c)t} \quad . \tag{2.61}$$

Here $q(0) > 0$ is any sufficiently small constant, h_1 and h_2 are arbitrary, and $\kappa(c)$ and $s(c)$ are set positive constants which may depend on c. These upper and lower functions in conjunction with the maximum principle easily establish various mean wavespeed results. For example, if for any $\underset{\sim}{c}$ and $\tilde{c}$ in $(c_1,\tilde{c}_2)$ we have

$$\underline{u}(0,x,\tilde{c}) \leq u(0,x) \leq \bar{u}(0,x,\underset{\sim}{c}) \quad \text{for all } x$$

where $\underline{u}$ and $\bar{u}$ are any of the upper and lower functions in (2.60), then the maximum principle implies that

$$\underline{u}(t,x-\tilde{c}t,\tilde{c}) \leq u(t,x) \leq \bar{u}(t,x-\underset{\sim}{c}t,\underset{\sim}{c}) \text{ for all } x \text{ all } t \geq 0$$

is also true, where $u(t,x)$ is the solution of equation (2.55) with initial condition $u(0,x)$. That is, if $u(t,x)$ is any solution of (2.55) whose initial condition $u(0,x)$ satisfies

$$\begin{aligned}\phi(x-h_2,\tilde{c}) - q(0)\cdot\left[\phi(x-h_2,\tilde{c}) - \phi_-\right] &\leq u(0,x)\\ &\leq \phi(x+h_1,\underset{\sim}{c}) + q(0)\cdot\left[\phi(x+h_1,\underset{\sim}{c}) - \phi_-\right] \text{ for all } x\end{aligned} \tag{2.62}$$

for any $\tilde{c}$, $\underset{\sim}{c}$ in $(c_1,\tilde{c}_2)$, for any h_1 and h_2, and for some sufficiently small $q(0) > 0$, then

$$\begin{aligned}\phi(x-\tilde{c}t-h_2,\tilde{c}) - q(t,\tilde{c})\cdot\left[\phi(x-\tilde{c}t-h_2,\tilde{c}) - \phi_-\right] &\leq u(t,x)\\ &\leq \phi(x-\underset{\sim}{c}t+h_1,\underset{\sim}{c}) + q(t,\underset{\sim}{c})\cdot\left[\phi(x-\underset{\sim}{c}t+h_1,\underset{\sim}{c}) - \phi_-\right]\\ &\qquad \text{for all } x, \text{ all } t \geq 0 \quad .\end{aligned} \tag{2.63}$$

Relation (2.63) implies that $u(t,x)$ cannot travel with a mean wavespeed larger than $\tilde{c}$ nor with a mean wavespeed smaller than $\underset{\sim}{c}$. This is clear from the illustration in figure (8) below.

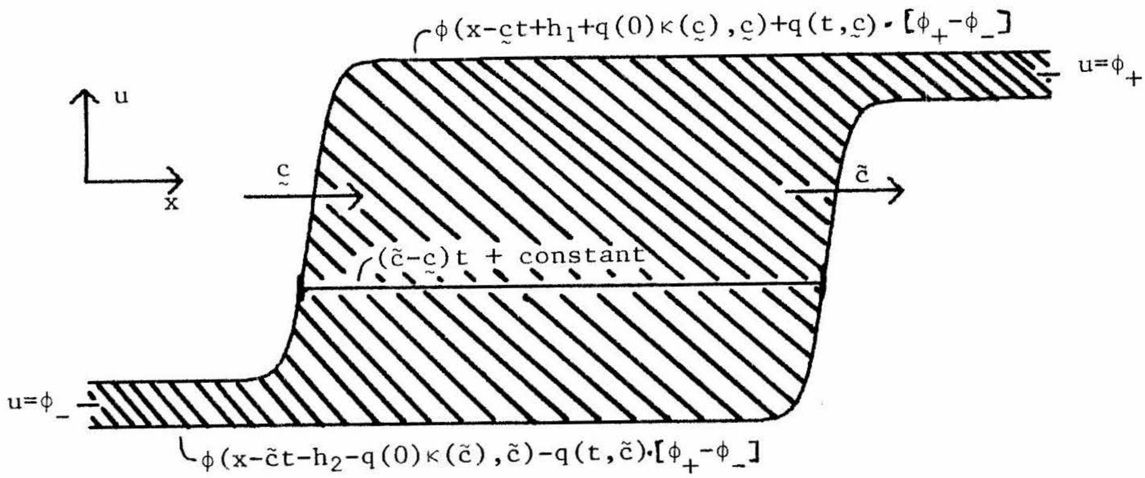


Figure (8): Since the functions bounding the shaded region move with speed $\underset{\sim}{c}$ and $\tilde{c}$, since $q(t,\tilde{c})\to 0$ and $q(t,\underset{\sim}{c})\to 0$ as $t\to+\infty$, and since $u(t,x)$ remains in the shaded region for all $t\geq 0$, $u(t,x)$ cannot travel with a mean wavespeed c outside the interval $(\underset{\sim}{c},\tilde{c})$.

We expand this observation into the following theorem.

<u>Theorem 2.8 (N → S):</u> Suppose that $u(t,x) \equiv \phi(x-c_0t,c_0)$ is a bounded monotonic solution of

$$u_t = f(u_{xx},u_x,u) \qquad f_1 > 0 \quad , \tag{2.55}$$

that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ is a node, that $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ is a saddle point, and that $\phi(x,c_0)$ decays to ϕ_- at the usual rate as $x \to -\infty$.

Define the positive exponential decay constant $\lambda(c)$ by

$$\lambda(c) \equiv \frac{-(f_2(0,0,\phi_-)+c) - \sqrt{(f_2(0,0,\phi_-)^2 - 4f_1(0,0,\phi_-)f_3(0,0,\phi_-)}}{2f_1(0,0,\phi_-)}$$

for all $c \leq c_{max}$, and define c_1 and c_2 as in the previous theorem.

Furthermore, set $\tilde{c}_2 \equiv \min\{c_2, c_{max}\}$.

Then if $u(t,x)$ is any solution of (2.55) whose initial condition $u(0,x)$ is smooth and satisfies

$$\begin{aligned}
&\phi_+ - q_0 \le u(0,x) \le \phi_+ + q_0 && \text{for all } x > x_0 \text{ for any } x_0 \\
&\phi_- < u(0,x) \le \phi_+ + q_0 && \text{for all } x \text{ if } \phi(x,c_0) \text{ is increasing in } x, \\
&\phi_+ - q_0 \le u(0,x) < \phi_- && \text{for all } x \text{ if } \phi(x,c_0) \text{ is decreasing in } x,
\end{aligned} \tag{2.64}$$

then we can conclude the following:

(1) if for any c in $(c_1, \tilde{c}_2)$ there is an $\alpha > 0$ such that

$$e^{-\lambda(c)x} \, |u(0,x) - \phi_-| > \alpha \quad \text{for all } x < 0$$

and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ cannot travel with mean wavespeed larger than c;

(2) if for any c in $(c_1, \tilde{c}_2)$ there is a $\beta > 0$ such that

$$e^{-\lambda(c)x} \, |u(0,x) - \phi_-| < \beta \quad \text{for all } x < 0$$

and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ cannot travel with mean wavespeed smaller than c;

(3) if for any c in $(c_1, \tilde{c}_2)$ there is an $\alpha > 0$ and a $\beta > 0$ such that

$$\alpha < e^{-\lambda(c)x} \, |u(0,x) - \phi_-| < \beta \quad \text{for all } x < 0$$

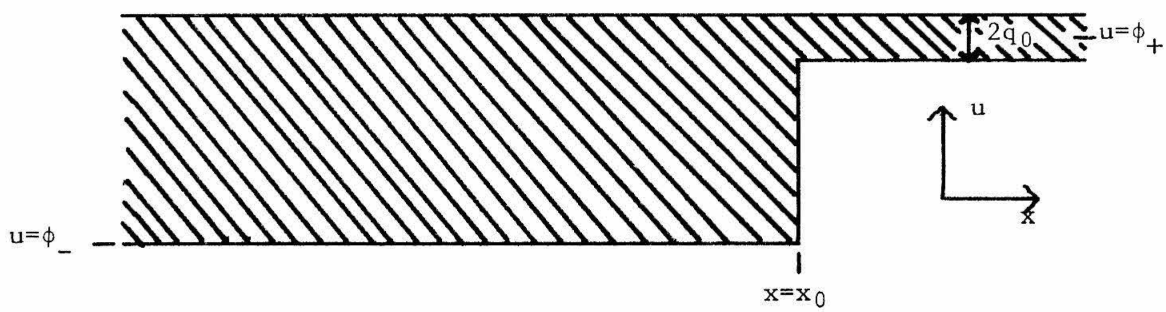
and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ travels with mean wavespeed c and has finite dispersion; and

(4) if for any c in $(c_1, \tilde{c}_2)$

$$\begin{aligned}
&\lim_{x \to -\infty} e^{-(\lambda(c)-\mu)x} \, |u(0,x) - \phi_-| = 0 \quad \text{and} \\
&\lim_{x \to -\infty} e^{-(\lambda(c)+\mu)x} \, |u(0,x) - \phi_-| = +\infty
\end{aligned} \tag{2.65}$$

hold for all $\mu > 0$, and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ travels with mean wavespeed c but may not have finite dispersion.

Roughly speaking, theorem 2.8 (N → S) supposes that $u(0,x)$ is any smooth function which is in a region like the one shaded below. It then concludes that if $u(0,x)$ decays to



the node ϕ_- exponentially as $x \to -\infty$, then the mean wavespeed of the solution $u(t,x)$ of (2.55) is determined only by the exponential decay constant.

Proof: We prove the above theorem only for the case of $\phi(x,c_0)$ increasing in x. The proof when $\phi(x,c_0)$ is decreasing is similar. By our hypotheses we can apply theorem 2.7 (N → S). We thus know that for each c in $(c_1,\tilde{c}_2)$ there exists a bounded monotonic N → S type solution $u(t,x) = \phi(x-ct,c)$ of (2.55). Moreover, $\phi(+\infty,c) = \phi_+$ and $\phi(x,c)$ decays to ϕ_- at the usual rate as $x \to -\infty$. Finally, for each of these monotonic traveling waves there are corresponding upper and lower functions $\overline{u}(t,x-ct,c)$ and $\underline{u}(t,x-ct,c)$ given by

$$\begin{aligned} \overline{u}(t,x,c) &\equiv \phi(x+h_1+h(t,c),c) + q(t,c)\cdot[\phi(x+h_1+h(t,c),c)-\phi_-] \\ \underline{u}(t,x,c) &\equiv \phi(x-h_2-h(t,c),c) - q(t,c)\cdot[\phi(x-h_2-h(t,c),c)-\phi_-] \end{aligned} \tag{2.60}$$

where

$$h(t,c) = q(0)\cdot\kappa(c)(1 - e^{-s(c)t}), q(t,c) \equiv q(0)e^{-s(c)t} \quad . \tag{2.61}$$

To prove part (1), we note that when (2.64) and the assumptions of part (1) are satisfied, then we can bound $u(0,x)$ by

$$\phi(x-h_2,c) - q(0)\cdot\left[\phi(x-h_2,c)-\phi_-\right] \le u(0,x) \le \phi_+ + q(0)\cdot\left[\phi_+-\phi_-\right] \quad \text{for all } x \tag{2.66}$$

for any $q(0) > q_0\left[\phi_+-\phi_-\right]^{-1}$ by taking h_2 sufficiently large. We note that since ϕ_+ is a saddle point, $f(0,0,\phi_+) = 0$ and $f_3(0,0,\phi_+) < 0$. Thus, if we define $\eta = \frac{1}{2} f_3(0,0,\phi_+) < 0$ and

$$\bar{u}(t,x) \equiv \phi_+ + q(0)\cdot\left[\phi_+-\phi_-\right]e^{\eta t} \quad ,$$

then for $q(0) > 0$ sufficiently small

$$\begin{aligned}\bar{u}_t - f(\bar{u}_{xx},\bar{u}_x,\bar{u}) &= q(0)\cdot\left[\phi_+-\phi_-\right]\eta e^{\eta t} - f(0,0,q(0)\cdot\left[\phi_+-\phi_-\right]e^{\eta t}) \\ &\ge 0 \quad \text{for all } t \ge 0 \quad .\end{aligned}$$

Hence, $\bar{u}$ is an upper function for $q(0) > 0$ sufficiently small. Thus, (2.66) bounds the initial condition $u(0,x)$ by the lower function $\underline{u}(0,x,c)$ (see equation (2.60)) and the newly defined upper function $\bar{u}(0,x)$. Using the maximum principle, we conclude

$$\begin{aligned}\phi(x-ct-h_2-q(0)\kappa(c),c) - q(t,c)\cdot\left[\phi_+-\phi_-\right] &\le u(t,x) \\ &\le \phi_+ + q(0)\cdot\left[\phi_+-\phi_-\right]e^{\eta t} \text{ for all } x \text{ and all } t \ge 0,\end{aligned} \tag{2.67}$$

where $q(t,c)$ is given in (2.61), when $q(0) > 0$ is sufficiently small. Since $q(0)$ can be any constant larger than $q_0\left[\phi_+-\phi_-\right]^{-1}$, it can be taken arbitrarily small by taking $q_0 > 0$ arbitrarily small. Thus, for q_0 sufficiently small relation (2.67) holds. This is illustrated in figure (9) below, where we see that $u(t,x)$ cannot travel with mean wavespeed faster than c.

Part (2) is proved in a manner very similar to part (1). In fact we find that

$$\phi_- \le u(t,x) \le \phi(x-ct+h_1+q(0)\kappa(c),c) + q(t,c)\cdot\left[\phi_+-\phi_-\right] \text{ for all } x \text{ and all } t \ge 0,$$

which is illustrated in figure (10). We conclude for this case that $u(t,x)$ cannot travel with mean wavespeed slower than c.

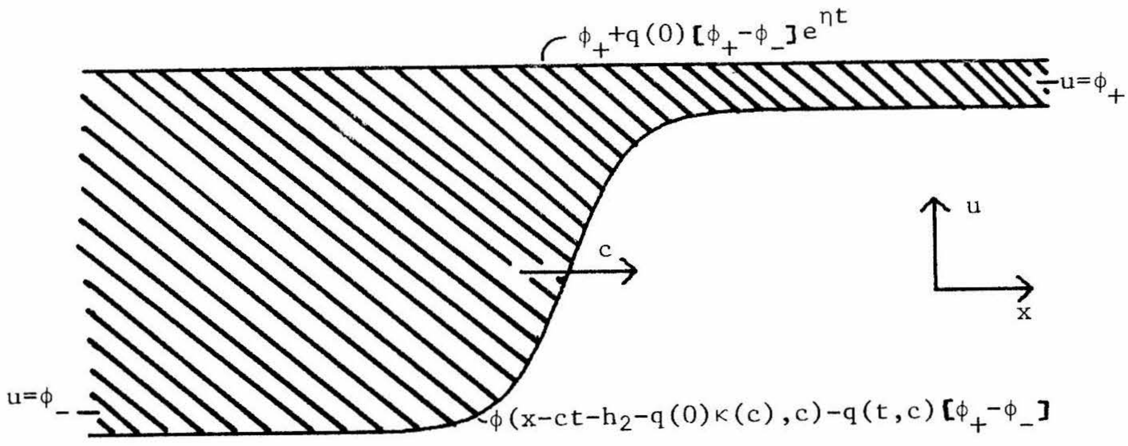


Figure (9): Since $q(0)\cdot[\phi_+-\phi_-]e^{\eta t}\to 0$ and $q(t,c)\to 0$ as $t\to+\infty$, the fact that $u(t,x)$ remains in the shaded area for all $t\geq 0$ implies it cannot travel with mean wavespeed faster than c.

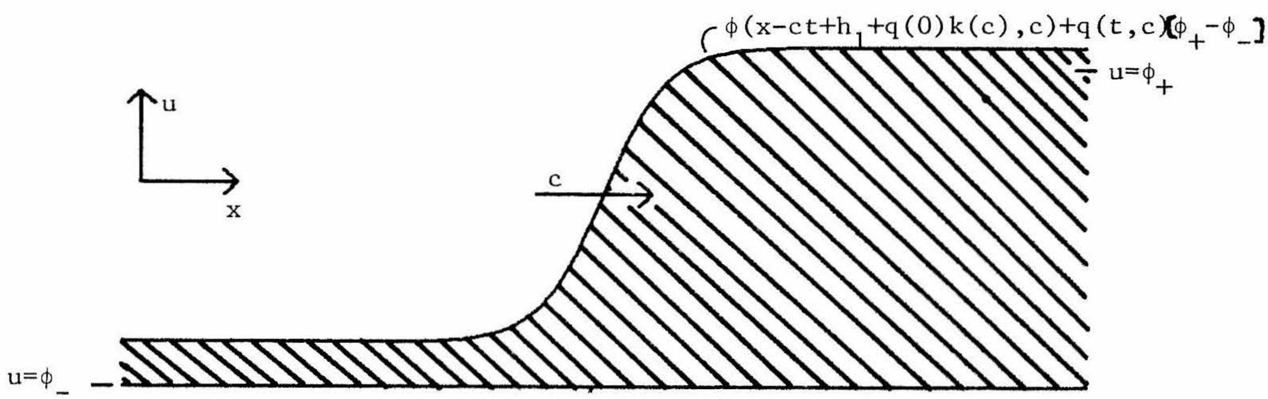


Figure (10): Since $q(t,c)\to 0$ as $t\to+\infty$ and since $u(t,x)$ remains in the shaded region for all $t\geq 0$, $u(t,x)$ cannot travel with wave-speed slower than c.

To prove part (3), we note that when $u(0,x)$ satisfies (2.64) and the assumptions of part (3), then we can bound $u(0,x)$ by

$$\phi(x-h_2,c)-q(0)\cdot[\phi(x-h_2,c) - \phi_-] \leq u(0,x) \leq \phi(x+h_1,c) + q(0)\cdot[\phi(x+h_1,c)-\phi_-] \quad \text{for all } x \, , \tag{2.68}$$

for any $q(0) > q_0[\phi_+-\phi_-]^{-1}$ by taking h_1 and h_2 sufficiently large. For $q(0) > 0$ sufficiently small, relation (2.68) bounds the initial condition $u(0,x)$ by the lower and upper functions $\underline{u}(0,x,c)$ and $\overline{u}(0,x,c)$ (see equation (2.60)). Using the maximum principle and the expressions for $\underline{u}(t,x,c)$ and $\overline{u}(t,x,c)$, we find that

$$\phi(x-ct-h_2-q(0)\kappa(c),c) - q(t,c)\cdot[\phi_+-\phi_-] \leq u(t,x)$$
$$\leq \phi(x-ct+h_1+q(0)\kappa(c),c) + q(t,c)\cdot[\phi_+-\phi_-]$$
$$\text{for all } x \text{ and all } t \geq 0 \ . \qquad (2.69)$$

This is illustrated in figure (11). We conclude that $u(t,x)$ travels with mean wavespeed c and has finite dispersion. The phrase "has finite dispersion" is used here and in the statement of the theorem to mean precisely that the distance between the lower and upper functions which bound $u(t,x)$ in (2.69) is limited to no more than $2q(0)\kappa(c) + h_1+h_2$, which is <u>finite</u>. This is in contrast to part (4), where we are only able to show that the distance between these functions grows no faster than $o(t)$.

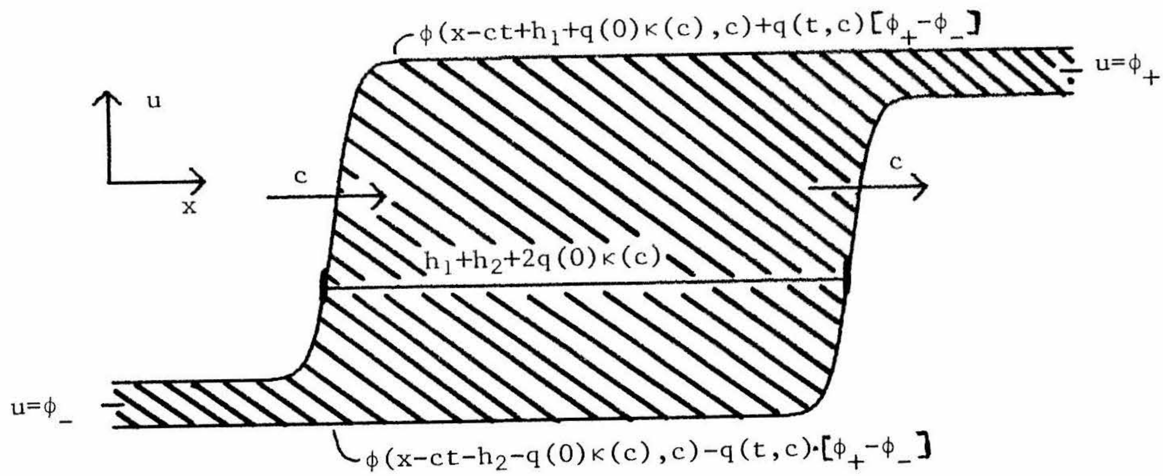


Figure (11): Since $q(t,c)\to 0$ as $t\to\infty$, and since $u(t,x)$ must remain in the shaded region for all $t\geq 0$, $u(t,x)$ must travel with mean wavespeed c. Furthermore, the distance between these two bounding curves is constant.

We will delay the proof of part (4) until Chapter V. There it is shown that when $u(0,x)$ satisfies (2.64) and the assumptions of part (4), then $u(t,x)$ can be bounded by curves which move with asymptotic speed c. Specifically, the midpoints of the leading and trailing bounding curves are given by $x = ct+o(t)$ and $x = ct-o(t)$, as illustrated in figure (12). Thus, the curves move with asymptotic speed c but the distance between them grows in time as $o(t)$. We therefore cannot conclude "finite dispersion" as in part (3).

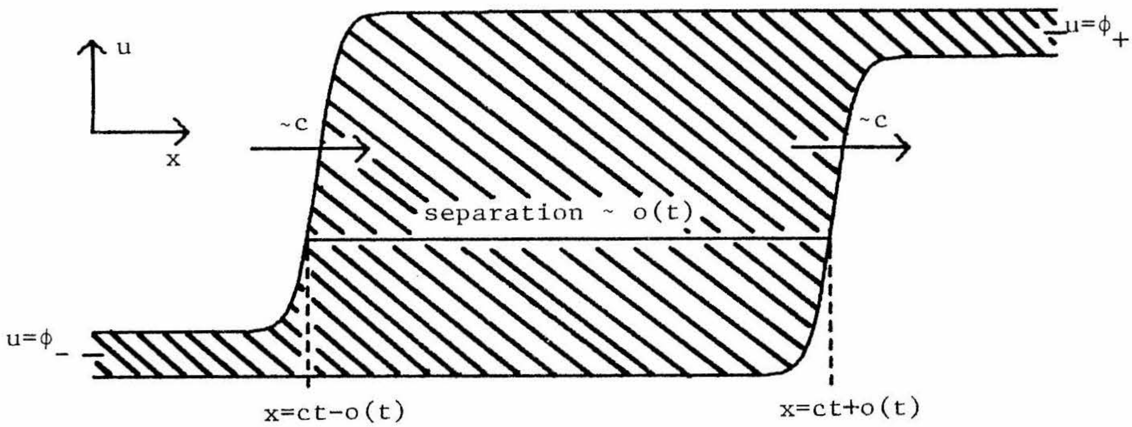


Figure (12): Since the leading and trailing curves move with asymptotic speed c, $u(t,x)$ must also move with mean speed c since it remains in the shaded area for all $t \geq 0$. However, the separation between the curves increases like $o(t)$ as t increases, and so finite dispersion has not been shown.

The establishment of theorem 2.8 $(N \to S)$ completes this presentation of the $N \to S$ case. We now continue to the other cases.

<u>Case III: $S \to N$.</u> Suppose that $u(t,x) \equiv \phi(x,c_0)$ is a bounded monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + c_0 u_x$$

and suppose that $\phi = \phi(-\infty, c_0)$, $v = 0$ is a saddle point and $\phi = \phi(+\infty, c_0)$, $v = 0$ is a node of

$$\phi_x = v$$
$$f(v_x, v, \phi) + cv = 0 \quad .$$

This case is materially the same as the $N \to S$ case already treated. In fact by substituting $-x$ for x we can reduce the $S \to N$ case to the $N \to S$ case. Therefore, we will continue on to the $N \to N$ case.

Case IV: $N \to N$. We now treat the final case. Suppose $u(t,x) \equiv \phi(x, c_0)$ is a bounded monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{2.71}$$

at $c = c_0$, and suppose $\phi = \phi(-\infty, c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty, c_0) \equiv \phi_+$, $v = 0$ are both nodes of

$$\begin{aligned} \phi_x &= v \\ f(v_x, v, \phi) + cv &= 0 \end{aligned} \tag{2.72}$$

at $c = c_0$. We also assume (without loss) that $\phi(x, c_0)$ is increasing. Then the phase-plane looks like the illustration in figure (13) below.

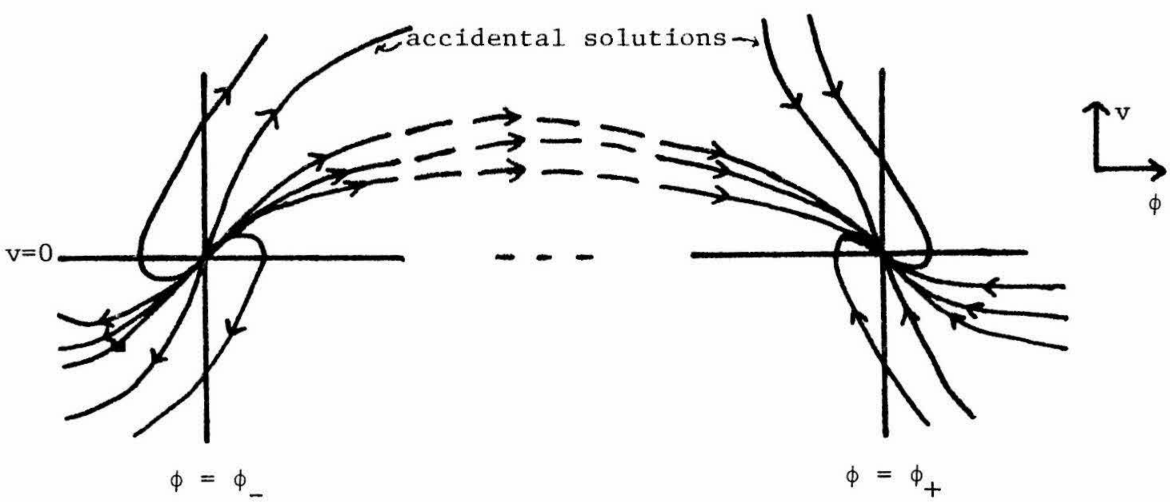


Figure (13)

Consider the solutions $\phi = \tilde{\phi}(x,c_0,v_0)$, $v = \tilde{v}(x,c_0,v_0)$ of system (2.72) defined by the initial conditions

$$\tilde{\phi}(x_0,c_0,v_0) = \phi(x_0,c_0)$$
$$\tilde{v}(x_0,c_0,v_0) = v_0$$

for any fixed finite x_0. Since solutions of differential equations are continuous relative to initial conditions (see e.g. reference [6]), for any x_1 we can make

$$|\tilde{\phi}(x_0+x_1,c_0,v_0) - \phi(x_0+x_1,c_0)| + |\tilde{v}(x_0+x_1,c_0,v_0) - \phi_x(x_0+x_1,c_0)|$$

as small as we wish by taking v_0 sufficiently near $\phi_x(x_0,c_0)$. Since we can take x_1 as large or as small as we like, the attractive nature of the node at $\phi = \phi_-$, $v = 0$ (as $x \to -\infty$) and of the node at $\phi = \phi_+$, $v = 0$ (as $x \to +\infty$) guarantees that

$$\tilde{\phi}(x,c_0,v_0) \to \phi_- \quad \text{as} \quad x \to -\infty$$
$$\tilde{\phi}(x,c_0,v_0) \to \phi_+ \quad \text{as} \quad x \to +\infty$$

for v_0 in $[v_-,v_+]$ for some $v_- < \phi_x(x_0,c_0) < v_+$. Further, there is a $\tilde{v}_-$, $\tilde{v}_+$ ($\tilde{v}_- < \phi_x(x_0,c_0) \leq \tilde{v}_+$) such that $\tilde{\phi}(x,c_0,v_0)$ is monotone (as well as having $\phi(-\infty,c_0,v_0) = \phi_-$ and $\tilde{\phi}(+\infty,c_0,v_0) = \phi_+$) for all v_0 in $(\tilde{v}_-, \tilde{v}_+]$.

This result is clear from the phase plane considerations illustrated in figure (14) below. In particular for any $\tilde{\phi}_- > \phi_-$ near enough to ϕ_-, the phase plane directors point down for all ϕ in $(\phi_-,\tilde{\phi}_-]$. Also the horizontal components of the directors point in the positive direction whenever $v > 0$. This means that any solution $\tilde{\phi}(x,c_0,v_0)$ which crosses the $\phi = \tilde{\phi}_-$ line at a positive point v which is no larger than the crossing point v of the accidental solution (i.e. the solution of (2.72) which decays to ϕ_- at the accidental rate as $x \to -\infty$), then

$\tilde{\phi}(x,c_0,v_0)$ must decrease monotonically to ϕ_- as x decreases to $-\infty$. Similarly, there is a $\tilde{\phi}_+ < \phi_+$ such that if $\tilde{\phi}(x,c_0,v_0)$ crosses the $\phi = \tilde{\phi}$ line at a positive point v under the accidental solution (i.e. the solution of (2.72) which decays to ϕ_+ at the accidental rate as $x \to +\infty$), then $\tilde{\phi}(x_0,c_0,v_0)$ must increase monotonically to ϕ_+ as x increases to $+\infty$. Since $\tilde{v}(x,c_0,v_0)$ can be made arbitrarily near to $\phi_x(x,c_0)$ over any finite interval by taking v_0 near to $\phi_x(x_0,c_0)$, $\tilde{\phi}(x,c_0,v_0)$ must be monotonic for at least a limited range of v_0 about $\phi_x(x_0,c_0)$. This is illustrated in figure (14).

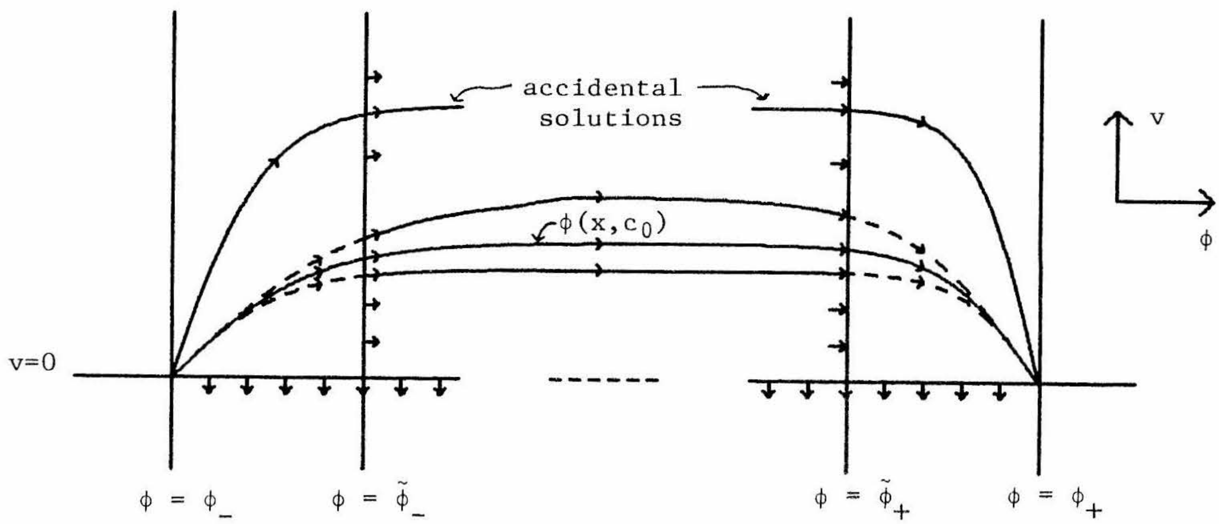


Figure (14)

From the phase plane we can find the extremal monotonic solutions of (2.72) at $c = c_0$. From figure (14) we see that the largest v_0 for which $\tilde{\phi}(x,c_0,v_0)$ is a monotonic solution is the least value $v_0=v_2$ for which $\tilde{\phi}(x,c_0,v_2)$ decays at the accidental rate as either $x \to -\infty$ or $x \to +\infty$. If v_0 is slightly larger, then $\tilde{\phi}(x,c_0,v_0)$ is non-monotonic. Similarly, as v_0 decreases $\tilde{\phi}(x,c_0,v_0)$ remains a monotonic solution until the value of v_0 (which we define to be v_1) for which the phase plane trajectory of $\tilde{\phi}(x,c_0,v_1)$ intersects the $v = 0$ curve between $\phi = \phi_-$ and $\phi = \phi_+$. Since $\tilde{v}(x,c_0,v_1) \geq 0$ for all x and since $\tilde{v}(x,c_0,v_1) = 0$ when $\tilde{\phi}(x,c_0,v_1) = \phi_0$ for some $\phi_0 \in (\phi_-,\phi_+)$, $\phi = \phi_0$, $v = 0$ must be a singular point. From the illustration in figure (15), we see that $\phi = \phi_0$, $v = 0$ is a saddle point (or may be a coalescence of multiple singular points as an accidental case). Thus, when v_0 has decreased to v_1, $\tilde{\phi}(x,c_0,v_0)$ has bifurcated from a monotonic solution with $\tilde{\phi}(-\infty,c_0,v_0) = \phi_-$ and $\tilde{\phi}(+\infty,c_0,v_0) = \phi_+$ into at least two distinct monotonic solutions. Usually as v_0 decreases to v_1, $\tilde{\phi}(x,c_0,v_0)$ becomes two monotonic solutions $\phi_1(x,c_0)$ and $\phi_2(x,c_0)$ with $\phi_1(-\infty,c_0) = \phi_-$, $\phi_1(+\infty,c_0) = \phi_0$, $\phi_2(-\infty,c_0) = \phi_0$, and $\phi_2(+\infty,c_0) = \phi_+$, and with $\phi = \phi_0$, $v = 0$ being a saddle point. Thus the monotonic $N \rightarrow N$ type solution almost always has a $N \rightarrow S$ and a $S \rightarrow N$ type solution as the limiting case, as is illustrated in figure (15).

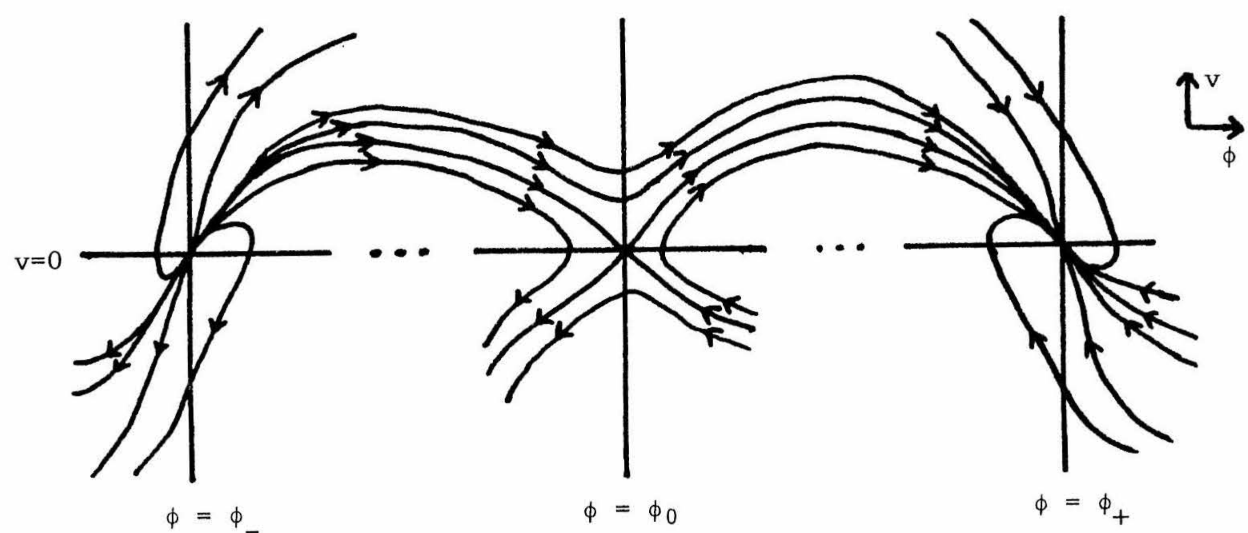


Figure (15)

The other possibilities are ϕ_0 being the coalescence of multiple singular points or the possibility of the limiting case being more than two separate solutions. This latter accidental case is illustrated in figure (16). As illustrated, the intermediate singular points are saddle points or coalesced singular points.

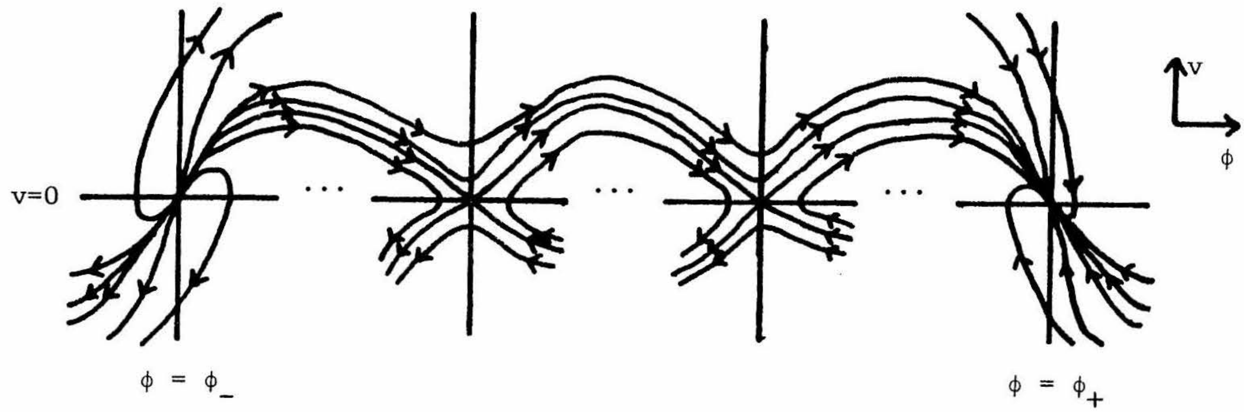


Figure (16)

Thus in brief, if $\phi(x,c_0)$ is a monotonic $N \to N$ type solution of (2.72) at $c = c_0$, then there is a continuously differentiable family of solutions. One limiting member of this family is a solution which decays at the accidental rate as $x \to -\infty$ or $x \to +\infty$. The other limiting member is at least two separate monotone solutions which are usually a $N \to S$ and $S \to N$ pair of waves.

We now consider solutions at wave velocities $c = c_1$ near c_0. Similar to the $N \to S$ case, continuity arguments can be used to show that a monotone solution $\phi(x,c_1)$ exists with $\phi(-\infty,c_1) = \phi_-$ and $\phi(+\infty,c_1) = \phi_+$. Since one monotone solution at $c = c_1$ exists, the previous arguments show that a family of solutions exists at $c = c_1$. One limiting member of this family for $c = c_1$ decays at the accidental rate as $x \to -\infty$ or $x \to +\infty$, and the other limiting member is at least two distinct solutions.

This characterization of the solution family at fixed values of c determines the largest and smallest values of c for which monotone solutions $\phi(x,c)$ (with $\phi(-\infty,c) = \phi_-$ and $\phi(+\infty,c) = \phi_+$) exist. As c increases or decreases from c_0, monotone solutions continue to exist until either (1) an accidentally decaying solution from $\phi = \phi_-$, $v = 0$ or $\phi = \phi_+$, $v = 0$ intersects the $v = 0$ axis at a singular point $\phi = \phi_0$, $v = 0$ for some ϕ_0 in (ϕ_-,ϕ_+), or (2) $\phi = \phi_-$, $v = 0$ or $\phi = \phi_+$, $v = 0$ changes from a node to a spiral point.

We summarize this discussion in the next theorem.

<u>Theorem 2.7 (N → N):</u> Suppose $u(t,x) \equiv \phi(x,c_0)$ is a bounded monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{2.71}$$

at $c = c_0$, and also suppose that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ are both nodes of

$$\begin{aligned} \phi_x &= v \\ f(v_x,v,\phi) + cv &= 0 \end{aligned} \tag{2.72}$$

at $c = c_0$. Then there is an interval (c_1,c_2) such that for each c in (c_1,c_2) there exists a continuously differentiable (in c and α) family of monotonic solutions $u(t,x) = \Phi(x,c,\alpha)$, $0 < \alpha \leq 1$ of (2.71). For $0 < \alpha \leq 1$, $\Phi(-\infty,c,\alpha) = \phi_-$ and $\Phi(+\infty,c,\alpha) = \phi_+$ for c in (c_1,c_2). Moreover, for $\phi(x,c_0)$ increasing (decreasing) the phase plane trajectories of $\Phi(x,c,\alpha)$ are increasing (decreasing) in α. At $\alpha = 1$, $\Phi(x,c,1)$ decays at the accidental rate as either $x \to -\infty$ or $x \to +\infty$. At $\alpha = 0$, the phase plane trajectory corresponds to at least two distinct solutions. Finally, the limiting values c_1 and c_2 of c are either

$$c_1 = c_{min} \equiv 2\sqrt{f_1(0,0,\phi_+)f_3(0,0,\phi_+)} - f_2(0,0,\phi_+)$$
$$c_2 = c_{max} \equiv -2\sqrt{f_1(0,0,\phi_-)\ f_3(0,0,\phi_-)} - f_2(0,0,\phi_-)$$

or (when they exist) the points $c_1 \in (c_{min}, c_0)$ and $c_2 \in (c_0, c_{max})$ nearest to c_0 for which the trajectory of an accidentally decaying solution from $\phi = \phi_-$, $v = 0$ or $\phi = \phi_+$, $v = 0$ intersects the $v = 0$ line at a singular point $\phi = \phi_0$, $v = 0$ with $\phi_0 \in (\phi_-, \phi_+)$.

This theorem shows the sharpness of the stability results obtained in theorem (2.5) of section (2.2) for the $N \to N$ case. For example, suppose that $\phi(x,c_0)$ is a monotonic $N \to N$ type steady state solution of (2.71) at $c = c_0$ which decays at, say, the accidental rate as $x \to -\infty$ and at the usual rate as $x \to +\infty$. From the above theorem, we know that there are solutions $\phi(x,c)$ (continuous in c) which decay to $\phi(-\infty,c_0)$ at the accidental rate as $x \to -\infty$ and decay to $\phi(+\infty,c_0)$ at the usual rate as $x \to +\infty$ for an interval of speeds c including c_0. Now theorem (2.5) says that $u(t,x) \equiv \phi(x,c_0)$ is stable to perturbations which decay like $\phi_x(x,c_0)$ as $x \to -\infty$ and $x \to +\infty$. As in the $N \to S$ case, if the class of perturbations is enlarged to include those which decay at slightly slower exponential rates as $x \to \pm\infty$, then the perturbed initial condition

$$u(0,x) = \phi(x,c_1)$$

would be allowed for c_1 near enough c_0. Since the resulting solution of (2.71) at $c = c_0$ is $u(t,x) = \phi(x-(c_1-c_0)t,c_1)$, $\phi(x,c_0)$ is unstable to this perturbation. Thus the asymptotic decay conditions on the allowed perturbations cannot be significantly weakened in theorem (2.5) for this case. Similarly, theorem (2.5) is sharp for the case of $\phi(x,c_0)$ decaying at the usual rate as $x \to -\infty$ and at the accidental rate as $x \to +\infty$ and for the case of $\phi(x,c_0)$ decaying at the usual rates as $x \to -\infty$ and as $x \to +\infty$.

We now introduce the mean wavespeed/initial condition results for this case. Since this result contains no essentially new ideas, we simply quote it.

Theorem 2.8 $(N \to N)$: Suppose that $u(t,x) = \phi(x-c_0t,c_0)$ is a bounded monotonic solution of

$$u_t = f(u_{xx},u_x,u) \tag{2.73}$$

and that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ are both nodes of the system

$$\phi_x = v$$
$$f(v_x,v,\phi) + cv = 0$$

at $c = c_0$.

Define the exponential rate constants $\lambda^-(c)$ and $\lambda^+(c)$ by

$$\lambda^-(c) \equiv \frac{-(f_2(0,0,\phi_-)+c) - \sqrt{(f_2(0,0,\phi_-)+c)^2 - 4f_1(0,0,\phi_-)\,f_3(0,0,\phi_-)}}{2f_1(0,0,\phi_-)}$$

$$\lambda^+(c) \equiv \frac{-(f_2(0,0,\phi_+)+c) + \sqrt{(f_2(0,0,\phi_+)+c)^2 - 4f_1(0,0,\phi_+)\,f_3(0,0,\phi_+)}}{2f_1(0,0,\phi_+)}$$

for all c in $[c_{min},c_{max}]$, and define c_1 and c_2 as in the previous theorem.

Suppose that $u(t,x)$ is any solution of (2.73) whose initial condition $u(0,x)$ is smooth and satisfies

$$\min\{\phi_-,\phi_+\} < u(0,x) < \max\{\phi_-,\phi_+\} \quad \text{for all } x .$$

Then:

(1) if for any c in (c_1,c_2) there is an $\alpha > 0$ and a $\beta > 0$ such that

$$\alpha < e^{-\lambda^-(c)x} |u(0,x)-\phi_-| \quad \text{for all} \quad x \leq 0 \quad \text{and}$$

$$\beta > e^{-\lambda^+(c)x} |u(0,x)-\phi_+| \quad \text{for all} \quad x \geq 0$$

then $u(t,x)$ cannot travel with mean wavespeed larger than c;

(2) if for any c in (c_1,c_2) there is an $\alpha > 0$ and a $\beta > 0$ such that

$$\alpha > e^{-\lambda^-(c)x} |u(0,x)-\phi_-| \quad \text{for all} \quad x \leq 0 \quad \text{and}$$
$$\beta < e^{-\lambda^+(c)x} |u(0,x)-\phi_+| \quad \text{for all} \quad x \geq 0$$

then $u(t,x)$ cannot travel with mean wavespeed smaller than c;

(3) if for any c in (c_1,c_2) there are positive constants $\alpha,\beta,\gamma,\delta$ such that

$$\alpha < e^{-\lambda^-(c)x} |u(0,x)-\phi_-| < \beta \quad \text{for all} \quad x \leq 0$$
$$\gamma < e^{-\lambda^+(c)x} |u(0,x)-\phi_+| < \delta \quad \text{for all} \quad x \geq 0$$

then $u(t,x)$ travels with mean wavespeed c and has finite dispersion; and

(4) if for any c in (c_1,c_2) we have

$$\lim_{x\to-\infty} e^{-(\lambda^-(c)-\mu)x} |u(0,x)-\phi_-| = 0 \qquad \lim_{x\to-\infty} e^{-(\lambda^-(c)+\mu)x} |u(0,x)-\phi_-| = +\infty$$

$$\lim_{x\to+\infty} e^{-(\lambda^+(c)-\mu)x} |u(0,x)-\phi_+| = +\infty \quad \lim_{x\to+\infty} e^{-(\lambda^+(c)+\mu)x} |u(0,x)-\phi_+| = 0$$

for all $\mu > 0$, then $u(t,x)$ travels with mean wavespeed c (but may not have finite dispersion).

Roughly speaking, the above theorem shows that if $u(0,x)$ decays to ϕ_- like $\alpha e^{\lambda^-(c)x}$ and to ϕ_+ like $\beta e^{\lambda^+(c)x}$ for some c in (c_1,c_2), then $u(t,x)$ must propagate with mean wavespeed c. One naturally wonders how solutions of (2.73) behave when $u(0,x)$ decays to ϕ_- like $\alpha e^{\lambda^-(c_-)x}$ as $x \to -\infty$ and to ϕ_+ like $\beta e^{\lambda^+(c_+)x}$ as $x \to +\infty$, but $c_- \neq c_+$. This question is easily answered when $c_1 < c_- < c_+ < c_2$. We will show that in the general case $u(t,x)$ will evolve into a $N \to S$ type traveling wave of speed c_- (which goes from ϕ_- at $x = -\infty$ to ϕ_0 at $x = +\infty$) and into a $S \to N$ type traveling wave of speed c_+ (which goes from ϕ_0 at $x = -\infty$ to ϕ_+ at $x = +\infty$).

We consider only the common case where the phase plane trajectories corresponding to $\tilde{\phi}(x,c_-,\alpha)$ and $\tilde{\phi}(x,c_+,\alpha)$ at $\alpha = 0$ both intersect the $v = 0$ line at the single saddle point $\phi = \phi_0$, $v = 0$ with $\phi_- < \phi_0 < \phi_+$. This is illustrated in figure (15). Consider the solutions

$$\phi(x,c_-) \equiv \tilde{\phi}(x,c_-,\alpha_-)$$

$$\phi(x,c_+) \equiv \tilde{\phi}(x,c_+,\alpha_+)$$

for any fixed α_- and α_+ in $(0,1)$. Let $\phi^-_{NS}(x,c_-)$ be the monotonic $N \to S$ type solution at $c = c_-$ with

$$\phi^-_{NS}(-\infty,c_-) = \phi_- \qquad \phi^-_{NS}(+\infty,c_-) = \phi_0 \quad .$$

Also let $\phi^-_{SN}(x,c_-)$ be the monotonic $S \to N$ type solution at $c = c_-$ with

$$\phi^-_{SN}(-\infty,c_-) = \phi_0 \qquad \phi^-_{SN}(+\infty,c_-) = \phi_+ \quad .$$

Note that the phase plane trajectories of $\phi^-_{NS}(x,c_-)$ and $\phi^-_{NS}(x,c_-)$ correspond to the limiting trajectory of $\tilde{\phi}(x,c_-,\alpha)$ at $\alpha = 0$. Similarly let $\phi^+_{NS}(x,c_+)$ and $\phi^+_{SN}(x,c_+)$ be the monotonic $N \to S$ and $S \to N$ type solutions at $c = c_+$ with

$$\phi^+_{NS}(-\infty,c_+) = \phi_- \qquad \phi^+_{NS}(+\infty,c_+) = \phi_0$$

$$\phi^+_{SN}(-\infty,c_+) = \phi_0 \qquad \phi^+_{SN}(+\infty,c_+) = \phi_+ \quad .$$

Now suppose that $u(t,x)$ is any solution of (2.73) whose initial condition

$u(0,x)$ is smooth and satisfies

$$\phi_- < u(0,x) < \phi_+ \quad \text{for all } x$$
$$\alpha_1 < e^{-\lambda^-(c_-)x}\,|u(0,x)-\phi_-| < \alpha_2 \quad \text{for all } x \le 0$$
$$\beta_1 < e^{-\lambda^+(c_+)x}\,|u(0,x)-\phi_+| < \beta_2 \quad \text{for all } x \ge 0$$

for some constants $0 < \alpha_1 < \alpha_2$ and $0 < \beta_1 < \beta_2$. By selecting h_1, h_2, h_3, and h_4 sufficiently large, we can guarantee that

$$\phi^-_{NS}(x-h_1,c_-) \le u(0,x) \le \phi^+_{SN}(x+h_4,c_+) \quad \text{and}$$
$$\phi(x-h_2,c_+) \le u(0,x) \le \phi(x+h_3,c_-)$$

hold for all x. Thus the maximum principle implies that for all $t \ge 0$ the solution $u(t,x)$ must satisfy

$$\begin{aligned} &\phi^-_{NS}(x-c_-t-h_1,c_-) \le u(t,x) \le \phi^+_{SN}(x-c_+t+h_4,c_+) \quad \text{and} \\ &\phi(x-c_+t-h_2,c_+) \le u(t,x) \le \phi(x-c_-t+h_3,c_-) \end{aligned} \tag{2.74}$$

for all x. The bounds of (2.74) on $u(t,x)$ are illustrated in figure (17) below for t quite large. The implication of the maximum principle is that $u(t,x)$ must remain in the shaded area for all $t \ge 0$. Clearly the solution $u(t,x)$ has evolved into two stacked waves as claimed.

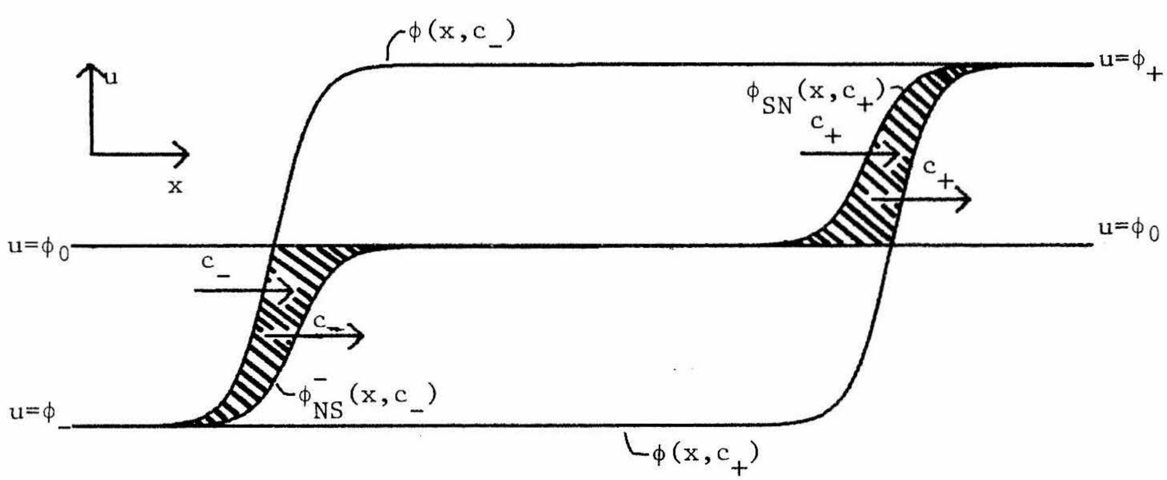


Figure (17)

This completes this presentation of the mean wavespeed/initial condition results. As a brief summary, we found in the $S \to S$ case that there is a single traveling wave $u(t,x) = \phi(x-c_0t)$ at a single fixed wavespeed c_0, in the $N \to S$ and $S \to N$ cases that at each speed c in a range of wavespeeds there is a single traveling wave $u(t,x) = \phi(x-ct,c)$, and in the $N \to N$ case at each speed c in a range of wavespeeds there is a family of solutions $u(t,x) = \phi(x-ct,c,\alpha)$. For each of these cases, the mean wavespeed of a solution $u(t,x)$ is determined mainly by the asymptotic decay rate of $u(0,x)$ as $x \to -\infty$ (if $\phi(-\infty)$ is a node) and as $x \to +\infty$ (if $\phi(+\infty)$ is a node).

This finishes this presentation of our basic results. In the next chapter we will develop the mathematical tools and assumptions needed to rigorously establish these and other results in subsequent chapters.

Chapter III

MATHEMATICAL PRELIMINARIES

In this chapter we develop the mathematical tools needed to rigorously prove the results in subsequent chapters. In this chapter, we will work with a class of equations general enough to contain (as special cases) all sets of equations we will consider later. This class will include parabolic systems of equations which contain multiple dependent variables, multiple independent variables, and even some integral terms.

Specifically, in section (3.1) we introduce some notation we will use to simplify our exposition. In section (3.2) we will modify the equations to prevent infinities from arising and discuss the physical consequences of this modification. Section (3.3) is devoted to developing the necessary preliminary mathematical results. Specifically, these are the maximum principle, the uniformity lemma, and the asymptotic state theorem. In the last section, (3.4), we collect the set of hypotheses we will use in deriving our results. These hypotheses are of three types: smoothness conditions on the equations, parabolicity requirements for the equations, and existence assumptions for solutions of the initial value problem. In section (3.4) we will also briefly discuss how the hypotheses of the section fit together with the mathematical theorems of section (3.3).

3.1 Notation. In this short section we introduce some notation which will simplify our exposition. In this chapter we will work with equations which we can write as

$$u_t^{(k)} = F^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds)$$

$$k = 1,2,\ldots, \nu \quad , \quad (3.1)$$

and in subsequent chapters we will work with special cases of this system of equations. Here, ~ denotes a vector of dependent variables such as $\tilde{u} \equiv (u^{(1)}, u^{(2)}, u^{(3)}, \ldots u^{(\nu)})$, $\rightarrow$ denotes a vector of independent variables such as $\vec{x} \equiv (x_1, x_2, \ldots, x_n)$, Y and T are finite positive constants, and the dimensions ν and n are fixed positive integers. We use the notation that

$$\phi_i^{(k)} \equiv \frac{\partial \phi^{(k)}}{\partial x_i}, \quad \phi_{ij} \equiv \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \quad \text{etc.}$$

Also, whenever <u>dummy</u> indices are used in a function argument (like ij, i, and q but <u>not</u> k in equation (3.1)), we mean that the function depends on the subscripted variable with all possible indices. For example, equation (3.1) can be written in full as

$$u_t^{(k)} = F^{(k)}(u_{11}^{(k)}, u_{12}^{(k)}, \ldots, u_{1n}^{(k)}, u_{21}^{(k)}, \ldots, u_{nn}^{(k)}, u_1^{(k)}, \ldots, u_n^{(k)}, \quad \text{etc.}).$$

Besides the above notation, for the derivatives of

$$F^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s, \vec{y}, \tilde{u}(t-s, \vec{x}-\vec{y}))d\vec{y}ds)$$

and of

$$G^{(q)}(s, \vec{y}, \tilde{u}(t-s, \vec{x}-\vec{y}))d\vec{y}ds)$$

we will use the following expressions:

$$F_{1mn}^{(k)} \equiv \frac{\partial F^{(k)}}{\partial u_{mn}^{(k)}}, \quad F_{2m}^{(k)} \equiv \frac{\partial F^{(k)}}{\partial u_m^{(k)}}, \quad F_{3m}^{(k)} \equiv \frac{\partial F^{(k)}}{\partial u^{(m)}},$$

$$F_{4m}^{(k)} \equiv \frac{\partial F^{(k)}}{\partial \left[\int_0^T \int_{||\vec{y}||<Y} G^{(m)}(s, \vec{y}, \tilde{u}(t-s, \vec{x}-\vec{y}))d\vec{y}ds\right]}$$

$$G_{3m}^{(q)} \equiv \frac{\partial G^{(q)}}{\partial u^{(m)}} \quad .$$

For brevity, we will often use notation like

$$F^{(k)}([\tilde{u}]) \equiv F^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s, \vec{y}, \tilde{u}(t-s, \vec{x}-\vec{y}))d\vec{y}ds)$$

for functions like $F^{(k)}$ when their arguments are obvious. This is most of the notation we will use. The rest of the notation will be introduced as needed, and we now continue on to the modification of the given equations, equations (3.1).

3.2 Modification of the original equations. Basically, in this section we modify the given equations (3.1) so that infinities in $u^{(k)}, u_i^{(k)}$, and $u_{ij}^{(k)}$ can never develop as time progresses. We will first discuss the reasons for modifying the equations and the mathematical effects of the modification. We then present the actual modification of the equations.

It may be possible that, for solutions $\tilde{u}(t,\vec{x})$ of equations (3.1), infinities in $u^{(k)}, u_i^{(k)}$, or $u_{ij}^{(k)}$ may arise even for very nice initial conditions. For our purposes, infinities in the $u^{(k)}$ are of no concern. For example, the stability of a steady state $\phi(\vec{x})$ to the perturbation $\tilde{u}(0,\vec{x}) - \phi(\vec{x})$ is decided long before one of the $u^{(k)}$ becomes infinite. However, the possibility that the $u^{(k)}$ remain finite and some of the $u_i^{(k)}$ or $u_{ij}^{(k)}$ may become infinite for some equations, presents us with mathematical problems. The first of these problems is how to continue the solution $\tilde{u}(t,\vec{x})$ to times after the infinity occurs. The second is that all our derivations will require the maximum principle, and to prove the maximum principle we must require that for any $T_0 > 0$, $F_{1ij}^{(k)}([\tilde{u}])$, $F_{2i}^{(k)}([\tilde{u}])$, $F_{3i}^{(k)}([\tilde{u}])$, $F_{4i}^{(k)}([\tilde{u}])$, and $G_{3i}^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))$ are bounded for all x, all $t \in [0,T_0]$, all $\vec{y}$ with $||\vec{y}|| < Y$, and all $s \in [0,T]$. The last problem is that in deriving our instability results, we will need the result that if a solution $\tilde{u}(t,\vec{x})$ of (3.1) increases (decreases) to a bounded function $\phi(\vec{x})$ as $t \to \infty$, then $\phi(\vec{x})$ is a steady state solution of (3.1).

These three problems are strictly mathematical in origin. Physically, dependent variables and their derivatives attain <u>absolute</u> infinity only in rare circumstances, and so physical continuation is assured. Also, physically the derivatives of $F^{(k)}$ and $G^{(q)}$ should be bounded, since almost all physical systems will eventually reach saturation levels as $|\tilde{u}|$, $|\tilde{u}_i|$, and $|\tilde{u}_{ij}|$ are increased. The last problem is also mathematical in origin, since one expects that the only time-independent states which physical systems can evolve into are steady states. Thus, if for some reasonable initial conditions some equations of the form (3.1) have solutions $\tilde{u}$, $\tilde{u}_i$, and $\tilde{u}_{ij}$ which develop infinities, then the mathematical formulation is inadequate. In this regard, note that long before an infinity is reached, other terms (representing excluded physical effects) should be included in any equation modeling a physical system.

The most satisfactory way to resolve these problems would be by proving for all equations of interest (of the form (3.1)), that $\tilde{u}$, $\tilde{u}_i$, and $\tilde{u}_{ij}$ are bounded uniformly for all time when $\tilde{u}$ has reasonable initial conditions. This would immediately eliminate the first and (with smooth functions $F^{(k)}$) second problems. The steady state result needed to resolve the last problem is also an easy consequence of the uniform boundedness of $\tilde{u}_i$ and $\tilde{u}_{ij}$. An alternative way of resolving these mathematical problems is to <u>assume</u> the needed boundedness results for the equations of interest. I.e., we could assume that when $\tilde{u}$ is bounded uniformly in time, then $\tilde{u}_i$ and $\tilde{u}_{ij}$ are also - at least when reasonable initial conditions are used. This assumption is sufficient for our needs. It is not an unreasonable assumption since when $\sum_{ij} F^{(k)}_{1ij}\, \xi_i\, \xi_j > 0$ for all arguments of $F^{(k)}$, all k, and all $\vec{\xi} \neq \vec{0}$ (which we will assume later), then the equations have positive diffusion which tends to smooth out the solutions.

We will not use either of the above alternatives. Instead we will use the technical device of modifying the original equations. The modified equations depend on an arbitrary fixed positive constant M, and can be written as

$$u_t^{(k)} = F_M^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G_M^{(q)}(s, \vec{y}, \tilde{u}(t-s, \vec{x}-\vec{y}))d\vec{y}ds) \ , \ k=1,\ldots,\nu, \tag{3.2}$$

where $F_M^{(k)}$ and $G_M^{(q)}$ are modified functions similar to $F^{(k)}$ and $G^{(q)}$. These equations are modified so that

(1) the modified equations are identical to the original whenever $|u^{(k)}| < M, |u_i^{(k)}| < M$ and $|u_{ij}^{(k)}| < M$ for all i,j,k for any prechosen arbitrarily large constant $M > 0$,

(2) for some $\tilde{M}(M) > M$, whenever $|u^{(k)}| > \tilde{M}$, $|u_i^{(k)}| > \tilde{M}$, or $|u_{ij}^{(k)}| > \tilde{M}$ for any i,j, then equation k becomes a heat equation with constant coeficients:

$$u_t^{(k)} = \alpha \sum_i u_{ii}^{(k)} \qquad 0 < \alpha < \infty \ ,$$

(3) the transitions from the original equations to the heat equations are smooth and all useful properties of the original equations are retained. These properties are:

(3a) All the modified functions $F_M^{(k)}$ and $G_M^{(q)}$ retain all the smoothness properties of the original functions $F^{(k)}$ and $G^{(q)}$ for all k,q and any $M > 0$;

(3b) If originally $\sum_{ij} F_{1ij}^{(k)} \xi_i \xi_j > 0$ for all arguments and all $\vec{\xi} \neq \vec{0}$, then the modified equations have the same property for any $M > 0$;

(3c) If originally $F_{3\ell}^{(k)} \geq 0$ (≤ 0) for all arguments, then the modified equations have the same property for any $M > 0$;

(3d) If originally $F_{4i}^{(k)} \geq 0$ (≤ 0) for all arguments, then the same is true for the modified equations for any $M > 0$; and

(3e) If $G_{3i}^{(q)} \geq 0$ (≤ 0) for all arguments, then the same is true for the modified equations for any $M > 0$.

The modification of the equations to the heat equation is certainly somewhat artificial and arbitrary. However, all subsequent results will hold for <u>all</u> M sufficiently large. As M approaches infinity, the modified system becomes a very good approximation to the original system of equations. Note that for all solutions $\tilde{u}(t,\vec{x})$ of the original system of equations which have $\tilde{u}$, $\tilde{u}_i$, and $\tilde{u}_{ij}$ bounded for all $t \geq 0$ and all $\vec{x}$, the modification is irrelevant. Thus all our results about the modified equations will be directly applicable to these solutions. Moreover, our results about the modified equations are directly applicable to all solutions $\tilde{u}(t,\vec{x})$ of the original system of equations for all t until $\tilde{u}$, $\tilde{u}_i$, or $\tilde{u}_{ij}$ becomes unbounded. Thus, modification of the equations is a superior alternative to proving bounds on $\tilde{u}$, $\tilde{u}_i$, and $\tilde{u}_{ij}$ in the limited sense that whenever bounds can be proven for solutions of the original equations, the modified equations reduce to these equations for M sufficiently large.

We now place mild smoothness conditions on the original equations

$$u_t^{(k)} = F^{(k)}\left(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))\,d\vec{y}\,ds\right) \quad k=1,\ldots,\nu \, . \tag{3.1}$$

Specifically, we now assume that

H1: For all q, all $j = 1,\ldots,\nu$, all $s \in [0,T]$, and all $||\vec{y}|| \leq Y$, $G_{3_j}^{(q)}$ exists and is continuous in all arguments; and

H2: For all i,j,k, and ℓ, $F^{(k)}_{1ij}$, $F^{(k)}_{2i}$, $F^{(k)}_{3i}$, and $F^{(k)}_{4\ell}$ exist and are continuous in all arguments.

Without losing the important properties (3a)-(3e) (when they occur in the original equation), the modified equations also have the properties:

(4) for any continuous function ϕ and any continuous and bounded function Ψ,

$$\int_0^T \int_{||\vec{y}||<Y} G^{(q)}_M(s,\vec{y},\phi(t-s,\vec{x}-\vec{y}))d\vec{y}ds \quad \text{and}$$

$$\int_0^T \int_{||\vec{y}||<Y} G^{(q)}_{M,3i}(s,\vec{y},\phi(t-s,\vec{x}-\vec{y})) \cdot \Psi(t-s,\vec{x}-\vec{y})d\vec{y}ds$$

are both bounded independently of $t > 0$, $\vec{x}$, and ϕ;

(5) the derivatives $F^{(k)}_{M,1ij}$, $F^{(k)}_{M,2i}$, $F^{(k)}_{M,3i}$, and $F^{(k)}_{M,4q}$ are all bounded uniformly in all arguments;

(6) if $u^{(k)}$, $u^{(k)}_i$, $u^{(k)}_{ij}$ are all bounded at each time $t > 0$, then $u^{(k)}$ being bounded for $t \le 0$ implies that $u^{(k)}$, $u^{(k)}_i$, and $u^{(k)}_{ij}$ are all bounded <u>uniformly</u> for $(\vec{x},t) \in \mathbb{R}^n \times [0,\infty)$; and finally

(7) if $\sum_{ij} F^{(k)}_{1ij}\, \xi_i\, \xi_j > 0$ for all arguments and all $\vec{\xi} \neq \vec{0}$, then for any $M > 0$ there is a $\delta_M > 0$ such that

$$\sum_{ij} F^{(k)}_{M,1ij}\, \xi_i\, \xi_j \ge \delta_M > 0$$

for all arguments of $F^{(k)}_M$ and all $\vec{\xi}$ with $\sum_i \xi_i\, \xi_i = 1$.

Of these preceding properties, properties (1), (2), (3), (4), (5), and (7) will follow from inspection of the modified equations. Property (6) is the conclusion of the uniformity lemma, which is stated and proved in section (3.3).

We now present the actual modified equations. Note that any modification of the equations (3.1) which satisfies conditions (1) through

(7) is as good (for our purposes) as the particular modification we use. The virtue of the particular modification we use is that it works for all equations of the form (3.1).

We first define some needed auxillary functions. Let M and $\tilde{M}$ be any constants with $0 < M < \tilde{M} < +\infty$. Define $H_M^{\tilde{M}}(x)$ by

$$H_M^{\tilde{M}}(x) \equiv \begin{cases} 0 & |x| \geq \tilde{M} \\ 1 - \dfrac{k}{\tilde{M}-M} \displaystyle\int_{M^+}^{x} \exp\{-\dfrac{\tilde{M}-M}{(y-M)(\tilde{M}-y)}\}dy & M < x < \tilde{M} \\ 1 & |x| \leq M \\ H_M^{\tilde{M}}(-x) & x \leq -M \end{cases}$$

where $k \equiv \left[\int_0^1 \exp\{-\frac{1}{y(1-y)}\}\, dy\right]^{-1}$. For brevity, let $H_M(x)$ represent $H_M^{2M}(x)$. Note that $H_M^{\tilde{M}}(x)$ is an even, non-negative function in C^∞ with $H_M^{\tilde{M}}(x) \equiv 1$ for $|x| \leq \tilde{M}$, with $H_M^{\tilde{M}}(x) \equiv 0$ for $|x| \geq \tilde{M}$, and with $0 \geq \operatorname{sgn}(x) \frac{d}{dx} H_M^{\tilde{M}}(x) \geq -\frac{4}{\tilde{M}-M}$. This function is illustrated in figure (1) below.

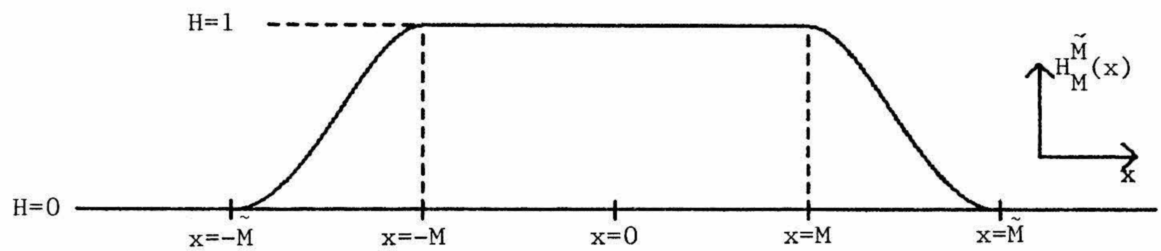


Figure (1)

By an overhead bar $(\overline{\qquad}^{M})$ we denote a quantity which "saturates" in the interval $(M,2M)$. Specifically,

$$\overline{\chi}^M \equiv \int_0^{\chi} H_M^{2M}(\phi)\, d\phi \quad \text{and}$$

$$\overline{\vec{\chi}}^M \equiv (\overline{\chi^{(1)}}^M, \ldots, \overline{\chi^{(\nu)}}^M) \quad .$$

The quantity $\overline{\chi}^M$ as a function of χ is illustrated in figure (2) below.

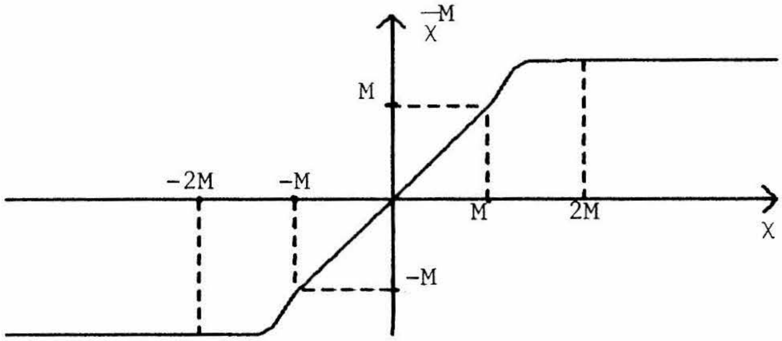


Figure (2)

We also need a way to "switch-on" quantities over intervals of interest. Thus, define $\mathcal{H}_M^{\tilde{M}}(x)$ as an even, non-negative, element of C^∞ such that

$$\mathcal{H}_M^{\tilde{M}}(x) \equiv \begin{Bmatrix} 0 & |x| \le M \\ 1 & |x| \ge \tilde{M} \end{Bmatrix} ,$$

$\int_M^{\tilde{M}} \mathcal{H}_M^{\tilde{M}}(x)\,dx = \tilde{M}$, and $0 \le \mathcal{H}_M^{\tilde{M}}(x) \le \frac{4\tilde{M}}{\tilde{M}-M}$, where $\tilde{M} > M$. Thus a quantity

$$\underline{\chi}_M \equiv \int_0^\chi \mathcal{H}_M^{2M}(\phi)\,d\phi$$

switches smoothly from being identically zero when $|\chi| < M$ to being χ when $|\chi| \ge 2M$.

Finally, for $0 < M < \tilde{M} < N < \tilde{N}$, we define the function $\mathcal{H}_{M,N}^{\tilde{M},\tilde{N}}(x)$ as an even element in C^∞ with

$$\mathcal{H}_{M,N}^{\tilde{M},\tilde{N}}(x) \equiv \begin{Bmatrix} \mathcal{H}_M^{\tilde{M}}(x) & \text{for} & |x| \le N \\ 0 & \text{for} & |x| \ge \tilde{N} \end{Bmatrix} ,$$

with $\int_0^{\tilde{N}} \mathcal{H}_{M,N}^{\tilde{M},\tilde{N}}(x)\,dx = 0$ and with $1 \ge \mathcal{H}_{M,N}^{\tilde{M},\tilde{N}}(x) \ge -\frac{4N}{\tilde{N}-N}$ for $N \le |x| \le \tilde{N}$.
We will not take the time to actually construct such $\mathcal{H}_M^{\tilde{M}}(x)$ and $\mathcal{H}_{M,N}^{\tilde{M},\tilde{N}}(x)$

since we are certain they exist (since we can draw them).

We now give the modified equations. Recall that we assume that the smoothness conditions H1 and H2 are satisfied. Let $M > 0$ be given. Then let

$$F_M^{\max} \equiv \sup_{\substack{0 \le |u_{ij}^{(k)}| \le 24M \\ 0 \le |u_i^{(k)}| \le 2M \\ 0 \le |u^{(\ell)}| \le 2M \\ \text{all } i,j,k,\ell}} \{F^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds)\} \quad ,$$

let $C \equiv \frac{1}{2M} F_M^{\max}$, let $N \equiv 48M\{(4n+8)(8n+1)+1\}$, and let $P = 2n(n+1)\times(n+2)(8n+1)N+N$, where n is the number of independent spatial variables: $\vec{x} = (x\ ,\dots,x_n)$. We define

$$G_M^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y})) \equiv G^{(q)}(s,\vec{y},\vec{\tilde{u}}^M(t-s,\vec{x}-\vec{y})) \quad ,$$

and the modified equations are

$$\begin{aligned}
u_t^{(k)} &= F_M^{(k)}([\tilde{u}]) \qquad k = 1,2,\dots,\nu \qquad\qquad (3.2)\\
&\equiv F_M^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G_M^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds) \quad k = 1,\dots,\nu \\
&\equiv H_M(u^{(k)})\Big[\prod_i H_M(u_i^{(k)})\Big]\Big[\prod_i H_{12M}(u_{ii}^{(k)})\Big]\Big[\prod_{i>j} H_{12M}(2u_{ii}^{(k)}+2u_{jj}^{(k)}+u_{ij}^{(k)}+u_{ji}^{(k)})\Big] \\
&\quad \times F^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \vec{\tilde{u}}^M, \int_0^T \int_{||\vec{y}||<Y} G_M^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds) \\
&\quad + C\cdot\left\{\sum_i \int_0^{u_{ii}^{(k)}} H_{M,48M}^{2M,N}(\phi)d\phi + \sum_{i>j}\int_0^{2u_{ii}^{(k)}+2u_{jj}^{(k)}+u_{ij}^{(k)}+u_{ji}^{(k)}} H_{6M,48M}^{12M,N}(\phi)d\phi\right\} \\
&\quad \times \Big[\prod_i H_N^P(u_{ii}^{(k)})\Big]\Big[\prod_{i>j} H_N^P(2u_{ii}^{(k)}+2u_{jj}^{(k)}+u_{ij}^{(k)}+u_{ji}^{(k)})\Big]\cdot\Big[H_M(u^k)\Big]\cdot\Big[\prod_i H_M(u_i^k)\Big] \\
&\quad + \frac{C}{8n+\frac{1}{2}}\Big\{1-H_M(u^{(k)})\Big[\prod_i H_M(u_i^{(k)})\Big]\Big[\prod_i H_{24M}u_{ii}^{(k)}\Big] \\
&\quad \times\Big[\prod_{i>j} H_{24M}(2u_{ii}^{(k)}+2u_{jj}^{(k)}+u_{ij}^{(k)}+u_{ji}^{(k)})\Big]\Big\}\cdot\Big(\sum_i u_{ii}^{(k)}\Big) \quad k = 1,\dots,\nu \ .
\end{aligned}$$

These are the modified equations we will use. They are exceedingly complicated, and for any specific system of equations (of the form (3.1)) much simpler modifications can be constructed which will also satisfy conditions (1) through (7). We will only use the facts that the modified equations can be written as

$$u_t^{(k)} = F_M^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G_M^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds) \quad k=1,\ldots,\nu \tag{3.2}$$

and satisfy properties (1), (2), (3), (4), (5), and (7) for all $M > 0$. In the next section, we will show that solutions of the modified equations satisfy the maximum principle, that the solutions satisfy property (6) (the uniformity lemma), and that the only time independent states which a solution can evolve into are steady state solutions of (3.2).

3.3 The maximum principle, uniformity lemma, and asymptotic state theorem.

In this section we will develop some general mathematical results about systems of the form

$$u_t^{(k)} = F_M^{(k)}([\tilde{u}]) \qquad k = 1,\ldots,\nu \quad . \tag{3.2}$$

These results will be basic to all subsequent derivations.

We will first state and prove the maximum principle, since this will be the primary tool needed in subsequent chapters. We will then prove the uniformity lemma, which essentially states that for a solution $\tilde{u}$ of (3.2), if $\tilde{u}$, $\tilde{u}_i$, and $\tilde{u}_{ij}$ are bounded for all x at each $t > 0$, then $\tilde{u}$, $\tilde{u}_i$, and $\tilde{u}_{ij}$ are bounded uniformly for all x and all $t > 0$. Thus, this lemma can be used to extend existence results stating that $\tilde{u}$, $\tilde{u}_i$, and $\tilde{u}_{ij}$ are bounded for $0 \le t \le T_0$ for any T_0, to uniform bounds. The last result we will prove in this section is the asymptotic state theorem. This theorem shows that when any solution $\tilde{u}$ of (3.2) with $\tilde{u}_t$ continuous and

$\tilde{u}, \tilde{u}_1$, and $\tilde{u}_{11}$ uniformly bounded, has $\tilde{u}(t,\vec{x}) \to \tilde{\tau}(\vec{x})$ monotonically as $t \to \infty$, then $\tilde{\tau}(\vec{x})$ is a steady state solution of (3.2) whenever there is only a single spatial dimension.

We now state and prove the maximum principle. In order to do so concisely, we introduce the function spaces C_x^i and C_t^j. The first class is defined as all functions whose i-th order spatial derivatives are all continuous. The second class is similarly defined as all functions whose j-th order time derivatives are continuous.

The maximum principle we will state is very similar to the maximum principle in section (2.1). However, there are two major differences. The first is that the new maximum principle is for the more general equations (3.2). The other is that the new maximum principle will hold for arbitrary spatial domains Ω in $\mathbb{R}^n$.

Theorem 3.1 (The maximum principle): Let Ω be any domain (open connected set) in $\mathbb{R}^n$, let $\bar{\Omega}$ be its closure, and let $T_0 > 0$ be any fixed constant. Suppose that:

A1 The equations (3.1) satisfy the smoothness conditions H1 and H2;

A2 The equations (3.1) form a parabolic system - that is, for all arguments

$$\sum_{ij} F_{1ij}^{(k)} \xi_i \xi_j > 0 \quad \text{for all } k, \text{ all } \vec{\xi} \neq 0 ,$$

$$F_{3\ell}^{(k)} \geq 0 \quad \text{for all } \ell \neq k, \text{ all } k, \text{ and}$$

$$F_{4q}^{(k)} G_{3i}^{(q)} \geq 0 \quad \text{for all } k, q, \text{ and } i$$

are satisfied;

A3 The functions $\tilde{u}(t,\vec{x})$ and $\tilde{v}(t,\vec{x})$ are in $C_x^2 \cap C_t^1$ for all $(t,\vec{x})$ in

$(0,T_0] \times \Omega$, are in C^0 for all $(t,\vec{x})$, and are bounded uniformly for all $(t,\vec{x})$ in $[-T,T_0] \times \bar{\Omega}$.

Then if

$$u_t^{(k)} - F_M^{(k)}([\tilde{u}]) \geq v_t^{(k)} - F_M^{(k)}([\tilde{v}]) \quad \text{for all } k, \text{ all } \vec{x} \text{ in } \Omega, \text{ and all} \tag{3.3}$$

t in $(0,T_0]$ for any fixed $M > 0$, and if

$$u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x}) \quad \text{for all } k, \text{ all } (t,\vec{x}) \text{ in } (-\infty,0] \times \mathbb{R}^n \text{ and for all} \tag{3.4}$$

$(t,\vec{x})$ in $[0,T_0] \times (\mathbb{R}^n - \Omega)$, then

$$u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x}) \quad \text{for all } k \text{ and for all } (t,\vec{x}) \text{ in } [0,T_0] \times \bar{\Omega} \tag{3.5}$$

as well.

Thus, in rough terms this theorem states that when $\tilde{u}$ and $\tilde{v}$ satisfy the differential inequalities in (3.3) in some domain Ω for $t > 0$, when $u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x})$ for $\vec{x}$ in $\mathbb{R}^n$ and $t \leq 0$, and when $u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x})$ for $t > 0$ and $\vec{x}$ outside of Ω, then $u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x})$ in Ω for all $t \geq 0$ as well. Note that in a mathematical sense we have assumed too much by requiring $u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x})$ for all $t \geq 0$ and $\vec{x} \notin \Omega$ and for all $t \leq 0$ and $\vec{x} \in \mathbb{R}^n$ Clearly the values of $\tilde{u}(t,\vec{x})$ and $\tilde{v}(t,\vec{x})$ are irrelevant when $t < -T$ and when $||\vec{x}-\vec{x}_0||>Y$ for all $\vec{x}_0$ in Ω, since the integral terms of inequality (3.3) are

$$\int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds \quad \text{and}$$

$$\int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s,\vec{y},\tilde{v}(t-s,\vec{x}-\vec{y}))d\vec{y}ds \quad .$$

However, these extra assumptions make the exposition easier and do not affect the results in subsequent chapters in any way.

We now prove the maximum principles by using extensions of the material in Chapter III of reference [1].

<u>Proof</u>: The proof of this theorem is very similar to the proof of the previous maximum principle in section (2.1). As before, we will prove the theorem by defining $\tilde{h} \equiv \tilde{u} - \tilde{v}$, using the mean value theorem to convert the nonlinear system of inequalities (3.3) into a "linear" system of inequalities in $\tilde{h}$, and finally showing that the linear inequalities imply that $h^{(k)} \geq 0$ for each k.

Define $\tilde{h} \equiv \tilde{u} - \tilde{v}$, and the function of Θ

$$H^{(k)}([\tilde{v}],[\tilde{h}],\Theta) \equiv F_M^{(k)}([\tilde{v}+\Theta\tilde{h}]) - F_M^{(k)}([\tilde{v}]) \qquad k = 1,\ldots, \nu \ .$$

Note that (3.3) can be written as

$$h_t^{(k)} \geq H^{(k)}([\tilde{v}],[\tilde{h}],1) - H^{(k)}([\tilde{v}],[\tilde{h}],\ 0) \qquad k = 1,\ldots, \nu \ . \quad (3.6)$$

The derivative of $H^{(k)}$ is

$$\frac{\partial}{\partial\Theta} H^{(k)}([\tilde{v}],[\tilde{h}],\Theta) = \sum_{ij} F_{M,1ij}^{(k)} h_{ij}^{(k)} + \sum_{i} F_{M,2i}^{(k)} h_i^{(k)} + \sum_{\ell} F_{M,3\ell}^{(k)} h^{(\ell)}$$

$$+ \sum_{q,\ell} F_{M,4q}^{(k)} \int_0^T \int_{||\vec{y}||<Y} G_{M,3\ell}^{(q)}(s,\vec{y},\tilde{v}(t-s,\vec{x}-\vec{y})$$

$$+ \Theta\tilde{h}(t-s,\vec{x}-\vec{y})) \cdot h^{\ell}(t-s,\vec{x}-\vec{y})d\vec{y}ds \quad ,$$

where the arguments of the $F_{M,1ij}^{(k)}$, $F_{M,2i}^{(k)}$, $F_{M,3\ell}^{(k)}$, and $F_{M,4q}^{(k)}$ are

$$v_{ij}^{(k)} + \Theta h_{ij}^{(k)},\ v_i^{(k)} + \Theta h_i^{(k)},\ \tilde{v} + \Theta\tilde{h},$$

$$\int_0^T \int_{||\vec{y}||<Y} G_M^{(q)}(s,\vec{y},\tilde{v}(t-s,\vec{x}-\vec{y})+\Theta\tilde{h}(t-s,\vec{x}-\vec{y}))d\vec{y}ds \ . \qquad (3.7)$$

From the mean value theorem and (3.6), we conclude that for each k, t, and $\vec{x}$ there is $\Theta(k,t,\vec{x})$ in $[0,1]$ such that $h_t^{(k)} \geq \frac{\partial}{\partial\Theta} H^{(k)}([\tilde{v}],[\tilde{h}],\Theta)|_{\Theta=\Theta(k,t,\vec{x})}$. That is,

$$h_t^{(k)} \geq \sum_{ij} F_{M,1ij}^{(k)} h_{ij}^{(k)} + \sum_i F_{M,2i}^{(k)} h_i^{(k)} + \sum_\ell F_{M,3\ell}^{(k)} h^{(\ell)}$$
$$+ \sum_{q,\ell} F_{M,4q}^{(k)} \int_0^T \int_{||\vec{y}||<Y} G_{M,3\ell}^{(q)}(s,\vec{y},\tilde{v}(t-s,\vec{x}-\vec{y}) + \Theta(k,t,\vec{x})\ \tilde{h}(t-s,\vec{x}-\vec{y}))$$
$$\times\ h^{(\ell)}(t-s,\vec{x}-\vec{y})d\vec{y}ds \qquad k = 1,\ldots\nu \qquad (3.8)$$

where the arguments of $F_{M,1ij}^{(k)}$, of $F_{M,2i}^{(k)}$, of $F_{M,3i}^{(k)}$, and of $F_{M,4q}^{(k)}$ are given by (3.7) with $\Theta = \Theta(k,t,\vec{x})$. From our assumptions, $F_{M,1ij}^{(k)}$, $F_{M,2i}^{(k)}$, $F_{M,3i}^{(k)}$, $F_{M,4q}^{(k)}$, and $G_{M,3\ell}^{(q)}$ are uniformly bounded. Moreover $\sum_{ij} F_{M,1ij}^{(k)} \xi_i \xi_j > 0$ for all $\vec{\xi} \neq \vec{0}$, $F_{M,3\ell}^{(k)} \geq 0$ for all $\ell \neq k$, and $F_{M,4q}^{(k)}\, G_{M,3\ell}^{(q)} \geq 0$ for all q and ℓ.

We will prove the maximum principle by showing that each $h^{(k)} \geq 0$ whenever the inequalities

$$h_t^{(k)} \geq \sum_{ij} \alpha_{ij}^{(k)} h_{ij}^{(k)} + \sum_i \beta_i^{(k)} h_i^{(k)} + \sum_\ell \gamma_\ell^{(k)} h^{(\ell)}$$
$$+ \sum_{q,\ell} \int_0^T \int_{||\vec{y}||<Y} g_\ell^{(q)}(k,t,\vec{x},s,\vec{y})\ h^{(\ell)}(t-s,\vec{x}-\vec{y})d\vec{y}ds \qquad k = 1,\ldots,\nu \qquad (3.9)$$

are satisfied. Here $\alpha_{ij}^{(k)}$, $\beta_i^{(k)}$, $\gamma_\ell^{(k)}$ are any functions of $(t,\vec{x})$ which are uniformly bounded for $(t,\vec{x})$ in $[0,T_0] \times \bar{\Omega}$, $g_\ell^{(q)}$ is any function which is uniformly bounded for $(t,\vec{x}) \in [0,T_0] \times \bar{\Omega}$, $0 \leq s \leq T_0$, and for $||\vec{y}|| \leq Y$. These functions are also required to satisfy $\sum_{ij} \alpha_{ij}^{(k)}\ \xi_i\ \xi_j > 0$ for all $\vec{\xi} \neq \vec{0}$, $\gamma_\ell^{(k)} \geq 0$ for all $\ell \neq k$, and $g_\ell^{(q)} \geq 0$. Showing all $h^{(k)} \geq 0$ will immediately establish the maximum principle because (3.8) is a special case of (3.9).

Let $r \equiv ||\vec{x}|| = \sqrt{\sum_i x_i x_i}$, and define

$$\tilde{w} = \tilde{h}e^{-\eta t} \text{ sech } pr \qquad (3.10)$$

where p is any fixed positive constant and $\eta > 0$ will be selected later. From (3.10) we find

$$h^{(k)} = w^{(k)}\ e^{\eta t} \cosh pr$$
$$h_t^{(k)} = (w_t^{(k)} + \eta w^{(k)}) e^{\eta t} \cosh pr$$
$$h_i^{(k)} = (w_i^{(k)} + pw^{(k)} \frac{x_i}{r} \tanh pr) e^{\eta t} \cosh pr$$
$$h_{ij}^{(k)} = (w_{ij}^{(k)} + \frac{p}{r}\left[w_i^{(k)} x_j + w_j^{(k)} x_i\right] \tanh pr + \frac{p}{r}\left[\delta_{ij} - \frac{x_i x_j}{r^2}\right] w^{(k)} \tanh pr$$
$$+ \frac{p^2}{r^2} x_i x_j\ w^{(k)}) e^{\eta t} \cosh pr \quad .$$

We substitute these expressions into (3.9), and our differential inequalities become

$$w_t^{(k)} = \sum_{ij} \alpha_{ij}^{(k)}\ w_{ij}^{(k)} + \sum_i \left\{\beta_i^{(k)} + \sum_j \left[(\alpha_{ij}^{(k)} + \alpha_{ji}^{(k)}) \frac{p}{r} x_j \tanh pr\right]\right\} w_i^{(k)}$$
$$+ \sum_{\ell \neq k} \gamma_\ell^{(k)}\ w^{(\ell)}$$
$$+ w^{(k)}\left[-\eta + \gamma_k^{(k)} + \left\{\frac{p}{r} \sum_i x_i \beta_i^{(k)} + p \sum_i \frac{1}{r} \alpha_{ii}^{(k)} - p \sum_{ij} \frac{1}{r^3} \alpha_{ij}^{(k)} x_i x_j\right\} \tanh pr\right.$$
$$\left. + \frac{p^2}{r^2} \sum_{ij} \alpha_{ij}^{(k)}\ x_i x_j\right]$$
$$+ \sum_{q,\ell} \int_0^T \int_{||\vec{y}||<Y} g_\ell^{(q)}(k,t,\vec{x},s,\vec{y})\ \frac{\cosh p||\vec{x}-\vec{y}||}{\cosh pr}\ w^{(\ell)}(t-s,\vec{x}-\vec{y})\, d\vec{y}\, ds$$
$$k = 1,\ldots,\nu \quad . \tag{3.11}$$

Let $S > 0$ be large enough so that

$$S > \sum_{q,\ell} \int_0^T \int_{||\vec{y}||<Y} g_\ell^{(q)}(k,t,\vec{x},s,\vec{y})\ \frac{\cosh p||\vec{x}-\vec{y}||}{\cosh pr}\ d\vec{y}\, ds$$

for all $(t,\vec{x})$ in $\left[0,T_0\right] \times \overline{\Omega}$ and all k. Let η be chosen large enough so

$$\eta > \sum_{\ell \neq k} \gamma_\ell^{(k)} + |\gamma_k^{(k)}| + \frac{p^2}{r^2} \sum_{ij} \alpha_{ij}^{(k)}\ x_i x_j$$
$$+ \tanh pr \left\{p\left(\sum_i \frac{1}{r}\alpha_{ii}^{(k)} - \sum_{ij} \frac{1}{r^3}\alpha_{ij}^{(k)}\ x_i x_j\right) + \frac{p}{r} \sum_i x_i \beta_i^{(k)}\right\} + S+1$$
$$\text{for all } k \tag{3.12}$$

for all $(t,\vec{x})$ in $\left[0,T_0\right] \times \overline{\Omega}$.

Let $B_R \equiv \{\forall \vec{x}:\ ||\vec{x}|| < \mathbb{R}\}$. We note that $h^{(k)}(t,\vec{x}) \geq 0$ and hence $w^{(k)}(t,\vec{x}) \geq 0$ for $t \leq 0$ and for $(t,\vec{x}) \in \left[0,T_0\right] \times (\mathbb{R}^n - \Omega)$ for all

k due to our assumptions. We now show that this and (3.11) implies that $w^{(k)}(t,\vec{x}) \geq 0$ for all $(t,\vec{x})$ in $[0,T_0] \times \bar{\Omega}$. Suppose that $w^{(k)}(t,\vec{x}) < -\varepsilon < 0$ for some k and some $(t,\vec{x}) \in [0,T_0] \times \bar{\Omega}$. Since $w^{(k)}(t,\vec{x}) \to 0$ as $||\vec{x}|| \to \infty$, there is a $R(\varepsilon)$ such that $|w^{(k)}(t,\vec{x})| < \varepsilon/2$ for all $(t,\vec{x})$ in $[0,T_0] \times \bar{\Omega}$ with $||\vec{x}|| \geq R(\varepsilon)$. Define $w_{min}(t,\vec{x}) \equiv \min_k w^{(k)}(t,\vec{x})$. Since $w_{min}(t,\vec{x}) \geq -\varepsilon/2$ for $t = 0$, $\vec{x} \in \bar{\Omega}$ and for $t \in [0,T_0]$, $\vec{x} \in \partial[\Omega \cap B_R]$, and since $w_{min}(t,\vec{x}) \leq -\varepsilon$ for some $(t,\vec{x})$ in $[0,T_0] \times [\bar{\Omega} \cap B_R]$, w_{min} has a minimum in $(t,\vec{x}) \in [0,T_0] \times (\Omega \cap B_R)$. At this point $(t_m,\vec{x}_m)$, there is a k such that $w^{(k)}(t,\vec{x}) = w_{min}(t,\vec{x})$ and $w^{(k)}(t,\vec{x})$ is at a relative minimum at $(t_m,\vec{x}_m)$. Thus, at this point we have

$$w_t^{(k)} \leq 0, \quad \sum_{ij} \alpha_{ij}^{(k)} w_{ij}^{(k)} \geq 0, \quad w_i^{(k)} = 0, \quad w^{(k)} \leq -\varepsilon, \text{ and } w^{(\ell)} \geq w^{(k)} \text{ for all } \ell. \tag{3.13}$$

However, from the definition of η in (3.12) we see that substitution of (3.13) into the differential inequality in (3.11) implies that

$$w_t^{(k)} \leq 0 \quad \text{and} \quad w_t^{(k)} \geq \varepsilon \quad .$$

This is a contradiction, and so $w^{(k)} \geq -\varepsilon$ for all $(t,\vec{x})$ in $[0,T_0] \times \bar{\Omega}$. Since $\varepsilon > 0$ is arbitrary, $w^{(k)} \geq 0$ for all k and all $(t,\vec{x})$ and hence $h^{(k)}$ is also. Thus the maximum principle is established.

We will almost always use the maximum principle with $\Omega \equiv \mathbb{R}^n$. For this case the theorem requirement (3.4) simplifies since $\mathbb{R}^n - \Omega = \emptyset$.

Note that the differential inequalities are only required to be satisfied for $(t,\vec{x})$ in $(0,T_0] \times \Omega$. Outside of $(0,T_0] \times \Omega$ the functions $\tilde{u}(t,\vec{x})$ and $\tilde{v}(t,\vec{x})$ are only required to be continuous. This corresponds to the fact that to find solutions of

$$\tilde{u}_t = \tilde{F}_M([\tilde{u}]) \qquad (t,\vec{x}) \in (0,T_0) \times \Omega \tag{3.2}$$

when the $\tilde{F}_M$ contain integral terms, one must supply $\tilde{u}(t,x)$ for $(t,\vec{x})$ outside of $(0,T_0] \times \Omega$ as initial conditions.

When no integral terms are present in the $\tilde{F}$, the requirement of (3.4) can be simplified. For the theorem to hold in this case, it is sufficient to require

$$\begin{aligned} &u^{(k)}(0,\vec{x}) \geq v^{(k)}(0,\vec{x}) \quad \text{for all } k, \text{ all } \vec{x} \in \Omega, \text{ and} \\ &u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x}) \quad \text{for all } k, \text{ all } t \in [0,T_0], \text{ all } x \in \partial\Omega . \end{aligned} \tag{3.4'}$$

This corresponds to the fact that to find solutions of

$$\tilde{u}_t = \tilde{F}_M([\tilde{u}]) \qquad (t,\vec{x}) \in (0,T_0) \times \Omega \tag{3.2}$$

when $\tilde{F}_M$ contains no integrals, $\tilde{u}(t,\vec{x})$ only needs to be prescribed on the initial and lateral boundaries.

Before we continue on to the uniformity lemma, we note three immediate extensions of the maximum principle. The first extension is that the maximum principle will remain valid if the $\tilde{F}_M$ is allowed to depend on integrals over only time $\int_0^T G^{(q)}(s,\tilde{u}(t-s,\vec{x}))ds$ and on integrals over only space $\int_{||\vec{y}||<Y} G^{(q)}(\vec{y},\tilde{u}(t,\vec{x}-\vec{y}))d\vec{y}$. We will not pursue this further except to note that all subsequent results which are valid when $\tilde{F}$ depends on integrals over time and space remain valid when $\tilde{F}$ also depends on integrals only over time and integrals only over space.

The second extension is that the maximum principle remains true if time is discrete; that is if $t, T,$ and T_0 are replaced by integers, $u_t^{(k)}(t,\vec{x})$ is replaced by $u^{(k)}(t+1,\vec{x}) - u^{(k)}(t,\vec{x})$, and the integrals over time are replaced by the appropriate sums. The proof of the maximum principle for this case is essentially identical to the proof presented above. Note that there are potential physical applications for discrete time systems since, for example, some ecologists measure things yearly and some geneticists do things in terms of generations. However, it would perhaps be (physically) surprising if these discrete time models included local operators like $\frac{\partial}{\partial x_i}$. We shall not pursue discrete time systems further in subsequent chapters, except to note that for all subsequent results about continuous time systems there are analogous results for discrete time systems.

The last extension we consider is that the maximum principle remains valid if $(0,T_0) \times \Omega$ is replaced by an arbitrary domain D in $\mathbb{R}^{n+1}$. In proving the uniformity lemma, we shall use this extension in a case where no integrals are present. For this case, the requirement of (3.4) is

$$u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x}) \text{ for } (t,\vec{x}) \in \partial D \cap \{(t,\vec{x}): 0 \leq t < T_0\} \quad \text{and}$$
$$\text{for } (t,x) \in \overline{D} \cap \{(t,\vec{x}): t = 0\}, \quad k = 1,\ldots,\nu \ .$$

The conclusion of the theorem is

$$u^{(k)}(t,\vec{x}) \geq v^{(k)}(t,\vec{x}) \text{ for } (t,\vec{x}) \in \overline{D} \cap \{(t,\vec{x}): 0 \leq t \leq T_0\} \quad ,$$

and the proof is virtually identical to the one given.

This completes our discussion of the maximum principle. We now continue by stating and proving the uniformity lemma.

Theorem 3.2 (The uniformity lemma): Let $D(T_0)$ be the domain $\{(t,\vec{x}): 0 < t < T_0\}$. Suppose for each $T_0 > 0$

A1 that the equations

$$u_t^{(k)} = F^{(k)}([\tilde{u}]) \qquad k = 1,\ldots,\nu \tag{3.1}$$

satisfy the smoothness conditions H1 and H2;

A2 that $\tilde{u}(t,\vec{x}) \in C_t^1 \cap C_x^2$ and solves

$$u_t^{(k)} = F_M^{(k)}([\tilde{u}]) \quad k = 1,\ldots,\nu, \text{ for } (t,\vec{x}) \in \overline{D(T_0)} \text{ for some } M > 0; \text{ and} \tag{3.2}$$

A3 that the $u^{(k)}, u_i^{(k)}$, and $u_{ij}^{(k)}$ are all bounded and are locally Hoelder continuous with exponent ε in t and $\vec{x}$ (for some $\varepsilon > 0$) for all $(t,\vec{x}) \in \overline{D(T_0)}$.

Then

$$\begin{aligned}
|u^{(k)}(t,\vec{x})| &\le \max\{\sup|u^{(k)}(0,\vec{x})|,\ 2M\} \equiv U^{(k)} \\
|u_i^{(k)}(t,\vec{x})| &\le \max\{\sup|u_i^{(k)}(0,\vec{x})|,\ 2M\} \equiv U_i^{(k)}, \quad \text{and} \\
|u_{ij}^{(k)}(t,\vec{x})| &\le \max\{\sup|u_{ij}^{(k)}(0,\vec{x})|,\ 2\tilde{M}(M)\} \equiv U_{ij}^{(k)}
\end{aligned} \tag{3.14}$$

$$\text{for all} \quad (t,\vec{x}) \in D(+\infty) \quad .$$

Thus, we see that when $\tilde{u}$, $\tilde{u}_i$, and $\tilde{u}_{ij}$ are bounded for each $\overline{D(T_0)}$, they are also uniformly bounded over $D(+\infty)$ as well. This will be needed in the proof of the asymptotic state theorem.

Proof: Suppose $(t_0,\vec{x}_0)$ is any point with $|u^{(k)}(t_0,\vec{x}_0)| > U^{(k)}$. Let $T_0 > t_0$ be chosen. Define $D^{(k)}(T_0) = D(T_0) \cap \{\text{all } (t,x) \text{ for which } |u^{(k)}(t,\vec{x})| > U^{(k)}\}$. Further, let $D^{(k)'}(T_0)$ be the largest connected component of $D^{(k)}(T_0)$ containing $(t_0,\vec{x}_0)$. Since $|u^{(k)}(t,x)| \le U^{(k)}$ on all boundaries of $\overline{D^{(k)'}(T_0)}$, since the k^{th} equation of (3.2) is the heat equation $u_t^{(k)} = \alpha(\sum_j u_{jj}^{(k)})$ for $(t,\vec{x})$ in $D^{(k)'}(T_0)$, and since this heat equation has $u^{(k)} \equiv U^{(k)}$ and $u^{(k)} \equiv -U^{(k)}$ as solutions, the maximum principle implies that $-U^{(k)} \le u^{(k)}(t,\vec{x}) \le U^{(k)}$ for all $(t,\vec{x})$ in $\overline{D^{(k)'}(T_0)}$. This contradicts $|u^{(k)}(t_0,\vec{x}_0)| > U^{(k)}$. Thus $|u^{(k)}(t,\vec{x})| \le U^{(k)}$ for all $t \ge 0$ and all $\vec{x}$.

Similarly, let $(t_0,\vec{x}_0)$ be any point with $|u_i^{(k)}(t_0,\vec{x}_0)| > U_i^{(k)}$. As before, let $T_0 > t_0$ be chosen, define $D_i^{(k)}(T_0) = D(T_0) \cap \{\forall(t,\vec{x}): |u_i^{(k)}(t,x)| > U_i^{(k)}\}$, and let $D_i^{(k)'}(T_0)$ be the largest connected component of $D_i^{(k)}(T_0)$ which contains $(t_0,\vec{x}_0)$. Since $u^{(k)}$, $u_i^{(k)}$, and $u_{ii}^{(k)}$ are locally Hoelder continuous (with exponent $\varepsilon > 0$) for $(t,\vec{x}) \in D_i^{(k)}(T_0)$, and since in this region $u^{(k)}$ satisfies a heat equation $u_t^{(k)} = \alpha(\sum_j u_{jj}^{(k)})$, the function $u_t^{(k)}$ must also be Hoelder continuous in this region. Thus, $u^{(k)} \in C_t^\infty \cap C_x^\infty$ for $(t,\vec{x})$ in $D_i^{(k)'}(T_0)$. (This follows from theorem 13, Chapter III, reference [7]). Therefore

$$u_t^{(k)} = \alpha(\sum_j u_{jj}^{(k)}) \quad \text{for} \quad (t,\vec{x}) \in D_i^{(k)'}(T_0)$$

implies that $v^{(k)} \equiv u_i^{(k)}$ satisfies

$$v_t^{(k)} = \alpha(\sum_j v_{jj}^{(k)}) \quad \text{for} \quad (t,\vec{x}) \in D_i^{(k)'}(T_0) \quad .$$

Since $|v^{(k)}| \equiv |u_i^{(k)}| \leq U_i^{(k)}$ on all boundaries of $\overline{D_i^{(k)'}(T_0)}$, as before the maximum principle implies

$$|u_i^{(k)}| \leq U_i^{(k)} \quad \text{for all} \quad (t,\vec{x}) \in \overline{D_i^{(k)'}(T_0)} \quad .$$

This contradicts $u_i^{(k)}(t_0,x_0) > u_i^{(k)}$, and so $|u_i^{(k)}(t_0,x_0)| \leq U_i^{(k)}$ for all $t \geq 0$ and all $\vec{x}$.

The bounds on the second spatial derivatives of $u^{(k)}$ are shown <u>by virtually the same argument as the one given for its first derivatives.</u>

We defer discussing the uniformity lemma until section (3.4) where we will discuss how the maximum principle, the uniformity lemma, and the asymptotic state theorem will be used in conjunction with the hypotheses we shall assume. Therefore we now state (and prove) the asymptotic state theorem.

<u>Theorem 3.3 (Asymptotic state theorem):</u> Suppose there is only one spatial dimension; i.e. $n = 1$. Suppose for some $T_0 > 0$

A1 that the equations

$$u_t^{(k)} = F^{(k)}([\tilde{u}]) \qquad k = 1,\ldots,\nu \tag{3.1}$$

satisfy the smoothness conditions H1 and H2;

A2 that the functions $F^{(k)}(u_{xx}^{(k)}, u_x^{(k)}, \tilde{u}, \int_0^T \int_{|y|<Y} G^{(q)}(s,y,\tilde{u}(t-s,x-y))dyds)$

satisfy $F^{(k)}_{1ii} > 0$ at $i = 1$ for all k and all arguments of $F^{(k)}$; and A3 that the functions $\tilde{u}(t,x) \in C^2_x \cap C^1_t$ satisfy

$$u^{(k)}_t = F^{(k)}_M([\tilde{u}]) \qquad k = 1,\ldots,\nu \tag{3.2}$$

for some $M > 0$; $\tilde{u}$, $\tilde{u}_x$, $\tilde{u}_{xx}$ are all uniformly bounded; and each $u^{(k)}_t$ is either ≥ 0 or ≤ 0 for all x and all $t \geq T_0$.

Then if $\lim_{t\to\infty} u^{(k)}(t,x) = \tau^{(k)}(x)$ exists pointwise for all k and all x in an open interval I, then $\tilde{\tau}(x)$ is in C^2_x and solves the steady state equations $F^{(k)}_M([\tilde{\tau}]) = 0$ $(k = 1,\ldots,\nu)$ for all x in I.

Thus, when there is only a single spatial dimension and when there is a solution u with $\tilde{u}$, $\tilde{u}_x$, $\tilde{u}_{xx}$ bounded uniformly, then if $\tilde{u}$ evolves monotonically into a vector function $\tilde{\tau}(x)$ as $t\to\infty$ we can conclude that $\tilde{\tau}(x)$ is a steady state solution.

Proof: For clarity we will use t and x subscripts to denote partial derivatives in this proof. Let x_0 be any point in I and let $R > 0$ be small enough for $[x_0-R,\ x_0+R] \subseteq I$. Define $K_R \equiv [x_0-R, x_0+R]$. Since $|u^{(k)}_x|$ and $|u^{(k)}_{xx}|$ are bounded, $u^{(k)}(t,x)$ and $u^{(k)}_x(t,x)$ are equicontinuous (parametrized by t) on K_R. Thus we can select a sequence t_1, $t_2,\ldots$ such that

(1) $T_0 < t_1 < t_2 < \ldots$

(2) $t_n \to \infty$ as $n \to \infty$, and

(3) $u^{(k)}(t_n,x)$ and $u^{(k)}_x(t_n,x)$ converge uniformly for all k and for all $x \in K_R$ as $n \to \infty$.

This is a consequence of the Ascoli-Arzelas theorem (see e.g. Chapter VII, reference [8]). However, since each $u^{(k)}(t,x)$ is monotonic in t, $u^{(k)}_t(t,x) \to 0$ pointwise (and hence uniformly) as $t \to \infty$ for all x in K_R.

Thus, we have $\tilde{u}(t,x)$ converging uniformly as $t \to \infty$, $\tilde{u}_x(t_n,x)$ converging uniformly as $n \to \infty$, and $\tilde{u}_t(t,x)$ converging uniformly as $t \to \infty$ for all x in K_R.

We now utilize assumption A3. By the mean value theorem we know that

$$\begin{aligned} u_t^{(k)}(t_n,x)-u_t^{(k)}(t_m,x) &= F_{M,1}^{(k)}\left[u_{xx}^{(k)}(t_n,x) - u_{xx}^{(k)}(t_m,x)\right] \\ &+ F_{M,2}^{(k)}\left[u_x^{(k)}(t_n,x) - u_x^{(k)}(t_m,x)\right] \\ &+ \sum_{\ell} F_{M,3\ell}^{(k)}\left[u^{(\ell)}(t_n,x) - u^{(\ell)}(t_m,x)\right] \qquad (3.15) \\ &+ \sum_{q,\ell} F_{M,4q}^{(k)} \int_0^T \int_{|y|<Y} G_{M,3\ell}^{(q)}(s,y)\left[u^{(\ell)}(t_n-s,x-y)\right. \\ &\qquad\qquad \left. - u^{(\ell)}(t_m-s,x-y)\right] dyds \end{aligned}$$

for some arguments of $F_{M,1}^{(k)}$, $F_{M,2}^{(k)}$, $F_{M,3\ell}^{(k)}$, $F_{M,4q}^{(k)}$, and $G_{M,3\ell}^{(q)}$. Since the functions $F_{M,2}^{(k)}$, $F_{M,3\ell}^{(k)}$, $F_{M,4q}^{(k)}$, and $G_{M,3\ell}^{(q)}$ are bounded independently of their arguments, and since there is a $\delta_M > 0$ for which $F_{M,1}^{(k)} \geq \delta_M > 0$ for all arguments and all k, equation (3.15) and the uniform convergence of $u_t^{(k)}(t_n,x)$, of $u_x^{(k)}(t_n,x)$, and of $\tilde{u}(t,x)$ as $n \to \infty$ and $t \to \infty$ imply that $u_{xx}^{(k)}(t_n,x)$ is a uniform Cauchy sequence as $n \to \infty$ for x in K_R. Thus $u_{xx}^{(k)}$ converges uniformly as $n \to \infty$. Hence $\tilde{\tau}(x)$ is in C_x^2 and has bounded first and second derivatives. Thus $\tilde{\tau}(x)$ solves the steady state equation for all x in K_R, and hence all x in I.

The asymptotic state theorem only holds when there is a single spatial dimension. When more than one spatial dimension is present, (3.14) changes to include the term $\sum_{ij} F_{M,1ij}^{(k)}(u_{ij}^{(k)}(t_n,\vec{x}) - u_{ij}^{(k)}(t_m,\vec{x}))$. Although all other terms in (3.14) still go to zero as $m,n \to \infty$, we cannot conclude that $u_{ij}^{(k)}(t_n,\vec{x}) - u_{ij}^{(k)}(t_m,\vec{x})$ all go to zero separately as m,n go to infinity. Note that if all the $u_{ij}^{(k)}(t,\vec{x})$ were equicontinuous in t (which is true when, for example, all third order spatial derivatives are uniformly bounded), then the Ascoli-Arzelas theorem would ensure that $\tilde{\tau} \in C_x^2$ and that $\tilde{\tau}$ solves the steady state equations.

We defer discussing this result further until the next section, where we will discuss how all three results in this section will be used in conjunction with the hypotheses of the next section.

3.4 General assumptions. This section provides a convenient place to collect the set of hypotheses we will assume to hold in all the derivations in subsequent chapters. We will first state these assumptions and then briefly discuss how they fit together with the results of section (3.3).

The first three hypotheses are about the form of the equations

$$u_t^{(k)} = F^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds),\ k=1,\ldots,\nu\ . \tag{3.1}$$

Namely, the first two are smoothness assumptions about the $G^{(q)}$ and the $F^{(k)}$, and the third assumption is that equations (3.1) form a parabolic system. The other two assumptions will guarantee existence of solutions to the initial value problem.

The first two hypotheses are:

H1: For all q, all j, all $s \in [0,T]$, all $\vec{y}$ with $||\vec{y}|| \leq Y$, the derivatives $G_{3j}^{(q)}$ exist and are continuous in all arguments. Moreover, for some $\alpha > 0$, $G_{3j}^{(q)}(s,\vec{y},\tilde{u})$ is locally Hoelder continuous with exponent α in the arguments $\tilde{u}$.

H2: For all i,j,k,ℓ,q and for all arguments, the derivatives $F_{1ij}^{(k)}$, $F_{2i}^{(k)}$, $F_{3\ell}^{(k)}$, and $F_{4q}^{(k)}$ exist and (for some $\alpha > 0$) are locally Hoelder continuous with exponent α in all arguments.

These hypotheses are slightly stronger than the smoothness hypotheses we used previously. We require this slightly stronger smoothness because we will often need to write a solution $\tilde{u}$ of (3.1) as $\tilde{u}(t,\vec{x}) = \tilde{v}(t,\vec{x}) + \varepsilon\,\tilde{\vec{\phi}}(t,\vec{x})$, need to then expand (3.1) as

$$
\begin{aligned}
v_t^{(k)} + \varepsilon\phi_t^{(k)} = F^{(k)}([\tilde{v}]) + \varepsilon\Bigg\{ & \sum_{ij} F_{1ij}^{(k)}\,\phi_{ij}^{(k)} + \sum_i F_{2i}^{(k)}\,\phi_i^{(k)} + \sum_{\ell} F_{3\ell}^{(k)}\phi^{(\ell)} \\
& + \sum_{q,\ell} F_{4q}^{(k)} \int_0^T \int_{||\vec{y}||<Y} G_{3\ell}^{(q)}(s,\vec{y},\tilde{v}(t-s,\vec{x}-\vec{y})) \\
& \phi^{(\ell)}(t-s,\vec{x}-\vec{y})\,d\vec{y}ds\Bigg\} \\
& + \text{h.o.t.s} \qquad k = 1,\ldots,\nu \quad ,
\end{aligned}
$$

and then finally conclude that the "h.o.t.s" are terms which are of higher than linear order in ε. The Hoelder continuity ensures that the "higher order terms" are of higher than linear order in ε.

The third hypothesis ensures that equations (3.1) form a parabolic system. It is:

H3: The functions

$$F^{(k)}(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))\,d\vec{y}ds)$$

satisfy

$$\sum_{ij} F_{1ij}^{(k)}\,\xi_i\,\xi_j > 0 \quad \text{for all } \vec{\xi} \neq \vec{0} \quad ,$$

$$F_{3\ell}^{(k)} \geq 0 \quad \text{for all } \ell \neq k, \text{ and}$$

$$F_{4q}^{(k)}\,G_{3\ell}^{(q)} \geq 0 \quad \text{for all } q \text{ and } \ell \quad ,$$

for all k and all arguments of $F^{(k)}$ and $G^{(q)}$.

Note that this is precisely one of the requirements of the maximum principle.

The last two hypotheses we will need are concerned with the existence of solutions of the initial value problem. At the present state of the theory of partial differential equations, we cannot generally prove the existence of solutions of

$$u_t^{(k)} = F_M^{(k)}([\tilde{u}]) \qquad k = 1,\ldots,\nu \tag{3.2}$$

for reasonably general initial conditions. We therefore must assume this existence.

H4: For all $M > 0$ sufficiently large, we assume that for every initial condition $\tilde{u}(t,\vec{x})$ (defined for all x and all $t \leq 0$) which satisfies

(1) $\tilde{u}(t,\vec{x})$ is bounded and uniformly Hoelder continuous with exponent α in $\vec{x}$ and t for some $\alpha > 0$

(2) $u^{(k)}(0,\vec{x})$, $u_i^{(k)}(0,\vec{x})$, and $u_{ij}^{(k)}(0,\vec{x})$ are bounded for all $\vec{x}$, and

(3) $u_{ij}^{(k)}(0,\vec{x})$ is uniformly Hoelder continuous with exponent α in $\vec{x}$, then the system of equations

$$u_t^{(k)} = F_M^{(k)}\left(u_{ij}^{(k)}, u_i^{(k)}, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G_M^{(q)}(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds\right) \quad k=1,\ldots,\nu \tag{3.2}$$

has a solution $\tilde{u}(t,\vec{x})$ for all $t \geq 0$ which satisfies the conditions

(4) $\tilde{u}(t,\vec{x})$ agrees with the initial conditions for $t \leq 0$,

(5) for any $T_0 > 0$, $u^{(k)}(t,\vec{x})$, $u_i^{(k)}(t,\vec{x})$, and $u_{ij}^{(k)}(t,x)$ exist and are bounded for $0 \leq t \leq T_0$, and

(6) $u_{ij}^{(k)}(t,\vec{x})$ is locally Hoelder continuous with exponent $\varepsilon(T_0)$ (for some $\varepsilon(T_0) > 0$) in $\vec{x}$ and t for all $\vec{x}$ and all t in $[0,T_0]$, for any $T_0 > 0$.

H5: If there is a single spatial dimension (i.e. $n = 1$), then for all $M > 0$ sufficiently large, we assume that for any c, any x_0, x_1 with $x_0 \neq x_1$, and for every initial condition $\tilde{u}(t,x)$ that satisfies conditions (1), (2) and (3) of H4 at all points <u>but</u> $x = x_0$ and $x = x_1$ and that is continuous at $x = x_0$ and $x = x_1$ for $t \leq 0$ (but in general is not differentiable there), there is a solution $\tilde{u}(t,x)$ of the system of equations

$$u_t^{(k)} = F_M^{(k)}\left(u_{xx}^{(k)}, u_x^{(k)}, \tilde{u}, \int_0^T \int_{|y|<Y} G_M^{(q)}(s,y,\tilde{u}(t-s,x-y))dyds\right) + cu_x^{(k)}$$

$$k = 1, \ldots, \nu \tag{3.16}$$

for all $t \geq 0$ which satisfies the conditions

(4) $\tilde{u}(t,x)$ agrees with the initial conditions when $t \leq 0$,

(5) for any $T_0 > \delta > 0$, $u^{(k)}(t,x)$, $u_x^{(k)}(t,x)$, and $u_{xx}^{(k)}(t,x)$ exist and are bounded for $\delta \leq t \leq T_0$

(6) for any $T_0 > \delta > 0$, $u_{xx}^{(k)}(t,x)$ is locally Hoelder continuous with exponent $\varepsilon(\delta, T_0)$ (for some $\varepsilon(\delta, T_0) > 0$) in t and x for all x and all t in $[\delta, T_0]$.

Hypothesis H4 assumes the existence of satisfactory solutions of (3.2) when the initial conditions are smooth. Hypothesis H5 also assumes the existence of satisfactory solutions of (3.2) in terms of a coordinate system which travels with speed c. It also permits the initial conditions to have discontinuous derivatives at $x = x_0$ and $x = x_1$. We need this last existence assumption because in the proof of the instability of non-monotonic waves (in one spatial dimension), the perturbed initial condition in general has two jumps in its derivative due to the bulge added to the non-perturbed wave.

We now briefly discuss how the hypotheses, the modification of the equations, and the mathematical results of section (3.3) fit together. First, if hypotheses H1, H2 and H3 about the functions $F^{(k)}$ and $G^{(q)}$ are assumed to hold, then the modified functions $F_M^{(k)}$ and $G_M^{(q)}$ have the properties

P1: For any particular $M > 0$, for all q and j, and for all arguments, the derivatives $G^{(q)}_{M,3j}(s,\vec{y},\tilde{u})$ exist, are uniformly bounded, and are uniformly Hoelder continuous with exponent α in the argument $\tilde{u}$ (where α is the same as in H1).

P2: For all i,j,k,ℓ, for all arguments, and for any particular $M > 0$, the derivatives $F^{(k)}_{M,1ij}$, $F^{(k)}_{M,2i}$, $F^{(k)}_{M,3\ell}$, and $F^{(k)}_{M,4\ell}$ exist, are uniformly bounded, and are uniformly Hoelder continuous with exponent α in all arguments (where α is the same as in H2).

P3: For any particular $M > 0$, there is a $\delta_M > 0$ such that the functions

$$F^{(k)}_M(u^{(k)}_{ij}, u^{(k)}_i, \tilde{u}, \int_0^T \int_{||\vec{y}||<Y} G^{(q)}_M(s,\vec{y},\tilde{u}(t-s,\vec{x}-\vec{y}))d\vec{y}ds)$$

satisfy

$$\sum_{ij} F^{(k)}_{M,1ij}\ \xi_i\ \xi_j \geq \delta_M > 0 \qquad \text{for all } \vec{\xi} \text{ with } \sum_i \xi_i\ \xi_i = 1$$

$$F^{(k)}_{M,3\ell} \geq 0 \qquad \text{for all } \ell \neq k$$

$$F^{(k)}_{M,4q}\ G^{(q)}_{M,3\ell} \geq 0 \qquad \text{for all } q \text{ and } \ell,$$

for all k and all arguments of $F^{(k)}_M$ and $G^{(k)}_M$.

Thus the modifications change local Hoelder continuity into uniform Hoelder continuity in H1 and H2, and also places a lower bound δ_M on the "local diffusion constant" in H3.

We now look for the net effects of the existence assumption H4. To simplify exposition, define

$S \equiv \{$all vector functions $\tilde{u}(t,\vec{x})$ defined for $t \leq 0$ which satisfy the smoothness conditions (1), (2), and (3) of hypothesis H4$\}$.

Assume H1, H2, H3, and H4 are satisfied. From H4, for any initial condition

$\tilde{u}(t,\vec{x}) \in S$ there is a solution of

$$u_t^{(k)} = F_M^{(k)}([\tilde{u}]) \quad \text{for} \quad t > 0, \quad k = 1,\ldots,\nu \qquad (3.2)$$

with properties (4), (5), and (6). However, the uniformity lemma then implies that $\tilde{u}(t,\vec{x})$, $\tilde{u}_i(t,\vec{x})$, and $\tilde{u}_{ij}(t,\vec{x})$ are uniformly bounded for all $t \geq 0$ and all $\vec{x}$. Thus, the net effect is that whenever the initial conditions $\tilde{u}(t,\vec{x})$ for $t \leq 0$ are smooth enough to be in S, then assuming H1, H2, H3, and H4 implies that a solution $\tilde{u}(t,\vec{x})$ of (3.2) for $t > 0$ exists which both matches the initial conditions for $t \leq 0$ and has $\tilde{u}(t,\vec{x})$, $\tilde{u}_i(t,\vec{x})$, and $\tilde{u}_{ij}(t,\vec{x})$ uniformly bounded for all $t \geq 0$ and all x.

Now suppose we have constructed continuous bounded functions $\bar{\tilde{u}}(t,\vec{x})$ and $\underline{\tilde{u}}(t,\vec{x})$ which are in $C_x^2 \cap C_t^1$ for $t > 0$ and also satisfy

$$\bar{u}_t^{(k)} \geq F_M^{(k)}([\bar{\tilde{u}}]) \qquad \text{for all } t > 0 \qquad k = 1,\ldots,\nu$$

$$\underline{u}_t^{(k)} \leq F_M^{(k)}([\underline{\tilde{u}}]) \qquad \text{for all } t > 0 \qquad k = 1,\ldots,\nu \ .$$

Consider any initial condition $\tilde{u}(t,\vec{x})$ for $t \leq 0$ smooth enough to be in S and which also satisfies

$$\bar{u}^{(k)}(t,\vec{x}) \geq u^{(k)}(t,\vec{x}) \qquad \text{for all } \vec{x}, \text{ all } t \leq 0, \qquad k = 1,\ldots,\nu \ .$$

Then H4 implies that there exists a solution $\tilde{u}(t,\vec{x})$ of (3.2) for $t \geq 0$ which matches the initial conditions for $t \leq 0$ and is bounded for all x and all t in $[0,T_0]$ for any $T_0 > 0$. The hypotheses of the maximum principle are thus satisfied, and

$$\bar{u}^{(k)}(t,\vec{x}) \geq u^{(k)}(t,\vec{x}) \quad \text{for all} \quad t \geq 0 \ .$$

Thus, the net effect is that whenever $\tilde{u}(t,\vec{x})$ is smooth enough to belong in S, then assuming that H1, H2, H3, and H4 are satisfied implies that there is a solution $\tilde{u}(t,\vec{x})$ of (3.2) for $t \geq 0$ which matches the initial conditions for $t \leq 0$. Moreover, whenever $\bar{u}^{(k)}(t,\vec{x}) \geq u^{(k)}(t,\vec{x})$ for all $\vec{x}$ and all k holds for all $t \leq 0$, then it holds for all $t \geq 0$ as well.

Similarly, if $u^{(k)}(t,\vec{x}) \geq \underline{u}^{(k)}(t,\vec{x})$ for all $\vec{x}$ and all k holds for $t \leq 0$, then it holds for $t \geq 0$ as well.

We now briefly look at the net effects of the existence assumption H5 when there is only one spatial dimension. Define S' as the set of all functions $\tilde{u}(t,x)$ defined for $t \leq 0$ which satisfy the initial condition requirements of H5. Assume that H1, H2, H3, and H5 are satisfied. Let $\tilde{u}(t,x)$ be in S' for $t \leq 0$. Then H5 guarantees a solution of (3.2) exists for $t \geq 0$ which satisfies properties (4), (5), and (6) of H5. The uniformity lemma then shows that $\tilde{u}(t,x)$, $\tilde{u}_x(t,x)$, and $\tilde{u}_{xx}(t,x)$ are uniformly bounded for all $t \geq \delta$ for any $\delta > 0$. Property (6) of H5 and the results of the uniformity lemma can then be used in the asymptotic state theorem. This shows that if $u^{(k)}(t,x) \to \tau^{(k)}(x)$ monotonically as $t \to \infty$ for each k, then $\tilde{\tau}(x)$ is a steady state solution of (3.2):

$$F_M^{(k)}\left(\left[\tilde{\tau}\right]\right) = 0 \qquad k = 1,\ldots,\nu \quad .$$

Thus, the net effect is that when there is a single spatial dimension and when the initial conditions $\tilde{u}(t,x) \in S'$ for $t \leq 0$, then assuming H1, H2, H3, and H5 implies that there is a solution $\tilde{u}(t,x)$ of (3.2) for $t \geq 0$ which matches the initial conditions for $t \leq 0$, and for which $\tilde{u}(t,x)$, $\tilde{u}_x(t,x)$, and $\tilde{u}_{xx}(t,x)$ are uniformly bounded for all $t \geq \delta$ (for any $\delta > 0$). Moreover, if

$$u^{(k)}(t,x) \to \tau^{(k)}(x) \text{ monotonically as } t \to \infty \text{ for each } k,$$

then $\tilde{\tau}(x)$ is a steady state solution of (3.2).

Now suppose $\tilde{\overline{u}}(t,x)$ and $\underline{\tilde{u}}(t,x)$ are the same upper and lower functions defined previously. The net effect of assuming H1, H2, H3, and H5 is that whenever $\tilde{u}(t,x)$ is in S' for $t \leq 0$, then there is a solution $\tilde{u}(t,x)$ of (3.2) for $t \geq 0$ which matches the initial conditions for $t \leq 0$. Moreover, as before, if

$$\overline{u}^{(k)}(t,x) \geq u^{(k)}(t,x) \quad \text{for all } x \text{ and all } k$$

holds for all $t \leq 0$, then it is true for all $t \geq 0$ also. Similarly, if

$$u^{(k)}(t,x) \geq \underline{u}^{(k)}(t,x) \quad \text{for all } x \text{ and all } k$$

holds for all $t \leq 0$, then it is true for all $t \geq 0$ as well.

As a final remark for this chapter, let us note that we will not prove the individual results in subsequent chapters under the most general possible hypotheses. Instead we will tend to use the same overall hypotheses for all the results in each chapter. We also will not use the most general possible overall hypotheses. We will sacrifice mathematical (but not physical) generality to gain mathematical and expositional simplicity. We shall also occassionally limit the generality of the systems of equations we treat in order to prevent undue proliferation in the possible results. Note that an undue proliferation of possible outcomes often suggests that the optimal approach is to treat each specific problem separately.

This completes this chapter on mathematical preliminaries. In subsequent chapters we will apply these results to study the stability/instability of traveling waves and the connection between the initial conditions and the mean wavespeeds. This will be done for the classes of equations

$$u_t = F(u_{xx}, u_x, u)$$

(and generalizations to multiple spatial dimensions) in Chapters IV and V, for

$$u_t = F(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} G^{(q)}(s,y,u(t-s,x-y))dyds)$$

(and generalizations to multiple spatial dimensions) in Chapter VI, and also for

$$u_t^{(k)} = F(u_{xx}^{(k)}, u_x^{(k)}, \tilde{u}) \qquad k = 1, \ldots, \nu$$

(and generalizations to multiple spatial dimensions) in Chapter VII. The results of Chapter II will be included in Chapters IV and V, and Chapter VIII will examine physical examples of each of the above types of systems.

Chapter IV

STABILITY FOR THE SIMPLEST CASE

In this chapter we deal mainly with parabolic equations which contain only one dependent variable, contain only one spatial variable, and contain no integrals. Throughout this chapter we will assume that the hypotheses H2 (smoothness of the equation), H3 (parabolicity of the equation), and H4 and H5 (existence of solutions to the initial value problem) are satisfied. We will also assume that a very large $M > 0$ has been chosen and we will work with the resulting specific equation

$$u_t = f(u_{xx},u_x,u)\ ,\ f_1 > 0, \quad \text{where} \quad f(u_{xx},u_x,u) \equiv F_M(\,\tilde{u}\,) \quad . \tag{4.1}$$

In this chapter we will almost exclusively be concerned with equation (4.1) over the domain $\{\forall(t,x)$ with x in $\mathbb{R}$ and $t \geq 0\}$. Specifically, in this chapter we will determine the stability/instability of very nearly every bounded traveling wave (and steady state) solution $u(t,x) \equiv \phi(x-ct)$. Thus some of the material in this present chapter is duplicated in chapter II. In this chapter we will treat only the traveling waves and steady states $u(t,x) \equiv \phi(x-ct)$ for which $|\phi(x-ct| < M$ for all x. This is sufficient since each bounded traveling wave or steady state $\phi(x-ct)$ satisfies $|\phi(x-ct)| < M$ for all x for all M sufficiently large. In addition, in this chapter we will determine the stability of all steady state solutions $u(t,x) = \phi(x)$ to the finite domain-boundary value problem

$$u_t = f(u_{xx},u_x,u) \quad \text{for} \quad 0 \leq x \leq 1\ ,\ t \geq 0$$

$$u(t,x) \equiv A \quad \text{at} \quad x = 0\ ,\ u(t,x) \equiv B \quad \text{at} \quad x = 1 \quad ,$$

where A and B are any given fixed constants.

This chapter has been organized into several short sections.

Basically the stability of monotonic traveling wave (and steady state) solutions of (4.1) over the infinite spatial domain is treated in sections (4.1) through (4.11). The instability of non-monotonic traveling wave (and steady state) solutions is treated in sections (4.12) through (4.17). Section (4.18) is used to discuss the stability/instability of the steady state solutions of the finite spatial domain-boundary value problem. Finally, the last section, (4.19), is used to summarize this chapter in broad terms.

To be more specific, in section (4.1) we discuss the phase plane for traveling wave solutions $u(t,x) = \phi(x-ct)$ of equation (4.1); that is, the phase plane of the system

$$\phi_x = v$$
$$f(v_x, v, \phi) + cv = 0 \quad .$$

In this discussion we pay particular attention to merged singular points as well as ordinary ones. In section (4.2) the stability for constant steady states is derived.

In sections (4.3) through (4.6) we actually derive the stability of non-constant monotonic traveling waves (and steady states). In section (4.3) we discuss the nature of these monotonic waves, especially the asymptotic (as $x \to \pm \infty$) nature. Next, the basic stability results are obtained in section (4.4). Then, better upper and lower functions are developed in section (4.5), and in section (4.6) these bounding functions are used to obtain our final sharp stability results for monotonic traveling wave (and steady state) solutions of (4.1).

The short sections (4.7) through (4.11) are used to discuss topics related to the stability of monotone waves. In section (4.7) we extend the stability results to the cases where either $\phi(-\infty)$ or $\phi(+\infty)$ (or both) are not nodes, saddle points, nor merged combinations of nodes and saddle points. Section (4.8) is used to describe explicitly how the stability of a monotonic wave depends on the function f. Next, in section (4.9) we compare our stability results with those obtainable by conventional eigenanalysis/variational methods. In a similar vein, in section (4.10) we show how the stability class for a monotonic wave $\phi(x-ct)$ splits the generalized null space of equation (4.1) linearized about ϕ. Finally, in section (4.11) we consider the extension of our stability results to multiple spatial dimensions. There we find that all our stability results are easily extended to monotonic plane waves (in higher spatial dimensions), and also that our methods are applicable to other types of monotonic traveling waves in multiple spatial dimensions.

We begin our derivation of the instability of non-monotonic waves in section (4.12). We state and prove the instability theorem in this section using lemmas which are proved in section (4.13). In section (4.12) we find that every traveling wave and steady state solution of equation (4.1) which has at least two relative extrema is extremely unstable. We also find that most traveling wave and steady state solutions which have exactly one relative extremum are also unstable. However, there is a type of traveling wave and steady state solution with exactly one extremum for which we cannot determine the stability or instability. We discuss this indeterminate case in section (4.14), where we are able to characterize which of those solutions are stable and which are unstable.

In sections (4.15) through (4.17) we discuss topics related to the instability of non-monotonic waves. We point out how the instability proof can be adapted to strengthen the instability results for some types of constant steady states in section (4.15). We use section (4.16) to comment on some other uses of the hair-trigger effect, and in section (4.17) we extend the instability results (in a limited sense) to the multiple spatial dimensions case.

In section (4.18) we treat the stability/instability of steady state solutions to the finite domain boundary value problem

$$u_t = f(u_{xx}, u_x, u) \qquad 0 \leq x \leq 1 \ , \ t \geq 0$$
$$u(t,x) \equiv A \quad \text{at} \quad x = 0 \qquad u(t,x) \equiv B \quad \text{at} \quad x = 1 \quad ,$$

where A and B are fixed constants. We again find that steady state solutions with at least two relative extrema are unstable, solutions with exactly one relative extremum can be stable or unstable, and solutions with no relative extrema are stable.

Finally, we will conclude this chapter in section (4.19) with some general overall remarks.

4.1 Singular points in the phase plane. As in Chapters I and II, we begin by converting traveling wave solutions $u(t,x) \equiv \phi(x-ct)$ of

$$u_t = f(u_{xx}, u_x, u) \tag{4.1}$$

into steady state solutions $u(t,x) \equiv \phi(x)$ of

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad . \tag{4.2}$$

We do this by switching to a new coordinate system t', x' which travels with speed c relative to the original stationary coordinates:

$$t' = t \qquad x' = x - ct \quad . \tag{4.3}$$

For convenience we drop the prime superscripts on the t' and x' and thus obtain (4.2). There is no possibility of confusion if we remember that the parameter c in (4.2) can be used to obtain the original stationary coordinates by

$$t_{stat} = t \qquad x_{stat} = x + ct \quad .$$

Thus, instead of studying the traveling wave solutions of (4.1), we have chosen to study the steady states of (4.2) for each value of c.

A typical method of examining the steady state solutions of (4.2) (at any fixed value of c) is to go to the phase plane. We write (4.2) as the equivalent first order system

$$\begin{aligned} \phi_x &= v \\ f(v_x, v, \phi) + cv &= 0 \quad , \end{aligned} \tag{4.4}$$

and note that the phase plane is the graphical representation of the solutions $\phi_x(\phi,v)$, $v_x(\phi,v)$ of (4.4).

The crucial points $\phi = \phi_0$, $v = v_0$ in the phase plane are the singular points of (4.4); i.e. the points (ϕ_0, v_0) for which (4.4) implies that $\phi_x = v_x = 0$ when $\phi = \phi_0$, $v = v_0$. From (4.4) we see that $\phi = \phi_0$, $v = v_0$ is a singular point if and only if

$$v_0 = 0 \qquad f(0,0,\phi_0) = 0 \quad .$$

Thus $\phi = \phi_0$, $v = 0$ is a singular point for all values of c if it is a singular point at any speed $c = c_0$.

These singular points are crucial because if $u(t,x) \equiv \phi(x)$ is a monotonic steady state solution of (4.2), then $\phi = \phi(-\infty)$, $v = \phi_x(-\infty)$ and $\phi = \phi(+\infty)$, $v = \phi_x(+\infty)$ must be singular points. Moreover, as we discovered in Chapter II the stability of a monotonic steady state solution $u(t,x) \equiv \phi(x)$ depends heavily on which types of singular points $\phi(-\infty)$ and $\phi(+\infty)$ are.

The usual classification of singular points is as nodes, spiral points, and saddle points. We note that the usual definitions of these types of singular points are that

$\phi = \phi_0$, $v = 0$ is a node of (4.4) when $f(0,0,\phi_0) = 0$, $f_3(0,0,\phi_0) > 0$ and for values of c such that $|f_2(0,0,\phi_0) + c| \geq 2\sqrt{f_1(0,0,\phi_0)f_3(0,0,\phi_0)}$,

$\phi = \phi_0$, $v = 0$ is a spiral point of (4.4) when $f(0,0,\phi_0) = 0$, $f_3(0,0,\phi_0) > 0$ and for values of c such that $|f_2(0,0,\phi_0) + c| < 2\sqrt{f_1(0,0,\phi_0)f_3(0,0,\phi_0)}$,

$\phi = \phi_0$, $v = 0$ is a saddle point of (4.4) when $f_3(0,0,\phi_0) < 0$ for all values of c.

Note that these definitions include centers as a special case of spiral points. We will now extend the definitions of nodes and saddle points to cover the case where $f_3(0,0,\phi_0) = 0$ and which can be thought of as two or more singular points merged together at $\phi = \phi_0$, $v = 0$.

We first define a singular point $\phi = \phi_0$, $v = 0$ of system (4.4) to be regular when $f(0,0,\phi)$ has a zero of order m at $\phi = \phi_0$ for some positive integer m. Specifically, for m a positive integer we define the point $\phi = \phi_0$, $v = 0$ to be a <u>regular singular point of order m</u> if and only if there exists a $\mu \neq 0$ and a $q > 0$ such that

$$\begin{aligned} f(0,0,\phi) &= \mu(\phi-\phi_0)^m + 0(|\phi-\phi_0|)^{m+q}) \quad \text{as} \quad \phi \to \phi_0 \\ f_3(0,0,\phi) &= \mu m(\phi-\phi_0)^{m-1} + 0(|\phi-\phi_0|^{m+q-1}) \quad \text{as} \quad \phi \to \phi_0 \quad . \end{aligned} \tag{4.5}$$

We now extend the definition of 'node' and 'saddle point' to cover all possible regular singular points. We first realize that a higher order singular point $(\phi_0,0)$ can behave differently for $\phi \gtrsim \phi_0$ than for $\phi \lesssim \phi_0$. For example, it can be node-like for $\phi \gtrsim \phi_0$ and saddle-like for $\phi \lesssim \phi_0$. Hence we shall use both a $+$ designation (for the behavior when $\phi \gtrsim \phi_0$) and a $-$ designation (for the behavior when $\phi \lesssim \phi_0$) for a singular point $(\phi_0,0)$.

Note that ordinary first order nodes (and spiral points) have $f(0,0,\phi) > 0$ for $\phi \gtrsim \phi_0$ and have $f(0,0,\phi) < 0$ for $\phi \lesssim \phi_0$. Note also that the range of values of c for which $\phi = \phi_0$, $v = 0$ can be a spiral point (or center) collapses to nothing as $f_3(0,0,\phi_0) \to 0$. Moreover, first order saddle points have $f(0,0,\phi) < 0$ for $\phi \gtrsim \phi_0$ and have $f(0,0,\phi) > 0$ for $\phi \lesssim \phi_0$. We therefore define a regular singular point of order m $(m \geq 2)$ to be a + node (+ saddle) if $\mu(\phi-\phi_0)^m > 0$ (if $\mu(\phi-\phi_0)^m < 0$) for $\phi > \phi_0$. Similarly, we define a regular singular point of order m $(m \geq 2)$ to be a - node (- saddle) if $\mu(\phi-\phi_0)^m < 0$ (if $\mu(\phi-\phi_0)^m > 0$) for $\phi < \phi_0$. For completeness, if $(\phi_0, 0)$ is an ordinary first order node (first order saddle) we will designate it as a - node and as a + node (as a - saddle and as a + saddle). For brevity, we will designate a regular singular point as a + N, a - N, a + S, or a - S to denote that it is a + node, a - node, a + saddle, or a - saddle, respectively. When we need to refer to the behavior of $f(0,0,\phi)$ for both $\phi \gtrsim \phi_0$ and $\phi \lesssim \phi_0$, we will designate a regular singular point $(\phi_0, 0)$ as a N, S, NS, or SN if it is both a - N and a + N, both a - S and a + S, both a - N and a + S, or both a - S and a + N, respectively.

This completes our discussion of singular points in the phase plane. Although we have not shown that these extensions of the definitions of node and saddle point are reasonable, if one solves the asymptotic formula

$$f_1(0,0,\phi_0)(\phi-\phi_0)_{xx} + (f_2(0,0,\phi_0) + c)(\phi-\phi_0)_x + \mu(\phi-\phi_0)^m = 0$$

and plots the resulting solutions in the phase plane near $\phi = \phi_0$, $\phi_x \equiv v = 0$, one sees that locally ϕ_0 behaves exactly like an ordinary node (like a saddle) for $\phi \gtrsim \phi_0$ when it is a + N (when it is a + S).

Similarly it behaves exactly like an ordinary node (like a saddle) for $\phi \lesssim \phi_0$ when it is a $-$ N (when it is a $-$ S).

Although it is not mathematically necessary, subsequently we will usually work only with regular singular points. This simplifies both the mathematical details and the exposition. Thus we will often use the following hypothesis

H6: All singular points $\phi = \phi_0$, $v = 0$ with $|\phi_0| < M$ of the system

$$\begin{aligned} \phi_x &= v \\ f(v_x, v, \phi) + cv &= 0 \end{aligned} \tag{4.4}$$

are regular.

Since there are physically interesting examples where the singular points of (4.4) are not regular (notably equations like Burger's equation where $f(0,0,\phi) \equiv 0$ for all ϕ), often we will point out generalizations of our theorems in remarks following the theorems.

We now use these definitions in deriving the stability of constant steady states. Later they will prove important in determining the stability of monotone waves.

<u>4.2 Stability definitions. Stability of constant steady states</u>. The stability of constant steady states is very simple. We will therefore simply state and prove the result. However, in order to state the result precisely, we need to reintroduce the definitions of C^w-stability and $\mathcal{C}^w$-stability. These definitions will be exactly like those used in Chapter II, <u>except</u> that the perturbations of the initial conditions will now be required to satisfy the smoothness conditions needed by the existence assumption H4. Specifically:

A function $\Psi(x)$ is defined to be in the class H_x^2 if and only if $\Psi(x)$ is defined for all x in $(-\infty, \infty)$, $\Psi(x)$ is twice differentiable everywhere, $\Psi(x)$, $\Psi'(x)$, and $\Psi''(x)$ are bounded, and $\Psi''(x)$ is uniformly Hoelder continuous with some exponent $\alpha > 0$.

Let $w(x)$ be any continuous function with $w(x) \geq 1$ for all x. Then any steady state solution $u(t,x) \equiv \phi(x)$ of equation

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{4.2}$$

is defined to be <u>C^w-stable</u> if and only if given any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that every solution $u(t,x)$ of (4.2) satisfies

$$|\{u(t,x) - \phi(x)\}w(x)| \leq \varepsilon \quad \text{for all } x \text{ and all } t > 0 \tag{4.6}$$

whenever the initial conditions $u(0,x)$ are in H_x^2 and satisfy

$$|\{u(0,x) - \phi(x)\}w(x)| \leq \delta(\varepsilon) \quad \text{for all } x \quad . \tag{4.7}$$

Similarly, $\phi(x)$ is defined to be <u>$\mathcal{C}^w$-stable</u> if and only if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that every solution $u(t,x)$ of (4.2) satisfies

$$|u(t,x) - \phi(x)| \leq \varepsilon \quad \text{for all } x \text{ and all } t > 0 \tag{4.8}$$

whenever the initial conditions $u(0,x)$ are in H_x^2 and satisfy relation (4.7).

These stability definitions are precisely the stability definitions of Chapter II, except that the phrases "the initial conditions $u(0,x)$ are smooth" have been replaced with "the initial conditions $u(0,x)$ are in H_x^2". Thus the physical interpretations of these stability definitions remains the same as in Chapter II. Note also that H_x^2 has been defined so that $u(0,x) \in H_x^2$ is exactly the smoothness condition needed by the existence hypotheses H4.

With these new definitions, we easily state and prove the stability results for constant steady states.

Theorem 4.1: Suppose that hypotheses H2, H3, H4, and H6 are satisfied. Suppose further that $u(t,x) \equiv \phi_0$ is a constant steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{4.2}$$

and that $|\phi_0| < M$. Then $\phi = \phi_0$, $v = 0$ is a regular singular point of order m (for some positive integer m), and

(1) if $\phi = \phi_0$, $v = 0$ is a $+ S$ then $u(t,x) \equiv \phi_0$ is C^w-stable with $w(x) \equiv 1$ if the perturbations are restricted to be non-negative;

(2) if $\phi = \phi_0$, $v = 0$ is a $- S$ then $u(t,x) \equiv \phi_0$ is C^w-stable with $w(x) \equiv 1$ if the perturbations are restricted to be non-positive;

(3) if $\phi = \phi_0$, $v = 0$ is a S then $u(t,x) \equiv \phi_0$ is C^w-stable with $w(x) \equiv 1$;

(4) if $\phi \equiv \phi_0$, $v = 0$ is a $+ N$ type singular point of order m or is a spiral point or center, then $\phi(x)$ is $\mathfrak{C}^w$-unstable with $w(x) \equiv 1$. Further, if it is a $+ N$ of order 1, a spiral point, or a center then it is $\mathfrak{C}^w$-unstable with $w(x) \equiv 1 + e^{-\kappa x} + e^{+\kappa x}$ for $\kappa > 0$ sufficiently small, and if it is a $+ N$ of order 2 then it is $\mathcal{L}^2$ unstable. Moreover, u has these instabilities even if we restrict the perturbations to be non-negative; and

(5) if $\phi = \phi_0$, $v = 0$ is a $- N$ type singular point of order m or a spiral point or center, then $u(t,x) \equiv \phi_0$ has the same instability as in case (4) except that the perturbations can now only be restricted to being non-positive.

Thus, in rough terms constant steady states are very stable if they are saddle points and are unstable if they are nodes, spiral points, or centers. Furthermore, the weaker (i.e., higher order) the node the weaker the instability.

Proof: To prove the stability parts of the theorem, we will define appropriate smooth upper and lower functions $\overline{u}(t,x)$ and $\underline{u}(t,x)$. The maximum principle will then yield stability.

For part (1), define $\overline{u}(h,t,x)$ and $\underline{u}(t,x)$ by

$$\overline{u}(h,0,x) \equiv \phi_0+h, \quad \overline{u}_t(h,t,x) \equiv f(0,0,\overline{u}(h,t,x)) \quad (h>0), \quad \underline{u}(t,x) \equiv \phi_0 . \tag{4.9}$$

Note that $\overline{u}$ and $\underline{u}$ are both solutions of equation (4.2). Moreover, since $\phi = \phi_0$, $v = 0$ is a $+S$ then $f(0,0,\overline{u})$ and hence $\overline{u}_t$ are both negative for all $h > 0$ sufficiently small. Thus for all $h > 0$ small enough $\overline{u}_t < 0$ for all $t > 0$. Now suppose that $u(0,x)$ is any initial condition in H^2_x. Then the solution $u(t,x)$ of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{4.2}$$

exists for all $t \geq 0$ and has u, u_x, and u_{xx} bounded. If $u(0,x)$ also satisfies

$$\phi_0 \leq u(0,x) \leq \phi_0 + h \quad \text{for all} \quad x$$

for any $h > 0$ small enough, we conclude from the maximum principle that

$$\phi_0 \equiv \underline{u}(t,x) \leq u(t,x) \leq \overline{u}(h,t,x) < \overline{u}(h,0,x) \equiv \phi_0 + h \quad \text{for all} \quad x$$

is satisfied for all $t > 0$. Thus $u(t,x) \equiv \phi_0$ is C^w-stable with $w(x) \equiv 1$ if the perturbations $u(0,x) - \phi_0$ are restricted to be non-negative.

Part (2) is proved similarly.

To prove part (3), we define $\overline{u}(t,h,x)$ and $\underline{u}(t,h,x)$ for $h > 0$ by

$$\begin{aligned} \overline{u}(h,0,x) &\equiv \phi_0 + h \qquad \overline{u}_t = f(0,0,\overline{u}) \\ \underline{u}(h,0,x) &\equiv \phi_0 - h \qquad \underline{u}_t = f(0,0,\underline{u}) \quad . \end{aligned} \tag{4.10}$$

Since $\phi = \phi_0$, $v = 0$ is an S, we conclude that whenever $h > 0$ is small enough then $\overline{u}_t(h,0,x) < 0$ for all $t > 0$ and $\underline{u}_t(h,0,x) > 0$ for all $t > 0$. Thus, the existence assumption H4 together with the maximum principle imply that whenever $u(0,x)$ is in H_x^2 and

$$\phi_0 - h \leq u(0,x) \leq \phi_0 + h \quad \text{for all} \quad x$$

is satisfied for any $h > 0$ small enough, then the solution $u(t,x)$ of (4.2) with initial condition $u(0,x)$ exists and satisfies

$$\phi_0 - h \equiv \underline{u}(h,0,x) < \underline{u}(h,t,x) \leq u(t,x) \leq \overline{u}(h,t,x) < \overline{u}(h,0,x) \equiv \phi_0 + h$$
$$\text{for all} \quad x$$

and for all $t > 0$. Thus $u(t,x) \equiv \phi_0$ is C^w-stable with $w(x) \equiv 1$.

We now prove part (4), first dealing with the case that $\phi = \phi_0$, $v = 0$ is a node of order 1, a spiral point, or a center. In these cases, $f_3(0,0,\phi_0) \equiv \mu > 0$. Thus define

$$\underline{u}(h,t,x) \equiv \phi_0 + he^{\mu t/4} \operatorname{sech} \kappa x$$

where $h > 0$ and $\kappa > 0$ are at our disposal. Define

$$A \equiv \sup_{\alpha,\beta,\gamma} |f_1(\alpha,\beta,\gamma)| \ , \ B \equiv \sup_{\alpha,\beta,\gamma} |f_2(\alpha,\beta,\gamma) + c| \quad ,$$

and let $\kappa > 0$ be any constant small enough so that

$$A\kappa^2 < \mu/4 \ , \ B\kappa < \mu/4 \quad .$$

Then,

$$\underline{u}_t - f(\underline{u}_{xx},\underline{u}_x,\underline{u}) - c\underline{u}_x \leq \mu/4(\underline{u}-\phi_0) - f_1\underline{u}_{xx} - (f_2+c)\underline{u}_x - \mu(\underline{u}-\phi_0) + \text{h.o.t.s.}$$

where $f_1 \equiv f_1(0,0,\phi_0)$, $f_2 \equiv f_2(0,0,\phi_0)$, and "h.o.t.s" are "higher order terms". Substituting for $\underline{u}$, we find that

$$\underline{u}_t - f(\underline{u}_{xx},\underline{u}_x,\underline{u}) - c\underline{u}_x \leq -\frac{\mu h}{4} e^{\mu t/4} \operatorname{sech} \kappa x + \text{h.o.t.s} \quad . \tag{4.11}$$

Since f_1, f_2, and f_3 are uniformly Hoelder continuous with some non-zero exponent, there exists an h_0 such that

$$\underline{u}_t - f(\underline{u}_{xx},\underline{u}_x,\underline{u}) - c\underline{u}_x \leq 0 \quad \text{for all} \quad x \tag{4.12}$$

whenever $h > 0$ and $t \geq 0$ are such that $0 < he^{\mu t/4} < h_0$.

In order to see that (4.12) implies instability, define $u(h,t,x)$ as the solutions of (4.2) with $u(h,0,x) = \underline{u}(h,0,x)$ for $0 < h < h_0$. Since the $u(h,0,x)$ are in H_x^2, the existence assumption H4 and the maximum principle together imply that $u(h,t,x)$ exists and

$$\underline{u}(h,t,x) \leq u(h,t,x) \quad \text{for all} \quad x$$

whenever $0 < h < h_0$ and for all $t \geq 0$ such that $0 < he^{\mu t/4} < h_0$. Since $\underline{u}(h,t,x) = \phi_0 + he^{\mu t/4}$ sech κx and $u(h,0,x) = \phi_0 + h$ sech κx, $u(t,x) \equiv \phi_0$ is $\mathfrak{C}^w$-unstable with $w(x) \equiv 1 + e^{\kappa x} + e^{-\kappa x}$, even if the perturbations are restricted to be non-negative.

Suppose now that $\phi = \phi_0$, $v = 0$ is a $+$ N of order $m \geq 2$. Consider

$$\underline{u}(t_0,x_0,t,x) \equiv \phi_0 + \{(t_0-t)\sqrt{x^2+x_0^2}\}^{-\frac{1}{m-1}}\sqrt{x^2+x_0^2}^{\,\eta} \tag{4.13}$$

where η is a fixed constant in $(0, \frac{1}{m-1})$. Let $t_0 > 0$ be any fixed positive number. By an analysis similar to the preceding case, there is an $h_0 > 0$ such that

$$\underline{u}_t - f(\underline{u}_{xx},\underline{u}_x,\underline{u}) - c\underline{u}_x \leq 0$$

whenever $t_0 > t \geq 0$ and $x_0 > 0$ are such that

$$(t_0-t)^{-\frac{1}{m-1}}|x_0|^{\eta-\frac{1}{m-1}} < h_0 .$$

Since x_0 can be arbitrarily large (and thus $|\underline{u}(t_0,x_0,t,x) - \phi_0|$ is arbitrarily small), defining $u(t_0,x_0,t,x)$ as the solution of (4.2) with initial condition $u(t_0,x_0,0,x)$ shows that $u(t,x) \equiv \phi_0$ is $\mathfrak{C}^w$-unstable with $w(x) \equiv 1$ as in the preceding case. Note that when $m = 2$, $u(t_0,x_0,0,x) - \phi_0$ is $\mathcal{L}^2$ integrable when η is chosen to be in $(0,\frac{1}{4})$.

This completes part (4).

Part (5) is proved similarly.

This completes the stability picture for constant steady states $u(t,x) \equiv \phi_0$ when ϕ_0 is a regular singular point. Note that equations (4.9) and (4.10) show that when ϕ_0 is + S, a − S, or a S then small non-negative perturbations, small non-positive perturbations, and all perturbations (respectively) decay exponentially in time (if $m = 1$), or like $t^{-\frac{1}{m-1}}$ (if $m > 1$). Similarly, perturbations about ϕ_0 grow exponentially in time (if $m = 1$) or like $(t_0-t)^{-\frac{1}{m-1}}$ (if $m > 1$) when the perturbations are restricted to a node-like region.

Lastly, we note that the stability of a constant steady state solution $u(t,x) \equiv \phi_0$ when ϕ_0 is not a <u>regular</u> singular point can be found by the same technique. This results in $u(t,x) \equiv \phi_0$ being

1) C^w-stable for positive perturbations if $f(0,0,\phi) \leq 0$ for all ϕ in (ϕ_0,ϕ_0+h) for some $h > 0$,

2) $\not{C}^w$-unstable for positive perturbations if $f(0,0,\phi) > 0$ for all ϕ in (ϕ_0,ϕ_0+h) for some $h > 0$,

3) C^w-stable for negative perturbations if $f(0,0,\phi) \geq 0$ for all ϕ in (ϕ_0-h,ϕ_0) for some $h > 0$,

4) $\not{C}^w$-unstable for negative perturbations if $f(0,0,\phi) < 0$ for all ϕ in (ϕ_0-h,ϕ_0) for some $h > 0$, and

5) C^w-stable if $f(0,0,\phi) \leq 0$ for all ϕ in (ϕ_0,ϕ_0+h) and if $f(0,0,\phi) \geq 0$ for all ϕ in (ϕ_0-h,ϕ_0) for some $h > 0$.

In the above, $w(x) \equiv 1$.

This concludes our analysis of constant steady states. In the next section, we begin our analysis of monotone steady states.

4.3 Nature of monotone traveling waves. For this section we assume that $u(t,x) \equiv \phi(x)$ is a bounded non-constant monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad . \tag{4.2}$$

From this assumption we will deduce some facts about $\phi(x)$. These facts will be needed in the actual derivation of the stability results for non-constant bounded monotonic steady state solutions of (4.2).

We first note that since $u(t,x) \equiv \phi(x)$ is monotonic and non-constant, $\phi_x(x) \neq 0$ for all x. This is because $\phi(x)$ being monotonic implies that either $\phi_x(x) \geq 0$ for all x or $\phi_x(x) \leq 0$ for all x. Hence if $\phi_x(x) = 0$ at some $x = x_0$ then we would have $\phi_x(x_0) = \phi_{xx}(x_0) = 0$. Thus, equation (4.2) would imply that

$$f(0,0,\phi(x_0)) = 0$$

and hence that $u(t,x) \equiv \phi(x_0)$ is a constant steady state solution of equation (4.2). By the uniqueness of solutions of ordinary differential equations, this implies that $\phi(x) \equiv \phi(x_0)$ for all x. Hence $\phi(x)$ being a non-constant monotonic steady state solution of equation (4.2) implies that $\phi_x(x) \neq 0$ for all x.

We now note that ϕ_x, ϕ_{xx}, and ϕ_{xx}/ϕ_x are all bounded. Specifically, because equation (4.2) reduces to a heat equation whenever $|\phi_x| > 2M$, either $|\phi_x(x)| < 2M$ for all x or $|\phi(x)|$ grows linearly for all x sufficiently large or small. Since this latter case violates the boundedness of $\phi(x)$, we see that ϕ_x is bounded for all x. The function ϕ_{xx} is also clearly bounded. This is because ϕ and ϕ_x are bounded, because $f_1(\alpha,\beta,\gamma) \geq \delta_M > 0$ for some $\delta_M > 0$ and for all arguments α,β,γ, and finally because

$$f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0 \quad . \tag{4.14}$$

We already know that ϕ_{xx}/ϕ_x is bounded for all x in any finite interval and that ϕ_{xx}/ϕ_x is continuous. Since $\phi \to \phi(+\infty)$ and $\phi_x \to 0$ as $x \to +\infty$, equation (4.14) implies that $\phi_{xx} \to 0$ as $x \to +\infty$ as well. Thus, asymptotically the formula

$$f_1\ \phi_{xx} + (f_2+c)\phi_x + f_3(\phi-\phi(+\infty)) = 0 \quad \text{as} \quad x \to +\infty$$

must be satisfied, where the arguments of f_1, f_2, and f_3 are $(0,0,\phi(+\infty))$. We note that solving this equation shows that ϕ_{xx}/ϕ_x remains bounded as $x \to +\infty$ because $f_1 > 0$ and f_2 and f_3 are finite. Similarly ϕ_{xx}/ϕ_x remains bounded as $x \to -\infty$, and is therefore bounded for all x.

We also note that solving the asymptotic equation shows that $|\phi_{xx}|$ is decreasing for all x and all $-x$ sufficiently large.

The rest of this section treats the asymptotic nature (as $x \to -\infty$ and $x \to +\infty$) of monotonic solutions of the steady state equation, (4.14). This is done only for the cases where $\phi(-\infty)$ and $\phi(+\infty)$ are both regular singular points. These results are contained in table 4.1, and are derived by solving the asymptotically valid formula

$$f_1\ \phi_{xx} + (f_2+c)\phi_x + \mu(\phi-\phi_0)^m = 0 \quad . \tag{4.15}$$

Here, ϕ_0 is the appropriate one of $\phi(-\infty)$ and $\phi(+\infty)$, $f_1 = f_1(0,0,\phi_0)$, $f_2 = f_2(0,0,\phi_0)$, m is the order of the singular point $\phi = \phi_0$, $v = 0$, and μ is the correct coefficient. For brevity table 4.1 contains only the asymptotic decay rates of $\phi(x)$ as $x \to -\infty$ and $x \to +\infty$ for the case of $\phi(x)$ being monotonically increasing. This is sufficient since replacing x by $-x$ changes a decreasing function into an increasing function. Also for brevity, the table only contains the asymptotic nature of ϕ. The asymptotic decay rates of ϕ_x can correctly be obtained by formally differentiating the asymptotic formulas for ϕ.

Table 4.1

Part 1: Asymptotic nature of $\phi(x)$ as $x \to +\infty$, $\phi'(x) > 0$ everywhere

Asymptotic equation: $f_1\,\phi_{xx} + (f_2+c)\phi_x + \mu|\phi-\phi(+\infty)|^m = 0$

$$\lambda_{1,2} \equiv \left\{ \begin{array}{ll} \dfrac{-(f_2+c)\pm\sqrt{(f_2+c)^2+4\mu f_1}}{2f_1} & \text{if} \quad m = 1 \\ \\ \dfrac{-(f_2+c) \pm |f_2+c|}{2f_1} & \text{if} \quad m = 2,3,\ldots \end{array} \right\}, \text{where } \lambda_1 \le \lambda_2$$

Case 1: $\phi = \phi(+\infty)$, $v = 0$ is a S^-. Requirements: $\mu > 0$

if $m = 1$ $\qquad \phi \sim \phi(+\infty) + a_0 e^{\lambda_1 x} + O(e^{(\lambda_1-\delta)x})$

if $m>1$, $f_2+c>0$ $\qquad \phi \sim \phi(+\infty) + a_0 e^{\lambda_1 x} + O(e^{(\lambda_1-\delta)x})$

if $m>1$, $f_2+c<0$ $\qquad \phi \sim \phi(+\infty) + a_0 x^{-\frac{1}{m-1}} + O(x^{-\frac{1}{m-1}-\delta})$

if $m>1$, $f_2+c=0$ $\qquad \phi \sim \phi(+\infty) + a_0 x^{-\frac{2}{m-1}} + O(x^{-\frac{2}{m-1}-\delta})$

for some constants $a_0 < 0$ and $\delta > 0$.

Case 2: $\phi = \phi(+\infty)$, $v = 0$ is a N^-. Requirements: $\mu < 0$, $(f_2+c) > 0$,

and when $m = 1$ $(f_2+c)^2 \ge -4\mu f_1$

if $m = 1$, $(f_2+c)^2 > -4\mu f_1$ $\qquad \phi \sim \phi(+\infty)+a_0 e^{\lambda_2 x}+O(e^{(\lambda_2-\delta)x})$ (usual)

or $\phi \sim \phi(+\infty)+a_0 e^{\lambda_1 x}+O(e^{(\lambda_1-\delta)x})$ (accidental)

if $m = 1$, $(f_2+c)^2 = -4\mu f_1$ $\qquad \phi \sim \phi(+\infty)+a_0 x e^{\lambda_2 x}+O(e^{\lambda_2 x})$ (usual)

or $\phi \sim \phi(+\infty)+a_0 e^{\lambda_2 x}+O(e^{(\lambda_2-\delta)x})$ (accidental)

if $m > 1$, $(f_2+c)>0$ $\qquad \phi \sim a_0 x^{-\frac{1}{m-1}} + O(x^{-\frac{1}{m-1}-\delta})$ (usual)

or $\phi \sim a_0 e^{\lambda_1 x} + O(e^{(\lambda_1-\delta)x})$ (accidental)

for some constants $a_0 < 0$ and $\delta > 0$.

Other cases: No solution decays to $\phi = \phi(+\infty)$ as $x \to +\infty$ if $\mu < 0$,

$(f_2+c) \le 0$, and $m > 1$. No <u>monotone</u> solution decays to $\phi = \phi(+\infty)$ as $x \to +\infty$ when $\mu < 0$, $m = 1$, and $(f_2+c)^2 < -4\mu f_1$ or $(f_2+c) \le 0$. The case $\mu = 0$ for all m is an excluded irregular singular point case.

Part 2: Asymptotic nature of $\phi(x)$ as $x \to -\infty$, $\phi'(x) > 0$ everywhere

Asymptotic equation: $f_1\phi_{xx} + (f_2+c)\phi_x + \mu|\phi-\phi(-\infty)|^m = 0$

$$\lambda_{1,2} \equiv \left\{\begin{array}{ll} \dfrac{-(f_2+c) \pm\sqrt{(f_2+c)^2-4\mu f_1}}{2f_1} & \text{if} \quad m = 1 \\[2ex] \dfrac{-(f_2+c) \pm |f_2+c|}{2f_1} & \text{if} \quad m = 2,3,\ldots \end{array}\right\}, \text{ where } \lambda_1 \le \lambda_2$$

Case 1: $\phi = \phi(-\infty)$, $v = 0$ is a S^+. Requirements: $\mu < 0$.

if $m = 1$ $\qquad \phi \sim \phi(-\infty)+a_0e^{\lambda_2 x}+0(e^{(\lambda_2+\delta)x})$

if $m>1$, $(f_2+c)<0$ $\qquad \phi \sim \phi(-\infty)+a_0e^{\lambda_2 x}+0(e^{(\lambda_2+\delta)x})$

if $m>1$, $(f_2+c)>0$ $\qquad \phi \sim \phi(-\infty)+a_0|x|^{-\frac{1}{m-1}} + 0(|x|^{-\frac{1}{m-1}-\delta})$

if $m>1$, $(f_2+c)=0$ $\qquad \phi \sim \phi(-\infty)+a_0|x|^{-\frac{2}{m-1}} + 0(|x|^{-\frac{2}{m-1}-\delta})$

for some constants $a_0 > 0$ and $\delta > 0$.

Case 2: $\phi = \phi(-\infty)$, $v = 0$ is a N^+. Requirements: $\mu > 0$, $(f_2+c) < 0$ and when $m = 1$ $(f_2+c)^2 \ge 4\mu f_1$.

if $m = 1$, $(f_2+c)^2>4\mu f_1$ $\qquad \phi \sim \phi(-\infty)+a_0e^{\lambda_1 x}+0(e^{(\lambda_1+\delta)x})$ (usual)

or $\phi \sim \phi(-\infty)+a_0e^{\lambda_2 x}+0(e^{(\lambda_2+\delta)x})$ (accidental)

if $m = 1$, $(f_2+c)^2=4\mu f_1$ $\qquad \phi \sim \phi(-\infty)+a_0|x|e^{\lambda_1 x}+0(e^{\lambda_1 x})$ (usual)

or $\phi \sim \phi(-\infty)+a_0e^{\lambda_1 x}+0(e^{(\lambda_1+\delta)x}$ (accidental)

if $m > 1$, $(f_2+c)<0$ $\qquad \phi \sim \phi(-\infty)+a_0|x|^{-\frac{1}{m-1}}+0(|x|^{-\frac{1}{m-1}-\delta})$ (usual)

or $\phi \sim \phi(-\infty)+a_0e^{\lambda_2 x}+0(e^{(\lambda_2+\delta)x})$ (accidental)

for some constants $a_0 > 0$ and $\delta > 0$.

Other cases: No solution decays to $\phi = \phi(-\infty)$ as $x \to -\infty$ if $\mu > 0$, $(f_2 + c) \ge 0$, and $m > 1$. No <u>monotone</u> solution decays to $\phi = \phi(-\infty)$

when $\mu > 0$, $m = 1$, and $(f_2 + c)^2 < 4\mu f_1$ or $(f_2 + c) \geq 0$. The case $\mu=0$ for all m is an excluded irregular singular point case.

In the table above, there are two possible asymptotic natures listed when $\phi(x)$ decays to a node $\phi = \phi_0$, $v = 0$ as $x \to -\infty$ or as $x \to +\infty$. The slowest of these decays is labeled the "usual" decay and the more rapid is labeled the "accidental" decay. Whenever the conditions of case 2 of part 1 hold, there exists a solution $\Psi(x)$ of $f(\Psi_{xx},\Psi_x,\Psi) + c\Psi_x$ for which $\Psi(+\infty) = \phi_0$, for which $\Psi'(x) > 0$ for all x sufficiently large, and for which $\Psi(x)$ decays to ϕ_0 at the accidental rate as $x \to +\infty$. Any monotonically increasing solution $\phi(x)$ of $f(\phi_{xx},\phi_x,\phi) + c\phi_x$ which goes to ϕ_0 as $x \to +\infty$ must either be $\Psi(x+h)$ for some constant h, or it must decay at the usual rate. Similarly, there can be only one solution $\Psi(x)$ (modulo translations in x) of $f(\Psi_{xx},\Psi_x,\Psi) + c\Psi_x$ which decreases to a node at the accidental decay rate as $x \to -\infty$. Note that in this light, solutions $\phi(x)$ of $f(\phi_{xx},\phi_x,\phi)+c\phi_x$ which decay monotonically to a saddle point $\phi = \phi_0$, $v = 0$ as $x \to -\infty$ or as $x \to +\infty$ must be considered "accidentally decaying" solutions. This is because there can only be one monotonically increasing solution $\Psi(x)$ (modulo translations in x) which decays to a saddle point as $x \to -\infty$ or as $x \to +\infty$.

The asymptotic nature of monotonic solutions which decay to an irregular singular point as $x \to -\infty$ or $x \to +\infty$ can also be found by solving the appropriate asymptotic equation. We will not do this; instead we note that for any specific example it is a straightforward procedure.

Note that we have proven only a few of the general results concerning the behavior of solutions $\phi(x)$ of equations (4.14) contained in this section. However, we shall presume that these results are correct, noting for any <u>specific</u> solution $\phi(x)$ to any <u>specific</u> equation that verification of these results is trivial.

In the next three sections we will use these results to derive the stability results for monotonic waves. These sections will closely follow the developments in section (2.2).

<u>4.4 Basic stability results for monotone waves</u>. We now derive the basic stability result for monotone waves. The result is equivalent to theorem (2.2) in Chapter II.

We can easily demonstrate that a monotonic steady state solution $u(t,x) \equiv \phi(x)$ of

$$u_t = f(u_{xx}, u_x, u) + cu_x \qquad (4.2)$$

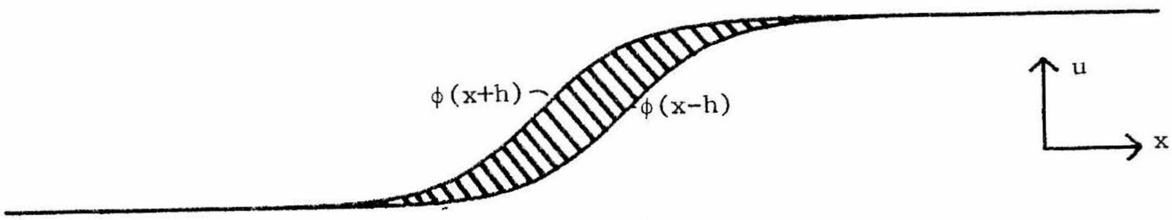
must possess at least a limited amount of stability. Suppose for example that $\phi(x)$ is monotonically increasing in x. Then for any $h > 0$ (no matter how small) $\phi(x-h)$ and $\phi(x+h)$ also solve (4.2). So, when $u(0,x)$ is any smooth initial condition with

$$\phi(x-h) \leq u(0,x) \leq \phi(x+h) \quad , \qquad (4.16)$$

then the maximum principle shows that

$$\phi(x-h) \leq u(t,x) \leq \phi(x+h) \qquad (4.17)$$

for all $t > 0$ as well. That is, $u(t,x)$ must remain in the shaded region for all $t \geq 0$ in the illustration below. To obtain a definite stability result, we need only identify the class of functions which can be bounded as in (4.16) and (4.17).



Theorem 4.2: Suppose that hypotheses H2, H3, and H4 are satisfied. If $u(t,x) \equiv \phi(x)$ is any bounded non-constant monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \ , \tag{4.2}$$

then it is a C^w-stable solution with $w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$.

Note that H6 (all singular points are regular) has not been assumed. Thus the theorem holds even when $\phi(-\infty)$ or $\phi(+\infty)$ is an irregular singular point.

Proof: The function $u(t,x) \equiv \phi(x)$ solves (4.2), and thus for any h, $u(t,x) \equiv \phi(h,x) \equiv \phi(x+h)$ does also. The existence assumption H4 and the maximum principle together show that whenever the initial conditions $u(0,x)$ are in H_x^2 and also satisfy

$$\phi(x-h) \le u(0,x) \le \phi(x+h) \quad \text{for all} \quad x \ , \tag{4.16}$$

then the solution $u(t,x)$ of (4.2) exists for all $t \ge 0$ and satisfies

$$\phi(x-h) \le u(t,x) \le \phi(x+h) \quad \text{for all} \quad x \tag{4.17}$$

for all $t \ge 0$. This implies C^w-stability with $w \equiv 1 + \frac{1}{|\phi'(x)|}$.
Specifically, from (4.17),

$$\left(1 + \frac{1}{|\phi'(x)|}\right)\cdot[\phi(x-h)-\phi(x)] \le \left(1 + \frac{1}{|\phi'(x)|}\right)\cdot[u(t,x)-\phi(x)]$$

$$\le \left(1 + \frac{1}{|\phi'(x)|}\right)[\phi(x+h)-\phi(x)] \quad .$$

Since $\phi''(x)$, $\phi'(x)$ and $\phi''(x)/\phi'(x)$ are bounded, and since $|\phi''(x)|$ is decreasing for all x and $-x$ sufficiently large, there is a $B > 0$ such that

$$(1 + \frac{1}{|\phi'(x)|})|\phi(x+h)-\phi(x)| \leq B|h| \quad \text{for all } h \text{ and all } x.$$

Thus for any $\varepsilon > 0$, $|u(t,x)-\phi(x)|(1 + \frac{1}{|\phi'(x)|}) < \varepsilon$ when we take $|h| < \varepsilon/B$. (We of course select the sign of h as $\mathrm{sgn}\{\phi'(x)\}$ so that $\phi(x+h) > \phi(x) > \phi(x-h)$). Since there is also a $\delta(|h|) > 0$ such that

$$|u(0,x) - \phi(x)| \ (1 + \frac{1}{|\phi'(x)|}) \leq \delta(|h|)$$

implies $\phi(x-h) \leq u(0,x) \leq \phi(x+h)$, C^w-stability of the solution $\phi(x)$ of (4.2) is established.

The theorem states that a monotonic steady state $\phi(x)$ is C^w-stable with $w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$. This means that the solution $u(t,x) \equiv \phi(x)$ is stable to small perturbations $u(0,x) - \phi(x)$ which decay asymptotically like $|\phi'(x)|$ as $x \to -\infty$ and as $x \to +\infty$. Note that when $\phi(-x)$ (when $\phi(+x)$) is a regular singular point, table 4.1 lists these asymptotic decay rates as $x \to -\infty$ (as $x \to +\infty$).

We proved the above stability theorem by using the <u>solutions</u> $\phi(x-h)$ and $\phi(x+h)$ as our upper and lower functions. In the next section (4.5), we will find better upper and lower functions when either $\phi(-\infty)$ or $\phi(+\infty)$ is a saddle point. This will be done by exploiting the differential inequalities allowed by the maximum principle. These upper and lower functions will then lead to our final (sharp) stability result for monotone steady state solutions of (4.2).

<u>4.5 Improved upper and lower functions.</u> In this section we exploit the differential inequalities allowed by the maximum principle to find better

upper and lower functions for equation (4.2). Recall that $\overline{u}(t,x)$ and $\underline{u}(t,x)$ are defined to be upper and lower functions (respectively) of the equation

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{4.2}$$

if and only if they satisfy the following differential inequalities:

$$\overline{u}_t - f(\overline{u}_{xx}, \overline{u}_x, \overline{u}) + c\overline{u}_x \geq 0 \tag{4.18a}$$

$$\underline{u}_t - f(\underline{u}_{xx}, \underline{u}_x, \underline{u}) + c\underline{u}_x \leq 0 \quad . \tag{4.18b}$$

Suppose that $u(t,x) \equiv \phi(x)$ is a bounded monotonic steady state solution of (4.2) and that one of $\phi(-\infty)$ and $\phi(+\infty)$ is a first order saddle point and the other is a regular singular point of order $m \geq 1$. In this case the following lemma yields upper and lower functions which are much better for our purposes than $\phi(x+h)$ and $\phi(x-h)$. Note that when <u>both</u> $\phi(-\infty)$ and $\phi(+\infty)$ are ordinary first order singular points the following lemma essentially reduces to lemma (2.3) of Chapter II. Note also that the lemma only <u>directly</u> considers the case of $\phi(x)$ being an increasing function of x. This is sufficient since the transformation $x \to -x$ will change any decreasing function to an increasing one.

<u>Lemma 4.3:</u> Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x)$ is a bounded non-constant monotonic steady state solution of equation (4.2). In particular, suppose that $\phi(x)$ is increasing in x. Define $\phi(-\infty) \equiv \phi_-$ and $\phi(+\infty) \equiv \phi_+$. Then

(1) if $\phi = \phi_-$, $v = 0$ is a regular singular point (of order $m \geq 1$) and $\phi = \phi_+$, $v = 0$ is an ordinary first order saddle point, then

$$\overline{u}(t,x) \equiv \phi(x+h(t)) + q(t)\cdot[\phi(x+h(t))-\phi_-]^n \quad \text{and} \tag{4.19a}$$

$$\underline{u}(t,x) \equiv \phi(x-h(t)) - q(t)\cdot[\phi(x-h(t))-\phi_-]^n \tag{4.19b}$$

are upper and lower functions (respectively) of equation (4.2). Here

$n \geq 1$ is defined by $[\phi(x)-\phi_-]^n/\phi'(x) \to a_-$ as $x \to -\infty$ where a_- is a positive constant, and

$$h(t) \equiv \alpha\kappa(1-e^{-st}) + h_0 \qquad q(t) \equiv \alpha e^{-st} \tag{4.20}$$

where s and κ are particular positive constants, h_0 is arbitrary, and $\alpha > 0$ is any sufficiently small constant.

(2) if $\phi = \phi_-$, $v = 0$ is an ordinary first order saddle point and $\phi = \phi_+$, $v = 0$ is a regular singular point (of order $m \geq 1$), then

$$\overline{u}(t,x) \equiv \phi(x+h(t)) + q(t)\cdot[\phi_+-\phi(x+h(t))]^n \quad \text{and} \tag{4.21a}$$

$$\underline{u}(t,x) \equiv \phi(x-h(t)) - q(t)\cdot[\phi_+-\phi(x-h(t))]^n \tag{4.21b}$$

are upper and lower functions (respectively) of equation (4.2). Here $n \geq 1$ is defined by $[\phi_+-\phi(x)]^n/\phi'(x) \to a_+$ as $x \to +\infty$ where a_+ is some positive constant, and $h(t)$ and $q(t)$ are defined as above.

When one of $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ is an ordinary first order saddle point and the other is a regular singular point (of some order $m \geq 1$), the above lemma provides new upper and lower functions. As x approaches the saddle point at either $x = -\infty$ or $x = +\infty$, $\overline{u}(0,x) - \phi(x)$ and $\phi(x) - \underline{u}(0,x)$ asymptote to positive constants. However, as x approaches the other singular point at either $x = +\infty$ or $x = -\infty$, $\overline{u}(0,x) - \phi(x)$ and $\phi(x) - \underline{u}(0,x)$ decay asymptotically like $\phi'(x)$ does, which is the same asymptotic decay rate that $\phi(x+h)-\phi(x)$ and $\phi(x)-\phi(x-h)$ decay at. Since $\overline{u}(0,x)-\phi(x)$ and $\phi(x)-\underline{u}(0,x)$ decay asymptotically no faster than $\phi(x+h)-\phi(x)$ and $\phi(x)-\phi(x-h)$ at the non-saddle point end and since $\overline{u}(0,x)-\phi(x)$ and $\phi(x)-\underline{u}(0,x)$ asymptote to positive constants at the saddle point end, these new upper and lower functions are much better for stability proofs than the $\phi(x+h)$ and $\phi(x-h)$ used previously.

In figures (1) and (2) below, we have sketched the new upper and lower functions at $t = 0$ and $t = +\infty$. Note that in these sketches we have used a value of h_0 for the upper functions which is $\Delta h > 0$ larger than the value used for the lower functions.

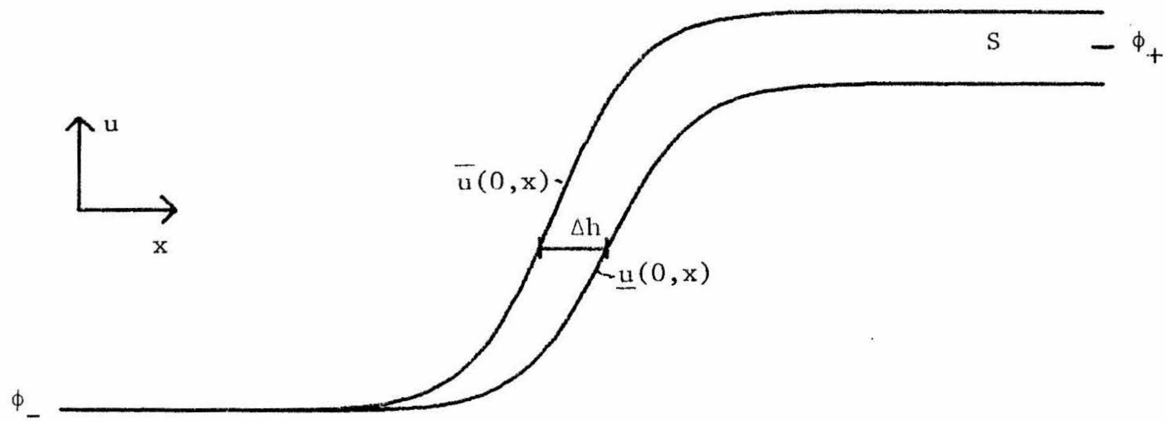


Figure (1a): The functions $\bar{u}(0,x)$ and $\underline{u}(0,x)$ from (4.19) when $\phi=\phi_+, v=0$ is a first order saddle point.

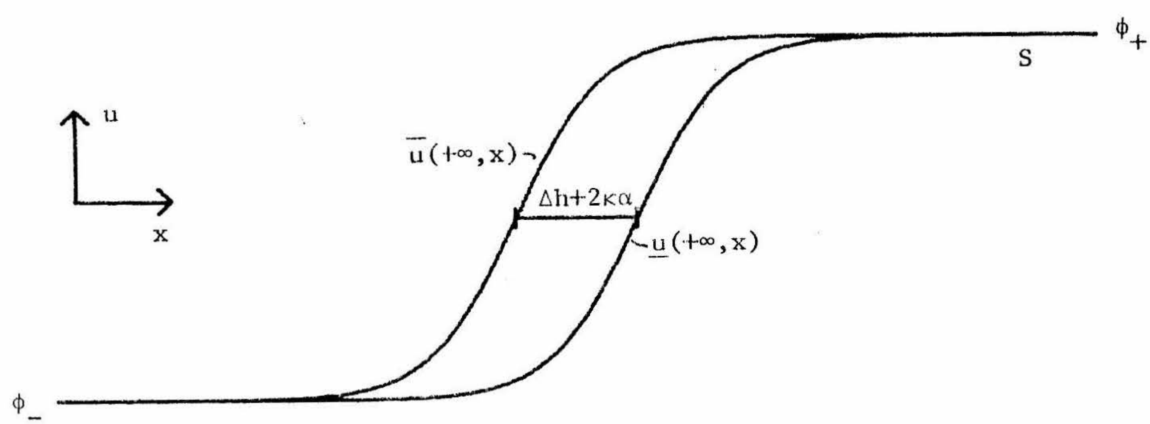


Figure (1b): The functions $\bar{u}(+\infty,x)$ and $\underline{u}(+\infty,x)$ from (4.19) when $\phi=\phi_+, v=0$ is a first order saddle point.

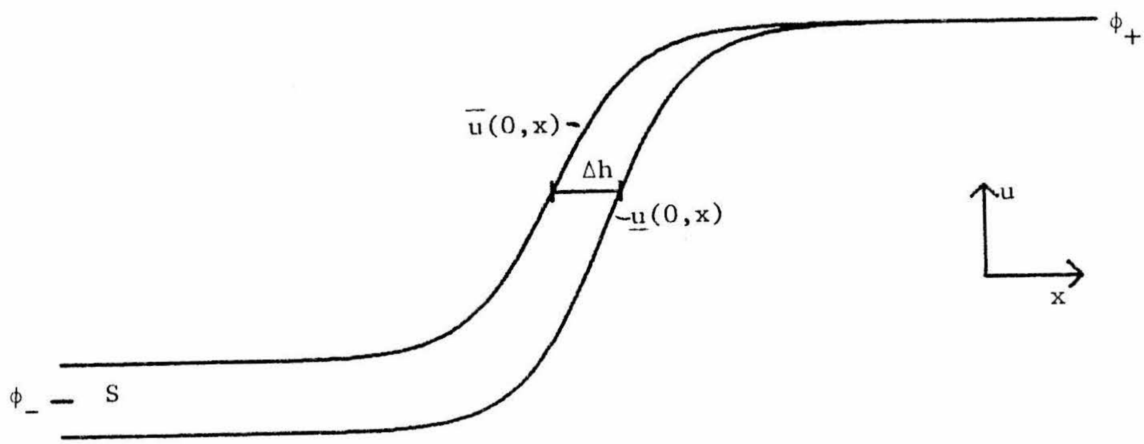


Figure (2a): The functions $\overline{u}(0,x)$ and $\underline{u}(0,x)$ from (4.21) when $\phi=\phi_-, v=0$ is a first order saddle point.

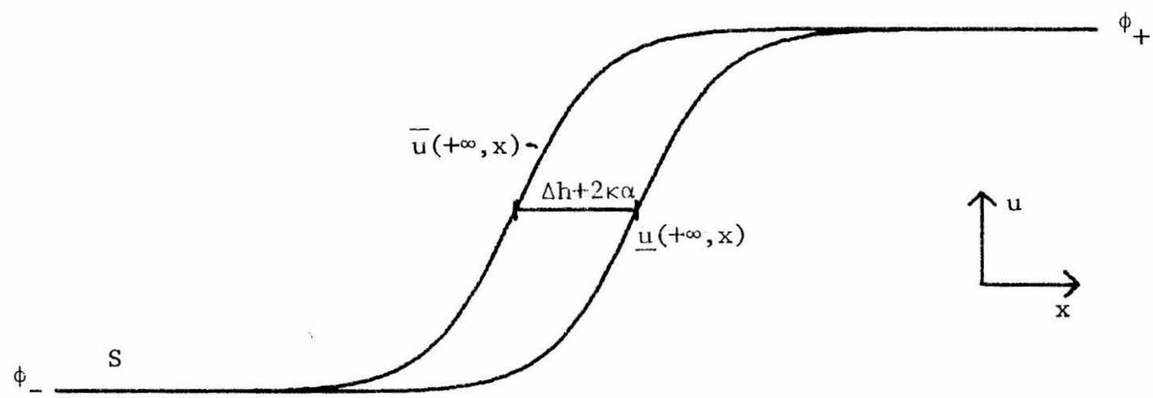


Figure (2b): The functions $\overline{u}(+\infty,x)$ and $\underline{u}(+\infty,x)$ from (4.21) when $\phi=\phi_-, v=0$ is a first order saddle point.

<u>Proof of lemma (4.3):</u> We prove only that $\overline{u}(t,x)$ in (4.19) is an upper function. The proofs of the other cases follow from similar calculations.

We will prove $\overline{u}$ to be an upper function of (4.2) by showing that $\overline{u}_t - f(\overline{u}_{xx},\overline{u}_x,\overline{u}) - c\overline{u}_x \geq 0$. Let us abbreviate $f_1 \equiv f_1(\phi_{xx},\phi_x,\phi)$, $f_2 = f_2(\phi_{xx},\phi_x,\phi)$, and $f_3 = f_3(\phi_{xx},\phi_x,\phi)$. We have

$$\bar{u}_t = \phi' h_t [1+nq(\phi-\phi_-)^{n-1}] + q_t(\phi-\phi_-)^n \quad , \tag{4.22a}$$

$$\bar{u} = \phi + q(\phi-\phi_-)^n \quad , \tag{4.22b}$$

$$\bar{u}_x = \phi'[1+nq(\phi-\phi_-)^{n-1}] \quad , \quad \text{and} \tag{4.22c}$$

$$\bar{u}_{xx} = \phi''[1+nq(\phi-\phi_-)^{n-1}] + \phi'\phi'\cdot n(n-1)q\cdot(\phi-\phi_-)^{n-2} \quad , \tag{4.22d}$$

where the argument of ϕ, ϕ', and ϕ'' is $x + h(t)$. We now substitute these in $\bar{u}_t - f(\bar{u}_{xx}, \bar{u}_x, \bar{u}) - c\bar{u}_x$ and expand in q.

We first consider the region $x + h(t) \geq x_0$ where $x_0 > 0$ is very large. We find

$$\begin{aligned} \bar{u}_t - f(\bar{u}_{xx},\bar{u}_x,\bar{u}) - c\bar{u}_x \geq{} & \phi' h_t + q_t(\phi-\phi_-)^n \\ & - f_1 q\{\phi'' n(\phi-\phi_-)^{n-1} + \phi'\phi' n(n-1)(\phi-\phi_-)^{n-2}\} \\ & - (f_2+c)q\phi'\cdot n(\phi-\phi_-)^{n-1} - f_3 q(\phi-\phi_-)^n \\ & + \text{h.o.}\{\phi'' q,\ \phi'\phi'(\phi-\phi_-)^{n-2}q, \phi' q,\ q(\phi-\phi_-)^n\} \end{aligned} \tag{4.23}$$

where h.o. stands for terms uniformly of higher algebraic order; that is, $\text{h.o.}\{s\,a(x), sb(x), sc(x), sd(x)\} \sim O(s^{1+\delta} \max\{a(x),b(x),c(x),d(x)\})$ uniformly in x (for $x > x_0$) for some $\delta > 0$ as $s \to 0$. This uniformity of the higher order terms comes from the uniform Hoelder continuity of f_1, f_2, and f_3. Since $f_3(+\infty) \equiv f_3(0,0,\phi_+) < 0$, we see that for some $x_0 > 0$ sufficiently large and for some $q^+ > 0$ sufficiently small, there exists an $N^+ > 0$ and an $s > 0$ such that whenever

$$0 \leq q \leq q^+,\ 0 \leq -q_t \leq sq, \text{ and } h_t \geq N^+ q \quad ,$$

then $\bar{u}_t - f(\bar{u}_{xx},\bar{u}_x,\bar{u}) - c\bar{u}_x \geq 0$ for all $x + h(t) \geq x_0$.

We now make a similar estimate for $x + h(t) \leq -x_0$. We again find

$$\bar{u}_t - f(\bar{u}_{xx},\bar{u}_x,\bar{u}) - c\bar{u}_x \geq \phi' h_t + q_t(\phi-\phi_-)^n \tag{4.24}$$
$$- f_1 q\{\phi'' n(\phi-\phi_-)^{n-1} + \phi'\phi' n(n-1)(\phi-\phi_-)^{n-2}\}$$
$$- (f_2+c)q\phi' n(\phi-\phi_-)^{n-1} - f_3 q(\phi-\phi_-)^n$$
$$+ \text{h.o.}\{\phi'' q, \phi'\phi'(\phi-\phi_-)^{n-2} q,\ \phi'\cdot q,\ q(\phi-\phi_-)^n\}$$

where h.o. again stands for uniformly higher order terms. Since ϕ''/ϕ' and $(\phi-\phi_-)^n/\phi'$ are both bounded as $x \to -\infty$, we see that there exists a $q^- > 0$, an $M^- > 0$, and an $N^- > 0$ such that $\bar{u}_t - f(\bar{u}_{xx},\bar{u}_x,\bar{u}) - c\bar{u}_x \geq 0$ for $x + h(t) \leq -x_0$ whenever

$$0 \leq q \leq q^- \quad \text{and} \quad h_t + M^- q_t \geq N^- q \quad .$$

The interior $|x + h(t)| \leq x_0$ is easily handled. We find that

$$\bar{u}_t - f(\bar{u}_{xx},\bar{u}_x,\bar{u}) - c\bar{u}_x \geq \phi' h_t + q_t(\phi-\phi_-)^n \tag{4.25}$$
$$- f_1 q\{\phi'' n(\phi-\phi_-)^{n-1} + \phi'\phi' n(n-1)(\phi-\phi_-)^{n-2}\}$$
$$- (f_2+c)q\phi' n(\phi-\phi_-)^{n-1} - f_3 q(\phi-\phi_-)^n$$
$$+ \text{h.o.}\{q\} \quad .$$

Since there is a $\delta > 0$ such that $\phi'(x) > \delta$ for all x in $[-x_0,x_0]$, and since ϕ'' and ϕ are bounded, there exists a $q^0 > 0$, an $M^0 > 0$, and an $N^0 > 0$ such that $\bar{u}_t - f(\bar{u}_{xx},\bar{u}_x,\bar{u}) - c\bar{u}_x \geq 0$ for all $|x+h(t)| \leq x_0$ whenever

$$0 \leq q \leq q^0 \quad \text{and} \quad h_t + M^0 q_t \geq N^0 q \quad .$$

Summarizing the results for the three regions, we have $\bar{u}_t - f(\bar{u}_{xx},\bar{u}_x,\bar{u}) - c\bar{u}_x \geq 0$ for all x whenever

$$0 \leq q \leq \min\{q^-,q^0,q^+\} \ ,\ 0 \leq -q_t \leq sq \ , \text{ and}$$
$$h_t \geq \max\{M^-,M^0\}(-q_t) + \max\{N^-,N^0,N^+\}$$

hold. Hence, we take

$$h(t) \equiv \alpha\kappa(1-e^{-st}) + h_0 \qquad q(t) \equiv \alpha e^{-st} \tag{4.20}$$

where $\kappa \equiv \max\{M^-,M^0\}s + \max\{N^-,N^0,N^+\}$, and note that $\bar{u}$ is an upper function for all $0 \leq \alpha \leq \min\{q^-, q^0, q^+\}$. This establishes the lemma.

Lemma (4.3) provides good upper and lower functions when either $\phi = \phi_-$, $v = 0$ or $\phi = \phi_+$, $v = 0$ is an ordinary first order saddle point. Since we were able to improve our upper and lower functions when at least one of $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ is a saddle point, one expects that still better upper and lower functions can be found when both $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are ordinary first order saddle points. The following lemma shows this to be so. Note that again we deal directly only with the case of $\phi(x)$ being an increasing function of x.

Lemma 4.4: Assume hypotheses H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x)$ is a bounded non-constant monotonic steady state solution of equation (4.2). In particular, suppose that $\phi(x)$ is increasing in x. Define $\phi(-\infty) \equiv \phi_-$ and $\phi(+\infty) \equiv \phi_+$. Then if $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are both ordinary first order saddle points, then

$$\overline{u}(t,x) \equiv \phi(x+h(t)) + |q(t)| \quad \text{and} \tag{4.26a}$$

$$\underline{u}(t,x) \equiv \phi(x-h(t)) - |q(t)| \tag{4.26b}$$

are upper and lower functions (respectively) of equation (4.2). Here,

$$h(t) \equiv \alpha\kappa(1-e^{-st}) + h_0 \qquad q(t) \equiv \alpha e^{-st} \tag{4.20}$$

where s and κ are particular positive constants, h_0 is arbitrary, and $\alpha > 0$ is any sufficiently small constant.

Proof: We will not formally prove lemma (4.4). Its proof follows from calculations very similar to the one which proved lemma (4.3).

Thus when both $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are ordinary first order saddle points, the above lemma provides new upper and lower functions. Since $\overline{u}(0,x) - \phi(x)$ and $\phi(x) - \underline{u}(0,x)$ both asymptote to positive constants as $x \to -\infty$ and as $x \to +\infty$, these new upper and lower functions are much better than the upper and lower functions contained in lemma (4.3) and are also much better than $\phi(x+h)$ and $\phi(x-h)$.

In Figure (3) we have sketched these new upper and lower functions at $t = 0$ and $t = +\infty$. Note that the values of h_0 used for $\overline{u}$ is $\Delta h > 0$ larger than the value of h_0 used for $\underline{u}$ in these sketches.

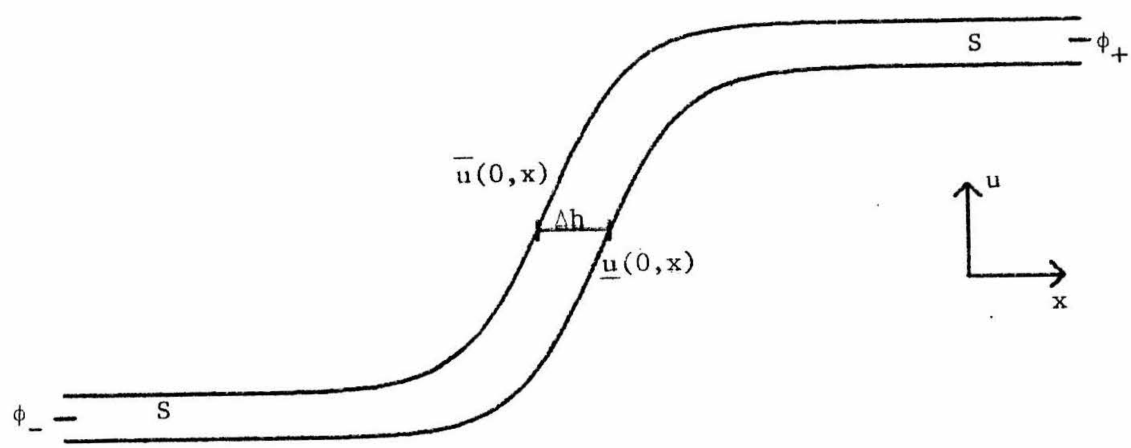


Figure (3a): The functions $\overline{u}(0,x)$ and $\underline{u}(0,x)$ from (4.26) when $\phi=\phi_-$, $v=0$ and $\phi=\phi_+$, $v=0$ are both first order saddle points.

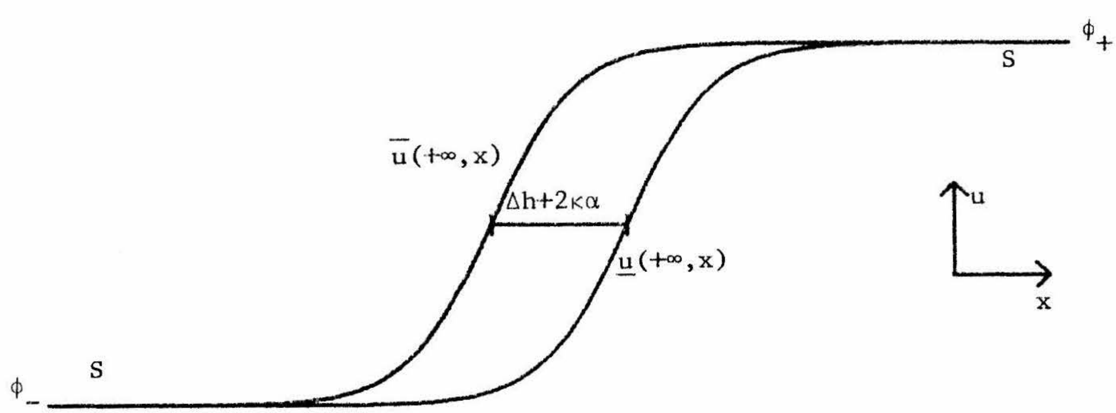


Figure (3b): The functions $\overline{u}(+\infty,x)$ and $\underline{u}(+\infty,x)$ from (4.26) when $\phi=\phi_-$, $v=0$ and $\phi=\phi_+$, $v=0$ are both first order saddle points.

The new upper and lower functions developed in lemmas (4.3) and (4.4) in conjunction with the maximum principle will immediately yield our main stability result for bounded non-constant monotonic steady state solutions $u(t,x) \equiv \phi(x)$ of

$$u_t = f(u_{xx},u_x,u) + cu_x \quad . \tag{4.2}$$

In the next section we will obtain our main stability result in exactly this way.

<u>4.6 The stability of monotone waves.</u> We now state and prove our final stability result for monotonic steady state solutions of

$$u_t = f(u_{xx},u_x,u) + cu_x \quad . \tag{4.2}$$

The maximum principle and the bounding functions developed in the previous section make this an easy task. In order to simplify the statement of the theorem, let us first define

$$r_+\{\phi'(x)\} \equiv \begin{Bmatrix} \phi'(x) & x \geq 0 \\ \phi'(0) & x \leq 0 \end{Bmatrix}, \quad r_-\{\phi'(x)\} \equiv \begin{Bmatrix} \phi'(0) & x \geq 0 \\ \phi'(x) & x \leq 0 \end{Bmatrix}.$$

Note that the following theorem directly treats decreasing as well as increasing steady states $u(t,x) \equiv \phi(x)$.

<u>Theorem 4.5 (The stability of monotone waves):</u> Assume that hypotheses H2, H3, and H4 are satisfied, and suppose that $u(t,x) \equiv \phi(x)$ is a bounded non-constant monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{4.2}$$

at some particular value of c. Let $\phi(-\infty) \equiv \phi_-$ and $\phi(+\infty) \equiv \phi_+$. Then $u(t,x) = \phi(x)$ is C^w-stable where

(1) if $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are both ordinary first order saddle points then $w(x) \equiv 1$;

(2) if $\phi = \phi_-$, $v = 0$ is a regular singular point (of order $m_- \geq 1$) and $\phi = \phi_+$, $v = 0$ is an ordinary first order saddle point then

$$w(x) \equiv 1 + \frac{1}{|r_-\{\phi'(x)\}|} \quad ;$$

(3) if $\phi = \phi_-$, $v = 0$ is an ordinary first order saddle point and $\phi = \phi_+$, $v = 0$ is a regular singular point (of order $m_+ \geq 1$) then

$$w(x) \equiv 1 + \frac{1}{|r_+\{\phi'(x)\}|} \quad ; \text{ and}$$

(4) if neither case (1), (2), nor (3) occur then

$$w(x) \equiv 1 + \frac{1}{|\phi'(x)|} \quad .$$

Proof: We prove this theorem only for the case of $\phi(x)$ being an increasing function of x. The proof when $\phi(x)$ is decreasing follows from transforming x to $-x$.

To prove part (1) we use the upper and lower functions contained in lemma (4.4). To prove parts (2) and (3) we use the upper and lower functions contained in lemma (4.3). The existence hypothesis H4 and the maximum principle together show that any initial condition $u(0,x)$ smooth enough to be in H_x^2 which is also bounded above by an upper function and bounded below by a lower function, has a solution $u(t,x)$ for all $t \geq 0$ that remains between the upper and lower functions. This immediately implies that $\phi(x)$ is stable because the parameters $\alpha > 0$ and $|h_0| > 0$ in the definitions of the upper and lower functions can be taken as small as we please. (See equations (4.19), (4.20), (4.21), and (4.26)). Inspection of the formulas for the upper and lower functions shows that the classes of perturbations bounded by these functions are the same as those allowed in the definition of C^w-stability, with $w(x)$ as given in the appropriate part (1), (2), or (3) of the theorem. Thus, parts (1), (2), and (3) of the theorem are established. Part (4) has already been proved in theorem (4.2).

As a rough summary of the stability theorem (4.5), we see that bounded non-constant monotonic steady state solutions $u(t,x) = \phi(x)$ of equation (4.2) are stable with respect to small smooth initial perturbations which are

(1) bounded as $x \to -\infty$ (as $x \to +\infty$) when $\phi(x)$ goes to a first order saddle point at $x = -\infty$ (at $x = +\infty$), and

(2) decay asymptotically no slower than $|\phi'(x)|$ as $x \to -\infty$ (as $x \to +\infty$) when $\phi(x)$ goes to a regular singular point which is not a first order saddle point at $x = -\infty$ (at $x = +\infty$).

Note that except for the question of whether $\phi(x)$ decays to a node at the usual or accidental rate, the slowest allowed asymptotic decay rate for perturbations (in the above theorem) depends only on the expansion of $f(\phi_{xx},\phi_x,\phi) + c\phi_x$ about $(\phi_{xx},\phi_x,\phi) = (0,0,\phi_-)$ and about $(\phi_{xx},\phi_x,\phi) = (0,0,\phi_+)$. This is because these expansions not only determine whether ϕ_- and ϕ_+ are first order saddle points, but also determine the asymptotic decay rates of $\phi'(x)$ as $x \to -\infty$ and as $x \to +\infty$. In particular, the asymptotic decay rates allowed for perturbations can be calculated immediately from table 4.1 for all cases.

Theorem (4.5) is our major stability result for monotone waves. Note that this result reduces to the stability result of theorem (2.5) in Chapter II when both $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are first order singular points. Recall that in section (2.4) we showed that these results are almost always sharp by constructing nearby traveling wave solutions which travel at slightly different speeds. Thus in the extremely common case of first order singular points theorem (4.5) is nearly always sharp. We will discuss the sharpness of theorem (4.5) in Chapter V, where this topic arises naturally as a by-product of the mean wavespeed/initial condition developement.

The next five short sections are used to discuss topics related to the stability results in this section. In the next section, section (4.7), we point out the extension of the stability results to the cases where $\phi = \phi_-$, $v = 0$ or $\phi = \phi_+$, $v = 0$ is an irregular singular point. In section (4.8) we show explicitly how the stability of a monotonic wave depends on f. Section (4.9) compares the stability results of theorem (4.5) with those obtainable by more conventional eigenanalysis/variational methods. As a related topic, in section (4.10) we show how the stability classes of theorem (4.5) split the generalized null space of equation (4.2) linearized about $\phi(x)$. Finally in section (4.11) we extend our results to higher spatial dimensions.

4.7 Irregular singular points. In the last section we found new stability results for monotonic steady states $u(t,x) \equiv \phi(x)$ of equation (4.2) when at least one of $\phi(-\infty)$ and $\phi(+\infty)$ is a first order saddle point and the other is a regular singular point. This was accomplished by using the bounding functions constructed in lemmas (4.3) and (4.4) along with the maximum principle. For the sake of mathematical completeness, in this section we briefly consider the case where one of $\phi(-\infty)$ and $\phi(+\infty)$ is a first order saddle point and the other is an irregular singular point. To improve on the stability results of theorem (4.2) for this case, we need to construct new upper and lower functions $\bar{u}$ and $\underline{u}$ like the ones in lemma (4.3) and then apply the maximum principle. We will not do this here. Instead we note that new upper and lower functions $\bar{u}$ and $\underline{u}$ very similar to the ones in lemma (4.3) can be constructed in this case of one of the singular points

$\phi(-\infty)$ and $\phi(+\infty)$ being irregular and the other one being a first order saddle point. Upon applying the maximum principle we find the generalization that theorem 4.5 remains true even if the regularity condition for the singular points in parts (2) and (3) is omitted.

4.8 The dependence of the stability on f. Theorem (4.5) shows that a bounded non-constant monotonic steady state solution $u(t,x) \equiv \phi(x)$ of

$$u_t = f(u_{xx}, u_x, u) + cu_x \qquad (4.2)$$

is stable to small smooth perturbations which decay asymptotically no slower than certain limiting asymptotic decay rates. For example, the limiting asymptotic decay rate as $x \to +\infty$ is determined completely by whether or not $\phi(+\infty)$ is a first order saddle point and by the asymptotic decay rate of $\phi(x)$ as $x \to +\infty$. However, expansion of the steady state equation about $(\phi_{xx}, \phi_x, \phi) = (0,0,\phi(+\infty))$ yields the asymptotic equation

$$f_1\, \phi_{xx} + (f_2+c)\phi_x + \mu(\phi-\phi(+\infty))^m = 0 \quad \text{for } x \text{ large} , \qquad (4.27)$$

where $f_1 \equiv f_1(0,0,\phi(+\infty))$ and $f_2 \equiv f_2(0,0,\phi(+\infty))$. This asymptotic equation can be used to determine both the asymptotic decay rate of $\phi(x)$ as $x\to+\infty$ and whether $\phi(+\infty)$ is a first order saddle point. Thus the quantities f_1, (f_2+c), μ, and m completely determine the limiting asymptotic decay rate (as $x \to +\infty$) allowed for perturbations by theorem (4.5), at least when it is known whether $\phi'(x)$ decays at the usual or accidental decay rate when $\phi(+\infty)$ is a node.

In table (4.2) we list these limiting asymptotic decay rates (as $x \to +\infty$) for the case of $\phi(x)$ being an increasing function of x. Note that in table 4.2 we use a slightly different μ than is used in equation (4.27). Namely, for table (4.2) we use a μ defined by

$$f_1\, \phi_{xx} + (f_2+c)\phi_x - \mu|\phi-\phi(+\infty)|^m = 0 \quad \text{for } x \text{ large} .$$

We use this definition because $\phi(x) < \phi(+\infty)$ for all x and we do not wish to change the sign of μ each time we change m. Also note that two possible limiting asymptotic decay rates are given whenever $\phi(+\infty)$ is a node. This comes from the two possible asymptotic decay rates (the usual and the accidental) of $\phi'(x)$ whenever $\phi(+\infty)$ is a node. Finally, the limiting asymptotic decay rates when $\phi(+\infty)$ is a saddle point are correctly listed as if $\phi'(x)$ decays to $\phi(+\infty)$ at the accidental rate.

Table 4.2

Slowest allowed asymptotic decay rate (as $x \to +\infty$) for perturbations.

Asymptotic equation: $f_1 \phi_{xx} + (f_2+c)\phi_x - \mu|\phi-\phi(+\infty)|^m = 0$ for x large

$\phi(x)$ increasing, $(f_2+c) > 0$

m	μ	Type of singular point	Decay rate when $\phi(x)$ decays at the accidental rate	Decay rate when $\phi(x)$ decays at the usual rate
1	+	N	$O(\exp \lambda_2 x)$	$O(\exp \lambda_1 x)$
>1	+	- N	$O(\exp \lambda_2^* x)$	$O(x^{-\frac{m}{m-1}})$
>1	-	- S	$O(\exp \lambda_2^* x)$	————
1	-	S	$O(1)$	————

$$\lambda_1 \equiv \frac{-(f_2+c)+\sqrt{(f_2+c)^2-4f_1\mu}}{2f_1} \qquad \lambda_2 \equiv \frac{-(f_2+c)-\sqrt{(f_2+c)^2-4\mu f_1}}{2f_1}$$

$$\lambda_2^* \equiv -\frac{f_2+c}{f_1}$$

$\phi(x)$ increasing, $(f_2+c) < 0$

m	μ	Type of singular point	Decay rate when $\phi(x)$ decays at the accidental rate	Decay rate when $\phi(x)$ decays at the usual rate
1	+	N	no steady state $\phi(x)$	no steady state $\phi(x)$
>1	+	-N	no steady state $\phi(x)$	no steady state $\phi(x)$
>1	-	-S	$0(x^{-\frac{m}{m-1}})$	———
1	-	S	$0(1)$	———

Note: When $(f_2+c) < 0$ and $\mu > 0$, then $\phi(+\infty)$ is an unstable node.

4.9 Comparison with eigenanalysis results. We now consider a conventional eigenanalysis method for finding the stability of bounded non-constant monotonic steady state solutions $u(t,x) \equiv \phi(x)$ of

$$u_t = f(u_{xx},u_x,u) + cu_x \quad . \tag{4.2}$$

The approach we will demonstrate is commonly used on special cases of (4.2), notably by Sattinger [2] on the class of equations

$$u_t = u_{xx} + \tilde{f}(u,u_x) \quad .$$

Our objective is to produce a table exactly like Table (4.2) which will list the slowest asymptotic decay rate (as $x \to +\infty$) of the perturbations allowed by the eigenanalysis calculations. This will allow easy comparison of the eigenanalysis results with the maximum principle results.

We begin by linearizing equation (4.2) about the steady state $\phi(x)$:

$$u(t,x) \equiv \phi(x) + \eta\tilde{\Psi}(t,x) \qquad 0 < \eta << 1 \quad , \tag{4.28a}$$

$$\tilde{\Psi}_t = \mathcal{L}\tilde{\Psi} + 0(\eta) \quad , \tag{4.28b}$$

$$\mathcal{L}\tilde{\Psi} \equiv f_1\tilde{\Psi}_{xx} + (f_2+c)\tilde{\Psi}_x + f_3\tilde{\Psi} \quad , \text{ where} \tag{4.28c}$$

$$f_i(x) \equiv f_i(\phi_{xx}(x),\phi_x(x),\phi(x)) \quad \text{for } i = 1,2,3 \quad . \tag{4.28d}$$

We now must determine the spectra of $\mathcal{L}$ when various Banach spaces are used as its domains. To do this, we define $\tilde{\Psi}(\lambda,t,x)$ and $\Psi(\lambda,x)$ by

$$\tilde{\Psi}_t = \mathcal{L}\tilde{\Psi}\ , \ \tilde{\Psi}(\lambda,0,x) = \Psi(\lambda,x), \text{ and } \lambda\Psi = f_1\Psi_{xx} + (f_2+c)\Psi_x+f_3\Psi. \qquad (4.29)$$

For computational convenience we introduce $\tau(\lambda,x)$, defined by

$$\Psi(\lambda,x) = \phi_x(x)\cdot\tau(\lambda,x) \quad , \qquad (4.30)$$

where $\phi(x)$ is the monotonic steady state. Note that in defining $\tau(\lambda,x)$ we have implicitly used the fact that $\phi(x)$ is monotonic; i.e. that $\phi_x(x) \neq 0$. Now in terms of τ the equation for Ψ in (4.29) becomes

$$\lambda\tau = f_1\,\tau_{xx} + [(f_2+c) + 2f_1\phi_{xx}/\phi_x]\tau_x \quad . \qquad (4.31)$$

This immediately yields the variational characterization

$$\lambda \equiv -\inf_{\tau\in\mathcal{A}} \frac{\displaystyle\int_{-\infty}^{\infty} \tau_x^2\ \phi_x^2\ \exp\{\int_0^x \frac{f_2(s)+c}{f_1(s)}\,ds\}dx}{\displaystyle\int_{-\infty}^{\infty}\frac{1}{f_1(x)}\ \tau^2\phi_x^2\ \exp\{\int_0^x \frac{f_2(s)+c}{f_1(s)}\,ds\}dx} \qquad (4.32)$$

for the largest eigenvalue λ of $\mathcal{L}$ when $\tau(\lambda,x)$ is restricted to the linear admissibility space $\mathcal{A}$. From (4.32) it is clear that the largest eigenvalue of $\mathcal{L}$ is non-positive, even when the admissibility space $\mathcal{A}$ is all functions τ for which both integrals converge. (This is clearly the largest space for which (4.32) remains valid.)

We now rephrase this result in terms of the eigenfunctions Ψ. Let $f_i^{\pm} \equiv f_i(\pm\infty)$ for $i = 1,2$. Also define the Banach space $\mathcal{B}$ to be all twice differentiable functions Ψ for which the integrals

$$\int_{-\infty}^{\infty} \Psi^2(x)\ \exp\Big[\int_0^x \frac{f_2(s)+c}{f_1(s)}\,ds\Big]dx \quad ,$$

$$\int_{-\infty}^{\infty} \Psi_x^2(x)\ \exp\Big[\int_0^x \frac{f_2(s)+c}{f_1(s)}\,ds\Big]dx \quad , \quad \text{and}$$

$$\int_{-\infty}^{\infty} \Psi_{xx}^2(x)\exp\Big[\int_0^x \frac{f_2(s)+c}{f_1(s)}\,ds\Big]dx$$

converge. Note that the convergence of the first two of these integrals is precisely equivalent to the convergence of the two integrals in (4.32). Also, note that these convergence conditions are essentially that $\Psi(x)$, $\Psi_x(x)$, $\Psi_{xx}(x)$ must decay to zero at least slightly faster than $\exp - \frac{f_2^- + c}{f_1^-} x$ as $x \to -\infty$ and at least slightly faster than $\exp - \frac{f_2^+ + c}{f_1^+} x$ as $x \to +\infty$. Thus, in terms of the eigenfunctions Ψ, our variational argument shows that $\lambda = 0$ is the largest possible eigenvalue of $\mathcal{L}$ with eigenfunction $\Psi(\lambda, x)$ in $\mathcal{B}$.

Note that this non-positivity of the spectrum of $\mathcal{L}$ over $\mathcal{B}$ cannot imply stability of $u(t,x) \equiv \phi(x)$ when $\mathcal{B}$ contains a generalized null function of $\mathcal{L}$. For example, if $\mathcal{L}^2\Psi = 0$ and $\mathcal{L}\Psi \neq 0$, then (4.28b) implies that

$$\tilde{\Psi}_t = \mathcal{L}\tilde{\Psi} \neq 0 \qquad \tilde{\Psi}_{tt} = \mathcal{L}^2\tilde{\Psi} = 0 \quad .$$

Consequently, the perturbation $u(t,x) - \phi(x)$ will grow linearly in time if $u(0,x) - \phi(x) \equiv \eta\Psi(x)$. These particular perturbations actually correspond to initially perturbing $u(t,x) = \phi(x) \equiv \phi(x,c)$ onto a nearby traveling wave $\phi(x-(\delta c)t, c+(\delta c))$ which travels with speed $c + \delta c$ (or in our current moving coordinate system, speed δc). This is an unstable perturbation since $\phi(x-(\delta c)t, c+(\delta c))$ travels with a speed δc different than $\phi(x) \equiv \phi(x,c)$ does, and hence it will drift away from $\phi(x)$ as t increases.

When the null space of $\mathcal{L}$ over the domain $\mathcal{B}$ is simple, then the non-positivity of the spectrum of $\mathcal{L}$ over $\mathcal{B}$ <u>should</u> imply that the steady state solution $u(t,x) \equiv \phi(x)$ of (4.2) is stable to all small perturbations in $\mathcal{B}$. To gauge the potential of the eigenanalysis method, let us assume that whenever the null space of $\mathcal{L}$ over $\mathcal{B}$ is simple, then the method can be used to show that $u(t,x) \equiv \phi(x)$ is stable for perturbations in $\mathcal{B}$. Table 4.3 shows the resulting limiting asymptotic decay rate (as $x \to +\infty$) for perturbations about $u(0,x) \equiv \phi(x)$. The cases with a * in table 4.3 denote an improvement over the asymptotic decay rates found by using the maximum principle contained in table 4.2. These cases are the cases when $\phi(x)$ decays at the accidental rate to a node at $x = +\infty$, and the case where $\phi(x)$ decays to a higher order saddle point (and $(f_2 + c) < 0$) at $x = +\infty$. These are all unusual cases, but it is clear that

Table 4.3

Slowest allowed asymptotic decay rate (as $x \to +\infty$) of perturbations

Asymptotic equation: $f_1 \phi_{xx} + (f_2+c)\phi_x - \mu|\phi-\phi(+\infty)|^m$ for x large

$\phi(x)$ increasing, $(f_2+c) > 0$

Type	m	μ	Decays rate when $\phi(x)$ decays at the accidental rate	Decay rate when $\phi(x)$ decays at the usual rate
N	1	+	$0(\exp - (\frac{f_2+c}{f_1}))$*	$0(\exp - (\frac{f_2+c}{f_1}))$
-N	>1	+	$0(\exp - (\frac{f_2+c}{f_1}))$*	$0(\exp - (\frac{f_2+c}{f_1}))$
-S	>1	-	$0(\exp - (\frac{f_2+c}{f_1}))$	————
S	1	-	$0(\exp - (\frac{f_2+c}{f_1}))$	————

$\phi(x)$ increasing, $(f_2+c) < 0$

Type	m	μ	Decay rate when $\phi(x)$ decays at the accidental rate	Decay rate when $\phi(x)$ decays at the usual rate
N	1	+	no steady state $\phi(x)$	no steady state $\phi(x)$
-N	>1	+	no steady state $\phi(x)$	no steady state $\phi(x)$
-S	>1	-	0(1)*	————
S	1	-	0(1)	————

Note: These results are <u>invalid</u> whenever there is a $\Psi(x)$ which decays at least as fast as $\exp\{-\frac{f_2^-+c}{f_1^-}\}$ (as $x \to -\infty$) and $\exp\{-\frac{f_2^++c}{f_1^+}\}$ (as $x \to +\infty$) and which satisfies $\mathcal{L}\Psi \neq 0$, $\mathcal{L}^2\Psi = 0$.

*Cases with a * denote improvement over results in Table 4.2.

genuine improvement in these cases may be possible when the null space of $\mathcal{L}$ over $\mathcal{B}$ is simple.

Before continuing, we point out that in Chapters II and V the maximum principle results are shown to be sharp in most cases. This is accomplished by constructing nearby traveling waves $\phi(x-(\delta c)t, c+(\delta c))$ which slightly violate the limiting asymptotic decay rates and which travel at speeds slightly different than c. We note that even in principle the eigenanalysis results can never contradict these sharpness results. For if $\mathcal{B}$ is large enough to contradict the sharpness results, then it must contain $\frac{\partial}{\partial(\delta c)}\phi(x, c+(\delta c))\Big|_{\delta c=0}$, and thus the null space of $\mathcal{L}$ over $\mathcal{B}$ is not simple.

<u>4.10. Splitting the null space.</u> In this section we wish to briefly show that the stability classes of theorem (4.5) can often split the null space of $\mathcal{L}$; that is, can include one null function and exclude another (generalized) null function. For brevity we will demonstrate this only for a single case.

Suppose that $\phi(x,c)$ is a monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{4.2}$$

at $c = c_0$, suppose that $\phi(-\infty,c_0)$ is an ordinary first-order saddle point, that $\phi(+\infty,c_0)$ is an ordinary first-order node, and that $\phi(x,c_0)$ decays to $\phi(+\infty,c_0)$ at the usual rate. From section (2.4) we know that this means that for an interval of values of c including c_0, there are monotone solutions $\phi(x,c)$ of (4.2) for which $\phi(-\infty,c) = \phi(-\infty,c_0)$ and which all decay to $\phi(+\infty,c_0)$ at the usual rate as $x \to +\infty$. Since then $\phi(x,c)$ $\sim O(\exp \lambda_1(c)x)$ as $x \to +\infty$, where $\lambda_1(c) \equiv \frac{1}{2f_1}\{-(f_2+c)+\sqrt{(f_2+c)^2-4f_1f_3}\}$

with $f_i \equiv f_i(0,0,\phi(+\infty,c_0))$, we find that

$$\phi_c(x,c_0) \equiv \frac{\partial}{\partial c}\,\phi(x,c)\big|_{c=c_0} \sim O(x \exp \lambda_1(c_0)x) \quad \text{as} \quad x \to +\infty \ .$$

We now consider the linear operator $\mathcal{L}$ defined in the previous section. The functions $\phi(x,c)$ all solve

$$f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0 \quad . \tag{4.33}$$

Therefore, differentiating (4.33) with respect to x and c shows that

$$\mathcal{L}\phi_x = 0 \qquad \mathcal{L}\phi_c = -\,\phi_x \quad ,$$

and hence $\mathcal{L}^2\phi_c = 0$ but $\mathcal{L}\phi_c \neq 0$. That is, ϕ_x is a null function of $\mathcal{L}$ and ϕ_c is a generalized null function.

Now, the perturbations allowed by theorem (4.5) for this case are all small perturbations which decay asymptotically at least as fast as ϕ_x; that is, decay asymptotically as fast as $\exp(\lambda_1(c)x)$ as $x \to +\infty$. This neatly includes the perturbation ϕ_x but excludes ϕ_c, even though they both belong to the generalized null space of $\mathcal{L}$ and only differ slightly in their asymptotics as $x \to +\infty$. Of course the exclusion of ϕ_c is necessary, since it is the linearization of the unstable initial perturbation $u(0,x) - \phi(x,c_0) = \phi(x,c_0+\delta c) - \phi(x,c_0)$.

<u>4.11 Extension of stability results to higher spatial dimensions.</u> We now generalize our stability results to monotone traveling wave solutions in multiple spatial dimensions. We will work only with two spatial variables $(\vec{x} = (x,y))$ in this section. However, it will be clear that our discussion will apply equally well when there are more than two spatial dimensions.

Suppose that $u(t,\vec{x}) \equiv \phi(\vec{x}-\vec{c}t)$ is a traveling wave solution of

$$u_t = f(u_{xx},u_{xy},u_{yy},u_x,u_y,u) \quad , \tag{4.34}$$

and that equation (4.34) is parabolic (i.e., satisfies hypothesis H3). By changing to the coordinate system which travels with velocity $\vec{c} \equiv (c_x,c_y)$, we can work with the steady state solution $u(t,\vec{x}) \equiv \phi(\vec{x})$ of

$$u_t = f(u_{xx},u_{xy},u_{yy},u_x,u_y,u) + c_x \cdot u_x + c_y \cdot u_y \tag{4.35}$$

instead.

In order to discuss stability, let us extend the definition of C^w-stability and $\mathfrak{C}^w$-stability to two spatial dimensions. Suppose $w(x,y)$ is any continuous function with $w(x,y) \geq 1$ for all x, y. Then, we define C^w-stability of steady state solutions $u(t,\vec{x}) = \phi(\vec{x})$ of (4.35) exactly as in the original one spatial dimension definition, <u>except</u> that it concerns solutions of (4.35) instead of (4.2) and that whenever the variable x appears in the original definition it should be replaced by $\vec{x} \equiv (x,y)$.

Our stability results are very easily generalized to traveling plane wave solutions of (4.34), which we can take to be steady state plane solutions of (4.35). Clearly, without loss of generality we can assume that our steady state plane wave solution of (4.35) is

$$u(t,x,y) \equiv \phi(x)$$

and is independent of y. Thus $u(t,x,y) = u(t,x) = \phi(x)$ also solves

$$u_t = f(u_{xx},0,0,u_x,0,0) + c_x \cdot u_x \equiv \tilde{f}(u_{xx},u_x,u) + c_x \cdot u_x \quad . \tag{4.36}$$

After a moment's reflection, it is clear that whenever $\overline{u}(t,x)$ and $\underline{u}(t,x)$ are upper and lower functions of the equation

$$u_t = \tilde{f}(u_{xx},u_x,u) + c_x \cdot u_x \quad ,$$

then $\overline{u}(t,x,y) \equiv \overline{u}(t,x)$ and $\underline{u}(t,x,y) \equiv \underline{u}(t,x)$ are upper and lower functions of equation (4.35). Thus the results in theorem's (4.2) and (4.5) remain true for plane waves if $C^{w(x)}$-stability is replaced by $C^{\overline{w}(x,y)}$-stability with $\overline{w}(x,y) \equiv w(x)$.

To summarize this result, if $u(t,x,y)$ is a traveling plane wave solution of (4.34), we change to a coordinate system which moves with the plane wave and which is oriented so that the plane wave depends only on x, not on y. Then, denoting the plane wave in this coordinate system by $u(t,x,y) = \phi(x)$ we find that if $\phi(x)$ is monotone in x, then it is stable to small perturbations which are bounded as $y \to \pm\infty$ and decay asymptotically no slower than the rates allowed by theorems (4.2) and (4.5) as $x \to \pm\infty$.

The maximum principle can also be applied to other types of "monotone" traveling waves. For example, suppose $u(t,x) \equiv \phi(x,y)$ is a steady state solution of (4.35), and that for some unit vector (e_x, e_y) the solution $\phi(x,y)$ satisfies

$$\phi(x+he_x, y+he_y) > \phi(x,y) \quad \text{for all} \quad h > 0 \quad .$$

Then clearly we can use $\phi(x+he_x, y+he_y)$ and $\phi(x-he_x, y-he_y)$ as upper and lower functions, thus proving that $\phi(x,y)$ has at least a limited amount of stability. Moreover, for some cases we could probably "improve" these upper and lower functions and better the stability results. We will not do this because of the difficulty in finding such monotonic waves (which are not plane waves) in physically interesting equations. Instead we simply note that if such a wave is discovered, then this approach to its stability can be used.

This completes our discussion of the stability of monotonic traveling waves and steady state solutions of

$$u_t = f(u_{xx}, u_x, u) \quad .$$

In the rest of this chapter we examine the other side of the picture: the instability of non-monotonic waves.

<u>4.12 Instability of non-monotonic waves.</u> In this section we show that very nearly all non-monotonic traveling waves and steady states are unstable.

Specifically, we shall show that all <u>non</u>-monotonic steady state solutions $u(t,x) \equiv \phi(x)$ of

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{4.2}$$

are

(1) unstable to <u>all</u> non-negative perturbations which are strictly positive in a fixed finite interval whenever $\phi(x)$ has at least two relative extrema,

(2) unstable to perturbations which decay like $|\phi'(x)|$ as $x \to -\infty$ and as $x \to +\infty$ whenever $\phi(x)$ has only a single relative extremum and either $\phi(-\infty)$ or $\phi(+\infty)$ is a saddle point.

The result in (1) is very strong since it shows that most non-monotonic waves are unstable, even to arbitrarily small perturbations of finite extent. The weaker result in (2) does not preclude the possibility that non-monotonic waves with a single relative extremum are stable to small perturbations which decay faster than $|\phi'(x)|$ as $x \to \pm\infty$.

The strongest motivation for these results is their correctness for <u>steady state</u> solutions of special equations. For example, they are correct for Fischer's equation [4] and other equations of the form $u_t = u_{xx} + h(u)$ [5]. Recall that Fischer's equation was used as a motivating example in section (2.3).

We will now state and prove these instability results precisely. In this following theorem (and afterward), note that whenever we speak of a non-monotonic function having n relative extrema, we are excluding the extrema at $x = \pm\infty$. I.e., there are n distinct <u>finite</u> values of x at which $\phi(x)$ has an extremum.

<u>Theorem 4.6 (Instability of non-monotonic waves):</u> Assume that hypotheses H2, H3, H4, and H5 are satisfied. Suppose also that $u(t,x) \equiv \phi(x)$ is a bounded <u>non</u>-monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad . \tag{4.2}$$

Then

(1) If there are at least two distinct finite values of x at which $\phi(x)$ has relative extrema, then there is a finite interval $[x_0, x_1]$ and a $\Delta > 0$ such that for any $\varepsilon > 0$ there is a $p(x)$ in H_x^2 satisfying

$$0 \leq p(x) \leq \varepsilon \quad \text{when} \quad x_0 < x < x_1$$

$$p(x) \equiv 0 \quad \text{when} \quad x \notin (x_0, x_1)$$

for which the solution $u(\varepsilon,t,x)$ of equation (4.2) with the initial condition

$$u(\varepsilon,0,x) = \phi(x) + p(x)$$

satisfies

$$u(\varepsilon,t,x) - \phi(x) > \Delta$$

for some x and some $t > 0$. Moreover, if $u(t,x)$ is <u>any</u> solution of (4.2) whose initial condition $u(0,x)$ is in H_x^2 and satisfies

$$u(\varepsilon,0,x) = \phi(x) + p(x) \leq u(0,x) \quad \text{for all} \quad x \quad ,$$

then

$$u(\varepsilon,t,x) \leq u(t,x) \quad \text{for all} \quad x \quad \text{and all} \quad t \geq 0 \quad .$$

Thus for some x and some $t > 0$

$$u(t,x) - \phi(x) > \Delta$$

(2) If there is only a single finite value of x, $x = x_e$, where $\phi(x)$ has a relative extremum and if $\phi(x)$ goes to a saddle point as $x \to -\infty$ or as $x \to +\infty$, then $u(t,x) \equiv \phi(x)$ is $\mathcal{C}^W$-unstable where

$$w(x) \equiv \left\{ \begin{array}{ll} 1 + \frac{1}{|\phi'(x)|} + \frac{1}{|\phi'(x_e+1)|} & x \leq x_e - 1 \\ 1 + \frac{1}{|\phi'(x_e-1)|} + \frac{1}{|\phi'(x_e+1)|} & x_e - 1 \leq x \leq x_e + 1 \\ 1 + \frac{1}{|\phi'(x_e-1)|} + \frac{1}{|\phi'(x)|} & x_e + 1 \leq x \end{array} \right\} .$$

Recall that by the phrase "goes to a saddle point as $x \to -\infty$" we mean that either

(a) $\phi(-\infty)$ is a saddle point,

(b) $\phi(-\infty)$ is a NS type singular point and $\phi'(x) > 0$ for all x sufficiently small, or

(c) $\phi(-\infty)$ is a SN type singular point and $\phi'(x) < 0$ for all x sufficiently small,

occurs. Similarly the phrase "goes to a saddle point as $x \to +\infty$" means that either

(a) $\phi(+\infty)$ is a saddle point,

(b) $\phi(+\infty)$ is a SN type singular point and $\phi'(x) > 0$ for all x sufficiently large, or

(c) $\phi(+\infty)$ is a NS type singular point and $\phi'(x) < 0$ for all x sufficiently large,

occurs. Thus we see that the requirement in part (2) of the theorem is that $\phi(x)$ goes to a saddle point or to the saddle point side of a singular point of mixed character as either $x \to -\infty$ or $x \to +\infty$.

Note that the $w(x)$ in part (2) of this theorem is essentially $1 + \frac{1}{|\phi'(x)|}$ modified so that it remains finite at $x = x_e$. The constants $\frac{1}{|\phi'(x_e+1)|}$ and $\frac{1}{|\phi'(x_e-1)|}$ were included in $w(x)$ only because we have defined $\mathfrak{C}^w$-stability for <u>continuous</u> $w(x)$.

As in section (2.3), the proof of this theorem is in three parts. The first part is selecting the appropriate initial conditions. The second part (the hair-trigger effect) is showing that the perturbed solution of (4.2) will only increase in time as it evolves into another steady state solution of (4.2). The last part is showing that the possible final steady states are bounded (independently of the initial conditions) away from the initial unperturbed steady state solution. In fact, for the perturbations we use we will be able to show that the final steady state is the smallest constant steady state which is larger than the initial conditions at all x.

In proving the above theorem we will rely very heavily on the two major properties of equation (4.2). Namely, in proving the hair-trigger effect of step two, we will use the maximum principle many times. In selecting the appropriate initial conditions (and in identifying the final steady state) we will strongly use the phase plane representation of the steady states of (4.2).

In this section we will prove only step two. For the first and third steps we will use the following two lemmas (which will be proved in the next section):

Lemma (4.7): Assume that hypotheses H2 and H3 are satisfied, and suppose that $u(t,x) \equiv \phi(x)$ is a bounded non-monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad . \tag{4.2}$$

(1) If $\phi(x)$ has relative extrema at least two distinct finite points x, then there are functions $\phi(x,\varepsilon)$, $x_-(\varepsilon)$, and $x_+(\varepsilon)$ (with $\phi(x,\varepsilon)$ in C_x^3) such that for all ε in $(0,\varepsilon_0)$ (for some $\varepsilon_0 > 0$) the following conditions are satisfied:

(a) $x_-(\varepsilon) < x_e < x_+(\varepsilon)$,

(b) $f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0$ for $\phi=\phi(x,\varepsilon)$ and all x in $[x_-(\varepsilon),x_+(\varepsilon)]$,

(c) $\phi(x,\varepsilon) > \phi(x)$ for all x in $(x_-(\varepsilon), x_+(\varepsilon))$,

(d) $\phi(x_-(\varepsilon),\varepsilon) = \phi(x_-(\varepsilon))$, $\phi(x_+(\varepsilon),\varepsilon) = \phi(x_+(\varepsilon))$,

(e) $\max\limits_{x_-(\varepsilon)\le x\le x_+(\varepsilon)} |\phi(x,\varepsilon)-\phi(x)| \to 0$ as $\varepsilon \to 0$, and

(f) $x_0 \le x_-(\varepsilon) < x_+(\varepsilon) \le x_1$ for some $-\infty < x_0 < x_1 < +\infty$.

Here, in condition (a) the point $x = x_e$ is any point where $\phi(x)$ has a relative extremum. For simplicity, when $\phi(x)$ has at least three extrema we will always take $x = x_e$ to be between two other extrema.

(2) If $\phi(x)$ has an extremum only at a single finite value of x and if $\phi(x)$ goes to a saddle point as either $x \to -\infty$ or as $x \to +\infty$, then there are functions $\phi(x,\varepsilon)$, $x_-(\varepsilon)$ and $x_+(\varepsilon)$ (with $\phi(x,\varepsilon)$ in C^3_x) such that for all $0 < \varepsilon < \varepsilon_0$ (for some $\varepsilon_0 > 0$) conditions (a), (b), (c), (d) and (e) are satisfied. Now however, either $x_-(\varepsilon) \to -\infty$ or $x_+(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, and we have

(f') $\max\limits_{\substack{|x|>|x_e|+1 \\ x_-(\varepsilon)\le x\le x_+(\varepsilon)}} \{|\phi(x,\varepsilon)-\phi(x)|\cdot(1 + \frac{1}{|\phi'(x)|})\} \to 0$ as $\varepsilon \to 0$.

Here the point $x = x_e$ is the single point where $\phi(x)$ has a relative extremum.

Lemma (4.8): Assume that hypotheses H2, H3, H4, and H5 are satisfied. Let $\phi(x)$ be any bounded non-monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \quad . \qquad (4.2)$$

If $\phi(x)$ has only a single relative extremum, assume also that $\phi(x)$ goes to a saddle point as either $x \to -\infty$ or $x \to +\infty$. Then, if $\tilde{\phi}(x)$ is any other steady state solution of (4.2) satisfying

$$\phi(x) \le \tilde{\phi}(x) \quad \text{for all} \quad x$$

then $\tilde{\phi}(x) \ge \phi_0$ for all x as well. Here ϕ_0 is the <u>least constant</u> steady state solution of (4.2) with $\phi(x) < \phi_0$ for all x. Thus,

$$\phi(x) < \phi_0 \le \tilde{\phi}(x) \quad \text{for all} \quad x$$

where ϕ_0 is the least solution of $f(0,0,\phi_0) = 0$ satisfying

$$\phi(x) < \phi_0 \quad \text{for all} \quad x \quad .$$

The basic situation is illustrated in Figure (4) below. We use lemma (4.7) to find the functions $\phi(x,\varepsilon)$, $x_-(\varepsilon)$, and $x_+(\varepsilon)$.

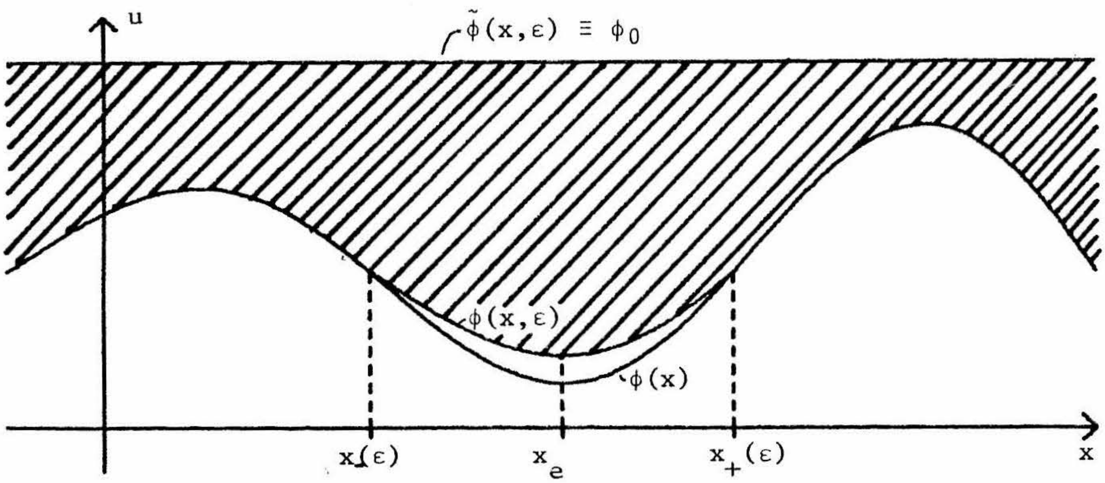


Figure (4)

These functions are utilized to form our perturbed initial conditions

$$u(\varepsilon,0,x) \equiv \left\{ \begin{array}{lll} \phi(x) & \text{for} & x \le x_-(\varepsilon) \\ \phi(x,\varepsilon) & \text{for} & x_-(\varepsilon) \le x \le x_+(\varepsilon) \\ \phi(x) & \text{for} & x_+(\varepsilon) \le x \end{array} \right\} . \quad (4.37)$$

We then prove that the solution $u(\varepsilon,t,x)$ of (4.2) (with initial condition $u(\varepsilon,0,x)$) is increasing in t, and that in fact $u(\varepsilon,t,x) \to \tilde{\phi}(x,\varepsilon)$ as $t \to +\infty$. Here, $\tilde{\phi}(x,\varepsilon)$ is the <u>smallest</u> steady state solution of (4.2) which satisfies $u(\varepsilon,0,x) \le \tilde{\phi}(x,\varepsilon)$ for all x. Since now $\tilde{\phi}(x,\varepsilon) \ge \phi(x)$ for all x and $\tilde{\phi}(x,\varepsilon) > \phi(x)$ at $x = x_e$, lemma (4.8) shows that

$$\phi(x) < \phi_0 \le \tilde{\phi}(x,\varepsilon) \quad \text{for all} \quad x,$$

where ϕ_0 is the smallest <u>constant</u> steady state solution of (4.2) satisfying $\phi(x) < \phi_0$ for all x. Further, the perturbed initial conditions $u(\varepsilon,0,x)$ which we use satisfy

$$u(\varepsilon,0,x) < \phi_0 \quad \text{for all} \quad x$$

when $\varepsilon > 0$ is sufficiently small. Since $\tilde{\phi}(x,\varepsilon)$ is the smallest possible steady state, we will then have $\tilde{\phi}(x,\varepsilon) \equiv \phi_0$ for all $\varepsilon > 0$ small enough. To summarize this, as is depicted in Figure (4) we have that the solution $u(\varepsilon,t,x)$ of equation (4.2) with the initial condition $u(\varepsilon,0,x)$ must satisfy

$$u(\varepsilon,+\infty,x) \equiv \phi_0 \quad \text{for all} \quad x$$

whenever $\varepsilon > 0$ is small enough. Moreover, if $u(t,x)$ is any solution of (4.2) with an initial condition satisfying

$$u(\varepsilon,0,x) \le u(0,x) \le \phi_0 \quad \text{for all} \quad x \quad ,$$

then the maximum principle implies that

$$u(\varepsilon,t,x) \le u(t,x) \le \phi_0 \quad \text{for all} \quad x \quad \text{and all} \quad t \ge 0 \quad .$$

In particular, $u(\varepsilon,+\infty,x) \equiv \phi_0$ and so $u(+\infty,x) \equiv \phi_0$ also. That is, <u>every solution $u(t,x)$ of equation (4.2) whose initial condition $u(0,x)$ is in the shaded region of Figure (4) must evolve to the constant steady state $u(+\infty,x) \equiv \phi_0$</u>.

<u>Proof of theorem (4.6):</u> Consider the initial conditions

$$u(\varepsilon,0,x) \equiv \left\{ \begin{array}{lll} \phi(x) & \text{for} & x \le x_-(\varepsilon) \\ \phi(x,\varepsilon) & \text{for} & x_-(\varepsilon) \le x \le x_+(\varepsilon) \\ \phi(x) & \text{for} & x_+(\varepsilon) \le x \end{array} \right\} \quad . \tag{4.37}$$

We note first that for all ε in $(0,\varepsilon_0)$ the existence assumption H5 guarantees that there is a solution $u(\varepsilon,t,x)$ of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{4.2}$$

for all $t > 0$ which has the initial value $u(\varepsilon,0,x)$ at $t = 0$. We will first use the maximum principle to show that $u(\varepsilon,h,x) \geq u(\varepsilon,0,x)$ for all x and for all $h \geq 0$. We will then use it to show that

$$u(\varepsilon,t+h,x) \geq u(\varepsilon,t,x) \quad \text{for all } x, \text{ all } h \geq 0, \text{ and all } t \geq 0 .$$

This last inequality shows $u(\varepsilon,t,x)$ to be non-decreasing in t and will be the key to the proof. We now prove these.

We note that $u(\varepsilon,0,x) \geq \phi(x)$ for all x. Since $u(\varepsilon,t,x)$ and $\phi(x)$ are both solutions of (4.2), the maximum principle implies that

$$u(\varepsilon,t,x) \geq \phi(x) \quad \text{for all } x \text{ and all } t \geq 0 . \qquad (4.38)$$

In particular, (4.38) implies that

$$u(\varepsilon,t,x_-(\varepsilon)) \geq \phi(x_-(\varepsilon)) \equiv \phi(x_-(\varepsilon),\varepsilon) \quad \text{for all } t \geq 0, \text{ and}$$

$$u(\varepsilon,t,x_+(\varepsilon)) \geq \phi(x_+(\varepsilon)) \equiv \phi(x_+(\varepsilon),\varepsilon) \quad \text{for all } t \geq 0 .$$

Since $u(\varepsilon,0,x) = \phi(x,\varepsilon)$ for all x in $[x_-(\varepsilon),x_+(\varepsilon)]$, since $u(\varepsilon,t,x_-(\varepsilon)) \geq \phi(x_-(\varepsilon),\varepsilon)$, since $u(\varepsilon,t,x_+(\varepsilon)) \geq \phi(x_+(\varepsilon),\varepsilon)$, and finally since $u(\varepsilon,t,x)$ and $\phi(x,\varepsilon)$ are both solutions of (4.2) for all $t > 0$ and all x in $[x_-(\varepsilon),x_+(\varepsilon)]$, the maximum principle implies that

$$u(\varepsilon,t,x) \geq \phi(x,\varepsilon) \quad \text{for all } x \text{ in } [x_-(\varepsilon),x_+(\varepsilon)] \text{ and all } t \geq 0 . \qquad (4.39)$$

From (4.38) and (4.39) we see that

$$u(\varepsilon,t,x) \geq u(\varepsilon,0,x) \quad \text{for all } x \text{ and all } t \geq 0 . \qquad (4.40)$$

Relation (4.40) will now imply that $u(\varepsilon,t,x)$ is non-decreasing in t. To see this, let $h \geq 0$ be any constant. Clearly $u(\varepsilon,t,x)$ being a solution of (4.2) implies that $u(\varepsilon,t+h,x)$ is also a solution. But (4.40) shows that

$$u(\varepsilon,t+h,x) \geq u(\varepsilon,t,x) \quad \text{for all } x \qquad (4.41)$$

is satisfied at $t = 0$, and the maximum principle shows that it therefore must be true for all $t > 0$ as well. Hence, we have that $u(\varepsilon,t,x)$ is non-decreasing in t. From this result, the proof of the theorem will now follow.

From the modification of the equations in Chapter III, we know that there is a $\tilde{M} > 0$ such that $f(0,0,\phi_1) = 0$ for all constants $\phi_1 \geq \tilde{M}$. (Note that this particular aspect of the modifications does not affect the stability of the <u>bounded</u> steady state $\phi(x)$. No modifications to $f(u_{xx},u_x,u)$ for extremely large values of u can affect the stability of a bounded steady state, since $\phi(x)$ will be known to be unstable long before the perturbations $u(t,x) - \phi(x)$ are so large that the modification of f for large values of u has any effect.) Let us take the constant ϕ_1 to be so large that $\phi_1 > \tilde{M}$ and

$$u(\varepsilon,0,x) \leq \phi_1 \quad \text{for all} \quad x \quad \text{and all} \quad \varepsilon \quad \text{in} \quad (0,\varepsilon_0) \quad .$$

Then since $f(0,0,\phi_1) = 0$, the function $u(t,x) \equiv \phi_1$ must be a constant steady state solution of (4.2). The maximum principle now implies that

$$u(\varepsilon,t,x) \leq \phi_1 \quad \text{for all} \quad x, \quad \text{all} \quad t \geq 0, \quad \text{and all} \quad \varepsilon \quad \text{in} \quad (0,\varepsilon_0) \quad .$$

Thus, for each ε and x the function $u(\varepsilon,t,x)$ is non-decreasing and bounded in t. The limit $u(\varepsilon,t,x) \to \tilde{\phi}(x,\varepsilon)$ as $t \to +\infty$ must therefore exist pointwise at each ε and x. From the uniformity lemma and the asymptotic state theorem of Chapter III, we conclude that $\tilde{\phi}(x,\varepsilon)$ is a steady state solution of (4.2).

So far we know that $u(\varepsilon,t,x)$ is non-decreasing in t (at each x and ε) and that $u(\varepsilon,+\infty,x) = \tilde{\phi}(x,\varepsilon)$, where $\tilde{\phi}(x,\varepsilon)$ is a bounded steady state solution of (4.2). We now use the maximum principle once more to identify this final steady state $\tilde{\phi}(x,\varepsilon)$ as the <u>least</u> steady state solution of (4.2) satisfying

$$u(\varepsilon,0,x) \leq \tilde{\phi}(x,\varepsilon) \quad \text{for all} \quad x \quad . \tag{4.42}$$

That is, if $\tilde{\tilde{\phi}}(x,\varepsilon)$ is <u>any</u> steady state solution of (4.2) satisfying

$$u(\varepsilon,0,x) \leq \tilde{\tilde{\phi}}(x,\varepsilon) \quad \text{for all} \quad x \quad , \tag{4.43}$$

then

$$\tilde{\phi}(x,\varepsilon) \leq \tilde{\tilde{\phi}}(x,\varepsilon) \quad \text{for all} \quad x \quad . \tag{4.44}$$

Indeed suppose that $\tilde{\tilde{\phi}}(x,\varepsilon)$ is a steady state solution which satisfies (4.43) and suppose also that $\tilde{\tilde{\phi}}(x,\varepsilon) < \tilde{\phi}(x,\varepsilon)$ at any point $x = \tilde{x}$. Since (4.43) holds, the maximum principle implies that

$$u(\varepsilon,t,x) \leq \tilde{\tilde{\phi}}(x,\varepsilon) \quad \text{for all} \quad x \quad \text{and all} \quad t > 0 \quad .$$

But now at $x = \tilde{x}$,

$$u(\varepsilon,t,\tilde{x}) \leq \tilde{\tilde{\phi}}(\tilde{x},\varepsilon) < \tilde{\phi}(\tilde{x}) \quad \text{for all} \quad t > 0 \quad ,$$

and this contradicts the definition of $\tilde{\phi}(x,\varepsilon) \equiv u(\varepsilon,+\infty,x)$ at $x = \tilde{x}$. Thus, all steady solutions $\tilde{\tilde{\phi}}(x,\varepsilon)$ satisfying (4.43) must also satisfy (4.44).

We are now very nearly done. From lemma (4.8) we easily conclude that

$$u(\varepsilon,+\infty,x) \equiv \tilde{\phi}(x,\varepsilon) \geq \phi_0 \quad \text{for all} \quad x \quad \text{and all} \quad \varepsilon \quad \text{in} \quad (0,\varepsilon_0) \quad ,$$

where ϕ_0 is the smallest <u>constant</u> steady state with

$$\phi(x) < \phi_0 \quad \text{for all} \quad x \quad .$$

In particular, for some x (namely $x = 0$),

$$u(\varepsilon,+\infty,x) - \phi(x) \geq \phi_0 - \phi(0) \quad .$$

We define the ε - independent quantity Δ by

$$\Delta = \tfrac{1}{2}\{\phi_0 - \phi(0)\}$$

and note for t sufficiently large that

$$u(\varepsilon,t,x) - \phi(x) \geq \Delta \text{ at } \quad x = 0 \quad .$$

To complete the proof, we need only to verify that the initial perturbations $u(\varepsilon,0,x) - \phi(x)$ satisfy the conditions set forth in theorem (4.6). However, inspection of the initial conditions $u(\varepsilon,0,x)$ defined in (4.37) and inspection of properties (e) and (f) or (e) and (f') of lemma (4.7) show that the functions $u(\varepsilon,0,x) - \phi(x)$ have all the properties claimed (for the the unstable perturbations of $u(t,x) \equiv \phi(x)$) by theorem (4.6) except one. This lack is that $u(\varepsilon,0,x) - \phi(x)$ is not differentiable twice at $x = x_-(\varepsilon)$ or at $x = x_+(\varepsilon)$, and is therefore not in H_x^2. We now remedy this.

Suppose that $u(0,x)$ is any function in H_x^2 satisfying

$$u(\varepsilon,0,x) \leq u(0,x) \leq \tilde{\phi}(x,\varepsilon) \quad \text{for all} \quad x \quad . \tag{4.45}$$

From the maximum principle, we find that the solution $u(t,x)$ of (4.2) (with initial condition $u(0,x)$) satisfies

$$u(\varepsilon,t,x) \leq u(t,x) \leq \tilde{\phi}(x,\varepsilon) \quad \text{for all} \quad x \quad \text{and all} \quad t \geq 0. \tag{4.46}$$

Hence, since $u(\varepsilon,t,x) \to \tilde{\phi}(x,\varepsilon)$ as $t \to \infty$, we must also have

$$u(+\infty,x) = \tilde{\phi}(x,\varepsilon) \quad \text{for all} \quad x \tag{4.47}$$

whenever (4.45) holds. We now do not need to use $u(\varepsilon,0,x)$ as our initial condition. Instead, for each $\varepsilon > 0$ we simply select a $u(0,x)$ in H_x^2 satisfying (4.45). We choose this $u(0,x)$ to both approximate $u(\varepsilon,0,x)$ as closely as we please for all x in $(x_-(\varepsilon) - 1,\ x_+(\varepsilon) + 1)$ and to be identically $\phi(x)$ for all x outside of $(x_-(\varepsilon)-1, x_+(\varepsilon)+1)$.

This establishes theorem (4.6). Note that the results contained in equations (4.45) and (4.47) are illustrated in Figure (5).

Theorem (4.6) very nearly completes the stability picutre for steady state solutions of

$$u_t = f(u_{xx},u_x,u) + cu_x \quad . \tag{4.2}$$

In summary, steady state solutions $\phi(x)$ of (4.2) which have at least two relative extrema are unstable, even to arbitrarily small perturbations of finite extent. Steady states $\phi(x)$ which have only a single relative extremum and which go to a saddle point as $x \to -\infty$ or as $x \to +\infty$ are unstable, at least to arbitrarily small perturbations which decay asymptotically like $\phi'(x)$ as $x \to -\infty$ and as $x \to +\infty$. The stability of steady states which have only a single relative extremum and which go to nodes both as $x \to -\infty$ and as $x \to +\infty$ has not been discussed yet. We treat this indeterminate case in section (4.14). Finally, steady states $\phi(x)$ which have no relative extrema (monotonic steady states) are stable, at least to small perturbations which decay asymptotically no slower than $\phi'(x)$ as $x \to \pm\infty$. (The precise stability of these monotonic steady states is given by theorem (4.5)). Thus the stability of steady state solutions is generic: it depends only on a few easily determined properties of the particular steady state and the particular equation.

In the next section, section (4.13), we will prove lemmas (4.7) and (4.8). Before continuing on to this section some further remarks are in order.

First, let us note that our stability picture is incomplete. In theorem (4.6) we have not determined the stability or instability of steady states $\phi(x)$ which have only a single relative extremum and which go to a node as $x \to -\infty$ and to another node as $x \to +\infty$. We discuss this indeterminate case in section (4.14). There we will be able to characterize which steady states of this case are stable and which are unstable. For any particular example of this indeterminate case, this characterization should provide a practical method for deciding the stability or instability of any particular steady state solution of any particular equation.

Second, the proofs of lemma (4.7), lemma (4.8), and theorem (4.6) can be extended to include some constant steady states $u(t,x) \equiv \phi_0$ as part of the "steady states with at least two relative extrema" case. Specifically, the constant steady state $u(t,x) \equiv \phi_0$ can be included whenever the singular point $\phi = \phi_0$, $v = 0$ is a spiral point or a center. Thus, these constant steady states are unstable even to arbitrarily small perturbations of finite extent. We will discuss this in section (4.15).

Third, note that we have actually proved much more than the fact that $u(t,x) \equiv \phi(x)$ is unstable. For the steady states $\phi(x)$ treated by theorem (4.6), we have actually shown that whenever $u(t,x)$ is a solution of (4.2) whose initial condition satisfies

$$u(\varepsilon,0,x) \leq u(0,x) \leq \tilde{\phi}(x,\varepsilon) \quad \text{for all } x \tag{4.45}$$

then $u(+\infty,x) \equiv \tilde{\phi}(x,\varepsilon)$. Since for all sufficiently small $\varepsilon > 0$ we have

$$u(\varepsilon,0,x) \leq \phi_0 \quad \text{for all } x$$

where ϕ_0 is the smallest <u>constant</u> steady state satisfying

$$\phi(x) < \phi_0 \quad \text{for all } x \quad ,$$

we can conclude that $\tilde{\phi}(x,\varepsilon) \equiv \phi_0$ for all x and all $\varepsilon > 0$ sufficiently small. Thus whenever any solution $u(t,x)$ of (4.2) satisfies

$$u(\varepsilon,0,x) \leq u(0,x) \leq \phi_0 \quad \text{for all } x$$

at $t = 0$, then $u(+\infty,x) \equiv \phi_0$. This is illustrated in Figure (5). From this we see that the proof of theorem (4.6) provides a potentially powerful technique for finding the final state $u(+\infty,x)$ as a function of the initial condition $u(0,x)$. This will be briefly discussed in section (4.16).

Fourth, let us note that we can extend these instability results to plane waves in higher spatial dimensions, although the results are weaker than the results in theorem (4.6). In section (4.17) we will discuss these extensions to multiple spatial dimensions.

Finally, the methods we have used to prove theorems (4.5) and

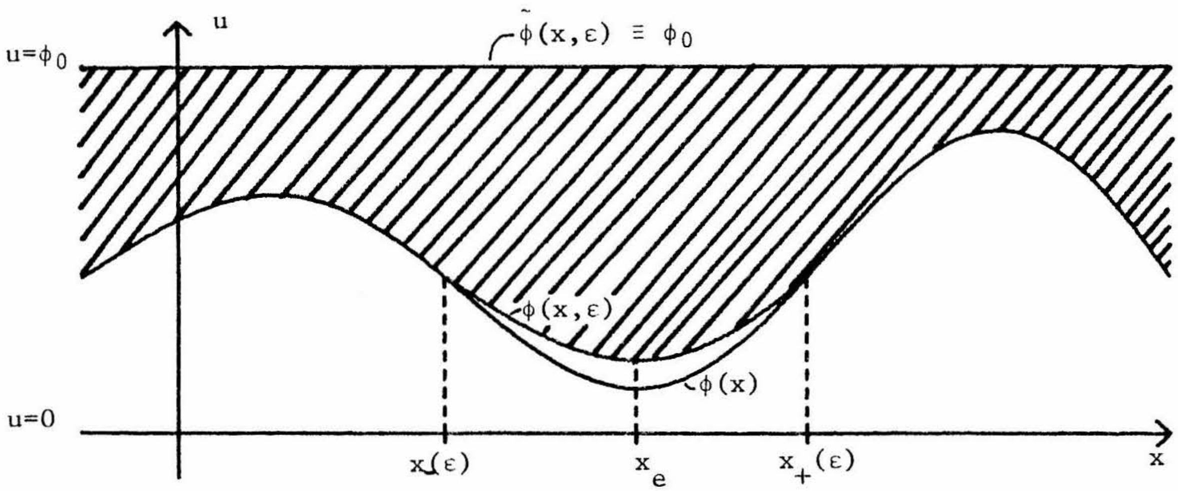


Figure (5): From (4.45) and (4.47) we see that any solution u(t,x) whose initial values u(0,x) are in the shaded region must evolve into the constant steady state ϕ_0 as $t\to+\infty$. That is, $u(+\infty,x) \equiv \phi_0$.

(4.6) can also be used on finite spatial domain-boundary value problems. Consider the following finite domain-boundary value problem:

$$u_t = f(u_{xx},u_x,u) \qquad 0 \le x \le 1 \ , \ t \ge 0$$
$$u(t,x) \equiv A \quad \text{at} \quad x = 0 \qquad u(t,x) \equiv B \quad \text{at} \quad x = 1$$

where A and B are fixed constants. In section (4.18) we will determine the stability of all steady state solutions $u(t,x) \equiv \phi(x)$ of this boundary value problem.

We now continue on to section (4.13) where we prove lemmas (4.7) and (4.8).

4.13 Proof of lemmas (4.7) and (4.8). In this section we will prove lemmas (4.7) and (4.8). We will prove these lemmas by using a key observation about the phase plane of the steady state solutions of (4.2). To prove these lemmas we will first suppose that $\phi(x)$ is any bounded non-monotonic solution of the steady state equation for (4.2), namely of the equation

$$f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0 \quad . \tag{4.48}$$

We will also suppose that $\phi(x)$ has a relative extrema at $x = x_e$, and then will consider the solutions $\phi(x,\varepsilon)$ of (4.48) which have the initial conditions

$$\phi(x_e,\varepsilon) = \phi(x_e) + \varepsilon \ , \ \phi_x(x_e,\varepsilon) = \phi_x(x_e) = 0 \quad .$$

The key to proving both lemmas is noting that the phase plane of

$$\begin{aligned} \phi_x &= v \\ f(v_x,v,\phi) + cv &= 0 \end{aligned} \tag{4.49}$$

implies that for all ε with $|\varepsilon|$ sufficiently small, $\phi(x,\varepsilon)$ and $\phi(x)$ must intersect at least once when $x > x_e$ and at least once when $x < x_e$. Note that this intersection property constitutes an oscillation (or comparison) result about the ordinary differential equation (4.48).

We start by noting that the phase plane of system (4.49) possesses the following properties:

(i) all singular points are on the $v = 0$ line,

(ii) the horizontal components of the phase plane trajectories are positive when $v > 0$ and negative when $v < 0$, and

(iii) the phase plane trajectories never cross (except at singular points).

Let $\phi(x)$ and $\phi(x,\varepsilon)$ be the solutions of (4.49) we defined above. Because the phase plane of (4.49) has properties (i), (ii), and (iii), the possible behaviors of $\phi(x)$ and $\phi(x,\varepsilon)$ are severely limited. For example, suppose that

(1) $\phi(x)$ is a bounded solution of (4.49) with, say, a relative

maximum at $x = x_e$,

(2) $\phi(x,\varepsilon)$ is the solution of (4.49) with initial condition

$$\phi(x_e,\varepsilon) = \phi(x_e) + \varepsilon \qquad \phi_x(x_e,\varepsilon) = \phi_x(x_e) = 0 \quad ,$$

and

(3) ε is, say, positive and small enough so that $\phi = \phi_0$, $v = 0$ is not a singular point for any ϕ_0 in $[\phi(x_e), \phi(x_e) + \varepsilon]$. When $\phi(x)$ and $\phi(x,\varepsilon)$ satisfy these three conditions, the phase plane of system (4.49) implies that exactly one of the following alternatives must be the case for $x > x_e$:

(1') $\phi(x)$ and $\phi(x,\varepsilon)$ both have at least one extrema for $x > x_e$. For this case, let $\tilde{x}$ be the least x larger than x_e at which $\phi(x)$ has a relative minimum and let $\tilde{x}(\varepsilon) > x_e$ be the least point $x > x_e$ at which $\phi(x,\varepsilon)$ has a relative minimum. Then $\phi(\tilde{x}) > \phi(\tilde{x}(\varepsilon),\varepsilon)$.

(2') $\phi(x)$ has at least one relative extrema for $x > x_e$ but $\phi(x,\varepsilon)$ has none for $x > x_e$. For this case, let $\tilde{x}$ again be the least $x > x_e$ at which $\phi(x)$ has a relative minimum. Then one of the following must happen

(a) $\phi(x,\varepsilon) \to -\infty$ as $x \to +\infty$, as

(b) $\phi(x,\varepsilon) \to \phi_0(\varepsilon)$ as $x \to +\infty$ where $\phi_0(\varepsilon)$ is a singular point and $\phi(\tilde{x}) > \phi_0(\varepsilon) \equiv \phi(+\infty,\varepsilon)$.

(3') $\phi(x,\varepsilon)$ has at least one relative extrema for $x > x_e$ but $\phi(x)$ has none. For this case we let $\tilde{x}(\varepsilon)$ be the least $x > x_e$ at which $\phi(x,\varepsilon)$ has a relative minimum. In this case, $\phi(x) \to \phi_0$ as $x \to +\infty$ where ϕ_0 is a singular point and $\phi(+\infty) \equiv \phi_0 > \phi(\tilde{x}(\varepsilon),\varepsilon)$.

(4') Neither $\phi(x)$ nor $\phi(x,\varepsilon)$ has a relative extrema for $x > x_e$. In this case $\phi(x) \to \phi_0$ as $x \to +\infty$ where ϕ_0 is a singular point, and one of the following must occur

(a) $\phi(x) \to \phi_0$ as $x \to +\infty$ and $\phi(x,\varepsilon) \to -\infty$ as $x \to +\infty$,

(b) $\phi(x) \to \phi_0$ as $x \to +\infty$ and $\phi(x,\varepsilon) \to \phi_0(\varepsilon)$ as $x \to +\infty$ where $\phi_0(\varepsilon)$ is a singular point and $\phi(+\infty) \equiv \phi_0 > \phi_0(\varepsilon) \equiv \phi(+\infty,\varepsilon)$, or

(c) $\phi(x) \to \phi_0$ as $x \to +\infty$ and $\phi(x,\varepsilon) \to \phi_0(\varepsilon)$ as $x \to +\infty$ where $\phi_0(\varepsilon) = \phi_0$.

The phase planes of these alternatives are illustrated in Figures (6) - (12).

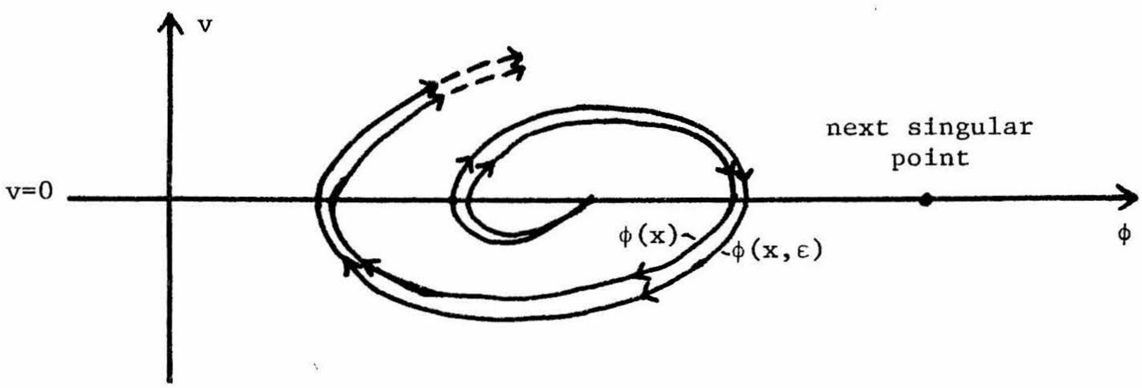


Figure (6): Since the maximum of $\phi(x,\varepsilon)$ at $x=x_e$ is larger than the maximum of $\phi(x)$ at $x=x_e$, if both $\phi(x,\varepsilon)$ and $\phi(x)$ have a relative minimum for $x>x_e$ then the next minimum of $\phi(x,\varepsilon)$ is smaller than the next minimum of $\phi(x)$. (Case 1').

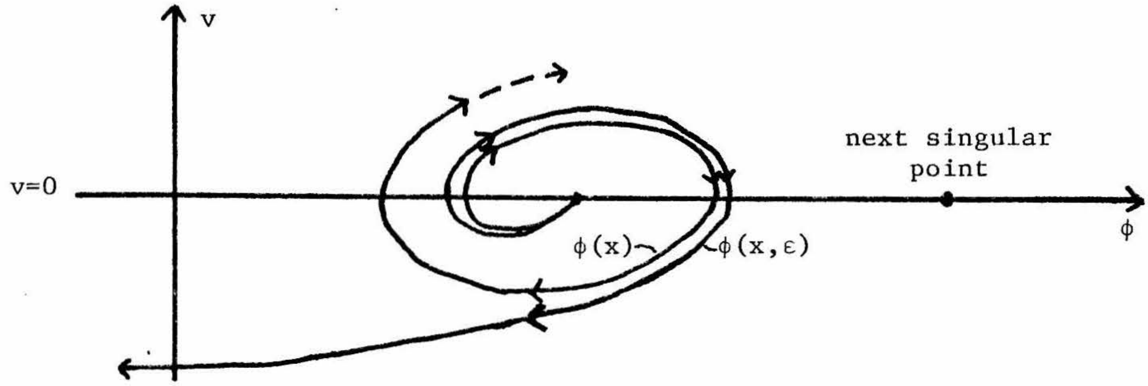


Figure (7): If $\phi(x)$ has a relative minimum at some $x>x_e$ and $\phi(x,\varepsilon)$ does not, then $\phi(x,\varepsilon)$ may go to $-\infty$ as $x\to+\infty$. (Case 2'a).

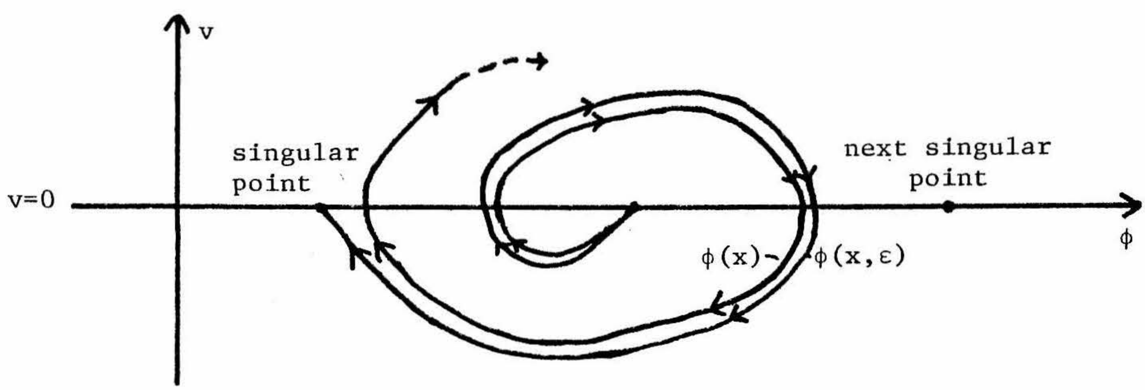


Figure (8): If $\phi(x)$ has a relative minimum at some first $\tilde{x}>x_e$ and $\phi(x,\varepsilon)$ has none for $x>x_e$, then $\phi(x,\varepsilon)$ may go to a singular point $\phi_0(\varepsilon)$ with $\phi(\tilde{x})>\phi_0(\varepsilon)\equiv\phi(+\infty,\varepsilon)$. (Case 2'b).

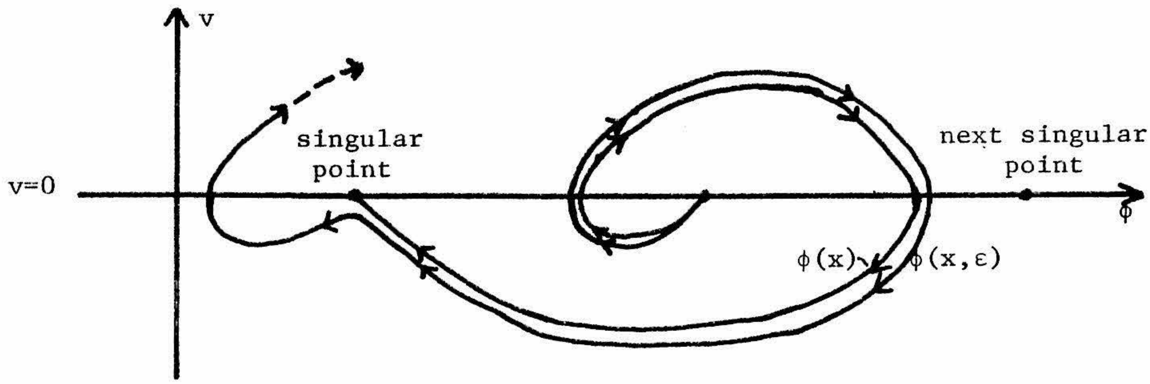


Figure (9): If $\phi(x,\varepsilon)$ has a relative minimum at some first $\tilde{x}(\varepsilon)>x_e$ and $\phi(x)$ has none for $x>x_e$, then $\phi(x)$ goes to a singular point ϕ_0 as $x\to+\infty$, and $\phi(+\infty)\equiv\phi_0>\phi(\tilde{x}(\varepsilon),\varepsilon)$. (Case 3').

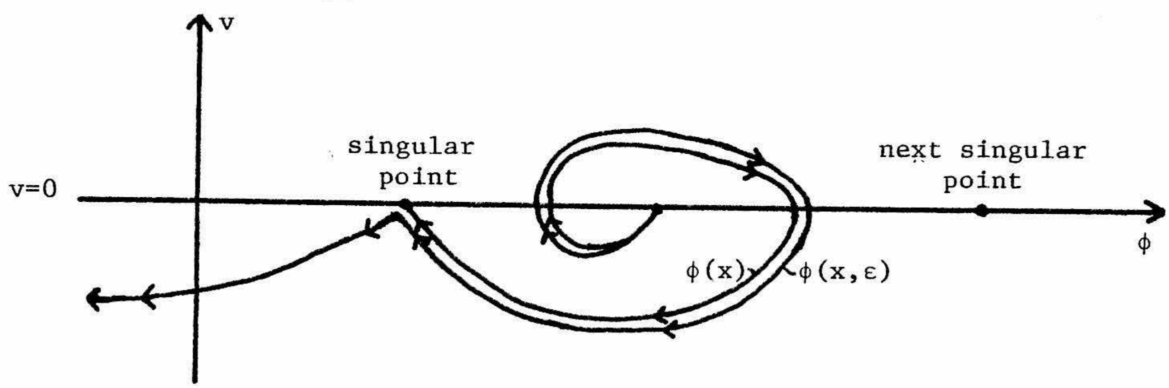


Figure (10): If neither $\phi(x)$ nor $\phi(x,\varepsilon)$ has a relative minimum for $x>x_e$, then $\phi(x)\to\phi_0$ as $x\to+\infty$ (where ϕ_0 is a singular point), and $\phi(x,\varepsilon)$ may go to $-\infty$ as $x\to+\infty$. (Case 4'a).

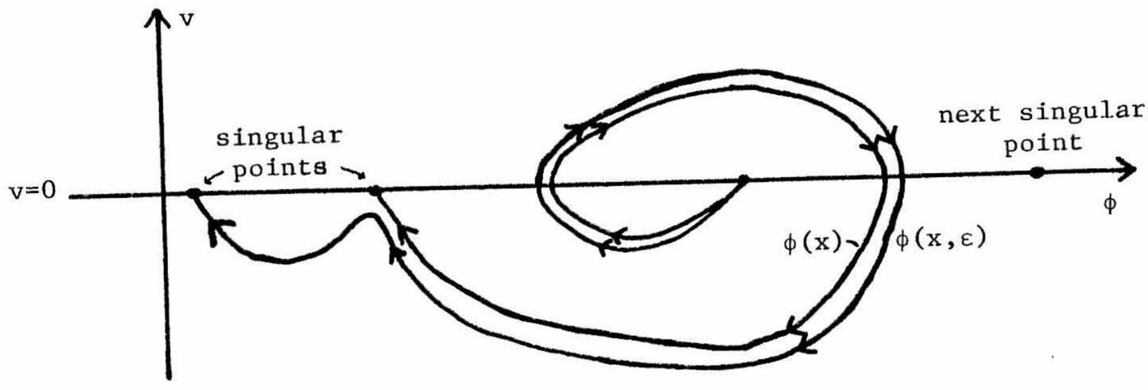


Figure (11): If neither $\phi(x)$ nor $\phi(x,\varepsilon)$ has a relative minimum for $x>x_e$, $\phi(+\infty) = \phi_0$ and $\phi(+\infty,\varepsilon) = \phi_0(\varepsilon)$ may occur, where ϕ_0 and $\phi_0(\varepsilon)$ are singular points and $\phi_0 > \phi_0(\varepsilon)$. (Case 4'b).

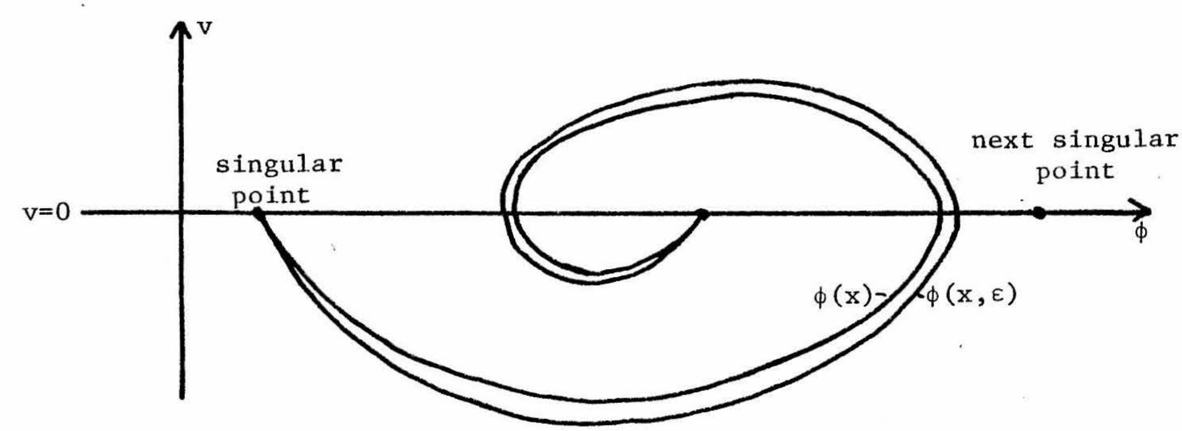


Figure (12): If neither $\phi(x)$ nor $\phi(x,\varepsilon)$ has a relative minimum for $x>x_e$, then $\phi(x)$ and $\phi(x,\varepsilon)$ may go to the <u>same</u> singular point as $x\to+\infty$. (Case 4'c).

There is a very simple way to summarize all of these alternatives. Let us allow the singular points $\phi(+\infty)$, $\phi(-\infty)$, $\phi(+\infty,\varepsilon)$, and $\phi(-\infty,\varepsilon)$ to be called extrema for the curves $\phi(x)$ and $\phi(x,\varepsilon)$, and suppose we automatically let $\phi = -\infty$ and $\phi = +\infty$ be a minimum and maximum (respectively) of ϕ. With these definitions, we see that since the <u>maximum</u> of $\phi(x,\varepsilon)$ at $x = x_e$ is <u>larger</u> than the maximum of $\phi(x)$ at $x = x_e$, the phase plane of (4.49) implies that either

(1") the <u>first minimum</u> of $\phi(x,\varepsilon)$ with $x > x_e$ must be <u>smaller</u> than the first minimum of $\phi(x)$ with $x > x_e$, or

(2") the <u>first minimum</u> of $\phi(x,\varepsilon)$ and of $\phi(x)$ with $x > x_e$ <u>occur at $x = +\infty$</u> and <u>$\phi(+\infty,\varepsilon) = \phi(+\infty)$</u>.

Similar alternatives occur for $x < x_e$. In particular, since the maximum $\phi(x_e,\varepsilon)$ is larger than the maximum $\phi(x)$, then either the last minimum of $\phi(x,\varepsilon)$ with $x < x_e$ is smaller than the last minimum of $\phi(x)$ with $x < x_e$, or both last minima occur at $x = -\infty$ and $\phi(-\infty,\varepsilon) = \phi(-\infty)$.

We can also interchange the roles of minima and maxima in the above alternatives and the results will remain valid. That is, if $\phi(x,\varepsilon)$ and $\phi(x)$ both have a relative minimum at $x = x_e$ and $\phi(x_e,\varepsilon) > \phi(x_e)$, then either the next maximum of $\phi(x,\varepsilon)$ is smaller than the next maximum of $\phi(x)$, or the next maximum of both $\phi(x,\varepsilon)$ and $\phi(x)$ occurs at $x = +\infty$ and $\phi(+\infty,\varepsilon) = \phi(+\infty)$.

Note finally that similar alternatives occur if we take $\varepsilon < 0$ but ε large enough so that there are no singular points in $[\phi(x_e)-\varepsilon,\phi(x_e)]$. In fact, taking $\varepsilon < 0$ is essentially equivalent to interchanging the roles of $\phi(x)$ and $\phi(x,\varepsilon)$ in the above lists of alternatives.

We now use the above lists of alternatives (and similar phase plane observations) to prove lemmas (4.7) and (4.8).

<u>Proof of lemma (4.7):</u> In proving lemma (4.7) we will need to consider

three separate cases. Namely, the cases of where the non-monotonic steady state $\phi(x)$ in the statement of the lemma has at least three relative extrema, has exactly two relative extrema, and has only one relative extremum. We begin with the simplest case, where $\phi(x)$ has at least three relative extrema.

Suppose that $\phi(x)$ is a bounded solution of the steady state equation (4.48), and suppose that $\phi(x)$ has at least three extrema. In particular, assume that $\phi(x)$ has a relative extrema at $x = x_e$, has at least one relative extrema when $x > x_e$ and has at least one when $x < x_e$. We finally suppose that $\phi(x)$ has a relative maximum at $x = x_e$ since the case of a relative minimum can be handled similarly. For notation, let $\tilde{x}_-$ be the largest value of $x < x_e$ at which $\phi(x)$ has a relative minimum, and let $\tilde{x}_+$ be the smallest value of $x > x_e$ at which $\phi(x)$ has a relative minimum. Thus, $\phi(x)$ has consecutive extrema at $x = \tilde{x}_-$, $x = x_e$, and $x = \tilde{x}_+$ with $\phi(\tilde{x}_-)$, $\phi(x_e)$, and $\phi(\tilde{x}_+)$ being a relative minimum, maximum, and minimum respectively.

Similar to before, we define $\phi(x,\varepsilon)$ for all $0 < \varepsilon < \varepsilon_2$ as the solutions of (4.48) with the initial conditions

$$\phi(x_e,\varepsilon) = \phi(x_e) + \varepsilon \qquad \phi_x(x_e,\varepsilon) = \phi_x(x_e) = 0 \quad , \qquad (4.50)$$

and we assume that $\varepsilon_2 > 0$ is small enough so that $\phi = \phi_0$, $v = 0$ is not a singular point for any ϕ_0 in $[\phi(x_e), \phi(x_e) + \varepsilon_2]$. Let us now note that $\phi(x,\varepsilon)$ and $\phi_x(x,\varepsilon)$ are both continuously differentiable in ε (see e.g. [6]). Thus, for all ε with $0 < \varepsilon < \varepsilon_1$ (for some ε_1 in $(0,\varepsilon_2)$), $\phi(x,\varepsilon)$ has a relative minimum at $x = \tilde{x}_-(\varepsilon)$ near $x = \tilde{x}_-$ and also has a relative minimum at $x = \tilde{x}_+(\varepsilon)$ near $x = \tilde{x}_+$. From the list of possible alternatives, we immediately see that

$$\phi(\tilde{x}_+) > \phi(\tilde{x}_+(\varepsilon),\varepsilon) \quad \text{and} \quad \phi(\tilde{x}_-) > \phi(\tilde{x}_-(\varepsilon),\varepsilon)$$

must be the case. Since $\phi(x_e,\varepsilon) = \phi(x_e) + \varepsilon > \phi(x_e)$, careful consideration of the list of possible alternatives shows that the curves $\phi(x)$ and $\phi(x,\varepsilon)$ intersect at least once when x is in $(\tilde{x}_-(\varepsilon),x_e)$ and also at least once when x is in $(x_e,\tilde{x}_+(\varepsilon))$. Hence, we let $x = x_-(\varepsilon)$ be the largest $x < x_e$ at which $\phi(x_-(\varepsilon),\varepsilon) = \phi(x_-(\varepsilon))$ and let $x = x_+(\varepsilon)$ be the smallest $x > x_e$ at which $\phi(x_e(\varepsilon),\varepsilon) = \phi(x_+(\varepsilon))$. Figure (13) illustrates the present situation.

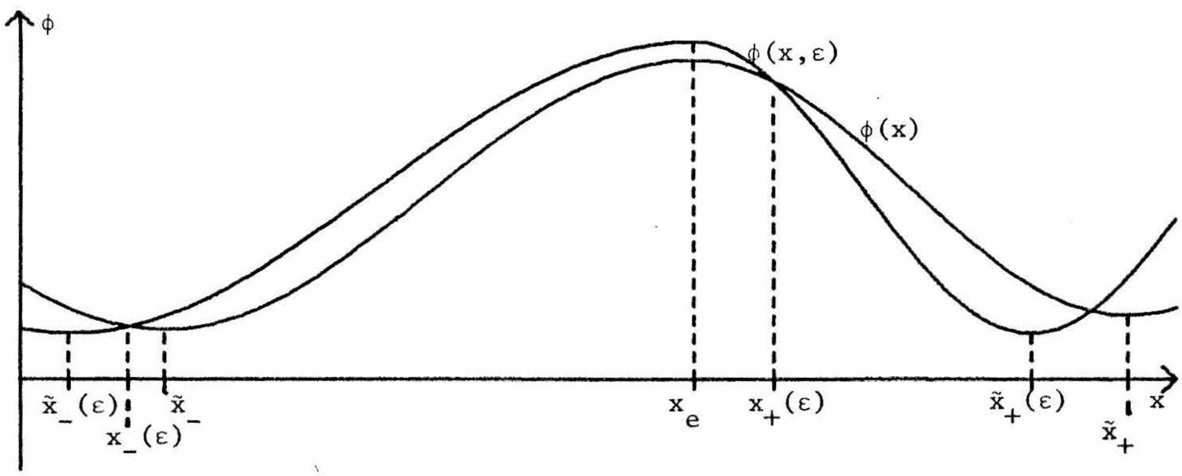


Figure (13)

It is clear that $\phi(x,\varepsilon)$, $x_-(\varepsilon)$, and $x_+(\varepsilon)$ possess the properties

(a) $x_-(\varepsilon) < x_e < x_+(\varepsilon)$

(b) $f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0$ for $\phi = \phi(x,\varepsilon)$,

(c) $\phi(x,\varepsilon) > \phi(x)$ for all x in $(x_-(\varepsilon),x_+(\varepsilon))$, and

(d) $\phi(x_-(\varepsilon),\varepsilon) = \phi(x_-(\varepsilon))$, $\phi(x_+(\varepsilon),\varepsilon) = \phi(x_+(\varepsilon))$

for all ε in $(0,\varepsilon_1)$. Moreover, let $x_0 < \tilde{x}_-$ and $x_1 > \tilde{x}_+$ be given. The uniform continuity of $\phi_x(x,\varepsilon)$ in ε when x is restricted to

$[x_0, x_1]$ shows that

$$x_0 \leq \tilde{x}_-(\varepsilon) \leq x_-(\varepsilon) < x_e < x_+(\varepsilon) \leq \tilde{x}_+(\varepsilon) \leq x_1$$

for all ε in $(0,\varepsilon_0)$ for any sufficiently small ε_0 in $(0,\varepsilon_1)$. Thus, for $0 < \varepsilon < \varepsilon_0$ we have

(f) $x_0 \leq x_-(\varepsilon) < x_+(\varepsilon) \leq x_1$.

Property (e) is also easily established. From property (f),

$$\max_{x_0 \leq x \leq x_1} |\phi(x,\varepsilon) - \phi(x)| \geq \max_{x_-(\varepsilon) \leq x \leq x_+(\varepsilon)} |\phi(x,\varepsilon) - \phi(x)| \ .$$

Thus, property (e) immediately follows from the uniform continuity of $\phi(x,\varepsilon)$ in ε when x is restricted to $[x_0, x_1]$, since this uniform continuity implies that

$$\max_{x_0 \leq x \leq x_1} |\phi(x,\varepsilon) - \phi(x)| \to 0 \quad \text{as} \quad \varepsilon \to 0 \ .$$

This establishes lemma (4.7) in the case where $\phi(x)$ has at least three extrema.

We now will establish the lemma in the case where $\phi(x)$ has exactly two relative extrema. This proof will be a slight variant of the preceding case.

Suppose that $\phi(x)$ is a bounded non-monotonic solution of (4.48) and suppose further that $\phi(x)$ has exactly two relative extrema, one at $x = x_e$ and one at $x = \tilde{x}_+$ with $\tilde{x}_+ > x_e$. Finally, suppose that $\phi(x)$ has a relative <u>maximum</u> at $x = x_e$ since the other case is handled in a similar manner.

For this present case we define $\tilde{\phi}(x,\varepsilon)$ for all $0 < \varepsilon < \varepsilon_3$ as the solutions of

$$f(\tilde{\phi}_{xx},\tilde{\phi}_x,\tilde{\phi}) + c\tilde{\phi}_x = 0 \qquad (4.48)$$

with the initial conditions

$$\tilde{\phi}(x_e,\varepsilon) = \phi(x_e) + \varepsilon \ , \ \tilde{\phi}_x(x_e,\varepsilon) = \phi_x(x_e) = 0 \ . \qquad (4.50)$$

Here $\varepsilon_3 > 0$ is any constant small enough so that $\phi = \phi_0$, $v = 0$ is <u>not</u> a singular point for any ϕ_0 in $[\phi(x_e), \phi(x_e) + \varepsilon_3]$. For this case we cannot use $\tilde{\phi}(x,\varepsilon)$ directly as the $\phi(x,\varepsilon)$ in the lemma. Instead, select $\tilde{x}_-$ as any point x smaller than x_e, and define

$$\phi(x,\varepsilon) = \begin{cases} \tilde{\phi}(x,\varepsilon) & \text{if } \phi(x) = \tilde{\phi}(x,\varepsilon) \text{ for some } x \in [\tilde{x}_-,x_e) \\ \tilde{\phi}(x+h(\varepsilon),\varepsilon) & \text{otherwise, where } h(\varepsilon)<0 \text{ is the largest constant} \\ & \text{such that } \tilde{\phi}(\tilde{x}_-+h(\varepsilon),\varepsilon) = \phi(\tilde{x}_-) \end{cases} .$$

Note that the uniform continuity of $\tilde{\phi}_x(x,\varepsilon)$ in ε (when x is restricted to compact sets) shows that $h(\varepsilon)$ exists for all $\varepsilon > 0$ small enough. This uniform continuity also shows that there is a $K > 0$ such that $-K\varepsilon < h(\varepsilon) < 0$ for all ε in $(0,\varepsilon_2)$, where ε_2 in $(0,\varepsilon_3)$ is any sufficiently small constant.

As in the previous case, for all $\varepsilon > 0$ sufficiently small $\tilde{\phi}(x,\varepsilon)$ has a relative minimum at $x = \tilde{x}_+(\varepsilon)$ near $\tilde{x}_+$. Also as before, we can therefore conclude that $\phi(x)$ and $\tilde{\phi}(x,\varepsilon)$ must intersect at least once for x in $(x_e,\tilde{x}_+(\varepsilon))$. Moreover since $\phi(x)$ has no relative extrema for x larger than $\tilde{x}_+$, we can also conclude that the curves $\phi(x,\varepsilon) \equiv \tilde{\phi}(x+h(\varepsilon),\varepsilon)$ and $\phi(x)$ intersect at least once when x is in $(x_e,\tilde{x}_+(\varepsilon) + h(\varepsilon)]$. We define $x = x_-(\varepsilon)$ and $x = x_+(\varepsilon)$ as the largest point $x < x_e$ and the smallest point $x > x_e$ at which the curves $\phi(x,\varepsilon)$ and $\phi(x)$ intersect. This present situation is depicted in Figure (14) below. Properties (a) through (f) can now be verified in a manner similar to the previous case, and this establishes lemma (4.7) in the two extrema case.

We now treat the final case, where the non-monotonic steady state solution $\phi(x)$ of (4.48) has only a single relative extremum and $\phi(x)$ goes to a saddle point as $x \to -\infty$ or as $x \to +\infty$. We treat this case in a

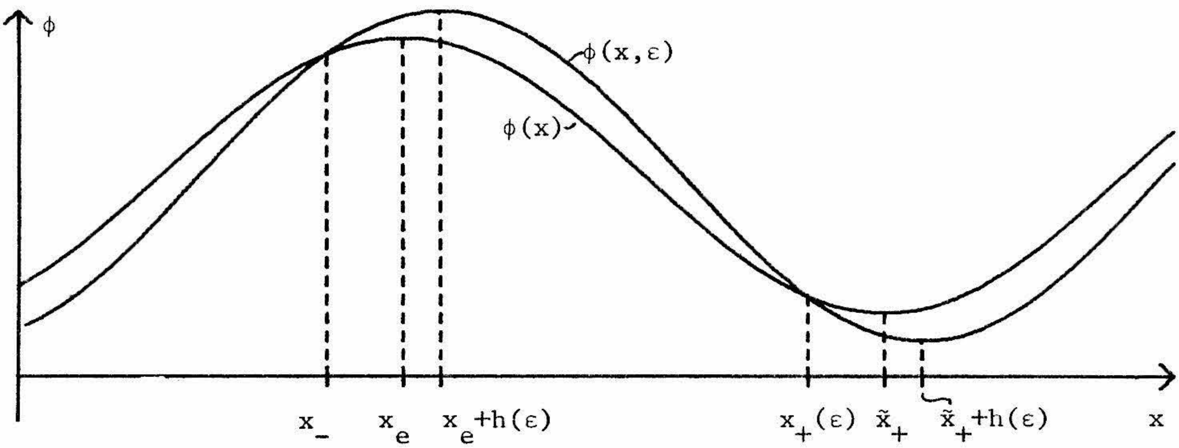


Figure (14)

manner very similar to the previous case. For this case we assume that the relative extremum of $\phi(x)$ occurs at $x = x_e$ and again assume without loss that it is a relative maximum. Moreover, we assume that $\phi(x)$ goes to a saddle point as $x \to +\infty$, since the other case can be handled similarly.

For $0 < \varepsilon < \varepsilon_1$ we again define $\tilde{\phi}(x,\varepsilon)$ as the solutions of

$$f(\tilde{\phi}_{xx},\tilde{\phi}_x,\tilde{\phi}) + c\tilde{\phi}_x = 0 \tag{4.48}$$

with the initial conditions

$$\tilde{\phi}(x_e,\varepsilon) = \phi(x_e) + \varepsilon\,, \quad \tilde{\phi}_x(x_e,\varepsilon) = \phi_x(x_e) = 0 \quad ,$$

where $\varepsilon_1 > 0$ is small enough so that there are no singular points in $[\phi(x_e),\phi(x_e) + \varepsilon_1]$. Select $\tilde{x}_-$ as any point x smaller than x_e, and define

$$\phi(x,\varepsilon) \equiv \left\{\begin{array}{ll} \tilde{\phi}(x,\varepsilon) & \text{if } \phi(x) = \tilde{\phi}(x,\varepsilon) \text{ for some } x \in [\tilde{x}_-,x_e) \\ \tilde{\phi}(x+h(\varepsilon),\varepsilon) & \text{otherwise, where } h(\varepsilon) < 0 \text{ is the largest} \\ & \text{constant such that } \tilde{\phi}(\tilde{x}_- + h(\varepsilon),\varepsilon) = \phi(\tilde{x}_-) \end{array}\right\}.$$

Note that as in the previous case such an $h(\varepsilon)$ exists and satisfies $-K\varepsilon < h(\varepsilon) < 0$ for some $K > 0$ for all ε in $(0,\varepsilon_0)$ when $\varepsilon_0 > 0$ is small enough. We now examine the list of possible alternatives for $x > x_e$.

In particular we note that case (4'c) <u>cannot</u> occur because $\phi(x)$ goes to a saddle as $x \to +\infty$. We see from the remaining alternatives that $\phi(x,\varepsilon)$ intersects $\phi(x)$ at least once for $x > x_e$ for all ε in $(0,\varepsilon_0)$. We define $x = x_-(\varepsilon)$ as the largest point $x < x_e$ at which $\phi(x,\varepsilon)$ and $\phi(x)$ intersect. Similarly, we define $x = x_+(\varepsilon)$ as the least point $x > x_e$ at which $\phi(x,\varepsilon)$ and $\phi(x)$ intersect. One now verifies that conditions (a), (b), (c), (d), (e), and (f') are satisfied, and this establishes the lemma.

<u>Proof of lemma 4.8:</u> Suppose that $\phi(x)$ is any bounded non-monotonic solution of

$$f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0 \quad , \tag{4.59}$$

and let $\tilde{\phi}(x)$ be any other solution of (4.59) with

$$\phi(x) \leq \tilde{\phi}(x) \quad \text{for all} \quad x \quad . \tag{4.60}$$

Also let $\phi = \phi_0$ be the smallest constant solution of (4.59) with

$$\phi(x) < \phi_0 \quad \text{for all} \quad x \quad .$$

To prove lemma (4.8) we will first show that $\tilde{\phi}(x) \geq \phi_0$ for some x, and then show that $\tilde{\phi}(x) \geq \phi_0$ for all x.

We now show that $\tilde{\phi}(x) \geq \phi_0$ for some x. To show this, we will consider the two separate cases of $\phi(x)$ having at least two relative extrema and of $\phi(x)$ having exactly one relative extremum.

Suppose that $\phi(x) \leq \tilde{\phi}(x)$ for all x, suppose also that $\tilde{\phi}(x) < \phi_0$ for all x, and finally suppose that $\phi(x)$ has at least two relative extrema. Select $x = x_e$ and $x = x_+ > x_e$ so that $\phi(x)$ has a relative extremum at each of these points and so that $\phi(x)$ has no relative extrema between $x = x_e$ and $x = x_+$. Let us also assume that these relative extrema are a maximum (at $x = x_e$) and a minimum (at $x = x_+ > x_e$) since the other case can be handled similarly. Since $\tilde{\phi}(x) \geq \phi(x)$ for all x

and since $\tilde{\phi}(x) \not\equiv \phi(x)$, we see that $\tilde{\phi}(x) > \phi(x)$ for all x and so

$$\phi(x) < \tilde{\phi}(x) < \phi_0 \quad \text{for all} \quad x \quad .$$

In particular, $\phi(x_e) < \tilde{\phi}(x_e)$. Thus let $x = \tilde{x}_e$ be the point nearest to $x = x_e$ where $\tilde{\phi}(x)$ has a relative maximum. Clearly

$$\phi(x_e) < \tilde{\phi}(\tilde{x}_e) < \phi_0 \quad .$$

From the list of possible alternatives we see that $\phi(x)$ and $\tilde{\phi}(x)$ must intersect for some $x > \min(x_e, \tilde{x}_e)$. Thus,

$$\phi(x) \leq \tilde{\phi}(x) < \phi_0 \quad \text{for all} \quad x$$

is not possible in this case.

Suppose now that $\phi(x) \leq \tilde{\phi}(x)$ for all x that $\tilde{\phi}(x) < \phi_0$ for all x, and also that $\phi(x)$ has only a single extremum at $x = x_e$. Let us also assume that this extremum is a maximum since the other case can be handled in a similar manner. As before we have

$$\phi(x) < \tilde{\phi}(x) < \phi_0 \quad \text{for all} \quad x \quad .$$

Also as in the previous case, an examination of the list of possible alternatives shows that $\tilde{\phi}(x)$ and $\phi(x)$ must intersect unless $\tilde{\phi}(x)$ has only a single extremum, $\tilde{\phi}(x)$ and $\phi(x)$ both decay to the same singular point at $x = -\infty$, and $\tilde{\phi}(x)$ and $\phi(x)$ both decay to the same singular point at $x = +\infty$. However, $\phi(x)$ goes to a saddle point as $x \to -\infty$ or as $x \to +\infty$ and so either $\tilde{\phi}(x)$ has more than a single extremum or one of $\phi(+\infty) \neq \tilde{\phi}(+\infty)$ and $\phi(-\infty) \neq \tilde{\phi}(-\infty)$ occurs. Thus $\tilde{\phi}(x)$ and $\phi(x)$ must intersect at least once, and so

$$\phi(x) \leq \tilde{\phi}(x) < \phi_0 \quad \text{for all} \quad x$$

is not possible either.

So far we have shown that whenever $\phi(x)$ any non-monotonic solution of

$$f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0 \qquad (4.59)$$

which satisfies the hypotheses of lemma (4.8), then if $\tilde{\phi}(x)$ is any <u>other</u> solution of (4.59) such that

$$\tilde{\phi}(x) \geq \phi(x) \quad \text{for all} \quad x \qquad (4.61)$$

then

$$\tilde{\phi}(x) \geq \phi_0 \quad \text{at some} \quad x \quad , \qquad (4.62)$$

where ϕ_0 is the least constant steady state solution satisfying

$$\phi(x) < \phi_0 \quad \text{for all} \quad x \quad . \qquad (4.63)$$

We will now complete the proof of the lemma by showing that (4.61) and (4.62) together imply that either

$$\tilde{\phi}(x) \equiv \phi_0 \qquad (4.64)$$

or

$$\tilde{\phi}(x) > \phi_0 \quad \text{for all} \quad x \qquad (4.65)$$

occurs. This final step in the proof will follow from the minimality of the final steady states $u(\varepsilon,+\infty,x) \equiv \tilde{\phi}(x,\varepsilon)$ in the hair-trigger effect.

Suppose first that $\tilde{\phi}(x) \not\equiv \phi_0$. Then (4.62) implies that

$$\tilde{\phi}(x) > \phi_0 \quad \text{at some} \quad x \quad . \qquad (4.66)$$

We now assume that

$$\tilde{\phi}(x) < \phi_0 \quad \text{at some} \quad x \qquad (4.67)$$

also, since otherwise lemma (4.8) would be satisfied. From relations (4.61), (4.66), and (4.67) we now obtain a contradiction. From the proof of the hair-trigger effect in theorem (4.6), we know that there exists a solution $\tilde{\phi}(x,\varepsilon)$ of (4.59) with

$$u(\varepsilon,0,x) \leq \tilde{\phi}(x,\varepsilon) \quad \text{for all} \quad x \quad , \qquad (4.68)$$

and with

$$\tilde{\phi}(x,\varepsilon) \leq \tilde{\tilde{\phi}}(x,\varepsilon) \quad \text{for all} \quad x \qquad (4.69)$$

satisfied whenever $\tilde{\tilde{\phi}}(x,\varepsilon)$ is any other solution of (4.59) such that

$$u(\varepsilon,0,x) \leq \tilde{\tilde{\phi}}(x,\varepsilon) \quad \text{for all} \quad x \tag{4.70}$$

is satisfied. Since (4.61) shows that $\phi(x)<\tilde{\phi}(x)$ for all x, an examination of the initial conditions $u(\varepsilon,0,x)$ (given in equation (4.37)) shows that

$$u(\varepsilon,0,x) < \tilde{\phi}(x) \quad \text{for all} \quad x$$

whenever $\varepsilon > 0$ is small enough. Moreover, for $\varepsilon > 0$ small enough we also have

$$u(\varepsilon,0,x) < \phi_0 \quad \text{for all} \quad x \quad .$$

Thus equations (4.68), (4.69), and (4.70) imply that there is a solution $\tilde{\phi}(x,\varepsilon)$ of (4.59) satisfying

$$\phi(x) \leq u(\varepsilon,0,x) \leq \tilde{\phi}(x,\varepsilon) \qquad \text{for all} \quad x \tag{4.71a}$$

$$\tilde{\phi}(x,\varepsilon) \leq \phi_0 \qquad \text{for all} \quad x, \text{ and} \tag{4.71b}$$

$$\tilde{\phi}(x,\varepsilon) \leq \tilde{\phi}(x) \qquad \text{for all} \quad x \tag{4.71c}$$

for any $\varepsilon > 0$ small enough. Since $\tilde{\phi}(x) < \phi_0$ at some x, $\tilde{\phi}(x,\varepsilon) \not\equiv \phi_0$. But since $\tilde{\phi}(x,\varepsilon)$ is a solution of (4.59) and since

$$\phi(x) \leq \tilde{\phi}(x,\varepsilon) \quad \text{for all} \quad x \quad ,$$

we have shown in the first part of the proof that either

$$\tilde{\phi}(x,\varepsilon) \equiv \phi_0 \quad , \quad \text{or}$$

$$\tilde{\phi}(x,\varepsilon) > \phi_0 \qquad \text{for some} \quad x \quad .$$

This contradicts (4.71b). Hence either $\tilde{\phi}(x) \equiv \phi_0$ or $\tilde{\phi}(x) > \phi_0$ for all x, and we have now established lemma (4.8).

Thus we have established lemmas (4.7) and (4.8). Note that in proving lemma (4.8) we have also essentially proved an oscillation theorem for the ordinary differential equation

$$f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0 \quad , \quad f_1 > 0 \quad . \tag{4.59}$$

To state this result explicitly, let $\phi(x)$ be any bounded non-monotonic solution of (4.59). If $\phi(x)$ has only a single extremum assume that $\phi(x)$ goes to a saddle as either $x \to -\infty$ or as $x \to +\infty$. Let ϕ_0^- and ϕ_0^+ (respectively) be the largest and smallest constant solutions of (4.59) which satisfy

$$\phi_0^- < \phi(x) < \phi_0^+ \quad \text{for all} \quad x \quad .$$

Lemma (4.8) directly implies that there is no solution $\tilde{\phi}(x)$ of (4.59) which satisfies

$$\phi(x) \leq \tilde{\phi}(x) \quad \text{for all } x \text{ and} \quad \tilde{\phi}(x) < \phi_0^+ \quad \text{for some} \quad x \ .$$

By transforming $\phi \to -\phi$ one sees that lemma (4.8) also implies that there is no solution $\tilde{\phi}(x)$ of (4.59) which satisfies

$$\tilde{\phi}(x) \leq \phi(x) \quad \text{for all} \quad x \quad \text{and} \quad \phi_0^- < \tilde{\phi}(x) \quad \text{for some} \quad x \ .$$

Thus we have shown that if $\tilde{\phi}(x)$ is <u>any</u> solution of (4.59) such that

$$\phi_0^- < \tilde{\phi}(x) < \phi_0^+ \quad \text{for some} \quad x$$

is satisfied, then $\phi(x)$ and $\tilde{\phi}(x)$ must intersect at at least one point x. This clearly demonstrates that lemma (4.8) is an oscillation result about solutions of (4.59).

The establishment of lemmas (4.7) and (4.8) in this section nearly completes our treatment of the instability of non-monotonic waves. In the next section, section (4.14), we complete our treatment by discussing the indeterminate case.

<u>4.14 Stability/instability in the indeterminate case.</u> For this section we assume that $\phi(x)$ is a bounded non-monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \quad , \tag{4.2}$$

that $\phi(x)$ has a single relative extremum which occurs at $x = x_e$, and that $\phi(x)$ goes to a node as $x \to -\infty$ and to a node as $x \to +\infty$. Moreover, we will assume that $\phi(x)$ has a maximum at $x = x_e$ since the case of $\phi(x)$ having a minimum is handled in a similar manner. In this section we will determine when $u(t,x) \equiv \phi(x)$ is unstable. To do this we again define $\tilde{\phi}(x,\varepsilon)$ as the solution of the steady state equation

$$f(\tilde{\phi}_{xx},\tilde{\phi}_x,\tilde{\phi}) + c\tilde{\phi}_x = 0 \tag{4.59}$$

with the initial condition

$$\tilde{\phi}(x_e,\varepsilon) = \phi(x_e) + \varepsilon \qquad \tilde{\phi}_x(x_e,\varepsilon) = \phi_x(x_e) = 0 \quad .$$

The stability or instability of $u(t,x) = \phi(x)$ will essentially depend on the intersection properties of $\phi(x)$ and $\tilde{\phi}(x,\varepsilon)$.

As our first case, suppose that $\phi(x)$ decays at the accidental rate as, say, $x \to +\infty$. Then for any $\varepsilon > 0$ (no matter how small) the phase plane alternative (4'c) cannot occur. An examination of the proofs of lemma (4.7), lemma (4.8), and theorem (4.6) shows that the only use made of the hypotheses that $\phi(x)$ goes to a saddle point as either $x \to +\infty$ or as $x \to -\infty$ was to eliminate phase plane alternative (4'c) for either $x > x_e$ or $x < x_e$. Thus the proofs given for lemma (4.7), lemma (4.8), and theorem (4.6) work equally well if $\phi(x)$ decays to a node at the accidental rate as either $x \to -\infty$ or as $x \to +\infty$. To summarize this, if $u(t,x) \equiv \phi(x)$ is a bounded non-monotonic steady state solution of (4.2) such that

(i) $\phi(x)$ has a single extremum at $x = x_e$,

(ii) $\phi(x)$ goes to a node as $x \to -\infty$ and goes to a node as $x \to +\infty$,

(iii) $\phi(x)$ decays at the accidental rate as $x \to -\infty$ or $x \to +\infty$,

then $u(t,x) \equiv \phi(x)$ is $\mathbb{C}^w$-unstable where $w(x)$ is given by

$$w(x) \equiv \left\{ \begin{array}{ll} 1 + \dfrac{1}{|\phi'(x)|} + \dfrac{1}{|\phi'(x_e+1)|} & x \le x_e - 1 \\ 1 + \dfrac{1}{|\phi'(x_e-1)|} + \dfrac{1}{|\phi'(x_e+1)|} & x_e - 1 \le x \le x_e + 1 \\ 1 + \dfrac{1}{|\phi'(x_e-1)|} + \dfrac{1}{|\phi'(x)|} & x_e + 1 \le x \end{array} \right\} . \quad (4.72)$$

We now consider the remaining case. Namely the case that

(i) $\phi(x)$ has a single extremum at $x = x_e$

(ii) $\phi(x)$ goes to a node as $x \to -\infty$ and goes to a node as $x \to +\infty$, and

(iii) $\phi(x)$ decays at the usual rate as $x \to -\infty$ and as $x \to +\infty$.

Define the function

$$\phi_\varepsilon(x) \equiv \frac{\partial}{\partial\varepsilon} \tilde{\phi}(x,\varepsilon) \Big|_{\varepsilon=0} ,$$

and note that

$$\phi_\varepsilon(x) + h\phi_x(x) \equiv \frac{\partial}{\partial\varepsilon} \tilde{\phi}(x+h\varepsilon,\varepsilon) \Big|_{\varepsilon=0} .$$

We will show that if there is an h such that

$$\phi_\varepsilon(x) + h\phi_x(x) > 0 \quad \text{for all } x$$

then $u(t,x) \equiv \phi(x)$ is C^w-stable where $w(x)$ is given in (4.72). Moreover, if such an h does not exist we will show that $u(t,x) \equiv \phi(x)$ is usually $\not{C}^w$-unstable.

Suppose first that $\phi_\varepsilon(x) > 0$ for all x. Then for all ε in $(0,\varepsilon_0)$ (for some $\varepsilon_0 > 0$ sufficiently small) we have that

$$\tilde{\phi}(x,-\varepsilon) < \phi(x) < \tilde{\phi}(x,\varepsilon) \quad \text{for all } x .$$

Since $u(\varepsilon,t,x) \equiv \tilde{\phi}(x,\varepsilon)$ and $u(-\varepsilon,t,x) \equiv \tilde{\phi}(x,-\varepsilon)$ are both solutions of (4.2), the maximum principle implies that every solution $u(t,x)$ of (4.2) whose initial condition $u(0,x)$ is in H_x^2 and satisfies

$$\tilde{\phi}(x,-\varepsilon) < u(0,x) < \tilde{\phi}(x,\varepsilon) \quad \text{for all } x ,$$

must satisfy

$$\tilde{\phi}(x,-\varepsilon) < u(t,x) < \tilde{\phi}(x,\varepsilon) \quad \text{for all } x \text{ and all } t \ge 0 . \quad (4.73)$$

Since $\varepsilon > 0$ can be as small as we please, relation (4.73) implies that the steady state $\phi(x)$ is C^w-stable with $w(x)$ given by (4.72).

Suppose now that either $\phi_\varepsilon(x) > 0$ for all $x > x_e$ and $\phi_\varepsilon(x)<0$ for some $x < x_e$, or $\phi_\varepsilon(x) > 0$ for all $x < x_e$ and $\phi_\varepsilon(x) < 0$ for some $x > x_e$. Recalling the assumption that $\phi(x)$ has a relative maximum at $x = x_e$, when $\phi_\varepsilon(x) > 0$ for all $x > x_e$ and $\phi_\varepsilon(x) < 0$ for some $x<x_e$ we increase h from zero until either

(1) $\phi_\varepsilon(x) + h\phi_x(x) > 0$ for all x, or

(2) $\phi_\varepsilon(x) + h\phi_x(x) < 0$ for some $x < x_e$ and for some $x > x_e$.

Similarly if $\phi_\varepsilon(x) > 0$ for all $x < x_e$ and $\phi_\varepsilon(x) < 0$ for some $x > x_e$, then we decrease h from zero until either case (1) or case (2) occurs. Suppose that case (1) occurs. Then there is an $\varepsilon_0 > 0$ such that for all ε in $(0,\varepsilon_0)$,

$$\tilde{\phi}(x-h\varepsilon,\varepsilon) < \phi(x) < \tilde{\phi}(x+h\varepsilon,\varepsilon) \quad \text{for all } x \quad .$$

Similar to the preceding case, using $\tilde{\phi}(x-h\varepsilon,\varepsilon)$ and $\tilde{\phi}(x+h\varepsilon,\varepsilon)$ with the maximum principle implies that $u(t,x) \equiv \phi(x)$ is C^w-stable where $w(x)$ is defined in (4.72).

Suppose now that either case (2) occurs or that $\phi_\varepsilon(x) < 0$ for some $x < x_e$ and for some $x > x_e$. When either of these occurs there is an h such that $\phi_\varepsilon(x) + h\phi_x(x) < 0$ for some $x < x_e$ and some $x > x_e$. Therefore, for all ε in $(0,\varepsilon_0)$ (for some $\varepsilon_0 > 0$) the curve $\tilde{\phi}(x+h\varepsilon,\varepsilon)$ intersects $\phi(x)$ at least once when $x > x_e$ and at least once when $x < x_e$ Hence we define $\phi(x,\varepsilon) \equiv \tilde{\phi}(x+h\varepsilon,\varepsilon)$, define $x_-(\varepsilon)$ as the largest $x < x_e$ at which $\phi(x_-(\varepsilon),\varepsilon) = \phi(x_-(\varepsilon))$, and define $x_+(\varepsilon)$ as the smallest $x > x_e$ at which $\phi(x_+(\varepsilon),\varepsilon) = \phi(x_+(\varepsilon))$. It is easily seen that $\phi(x,\varepsilon)$, $x_-(\varepsilon)$, and $x_+(\varepsilon)$ satisfy all the conditions of part (2) in lemma (4.7). We can now apply the hair-trigger argument used in the proof of theorem (4.6). This shows that the solutions $u(\varepsilon,t,x)$ of (4.2) with the initial conditions given in (4.37) are non-decreasing in time and that

$$u(\varepsilon,+\infty,x) = \phi_\infty(\varepsilon,x) \quad ,$$

where $\phi_\infty(\varepsilon,x)$ is a steady state solution of (4.2) satisfying

$$\phi(x) < \phi_\infty(\varepsilon,x) \quad \text{for all} \quad x \quad .$$

We now show that $\phi_\infty(\varepsilon,x) - \phi(x) \geq \varepsilon_0$ for some x and for all ε in $(0,\varepsilon_0)$. To see this, note that for all ε in $(0,\varepsilon_0)$ there is an h such that $\tilde{\phi}(x+h\varepsilon,\varepsilon)$ intersects $\phi(x)$ when $x < x_e$ and also when $x > x_e$. Since $\phi(x)$ is increasing for $x < x_e$ and is decreasing for $x > x_e$, for any h and for any ε in $(0,\varepsilon_0)$ the curves $\phi(x)$ and $\tilde{\phi}(x+h\varepsilon,\varepsilon)$ intersect somewhere. Thus $\phi_\infty(\varepsilon,x)$ <u>cannot</u> be $\tilde{\phi}(x+h\varepsilon,\varepsilon)$ for any h and for any ε in $(0,\varepsilon_0)$. Therefore

$$\phi_\infty(\varepsilon,x) - \phi(x) \geq \varepsilon_0 \quad \text{for all} \quad \varepsilon \quad \text{in} \quad (0,\varepsilon_0) \quad ,$$

and $\mathfrak{C}^w$-instability is established.

This completes our stability/instability analysis for the indeterminate case. To summarize the results, we have assumed that

(i) $\phi(x)$ has a single extremum at $x = x_e$, and

(ii) $\phi(x)$ goes to a node as $x \to -\infty$ and goes to a node as $x \to +\infty$.

Then, if $\phi(x)$ decays at the accidental rate as $x \to -\infty$ or as $x \to +\infty$ we have shown that $u(t,x) \equiv \phi(x)$ is $\mathfrak{C}^w$-unstable (where $w(x)$ is given by (4.72)). If $\phi(x)$ decays at the usual rate as $x \to -\infty$ and as $x \to +\infty$ a more complicated stability picture occurs. In this case if $\phi_\varepsilon(x) > 0$ for all x then $u(t,x) \equiv \phi(x)$ is C^w-stable. If $\phi_\varepsilon(x) < 0$ for some $x < x_e$ and for some $x > x_e$ then $u(t,x) \equiv \phi(x)$ is $\mathfrak{C}^w$-unstable.

Finally, if neither of the above occurs we increase or decrease the parameter h in $\phi_\varepsilon(x) + h\phi_x(x)$ until either

(1) $\phi_\varepsilon(x) + h\phi_x(x) > 0$ for all x, or

(2) $\phi_\varepsilon(x) + h\phi_x(x) < 0$ for some $x < x_e$ and for some $x > x_e$

occurs. In case (1) $u(t,x) \equiv \phi(x)$ is C^W-stable, and in case (2) it is C^W-unstable.

For any specific steady state $\phi(x)$ of any <u>specific</u> equation

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad , \tag{4.2}$$

it appears that the above stability criterion is impractical unless one can solve for $\phi(x)$ and $\phi(x,\varepsilon)$ analytically. However, even if $\phi(x)$, $\phi_x(x)$, and $\phi_\varepsilon(x)$ can only be found by numerically solving the equations

$$\begin{aligned} \phi_x &= v \\ f(v_x, v, \phi) + cv &= 0 \quad , \end{aligned} \tag{4.74}$$

the above stability criterion should be a practical method for determining the stability of $u(t,x) \equiv \phi(x)$ in this indeterminate case. This is because one knows (from equations (4.74)) the asymptotic behavior as $x \to \pm\infty$ of $\phi(x)$, $\phi_x(x)$, and $\phi_\varepsilon(x)$ to within some unknown coefficients. Thus one needs to numerically solve for $\phi(x)$, $\phi_x(x)$, and $\phi_\varepsilon(x)$ only over a region large enough so that the asymptotics are valid outside of the region. From this calculation one can find the unknown coefficients in the asymptotic formulas for $\phi(x)$, $\phi_x(x)$, and $\phi_\varepsilon(x)$. The calculation will explicitly show when $\phi_\varepsilon(x) + h\phi_x(x)$ is positive or negative in a large region, and the asymptotics will show the same outside of the large region. Thus <u>in principle</u> the stability criterion developed in this section is a practical way to determine the stability of the steady states $\phi(x)$ which belong to the indeterminate case. On the other hand, these numerical calculations may be so delicate that the implementation of the stability criterion is very difficult.

This completes the analysis in this section. However it is possible that the stability case never occurs. That is, we can conjecture that if the bounded steady state solution $u(t,x) \equiv \phi(x)$ satisfies the conditions

(i) $\phi(x)$ has a single relative extremum at some $x = x_e$,

(ii) $\phi(x)$ goes to a node as $x \to -\infty$ and to a node as $x \to +\infty$,

and

(iii) $\phi(x)$ decays at the usual rate as $x \to -\infty$ and as $x \to +\infty$,

then there exists an h such that

$$\phi_\varepsilon(x) + h\phi_x(x) < 0 \quad \text{for some} \quad x < x_e \quad \text{and for some} \quad x > x_e \quad .$$

The direct implication of this conjecture is that $u(t,x) \equiv \phi(x)$ is $\mathcal{C}^w$-unstable. Thus, if this conjecture is true then every bounded steady state solution of (4.2) which has a single relative extremum would be C^w-unstable.

This section completes our analysis of the instability of non-monotonic waves. In the next three sections we briefly discuss related topics. Specifically in the next section, section (4.15), we will extend our instability results to some constant steady states. In section (4.16) we will comment on the potential applications of our techniques to the final state problem. In section (4.17) we will extend our results to non-monotonic traveling plane waves in multiple spatial dimensions.

4.15 Instability of some constant steady states. In this section we suppose that $u(t,x) \equiv \phi_0$ is a center or a spiral point of the phase plane of

$$f(\phi_{xx},\phi_x,\phi) + c\phi_x = 0 \quad . \tag{4.59}$$

We now note that if we select any point x to be $x = x_e$ and if we define $\phi(x,\varepsilon)$ as the solution of (4.59) with initial condition

$$\phi(x_e,\varepsilon) = \phi_0 + \varepsilon \qquad \phi_x(x_e,\varepsilon) = 0 \quad ,$$

then there is a $\Delta > 0$ such that

$$\phi(x,\varepsilon) = \phi_0$$

for at least one x in $(x_e - \Delta, x_e)$ and one x in $(x_e, x_e + \Delta)$ for all ε sufficiently small. This is precisely the same behavior that $\phi(x,\varepsilon)$ possessed when $\phi(x)$ was a solution of (4.59) with at least three extrema (instead of $\phi(x)$ being ϕ_0, a constant steady state). Indeed, if we examine the proofs of lemma (4.7), lemma (4.8), and theorem (4.6), we see that they can be easily extended to include $\phi(x) \equiv \phi_0$ as part of the "steady state with at least two extrema" case. Thus if the singular point ϕ_0 is a spiral or a center of (4.59), then the steady state solution $u(t,x) \equiv \phi_0$ of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{4.2}$$

has the same instability as in case (1) of theorem (4.6). Namely, it is unstable to arbitrarily small perturbations of a finite extent. This is a distinct improvement over the instability results in theorem (4.1) for these cases.

At first glance these instability results may seem surprising since a singular point $\phi = \phi_0$, $v = 0$ is a spiral point or a center of (4.59) precisely when

$$f_3(0,0,\phi_0) > 0 \quad \text{and}$$

$$-\,2\sqrt{f_1(0,0,\phi_0)f_3(0,0,\phi_0)} - f_2(0,0,\phi_0) < c$$

$$< 2\sqrt{f_1(0,0,\phi_0)f_3(0,0,\phi_0)} - f_2(0,0,\phi_0) \quad .$$

Thus, apparently stability of a constant steady state depends on the wavespeed c. However, even though a constant steady state is unchanged by shifting to a different moving coordinate system, its stability can change. This is because we measure its stability relative to the moving coordinate system and because perturbations which grow in one coordinate system may not grow in another. It is easy to see how this can come about. Consider a monotonic traveling wave solution $u(t,x) = \phi(x-c_0t)$ of

$$u_t = f(u_{xx}, u_x, u) \quad , \tag{4.1}$$

which is our equation in the original stationary coordinate system. For any $c < c_0$ we see that $u(t,x) \to \phi(-\infty)$ as $t \to +\infty$ with $x - ct$ fixed, and for any $c > c_0$ we see that $u(t,x) \to \phi(+\infty)$ as $t \to +\infty$ with $x-ct$ fixed. Thus in a moving coordinate system the behavior of a solution as $t \to \infty$ at a fixed point in the coordinate system in general depends on the speed of the coordinate system.

<u>4.16 Application to the final state problem.</u> In this section we will comment on some applications of the hair-trigger effect to determining the final state. Specifically, if $u(t,x)$ is any solution of

$$u_t = f(u_{xx}, u_x, u) + cu_x \tag{4.2}$$

then we would like to determine $u(+\infty,x)$ in terms of $u(0,x)$.

Suppose that $u(t,x) \equiv \phi(x)$ is any bounded non-monotonic steady state solution of (4.2) with at least two relative extrema. Let $x = x_e$ be any point where an extrema of $\phi(x)$ occurs. In proving the hair-trigger effect we constructed the initial condition

$$u(\varepsilon,0,x) \equiv \left\{ \begin{array}{ll} \phi(x) & x \leq x_-(\varepsilon) \\ \phi(x,\varepsilon) & x_-(\varepsilon) \leq x \leq x_+(\varepsilon) \\ \phi(x) & x_+(\varepsilon) \leq x \end{array} \right\}$$

for all $\varepsilon > 0$ sufficiently small. Here $\phi(x,\varepsilon)$ is either the steady state solution of (4.2) with

$$\phi(x_e,\varepsilon) = \phi(x_e) + \varepsilon \qquad \phi_x(x_e,\varepsilon) = \phi_x(x_e) = 0$$

or is an $O(\varepsilon)$ translate of this steady state. We then showed that if $u(t,x)$ is any solution of (4.2) whose initial condition $u(0,x)$ is smooth and satisfies

$$u(\varepsilon,0,x) \leq u(0,x) \leq \phi_0^+ \text{ for all } x \text{ and any } \varepsilon > 0 \text{ sufficiently small,} \tag{4.80}$$

then

$$u(t,x) \to \phi_0^+ \quad \text{as} \quad t \to +\infty \quad \text{at each } x .$$

Here ϕ_0^+ is the least <u>constant</u> steady state solution of (4.2) satisfying

$$\phi(x) < \phi_0^+ \quad \text{for all } x.$$

This is illustrated in Figure (15) below. We see that any solution $u(t,x)$ of (4.2) whose initial condition is at least as large as $u(\varepsilon,0,x)$ (which is $\phi(x)$ with a small additional positive bulge) and is no larger than ϕ_0^+, must go to the constant steady state ϕ_0^+ as $t \to +\infty$.

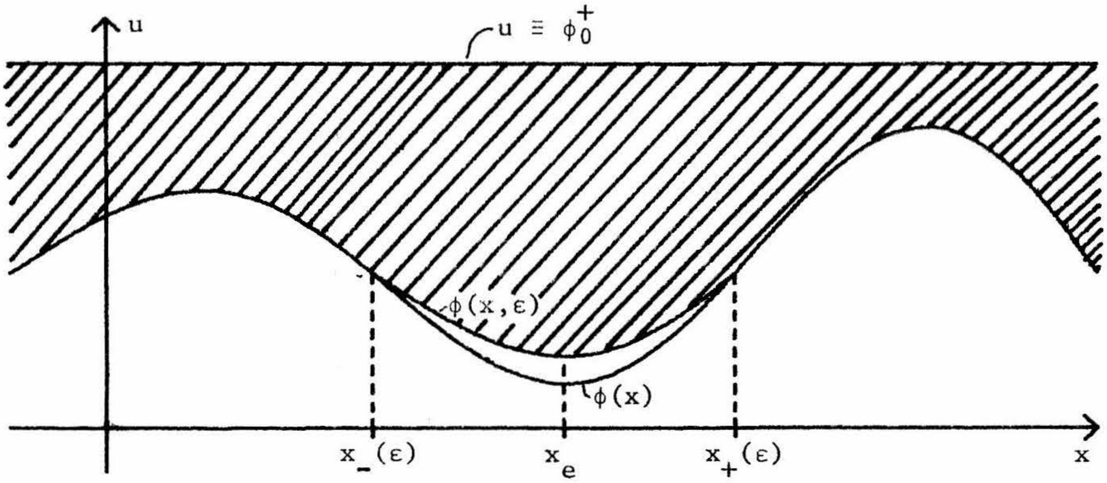


Figure (15): Any solution $u(t,x)$ whose initial condition $u(0,x)$ is in the shaded region must evolve into ϕ_0^+; that is, $u(+\infty,x) \equiv \phi_0^+$.

Now in proving theorem (4.6) we always considered ε slightly positive and we proved a positive hair-trigger effect as depicted in Figure

(15). However, we could have equally chosen ε to be slightly negative and proved a negative hair-trigger effect. Specifically, we could have used the initial condition $u(\varepsilon,0,x)$ defined as before but with ε slightly negative. This would lead to $u(\varepsilon,0,x)$ being essentially $\phi(x)$ with a small negative bulge $\phi(x,\varepsilon) - \phi(x)$ of finite extent. We would then find that if $u(t,x)$ is any solution of (4.2) whose initial condition $u(0,x)$ satisfies

$$\phi_0^- \le u(0,x) \le u(\varepsilon,0,x) \text{ for all } x \text{ and for any } -\varepsilon > 0 \text{ sufficiently small,} \tag{4.81}$$

then

$$u(t,x) \to \phi_0^- \quad \text{as} \quad t \to +\infty \quad \text{at each} \quad x .$$

Here ϕ_0^- is the largest <u>constant</u> steady state solution of (4.2) which satisfies

$$\phi_0^- < \phi(x) \quad \text{for all} \quad x \quad .$$

This negative hair-trigger effect is illustrated in Figure (16) below.

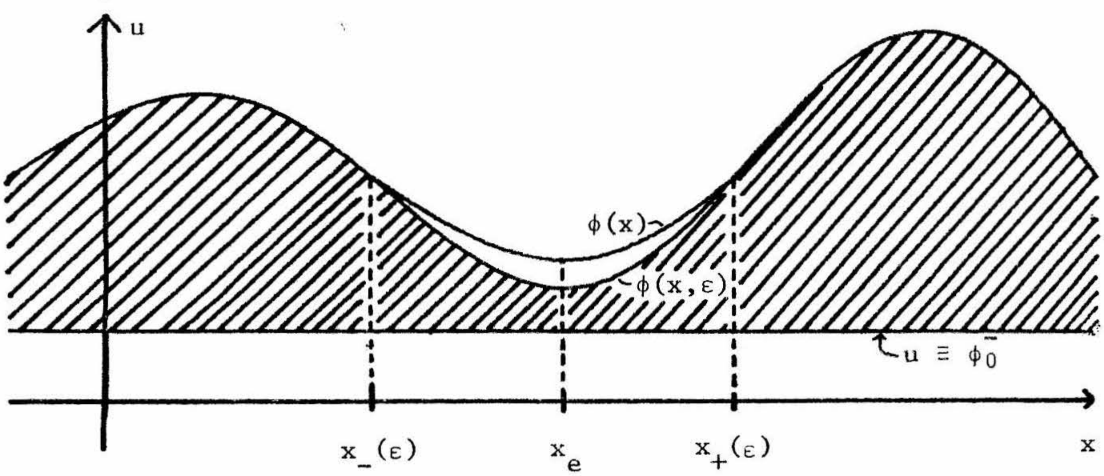


Figure (16): Any solution $u(t,x)$ whose initial condition $u(0,x)$ is in the shaded region will evolve into ϕ_0^-; that is, $u(+\infty,x) \equiv \phi_0^-$.

Between (4.80) and (4.81) we have found the final steady state $u(+\infty,x)$ for a large class of initial conditions. Further, we can start with any non-monotonic steady state which has at least two extrema and even some constant steady states (as discussed in the preceding section). Moreover, we can use any relative extremum of a non-monotonic steady state for $x = x_e$. By using all of the extrema of all these steady states, we can find the final steady states $u(+\infty,x)$ for a very large class of initial conditions. Thus we have a potentially very powerful method for finding the final steady states as a function of the initial conditions. We will not follow this further except to introduce the following cautionary remark. The hair-trigger effect can be used to find $u(+\infty,x)$, but it does not show how $u(t,x)$ approaches $u(+\infty,x)$. Thus $u(+\infty,x) \equiv \lim_{t\to+\infty} u(t,x)$ (x fixed) may not be the same as $u(+\infty,x) \equiv \lim_{t\to+\infty} u(t,x-ct)$ ($x-ct$ fixed). That is, $u(+\infty,x)$ may not be the same in different moving coordinate systems.

4.17 Instability in higher spatial dimensions. In this section we assume that there are two or more spatial variables. We will discuss the direct extension of the instability results to traveling plane waves. Actually we will only deal with two spatial variables ($\vec{x} \equiv (x,y)$), but the generalization to three or more spatial variables will be readily apparent.

Suppose that $u(t,\vec{x}) = \phi(\vec{x}-\vec{c}t)$ is a traveling plane wave solution of

$$u_t = f(u_{xx},u_{xy},u_{yy},u_x,u_y,u) \quad , \tag{4.82}$$

and suppose that equation (4.82) is parabolic (i.e., satisfies hypothesis (H3). Let us change to a coordinate system which is oriented so that ϕ depends only on x (where $\vec{x} \equiv (x,y)$). Let us also transform to a coordinate system which travels with velocity $\vec{c} = (c_x,0)$. In terms of this coordinate system, our equation is now

$$u_t = \tilde{f}(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u) + c_x u_x \tag{4.83}$$

and our traveling plane wave solution is now the steady plane wave

$$u(t,\vec{x}) \equiv u(t,x,y) = \phi(x) \quad . \tag{4.84}$$

Thus, $u(t,\vec{x}) \equiv u(t,x,y) = \phi(x)$ solves

$$u_t = \tilde{f}(u_{xx}, 0, 0, u_x, 0, u) + c_x u_x \equiv \tilde{\tilde{f}}(u_{xx}, u_x, u) + c_x u_x \quad . \tag{4.85}$$

Clearly if we restrict the initial perturbations $u(0,\vec{x}) - \phi(x)$ to depend only on x, then $u(t,x,y) = u(t,x)$ is identical to the solution of

$$u_t = \tilde{\tilde{f}}(u_{xx}, u_x, u) + c_x u_x \quad . \tag{4.86}$$

In particular, if the plane wave solution $\phi(x)$ is non-monotonic then the solution $u(t,x,y) = \phi(x)$ of (4.83) has exactly the instability described in theorem (4.6) where the unstable initial perturbations do not depend on y. Moreover, let $u(\varepsilon,0,x,y) \equiv u(\varepsilon,0,x)$ be the initial conditions used in the proof of theorem (4.6). It was shown that

$$u(\varepsilon,t,x,y) = u(\varepsilon,t,x) \to \phi_0 \quad \text{as} \quad t \to +\infty \quad \text{for all} \quad x \ ,$$

where ϕ_0 is the smallest singular point of equation (4.86) with

$$\phi(x) < \phi_0 \quad \text{for all} \quad x \quad .$$

Let us now use the maximum principle for the full equation (4.83). We immediately find that if $u(t,\vec{x}) \equiv u(t,x,y)$ is any solution of (4.83) whose initial condition $u(0,x,y)$ satisfies

$$u(\varepsilon,0,x) \le u(0,x,y) \le \phi_0 \quad \text{for all} \quad x \quad \text{and all} \quad y \ , \tag{4.87}$$

then

$$u(t,x,y) \to \phi_0 \ \text{as} \ t \to +\infty \quad \text{at each} \quad x \ \text{and} \ y \quad .$$

Thus the plane wave solution $\phi(x)$ is unstable to all initial perturbations $u(0,\vec{x}) - \phi(\vec{x})$ which satisfy (4.87) for any $\varepsilon > 0$ sufficiently small.

Note that this instability result for plane waves is weaker than the corresponding result for a single spatial dimension. This is because a finite x-interval in one dimension is a finite spatial region, while a finite x-interval in two or more dimensions is an infinite spatial region. Thus, in more than a single spatial dimension we have not shown that non-monotonic traveling plane waves are unstable to arbitrarily small perturbations which are non-zero only in a finite spatial region.

4.18 Stability/instability of steady solutions in finite domains. In this section we consider the following finite spatial domain-boundary value problem

$$u_t = f(u_{xx}, u_x, u) \qquad 0 \le x \le 1 \tag{4.88a}$$

$$u(t,x) = A \quad \text{at} \quad x = 0 \qquad u(t,x) = B \quad \text{at } x = 1 \ . \tag{4.88b}$$

Specifically, we will determine the stability or instability of steady state solutions $u(t,x) \equiv \phi(x)$ of problem (4.88) for all major cases. Since no essentially new ideas are involved in this section, we will be extremely brief.

First we define stability and instability appropriately for this problem. Suppose that $u(t,x) \equiv \phi(x)$ is a solution of problem (4.88). Then if for every $\varepsilon > 0$ there is a $\delta > 0$ such that all solutions $u(t,x)$ of problem (4.88) satisfy

$$|u(t,x) - \phi(x)| < \varepsilon \quad \text{for all} \quad x \quad \text{in} \quad (0,1), \quad \text{all} \quad t > 0 \tag{4.89}$$

whenever their initial conditions $u(0,x)$ are in H_x^2 and satisfy

$$\begin{gathered} |u(0,x) - \phi(x)| < \delta \quad \text{for all } x \text{ in } (0,1) \\ u(0,0) = A \qquad u(0,1) = B \ , \end{gathered} \tag{4.90}$$

we define $\phi(x)$ to be a stable solution of (4.88). If a solution $u(t,x) \equiv \phi(x)$ is not stable we define it to be unstable. Note that these definitions of stability and instability are exactly equivalent to the definitions of C^w-stability and C^w-instability over the finite interval $[0,1]$, at least when $w(x) \geq 1$ and $w(x)$ is bounded over $[0,1]$.

For our first case let us assume that $\phi(x)$ is strictly monotonic; that is, assume that $\phi'(x) \neq 0$ for all $0 \leq x \leq 1$. To be definite, let us assume that $\phi(x)$ is increasing since the analysis when $\phi(x)$ is decreasing is very similar. Now for any $h > 0$ sufficiently small

$$\phi(x-h) < \phi(x) < \phi(x+h) \quad \text{for all} \quad x \quad \text{in} \quad [0,1] \quad .$$

Let $u(t,x)$ be any solution of problem (4.88) whose initial condition $u(0,x)$ is in H_x^2 and satisfies

$$\begin{aligned} &\phi(x-h) \leq u(0,x) \leq \phi(x+h) \quad \text{for all} \quad x \quad \text{in} \quad [0,1] \\ &u(0,0) = A \qquad u(0,1) = B \quad . \end{aligned} \tag{4.91}$$

Since

$$\begin{aligned} \phi(-h) < u(t,0) = A < \phi(+h) \quad &\text{for all} \quad t > 0 \\ \phi(1-h) < u(t,1) = B < \phi(1+h) \quad &\text{for all} \quad t > 0 \quad , \end{aligned}$$

and since (4.91) is satisfied, the maximum principle implies that

$$\phi(x-h) \leq u(t,x) \leq \phi(x+h) \quad \text{for all } x \text{ in } [0,1], \text{ for all } t \geq 0 \quad .$$

Because $h > 0$ is as small as we please we see that $u(t,x) \equiv \phi(x)$ is a stable solution of problem (4.88). Similarly, if $\phi(x)$ is strictly decreasing then $u(t,x) \equiv \phi(x)$ is stable.

For our second case let us assume that $\phi(x)$ has at least two relative extrema in $(0,1)$. A close examination of the proof of lemma (4.7) shows that since at least two extrema are in $(0,1)$, we can find a $\phi(x,\varepsilon)$, a $x_-(\varepsilon)$, and a $x_+(\varepsilon)$ which satisfy conditions (a), (b), (c), (d), and (e) of lemma (4.7) and such that

$$0 < x_-(\varepsilon) < x_+(\varepsilon) < 1$$

for all ε in $(0,\varepsilon_0)$ for some $\varepsilon_0 > 0$ sufficiently small. Even though

$[0,1]$ is a finite domain, we can use the boundary conditions (4.88b) and the maximum principle to prove a hair-trigger effect. Specifically, if $u(\varepsilon,t,x)$ is defined as the solution of problem (4.88) with the initial condition

$$u(\varepsilon,0,x) = \begin{Bmatrix} \phi(x) & 0 \le x \le x_-(\varepsilon) \\ \phi(x,\varepsilon) & x_-(\varepsilon) \le x \le x_+(\varepsilon) \\ \phi(x) & x_+(\varepsilon) \le x \le 1 \end{Bmatrix} ,$$

then a proof extremely similar to the one used in proving theorem (4.6) shows that $u(\varepsilon,t,x)$ is non-decreasing in t and that

$$u(\varepsilon,+\infty,x) \equiv \phi_\infty(\varepsilon,x) \qquad 0 \le x \le 1 \quad .$$

Here, $\phi_\infty(\varepsilon,x)$ is the least steady state solution of problem (4.88) with

$$u(\varepsilon,0,x) \le \phi_\infty(\varepsilon,x) \quad \text{for all} \quad x \quad \text{in} \quad [0,1] \quad .$$

To finish the proof of instability, define $\tilde{\phi}(x,\varepsilon)$ as the solution of

$$f(\tilde{\phi}_{xx},\tilde{\phi}_x,\tilde{\phi}) = 0$$

with the initial condition

$$\tilde{\phi}(x_e,\varepsilon) = \phi(x_e) + \varepsilon \qquad \tilde{\phi}_x(x_e,\varepsilon) = \phi_x(x_e) = 0 \quad ,$$

where $x = x_e$ is any point in $(0,1)$ at which $\phi(x)$ has an extremum. We note that because of the uniform continuity of $\tilde{\phi}(x,\varepsilon)$ and $\tilde{\phi}_x(x,\varepsilon)$ in ε (for x in $[0,1]$), the phase plane alternatives imply that $\tilde{\phi}(x+h,\varepsilon)$ intersects $\phi(x)$ at least once in $(0,1)$ for all ε in $(0,\varepsilon_0')$ and all h in $(-h_0,h_0)$ for some $\varepsilon_0' > 0$ and $h_0 > 0$ sufficiently small. Hence $\phi_\infty(\varepsilon,x)$ is not $\tilde{\phi}(x+h,\varepsilon)$ for any ε in $(0,\varepsilon_0)$ and h in $(-h_0,h_0)$. Thus,

$$\lim_{\varepsilon\to 0} \max_{0\le x\le 1} (\phi_\infty(\varepsilon,x) - \phi(x)) \nrightarrow 0 \quad .$$

Therefore $u(t,x) \equiv \phi(x)$ is unstable whenever it has at least two relative extrema in $(0,1)$.

For our last major case let us assume that $\phi(x)$ has exactly one relative extremum in $(0,1)$. As before, define $\tilde{\phi}(x,\varepsilon)$ as the solution of

$$f(\tilde{\phi}_{xx},\tilde{\phi}_x,\tilde{\phi}) = 0$$

with initial condition

$$\tilde{\phi}(x_e,\varepsilon) = \phi(x_e) + \varepsilon \qquad \tilde{\phi}_x(x_e,\varepsilon) = \phi_x(x_e) = 0 \quad .$$

Here $x = x_e$ is the single point in $(0,1)$ where $\phi(x)$ has a maximum. Define

$$\phi_\varepsilon(x) \equiv \frac{\partial}{\partial\varepsilon} \tilde{\phi}(x,\varepsilon) \Big|_{\varepsilon=0} \quad ,$$

and note that

$$\phi_\varepsilon(x) + h\phi_x(x) \equiv \frac{\partial}{\partial\varepsilon} \tilde{\phi}(x+h\varepsilon,\varepsilon) \Big|_{\varepsilon=0} \quad .$$

Suppose that there is an h such that

$$\phi_\varepsilon(x) + h\phi_x(x) > 0 \quad \text{for all} \quad x \quad \text{in} \quad [0,1] \quad .$$

Then, for all ε in $(0,\varepsilon_0)$ (for some $\varepsilon_0 > 0$ sufficiently small)

$$\tilde{\phi}(x-h\varepsilon,-\varepsilon) \le \phi(x) \le \tilde{\phi}(x+h\varepsilon,\varepsilon) \quad \text{for all } x \text{ in } [0,1] \quad .$$

Similar to the case where $\phi(x)$ was strictly monotonic, we can now use $\tilde{\phi}(x-h\varepsilon,-\varepsilon)$, $\tilde{\phi}(x+h\varepsilon,\varepsilon)$, and the maximum principle to conclude that $u(t,x) \equiv \phi(x)$ is stable for this case.

Suppose now that there is an h such that

$$\phi_\varepsilon(x) + h\phi_x(x) < 0 \quad \text{for some } x \text{ in } (0,x_e) \text{ and for some } x \text{ in } (x_e,1) \quad .$$

Then for all ε in $(0,\varepsilon_0)$ (for some ε_0 sufficiently small) the curves $\tilde{\phi}(x+h\varepsilon,\varepsilon)$ and $\phi(x)$ intersect at least once in $(0,x_e)$ and at least once in $(x_e,1)$. We can now define $\phi(x,\varepsilon) \equiv \tilde{\phi}(x+h\varepsilon,\varepsilon)$, define $x_-(\varepsilon)$ as the largest x in $(0,x_e)$ such that $\phi(x_-(\varepsilon),\varepsilon) = \phi(x_-(\varepsilon))$, and define $x_+(\varepsilon)$ as the smallest x in $(x_e,1)$ such that $\phi(x_+(\varepsilon),\varepsilon) = \phi(x_+(\varepsilon))$. Similar to the case where $\phi(x)$ has two relative extrema in $(0,1)$, we can use these $\phi(x,\varepsilon)$, $x_-(\varepsilon)$, and $x_+(\varepsilon)$ to prove a hair-trigger effect. From this hair-trigger effect we can then deduce that $u(t,x) \equiv \phi(x)$ is unstable for this case.

We now summarize the stability results of this section. Suppose that we are given the following boundary value problem:

$$u_t = f(u_{xx}, u_x, u) \qquad 0 \leq x \leq 1 \tag{4.88a}$$

$$u(t,x) = A \text{ at } x = 0 \qquad u(t,x) = B \text{ at } x = 1 \quad , \tag{4.88b}$$

where A and B are given constants. Suppose that $u(t,x) \equiv \phi(x)$ is a steady solution of this problem. Then:

(1) if $\phi'(x) \neq 0$ for all x in $[0,1]$ then $u(t,x) \equiv \phi(x)$ is stable;

(2) if $\phi(x)$ has at least two relative extrema in the interval $0 < x < 1$ then $u(t,x) \equiv \phi(x)$ is unstable;

(3) if $\phi(x)$ has exactly one extrema in the interval $0 < x < 1$ and if for some h, $\phi_\varepsilon(x) + h\phi_x(x) > 0$ for all x in $[0,1]$, then $u(t,x) \equiv \phi(x)$ is stable; and finally

(4) if $\phi(x)$ has exactly one extrema in the interval $0 < x < 1$ (which occurs at $x = x_e$) and if for some h, $\phi_\varepsilon(x) + h\phi_x(x) < 0$ for some x in $(0,x_e)$ and for some x in $(x_e,1)$, then $u(t,x) \equiv \phi(x)$ is unstable.

This completes our stability analysis of steady solutions of problem (4.88). Note that we have treated the major cases of problem (4.88), but that we have not treated some minor cases here. Note also that similar techniques can be used to establish the stability of steady solutions of problems like

$$u_t = f(u_{xx}, u_x, u) \qquad 0 \leq x \leq +\infty \tag{4.92a}$$

$$u(t,x) \equiv A \text{ at } x = 0 \quad . \tag{4.92b}$$

We will not pursue this line of inquiry however.

In the next section we conclude this chapter with some general remarks.

4.19 Some general comments. In this final section of Chapter IV we make some general comments about the material in this chapter.

First, the main purpose of this chapter is to provide the means to determine the precise stability of any traveling wave solution $u(t,x) \equiv \phi(x-ct)$ of

$$u_t = f(u_{xx}, u_x, u) \tag{4.1}$$

by inspection. Together, theorems (4.5) and (4.6) come very close to doing precisely this. In fact, in Chapter V we will show that the stability results for monotone waves contained in theorem (4.5) are sharp in almost all cases, including all non-accidental cases where $\phi(-\infty)$ and $\phi(+\infty)$ are both order one singular points. Also, for non-monotonic waves $\phi(x-ct)$ we can hardly expect better results than theorem (4.6) gives for the "at least two relative extrema" case. However, for some monotone waves $\phi(x-ct)$ which either decay to a node at $\phi(-\infty)$ or to a node at $\phi(+\infty)$ at the accidental rate and for some monotone waves $\phi(x-ct)$ where either $\phi(-\infty)$ or $\phi(+\infty)$ is not a first order singular point, some improvements in the stability results might be possible. Also, one may be able to improve on the results in theorem (4.6) for the single relative extremum type of traveling wave solutions $u(t,x) = \phi(x-ct)$. Although we will not pursue this topic, let us note that in proving theorem (4.6) we always used solutions of

$$u_t - f(u_{xx}, u_x, u) - cu_x = 0 \tag{4.2}$$

for our upper and lower functions. Since we never took advantage of the differential inequalities allowed by the maximum principle, perhaps we could improve the results in theorem (4.6) by utilizing these allowed inequalities.

A major advantage of theorems (4.5) and (4.6) is that these theorems are generic. That is, the stability results contained in these theorems depend only on a few easily determined characteristics of the traveling wave solutions. The ease of the application of these results and the sharpness of these results make theorems (4.5) and (4.6) very useful.

A disadvantage of the stability theory developed in this chapter is the difficulty of the application of the stability criteria developed in section (4.14) for the indeterminate case. However, if someone shows that the stability case never occurs (which is not an unreasonable conjecture) then this disadvantage would immediately disappear.

The stability results for monotonic traveling waves in a single spatial dimension have direct extensions to monotonic traveling plane waves in multiple spatial dimensions, as was discussed in section (4.11). However, the direct extension of the instability results of theorem (4.6) to non-monotonic traveling plane waves in higher spatial dimensions significantly weakens the instability results, as is discussed in section (4.17). Specifically, in multiple spatial dimensions we no longer have instability for arbitrarily small perturbations of finite extent. One could plausibly conjecture that these plane waves are indeed unstable to arbitrarily small perturbations of finite extent. However the lack of a phase plane for solutions of

$$f(\phi_{xx},\phi_{xy},\phi_{yy},\phi_x,\phi_y,\phi) + c_x\phi_x + c_y\phi_y = 0$$

means that the proof of any such conjecture would differ significantly from the proof used for the one spatial dimension case. We will meet this same problem again in Chapter VI and in Chapter VII where we extend the stability results to some equations containing integrals and to some systems of equations. There the lack of a phase plane prevents us from extending the instability results to these new classes of equations.

In section (4.18) we used our techniques to find the stability/instability of steady solutions of the boundary value problem

$$u_t = f(u_{xx}, u_x, u) \qquad 0 \leq x \leq 1$$
$$u(t,x) = A \quad \text{at} \quad x = 0 \qquad u(t,x) = B \quad \text{at} \quad x = 1 \quad .$$

We also noted in section (4.18) that we could extend these stability/instability results to the steady state solutions of the one-sided boundary value problem

$$u_t = f(u_{xx}, u_x, u) \qquad 0 \leq x \leq +\infty$$
$$u(t,x) = A \quad \text{at} \quad x = 0 \quad .$$

However there is no readily apparent extension of these results to the stability of steady state solutions of mixed boundary condition problems such as

$$u_t = f(u_{xx}, u_x, u) \qquad 0 \leq x \leq 1$$
$$g_1(u, u_x) = 0 \quad \text{at} \quad x = 0 \qquad g_2(u, u_x) = 0 \quad \text{at} \quad x = 1 \quad .$$

Apparently finding such an extension involves using at least one new idea.

This completes this chapter on the stability and instability of traveling wave solutions of

$$u_t = f(u_{xx}, u_x, u) \quad . \tag{4.1}$$

In the next chapter we will explore a related topic. Namely we will explore the connection between the mean wavespeed of a solution $u(t,x)$ of (4.1) and its initial condition $u(0,x)$

Chapter V

MEAN WAVESPEED AND THE INITIAL CONDITIONS

In this chapter we again deal almost exclusively with parabolic equations which contain only one dependent variable, contain only one independent variable, and contain no integrals. Throughout this chapter we will assume that the hypotheses H2 (smoothness of the equation), H3 (parabolicity of the equation), and H4 (existence of solutions to the initial value problems) are satisfied. We also assume a very large $M > 0$ has been chosen, and as in the previous chapter we therefore work with the resulting specific equation

$$u_t = f(u_{xx}, u_x, u) \qquad f_1 > 0 \tag{5.1}$$

where $f(u_{xx}, u_x, u) \equiv F_M^{(1)}([\tilde{u}])$.

In this chapter we will establish connections between the mean wavespeed of solutions $u(t,x)$ of (5.1) and their initial conditions $u(0,x)$. This topic was discussed in section (2.4), and thus much of the material in this present chapter is duplicated there. Specifically, in this chapter we consider equations of the form (5.1) which admit non-constant bounded <u>monotonic</u> solutions

$$u(t,x) \equiv \phi(x-ct,c) \tag{5.2}$$

for some values of c (which may be zero), since these are the non-trivial stable traveling wave solutions of (5.1). We will first determine when the existence of a monotonic solution $\phi(x-ct,c)$ of (5.1) at a particular wavespeed c implies the existence or non-existence of other nearby traveling wave solutions, both at the same and at slightly different wavespeeds c. We then use these existence results and the maximum principle to establish the connection between the mean wavespeed of solutions $u(t,x)$ of (5.1) and their initial conditions $u(0,x)$.

In order to see how such mean wavespeed/initial condition results can be obtained, we consider a motivating example. Let $u(t,x) \equiv \phi(x-ct,c)$ be an increasing traveling wave solution of (5.1). Then for any h_1, $h_2 > 0$ (no matter how large) $\phi(x-ct + h_2, c)$ and $\phi(x-ct - h_1, c)$ are also solutions. The maximum principle therefore implies that all solutions $u(t,x)$ of (5.1) with initial conditions $u(0,x)$ satisfying

$$\phi(x-h_1,c) \leq u(0,x) \leq \phi(x+h_2,c) \quad \text{for all } x \; , \tag{5.2}$$

must also satisfy

$$\phi(x-h_1-ct,c) \leq u(t,x) \leq \phi(x+h_2-ct,c) \quad \text{for all } x, \text{ all } t \geq 0 \; . \tag{5.3}$$

This is illustrated in Figure (1) below, where the implication of the maximum principle is that all solutions of (5.1) which are initially in the shaded region will remain in the shaded region for all $t \geq 0$. It is apparent from Figure (1) that relation (5.3) implies that $u(t,x)$ travels

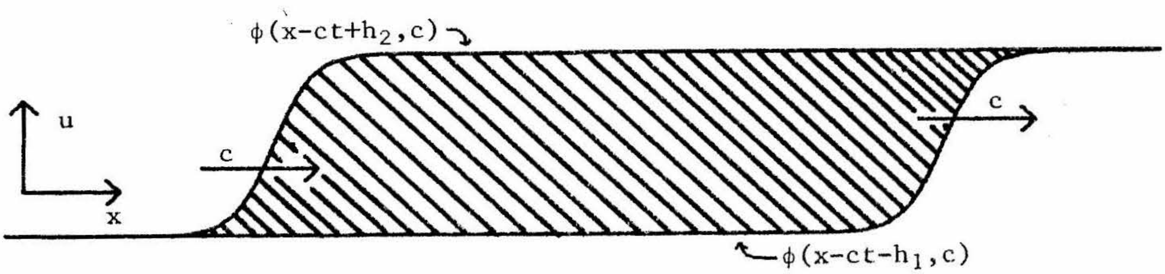


Figure (1)

with mean wavespeed c in an appropriate sense. Moreover, h_1 and h_2 can be arbitrarily large. Thus the main restrictions on which initial conditions can be bounded as in (5.2) are asymptotic in nature. Furthermore, it is clear that stronger results can be obtained by using the upper and lower functions constructed in section (4.5). Finally, when the existence of $\phi(x-ct,c)$ implies the existence of $\phi(x-c't,c')$ for some c' near c, it is clear that similar results can be obtained showing which $u(t,x)$ travel with mean wavespeed c'.

In this chapter we will consider only the four main types of monotonic traveling waves. Specifically, we will consider only the $S \to S$, $N \to S$, $S \to N$, and $N \to N$ types of monotonic traveling waves $\phi(x-ct,c)$ where $\phi(-\infty,c)$ and $\phi(+\infty,c)$ are first order singular points. Although similar results can be easily obtained for any specific example when $\phi(-\infty)$ and/or $\phi(+\infty)$ are higher order singular points, to retain simplicity we will develop the general theory only for first order singular points. We will consider the $S \to S$, $N \to S$, and $N \to N$ cases in sections (5.1), (5.2), and (5.3), respectively. For each case, assuming a single monotone traveling wave solution exists we will first determine the existence or non-existence of nearby monotonic waves traveling with the same and nearly the same wavespeeds. We will then use the maximum principle and the upper and lower functions constructed in section (4.5) to obtain the mean wavespeed/initial condition results. In the other three sections in this chapter, sections (5.4) through (5.6), we will briefly discuss related topics. Specifically, in section (5.4) we will use the wavespeed results to show the sharpness of the stability results contained in theorem (4.5). In section (5.5) we will discuss the extension of the wavespeed/initial condition results to traveling plane waves in multiple spatial dimensions. Finally, we end this chapter in section (5.6) with some concluding remarks.

We now begin this program with the simplest case, namely the case where $u(t,x) \equiv \phi(x-c_0t,c_0)$ is a monotonic $S \to S$ type traveling wave solution of (5.1).

5.1 Saddle-saddle case. In this section, we assume that $u(t,x) \equiv \phi(x,c_0)$ exists and is a bounded monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{5.4}$$

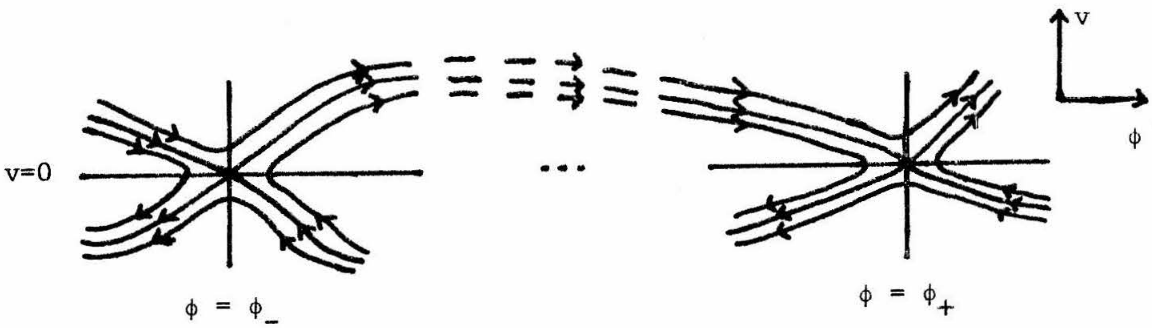
at wavespeed $c = c_0$. We also assume that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ are both first order saddle points of the system

$$\begin{aligned} \phi_x &= v \\ f(v_x,v,\phi) + cv &= 0 \end{aligned} \tag{5.5}$$

at $c = c_0$. In addition we will assume that $\phi(x,c_0)$ is increasing, since the analysis for $\phi(x,c_0)$ decreasing is very similar. In this section we will first show that for any value of c (including c_0), there can be only one such solution $\phi(x,c)$ modulo translations in x. We then will establish the mean wavespeed/initial condition result for this case. As a by-product of this last result, we will find that there can be only one speed c_0 at which a solution $\phi(x,c_0)$ (with the above properties) can exist for this case. To complete this section, we will then summarize these results in two theorems. We now do this.

Let $\phi(x,c_0)$ be the monotonic steady state solution of equation (5.4) at $c = c_0$ with all the properties assumed above. With these assumptions, the phase plane of (5.5) at $c = c_0$ must look like the illustration below. Since $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ are saddle points at $c = c_0$, they are saddle points for all values of c. Hence for each c there exists functions $\Psi_-(x,c)$ and $\Psi_+(x,c)$ such that every steady state solution $\phi(x,c)$ of equation (5.4) with $\phi(-\infty,c) = \phi_-$ and with $\phi_x(x,c) > 0$ for all x sufficiently small, must be

$$\phi(x,c) \equiv \Psi_-(x+h,c) \quad \text{for all} \quad x$$



for some constant h. Similarly, if $u(t,x) = \phi(x,c)$ solves equation (5.4), if $\phi(+\infty,c) = \phi_+$, and if $\phi_x(x,c) > 0$ for all x sufficiently large, then

$$\phi(x,c) \equiv \Psi_+(x+h,c) \quad \text{for all} \quad x$$

for some constant h. Therefore, for any c there is at most one steady state solution $\phi(x,c)$ of (5.4) (modulo translations in x) which is both monotonic and goes from $\phi(-\infty,c) = \phi_-$ to $\phi(+\infty,c) = \phi_+$. One sees that finding a value of c_0 for c at which such a monotonic steady state solution exists, is equivalent to finding a c_0 for which

$$\Psi_-(x+h,c_0) \equiv \Psi_+(x,c_0) \quad \text{for all} \quad x \quad \text{and for some} \quad h \quad .$$

Even though this can only occur accidentally at any given wavespeed c_0, the existence of a $S \to S$ wave for some wavespeed c cannot be regarded as an accidental occurrence.

We now establish the mean wavespeed/initial condition result for this case. Consider equation (5.4) at $c = 0$, specifically

$$u_t = f(u_{xx},u_x,u) \quad . \tag{5.6}$$

This is the given equation in terms of the original stationary coordinate system. Let $\phi(x,c_0)$ be the steady state solution of (5.4) at $c = c_0$ as described above: $\phi(-\infty,c_0) = \phi_-$; $\phi(+\infty,c_0) = \phi_+$; $\phi = \phi_-$, $v = 0$ and $\phi = \phi_+$, $v = 0$ both saddle points; and $\phi(x,c_0)$ monotonically increasing in x. Then $u(t,x) = \phi(x-c_0t,c_0)$ solves equation (5.6). We now utilize the upper and lower functions of lemma (4.4) and the maximum principle. This immediately shows that if $\overline{u}(t,x)$ and $\underline{u}(t,x)$ are any of the upper and lower functions given in lemma (4.4), then

$$\underline{u}(0,x) \leq u(0,x) \leq \overline{u}(0,x) \quad \text{for all} \quad x \tag{5.7}$$

implies that

$$\underline{u}(t,x-c_0t) \leq u(t,x) \leq \overline{u}(t,x-c_0t) \quad \text{for all} \quad x \quad \text{and all} \quad t \geq 0 \tag{5.8}$$

for any solution $u(t,x)$ of (5.6). Substituting for $\overline{u}$ and $\underline{u}$ from lemma (4.4), we find that for any $q(0) > 0$ small enough and for any h_1 and h_2, all solutions $u(t,x)$ of (5.6) whose initial conditions $u(0,x)$ are in H_x^2 and satisfy

$$\phi(x-h_1,c_0) - q(0) \leq u(0,x) \leq \phi(x+h_2,c_0) + q(0) \quad \text{for all} \quad x \tag{5.9}$$

must satisfy

$$\phi(x-c_0t-h_1-\kappa q(0),c_0) - q(t) \leq u(t,x) \leq \phi(x-c_0t+h_2+\kappa q(0),c_0) + q(t)$$
$$\text{for all} \quad x, \quad \text{all} \quad t \geq 0 \; . \tag{5.10}$$

Here $q(t)$ is defined by equation (4.20) and thus $q(t) \to 0$ monotonically as $t \to +\infty$. We illustrate the bounds of relation (5.10) on $u(t,x)$ in Figure (2) below. From this illustration it is clear that whenever $u(0,x)$ satisfies (5.9) for any $q(0) > 0$ small enough and some h_1 and h_2, then the resulting bounds of (5.10) on the solution $u(t,x)$ imply that $u(t,x)$ travels with mean wavespeed c_0 in an appropriate sense.

Thus when $u(0,x)$ can be bounded as in (5.9) we have found that the resulting solution $u(t,x)$ must travel with mean wavespeed c_0. To

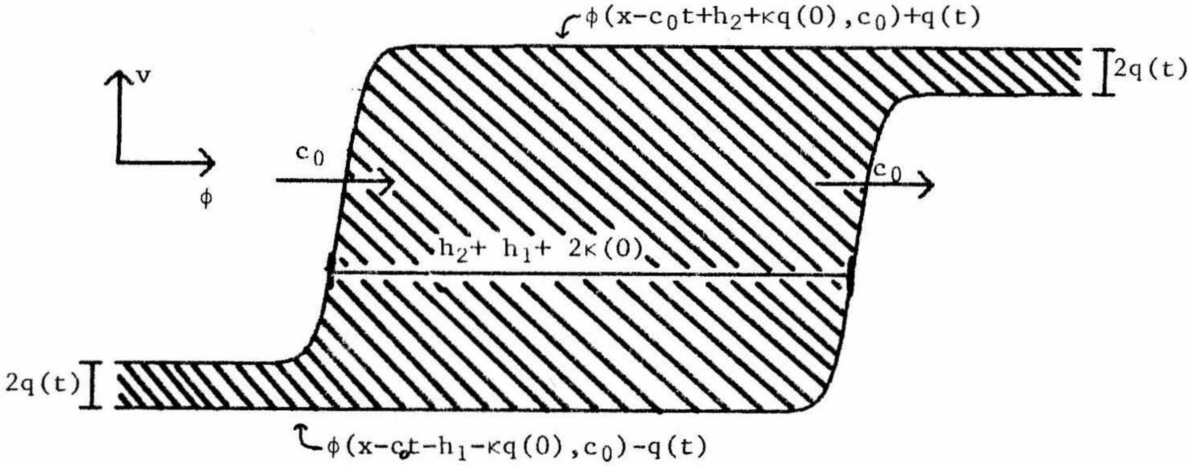


Figure (2): Since both of the functions bounding the shaded region move with speed c_0, since $q(t)\to 0$ as $t\to+\infty$, and since $u(t,x)$ must be in the shaded region for all $t\geq 0$, $u(t,x)$ must travel with mean wavespeed c_0.

obtain the mean wavespeed/initial condition result, we need only identify the class of initial conditions which can be bounded by (5.9). We note that (5.9) is satisfied for a particular $q(0) > 0$ and some h_1 and h_2 sufficiently large whenever the conditions

$$\begin{aligned} \phi_- - \alpha' &< u(0,x) < \phi_+ + \alpha' && \text{for all } x \\ \phi_- - \alpha' &< u(0,x) < \phi_- + \alpha' && \text{for all } x < -x_0 \\ \phi_+ - \alpha' &< u(0,x) < \phi_+ + \alpha' && \text{for all } x > x_0 \end{aligned} \qquad (5.11)$$

are satisfied for any $x_0 > 0$ and any α' in $(0,q(0))$. Therefore, whenever $u(0,x)$ is in H_x^2 and satisfies conditions (5.11) for any sufficiently small $\alpha' > 0$ and for any $x_0 > 0$, then the resulting solution $u(t,x)$ of

$$u_t = f(u_{xx},u_x,u) \qquad (5.6)$$

travels with mean wavespeed c_0. This immediately implies that <u>there is at most one speed c_0 for which a monotone solution $\phi(x-c_0t,c_0)$ (with</u>

$\phi(-\infty,c_0) = \phi_-$ and $\phi(+\infty,c_0) = \phi_+)$ exists.

In summary we have shown the following:

Theorem 5.1 $(S \to S)$: Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x-c_0t,c_0)$ is a bounded monotonic traveling wave (or steady state) solution of

$$u_t = f(u_{xx},u_x,u) \quad , \tag{5.6}$$

and also that $\phi = \phi(-\infty,c_0)$, $v = 0$ and $\phi = \phi(+\infty,c_0)$, $v = 0$ are both order one saddle points of system (5.5) at $c = c_0$. Then if $u(t,x) = \tilde{\phi}(x-\tilde{c}t,\tilde{c})$ is any other monotonic traveling wave (or steady state) solution of (5.6) with $\tilde{\phi}(-\infty,\tilde{c}) = \phi(-\infty,c_0)$ and $\tilde{\phi}(+\infty,\tilde{c}) = \phi(+\infty,c_0)$, then

$$\tilde{\phi}(\tilde{x}-\tilde{c}t,\tilde{c}) \equiv \phi(x-c_0t+h,c_0) \quad \text{for all } x, \text{ all } t \geq 0$$

for some h. In particular $\tilde{c} = c_0$.

Theorem 5.2 $(S \to S)$: Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x-c_0t,c_0)$ is a monotonic bounded traveling wave (or steady state) solution of (5.6), and also suppose that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ are both order one saddle points of system (5.5) at $c = c_0$. Then if $u(t,x)$ is any solution of (5.6) whose initial condition $u(0,x)$ is in H^2_x and satisfies

$$\phi_- - \alpha' < u(0,x) < \phi_- + \alpha' \quad \text{for all } x < -x_0$$

$$\phi_+ - \alpha' < u(0,x) < \phi_+ + \alpha' \quad \text{for all } x > +x_0$$

$$\min\{\phi_-,\phi_+\} - \alpha' < u(0,x) < \max\{\phi_-,\phi_+\} + \alpha' \quad \text{for all } x$$

for any $\alpha' > 0$ small enough and any $x_0 > 0$, then $u(t,x)$ travels with mean wavespeed c_0.

Note that we have established these theorems only in the case where $\phi(x,c_0)$ is increasing in x. However, a similar analysis to the one presented will establish these two theorems for the case of $\phi(x,c_0)$ decreasing.

Roughly speaking, theorem 5.1 $(S \to S)$ shows that if ϕ_- and ϕ_+ are both order one saddle points, then there is at most one wavespeed $c = c_0$ at which a monotonic traveling wave $u(t,x) = \phi(x-c_0t,c_0)$ with $\phi(-\infty,c_0) = \phi_-$ and $\phi(+\infty,c_0) = \phi_+$ can exist. Furthermore, if such a traveling wave exists, all other similar traveling waves $u(t,x) = \tilde{\phi}(x-c_0t,c_0)$ are translates of $\phi(x-c_0t,c_0)$. Finally, from theorem 5.2 $(S \to S)$ we see that if such a traveling wave exists, then any solution $u(t,x)$ of equation (5.6) must travel with mean wavespeed c_0 whenever its initial conditions remotely resemble the traveling wave, as is illustrated in Figure (3) below.

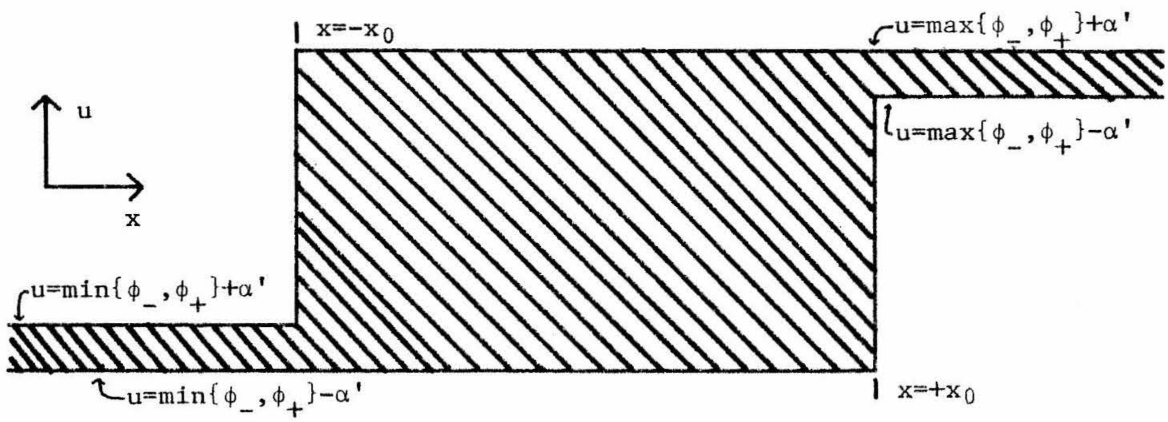


Figure (3): If $u(0,x)$ is contained in any region like the one shaded above, then $u(t,x)$ must move with mean wavespeed c_0. (See theorem 5.2 $(S \to S)$).

This completes the $S \to S$ case. We continue in the next section by analyzing the $N \to S$ case.

5.2 Node-saddle case. In this section we assume that $u(t,x) \equiv \phi(x,c_0)$ exists and is a bounded monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{5.4}$$

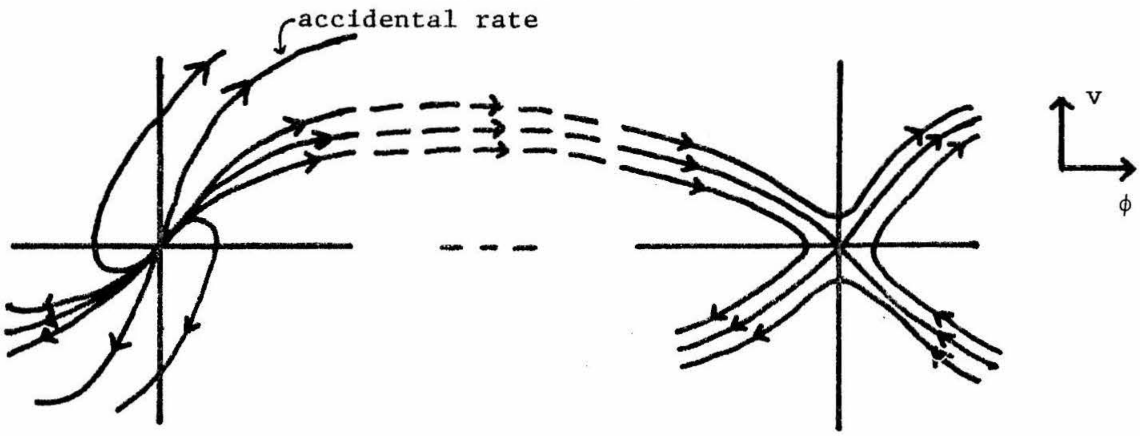
at $c = c_0$. We also assume that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ is a first order node and that $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ is a first order saddle point of the system

$$\begin{aligned} \phi_x &= v \\ f(v_x,v,\phi) + cv &= 0 \end{aligned} \tag{5.5}$$

at $c = c_0$. Finally in this section we will assume that $\phi(x,c_0)$ is increasing, since the analysis for $\phi(x,c_0)$ decreasing proceeds similarly.

In this section we will first use a continuity argument to show that if $\phi(x,c_0)$ decays to the node ϕ_- at the usual asymptotic rate as $x \to -\infty$, then for each wavespeed c' in at least a small range (c_1,c_2) about c_0 there is a monotonic steady state solution $\phi(x,c')$ of (5.4) at $c = c'$. Furthermore, we will find that $\phi(-\infty,c') = \phi(-\infty,c_0) \equiv \phi_-$, that $\phi(+\infty,c') = \phi(+\infty,c_0) \equiv \phi_+$, and that $\phi(x,c')$ decays to ϕ_- at the usual rate as $x \to -\infty$. As a by-product of this analysis, we will find that at any given wavespeed c there is at most one such solution of (5.4) (modulo translations in x). Then, by examining how the continuity argument can fail for c sufficiently far from c_0, we will be able to identify c_1 and c_2. We will then summarize these results in a theorem. Finally, we will quote and prove the mean wavespeed/initial value results for this case. We now carry out this program.

Let $\phi(x,c_0)$ be the monotonic steady state solution of (5.4) at $c = c_0$ with all of the properties assumed above. Then, the phase plane of system (5.5) at $c = c_0$ must look like



Since $\phi = \phi_+$, $v = 0$ is a saddle point at $c = c_0$, it is a saddle point at each value of c. Similarly, since $\phi = \phi_-$, $v = 0$ is an unstable node at $c = c_0$, it is an unstable node at each value of $c \le c_{max}$, where c_{max} is

$$c_{max} \equiv -2\sqrt{f_1(0,0,\phi_-)f_3(0,0,\phi_-)} - f_2(0,0,\phi_-) \quad .$$

We now show that the existence of the monotonic steady state $\phi(x,c_0)$ of (5.4) at $c = c_0$ implies that similar monotonic steady state solutions of (5.4) exist for all $c \le c_{max}$ with c near enough to c_0. Since $\phi = \phi_+$ is a first order saddle point of system (5.5) for each c, at each c there is a solution $\Psi(x,c)$ of (5.5) such that $\Psi(x,c) \to \phi_+$ as $x \to +\infty$ and such that $\Psi(x,c)$ is increasing for all x sufficiently large. Moreover, there can only be one such solution (modulo translations in x). Thus, for each $c \le c_{max}$ there is at most one steady state solution $\phi(x,c)$ of (5.4) (modulo translations in x) which is monotone and which goes from $\phi(-\infty,c) = \phi_-$ to $\phi(+\infty,c) = \phi_+$. (Of course, for $c > c_{max}$ the point $\phi = \phi_-$, $v = 0$ is no longer an unstable node, and so no such solutions can exist for $c > c_{max}$).

By using the translational freedom in x for each c in the definition of $\Psi(x,c)$, we can make $\Psi(x,c)$ and $v(x,c) \equiv \frac{\partial}{\partial x}\Psi(x,c)$ both be continuously differentiable in c. (This is an implication of Chapter 13 reference [6], for example). Moreover, by further translation of $\Psi(x,c)$, we can in addition set $\Psi(x,c_0) \equiv \phi(x,c_0)$.

Let $\tilde{\phi}_+ < \phi_+$ be selected such that $f_3(0,0,\phi) < 0$ for all ϕ in $[\tilde{\phi}_+,\phi_+]$, and let $x_+(c)$ be defined by

$$\Psi(x,c) \geq \tilde{\phi}_+ \quad \text{for all} \quad x \geq x_+(c) \quad .$$

From the phase plane of system (5.5), one realizes that

$$v(x,c) \equiv \frac{\partial}{\partial x} \Psi(x,c) > 0 \quad \text{for all} \quad x \geq x_+(c) \quad .$$

Let $\tilde{c}_1,\tilde{c}_2$ with $\tilde{c}_1 < c_0 < \tilde{c}_2$ be selected, and let x_+ be defined by

$$x_+ \equiv \max_{\tilde{c}_1 \leq c \leq \tilde{c}_2} \{x_+(c)\} \quad .$$

We have observed from the phase plane that $\Psi(x,c)$ is monotone for all $x \geq x_+$ when c is in $(\tilde{c}_1,\tilde{c}_2)$. Suppose a constant x_- with $x_- < x_+$ is selected. No matter how small x_- is, the uniform continuity of $v(x,c)$ in c when x is restricted to the interval $[x_-,x_+]$ shows that for some $\tilde{\tilde{c}}_1$ in $[\tilde{c}_1,c_0)$ and some $\tilde{\tilde{c}}_2$ in $(c_0,\tilde{c}_2]$, the function $v(x,c) > 0$ for all x in $[x_-,x_+]$ when c is in $(\tilde{\tilde{c}}_1,\tilde{\tilde{c}}_2)$. Hence we now know that for any x_- (no matter how small) there is a $\tilde{\tilde{c}}_1 < c_0$ and a $\tilde{\tilde{c}}_2 > c_0$ such that $\Psi(x,c)$ is monotonic for all $x \geq x_-$ when c is in $(\tilde{\tilde{c}}_1,\tilde{\tilde{c}}_2)$.

Now let us define $\tilde{\phi}_- > \phi_-$ such that $f_3(0,0,\phi) > 0$ for all ϕ in $[\phi_-,\tilde{\phi}_-]$. By selecting x_- sufficiently small and selecting $\tilde{\tilde{c}}_1 < c_0$ and $\tilde{\tilde{c}}_2 > c_0$ sufficiently near c_0, the uniform continuity of $\Psi(x,c)$ in c for x in $[x_-,x_+]$ shows that $\Psi(x,c) \equiv \tilde{\phi}_-$ at exactly one point $x = x_-(c)$ with $x = x_-(c)$ in $[x_-,x_+]$. We will now show for all $c \leq c_{max}$, c sufficiently near c_0, that $\Psi(x,c)$ decays to ϕ_- monotonically for $x < x_-(c)$ and decays to ϕ_- at the usual rate as $x \to -\infty$.

Consider the phase plane of system (5.5) near $\phi = \phi_-$ at any value of $c \leq c_{max}$, as is illustrated in Figure (4) below. Let us examine the phase plane trajectories of all solutions of system (5.5) which decrease from $\phi = \tilde{\phi}_-$ to $\phi = \phi_-$ at the usual rate as $x \to -\infty$. We see that all these trajectories must cross the $\phi = \tilde{\phi}_-$ line at a positive point v which is smaller than the point $v = v_a(c)$ at which the accidental solution (i.e. the solution which decays to $\phi = \phi_-$ at the accidental rate as $x \to -\infty$) crosses the $\phi = \tilde{\phi}_-$ line. Conversely, as illustrated in Figure (4), any solution of (5.5) which crosses the $\phi = \tilde{\phi}_-$ line at a positive point $v < v_a(c)$ must decrease monotonically from $\phi = \tilde{\phi}_-$ to $\phi = \phi_-$ at the usual rate as x decreases to $-\infty$.

Now we have already shown that whenever c is in $(\tilde{\tilde{c}}_1, \tilde{\tilde{c}}_2)$ then $\Psi(x,c)$ is monotonic for $x \geq x_-(c)$ and $\Psi(+\infty,c) = \phi_+$, where

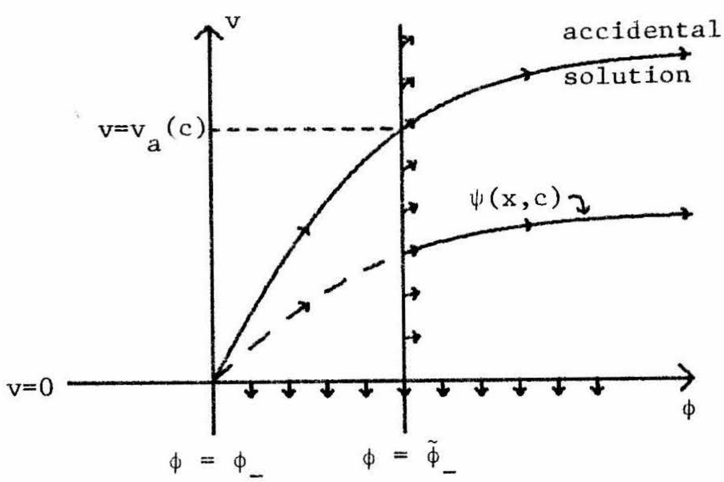


Figure (4): Phase plane of system (5.5) near $\phi=\phi_-$ and at any $c \leq c_{max}$. If the phase plane trajectory of $\Psi(x,c)$ intersects the $\tilde{\phi}=\phi_-$ line at any positive point v below the crossing point $v_a(c)$ of the trajectory of the solution which decays to ϕ_- at the accidental rate, then $\Psi(x,c)$ must decay monotonically to ϕ_- at the usual rate as $x \to -\infty$. This is because the phase plane directors on the $v=0$ line point downward for ϕ in $(\phi_-,\tilde{\phi}_-]$, and because the horizontal components of the phase plane directors on the $\phi=\tilde{\phi}_-$ line are positive for $v>0$.

$x_-(c)$ has been defined as the point x at which

$$\Psi(x_-(c),c) = \tilde{\phi}_- \quad .$$

Thus, to conclude that $\Psi(x,c')$ is a monotonic steady state solution of (5.4) at $c = c'$ (with $\Psi(-\infty,c') = \phi_-$, with $\Psi(+\infty,c') = \phi_+$, and with $\Psi(x,c')$ decaying to ϕ_- at the usual rate as $x \to -\infty$) for any c' in $(\tilde{\tilde{c}}_1,\tilde{\tilde{c}}_2) \bigcap (-\infty,c_{max}]$, we now need only to show that

$$v(x_-(c'),c') \equiv \frac{\partial}{\partial x} \Psi(x,c')\big|_{x=x_-(c')} < v_a(c') \quad .$$

However, at $c = c_0$ we have that $v(x_-(c_0),c_0) < v_a(c_0)$. Moreover, $v(x_-(c),c)$ and $v_a(c)$ are continuous in c for $c \le c_{max}$. Thus for some c_1 in $[\tilde{\tilde{c}}_1,c_0)$ and some c_2 in $(c_0,\tilde{\tilde{c}}_2]$, both sufficiently near c , we can conclude that $v(x_-(c'),c') < v_a(c')$ for all c' in $(c_1,c_2) \bigcap (-\infty,c_{max}]$, as is needed.

Thus, for some $c_1 < c_0$ and some $c_2 > c_0$ we have shown by a continuity argument that for each c in $(c_1,c_2) \bigcap (-\infty,c_{max}]$, there is a traveling wave solution $u(t,x) = \Psi(x-ct,c)$ of

$$u_t = f(u_{xx},u_x,u) \qquad (5.6)$$

which is monotone, which decays to $\Psi(-\infty,c) = \phi_-$ at the usual rate, and which has $\Psi(+\infty,c) = \phi_+$. Furthermore, we have shown that at each wavespeed c there is at most one such traveling wave solution (modulo translations in x) and also have shown that the functions $\Psi(x,c)$ and $\frac{\partial}{\partial x} \Psi(x,c)$ are continuously differentiable in c. We now identify the extremal wavespeeds c_1 and c_2.

Let us examine the continuity arguments used to show the existence of the traveling wave solutions $u(t,x) = \Psi(x-ct,c)$ of (5.6).

Suppose that $c_1 > -\infty$, so that monotonic traveling wave solutions $\phi(x-ct,c)$ (with the properties described above) exist for $c > c_1$, but not for $c < c_1$. From the continuity arguments we see that either

(1) there is a monotone solution $u(t,x) = \Psi(x-c_1 t,c_1)$ which decays to $\Psi(-\infty,c_1) = \phi_-$ at the <u>accidental</u> rate as $x \to -\infty$ and which has $\Psi(+\infty,c_1) = \phi_+$, or

(2) the phase plane trajectory of system (5.5) at $c = c_1$ which corresponds to the monotone wave $\Psi(x,c)$ intersects (but does not cross) the $v = 0$ axis at at least one point ϕ_0 in (ϕ_-,ϕ_+), as illustrated in Figure (5) below.

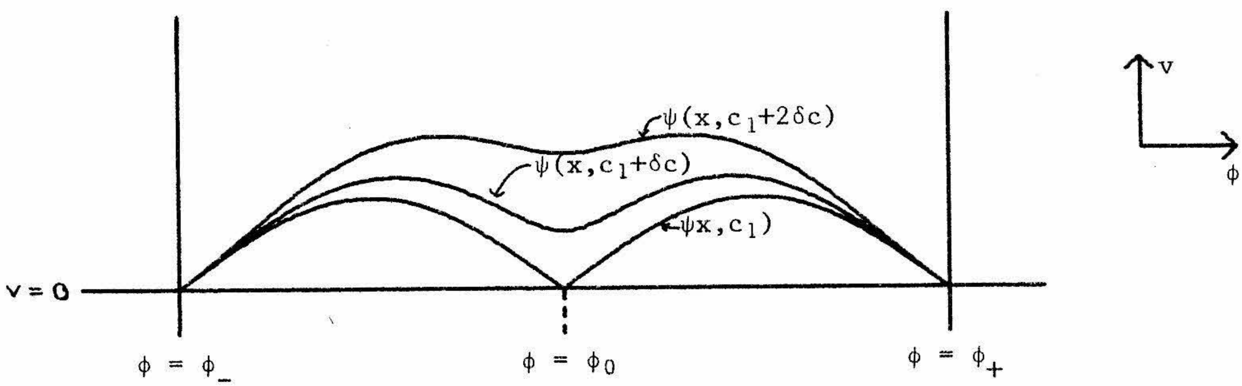


Figure (5): Phase plane trajectories of $\phi=\Psi(x,c)$, $v=\frac{\partial}{\partial x}\Psi(x,c)$ at $c=c_1$ and at two values of c slightly larger than c_1. As c decreases to c_1, the phase plane trajectory which emanates from the saddle point $\phi=\phi_+$, $v=0$ approaches the line $v=0$ at a point $\phi=\phi_0$, and when $c=c_1$ the trajectory intersects (but does not cross) the line $v=0$ at $\phi=\phi_0$. Thus $\phi=\phi_0$, $v=0$ must be a singular point. Note that the phase plane trajectories of $\Psi(x,c_1+2\delta c)$, $\Psi(x,c_1+\delta c)$, and $\Psi(x,c_1)$ belong to the distinct phase planes of system (5.5) at $c=c_1+2\delta$, $c=c_1+\delta c$, and $c=c_1$, respectively.

The first possibility (of $\Psi(x,c)$ decaying at the accidental rate as $x \to -\infty$) is straightforward, and so we consider the second possibility. Therefore, suppose that the second possibility occurs, and suppose further that the phase plane trajectory at $c = c_1$ intersects (but does not cross) the $v = 0$ line at only a single point $\phi = \phi_0$, $v = 0$. Since the phase plane trajectory intersects the line $v = 0$ at $\phi = \phi_0$ and does not cross this line, $\phi = \phi_0$, $v = 0$ is a singular point. Moreover, as illustrated in Figure (5), a trajectory both enters and leaves this singular point. Thus $\phi = \phi_0$, $v = 0$ is either an ordinary first order saddle point or is a higher order singular point. Clearly in this case as c goes to c_1, $\Psi(x,c)$ evolves into two separate monotonic traveling waves: a traveling wave $u(t,x) \equiv \Psi_1(x-c_1t,c_1)$ which has $\Psi_1(-\infty,c_1) = \phi_-$ and has $\Psi_1(+\infty,c_1) = \phi_0$, and a traveling wave $u(t,x) \equiv \Psi_2(x-c_1t,c_1)$ which has $\Psi_2(-\infty,c_1) = \phi_0$ and has $\Psi_2(+\infty,c_1) = \phi_+$. Also, typically the intermediate singular point $\phi = \phi_0$, $v = 0$ is an ordinary saddle point, although it can also be a higher order singular point. Thus in this case, as the wavespeed c decreases to c_1 the single monotonic traveling wave $\phi(x-ct,c)$ bifurcates into two distinct monotonic traveling waves.

It could happen that as c approaches c_1, the trajectory of $\Psi(x,c)$ intersects the $v = 0$ axis simultaneously at several different singular points $\phi_0^{(1)}, \phi_0^{(2)}, \ldots, \phi_0^{(m)}$ in (ϕ_-,ϕ_+) when $c = c_1$. This is illustrated in Figure (6) below. Again, each of the singular points $\phi_0^{(i)}$ is either an ordinary first order saddle point or is a higher order singular point. Clearly as c goes to c_1 the traveling wave $u(t,x) = \Psi(x-ct,c)$ evolves into the $m + 1$ monotonic traveling waves $u(t,x) = \Psi_i(x-c_1t,c_1)$, $i = 1,\ldots m$. Moreover, $\Psi_i(-\infty,c_1) = \phi_0^{(i-1)}$ and $\Psi_i(+\infty,c_1) = \phi_0^{(i)}$ where $\phi_0^{(0)}$ and $\phi_0^{(m+1)}$ have been defined as ϕ_- and ϕ_+. Thus in this case, at $c = c_1$ the single monotonic traveling wave

$u(t,x) = \Psi(x-ct,c)$ bifurcates into the $(m+1)$ monotonic traveling waves $u(t,x) = \Psi_i(x-c_1t,c_1)$.

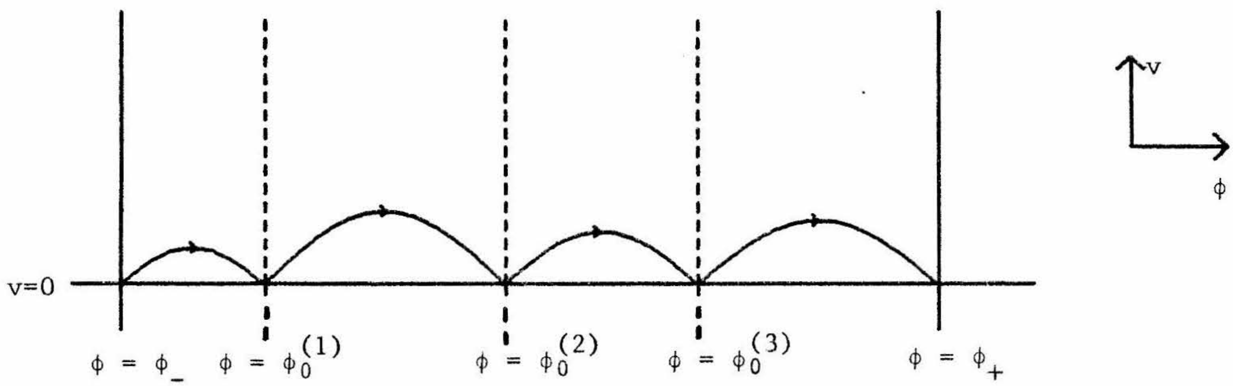


Figure (6): At $c=c_1$, the phase plane trajectory emanating from the saddle point a $\phi=\phi_+$ may intersect (but not cross) the v=0 line at several different singular points before reaching $\phi=\phi_-$.

In summary, so far we have found that if $c_1 > -\infty$, then either $\Psi(x,c_1)$ decays to ϕ_- at the accidental rate as $x \to -\infty$ or that at $c = c_1$ there are two or more monotonic waves corresponding to the single monotonic wave $\Psi(x,c)$ for $c > c_1$. Similarly, if $c_2 < c_{max}$ then either $\Psi(x,c_2)$ is a monotonic traveling wave with $\Psi(+\infty,c_2) = \phi_+$ and which decays to ϕ_- at the accidental rate as $x \to -\infty$, or $\Psi(x,c)$ utilizes intermediate singular points to bifurcate into two or more monotonic traveling waves as c goes to c_2. Thus at both $c = c_1$ and $c = c_2$ there can be two possible types of behavior of $\Psi(x,c)$. We now will use the maximum principle and find which behavior occurs at $c = c_1$ and which occurs at $c = c_2$.

We first consider $c = c_2$. We will now show that $\Psi(x,c)$ cannot evolve into two or more monotonic traveling waves as c goes to c_2.

Indeed, suppose that it does. Let these monotonic traveling wave solutions be $u(t,x) = \Psi_i(x-c_2t,c_2)$, where

$$\begin{aligned}\phi_- &\equiv \phi_0^{(0)} = \Psi_1(-\infty,c_2) < \Psi_1(+\infty,c_2) \equiv \phi_0^{(1)} \equiv \Psi_2(-\infty,c_2) < \Psi_2(+\infty,c_2) \\ &\equiv \phi_0^{(2)} \equiv \Psi_3(-\infty,c_2) < \dots \equiv \phi_0^{(m)} \equiv \Psi_{m+1}(-\infty,c_2) < \Psi_{m+1}(+\infty,c_2) \\ &\equiv \phi_0^{(m+1)} \equiv \phi_+ \quad .\end{aligned}$$

Consider the last monotonic traveling wave (with speed c_2) $u(t,x) = \Psi_{m+1}(x-c_2t,c_2)$, and let $\tilde{u}(t,x) = \Psi(x-\tilde{c}t,\tilde{c})$ be any of the monotonic traveling waves with $\tilde{c}$ in (c_1,c_2). We recall that $\Psi(-\infty,\tilde{c}) = \phi_-$ and that $\Psi_{m+1}(-\infty,c_2) = \phi_0^{(m)} > \phi_-$. Moreover, the asymptotic decay of $\Psi(x,\tilde{c})$ to the saddle point ϕ_+ is given by

$$\Psi(x,\tilde{c}) \sim \phi_+ - a(\tilde{c})e^{k(\tilde{c})x} \quad \text{as} \quad x \to +\infty ,$$

and that of $\Psi_{m+1}(x,c_2)$ is given by

$$\Psi_{m+1}(x,c_2) \sim \phi_+ - a(c_2)e^{k(c_2)x} \quad \text{as} \quad x \to +\infty ,$$

where $a(\tilde{c})$ and $a(c_2)$ are some positive constants, and where $k(\tilde{c})$ and $k(c_2)$ are given by

$$k(c) = \frac{-(f_2(0,0,\phi_+)+c)-\sqrt{(f_2(0,0,\phi_+)+c)^2-4f_1(0,0,\phi_+)f_3(0,0,\phi_+)}}{2f_1(0,0,\phi_+)}$$

at $c = \tilde{c}$ and $c = c_2$. Note that $c_2 > \tilde{c}$ implies that $|k(c_2)| > |k(\tilde{c})|$. This means that for some h sufficiently large

$$\Psi_{m+1}(x+h,c_2) > \Psi(x,\tilde{c}) \quad \text{for all} \quad x \quad .$$

But the maximum principle now implies that

$$\Psi_{m+1}(x-c_2t+h,c_2) \geq \Psi(x-\tilde{c}t,\tilde{c}) \quad \text{for all} \quad x, \quad \text{all} \quad t > 0 \quad .$$

This is illustrated in the sketch below, and is clearly nonsense since $c_2 > \tilde{c}$. Thus as c goes to c_2, the monotonic wave $\Psi(x,c)$ cannot evolve into two or more monotonic traveling waves. Therefore, if $c_2 < c_{max}$ then $u(t,x) = \Psi(x-c_2t,c_2)$ exists and is a monotonic traveling wave with $\Psi(+\infty,c_2) = \phi_+$ and which decays to ϕ_- at the <u>accidental</u> rate as $x \to -\infty$.

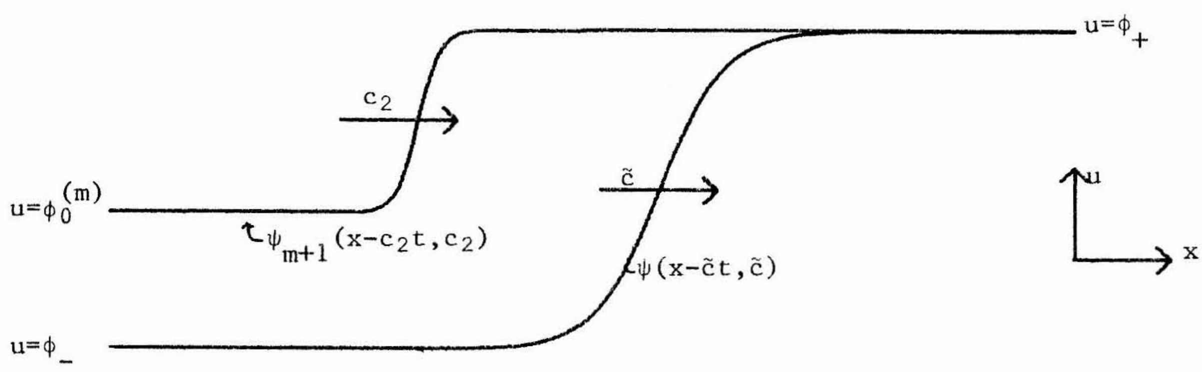


Figure (7): The maximum principle requires $\Psi_{m+1}(x-c_2t+h,c_2) \geq \Psi(x-\tilde{c}t,\tilde{c})$ for all x and all $t \geq 0$. However $\Psi_{m+1}(x-c_2t+h,c_2)$ travels with wavespeed c_2, which is larger than the wavespeed $\tilde{c}$ of $\Psi(x-\tilde{c}t,\tilde{c})$. Thus this is impossible.

Now let us briefly consider $c = c_1$. For this case we first assume that $u(t,x) = \Psi(x-c_1t,c_1)$ exists with $\Psi(+\infty,c_1) = \phi_+$ and with $\Psi(x,c_1)$ decaying at the accidental rate to ϕ_- as $x \to -\infty$. Similar to the preceding case, we can use the maximum principle to establish a contradiction. Thus as c goes to c_1, $\Psi(x,c)$ must evolve into two or more monotonic traveling waves. Moreover, let $u(t,x) = \Psi_1(x-c_1t,c_1)$ be the monotonic wave at $c = c_1$ with $\Psi_1(+\infty,c_1) = \phi_0^{(1)}$ and with $\Psi_1(-\infty,c_1) = \phi_-$. Then if $\Psi_1(x-c_1t,c_1)$ decayed to ϕ_- at the <u>accidental</u> rate as $x \to -\infty$, we could use the maximum principle to establish a contradiction similar to the contradiction in the $c = c_2$ case. Thus, $\Psi_1(x,c_1)$ must decay to ϕ_- at the <u>usual</u> rate as $x \to -\infty$.

In particular, note that if there are no singular points $\phi = \phi_0$, $v = 0$ with $\phi_- < \phi_0 < \phi_+$, then $\Psi(x,c)$ cannot bifurcate into two or more traveling waves because each of these traveling waves requires an intermediate singular point. Thus $c_1 = -\infty$ in this case. That is, when $f(0,0,\phi) = 0$ has no solutions for $\phi_- < \phi < \phi_+$, then $c_1 = -\infty$ and hence monotonic traveling waves $u(t,x) = \Psi(x-ct,c)$ with $\Psi(+\infty,c) = \phi_+$ and with $\Psi(x,c)$ decaying to ϕ_- at the usual rate as $x \to -\infty$ exist for all $c \leq c_2$.

We now summarize these results in the following theorem.

Theorem 5.1 $(N \to S)$: Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x-c_0t,c_0)$ is a bounded monotonic traveling wave (or steady state when $c_0 = 0$) solution of

$$u_t = f(u_{xx},u_x,u) \quad . \tag{5.6}$$

Suppose further that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ is an ordinary first order node and that $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ is an ordinary first order saddle point of the system

$$\begin{aligned} \phi_x &= v \\ f(v_x,v,\phi) + cv &= 0 \end{aligned} \tag{5.5}$$

at $c = c_0$. Finally suppose that $\phi(x,c_0)$ decays to ϕ_- at the usual rate as $x \to -\infty$. Then there is a c_1 and a c_2 with

$$-\infty \leq c_1 < c_0 < c_2$$

such that for each c' in $(c_1,c_2) \cap (-\infty,c_{max}]$ there exists a $\phi(x,c')$ satisfying the following conditions:

(1) $\phi(x,c')$, $\phi_x(x,c')$ are continuously differentiable in c',

(2) $\phi(x,c')$ is monotonic in x,

(3) $u(t,x) \equiv \phi(x-c't,c')$ solves equation (5.6),

(4) $\phi(-\infty,c') = \phi_-$ and $\phi(+\infty,c') = \phi_+$, and

(5) $\phi(x,c')$ decays to ϕ_- at the usual rate as $x \to -\infty$.

Also, if $\phi_1(x,c')$ and $\phi_2(x,c')$ are any functions satisfying (2), (3) and (4) at some c', then $\phi_1(x+h,c') = \phi_2(x,c')$ for all x for some h at that value of c'.

Moreover, if $c_2 < c_{max}$ then there is a traveling wave solution $u(t,x) = \phi(x-c_2t,c_2)$ of (5.6) which satisfies conditions (2), (3), and (4), but which decays to ϕ_- at the <u>accidental</u> rate as $x \to -\infty$.

Similarly, if $c_1 > -\infty$ then there are one or more values of ϕ, $\phi_0^{(1)}$, $\phi_0^{(2)}$, .. $\phi_0^{(m)}$ such that

(1) $\min\{\phi_-,\phi_+\} \equiv \phi_0^{(0)} < \phi_0^{(1)} < \phi_0^{(2)} < \ldots < \phi_0^{(m)} < \phi_0^{(m+1)} \equiv \max\{\phi_-,\phi_+\}$;

(2) for each $i = 1,\ldots,m$ $\phi = \phi_0^{(i)}$, $v = 0$ is a singular point, and if it is first order then it is a saddle point of system (5.5) at $c = c_1$;

(3) if $\phi(x,c_0)$ is increasing then there are $(m + 1)$ traveling wave solutions $u_i(t,x) = \phi_i(x-c_1t,c_1)$ of (5.6) such that

(a) $\phi_i(x,c_1)$ is increasing in x,

(b) $\phi_i(-\infty,c_1) = \phi_0^{(i-1)}$ and $\phi_i(+\infty,c_1) = \phi_0^{(i)}$, and

(c) $\phi_1(x,c_1)$ decays to ϕ_- at the usual rate as $x \to -\infty$;

(4) if $\phi(x,c_0)$ is decreasing then there are $(m + 1)$ traveling wave solutions $u_i(t,x) = \phi_i(x-c_1t,c_1)$ of (5.6) such that

(a) $\phi_i(x,c_1)$ is decreasing in x,

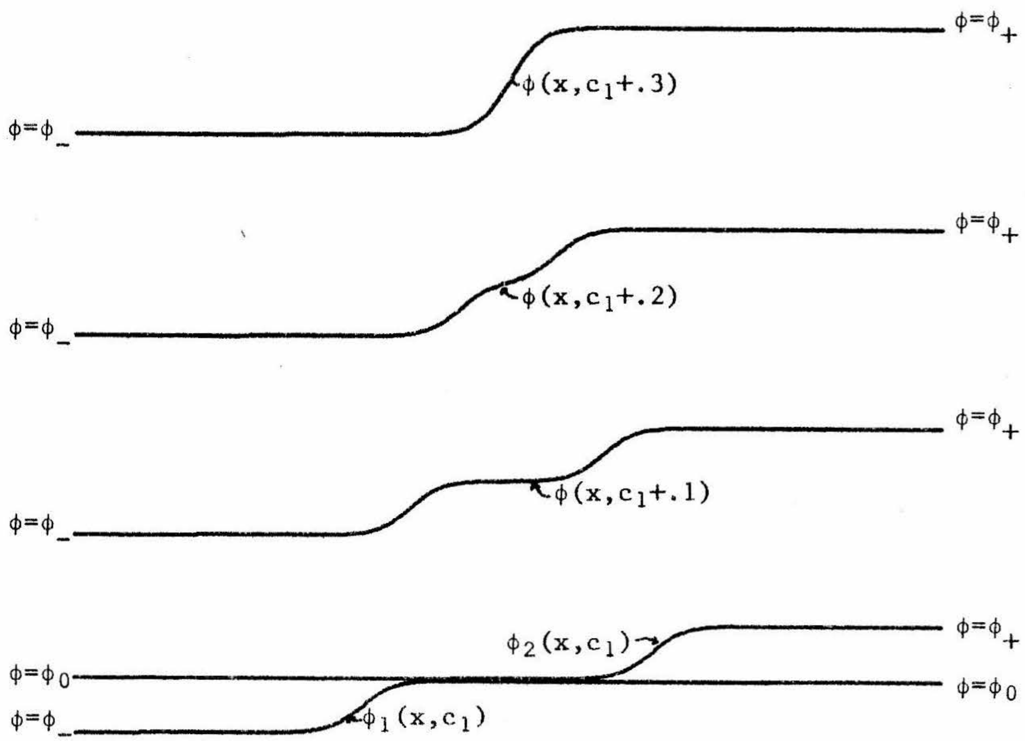
(b) $\phi_i(-\infty,c_1) = \phi_0^{(m+2-i)}$ and $\phi_i(+\infty,c_1) = \phi_0^{(m+1-i)}$

(c) $\phi_1(x,c_1)$ decays to ϕ_- at the usual rate as $x \to -\infty$.

In particular, if $f(0,0,\phi) \neq 0$ for all ϕ in (ϕ_-,ϕ_+), then $c_1 = -\infty$.

Thus, roughly speaking the above theorem shows that if

$u(t,x) = \phi(x-c_0t)$ is a $N \to S$ type monotonic wave which decays to the node at the usual rate as $x \to -\infty$, then similar monotonic $N \to S$ type traveling wave solutions $u(t,x) = \phi(x-ct,c)$ exist for all wavespeeds $c > c_0$ until c reaches c_{max} (where the unstable node changes to an unstable spiral point) or until c reaches a point where $u(t,x) = \phi(x-ct,c)$ decays to the node at the accidental rate as $x \to -\infty$. Similarly, monotonic $N \to S$ type traveling wave solutions $u(t,x) = \phi(x-ct,c)$ exist for all wavespeeds $c < c_0$ until c reaches $-\infty$ or until c reaches a point where $u(t,x) = \phi(x-ct,c)$ bifurcates into at least two distinct traveling waves. In the following sketch, a typical evolution of $\phi(x,c)$ into two distinct waves as c goes to c_1 is depicted.



When $c_1 > -\infty$, the general situation <u>can</u> be complicated. However the <u>typical</u> situation is very simple when $c_1 > -\infty$. Typically one does not expect the phase plane trajectory at $c = c_1$ to intersect the $v = 0$ line at more than one singular point $\phi = \phi_0$ in (ϕ_-, ϕ_+). Moreover, one also expects that the singular point $\phi = \phi_0$, $v = 0$ will be first order, and thus it must be a saddle point. Therefore the typical situation is the following. The fastest monotonic traveling wave solution $u(t,x) = \phi(x-c_2t, c_2)$ with $\phi(-\infty, c_2) = \phi_-$ and with $\phi(+\infty, c_2) = \phi_+$ either occurs at $c_2 < c_{max}$ and has $\phi(x,c_2)$ decaying to ϕ_- at the accidental rate as $x \to -\infty$, or occurs at $c_2 = c_{max}$ where the accidental and usual decay rates are nearly equal. For $c_1 < c < c_2$, the $N \to S$ type monotonic traveling wave solutions $u(t,x) = \phi(x-ct,c)$ all have $\phi(+\infty, c) = \phi_+$ and all decay to ϕ_- at the usual decay rate as $x \to -\infty$. This either occurs for all $c < c_2$ (i.e. $c_1 = -\infty$), or at some $c_1 > -\infty$ the monotonic $N \to S$ type solution $u(t,x) = \phi(x-ct,c)$ bifurcates (typically) into another $N \to S$ type monotonic traveling wave $u(t,x) = \phi_1(x-c_1t, c_1)$ and into a $S \to S$ type monotonic traveling wave $u(t,x) = \phi_2(x-c_1t, c_1)$. Moreover, $\phi_1(x,c_1)$ decays to ϕ_- at the usual rate as $x \to -\infty$ and $\phi_1(+\infty, c_1) = \phi_0$. Also, $\phi_2(-\infty, c_1) = \phi_0$ and $\phi_2(+\infty, c_1) = \phi_+$. Here ϕ_0 is some saddle point in (ϕ_-, ϕ_+).

Clearly we can apply the theorem to the secondary $N \to S$ type monotonic traveling wave $u(t,x) = \phi_1(x-c_1t, c_1)$. This shows that $N \to S$ type monotonic traveling waves similar to $\phi_1(x,c_1)$ exist for all c near enough c_1. In particular, these monotonic waves exist for all $c < c_1$ until a bifurcation of this $N \to S$ type monotonic traveling wave into a $N \to S$, $S \to S$ pair of monotonic traveling waves occurs. Note that these bifurcations cannot continue indefinitely. There must be a last bifurcation because there is only a finite number of saddle points in the interval (ϕ_-,ϕ_+).

We now utilize the results in theorem 5.1 $(N \to S)$, the upper and lower functions constructed in lemma (4.3), and the maximum principle. Together these yield the following mean wavespeed/initial condition result:

<u>Theorem 5.2 $(N \to S)$:</u> Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $u(t,x) = \phi(x-c_0t,c_0)$ is a bounded monotonic solution of

$$u_t = f(u_{xx},u_x,u) \qquad f_1 > 0 \quad , \tag{5.6}$$

that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ is a first order node and that $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ is a first order saddle point of system (5.5) at $c = c_0$. Finally suppose that $\phi(x,c_0)$ decays to ϕ_- at the usual rate as $x \to -\infty$.

Define the positive exponential decay constants $\lambda(c)$ by

$$\lambda(c) \equiv \frac{-(f_2(0,0,\phi_-)+c)-\sqrt{(f_2(0,0,\phi_-)+c)^2-4f_1(0,0,\phi_-)f_3(0,0,\phi_-)}}{2f_1(0,0,\phi_-)} \tag{5.12}$$

for all $c \leq c_{max}$, and define c_1 and c_2 as in the previous theorem. Furthermore, define $\tilde{c}_2 \equiv \min\{c_2,c_{max}\}$.

Then if $u(t,x)$ is any solution of (5.6) whose initial condition $u(0,x)$ is in H^2_x and satisfies

$$\phi_+ - q_0 \leq u(0,x) \leq \phi_+ + q_0 \text{ for all } x > x_0 \text{ for any } x_0 \ , \tag{5.13}$$

$$\phi_- < u(0,x) \leq \phi_+ + q_0 \text{ for all } x \text{ if } \phi(x,c_0) \text{ is increasing in } x, \text{ and} \tag{5.14}$$

$$\phi_+ - q_0 \leq u(0,x) < \phi_- \text{ for all } x \text{ if } \phi(x,c_0) \text{ is decreasing } x, \tag{5.15}$$

then we can conclude the following:

(1) if for any c in $(c_1,\tilde{c}_2)$ there is an $\alpha > 0$ such that

$$e^{-\lambda(c)x}|u(0,x) - \phi_-| > \alpha \quad \text{for all} \quad x < 0 \tag{5.16}$$

and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ cannot travel with mean wavespeed larger than c;

(2) if for any c in $(c_1,\tilde{c}_2)$ there is a $\beta > 0$ such that

$$e^{-\lambda(c)x}|u(0,x) - \phi_-| < \beta \quad \text{for all} \quad x < 0 \tag{5.17}$$

and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ cannot travel with mean wavespeed smaller than c;

(3) if for any c in $(c_1,\tilde{c}_2)$ there is an $\alpha > 0$ and a $\beta > 0$ such that

$$\alpha < e^{-\lambda(c)x}|u(0,x) - \phi_-| < \beta \quad \text{for all} \quad x < 0 \tag{5.18}$$

and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ travels with mean wavespeed c and has finite dispersion; and

(4) if for any c in $(c_1,\tilde{c}_2)$

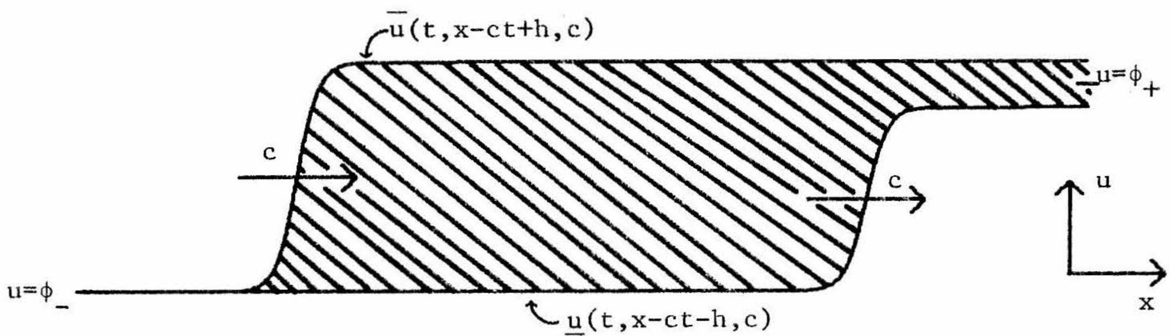
$$\begin{aligned} &\lim_{x\to-\infty} e^{(-\lambda(c)+\mu)x}|u(0,x) - \phi_-| = 0 \quad \text{for all} \quad \mu > 0 \ , \\ &\lim_{x\to-\infty} e^{(-\lambda(c)-\mu)x}|u(0,x) - \phi_-| = +\infty \quad \text{for all} \quad \mu > 0 \ , \quad \text{and} \end{aligned} \tag{5.19}$$

if $q_0 > 0$ is sufficiently small, then $u(t,x)$ travels with mean wavespeed c but may not have finite dispersion.

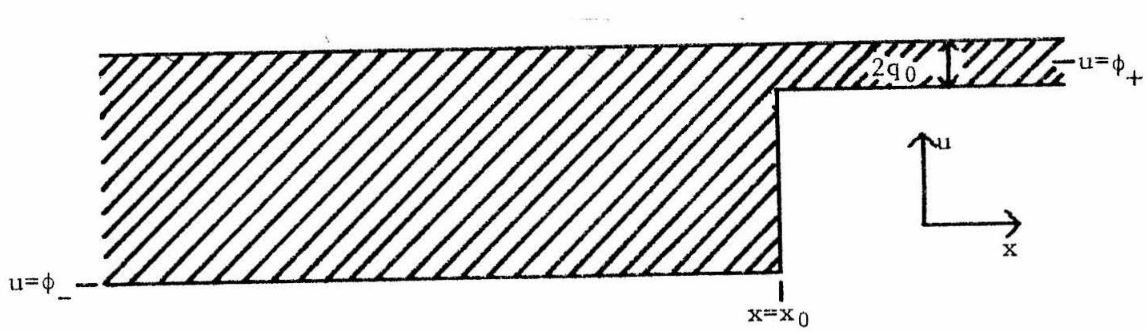
The meaning of the phrase "has finite dispersion" will be established in the proof.

The basic situation is the following. We have assumed the existence of a single monotonic wave $u(t,x) = \phi(x-c_0t,c_0)$ with $\phi(-\infty,c_0) \equiv \phi_-$ being a node and $\phi(+\infty,c_0) \equiv \phi_+$ being a saddle point. Theorem 5.1 $(N \to S)$ then shows that for a range $(c_1,\tilde{c}_2)$ of wavespeeds c there are similar monotonic $N \to S$ type traveling waves $u(t,x) = \phi(x-ct,c)$, which all have $\phi(-\infty,c) = \phi_-$ and $\phi(+\infty,c) = \phi_+$. For each of these $N \to S$ type waves, lemma (4.3) yields upper and lower functions $\overline{u}(t,x-ct,c)$

and $\underline{u}(t,x-ct,c)$ of equation (5.6), like the ones shown in the following sketch. The use of the maximum prinicple and these upper and lower functions will yield theorem (5.2) $(N \to S)$.



Roughly speaking, theorem 5.2 $(N \to S)$ supposes that $u(0,x)$ is any smooth function which is in a region like the one shaded below.



It then concludes that if $u(0,x)$ decays to the node ϕ_- exponentially as $x \to -\infty$, then the mean wavespeed of the solution $u(t,x)$ of (5.6) is determined only by the exponential decay constant.

Proof of theorem (5.2) ($N \to S$): We prove the theorem only for the case where $\phi(x,c_0)$ is increasing in x, and note that the proof when $\phi(x,c_0)$ is decreasing is very similar. Since $u(t,x) = \phi(x-c_0t,c_0)$ is a bounded monotonic $N \to S$ type solution of (5.6), we can apply theorem 5.1 ($N \to S$) and conclude the existence of similar monotonic traveling waves $u(t,x) = \phi(x-ct,c)$ for $c_1 < c < \tilde{c}_2$.

We now recall the upper and lower functions constructed in lemma (4.3). Namely, for each c in (c_1,c_2) there is a family of upper functions $\overline{u}(t,x-ct,c,q(0),h_0)$ and a family of lower functions $\underline{u}(t,x-ct,c,q(0),h_0)$ of the equation

$$u_t = f(u_{xx},u_x,u) \quad . \tag{5.6}$$

For the case at hand these upper and lower functions are

$$\overline{u}(t,x-ct,c,q(0),h_0) \equiv \phi(x-ct+h(t,c),c) + q(t,c)\cdot[\phi(x-ct+h(t,c),c) - \phi_-] \tag{5.20}$$

$$\underline{u}(t,x-ct,c,q(0),h_0) \equiv \phi(x-ct-h(t,c),c) - q(t,c)\cdot[\phi(x-ct-h(t,c),c) - \phi_-] , \tag{5.21}$$

where $h(t,c)$ and $q(t,c)$ are

$$h(t,c) \equiv q(0)\cdot\kappa(c)(1-e^{-s(c)t}) + h_0 \qquad q(t,c) \equiv q(0)e^{-s(c)t} \tag{5.22}$$

for some positive constants $\kappa(c)$ and $s(c)$ (which in general depend on c), where h_0 is arbitrary, and where $q(0) > 0$ is any sufficiently small constant. Moreover, since the dependence of $u_t - f(u_{xx},u_x,u) - cu_x$ on c is continuous and since $\phi(x,c)$, $\phi_x(x,c)$, and hence $\phi_{xx}(x,c)$ are also continuous in c, from the proof of lemma (4.3) we see that both $\kappa(c)$ and $s(c)$ can be taken to be continuous in c for $c_1 < c < c_2$. Also the proof shows that for $c_1 < c < c_2$ there is a continuous $q_{max}(c) > 0$ such that $\overline{u}(t,x-ct,c,q(0),h_0)$ and $\underline{u}(t,x-ct,c,q(0),h_0)$ are upper and lower functions of equation (5.6) for all $0 < q(0) < q_{max}(c)$ at each c in (c_1,c_2).

To prove part (1), we note that when relations (5.13), (5.14), (5.15), and (5.16) are satisfied, then we can bound $u(0,x)$ by

$$\begin{aligned}\underline{u}(0,x,c,q(0),h_0) &\equiv \phi(x-h_0,c) - q(0)\cdot[\phi(x-h_0,c) - \phi_-] \\ &\leq u(0,x) \leq \phi_+ + q(0)\cdot[\phi_+-\phi_-]\end{aligned} \tag{5.23}$$

for any $q(0) > q_0[\phi_+-\phi_-]^{-1}$ by taking h_0 sufficiently large. This is because theorem 5.1 $(N \to S)$ shows that $\phi(x,c)$ decays to ϕ_- at the usual rate as $x \to -\infty$; i.e., that

$$\phi(x,c) \sim \phi_- + ae^{\lambda(c)x} + O(e^{(\lambda(c)+\delta)x}) \quad \text{as} \quad x \to -\infty$$

for some positive a and δ, where $\lambda(c)$ is given by (5.12). We note that since ϕ_+ is a saddle point, if we define $\eta \equiv \frac{1}{2} f_3(0,0,\phi_+)$ then $\eta < 0$. Moreover,

$$\overline{u}(t,x) \equiv \phi_+ + q(0)\cdot[\phi_+-\phi_-]e^{\eta t} \tag{5.24}$$

is an x-independent upper function of equation (5.6) for all $t \geq 0$ whenever $q(0) > 0$ is sufficiently small. Since $q(0)$ can be taken as any constant larger than $q_0[\phi_+-\phi_-]^{-1}$, by taking q_0 sufficiently small we can take $q(0)$ to be small enough so that $\underline{u}(t,x-ct,c,q(0),h_0)$ and $\overline{u}(t,x)$ are a lower and upper function (respectively) of equation (5.6). The maximum principle implies that

$$\underline{u}(t,x-ct,c,q(0),h_0) \leq u(t,x) \leq \overline{u}(t,x) \quad \text{for all} \quad x \quad \text{and all} \quad t \geq 0 \ .$$

This yields

$$\begin{aligned}&\phi(x-ct-h_0-q(0)\kappa(c),c) - q(t,c)[\phi_+-\phi_-] \\ &\qquad \leq u(t,x) \leq \phi_+ + q(0)e^{\eta t}[\phi_+-\phi_-] \text{ for all } x, \text{ all } t \geq 0 ,\end{aligned} \tag{5.25}$$

where $q(t,c)$ is given in (5.22). This relation is illustrated in Figure (8) below, and we see that $u(t,x)$ cannot travel with mean wavespeed larger than c.

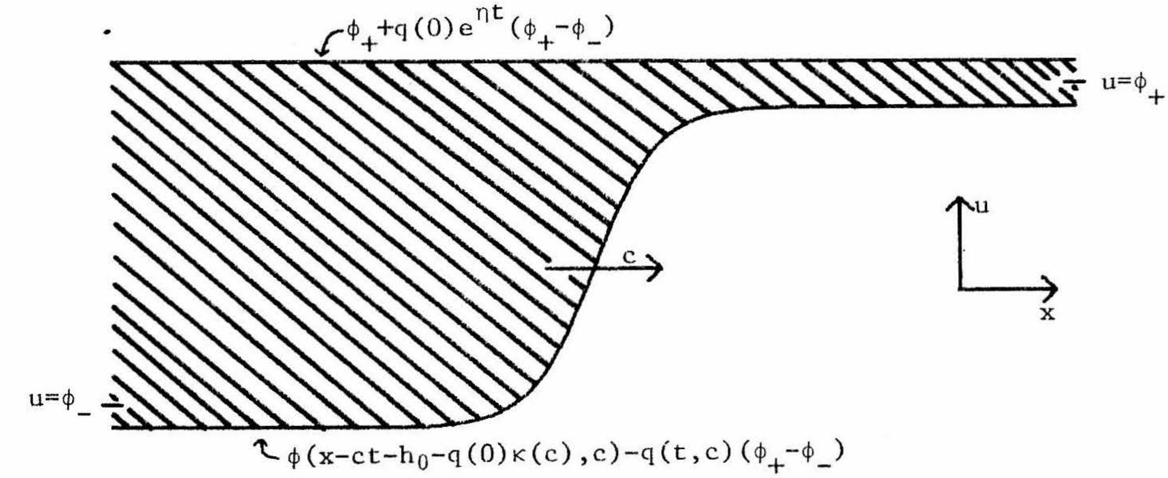


Figure (8): Since $q(0)e^{\eta t}\to 0$ and $q(t,c)\to 0$ as $t\to+\infty$, the fact that $u(t,x)$ remains in the shaded area for all $t\geq 0$ implies that it cannot travel with mean wavespeed larger than c.

Part (2) is proved in a manner very similar to part (1). In fact we find that

$$\phi_- \leq u(t,x) \leq \phi(x-ct+h_0+q(0)\kappa(c),c)+q(t,c)[\phi_+-\phi_-] \text{ for all } x, \text{ all } t \geq 0 \quad .$$

This relationship is illustrated in Figure (9) below, and we see that $u(t,x)$ cannot travel with mean wavespeed smaller than c.

To prove part (3), we note that when $u(0,x)$ satisfies (5.13), (5.14), (5.15), and (5.18), then we can bound $u(0,x)$ by

$$\phi(x-h_2,c)-q(0)[\phi(x-h_2,c)-\phi_-] \leq u(0,x) \leq \phi(x+h_1,c)+q(0)[\phi(x+h_1,c)-\phi_-] \quad (5.26)$$

for all x, for any $q(0) > q_0[\phi_+-\phi_-]^{-1}$ by taking h_1 and h_2 large enough. For $0 < q(0) < q_{max}(c)$, relation (5.26) bounds $u(0,x)$ by the upper and lower functions $\overline{u}(0,x,c,q(0),h_1)$ and $\underline{u}(0,x,c,q(0),h_2)$. Thus when $q_0 < [\phi_+-\phi_-]q_{max}(c)$, (5.25) bounds $u(0,x)$ by upper and lower functions at $t = 0$ and so the maximum principle implies that

$$\underline{u}(t,x-ct,c,q(0),h_2) \leq u(t,x) \leq \overline{u}(t,x-ct,c,q(0),h_1) \quad .$$

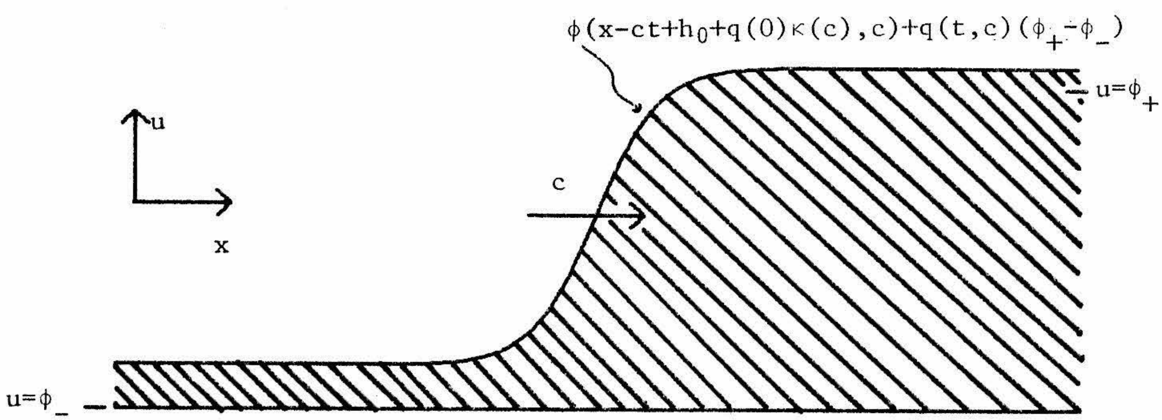


Figure (9): Since $q(t,c)\to 0$ as $t\to+\infty$, the fact that $u(t,x)$ remains in the shaded area for all $t\geq 0$ implies that it cannot have mean wavespeed smaller than c.

We thus conclude that

$$\phi(x-ct-h_2-q(0)\kappa(c),c)-q(t,c)[\phi_+-\phi_-] \leq u(t,x) \tag{5.27}$$

$$\leq \phi(x-ct+h_1+q(0)\kappa(c),c)+q(t,c)[\phi_+-\phi_-] \text{ for all } x, \text{ all } t \geq 0 .$$

This relationship is illustrated in Figure (10). We conclude that $u(t,x)$ travels with mean wavespeed c and has finite dispersion. The phase "has finite dispersion" is used here and in the statement of the theorem to mean precisely that the distance between the lower and upper functions which bound $u(t,x)$ in (5.27) is limited to no more than $h_1 + h_2 + 2q(0)\cdot\kappa(c)$, which is <u>finite</u>. This is in contrast to part (4), where we will only be able to show that the distance between these functions grows no faster than $o(t)$.

To prove part (4) we will need to adopt a slightly different strategy. Instead of bounding the initial condition $u(0,x)$ by a single

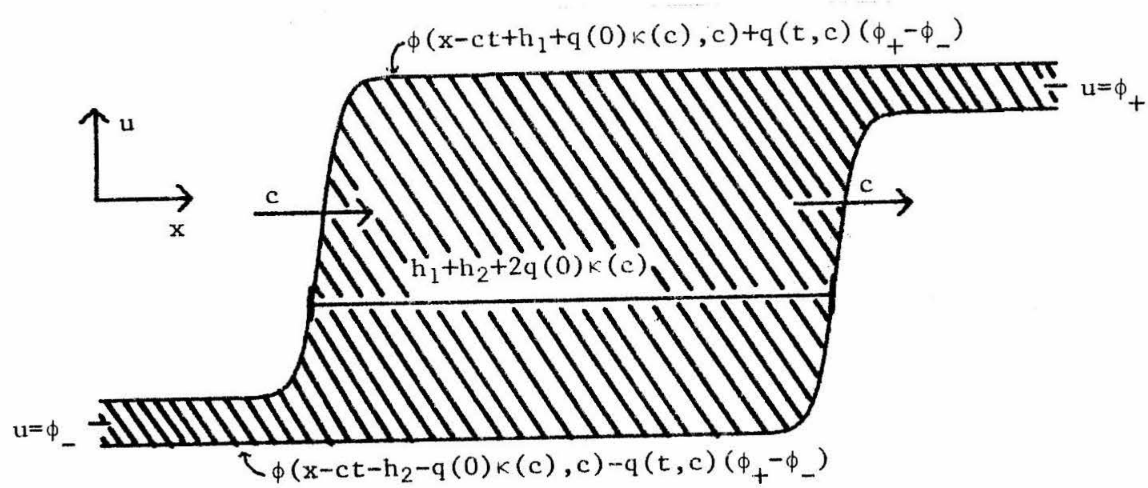
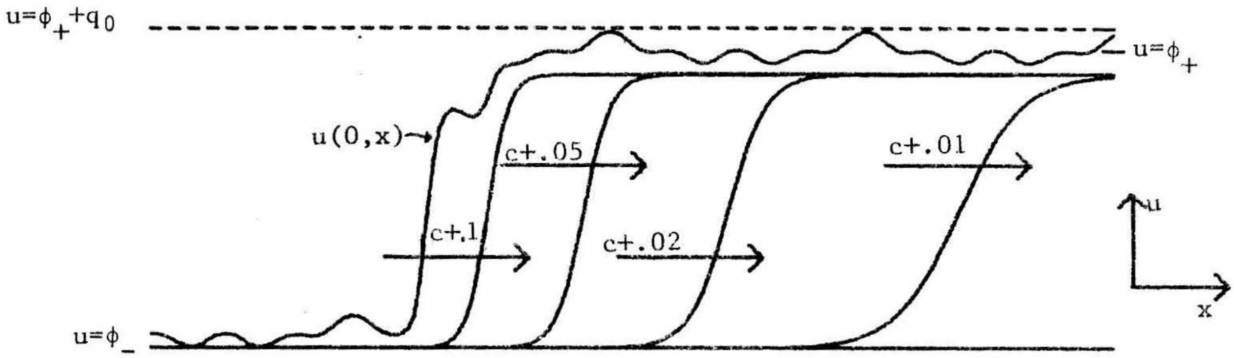


Figure (10): Since $q(t,c)\to 0$ as $t\to+\infty$, and since $u(t,x)$ remains in the shaded region for all $t\geq 0$, $u(t,x)$ must travel with mean wavespeed c. Furthermore, the distance between the two bounding curves remains constant.

upper function behind it and a single lower function in front of it, we will need to bound $u(0,x)$ by a series of upper and lower functions each moving with different velocities. In fact, we will bound $u(0,x)$ in front by one lower function $\underline{u}(t,x-\tilde{c}t,\tilde{c},q(0),h_0)$ for each $\tilde{c}$ in an interval $(c,c+\delta c)$ for some δc. We will find that the nearer $\tilde{c}$ is to c, the farther to the right we will need to place the lower function $\underline{u}(t,x-\tilde{c}t,\tilde{c},q(0),h_0)$ initially in order to bound $u(0,x)$ below. This is shown in the sketch on the next page. Since any one of these lower functions travels slower than the lower functions to its left and faster than the lower functions to its right, as time progresses the lower function is overtaken by all the faster lower functions to its left and it overtakes the slower lower functions to its right. Similar behavior occurs for the upper functions bounding $u(0,x)$ on its left. To prove part (4), at



each time $t > 0$ we will select the lower function in front of $u(t,x)$ which is farthest to the left, and the upper function behind $u(t,x)$ which is farthest to the right. We will show that the distance between these optimal bounding functions grows no faster than $o(t)$ and that they are centered at $ct + o(t)$.

Assume that $u(0,x)$ satisfies (5.13), (5.14), (5.15), and (5.19) where we will select the $q_0 > 0$ later. Let $\mu_0 > 0$ be so small that

$$\lambda(c_1) < \lambda(c) - \mu_0 < \lambda(c) < \lambda(c) + \mu_0 < \lambda(\tilde{c}_2) \quad .$$

Let $\overline{c}_1$ and $\overline{c}_2$ be the wavespeeds in $(c_1, \tilde{c}_2)$ defined by

$$\lambda(\overline{c}_1) = \lambda(c) - \mu_0 \ , \quad \lambda(\overline{c}_2) = \lambda(c) + \mu_0 \quad .$$

Now select an $x_0 > 0$ so large that

$$\begin{aligned} &\overline{u}(0,x+x_0,\tilde{c},q(0),0) \geq \exp\{\lambda(\tilde{c})x\} \quad \text{for all} \quad x \leq 0 \quad \text{and} \\ &\underline{u}(0,x-x_0,\tilde{c},q(0),0) \leq \exp\{\lambda(\tilde{c})x\} \quad \text{for all} \quad x \leq 0 \end{aligned} \tag{5.28}$$

hold for all $\tilde{c}$ in $[\overline{c}_1, \overline{c}_2]$ and for all $q(0)$ in $(0, \tilde{q}_{max}]$, where $\tilde{q}_{max}$ is given by

$$\tilde{q}_{max} \equiv \min_{\overline{c}_1 \leq \tilde{c} \leq \overline{c}_2} \{q_{max}(\tilde{c})\} \quad .$$

Select $\tilde{q}$ as the constant $\frac{1}{2}\tilde{q}_{max}$, and choose the constant q_0 in expressions (5.13), (5.14), and (5.15) to be $\frac{1}{4}\tilde{q}_{max}$. Finally select an x_1 so large that both

$$\underline{u}(0,x-x_0-x_1,\tilde{c},\tilde{q},0) \le u(0,x) \le \overline{u}(0,x+x_0+x_1,\tilde{c},\tilde{q},0) \quad \text{for all } x \ge 0, \quad \text{and} \tag{5.29}$$

$$u(0,x) \le \overline{u}(0,x_0+x_1,\tilde{c},\tilde{q},0) \qquad \text{for all} \quad x \tag{5.30}$$

hold for all $\tilde{c}$ in $[\overline{c}_1,\overline{c}_2]$.

We define $\overline{c}(\mu)$ and $\underline{c}(\mu)$ by

$$\lambda(\overline{c}(\mu)) = \lambda(c) - \mu \quad , \quad \lambda(\underline{c}(\mu)) = \lambda(c) + \mu \quad \text{for } 0 \le \mu \le \mu_0 \quad .$$

For each μ in $0 < \mu \le \mu_0$ we will find an $\overline{h}(\mu) \ge 0$ and an $\underline{h}(\mu) \ge 0$ such that

$$\underline{u}(0,x-x_0-x_1,\underline{c}(\mu),\tilde{q},\underline{h}(\mu)) \le u(0,x) \le \overline{u}(0,x+x_0+x_1,\overline{c}(\mu),\tilde{q},\overline{h}(\mu)) \text{ for all } x \le 0 \, . \tag{5.31}$$

Since (5.29) and (5.30) also hold we will then know that (5.31) holds for all x, and we will be able to then apply the maximum principle. We now find these $\overline{h}(\mu)$ and $\underline{h}(\mu)$.

Define $s(x)$, $\overline{s}(x)$, and $\underline{s}(x)$ for $x \le 0$ by

$$s(x) \equiv (u(0,x)-\phi_-)e^{-\lambda(c)x} \quad , \quad \overline{s}(x) \equiv \max_{x \le \tilde{x} \le 0} \{s(\tilde{x})\} \quad , \quad \text{and}$$

$$\underline{s}(x) \equiv \min_{x \le \tilde{x} \le 0} \{s(\tilde{x})\} \quad .$$

Note that $\underline{s}(x) \le s(x) \le \overline{s}(x)$, that $\underline{s}(x)$ is non-decreasing in x and that $\overline{s}(x)$ is non-increasing in x. Define

$$\overline{M}(\mu) = \max_{-\infty < x \le 0} \{\overline{s}(x)e^{\mu x}\} \quad \text{for} \quad 0 < \mu \le \mu_0$$

and define

$$\overline{h}(\mu) = \max\{0, \frac{1}{\lambda-\mu_0} \log \overline{M}(\mu)\} \quad .$$

Similarly, define

$$\underline{M}(\mu) = \min_{-\infty < x \le 0} \{\underline{s}(x)e^{-\mu x}\} \quad \text{for} \quad 0 < \mu \le \mu_0$$

and

$$\underline{h}(\mu) = \max\{0, \frac{-1}{\lambda-\mu_0} \log \underline{M}(\mu)\} \quad .$$

We note that with these definitions,

$$\underline{u}(0,x-x_0-x_1,\underline{c}(\mu),\tilde{q},\underline{h}(\mu)) \leq u(0,x)$$
$$\leq \overline{u}(0,x+x_0+x_1,\overline{c}(\mu),\tilde{q},\overline{h}(\mu)) \text{ for all } x, \text{ all } 0 < \mu \leq \mu_0 .$$

We thus can apply the maximum principle for each μ. This yields

$$\underline{u}(t,x-\underline{c}(\mu)t-x_0-x_1,\underline{c}(\mu),\tilde{q},\underline{h}(\mu)) \leq u(t,x) \tag{5.33}$$
$$\leq \overline{u}(t,x-\overline{c}(t)+x_0+x_1,\overline{c}(\mu),\tilde{q},\overline{h}(\mu)) \text{ for all } x, \text{ all } t \geq 0 ,$$
$$\text{all } 0 < \mu \leq \mu_0 .$$

Let the maximum of the quantity $\kappa(\tilde{c})$ (which appears in the definitions of the upper and lower functions) be

$$\kappa = \max_{\overline{c}_1 \leq c \leq \overline{c}_2} \kappa(\tilde{c}) \quad ,$$

and define the constant α by

$$\alpha = x_0+x_1+\tilde{q}\kappa \quad .$$

From the bounds (5.33) on $u(t,x)$, we conclude that

$$\phi(x-\underline{c}(\mu)t-\underline{h}(\mu)-\alpha,\underline{c}(\mu))-q(t,\underline{c}(\mu))[\phi_+-\phi_-] \leq u(t,x) \tag{5.34}$$
$$\leq \phi(x-\overline{c}(\mu)t+\overline{h}(\mu)+\alpha,\overline{c}(\mu))+q(t,\overline{c}(\mu))[\phi_+-\phi_-] \text{ for all } x, \text{ all } t \geq 0$$

holds for all $0 < \mu \leq \mu_0$. (See (5.20), (5.21), and (5.22)).

The lower functions are roughly positioned at

$$a(t,\mu) = \underline{c}(\mu)t + \underline{h}(\mu) + \alpha$$

and the upper functions are roughly positioned at

$$b(t,\mu) = \overline{c}(\mu)t - \overline{h}(\mu) - \alpha \quad .$$

At any given time t, we now roughly minimize $a(t,\mu)$ and maximize $b(t,\mu)$ over μ. Clearly at each $t > 0$ these functions have a unique minimum and maximum (respectively) in $0 < \mu \leq \mu_0$ since $\underline{c}(\mu)$ decreases with μ, since $\overline{c}(\mu)$ increases with μ, and since $\overline{h}(\mu)$ and $\underline{h}(\mu)$ are both non-increasing in μ. We now assume that worst case: $\overline{h}(\mu)$ and $\underline{h}(\mu)$ both go to $+\infty$ as $\mu \to 0$. We first handle $a(t,\mu)$. We have

$$a(t,\mu) = ct + (\underline{c}(\mu)-c)t - \frac{1}{\lambda-\mu_0} \log \underline{s}(\tilde{x}(\mu))\exp\{-\mu\tilde{x}(\mu)\} + \alpha$$

where $\tilde{x}(\mu)$ is the least value of $x \leq 0$ at which $s(x)e^{-\mu x}$ is at its minimum. To minimize $a(t,\mu)$, we select $\mu = \mu(t)$ such that $\tilde{x}(\mu)$ is the largest value in $\{\tilde{x}(\mu): 0 < \mu < \mu_0\}$ satisfying

$$\tilde{x}(\mu) \leq - \frac{\lambda-\mu_0}{\mu} (\underline{c}(\mu)-c)t \sim - (\lambda-\mu_0) \frac{dc(0)}{d\mu} t \quad . \tag{5.35}$$

Furthermore, define $T(t)$ as the value of t for which equality holds in (5.35). Utilizing (5.35), we have

$$a(t,\mu(t)) \leq ct - \frac{1}{\lambda-\mu_0} \log \underline{s}(- \frac{\lambda-\mu_0}{\mu} (\underline{c}(\mu)-c)T(t)) + \alpha \ ,$$

and by our hypotheses this implies

$$a(t,\mu(t)) \leq ct + o(T(t)) \quad \text{as} \quad t \to +\infty \ .$$

Thus, choose a sequence $t_1, t_2, \ldots$ such that

$$t_n = T(t_n) \ , \ t_n \to +\infty \quad \text{as} \quad n \to +\infty \ .$$

We have $a(t,\mu(t))$ non-decreasing in t and also have

$$a(t_n,\mu(t_n)) \leq ct_n + o(t_n) \quad .$$

Similarly, we can show that there is a roughly optimal $\mu = \tilde{\mu}(t)$ for the lower functions, that $h(t,\tilde{\mu}(t))$ is non-increasing in t, and that for a sequence $t = \tilde{t}_n$ with $\tilde{t}_n \to \infty$ as $n \to +\infty$

$$b(\tilde{t}_n,\mu(\tilde{t}_n)) \geq c\tilde{t}_n - o(\tilde{t}_n) \quad .$$

Thus we conclude that $u(t,x)$ must travel with mean wavespeed c. Note however that the separation between the upper and lower functions in general grows as $\sim o(t)$, and so we cannot conclude that $u(t,x)$ has finite dispersion. This establishes part (4) of the theorem.

Thus, we see that if the initial condition $u(0,x)$ decays to ϕ_- exponentially with exponential rate constant $\lambda(c)$, and if

$$\lambda(c_1) < \lambda(c) < \lambda(\tilde{c}_2) \quad ,$$

then $u(t,x)$ has mean wavespeed c. One naturally wonders how $u(t,x)$ behaves when $u(0,x)$ decays to ϕ_- at an asymptotically slower rate than $e^{\lambda(c_1)x}$. We will briefly consider this question for the two cases of $c_1 = -\infty$ and $c_1 > -\infty$.

Suppose that $c_1 = -\infty$, and so monotone traveling wave solutions $u(t,x) = \phi(x-ct,c)$ with $\phi(-\infty,c) = \phi_-$ and $\phi(+\infty,c) = \phi_+$ exist for all $c < \tilde{c}_2$. Note that $\lambda(c) \to 0$ as $c \to -\infty$. Therefore, consider any initial condition $u(0,x)$ which satisfies conditions (5.13), (5.14), and (5.15) of theorem 5.2 $(N \to S)$, and which also decays to ϕ_- <u>algebraically</u> (rather than exponentially). Then for any c (no matter how small) we can conclude from theorem 5.2 $(N \to S)$ that $u(t,x)$ cannot travel with a mean wavespeed larger than c. We do <u>not</u> conclude that $u(t,x)$ "travels to the left with infinite wavespeed" however. We simply say that in <u>any</u> coordinate system

$$t' = t \qquad x' = x-ct \quad ,$$

$u(t,x)$ satisfies

$$u(t,x) \to \phi_+ \quad \text{as} \quad t' \to +\infty \quad \text{at any fixed} \quad x' \quad .$$

Thus $u(t,x)$ does not behave very much like a "wave" in this case.

Suppose now that $c_1 > -\infty$. We know from theorem 5.1 $(N \to S)$ that at $c = c_1$ the monotonic traveling wave $u(t,x) = \phi(x-ct,c)$ bifurcates into at least two traveling waves. We consider only the typical case: at wavespeed $c = c_1$ the traveling wave $u(t,x) = \phi(x-ct,c)$ splits exactly into the monotonic waves

$$u(t,x) = \phi_1(x-ct,c) \qquad c_1' < c \leq c_1 \quad , \tag{5.36}$$

$$u(t,x) = \phi_2(x-c_1t,c_1) \quad , \tag{5.37}$$

where $\phi_1(-\infty,c) = \phi_-$ $\quad \phi_1(+\infty,c) = \phi_0$

$\phi_2(-\infty,c_1) = \phi_0$ $\quad \phi_2(+\infty,c_1) = \phi_+$

and where $\phi = \phi_0$, $v = 0$ is an order one saddle point. Note that in (5.36), if the smallest wavespeed $c_1^!$ is not $-\infty$ then it is a secondary bifurcation point.

Assume now that $u(0,x)$ satisfies the conditions (5.13), (5.14), and (5.15) of theorem 5.2 $(N \to S)$ for some sufficiently small $q_0 > 0$, that $\alpha < |u(0,x) - \phi_-|e^{-\lambda(c)x} < \beta$ for all $x \leq 0$ for some positive α and β, and that $\lambda(c_1^!) < \lambda(c) < \lambda(c_1)$. We will show that $u(t,x)$ evolves into two pieces: a piece bounded by ϕ_- and ϕ_0 which travels with a mean wavespeed no faster than $c < c_1$, and a piece bounded by ϕ_0 and ϕ_+ which travels with mean wavespeed c_1. Indeed, from lemmas (4.3) and (4.4) we use

(1) the upper functions $\overline{u}_2(t,x-c_1t,c_1,q_0,h_0)$ corresponding to the $S \to S$ wave $u(t,x) = \phi_2(x-c_1t,c_1)$,

(2) the lower functions $\underline{u}_1(t,x-ct,c,q_0,h_0)$ corresponding to the secondary $N \to S$ wave $u(t,x) = \phi_1(x-ct,c)$, and

(3) the lower functions $\underline{u}(t,x-\tilde{c}t,\tilde{c},q_0,h_2(\tilde{c}))$ corresponding to the primary $N \to S$ wave $u(t,x) = \phi(x-\tilde{c}t,\tilde{c})$ with $\tilde{c} > c_1$. By selecting h_0, h_1, and $h_2(\tilde{c})$ sufficiently large, we can bound the initial condition $u(0,x)$ by

$\underline{u}_1(0,x,c,q_0,h_1) \leq u(0,x) \leq \overline{u}_2(0,x,c_1,q_0,h_0)$ for all x, and

$\underline{u}(0,x,\tilde{c},q_0,h_2(\tilde{c})) \leq u(0,x)$ for all x, all $\tilde{c}$ in $(c_1,\tilde{c}_2)$.

Thus, the maximum principle yields

$\underline{u}_1(t,x-ct,c,q_0,h_1) \leq u(t,x) \leq \overline{u}_2(t,x-c_1t,c_1,q_0,h_0)$ for all x, all $t \geq 0$, and

$\underline{u}(t,x-\tilde{c}t,\tilde{c},q_0,h_2(\tilde{c})) \leq u(t,x)$ for all x, all $t \geq 0$, all $\tilde{c} \in (c_1,c_2)$.

This is illustrated in Figure (11) below for large t. We conclude that $u(t,x)$ evolves into two stacked waves.

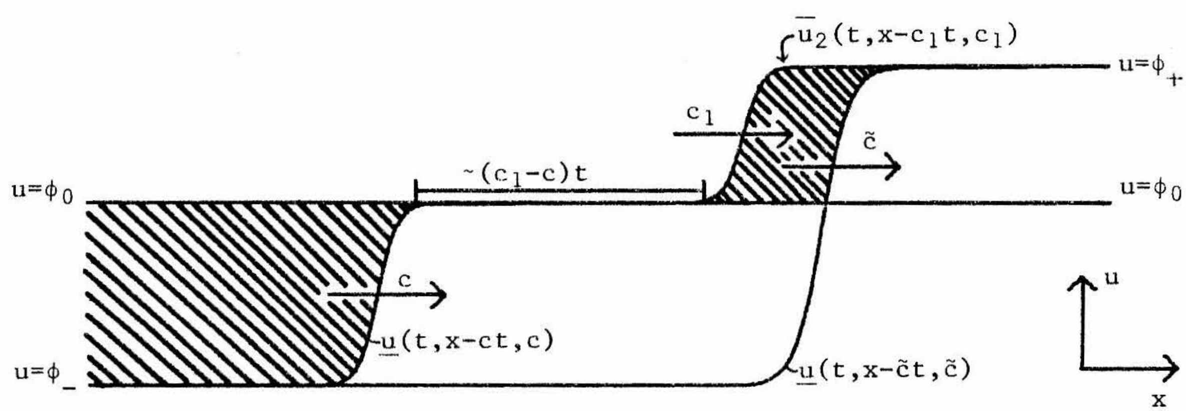


Figure (11): Since $\tilde{c}>c_1$ can be taken as close to c_1 as we like, since $c_1>c$, and since $u(t,x)$ must remain in the shaded region for all $t\geq 0$, we conclude that it evolves into two stacked waves.

This completes our presentation of the $N \to S$ case. We will not present the $S \to N$ case since it is very similar to the $N \to S$ case that we have discussed. (In fact, substitution of $-x$ for x will convert the $S \to N$ case into a $N \to S$ case). We therefore will continue in the next section by considering the $N \to N$ case.

5.3 Node-node case. In this section we treat the final case. Specifically, we assume that $u(t,x) = \phi(x,c_0)$ is a bounded monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{5.38}$$

at $c = c_0$. We also assume that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ are both first order nodes of the system

$$\begin{aligned} \phi_x &= v \\ f(v_x, v, \phi) + cv &= 0 \end{aligned} \tag{5.39}$$

at $c = c_0$. Finally, we also will assume $\phi(x,c_0)$ to be increasing in x since the analysis for the case of $\phi(x,c_0)$ decreasing proceeds similarly.

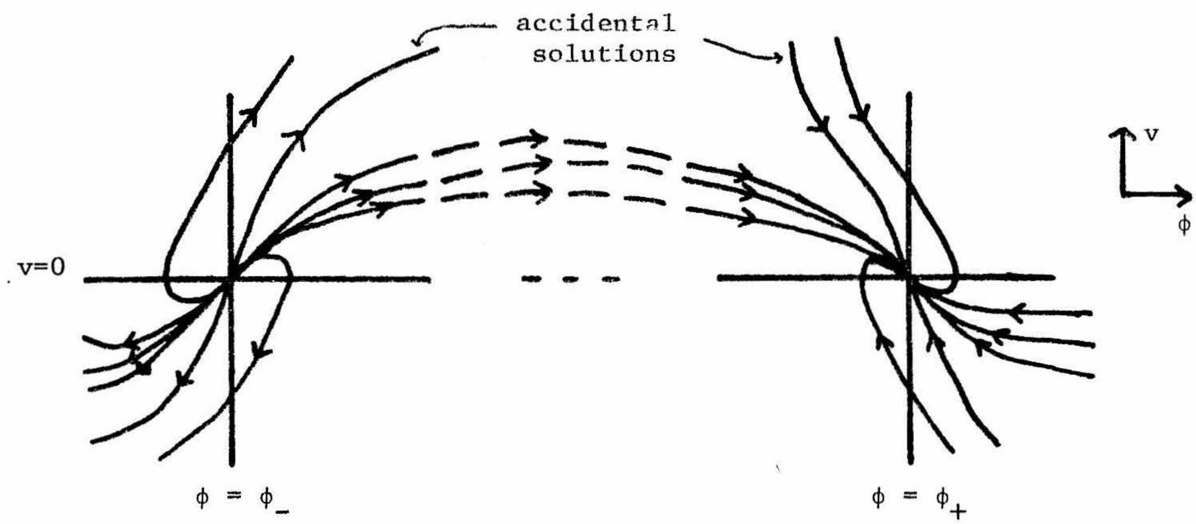
In this section we will first show that the existence of $\phi(x,c_0)$ implies the existence of a continuous family of similar solutions $\phi(x,c_0,v)$ at the same wavespeed c_0. By using the phase plane of system (5.39) we will be able to characterize this family of solution by finding its limiting members. Next, we will find that since <u>some</u> members of the family $\phi(x,c_0,v)$ decay to ϕ_- and ϕ_+ at the <u>usual</u> rate as $x \to -\infty$ and as $x \to +\infty$, then there must exist families of solutions $\phi(x,\tilde{c},v)$ similar to $\phi(x,c_0,v)$ at $c = \tilde{c}$ for all $\tilde{c}$ near c_0. We will then identify the slowest and the fastest wavespeeds $\tilde{c}$ at which $N \to N$ type monotonic traveling waves exist. We will summarize these existence results in theorem 5.1 $(N \to N)$. Finally, we will quote the mean wavespeed/initial condition results for this case.

The phase plane of system (5.39) looks something like the illustration below. Consider the solutions $\phi = \tilde{\phi}(x,c_0,v_0)$, $v = \tilde{v}(x,c_0,v_0)$ of system (5.39) at $c = c_0$ which are defined by the initial conditions

$$\begin{aligned} \tilde{\phi}(x_0,c_0,v_0) &= \phi(x_0,c_0) \\ \tilde{v}(x_0,c_0,v_0) &= v_0 \end{aligned}$$

for any fixed finite x_0. Since solutions of differential equations are continuous relative to initial conditions (see e.g. reference [6]), for any x we can make

$$|\tilde{\phi}(x_0+x_1,c_0,v_0) - \phi(x_0+x_1,c_0)| + |\tilde{v}(x_0+x_1,c_0,v_0) - \phi_x(x_0+x_1,c_0)|$$



as small as we wish by taking v_0 sufficiently near $\phi_x(x_0,c_0)$. Since we can take x_1 as large or small as we like, the attractive nature of the node at $\phi = \phi_-$, $v = 0$ (as $x \to -\infty$) and of the node at $\phi = \phi_+$, $v = 0$ (as $x \to +\infty$) guarantees that

$$\tilde{\phi}(x,c_0,v_0) \to \phi_- \text{ as } x \to -\infty$$

$$\tilde{\phi}(x,c_0,v_0) \to \phi_+ \text{ as } x \to +\infty$$

for v_0 in $[v_-,v_+]$ for some $v_- < \phi_x(x_0,c_0) < v_+$. Further, there is a $\tilde{v}_-$, $\tilde{v}_+$ ($\tilde{v}_- < \phi_x(x_0,c_0) \leq \tilde{v}_+$) such that $\tilde{\phi}(x,c_0,v_0)$ is monotone (as well as $\tilde{\phi}(-\infty,c_0,v_0) = \phi_-$ and $\tilde{\phi}(+\infty,c_0,v_0) = \phi_+$) for all v_0 in $(\tilde{v}_-,\tilde{v}_+]$.

These results are clear from the phase plane considerations illustrated in Figure (12) below. In particular for any $\tilde{\phi}_- > \phi_-$ near enough to ϕ_-, the phase plane directors point downward for all ϕ in $(\phi_-,\tilde{\phi}_-]$. Also horizontal components of the directors always point in the positive direction whenever $v > 0$. This means that any solution $\tilde{\phi}(x,c_0,v_0)$ which crosses the $\phi = \tilde{\phi}_-$ line at a positive point v which is no larger than the crossing point v of the accidental solution (i.e. the solution of (5.39) which decays to ϕ_- at the accidental rate as $x \to -\infty$), then

$\tilde{\phi}(x,c_0,v_0)$ must decrease monotonically to ϕ_- as x decreases to $-\infty$. Similarly there is a $\tilde{\phi}_+ < \phi_+$ such that if $\tilde{\phi}(x,c_0,v_0)$ crosses the $\phi = \tilde{\phi}_+$ line at a positive point v under the accidental solution (i.e. the solution of (5.39) which decays to ϕ_+ at the accidental rate as $x \to +\infty$), then $\tilde{\phi}(x_0,c_0,v_0)$ must increase monotonically to ϕ_+ as x increases to $+\infty$. Since $\tilde{v}(x,c_0,v_0)$ can be made arbitrarily near $\tilde{v}(x,c_0,v_0)$ over any finite interval by taking v_0 near to $\phi_x(x_0,c_0)$, $\tilde{\phi}(x,c_0,v_0)$ must be monotonic for at least a limited range of v_0 about $\phi_x(x_0,c_0)$. This is illustrated in Figure (12).

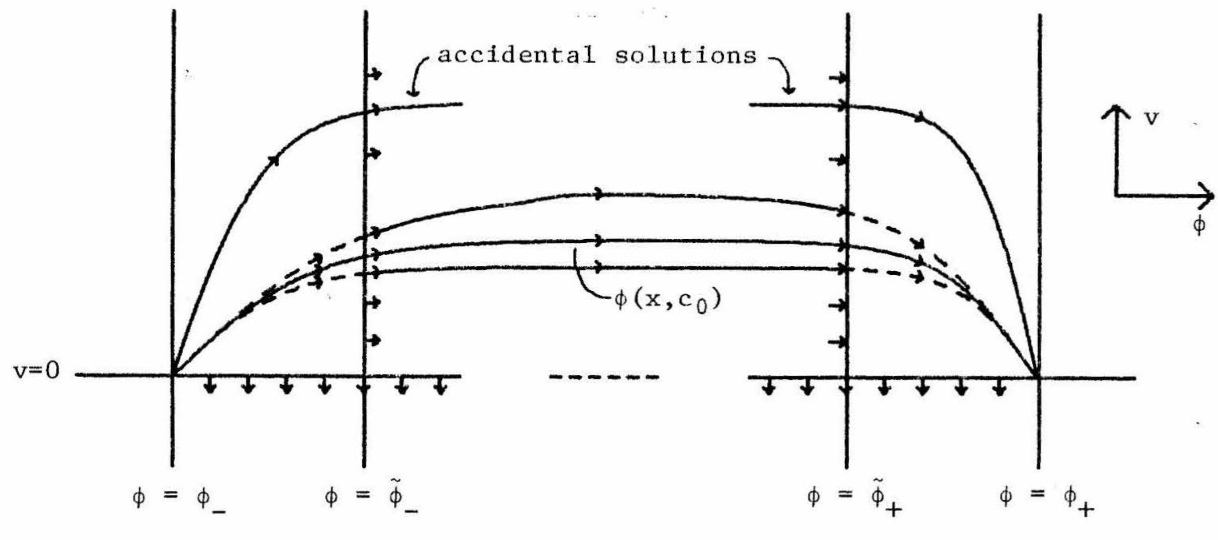


Figure (12)

From the phase plane we can easily find the extremal monotonic solution of (5.39) at $c = c_0$. From Figure (12) we see that the largest v_0 for which $\tilde{\phi}(x,c_0,v_0)$ is a monotonic solution is the least value of $v_0 = v_2$ for which $\tilde{\phi}(x,c_0,v_0)$ decays at the accidental rate as either

$x \to -\infty$ or $x \to +\infty$. For v_0 sightly larger, $\tilde{\phi}(x,c_0,v_0)$ is non-monotonic. Similarly, as v_0 decreases, $\tilde{\phi}(x,c_0,v_0)$ remains a monotonic solution until the value of v_0 (which we define to be v_1) for which the phase plane trajectory of $\tilde{\phi}(x,c_0,v_0)$ intersects the $v = 0$ curve between $\phi = \phi_-$ and $\phi = \phi_+$. Since $\tilde{v}(x,c_0,v_0) \geq 0$ for all x and since $\tilde{v}(x,c_0,v_0) = 0$ when $\tilde{\phi}(x,c_0,v_0) = \phi_0$ for some ϕ_0 in (ϕ_-,ϕ_+), then $\phi = \phi_0$, $v = 0$ must be a singular point. From the illustration in Figure (13), we see that $\phi = \phi_0$, $v = 0$ must be either a first order saddle point or a higher order singular point. Thus, when v_0 has decreased to v_1, $\tilde{\phi}(x,c_0,v_0)$ has bifurcated from a monotonic $N \to N$ solution into at least two monotonic solutions. Usually as v_0 decreases to v_1, $\tilde{\phi}(x,c_0,v_0)$ goes into two monotonic solutions: a $N \to S$ type solution $\phi_1(x,c_0)$ with $\phi_1(-\infty,c_0) = \phi_-$ and $\phi_1(+\infty,c_0) = \phi_0$, and a $S \to N$ type solution $\phi_2(x,c_0)$ with $\phi_2(-\infty,c_0) = \phi_0$ and $\phi_2(+\infty,c_0) = \phi_+$, where $\phi = \phi_0$, $v = 0$ is a first order saddle point. Thus the $N \to N$ type solutions almost always have a $N \to S$ and a $S \to N$ type solution as the limiting case, as is illustrated in Figure (13) below. The other possibilities are the possibility that $\phi = \phi_0$, $v = 0$ is a higher order singular point or the possibility that at $v_0 = v_1, \phi(x,c_0,v_0)$ bifurcates into more than two separate monotonic solutions. This latter case is illustrated in Figure (14). As illustrated, the intermediate singular points are saddle points or higher order singular points.

In brief, if $\phi(x,c_0)$ is a monotonic $N \to N$ type solution of (5.39) at $c = c_0$, then there is a continuous family of similar monotonic $N \to N$ type solutions. One limiting member of this family is a solution which decays at the accidental rate as $x \to -\infty$ or as $x \to +\infty$. The other

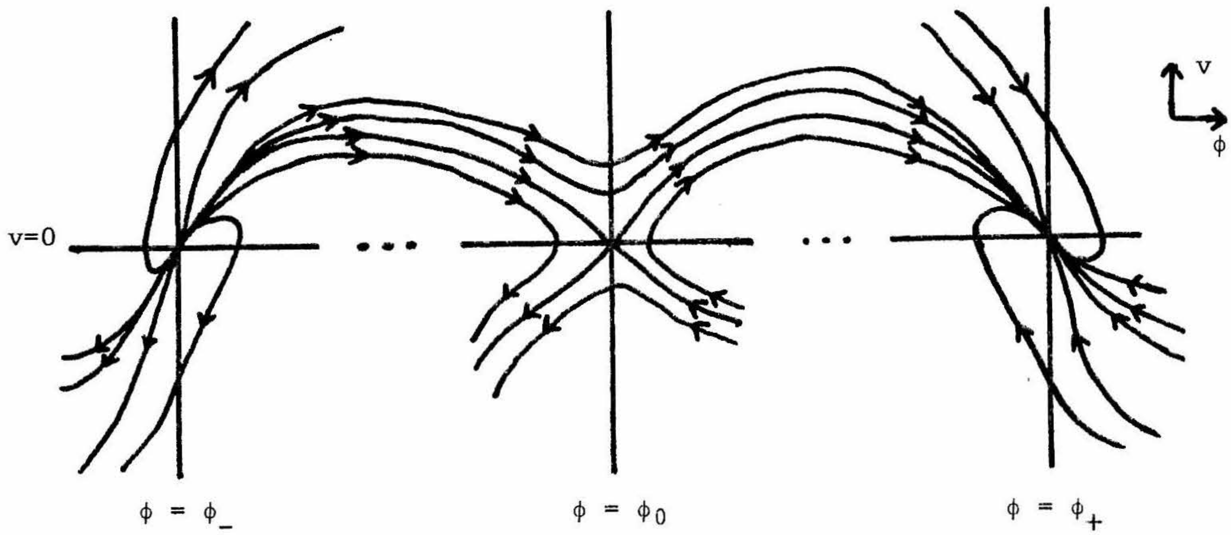


Figure (13)

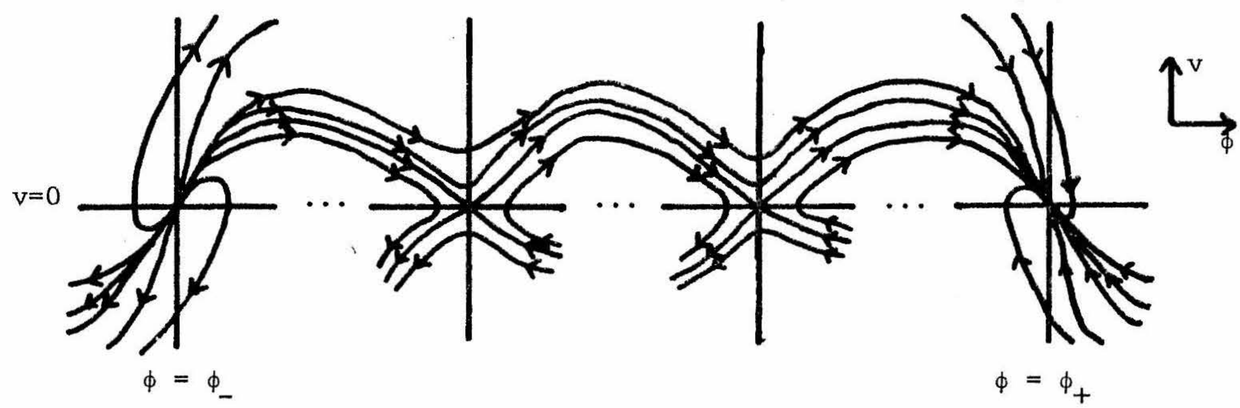


Figure (14)

limiting "member" is at least two monotonic solutions which are usually a $N \to S$ and a $S \to N$ pair of waves.

We now consider solutions at wave velocities $c = \tilde{c}$ near c_0. Similar to the $N \to S$ case, continuity arguments can be used to show that a monotonic solution $\phi(x,\tilde{c})$ exists with $\phi(-\infty,\tilde{c}) = \phi_-$ and $\phi(+\infty,\tilde{c}) = \phi_+$. Since one monotonic solution at $c = \tilde{c}$ exists, the previous arguments show that a family of solutions exists at $c = \tilde{c}$. One limiting member decays at the accidental rate as $x \to -\infty$ or $x \to +\infty$, and the other "member" is at least two separate solutions.

This characterization of the solution family at fixed values of c determines the smallest and largest values of c for which monotonic solutions $\phi(x,c)$ (with $\phi(-\infty,c) = \phi_-$ and with $\phi(+\infty,c) = \phi_+$) exist. As c increases (or decreases) from c_0, monotonic solutions continue to exist until either

(1) an accidentally decaying solution from $\phi = \phi_-$, $v = 0$ or $\phi = \phi_+$, $v = 0$ intersects the $v = 0$ axis at a singular point $\phi = \phi_0$, $v = 0$ with $\phi_- < \phi_0 < \phi_+$, or

(2) $\phi = \phi_-$, $v = 0$ or $\phi = \phi_+$, $v = 0$ changes from a node to a spiral point.

We summarize this discussion in the theorem below.

Theorem 5.1 ($N \to N$): Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $u(t,x) = \phi(x,c_0)$ is a bounded monotonic steady state solution of

$$u_t = f(u_{xx},u_x,u) + cu_x \tag{5.38}$$

at $c = c_0$, and also suppose that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ are both first order nodes of

$$\begin{aligned} \phi_x &= v \\ f(v_x,v,\phi) + cv &= 0 \end{aligned} \tag{5.39}$$

at $c = c_0$. Then there is an interval (c_1,c_2) such that for any $\tilde{c}$ in (c_1,c_2) there exists a continuously differentiable (in c and α) family of monotonic solutions $u(t,x) = \tilde{\phi}(x,\tilde{c},\alpha)$ $0 < \alpha \leq 1$ of (5.38) at $c = \tilde{c}$. For $0 < \alpha \leq 1$, $\phi(-\infty,\tilde{c},\alpha) = \phi_-$ and $\tilde{\phi}(+\infty,\tilde{c},\alpha) = \phi_+$ for all $\tilde{c}$ in

(c_1,c_2). Moreover, if $\phi(x,c_0)$ is increasing (decreasing) in x then the phase plane trajectories are increasing (decreasing) in α. At $\alpha = 1$, $\phi(x,\tilde{c},\alpha)$ decays at the accidental rate as $x \to -\infty$ or as $x \to +\infty$. For $0 < \alpha < 1$, $\phi(x,\tilde{c},\alpha)$ decays at the usual rate as $x \to -\infty$ and as $x \to +\infty$. At $\alpha = 0$, the phase plane trajectory corresponds to at least two distinct monotonic steady state solutions of (5.38) at $c = \tilde{c}$. Finally the limiting wavespeeds c_1 and c_2 are either

$$c_1 = c_{min} \equiv 2\sqrt{f_1(0,0,\phi_+)f_3(0,0,\phi_+)} - f_2(0,0,\phi_+)$$

$$c_2 = c_{max} \equiv -2\sqrt{f_1(0,0,\phi_-)f_3(0,0,\phi_-)} - f_2(0,0,\phi_-)$$

or (when they exist) the points c_1 in (c_{min},c_0) and c_2 in (c_0,c_{max}) nearest to c_0 for which the trajectory of an accidentally decaying solution from $\phi = \phi_-$, $v = 0$ or from $\phi = \phi_+$, $v = 0$ intersects the $v = 0$ line at a singular point $\phi = \phi_0$, $v = 0$ with $\phi_- < \phi_0 < \phi_+$.

The above theorem summarizes the discussion preceding it. Roughly speaking, it shows that if a single monotonic $N \to N$ type traveling wave $u(t,x) = \phi(x-c_0t,c_0)$ exists for some wavespeed c_0, then for each wavespeed $\tilde{c}$ near enough to c_0 there is a family of similar monotonic $N \to N$ type traveling waves $u(t,x) = \phi(x-\tilde{c}t,\tilde{c},\alpha)$. Note that in the theorem we have used a different parametrization of the family of solutions than was used in the preceding discussion.

We now present the mean wavespeed/initial condition results for this case. Since the proof of this next theorem is very similar to that of theorem (5.2) $(N \to S)$ and contains no new ideas, we will not present the proof.

<u>Theorem 5.2 $(N \to N)$:</u> Assume that hypotheses H2, H3, and H4 are satisfied.

Suppose that $u(t,x) = \phi(x-c_0t,c_0)$ is a bounded monotonic solution of

$$u_t = f(u_{xx},u_x,u) \qquad f_1 > 0 \tag{5.40}$$

and that $\phi = \phi(-\infty,c_0) \equiv \phi_-$, $v = 0$ and $\phi = \phi(+\infty,c_0) \equiv \phi_+$, $v = 0$ are both first order nodes of system (5.39) at $c = c_0$.

Define the exponential rate constants $\lambda^-(c)$ and $\lambda^+(c)$ by

$$\lambda^-(c) \equiv \frac{-(f_2(0,0,\phi_-)+c)-\sqrt{(f_2(0,0,\phi_-)+c)^2-4f_1(0,0,\phi_-)f_3(0,0,\phi_-)}}{2f_1(0,0,\phi_-)}$$

$$\lambda^+(c) \equiv \frac{-(f_2(0,0,\phi_+)+c)+\sqrt{(f_2(0,0,\phi_+)+c)^2-4f_1(0,0,\phi_+)f_3(0,0,\phi_+)}}{2f_1(0,0,\phi_+)}$$

for all c in $[c_{min},c_{max}]$, and define c_1 and c_2 as in the previous theorem.

Suppose that $u(t,x)$ is any solution of (5.40) whose initial condition $u(0,x)$ is in H_x^2 and satisfies

$$\min\{\phi_-,\phi_+\} < u(0,x) < \max\{\phi_-,\phi_+\} \quad \text{for all } x \ .$$

Then

(1) if for any c in (c_1,c_2) there is an $\alpha > 0$ and a $\beta > 0$ such that

$$\alpha < e^{-\lambda^-(c)x}|u(0,x)-\phi_-| \quad \text{for all } x \le 0 \text{ and}$$
$$\beta > e^{-\lambda^+(c)x}|u(0,x)-\phi_+| \quad \text{for all } x \ge 0$$

then $u(t,x)$ cannot travel with mean wavespeed larger than c;

(2) if for any c in (c_1,c_2) there is an $\alpha > 0$ and a $\beta > 0$ such that

$$\alpha > e^{-\lambda^-(c)x}|u(0,x)-\phi_-| \quad \text{for all } x \le 0 \text{ and}$$
$$\beta < e^{-\lambda^+(c)x}|u(0,x)-\phi_+| \quad \text{for all } x \ge 0$$

then $u(t,x)$ cannot travel with mean wavespeed smaller than c;

(3) if for any c in (c_1,c_2) there are positive constants $\alpha,\beta,\gamma,$ δ such that

$$\alpha < e^{-\lambda^-(c)x}|u(0,x)-\phi_-| < \beta \quad \text{for all} \quad x \le 0 \quad \text{and}$$

$$\gamma < e^{-\lambda^+(c)x}|u(0,x)-\phi_+| < \delta \quad \text{for all} \quad x \ge 0$$

then $u(t,x)$ travels with mean wavespeed c and has finite dispersion, and

(4) if for any c in (c_1,c_2) we have

$$\lim_{x\to-\infty} e^{-(\lambda^-(c)-\mu)x}|u(0,x)-\phi_-| = 0\ , \quad \lim_{x\to-\infty} e^{-(\lambda^-(c)+\mu)x}|u(0,x)-\phi_-| = +\infty,$$

$$\lim_{x\to+\infty} e^{-(\lambda^+(c)-\mu)x}|u(0,x)-\phi_+| = +\infty, \quad \lim_{x\to+\infty} e^{-(\lambda^+(c)+\mu)x}|u(0,x)-\phi_+| = 0$$

for all $\mu > 0$, then $u(t,x)$ travels with mean wavespeed c (but may not have finite dispersion).

Roughly speaking, the above theorem shows that if $u(0,x)$ decays to ϕ_- like $\alpha e^{\lambda^-(c)x}$ and to ϕ_+ like $\beta e^{\lambda^+(c)x}$ for any c in (c_1,c_2), then $u(t,x)$ must travel with mean wavespeed c. One naturally wonders how solutions $u(t,x)$ of (5.40) behave when $u(0,x)$ decays to ϕ_- like $\alpha e^{\lambda^-(c_-)x}$ as $x \to -\infty$ and to ϕ_+ like $\beta e^{\lambda^+(c_+)x}$ as $x \to +\infty$ if $c_- \neq c_+$. This question is easily answered when $c_1 < c_- < c_+ < c_2$. We will now show that typically $u(t,x)$ will evolve into a $N \to S$ type traveling wave of speed c_- (which goes from ϕ_- at $x = -\infty$ to ϕ_0 at $x = +\infty$) and into a $S \to N$ type traveling wave of speed c_+ (which goes from ϕ_0 at $x = -\infty$ to ϕ_+ at $x = +\infty$), where $\phi = \phi_0$, $v = 0$ is a saddle point of system (5.39).

We consider only the typical case, where the phase plane trajectories of $\Phi(x,c_-,0)$ and of $\Phi(x,c_+,0)$ both intersect the $v = 0$ at the same single first order saddle point $\phi = \phi_0$, $v = 0$ with $\phi_- < \phi_0 < \phi_+$, as illustrated in Figure (13). Consider the solutions $\phi(x,c_-) \equiv \Phi(x,c_-,\alpha)$ and $\phi(x,c_+) = \Phi(x,c_+,\alpha_+)$ for any α_- and α_+ in $(0,1)$. Further, let $\phi^-_{NS}(x,c_-)$ be the monotonic $N \to S$ solution at $c = c_-$ and let $\phi^+_{SN}(x,c_+)$ be the monotonic $S \to N$ solution at $c = c_+$. Note that

$$\phi^-_{NS}(-\infty,c_-) = \phi_- \qquad \phi^-_{NS}(+\infty,c_-) = \phi_0 \quad ,$$

$$\phi^+_{SN}(-\infty,c_+) = \phi_0 \qquad \phi^+_{SN}(-\infty,c_+) = \phi_+ \quad ,$$

$$\phi(-\infty,c_-) = \phi_- \qquad \phi(+\infty,c_-) = \phi_+ \quad ,$$

$$\phi(-\infty,c_+) = \phi_- \qquad \phi(+\infty,c_+) = \phi_+ \quad ,$$

and that ϕ^-_{NS} and ϕ^+_{SN} correspond to portions of the limiting trajectories of $\Phi(x,c_-,\alpha)$ and $\Phi(x,c_+,\alpha)$ at $\alpha = 0$. Suppose now that $u(t,x)$ is any solution of (5.40) whose initial condition $u(0,x)$ is in H^2_x and satisfies

$$\phi_- < u(0,x) < \phi_+ \qquad \text{for all } x \quad ,$$

$$\alpha_1 < e^{-\lambda^-(c_-)x}|u(0,x)-\phi_-| < \alpha_2 \quad \text{for all } x \le 0, \text{ and}$$

$$\beta_1 < e^{-\lambda^+(c_+)x}|u(0,x)-\phi_+| < \beta_2 \quad \text{for all } x \ge 0 \quad ,$$

for some positive constants α_1, α_2, β_1, and β_2. Suppose also that $c_1 < c_- < c_+ < c_2$. By selecting h_1, h_2, h_3, and h_4 sufficiently large, we can guarantee that

$$\phi^-_{NS}(x-h_1,c_-) \le u(0,x) \le \phi^+_{SN}(x+h_2,c_+) \quad \text{for all } x, \text{ and}$$

$$\phi(x-h_3,c_+) \le u(0,x) \le \phi(x+h_4,c_-) \quad \text{for all } x.$$

Thus the maximum principle implies that

$$\phi^-_{NS}(x-c_-t-h_1,c_-) \le u(t,x) \le \phi^+_{SN}(x-c_+t+h_2,c_+) \quad \text{for all } x, \text{ all } t \ge 0 \tag{5.41}$$

$$\phi(x-c_+t-h_3,c_+) \le u(t,x) \le \phi(x-c_-t-h_4,c_-) \quad \text{for all } x, \text{ all } t \ge 0 \quad .$$

The bounds on $u(t,x)$ given by relations (5.41) are illustrated in Figure (15) below for large t. The implication of the maximum principle is that $u(t,x)$ must remain in the shaded area for all $t \ge 0$. Clearly for large

t, $u(t,x)$ has evolved into two stacked waves, one moving with speed c_- and one moving with speed c_+.

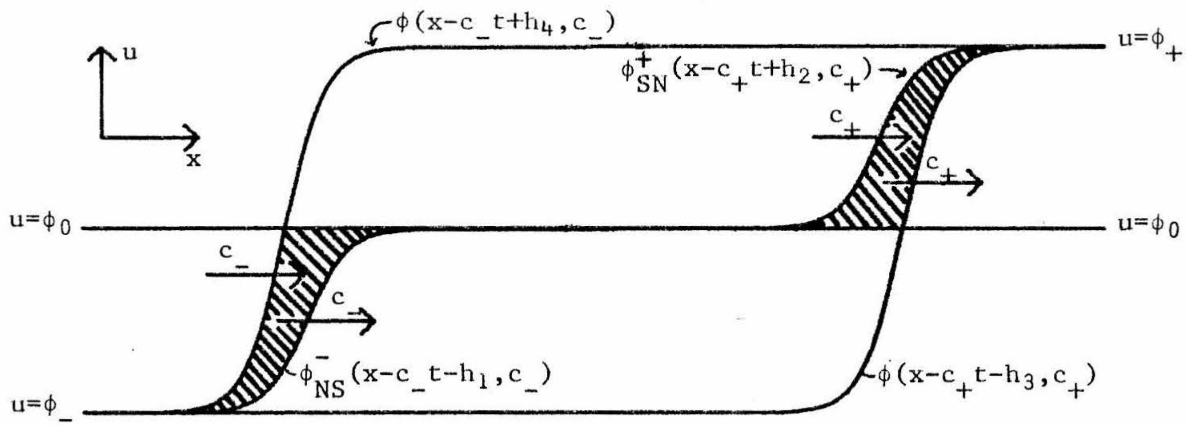


Figure (15)

This completes our presentation of the mean wavespeed/initial conditions for this case, and thus for all cases. As a rough summary we found that in the $S \to S$ case there is a single traveling wave $\phi(x-c_0t,c_0)$, in the $N \to S$ and $S \to N$ cases there is a single traveling wave $\phi(x-ct,c)$ at each wavespeed c in a range of wavespeeds, and in the $N \to N$ case there is a family of solutions $\phi(x-ct,c,\alpha)$ at each speed c in a range of wavespeeds. For each of these cases, the mean wavespeed of any solution $u(t,x)$ is determined mainly by the asymptotic decay rate of $u(0,x)$ as $x \to -\infty$ (if $\phi(-\infty)$ is a node) and the asymptotic decay rate of $u(0,x)$ as $x \to +\infty$ (if $\phi(+\infty)$ is a node).

In the remaining sections of this chapter, we briefly discuss topics related to the mean wavespeed/initial condition results. In the next section, section (5.4), we use the wavespeed results to show the sharpness of the stability results contained in theorem (4.5). In section (5.5) we discuss how the mean wavespeed/initial condition results can be extended to include traveling monotonic plane waves in higher spatial dimensions. Finally, we conclude this chapter with some closing remarks in section (5.6).

5.4 Sharpness of the stability results for monotonic waves. In theorem (4.5) we obtained our major stability results for bounded monotonic traveling wave (or steady state) solutions of

$$u_t = f(u_{xx}, u_x, u) \quad . \tag{5.40}$$

Roughly speaking, that theorem shows that any bounded monotonic (in x) solution $u(t,x) = \phi(x-c_0t, c_0)$ is stable to smooth initial perturbations $p(x) = u_p(0,x) - \phi(x,c_0)$ which are small and

(1) bounded as $x \to -\infty$ (as $x \to +\infty$) if $\phi = \phi(-\infty, c_0)$, $v = 0$ (if $\phi = \phi(+\infty, c_0)$, $v = 0$) is a first order saddle point, and

(2) decay asymptotically no slower than the same exponential rate that $\phi_x(x,c_0)$ decays at as $x \to -\infty$ (as $x \to +\infty$) if $\phi = \phi(-\infty, c_0)$, $v = 0$ (if $\phi = \phi(+\infty, c_0)$, $v = 0$) is a first order node.

We will use the existence results of theorems 5.1 $(N \to S)$ and theorem 5.1 $(N \to N)$ to show that these stability results are sharp in most cases.

Suppose that we start with a bounded monotonic traveling wave solution $u(t,x) = \phi(x-c_0t, c_0)$ of (5.40), and suppose that we can use either theorem 5.1 $(N \to S)$, the equivalent result for the $(S \to N)$ case, or theorem 5.1 $(N \to N)$, to show that for any $\tilde{c}$ near enough c_0 there is a traveling wave solution $u(t,x) = \phi(x-\tilde{c}t, \tilde{c})$ with wavespeed $\tilde{c}$. We note that these theorems show that $\phi(x,c)$ is differentiable in c, and so the perturbation

$$p(x,\tilde{c}) = \phi(x,\tilde{c}) - \phi(x,c_0)$$

can be made as small as we wish by taking $\tilde{c}$ near enough to c_0. Moreover, each $\phi(x,\tilde{c})$ decays to $\phi(-\infty,c_0)$ and to $\phi(+\infty,c_0)$ exponentially, and with exponential rate constants which depend continuously on $\tilde{c}$. Thus for $\tilde{c}$ near c_0, $p(x,c)$ decays at an asymptotic rate which is slightly slower than that allowed by theorem (4.5). Finally, if the initial conditions of $u(t,x) = \phi(x-c_0t,c_0)$ are perturbed by $p(x,\tilde{c})$ then the resulting perturbed solution is $u_p(t,x) = \phi(x-\tilde{c}t,\tilde{c})$. The difference in velocities of $\phi(x-\tilde{c}t,\tilde{c})$ and $\phi(x-c_0t,c_0)$ imply that $u_p(t,x)$ drifts away from $u(t,x)$ linearly in time, and so $u(t,x) = \phi(x-c_0t,c_0)$ is unstable relative to the initial perturbations $p(x,\tilde{c})$.

In summary, whenever $u(t,x) = \phi(x-c_0t,c_0)$ is such that we can use our theorems to show the existence of similar traveling waves $u(t,x) = \phi(x-\tilde{c}t,\tilde{c})$ for all $\tilde{c}$ near c_0, then the initial perturbation $p(x,\tilde{c}) \equiv \phi(x,\tilde{c}) - \phi(x,c_0)$ $(\tilde{c} \neq c_0)$

(1) can be made as small as we wish by taking $\tilde{c}$ near c_0,

(2) can be made to violate the asymptotic decay restrictions of theorem (4.5) on perturbations by as slight a margin as we wish,

(3) is an unstable perturbation.

Thus, for these cases theorem (4.5) is sharp.

For the following table, table 5.1, we have assumed that $u(t,x) = \phi(x-c_0t)$ is a bounded monotonic traveling wave or steady state solution of

$$u_t = f(u_{xx},u_x,u) \quad , \tag{5.40}$$

and that $\phi = \phi(-\infty)$, $v = 0$ and $\phi = \phi(+\infty)$, $v = 0$ are first order singular

Table (5.1)

Sharpness of the stability results of theorem (4.5) for any bounded monotonic solution $u(t,x) = \phi(x-c_0t)$.

Behavior of $\phi(x)$ for $x << -1$ and $x >> -1$				Sharpness of theorem (4.5)
type singular point at $-\infty$	asymptotic decay rate as $x \to -\infty$	type singular point at $+\infty$	asymptotic decay rate as $x \to +\infty$	sharpness
S	___	S	___	sharp
N	usual	S	___	sharp
N	accidental	S	___	?
S	___	N	usual	sharp
S	___	N	accidental	?
N	usual	N	usual	sharp
N	usual	N	accidental	sharp
N	accidental	N	usual	sharp
N	accidental	N	accidental	?

points. We have listed all possible cases, and denoted the ones for which the above arguments show that the stability results contained in theorem (4.5) are sharp.

Table 5.1 is the main result of this section. In particular we note that it shows that in all non-accidental cases where $\phi = \phi(-\infty)$, $v = 0$ and $\phi = \phi(+\infty)$, $v = 0$ are first order singular points theorem (4.5) is sharp. However, in some accidental cases where $\phi = \phi(-\infty)$, $v = 0$ and $\phi = \phi(+\infty)$, $v = 0$ are first order singular points and in all cases where at least one of $\phi = \phi(-\infty)$, $v = 0$ and $\phi = \phi(+\infty)$, $v = 0$ is not a first order singular point, the sharpness of theorem (4.5) remains open to question.

5.5 The mean wavespeed of plane waves. In this section we briefly discuss the direct extension of the results in this chapter to bounded monotonic traveling plane wave solutions when more than a single spatial dimension is present. Actually we will work only with two spatial variables ($\vec{x} \equiv (x,y)$) but it will be clear that our discussion applies equally well when more than two spatial variables are present.

Suppose that

$$u_t = f(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u) \tag{5.42}$$

is a parabolic equation (i.e., satisfies hypothesis H3) and that $u(t,\vec{x}) = \phi(\vec{x}-\vec{c}t)$ is a bounded monotonic traveling plane wave solution of (5.42). Since we can rotate the x-y axes without destroying the parabolicity of equation (5.42), let us take (without loss of generality) our monotonic plane waves to be

$$u(t,\vec{x}) \equiv u(t,x,y) = \phi(x-ct) \quad .$$

This plane wave is a solution of

$$u_t = f(u_{xx},0,0,u_x,0,u) \equiv \tilde{f}(u_{xx},u_x,u) \quad .$$

Clearly whenever $u(t,x)$ is a solution of

$$u_t = \tilde{f}(u_{xx},u_x,u) \tag{5.43}$$

then $u(t,\vec{x}) \equiv u(t,x,y) \equiv u(t,x)$ is a solution of (5.42). Moreover, if $\overline{u}(t,x)$ and $\underline{u}(t,x)$ are any upper and lower functions of (5.43) then $\overline{u}(t,x,y) \equiv \overline{u}(t,x)$ and $\underline{u}(t,x,y) \equiv \underline{u}(t,x)$ are also upper and lower functions of equation (5.42).

Thus whenever $u(t,x,y) \equiv \phi(x-ct)$ is a bounded monotonic (in x) plane wave solution of (5.42), all the existence and mean wavespeed results in this chapter about the equation

$$u_t = \tilde{f}(u_{xx},u_x,u) \tag{5.43}$$

apply equally well to the equation

$$u_t = f(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u) \quad . \qquad (5.42)$$

Note however that the mean wavespeed results can be regarded as being stronger for plane waves than for waves in one spatial dimension. This is because whenever we concluded "$u(t,x)$ travels with mean wavespeed c" in one spatial dimension, we can conclude that "$u(t,x,y)$ travels with mean wavespeed $\vec{c} \equiv (c,0)$" in two spatial dimensions. In conclusion, the mean wavespeed/initial condition results immediately generalize to multiple spatial dimensions.

5.6 Conclusions. In this chapter we established some results about the qualitative behavior of solutions of

$$u_t = f(u_{xx}, u_x, u) \qquad (5.44)$$

by utilizing the maximum principle, the upper and lower functions constructed in Chapter IV, and the phase plane for traveling wave solutions of (5.44). These results are not exhaustive (in the mathematical sense). Many similar results can be found by using the same techniques. For example, one can look for results about monotonic waves which have higher order singular points at either $x = -\infty$ or $x = +\infty$. However, without analyzing specific physical examples we cannot be sure of the utility of these extensions. Therefore, we will not puruse any of the extensions to the results presented here.

In Chapter IV we used the maximum prinicple to establish theorems which would allow us to determine the stability or instability of any traveling wave (or steady state) solution of

$$u_t = f(u_{xx}, u_x, u) \qquad f_1 > 0$$

by inspection. In this chapter we used the maximum principle to establish connections between the initial condition $u(0,x)$ and the mean wavespeed of the resulting solution $u(t,x)$ of equation (5.44). These two chapters complete our development of the general theory of equations like (5.44). In the next two chapters we will directly extend many of our results to the other types of equations for which the maximum principle holds.

Chapter VI

EXTENSION TO NON-LOCAL OPERATORS

In this chapter we extend some of the results of Chapters IV and V to parabolic equations which contain integrals. We will consider only direct extensions, and we will be as brief as possible. Basically we will find that the stability results for monotonic waves and the mean wavespeed/initial condition results are still valid. However, we will not be able to prove the instability of non-monotonic waves.

To be more specific, in this chapter we will treat equations of the form

$$u_t = F(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} G(s, y, u(t-s, x-y))dyds) \qquad (6.1')$$

where $T > 0$ and $Y > 0$ are fixed constants. Throughout this chapter we will assume that hypotheses H1 and H2 (smoothness of equation (6.1')), H3 (parabolicity of equation (6.1'), and H4 (existence of solutions of the initial value problem) are satisfied. We also assume that a very large $M > 0$ has been chosen, and we work with the resulting equation

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s, y, u(t-s, x-y))dyds) \quad , \qquad (6.1)$$

where $f \equiv F_M$ and $g \equiv G_M$.

Briefly, in the first section of this chapter we will derive the stability results for monotonic waves. In section (6.2) we will discuss the instability results for non-monotonic waves. We will derive the mean wavespeed/initial condition results in section (6.3). Finally, we will use the last section, section (6.4), to express some general remarks. We now start with the stability of monotonic waves.

<u>6.1 Stability of monotonic traveling waves.</u> In this section we extend

the stability results to monotonic traveling wave (and steady state) solutions of equation (6.1). To do this we need to first redefine our stability concepts and the concepts of "nodes" and "saddle points" appropriately. We begin with our stability concepts.

Let $w(x)$ be any continuous function with $w(x) \geq 1$ for all x. Then any steady state solution $u(t,x) \equiv \phi(x)$ of the equation

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y+cs))dyds) + cu_x \quad (6.2)$$

is defined to be <u>C^w-stable</u> if and only if given any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that every solution $u(t,x)$ of (6.2) satisfies

$$|\{u(t,x) - \phi(x)\}w(x)| \leq \varepsilon \text{ for all } x \text{ and all } t > 0 \quad (6.3)$$

whenever the initial conditions $u(t,x)$ $(t \leq 0)$ satisfy

(i) $u(t,x)$ is bounded and uniformly Hoelder continuous (with some exponent $\alpha > 0$) in t and x for $t \leq 0$,

(ii) $u(0,x)$ is in H_x^2, and

(iii) $|\{u(t,x) - \phi(x)\}w(x)| \leq \delta(\varepsilon)$ for all x, all $t \leq 0$.

Similarly, $\phi(x)$ is defined to be <u>$\mathfrak{C}^w$-stable</u> if and only if given any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that every solution $u(t,x)$ of (6.2) satisfies

$$|u(t,x) - \phi(x)| \leq \varepsilon \text{ for all } x \text{ and all } t > 0 \quad (6.4)$$

whenever the initial conditions $u(t,x)$ $(t \leq 0)$ satisfy (i), (ii), and (iii). A solution $u(t,x) \equiv \phi(x)$ which is not C^w-stable will be called <u>C^w-unstable,</u> and if it is not $\mathfrak{C}^w$-stable it will be called <u>$\mathfrak{C}^w$-unstable</u>.

These stability definitions are very similar to those used previously. The only major change is that the requirements on the perturbed initial condition $u(t,x) - \phi(x)$ must now be satisfied for $t < 0$ as well as for $t = 0$.

We now define the concepts of "singular point", "node", and "saddle point" appropriately. Note that the steady state equation of (6.2),

$$f(\phi_{xx},\phi_x,\phi,\int_0^T\int_{|y|<Y} g(s,y,\phi(x-y+cs))dyds) + c\phi_x = 0 \quad ,(6.5)$$

has no phase plane representation. Therefore, our definitions of "singular point", "node", and "saddle point" will not refer to any phase plane representation. Instead we will define these terms so that the results in this chapter are analogous to those in the previous chapters. Thus, we define ϕ_0 to be a <u>singular point</u> if and only if

$$f(0,0,\phi_0,\int_0^T\int_{|y|<Y} g(s,y,\phi_0)dyds) = 0 \quad , \qquad (6.6)$$

and to be a <u>regular singular point of order one</u> if and only if

$$f(0,0,\phi_0+\eta,\int_0^T\int_{|y|<Y} g(s,y,\phi_0+\eta)dyds) = \mu\eta + O(\eta^{1+\Delta}) \text{ as } \eta\to 0 \quad (6.7)$$

where $\mu \neq 0$ and Δ is some positive constant. If $\mu > 0$ then we define ϕ_0 to be a <u>first order node</u>, and if $\mu < 0$ we define ϕ_0 to be a <u>first order saddle point</u>. For simplicity we will treat only first order singular points in most of this chapter.

With the above definitions, the extension of the stability results contained in sections (4.1) through (4.11) is now very easy. We start with the stability of constant steady state solutions $u(t,x) \equiv \phi_0$ of (6.2). This result is:

<u>Theorem 6.1:</u> Assume that hypotheses H1, H2, H3, and H4 are satisfied. Suppose further that $u(t,x) \equiv \phi_0$ is a constant steady state solution of

$$u_t = f(u_{xx},u_x,u,\int_0^T\int_{|y|<Y} g(s,y,u(t-s,x-y+cs))dyds) + cu_x \quad . \ (6.2)$$

Then

(1) if ϕ_0 is a first order saddle point then $u(t,x) \equiv \phi_0$ is C^w-stable with $w(x) \equiv 1$, and

(2) if ϕ_0 is a first order node then $u(t,x) \equiv \phi_0$ is $\mathcal{C}^w$-unstable with $w(x) = 1 + e^{+\kappa x} + e^{-\kappa x}$ for any $\kappa > 0$ sufficiently small.

Proof: Very similar to the proofs of parts (3) and (4) of theorem (4.1).

Thus, again saddle points are stable constant solutions and nodes are unstable constant solutions.

If we were to closely follow the treatment in Chapter IV, we would now find the asymptotic behavior of the steady state solutions of equation (6.2). We will not do this. Instead all asymptotic behavior results that we need will be assumed. However, verification of these asymptotic assumptions will be very easy for any specific solution of any specific equation. We now continue to the stability results for monotonic waves. We begin with the basic stability result.

Theorem 6.2: Assume that hypotheses H1, H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x)$ is a bounded strictly monotonic steady state solution of equation (6.2), that $\phi''(x)/\phi'(x)$ is bounded, and that $|\phi''(x)|$ is decreasing for all x sufficiently large and for all x sufficiently small. Then $u(t,x) \equiv \phi(x)$ is C^w-stable with

$$w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$$

Proof: The proof is very similar to that of theorem (4.2). Since $\phi(x-h)$ and $\phi(x+h)$ are both solutions of equation (6.2) for any h, the maximum principle implies that whenever

$$\phi(x-h) \le u(t,x) \le \phi(x+h) \quad \text{for all } x \tag{6.8}$$

is satisfied for $t \le 0$, then it remains satisfied for all $t \ge 0$. Thus $u(t,x) \equiv \phi(x)$ possesses a class of perturbations for which it is stable.

With the assumptions of the theorem, we can identify the stability class as including the class allowed by the definition of C^w-stability with $w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$. Thus the theorem is established.

We see that strictly monotonic steady state solutions of equation (6.2) have at least a limited stability, exactly as occurred in Chapter IV. Note that regardless of whether $\phi''(x)/\phi'(x)$ is bounded, whether $\phi'(x) \neq 0$ for all x, or whether $|\phi''(x)|$ is decreasing for all x sufficiently large and sufficiently small, the maximum principle implies that if

$$\phi(x-h) \leq u(t,x) \leq \phi(x+h) \quad \text{for all} \quad x \tag{6.8}$$

is satisfied for $t \leq 0$ then it remains satisfied for all $t \geq 0$ as well. Thus <u>every</u> monotonic steady state solution of equation (6.2) possesses a limited stability. However, whenever one of the hypotheses of theorem is violated we cannot identify the stability as C^w-stability with $w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$. Instead $u(t,x) \equiv \phi(x)$ would be C^w-stable with some other $w(x)$. Note that for any <u>particular</u> steady state solution $u(t,x) \equiv \phi(x)$ of (6.2) it is very easy to identify the precise stability implied by relation (6.8).

As in Chapter IV, we now improve our basic stability results by constructing better upper and lower functions. Indeed, the estimates used in constructing the upper and lower functions contained in lemmas (4.3) and (4.4) are valid in the present situation. Therefore the following two lemmas can be established by proofs very similar to the ones used to prove lemmas (4.3) and (4.4).

<u>Lemma 6.3:</u> Assume that hypotheses H1, H2, H3, and H4 are satisfied.

Suppose that $u(t,x) \equiv \phi(x)$ is a bounded strictly monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y+cs))dyds) + cu_x. \quad (6.2)$$

Suppose further that $\phi''(x)/\phi'(x)$ is bounded for all x, that $|\phi''(x)|$ is decreasing for all x sufficiently large and for all x sufficiently small, that $[\phi(x)-\phi(+\infty)]/\phi'(x)$ is bounded for all $x \geq 0$, and that $[\phi(x)-\phi(-\infty)]/\phi'(x)$ is bounded for all $x \leq 0$. Define $\phi(+\infty) \equiv \phi_+$ and $\phi(-\infty) \equiv \phi_-$. Then

(1) if ϕ_+ is a first order saddle point then

$$\overline{u}(t,x) \equiv \phi(x+h(t)) + q(t)\cdot[\phi(x+h(t))- \phi_-] \text{ and} \quad (6.9a)$$

$$\underline{u}(t,x) \equiv \phi(x-h(t)) - q(t)\cdot[\phi(x-h(t))- \phi_-] \quad (6.9b)$$

are upper and lower functions (respectively) of equation (6.2). Here,

$$h(t) \equiv \alpha\kappa(1-e^{-st}) + h_0 \qquad q(t) \equiv \alpha e^{-st} \quad (6.10)$$

where s and κ are particular positive constants, h_0 is arbitrary, and α is any constant with the same sign as $\phi'(x)$ and with sufficiently small magnitude.

(2) if ϕ_- is a first order saddle point then

$$\overline{u}(t,x) \equiv \phi(x+h(t)) + q(t)[\phi_+-\phi(x+h(t))] \text{ and} \quad (6.11a)$$

$$\underline{u}(t,x) \equiv \phi(x-h(t)) - q(t)[\phi_+-\phi(x-h(t))] \quad (6.11b)$$

are upper and lower functions (respectively) of equation (6.2). Here h(t) and q(t) are defined as in the preceding case.

Lemma 6.4: Assume that hypotheses H1, H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x)$ is a bounded strictly monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y+cs))dyds) + cu_x. \quad (6.2)$$

Suppose further that $\phi''(x)/\phi'(x)$ is bounded for all x, that $|\phi''(x)|$ is decreasing for all x sufficiently large and for all x sufficiently small, that $[\phi(x)-\phi(+\infty)]/\phi'(x)$ is bounded for all $x \geq 0$, and that $[\phi(x)-\phi(-\infty)]/\phi'(x)$ is bounded for all $x \leq 0$. If $\phi(-\infty)$ and $\phi(+\infty)$ are both first order saddle points then

$$\overline{u}(t,x) \equiv \phi(x+h(t)) + |q(t)| \text{ and} \tag{6.12a}$$

$$\underline{u}(t,x) \equiv \phi(x-h(t)) - |q(t)| \tag{6.12b}$$

are upper and lower functions (respectively) of equation (6.2). Here,

$$h(t) \equiv \alpha\kappa(1-e^{-st}) + h_0 \qquad q(t) \equiv \alpha e^{-st} \tag{6.10}$$

where s and κ are particular positive constants, h_0 is arbitrary, and α is any constant with the same sign as $\phi'(x)$ and with sufficiently small magnitude.

Basically lemmas (6.3) and (6.4) show that equation (6.2) possesses upper and lower functions which are very similar to the ones used extensively in Chapters IV and V. These upper and lower functions of equation (6.2) look very much like the upper and lower functions of equation (4.2), which are sketched in Figures (1), (2), and (3) of Chapter IV.

In lemmas (6.3) and (6.4) we assumed that the strictly monotonic steady state $u(t,x) = \phi(x)$ satisfies the conditions

(a) $\phi''(x)/\phi'(x)$ is uniformly bounded,

(b) $|\phi''(x)|$ is decreasing for all x sufficiently large and all x sufficiently small,

(c) $[\phi(x)-\phi(+\infty)]/\phi'(x)$ is bounded for all $x \geq 0$, and

(d) $[\phi(x)-\phi(-\infty)]/\phi'(x)$ is bounded for all $x \leq 0$.

Note that whenever $u(t,x) \equiv \phi(x)$ is a bounded strictly monotonic steady state solution of equation (6.2) which has

(i) $\phi(x)-\phi(+\infty)$, $\phi'(x)$, and $\phi''(x)$ all decaying asymptotically at the same exponential rate as $x \to +\infty$, and

(ii) $\phi(x)-\phi(-\infty)$, $\phi'(x)$, and $\phi''(x)$ all decaying asymptotically at the same exponential rate as $x \to -\infty$,

then conditions (a), (b), (c), and (d) are satisfied. Fortunately, exponential decay as $x \to +\infty$ and as $x \to -\infty$ is almost always the case, and so conditions (a), (b), (c), and (d) are not very restrictive. Moreover, whenever some of the conditions (a), (b), (c), and (d) are violated by a specific steady state, it should be possible to construct upper and lower functions similar to the ones given by lemma (6.3) or lemma (6.4).

We now use the upper and lower functions constructed in lemmas (6.3) and (6.4) in conjunction with the maximum principle. This immediately yields our final stability result for monotonic steady state solutions of equation (6.2). In order to state these results concisely, recall the definitions

$$r_+\{\phi'(x)\} \equiv \begin{Bmatrix} \phi'(x) & x \geq 0 \\ \phi'(0) & x \leq 0 \end{Bmatrix} , \quad r_-\{\phi'(x)\} \equiv \begin{Bmatrix} \phi'(0) & x \geq 0 \\ \phi'(x) & x \leq 0 \end{Bmatrix} .$$

With these definitions, our stability results are:

<u>Theorem 6.5 (The stability of monotone waves):</u> Assume that hypotheses H1, H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x)$ is a bounded strictly monotonic steady state solution of

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y+cs))dyds) + cu_x \quad . \quad (6.2)$$

Suppose further that $\phi''(x)/\phi'(x)$ is uniformly bounded, that $|\phi''(x)|$ is decreasing for all x sufficiently large and for all x sufficiently small, that $[\phi(x)-\phi(+\infty)]/\phi'(x)$ is bounded for all $x \geq 0$, and finally that $[\phi(x)-\phi(-\infty)]/\phi'(x)$ is bounded for all $x \leq 0$. Then $u(t,x) \equiv \phi(x)$ is C^w-stable where

(1) if both $\phi = \phi(-\infty)$ and $\phi = \phi(+\infty)$ are first order saddle points, then $w(x) \equiv 1$;

(2) if $\phi = \phi(-\infty)$ is a first order saddle point but $\phi = \phi(+\infty)$ is not a first order saddle point, then $w(x) \equiv 1 + \frac{1}{|r_+\{\phi'(x)\}|}$,

(3) if $\phi = \phi(+\infty)$ is a first order saddle point but $\phi = \phi(-\infty)$ is not a first order saddle point, then $w(x) \equiv 1 + \frac{1}{|r_-\{\phi'(x)\}|}$, and

(4) if neither $\phi = \phi(-\infty)$ nor $\phi = \phi(+\infty)$ is a first order saddle point, then $w(x) \equiv 1 + \frac{1}{|\phi'(x)|}$.

The conclusions of theorem (6.5) are exactly the same as the conclusions of theorem (4.5) in Chapter IV. The inclusion of integrals in equations (6.1) and (6.2) has not altered the stability results for monotonic waves, except that it forces us to place mild restrictions on the asymptotic behavior of $\phi(x)$ as $x \to \pm\infty$.

In this section we have analyzed the stability of constant and strictly monotonic traveling wave solutions $u(t,x) = \phi(x-ct)$ of

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y))dyds) \quad . \ (6.1)$$

We changed to the coordinate system which moves with the same speed as the wave. This leads to the steady state solution $u(t,x) = \phi(x)$ of

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y+cs))dyds) + cu_x . \ (6.2)$$

We found in theorem (6.1) that if $\phi(x)$ is a constant steady state solution $(\phi(x) \equiv \phi_0)$, then $u(t,x) \equiv \phi_0$ has the same stability as was found in theorem (4.1) for constant steady state solutions of equations which do not contain integrals. We have also examined strictly monotonic steady state solutions $u(t,x) \equiv \phi(x)$ of equation (6.2). We found (in theorem (6.2), lemma (6.3), lemma (6.4), and theorem (6.5)) that if relatively mild restrictions were placed on the asymptotic (as $x \to \pm\infty$) behavior of $\phi(x)$, then the results in theorem (4.2), lemma (4.3), lemma (4.4), and theorem (4.5) about equation (4.2) remain valid for equation (6.2).

This completes this section on the stability of monotonic traveling waves. Note however that the stability results of theorem (6.5) can be extended to include monotonic traveling plane waves when more than a single spatial dimension is present, similar to the extension of theorem (4.5) discussed in section (4.11).

6.2 The instability of non-monotonic waves. Despite the title of this section, we have not been able to establish the instability of non-monotonic steady state solutions $u(t,x) = \phi(x)$ of

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y+cs))dyds) + cu_x. \quad (6.2)$$

Recall that the proof in Chapter IV of the instability of non-monotonic steady state solutions of

$$u_t = f(u_{xx}, u_x, u) + cu_x \quad (4.2)$$

is in three steps. The first step consists of constructing appropriate initial conditions $u(\varepsilon,0,x)$, which essentially are $\phi(x)$ with small additional bulges. Lemma (4.7) was used to construct these initial conditions. The second step is proving the hair-trigger effect, which shows that $u(\varepsilon,t,x)$ is increasing in t and that $u(\varepsilon,+\infty,x) \equiv \phi_\infty(\varepsilon,x)$. Here $\phi_\infty(\varepsilon,x)$ is the minimal steady state solution which satisfies

$$u(\varepsilon,0,x) \leq \phi_\infty(\varepsilon,x) \quad \text{for all } x .$$

The third and final step of the proof is showing that

$$\lim_{\varepsilon \to 0} \{\max_x \phi_\infty(\varepsilon,x) - \phi(x)\} \nrightarrow 0 \quad .$$

This step was accomplished by using lemma (4.8), which shows that for all $\varepsilon > 0$ sufficiently small $\phi_\infty(\varepsilon,x)$ is constant in x and ε.

The second step of the proof used in Chapter IV remains valid in the present situation. That is, there is a hair-trigger effect for equation (6.2). However lemmas (4.7) and (4.8) formed the first and third steps of the proof, and these lemmas were established by using the phase plane representation of the steady state solutions of equation (4.2). Since the steady states $u(t,x) = \phi(x)$ of equation (6.2) are the solutions of

$$f(\phi_{xx},\phi_x,\phi, \int_0^T \int_{|y|<Y} g(s,y,\phi(x-y+cs))dyds) + c\phi_x = 0 \quad , \qquad (6.13)$$

clearly there is no phase plane representation of the steady state solutions of (6.2). Therefore, our means of proving lemmas (4.7) and (4.8) cannot be used to extend the lemmas to the present situation.

Thus, the proofs used in Chapter IV cannot be used here. However, since the stability results for <u>monotonic</u> steady state solutions of equation (6.2) closely parallel the stability results for <u>monotonic</u> steady state solutions of equation (4.2), <u>an attractive conjecture is that the instability of non-monotonic steady state solutions $u(t,x) = \phi(x)$ of equation (6.2) is exactly the same as the instability of steady state solutions $u(t,x) = \phi(x)$ of equation (4.2).</u> Since the hair-trigger effect is valid for equation (6.2), one needs only to develop intersection results for the ordinary differential-integral equation

$$f(\phi_{xx},\phi_x,\phi, \int_0^T \int_{|y|<Y} g(s,y,\phi(x-y+cs))dyds) + c\phi_x = 0 \quad (6.13)$$

similar to the intersection results of lemma (4.7) and lemma (4.8). If such results were established, the instability of non-monotonic solutions would immediately follow.

We now continue to the next section, where we extend the mean wavespeed/initial condition results of Chapter V to the equation

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y))dyds) \quad . \qquad (6.1)$$

6.3 Mean wavespeed and the initial conditions. In this section we will extend the mean wavespeed/initial condition results of Chapter V to the equation

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y))dyds) \quad . \qquad (6.1)$$

Recall that in Chapter V we considered equations

$$u_t = f(u_{xx}, u_x, u) \qquad (5.1)$$

which have a non-constant bounded monotonic solution $u(t,x) = \phi(x-ct,c)$. For each major case of $\phi(x-ct,c)$ being a $S \to S$, a $N \to S$, a $S \to N$, and a $N \to N$ type monotonic wave, we determined

(1) when the existence of $\phi(x-ct,c)$ implies the existence or non-existence of similar monotonic waves at nearby wavespeeds,

(2) when the existence of $\phi(x-ct,c)$ implies the existence or non-existence of similar monotonic waves at the same wavespeed c, and

(3) the mean wavespeed of $u(t,x)$ in terms of $u(0,x)$.

We will not extend the existence and non-existence results for equation (5.1) to equation (6.1). Presumably for any specific equation of the form (6.1), one could establish existence/non-existence results by using knowledge of the asymptotic behavior of solutions of

$$f(\phi_{xx},\phi_x,\phi,\int_0^T \int_{|y|<Y} g(s,y,\phi(x-y+cs))dyds) + c\phi_x = 0 \quad (6.13)$$

in conjunction with continuity arguments. However, in this chapter we will only extend the mean wavespeed/initial condition results of Chapter V to equation (6.1). Since the proofs of these mean wavespeed results are very similar to the proofs of theorems (5.2) in Chapter V, we simply quote the mean wavespeed results here. These results are:

Theorem 6.6 $(S \to S)$: Assume that H1, H2, H3, and H4 are satisfied. Suppose that $u(t,x) \equiv \phi(x-ct)$ is a bounded strictly monotonic solution of

$$u_t = f(u_{xx},u_x,u,\int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y+cs))dyds) \quad .(6.1)$$

Define $\phi(-\infty) = \phi_-$ and $\phi(+\infty) = \phi_+$. Suppose that there is a $k^- > 0$ and a $k^+ < 0$ such that

$$\phi(x) = \phi_- + ae^{k^-x} + o(e^{(k^-+\delta)x}) \quad \text{as} \quad x \to -\infty, \qquad (6.14a)$$

$$\phi'(x) = k^-ae^{k^-x} + o(e^{(k^-+\delta)x}) \quad \text{as} \quad x \to -\infty, \qquad (6.14b)$$

$$\phi''(x) = (k^-)^2\, ae^{k^-x} + o(e^{(k^-+\delta)x}) \quad \text{as} \quad x \to -\infty, \qquad (6.14c)$$

$$\phi(x) = \phi_+ + be^{k^+x} + o(e^{(k^+-\delta)x} \quad \text{as} \quad x \to +\infty, \qquad (6.14d)$$

$$\phi'(x) = k^+be^{k^+x} + o(e^{(k^+-\delta)x}) \quad \text{as} \quad x \to +\infty, \text{ and} \qquad (6.14e)$$

$$\phi''(x) = (k^+)^2\, be^{k^+x} + o(e^{(k^+-\delta)x}) \quad \text{as} \quad x \to +\infty, \qquad (6.14f)$$

where a and b are some non-zero constants and δ is some positive constant. Finally suppose that ϕ_- and ϕ_+ are both first order saddle points.

If $\phi(x)$ satisfies these assumptions, then whenever $u(t,x)$ is any solution of equation (6.1) whose initial condition $u(t,x)$ $(t \le 0)$ is uniformly Hoelder continuous (with some exponent $\alpha > 0$), has $u(0,x)$ in H_x^2, and for all $t \le 0$ satisfies

$$\phi_- - \alpha' < u(t,x) < \phi_- + \alpha' \qquad \text{for all } x-ct \leq -x_0 \;, \quad (6.15a)$$

$$\phi_+ - \alpha' < u(t,x) < \phi_+ + \alpha' \qquad \text{for all } x-ct \geq +x_0 \;, \quad (6.15b)$$

$$\min\{\phi_-,\phi_+\} - \alpha' < u(t,x) < \max\{\phi_-,\phi_+\} + \alpha' \text{ for all } x \qquad (6.15c)$$

for any $\alpha' > 0$ sufficiently small and any $x_0 > 0$, then $u(t,x)$ must <u>propagate with mean wavespeed c.</u>

<u>Theorem 6.6 (N → S):</u> Assume that hypotheses H1, H2, H3, and H4 are satisfied. Suppose there is a c_1 and a $c_2 \geq c_1$ such that for each c in $[c_1,c_2]$, there exists a bounded strictly monotonic solution $u(t,x) = \phi(x-ct,c)$ of

$$u_t = f(u_{xx},u_x,u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y))dyds) \quad . \; (6.1)$$

Suppose also that for each c in $[c_1,c_2]$ that $\phi(-\infty,c) = \phi_-$, that $\phi(+\infty,c) = \phi_+$, and that $\phi(x,c)$ and $\frac{\partial}{\partial x}\phi(x,c)$ are continuously differentiable in c. Further, suppose that there is a continuous $k^-(c) > 0$ and $k^+(c) < 0$ such that for each c in $[c_1,c_2]$

$$\phi(x,c) = \phi_- + a(c)e^{k^-(c)x} + o(e^{(k^-(c)+\delta)x}) \qquad \text{as } x \to -\infty, \quad (6.16a)$$

$$\phi_x(x,c) = a(c)k^-(c)e^{k^-(c)x} + o(e^{(k^-(c)+\delta)x}) \qquad \text{as } x \to -\infty, \quad (6.16b)$$

$$\phi_{xx}(x,c) = a(c)(k^-(c))^2\, e^{k^-(c)x} + o(e^{(k^-(c)+\delta)x}) \qquad \text{as } x \to -\infty, \quad (6.16c)$$

$$\phi(x,c) = \phi_+ + b(c)e^{k^+(c)x} + o(e^{(k^+(c)-\delta)x}) \qquad \text{as } x \to +\infty, \quad (6.16d)$$

$$\phi_x(x,c) = b(c)k^+(c)e^{k^+(c)x} + o(e^{(k^+(c)-\delta)x}) \qquad \text{as } x \to +\infty, \quad (6.16e)$$

$$\phi_{xx}(x,c) = b(c)(k^+(c))^2\, e^{k^+(c)x} + o(e^{(k^+(c)-\delta)x}) \qquad \text{as } x \to +\infty, \quad (6.16f)$$

where $a(c)$ and $b(c)$ are non-zero constants and δ is some positive constant. Finally, suppose that ϕ_+ is a first order saddle point and that ϕ_- is not a first order saddle point.

If $\phi(x,c)$ satisfies the above assumptions, then whenever $u(t,x)$ is <u>any</u> solution of equation (6.1) whose initial condition $u(t,x)$ $(t \leq 0)$

is uniformly Hoelder continuous (with some exponent $\alpha > 0$), has $u(0,x)$ in H_x^2, and satisfies for all $t \leq 0$

$$\phi_+ - q_0 \leq u(t,x) \leq \phi_+ + q_0 \quad \text{for all} \quad x > x_0 \quad \text{and any} \quad x_0 \,, \tag{6.17a}$$

$$\phi_- \leq u(t,x) \leq \phi_+ + q_0 \quad \text{for all } x \text{ if } \phi(x,c) \text{ is increasing in } x, \text{ and} \tag{6.17b}$$

$$\phi_+ - q_0 \leq u(t,x) \leq \phi_- \qquad \text{for all } x \text{ if } \phi(x,c) \text{ is decreasing in } x, \tag{6.17c}$$

then we can conclude the following:

(1) if for any c in $[c_1,c_2]$ there is a $\beta > 0$ such that

$$e^{-k^-(c)y}|u(t,x)-\phi_-| > \beta \quad \text{for all } y \equiv x-ct < 0 \text{ and } t \leq 0$$

and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ cannot travel with mean wavespeed larger than c;

(2) if for any c in $[c_1,c_2]$ there is a $\beta > 0$ such that

$$e^{-k^-(c)y}|u(t,x)-\phi_-| < \beta \quad \text{for all } \; y \equiv x-ct < 0 \text{ and } t \leq 0$$

and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ cannot travel with mean wavespeed smaller than c;

(3) if for any c in $[c_1,c_2]$ there is a β_1, $\beta_2 > 0$ such that

$$\beta_1 < e^{-k^-(c)y}|u(t,x)-\phi_-| < \beta_2 \quad \text{for all } y \equiv x-ct < 0 \text{ and } t \leq 0$$

and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ must travel with mean wavespeed c and must have finite dispersion; and

(4) if for any c in (c_1,c_2)

$$\lim_{x\to-\infty} e^{(-k^-(c)+\mu)y}|u(t,x)-\phi_-| = 0 \quad \text{for all } t \leq 0 \text{ and all } \mu > 0 \;,$$

$$\lim_{x\to-\infty} e^{(-k^-(c)-\mu)y}|u(t,x)-\phi_-| = +\infty \text{ for all } t \leq 0 \text{ and all } \mu > 0 \;,$$

where $y \equiv x-ct$, and if $q_0 > 0$ is sufficiently small, then $u(t,x)$ travels with mean wavespeed c but may not have finite dispersion.

<u>Theorem 6.6 $(N \to N)$:</u> Assume that hypotheses H1, H2, H3, and H4 are satisfied. Suppose there is a c_1 and a $c_2 \geq c_1$ such that for each c in $[c_1, c_2]$ there exists a bounded strictly monotonic solution $u(t,x) = \phi(x-ct,c)$ of

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y))dyds) \quad . \quad (6.1)$$

Suppose also that for each c in $[c_1, c_2]$ that $\phi(-\infty,c) = \phi_-$, that $\phi(+\infty,c) = \phi_+$, and that $\phi(x,c)$ and $\frac{\partial}{\partial x}\phi(x,c)$ are continuously differentiable in c. Further, suppose that there is a continuous $k^-(c) > 0$ and $k^+(c) < 0$ such that for each c in $[c_1, c_2]$

$$\phi(x,c) = \phi_- + a(c)\, e^{k^-(c)x} + o(e^{(k^-(c)+\delta)x}) \quad \text{as } x \to -\infty, \quad (6.16a)$$

$$\phi_x(x,c) = a(c)k^-(c)\, e^{k^-(c)x} + o(e^{(k^-(c)+\delta)x}) \quad \text{as } x \to -\infty, \quad (6.16b)$$

$$\phi_{xx}(x,c) = a(c)(k^-(c))^2\, e^{k^-(c)x} + o(e^{(k^-(c)+\delta)x}) \quad \text{as } x \to -\infty, \quad (6.16c)$$

$$\phi(x,c) = \phi_+ + b(c)e^{k^+(c)x} + o(e^{(k^+(c)-\delta)x}) \quad \text{as } x \to +\infty, \quad (6.16d)$$

$$\phi_x(x,c) = b(c)k^+(c)\, e^{k^+(c)x} + o(e^{(k^+(c)-\delta)x}) \quad \text{as } x \to +\infty, \quad (6.16e)$$

$$\phi_{xx}(x,c) = b(c)(k^+(c))^2\, e^{k^+(c)x} + o(e^{(k^+(c)-\delta)x}) \quad \text{as } x \to +\infty, \quad (6.16f)$$

where $a(c)$ and $b(c)$ are non-zero constants and δ is some positive constant. Finally, suppose that neither ϕ_- nor ϕ_+ is a first order saddle point.

If $\phi(x,c)$ satisfies the above assumptions, then whenever $u(t,x)$ is <u>any</u> solution of equation (6.1) whose initial condition $u(t,x)$ $(t \leq 0)$ is uniformly Hoelder continuous (with some exponent $\alpha > 0$), has $u(0,x)$ in H_x^2, and satisfies

$$\min\{\phi_-,\phi_+\} < u(t,x) < \max\{\phi_-,\phi_+\} \quad \text{for all } x \text{ and all } t \leq 0 \quad ,(6.18)$$

then we can conclude the following:

(1) if for any c in $[c_1, c_2]$ there is a β_1, $\beta_2 > 0$ such that

$\beta_1 < e^{-k^-(c)y}|u(t,x)-\phi_-|$ for all $y \equiv x-ct < 0$ and $t \leq 0$, and

$\beta_2 > e^{-k^+(c)y}|u(t,x)-\phi_+|$ for all $y \equiv x-ct > 0$ and $t \leq 0$,

then $u(t,x)$ cannot travel with mean wavespeed larger than c;

(2) if for any c in $[c_1,c_2]$ there is a β_1, $\beta_2 > 0$ such that

$\beta_1 > e^{-k^-(c)y}|u(t,x)-\phi_-|$ for all $y \equiv x-ct < 0$ and $t \leq 0$,

$\beta_2 < e^{-k^+(c)y}|u(t,x)-\phi_+|$ for all $y \equiv x-ct > 0$ and $t \leq 0$,

then $u(t,x)$ cannot travel with mean wavespeed smaller than c;

(3) if for any c in $[c_1,c_2]$ there is a β_1, β_2, β_3, $\beta_4 > 0$ such that

$\beta_1 < e^{-k^-(c)y}|u(t,x)-\phi_-| < \beta_2$ for all $y \equiv x-ct < 0$ and $t \leq 0$,

$\beta_3 < e^{-k^+(c)y}|u(t,x)-\phi_+| < \beta_4$ for all $y \equiv x-ct > 0$ and $t \leq 0$,

then $u(t,x)$ travels with mean wavespeed c and has finite dispersion, and

(4) if for any c in (c_1,c_2)

$$\lim_{x\to-\infty} e^{-(k^-(c)-\mu)y}|u(t,x)-\phi_-| = 0, \quad \lim_{x\to-\infty} e^{-(k^-(c)+\mu)y}|u(t,x)-\phi_-| = +\infty,$$

$$\lim_{x\to+\infty} e^{-(k^+(c)-\mu)y}|u(t,x)-\phi_+| = +\infty, \quad \lim_{x\to+\infty} e^{-(k^+(c)+\mu)y}|u(t,x)-\phi_+| = 0$$

for all $\mu > 0$ and all $t \leq 0$ (where $y \equiv x-ct$), then $u(t,x)$ travels with mean wavespeed c (but may not have finite dispersion).

Basically theorems 6.6 ($S \to S$), ($N \to S$), and ($N \to N$) show that for large classes of initial conditions $u(t,x)$ ($t \leq 0$), the mean wavespeed of the resulting solution of equation (6.1) depends entirely on

(1) the asymptotic decay rate of $u(t,x)$ to ϕ_- as $x \to -\infty$ for $t \leq 0$ if ϕ_- is not a first order saddle point, and

(2) the asymptotic decay rate of $u(t,x)$ to ϕ_+ as $x \to +\infty$ for $t \leq 0$ if ϕ_+ is not a first order saddle point.

Theorems 6.6 $(S \to S)$, $(N \to S)$, and $(N \to N)$ are very similar to theorems 5.2 $(S \to S)$, $(N \to S)$, and $(N \to N)$ of Chapter V. The only major differences are that

(1) in theorems (6.6) conclusions are drawn about the mean wavespeeds of solutions of equation (6.1), whereas in theorems (5.2) conclusions are drawn for equation (5.1);

(2) in theorems (6.6) we needed to make assumptions about the asymptotic behavior (as $x \to \pm \infty$) of the monotonic traveling waves (see equations (6.14) and (6.16)), whereas in theorems (5.2) we knew that the monotonic traveling waves always satisfy these asymptotic assumptions;

(3) for theorems 6.6 $(N \to S)$ and $(N \to N)$ we needed to assume the existence of all the monotonic traveing waves we used, whereas in theorems 5.2 $(N \to S)$ and $(N \to N)$ we needed only to assume the existence of a single monotonic traveling wave; and

(4) the restrictions on the initial conditions in theorems (6.6) are for all $t \leq 0$, whereas the restrictions for theorems (5.2) are only for $t = 0$.

Let us note that for almost all equations of the form (6.1), in practice almost every traveling wave and steady state solution $u(t,x) = \phi(x-ct,c)$ satisfies the asymptotic restrictions of equations (6.14) and (6.16). Thus these asymptotic behavior requirements in theorems 6.6 $(S \to S)$, $(N \to S)$, and $(N \to N)$ are relatively innocuous.

This completes our presentation of the mean wavespeed/initial condition results for equation (6.1). We complete this chapter in the next section with some final remarks.

6.4 Some general remarks. We have found in this chapter that the stability results for monotonic traveling waves can be extended to equations of the form

$$u_t = f(u_{xx}, u_x, u, \int_0^T \int_{|y|<Y} g(s,y,u(t-s,x-y))dyds) \quad . \qquad (6.1)$$

We also found that the mean wavespeed/initial condition results are readily extended to equation (6.1). However, we are unable to extend the instability results for non-monotonic waves to equation (6.1).

In this chapter we developed only the most readily obtainable results about equation (6.1). Many potential areas of research have been ignored. For example, one could examine the asymptotic behavior (as $x \to \pm \infty$) of the monotonic traveling wave solutions of equation (6.1). This would determine when the hypotheses about the asymptotic behavior of monotone traveling waves in theorems (6.6) are satisfied. These asymptotic results could also potentially be used in conjunction with continuity arguments to prove existence/non-existence theorems about monotonic traveling wave solutions of equation (6.1), similar to theorems (5.1) about solutions of equation (5.1).

Establishing intersection results similar to lemmas (4.7) and (4.8) is another potentially interesting research area. Since equation (6.2) has a hair-trigger effect, whenever one can establish intersection results similar to those in lemmas (4.7) and (4.8) for any class of non-monotonic steady state solutions of (6.2), then instability of those steady states follows immediately.

This completes our treatment of equation (6.1). In the next chapter we treat a similar topic. Specifically we will extend the results of Chapters IV and V to some systems of equations.

Chapter VII

EXTENSION TO MULTIPLE DEPENDENT VARIABLES

In this chapter we extend some of the results of Chapters IV and V to a special class of systems of equations which contain no integrals. As in Chapter VI we will consider only direct extensions, and we will be as brief as possible. Basically we will find that the stability results for monotonic waves and the mean wavespeed/initial condition results are still valid. However, we will not be able to prove the instability of non-monotonic waves.

To be more specific, in this chapter we will treat systems of equations of the following form

$$u_t^{(i)} = F^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) \qquad i = 1,\dots,m \quad , \tag{7.1'}$$

where $F_1^{(i)} > 0$ and $F_{3j}^{(i)} \geq 0$ for all $j \neq i$ and all i, and where $\tilde{u} \equiv (u^{(1)},\dots,u^{(m)})$. Throughout this chapter we will assume that hypotheses H2 (smoothness of system (7.1')), H3 (parabolicity of system (7.1')), and H4 (existence of solutions to the initial value problem) are satisfied. We also assume that a very large $M > 0$ has been chosen, and we work with the resulting system of equations

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) \qquad i = 1,\dots,m \quad , \tag{7.1}$$

where $\tilde{f} \equiv \tilde{F}_M$.

Briefly, in the first section of this chapter we will derive the stability results for monotonic waves. In section (7.2) we will discuss the instability results for non-monotonic waves. We will derive the mean wavespeed/initial condition results in section (7.3). Finally, we will use the last section, section (7.4), to express some general remarks. We start with the stability of monotonic waves.

<u>7.1 Stability of monotonic traveling waves.</u> In this section we extend the stability results to monotonic traveling wave (and steady state) solutions of system (7.1). To do this we need to first appropriately redefine our stability concepts, the concept of a "node" and a "saddle point", and the concept of a monotonic traveling wave. We will also need to introduce some results from Perron-Frobenius matrix theory [9]. We begin by redefining our stability concepts appropriately.

Let $w(x)$ be any continuous function with $w(x) \geq 1$ for all x. Then any steady state solution $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ of the system

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) + cu_x^{(i)} \qquad i = 1,\dots,m \tag{7.2}$$

is defined to be <u>C^w-stable</u> if and only if given any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that every solution $\tilde{u}(t,x)$ of (7.2) satisfies

$$|\{u^{(i)}(t,x)-\phi^{(i)}(x)\}w(x)| \leq \varepsilon \quad \text{for all } x, \text{ all } t \geq 0, \text{ and } i = 1,\dots,m \tag{7.3}$$

whenever the initial conditions $\tilde{u}(0,x)$ are in H_x^2 and satisfy

$$|\{u^{(i)}(0,x)-\phi^{(i)}(x)\}w(x)| \leq \delta(\varepsilon) \quad \text{for all } x \text{ and } i = 1,\dots,m \ . \tag{7.4}$$

Similarly, $\tilde{\phi}(x)$ is defined to be <u>$\mathfrak{C}^w$-stable</u> if and only if given any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that every solution $\tilde{u}(t,x)$ of (7.2) satisfies

$$|u^{(i)}(t,x)-\phi^{(i)}(x)| \leq \varepsilon \quad \text{for all } x, \text{ all } t \geq 0, \text{ and } i = 1,\dots,m \tag{7.5}$$

whenever the initial conditions $\tilde{u}(0,x)$ are in H_x^2 and satisfy (7.4). A solution $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ which is not C^w-stable will be called <u>C^w-unstable</u>, and if it is not $\mathfrak{C}^w$-stable it will be called <u>$\mathfrak{C}^w$-unstable</u>.

We now define the concepts of "singular point", "node", and "saddle point". As in Chapter VI, there is no phase plane representation of the steady states $\tilde{u}(t,x) = \tilde{\phi}(x)$ of system (7.2). Since the definitions of "singular point", "node", and "saddle point" cannot refer to features of a phase plane, we will instead choose the definitions of these terms so that the results in this chapter are analogous to those in previous chapters. Thus, we define $\tilde{\phi}_0$ to be a <u>singular point</u> if and only if

$$\tilde{f}(0,0,\tilde{\phi}_0) = \tilde{0} \quad , \tag{7.6}$$

and to be a <u>regular singular point of order one</u> if and only if

$$\tilde{f}(0,0,\tilde{\phi}_0+\tilde{\eta}) = A\tilde{\eta} + o(||\tilde{\eta}||^{1+\Delta}) \quad \text{as} \quad ||\tilde{\eta}|| \to 0 \quad , \tag{7.7}$$

where the matrix A is <u>irreducible</u> (see reference [9]) and <u>nonsingular</u>. If all the eigenvalues of A have negative real parts then we define $\tilde{\phi}_0$ to be a <u>first order saddle point</u>, and if A has an eigenvalue with positive real part then we define $\tilde{\phi}_0$ to be a <u>first order node</u>.

We now define monotonicity appropriately. If $\tilde{\phi}(x)$ has either <u>all</u> $\phi^{(i)}(x)$ increasing for all x or has <u>all</u> $\phi^{(i)}(x)$ decreasing for all x, then we define $\tilde{\phi}(x)$ to be <u>monotone</u>. If in addition $\phi^{(i)}(x) \neq 0$ for all i and x, then $\tilde{\phi}(x)$ will be called <u>strictly monotonic</u>. If $\tilde{\phi}(x)$ has some $\phi^{(i)}(x)$ increasing for all x and some decreasing for all x, then we define $\tilde{\phi}(x)$ to be <u>quasi-monotonic</u>. If $\phi^{(i)}(x) \neq 0$ for all i and x and if $\tilde{\phi}(x)$ is quasi-monotonic, we will call $\tilde{\phi}(x)$ <u>strictly quasi-monotonic</u>.

We now introduce two needed results from Perron-Frobenius (PF) theory. Let $\tilde{\phi}_0$ be any regular singular point of order one, and define the matrix A by

$$A_{ij} = f^{(i)}_{3j}(0,0,\tilde{\phi}_0)$$

as in equation (7.7). By hypothesis H3, $A_{ij} \geq 0$ for all $i \neq j$. Also, by our definition of regular singular point, A is irreducible. Perron-Frobenius (PF) theory therefore implies that

(1) the eigenvalue of A with largest real part is real and simple, and

(2) the eigenvector corresponding to the (real) eigenvalue with largest real part can be taken to have all positive components. These results are discussed in reference [9] for example.

With the above definitions and the two results of PF-theory, the stability results for monotonic waves can now be readily extended to solutions of system (7.1). We begin with the stability results for constant steady state solutions $\tilde{u}(t,x) \equiv \tilde{\phi}_0$.

Theorem 7.1: Assume that hypotheses H2, H3, and H4 are satisfied. Suppose further that $\tilde{u}(t,x) \equiv \tilde{\phi}_0$ is a constant steady state solution of

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) + cu_x^{(i)} \qquad i = 1,\ldots,m \quad . \qquad (7.2)$$

Then

(1) if $\tilde{\phi}_0$ is a first order saddle point then $\tilde{u}(t,x) \equiv \tilde{\phi}_0$ is C^w-stable with $w(x) \equiv 1$, and

(2) if $\tilde{\phi}_0$ is a first order node then $\tilde{u}(t,x) \equiv \tilde{\phi}_0$ is C^w-unstable with $w(x) \equiv 1 + e^{-\kappa x} + e^{+\kappa x}$ for any $\kappa > 0$ sufficiently small.

Thus, once again saddle points are very stable constant solutions and nodes are unstable constant solutions.

We will prove theorem (7.1) in some detail, since the proof clearly illustrates how PF-theory is used with the maximum principle to obtain stability results. Recall that a function $\tilde{u}(t,x)$ is defined to be an upper function of system (7.2) when

$$u_t^{(i)} - f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) - cu_x^{(i)} \geq 0 \quad \text{for all } x \text{ and all } i = 1,\ldots,m \quad .$$

Similarly, $\tilde{u}(t,x)$ is a lower function of system (7.2) when

$$u_t^{(i)} - f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) - cu_x^{(i)} \leq 0 \quad \text{for all } x \text{ and all } i = 1,\dots,m \quad .$$

<u>Proof of theorem (7.1):</u> Define the matrix A by

$$A_{ij} = F_{3j}^{(i)}(0,0,\tilde{\phi}_0) \quad .$$

Let λ be the eigenvalue with largest real part of the matrix A, and let $\tilde{\alpha}$ be its corresponding eigenvector. From PF-theory, λ is real and we can assume that $\tilde{\alpha} \equiv (\alpha^{(1)},\dots,\alpha^{(m)})$ has $\alpha^{(i)} > 0$ for all i.

To prove part (1), assume that $\tilde{\phi}_0$ is a saddle point. Then $\lambda < 0$. Define

$$\tilde{B}(\varepsilon,t,x) \equiv \varepsilon\tilde{\alpha}e^{\lambda t/2} + \tilde{\phi}_0 \quad .$$

We calculate

$$\begin{aligned} & B_t^{(i)} - f^{(i)}(B_{xx}^{(i)}, B_x^{(i)}, \tilde{B}) - cB_x^{(i)} \\ &= \tfrac{1}{2}\varepsilon\lambda e^{\lambda t/2}\, \alpha^{(i)} - f^{(i)}(0,0,\tilde{\phi}_0 + \varepsilon e^{\lambda t/2}\tilde{\alpha}) \\ &= -\tfrac{1}{2}\varepsilon\lambda e^{\lambda t/2}\, \alpha^{(i)} + o(\varepsilon e^{\lambda t/2}) \quad . \end{aligned}$$

Thus, there is an $\varepsilon_0 > 0$ such that $\tilde{B}(\varepsilon,t,x)$ is an upper function and $\tilde{B}(-\varepsilon,t,x)$ is a lower function of system (7.2) for every ε in $(0,\varepsilon_0)$. The maximum principle now shows that any solution $\tilde{u}(t,x)$ of (7.2) whose initial condition $\tilde{u}(0,x)$ is in H_x^2 and satisfies

$B^{(i)}(-\varepsilon,0,x) \leq u^{(i)}(0,x) \leq B^{(i)}(\varepsilon,0,x)$ for all x, all i, and any ε in $(0,\varepsilon_0)$,

must also satisfy

$B^{(i)}(-\varepsilon,t,x) \leq u^{(i)}(t,x) \leq B^{(i)}(\varepsilon,t,x)$ for all x, all $t \geq 0$, and all i.

Thus, small bounded initial perturbations decay exponentially in time. Therefore, when $\tilde{\phi}_0$ is a saddle point of order one we have established that $\tilde{u}(t,x) \equiv \tilde{\phi}_0$ is C^w-stable with $w(x) \equiv 1$.

Part (2) is proved similarly. We note that since $\tilde{\phi}_0$ is a node, $\lambda > 0$. We now define

$$\tilde{B}(\varepsilon,t,x) \equiv \tilde{\phi}_0 + \varepsilon\tilde{\alpha}e^{\lambda t/4} \text{ sech } \kappa x \quad .$$

A short calculation shows that for any $\kappa > 0$ sufficiently small, there is an $\varepsilon_0 > 0$ such that $\tilde{B}(\varepsilon,t,x)$ is a <u>lower</u> function whenever $0 \le \varepsilon e^{\lambda t/4} \le \varepsilon_0$. Define $\tilde{u}(\varepsilon,t,x)$ as the solution of system (7.2) with the initial condition

$$\tilde{u}(\varepsilon,0,x) = \tilde{B}(\varepsilon,0,x) \quad .$$

The maximum principle now implies that for any ε in $(0,\varepsilon_0)$,

$$\phi_0^{(i)} < \phi_0^{(i)} + \varepsilon\alpha^{(i)}\, e^{\lambda t/4} \text{ sech } \kappa x \le u^{(i)}(\varepsilon,t,x)$$

for all x, all i, and all t such that $0 < \varepsilon e^{\lambda t/4} < \varepsilon_0$.

Therefore, when $\tilde{\phi}_0$ is a node of order one we have established that $\tilde{u}(t,x) \equiv \tilde{\phi}_0$ is $\mathfrak{C}^w$-unstable with $w(x) = 1 + e^{-\kappa x} + e^{\kappa x}$ for any $\kappa > 0$ sufficiently small.

<u>This completes the proof of theorem (7.1).</u>

The stability results in theorem (7.1) for constant solutions of system (7.2) are analogous to the results in theorem (4.2) about constant solutions of equation (4.2). Following the treatment in Chapter IV, we will now briefly derive the asymptotic behavior of the monotonic steady state solutions of system (7.2). We will not prove the correctness of the asymptotic results we obtain. Instead, we note that their correctness is a consequence of the material in Chapter 13 reference [6].

Suppose that $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ is a monotonic steady state solution of system (7.2). Define $\tilde{\phi}(-\infty) \equiv \tilde{\phi}_-$ and $\tilde{\phi}(+\infty) \equiv \tilde{\phi}_+$, and assume that both $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are regular singular points of order one. Also define the diagonal matrices $D_1^{\pm}$ and $D_2^{\pm}$ by

$$D^-_{1,ii} = f_1^{(i)}(0,0,\tilde{\phi}_-) \qquad D^-_{2,ii} = f_2^{(i)}(0,0,\tilde{\phi}_-)$$

$$D^+_{1,ii} = f_1^{(i)}(0,0,\tilde{\phi}_+) \qquad D^+_{2,ii} = f_2^{(i)}(0,0,\tilde{\phi}_+) \quad ,$$

define the matrices A^- and A^+ by

$$A^-_{ij} = f_{3j}^{(i)}(0,0,\tilde{\phi}_-) \quad \text{and} \quad A^+_{ij} = f_{3j}^{(i)}(0,0,\tilde{\phi}_+) \quad ,$$

and finally let I be the $m \times m$ identity matrix. Note that the regularity assumptions about $\tilde{\phi}_-$ and $\tilde{\phi}_+$ imply that A^- and A^+ are both irreduscible and non-singular. As $x \to -\infty$ and as $x \to +\infty$, $\tilde{\phi}(x)$ must satisfy the asymptotic equations

$$D^-_{1,ii}\,\phi^{(i)}_{xx}(x) + (D^-_{2,ii}+c)\phi^{(i)}_x(x) + \sum_j A^-_{ij}\{\phi^{(j)}(x)-\phi^{(j)}_-\} \sim 0 \tag{7.8a}$$

$$\text{as} \quad x \to -\infty, \quad i = 1,\dots,m$$

$$D^+_{1,ii}\phi^{(i)}_{xx}(x) + (D^+_{2,ii}+c)\phi^{(i)}_x(x) + \sum_j A^+_{ij}\{\phi^{(j)}(x)-\phi^{(i)}_+\} \sim 0 \tag{7.8b}$$

$$\text{as} \quad x \to +\infty, \quad i = 1,\dots,m \quad .$$

We conclude that the asymptotic behavior of $\tilde{\phi}(x)$ is given by

$$\tilde{\phi}(x) = \tilde{a}e^{k^-x} + o(e^{(k^-+\delta)x}) \quad \text{as} \quad x \to -\infty \quad \text{and} \tag{7.9a}$$

$$\tilde{\phi}(x) = \tilde{b}e^{k^+x} + o(e^{(k^+-\delta)x}) \quad \text{as} \quad x \to +\infty, \tag{7.9b}$$

where $\delta > 0$ is a positive constant, where k^- and k^+ are nonzero constants such that the matrices

$$A^-(k^-) \equiv (k^-)^2D_1^- + k^-(D_2^-+cI) + A^-$$

$$A^+(k^+) \equiv (k^+)^2D_1^+ + k^+(D_2^++cI) + A^+$$

are singular, and where $\tilde{a}$ and $\tilde{b}$ are nonzero null vectors of $A^-(k^-)$ and $A^+(k^+)$, respectively. Moreover, the asymptotic behavior of $\tilde{\phi}_x(x)$ and $\tilde{\phi}_{xx}(x)$ (as $x \to \pm\infty$) can be obtained by formally differentiating the expressions in equations (7.9).

Since $\tilde{\phi}(x)$ is monotonic, we see that k^- and k^+ are real. In fact $k^- > 0$ and $k^+ < 0$. Also since $\tilde{\phi}(x)$ being monotonic implies that either all $\phi^{(i)}(x)$ are increasing or all $\phi^{(i)}(x)$ are decreasing, either

$$a^{(i)} \geq 0 \quad \text{and} \quad b^{(i)} \leq 0 \quad \text{for all } i, \text{ or} \tag{7.10a}$$

$$a^{(i)} \leq 0 \quad \text{and} \quad b^{(i)} \geq 0 \quad \text{for all } i \ . \tag{7.10b}$$

However, we know from PF-theory that $A^-(k^-)$ and $A^+(k^+)$ each have a <u>simple</u> real eigenvalue λ^- and λ^+ whose corresponding eigenvectors have all positive components. Thus, $\lambda^- = \lambda^+ = 0$ since the non-zero null vectors of $A^-(k^-)$ and $A^+(k^+)$ satisfy either (7.10a) or (7.10b). This implies that all the components of the null vectors of $A^-(k^-)$ and $A^+(k^+)$ must be nonzero. That is, we now know that either

$$a^{(i)} > 0 \quad \text{and} \quad b^{(i)} < 0 \quad \text{for all } i, \text{ or} \tag{7.11a}$$

$$a^{(i)} < 0 \quad \text{and} \quad b^{(i)} > 0 \quad \text{for all } i \ . \tag{7.11b}$$

To summarize our brief asymptotic analysis, we have shown that whenever $\tilde{u}(t,x) = \tilde{\phi}(x)$ is a monotonic steady state solution of system (7.2) and whenever $\tilde{\phi}(-\infty) \equiv \tilde{\phi}_-$ and $\tilde{\phi}(+\infty) \equiv \tilde{\phi}_+$ are both regular singular points of order one, then the asymptotic behavior of $\tilde{\phi}(x)$ is given by

$$\tilde{\phi}(x) = \tilde{\phi}_- + \tilde{a}e^{k^-x} + o(e^{(k^-+\delta)x}) \quad \text{as } x \to -\infty, \tag{7.12a}$$

$$\tilde{\phi}_x(x) = k^-\tilde{a}e^{k^-x} + o(e^{(k^-+\delta)x}) \quad \text{as } x \to -\infty, \tag{7.12b}$$

$$\tilde{\phi}_{xx}(x) = (k^-)^2\,\tilde{a}e^{k^-x} + o(e^{(k^-+\delta)x}) \quad \text{as } x \to -\infty, \tag{7.12c}$$

$$\tilde{\phi}(x) = \tilde{\phi}_+ + \tilde{b}e^{k^+x} + o(e^{(k^+-\delta)x}) \quad \text{as } x \to +\infty, \tag{7.12d}$$

$$\tilde{\phi}_x(x) = k^+\tilde{b}e^{k^+x} + o(e^{(k^+-\delta)x}) \quad \text{as } x \to +\infty, \tag{7.12e}$$

$$\tilde{\phi}_{xx}(x) = (k^+)^2\,\tilde{b}e^{k^+x} + o(e^{(k^+-\delta)x}) \quad \text{as } x \to +\infty, \tag{7.12f}$$

where δ is some positive constant, where k^- and k^+ are some real constants with $k^- > 0$ and $k^+ < 0$, and where $\tilde{a}$ and $\tilde{b}$ are some real vectors with all nonzero components. In other words, $\phi^{(1)}(x)$, $\phi_x^{(1)}(x)$, $\phi_{xx}^{(1)}(x)$, $\phi^{(2)}(x), \ldots, \phi_{xx}^{(m)}(x)$ all decay asymptotically at the same exponential rate as $x \to -\infty$ and all decay at the same exponential rate as $x \to +\infty$.

We now develop the stability theory for monotonic steady state solutions of system (7.2). This development will parallel the treatment in sections (4.4) through (4.6) of Chapter IV and the treatment in section (6.1) of Chapter VI. We will not present the proofs of the remaining results in this section. The proofs are very similar to the proofs in sections (4.4) through (4.6) of Chapter IV. The only major change is the insertion of the asymptotic results of equations (7.12) at the appropriate points.

Theorem 7.2: Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ is a bounded strictly monotonic steady state solution of

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) + cu_x^{(i)} \qquad i = 1,\ldots,m \quad . \qquad (7.2)$$

Suppose that $\tilde{\phi}(-\infty)$ and $\tilde{\phi}(+\infty)$ are both regular singular points of order one. Define k^- as the (positive) exponential decay constant of $\tilde{\phi}(x)$ as $x \to -\infty$, and define k^+ as the (negative) exponential decay constant of $\tilde{\phi}(x)$ as $x \to +\infty$. Then $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ is C^w-stable with

$$w(x) \equiv 1 + \frac{1}{|\phi_x^{(1)}(x)|} \quad ,$$

or equivalently, with

$$w(x) \equiv 1 + e^{-k^- x} + e^{-k^+ x} \quad .$$

We see that strictly monotonic steady state solutions of system (7.2) have at least a limited stability. In the next two lemmas we will construct upper and lower functions of equation (7.2). These new upper and lower functions will allow us to improve our basic stability results.

Lemma 7.3: Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ is a bounded strictly monotonic steady state solution of system (7.2). Define $\tilde{\phi}(-\infty) \equiv \tilde{\phi}_-$ and $\tilde{\phi}(+\infty) \equiv \tilde{\phi}_+$, and assume that both $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are regular singular points of order one. Also, define the matrices A^- and A^+ by

$$A^-_{ij} \equiv f^{(i)}_{3j}(0,0,\tilde{\phi}_-) \qquad A^+_{ij} \equiv f^{(i)}_{3j}(0,0,\tilde{\phi}_+) \quad ,$$

and let $\tilde{\alpha}_-$ and $\tilde{\alpha}_+$ be the eigenvectors of A^- and A^+ (respectively) corresponding to the eigenvalues with largest real part. Finally, let $\tilde{\alpha}_-$ and $\tilde{\alpha}_+$ have all positive components.

Then

(1) if $\tilde{\phi}_+$ is a saddle point of order one then

$$\bar{u}^{(i)}(t,x) \equiv \phi^{(i)}(x+h(t)) + q(t)\alpha^{(i)}_+ \cdot [\phi^{(i)}(x+h(t))-\phi^{(i)}_-]/\ |\phi^{(i)}_+ - \phi^{(i)}_-| \tag{7.13a}$$

$i = 1,\ldots,m$ and

$$\underline{u}^{(i)}(t,x) \equiv \phi^{(i)}(x-h(t)) - q(t)\alpha^{(i)}_+ \cdot [\phi^{(i)}(x-h(t))-\phi^{(i)}_-]/\ |\phi^{(i)}_+ - \phi^{(i)}_-| \tag{7.13b}$$

$i = 1,\ldots,m$

are upper and lower functions (respectively) of system (7.2). Here,

$$h(t) \equiv \varepsilon\kappa(1-e^{-st}) + h_0 \ , \quad q(t) \equiv \varepsilon e^{-st} \quad , \tag{7.14}$$

where s and κ are particular positive constants, h_0 is arbitrary and ε is any constant with sufficiently small magnitude and with the same sign as $\phi^{(i)}_x(x)$.

(2) if $\tilde{\phi}_-$ is a saddle point of order one then

$$\bar{u}^{(i)}(t,x) \equiv \phi^{(i)}(x+h(t)) + q(t)\alpha^{(i)}_- \cdot [\phi^{(i)}_+ - \phi^{(i)}(x+h(t))]/\ |\phi^{(i)}_+ - \phi^{(i)}_-| \tag{7.15a}$$

$i = 1,\ldots,m$ and

$$\underline{u}^{(i)}(t,x) \equiv \phi^{(i)}(x-h(t)) - q(t)\alpha^{(i)}_- \cdot [\phi^{(i)}_+ - \phi^{(i)}(x-h(t))]/\ |\phi^{(i)}_+ - \phi^{(i)}_-| \tag{7.15b}$$

$i = 1,\ldots,m$

are upper and lower functions (respectively) of system (7.2). Here, $h(t)$

and $q(t)$ are defined as in the preceding case.

Lemma 7.4: Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ is a bounded strictly monotonic steady state solution of system (7.2). Define $\tilde{\phi}(-\infty) \equiv \tilde{\phi}_-$ and $\tilde{\phi}(+\infty) \equiv \tilde{\phi}_+$, and assume that both $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are first order saddle points. Also, define the matrices A^- and A^+ by

$$A^-_{ij} \equiv f^{(i)}_{3j}(0,0,\tilde{\phi}_-) \qquad A^+_{ij} \equiv f^{(i)}_{3j}(0,0,\tilde{\phi}_+) \quad ,$$

and assume that there is a vector $\tilde{\alpha}$ with all positive components such that

$$\sum_j A^-_{ij}\,\alpha^{(j)} < 0 \quad \text{and} \quad \sum_j A^+_{ij}\,\alpha^{(j)} < 0 \quad \text{for all} \quad i = 1,\dots,m \quad .$$

Then

$$\overline{u}^{(i)}(t,x) \equiv \phi^{(i)}(x+h(t)) + |q(t)\alpha^{(i)}| \qquad i = 1,\dots,m \quad \text{and} \qquad (7.16a)$$

$$\underline{u}^{(i)}(t,x) \equiv \phi^{(i)}(x-h(t)) - |q(t)\alpha^{(i)}| \qquad i = 1,\dots,m \qquad (7.16b)$$

are upper and lower functions (respectively) of system (7.2). Here,

$$h(t) \equiv \varepsilon\kappa(1-e^{-st}) + h_0 \quad , \quad q(t) \equiv \varepsilon e^{-st} \quad , \qquad (7.14)$$

where s and κ are particular positive constants, h_0 is arbitrary, and ε is any constant with sufficiently small magnitude and with the same sign as $\phi^{(1)}_x(x)$.

Theorem 7.5 (The stability of monotone waves): Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ is a bounded strictly monotonic steady state solution of

$$u^{(i)}_t = f^{(i)}(u^{(i)}_{xx}, u^{(i)}_x, \tilde{u}) + cu^{(i)}_x \qquad i = 1,\dots,m \quad . \qquad (7.2)$$

Define $\tilde{\phi}(-\infty) \equiv \tilde{\phi}_-$ and $\tilde{\phi}(+\infty) \equiv \tilde{\phi}_+$ and assume that both $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are regular singular points of order one. Also define the matrices A^- and A^+ by

$$A^-_{ij} = f^{(i)}_{3j}(0,0,\tilde{\phi}_-) \quad \text{and} \quad A^+_{ij} = f^{(i)}_{3j}(0,0,\tilde{\phi}_+) \quad .$$

Finally, let $k^- > 0$ and $k^+ < 0$ be defined by

$$\tilde{\phi}(x) \sim \tilde{a}e^{k^- x} \quad \text{as} \quad x \to -\infty \quad \text{and} \quad \tilde{\phi}(x) \sim \tilde{b}e^{k^+ x} \quad \text{as} \quad x \to +\infty$$

where $\tilde{a}$ and $\tilde{b}$ are non-zero vectors.

Then $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ is C^w-stable where

(1) if $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are both saddle points and if there is a vector $\tilde{\alpha}$ with all positive components such that

$$\sum_j A^-_{ij}\, \alpha^{(j)} < 0 \quad \text{and} \quad \sum_j A^+_{ij}\, \alpha^{(j)} < 0 \quad \text{for all} \quad i = 1,\ldots,m \;,$$

then $w(x) \equiv 1$;

(2) if $\tilde{\phi}_-$ is a saddle point then $w(x) \equiv 1 + \dfrac{1}{|r_+\{\phi^{(1)}_x(x)\}|}$, or equivalently, $w(x) \equiv 1 + e^{-k^+ x}$;

(3) if $\tilde{\phi}_+$ is a saddle point then $w(x) \equiv 1 + \dfrac{1}{|r_-\{\phi^{(1)}_x(x)\}|}$, or equivalently, $w(x) \equiv 1 + e^{-k^- x}$; and

(4) if neither $\tilde{\phi}_-$ nor $\tilde{\phi}_+$ is a saddle point then $w(x) \equiv 1 + \dfrac{1}{|\phi^{(1)}_x(x)|}$, or equivalently, $w(x) \equiv 1 + e^{-k^- x} + e^{-k^+ x}$.

Lemma (7.3), lemma (7.4), and theorem (7.5) are very similar to lemma (4.3), lemma (4.4), and theorem (4.5) in Chapter IV. The upper and lower functions contained in lemmas (7.3) and (7.4) are natural extensions of those developed in lemmas (4.3) and (4.4) of Chapter IV, and the stability results in theorem (7.5) are very similar to the results in theorem (4.5). In fact, basically theorem (7.5) shows that bounded strictly monotonic steady state solutions $\tilde{u}(t,x) = \tilde{\phi}(x)$ of system (7.2) are stable with respect to small initial perturbations which are

(1) bounded as $x \to -\infty$ (as $x \to +\infty$) when $\tilde{\phi}(x)$ goes to a first order saddle point at $x = -\infty$ (at $x = +\infty$), and

(2) decay asymptotically no slower than $\phi_x^{(1)}(x)$ as $x \to -\infty$ (as $x \to +\infty$) when $\tilde{\phi}(x)$ goes to a first order node at $x = -\infty$ (at $x = +\infty$).

There are two unexpected limitations of the stability theory developed by theorem (7.2), lemma (7.3), lemma (7.4), and theorem (7.5). The first is that we needed to assume a consistency relation between the matrices A^- and A^+. Namely, we needed to assume that there is a vector $\tilde{\alpha}$ with all positive components such that

$$\sum_j A_{ij}^- \alpha^{(j)} < 0 \quad \text{and} \quad \sum_j A_{ij}^+ \alpha^{(j)} < 0 \quad \text{for all} \quad i = 1,\ldots,m \ .$$

This condition could very well be unnecessary.

The second unexpected limitation of the theory is that it does not treat quasi-monotonic steady states $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$. (Recall that quasi-monotonic steady states are solutions $\tilde{\phi}(x)$ which have some $\phi^{(i)}(x)$ increasing for all x and some $\phi^{(i)}(x)$ decreasing for all x). Although this limitation is unexpected, some such general limitation is necessary. To see this, consider the following common situation. Let $m = 2$, and suppose that $\tilde{\phi}(x) \equiv (\phi^{(1)}(x), \phi^{(2)}(x))$ is a strictly quasi-monotonic steady state solution of

$$f^{(i)}(\phi_{xx}^{(i)}, \phi_x^{(i)}, \tilde{\phi}(x)) + c\phi_x^{(i)} = 0 \, , \; i = 1 \text{ and } 2 \ .$$

Suppose also that $\phi^{(1)}$, $\phi_x^{(1)}$, $\phi_{xx}^{(1)}$, $\phi^{(2)}$, $\phi_x^{(2)}$, and $\phi_{xx}^{(2)}$ all decay asymptotically at the same exponential rate as $x \to +\infty$ and that they all decay at the same rate as $x \to -\infty$. Finally, let us suppose that there is a $\delta > 0$ such that

$$f_{3j}^{(i)}(\phi_{xx}^{(i)}, \phi_x^{(i)}, \tilde{\phi}) \geq \delta > 0 \quad \text{for } j \neq i, \text{ all } i, \text{ and all } x \ .$$

Then a short calculation shows that

$$\underline{u}^{(i)}(t,x) \equiv \phi^{(i)}(x) + |\phi_x^{(i)}(x)|\varepsilon e^{st} \qquad i = 1 \text{ and } 2$$

is an lower function of (7.2) for all s sufficiently small and all ε and t such that $0 < \varepsilon e^{st} < \varepsilon_0$ (for some $\varepsilon_0 > 0$). Thus $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ is $\mathfrak{C}^w$-unstable with $w(x) \equiv 1 + \dfrac{1}{|\phi_x^{(1)}(x)|}$, and so we conclude that quasi-monotonic steady states are commonly less stable than monotonic steady states.

This completes our development of the stability theory for constant and monotonic steady state solutions of (7.2). In brief, the stability theory we have developed is very similar to that of section (4.2) and sections (4.4) through (4.6) of Chapter IV. However this stability theory is incomplete. We have not

(1) found whether the consistency relations between A^- and A^+ (in lemma (7.4) and theorem (7.5)) are necessary,

(2) discovered whether all quasi-monotonic steady states are unstable,

(3) treated the cases where $\tilde{\phi}(x)$ is monotonic but not strictly monotonic,

(4) treated the cases where either A^- or A^+ is reducible or singular, or

(5) extended the results to traveling plane wave solutions of (7.1).

Finally, note that for any particular steady state solution of any particular system of equations the methods we used in this section may very useful even when the general theory we developed is not applicable.

7.2 Instability of non-monotonic waves. We have not been able to establish the instability of non-monotonic steady state solutions $\tilde{u}(t,x) = \tilde{\phi}(x)$ of

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) + cu_x^{(i)} \quad . \qquad (7.2)$$

The problems involved in extending the instability results of theorem (4.6) to systems of equations are precisely the same problems that arose in section (6.2) when we discussed the possible extension of theorem (4.6) to equations containing integrals. Since these problems are discussed in section (6.2), we will not discuss them here.

The situation is very similar to that in section (6.2). An interesting conjecture is that the instability of non-monotonic steady state solutions $\tilde{u}(t,x) \equiv \tilde{\phi}(x)$ of system (7.2) is exactly the same as the instability of steady state solutions of equation (4.2). One can prove that system (7.2) has a hair-trigger effect, and so to prove this conjecture one needs only to develope intersection results for the ordinary differential system of equations

$$f^{(i)}(\phi_{xx}^{(i)},\phi_x^{(i)},\tilde{\phi}) + c\phi_x^{(i)} = 0 \qquad i = 1,\ldots,m$$

that are similar to lemmas (4.7) and (4.8). If such intersection results were established, the instability of non-monotonic steady state solutions would immediately follow.

We now continue to the next section, where we extend the mean wavespeed initial condition results of Chapter V to the system of equations

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)},u_x^{(i)},\tilde{u}) \quad . \tag{7.1}$$

7.3 Mean wavespeed and the initial conditions. In this section we will extend the mean wavespeed/initial condition results of Chapter V to the system of equations

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)},u_x^{(i)},\tilde{u}) \qquad i = 1,\ldots,m \quad . \tag{7.1}$$

Recall that in Chapter V we considered equations

$$u_t = f(u_{xx},u_x,u) \tag{5.1}$$

which have a non-constant bounded monotonic solution $u(t,x) = \phi(x-ct,c)$. For each major case of $\phi(x-ct,c)$ being a $S \to S$, a $N \to S$, a $S \to N$, and a $N \to N$ type monotonic wave, we determined

(1) when the existence of $\phi(x-ct,c)$ implies the existence or non-existence of similar monotonic waves at nearby wavespeeds,

(2) when the existence of $\phi(x-ct,c)$ implies the existence or non-existence of similar monotonic waves at the same wavespeed c, and

(3) the mean wavespeed of $u(t,x)$ in terms of $u(0,x)$.

We will not try to extend the existence/non-existence results for solutions of (5.1) to solutions of system (7.1). Presumably for any specific system of the form (7.1), one could establish existence/non-existence results by employing knowledge of the asymptotic behavior (as $x \to \pm\infty$) of solutions of

$$f^{(i)}(\phi^{(i)}_{xx}, \phi^{(i)}_{x}, \tilde{\phi}) + c\phi^{(i)}_{x} = 0 \qquad (7.17)$$

in conjunction with continuity arguments. However, in this chapter we will only extend the mean wavespeed/initial condition results of Chapter V to system (7.1). Since the proofs of these mean wavespeed results are very similar to the proofs of theorems (5.2) in Chapter V, we will simply quote (and not prove) the mean wavespeed/initial condition results here. These results are:

<u>Theorem 7.6 ($S \to S$):</u> Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that $\tilde{u}(t,x) \equiv \tilde{\phi}(x-ct)$ is a bounded strictly monotonic solution of

$$u^{(i)}_{t} = f^{(i)}(u^{(i)}_{xx}, u^{(i)}_{x}, \tilde{u}) \qquad i = 1,\ldots,m \quad . \qquad (7.1)$$

Define $\tilde{\phi}(-\infty) \equiv \tilde{\phi}_-$ and $\tilde{\phi}(+\infty) \equiv \tilde{\phi}_+$, and assume that both $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are first order saddle points. Also, define the matrices A^- and A^+ by

$$A^-_{ij} \equiv f^{(i)}_{3j}(0,0,\tilde{\phi}_-) \qquad A^+_{ij} \equiv f^{(i)}_{3j}(0,0,\tilde{\phi}_+)$$

and assume that there is a vector $\tilde{\alpha}$ with all positive components such that

$$\sum_j A^-_{ij}\,\alpha^{(j)} < 0 \quad \text{and} \quad \sum_j A^+_{ij}\,\alpha^{(j)} < 0 \quad \text{for all} \quad i = 1,\ldots,m \ .$$

If $\tilde{\phi}(x)$ satisfies these assumptions, then whenever $\tilde{u}(t,x)$ is any solution of system (7.1) whose initial condition $\tilde{u}(0,x)$ is in H^2_x and satisfies

$$\phi^{(i)}_- - \alpha' < u^{(i)}(0,x) < \phi^{(i)}_- + \alpha' \quad \text{for all} \quad i, \quad \text{all } x \leq - x_0 \quad , \qquad (7.18a)$$

$$\phi^{(i)}_+ - \alpha' < u^{(i)}(0,x) < \phi^{(i)}_+ + \alpha' \quad \text{for all} \quad i, \quad \text{all } x \geq + x_0 \ , \text{ and} \qquad (7.18b)$$

$$\min\{\phi^{(i)}_-,\phi^{(i)}_+\} - \alpha' < u^{(i)}(0,x) < \max\{\phi^{(i)}_-,\phi^{(i)}_+\} + \alpha' \quad \text{for all } i \text{ and all } x, \qquad (7.18c)$$

for any $\alpha' > 0$ sufficiently small and any x_0, then $u(t,x)$ must travel with mean wavespeed c.

Theorem 7.6 $(N \to S)$: Assume that hypotheses H2, H3, and H4 are satisfied. Suppose that there is a c_1 and a $c_2 \geq c_1$ such that for each c in $[c_1,c_2]$, there exists a bounded strictly monotonic solution $\tilde{u}(t,x) \equiv \tilde{\phi}(x-ct)$ of

$$u^{(i)}_t = f^{(i)}(u^{(i)}_{xx},u^{(i)}_x,\tilde{u}) \qquad i = 1,\ldots,m \ . \qquad (7.1)$$

Suppose also that for each c in $[c_1,c_2]$ that $\tilde{\phi}(-\infty,c) = \tilde{\phi}_-$, that $\tilde{\phi}(+\infty,c) = \tilde{\phi}_+$, and that $\tilde{\phi}(x,c)$ and $\frac{\partial}{\partial x}\tilde{\phi}(x,c)$ are continuously differentiable in c. Further, assume that $\tilde{\phi}_-$ is a regular singular point of order one and that $\tilde{\phi}_+$ is a first order saddle point. Finally, define $k^-(c) > 0$ by

$$\tilde{\phi}(x,c) = \tilde{\phi}_- + \tilde{a}(c)e^{k^-(c)x} + o(e^{(k^-(c)+\delta)x}) \quad \text{as} \quad x \to -\infty$$

where $\tilde{a}(c) \neq \tilde{0}$ and δ is positive, and assume that $k^-(c)$ is continuous in c.

If $\tilde{\phi}(x,c)$ satisfies the above assumptions, then whenever $\tilde{u}(t,x)$ is <u>any</u> solution of system (7.1) whose initial condition $\tilde{u}(0,x)$ is in H_x^2 and satisfies

$$\phi_+^{(i)} - q_0 \le u^{(i)}(0,x) \le \phi_+^{(i)} + q_0 \qquad i = 1,\ldots,m$$
$$\text{for all } x > x_0 \text{ for any } x_0 \quad , \tag{7.19a}$$

$$\phi_-^{(i)} \le u^{(i)}(0,x) \le \phi_+^{(i)} + q_0 \qquad i = 1,\ldots,m$$
$$\text{for all } x \text{ if } \tilde{\phi}(x,c) \text{ is increasing in } x, \text{ and} \tag{7.19b}$$

$$\phi_+^{(i)} - q_0 \le u^{(i)}(0,x) \le \phi_-^{(i)} \qquad i = 1,\ldots,m$$
$$\text{for all } x \text{ if } \tilde{\phi}(x,c) \text{ is decreasing in } x, \tag{7.19c}$$

then we can conclude the following:

(1) if for any c in $[c_1,c_2]$ there is a $\beta > 0$ such that $e^{-k^-(c)x}|u^{(i)}(0,x) - \phi_-^{(i)}| > \beta$ $(i = 1,\ldots,m)$ for all $x < 0$ and if $q_0 > 0$ is sufficiently small, then $\tilde{u}(t,x)$ cannot travel with mean wavespeed larger than c;

(2) if for any c in $[c_1,c_2]$ there is a $\beta > 0$ such that $e^{-k^-(c)x}|u^{(i)}(0,x) - \phi_-^{(i)}| < \beta$ $(i = 1,\ldots,m)$ for all $x < 0$ and if $q_0 > 0$ is sufficiently small, then $\tilde{u}(t,x)$ cannot travel with mean wavespeed smaller than c;

(3) if for any c in $[c_1,c_2]$ there is a β_1, $\beta_2 > 0$ such that $\beta_1 < e^{-k^-(c)x}|u^{(i)}(0,x)-\phi_-^{(i)}| < \beta_2$ $(i = 1,\ldots,m)$ for all $x < 0$ and if $q_0 > 0$ is sufficiently small, then $\tilde{u}(t,x)$ must travel with mean wavespeed c and have finite dispersion; and

(4) if for any c in (c_1,c_2)

$$\lim_{x\to-\infty} e^{(-k^-(c)+\mu)x}|u^{(i)}(0,x)-\phi_-^{(i)}| = 0 \quad (i = 1,\ldots,m) \text{ for all } \mu > 0 \quad ,$$

$$\lim_{x\to-\infty} e^{-(k^-(c)+\mu)x}|u^{(i)}(0,x)-\phi_-^{(i)}| = +\infty \quad (i = 1,\ldots,m) \text{ for all } \mu > 0 \quad ,$$

and if $q_0 > 0$ is sufficiently small, then $\tilde{u}(t,x)$ travels with mean wavespeed c but may not have finite dispersion.

Theorem 7.6 ($N \to N$): Assume that hypotheses H2, H3, and H4 are satisfied. Suppose there is a c_1 and a $c_2 \geq c_1$ such that for each c in $[c_1,c_2]$ there exists a bounded strictly monotonic solution $\tilde{u}(t,x) \equiv \tilde{\phi}(x-ct)$ of

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) \quad i = 1,\ldots,m \quad . \tag{7.1}$$

Suppose also that for each c in $[c_1,c_2]$ that $\tilde{\phi}(-\infty,c) = \tilde{\phi}_-$, that $\tilde{\phi}(+\infty,c) = \tilde{\phi}_+$, and that $\tilde{\phi}(x,c)$ and $\frac{\partial}{\partial x}\tilde{\phi}(x,c)$ are continuously differentiable in c. Further, assume that $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are both regular singular points of order one. Finally, define $k^-(c) > 0$ and $k^+(c) < 0$ by

$$\tilde{\phi}(x,c) = \tilde{\phi}_- + \tilde{a}(c)e^{k^-(c)x} + o(e^{(k^-(c)+\delta)x}) \quad \text{as} \quad x \to -\infty$$

$$\tilde{\phi}(x,c) = \tilde{\phi}_+ + \tilde{b}(c)e^{k^+(c)x} + o(e^{(k^+(c)-\delta)x}) \quad \text{as} \quad x \to +\infty$$

(where $\tilde{a}(c) \neq \tilde{0}$, $\tilde{b}(c) \neq \tilde{0}$, and δ is positive) and assume that $k^-(c)$ and $k^+(c)$ are continuous in c.

If $\tilde{\phi}(x,c)$ satisfies the above assumptions, then whenever $\tilde{u}(t,x)$ is any solution of system (7.1) whose initial condition $\tilde{u}(0,x)$ is in H_x^2 and satisfies

$$\min\{\phi_-^{(i)},\phi_+^{(i)}\} < u^{(i)}(0,x) < \max\{\phi_-^{(i)},\phi_+^{(i)}\} \quad i = 1,\ldots,m \quad \text{for all} \quad x$$

then we can conclude the following:

(1) if for any c in $[c_1,c_2]$ there is a $\beta_1, \beta_2 > 0$ such that

$\beta_1 < e^{-k^-(c)x}|u^{(i)}(0,x)-\phi_-^{(i)}|$ $(i = 1,\ldots,m)$ for all $x \leq 0$ and

$\beta_2 > e^{-k^+(c)x}|u^{(i)}(0,x)-\phi_+^{(i)}|$ $(i = 1,\ldots,m)$ for all $x \geq 0$,

then $\tilde{u}(t,x)$ cannot travel with mean wavespeed larger than c;

(2) if for any c in $[c_1,c_2]$ there is a $\beta_1, \beta_2 > 0$ such that

$\beta_1 > e^{-k^-(c)x}|u^{(i)}(0,x)-\phi_-^{(i)}|$ $(i = 1,\ldots,m)$ for all $x \leq 0$ and

$\beta_2 < e^{-k^+(c)x}|u^{(i)}(0,x)-\phi_+^{(i)}|$ $(i = 1,\ldots,m)$ for all $x \geq 0$,

then $\tilde{u}(t,x)$ cannot travel with mean wavespeed smaller than c;

(3) if for any c in $[c_1,c_2]$ there is a $\beta_1, \beta_2, \beta_3, \beta_4 > 0$ such that

$\beta_1 < e^{-k^-(c)x}|u^{(i)}(0,x)-\phi_-^{(i)}| < \beta_2$ $i = 1,\ldots,m$ for all $x \leq 0$ and

$\beta_3 < e^{-k^+(c)x}|u^{(i)}(0,x)-\phi_+^{(i)}| < \beta_4$ $i = 1,\ldots,m$ for all $x \geq 0$,

then $\tilde{u}(t,x)$ travels with mean wavespeed c and has finite dispersion; and

(4) if for any c in (c_1,c_2)

$$\lim_{x\to-\infty} e^{-(k^-(c)-\mu)x}|u^{(i)}(0,x)-\phi_-^{(i)}| = 0 \quad ,$$

$$\lim_{x\to-\infty} e^{-(k^-(c)+\mu)x}|u^{(i)}(0,x)-\phi_-^{(i)}| = +\infty \quad ,$$

$$\lim_{x\to+\infty} e^{-(k^+(c)-\mu)x}|u^{(i)}(0,x)-\phi_+^{(i)}| = +\infty \quad ,$$

$$\lim_{x\to+\infty} e^{-(k^+(c)+\mu)x}|u^{(i)}(0,x)-\phi_+^{(i)}| = 0$$

for all $i = 1,\ldots,m$ and for all $\mu > 0$, then $\tilde{u}(t,x)$ travels with mean wavespeed c (but may not have finite dispersion).

Basically, theorems 7.6 $(S \to S)$, $(N \to S)$, and $(N \to N)$ show that for large classes of initial conditions $\tilde{u}(0,x)$, the mean wavespeed of the solution $\tilde{u}(t,x)$ of system (7.1) depends entirely on

(1) the asymptotic decay rate of $\tilde{u}(0,x)$ to $\tilde{\phi}_-$ as $x \to -\infty$ if $\tilde{\phi}_-$ is not a first order saddle point, and

(2) the asymptotic decay rate of $\tilde{u}(0,x)$ to $\tilde{\phi}_+$ as $x \to +\infty$ if $\tilde{\phi}_+$ is not a first order saddle point.

Theorems 7.6 $(S \to S)$, $(N \to S)$, and $(N \to N)$ are very similar to theorems 5.2 $(S \to S)$, $(N \to S)$, and $(N \to N)$ of Chapter V. The major differences are that

(1) in theorems (7.6) conclusions are drawn about the mean wavespeeds of vector solutions $\tilde{u}(t,x)$ of system (7.1), whereas in theorems (5.2) the conclusions are drawn for solutions of equation (5.1);

(2) for theorems 7.6 $(N \to S)$ and $(N \to N)$ we needed to assume the existence of all the monotonic traveling waves we used, whereas in theorems 5.2 $(N \to S)$ and $(N \to N)$ we needed only to assume the existence of a single monotonic traveling wave; and

(3) the restrictions on the initial conditions $\tilde{u}(0,x)$ in theorems (7.6) are very similar to those on $u(0,x)$ in theorems (5.2) except that for theorem (7.6) the restrictions are for $u^{(i)}(0,x)$ for <u>all</u> i.

This completes our presentation of the mean wavespeed/initial condition results for system (7.1). We complete this chapter in the next section with some general remarks.

<u>7.4 Some general remarks.</u> We have found in this chapter that the stability results for monotonic traveling waves can be extended to some systems of equations of the form

$$u_t^{(i)} = f^{(i)}(u_{xx}^{(i)}, u_x^{(i)}, \tilde{u}) \qquad i = 1,\dots,m \quad . \tag{7.1}$$

We also found that the mean wavespeed/initial condition results are readily extended to system (7.1). However, we have been unable to extend the instability results for non-monotonic waves to system (7.1).

In this chapter we have developed only the most readily obtainable results about system (7.1). Our treatment of system (7.1) is correspondingly incomplete. Many extensions of the general theory should be possible. Of these, extending the instability results for non-monotonic traveling waves and steady states to solutions of system (7.1) is perhaps the most interesting.

The class of systems of equations of the form (7.1) (which satisfy hypotheses H3) is quite limited. This suggests that a better approach may be to treat each specific physical example separately. Instead of trying to use our methods to extend the general theory for systems like (7.1), perhaps we should try to utilize these methods to obtain specific results for each specific example that arises.

This completes our treatment of system (7.1). In the next chapter, Chapter VIII, we shall utilize the results in Chapters IV, V, VI, and VII on several examples.

Chapter VIII

EXAMPLES

In this chapter we apply the results of the preceding chapters on some illustrative examples. We begin by treating Burger's equation. In section (8.2) we examine Fischer's equation. An artificial (but illustrative) example is considered in section (8.3). We treat an equation containing integrals in section (8.4). Finally, our last example will be a reaction diffusion system, which will be examined in section (8.5). We now begin with Burger's equation.

8.1 Burger's equation. We have chosen to examine Burger's equation,

$$u_t = u_{xx} - uu_x \quad , \tag{8.1}$$

because it is the simplest non-trivial example of equations

$$u_t = f(u_{xx}, u_x, u) \tag{8.2}$$

which have $f(0,0,u) = 0$ for all u. This type of equation does not have a discrete set of stable (saddle point) and unstable (node, spiral point, and center) constant steady states. Instead, this type of equation has a continuum of neutrally stable constant steady states. Since the stability theorems (in Chapter IV) for monotonic waves $u(t,x) = \phi(x-ct)$ do not distinguish between $\phi(\pm \infty)$ being unstable and neutrally stable constant steady states, we do not expect the stability theorems to be sharp in this case. Thus, Burger's equation points out an important limitation of the stability theorems for monotonic waves.

In this section we first determine the monotonic traveling wave solutions of Burger's equation. Next, we apply the stability theorems directly to these solutions. We then exhibit some new upper and lower functions. By utilizing these upper and lower functions we find sharp stability results.

Burger's equation is also interesting because it possesses important unsteady solutions, such as the "single hump" solutions. As the last topic in this section, we use the maximum principle to show that these "single hump" solutions have at least a limited stability.

We begin by finding the bounded solutions of the form $u(t,x) = \phi(x-ct)$. These solutions must satisfy

$$\phi' = v$$
$$v' = v(\phi-c) \quad .$$

The phase plane representation of these solutions is illustrated in Figure (1) below, and the bounded solutions are given by

$$u(t,x,c,\alpha) = \phi(x-ct,c,\alpha) \quad \text{where} \quad \phi(x,c,\alpha) \equiv c - \alpha \tanh \tfrac{1}{2}\alpha x \ .$$

Neither $\phi(-\infty,c,\alpha)$ nor $\phi(+\infty,c,\alpha)$ is a first order saddle point, and so theorem (4.5) implies only that

$$u(t,x,c,\alpha) \equiv \phi(x-ct,c,\alpha)$$

is C^w-stable with $w = 1 + e^{\alpha(x-ct)} + e^{-\alpha(x-ct)}$. In other words, in the coordinate system which moves with speed c, theorem (4.5) shows that the traveling wave $\phi(x-ct,c,\alpha)$ is stable to small initial perturbations which decay asymptotically no slower than $e^{+|\alpha|x}$ as $x \to -\infty$ and no slower than $e^{-|\alpha|x}$ as $x \to +\infty$.

Since theorem (4.5) does not distinguish between $\phi(\pm\infty,c,\alpha)$ being unstable and neutrally stable constant steady states, we should be able to do better. Indeed, suppose that $u(t,x,c,\alpha) \equiv \phi(x-ct,c,\alpha)$ is any of the above monotonic traveling wave solutions. Then for any $\varepsilon > 0$ sufficiently small and any δ in $(0,1)$,

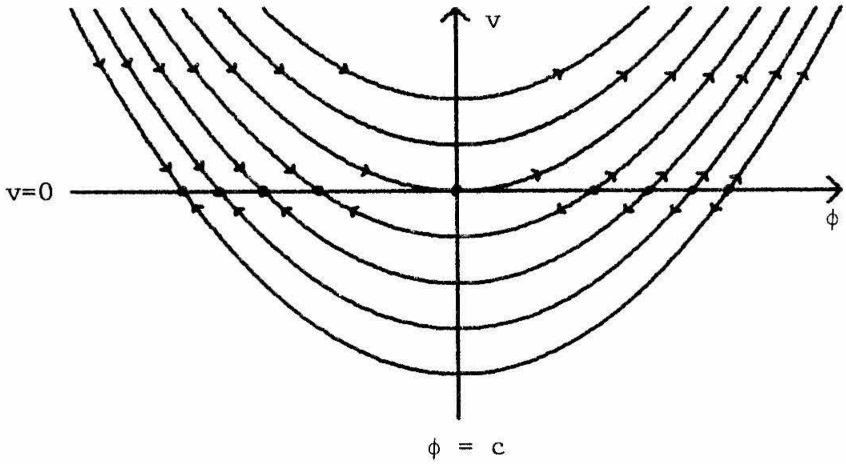


Figure (1): Phase plane representation of the solutions $u(t,x) = \phi(x-ct)$ of Burger's equation.

$$\overline{u}(t,x,c,\alpha,\varepsilon,\delta) \equiv \phi(x-ct+h(t)) + \eta(t) \operatorname{sech} \varepsilon(x-ct+h(t)) \quad \text{and} \qquad (8.3a)$$

$$\underline{u}(t,x,c,\alpha,\varepsilon,\delta) \equiv \phi(x-ct-h(t)) - \eta(t) \operatorname{sech} \varepsilon(x-ct-h(t)), \text{ where} \quad (8.3b)$$

$$\eta(t) = \delta\varepsilon^4 e^{-\varepsilon^3 t} \qquad h(t) = -2\delta\varepsilon(1-e^{-\varepsilon^3 t}) \quad , \qquad (8.3c)$$

are upper and lower functions (respectively) of Burger's equation. The maximum principle implies that if $u(t,x)$ is any solution of Burger's equation whose initial condition $u(0,x)$ satisfies

$$\underline{u}(0,x+h_0,c,\alpha,\varepsilon,\delta) \le u(0,x) \le \overline{u}(0,x+h_1,c,\alpha,\varepsilon,\delta) \quad \text{for all } x ,$$

then $u(t,x)$ must also satisfy

$$\underline{u}(t,x+h_0,c,\alpha,\varepsilon,\delta) \le u(t,x) \le \overline{u}(t,x+h_1,c,\alpha,\varepsilon,\delta) \quad \text{for all } x, \text{ all } t \ge 0 . \qquad (8.4)$$

From the expressions for $\underline{u}$ and $\overline{u}$ in equations (8.3), we see that relation (8.4) implies that

$$u(t,x,c,\alpha) \equiv \phi(x-ct,c,\alpha)$$

is C^w-stable with $w \equiv 1 + e^{+\varepsilon(x-ct)} + e^{-\varepsilon(x-ct)}$ for all $\varepsilon > 0$ sufficiently small. That is, in the coordinate system which moves with speed c, the traveling wave $\phi(x-ct,c,\alpha)$ is stable to small initial perturbations which decay asymptotically no slower than $e^{+\varepsilon x}$ as $x \to -\infty$ and $e^{-\varepsilon x}$ as $x \to +\infty$ for <u>any</u> $\varepsilon > 0$. This stability result is much stronger than the one obtained from theorem (4.5) since $\varepsilon > 0$ can be as small as we wish.

Since Burger's equation is of the form

$$u_t = f(u_{xx},u_x,u) \quad \text{with} \quad f(0,0,u) = 0 \quad \text{for all} \quad u \ ,$$

the stability theorems of Chapter IV yield rather poor results. However, by using the techniques of Chapter IV we found good (in fact sharp) stability results. One expects that any other specific equation of the form

$$u_t = f(u_{xx},u_x,u) \quad \text{with} \quad f(0,0,u) = 0 \quad \text{for all} \quad u$$

can be treated in the same manner. Namely, for any specific monotonic wave solution of any specific equation, one should be able to obtain sharp stability results by constructing appropriate upper and lower functions.

Besides the monotonic steadily progressing and steady solutions $\phi(x-ct,c,\alpha)$, Burger's equation also possesses important unsteady solutions. One class of these solutions are the triangular "single hump" waves. These waves are given by

$$u(t,x,c,\alpha,t_0) = c + \frac{1}{\sqrt{t+t_0}} R\left(\frac{x-ct}{2\sqrt{t+t_0}}, \quad \alpha\right) \ ,$$

where

$$R(\zeta,\alpha) \equiv \frac{\alpha e^{-\zeta^2}}{\sqrt{\pi}+\alpha\int_\zeta^\infty e^{-s^2} ds} \ ,$$

and where $t_0 > 0$. (See e.g. reference [10]). It is easy to use the maximum principle to show that these single hump waves have at least a limited stability. We note that $R_\alpha(\zeta,\alpha) > 0$ for all α and ζ. Hence

$$u(t,x,c,\alpha-\varepsilon,t_0) < u(t,x,c,\alpha,t_0) < u(t,x,c,\alpha+\varepsilon,t_0) \quad \text{for all} \quad \varepsilon > 0 \ .$$

Thus whenever $u(t,x)$ is <u>any</u> solution of Burger's equation whose initial conditions $u(0,x)$ satisfy

$$u(0,x,c,\alpha-\varepsilon,t_0) \le u(0,x) \le u(0,x,c,\alpha+\varepsilon,t_0) \quad \text{for all} \quad x \quad ,$$

then $u(t,x)$ must also satisfy

$$u(t,x,c,\alpha-\varepsilon,t_0) \le u(t,x) \le u(t,x,c,\alpha+\varepsilon,t_0) \quad \text{for all} \quad x \quad \text{and all} \quad t \ge 0 \; . \tag{8.5}$$

This is illustrated in Figure (2) below, where the implication of relation (8.5) is that $u(t,x)$ must remain in the shaded region for all $t \ge 0$. Clearly this implies that single hump waves have at least a limited stability

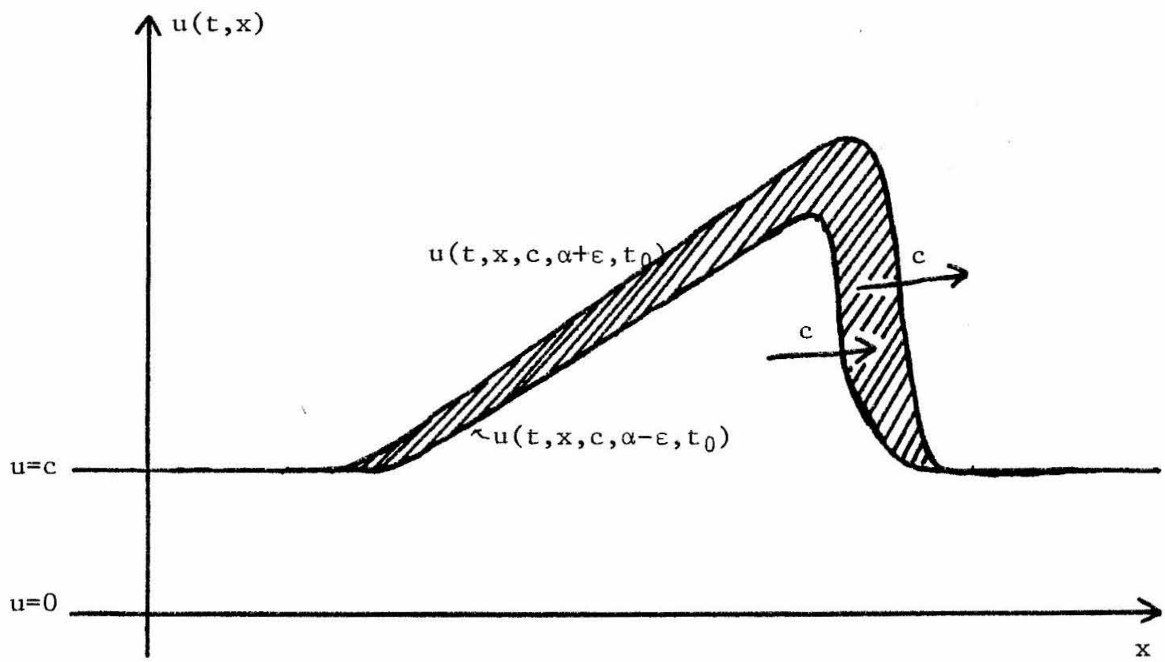


Figure (2): Relation (8.5) implies that $u(t,x)$ must remain in the shaded region for all $t \ge 0$.

This concludes our brief look at Burger's equation. Briefly, we found that the stability theorems of Chapter IV yield rather poor stability results for the monotonic wave solutions. However, by utilizing some upper and lower functions we were able to show that all bounded steadily progressing and steady solutions are stable to all small smooth initial perturbations which decay asymptotically no slower than $e^{\varepsilon x}$ as $x \to -\infty$ and $e^{-\varepsilon x}$ as $x \to +\infty$ for some $\varepsilon > 0$. We also found that the "single hump" solutions have at least a limited stability.

8.2 Fischer's equation. In this section we briefly treat Fischer's equation,

$$u_t = u_{xx} + u(1-u) \quad . \tag{8.6}$$

For each wavespeed c we will find the bounded solutions $u(t,x) = \phi(x-ct)$ of equation (8.6) by examining the phase plane of

$$\begin{aligned} \phi_x &= v \\ v_x &= -cv - \phi(1-\phi) \quad . \end{aligned} \tag{8.7}$$

We will then apply the stability and instability results of Chapter IV. We will not apply the mean wavespeed results of Chapter V since the application is simple and the results are uninteresting. We now carry out this program.

First suppose $c \leq -2$. Then $\phi = 0$, $v = 0$ is an unstable node and $\phi = 1$, $v = 0$ is a saddle point. The phase plane of (8.7) is roughly sketched in Figure (3) for c having any fixed value ≤ -2. As shown in the sketch, the only bounded traveling wave solutions $u(t,x) \equiv \phi(x-ct,c)$ are

(1) the constant traveling wave solution $u(t,x) \equiv \phi_0(x-ct,c) \equiv 0$,

(2) the constant traveling wave solution $u(t,x) \equiv \phi_1(x-ct,c) \equiv 1$,

(3) the monotonic traveling wave solution $u(t,x) \equiv \phi_{NS}(x-ct,c)$

labeled by a (*) in Figure (3).

The constant traveling wave $\phi_0(x-ct,c) \equiv 0$ is a node, and hence it is $\mathbb{C}^w$-unstable with $w(x) = 1 + e^{\kappa x} + e^{-\kappa x}$ for all $\kappa > 0$ sufficiently small. (Recall that we always define stability of a traveing wave in terms of a coordinate system which travels at the same velocity as the wave). On the other hand, the constant traveling wave $\phi_1(x-ct,c) \equiv 1$ is a saddle point, and hence it is C^w-stable with $w(x) \equiv 1$. Finally, the monotonic traveling wave $u(t,x) = \phi_{NS}(x-ct,c)$ decays at the <u>usual</u> rate to the node $\phi = 0, v = 0$ as $x \to -\infty$ and goes to the saddle point $\phi = 1, v = 0$ as $x \to +\infty$. In particular,

$$\phi_{NS}(x,c) \sim a(c)e^{k^-(c)x} \text{ as } x \to -\infty$$

where $a(c)$ is some positive constant and

$$k^-(c) = \tfrac{1}{2}\left[-c+\sqrt{c^2-4}\,\right] \quad .$$

Thus, theorem (4.5) implies that $u(t,x) \equiv \phi_{NS}(x-ct,c)$ is C^w-stable with $w(x) \equiv 1 + e^{-k^-(c)x}$. Moreover, since $\phi = 0, v = 0$ and $\phi = 1, v = 0$ are both first order singular points and since $\phi(x,c)$ decays at the usual rate as $x \to -\infty$, theorem 5.1 $(N \to S)$ shows that this stability result is sharp. Note that the stability of $\phi_{NS}(x-ct,c)$ <u>decreases</u> as c increases, and in particular that the monotonic traveling wave of speed $c = -2$ is the <u>least</u> stable.

We now consider $-2 < c < 0$. At these wavespeeds c, the point $\phi = 0, v = 0$ is an unstable spiral point and the point $\phi = 1, v = 0$ is a saddle point. The phase plane of system (8.7) is sketched for these values of c in Figure (4). As shown in the sketch, the only bounded traveling wave solutions $u(t,x) \equiv \phi(x-ct,c)$ are now

(1) the constant traveling wave $u(t,x) \equiv \phi_0(x-ct,c) \equiv 0$,

(2) the constant traveling wave $u(t,x) \equiv \phi_1(x-ct,c) \equiv 1$, and

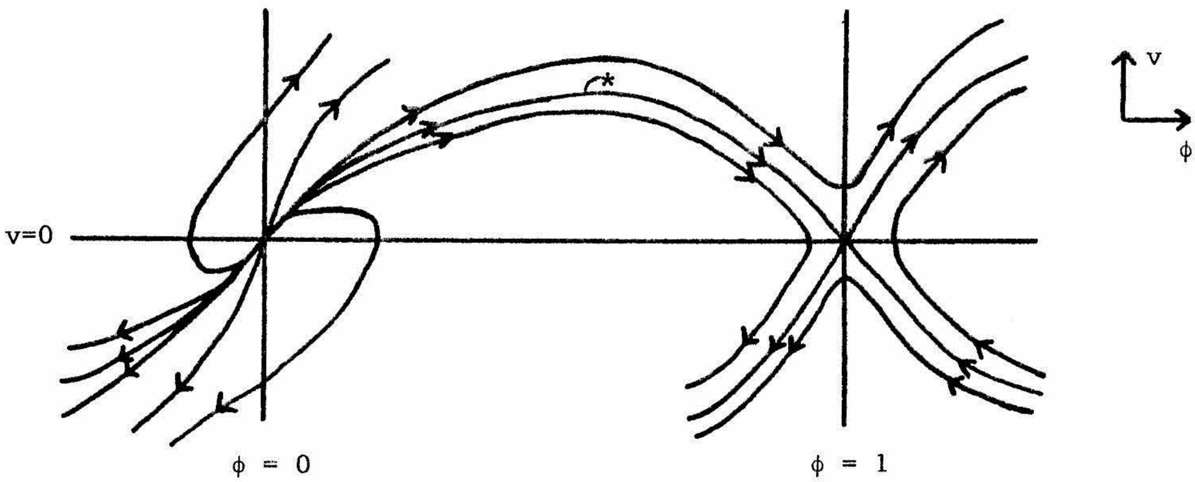


Figure (3): Rough sketch of the phase plane of system (8.7) for $c \leq -2$. Trajectory (*) represents the monotonic solution $\phi_{NS}(\overline{x})$.

(3) the non-monotonic traveling wave solution $u(t,x) \equiv \phi_{SpS}(x-ct,c)$ labeled by a (*) in Figure (4).

The constant traveling wave $\phi_0(x-ct,c) \equiv 0$ is a spiral point, and so it is very unstable at these wavespeeds (see section (4.15)). The constant traveling wave $\phi_1(x-ct,c) \equiv 1$ is a saddle point, and so it is C^w-stable with $w(x) \equiv 1$. Finally, the traveling wave solution $\phi_{SpS}(x-ct,c)$ goes to the spiral point $\phi = 0$, $v = 0$ as $x \to -\infty$ and goes to the saddle point $\phi = 1$, $v = 0$ as $x \to +\infty$. Since $\phi_{SpS}(x-ct,c)$ has an infinite number of relative extrema, theorem (4.6) shows that it is very unstable.

Finally, we consider $c = 0$. At this wavespeed, the point $\phi = 0$, $v = 0$ is a center and the point $\phi = 1$, $v = 0$ is a saddle point. The phase plane of system (8.7) is roughly sketched in Figure (5) below for $c = 0$. As shown in the sketch, the only bounded steady state solutions $u(t,x) \equiv \phi(x,0)$ are

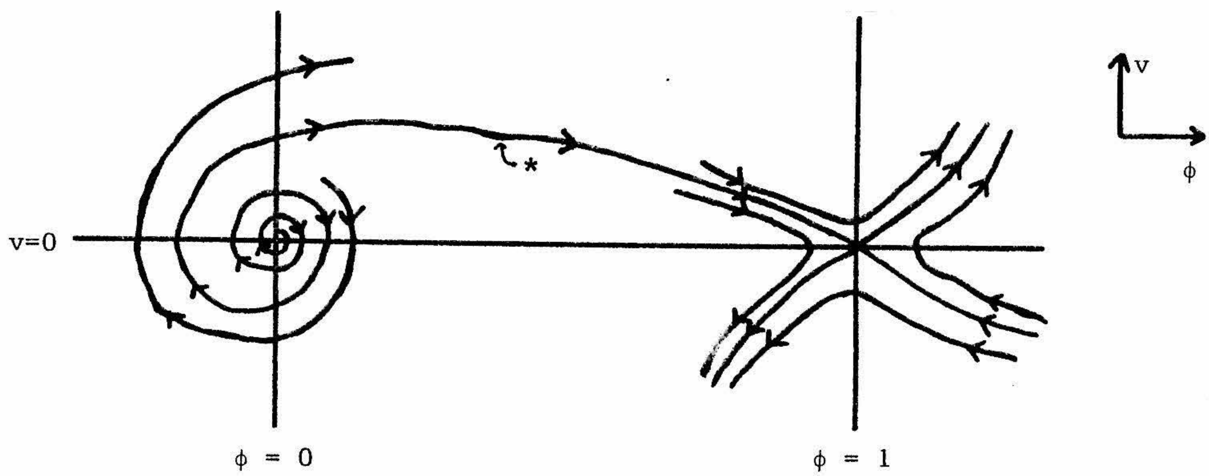


Figure (4): Rough sketch of the phase plane of system (8.7) for $-2<c<0$. Trajectory (*) represents the non-monotonic solution $\phi_{SpS}(x)$.

(1) the constant steady state $u(t,x) \equiv \phi_0(x,0) \equiv 0$,

(2) the constant steady state $u(t,x) \equiv \phi_1(x,0) \equiv 1$,

(3) the family of periodic steady state solutions $u(t,x) = \phi_{co}(x,0,\alpha)$, and

(4) the non-monotonic steady state solution $u(t,x) \equiv \phi_{SS}(x,0)$.

The constant steady states $\phi_0(x,0)$ and $\phi_1(x,1)$ have exactly the same stability as when $-2 < c < 0$. Therefore, we turn our attention to the periodic solutions $\phi_{co}(x,0,\alpha)$, which are represented in the phase plane in Figure (5) by the set of closed orbits. All these solutions have an infinite number of relative extrema, and so they are all very unstable. Now consider the non-monotonic steady state solution $u(t,x) \equiv \phi_{SS}(x,0)$. This solution has exactly one relative extrema and has $\phi_{SS}(-\infty,0) = \phi_{SS}(+\infty,0) = 1$. Since $\phi = 1$, $v = 0$ is a saddle point, and since

$$\phi_{SS}(x,0) \sim 1 - ae^{x} \quad \text{as} \quad x \to -\infty$$

$$\phi_{SS}(x,0) \sim 1 - be^{-x} \quad \text{as} \quad x \to +\infty$$

where a and b are some positive constants, part (2) of theorem (4.6) shows that $u(t,x) \equiv \phi_{SS}(x,0)$ is $\mathbb{C}^{w}$-unstable with $w(x) \equiv 1 + e^{-x} + e^{+x}$.

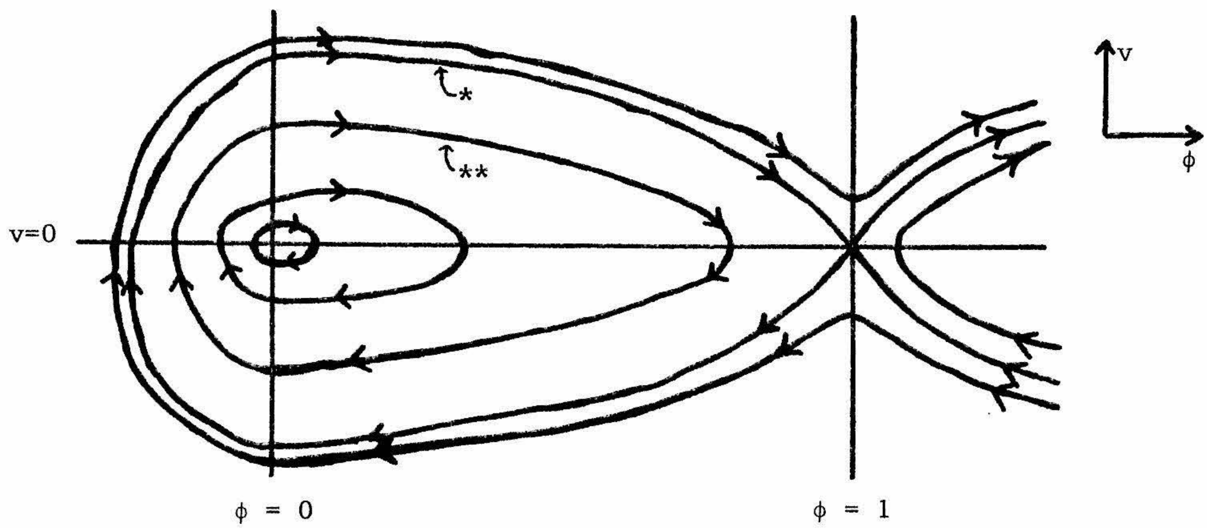


Figure (5): Rough sketch of the phase plane of system (8.7) at c=0. Trajectory (*) represent the non-monotonic saddle point - saddle point solution $\phi_{SS}(x,0)$. Trajectory (**) is a typical member of the family of periodic solutions $\phi_{co}(x,0,\alpha)$ represented by the closed orbits in the phase plane.

We will not consider the cases where $c > 0$. These cases can be reduced to the cases where $c < 0$ by utilizing the transformation $x \to -x$.

This completes our examination of Fischer's equation. In the next section we will examine an artificial (but illustrative) example.

<u>8.3 An illustrative example</u>. In this section we will examine the equation

$$u_t = u_{xx} + 4uu_x + \tfrac{1}{2}\cdot u(u-1)(u+1)(u-2)(u+2) \quad . \tag{8.8}$$

This equation is interesting because it possesses nearly every possible type of traveling wave and steady state solution. We will find the bounded solutions $u(t,x) = \phi(x-ct)$ of equation (8.8) by examining the phase plane of

$$\begin{aligned} \phi_x &= v \\ v_x &= -4\phi v - \tfrac{1}{2}\,\phi(\phi-1)(\phi+1)(\phi-2)(\phi+2) - cv \quad . \end{aligned} \tag{8.9}$$

We will then apply the results of Chapter IV to determine the stability of these solutions. For brevity we will not do this for all ranges of wavespeeds c. Instead, we will look at the steady states $(c = 0)$ and the traveling waves with wavespeed c in $(0, 8 - \sqrt{48})$. We will also ignore the relatively uninteresting constant solutions. We now do this.

First suppose $c = 0$. Then $\phi = -2$ and $\phi = +2$ are an unstable and a stable node, $\phi = -1$ and $\phi = +1$ are saddle points, and $\phi = 0$ is a center. The phase plane of system (8.9) at $c = 0$ is sketched in Figure (6). As labeled in the sketch, there are eight different types of bounded non-constant steady state solutions. We now consider these cases separately.

Case (1): The solution $u(t,x) = \phi^+_{SS}(x,0)$ is a monotonic (increasing) steady state with $\phi^+_{SS}(-\infty,0) = -1$ and $\phi^+_{SS}(+\infty,0) = +1$. Since both $\phi = -1$, $v = 0$ and $\phi = +1$, $v = 0$ are saddle points, $u(t,x) = \phi^+_{SS}(x,0)$ is C^w-stable with $w(x) \equiv 1$.

Case (2): The solution $u(t,x) = \phi^-_{SS}(x,0)$ is a monotonic (decreasing) steady state with $\phi^-_{SS}(-\infty,0) = +1$ and $\phi^-_{SS}(+\infty,0) = -1$. As above, $u(t,x) = \phi^-_{SS}(x,0)$ is C^w-stable with $w(x) \equiv 1$.

Case (3): The solution $u(t,x) = \phi_{NS}(x,0)$ is a monotonic steady

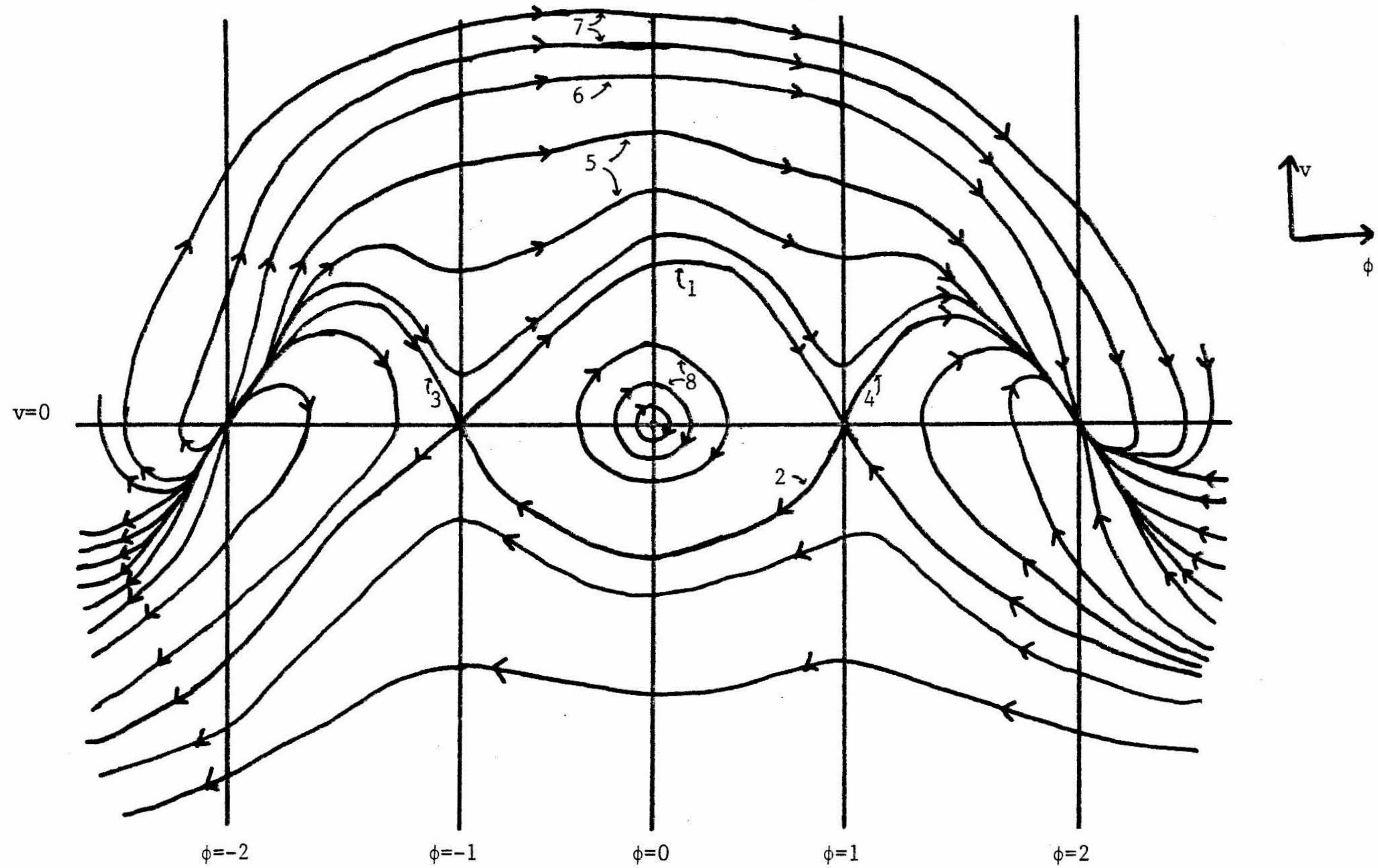


Figure 6: Phase plane of system (8.9) at c=0. The non-constant bounded solutions are (1) $\phi^{+}_{SS}(x,0)$, (2) ${}_{ac}\phi^{-}_{SS}(x,0)$, (3) $\phi_{NS}(x,0)$, (4) $\phi_{SN}(x,0)$, (5) the family $\phi_{NN}(x,0,\alpha)$, (6) $\phi_{NN}(x,0)$, (7) the family $\tilde{\phi}_{NN}(x,0,\alpha)$, and (8) the family $\phi_{co}(x,0,\alpha)$.

state which decays to the node $\phi = -2$, $v = 0$ at the usual rate as $x \to -\infty$ and which goes to the saddle point $\phi = -1$, $v = 0$ as $x \to +\infty$. Since $\phi_{NS}(x,0) \sim -2 + a(0)e^{2x}$ as $x \to -\infty$ (where $a(0) > 0$), $u(t,x) = \phi_{NS}(x,0)$ is C^w-stable with $w(x) = 1 + e^{-2x}$.

Case(4): The solution $u(t,x) = \phi_{SN}(x,0)$ is a monotonic steady state which goes to the saddle point $\phi = 1$, $v = 0$ as $x \to -\infty$ and which decays at the usual rate to the node $\phi = 2$, $v = 0$ as $x \to +\infty$. Since $\phi_{SN}(x,0) \sim 2 - a(0)e^{-2x}$ as $x \to +\infty$ (where $a(0) > 0$), $u(t,x) = \phi_{SN}(x,0)$ is C^w-stable with $w(x) \equiv 1 + e^{2x}$.

Case (5): The family $u(t,x,\alpha) = \phi_{NN}(x,0,\alpha)$ are all monotonic steady states which decay to the node $\phi = -2$, $v = 0$ at the usual rate as $x \to -\infty$ and which decay to the node $\phi = 2$, $v = 0$ at the usual rate as $x \to +\infty$. Thus,

$$\phi_{NN}(x,0,\alpha) \sim -2 + a(0,\alpha)e^{+2x} \quad \text{as} \quad x \to -\infty \quad \text{and}$$

$$\phi_{NN}(x,0,\alpha) \sim \quad 2 - b(0,\alpha)e^{-2x} \quad \text{as} \quad x \to +\infty$$

where $a(0,\alpha)$ and $b(0,\alpha)$ are some positive constants. Therefore, $u(t,x,\alpha) = \phi_{NN}(x,0,\alpha)$ is C^w-stable with $w(x) \equiv 1 + e^{-2x} + e^{+2x}$ for all α.

Case (6): The solution $u(t,x) = \phi^{ac}_{NN}(x,0)$ is a monotonic steady state which decays to the node $\phi = -2$, $v = 0$ at the accidental rate as $x \to -\infty$ and which decays to the node $\phi = +2$, $v = 0$ at the accidental rate as $x \to +\infty$. Thus,

$$\phi^{ac}_{NN}(x,0) \sim -2 + a(0)e^{6x} \quad \text{as} \quad x \to -\infty \quad \text{and}$$

$$\phi^{ac}_{NN}(x,0) \sim +2 - b(0)e^{-6x} \quad \text{as} \quad x \to +\infty$$

where $a(0)$ and $b(0)$ are some positive constants. Therefore, $u(t,x) = \phi^{ac}_{NN}(x,0)$ is C^w-stable with $w(x) \equiv 1 + e^{-6x} + e^{+6x}$.

Case (7): The family $u(t,x,\alpha) = \tilde{\phi}_{NN}(x,0,\alpha)$ are all non-monotonic steady states which have two relative extrema and which go to the node $\phi = -2$, $v = 0$ as $x \to -\infty$ and go to the node $\phi = 2$, $v = 0$ as $x \to +\infty$. Thus, $u(t,x,\alpha) = \tilde{\phi}_{NN}(x,0,\alpha)$ are very unstable for all α.

Case (8): The family $u(t,x,\alpha) = \phi_{co}(x,0,\alpha)$ are all periodic steady state solutions which are represented by the closed orbits around $\phi = 0$, $v = 0$. Since each steady state $u(t,x,\alpha) = \phi_{co}(x,0,\alpha)$ has an infinite number of relative extrema, it is very unstable.

Note that the phase plane at $c = 0$ is unusual in many respects. The wavespeed $c = 0$ is

(a) the unique wavespeed at which $S \to S$ waves exist,

(b) the unique wavespeed at which the solution which decays to the node $\phi = -2$, $v = 0$ at the accidental rate as $x \to -\infty$ also decays to the node $\phi = +2$, $v = 0$ at the accidental rate as $x \to +\infty$, and

(c) the unique wavespeed at which the limiting "member" of the monotonic $N \to N$ type solutions are the $N \to S$, $S \to S$, and $S \to N$ monotonic waves instead of being a $N \to S$ and a $S \to N$ wave.

We now consider wavespeeds c in $(0,8-\sqrt{48})$. As these speeds, $\phi = -2$, and $\phi = +2$ are still an unstable and a stable node, $\phi = -1$ and $\phi = +1$ are still saddle points, but $\phi = 0$ is now a stable spiral point. The phase plane of system (8.9) at any of these wavespeeds looks like the phase plane sketched in Figure (7) below. As labeled in the sketch, we will consider ten different types of bounded non-constant traveling wave solutions at each c. We now treat these cases separately.

Case (1): The solution $u(t,x,c) = \phi_{NS}(x-ct,c)$ is a monotonic wave which decays to the node $\phi = -2$, $v = 0$ at the usual rate as $x \to -\infty$

and which goes to the saddle point $\phi = -1$, $v = 0$ as $x \to +\infty$. It is correspondingly C^w-stable with $w(x) \equiv 1 + e^{-k_1^-(c)x}$ where

$$k_1^-(c) = +\tfrac{1}{2}\left[8-c-\sqrt{(c-8)^2-48}\right] . \tag{8.10}$$

Case (2): The solution $u(t,x,c) = \phi_{SN}(x-ct,c)$ is a monotonic wave which decays to the node $\phi = 2$, $v = 0$ at the usual rate as $x \to +\infty$ and which goes to the saddle point $\phi = 1$, $v = 0$ as $x \to -\infty$. Correspondingly, it is C^w-stable with $w(x) \equiv 1 + e^{-k_1^+(c)x}$ where

$$k_1^+(c) = -\tfrac{1}{2}\left[8+c-\sqrt{(c+8)^2-48}\right] . \tag{8.11}$$

Case (3): The solution $u(t,x,c) = \tilde{\phi}_{NS}(x-ct,c)$ is a monotonic wave which decays to the node $\phi = -2$, $v = 0$ at the usual rate as $x \to -\infty$ and which goes to the saddle point $\phi = 1$, $v = 0$ as $x \to +\infty$. Correspondingly, it is C^w-stable with $w(x) = 1 + e^{-k_1^-(c)x}$ where $k_1^-(c)$ is given in equation (8.10).

Case (4): The family $u(t,x,c,\alpha) = \phi_{NN}(x-ct,c,\alpha)$ are all monotonic waves which decay to the node $\phi = -2$, $v = 0$ at the usual rate as $x \to -\infty$ and which decay to the node $\phi = 2$, $v = 0$ at the usual rate as $x \to +\infty$. Correspondingly, they are C^w-stable with $w(x) \equiv 1 + e^{-k_1^-(c)x} + e^{-k_1^+(c)x}$ where $k_1^-(c)$ and $k_1^+(c)$ are defined in equations (8.10) and (8.11).

Case (5): The solution $u(t,x,c) = \phi_{NN}^{ac}(x-ct,c)$ is a monotonic wave which decays to the node $\phi = -2$, $v = 0$ at the accidental rate as $x \to -\infty$ and which decays to the node $\phi = 2$, $v = 0$ at the usual rate as $x \to +\infty$. Correspondingly, it is C^w-stable with $w(x) \equiv 1 + e^{-k_2^-(c)x} + e^{-k_1^+(c)x}$ where

$$k_2^-(c) = \tfrac{1}{2}\left[(8-c)+\sqrt{(c-8)^2-48}\right] . \tag{8.12}$$

Case (6): The family $u(t,x,c,\alpha) = \tilde{\phi}_{NN}(x-ct,c,\alpha)$ are all non-monotonic waves which have a <u>single</u> relative extrema, which go to the node

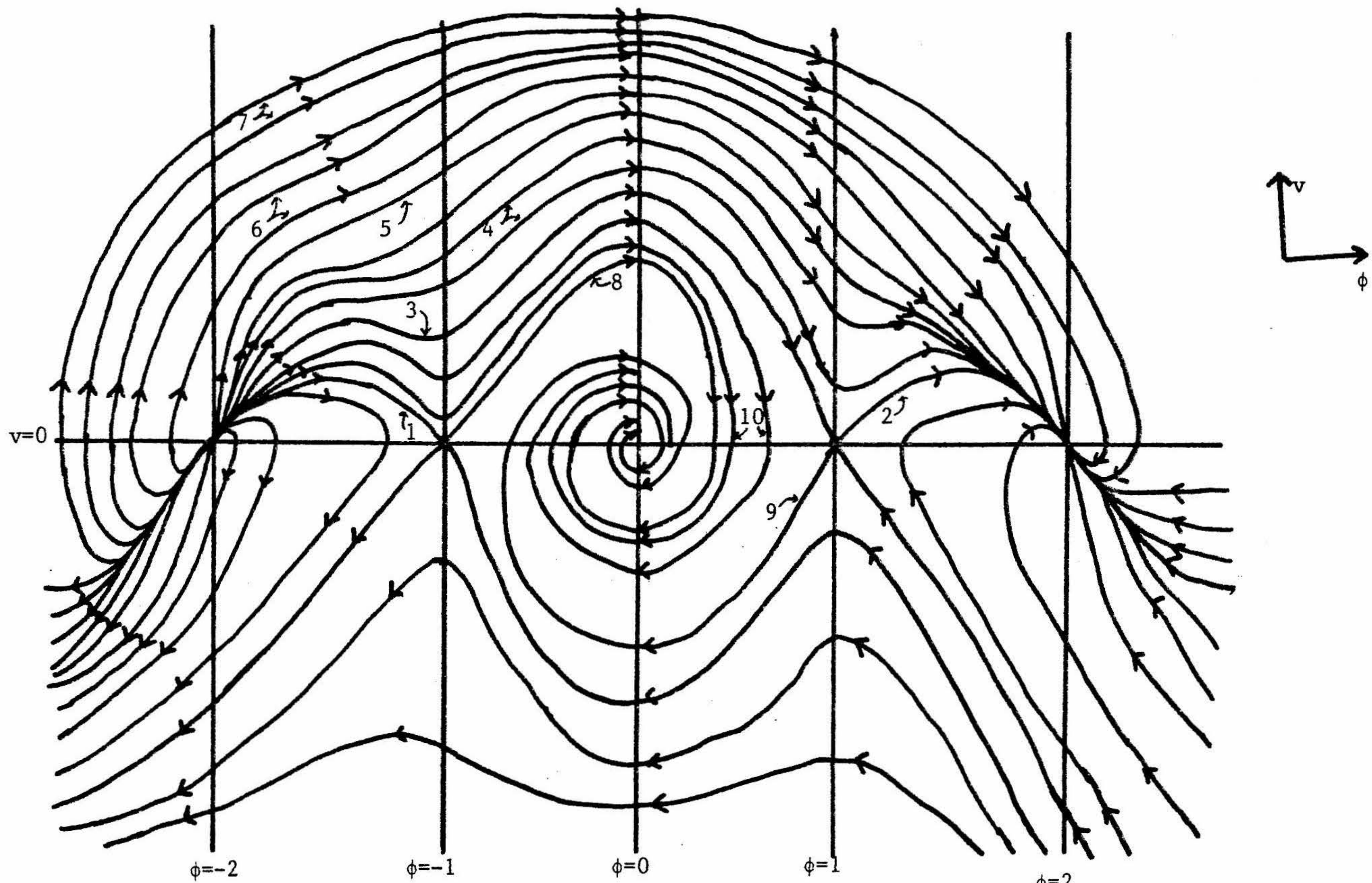


Figure 7: Phase plane of system (8.9) at c=0. The non-constant bounded solutions are (1) $\phi_{NS}(x,c)$, (2) $\phi_{SN}(x,c)$, (3) $\tilde{\phi}_{NS}(x,c)$, (4) the family $\phi_{NN}(x,c,\alpha)$, (5) $\phi_{NN}^{ac}(x,c)$, (6) the family $\tilde{\phi}_{NN}(x,c,\alpha)$, (7) the family $\tilde{\tilde{\phi}}_{NN}(x,c,\alpha)$, (8) $\phi_{SSp}(x,c)$, (9) $\tilde{\phi}_{SSp}(x,c)$, and (10) the family $\phi_{NSp}(x,c,\alpha)$.

$\phi = -2$, $v = 0$ as $x \to -\infty$, and which go to the node $\phi = 2$, $v = 0$ as $x \to +\infty$. This is the indeterminate case discussed in section (4.14). To determine whether these waves are C^w-stable or $\mathfrak{C}^w$-unstable (with $w(x) \equiv 1 + e^{-k_1^-(c)x} + e^{-k_1^+(c)x}$) one must determine their intersection properties.

Case (7): The family $u(t,x,c,\alpha) = \tilde{\tilde{\phi}}_{NN}(x-ct,c,\alpha)$ are all non-monotonic waves which have <u>two</u> relative extrema, which go to the node $\phi = -2$, $v = 0$ as $x \to -\infty$, and which go to the node $\phi = 2$, $v = 0$ as $x \to +\infty$. These are very unstable.

Case (8): The solution $u(t,x,c) = \phi_{SSp}(x-ct,c)$ goes to the saddle point $\phi = -1$, $v = 0$ as $x \to -\infty$ and goes to the spiral point $\phi = 0$, $v = 0$ as $x \to +\infty$. This wave has an infinite number of relative extrema and therefore is very unstable.

Case (9): The solution $u(t,x,c) = \tilde{\phi}_{SSp}(x-ct,c)$ goes to the saddle point $\phi = 1$, $v = 0$ as $x \to -\infty$ and goes to the spiral point $\phi = 0$, $v = 0$ as $x \to +\infty$. It has an infinite number of relative extrema and is therefore very unstable.

Case (10): The family $u(t,x,c,\alpha) = \phi_{NSp}(x-ct,c,\alpha)$ are all non-monotonic waves which go to the node $\phi = -2$, $v = 0$ as $x \to -\infty$ and which go to the spiral point $\phi = 0$, $v = 0$ as $x \to +\infty$. They all have an infinite number of relative extrema and are therefore all very unstable.

In summary, applications of the stability/instability results of Chapter IV immediately determine the stability or instability of each traveling wave and steady state solution, except the family of solutions $u(t,x,c,\alpha) = \tilde{\phi}_{NN}(x-ct,c,\alpha)$. These waves are part of the indeterminate case discussed in section (4.14). To determine the stability or instability of these waves, one needs to numerically discover whether the stability criterion of section (4.14) is satisfied.

The dependence of the $N \to S$ type monotonic wave $u(t,x,c) = \tilde{\phi}_{NS}(x-ct,c)$ on c is interesting. By comparing the phase planes in Figure (6) $(c = 0)$ and Figure (7) $(0 < c < 8 - \sqrt{48})$, one sees that as $c \to 0$ this $N \to S$ wave bifurcates into a $S \to S$ type wave and a $N \to S$ type wave. This bifurcation process is exactly as described in section (5.2) of Chapter V.

This completes our treatment of equation (8.8). In the next section we will utilize the results of Chapter VI to examine a specific equation which contains an integral.

8.4 A delayed Fischer's equation. In this section we briefly examine the equation

$$u_t = u_{xx} + \mu\cdot\int_0^T se^{-s/\Delta}\, u(t-s,x)ds - u^2 \quad , \tag{8.13}$$

where $\Delta > 0$, $T/\Delta >> 1$, and $\mu = \{\Delta^2[1-e^{-T/\Delta}-T/\Delta\; e^{-T/\Delta}]\}^{-1}$. Essentially this equation is a logistics equation with the growth term delayed and diffusion added. As such, it can provide a model of simple population processes. Note that this equation is not formally included in the class of equations treated in Chapter VI. Equation (8.13) has only the single integral $\int_0^T ds$, and in Chapter VI we treated equations which contained only double integrals $\int_0^T\int dyds$. This is not a difficulty since (as noted in Chapter VI) all the results pertaining to equations containing double integrals remain valid for equations containing a single integral.

We now examine the traveling wave and steady state solutions $u(t,x) = \phi(x-ct)$ of (8.13). These must solve

$$\phi_{xx} + c\phi_x + \mu\int_0^T se^{-s/\Delta}\phi(x+cs)ds - \phi^2 = 0 \quad . \tag{8.14}$$

Clearly the singular points of equation (8.14) are $\phi = 0$ and $\phi = 1$. According to the definitions in Chapter VI, the point $\phi = 0$ is a node and the point $\phi = 1$ is a saddle point. Thus the only constant traveling waves are

(1) $u(t,x) \equiv \phi_0(x-ct) \equiv 0$, which is $\mathfrak{C}^w$-unstable with $w(x) \equiv 1 + e^{-\kappa x} + e^{+\kappa x}$ for any $\kappa > 0$ sufficiently small, and

(2) $u(t,x) \equiv \phi_1(x-ct) \equiv 1$, which is C^w-stable with $w(x) \equiv 1$.

Finding monotonic non-constant solutions $u(t,x) = \phi(x-ct)$ of equation (8.13) is difficult. However, as $\Delta \to 0$ equation (8.13) goes to Fischer's equation. One therefore expects that at any specific c with $c < -2$ there is a monotonic $N \to S$ type solution $u(t,x,c,\Delta) \equiv \phi_{NS}(x-ct,c,\Delta)$ with $\phi_{NS}(-\infty,c,\Delta) = 0$, with $\phi_{NS}(+\infty,c,\Delta) = 1$, and continuous in Δ, for all $\Delta > 0$ sufficiently small. Presumably one could use a contraction argument to prove the existence of these solutions. We will not do this. Instead we will assume that these solutions exist, that they decay to $\phi = 0$ at the usual rate as $x \to -\infty$, and that they decay exponentially to $\phi = 1$ as $x \to +\infty$. These assumptions will allow us to apply our stability results.

Assume that $u(t,x,c,\Delta) \equiv \phi_{NS}(x-ct,c,\Delta)$ is a $N \to S$ type monotonic solution of (8.13) with the properties described above. From equation (8.14) we see that

$$\phi_{NS}(x,c,\Delta) \sim a(c,\Delta)e^{k_1(c,\Delta)} \text{ as } x \to -\infty$$

where $k_1(c,\Delta)$ is the smallest positive root of

$$k^2 + ck + \mu \cdot \int_0^T e^{-s/\Delta} e^{kcs} ds = 0 \quad , \qquad (8.15)$$

and where $a(c,\Delta)$ is a positive constant. From theorem (6.5) we can conclude that $u(t,x,c,\Delta) = \phi_{NS}(x-ct,c,\Delta)$ is C^w-stable with

$w(x) \equiv 1 + e^{-k_1(c,\Delta)x}$. Moreover, if $u(t,x,c,\Delta) \equiv \phi_{NS}(x-ct,c,\Delta)$ is a solution of (8.14) then so is $u(t,x,c,\Delta) \equiv \phi_{SN}(x-(-c)t,-c,\Delta)$, where the $S \to N$ type monotonic wave ϕ_{SN} is defined by

$$\phi_{SN}(x-ct,c,\Delta) = \phi_{NS}(-x-(-c)t,-c,\Delta) \quad .$$

From theorem (6.5) we conclude that $u(t,x,c,\Delta) \equiv \phi_{SN}(x-ct,c,\Delta)$ is C^w-stable with $w(x) \equiv 1 + e^{k_1(-c,\Delta)x}$.

We now summarize this example. We found that the only singular points are $\phi = 0$ and $\phi = 1$. Correspondingly, the only constant solutions are

(1) $u(t,x) \equiv \phi_0(x-ct) \equiv 0$ which is C^w-unstable with $w(x) \equiv 1 + e^{\kappa x} + e^{-\kappa x}$ for any $\kappa > 0$ sufficiently small, and

(2) $u(t,x) \equiv \phi_1(x-ct) \equiv 1$ which is C^w-stable with $w(x) \equiv 1$. We assumed that for each $c < -2$ there exists a bounded monotonic $N \to S$ type solution $u(t,x,c,\Delta) = \phi_{NS}(x-ct,c,\Delta)$ for all $\Delta > 0$ sufficiently small. We also assumed that these solutions decay to $\phi = 0$ like $e^{k_1(c,\Delta)x}$ as $x \to -\infty$. Then we found that

(3) the $N \to S$ wave $u(t,x,c,\Delta) \equiv \phi_{NS}(x-ct,c,\Delta)$ is C^w-stable with $w(x) \equiv 1 + e^{-k_1(c,\Delta)x}$ for $c < -2$, and that

(4) the $S \to N$ wave $u(t,x,c,\Delta) \equiv \phi_{NS}(-x+ct,-c,\Delta)$ is C^w-stable with $w(x) \equiv 1 + e^{k_1(-c,\Delta)x}$ for $c > 2$.

This completes our brief look at equation (8.13). In the next section we illustrate the results of Chapter VII by examining a system of equations which arises in chemical reaction theory.

<u>8.5 A reaction-diffusion system.</u> In this section we briefly analyze the system of equations

$$
\begin{aligned}
R_t &= R_{xx} - R\phi_x^2 + R - R^3 \\
R\phi_t &= R\phi_{xx} + 2R_x\phi_x \quad .
\end{aligned}
\tag{8.16}
$$

These equations can also be written as

$$
\begin{aligned}
U_t &= U_{xx} + (1-U^2-V^2)U \quad , \quad U = R\cos(\phi-\delta) \quad , \\
V_t &= V_{xx} + (1-U^2-V^2)V \quad , \quad V = R\sin(\phi-\delta) \quad ,
\end{aligned}
\tag{8.17}
$$

where δ is any constant. System (8.16) arises as the modulation equations of some reaction diffusion systems which are near bifurcation points [11]. System (8.17) provides an interesting example because it does not satisfy the prerequisites of the "general" theory of Chapter VII. Even though the results of Chapter VII do not apply, we will be able to find the stability or instability of all traveling wave solutions by utilizing the techniques developed in Chapters IV and VII.

To begin, let us note that system (8.17) is parabolic (satisfies hypothesis H3) only when $UV \leq 0$. Thus, whenever $UV > 0$ we will be unable to use the maximum principle. For each wavespeed c we will find the bounded solutions $U = U(x-ct)$, $V = V(x-ct)$ of system (8.17). We will then find their stability or instability.

First suppose that $c \leq -2$. To find the solutions $U(x-ct)$, $V(x-ct)$ of system (8.16), we will solve for the solutions $R(x-ct)$, $\phi(x-ct)$ of system (8.16). Therefore, consider

$$R'' - R\phi'^2 + cR' + R - R^3 = 0 \tag{8.18a}$$

$$R\phi'' + 2R'\phi' + cR\phi' = 0 \quad . \tag{8.18b}$$

From equation (8.18b) we find that if R is bounded then ϕ' must either be identically zero or must grow exponentially as $x \to \pm\infty$. Clearly the acceptable solutions are only those with $\phi' \equiv 0$. Equation (8.18a) now reduces to

$$R' = S$$
$$S' = -cS - R + R^3 \quad . \tag{8.19}$$

The phase plane of (8.19) is sketched in Figure (8) below for $c \le -2$. The bounded solutions are

(1) the constant traveling wave solution $R(t,x) \equiv R_0(x-ct,c) \equiv 0$,

(2) the constant traveling wave solution $R(t,x) \equiv R_1(x-ct,c) \equiv 1$,

and

(3) the monotonic traveling wave solution $R(t,x) = R_{NS}(x-ct,c)$

labeled by a * in Figure (8).

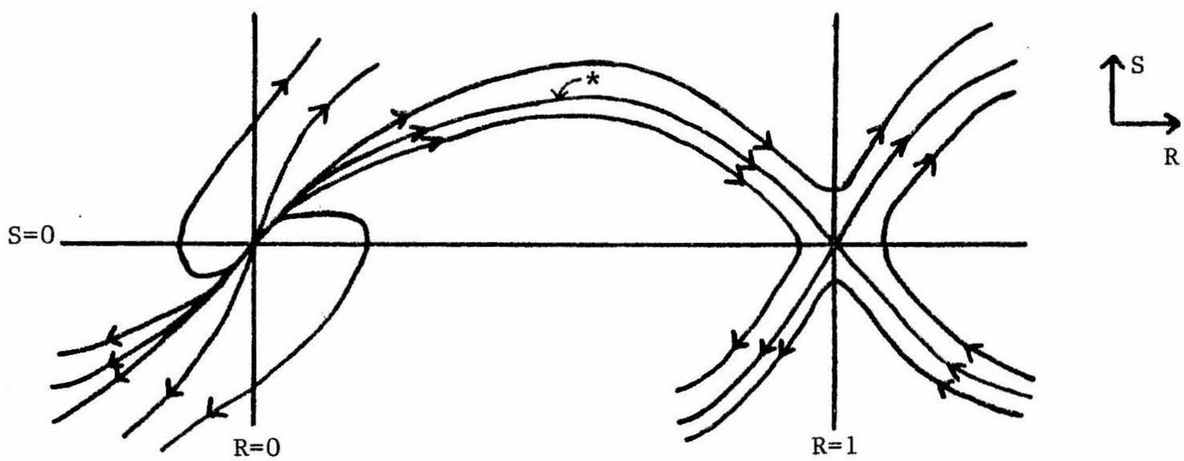


Figure (8): Phase plane representation of system (8.19) for $c \le -2$. Trajectory (*) represents the monotonic N→S solution R_{NS}.

With $\phi \equiv$ constant, system (8.16) reduces to the single equation

$$R_t = R_{xx} + R - R^3 \quad . \tag{8.20}$$

Clearly, since the constant traveling wave $R(t,x) = R_0(x-ct,c) \equiv 0$ is a

$\not{C}^w$-unstable solution of (8.20) with $w(x) \equiv 1 + e^{\kappa x} + e^{-\kappa x}$ (for any $\kappa > 0$ sufficiently small), $R(t,x) \equiv R_0(x-ct,c) \equiv 0$, ϕ = constant is a $\not{C}^w$-unstable solution of system (8.16) and $U(t,x) \equiv R_0(x-ct,c) \cos \phi \equiv 0$, $V(t,x) \equiv R_0(x-ct,c) \sin \phi \equiv 0$ is a $\not{C}^w$-unstable solution of system (8.17) with the same w.

We now consider the monotonic $N \to S$ type traveling wave solution $R(t,x) = R_{NS}(x-ct,c)$. Recall that at each c lemma (4.3) constructs upper functions $R(t,x) = \overline{R}(t,x-ct,c,\alpha)$ and lower functions $R(t,x) = \underline{R}(t,x-ct,c,\alpha)$ for all $0 < \alpha \leq \alpha_0(c)$ for some $\alpha_0(c) > 0$. These functions therefore satisfy

$$\begin{aligned} &\overline{R}_t - \overline{R}_{xx} - \overline{R} + \overline{R}^3 \geq 0 \\ &\underline{R}_t - \underline{R}_{xx} - \underline{R} + \underline{R}^3 \leq 0 \quad . \end{aligned} \tag{8.21}$$

We now examine the solutions R_{NS} and the implications of the upper and lower functions in terms of U and V.

In terms of U and V, the $N \to S$ type monotonic solution is

$$U(t,x) = U_{NS}(x-ct,c) \equiv R_{NS}(x-ct,c) \cos(\phi-\delta)$$

$$V(t,x) = V_{NS}(x-ct,c) \equiv R_{NS}(x-ct,c) \sin(\phi-\delta)$$

where ϕ and δ are arbitrary constants. Let us first choose $\delta = \phi + \pi/4$. Then the $N \to S$ type solutions are

$$U(t,x) = U_{NS}(x-ct,c) = \frac{1}{\sqrt{2}} R_{NS}(x-ct,c)$$

$$V(t,x) = V_{NS}(x-ct,c) = -\frac{1}{\sqrt{2}} R_{NS}(x-ct,c) \quad .$$

From the formulas in lemma (4.3) and a short calculation we learn that

$\overline{U}(t,x,c,\alpha) \equiv \frac{1}{\sqrt{2}} \overline{R}(t,x-ct,c,\alpha)$, $\overline{V}(t,x,c,\alpha) \equiv -\frac{1}{\sqrt{2}} \underline{R}(t,x-ct,c,\alpha)$ and

$\underline{V}(t,x,c,\alpha) \equiv \frac{1}{\sqrt{2}} \underline{R}(t,x-ct,c,\alpha)$, $\underline{V}(t,x,c,\alpha) \equiv -\frac{1}{\sqrt{2}} \overline{R}(t,x-ct,c,\alpha)$

are upper and lower functions (respectively) of system (8.16) for all $0 < \alpha \leq \tilde{\alpha}_0(c)$ for some $\tilde{\alpha}_0(c) > 0$. Moreover,

$$\overline{U}(t,x,c,\alpha) > \underline{U}(t,x,c,\alpha) > 0 \quad \text{and} \quad 0 > \overline{V}(t,x,c,\alpha) > \underline{V}(t,x,c,\alpha) \quad ,$$

and so we can apply the maximum principle. The maximum principle immediately implies that if $U(t,x)$, $V(t,x)$ is <u>any</u> solution of system (8.16) whose initial condition $U(0,x)$, $V(0,x)$ is smooth and satisfies

$$\frac{1}{\sqrt{2}}\,\underline{R}(0,x,c,\alpha) \leq U(0,x) \leq \frac{1}{\sqrt{2}}\,\overline{R}(0,x,c,\alpha)$$

$$-\frac{1}{\sqrt{2}}\,\overline{R}(0,x,c,\alpha) \leq V(0,x) \leq -\frac{1}{\sqrt{2}}\,\underline{R}(0,x,c,\alpha)$$

for all x and some α in $(0,\tilde{\alpha}(c))$, then $U(t,x)$, $V(t,x)$ must satisfy

$$\begin{aligned} \frac{1}{\sqrt{2}}\,\underline{R}(t,x-ct,c,\alpha) &\leq U(t,x) \leq \frac{1}{\sqrt{2}}\,\overline{R}(t,x-ct,c,\alpha) \\ -\frac{1}{\sqrt{2}}\,\overline{R}(t,x-ct,c,\alpha) &\leq V(t,x) \leq \frac{1}{\sqrt{2}}\,R(t,x-ct,c,\alpha) \end{aligned} \tag{8.21}$$

for all x and all $t \geq 0$. From the explicit expressions for $\overline{R}$ and $\underline{R}$ in lemma (4.3) and from the fact that

$$R_{NS} \sim a(c)e^{k_1(c)x} \text{ as } x \to -\infty$$

(where $a(c)$ is some positive constant and

$$k_1(c) \equiv +\tfrac{1}{2}\cdot[-c+\sqrt{c^2-4}]) \quad , \tag{8.22}$$

we see that (8.21) implies that the solution

$$\begin{aligned} U(t,x) &= \frac{1}{\sqrt{2}}\,R_{NS}(x-ct,c) \\ V(t,x) &= -\frac{1}{\sqrt{2}}\,R_{NS}(x-ct,c) \end{aligned} \tag{8.23}$$

is C^w-stable with $w(x) \equiv 1 + e^{-k_1(c)x}$. Moreover, system (8.16) is invariant to the addition of a constant to ϕ and equivalently, system (8.17) is invariant under transformation

$$\begin{pmatrix} U \\ V \end{pmatrix} \rightarrow \begin{pmatrix} \cos\delta' & -\sin\delta' \\ \sin\delta' & \cos\delta' \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

Therefore, each traveling wave solution

$$U(t,x) = U_{NS}(t,x,c,\delta) \equiv R_{NS}(x-ct,c) \cos \delta$$

$$V(t,x) = V_{NS}(t,x,c,\delta) \equiv R_{NS}(x-ct) \sin \delta$$

is also C^w-stable with $w(x) \equiv 1 + e^{-k_1(c)x}$.

There remains only one further class of solutions to examine. These are the constant traveling wave solutions

$$U(t,x) = U\ (t,x,c,\delta) \equiv \cos \delta$$

$$V(t,x) = V\ (t,x,c,\delta) \equiv \sin \delta .$$

By arguments similar to those used above, one can show that these solutions are C^w-stable with $w(x) \equiv 1$.

We now suppose that $-2 < c < 0$. As in the preceding case, all acceptable solutions $R = R(x-ct)$, $\phi = \phi(x-ct)$ must have $\phi' \equiv 0$, and so $R = R(x-ct)$ must satisfy

$$\begin{aligned} R_x &= S \\ S_x &= -cS - R + R^3 \end{aligned} \tag{8.19}$$

as in the preceding case. The phase plane of (8.19) for $-2 < c < 0$ is sketched in Figure (9) below. We see that the bounded solutions are

(1) the constant traveling wave solution $R(t,x) \equiv R_0(x-ct,c) \equiv 0$,

(2) the constant traveling wave solution $R(t,x) \equiv R_1(x-ct,c) \equiv 1$,

and

(3) the non-monotonic traveling wave solution $R(t,x) \equiv R_{SpS}(x-ct,c)$ labeled by a * in Figure (9).

With $\phi \equiv$ constant, system (8.16) reduces to the single equation

$$R_t = R_{xx} + R - R^3 \tag{8.20}$$

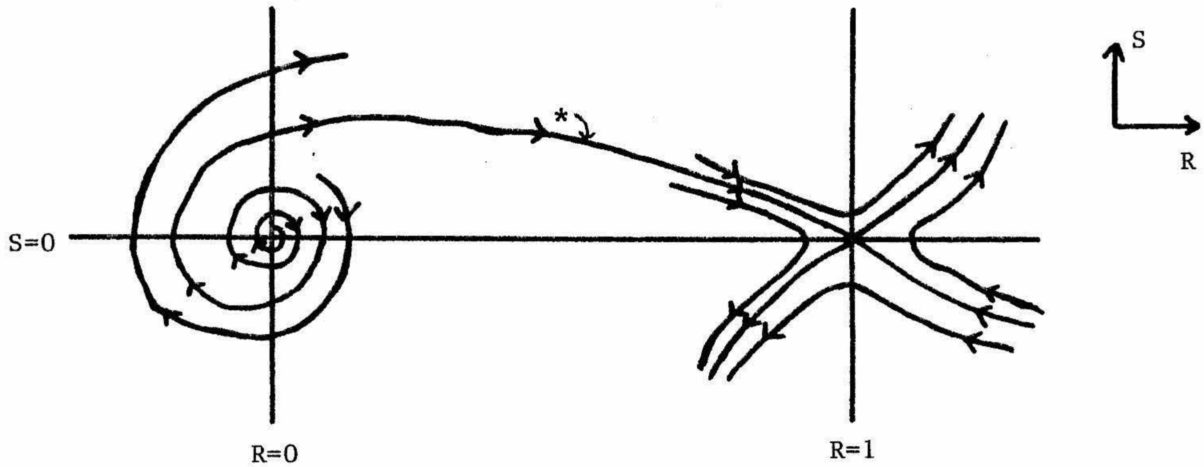


Figure (9): The phase plane of system (8.19) for $-2<c<0$. Trajectory (*) represents the non-monotonic solution R_{SpS}.

Consider the solution $R(t,x) = R_{SpS}(x-ct,c)$, $\phi(t,x) = \phi_0$ = a constant of system (8.16). Let us examine the effect of initially perturbing R only. We see that if we initially perturb R only then $\phi \equiv \phi_0$ for all x and all $t \geq 0$, and so R will be governed by equation (8.20). However, theorem (4.6) shows that $R_{SpS}(x-ct,c)$ is a very unstable solution of (8.20). Thus, the solution

$$R(t,x) = R_{SpS}(x-ct,c) \qquad \phi(t,x) \equiv \phi_0$$

of system (8.16) is very unstable. Correspondingly, the solution

$$U(t,x) = R_{SpS}(x-ct,c) \cos \phi_0 \qquad V(t,x) = R_{SpS}(x-ct,c) \sin \phi_0$$

of system (8.17) is also very unstable.

By a similar argument, one can show that the solution

$$R(t,x) = R_0(x-ct,c) \equiv 0 \qquad \phi(t,x) \equiv \phi_0$$

is a very unstable solution of system (8.16), and that the solution

$$U(t,x) = U_0(x-ct,c) \equiv 0 \qquad V(t,x) = V_0(x-ct,c) \equiv 0$$

is a very unstable solution of system (8.17).

Finally, as in the $c \leq -2$ case, the solution

$$R(t,x) = R_1(x-ct,c) \equiv 1 \qquad \phi(t,x) \equiv \phi_0$$

of system (8.16) and the solution

$$U(t,x) \equiv U_1(x-ct,c) \equiv \cos\phi_0 \qquad V(t,x) \equiv V_1(x-ct,c) \equiv \sin\phi_0$$

of system (8.17) are both C^w-stable with $w(x) \equiv 1$. This completes the stability picture for the $-2 < c < 0$ case.

The next wavespeed to examine is $c = 0$. However, all non-trivial steady state solutions $U(t,x) \equiv U(x)$, $V(t,x) \equiv V(x)$ either have $UV > 0$ for some x or have $UV \equiv 0$ for all x. The system (8.17) is not parabolic when $UV > 0$ for any x, and so we cannot utilize the maximum principle at $c = 0$. Therefore the stability of the steady state solutions of system (8.16) and (8.17) remains unresolved.

Finally, note that we do not need to consider solutions $U(x-ct,c)$, $V(x-ct,c)$ with $c > 0$. These solutions can be converted into solutions traveling with wavespeed $-c$ by utilizing the transformation $x \to -x$.

We now summarize this example. We considered the system of equations

$$\begin{aligned} U_t &= U_{xx} + (1-U^2-V^2)U \\ V_t &= V_{xx} + (1-U^2-V^2)V \end{aligned} \qquad (8.17)$$

We examined the stability of the traveling wave solutions $U(x-ct,c)$, $V(x-ct,c)$ for $c \leq 0$. When $c \leq -2$ we found that

(1) the solution $U(t,x) = U_0(x-ct,c) \equiv 0$, $V(t,x) = V_0(x-ct,c) \equiv 0$ is $\mathcal{C}^w$-unstable with $w(x) \equiv 1 + e^{\kappa x} + e^{-\kappa x}$ for all $\kappa > 0$ sufficiently small,

(2) the solution $U(t,x) = U_1(x-ct,c,\phi_0) \equiv \cos\phi_0$, $V(t,x) = V_1(x-ct,c,\phi_0) \equiv \sin\phi_0$ (where ϕ_0 is any constant) is C^w-stable with $w(x) \equiv 1$, and

(3) the solution $U(t,x) \equiv R_{NS}(x-ct,c)\cos\phi_0$, $V(t,x) = R_{NS}(x-ct,c) \sin\phi_0$ (where ϕ_0 is any constant and $R_{NS}(x-ct,c)$ is the solution of (8.19) which goes to the node $R = 0$ as $x \to -\infty$ and goes to the saddle point $R = 1$ as $x \to +\infty$) is C^w-stable with $w(x) \equiv 1 + e^{-k_1(c)x}$ where $k_1(c) \equiv \frac{1}{2}[-c+\sqrt{c^2-4}\,]$.

When $-2 < c < 0$ we found that

(1) the solution $U(t,x) = U_0(x-ct,c) \equiv 0$, $V(t,x) = V_0(x-ct,c) \equiv 0$ is very unstable,

(2) the solution $U(t,x) = U_1(x-ct,c,\phi_0) \equiv \cos\phi_0$, $V(t,x) = V_1(x-ct,c,\phi_0) \equiv \sin\phi_0$ (where ϕ_0 is any constant) is C^w-stable with $w(x) \equiv 1$, and

(3) the solution $U(t,x) = R_{SpS}(x-ct,c)\cos\phi_0$, $V(t,x) = R_{SpS}(x-ct,c) \sin\phi_0$ (where ϕ_0 is any constant and $R_{SpS}(x-ct,c)$ is the solution of (8.19) which goes to the spiral point $R = 0$ as $x \to -\infty$ and goes to the saddle point $R = 1$ as $x \to +\infty$) is very unstable.

For $c = 0$ we were not able to resolve the stability or instability of the solutions $U(t,x) \equiv U(x)$, $V(t,x) \equiv V(x)$. Finally, solutions $U(t,x) \equiv U(x-ct)$, $V(t,x) \equiv V(x-ct)$ with $c > 0$ can be reduced to solutions with $c < 0$ by employing the transformation $x \to -x$.

We conclude that even though system (8.17) does not satisfy the prerequisites of the general theory developed in Chapter VII, the techniques of the preceding chapters are still useful for determining the stability and instability of traveling waves.

This completes our analysis of this example. We conclude this chapter with some general remarks in the next section.

8.6 Some general remarks. In this chapter we applied the techniques and results of preceding chapters to several examples. We first looked at Burger's equation

$$u_t = u_{xx} - uu_x \quad . \tag{8.1}$$

This equation is representative of the class of equations $u_t = f(u_{xx}, u_x, u)$ where $f(0,0,u) \equiv 0$, and the weakness of the stability results of theorem (4.5) points out a shortcoming of our stability theorems for this class of equations. However, since we were able to obtain sharp stability results by using the techniques of Chapter IV, the weakness is only in the theorems and not in the approach we use. We also used our techniques to show that the "single hump" solutions of Burger's equation also have at least a limited stability. Thus Burger's equation illustrates how our techniques can be used to find the stability of some unsteady solutions.

We next examined Fischer's equation,

$$u_t = u_{xx} + u(1-u) \quad . \tag{8.6}$$

For this equation the theorems in Chapter IV immediately yielded sharp stability and instability results for every bounded traveling wave and steady state.

In section (8.3) we examined the equation

$$u_t = u_{xx} + 4uu_x + \tfrac{1}{2}\cdot u(u-1)(u+1)(u-2)(u+2) \quad . \tag{8.8}$$

Again, the results of Chapter IV immediately yielded sharp stability and instability results for almost every bounded traveling wave and steady state we examined. However, we found some non-monotonic $N \to N$ type solutions which each have a single relative extrema for this equation. These solutions are part of the indeterminate case treated in section (4.14). To find their stability, one must numerically determine whether the stability criterion of section (4.14) is satisfied.

In section (8.4) we examined a delayed Fischer's equation,

$$u_t = u_{xx} + \mu \cdot \int_0^T se^{-s/\Delta}\, u(t-s,x)ds - u^2 \quad . \tag{8.13}$$

Once monotonic (and constant) traveling wave and steady state solutions were found, it was easy to determine their stability by applying the results of Chapter VI. However, finding traveling wave and steady state solutions of equations containing integrals is generally difficult.

Our last example was the reaction diffusion system

$$\begin{aligned} R_t &= R_{xx} - R\phi_x^2 + R - R^3 \\ R\phi_t &= R\phi_{xx} + 2R_x\,\phi_x \quad , \end{aligned} \tag{8.16}$$

which can also be written as

$$\begin{aligned} U_t &= U_{xx} + (1-U^2-V^2)U \ , \ U = R\cos\phi \quad , \\ V_t &= V_{xx} + (1-U^2-V^2)V \ , \ V = R\sin\phi \end{aligned} \tag{8.17}$$

Even though this system does not satisfy the prerequisites of the general theory developed in Chapter VII, we were able to utilize the techniques of preceding chapters to determine the stability or instability of traveling wave solutions with $c \neq 0$. Thus, this example illustrates that the techniques we use are more powerful than the theorems we have developed.

This concludes this chapter of examples. We complete our presentation in the next chapter, where we discuss the material in the preceding chapters in general terms.

Chapter IX

CONCLUSIONS, CONJECTURES, AND CRITICAL REMARKS

In this final short chapter we will discuss the material contained in the previous chapters in broad terms. In section (9.1) we discuss the strengths of the results, and in section (9.2) we discuss some of the weaknesses. Finally, in the last section, section (9.3), we suggest some interesting areas for future research.

9.1 Strengths.

The main strengths of the results in preceding chapters are the largeness of the class of physical problems that can be treated, the ease of applying almost all the results, and the simplicity of the results. We now will briefly discuss each of these features separately.

The material in Chapter IV treats equations of the form $u_t = f(u_{xx}, u_x, u)$. Thus, the material in Chapter IV can be directly used to treat physical models which contain linear or nonlinear diffusion, transport, and source terms as shown below:

$$\underset{\substack{\text{growth/}\\ \text{decay}}}{u_t} = \underset{\substack{\text{linear or}\\ \text{nonlinear}\\ \text{diffusion}}}{(a(u)u_x)_x} + \underset{\substack{\text{transport}\\ \text{term}}}{c(u)u_x} + \underset{\substack{\text{source}\\ \text{term}}}{h(u)} \quad . \tag{9.1}$$

With the material in Chapter VI, we can treat physical models governed by equations like (9.1) with integral terms $\int_0^T \int_{|y|<Y} G(s,y,u(t-s,x-y))dyds$ included. However, these integral terms must be required to contribute positively to u_t when u is increased. Finally, with the material in Chapter VII we can treat physical models governed by a special type of systems of equations. By using the results of Chapters IV, V, VI, and VII, one can treat many physically interesting equations.

As demonstrated in Chapter VIII, almost all the results obtained in the preceding chapters are easy to apply. If one assumes the appropriate hypotheses, then the stability, instability, and mean wavespeed results depend only on easily determined quantities. The exception is stability in the indeterminate case treated in section (4.14), which can generally only be determined numerically.

Finally, most of the results contained in the preceding chapters are very simple. The stability or instability of any wave depends only on a few fundamental properties of the wave, such as the number of relative extrema and the nature of the wave at $x = \pm\infty$. Thus, the results demonstrate the generic nature of the stability of wave solutions to parabolic equations.

This completes our look at the strengths of the results. In the next section we briefly examine the weaknesses of the results.

9.2 Weaknesses. In this section we briefly examine the weaknesses of the results contained in preceding chapters. Basically these weaknesses are the unverifiable existence assumptions, the modification of the equations, and the incompleteness of some of the results. We will now briefly discuss each of these drawbacks.

The unverifiable nature of the existence assumptions H4 and H5 is a mathematical weakness of the results in earlier chapters. One can prove that solutions to the initial value problem for nonlinear parabolic equations exist in simple cases. However, for most equations one can only assume that solutions to the initial value problem exist.

The second weakness of the results contained in previous chapters is that they pertain to the modified equation; not to the original. However, this is not a serious weakness. One can view the results of previous chapters as determining the behavior of $u(t,x)$ for all $t \geq 0$ for different classes of initial conditions $u(0,x)$, and then interpreting this behavior in terms of stability and mean wavespeeds. Any solution $u(t,x)$ of the modified equation which has $|u|$, $|u_x|$, and $|u_{xx}|$ all smaller than M (where M is the arbitrarily large modification constant) is also the solution of the original equation. Thus we see that any solution of the original equation which has $|u|$, $|u_x|$, and $|u_{xx}|$ bounded will behave exactly as determined by our results about the modified equations. Moreover, for all $t > 0$ until $|u|$, $|u_x|$, or $|u_{xx}|$ becomes unbounded, any solution of the original equation must behave as predicted by the results in preceding chapters.

The other weaknesses of the results come from their incompleteness in some cases. We now discuss some of the more important of these weaknesses. First, the stability results for monotonic waves $\phi(x-ct)$ of Chapter IV do not distinguish between $\phi(-\infty)$ or $\phi(+\infty)$ being a higher order saddle point (which is a weakly stable constant steady state), being a singular point ϕ_0 with $f(0,0,\phi_0+\eta) = 0$ for all η near 0 (which is a neutrally stable constant steady state), and being a node (which is an unstable constant steady state). This suggests that a general improvement of our stability results for monotonic waves can be made when $\phi(-\infty)$ or $\phi(+\infty)$ is a higher order saddle point or a singular point with $f(0,0,\phi+\eta) \equiv 0$ for all η near 0. From our treatment of Burger's equation in section (8.1), we see that such an improvement would have significant applications.

There is also room for improvement in the indeterminate case treated in section (4.14). This case is the only case in Chapter IV where the stability or instability of a solution $\phi(x-ct)$ cannot be determined by inspection.

The lack of stability or instability results for non-monotonic solutions $\phi(x-ct)$ in Chapter VI and non-monotonic solutions $\tilde{\phi}(x-ct)$ in Chapter VII is a major drawback of the results for equations containing integrals and for systems of equations. Because of this lack, we cannot determine the stability of the majority of solutions $\phi(x-ct)$ and $\tilde{\phi}(x-ct)$.

Finally, the stability results developed in Chapter VII have been established only for a restricted class of parabolic systems. The narrowness of this class diminishes the utility of the results.

This completes this section, where we have touched on the major weaknesses of the results contained in the preceding chapters. In the final section we point out some interesting topics for future research.

9.3 Some potential research topics. In the preceding chapters many interesting topics for research arose which we did not pursue. In this final section we would like to suggest some of the more interesting of these topics.

First, general improvements in the stability results for monotonic waves $\phi(x-ct)$ in cases where either $\phi(-\infty)$ or $\phi(+\infty)$ is not a first order singular point should be possible. Of these cases, the physically most interesting case is when the equation $u_t = f(u_{xx},u_x,u)$ satisfies $f(0,0,\phi) \equiv 0$ for all ϕ.

Second, it would be very nice to resolve the indeterminate case treated in section (4.14). For example, if one could show that all solutions in the indeterminate case are unstable, then the stability of every solution $\phi(x-ct)$ of any equation $u_t = f(u_{xx}, u_x, u)$ would be determinable by inspection.

Third, existence and non-existence results for traveling wave solutions of equations containing integrals and of systems of equations (analogous to theorems (5.1)) would be interesting. One use of these results would be to establish the sharpness of the stability results in theorem (6.5) and theorem (7.5).

Finally, significant extensions to the class of systems that can be treated would be useful. The main limitation on the utility of the results in Chapter VII is the restrictiveness of the class of systems treated there.

This completes this chapter, where we have remarked on some of the strengths and weaknesses of our results and have touched on potential areas of research. In conclusion, we observe that many potential results remain which seem interesting and obtainable.

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